UM-SJTU JOINT INSTITUTE
DISCRETE MATHEMATICS
(VE203)

ASSIGNMENT 8

Name: Pan Chongdan
ID: 516370910121

Date: November 29, 2017

1 Q1

(i) **Input**: $a_1, \cdots, a_n, n$ unsorted elements

Output: All the $a_i, 1 \leq i \leq n$ in an increasing order.

```
for p = 1 to n − 1
x = a_{p+1};
    If a_p > a_{p+1} then
        i = 1; j = p;
        while i < j do
            m ← ⌈(i + j/2⌉;
            if x > a_m then i ← (m + 1);
            else j ← m;
        end while
        for k = p to i
            a_{k+1} = a_k;
        end for
        a_i = x;
    end if
end for
return (a_1 · · · a_{n+1});
```

(ii) For the Insertion Sort Algorithm, $n = \sum_1^7 = 28$

For the Binary Insertion Sort Algorithm, $n = 1+1+1+2+2+2+3 = 12$

(iii) $f(n) = \sum_1^{n-1} = \frac{n^2-n}{2}$, which is order n^2

(iv) $f(n)$ is $O(\log_2 n)$, which is faster than Insertion Sort.

2 Q2

(i) Assume $n = b^k, k = \log_b n$ then

$$f(n) = b^d \cdot f(b^{k-1}) + cb^{kd}$$

$$f(n) = b^d \cdot [b^d \cdot f(b^{k-2}) + cb^{kd-d}] + cb^{kd} = b^{2d} f(b^{k-2}) + 2cb^{kd}$$

$$f(n) = b^{2d} f(b^{k-2}) + 2cb^{kd} = b^{3d} f(b^{k-3}) + 3cb^{kd}$$

$$\cdots$$

$$f(n) = b^{kd} f(1) + kcn^d = f(1)n^d + cn^d \log_b n$$

(ii) Assume $n = b^k$, where $k = A + B, A \in \mathbb{Z}$ and $0 < B < 1$, then similarly to (i)

$$f(n) = f(\frac{n}{b^A})b^{Ad} + Acn^d$$

$$\lim_{n\to\infty} |\frac{f(n)}{n^d \log_b n}| = c$$

$$\therefore f \quad \text{is} \quad O(n^d \log_b(n))$$

(iii) Assume $n = b^k, k = \log_b n$ then

$$f(n) = a \cdot f(b^{k-1}) + cb^{kd}$$

$$f(n) = a \cdot [a \cdot f(b^{k-2}) + cb^{kd-d}] + cb^{kd} = a^2 f(b^{k-2}) + (1 + \frac{a}{b^d})cb^{kd}$$

$$\cdots$$

$$f(n) = a^k f(1) + [1 + \frac{a}{b^d} \cdots (\frac{a}{b^d})^{k-1}]cb^{kd} = a^k f(1) + \frac{a^k - b^{kd}}{ab^{kd-d} - b^{kd}} \cdot cn^d$$

$$f(n) = a^k f(1) + c \cdot \frac{b^d(n^d - a^k)}{b^d - a} = a^k \cdot (f(1) + \frac{cb^d}{a - b^d}) + c \cdot \frac{b^d n^d}{b^d - a}$$

$$f(n) = c_1 n^d + c_2 a^k$$

$$a^k = a^{\log_n b} = n^{\log_b a}$$

$$\therefore f(n) = c_1 n^d + c_2 n^{\log_b a}$$

(iv)

$$\lim_{n\to\infty} |\frac{f(n)}{n^d}| = |c_1 + c_2 n^{\log_b a - d}|$$

$$\because a < b^d \Rightarrow \log_b a - d < 0 \Rightarrow |c_1 + c_2 n^{\log_b a - d}| < |c_1 + c_2|$$

$$\therefore f \quad \text{is} \quad O(n^d)$$

(v) Similarly to last question,

$$\lim_{n\to\infty} |\frac{f(n)}{n^{log_b a}}| = |c_1 n^{d - \log_b a} + c_2|$$

$$\because a > b^d \Rightarrow \log_b a - d > 0 \Rightarrow |c_1 n^{d - \log_b a} + c_2| < |c_1 + c_2|$$

$$\therefore f \quad \text{is} \quad O(n^{\log_b a})$$

3 Q3

Since there is only on real root then

$$a_n = 2\alpha a_{n-1} - \alpha^2 a_{n-2}$$

$$2\alpha a_{n-1} - \alpha^2 a_{n-2} = 2\alpha(q_1 \alpha^{n-1} + q_2 n \alpha^{n-1}) - \alpha^2(q_1 \alpha^{n-2} + q_2 n \alpha^{n-2}) = q_1 \alpha^n + q_2 n \alpha^n = a_n$$

