UM-SJTU JOINT INSTITUTE
DISCRETE MATHEMATICS
(VE203)

ASSIGNMENT 3

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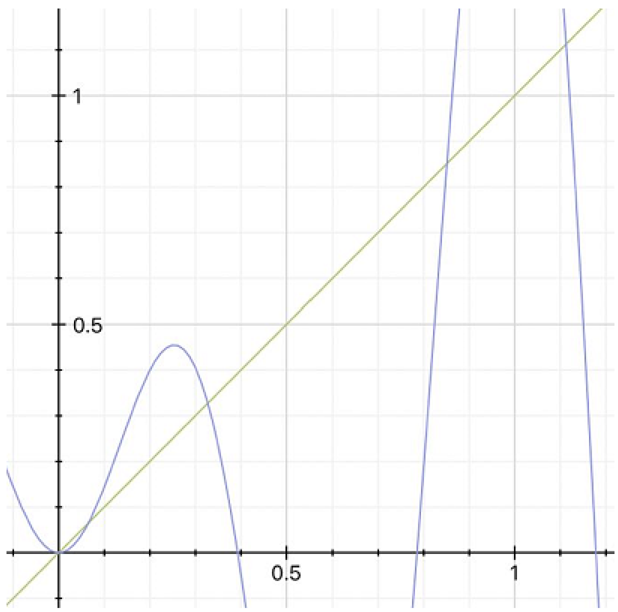
1 Q1

Assume $A = \{x \in L | x \preceq f(x)\}$, then $\forall x \in A, x \preceq f(x) \preceq f(f(x))$. Let $y = \vee A \Rightarrow f(y) = \vee A \Rightarrow y = f(y)$ Since $X \subseteq A$,y is the greatest upper bound in X. Similarly, there exists z is the least lower bound in X.

$\forall i \in I(x_i \in f(x))$,let $B = \{x \in L | \forall i \in I(x_i \preceq f(x) \preceq x)\}$. Assume $u = \wedge B, \forall x \in B, u \preceq x \Rightarrow f(u) \preceq f(x) \preceq x \Rightarrow f(u) \preceq u$.

$\forall x_i$ which is the lower bound of $B, x_i \preceq u, x_i \preceq u \Rightarrow x_i = f(x_i) \preceq f[f(u)] \preceq f(u) \preceq u, \therefore u \in B \Rightarrow u \preceq f(u) \Rightarrow u = f(u) \Rightarrow u \in X. \forall x_i, u$ is the least upper bound,similarly, we can find the greatest lower bound,so $(x, \preceq)$ is a complete lattice.

$f = 2xsin(8x)$, it has 4 points of intersection with the graph of $y = x$ during the interval [0,1].



2 Q2

Let $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ defined by

$$f(x, y) = x^2 + 2y(x \geq y)$$

$$f(x, y) = 2x + 1 + y^2(x < y)$$

So $f(0, 0) = 0, f(1, 0) = 1, f(0, 1) = 2, f(1, 1) = 3, f(2, 0) = 4, f(2, 1) = 5...$
$\therefore$ f is an injection and $|\mathbb{N} \times \mathbb{N}| \leq |\mathbb{N}|$ and $\mathbb{N} \times \mathbb{N}$ is countable.

Since Cantor function Function is bijection,so its inverse function satisfies $|\mathbb{N} \times \mathbb{N}| \geq |\mathbb{N}|$
$\therefore |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$

3 Q3

(i) Since every natural number has a unique factorisation into primes, assume $f = (0, a_0), (1, a_1)...(n, a_n)$. n means the $(n+1)_{th}$ prime and a_n is (n+1)'s power-1.
For example, assume $f = (0, 0), (1, 1) \to 2^{(0+1)} * 3^2 = 18$,so there exists a function g:$S \to \mathbb{N} \setminus \{0, 1\}$,which is countable because $\mathbb{N} \setminus \{0, 1\} \subset \mathbb{N}$. $\therefore S$ is countable and $|S| \leq |\mathbb{N}|$.

