
Question1 (5 points)

(a) (1 point) Let $\mathcal{C}$ be the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Evaluate the line integral

$$\oint_{\mathcal{C}} y(4x^2+y^2)\,dx + \oint_{\mathcal{C}} x(2x^2+3y^2)\,dy$$

Solution:

1M Let

$$P = y(4x^2+y^2) \qquad \text{and} \qquad Q = x(2x^2+y^2)$$

By Green's theorem, we have

$$\begin{aligned}\oint_C y(4x^2+y^2)\,dx + \oint_C x(2x^2+3y^2)\,dy &= \oint_C (P\mathbf{e}_x + Q\mathbf{e}_y)\cdot d\mathbf{r}\\ &= \iint_{\mathcal{D}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA\\ &= \iint_{\mathcal{D}} 2x^2\,dA\end{aligned}$$

To evaluate the double integral, we use the following substitution

$$x = ar\cos\theta \quad y = br\sin\theta.$$

The Jacobian of this substitution is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = abr$$

Therefore,

$$\begin{aligned}\oint_C y(4x^2+y^2)\,dx + \oint_C x(2x^2+3y^2)\,dy &= \int_0^1\int_0^{2\pi} 2a^2\cos^2\theta abr\,d\theta\,dr\\ &= a^3b\pi\end{aligned}$$

(b) (1 point) Let

$$\mathbf{F} = (\sin x)\mathbf{e}_x + (x+y)\mathbf{e}_y$$

Find the line integral of $\mathbf{F}$ around the perimeter of the rectangle with corners

$$(3,0),\ (3,5),\ (-1,5),\ (-1,0),$$

traversed in that order.

Solution:

1M Let $P=\sin x$ and $Q=x+y$, and $\mathcal{D}$ denote the rectangular region. Using Green's theorem, we have

$$\oint_{\partial\mathcal{D}} \mathbf{F}\cdot d\mathbf{r} = \iint_{\mathcal{D}}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA = \iint_{\mathcal{D}} dA = 4\cdot 5 = 20$$

(c) (1 point) Show that the line integral of

$$\mathbf{F} = x\mathbf{e}_y$$

around a positively oriented closed curve in the xy-plane measures the area of the region enclosed by the curve.

Solution:

1M The line integral can be converted into a double integral

$$\oint_{\mathcal{C}} \mathbf{F}\, d\mathbf{r} = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{\mathcal{D}} (1-0)\, dA$$

which is the area of the region $\mathcal{D}$ enclosed by the curve $\mathcal{C}$.

(d) (1 point) Use Green's theorem to find the area of the region enclosed by

one arch of the cycloid $x = t - \sin t$, $y = 1 - \cos t$.

Solution:

1M To find the area, we need to evaluate

$$\iint_D 1 dA$$

by converting it into a line integral using Green's theorem,

$$\iint_D 1 dA = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

We need to find components functions such that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

One such simple choice is to set

$$P = -y \qquad \text{and} \qquad Q = 0$$

According to the green theorem, we can have the equality

$$\begin{aligned}\iint_{\mathcal{D}} 1 dA = \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \int_{2\pi}^{0} P(x(t), y(t)) x'(t)\, dt \\ &= \int_{2\pi}^{0} -(1-\cos t)(1-\cos t)\, dt \\ &= 3\pi\end{aligned}$$

(e) (1 point) For the following scalar-valued function and the circle

$$f(x, y) = \ln(x^2 + y^2); \qquad x^2 + y^2 = a^2$$

Evaluate the flux integral

$$\oint_{\mathcal{C}} \nabla f \cdot \mathbf{n}\, ds$$

Solution:

1M The gradient filed is given by

$$\nabla f = P\mathbf{e}_x + Q\mathbf{e}_y = \frac{2x}{x^2+y^2}\mathbf{e}_x + \frac{2y}{x^2+y^2}\mathbf{e}_y$$

It is clear that the field and derivatives of its component functions are not defined at $(0,0)$, so we cannot actually use Green's theorem. We have to consider a parametrisation of the curve.

$$\mathbf{r}(\theta) = a\cos t\mathbf{e}_x + a\sin t\mathbf{e}_y \qquad \text{for} \quad 0 \le t \le 2\pi$$

