Vv255 Lecture 7

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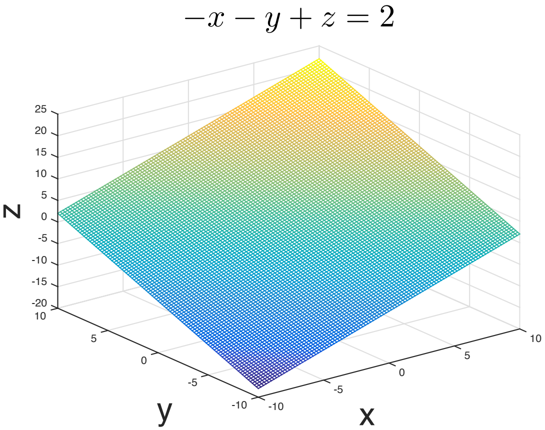
- Recall a plane is defined by

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \iff n_1 x + n_2 y + n_3 z - (n_1 x_0 + n_2 y_0 + n_3 z_0) = 0$$

where $\mathbf{r}$ and $\mathbf{r}_0$ are the position vectors of points (x, y, z) and (x_0, y_0, z_0) on the plane, respectively, and $\mathbf{n} = n_1 \mathbf{e}_x + n_2 \mathbf{e}_z + n_3 \mathbf{e}_z$ is the normal vector.



- With a simple rearrangement, we have

$$-x - y + z = 2 \implies z = x + y + 2 \tag{1}$$

- For every (x, y) in $\mathbb{R}^2$, there is one and only one z in $\mathbb{R}$ defined by Eq (1).

Q: Have you seen something similar before?

$$f : \mathbb{R}^2 \to \mathbb{R}$$

Definition

A function of two variables, denoted by,

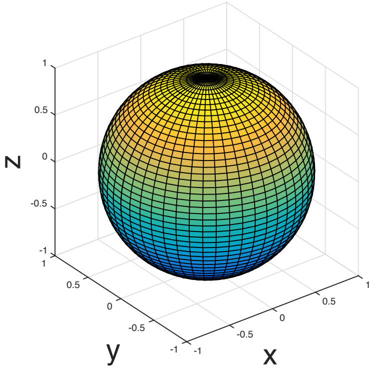
$$f(x, y)$$

is a rule that assigns each point (x, y) in some $\mathcal{D} \subset \mathbb{R}^2$ to a unique number in $\mathbb{R}$.

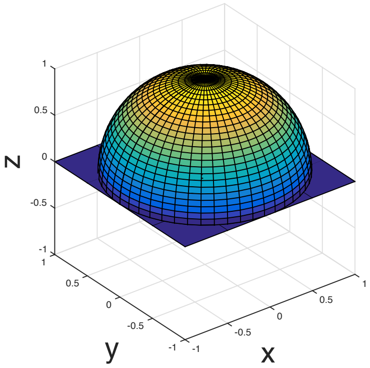
- The graphs of functions of two variables, $z = f(x, y)$, are surfaces in $\mathbb{R}^3$.

- However, some surface is not the graph of any function of two variables.

$$x^2 + y^2 + z^2 = 1$$



$$z = \sqrt{1 - x^2 - y^2}$$



Definition

If f is a function of two variables with domain $\mathcal{D}$, then the graph of f is the set of all points in such that $z = f(x, y)$ and (x, y) is in $\mathcal{D}$.

Matlab

```
%%%%%%%%%Shpere%%%%%%%%%%%%%%%%%%%%

>> sphere(50);

>> axis equal;
>> axis([-1 1 -1 1 -1 1]);
>> xlabel('x', 'fontsize', 30); ylabel('y', 'fontsize', 30); zlabel('z', 'fontsize', 30);

>> str = '$$x^2+y^2+z^2 = 1$$';
>> title(str, 'Interpreter','latex', 'fontsize',30, 'color','black');

%%%%%%%%%Hemishpere%%%%%%%%%%%%%%%%%%

>> [X,Y,Z] = sphere(50);
>> Z(Z < 0) = 0;
>> surf(X,Y,Z);
>> hold on;
>> axis equal;
>> axis([-1 1 -1 1 -1 1]);

>> x=-1:0.5:1; [X,Y] = meshgrid(x);
>> a=0; b=0; c=1; d=0;
>> Z=(d- a * X - b * Y)/c;
>> surf(X,Y,Z);
>> hold off;

>> xlabel('x', 'fontsize', 30); ylabel('y', 'fontsize', 30); zlabel('z', 'fontsize', 30);

>> str = '$$z = \sqrt{1-x^2-y^2}$$';
>> title(str, 'Interpreter','latex', 'fontsize',30, 'color','black');
```

Q: Can you picture the graph of the function

$$f(x,y) = 4x^2 + y^2$$

which is known as an elliptic paraboloid.

