VE203 DISCRETE MATHEMATICS

ORDER, LATTICES AND FUNCTION

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RELATION

- Definition
- A set R is called a relation if R only contains ordered pairs.
- dom $R = \{x \mid \exists x((x, y) \in R)\}$
- ran $R = \{y \mid \exists x((x, y) \in R)\}$

RELATION

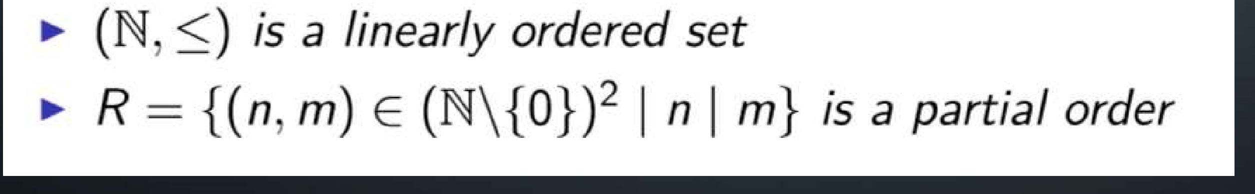
- **reflexive** $if\ for\ all\ a \in M, (a, a) \in R$
- **symmetric** $if\ for\ all\ a, b \in M, (a, b) \in R, then\ (b, a) \in R$
- **antisymmetric** $if\ for\ all\ a, b \in M, (a, b) \in R\ and\ (b, a) \in R, then\ a = b$
- **asymmetric** $if\ for\ all\ a, b \in M, (a, b) \in R, then\ (b, a) \notin R$
- transitive $if\ for\ all\ a, b, c \in M, (a, b) \in R\ and\ (b, c) \in R, then\ (a, c) \in R$

ORDER

- partial order
- reflexive, antisymmetric and transitive
- partial order set (poset)
- strict partially ordered set
- asymmetric and transitive

ORDER

- linear order or total order
- It should be a partially ordered set
- $if\ for\ all\ x, y \in M, (x, y) \in \mathrm{R}\ or\ (y, x) \in R$
- Why it is called linear?



- $(\mathbb{N}, \leq)$ *is a linearly ordered set*
- $R = \{(n, m) \in (\mathbb{N} \backslash \{0\})^2 \mid n \mid m\}$ *is a partial order*

ORDER

- well order
- It should be a linear ordered set
- $if\ \forall A \subseteq M, A \neq \emptyset, then\ there\ \exists x \in A, such\ that\ \forall y \in A, if\ (y, x) \in R, then\ y = x$
- $\exists x \forall y \in A, (x, y) \in R$
- every nonempty $A \subseteq M$ has a least element according to R

