Vv256 Lecture 8

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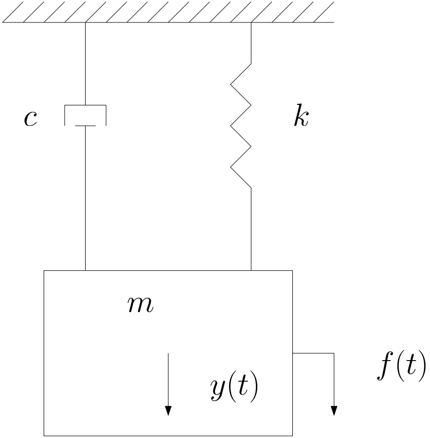
- Suppose we have a mass-spring system, $y(t)$



with mass m, damping coefficient c, spring constant k and an external force

$$f(t)$$

- Assume damping force is proportional to the velocity $\dot{y}$,

$$D(\dot{y}) = -c\dot{y}$$

and Hooke's law

$$R(y) = -ky$$

and Newton's second law, we have

$$m\ddot{y} = -ky - c\dot{y} + f \iff m\ddot{y} + c\dot{y} + ky = f$$

- When the spring is undamped $(c = 0)$ and unforced $(f(t) = 0)$, we have

$$m\ddot{y} + ky = 0 \iff \ddot{y} = -\frac{k}{m}y$$

- The roots for the corresponding characteristic equation are pure imaginary

$$r_{1,2} = \pm \mathrm{i}\sqrt{\frac{k}{m}} \implies y(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t)$$

where $\omega_n = \sqrt{k/m}$ is the natural frequency.

- If treat C_1 and C_2 as the Cartesian coordinates of a point

$$(C_1, C_2)$$

that is determined by the initial conditions

$$y(t_0) = y_0, \qquad \dot{y}(t_0) = y_1$$

then the point has a polar representation

$$C_1 = A\cos\theta \qquad \text{and} \qquad C_2 = A\sin\theta$$

and the general solution has the following form

$$\begin{aligned} y(t) = C_1\cos(\omega_n t) + C_2\sin(\omega_n t) &= A\cos\theta\cos(\omega_n t) + A\sin\theta\sin(\omega_n t) \\ &= A\cos(\omega_n t - \theta) \\ &= A\cos\left(\omega_n\left(t - \frac{\theta}{\omega_n}\right)\right) \end{aligned}$$

- Note A is the amplitude and θ is the phase of the oscillation.

- Now with damping, that is, $c > 0$,

$$m\ddot{y} + c\dot{y} + ky = 0 \iff \ddot{y} + \frac{c}{m}\dot{y} + \omega_n^2 y = 0$$

$$\iff \ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 0$$

where ζ is the damping ratio,

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}}$$

- The corresponding characteristic equation is given by

$$r^2 + 2\zeta\omega_n r + \omega_n^2 = 0$$

with roots

$$r_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

- The form of the solution thus depends on ζ

$$r_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

- If $\zeta = 1$, which is known as the critically damped case $c = c_c$, then

$$y(t) = C_1 e^{-\omega_n t} + C_2 t e^{-\omega_n t}$$

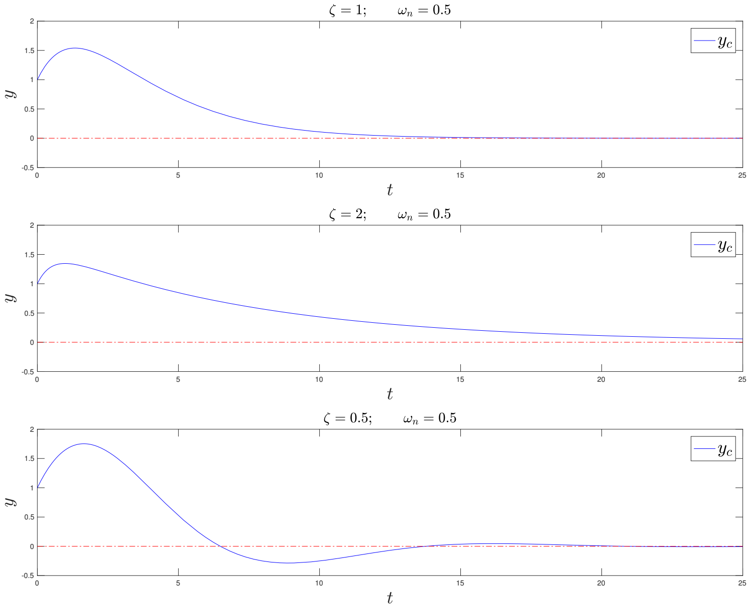
- If $\zeta > 1$, which is known as the overdamped case, then

