PROBABILISTIC METHODS IN ENGINEERING

VE401

Assignment IV

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Team number: 20

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Exercise 4.1

1) Since

$$\mathbf{A}\mathbf{X} = (X_1, X_2) = \begin{pmatrix} a_{11}X_1 + a_{12}X_2 \\ a_{21}X_1 + a_{22}X_2 \end{pmatrix}.$$

then

$$\begin{aligned} E[\mathbf{A}\mathbf{X}] = E\Big[\begin{pmatrix} a_{11}X_1 + a_{12}X_2 \\ a_{21}X_1 + a_{22}X_2 \end{pmatrix} \Big] &= \begin{pmatrix} E[a_{11}X_1 + a_{12}X_2] \\ E[a_{21}X_1 + a_{22}X_2] \end{pmatrix} \\ &= \begin{pmatrix} a_{11}E[X_1] + a_{12}E[X_2] \\ a_{21}E[X_1] + a_{22}E[X_2] \end{pmatrix} = \mathbf{A} \begin{pmatrix} E[X_1] \\ E[X_2] \end{pmatrix} = \mathbf{A}E[\mathbf{X}]. \end{aligned}$$

2)

$$\mathrm{Var}(\mathbf{A}\mathbf{X}) = \begin{pmatrix} \mathrm{Var}(a_{11}X_1 + a_{12}X_2) & \mathrm{Cov}(a_{11}X_1 + a_{12}X_2, a_{21}X_1 + a_{22}X_2) \\ \mathrm{Cov}(a_{21}X_1 + a_{22}X_2, a_{11}X_1 + a_{12}X_2) & \mathrm{Var}(a_{21}X_1 + a_{22}X_2) \end{pmatrix}$$

We denote that

$$\begin{aligned} & t = \mathrm{Cov}(X_1, X_2), \\ & \mathrm{Cov}(a_{21}X_1 + a_{22}X_2, a_{11}X_1 + a_{12}X_2) \\ =& E[(a_{21}X_1 + a_{22}X_2)(a_{11}X_1 + a_{12}X_2)] - E[a_{21}X_1 + a_{22}X_2]E[a_{11}X_1 + a_{12}X_2] \\ =& (a_{21}a_{12} + a_{22}a_{11})(E[X_1X_2] - E[X_1]E[X_2]) + a_{11}a_{21}(E[X_1^2] - E[X_1]^2) + a_{22}a_{12}(E[X_2^2] - E[X_2]^2) \\ =& (a_{21}a_{12} + a_{22}a_{11})\mathrm{Cov}(X_1, X_2) + a_{11}a_{21}\mathrm{Var}(X_1) + a_{22}a_{12}\mathrm{Var}(X_2) \\ =& (a_{21}a_{12} + a_{22}a_{11})t + a_{11}a_{21}\mathrm{Var}(X_1) + a_{22}a_{12}\mathrm{Var}(X_2), \end{aligned}$$

then

$$\mathrm{Var}(\mathbf{A}\mathbf{X}) = \begin{pmatrix} a_{11}^2\mathrm{Var}(X_1) + a_{12}^2\mathrm{Var}(X_2) + 2a_{11}a_{12}t & \mathrm{Cov}(a_{21}X_1 + a_{22}X_2, a_{11}X_1 + a_{12}X_2) \\ \mathrm{Cov}(a_{21}X_1 + a_{22}X_2, a_{11}X_1 + a_{12}X_2) & a_{21}^2\mathrm{Var}(X_1) + a_{22}^2\mathrm{Var}(X_2) + 2a_{21}a_{22}t \end{pmatrix}.$$

Also,

$$\begin{aligned} \mathbf{A}\mathrm{Var}(\mathbf{X}) &= \begin{pmatrix} a_{11}\mathrm{Var}(X_1) + a_{12}\mathrm{Cov}(X_1, X_2) & a_{11}\mathrm{Cov}(X_1, X_2) + a_{12}\mathrm{Var}(X_2) \\ a_{21}\mathrm{Var}(X_1) + a_{22}\mathrm{Cov}(X_1, X_2) & a_{21}\mathrm{Cov}(X_1, X_2) + a_{22}\mathrm{Var}(X_2) \end{pmatrix}. \\ \mathbf{A}^T &= \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}. \end{aligned}$$

Hence, we know

$$\mathrm{Var}(\mathbf{A}\mathbf{X}) = \mathbf{A}(\mathrm{Var}(\mathbf{X}))\mathbf{A}^T.$$

