On large cannonball numbers

Pazzaz

April 2019

If s is the number of sides, then the nth s-gonal number is

$$P(n,s) = \frac{n^2(s-2) - n(s-4)}{2}$$

Consider then the sum

$$S(k,s) = \sum_{n=1}^{k} P(n,s) = \frac{k(k+1)}{4}\left(\frac{(s-2)(2k+1)}{3} - (s-4)\right)$$

The question that this paper investigates is when this sum is equal to an s-gonal number for the same s, or when

$$S(k,s) = \frac{n^2(s-2) - n(s-4)}{2} \tag{1}$$

is true for some non-trivial values of n and k. Such values of $S(k,s)$ are called cannonball numbers.

Our main result is that (1) has solutions for all $s = 3p+2, p > 0$ and an expression that computes the explicit value of k that solves the equation. After computational tests, we conjectured that

$$k = \frac{s^2 - 4s - 2}{3}$$

To prove our conjecture, we prove that the corresponding n is an integer. First, we know that

$$n = \frac{\sqrt{8(s-2)P(n,s) + (s-4)^2} + (s-4)}{2(s-2)} \tag{2}$$

We also have that

$$S(\frac{s^2-4s-2}{3}, s) = \frac{(s^3 - 6s^2 + 3x + 19)(s^2 - 4s - 2)(s^2 - 4s + 1)}{162}$$

Let us now only consider the case $s = 3p+2$

$$S(\frac{s^2-4s-2}{3}, 3p+2) = \frac{\left(3p^3 - 3p + 1\right)\left(3p^2 - 2\right)\left(3p^2 - 1\right)}{2}$$

Now we can examine (2) but with $P(n,s) = S(\frac{s^2-4s-5}{3}+1, 3p+2)$.

$$\begin{aligned} n &= \frac{\sqrt{8(s-2)S(\frac{s^2-4s-2}{3}, 3p+2) + (s-4)^2} + (s-4)}{2(s-2)} \\ &= \frac{\sqrt{324p^8 - 648p^6 + 108p^5 + 396p^4 - 108p^3 - 63p^2 + 12p + 4} + 3p - 2}{6p} \end{aligned}$$

Now we prove the numerator is divisible by $6p$.

$$\begin{aligned} &\sqrt{324p^8 - 648p^6 + 108p^5 + 396p^4 - 108p^3 - 63p^2 + 12p + 4} + 3p - 2 \\ &\equiv \sqrt{3p^2 + 4} + 3p - 2 \\ &\equiv \sqrt{9p^2 - 12p + 4} + 3p - 2 \\ &\equiv \sqrt{(-3p+2)^2} + 3p - 2 \\ &\equiv -3p + 2 + 3p - 2 \quad (mod\ 6p) \\ &= 0 \end{aligned}$$

Therefore, the expression is always an integer, which means that n is always an integer. And so $S(\frac{s^2-4s-2}{3}, 3p+2)$ is always a cannonball number. ∎