Since this parametrisation gives a positively oriented curve with respect to the enclosed region, the outward unit normal is given by

$$\begin{aligned}\mathbf{n} = \frac{dy}{ds}\mathbf{e}_x - \frac{dx}{ds}\mathbf{e}_y &= \left(\frac{dy}{dt}\mathbf{e}_x - \frac{dx}{dt}\mathbf{e}_y\right)\frac{dt}{ds} \\ &= (a\cos t\mathbf{e}_x + a\sin t\mathbf{e}_y)\frac{1}{a} \\ &= \cos t\mathbf{e}_x + \sin t\mathbf{e}_y\end{aligned}$$

where s is the arc length parametrisation. Note the formula for the normal

$$\mathbf{n} = \frac{dy}{ds}\mathbf{e}_x - \frac{dx}{ds}\mathbf{e}_y$$

will have a negative sign if the parametrisation is not positively oriented with respect to the enclosed region. If you don't want to remember the formula, you could also obtain it from TNB, since this parametrisation clearly has zero tangential acceleration,

$$\mathbf{N} = \frac{\mathbf{r}''}{|\mathbf{r}''|} = -\cos t\mathbf{e}_x - \sin t\mathbf{e}_y$$

where $\mathbf{N}$ is the unit principal normal vector. And the outward unit normal is

$$\mathbf{n} = -\mathbf{N}$$

Hence the correct flux shall be

$$\begin{aligned}\oint_{\mathcal{C}} \nabla f \cdot \mathbf{n}\, ds &= \oint_{\mathcal{C}} \left(\frac{2a\cos t}{a^2}\mathbf{e}_x + \frac{2a\sin t}{a^2}\mathbf{e}_y\right)\cdot(\cos t\mathbf{e}_x + \sin t\mathbf{e}_y)\, ds \\ &= \oint_{\mathcal{C}} \frac{2}{a}\, ds = \int_0^{2\pi a} \frac{2}{a}\, ds = 4\pi\end{aligned}$$

which is different from the value when Green's theorem is blindly used.

$$\begin{aligned}\oint_{\mathcal{C}} \nabla f \cdot \mathbf{n}\, ds &= \iint_{\mathcal{D}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA \\ &= \iint_{\mathcal{D}} \left(\frac{2y^2-2x^2}{(x^2+y^2)^2} + \frac{2x^2-2y^2}{(x^2+y^2)^2}\right) dA \\ &= \iint_{\mathcal{D}} 0\, dA \\ &= 0\end{aligned}$$

Question2 (5 points)

(a) (1 point) Explain the meaning of the Laplacian of a scalar function φ.

Solution:

0M The Laplacian of a scalar function is the divergence of the gradient of it. For example, if the scalar function is the temperature at a point (x, y, z) in $\mathbb{R}^3$, then the gradient gives the direction and the magnitude of the maximum rate of increase in temperature. Roughly speaking, the divergence of it gives how the magnitude of this vector is changing in the nearby vicinity. A positive Laplacian at a point indicates that the temperature at this point is less than the average local value, the temperature has a net tendency to increase when moving away from this point. A negative Laplacian at a point indicates that the temperature at this point is higher than the average local value, the temperature has a net tendency to decrease when moving away from this point.

(b) (1 point) Show the curl of the velocity field of a rotating rigid body has the direction of the axis of the rotation, and its magnitude equals twice the angular speed of the rotation.

Solution:

0M Consider a rotating rigid body with the angular velocity

$$\boldsymbol{\omega} = 0\mathbf{e}_x + 0\mathbf{e}_y + |\boldsymbol{\omega}|\mathbf{e}_z$$

parallel to the z-axis.