Definition

The level curves or contour lines of a function $f(x,y)$ are the curves defined by

$$f(x,y) = k,$$

where k is a constant in the range of f.

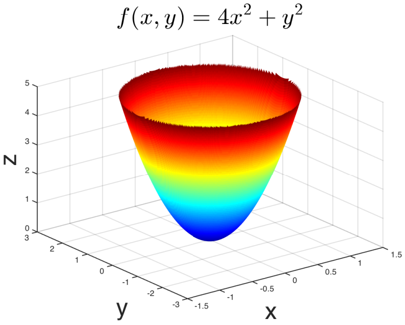
- Consider the level curves of the elliptic paraboloid

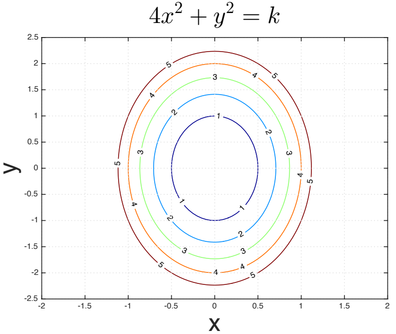
$$f(x,y) = k, \qquad \text{where} \quad f(x,y) = 4x^2 + y^2$$

for $k = 1, 2, 3, 4, 5$.

- We basically have give equations

$$4x^2 + y^2 = 1,\ 4x^2 + y^2 = 2,\ 4x^2 + y^2 = 3,\ 4x^2 + y^2 = 4,\ \text{and } 4x^2 + y^2 = 5$$





Q: What do those level curves represent?

Matlab

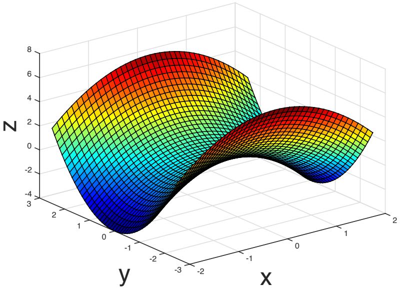
```
%%%%%%%%%%%%%%%%Paraboloid
%colormap(jet(64));
>> f = inline('4*x.^2+y.^2','x','y');
>> [X, Y] = meshgrid((-2:0.1:2), (-2.5:0.1:2.5));
>> Z = f(X, Y);
>> surf(X,Y,Z);

>> xlabel('x', 'fontsize', 30); ylabel('y', 'fontsize', 30);zlabel('z', 'fontsize', 30);
>> str = '$$f(x,y)=4x^2 + y^2$$';
>> title(str, 'Interpreter','latex', 'fontsize',30, 'color','black');

>> [X, Y] = meshgrid((-2:0.01:2),(-2.5:0.01:2.5));
>> Z = f(X,Y);
>> Z(Z>5*1^2)=NaN; % Any points above a certain Z value get removed.
>> han1 = surf(X,Y,Z);
>> set(han1,'edgecolor','none') % Avoids the entire surface appearing black
>> xlabel('x', 'fontsize', 30); ylabel('y', 'fontsize', 30);zlabel('z', 'fontsize', 30);
>> str = '$$f(x,y)=4x^2 + y^2$$';
>> title(str, 'Interpreter','latex', 'fontsize',30, 'color','black');

%%%%%%%%%%%%%%%%Contour
>> [k,h]=contour(x,y,z,[1,2,3,4,5]);
>> clabel(k,h); %this labels the c values on the contour plot
>> grid on;
>> xlabel('x', 'fontsize', 30); ylabel('y', 'fontsize', 30);
>> str = '$$4x^2 + y^2 = k$$';
>> title(str, 'Interpreter','latex', 'fontsize',30, 'color','black');
```

Q: Can you picture the graph of the function

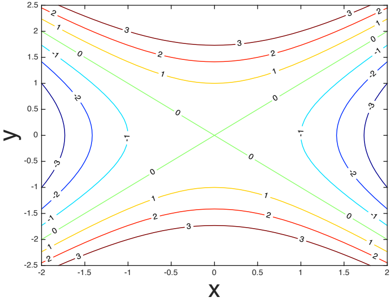
$$f(x, y) = y^2 - x^2$$

which is known as a hyperbolic paraboloid.