$$y(t) = C_1 e^{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)t} + C_2 e^{\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)t}$$

- If $\zeta < 1$, which is known as the underdamped case, then

$$y(t) = e^{-\zeta\omega_n t}\left(C_1 \cos\left(t\omega_n\sqrt{1-\zeta}\right) + C_2 \sin\left(t\omega_n\sqrt{1-\zeta}\right)\right)$$

Q: How will the solution in each case behave as $t \to \infty$?



$\zeta = 1; \qquad \omega_n = 0.5$
y
t
y_c
$\zeta = 2; \qquad \omega_n = 0.5$
y
t
y_c
$\zeta = 0.5; \qquad \omega_n = 0.5$
y
t
y_c
2
1.5
1
0.5
0
-0.5
0
5
10
15
20
25

- Now consider harmonically forced vibration, that is, we have the following

$$\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = \frac{F}{m}\cos\omega_d t$$

where ω_d is the driving frequency.

- Assuming a particular solution of it takes the following form

$$y_p(t) = C_1 \cos\omega_d t + C_2 \sin\omega_d t$$

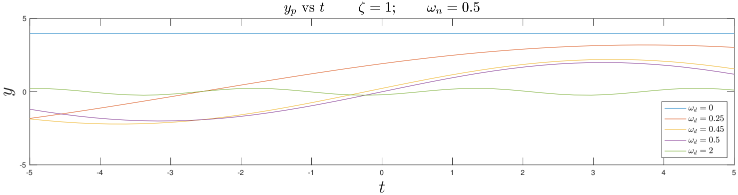
and substituting y_p, $\dot{y}_p$ and $\ddot{y}_p$ into the differential equation, we have

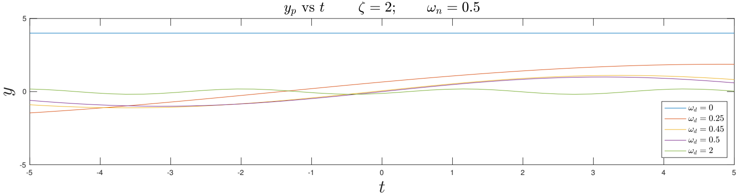
$$\left(-C_1\omega_d^2\cos\omega_d t - C_2\omega_d^2\sin\omega_d t\right) + 2\zeta\omega_n\left(-C_1\omega_d\sin\omega_d t + C_2\omega_d\cos\omega_d t\right)$$
$$+\,\omega_n^2\left(C_1\cos\omega_d t + C_2\sin\omega_d t\right) = \frac{F}{m}\cos\omega_d t$$

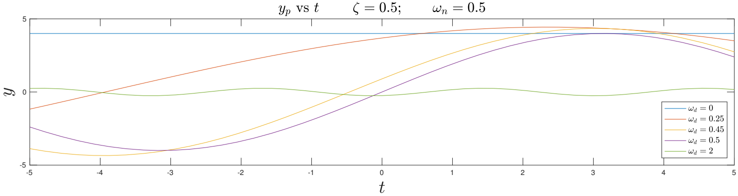
Equating coefficients and solving for C_1 and C_2, we have

$$y_p(t) = \frac{F}{m\left(B_1^2 + B_2^2\right)}\left(B_1\cos\omega_d t + B_2\sin\omega_d t\right) \quad \text{where} \quad \begin{aligned} B_1 &= \omega_n^2 - \omega_d^2 \\ B_2 &= 2\zeta\omega_d\omega_n \end{aligned}$$

