Question1 (1 points)

Find the point on the parabola

$$x = t, \qquad y = t^2 \qquad \text{for} \quad -\infty < t < \infty$$

closest to the point $(2, 1/2)$.

Solution:

1M The distance from a point on the parabola to the point $(2, 1/2)$ will be minimized if

$$f(t) = (x-2)^2 + (y-1/2)^2 = (t-2)^2 + (t^2-1/2)^2$$

is minimized. We simply differentiate f with respect to t,

$$f'(t) = 2t + 4t\left(t^2 - \frac{1}{2}\right) - 4 \implies f''(t) = 12t^2$$

Setting $f' = 0$, and by the second derivative test, we can conclude that

$$t = 1 \implies f(1) = \frac{5}{2}$$

is a local minimum. Since the function f is decreasing on $(-\infty, 1)$ and increasing on $(1, \infty)$. It is also the global minimum that we are looking for

$$\left(1, \frac{5}{2}\right)$$

Question2 (1 points)

Find the volume swept out by revolving the region bounded by one arch of the curve

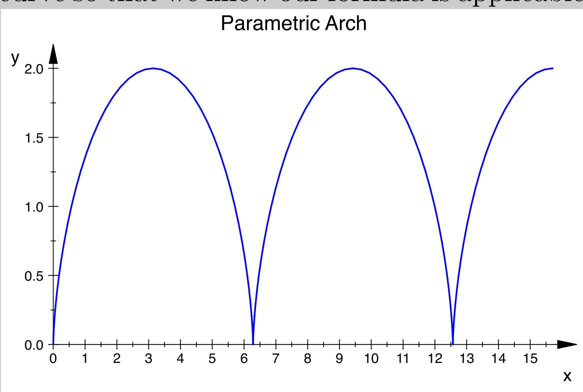
$$x = t - \sin t, \qquad y = 1 - \cos t$$

and the x-axis about the x-axis.

Solution:

1M It is essential to sketch a graph for the curve so that we know our formula is applicable

$$\text{Volume} = \int_0^{2\pi} \pi y^2 \, dx$$



applying u-substitution formula in reverse $\int_\alpha^\beta f(g(t))g'(t)\,dt = \int_a^b f(x)\,dx$ with

$$x = g(t) = t - \sin t \implies g'(t) = 1 - \cos t$$

Thus the volume can be evaluated by the following

$$\text{Volume} = \pi\int_0^{2\pi} y(t)^2 g'(t)\,dt = \pi\int_0^{2\pi}(1-\cos t)^2(1-\cos t)\,dt = 5\pi^2$$

Question3 (1 points)

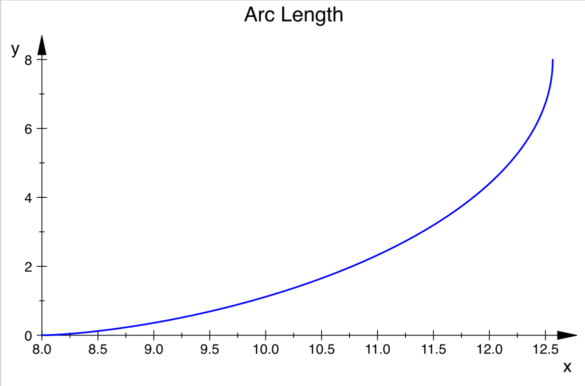
Find the length of the curve

$$x = 8\cos t + 8t\sin t, \qquad y = 8\sin t - 8t\cos t, \qquad 0 \le t \le \pi/2$$

Solution:

