Vv256 Lecture 13

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Definition

The following equation is known as Euler's equation.

$$t^2\ddot{y} + at\dot{y} + by = 0$$

where a and b are constants.

- It is clear that $t = 0$ is a regular singular point

$$P(t) = \frac{a}{t} \implies p(t) = a$$
$$Q(t) = \frac{b}{t^2} \implies q(t) = b$$

- The indicial equation is given by

$$F(r) = 0 \implies r(r-1) + p_0 r + q_0 = 0 \implies r(r-1) + ar + b = 0$$

- The solution $\varphi(t, r)$ must satisfy the following for all $n \in \mathbb{N}_1$,

$$G_n(r; c_0, c_1, \cdots, c_n) = 0$$

- From our early derivation, we have

$$G_n(r; c_0, c_1, \cdots, c_n) = (n+r)(n+r-1)c_n + \sum_{k=0}^{n} \Big[(k+r)p_{n-k} + q_{n-k}\Big] c_k$$

$$G_n = 0 \implies c_n = 0 \qquad \text{for} \quad n \in \mathbb{N}_1$$

since p_k and q_k are zero for all $k \in \mathbb{N}_1$ in this case. Hence

$$\varphi(t, r) = c_0 t^r \implies \phi_1(t) = t^{r_1} \qquad \text{and} \qquad \phi_2(t) = t^{r_2}$$

where r_1 and r_2 are distinct solutions of

$$F(r) = 0 \iff r^2 + (a-1)r + b = 0$$

- If we have repeated roots $r = r_1 = r_2$, then

$$\phi_1 = t^{r_1} \implies \mathcal{L}\left[\left.\frac{\partial \varphi}{\partial r}\right|_{r=r_1}\right] = 0 \implies \phi_2 = \frac{1}{c_0}\left.\frac{\partial \varphi}{\partial r}\right|_{r=r_1} = t^{r_1}\ln(t)$$

which is consistent with our early derivation.

Definition

For $\lambda \geq 0$, the following equation is known as Bessel's equation of order λ.

$$t^2\ddot{y} + t\dot{y} + (t^2 - \lambda^2)y = 0$$

- It occurs in advanced studies in physics, and engineering.
- In general, it is not possible to obtain closed form solutions to this equation.
- Note the $t = 0$ is a regular singular point, because tP and t^2Q are analytic

$$\begin{aligned} t^2\ddot{y} + t\dot{y} + (t^2 - \lambda^2)y = 0 &\implies \ddot{y} + \overbrace{\frac{1}{t}}^{P}\dot{y} + \overbrace{\frac{(t^2 - \lambda^2)}{t^2}}^{Q}y = 0 \\ &\implies p = tP = 1 \\ &\implies q = t^2Q = t^2 - \lambda^2 \end{aligned}$$

- Since the indicial equation always takes the form

$$r(r-1) + p_0 r + q_0 = 0 \implies r(r-1) + 1 \cdot r - \lambda^2 = 0 \implies r_{1,2} = \pm\lambda$$

1. Hence, provided that

$$r_1 - r_2 = \lambda + \lambda = 2\lambda \quad \text{is not an integer},$$

there will exits two linearly independent Frobenius solutions.

- Recall the recurrence relation in general is given by

$$(n+r)(n+r-1)c_n + \sum_{k=0}^{n} \Big[(k+r)p_{n-k} + q_{n-k}\Big] c_k = 0 \qquad \text{for all } n \geq 1.$$

- When $n = 1$, we have

$$\begin{aligned}(1+r)rc_1 + \Big((0+r)\cdot 0 + 0\Big)c_0 + \Big((1+r)\cdot 1 - \lambda^2\Big)c_1 &= 0 \\ \Big((1+r)^2 - \lambda^2\Big)c_1 &= 0\end{aligned}$$

Q: What will we have for $n \geq 2$?

$$\Big((n+r)^2 - \lambda^2\Big)c_n = -c_{n-2} \qquad \text{for } n \geq 2.$$

- So if we use the root $r = \lambda$ of the indicial equation, we have

$$
\begin{aligned}
\left((1+r)^2 - \lambda^2\right)c_1 = 0 \quad &\implies \left((1+\lambda)^2 - \lambda^2\right)c_1 = 0 \implies c_1 = 0 \\
\left((n+r)^2 - \lambda^2\right)c_n = -c_{n-2} \quad &\implies \left(n^2 + 2n\lambda\right) c_n = -c_{n-2} \\
&\implies c_n = \frac{-c_{n-2}}{n(2\lambda+n)} \qquad \text{for } n \geq 2.
\end{aligned}
$$

