Vv156 Lecture 1

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- To warm up we will go over some elementary notions in mathematics.

Definition

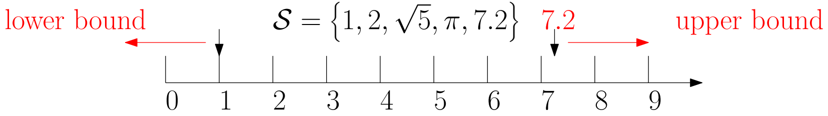
Let $\mathcal{S} \subset \mathbb{R}$ be a set of real numbers.

A real number $M \in \mathbb{R}$ is said to be an upper bound of $\mathcal{S}$ if

$$x \leq M \text{ for every } x \in \mathcal{S}$$

A real number $m \in \mathbb{R}$ is said to be a lower bound of $\mathcal{S}$ if

$$x \geq m \text{ for every } x \in \mathcal{S}$$



- A set is said to be bounded from above if it has an upper bound, bounded from below if it has a lower bound, and bounded if it has both.
- For example, the set of natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

is bounded from below by any $m \leq 1$.

- However, the set $\mathbb{N}$ is clearly not bounded from above, so $\mathbb{N}$ is unbounded.

Q: Is the set of reciprocals of the natural numbers bounded?

$$\mathcal{S} = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$$

Q: Can you think of an equivalent condition for a set $\mathcal{S}$ to be bounded?

- A set $\mathcal{S}$ is bounded if there exists $W \in \mathbb{R}$ such that

$$|x| \leq W \qquad \text{for every } x \in \mathcal{S}.$$

Definition

Let $\mathcal{S} \subset \mathbb{R}$ be a set of real numbers.

If $M \in \mathbb{R}$ is an upper bound of $\mathcal{S}$ such that

$$M \leq M^* \text{ for every upper bound } M^* \text{ of } \mathcal{S},$$

then M is called the supremum or least upper bound of $\mathcal{S}$, denoted

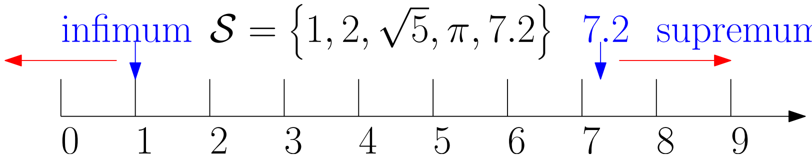
$$M = \sup(\mathcal{S})$$

If $m \in \mathbb{R}$ is a lower bound of $\mathcal{S}$ such that

$$m \geq m^* \text{ for every lower bound } m^* \text{ of } \mathcal{S},$$

then m is called the infimum or greatest lower bound of $\mathcal{S}$, denoted

$$m = \inf(\mathcal{S})$$



Q: Find the supremum and infimum of the following set $\mathcal{S}$, do they belong to $\mathcal{S}$?

$$\mathcal{S} = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$$

Q: Find the supremum of the set of real numbers

$$\mathcal{S} = \{x \in \mathbb{R} \mid x \leq \sqrt{2}\}$$

Q: Find the supremum of the set of rational numbers

$$\mathcal{S} = \{x \in \mathbb{Q} \mid x \leq \sqrt{2}\}$$

- I will assume everyone is familiar with open, closed and half-closed intervals

$$(a, b), \qquad [a, b], \qquad (a, b], \qquad \text{and} \qquad [a, b)$$

Definition

A set $\mathcal{V} \subset \mathbb{R}$ is a neighbourbood of a point $x \in \mathbb{R}$ if there exists some $\delta > 0$

$$(x - \delta, x + \delta) \subset \mathcal{V}$$

The open interval $(x - \delta, x + \delta)$ is called a δ-neighbourhood of x.

Q: Can a closed interval be a neighbourhood?

- For example, suppose

$$a < x < b$$

then the closed interval

$$\mathcal{V} = [a, b] \quad \text{is a neighbourhood of } x$$

since $\mathcal{V}$ contains the interval $(x - \delta, x + \delta)$ for sufficiently small $\delta > 0$.