1M It is essential to sketch a graph for the curve so that we know our formula is applicable

$$\text{Arc Length} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$



applying u-substitution formula in reverse $\int_\alpha^\beta f(g(t))g'(t)\,dt = \int_a^b f(x)\,dx$ with

$$x = g(t) = 8\cos t + 8t\sin t \implies g'(t) = 8t\cos t$$

Thus the length can be evaluated by the following

$$\text{Arc Length} = \int_0^{\pi/2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\frac{dx}{dt}\,dt$$

$$= \int_0^{\pi/2}\sqrt{1+\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\frac{dx}{dt}\,dt = \int_0^{\pi/2}\sqrt{\left(\frac{dy}{dt}\right)^2+\left(\frac{dx}{dt}\right)^2}\,dt$$

$$= \int_0^{\pi/2}\sqrt{(8t\sin t)^2+(8t\cos t)^2}\,dt = \pi^2$$

Question4 (1 points)

Find the coordinates of the centroid of the curve

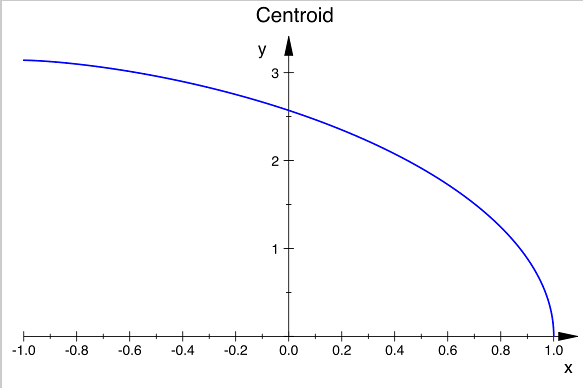
$$x = \cos t, \qquad y = t + \sin t, \qquad 0 \le t \le \pi$$

Solution:

1M It is essential to sketch a graph for the curve so that we know our formula is applicable

$$\bar{x} = \frac{\displaystyle\int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}{\displaystyle\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$$

$$\bar{y} = \frac{\displaystyle\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}{\displaystyle\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$$



applying u-substitution formula in reverse $\displaystyle\int_\alpha^\beta f\big(g(t)\big)g'(t)\,dt = \int_a^b f(x)\,dx$ with

$$x = g(t) = \cos t \implies g'(t) = -\sin t$$

Thus the centroid can be evaluated by the following

$$\bar{x} = \frac{\displaystyle\int_0^\pi \cos t\sqrt{\sin^2 t + (1+\cos t)^2}\,dt}{\displaystyle\int_0^\pi \sqrt{\sin^2 t + (1+\cos t)^2}\,dt} = \frac{\displaystyle\int_0^\pi \cos t\sqrt{\cos t + 1}\,dt}{\displaystyle\int_0^\pi \sqrt{\cos t + 1}\,dt} = \frac{1}{3}$$

$$\bar{y} = \frac{\displaystyle\int_0^\pi (t+\sin t)\sqrt{\sin^2 t + (1+\cos t)^2}\,dt}{\displaystyle\int_0^\pi \sqrt{\sin^2 t + (1+\cos t)^2}\,dt} = \frac{\displaystyle\int_0^\pi (t+\sin t)\sqrt{\cos t + 1}\,dt}{\displaystyle\int_0^\pi \sqrt{\cos t + 1}\,dt} = \pi - \frac{4}{3}$$

Question5 (1 points)

Find the area of the region bounded by the spiral

$$r = \theta \qquad \text{for} \quad 0 \le \theta \le \pi$$

Solution:

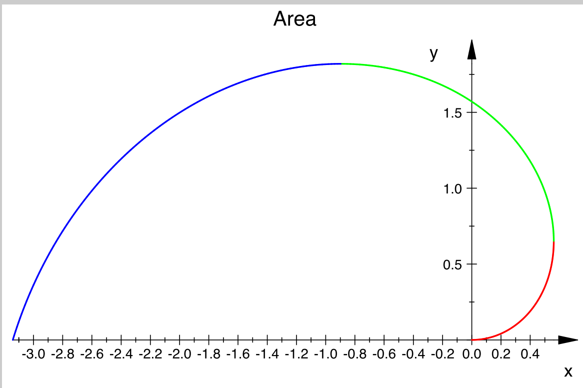
1M We actually need to be careful since the curve cannot be defined by a single $y = f(x)$

$$\text{Area} = \int_{-\pi}^{a} y_2\,dx - \int_0^a y_1\,dx$$

where y_1 is the red portion

$$y_1 = \theta\sin\theta \quad \text{for} \quad 0 \le \theta \le \theta_1$$
$$y_2 = \theta\sin\theta \quad \text{for} \quad \theta_1 \le \theta \le \pi$$

and y_2 is made of blue and green.