There also exists a injection from $\mathbb{N} \to S, n \to \{(0, 0), (1, 1), (2, 2)...(n, n)\}$, so $|\mathbb{N}| = |S|$

(ii) $f \preceq_1 f$ because $\forall i \in A((f(i) = f(i)) \wedge ([n] = [n])), \therefore (S, \preceq)$ is symmetric.

Assume $f \preceq_1 g$ and $\forall i \in A((f(i) = g(i)) \wedge ([n] \preceq_1 [m]))$, it's clearly $g \not\preceq_1 f$ because $\forall i \in A((f(i) = g(i))\text{and}([n] \nsubseteq [m]))$.
Assume $f \preceq_1 g$ and $(\exists i \in A)(f(i) < g(i) \wedge (\forall j < i)(f(i) = g(j)))$ then the i_g satisfying $g \preceq f$ is bigger than i, but $g(i) \neq f(i) \therefore (S, \preceq_1)$ is antisymmetric.

Assume $(f \preceq_1 g$ with $j = j_1) \wedge (g \preceq_1 h$ with $j = j_2)$ and $h : [k] \to \mathbb{N}$,then$f \preceq_1 h$ with $j_=\text{min}\{j_1, j_2\} \therefore (S, \preceq_1)$ is transitive and it's a partial order.
It's a linear order as long as we choose i as 0, then $\forall f, g$ has such relationship.
It;s also a well order because I can find the least element as $\{(0, 0)\}$
It's not a chain complete or a lattice because $f = \{(0, 1), (1, 2), (2, 5)\}, g = \{(0, 5), (1, 3), (2, 4)\}$, f, g doesn't have a upper bound.

(iii) $\because (f(i) = f(i)) \wedge ([n] = [n]) \therefore f \preceq_2 f$
Assume $(f \preceq_2 g) \wedge ([n] \subset [m])$ then $g \not\preceq_2 g$ because $[m] \nsubseteq [m]$.
Assume $f \preceq_2 g$ and $g \preceq_2 h : [k] \to \mathbb{N}$ then $f \preceq_2 h$ because $([n] \subseteq [m] \subseteq [k]) \wedge (f(i) \leq g(i) \leq h(i))$
$\therefore (S, \preceq_2)$ is a partial order.
It's not a linear order if $f = (0, 1), (1, 5), g = (0, 5), (1, 2)$ then they have no relationship.
It's not a chain complete because the chain may not have a upper bound.
$\forall f, g \in S, f = \{(0, a_0), (1, a_1)...(n, a_n)\}$ and $g = \{(0, b_0), (1, b_1)...(m, b_m)\}$.
Let c_n =max(a_n, b_n),d =max(m, n), then $h = \{(0, c_0), (1, c_1)...(d, c_d)\}$ is the least upper bound.
Similarly, I can find the greatest lower bound,so it's a complete lattice.

4 Q4

$$f(x, y, z) = \frac{1}{2}(\pi(x, y) + z)(\pi(x, y) + z + 1) + z$$

5 Q5

(i) $\forall A, B, C \in X, A \subseteq A$, so $(X, \subseteq)$ is reflexive.
if $A \subseteq B$, since $A \neq B \therefore B \nsubseteq A$, so$(X, \subseteq)$ is antisymmetric.
if $A \subseteq B, B \subseteq C$, then $A \subseteq C$, so$(X, \subseteq)$ is transitive.
$\therefore (X, \subseteq)$ is a partial order.
if $A = \{(1,2),(2,3)\}, B = \{(1,3),(2,4)\}$ then $A \nsubseteq B$ and $B \nsubseteq A$, so $(X, \subseteq)$ is not a linear order.
A and B don't have a greatest lower bound, so $(X, \subseteq)$ is not a lattice.
$(X, \subseteq)$ is chain complete because $\forall A \in X$,if A is a linear order, then$\vee A$ is the least upper bound.

