
Question1 (8 points)

Find the derivative y', show all your workings.

(a) (1 point) $y = x(\ln x - 1)$
(b) (1 point) $y = \sinh x$
(c) (1 point) $y = \cosh x$
(d) (1 point) $y = \tanh x$
(e) (1 point) $y = 2^{\cos x + \ln x}$
(f) (1 point) $y = \arccos(3x^2)$
(g) (1 point) $y = -\ln(\cos x)$
(h) (1 point) $y = x \arcsin x + \sqrt{1-x^2}$

where $\sinh x$, $\cosh x$ and $\tanh x$ are known as the hyperbolic functions, that are defined by

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right), \qquad \cosh x = \frac{1}{2}\left(e^x + e^{-x}\right) \qquad \text{and} \qquad \tanh = \frac{\sinh x}{\cosh x}$$

Solution:

(a) The product rule and the derivative of a logarithmic function

$$y' = \ln x - 1 + x\frac{1}{x} = \ln x$$

(b) The derivative of an exponential function

$$y' = \frac{1}{2}\left(e^x + e^{-x}\right) = \cosh x$$

(c) Similar to the last part

$$y' = \frac{1}{2}\left(e^x - e^{-x}\right) = \sinh x$$

(d) Use part (b) and (c), and the quotient rule, we have

$$y' = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = 1 - \tanh^2 x$$

(e) Use the chain rule

$$y' = \frac{d}{dx}\left(2^{\cos x + \ln x}\right) = \frac{d}{dx}\left(e^{(\cos x + \ln x)\ln 2}\right) = 2^{\cos x + \ln x}\left(-\sin x + \frac{1}{x}\right)\ln 2$$

(f) Use the chain rule

$$y' = \frac{6x}{\sqrt{1-9x^4}}$$

(g) Use the chain rule

$$y' = \frac{-1}{\cos x}\cdot(-\sin x) = \tan x$$

(h) By the chain rule and the product rule,

$$y' = \arcsin x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = \arcsin x$$

Question2 (1 points)

Use implicit differentiation to find y' for

$$e^{x+y} + \cos(xy) = 0$$

You may assume that y' exists.

Solution:

1M Differentiate implicitly, we have

$$e^{x+y}(1 + \frac{dy}{dx}) - \sin(xy)(y + x\frac{dy}{dx}) = 0 \implies y' = \frac{y\sin(xy) - e^{x+y}}{e^{x+y} - x\sin(xy)}$$

Question3 (3 points)

Suppose that f is a function with the properties:

1. f is differentiable everywhere
2. $f(x+y) = f(x)f(y)$ for all values of x and y
3. $f(0) \neq 0$
4. $f'(0) = 1$

(a) (1 point) Show that $f(0) = 1$.

Solution:

1M Consider the differentiability of f at $x = 0$, we known $f'(0) = 1$, so use it as A

$$f(0+h) = f(0) + Ah + \varepsilon(h)$$

from being differentiable at $x = 0$, we known the following limit must exit

$$\begin{aligned}
\lim_{h\to 0}\frac{\varepsilon(h)}{h} = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h} - 1 &= 0 \\
\lim_{h\to 0}\frac{f(0)f(h) - f(0)}{h} - 1 &= 0 \qquad \text{Property 2} \\
f(0)\cdot\lim_{h\to 0}\frac{f(h) - 1}{h} - 1 &= 0 \qquad \text{Property 3}
\end{aligned}$$

this limit can only exist if

$$\lim_{h\to 0} f(h) = 1$$

since f is differentiable everywhere and thus continuous everywhere

$$f(0) = 1$$

(b) (1 point) Show that $f(x) > 0$ for all values of x.