Q: Is this always valid? What is the condition for this solution to be valid?


y_p vs t $\zeta = 1$; $\omega_n = 0.5$
y
t
5
0
-5
-5
-4
-3
-2
-1
0
1
2
3
4
5
$\omega_d = 0$
$\omega_d = 0.25$
$\omega_d = 0.45$
$\omega_d = 0.5$
$\omega_d = 2$


y_p vs t $\zeta = 2$; $\omega_n = 0.5$
y
t
5
0
-5
-5
-4
-3
-2
-1
0
1
2
3
4
5
$\omega_d = 0$
$\omega_d = 0.25$
$\omega_d = 0.45$
$\omega_d = 0.5$
$\omega_d = 2$


y_p vs t $\zeta = 0.5$; $\omega_n = 0.5$
y
t
5
0
-5
-5
-4
-3
-2
-1
0
1
2
3
4
5
$\omega_d = 0$
$\omega_d = 0.25$
$\omega_d = 0.45$
$\omega_d = 0.5$
$\omega_d = 2$

- When $\zeta = 0$ and $\omega_d = \omega_n$, that is, resonance in the undamped case

$$\ddot{y} + \omega_n^2 y = \frac{F}{m} \cos \omega_n t$$

- The correct assumption about the form of a particular solution is

$$y_p(t) = t \left(C_1 \cos \omega_n t + C_2 \sin \omega_n t \right)$$

- Substituting into the equation, we have

$$2\omega_n \left(C_2 \cos(\omega_n t) - C_1 \sin(\omega_n t) \right) = \frac{F}{m} \cos \omega_n t$$

which implies the following solution

$$y_p = \frac{1}{2} \frac{F}{\sqrt{mk}} t \sin \omega_n t \implies y = B_1 \cos \omega_n t + B_2 \sin \omega_n t + y_p$$

where B_1 and B_2 are arbitrary constants.

Q: This is clearly unbounded, how about when $\zeta \neq 0$ while $\omega_d = \omega_n$?

- The particular solution to

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \frac{F}{m}\cos\omega_n t$$

is given by

$$y_p(t) = \frac{F}{m\left(B_1^2 + B_2^2\right)}\left(B_1\cos\omega_d t + B_2\sin\omega_d t\right) \quad \text{where} \quad \begin{aligned} B_1 &= \omega_n^2 - \omega_d^2 \\ B_2 &= 2\zeta\omega_d\omega_n \end{aligned}$$
$$= \frac{F}{2m\zeta\omega_n^2}\sin\omega_n t$$

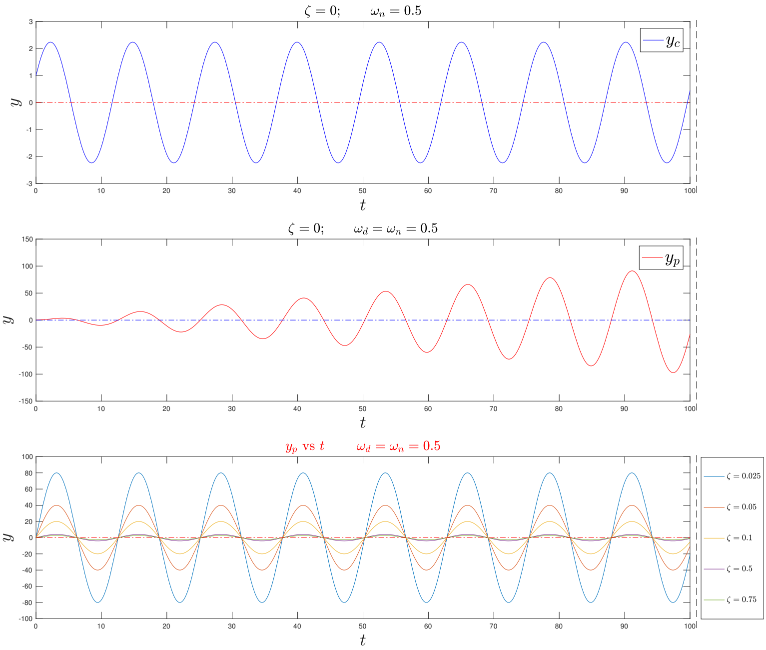
- Under a constant forcing, which is effectively $\omega_d = 0$, we have