(ii) $\forall f_1, f_2 \in X$, if $f_1 \preceq f_2$ and $f_1 = \{(a_1,b_1),(a_2,b_2)...(a_n,b_n)\}, f_2 = \{(a_1,b_1)...(a_n,b_n)...(a_m,b_m)\}$ $(m > n)$
$g_1 = \{(a_1,b_1{+}1),(a_2,b_2{+}1)...(a_n,b_n{+}1)\}, g_2 = \{(a_1,b_1{+}1)...(a_n,b_n{+}1)...(a_m,b_m{+}1)\}$
$\therefore g_1 \preceq g_2 \Rightarrow \phi_1$ is order preserving and it has a fixed point when $f = \emptyset$.

(iii) Assume $f_1 = \{(1,1),(2,2)\}, f_2 = \{(1,1),(2,2),(3,3)\}$, so $f_1, f_2 \in X$ and $f_1 \preceq f_2$. Then $g_1 = \{(0,0),(1,1),(2,2),(3,0),(4,0)...\}, g_2 = \{(0,0),(1,1),(2,2),(3,3),(4,0)...\}$
$g_1 \npreceq g_2 \Rightarrow \phi_2$ is not order preserving and it doesn't has a fixed point because $f_n \neq g_n$.

(iv) Assume $f_1 = \{(1,1),(2,2)\}, f_2 = \{(1,1),(2,2),(3,3)\}$, so $f_1, f_2 \in X$ and $f_1 \preceq f_2$. Then $g_1 = \{(1,0),(2,2),(3,3),(4,0),...\}, g_2 = \{(1,0),(2,2),(3,3),(4,4),(5,0)...\}$
$g_1 \npreceq g_2 \Rightarrow \phi_3$ is not order preserving.
If $f = \{(1,0),(2,1),(3,2),...(n,n-1)...\}$,then $g = \{(1,0),(2,1),(3,2),...(n,n-1)...\}$
So it has a fixed point.

(v) X is not countable.

$\forall M \in P(\mathbb{N}), M = \{N_0, N_1, N_2...\}$, there exists function $f : P(\mathbb{N}) \to X$ such that $f(M) = \{(N_0,0),(N_1,0),(N_2,0)...\}$
Since $N_0 \neq N_1 \neq N_2, f(M)$ is a function $\in X$ and f is a injection$\therefore |P(\mathbb{N})| \leq X$
$\because |\mathbb{N}| < |P(\mathbb{N})| \therefore |\mathbb{N}| < |X|$ and X is uncountable.

6 Q6

Since every non-zero rational number can be represented in the form $\frac{(-1)^k a}{b}$ where $k \in 0, 1$ and $a, b \in \mathbb{N}$ with $a, b \neq 0$ and the greatest common divisor of a and b is 1, let $Q = f(a,b,k)$ and assume $0 = f(0,1,0)$, so there exists an injection $f : \mathbb{Q} \to \mathbb{N} \times \mathbb{N} \times \{0,1\}$

Note that the cardinality of the range of f is actually smaller than $|\mathbb{N} \times \mathbb{N} \times \{0,1\}|$ because the greatest common divisor of a and b is 1.
$\therefore |\mathbb{Q}| \leq |\mathbb{N} \times \mathbb{N} \times \{0,1\}| \leq |\mathbb{N} \times \mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$,so $\mathbb{Q}$ is countable.
Since $\mathbb{N} \subset \mathbb{Q}$, there also exists an injection $g : \mathbb{N} \to \mathbb{Q}, g(x) = x$, so $|\mathbb{N}| \leq |\mathbb{Q}|$
$\therefore |\mathbb{N}| = |\mathbb{Q}|$

7 Q7

$\frac{x(x+1)}{2} \leq 223 + 1 \Rightarrow x < 21$,since $223 = \frac{20*21}{2} + 13$
$\therefore x + y = 20, x = 7, y = 13$