0M Let P be a point on the rigid body, it is clear that the velocity at P is orthogonal to both the position vector $\mathbf{r}$ of P and the angular velocity $\boldsymbol{\omega}$. Suppose d is the distance between the centre of rotation to P, then the angular speed is

$$\begin{aligned}|\boldsymbol{\omega}| = \frac{|\mathbf{v}|}{d} \implies |\mathbf{v}| = |\boldsymbol{\omega}|\, d = |\boldsymbol{\omega}||\mathbf{r}|\sin\theta &= |\boldsymbol{\omega} \times \mathbf{r}| \\ \implies \mathbf{v} &= \boldsymbol{\omega} \times \mathbf{r} \\ &= -|\boldsymbol{\omega}|y\mathbf{e}_x + |\boldsymbol{\omega}|x\mathbf{e}_y + 0\mathbf{e}_z\end{aligned}$$

0M By a direction computation of the curl of the velocity field, we have

$$\operatorname{curl}(\mathbf{v}) = 2|\boldsymbol{\omega}|\mathbf{e}_z = 2\boldsymbol{\omega}$$

0M This conclusion will not change under any orthonormal change of coordinates.

(c) (1 point) For $f = z^2(x^2 + y^2 + z^2)^2$, find $\nabla \cdot (\nabla f)$ and $\nabla \times (2f\nabla f)$.

Solution:

0M Use cylindrical coordinates,

$$f(x, y, z) = f^*(r, \theta, z) = z^2(r^2 + z^2)^2$$

0M The gradient is

$$\nabla f = 4rz^2(r^2 + z^2)\mathbf{e}_r + \Big(2z(r^2 + z^2)^2 + 4z^3(r^2 + z^2)\Big)\mathbf{e}_z$$

0M Thus the divergence is

$$\begin{aligned}\nabla \cdot \nabla f &= 8r^2z^2 + 8z^2(r^2+z^2) + 20z^2(r^2+z^2) + 2(r^2+z^2)^2 + 8z^4 \\ &= 2r^4 + 40r^2z^2 + 38z^4 \\ &= 2(x^2+y^2)^2 + 40(x^2+y^2)z^2 + 38z^4\end{aligned}$$

0M The curl of $2f\nabla f$ is

$$\nabla \times (2f\nabla f) = \nabla \times (\nabla f^2) = \mathbf{0}$$

it is zero for it is the curl of a gradient field, which must be irrotational.

Maxwell's equations relating the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ as they vary with time in a region containing no charge and no current can be stated as follows:

$$\operatorname{div}\mathbf{E} = 0 \qquad \operatorname{div}\mathbf{H} = 0 \qquad \operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \qquad \operatorname{curl}\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

where c is the speed of light. Use these equations to prove the following relationships.

(d) (1 point) $\nabla \times (\nabla \times \mathbf{E}) = -\dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{E}}{\partial t^2}$

Solution:

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left(-\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}\right) = -\frac{1}{c}\nabla \times \left(\frac{\partial \mathbf{H}}{\partial t}\right) = -\frac{1}{c}\frac{\partial}{\partial t}\nabla \times \mathbf{H} &= -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right) \\ &= -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}\end{aligned}$$

(e) (1 point) $\nabla^2 \mathbf{H} = \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{H}}{\partial t^2}$

Solution:

0M The Laplacian of a vector field is defined to be the vector of the Laplacian of the component functions of the vector field.

$$\mathbf{F} = F_x\mathbf{e}_x + F_y\mathbf{e}_y + F_z\mathbf{e}_z \implies \nabla^2\mathbf{F} = (\nabla^2 F_x)\mathbf{e}_x + (\nabla^2 F_y)\mathbf{e}_y + (\nabla^2 F_z)\mathbf{e}_z$$

0M Similarly, the gradient of a vector field is the vector of the gradient of the component functions of the vector field.

$$\nabla\mathbf{F} = (\nabla F_x)\mathbf{e}_x + (\nabla F_y)\mathbf{e}_y + (\nabla F_z)\mathbf{e}_z$$

0M We also need an identity for the curl of the curl of a vector field $\mathbf{F}$.

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla \cdot (\nabla \mathbf{F})$$

although very tedious, this identity can be proved directly from definitions.