Definition

A set $\mathcal{S} \subset \mathbb{R}$ is open in $\mathbb{R}$ if for every $x \in \mathcal{S}$ there exists a $\delta > 0$ such that

$$(x - \delta, x + \delta) \subset \mathcal{S}$$

A set $\mathcal{S} \subset \mathbb{R}$ is open if every $x \in \mathcal{S}$ has a neighbourhood $\mathcal{V}$ such that $\mathcal{V} \subset \mathcal{S}$.

Q: Are rational numbers an open set in $\mathbb{R}$?

Q: Is an arbitrary union of open sets open?

Q: Is an intersection of open sets open?

- For example, the interval

$$\mathcal{I}_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$$

is open for every $n \in \mathbb{N}$, but

$$\bigcap_{n=1}^{\infty} \mathcal{I}_n = \{0\}$$

is clearly not open.

- Another way to state the definition of open set is in terms of interior points.

Definition

Let $\mathcal{S} \subset \mathbb{R}$ be a set of real numbers. A point $x \in \mathcal{S}$ is an interior point of $\mathcal{S}$ if

$$\text{there is a } \delta > 0 \text{ such that } (x-\delta, x+\delta) \subset \mathcal{S}$$

A point $x \in \mathbb{R}$ is a boundary point of $\mathcal{S}$ if every neighbourhood

$$(x-\delta, x+\delta)$$

contains at least one point in $\mathcal{S}$ and at least one point not in $\mathcal{S}$.

- Therefore a set is open if and only if every point in the set is an interior point.

Q: Find the interior point/s of a set of irrational numbers.

Q: Find the boundary point/s of $\mathcal{S} = [1, 2)$.

Q: Is $\mathbb{R}$ open in $\mathbb{R}$? Is the empty set $\emptyset$ open in $\mathbb{R}$?

- Closed sets are complements of open sets.

Definition

A set $\mathcal{S} \subset \mathbb{R}$ is closed if $\mathcal{S}^{\complement} = \{x \in \mathbb{R} \mid x \notin \mathcal{S}\}$ is open.

Q: Is the interval $[a, \infty)$ closed?

Q: Is an arbitrary intersection of closed sets closed?

Q: Is an arbitrary union of closed sets closed?

- For example, suppose

$$\mathcal{I}_n = \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$$

then the union of the $\mathcal{I}_n$ is an open interval

$$\bigcup_{n=1}^{\infty} \mathcal{I}_n = (0, 1)$$

Q: Is $\mathbb{R}$ closed in $\mathbb{R}$? Is the empty set $\emptyset$ closed in $\mathbb{R}$?

Definition

A point $x \in \mathbb{R}$ is a limit point of $\mathcal{S} \subset \mathbb{R}$ if for every neighbourhood $(x-\delta, x+\delta)$ contains a point in $\mathcal{S}$ other than x itself.

- A limit point of a set $\mathcal{S}$ is a point in $\mathbb{R}$ that has points in $\mathcal{S}$ arbitrarily close to it.

Q: Is every point of every open set $\mathcal{S} \subset \mathbb{R}$ a limit point of $\mathcal{S}$?

Q: Is every limit point an interior point?

- For example, consider the closed set

$$\mathcal{S} = [0, 1]$$

  points 0 and 1 are limit points but not interior points of $\mathcal{S}$.

Q: Is the point 0 a boundary point of $\{0\}$? Is the point 0 a limit point of $\{0\}$?

Q: Is the point 0 a boundary point of $\mathcal{I} = [-1, 1]$? Is it a limit point of $\mathcal{I}$?

- A limit point of a set is either an interior point or a boundary point of the set.

Q: Find the limit point/s of the set $\mathbb{N}$ of natural numbers.

Definition

Let $\mathcal{S} \subset \mathbb{R}$. A point $x \in \mathbb{R}$ is an isolated point of $\mathcal{S}$ if there exists $\delta > 0$ such that

$$x \text{ is the only point belonging to } \mathcal{S} \text{ in the neighbourhood } (x - \delta, x + \delta).$$

- An isolated point of a set is a point in the set that does not have other points in the set arbitrarily close to it.
- Unlike limit points, isolated points are required to belong to the set.
- Every point $x \in \mathcal{S}$ is either a limit point or an isolated point.
- Clearly the set of natural numbers $\mathbb{N}$ contains only isolated points.

Q: Is it true that every interval has no isolated points?

Q: Find the isolated points for

$$\mathcal{S} = \{0\} \cup \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$$

Definition

A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.