Q1. Let $(L, \preceq)$ be a complete lattice and let $f : L \longrightarrow L$ be an order-preserving function. Prove that if $X = \{x \in L \mid f(x) = x\}$, then $(X, \preceq)$ is a complete lattice. In assignment 2 you showed that $([0, 1], \leq)$ (the unit interval with the usual order) is a complete lattice. Define an order-preserving function $f : [0, 1] \longrightarrow [0, 1]$ such that the set of fixed points of f endowed with the usual order $(\leq)$ on $\mathbb{R}$ is a linear order with exactly 4 elements.
(3+2 marks)

Q2. Without using Cantor's Pairing Function, prove that $\mathbb{N} \times \mathbb{N}$ is countable. Use the Schröder-Bernstein Theorem to show that $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$.
(2 marks)

Q3. For all $n \in \mathbb{N}$, use $[n]$ to denote the set $\{0, \ldots, n - 1\}$, i.e. the set of predecessors of n in $\mathbb{N}$ with the usual ordering. Define

$$S = \{f \mid (\exists n \in \mathbb{N})(f \text{ is a function with dom } f = [n] \text{ and ran } f \subseteq \mathbb{N})\}$$

(i) Use the fact that every natural number has a unique factorisation into primes to show that S is countable. Is $|S| = |\mathbb{N}|$?

(ii) Define $\preceq_1 \subseteq S \times S$ by: $f \preceq_1 g$ where $f : [n] \longrightarrow \mathbb{N}$ and $g : [m] \longrightarrow \mathbb{N}$, and $A = [n] \cap [m]$ if and only if

$$(\exists i \in A)(f(i) < g(i) \wedge (\forall j < i)(f(j) = g(j))) \text{ or } (\forall i \in A)(f(i) = g(i)) \wedge ([n] \subseteq [m]),$$

where $\leq$ and $<$ are the usual orders on $\mathbb{N}$. Is $(S, \preceq_1)$ a partial order? Is $(S, \preceq_1)$ a linear order? Is $(S, \preceq_1)$ chain complete? Is $(S, \preceq_1)$ a lattice? Is $(S, \preceq_1)$ a well-order? Prove your answers.

(iii) Define $\preceq_2 \subseteq S \times S$ by: $f \preceq_2 g$ where $f : [n] \longrightarrow \mathbb{N}$ and $g : [m] \longrightarrow \mathbb{N}$ if and only if

$$[n] \subseteq [m] \text{ and } (\forall i \in [n])(f(i) \leq g(i))$$

Is $(S, \preceq_2)$ a partial order? Is $(S, \preceq_2)$ chain complete? Is $(S, \preceq_2)$ a lattice? Is $(S, \preceq_2)$ a linear order? Prove your answers.

(3+2+2 marks)

Q4. Give an explicit formula that defines a bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$. You do not have to prove that this formula works!
(1 mark)

Q5. Define

$$X = \{R \in \mathcal{P}(\mathbb{N} \times \mathbb{N}) \mid R \text{ is a function}\}.$$

(i) Is $(X, \subseteq)$ a partial order? Is $(X, \subseteq)$ chain complete? Is $(X, \subseteq)$ a lattice? Is $(X, \subseteq)$ a linear order? Prove your answers.

(ii) Define $\Phi_1 : X \longrightarrow X$ such that Φ_1 takes input $f \in X$ and returns $g : \text{dom } f \longrightarrow \mathbb{N}$ such that for all $n \in \text{dom } f$, $g(n) = f(n) + 1$. Is $\Phi_1 : (X, \subseteq) \longrightarrow (X, \subseteq)$ order-preserving? Does $\Phi_1 : X \longrightarrow X$ have a fixed point? Prove your answers.

(iii) Define $\Phi_2 : X \longrightarrow X$ such that Φ_2 takes input $f \in X$ and returns $g : \mathbb{N} \longrightarrow \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \in \text{dom } f$, then $g(n) = f(n) + 1$ and if $n \notin \text{dom } f$, then $g(n) = 0$. Is $\Phi_2 : (X, \subseteq) \longrightarrow (X, \subseteq)$ order-preserving? Does $\Phi_2 : X \longrightarrow X$ have a fixed point? Prove your answers.

(iv) Define $\Phi_3 : X \longrightarrow X$ such that Φ_3 takes input $f \in X$ and returns $g : \mathbb{N} \longrightarrow \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \in \text{dom } f$, then $g(n+1) = f(n)+1$ and if $n \notin \text{dom } f$, then $g(n+1) = 0$. Is $\Phi_3 : (X, \subseteq) \longrightarrow (X, \subseteq)$ order-preserving? Does $\Phi_3 : X \longrightarrow X$ have a fixed point? Prove your answers.

(v) Is X countable? Prove your answer.

(2+2+2+2+2 marks)

Q6. Every nonzero rational number has a unique representation in the form $\frac{(-1)^k a}{b}$ where $k \in \{0, 1\}$ and $a, b \in \mathbb{N}$ with $a, b \neq 0$ and the greatest common divisor of a and b is 1. Use this to show that $\mathbb{Q}$ is countable. Prove that $|\mathbb{Q}| = |\mathbb{N}|$.
(3 marks)

Q7. Let $\pi(x, y)$ be Cantor's pairing function. Find $x, y \in \mathbb{N}$ such that $\pi(x, y) = 223$. You may do this question however you wish.
(1 mark)