Q1. Let $(G, \cdot)$ be a group and let $H \leq G$.

(i) Prove that for all $x \in H$, $x^{-1} \in H$

(ii) Prove that for all $x, y \in H$, $x \cdot y \in H$

(2 marks)

Q2. Find all of the subgroups of D_4.
(4 marks)

Q3. Prove that if $(G, \cdot)$ is a group of order 10 then there exists $g \in G$ with order 2.
(2 marks)

Q4. Construct an infinite group $(G, \cdot)$ that satisfies the following:

(i) Every element of $(G, \cdot)$ has finite order

(ii) For all $n \in \mathbb{N}$, there exists $m > n$ and $g \in G$ such that g has order m

(3 marks)

Q5. Show that $\langle (0123), (01) \rangle = S_4$.
(3 marks)

Q6. Does there exist $a, b \in A_5$ such that $\langle a, b \rangle = A_4$? Justify your answer.
(3 marks)

Q7. Let $(G, \cdot)$ be a group and let $H \leq G$. Define

$$X = \{aH \mid a \in G\}$$

I.e. X is the set of left cosets of H. Define $\star : X \times X \longrightarrow X$ by: for all $a, b \in G$,

$$(aH) \star (bH) = (a \cdot b)H.$$

(i) We say that H is *normal* if for all $h \in H$ and for all $g \in G$, $ghg^{-1} \in H$. Prove that if H is normal, then $(X, \star)$ is a group.

(ii) Find an example of a group $(G, \cdot)$ and $H \leq G$ such that $(X, \star)$ is not a group.

(5 marks)

Q8. Let $(G, \cdot)$ be a group.

(i) Let $a, b \in G$. Prove by induction that if $ab = ba$, then $(ab)^n = a^n b^n$

(ii) Prove that if $(G, \cdot)$ is abelian and H is the set of all elements of G that have finite order, then $H \leq G$

(4 marks)

Q9. Let G be the set of 2×2 invertible real matrices and let $\cdot$ be matrix multiplication. Show that $(G, \cdot)$ is a group. Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Find the orders of A, B and $A \cdot B$.
(4 marks)