Assignment 8

5.1 – The Properties of Determinants: 1, 2, 3, 8, 13

Questions 1–12 are about the rules for determinants.

1 If a 4 by 4 matrix has $\det A = \frac{1}{2}$, find $\det(2A)$ and $\det(-A)$ and $\det(A^2)$ and $\det(A^{-1})$.

$\det(2A) = 2^4 \cdot \frac{1}{2}$ $\boxed{= 8}$

$\det(-A) = -1^4 \cdot \frac{1}{2}$ $\boxed{= \frac{1}{2}}$

$\det(A^2) = (\frac{1}{2})^2$ $\boxed{= \frac{1}{4}}$

$\det(A^{-1}) = \frac{1}{\frac{1}{2}}$ $\boxed{= 2}$

2 If a 3 by 3 matrix has $\det A = -1$, find $\det(\frac{1}{2}A)$ and $\det(-A)$ and $\det(A^2)$ and $\det(A^{-1})$.

$\det(\frac{1}{2}A) = \frac{1}{2}^3 \cdot -1$

$= -\frac{1}{8}$

$\det(-A) = -1^3 \cdot -1$ $\boxed{= 1}$

$\det(A^2) = (-1)^2$ $\boxed{= 1}$

$\det(A^{-1}) = \frac{1}{-1}$

$= -1$

3 True or false, with a reason if true or a counterexample if false:

(a) The determinant of $I + A$ is $1 + \det A$.

False $I + I \neq 1 + 1$

(b) The determinant of ABC is $|A|\,|B|\,|C|$.

True Product rule extends

(c) The determinant of $4A$ is $4|A|$.

False $\det(4A) = 4^n \det A$

(d) The determinant of $AB - BA$ is zero. Try an example with $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

False $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$AB - BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

8 Prove that every orthogonal matrix ($Q^TQ = I$) has determinant 1 or -1.

(a) Use the product rule $|AB| = |A|\,|B|$ and the transpose rule $|Q| = |Q^T|$.

$Q^TQ = I \Rightarrow |Q^T||Q| = |Q|^2 = 1 \Rightarrow |Q| = \pm 1$

(b) Use only the product rule. If $|\det Q| > 1$ then $\det Q^n = (\det Q)^n$ blows up. How do you know this can't happen to Q^n?

stays orthogonal so det can't blow up.

Questions 13–27 use the rules to compute specific determinants.

13 Reduce A to U and find $\det A$ = product of the pivots:

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Leftarrow A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix}. \Rightarrow U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & 0 & -2/3 \end{bmatrix}$$

Product of 1,1,1 $= \boxed{1}$

Product of $1, -2, -2/3 = \boxed{3}$

5.2 – Permutations and Cofactors: 12, 13, 16

Problems 11–22 use cofactors $C_{ij} = (-1)^{i+j} \det M_{ij}$. Remove row i and column j.

12 Find the cofactor matrix C and multiply A times C^T. Compare AC^T with A^{-1}:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

det $= 2(4-1) +$
$1(-2-0) +$
$1(-2-0) +$
$2(4-0) +$
$1(-2-0) +$
$1(-2-0) +$
$2(4-1)$

$$\Rightarrow C = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 3 \end{bmatrix}$$

$$\overset{A}{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}} \times \overset{C^T}{\begin{bmatrix} 3 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 3 \end{bmatrix}} = \begin{bmatrix} 8 & -8 & 2 \\ -7 & 12 & -7 \\ 2 & -8 & 8 \end{bmatrix}$$

13 The n by n determinant C_n has 1's above and below the main diagonal:

$$C_1 = |0| \quad C_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \quad C_3 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \quad C_4 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}.$$

(a) What are these determinants C_1, C_2, C_3, C_4?

$C_1 = 0$ $\qquad$ $C_3 = 0$

$C_2 = -1$ $\qquad$ $C_4 = 1$

(b) By cofactors find the relation between C_n and C_{n-1} and C_{n-2}. Find C_{10}.