Here θ_1 gives the maximum x-coordinate

$$\theta_1 \cos\theta_1$$

Simplifying the sum and applying u-substitution, we have

$$\text{Area} = \int_{-\pi}^{0} y\,dx = \int_{\pi}^{0} y(\theta)x'(t)\,d\theta = -\int_{0}^{\pi} \theta\sin\theta\big(\cos\theta - \theta\sin\theta\big)\,d\theta = \frac{1}{6}\pi^3$$

However, there is a better way to find the area defined by a polar function $r = f(\theta)$,

$$\text{Area} = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta$$

provided that the polar function is continuous. You can find the derivation of the formula in your textbook. Thus, for this question, we could have done the following

$$\text{Area} = \frac{1}{2}\int_{0}^{\pi} \theta^2\,d\theta = \frac{1}{6}\pi^3$$

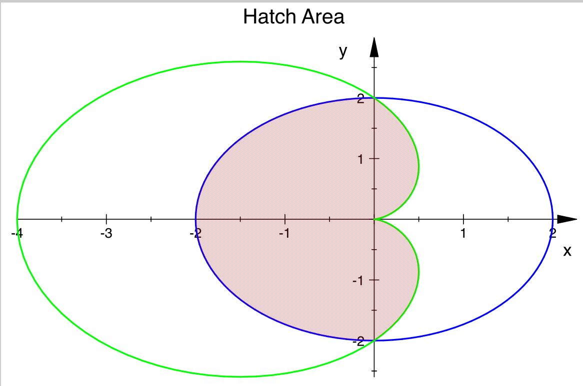
Question6 (1 points)
Find the area of the region shared by two curves defined by polar equations

$$r = 2 \qquad \text{and} \qquad r = 2(1-\cos\theta).$$

Solution:

1M Applying the formula $\frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta$ given in the last question two both curves, we have

$$\begin{aligned} \text{A} &= 2 \times \text{Area above } x\text{-axis} \\ &= \int_{0}^{\pi/2} 4(1-\cos\theta)^2\,d\theta + \int_{\pi/2}^{\pi} 4\,d\theta \\ &= 5\pi - 8 \end{aligned}$$



Question7 (3 points)
Evaluate the integral. Show all your workings.

(a) (1 point) $\displaystyle\int_0^\infty \frac{dv}{(1+v^2)(1+\arctan v)}$

Solution:

1M By definition, we have

$$\begin{aligned}\int_0^\infty \frac{dv}{(1+v^2)(1+\arctan v)} &= \lim_{b\to\infty}\int_0^b \frac{dv}{(1+v^2)(1+\arctan v)}\\ &= \lim_{b\to\infty}\int_1^{1+\arctan b}\frac{1}{u}\,du\\ &= \lim_{b\to\infty}(\ln(1+\arctan b)-\ln 1) = \ln(1+\frac{\pi}{2})\end{aligned}$$

(b) (1 point) $\displaystyle\int_{-1}^{4}\frac{dx}{\sqrt{|x|}}$

Solution:

1M By definition, we have

$$\begin{aligned}\int_{-1}^4 \frac{dx}{\sqrt{|x|}} &= \int_{-1}^0\frac{dx}{\sqrt{-x}}+\int_0^4\frac{dx}{\sqrt{x}}\\ &= \lim_{b\to 0^-}\int_{-1}^b\frac{dx}{\sqrt{-x}}+\lim_{a\to 0^+}\int_b^4\frac{dx}{\sqrt{x}}\\ &= \lim_{b\to 0^-}\left(-2\sqrt{-b}+2\sqrt{1}\right)+\lim_{a\to 0^+}\left(2\sqrt{4}-2\sqrt{b}\right)=6\end{aligned}$$