Solution:

1M By property 2, for arbitrary $z \in \mathbb{R}$,

$$f(z) = f\left(\frac{z}{2} + \frac{z}{2}\right) = f\left(\frac{z}{2}\right)\cdot f\left(\frac{z}{2}\right) = \left(f\left(\frac{z}{2}\right)\right)^2 \geq 0$$

we still need to eliminate the possibility of the equality. Suppose it is possible to have a number y such that

$$f(y) = 0$$

Let x be an arbitrary real number, then

$$f(x) = f(x - y + y) = f(x - y)f(y) = f(x - y) \cdot 0 = 0$$

Since x is arbitrary, it contradicts the fact $f(0) = 1$, hence no such y exists and

$$f(x) > 0 \qquad \text{for all } x.$$

Note here we restrict ourselves in real. It is a different story in complex.

(c) (1 point) Use the definition of derivative to show that $f'(x) = f(x)$ for all x.

Solution:

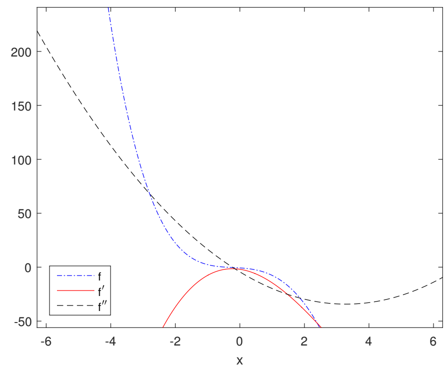
1M By definition, we have

$$f' = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f(x) \lim_{h\to 0} \frac{f(h) - 1}{h} = f(x) \lim_{h\to 0} \frac{f(h) - f(0)}{h}$$
$$= f(x)f'(0) = f(x)$$

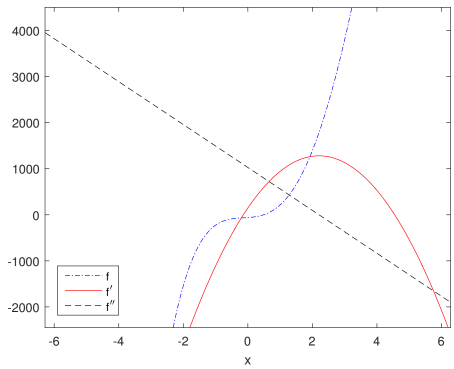
Question4 (2 points)

Is it possible to have $f(x)$ indicated by the following pictures? If not, explain why not.

(a) (1 point) f, f' and f'' are shown



(b) (1 point) f, f' and f'' are shown



Solution:

(a) There is no contradiction. Hence it is possible.

(b) The function is always increasing, however, the first derivative indicated by the graph is negative before $x = 2$, thus it is not possible.

Question5 (2 points)

(a) (1 point) Is there a differentiable function defined on $(-1, 1)$ which has no relative maximum but has got an absolute maximum? If there is one, sketch it, if not, explain.

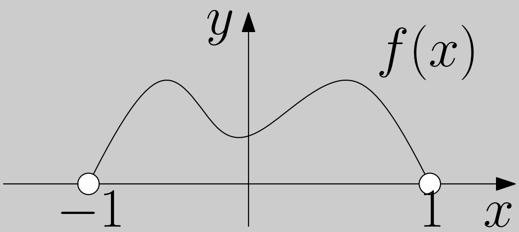
Solution:

1M No. It is not possible. Absolute maxima occur only at critical points or boundary points. Since the boundary points are not in the domain. It can only take place at the critical points, which will also give a relative maximum if it is an absolute maximum.

(b) (1 point) Is there a differentiable function defined on $(-1,1)$ which has no absolute minimum but has got a relative minimum? If there is one, sketch it, if not, explain.

Solution:

1M Yes, it is possible. For example,



Question6 (4 points)

The three cases in the first derivative test cover the situations one commonly encounters but do not exhaust all possibilities. Consider the function f, g, and h whose values at 0 are all 0 and, for $x \neq 0$,

$$f(x) = x^4 \sin\frac{1}{x}, \qquad g(x) = x^4\left(2+\sin\frac{1}{x}\right), \qquad h(x) = x^4\left(-2+\sin\frac{1}{x}\right)$$

(a) (1 point) Show that 0 is a critical number of all three functions but their derivatives change sign infinitely often on both sides of 0.