3) Since X_1 and X_2 follow independent normal distributions, then $\varrho_X = 0$.

$$f_X(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right]}.$$

$$\Sigma_X = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}, \quad \Sigma_X^{-1} = \frac{1}{\sigma_1^2\sigma_2^2} \begin{pmatrix} \sigma_2^2 & 0 \\ 0 & \sigma_1^2 \end{pmatrix} = \begin{pmatrix} \sigma_1^{-2} & 0 \\ 0 & \sigma_2^{-2} \end{pmatrix},$$

Hence,

$$\sqrt{\det \Sigma_X} = \sqrt{\sigma_1^2\sigma_2^2} = \sigma_1\sigma_2.$$

$$\Sigma_X^{-1}(x - \mu_X) = \begin{pmatrix} \sigma_1^{-2} & 0 \\ 0 & \sigma_2^{-2} \end{pmatrix} \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix} = \begin{pmatrix} \dfrac{x_1 - \mu_1}{\sigma_1^2} \\ \dfrac{x_2 - \mu_2}{\sigma_2^2} \end{pmatrix}.$$

Then,

$$\left\langle x - \mu_X, \Sigma_X^{-1}(x - \mu_X) \right\rangle = \frac{(x_1 - \mu_1)^2}{\sigma_1^2} + \frac{(x_2 - \mu_2)^2}{\sigma_2^2}.$$

Thus,

$$f_X(x) = f_X(x_1, x_2) = \frac{1}{2\pi\sqrt{\det \Sigma_X}} e^{-\frac{1}{2}\left\langle x-\mu_X, \Sigma_X^{-1}(x-\mu_X)\right\rangle}.$$

4) Since $\mathbf{A}$ is invertible, then $X = A^{-1}Y$.

$$f_Y(y) = f_X(x)|\det(A^{-1})| = f_X(A^{-1}y)|\det(A^{-1})|.$$

$$\begin{aligned} \det \Sigma_Y = \det \begin{pmatrix} \mathrm{Var}(Y_1) & \mathrm{Cov}(Y_1, Y_2) \\ \mathrm{Cov}(Y_2, Y_1) & \mathrm{Var}(Y_2) \end{pmatrix} &= \sigma_{Y_1}^2\sigma_{Y_2}^2 - \mathrm{Cov}(Y_1, Y_2)^2 \\ &= (a_{11}^2\sigma_1^2 + a_{12}^2\sigma_2^2)(a_{21}^2\sigma_1^2 + a_{22}^2\sigma_2^2) - (a_{11}a_{21}\sigma_1^2 + a_{22}a_{12}\sigma_2^2)^2 \\ &= (a_{11}a_{22} - a_{12}a_{21})^2\sigma_1^2\sigma_2^2 \\ &= \det(A)^2 \det(\Sigma_X). \end{aligned}$$

Hence,

$$\sqrt{|\det \Sigma_Y|} = \sqrt{|\det(A)^2 \det(\Sigma_X)|},$$

$$\sqrt{\det \Sigma_X} = \frac{1}{|\det(A)|}\sqrt{|\det \Sigma_Y|}.$$

$$\Sigma_Y^{-1}(y - \mu_Y) = \mathrm{Var}(Y)^{-1} \begin{pmatrix} y_1 - \mu_{Y_1} \\ y_2 - \mu_{Y_2} \end{pmatrix}$$

$$= (A^T)^{-1}\Sigma_X^{-1}A^{-1} \begin{pmatrix} y_1 - \mu_{Y_1} \\ y_2 - \mu_{Y_2} \end{pmatrix}$$

Since

$$\begin{pmatrix} y_1 - \mu_{Y_1} \\ y_2 - \mu_{Y_2} \end{pmatrix} = \begin{pmatrix} a_{11}X_1 + a_{12}X_2 - (a_{11}\mu_1 + a_{12}\mu_2) \\ a_{21}X_1 + a_{22}X_2 - (a_{21}\mu_1 + a_{22}\mu_2) \end{pmatrix} = A \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix},$$

then

$$\Sigma_Y^{-1}(y - \mu_Y) = (A^T)^{-1}\Sigma_X^{-1}(x - \mu_X).$$

$$\begin{aligned} \langle y - \mu_Y, \Sigma_Y^{-1}(y - \mu_Y)\rangle &= \langle A(x - \mu_X), (A^T)^{-1}\Sigma_X^{-1}(x - \mu_X)\rangle \\ &= (x - \mu_X)^T A^T (A^T)^{-1}\Sigma_X^{-1}(x - \mu_X) \\ &= \langle x - \mu_X, \Sigma_X^{-1}(x - \mu_X)\rangle. \end{aligned}$$

