On large cannonball numbers

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April 2019

If s is the number of sides, then the nth s-gonal number is

$$P(n,s) = \frac{n^2(s-2) - n(s-4)}{2}$$

Consider then the sum

$$S(k,s) = \sum_{n=1}^{k} P(n,s) = \frac{k(k+1)}{4}\left(\frac{(s-2)(2k+1)}{3} - (s-4)\right)$$

The question that this paper investigates is when this sum is equal to an s-gonal number for the same s, or when

$$S(k,s) = \frac{n^2(s-2) - n(s-4)}{2} \tag{1}$$

is true for some non-trivial values of n and k. Such values of $S(k,s)$ are called cannonball numbers.

Our main result is that (1) has solutions for all $s = 3p+2, p > 0$ and an expression that computes the explicit value of k that solves the equation. After computational tests, we conjectured that

$$k = \frac{s^2 - 4s - 2}{3}$$

To prove our conjecture, we prove that the corresponding n is an integer. First, we know that

$$n = \frac{\sqrt{8(s-2)P(n,s) + (s-4)^2} + (s-4)}{2(s-2)} \tag{2}$$

We also have that

$$S(\frac{s^2-4s-2}{3}, s) = \frac{(s^3 - 6s^2 + 3x + 19)(s^2 - 4s - 2)(s^2 - 4s + 1)}{162}$$

Let us now only consider the case $s = 3p+2$

$$S(\frac{s^2-4s-2}{3}, s) = S(3p^2-2, 3p+2) = \frac{\left(3p^3 - 3p + 1\right)\left(3p^2 - 2\right)\left(3p^2 - 1\right)}{2}$$

Now we can examine (2) but with $P(n,s) = S(3p^2-2, 3p+2)$.

$$
\begin{aligned}
n &= \frac{\sqrt{8(s-2)S(3p^2-2,3p+2)+(s-4)^2}+(s-4)}{2(s-2)} \\
&= \frac{\sqrt{324p^8-648p^6+108p^5+396p^4-108p^3-63p^2+12p+4}+3p-2}{6p} \\
&= \frac{18p^4-18p^2+6p}{6p} \\
&= 3p^3-3p+1
\end{aligned}
$$

Therefore, the expression is always an integer, which means that n is always an integer. And so $S(3p^2-2, 3p+2)$ is always a cannonball number. ∎