ORDER

Example

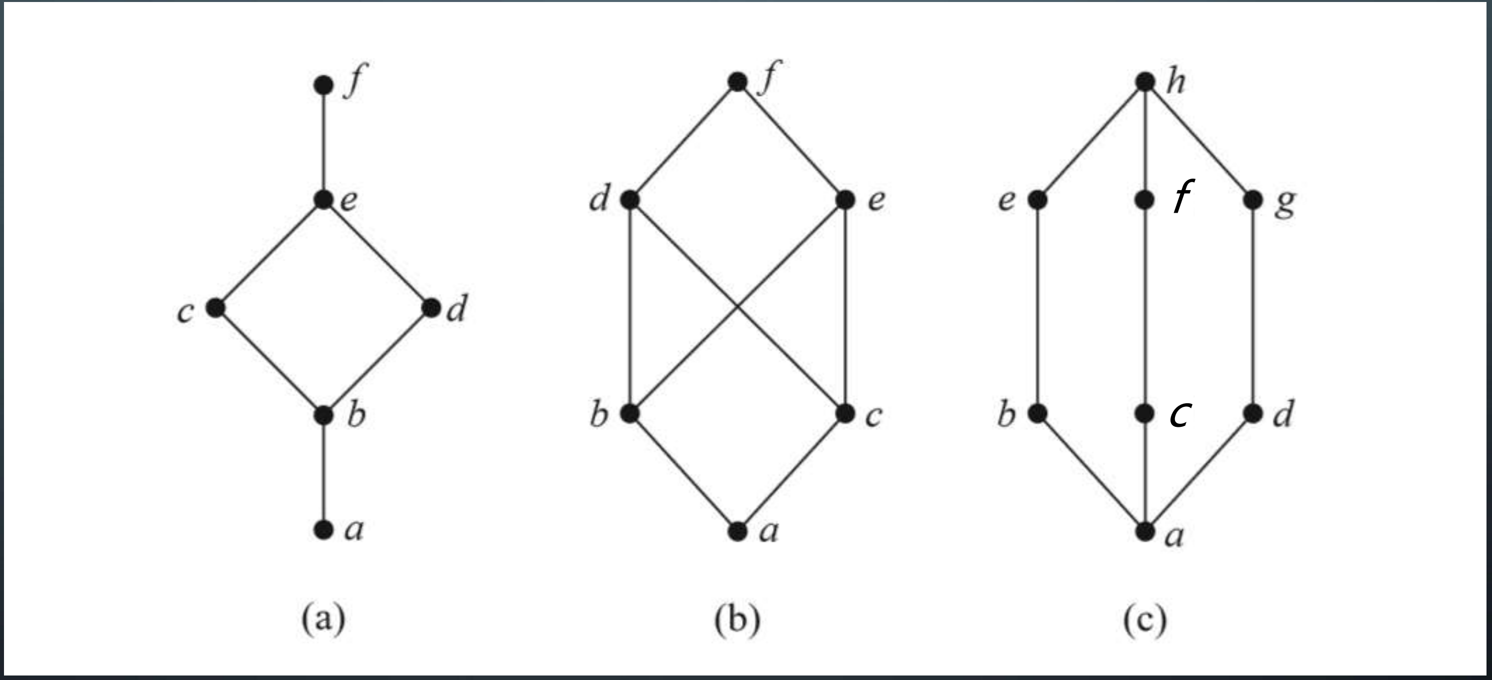
- *If R is a linear order on M and M is finite then R is a well-order*
- *The linear order $\leq$ on $\mathbb{N}$ is a well-order*
- *The linear order $\leq$ on $\mathbb{R}$ is not a well-order*
- *The linear order $\geq$ on $\mathbb{N}$ is not a well-order*

LATTICES

$$(\boldsymbol{x}, \boldsymbol{y}) \equiv \boldsymbol{x} \preccurlyeq \boldsymbol{y}$$

- Let $(L, \preccurlyeq)$ $be\ a\ poset\ and\ set\ S \subseteq L.$
- We said $x \in L$ is an upper bound $on\ S\ if\ \forall y \in S, y \preccurlyeq x$
- We said $x \in L$ is a lower bound $on\ S\ if\ \forall y \in S, x \preccurlyeq y$
- **CHANGE IN DEFINITION!**
- $x \in L$ is a least upper bound (l.u.b) $on\ S\ if\ \forall y \in L, y\ is$ upper bound, $x \preccurlyeq y$
- $y \in L$ is a greatest lower bound (g.l.b) $on\ S\ if\ \forall y \in L, y\ is$ upper bound, $y \preccurlyeq x$

LATTICES



Hasse Diagram

LATTICES

- Determine whether $(P(S), \subseteq)$ is a lattice where S is a set.
- Solution: Let A and B be two subsets of S. The least upper bound and the greatest lower bound of A and B are $A \cup B$ and $A \cap B$, respectively, as the reader can show. Hence, $(P(S), \subseteq)$ is a lattice.

LATTICES

- Complete Lattices
- $\forall X \subseteq L$, X has both a least upper bound and a greatest lower bound.
- Denote l.u.b as $\vee X$, g.l.b as $\wedge X$.

