Q1. Find a closed formula for the sequences generated by the following generating functions:

(i) $G(x) = x - 1 + \frac{1}{1-3x}$

(ii) $G(x) = e^{3x^2} - 1$

(iii) $G(x) = \frac{x}{1+x+x^2}$

(3 marks)

Q2. Let $G = (V, E)$ be the Petersen graph. Show that G does not have a Hamilton circuit, but for all $v \in V$, $G - v$ has a Hamilton circuit.
(3 marks)

Q3. Can you find a graph G of order $n \geq 3$ such that G does not have a Hamilton circuit and the degree of every vertex of G is $\geq \frac{n-1}{2}$?
(1 mark)

Q4. Determine if the following degree sequences are graphical and, if so, draw a graph that realises the degree sequence.

(i) $3 \geq 3 \geq 2 \geq 2 \geq 2$

(ii) $4 \geq 4 \geq 3 \geq 2 \geq 1$

(iii) $4 \geq 3 \geq 3 \geq 2 \geq 2$

(iv) $3 \geq 3 \geq 3 \geq 2 \geq 2$

(4 marks)

Q5. Prove that a graph of order n and size e must have at least $n - e$ connected components.
(2 marks)

Q6. Find the least number of colours required to colour the vertices of the Petersen graph such that no two vertices with the same colour are connected by an edge.
(2 marks)

Q7. Let X be a set (you may assume that it is finite in this exercise) and let $\mathcal{X} \subseteq \mathcal{P}(X)$. Define the **intersection graph** of $\mathcal{X}$ to be the graph $G_{\mathcal{X}} = (\mathcal{X}, E_{\mathcal{X}})$ such that

$$E_{\mathcal{X}} = \{\{x, y\} \in \mathcal{P}_2(\mathcal{X}) \mid x \cap y \neq \emptyset\}$$

(i) Let $H = (V, F)$ be a finite graph. Prove that there exists a set X and $\mathcal{X} \subseteq \mathcal{P}(X)$ such that $H \cong G_{\mathcal{X}}$.

(ii) For a finite graph H define

$$s(H) = \min\{|X| \mid \text{there exists } \mathcal{X} \subseteq \mathcal{P}(X) \text{ with } H \cong G_{\mathcal{X}}\}$$

Compute $s(H)$ when H is a cycle of order n.

(5 marks)

Q8.

(i) Consider a recurrence relation in the form

$$f(n)a_n = g(n)a_{n-1} + h(n)$$

for $n \geq 1$, and with $a_0 = C$. Show that this recurrence relation can be reduced to a recurrence relation of the form

$$b_n = b_{n-1} + Q(n)h(n)$$

where

$$b_n = g(n+1)Q(n+1)a_n \text{ and } Q(n) = \frac{f(1)f(2)\cdots f(n-1)}{g(1)g(2)\cdots g(n)}$$

(ii) Use (i) to solve the original recurrence relation and obtain

$$a_n = \frac{C + \sum_{i=1}^{n} Q(i)h(i)}{g(n+1)Q(n+1)}$$

(iii) Find an expression for the sequence (a_n) that satisfies

$$a_n = (n+3)a_{n-1} + n \text{ with } a_0 = 1$$

(6 marks)