- This implies that all of the odd coefficients zero,

$$c_{2k+1} = 0 \qquad k = 0, 1, 2, \ldots$$

- Now consider the even coefficients, we have

$$c_2 = \frac{-c_0}{2(2\lambda+2)}; \qquad c_4 = \frac{-c_2}{4(2\lambda+4)} = \frac{c_0}{2(2\lambda+2)\cdot 4(2\lambda+4)}$$

- Thus the general even coefficient is given by

$$c_{2k} = \frac{(-1)^k c_0}{2\cdot 4 \cdots (2k) \cdot (2\lambda+2)\cdot(2\lambda+4)\cdots(2\lambda+2k)} \qquad k \in \mathbb{N}_1$$

- With some simplification, we obtain the first linearly independent solution

$$\phi_1 = c_0 t^{\lambda} \left(1 + \sum_{k=1}^{\infty} c_{2k} t^{2k} \right)$$

where

$$\begin{aligned} c_{2k} &= \frac{(-1)^k}{2 \cdot 4 \cdots (2k) \cdot (2\lambda + 2) \cdot (2\lambda + 4) \cdots (2\lambda + 2k)} \\ &= \frac{(-1)^k}{2^{2k} k! (\lambda + 1) \cdot (\lambda + 2) \cdots (\lambda + k)} \end{aligned}$$

Q: Why can we expect the following to be the 2nd linearly independent solution

$$\phi_2 = c_0^* t^{-\lambda} \left(1 + \sum_{k=1}^{\infty} c_{2k} t^{2k} \right)$$

where

$$c_{2k}^* = \frac{(-1)^k}{2^{2k} k! (-\lambda + 1) \cdot (-\lambda + 2) \cdots (-\lambda + k)}$$

- Note c_0 or c_0^* are arbitrarily constants, a special choice can lead to special

$$\phi_1 \qquad \text{and} \qquad \phi_2$$

which are called the Bessel functions of the first kind of order

$$\lambda \qquad \text{and} \qquad -\lambda,$$

respectively, and we denote them using

$$J_\lambda \qquad \text{and} \qquad J_{-\lambda}$$

- We will delay the discussion of what exactly those special choices are

$$c_0 \qquad \text{and} \qquad c_0^*$$

- For now, let us focus on the fact that

$$J_\lambda(t) \qquad \text{and} \qquad J_{-\lambda}(t)$$

are two special linearly independent solutions to Bessel's equation.

- Therefore, provided that

$$r_1 - r_2 = 2\lambda \quad \text{is not an integer},$$

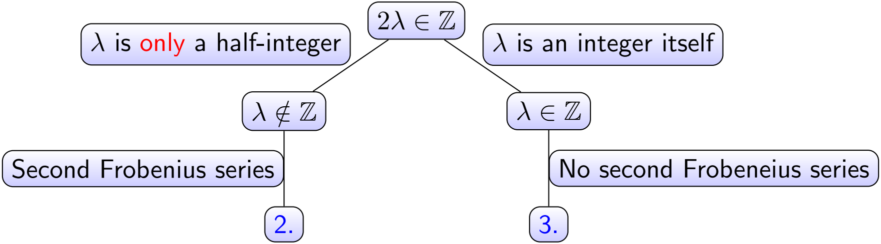
the following is the general solution

$$y = C_1 J_\lambda + C_2 J_{-\lambda}$$

and it can be shown that this general solution is valid for

$$t \in (0, \infty)$$

- Now let us consider what happens when



2. This means λ is not an integer and can only take the following values

$$\frac{2k+1}{2} \qquad k = 0, 1, 2, \dots$$

- By inspecting the recurrence relation, when $-\lambda$ is used,

$$c_n = \frac{-c_{n-2}}{n(2\lambda + n)} \qquad \text{for } n \geq 2.$$

there still are nonzero c_n that satisfies the recurrence relation,

$$c_n = \frac{-c_{n-2}}{n(-2\lambda + n)} \qquad \text{for } n \geq 2.$$

so a second linearly independent Frobenius solution exists, in fact, given by

$$J_{-\lambda}$$

- Thus, provided that λ is not an integer, the general solution is given by

$$y = C_1 J_\lambda + C_2 J_{-\lambda}$$

3. According to Frobenius, when $\lambda = k$ is an positive integer, we have

$$\phi_2 = CJ_k \ln t + t^{-k} \sum_{n=0}^{\infty} c_n^* t^n$$

as the second linearly independent solution, where C and c_n^* can be found by substituting ϕ_2 into the equation and equating coefficients. Warning!

