
Question1 (2 points)

Consider the following differential operator

$$p(\mathcal{D}) = \mathcal{D}^2 + 2\mathcal{D} - 15$$

(a) (1 point) Find the transfer function for

$$p(\mathcal{D})[y] = f(t); \qquad y(0) = 0, \quad \dot{y}(0) = 0$$

(b) (1 point) Find the Green's function $G(t;0)$ associated with

$$p(\mathcal{D})[y] = f(t); \qquad y(0) = 0, \quad \dot{y}(0) = 0$$

Question2 (1 points)

Discuss whether the set of all functions $f\colon [a,b] \to \mathbb{R}$, where $a, b \in \mathbb{R}$, such that

$$\left(\int_a^b f^2\,dx\right) < \infty$$

is a vector space under the usual addition and scalar multiplication.

Question3 (1 points)

Determine whether the set of solutions to the following equation is a vector space.

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0, \qquad \text{where } g \text{ and } l \text{ are two real constants.}$$

Question4 (1 points)

Given a matrix $\mathbf{M}$ of $n \times n$, and let

$$\mathcal{H} = \left\{\mathbf{A} \mid \mathbf{AM} = \mathbf{MA}\right\}$$

Show that $\mathcal{H}$ is a subspace of the vector space of all matrices of $n \times n$.

Question5 (1 points)

Find a basis for $\operatorname{span}(\mathcal{S})$ where $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ and

$$\mathbf{v}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}-1\\1\\4\end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix}3\\3\\2\end{bmatrix} \quad \text{and} \quad \mathbf{v}_4 = \begin{bmatrix}-2\\-4\\-6\end{bmatrix}$$

Question6 (1 points)

Find the characteristics equation for an arbitrary 2×2 matrix $\mathbf{A} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$.

Question7 (1 points)

Compute $\det(\mathbf{A})$, where $\mathbf{A} = \begin{bmatrix}3 & -7 & 8 & 9 & 6\\ 0 & 2 & -5 & 7 & 3\\ 0 & 0 & 1 & 5 & 0\\ 0 & 0 & 2 & 4 & -1\\ 0 & 0 & 0 & -2 & 0\end{bmatrix}$.

Question8 (2 points)

Find the eigenvalues and the corresponding eigenvectors of the matrix $\mathbf{A} = \begin{bmatrix}7 & 0 & -3 & 0\\ -9 & -2 & 3 & 0\\ 18 & 0 & -8 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$.

Question9 (0 points)

(a) Show that the Green's function associated with the damped spring-mass system

$$m\ddot{y} + c\dot{y} + ky = f(t), \qquad y(0) = 0, \qquad \dot{y}(0) = 0$$

for each of the three cases of damping is as follows. And write down the solutions.

i. (1 point (bonus)) Underdamped:

$$G(t; a) = \frac{1}{\mu} e^{-c(t-a)/2m} \sin\Big((t-a)\mu\Big) \qquad \text{where} \qquad \mu = \frac{(4mk - c^2)^{1/2}}{2m}$$

ii. (1 point (bonus)) Critically damped:

$$G(t; a) = (t-a)e^{-c(t-a)/2m}$$

iii. (1 point (bonus)) Overdamped:

$$G(t; a) = \frac{1}{\nu} e^{-c(t-a)/2m} \sinh\Big((t-a)\nu\Big) \qquad \text{where} \qquad \nu = \frac{(c^2 - 4mk)^{1/2}}{2m}$$

(b) (1 point (bonus)) Solve the following initial-value problems.

$$t^2\ddot{y} + t\dot{y} + 4y = \sin(\ln t); \qquad y(1) = 1, \quad \dot{y}(1) = 0$$

(c) (1 point (bonus)) Show that if $\{\mathbf{u}, \mathbf{v}\}$ is a basis for a vector space over $\mathbb{C}$, then so is

$$\left\{ \frac{1}{2}(\mathbf{u} + \mathbf{v}), \frac{1}{2\mathrm{i}}(\mathbf{u} - \mathbf{v}) \right\}$$

(d) (1 point (bonus)) Show the following is true for a finite-dimensional vector space $\mathcal{V}$.

$$\dim(\mathcal{U}_1 \cup \mathcal{U}_2) = \dim \mathcal{U}_1 + \dim \mathcal{U}_2 - \dim(\mathcal{U}_1 \cap \mathcal{U}_2)$$

where $\mathcal{U}_1$ and $\mathcal{U}_2$ are two subspaces of $\mathcal{V}$.

(e) (1 point (bonus)) Here is a question for those who are taking discrete mathematics. A ring is a set $\mathcal{K}$, together with addition and multiplication

$$(x, y) \to x + y$$
$$(x, y) \to xy$$

such that $\mathcal{K}$ is commutative group under addition, multiplication is associative and distributive with respect to addition. If $xy = yx$ for all x and y, the ring $\mathcal{K}$ is commutative. If there is an element 1 in $\mathcal{K}$ such that $1x = x1 = x$ for each x, $\mathcal{K}$ is a ring with identity. Let $\mathcal{K}$ be a commutative ring. A nonempty set $\mathcal{M}$ is module over $\mathcal{K}$ if $\mathcal{M}$ is commutative group under addition, and there is a scalar multiplication, that is, for each element in $\mathcal{K}$ and each element in $\mathcal{M}$, there is an element in $\mathcal{M}$. Furthermore, the scalar multiplication is associative and distributive. Discuss the similarity between a vector space over a field and a module over a ring. Discuss how to define a determinant

$$D\colon \mathcal{M}^n \to \mathcal{K} \qquad \text{where} \quad \mathcal{M} = \mathcal{K}^n$$

and $\mathcal{K}$ is commutative and thus $\mathcal{M}^n$ is the ring of all $n \times n$ matrices. Show D is unique.