LATTICES

- If $(L, \preceq)$ is a finite lattice then $(L, \preceq)$ is complete
- If A is any set, then $(\mathcal{P}(A), \subseteq)$ is a complete lattice. If $X \subseteq \mathcal{P}(A)$, then

$$\bigvee X = \bigcup X \text{ and } \bigwedge X = \bigcap X$$

- The lattice $(\mathbb{Q}, \leq)$ is not complete. One reason is that $\{x \in \mathbb{Q} \mid x^2 \leq 2\}$ has no l.u.b.
- Is the lattice $(\mathbb{R}, \leq)$ complete?

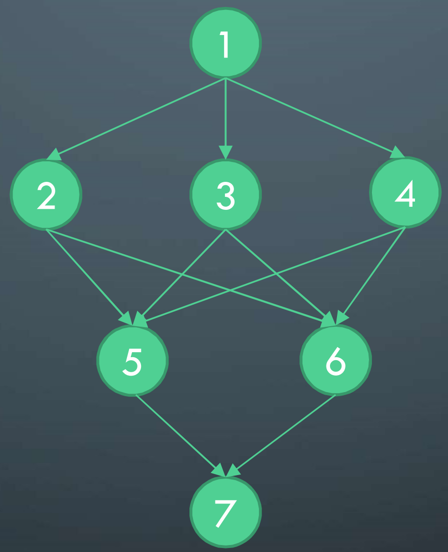
LATTICES

- If $(L, \preceq)$ is a complete lattice, then $(L, \preceq)$ has a maximal element given by $\bigvee L$. This maximal element is sometimes denoted $\mathbb{1}$.
- If $(L, \preceq)$ is a complete lattice, then $(L, \preceq)$ has a minimal element given by $\bigwedge L$. This minimal element is sometimes denoted .

\mathbbm{1}

\mathbb{O}
Maybe?

LATTICE VS. COMPLETE LATTICE



WHY WE NEED LATTICES

- Suppose that a project is made up of 20 different tasks. Some tasks can be completed only after others have been finished. How can an order be found for these tasks?

ALGORITHM 1 Topological Sorting.

procedure *topological sort* ($(S, \preccurlyeq)$: finite poset)
$k := 1$
while $S \neq \emptyset$
  $a_k := $ a minimal element of S {such an element exists by Lemma 1}
  $S := S - \{a_k\}$
  $k := k + 1$
return $a_1, a_2, \ldots, a_n$ {$a_1, a_2, \ldots, a_n$ is a compatible total ordering of S}

CHAIN COMPLETE POSETS

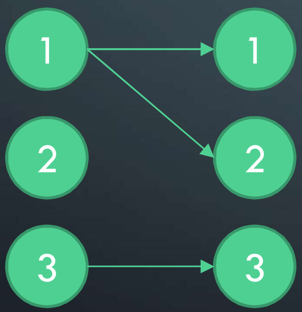
- Let $(P, \preccurlyeq)$ be a partial order. We say that $X \subseteq P$ is a chain if $(X, \preccurlyeq)$ is a linear order.
- If $(P, \preccurlyeq)$ is a linear order, then every $X \subseteq P$ is a chain.
- Let $(P, \preccurlyeq)$ be a partial order. We say that $(P, \preccurlyeq)$ is chain complete if for all $X \subseteq P$, if X is a chain then X has a least upper bound.
- This definition ensures the existence of least elements. Why?
- Empty set is also a chain!

CHAIN

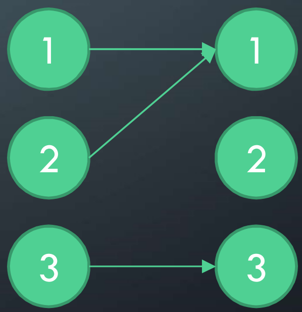
- *Every finite poset is chain complete*
- *Every complete lattice is a chain complete poset*