4 Q4

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$
$$\lambda_1 = 2, \lambda_2 = 1, \lambda_1 = -1$$
$$\therefore a_n = q_1 2^n + q_2 + q_3(-1)^n$$
$$3 = q_1 + q_2 + q_3$$
$$6 = 2q_1 + q_2 - q_3$$
$$0 = 4q_1 + q_2 + q_3$$
$$q_1 = -1, q_2 = 6, q_3 = -2$$
$$a_n = -2^n + 6 - 2(-1)^n$$

5 Q5

$$\lambda^2 - 5\lambda + 6 = 0$$
$$\lambda_1 = 2, \lambda_2 = 3$$
$$p_n = bn2^n + cn^2 + dn + e$$
$$p_n = 5p_{n-1} - 6p_{n-2}$$
$$b = -2, c = 1, d = \frac{15}{2}, e = \frac{67}{4}$$
$$a_n = q_1 2^n + q_2 3^n - 2n2^n + n^2 + \frac{15}{2}n + \frac{67}{4}$$
$$q_1 = -33, q_2 = \frac{65}{4}$$
$$a_n = -33 \cdot 2^n + \frac{65}{4} \cdot 3^n - 2n2^n + n^2 + \frac{15}{2}n + \frac{67}{4}$$

6 Q6

$$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$$
$$\lambda_1 = 3, \lambda_2 = \lambda_3 = 2$$
$$p_n = kn4^n + b4^n$$
$$kn4^n + b4^n = 7k(n-1)4^{n-1} - 16k(n-2)4^{n-2} + 12k(n-3)4^{n-3} + n4^n + 3b4^n$$
$$k = 16, b = -80$$
$$a_n = q_1 3^n + q_2 2^n + q_3 n2^n + 16n4^n - 80 \cdot 4^n$$
$$a_n = 49 \cdot 3^n + 28 \cdot 2^n + 27.5n2^n + 16n4^n + \frac{5}{2}4^n$$

7 Q7

$$a_{n+1} = a_n + (n+1)^4$$

$$a_n = an^5 + bn^4 + cn^3 + dn^2 + en + f$$

$$a(n+1)^5 + (b-1)(n+1)^4 + c(n+1)^3 + d(n+1)^2 + e(n+1) + f = an^5 + bn^4 + cn^3 + dn^2 + en + f$$

$$a = \frac{1}{5}, b = \frac{1}{2}, c = \frac{1}{3}, d = 0, e = -\frac{1}{30}, f = 0$$

$$a_n = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$

8 Q8

$$a_n - b_n = 2a_{n-1} \Rightarrow b_n = a_n - 2a_{n-1} \Rightarrow a_n = 5a_{n-1} - 4a_{n-2}$$

$$\lambda^2 - 5\lambda + 4 = 0$$

$$\lambda_1 = 4, \lambda_2 = 1$$

$$a_n = q_1 4^n + q_2 \Rightarrow a_n = 2 \cdot 4^n - 1$$

$$\therefore b_n = 4^n + 1$$

9 Q9

$$\lambda^2 - 2\lambda + 2 = 0$$

$$\lambda_1 = 1 + i, \lambda_2 = 1 - i$$

$$p_n = k3^n \Rightarrow k = \frac{2}{3}k - \frac{2}{9}k + 1 \Rightarrow k = \frac{9}{5} \Rightarrow p_n = \frac{9}{5} \cdot 3^n$$

$$a_n = (\sqrt{2})^n (q_1 \cos\frac{n\pi}{4} + q_2 \sin\frac{n\pi}{4}) + \frac{9}{5} \cdot 3^n \Rightarrow q_1 = -\frac{4}{5}, q_2 = -\frac{13}{5}$$

$$a_n = (\sqrt{2})^n (-\frac{4}{5}\cos\frac{n\pi}{4} - \frac{13}{5}\sin\frac{n\pi}{4}) + \frac{9}{5} \cdot 3^n$$