Question1 (3 points)
Consider the following function.

$$f(x,y) = \frac{\sqrt{y-x^2}}{1-x} + \ln(2+x^3-y)$$

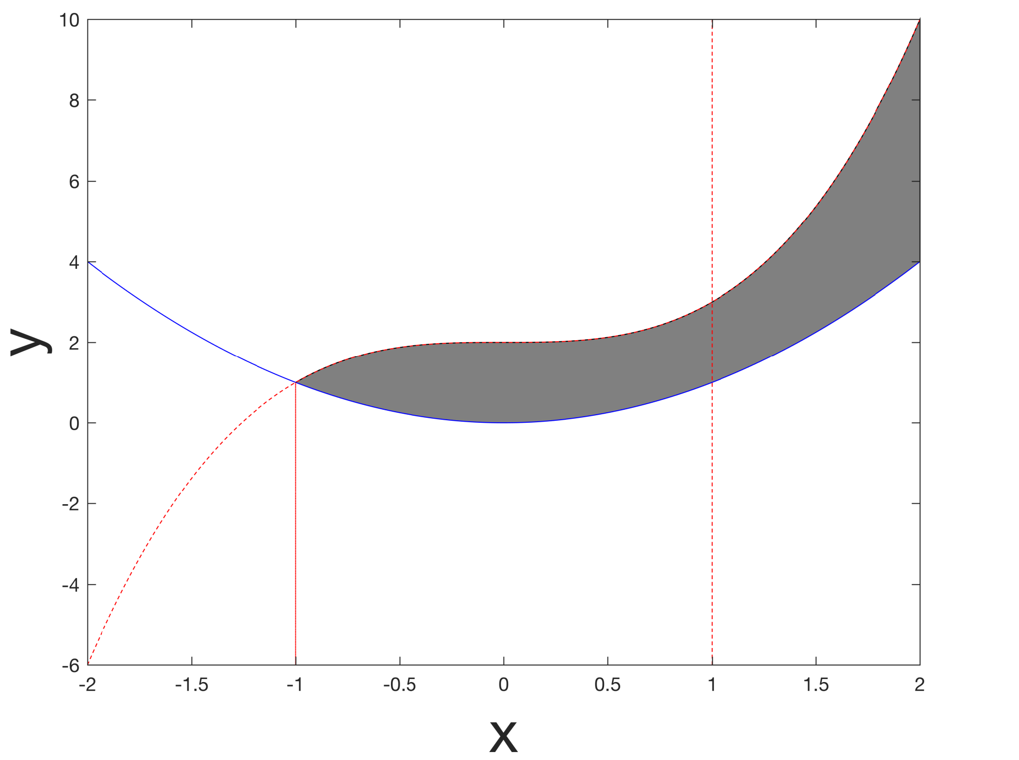
(a) (1 point) Sketch the domain of the function.

Solution:

1M We have the following conditions from the function ,

$$y \geq x^2; \qquad x \neq 1; \qquad y < x^3 + 2$$

The domain is the intersection of the three regions



(b) (1 point) Find the minimum of $f(x,y)$ along the path $y = x^2$ for $0 < x < 1$.

Solution:

1M Since we are finding the minimum of $f(x,y)$ on the the set

$$\{(x,y) \mid y = x^2\}$$

we are essentially finding the $x \in (0,1)$ such that

$$g(x) = f(x, y = x^2) = \ln(2 + x^3 - x^2)$$

attains its minimum. Since logarithmic function is monotonically increasing, we simply need to find the minimum of

$$h(x) = 2 + x^3 - x^2 \implies h'(x) = 3x^2 - 2x \implies h''(x) = 6x - 2$$

which can be shown to have a global minimum at $\frac{2}{3}$ in the open interval $(0, 1)$. Hence the minimum $f(x, y)$ along the path $y = x^2$ in $(0, 1)$ is

$$f\left(\frac{2}{3}, \frac{4}{9}\right) = \ln\left(\frac{50}{27}\right)$$

Question2 (5 points)

Evaluate each of the following limits, if it exists.

