Vv256 Lecture 1

Dr Jing Liu

UM-SJTU Joint Institute

September 11, 2017

Defintion

A differential equation is an equation of

an unknown function and its derivatives.

- For some unknown function

$$y = y(t)$$

the following is an example of differential equations

$$\frac{dy}{dt} = y^2 \iff \dot{y} = y^2 \iff y' = y^2$$

Q: What does the above equation say regarding the unknown function

$$y = y(t)$$

- A differential equation provides a way to specify/define a function just like an algebraic equation specifies a number

$$x^3 = 1$$

Q: Can you think of a very simple function $y(t)$ that satisfies

$$\dot{y} = y^2$$

- The function that is identically zero clearly satisfies the above equation.

$$y(t) = 0$$

Q: Is there any other function that satisfies the differential equation?

- Since the differential equation is in the form of

$$\dot{y} = \Phi(t, y) = y^2$$

we can graphically investigate all functions that satisfy the equation.

$$\underbrace{\dot{y}}_{\text{The slope of some unknown function}} = \underbrace{\Phi(t, y)}_{\text{Th value of the slope at } (t,y)}$$

- Specifically, we construct a vector field,

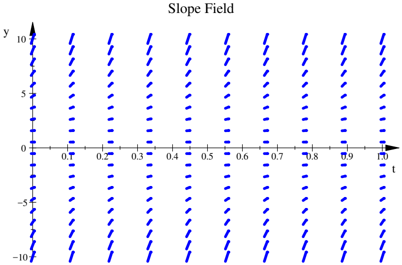
$$\mathbf{F} = 1\mathbf{e}_t + \Phi(t, y)\mathbf{e}_y$$

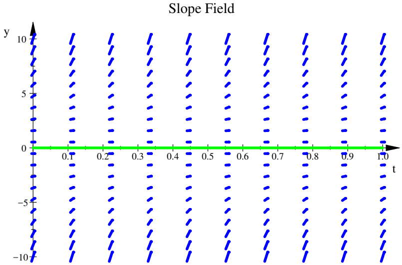
then the graph of the $\mathbf{F}$ shows the family of functions satisfy the equation

$$\dot{y} = \Phi(t, y)$$

Definition

The vector field $\mathbf{F} = 1\mathbf{e}_t + \Phi(t, y)\mathbf{e}_y$ is known as a slope field or a direction field.





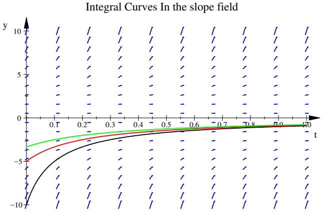
- Matlab can solve this simple equation symbolically, the output is given as

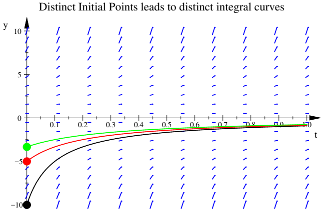
$$\left\{0,\ -\frac{1}{C+t}\right\}$$

where $y = 0$ is known as the trivial solution, and the second part

$$y = -\frac{1}{C+t} \qquad \text{where } C \text{ is an arbitrary constant.}$$

represents a family of solutions, which seem to have different initial points.





- Note a solution to a differential equation is a function instead of a number.

- If we demand the solution to have value $y = -10$ at $t = 0$ while satisfying

$$\dot{y} = y^2$$

we can narrow it down to a single solution in this case.

Definition

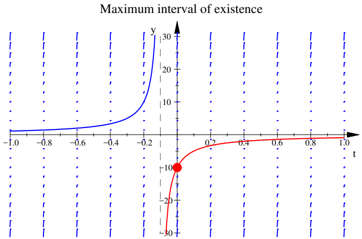
A differential equation with conditions for the value of the unknown function and possibly its derivatives at one particular point in the domain is known as an initial value problem.

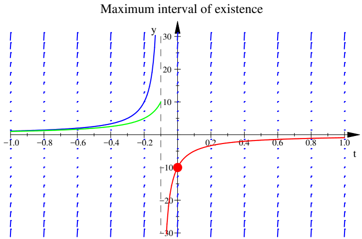
- For example, the following is an initial value problem (IVP)

$$\dot{y} = y^2, \qquad y(0) = -10$$

Q: What is the domain of the solution $y = -\dfrac{1}{0.1 + t}$ to the initial value problem

$$\dot{y} = y^2, \qquad y(0) = -10$$





- The branch in blue certainly satisfies the differential equation, but it is not relevant to the initial condition. For the green branch is just as good

$$y(t) = \begin{cases} -\frac{1}{0.1+t} & \text{for} \quad t > -0.1 \\ -\frac{1}{t} & \text{for} \quad t \leq -0.1 \end{cases}$$

  that is, a piecewise function is just as good as a function that can be written using a single formula. If we allow such functions, the solution is not unique.
- However, if we consider a smaller interval, e.g. $t > 0$, the solution is unique.
- So the domain of a solution might be smaller than the usual set in which the expression is defined.

