Vv156 Lecture 18

Dr Jing Liu

UM-SJTU Joint Institute

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- We are to develop a general method for solving integrals of the form

$$\boxed{\int f(x)g(x)\,dx}$$

- Suppose $G(x)$ an antiderivative of $g(x)$, that is, $\boxed{G'(x) = g(x)}$, and consider

$$\frac{d}{dx}\Big[f(x)G(x)\Big] = f(x)\cdot \boxed{G'(x)} + f'(x)\cdot G(x) = f(x)\cdot \boxed{g(x)} + f'(x)\cdot G(x)$$

- This states that $f(x)G(x)$ is an antiderivative of RHS, so in integral notation,

$$f(x)G(x) = \int \Big[f(x)g(x) + f'(x)G(x)\Big]\,dx = \boxed{\int f(x)g(x)\,dx} + \int f'(x)G(x)\,dx$$

Theorem

Suppose f and g are continuous, and f' is continuous, then

$$\boxed{\int f(x)g(x)\,dx} = f(x)G(x) - \int f'(x)G(x)\,dx, \qquad \text{where} \quad G'(x) = g(x)$$

- The application of the last theorem is called integration by parts.
- The idea behind integration by parts is to choose one of the factors so that

$$\int f(x)\, g(x)\, dx = f(x)\, G(x) - \int f'(x)\, G(x)\, dx, \quad \text{where} \quad G'(x) = g(x)$$

it becomes "simpler" when differentiated, while antiderivatives of the other factor are readily available.

Exercise

Use integration by parts to find

$$\int x^3 \ln x \, dx$$

- An antiderivative of $\ln x$ can be found using integration by parts.

- Suppose f is a differentiable function, then

$$\begin{aligned}\int f(x)\,dx &= \int f(x)g(x)\,dx, && \text{where} \quad g(x)=1\\ &= f(x)G(x)-\int f'(x)G(x)\,dx, && \text{where} \quad G'(x)=g(x)\\ &= f(x)x-\int xf'(x)\,dx\end{aligned}$$

- Therefore the integral $\int \ln x\,dx$ can be easily determined,

$$\begin{aligned}\int \ln x\,dx &= x\ln x-\int x\frac{1}{x}\,dx\\ &= x\ln x-\int 1\,dx\\ &= x\ln x-x+c=x(\ln x-1)+c\end{aligned}$$

- The theorem of integration by parts is also true for definite integral, again $f(x)G(x)$ is an antiderivative of $f(x)\,g(x) + f'(x)G(x)$ by the chain rule.
- Note the sum is a continuous function under our hypotheses, so FTC states

$$\int_a^x \Big(f(t)\cdot g(t) + f'(t)\cdot G(t)\Big)\,dt \qquad \text{is an antiderivative as well.}$$

- Hence the two functions are equal up to an additive constant c,

$$f(x)G(x) + c = \int_a^x \Big(f(t)g(t) + f'(t)G(t)\Big)\,dt$$

- For $x = a$, we see that

$$f(a)G(a) + c = \int_a^a \Big(f(t)g(t) + f'(t)G(t)\Big)\,dt = 0 \implies c = -f(a)G(a)$$

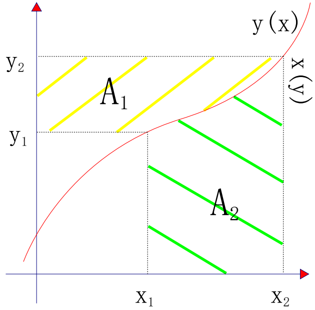
- So if $x = b$, and replace the dummy variable t,

$$\int_a^b f(x)g(x)\,dx + \int_a^b f'(x)G(x)\,dx = f(b)G(b) - f(a)G(a) = \Big[f(x)G(x)\Big]_a^b$$

$$\int_a^b f(x)g(x)\,dx = \Big[f(x)G(x)\Big]_a^b - \int_a^b f'(x)G(x)\,dx$$

- To understand it geometrically, consider the area under a smooth injective curve

$$A_1 = \int_{y_1}^{y_2} x(y)\,dy; \quad A_2 = \int_{x_1}^{x_2} y(x)\,dx$$



- However, the area can also be evaluated as

$$A_1 + A_2 = x_2y_2 - x_1y_1 = \Big[x \cdot y(x)\Big]_{x_1}^{x_2}$$

$$\boxed{\int_{y_1}^{y_2} x(y)\,dy} + \int_{x_1}^{x_2} y(x)\,dx = \Big[x \cdot y(x)\Big]_{x_1}^{x_2}$$

- If $y = f(x) \implies \boxed{\int_{y_1}^{y_2} x(y)\,dy} = \int_{x_1}^{x_2} xf'(x)\,dx$ by the method of substitution.

- And if we introduce $G(x) = x$ and $g(x) = G'(x) = 1$, then

$$\int_{x_1}^{x_2} f(x)g(x)\,dx = \Big[f(x)G(x)\Big]_{x_1}^{x_2} - \boxed{\int_{x_1}^{x_2} f'(x)G(x)\,dx}$$

- Integration by parts is very useful, what we need is to determine $f(x)$ and $g(x)$.
- One strategy often works for choosing $f(x)$ and $g(x)$ is known as LIATE.

> *choose $f(x)$ to be the function whose category occurs earlier in the following list and take $g(x)$ to be the remaining factor.*

Logarithmic, Inverse trigonometric, Algebraic, Trigonometric, Exponential

Exercise

Use integration by parts to evaluate

(a) $\int x \cos x \, dx.$

(b) $\int x^2 e^{-x} \, dx.$

(c) $\int e^x \cos x \, dx.$