Solution:

1M For $x \neq 0$, the derivative functions are

$$\begin{aligned}
f'(x) &= 4x^3\sin\frac{1}{x} - x^2\cos\frac{1}{x} = x^2\underbrace{\left(4x\sin\frac{1}{x} - \cos\frac{1}{x}\right)}_{A}\\
g'(x) &= 4x^3\left(\sin\frac{1}{x}+2\right) - x^2\cos\frac{1}{x} = x^2\underbrace{\left(4x\sin\frac{1}{x} + 8x - \cos\frac{1}{x}\right)}_{B}\\
h'(x) &= 4x^3\left(\sin\frac{1}{x}-2\right) - x^2\cos\frac{1}{x} = x^2\underbrace{\left(4x\sin\frac{1}{x} - 8x - \cos\frac{1}{x}\right)}_{C}
\end{aligned}$$

If we consider the limit of each, we need the followings

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0 \qquad \text{and} \qquad \lim_{x\to 0} x\cos\frac{1}{x} = 0$$

We have shown the first limit to be zero in class by the squeeze theorem, the second limit can be shown to be zero in a similar fashion. Then the limits are

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0}\left(4x^3\sin\frac{1}{x} - x^2\cos\frac{1}{x}\right)$$
$$= 4\left(\lim_{x\to 0} x^2\right)\cdot\left(\lim_{x\to 0} x\sin\frac{1}{x}\right) - \left(\lim_{x\to 0} x\right)\cdot\left(\lim_{x\to 0} x\cos\frac{1}{x}\right) = 0$$

$$\lim_{x\to 0} g'(x) = 0 \qquad \text{and} \qquad \lim_{x\to 0} h'(x) = 0$$

Now consider the limit of A, B and C, the first term of A and the first two terms of B and C are approaching zero as $x \to 0$, however, $\cos\frac{1}{x}$ takes value between -1 and 1 for $x \in (-\delta, \delta)$, where $\delta > 0$. No matter how small δ is, the value of $\cos\frac{1}{x}$ is oscillating between -1 and 1. Hence the derivatives change sign infinitely on both sides of 0 for x^2 is always positive.

(b) (3 points) Show that f has neither a local maximum nor a local minimum at 0, g has a local minimum, and h has a local maximum.

Solution:

1M For f, we need to show there must be numbers x_1 and x_2 such that

$$f(0) < f(x_1) \qquad \text{and} \qquad f(0) > f(x_2)$$

specifically, we need to find x_1 and x_2 in every δ-neighbourhood of 0 such that

$$f(x_1) > 0 \qquad \text{and} \qquad f(x_2) < 0$$

Because sine is an odd function, for every $x \in (-\delta, \delta)$ that is not equal to

$$\frac{1}{k\pi} \qquad \text{where} \qquad k \in \mathbb{Z}$$

the sine function is either positive or negative,

$$\sin\left(\frac{1}{x}\right) > 0 \qquad \text{or} \qquad \sin\left(\frac{1}{x}\right) < 0$$

When

$$\sin\frac{1}{x} < 0 \implies \sin\frac{-1}{x} > 0$$
$$\implies f(x) = x^4\sin\frac{1}{x} < 0 \implies f(-x) = (-x)^4\sin\frac{-1}{x} > 0$$

Similarly, if $f(x) > 0$, then $f(-x) < 0$. Since every δ-neighbourhood of 0 has

$$\text{both} \qquad x^* \neq \frac{1}{k\pi} \qquad \text{and} \qquad -x^* \qquad \text{for all} \qquad k \in \mathbb{Z}$$

Hence x^* and $-x^*$ are the values of x_1 and x_2 that we are looking for such that

$$f(x_1) > 0 \qquad \text{and} \qquad f(x_2) < 0$$

Therefore f has neither a local maximum nor a local minimum at 0.

