UM-SJTU JOINT INSTITUTE
DISCRETE MATHEMATICS
(VE203)

ASSIGNMENT 5

Name: Pan Chongdan
ID: 516370910121

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1 Q1

(i) $e \cdot x^{-1} \in H \Rightarrow x^{-1} \in H$

(ii) $y^{-1} \in H \Rightarrow x \cdot (y^{-1})^{-1} \in H \Rightarrow x \cdot y \in H$

2 Q2

1. {e}
2. {e,(01)(23)}
3. {e,(02)(13)}
4. {e,(13)}
5. {e,(30)(12)}
6. {e,(20)}
7. {e,(0123),(0321)}
8. {e,(01)(23),(02)(13),(13),(30)(12),(20),(0123),(0321)}
9. {e,(01)(23),(02)(13),(30)(12))}
10. {e,(20)(13),(0123),(0321)}

3 Q3

G's order is 10 $\Rightarrow \forall g \in G, g^{10} = e \Rightarrow (g^5)^2 = e$
$\because g^5 \in G \therefore g^5$'s order is 2.

4 Q4

$(G, \cdot) = S_n$,where n is infinite $\forall m, (012 \cdots m)^m = e$

5 Q5

$\mathbf{(0123)}^{-1} = \mathbf{(3210)}, \mathbf{(01)}(0123) = \mathbf{(123)}, (01)(3210) = \mathbf{(203)}, (203)(123) = \mathbf{(310)}$
$(123)(203) = \mathbf{(012)}, (123)^{-1} = \mathbf{(132)}, (203)^{-1} = \mathbf{(023)}, (310)^{-1} = \mathbf{(301)}, (012)^{-1} = \mathbf{(021)}$
$(01)(310) = \mathbf{(30)}, (01)(301) = \mathbf{(31)}, (01)(012) = \mathbf{(12)}, (01)(021) = \mathbf{(02)}$
$(02)(310) = \mathbf{(3120)}, (3120)^{-1} = \mathbf{(0213)}, (02)(301) = \mathbf{(3201)}, (3201)^{-1} = \mathbf{(1023)}, (02)(023) = \mathbf{(23)}, \mathbf{(01)(23)}, \mathbf{(02)(13)}, \mathbf{(03)(12)}, \mathbf{e}$

So there are total 24=4! elements in ¡(0123),(01)¿ and it's equal to S_4

6 Q6

Assume $a = (01)(02) = \mathbf{(021)}, b = \mathbf{(01)(23)}, a^{-1} = \mathbf{(012)}, a \cdot b = \mathbf{(123)}$
$(123)^{-1} = \mathbf{(321)}, (01)(23) = \mathbf{(013)}, (013)^{-1} = \mathbf{(031)}$
$(01)(23)(031) = \mathbf{(023)}, (023)^{-1} = \mathbf{(032)}, (012)(013) = \mathbf{(13)(02)}, (012)(032) = \mathbf{(03)(12)}, \mathbf{e}$
There are total 12 elements for A_4, so there exists

7 Q7

1. Assume $ghg^{-1} \notin H, a \in H, (ghg^{-1})a = b \in H$
   Then $(ghg^{-1})aa^{-1} = ba^{-1} = ghg^{-1}$
   $\because ba^{-1} \in H \therefore$ there is a contradiction for $ghg^{-1} \notin H$ if $(ghg^{-1})a \in H$
   $\therefore (ghg^{-1})a \notin H \Rightarrow (ghg^{-1})H \neq H$
   $\because (ghg^{-1}) = (gH) \star (hH) \star (g^{-1}H) = (gH) \star (g^{-1}H) = eH = H$
   So $\star$ is invalid if $ghg^{-1} \notin H$ and H must be normal
   If H is normal and $\star$ is valid:
   $[(aH) \star (bH)] \star (cH) = (a \cdot b)H \star (cH) = [(a \cdot b) \cdot c]H$
   $(aH) \star [(bH) \star (cH)] = aH \star [(b \cdot c]H) = [a \cdot (b \cdot c)]H = [(a \cdot b) \cdot c]H$
   $\therefore [(aH) \star (bH)] \star (cH) = (aH) \star [(bH) \star (cH)]$
   $\star$ is associative of X
   $(aH) \star (eH) = (a \cdot e)H = aH$
   eH is the identify of X

2. Assume $G = S_3, H = S_2, g = (02), h = (01) \Rightarrow ghg^{-1} = (02)(01)(02) = (21) \notin H$
   $(02)H = (012)H = \{(02), (012))\}, (21)H = (021)H = \{(12), (021)\}$

   According to the definition of $\star, (02)H \star (21)H = (021)H, (012)H \star (021)H = H$, which is a contradiction and $\star$ is not a function
   So $(X, \star)$ is not a group.

8 Q8

(i) When n=1,it's clear that $ab = ab$,assume $(ab)^m = a^m b^m$
when n=m+1,$(ab)^{m+1} = a^m b^m ab = a^m b^{m-1} ab^2 = \cdots = a^{m+1} b^{m+1}$
$\therefore (ab)^n = a^n b^n$

(ii) $\forall a, b \in H$, assume a, b's order are m, n.
$(ab)^{mn} = a^{mn} b^{mn} = e^(n + m) = e \Rightarrow ab \in H, (ab)^{-1} = (ab)^{mn-1}$.
$\because (ab)^{mn-1} = a^{mn-1} b^{mn-1} \therefore (ab)^{mn-1} \in H$
$\therefore H$ is a group and $H \leq G$

9 Q9

1.
$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$A^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$
$$A^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = e$$
   A's order is 3

2.
$$B^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$B^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = e$$
   B's order is 4

3. Assume
$$C = A \cdot B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$
$$C^n = \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix}$$
$$C^n * C = \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -(n+1) & 1 \end{pmatrix}$$
   $\therefore A \cdot B$ has infinite order.