Finally we plug in the terms,

$$\begin{aligned} f_Y(y) &= f_X(A^{-1}y)\det(A^{-1}) \\ &= \frac{|\det(A)|}{2\pi\sqrt{|\det \Sigma_Y|}} e^{-\frac{1}{2}\langle y-\mu_Y, \Sigma_Y^{-1}(y-\mu_Y)\rangle} |\det(A^{-1})| \\ &= \frac{1}{2\pi\sqrt{|\det \Sigma_Y|}} e^{-\frac{1}{2}\langle y-\mu_Y, \Sigma_Y^{-1}(y-\mu_Y)\rangle}. \end{aligned}$$

5) We denote that $\varrho = \mathrm{Cov}(Y_1, Y_2)/(\sigma_{Y_1}\sigma_{Y_2})$, then

$$\begin{aligned} 1 - \varrho^2 = 1 &- \frac{\mathrm{Cov}^2(Y_1, Y_2)}{\sigma_{Y_1}^2 \sigma_{Y_2}^2} \\ &= \frac{\sigma_{Y_1}^2\sigma_{Y_2}^2 - \mathrm{Cov}^2(Y_1, Y_2)}{\sigma_{Y_1}^2\sigma_{Y_2}^2} \\ &= \frac{\det \Sigma_Y}{\sigma_{Y_1}^2\sigma_{Y_2}^2} \end{aligned}$$

Hence,

$$\sqrt{|\det \Sigma_Y|} = \sigma_{Y_1}\sigma_{Y_2}\sqrt{1 - \varrho^2}.$$

Also,

$$\begin{aligned} &\langle y - \mu_Y, \Sigma_Y^{-1}(y - \mu_Y)\rangle \\ =&\frac{1}{1-\varrho^2}\frac{1}{\sigma_{Y_1}^2\sigma_{Y_2}^2}\langle y - \mu_Y, \Sigma_Y^*(y - \mu_Y)\rangle \\ =&\frac{1}{1-\varrho^2}\frac{1}{\sigma_{Y_2}^2\sigma_{Y_2}^2}(\sigma_{Y_1}^2(y_1 - \mu_{Y_1})^2 - 2\mathrm{Cov}(Y_1, Y_2)(y_1 - \mu_{Y_1})(y_2 - \mu_{Y_2}) + \sigma_{Y_1}^2(y_2 - \mu_{Y_2})^2) \\ =&\frac{1}{1-\varrho^2}[\frac{(y_1 - \mu_{Y_1})^2}{\sigma_{Y_1}^2} - 2\varrho(\frac{y_1 - \mu_{Y_1}}{\sigma_{Y_1}})(\frac{y_2 - \mu_{Y_2}}{\sigma_{Y_2}}) + \frac{(y_2 - \mu_{Y_2})^2}{\sigma_2^2}] \end{aligned}$$

Hence,

$$f_Y(y_1,y_2)=\frac{1}{2\pi\sigma_{Y_1}\sigma_{Y_2}\sqrt{1-\varrho^2}}e^{-\frac{1}{2(1-\varrho^2)}\left[\frac{(y_1-\mu_{Y_1})^2}{\sigma_{Y_1}^2}-2\varrho(\frac{y_1-\mu_{Y_1}}{\sigma_{Y_1}})(\frac{y_2-\mu_{Y_2}}{\sigma_{Y_2}})+\frac{(y_2-\mu_{Y_2})^2}{\sigma_2^2}\right]}.$$

Exercise 4.2

$$E[Y]=E\Bigg[\begin{pmatrix}a_{11}X_1+a_{12}X_2\\a_{21}X_1+a_{22}X_2\end{pmatrix}\Bigg]=\begin{pmatrix}a_{11}E[X_1]+a_{12}E[X_2]\\a_{21}E[X_1]+a_{22}E[X_2]\end{pmatrix}=A\begin{pmatrix}E[X_1]\\E[X_2]\end{pmatrix}=AE[X].$$

$$\begin{aligned}\mathrm{Var}(Y)&=\begin{pmatrix}\mathrm{Var}(Y_1)&\mathrm{Cov}(Y_1,Y_2)\\\mathrm{Cov}(Y_2,Y_1)&\mathrm{Var}(Y_2)\end{pmatrix}=\begin{pmatrix}(a_{11}^2+a_{12}^2)\sigma^2&(a_{11}a_{21}+a_{22}a_{12})\sigma^2\\(a_{11}a_{21}+a_{22}a_{12})\sigma^2&(a_{21}^2+a_{22}^2)\sigma^2\end{pmatrix}\\&=\sigma^2\begin{pmatrix}a_{11}^2+a_{12}^2&a_{11}a_{21}+a_{22}a_{12}\\a_{11}a_{21}+a_{22}a_{12}&a_{21}^2+a_{22}^2\end{pmatrix}.\end{aligned}$$