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \frac{F}{m} \implies y_p = \frac{F}{m\omega_n^2}$$

- The amplitude is larger by comparison when the damping ratio is small,

$$\zeta < \frac{1}{2}$$

- For really small ζ, the amplitude may be really large, but clearly bounded.


$\zeta = 0; \quad \omega_n = 0.5$
y_c
y
t
$\zeta = 0; \quad \omega_d = \omega_n = 0.5$
y_p
y
t
y_p vs t $\quad \omega_d = \omega_n = 0.5$
$\zeta = 0.025$
$\zeta = 0.05$
$\zeta = 0.1$
$\zeta = 0.5$
$\zeta = 0.75$
y
t

- Recall for a forcing function

$$e^{\lambda t} P(t) \cos \omega_d t$$

the particular solution will take the form

$$y_p(t) = e^{\lambda t}\Big(p(t)\cos\omega_d t + q(t)\sin\omega_d t\Big)$$

where P is a polynomial, and p and q are polynomials of the same degree.

- In general, we could try variation of parameters to solve

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = f(t)$$

- So far we have focused on linear vibration, which can be described by

$$\alpha(t)\ddot{y} + \beta(t)\dot{y} + \gamma(t)y = f(t)$$

- One way to deviate from it is to give up Hooke's law

$$R(t) = -ky$$

- One of many possible assumption on restoring force is

$$R(t) = -ky - ly^3$$

where k and l are constants.

- Notice that Hooke's law is being assumed if

$$l = 0$$

- If $l > 0$, the oscillator is said to hard, and if $L < 0$, it is soft.
- Suppose $m = 1$, and there is no damping, then we have the following

$$\ddot{y} + ky + ly^3 = f(t)$$

- This is sometimes used as an approximation for the pendulum equation

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0$$

- Instead of using linear approximation,

$$\sin\theta \approx \theta \implies \ddot{\theta} + \frac{g}{L}\theta = 0$$

it uses

$$\sin\theta \approx \theta - \frac{\theta^3}{3!} \implies \ddot{\theta} + \frac{g}{L}\left(\theta - \frac{\theta^3}{6}\right) = 0$$

- In general, the homogeneous equation

$$\ddot{y} + py + qy^3 = 0 \qquad \text{where } p \text{ and } q \text{ are constants.}$$

cannot be easily solved, but can be studied by considering another equation.

- Using the integrating factor $\mu = \dot{y}$, we have

$$\dot{y}\left(\ddot{y} + py + qy^3\right) = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}\left(\dot{y}\right)^2 + \frac{1}{2}py^2 + \frac{1}{4}qy^4\right) = 0$$

$$\frac{1}{2}\left(\dot{y}\right)^2 + \frac{1}{2}py^2 + \frac{1}{4}qy^4 = C$$

- The following equation is separable

$$\left(\dot{y}\right)^2 + py^2 + \frac{1}{2}qy^4 = 2C$$

  but the integral isn't trivial!

- However, if $\dot{y}(t_0) = y_0$ is given, then C can be solved and

$$\dot{y} = \Phi(t, y) = \pm\sqrt{2C - py^2 - \frac{1}{2}qy^4}$$

  is autonomous and can be studied using a slope field.