(a) (1 point)

$$\lim_{(x,y,z)\to(1,-1,-1)} \arcsin\left(\frac{2xy + yz}{x^2 + z^2}\right)$$

Solution:

1M In class, everything is stated for functions of two variables. However, those laws clearly also hold for functions of more than two variables. Since the function

$$h(x, y, z) = \frac{2xy + yz}{x^2 + z^2}$$

is a rational function and $(1, -1, -1)$ is clearly in its domain,

$$h(1, -1, -1) = -\frac{1}{2}$$

The function

$$g(u) = \arcsin(u)$$

is continuous, and

$$g\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

Hence the limit is in the domain of it.

$$\lim_{(x,y,z)\to(1,-1,-1)} \arcsin\left(\frac{2xy + yz}{x^2 + z^2}\right) = -\frac{\pi}{6}$$

(b) (1 point)

$$\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x - y}$$

Solution:

1M This is also a rational function, however, $(0, 0)$ is not the domain. If we factor the numerator, we have the following result

$$\begin{aligned}\frac{x^3-y^3}{x-y} &= \frac{(x-y)(x^2+xy+y^2)}{x-y}\\ &= x^2+xy+y^2 \qquad \text{when } x \neq y\end{aligned}$$

According to the sixth limit law, we have

$$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x-y} = \lim_{(x,y)\to(0,0)} x^2+xy+y^2 = 0$$

You might be sceptical here. Since the law need the equality of the two functions for the entire deleted neighbourhood of (a,b). It seems that we do not have the equality on the line

$$y = x$$

However, it is points that are in the domain that matters. Points along $y = x$ have no function value, thus cannot affect the limit of the function anywhere. The limit of the function at any point along $y = x$ are decided by nearby values of $f(x,y)$ where f is defined. Let me also use this opportunity to point out implicitly in the definition of limit of $f(x,y)$, we also only consider those points (x,y) inside the domain of f, since $f(x,y)$ has no value outside the domain, any point outside the domain does not need to satisfy

$$|f(x,y)-L| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$$

(c) (1 point)

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

Solution:

1M Limit along the path $x = 0$, we have

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } x=0}} \frac{x^2-y^2}{x^2+y^2} = -1$$

Limit along the path $y = 0$, we have

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=0}} \frac{x^2-y^2}{x^2+y^2} = 1$$

The two limits are not equal, so this limit does not exist.

(d) (1 point)

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$$

Solution:

1M It is easier to consider the limit in polar coordinates.

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

Since

$$r = \sqrt{x^2+y^2}$$

We have $(x,y) \to (0,0)$ if and only if $r \to 0^+$. Thus

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = \lim_{r\to 0^+} \frac{(r^2\cos^2\theta)\cdot(r\sin\theta)}{r^2} = \lim_{r\to 0^+} r\cos^2\theta\sin\theta = 0$$

(e) (1 point)

$$\lim_{(x,y)\to(1,0)} \frac{(x-1)^2\ln x}{(x-1)^2+y^2}$$

Solution:

1M Since

$$\frac{(x-1)^2}{(x-1)^2+y^2} \leq \frac{(x-1)^2+y^2}{(x-1)^2+y^2} = 1$$

we have the following inequality,

$$-|\ln x| \leq \frac{(x-1)^2\ln x}{(x-1)^2+y^2} \leq |\ln x|$$

Because

$$\lim_{(x,y)\to(1,0)} \ln x = \lim_{x\to 1} \ln x = 0 \implies \lim_{(x,y)\to(1,0)} |\ln x| = \lim_{(x,y)\to(1,0)} -|\ln x| = 0$$

then, by the squeeze theorem, we have

$$\lim_{(x,y)\to(1,0)} \frac{(x-1)^2\ln x}{(x-1)^2+y^2} = 0$$

Question3 (5 points)

(a) (2 points) Consider the following function,

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}.$$

Shown f is NOT continuous despite the fact that it is continuous in both x-direction and y-direction, and then find all directions that f is not continuous.

