Q1. The **Binary Insertion Sort Algorithm** is a variation of the Insertion Sort Algorithm that uses a binary search technique rather than a linear search technique to insert the i^{th} element in the correct place among the previously sorted elements.

(i) Express the Binary Insertion Sort Algorithm in pseudocode.

(ii) Compare the number of comparisons of elements used by the Insertion Sort Algorithm and the Binary Insertion Sort Algorithm when sorting the list $(7, 4, 3, 8, 1, 5, 4, 2)$.

(iii) Show that the Insertion Sort Algorithm uses $O(n^2)$ comparisons of elements.

(iv) Find the complexity of the Binary Insertion Sort Algorithm. Is it significantly faster than Insertion Sort?

(8 marks)

Q2. Let $a, b \in \mathbb{N}\backslash\{0\}$ with $b > 1$ and let $c, d \in \mathbb{R}$ with $c, d > 0$. Let f be an increasing function satisfying the recurrence relation

$$f(n) = a \cdot f\left(\frac{n}{b}\right) + cn^d$$

The goal of this exercise is to prove the Master Theorem.

(i) Show that if $a = b^d$ and n is a power of b, then $f(n) = f(1)n^d + cn^d \log_b(n)$.

(ii) Show that if $a = b^d$, then f is $O(n^d \log_b(n))$.

(iii) Show that if $a \neq b^d$ and n is a power of b, then

$$f(n) = c_1 n^d + c_2 n^{\log_b(a)} \text{ where } c_1 = \frac{b^d c}{b^d - a} \text{ and } c_2 = f(1) + \frac{b^d c}{a - b^d}$$

(iv) Show that if $a < b^d$, then f is $O(n^d)$.

(v) Show that if $a > b^d$, then f is $O(n^{\log_b(a)})$.

(5 marks)

Q3. Prove that if (a_n) is a sequence that satisfies a linear homogeneous recurrence relation of degree 2 whose characteristic polynomial has only one real root α, then there exists $q_1, q_2 \in \mathbb{R}$ such that for all $n \in \mathbb{N}$,

$$a_n = q_1 \alpha^n + q_2 n \alpha^n$$

(4 marks)

Q4. Find an expression for the terms of the sequence (a_n) that satisfy

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$$

with $a_0 = 3$, $a_1 = 6$ and $a_2 = 0$.
(2 marks)

Q5. Find an expression for the terms of the sequence (a_n) that satisfy

$$a_n = 5a_{n-1} - 6a_{n-2} + 2^n + 2n^2 + n$$

with $a_0 = 0$, $a_1 = 4$.
(3 marks)

Q6. Find an expression for the terms of the sequence (a_n) that satisfy

$$a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3} + n4^n$$

with $a_0 = -3$, $a_1 = 2$, $a_2 = 5$.
(2 marks)

Q7. For all $n \in \mathbb{N}\backslash\{0\}$, let

$$a_n = \sum_{i=1}^{n} i^4$$

By finding a recurrence relation that (a_n) satisfies and solving that recurrence relation, find an expression for the terms of the sequence (a_n).
(2 marks)

Q8. Solve the simultaneous recurrence relations

$$\begin{aligned} a_n &= 3a_{n-1} + 2b_{n-1} \\ b_n &= a_{n-1} + 2b_{n-1} \end{aligned}$$

with $a_0 = 1$ and $b_0 = 2$.
(3 marks)

Q9. Find an expression for the terms of the sequence (a_n) that satisfy

$$a_n = 2a_{n-1} - 2a_{n-2} + 3^n$$

with $a_0 = 1$, $a_1 = 2$.
(2 marks)