UM-SJTU JOINT INSTITUTE
DISCRETE MATHEMATICS
(VE203)

ASSIGNMENT 4

Name: Pan Chongdan
ID: 516370910121

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1 Q1

(i) (147)(258)(036),the order is 3

(ii) (014)(28)(36),the order is 6

(iii) (132)(45),the order is 6

(iv) (13)(56),the order is 2

2 Q2

(i) (1256)(12439)=(16)(15)(29)(23)(24),it's odd.

(ii) (08)(06)(04)(02)(19)(17)(15)(13),it's even.

(iii) (0124)(2198)(132568)=(04)(06)(05)(02)(03)(01)(89),it's odd.

(iv) (120)(94567)(0427)=(04)(09)(02)(01)(07)(06)(05),it's odd.

3 Q3

According to principle of induction: $\binom{1}{0} = \binom{1}{1} = 1 = \frac{1!}{1!0!} = \frac{n!}{(n-k)!k!}$
$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} = \frac{n!}{(n-k)!k!} + \frac{n!}{((n-k+1)!(k-1)!}$
$= \frac{n!}{(n-k)!(k-1)!}(\frac{1}{k} + \frac{1}{n-k+1}) = \frac{n!}{(n-k)!(k-1)!} \times \frac{n+1}{k(n-k+1)} = \frac{(n+1)!}{(n+1-k)!k!}$

4 Q4

$$56 - 8m - n = 31$$

$$m + n = 4$$

$$m = 3, n = 1$$

$$\binom{7}{m} \times \binom{56-8m}{n} = 1120$$

5 Q5

$$a_n = \binom{9}{n} \times (\frac{4}{3})^n$$

The biggest term is $a_5 = \frac{14336}{27}$

6 Q6

According to the principle of induction: $|S_1| = 1$, assume $|S_n| = n!$, and $f_1(x)$ is an arbitrary element of S_n

$$f_1(x) = \begin{cases} f(k_i) = k_{i+1}, i < m \\ f(k_m) = k_1, \\ f(x) = x, & x \notin k_i \cup k_m \end{cases} \tag{1}$$

Assume the n+1 element is S and there are ax such that $f(x_a) = x_a$.It's obvious that $a + m = n$. If we let f(s)=s, there is one way. If we let $f(x_a) = S$ and $f(S) = x_a$ there are a ways. If we let $f(k_i) = S$ and $f(S) = k_{i+1}$ there are m_1 ways. If we let $f(k_m) = S$ and $f(S) = k_1$, there is one way. So there are total n+1 ways. So for any cycle, it can become $n + 1$ new different cycles. So $|S_{n+1}| = |S_n \times (n + 1)| \therefore |S_{n+1}| = n + 1$

7 Q7

(i) Assume there is another identity f, then $e \cdot f = f = e$ but $f \neq e$, so f doesn't exist.

(ii) $\forall x$, assume there exists another inverse x^{-2}, then $(x^{-1} \cdot x) \cdot x^{-2} = x^{-2} \because (x^{-1} \cdot x) \cdot x^{-2} = x^{-1} \cdot (x \cdot x^{-2}) = x^{-1}$ So $x^{-1} = x^{-2}$, which is a contradiction.

8 Q8

(i) $\forall$ 2-cycle (ab), it can be represented as the product of an odd number of adjacent 2-cycles.If $a < b$,then

$$(ab) = [b(b{-}1)][(b{-}1)(b{-}2)] \cdots [(a{+}2)(a{+}1)][a(a{+}1)][(a{+}1)(a{+}2)] \cdots [(b{-}2)(b{-}1)][(b{-}1)b]$$

Since the product of two odd number is odd and the product of an odd number and an even number is even,so if σ can be written as a product of an even number of 2-cycles, then σ can be written as an even number of adjacent 2-cycles.

(ii) if $\sigma(m) = p$,let $(pq)\sigma = \sigma_1$ then $\sigma_1(m) = \sigma(m) + 1 = q$, similarly, if $\sigma(n) = q$, then $\sigma_1(n) = \sigma(n) - 1 = p$.
if $m > n$ then $(m, n) \in \{(k, l) \in [n] \times [n] | (k < l) \wedge (\sigma(l) < \sigma(k))\}$ and $(m, n) \notin \{(k, l) \in [n] \times [n] | (k < l) \wedge (\sigma_1(l) < \sigma_1(k))\} \therefore P(\sigma_1) = P(\sigma) - 1$
if $m < n$,$P(\sigma_1) = P(\sigma) + 1$
if m, n both don't exist, then $(p, q) \notin \{(k, l) \in [n] \times [n] | (k < l) \wedge (\sigma_1(l)]\sigma_1(k))\} \therefore P(\sigma_1) = P(\sigma) + 1$
So $P((pq)\sigma) = P(\sigma) \pm 1$

(iii) Assume σ is both even and odd then

$$\sigma = a_1 \cdot a_2 \cdot ... \cdot a_m = b_1 \cdot b_2 \cdot ... \cdot b_n$$

where m is odd and n is even
$\sigma^{-1} = b_n \cdot b_{n-1} \cdot ... \cdot b_1$
$e = \sigma \cdot \sigma^{-1} = a_1 \cdot a_2 \cdot ... \cdot a_m \cdot b_n \cdot b_{n-1} \cdot ... \cdot b_1$,so e is odd, which is a contradiction.

(iv) Since e is even, $e \in A_n$, $\forall$ bijection $B \in A_n$ if we change $f(x) = y$ then we exchange all pairs of x, y in B and forms a new bijection b, which is also even. $\therefore B^{-1} = b \in A_n$, A_n is a subgroup of S_n

(v) $\forall$ even $\sigma \in S_n, \sigma = a_1 \cdot a_2 \cdot ... \cdot a_m$,m is even,$\sigma_1 = (01) \cdot a_1 \cdot a_2 \cdot ... \cdot a_m$ is odd, and it's a bijection between σ and σ_1. Since $\sigma_1 \in S_n$, the number of even σ and odd σ is same in S_n.
$\therefore |A_n| = \frac{|S_n|}{2} = \frac{n!}{2}$

9 Q9

(i)

$$y^4 = y^2 \cdot y^2 = xyx^{-1} \cdot xyx^{-1} = xy^2x^{-1} = x^2yx^{-2}$$
$$y^8 = y^4 \cdot y^4 = x^2yx^{-2} \cdot x^2yx^{-2} = x^2y^2x^{-2} = x^3yx^{-3}$$

Similarly,

$$y^{16} = x^4yx^{-4}, y^{32} = x^5yx^{-5}$$

(ii)

$$yx^{-5} = y^{32}$$
$$yx^{-5}x^5 = y^{32}x^5$$
$$y = y^{32}$$

If $y's$ order is $n < 31$ then $y^n = e, x^5yx^{-5} = y = y^{32-n} \Rightarrow y^{31-n} = e$
$\therefore (n < 16) \wedge (n|(31 - n)) \Rightarrow n = 1$
$\because y \neq 1 \therefore n = 32$ and y's order is 31.