1M For g, we need to show, for some $\delta > 0$

$$g(x) \geq g(0) \qquad \text{for all} \qquad x \in (-\delta, \delta)$$

In fact, for all $x \neq 0$, we have

$$\sin\frac{1}{x} \geq -1 \implies 2 + \sin\frac{1}{x} \geq 1 \implies x^4\left(2 + \sin\frac{1}{x}\right) \geq x^4 \implies g(x) \geq x^4 > 0$$

Since

$$g(0) = 0$$

this shows it is actually the minimum for every neighbourhood of 0.

1M The argument is essentially the same for h.

Question7 (2 points)
Use l'Hôpital's rule to find the limits.

(a) (1 point) $\lim_{x\to 1^+}\left(\frac{1}{x-1} - \frac{1}{\ln x}\right)$

(b) (1 point) $\lim_{x\to(\pi/2)^-} (\tan x)^{(\pi/2)-x}$

Solution:

(a) Combine the two terms, we have an indeterminate form of $0/0$.

$$\begin{aligned}
\lim_{x\to 1^+}\left(\frac{1}{x-1} - \frac{1}{\ln x}\right) &= \lim_{x\to 1^+}\frac{\ln x - x + 1}{(x-1)\ln x} \\
&\overset{\text{LH}}{=} \lim_{x\to 1^+}\frac{1/x - 1}{\ln x + (x-1)/x} \\
&= \lim_{x\to 1^+}\frac{1-x}{x\ln x + (x-1)} \\
&\overset{\text{LH}}{=} \lim_{x\to 1^+}\frac{-1}{\ln x + 2} = -\frac{1}{2}
\end{aligned}$$

(b) Consider the logarithmic transformation, and consider the limit of $\ln y$.

$$\begin{aligned}
y = (\tan x)^{(\pi/2)-x} &\implies \ln y = \left(\frac{\pi}{2} - x\right)\ln(\tan x) \\
&\implies \lim_{x\to(\pi/2)^-}\ln y = \lim_{x\to(\pi/2)^-}\left(\frac{\pi}{2} - x\right)\ln(\tan x) \\
&= \lim_{x\to(\pi/2)^-}\frac{\left(\frac{\pi}{2} - x\right)}{\frac{1}{\ln(\tan x)}} \qquad \frac{0}{0} \\
&= \lim_{x\to(\pi/2)^-}\frac{\Big(\ln(\tan x)\Big)^2\tan x}{\tan^2 x + 1} \qquad \text{LH}
\end{aligned}$$

Let $T = \tan x$, then $T \to \infty$ as $x \to (\pi/2)^-$. With a few applications of LH

$$\begin{aligned}
\lim_{x\to(\pi/2)^-}\ln y = \lim_{T\to\infty}\frac{T\Big(\ln(T)\Big)^2}{T^2+1} = \lim_{T\to\infty}\frac{(\ln T)^2 + 2\ln T}{2T} &= \lim_{T\to\infty}\frac{\ln T + 1}{T} \\
&= \lim_{T\to\infty}\frac{1}{T} = 0
\end{aligned}$$

Since the exponential function is continuous,

$$\lim_{x\to(\pi/2)^-} y = \lim_{x\to(\pi/2)^-} e^{\ln y} = \exp\left(\lim_{x\to(\pi/2)^-} \ln y\right) = \exp(0) = 1$$

Question8 (1 points)
Show the inequality $\dfrac{\arctan x_2 - \arctan x_1}{x_2 - x_1} \leq 1$ is true when $x_2 > x_1$.

Solution:

1M Consider the following function

$$f(x) = \arctan x$$

Since f is continuous and differentiable on $\mathbb{R}$. The MVT is applicable, that is, there is a number a between any x_1 and x_2 such that

$$\frac{\arctan x_2 - \arctan x_1}{x_2 - x_1} = f'(a)$$

However,

$$\begin{aligned} f' = \frac{1}{1+a^2} \implies f' \leq 1 \implies & \frac{\arctan x_2 - \arctan x_1}{x_2 - x_1} \leq 1 \\ \implies & \arctan x_2 - \arctan x_1 \leq x_2 - x_1 \end{aligned}$$

Since $x_2 > x_1$.