Solution:

1M Since $\dfrac{xy}{x^2+y^2}$ is a rational function, it is continuous inside its domain. It can only be not continuous at the origin. To show $f(x,y)$ is continuous at $(0,0)$ along x-direction, we consider

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=0}} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x\cdot 0}{x^2+0^2} = \lim_{x\to 0} 0 = 0$$

which is equal to the function value at the origin

$$f(0,0)=0$$

thus continuous in this direction. Since the function is symmetric in terms of x and y. So it is also continuous along y-axis. To show it is not continuous at $(0,0)$, we consider the direction given by

$$\mathcal{C}\colon \mathbf{r}(t) = t(\mathbf{e}_x + \mathbf{e}_y)$$

The limit of the value along $\mathcal{C}$ is given by

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } \mathcal{C}}} \frac{xy}{x^2+y^2} = \lim_{t\to 0} \frac{t\cdot t}{t^2+t^2} = \lim_{t\to 0} \frac{1}{2} = \frac{1}{2}$$

Not being zero means it is not equal to the function value at $(0,0)$, so it is not continuous along $\mathcal{C}$.

1M Now consider the direction given by

$$\mathcal{C}\colon \mathbf{r}(t) = t(a\mathbf{e}_x + b\mathbf{e}_y) \qquad \text{where } a \text{ and } b \text{ are contants.}$$

then the limit

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } \mathcal{C}}} \frac{xy}{x^2+y^2} = \lim_{t\to 0} \frac{abt^2}{(a^2+b^2)t^2} = \lim_{t\to 0} \frac{ab}{a^2+b^2} = \frac{ab}{a^2+b^2}$$

is zero only if $a=0$ or $b=0$. So it is not continuous in every other direction.

(b) (3 points) It is even possible to construct examples which are continuous in every linear direction at a particular point but not continuous at the point. Consider

$$g(x,y) = \begin{cases} \dfrac{y}{x^2}\left(1-\dfrac{y}{x^2}\right) & \text{if} \quad 0<y<x^2 \\ 0 & \text{if} \quad y\le 0 \text{ or } y\ge x^2 \end{cases}$$

Explain why g is continuous at all points other than $(0,0)$. Show that g is continuous in every linear direction at $(0,0)$. Then show that g is NOT continuous at $(0,0)$ by finding a curve approaching the origin along which the limit at the origin is not zero.

Solution:

1M For the two dimensional open region $0<y<x^2$, $g(x,y)$ is essentially a rational function, so continuous inside its domain. For the two dimensional open regions

$y < 0$ and $y > x^2$, $g(x, y)$ is identically zero, which is clearly continuous. This left us the boundaries of those open regions, where things may go wrong

$$y = 0 \qquad \text{and} \qquad y = x^2$$

The function $p(x, y) = 0$ is continuous. Excluding points on the path

$$x = 0$$

the function $q(x, y) = \frac{y}{x^2}\left(1 - \frac{y}{x^2}\right)$ is continuous, and for the boundaries

$$q(x, y) = \frac{0}{x^2}\left(1 - \frac{0}{x^2}\right) = 0 = p(x, y)$$
$$q(x, y) = \frac{x^2}{x^2}\left(1 - \frac{x^2}{x^2}\right) = 0 = p(x, y)$$

So we only need to check the intersection of the path $x = 0$ with the boundaries

$$y = 0 \qquad \text{and} \qquad y = x^2$$

which means it can only be not continuous at the origin $(0, 0)$.

1M Let us consider all possible linear directions

$$\mathcal{C} \colon \mathbf{r}(t) = t(a\mathbf{e}_x + b\mathbf{e}_y) \qquad \text{where } a \text{ and } b \text{ are contants.}$$

No matter the values of a and b, it is always possible to find some sufficiently small neighbourhood of $t = 0$, in which all t values satisfy either

$$y \le 0 \iff bt \le 0 \qquad \text{or} \qquad y \ge x^2 \iff bt \ge a^2t^2$$

In other words, as $t \to 0$, moving on any direction given by $\mathcal{C}$ always end up in a region where $g(x, y)$ only takes the value 0. So along any linear direction defined by $\mathcal{C}$ gives

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } \mathcal{C}}} g(x, y) = \lim_{t\to 0} g(x(t), y(t)) = \lim_{t\to 0} 0 = 0 = g(0, 0)$$

which means the function is continuous in every linear direction.