$$\begin{aligned}\phi_2(t) = \frac{1}{2^{k-1}(k-1)!} J_k \ln x + t^{-k} \left(1 + \sum_{n=1}^{k-1} \frac{t^{2n}}{2^{2n} n!(k-1)\cdots(k-n)}\right) \\ + \frac{d_0}{2^k (k-1)!} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) t^k \\ + \sum_{m=1}^{\infty} \frac{d_{2m}}{2^k (k-1)!} \left[\left(1 + \frac{1}{2} + \cdots \frac{1}{m}\right) + \left(1 + \frac{1}{2} + \cdots \frac{1}{k+m}\right)\right] t^{k+2m}\end{aligned}$$

where $d_{2m} = \dfrac{(-1)^m}{2^{2m+k} m!(m+k)!}$.

- Now let us consider the special choices of c_0 that in the first place defines

$$J_\lambda$$

- The standard definition is to choose

$$c_0 = \frac{1}{2^\lambda \int_0^\infty x^\lambda e^{-x}\, dx} = \frac{1}{2^\lambda \int_0^\infty x^{\lambda+1-1} e^{-x}\, dx}$$

where the improper integral is known as the gamma function, denoted by,

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx$$

- Hence, using this notation, we have

$$J_\lambda = \frac{1}{2^\lambda \Gamma(1+\lambda)} t^\lambda \left(1 + \sum_{k=1}^{\infty} c_{2k} t^{2k}\right)$$

- Like any improper integral, we need to check its convergence

$$\Gamma(z) = \lim_{b\to\infty} \int_0^b x^{z-1} e^{-x}\, dx$$

- For every $s > 0$, the improper integral

$$\int_0^\infty e^{-sx}\, dx = \lim_{b\to\infty} \int_0^b e^{-sx}\, dx = \lim_{b\to\infty} \frac{e^{-sb} - 1}{-s} = \frac{1}{s}$$

- For every natural number n, consider the following limit

$$\lim_{x\to\infty} \frac{x^{n-1}}{e^{x/2}} = 0 \implies \text{there exists } M > 0 \text{ such that for all } x > M$$

$$\left|\frac{x^{n-1}}{e^{x/2}}\right| < \epsilon = 1$$

$$0 \le x^{n-1} \le e^{x/2}$$

- This implies for $x > M$,

$$0 \le e^{-x} x^{n-1} \le e^{-x/2}$$

- By the comparison test,

$$\Gamma(z)$$

is convergent for every natural number z.

- Let $z \geq 1$ be any real number, $\lfloor z \rfloor$ be the floor function of z, that is

$$\lfloor z \rfloor \leq z < \lfloor z \rfloor + 1 \implies z - 1 < \lfloor z \rfloor$$

- Thus

$$0 \leq e^{-x} x^{z-1} \leq e^{-x} x^{\lfloor z \rfloor}$$

- By the comparison test again,

$$\Gamma(z)$$

is convergent for $z \geq 1$.

- Lastly, for $0 < z < 1$, it can be shown that

$$\Gamma(z)$$

is convergent as well, which I will leave it to you to complete.

- Hence the gamma function $\Gamma(z)$ is convergent for

$$z > 0$$

- Moreover, for every $z > 0$, it can be shown by integration by parts,

$$\Gamma(z+1) = z\Gamma(z) = z(z-1)\Gamma(z-1)\cdots$$

- For this reason the gamma function is often called the generalized factorial.

$$\Gamma(n+1) = n!$$

- With this property, the Bessel function of the first kind can be written as

$$J_\lambda(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\lambda+n)} \left(\frac{t}{2}\right)^{2n+\lambda}$$

- Now if we introduce a parameter s to the definition of the gamma function

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx \implies \int_0^\infty x^{z-1} e^{-sx}\, dx$$

- By using the substitution

$$u = sx$$

we have

$$\int_0^\infty x^{z-1} e^{-sx}\, dx = \int_0^\infty \left(\frac{u}{s}\right)^{z-1} e^{-u} \frac{1}{s}\, du = \frac{1}{s^z} \int_0^\infty u^{z-1} e^{-u}\, du = \frac{\Gamma(z)}{s^z}$$

- Use this on integer values of z, we have

$$\int_0^\infty (2 - 3x + 5x^2) e^{-sx}\, dx = \frac{2}{s} - \frac{3}{s^2} + \frac{10}{s^3}$$

Q: If $f(x)$ is a continuous function on $[0, \infty)$, such that

$$\int_0^\infty f(x) e^{-sx}\, dx = -\frac{3}{s} + \frac{10}{s^2} = F(s)$$

Is $f(x)$ unique? What is there between $f(x)$ and $F(s)$?