(c) (1 point) $\displaystyle\int_{-1}^{\infty}\frac{dx}{x^2+5x+6}$

Solution:

1M By definition, we have

$$\begin{aligned}\int_{-1}^\infty\frac{dx}{x^2+5x+6} &= \lim_{b\to\infty}\int_{-1}^b\frac{dx}{(x+2)(x+3)}\\ &= \lim_{b\to\infty}\left(\int_{-1}^b\frac{1}{x+2}\,dx-\int_{-1}^b\frac{1}{x+3}\,dx\right)\\ &= \lim_{b\to\infty}\left([\ln(x+2)]_{-1}^b-[\ln(x+3)]_{-1}^b\right)\\ &= \lim_{b\to\infty}\left(\ln(b+2)-\ln 1-\ln(b+3)+\ln(2)\right)\\ &= \lim_{b\to\infty}\left(\ln\frac{b+2}{b+3}+\ln 2\right)=\ln 2\end{aligned}$$

Question8 (2 points)

Testing for Convergence

(a) (1 point)

$$\int_0^{\pi/2}\tan\theta\,d\theta$$

Solution:

1M By definition, we can easily reach the conclusion that it is divergent.

$$\int_0^{\pi/2} \tan\theta \, d\theta = \lim_{b\to\pi/2^-} \int_0^b \tan\theta \, d\theta = \lim_{b\to\pi/2^-} \left(-\ln(\cos b) + \ln(\cos 0)\right)$$
$$= -\lim_{b\to\pi/2^-} \ln(\cos b) = \infty$$

(b) (1 point)

$$\int_1^\infty \frac{dx}{x\left(\sqrt{\ln(x)} + \ln^2(x)\right)}$$

Solution:

1M By definition, we need to consider both improper integrals

$$\begin{aligned} A &= \int_1^\infty \frac{dx}{x(\sqrt{\ln(x)} + \ln^2(x))} \\ &= \underbrace{\int_1^2 \frac{dx}{x(\sqrt{\ln(x)} + \ln^2(x))}}_{A_1} + \underbrace{\int_2^\infty \frac{dx}{x(\sqrt{\ln(x)} + \ln^2(x))}}_{A_2} \end{aligned}$$

Let us begin with A_2. Since $\sqrt{\ln(x)} \geq 0$ for $x \geq 1$, we have

$$\frac{1}{x(\sqrt{\ln(x)} + \ln^2(x))} \leq \frac{1}{x\ln^2 x}$$

Using integration by parts, we have

$$B_2 = \int_2^\infty \frac{1}{x\ln^2 x}\, dx = \lim_{b\to\infty} \int_2^\infty \frac{1}{x\ln^2 x}\, dx = -\lim_{b\to\infty}\left[\frac{1}{\ln x}\right]_2^\infty = \frac{1}{\ln 2}$$

Hence, by the comparison test, A_2 is convergent.

A_1 is improper because the integrand has an essential discontinuity at $x = 1$,

$$\frac{1}{x\left(\sqrt{\ln(x)} + \ln^2(x)\right)} \to \infty \qquad \text{as} \qquad x \to 1^+$$

Essentially if it grows too quick, then the integral will not be finite, however,

$$\int_1^2 \frac{dx}{x\sqrt{\ln(x)}} = \lim_{a\to 1^+} \int_a^2 \frac{dx}{x\sqrt{\ln(x)}} = \lim_{a\to 1^+} \left[2\sqrt{\ln x}\right]_a^2 = 2\sqrt{\ln 2}$$

which shows $\dfrac{1}{x\sqrt{\ln x}}$ is not growing too fast, if we consider

$$\lim_{x\to 1^+} \frac{x\sqrt{\ln x}}{x(\sqrt{\ln(x)} + \ln^2(x))} \overset{\text{LH}}{=} \lim_{x\to 1^+} \frac{\frac{1}{2\sqrt{\ln(x)}} + \sqrt{\ln(x)}}{\frac{2\ln(x)}{x} + \frac{1}{2\sqrt{\ln(x)}} + \sqrt{\ln(x)}}$$
$$= \lim_{x\to 1^+} \left(1 - \frac{4\ln(x)^{\frac{3}{2}}}{\left(4\ln(x)^{\frac{3}{2}} + x\,(2\ln(x) + 1)\right)}\right) = 1$$

Thus the functions in the numerator and the denominator are approaching zero at a "similar" rate. Hence we can conclude A_1 is also convergent, therefore A is convergent. This is known as the limit comparison test for improper integrals.