FUNCTION

- Let $f \subseteq A \times B$ (so f is a relation). We say that f is a function, and write $f : A \longrightarrow B, if\ \mathrm{dom}\, f = A\ and\ \forall x \in A\ and\ \forall y, z \in B, if\ (x, y) \in f\ and\ (x, z) \in f, then\ y = z.$



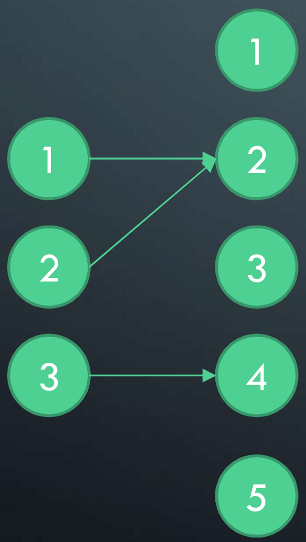
Not function!

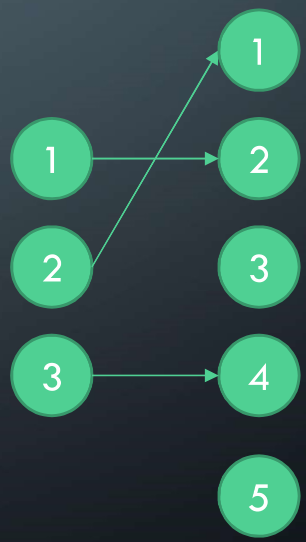


Function!

INJECTIVE FUNCTION (ONE TO ONE)

- Let $f : A \longrightarrow B$ be a function.
- $if\ \forall x, y \in A\ and\ \forall z \in B, if\ (x, z) \in f\ and\ (y, z) \in f, then\ x\ = y.$

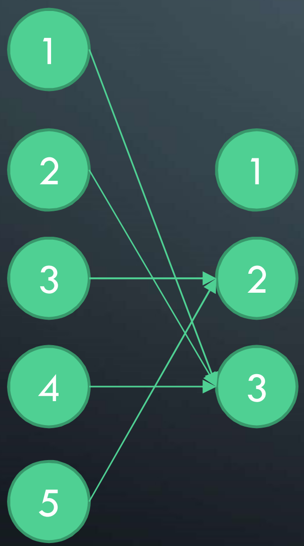


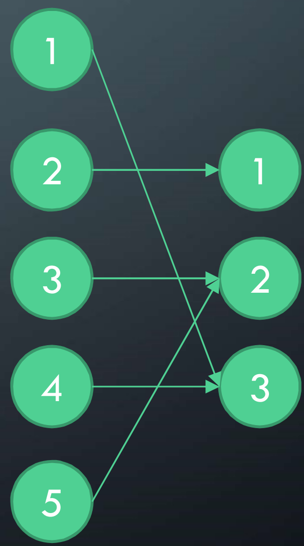


injective

SURJECTIVE FUNCTION (ONTO)

- Let $f : A \longrightarrow B$ be a function.
- $if\ \forall y \in B\ , \exists \mathrm{x} \in A, such\ that\ (x, y) \in f$





surjuective

BIJECTION

- If a function $f : A \longrightarrow B$ is both injective and surjective, then we say that f is a bijection.

COMPOSING FUNCTION

- Let f and g be functions with $\operatorname{ran} f \subseteq \operatorname{dom} g$. Then define $g \circ f$ to be the relation
- $g \circ f = \{(x, y) | \exists z((x, z) \in f \wedge (z, y) \in g)\}$
- If f and g are functions with $\operatorname{ran} f \subseteq \operatorname{dom} g$ then $g \circ f$ is a function

INVERSE FUNCTION

- Let $f \subseteq A \times B$ be a function. The inverse of f, written f^{-1} is the relation
- $f^{-1} = \{(x, y) \in B \times A | (y, x) \in f\}$
- identity function
- $\mathrm{id}_A = \{(x, y) \in A \times A | x = y\}$
- $f^{-1} \circ f = f \circ f^{-1} = \mathrm{id}_A$ if f is injective

TIKZ

- http://www.texample.net/tikz/examples/