$C_n = -C_{n-2}$ $\qquad$ $C_{10} = -C_8 = +C_6 = -C_4 \Rightarrow \boxed{-1}$

16 F_n is the determinant of the $1, 1, -1$ tridiagonal matrix of order n:

$$F_2 = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \quad F_3 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 3 \quad F_4 = \begin{vmatrix} 1 & -1 & & \\ 1 & 1 & -1 & \\ & 1 & 1 & -1 \\ & & 1 & 1 \end{vmatrix} \neq 4.$$

Expand in cofactors to show that $F_n = F_{n-1} + F_{n-2}$. These determinants are *Fibonacci numbers* $1, 2, 3, 5, 8, 13, \ldots$. The sequence usually starts $1, 1, 2, 3$ (with two 1's) so our F_n is the usual F_{n+1}.

1,1 cofactor of the nxn matrix is F_{n-1}.

1,2 cofactor has 1 in col 1 with cofactor F_{n-2}.

Multiply by $(-1)^{1+2}$ and (-1) from 1,2 entry to find $F_n = F_{n-1} + F_{n-2}$.

Fibonacci numbers.

5.3 – Cramer's Rule, Inverses, and Volumes: 2, 16, 19

2 Use Cramer's Rule to solve for y (only). Call the 3 by 3 determinant D:

$$y = \frac{\begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \boxed{\frac{c}{(ad - bc)}}$$

a. $2x_1 + 5x_2 = 1$
$x_1 + 4x_2 = 2$

$$y = \frac{\det(B_2)}{\det(A)} = \frac{(fg - id)}{D}$$

b. $2x_1 + x_2 = 1$
$x_1 + 2x_2 + x_3 = 0$
$x_2 + 2x_3 = 0$

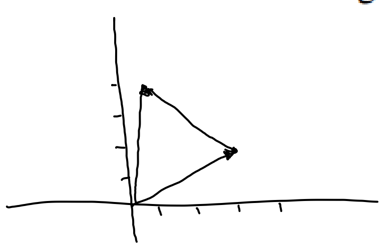
B_2 with $(1, 0, 0)$ in col 2 has det $B_2 = fg - id$

Problems 16–26 are about area and volume by determinants.

16 (a) Find the area of the parallelogram with edges $\boldsymbol{v} = (3, 2)$ and $\boldsymbol{w} = (1, 4)$.

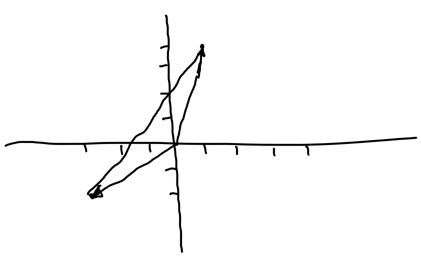
$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = A \qquad \det(A) = 3 \cdot 4 - 2 \cdot 1 = 12 - 2 \boxed{= 10}$$

(b) Find the area of the triangle with sides $\boldsymbol{v}$, $\boldsymbol{w}$, and $\boldsymbol{v} + \boldsymbol{w}$. Draw it.



$$\text{Area} = \frac{12 - 2}{2} = \boxed{5}$$

(c) Find the area of the triangle with sides $\boldsymbol{v}$, $\boldsymbol{w}$, and $\boldsymbol{w} - \boldsymbol{v}$. Draw it.



$$\text{Area} = \frac{12 - 2}{2} = \boxed{5}$$

19 The parallelogram with sides $(2, 1)$ and $(2, 3)$ has the same area as the parallelogram with sides $(2, 2)$ and $(1, 3)$. Find those areas from 2 by 2 determinants and say why they must be equal. (I can't see why from a picture. Please write to me if you do.)

$$\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} = 6 - 2 \boxed{= 4} \qquad \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} = 6 - 2 \boxed{= 4}$$

Transpose has same determinant.