Question9 (1 points)
Find all the real solutions to the equation $2^x + 5^x = 3^x + 4^x$. Justify your answers.

Solution:

1M We can see that $x = 0$ and $x = 1$ are two possible solutions. To find out whether there is any other solution, we rewrite the equation as

$$5^x - 4^x = 3^x - 2^x \tag{1}$$

Now, we consider a function

$$f(t) = t^x, \qquad \text{where } x \text{ is a constant.}$$

Clearly $f(t)$ is a continuous and differentiable on $(0, \infty)$. Thus we can apply MVT, there exists $t_1 \in (2, 3)$ such that

$$f'(t_1) = \frac{f(3) - f(2)}{3 - 2} \implies x t_1^{x-1} = 3^x - 2^x \tag{2}$$

Similarly, there exists $t_2 \in (4, 5)$ such that

$$x t_2^{x-1} = 5^x - 4^x \tag{3}$$

Combine (1), (2) and (3), we have

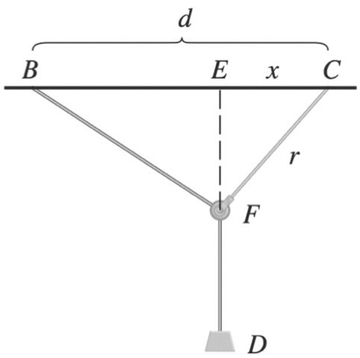
$$xt_1^{x-1} = xt_2^{x-1} \implies \begin{array}{l} x(t_1^{x-1} - t_2^{x-1}) = 0 \implies x = 0 \\ \left(\dfrac{t_1}{t_2}\right)^{x-1} = 1 \implies (x-1)\ln\left(\dfrac{t_1}{t_2}\right) = \ln 1 = 0 \implies x = 1 \end{array}$$

Since $t_1 \neq t_2$. Therefore, $x = 0$ and $x = 1$ are the only solution.

Question10 (1 points)

One of the problems posed by the Marquis de L'Hospital in his calculus textbook *Analyse des Infiniment Petits* concerns a pulley that is attached to the ceiling of a room at a point C by a rope of length r. At another point B on the ceiling, at a distance d from C (where $d > r$), a rope of length l is attached and passed through the pulley F and connected to a weight W. The weight is released and comes to rest at its equilibrium position D. As L'Hospital argued, this happens when the distance $|ED|$ is maximized. Show that when the system reaches equilibrium, the value of x is

$$\frac{r}{4d}\left(r + \sqrt{r^2 + 8d^2}\right)$$



Solution:

1M Firstly l'Hôpital's argument is based on the fact that the equilibrium would take place when the gravitational potential energy is minimized, and that happens when $|ED|$ is maximized. Let $a = |EF|$ and $b = |BF|$, thus

$$|ED| = l - b + a$$

By the Pythagorean theorem, the distance $|ED|$ is given by

$$\begin{aligned} f(x) = |ED| &= l - \sqrt{(d-x)^2 + r^2 - x^2} + \sqrt{r^2 - x^2} \\ &= l - \left[(d-x)^2 + r^2 - x^2\right]^{1/2} + \left(r^2 - x^2\right)^{1/2} \end{aligned}$$