1M However, if we consider a simple quadratic path

$$y = \frac{1}{2}x^2$$

since $0 < \frac{1}{2}x^2 < x^2$ for $x \ne 0$, we have

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=\frac{1}{2}x^2}} g(x, y) = \lim_{t\to 0} \frac{\frac{1}{2}t^2}{t^2}\left(1 - \frac{\frac{1}{2}t^2}{t^2}\right) = \frac{1}{4} \ne 0$$

thus it is not continuous.

Question4 (2 points)

(a) (1 point) Verify that the function $z = \ln(e^x + e^y)$ satisfies the following

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$$

and

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$$

Solution:

1M This is just a matter of differentiation and substitution.

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{e^x}{e^x + e^y} + \frac{e^y}{e^x + e^y} = 1$$

For the second order derivatives,

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 &= \frac{e^x(e^x+e^y) - e^x e^x}{(e^x+e^y)^2}\frac{e^y(e^x+e^y) - e^y e^y}{(e^x+e^y)^2} - \left(-\frac{e^x e^y}{(e^x+e^y)^2}\right)^2 \\ &= \frac{e^x e^y}{(e^x+e^y)^2}\frac{e^y e^x}{(e^x+e^y)^2} - \frac{e^{2x}e^{2y}}{(e^x+e^y)^4} \\ &= 0 \end{aligned}$$

(b) (1 point) You are told that there is a function f whose partial derivatives are

$$f_x(x,y) = x + 4y \qquad \text{and } f_y(x,y) = 3x - y$$

Should you believe it? Justify your answer.

Solution:

1M It is clear that f_x, f_y, f_{xy} and f_{yx} are continuous. So $f_{xy} = f_{yx}$ by the mixed derivative theorem. However, the actual computation reveals that

$$f_{xy} = 4 \qquad f_{yx} = 3$$

thus it is not possible. Since it contradicts the mixed derivative theorem.

- Let me take this opportunity to show you the proof for the mixed derivative theorem, which states

> If $f(x,y)$ and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined throughout an open region containing a point (a,b) and are all continuous at (a,b), then
>
> $$f_{xy}(a,b) = f_{yx}(a,b)$$

With the given hypothesis, the point (a,b) is an interior point of the open region, and there exists a closed rectangular region $\mathcal{R}$ containing (a,b) inside the open region. Let h and k be numbers such that the point $(a+h, b+k)$ is also in $\mathcal{R}$. Let us break the increment in f into two levels, and use the following notation,

$$\Delta F = F(a+h) - F(a) \qquad \text{where} \qquad F(x) = f(x, b+k) - f(x,b)$$

Since f is continuous, then F must also be continuous including on the boundary of $\mathcal{R}$. Since f has continuous partial derivatives, F must also be differentiable at any interior point of $\mathcal{R}$. Thus we invoke the Mean Value Theorem upon F, that is, there exists c between a and $a+h$ such that

$$F'(c_1) = \frac{\Delta F}{h} \implies \Delta F = h\left[f_x(c_1, b+k) - f_x(c_1, b)\right]$$

Let $g(y) = f_x(c_1, y)$. Since f_x and f_{xy} are continuous, we can again use MVT

$$g'(d_1) = \frac{g(b+k) - g(b)}{k} \implies \Delta F = hkf_{xy}(c_1, d_1)$$

for some (c_1, d_1) in the interior of $\mathcal{R}$. If we break the increment in a differently,

$$\Delta G = G(b+k) - G(b) \qquad \text{where} \qquad G(y) = f(a+h, y) - f(a, y)$$

and also apply MVT twice, we have

$$\Delta G = khf_{yx}(c_2, d_2)$$

for some (c_2, d_2) in the interior of $\mathcal{R}$. Notice

$$\begin{aligned}
\Delta F &= f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b) \\
&= f(a+h, b+k) - f(a, b+k) - f(a+h, b) + f(a, b) \\
&= \Delta G
\end{aligned}$$