Question9 (1 points)

Suppose the following improper integral is convergent for all $x > 0$.

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \qquad x > 0$$

Prove that if n is a natural number, then

$$\Gamma(n+1) = n!$$

Solution:

1M By integration by parts,

$$\int_0^N t^x e^{-t}\,dt = -t^x e^{-t}\Big|_0^N + x\int_0^N e^{-t}t^{x-1}\,dt = -N^x e^{-N} + x\int_0^N e^{-t}t^{x-1}\,dt$$

We have seen that $\lim\limits_{N\to\infty} N^x e^{-N} = \lim\limits_{N\to\infty} \dfrac{N^x}{e^N} = 0$ for all $x > 0$,

$$\begin{aligned}\Gamma(x+1) = \int_0^\infty t^{x-1}e^{-t}dt &= \lim_{N\to\infty}\int_0^N t^x e^{-t}\,dt \\ &= \lim_{N\to\infty}\left(-N^x e^{-N} + x\int_0^N e^{-t}t^{x-1}\,dt\right) \\ &= x\lim_{N\to\infty}\int_0^N e^{-t}t^{x-1} \\ &= x\Gamma(x)\end{aligned}$$

Lastly, evaluate the following integral,

$$\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1$$

and use induction we can show the given statement is true.

Question10 (1 points)

Find the values of p for which each integral converges.

$$\int_1^2 \frac{dx}{x(\ln x)^p}$$

Solution:

1M Using integration by parts,

$$\int \frac{dx}{x(\ln x)^p} = \begin{cases} \ln(\ln(x)) & \text{if } p = 1 \\ -\dfrac{\ln(x)^{1-p}}{(p-1)} & \text{if } p \neq 1 \end{cases}$$

So for each case, we have

$$\int_1^2 \frac{dx}{x(\ln x)^p} = \lim_{a\to 1^+}\int_a^2 \frac{dx}{x(\ln x)^p} = \begin{cases} \lim\limits_{a\to 1^+} \left[\ln\left(\ln\left(x\right)\right)\right]_a^2 & \text{if } p = 1 \\ \lim\limits_{a\to 1^+} \left[-\dfrac{\ln\left(x\right)^{1-p}}{(p-1)}\right]_a^2 & \text{if } p \neq 1 \end{cases}$$

$$= \begin{cases} \lim\limits_{a\to 1^+} \left(\ln\left(\ln(2)\right) - \ln\left(\ln(a)\right)\right) & \text{if } p = 1 \\ \lim\limits_{a\to 1^+} -\dfrac{\ln\left(2\right)^{1-p}}{(p-1)} + \dfrac{\ln\left(a\right)^{1-p}}{(p-1)} & \text{if } p \neq 1 \end{cases}$$

$$= \begin{cases} -\dfrac{\ln\left(2\right)^{1-p}}{(p-1)} & \text{if } p < 1 \\ \infty & \text{if } p = 1 \\ \infty & \text{if } p > 1 \end{cases}$$

Question11 (1 points)
Let

$$S(x) = \int_0^x |\cos t|\ dt$$

Find

$$\lim_{x\to+\infty} \frac{S(x)}{x}$$

Solution:

1M First, notice that when $n\pi \leq x < (n+1)\pi$, since $|\cos x| \geq 0$, we have

$$\int_0^{n\pi} |\cos x| dx \leq S(x) < \int_0^{(n+1)\pi} |\cos x| dx.$$

Also, $|\cos x|$ is a function with period of π, so

$$\int_0^{n\pi} |\cos x| dx = n\int_0^{\pi} |\cos x| dx = 2n$$

So, when $n\pi \leq x < (n+1)\pi$, $2n \leq S(x) < 2(n+1)$ and

$$\frac{2n}{(n+1)\pi} \leq \frac{S(x)}{x} < \frac{2(n+1)}{n\pi}.$$

When $x \to +\infty$, applying the squeeze theorem, we have

$$\lim_{x\to+\infty} \frac{S(x)}{x} = \frac{2}{\pi}.$$

Question12 (1 points)
Find

$$\int_0^{\infty} \frac{dx}{(1+x^2)^n}$$

where n is a positive integer.

Solution:

1M Let

$$I_n = \int_0^\infty \frac{dx}{(1+x^2)^n}$$

If we consider the difference between

$$\begin{aligned}I_{n+1} - I_n &= \int_0^\infty \frac{dx}{(1+x^2)^{n+1}} - \int_0^\infty \frac{dx}{(1+x^2)^n} \\ &= \int_0^\infty \frac{-x^2}{(1+x^2)^{n+1}} dx \\ &= \lim_{b\to\infty} \frac{x}{2(1+x^2)^n}\bigg|_0^b - \frac{1}{2}\int_0^\infty \frac{dx}{(1+x^2)^n} \\ &= -\frac{1}{2} I_n\end{aligned}$$

Therefore, we can get the recursive relation:

$$I_{n+1} = \frac{1}{2} I_n$$

Now consider the situation $n = 1$, we can easily obtain that:

$$I_1 = \int_0^\infty \frac{dx}{(1+x^2)} = \lim_{b\to\infty} \arctan(x)\bigg|_0^b = \frac{\pi}{2}$$

Then by using the recursive relation, we can get:

$$I_n = \frac{\pi}{2^n}$$

Question13 (1 points)

Let

$$x = 0.9999\ldots$$

Determine whether $x < 1$, $x = 1$ or $x > 1$. Justify your answer.

Solution:

1M Expand it

$$x = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots = \sum_{n=1}^\infty \frac{9}{10^n} = \frac{0.9}{1-0.1} = \frac{0.9}{0.9} = 1$$

Question14 (3 points)

(a) (1 point) Determine whether the series with partial sum $s_n = \dfrac{n}{3n-1}$ is convergent.

Solution:
$\lim_{n\to\infty} s_n = \frac{1}{3}$, hence it converges.

(b) (1 point) Determine whether the series $\sum_{n}^{\infty} \frac{n}{3n-1}$ is convergent.

Solution:
$\lim_{n\to\infty} \frac{n}{3n-1} = \frac{1}{3}$, hence it diverges.

(c) (1 point) Find all values of x for which the series converges, and what it converges to.

$$1 - e^{-x} + e^{-2x} - e^{-3x} + e^{-4x} - e^{-5x} + e^{-6x} - \cdots$$

Solution:
This is a geometric series with common ration of $-e^{-x}$, so

$$1 - e^{-x} + e^{-2x} - e^{-3x} + e^{-4x} - e^{-5x} + e^{-6x} - \cdots = \frac{1}{1-(-e^{-x})} = \frac{1}{1+e^{-x}}$$

this converges if $e^{-x} < 1 \implies -x < 0$, hence $x > 0$ are the values for which the series converges.

Question15 (2 points)
Determine whether the series converges. If so, find the sum.

(a) (1 point) $\ln\frac{1}{2} + \ln\frac{2}{3} + \ln\frac{3}{4} + \cdots + \ln\frac{k}{k+1} + \cdots$

Solution:
$s_n = -\ln(n+1)$; $\lim_{n\to\infty} s_n = -\infty$, so this series diverges.

