

# On large cannonball numbers

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If  $s$  is the number of sides, then the  $n$ th  $s$ -gonal number is

$$P(n, s) = \frac{n^2(s-2) - n(s-4)}{2}$$

Consider then the sum

$$S(k, s) = \sum_{n=1}^k P(n, s) = \frac{k(k+1)}{4} \left( \frac{(s-2)(2k+1)}{3} - (s-4) \right)$$

The question that this paper investigates is when this sum is equal to an  $s$ -gonal number for the same  $s$ , or when

$$S(k, s) = \frac{n^2(s-2) - n(s-4)}{2} \quad (1)$$

is true for some non-trivial values of  $n$  and  $k$ . Such values of  $S(k, s)$  are called cannonball numbers.

Our main result is that (1) has solutions for all  $s = 3p + 2, p > 0$  and an expression that computes the explicit value of  $k$  that solves the equation. After computational tests, we conjectured that

$$k = \frac{s^2 - 4s - 2}{3}$$

To prove our conjecture, we prove that the corresponding  $n$  is an integer. First, we know that

$$n = \frac{\sqrt{8(s-2)P(n, s) + (s-4)^2} + (s-4)}{2(s-2)} \quad (2)$$

We also have that

$$S\left(\frac{s^2 - 4s - 2}{3}, s\right) = \frac{(s^3 - 6s^2 + 3s + 19)(s^2 - 4s - 2)(s^2 - 4s + 1)}{162}$$

Let us now only consider the case  $s = 3p + 2$

$$S\left(\frac{s^2 - 4s - 2}{3}, s\right) = S(3p^2 - 2, 3p + 2) = \frac{(3p^3 - 3p + 1)(3p^2 - 2)(3p^2 - 1)}{2}$$

Now we can examine (2) but with  $P(n, s) = S(3p^2 - 2, 3p + 2)$ .

$$\begin{aligned} n &= \frac{\sqrt{8(s-2)S(3p^2-2, 3p+2) + (s-4)^2} + (s-4)}{2(s-2)} \\ &= \frac{\sqrt{324p^8 - 648p^6 + 108p^5 + 396p^4 - 108p^3 - 63p^2 + 12p + 4} + 3p - 2}{6p} \end{aligned}$$

Now we prove the numerator is divisible by  $6p$ .

$$\begin{aligned} &\sqrt{324p^8 - 648p^6 + 108p^5 + 396p^4 - 108p^3 - 63p^2 + 12p + 4} + 3p - 2 \\ &\equiv \sqrt{3p^2 + 4} + 3p - 2 \\ &\equiv \sqrt{9p^2 - 12p + 4} + 3p - 2 \\ &\equiv \sqrt{(-3p + 2)^2} + 3p - 2 \\ &\equiv -3p + 2 + 3p - 2 \pmod{6p} \\ &= 0 \end{aligned}$$

Therefore, the expression is always an integer, which means that  $n$  is always an integer. And so  $S(3p^2 - 2, 3p + 2)$  is always a cannonball number. ■