Thus,

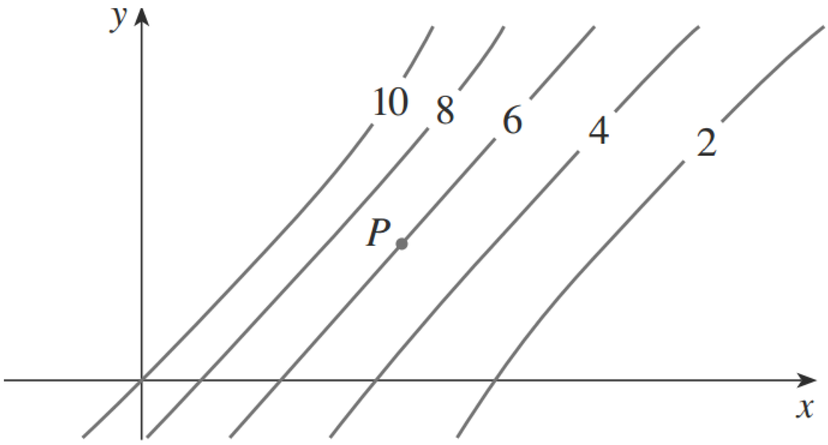
$$hkf_{xy}(c_1, d_1) = khf_{yx}(c_2, d_2) \implies f_{xy}(c_1, d_1) = f_{yx}(c_2, d_2)$$

for some (c_1, d_1) and (c_2, d_2) in the interior of $\mathcal{R}$. If we let $\mathcal{R}$ to shrink, that is, the sides of $\mathcal{R}$ approach zero, $\epsilon_x \to 0$ and $\epsilon_y \to 0$, then both h and k are approaching zero. Since f_{xy} and f_{yx} are continuous, $f_{xy}(c_1, d_1) \to f_{xy}(a, b)$ and $f_{yx}(c_2, d_2) \to f_{yx}(a, b)$ as $(h, k) \to (0, 0)$, which gives us the result,

$$f_{xy}(a, b) = f_{yx}(a, b)$$

There are different versions of this theorem, some with significantly weaker hypotheses than ours. However, this is sufficient for our purpose.

Question5 (1 points)
Some level curves of a functions $f(x, y)$ are shown below. Determine whether each of the partial derivatives $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ is positive or negative at P. Justify your answers.



Solution:

- If we fix y and allow x to vary, the level curves indicate that the value of f decreases as we move through P in the positive x-direction, so f_x is negative at P.
- If we fix x and allow y to vary, the level curves indicate that the value of f increases as we move through P in the positive y-direction, so f_y is positive at P.
- Since $f_{xx} = \frac{\partial}{\partial x} f_x$, so if we fix y and allow x to vary, f_{xx} is the rate of change of f_x as x increases. Note that at points to the right of P the level curves are spaced father apart (in x-direction) than at points to the left of P, demonstrating that f decreases less quickly with respect to x to the right of P. So as we move through P in the positive x-direction the (negative) value of f_x increases, hence f_{xx} is positive at P.
- Since $f_{xy} = \frac{\partial}{\partial y} f_x$, so if we fix x and allow y to vary, f_{xy} is the rate of change of f_x as y increases. The level curves are closer together at points above P than at those below P, meaning that f decreases more quickly with respect to x for y values above P. So as we move through P in the positive direction, the (negative) value of f_x decreases, hence f_{xy} is negative.
- Since $f_{yy} = \frac{\partial}{\partial y} f_y$, so if we fix x and allow y to vary, f_{yy} is the rate of change of f_y as y increases. The level curves are closer together (in the y-direction) at points above P than at those below P, demonstrating that f increases more quickly with respect to y above P. So as we move through P in the positive y-direction the (positive) value of f_y increases, hence f_{yy} is positive at P.

Question6 (4 points)
Consider the function

$$f(x, y) = \sqrt{|xy|}$$

(a) (1 point) Show $f_x(0,0)$ and $f_y(0,0)$ exist.

Solution:

1M Consider the definition of the partial derivative of f with respect to x at $(0,0)$,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\sqrt{|(0+h)(0)|} - \sqrt{|0\cdot 0|}}{h} = \lim_{h\to 0} 0 = 0$$

The function f is symmetric in terms of x and y, so $f_y(0,0) = 0$.

