**Q1.** Write the following bijections as products of disjoint cycles and state their order in the group $S_9$:

(i) $f : [9] \longrightarrow [9]$ defined by: for all $n \in [9]$,

$$f(n) = \begin{cases} n+3 & \text{if } n+3 < 9, \\ n+3-9 & \text{if } n+3 \geq 9 \end{cases}$$

(ii) (13)(203)(16)(38)(14)(234)

(iii) (1203)(245)(231)(105)

(iv) (45)(123)(456)(12)

**(4 marks)**

**Q2.** Write the following bijections as products of 2-cycles and state whether they are even or odd:

(i) (1256)(12439)

(ii) $f : [9] \longrightarrow [9]$ defined by: for all $n \in [9]$,

$$f(n) = \begin{cases} n+2 & \text{if } n+2 < 9, \\ n+2-9 & \text{if } n+2 \geq 9 \end{cases}$$

(iii) (0124)(2198)(132568)

(iv) (120)(94567)(0427)

**(4 marks)**

**Q3.** Prove that

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

**(2 marks)**

**Q4.** What is coefficient of $x^{31}y^4$ in the expansion of $((x+y)^8 + y)^7$?
**(2 mark)**

**Q5.** Find the greatest term in the expansion of $(1+4x)^9$ when $x = \frac{1}{3}$.
**(2 marks)**

**Q6.** Prove that $|S_n| = n!$.
**(2 marks)**

**Q7.** Let $(G, \cdot)$ be a group.

(i) Prove that the identity element of $(G, \cdot)$ is unique.

(ii) Prove that for all $x \in G$, $x$ has a unique inverse.

**(2 marks)**

**Q8.** Let $n \geq 3$ and consider $S_n$.

(i) We say that a 2-cycle $(pq)$ is *adjacent* if $p = k$ and $q = k + 1$. Prove that for all $\sigma \in S_n$, if $\sigma$ can be written as an odd number of 2-cycles, then $\sigma$ can be written as an odd number of adjacent 2-cycles, and if $\sigma$ can be written as a product of an even number of 2-cycles, then $\sigma$ can be written as an even number of adjacent 2-cycles.

(ii) For all $\sigma \in S_n$, define

$$P(\sigma) = |\{(k, l) \in [n] \times [n] \mid (k < l) \wedge (\sigma(l) < \sigma(k))\}|$$

Prove that if $(pq)$ is an adjacent cycle and $\sigma \in S_n$, then $P((pq)\sigma) = P(\sigma) \pm 1$.

(iii) Prove that no $\sigma \in S_n$ is both even and odd.

(iv) The *Alternating Group* on $[n]$, denoted $A_n$, is the set of all even bijections in $S_n$. Prove that $A_n$ is a subgroup of $S_n$.

(v) Prove that $|A_n| = \frac{n!}{2}$.

**(12 marks)**

**Q9.** Let $(G, \cdot)$ be a group. Let $x, y \in G$ be such that $xyx^{-1} = y^2$ and $y \neq e$.

(i) Show that $x^5yx^{-5} = y^{32}$.

(ii) If the order of $x$ is 5, then what is the order of $y$? Justify your answer.

**(6 marks)**