Q: Having an essential discontinuity is problematic, and we may have to reduce the domain, but how about a solution that is everywhere continuous?

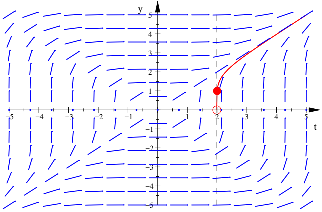
- For example, the initial value problem

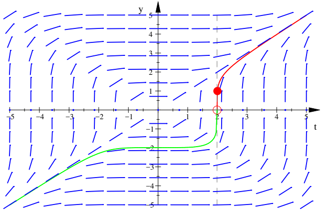
$$\dot{y} = \frac{t^4}{y^4}, \qquad y(2) = 1$$

is known to have the following solution, which is continuous everywhere,

$$y = \sqrt[5]{t^5 - 31}$$

Q: Why we have to exclude $t = \sqrt[5]{31}$.





- So we expect the following to be reasonable over an open interval.

Definition

A function

$$y = y(t)$$

defined on an interval $a < t < b$, is called a solution of the differential equation

$$\dot{y} = \Phi(t, y)$$

provided that y is a differentiable function of t on the interval

$$a < t < b$$

and the equation is defined and satisfied for every t in $a < t < b$.

- Note we say “a” solution rather than “the” solution. A differential equation, if it has a solution at all, usually has more than one solution.
- The solution to an initial value problem is a particular solution that satisfies the initial condition as well as satisfying the differential equation.

Exercise

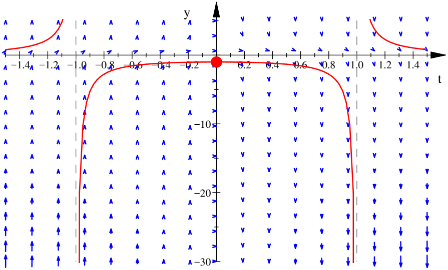
(a) Verify that $y = \dfrac{1}{t^2 + c}$, where c is an arbitrary constant, is a solution to

$$\dot{y} + 2ty^2 = 0$$

(b) Find the particular solution that satisfies $y(0) = -1$.

(c) Find the maximum interval on which the solution you found above is valid.

Maximum interval of existence



- To study how to solve differential equations, it is essential to classify them since different classes of equations often need to be solve differently.

Defintion

An ordinary differential equation (ODE) contains only derivatives of one or more dependent variables with respect to a single independent variable.

Q: Are the following equations ODEs?

$$\dot{y} + 5y = e^t \qquad \text{and} \qquad \dot{x} + \dot{y} = 2x + y$$

- ODEs are different from partial differential equations (PDEs), which involve partial derivatives of two or more independent variables.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{and} \qquad \operatorname{div}(\rho \mathbf{v}) + \frac{\partial \rho}{\partial t} = 0$$

Defintion

The order of a differential equation is the order of the highest-order derivative that occurs in the equation

Q: What is the order of the following differential equations?

$$\dddot{y} + 2e^t\ddot{y} + y\dot{y} = t^4 \qquad \text{and} \qquad (\dot{y})^2 + t(\dot{y})^3 + 4y = 0$$

- We will first consider first-order equations

$$\alpha(t)\dot{y} + \beta(t)y = \gamma(t), \qquad \dot{y}y = 2, \qquad \dot{y} = y^2 \quad \text{and} \quad (\dot{y})^2 = 4y$$

  which might be linear or nonlinear.

- A linear first-order equation involves only $\alpha(t)y$ and $\beta(t)\dot{y}$ "by themselves".
- If any term involves a function of y or $\dot{y}$ in the equation, then it is nonlinear.

Defintion

An nth-order ordinary differential equation

$$F(t, y, y', y'', \ldots, y^{(n)}) = f(t)$$

is said to be linear if F is a linear function in terms of y, y', ..., $y^{(n)}$.

- A function F is said to be linear in terms of y, y', ..., $y^{(n)}$ if and only if

$$F(t, ay, ay', \ldots, ay^{(n)}) = aF(t, y, y', \ldots, y^{(n)})$$

$$F(t, y_1 + y_2, \ldots, y_1^{(n)} + y_2^{(n)}) = F(t, y_1, y_1', \ldots, y_1^{(n)}) + F(t, y_2, y_2', \ldots, y_2^{(n)})$$

Q: Which of the followings is a linear differential equation?

1. $\ddot{y} + \dfrac{1}{\dot{y}} = 0$
2. $\ddot{y} + \sin y = 0$
3. $t^2\ddot{y} + yt^2 \sin t = t \cos t$
4. $\dddot{y} = \exp(\ddot{y})$