(b) (1 point) Is $f(x, y)$ continuous at $(0, 0)$?

Solution:

1M We could invoke theorems on compositions of continuous functions. However, let us see how we can prove it using the definition and the $\epsilon - \delta$ definition of limit. By definition, it is continuous if

$$\lim_{(x,y)\to(0,0)} f(x, y) = f(0, 0)$$

The function is clearly zero at $(0, 0)$, we need to check whether the limit is zero

$$\lim_{(x,y)\to(0,0)} \sqrt{|xy|}$$

We need to show, for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$\left|\sqrt{|xy|} - 0\right| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta$$

Since

$$(|x| - |y|)^2 \geq 0 \implies \frac{x^2 + y^2}{2} \geq |xy| \implies \sqrt{\frac{x^2 + y^2}{2}} \geq \sqrt{|xy|}$$

which means

$$\begin{aligned} \sqrt{|xy|} < \epsilon \qquad &\text{whenever} \qquad 0 < \sqrt{\frac{x^2 + y^2}{2}} < \epsilon \\ \sqrt{|xy|} < \epsilon \qquad &\text{whenever} \qquad 0 < \sqrt{x^2 + y^2} < \sqrt{2}\epsilon \end{aligned}$$

thus any $\delta \leq \sqrt{2}\epsilon$ is a valid choice. Therefore, it is continuous at $(0, 0)$.

(c) (1 point) Is there an arbitrary open region containing $(0, 0)$ in which f_x is continuous?

Solution:

1M The natural domain of $f(x, y)$ is the entire xy-plane. In the first quadrant,

$$f(x, y) = \sqrt{|xy|} = \sqrt{xy}$$

the partial derivatives at interior points of the first quadrant are given by

$$f_x(x, y) = \frac{1}{2}\sqrt{\frac{y}{x}}$$

If we consider the limit of f_x as (x, y) approaches $(0, 0)$ along $y = x$, we have

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } x=y}} \frac{1}{2}\sqrt{\frac{y}{x}} = \frac{1}{2}$$

which is not equal to

$$f_x(0, 0) = 0$$

Thus there is not possible to have

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = f_x(0,0)$$

which means f_x is not continuous at $(0,0)$. So let alone having the open region. Let us do more than what the question has asked for. In the second quadrant,

$$f(x,y) = \sqrt{|xy|} = \sqrt{-xy} \implies f_x(x,y) = -\frac{1}{2}\sqrt{-\frac{y}{x}}$$

Consider a point $(0, y_0)$ inside this quadrant, the limit fails to exist for any y_0

$$\lim_{\substack{(x,y)\to(0,y_0)\\ \text{along } y=y_0}} f_x(x,y)$$

Since

$$\lim_{\substack{(x,y)\to(0^+,y_0)\\ \text{along } y=y_0}} f_x(x,y) = \lim_{x\to 0^+} \frac{1}{2}\sqrt{\frac{y_0}{x}} = \infty$$

while

$$\lim_{\substack{(x,y)\to(0^-,y_0)\\ \text{along } y=y_0}} f_x(x,y) = \lim_{x\to 0^-} -\frac{1}{2}\sqrt{-\frac{y_0}{x}} = -\infty$$

which shows it is not even possible to have a deleted neighbourhood of $(0,0)$ in which f_x is continuous.

(d) (1 point) Is $f(x,y)$ differentiable at $(0,0)$?

Solution:

1M By definition, $f(x,y)$ is differentiable $(0,0)$ if and only if $f_x(0,0)$ and $f_y(0,0)$ both exists, and the following limit must be zero

$$\begin{aligned} L &= \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{f(0+\Delta x, 0+\Delta y) - f(0,0) - f_x(0,0)\Delta x - f_y(0,0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \\ &= \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\sqrt{\Delta x \Delta y}}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \end{aligned}$$

However, if we consider the limit along the path $\Delta x = \Delta y$, the limit is equal to $\frac{\sqrt{2}}{2}$. Therefore it is not differentiable at $(0,0)$.