(b) (1 point) $\ln\left(1-\frac{1}{4}\right) + \ln\left(1-\frac{1}{9}\right) + \ln\left(1-\frac{1}{16}\right) + \cdots + \ln\left(1-\frac{1}{(k+1)^2}\right) + \cdots$

Solution:

$$\ln\left(1-\frac{1}{(k+1)^2}\right) = \ln\frac{k(k+2)}{(k+1)^2} = \ln\frac{k}{k+1} - \ln\frac{k+1}{k+2}$$

$$s_n = \ln\frac{1}{2} + \sum_{k=2}^{n}\left[-\ln\frac{k}{k+1} + \ln\frac{k}{k+1}\right] - \ln\frac{n+1}{n+2} = \ln\frac{1}{2} - \ln\frac{n+1}{n+2}$$

Therefore $\lim_{n\to\infty} s_n = -\ln 2$, so this series converges to $-\ln 2$.

Question16 (6 points)
Use appropriate tests or theorems to determine convergence or divergence.

(a) (1 point) $\sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}}$

Solution:
Converges. Integral test.

(b) (1 point) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

Solution:
Diverges. Limit comparison test with $\sum_{n=2}^{\infty} \frac{1}{n}$

(c) (1 point) $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{n+3}{n^4+4}}$

Solution:
Converges. Comparison test with $\sum \sqrt{\frac{n+4n}{n^4+0}} = \sqrt{5} \sum \frac{1}{n^{3/2}}$

(d) (1 point) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2(n+2)!}{n!3^{2n}}$

Solution:
Converges. Ratio test

(e) (1 point) $\displaystyle\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^{n^2}$

Solution:
Converges. Root test, $\lim_{n\to\infty} (1 + x/n)^n = e^x$,

(f) (1 point) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}+1}{n+1}$

Solution:
Converges. Since it can be easily shown that it converges absolutely.

Question17 (3 points)
Prove the Ratio test.

$$
\begin{aligned}
\text{If} \quad & \lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1 \\
& \implies \text{Absolutely convergent} \\
& \implies \text{Convergent} \\
\\
\text{If} \quad & \lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1 \\
& \implies \text{divergent} \\
\\
\text{If} \quad & \lim_{n\to\infty} \sqrt[n]{|a_n|} = 1 \\
& \implies \text{Inconclusive}
\end{aligned}
$$

Solution:

0M Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$ and we need to show $\sum a_n$ is absolutely convergent.

- Consider some number r such that $L < r < 1$, then there is a number N such that if $n \geq N$,

$$\left|\frac{a_{n+1}}{a_n}\right| < r \implies |a_{n+1}| < r\,|a_n|$$

$$\begin{aligned} |a_{N+1}| &< r\,|a_N| \\ |a_{N+2}| &< r\,|a_{N+1}| < r^2\,|a_N| \\ &\vdots \\ |a_{N+k}| &< r\,|a_{N+k-1}| < r^k\,|a_N| \end{aligned}$$

- Since $\sum_{k=0}^{\infty} |a_N|\, a^k$ converges for $0 < r < 1$, by the comparison test the following series is convergent

$$\sum_{n=N+1}^{\infty} |a_n| = \sum_{k=1}^{\infty} |a_{N+k}|$$

- Therefore $\sum\limits_{n=1}^{\infty} |a_n|$, which is sum of the above series and a finite value, is convergent.

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N} |a_n| + \sum_{n=N+1}^{\infty} |a_n|$$

- Next, suppose $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ and we need to show $\sum a_n$ is divergent, in this case, there is a number N such that if $n \geq N$.

$$\left|\frac{a_{n+1}}{n}\right| > 1 \implies |a_{n+1}| > |a_n| \implies \lim_{n\to\infty} |a_n| \neq 0 \implies \lim_{n\to\infty} a_n \neq 0$$

- Hence by the divergence test, $\sum\limits_{n=1}^{\infty} |a_n|$ is divergent.

- Finally, we need to show when $L = 1$, the series has any of the three possibilities. We can demonstrate it by considering three cases, one for each scenario

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\text{absolutely convergent}} \qquad \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{n}}_{\text{conditionally convergent}} \qquad \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{divergent}}$$