Note this is a maximization problem of a continuous function $f(x)$ on the closed interval $[0, r]$. The extreme-value theorem guarantees the existence of the absolute maximum. Consider the derivative function of $f(x)$

$$\begin{aligned} f'(x) &= -\frac{1}{2}\left[(d-x)^2 + r^2 - x^2\right]^{-1/2} \cdot (-2(d-x) - 2x) + \frac{1}{2}\left(r^2 - x^2\right)^{-1/2} \cdot (-2x) \\ &= d\left[(d-x)^2 + r^2 - x^2\right]^{-1/2} - x\left(r^2 - x^2\right)^{-1/2} \\ &= \frac{d\left(r^2 - x^2\right)\left[(d-x)^2 + r^2 - x^2\right]^{1/2} - x\left(r^2 - x^2\right)^{1/2}\left[(d-x)^2 + r^2 - x^2\right]}{\left((d-x)^2 + r^2 - x^2\right)\left(r^2 - x^2\right)} \end{aligned}$$

Look for critical points, firstly when f' is undefined,

$$\left((d-x)^2+r^2-x^2\right)\left(r^2-x^2\right)=0 \implies x_1=r \qquad \text{and} \qquad x_2=\frac{d^2+r^2}{2d}$$

With some manipulation, we see

$$x_2-r=\frac{d^2+r^2}{2d}-\frac{2dr}{2d}=\frac{(d-r)^2}{2d}>0 \qquad \text{since } d>r.$$

So $x_2>r$ and $x_2\notin[0,r]$, thus it is not the local maximum we are looking for

$$x\neq\frac{d^2+r^2}{2d}$$

Then let us look for stationary points, $f'=0$

$$\begin{aligned} d^2\left(r^2-x^2\right)^2\left[(d-x)^2+r^2-x^2\right] &= x^2\left(r^2-x^2\right)\left[(d-x)^2+r^2-x^2\right]^2 \\ \implies d^2r^2-d^2x^2 &= x^2d^2-2dx^3+r^2x^2 \\ \implies (x-d)\left[2dx^2-r^2(x+d)\right] &= 0 \end{aligned}$$

$$\implies x_3=d, \qquad x_4=\frac{r^2-\sqrt{r^4+8d^2r^2}}{4d} \qquad \text{and} \qquad x_5=\frac{r^2+\sqrt{r^4+8d^2r^2}}{4d}$$

Recall $d>r$, so $x_3>r$ and $x_2\notin[0,r]$. And notice x_4 is clearly negative, so $x_4\notin[0,r]$. Hence the only valid candidate is

$$x=x_5=\frac{r^2+\sqrt{r^4+8d^2r^2}}{4d}=\frac{r}{4d}\left(r+\sqrt{r^2+8d^2}\right)$$

Recall this is a maximization problem of a continuous function $f(x)$ on a closed interval $[0,r]$. The extreme-value theorem guarantees the existence of the absolute maximum. We need to compare the function values at

$$x_1=r \qquad x_5=\frac{r}{4d}\left(r+\sqrt{r^2+8d^2}\right) \qquad \text{and} \qquad x_6=0$$

We could be brutal and evaluate f at all three values, and try to establish some inequalities and decide amongst the three. However, if we consider f' near 0

$$\lim_{x\to0^+}f'(x)=f'(0)=d[d^2+r^2]^{-1/2}>0$$

it will show there is an interval $[0,\delta)$ for some $\delta>0$, where the function is strictly increasing. Similarly, if we consider the derivative of f near r,

$$\lim_{x\to r^-}f'(x)=\lim_{x\to r^-}\left\{d\left[(d-x)^2+r^2-x^2\right]^{-1/2}-x\left(r^2-x^2\right)^{-1/2}\right\}=-\infty$$

it will show that there is an interval (δ,r) for some $0<\delta<r$, where the function is strictly decreasing. Being the only critical point between $x=0$ and $x=r$, x_5 must be a local maximum, that is, the function is strictly increasing on $[0,x_5]$ and is strictly decreasing on $[x_5,r]$ in this case. Hence $f(x_5)$ must be bigger than both $f(x_1)$ and $f(x_6)$. Therefore the absolute maximum and the equilibrium occur at

$$x=x_5=\frac{r}{4d}\left(r+\sqrt{r^2+8d^2}\right)$$

Note it depends neither on the weight W nor the length l.