Since $A^T=A^{-1}$, then

$$AA^T=I=\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}a_{11}^2+a_{12}^2&a_{11}a_{21}+a_{22}a_{12}\\a_{11}a_{21}+a_{22}a_{12}&a_{21}^2+a_{22}^2\end{pmatrix}.$$

Hence,

$$\mathrm{Var}(Y)=\sigma^2\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

Exercise 4.3

Denote that $x=z_{\alpha_1}$ and $y=z_{\alpha_2}$. Then we obtain

$$\begin{aligned}&\Phi(-x)+\Phi(-y)=\alpha,\\&g(x,y):=\Phi(-x)+\Phi(-y)-\alpha=0.\end{aligned}$$

Now we want to calculate the **conditional extreme** values of

$$f(x,y):=\frac{(x+y)\sigma}{\sqrt{n}}.$$

Thus we have the **partial differentiations** of $F(x,y,\lambda)=f(x,y)+\lambda g(x,y)$.

$$\begin{cases}F_x=\dfrac{\sigma}{\sqrt{n}}-\lambda f_N(-x)=0\\F_y=\dfrac{\sigma}{\sqrt{n}}-\lambda f_N(-y)=0\\F_\lambda=\Phi(-x)+\Phi(-y)-\alpha=0\end{cases},$$

where $f_N(\cdot)$ is the density of a standard normal distribution.

We find that $x = y$, which means $z_{\alpha_1} = z_{\alpha_2}$. Also, $\alpha_1 + \alpha_2 = \alpha$.

Thus $\alpha_1 = \alpha_2 = \alpha/2$.

Exercise 4.4

Since $(n-1)s^2/sigma_2$ follows a chi-squared distribution, then

$$\begin{aligned}1-\alpha &= P[\chi^2_{1-\alpha/2,n-1} \leq (n-1)s^2\sigma^2 \leq \chi^2_{\alpha/2,n-1}] \\ &= P\left[\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}\right] \\ &= P\left[\sqrt{\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}}} \leq \sigma \leq \sqrt{\frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}}\right].\end{aligned}$$

Since $\alpha = 0.05$, then

$$\begin{aligned}\chi^2_{0.025,50} &= 71.42, \\ \chi^2_{0.975,50} &= 32.36.\end{aligned}$$

Therefore,

$$\begin{aligned}\sqrt{\frac{50 \cdot 0.37^2}{71.42}} \leq &\sigma \leq \sqrt{\frac{50 \cdot 0.37^2}{32.36}} \\ 0.31 \leq &\sigma \leq 0.46.\end{aligned}$$

Hence, the 95% two-sided confidence interval for σ is [0.31,0.46].

Exercise 4.5

1) We consider the exclusive situations. If all the samples are greater than M or less than M, then M will not fall between X_{min} and X_{max}.

   Since $F(M) = \frac{1}{2}$, which means the probability that a sample is less or greater than M is both $\frac{1}{2}$.

   Hence the probability that all the samples are greater than M or less than M is

$$\left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^{n-1}.$$

   Therefore, the probability that M falls between X_{min} and X_{max} is

$$P[X_{min} \leq M \leq X_{max}] = 1 - \left(\frac{1}{2}\right)^{n-1}.$$

2) For $P[X_{k+1} \leq M \leq X_{n-k}]$, we can also consider the exclusive situations. In this case, the probability that the samples X_{k+1} to X_{n-k} are greater than M or less than M is

$$\sum_{x=1}^{k} \binom{n}{x} \left(\frac{1}{2}\right)^{x} \left(\frac{1}{2}\right)^{n-x} + \sum_{x=1}^{k} \binom{n}{n-x} \left(\frac{1}{2}\right)^{x} \left(\frac{1}{2}\right)^{n-x} = \sum_{x=1}^{k} \binom{n}{x} \left(\frac{1}{2}\right)^{n-1}.$$

$$P[X_{k+1} \leq M \leq X_{n-k}] = 1 - \sum_{x=1}^{k} \binom{n}{x} \left(\frac{1}{2}\right)^{n-1}.$$