$f(x, y) = y^2 - x^2$



$y^2 - x^2 = k$



Matlab

```
%Hyperbolic Paraboloid

>> f = inline('y.^2-x.^2','x','y');
>> [X, Y] = meshgrid((-2:0.1:2),(-2.5:0.1:2.5));
>> Z = f(X,Y);
>> surf(X,Y,Z)

>> xlabel('x', 'fontsize', 30);
>> ylabel('y', 'fontsize', 30);
>> zlabel('z', 'fontsize', 30);
>> str = '$$f(x,y)= y^2 - x^2$$';
>> title(str, 'Interpreter','latex', 'fontsize',30, 'color','black');

%%%%%%%%%%%%%%%%Contour

>> [k,h]=contour(X,Y,Z,[-3,-2,-1,0,1,2,3]);
>> clabel(k,h); %this labels the c values on the contour plot
>> xlabel('x', 'fontsize', 30);
>> ylabel('y', 'fontsize', 30);
>> str = '$$y^2 -x^2 = k$$';
>> title(str, 'Interpreter','latex', 'fontsize',30, 'color','black');
```

- Of course, the domain of a function of two variables may not be $\mathbb{R}^2$.

$$f(x,y) = \frac{1}{xy}$$

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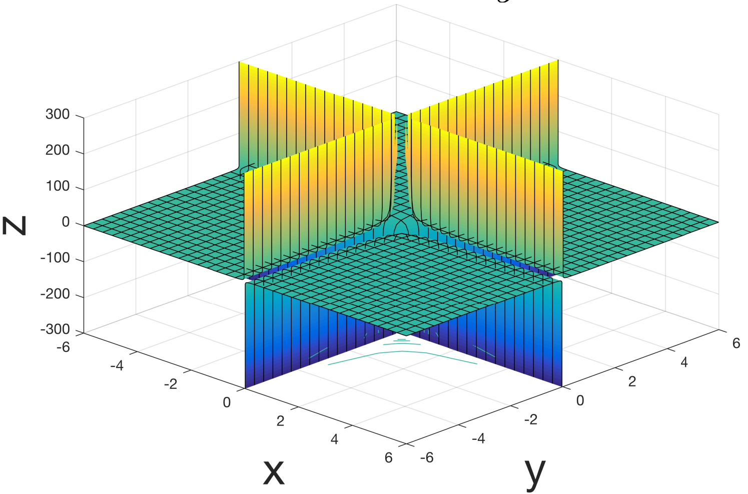
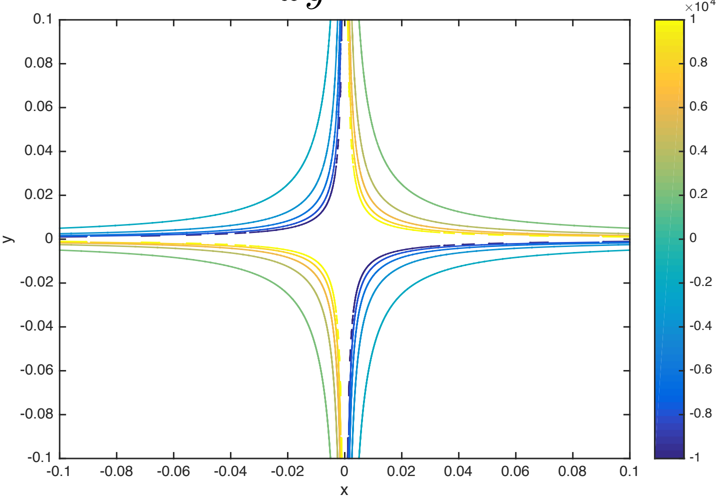


$$\frac{1}{xy} = k$$



0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
y
-0.1
-0.08
-0.06
-0.04
-0.02
0
0.02
0.04
0.06
0.08
0.1
x
×10^4
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1

Matlab

```
>> ezsurfc('1/(x*y)');

>> xlabel('x', 'fontsize', 30);
>> ylabel('y', 'fontsize', 30);
>> zlabel('z', 'fontsize', 30);

>> str = '$$f(x,y) = \frac{1}{xy}$$';
>> title(str, 'Interpreter','latex', 'fontsize',30, 'color','black');

>> ezcontour('1/(x*y)',[-0.1,0.1],500);
>> colorbar;

>> str = '$$\frac{1}{xy} = k$$';
>> title(str, 'Interpreter','latex', 'fontsize',30, 'color','black');
```

Exercise

Find the natural domain of

$$f(x,y) = \sqrt{y+1} + \ln(x^2 - y)$$