
Question1 (1 points)

Suppose $\mathbf{A}$ is a 3×3 matrix, and its eigenvalues are

$$\lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = 0,$$

and the corresponding eigenvectors are

$$\mathbf{x_1} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{x_2} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{x_3} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

respectively. Find the matrix $\mathbf{A}$.

Question2 (1 points)

Find the eigenvalues and the corresponding eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

Question3 (1 points)

Prove the following theorem.

> Suppose $\mathbf{A}$ is an $n \times n$ matrix, and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are eigenvectors for $\mathbf{A}$ with distinct eigenvalues, then these vectors are linearly independent.

Question4 (1 points)

Plot an informative phase portrait for the following system

$$\dot{R} = J$$
$$\dot{J} = -\sin R - \frac{1}{5}J$$

for $-10 \leq R \leq 10$ and $-12 \leq J \leq 12$.

Question5 (1 points)

Solve the first-order system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ by elimination, where

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 1 \\ -3 & 1 & -2 \\ -4 & 0 & -1 \end{bmatrix}$$

Question6 (1 points)

Prove the following theorem.

> Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices, then the matrix product
>
> $$\mathbf{AB} \quad \text{and} \quad \mathbf{BA}$$
>
> have the same eigenvalues.

Question7 (1 points)

Determine whether $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{bmatrix}$ is singular. If $\mathbf{A}$ is not singular, find $\mathbf{A}^{-1}$.

Question8 (2 points)

Prove the following theorem.

> Let $\mathbf{A}$ be an $n \times n$ matrix, then
>
> 1. λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$ if $\mathbf{A}$ is invertible and λ is an eigenvalue of $\mathbf{A}$.
> 2. $\mathbf{A}$ is singular if and only if $\lambda = 0$ is an eigenvalue of $\mathbf{A}$.

Question9 (1 points)

Suppose $\lambda_1 = 4$ and $\lambda_2 = -3$ are the eigenvalues of an unknown 2×2 matrix

$$\mathbf{A}$$

and the corresponding eigenvectors, respectively, are

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Find the eigenvalues and and the corresponding eigenspaces for the matrix

$$\mathbf{A}^3$$

Question10 (1 points)

Verify that the following matrix is diagonalizable

$$\mathbf{A} = \begin{bmatrix} 3 & -2 & -2 \\ -3 & -2 & -6 \\ 3 & 6 & 10 \end{bmatrix}$$

and find an invertible matrix $\mathbf{P}$ such that the following is a diagonal matrix

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

Question11 (1 points)

Find the general solution of the following system,

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & 2+\mathrm{i} \\ -1 & -1-\mathrm{i} \end{bmatrix} \mathbf{x}$$

Question12 (1 points)

Solve the given initial value problem, then describe the behaviour of the solution as $t \to \infty$.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 7 \\ 5 \\ 5 \end{bmatrix}$$

Question13 (1 points)

Find the solution of

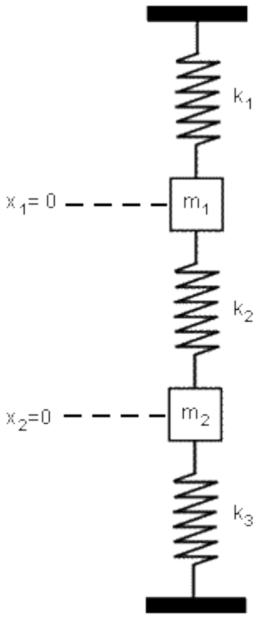
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

where the coefficient matrix is $\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 & 0 & 1 \\ -5 & 1 & -4 & 0 & 0 \\ 0 & -2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ which can be diagonalized by

$$\mathbf{P} = \begin{bmatrix} 2 & -1 & -2 & -8 & 0 \\ -10 & 1 & 1 & 15 & 0 \\ -5 & 1 & 2 & 10 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Question14 (1 points)

Consider the spring–mass system beside. The two masses, $m_1 = m_2 = 1$, are constrained by the three springs whose constants are $k_1 = k_2 = k_3 = 1$. If no external force is on the system and no damping force is present, By Newton's second law we can write the following equations for the coordinates x_1 and x_2 of the two masses:



$$\begin{aligned} m_1 \frac{d^2 x_1}{dt^2} &= k_2(x_2 - x_1) - k_1 x_1 \\ &= -(k_1 + k_2)x_1 + k_2 x_2 \end{aligned}$$

$$\begin{aligned} m_2 \frac{d^2 x_2}{dt^2} &= -k_3 x_2 - k_2(x_2 - x_1) \\ &= k_2 x_1 - (k_2 + k_3)x_2. \end{aligned}$$

Solve the system with the initial conditions

$$x_1(0) = 0, \qquad \dot{x_1}(0) = -1, \qquad x_2(0) = 0, \qquad \dot{x_2}(0) = 1$$

Question15 (1 points)

Given the following

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{4t} & 0 \\ te^{2t} & 0 & e^{2t} \end{bmatrix} \qquad \text{for} \qquad \mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Solve the following initial-value problem

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Question16 (1 points)

Find the matrix exponential $e^{\mathbf{B}t}$ for

$$\mathbf{B} = \begin{bmatrix} 9 & -5 \\ 0 & 1 \end{bmatrix}$$

Question17 (3 points)

Solve the following initial value problems

(a) (1 point)

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t \\ 0 \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(b) (1 point)

$$\dot{\mathbf{x}} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 4e^{2t} \\ 4e^{4t} \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(c) (1 point)

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Question18 (0 points)

(a) (1 point (bonus)) Find a 2×2 matrix $\mathbf{A}$ such that $\mathbf{Ax}$ is the orthogonal projection of

$$\mathbf{x} \in \mathbb{R}^2$$

onto the line spanned by

$$\mathbf{y} = 4\mathbf{e}_1 + 3\mathbf{e}_2$$

(b) (1 point (bonus)) Show $\det(\mathbf{A}) = P$, where P is the product of all eigenvalues of $\mathbf{A}$.

(c) (1 point (bonus)) The trace of a square matrix $\mathbf{A}$, denoted by $\operatorname{tr}(\mathbf{A})$, is the sum of the diagonal entries. Show $\operatorname{tr}(\mathbf{A}) = S$, where S is the sum of all eigenvalues of $\mathbf{A}$.

(d) (1 point (bonus)) Consider the following system

$$\frac{d}{dt}\begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} -\dfrac{R_1}{L} & -\dfrac{1}{L} \\ \dfrac{1}{C} & -\dfrac{1}{CR_2} \end{bmatrix} \begin{bmatrix} I \\ V \end{bmatrix}.$$

where I and V are unknown current and voltage in terms of t, R_1 and R_2 are resistance, L and C are inductance and capacitance, respectively. Find a condition that R_1, R_2, C, and L must satisfy such that the eigenvalues are distinct and real. Show under this condition $I \to 0$ and $V \to 0$ as $t \to \infty$ regardless of the initial conditions.

(e) (10 points (bonus)) Complete the IDEA survey for Vv256 before Friday 1st of Dec.
Link: Vv256