0M Using those facts, we can easily show

$$\begin{aligned}\nabla^2\mathbf{H} = \nabla \cdot \nabla\mathbf{H} = \nabla(\nabla \cdot \mathbf{H}) - \nabla \times (\nabla \times \mathbf{H}) &= \nabla 0 - \left(-\frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}\right) \\ &= \frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}\end{aligned}$$

Question3 (5 points)

(a) (1 point) Show that the following parametric equations represent an ellipsoid.

$$x = a\sin u\cos v, \quad y = b\sin u\sin v, \quad z = c\cos u, \quad \text{where} \quad 0 \le u \le \pi, 0 \le v \le 2\pi$$

Solution:

0M This is just a matter of covering back to the familiar implicit form,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \sin^2 u\cos^2 v + \sin^2 u\sin^2 v + \cos^2 u = 1$$

(b) (1 point) Find the area of the surface the part of the sphere

$$x^2 + y^2 + z^2 = b^2$$

that lies inside the cylinder

$$x^2 + y^2 = a^2, \qquad \text{where } 0 \le a \le b.$$

Solution:

0M We have the following vector-valued function for the surface,

$$\mathbf{r}(\phi, \theta) = b\sin\phi\cos\theta\mathbf{e}_x + b\sin\phi\sin\theta\mathbf{e}_y + b\cos\phi\mathbf{e}_z$$

where $0 \le \phi \le \sin^{-1}(a/b)$ and $0 \le \theta \le 2\pi$.

0M Evaluation

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{bmatrix} b^2\sin^2\phi\cos\theta \\ b^2\sin^2\phi\sin\theta \\ b^2\sin(2\phi)/2 \end{bmatrix} \implies A(S) = 2\int_0^{2\pi}\int_0^{\sin^{-1}(a/b)} |\mathbf{r}_\phi \times \mathbf{r}_\theta|\, d\phi\, d\theta$$

$$= 2\int_0^{2\pi}\int_0^{\sin^{-1}(a/b)} b^2\sin\phi\, d\phi\, d\theta$$

$$= 4\pi b\left(b - \sqrt{b^2 - a^2}\right)$$

(c) (1 point) Evaluate the surface integral

$$\iint_{\mathcal{S}} \left(x^2 + y^2\right)\, dS$$

where $\mathcal{S}$ is the surface with the vector equation

$$\mathbf{r}(u, v) = 2uv\mathbf{e}_x + (u^2 - v^2)\mathbf{e}_y + (u^2 + v^2)\mathbf{e}_z, \qquad u^2 + v^2 \le 1$$

Solution:

0M Find the magnitude of the cross product between partial derivatives,

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{(4u^2 - 4v^2)^2 + (4u^2 + 4v^2)^2 + 64\,(uv)^2}$$

0M Use polar coordinates $u = r\cos\theta$ and $v = r\sin\theta$, we have

$$\iint_{\mathcal{S}} \left(x^2 + y^2\right)\, dS = \int_0^{2\pi}\int_0^1 4\sqrt{2}r^7\, dr\, d\theta = \pi\sqrt{2}$$

(d) (1 point) Integrate $G(x, y, z) = y + z$ over the surface of the wedge in the first octant bounded by the coordinate planes and the planes $x = 2$ and $y + z = 1$.

Solution:

0M This wedge consists of five pieces, we have to compute over each one at a time.

0M Let $\mathcal{S}_1$ be the face of the wedge in the xz-plane, which can be defined explicitly

$$y = \phi(x, z) = 0 \qquad \text{where} \quad 0 \leq x \leq 2 \quad \text{and} \quad 0 \leq z \leq 1$$
$$\implies \mathbf{r}(u, v) = u\mathbf{e}_x + 0\mathbf{e}_y + v\mathbf{e}_z \implies |\mathbf{r}_u \times \mathbf{r}_v| = 1$$
$$\iint_{\mathcal{S}_1} G\,dS = \iint_{\mathcal{D}} (y + z)\,|\mathbf{r}_u \times \mathbf{r}_v|\;dA = \int_0^1 \int_0^2 v\,du\,dv = 1$$

0M Similarly, we can obtain the integral over the face in xy-plane

$$\iint_{\mathcal{S}_2} G\,dS = 1$$

and over the face in yz-plane.

$$\iint_{\mathcal{S}_3} G\,dS = \frac{1}{3}$$

and over the face in the plane $x = 2$,

$$\iint_{\mathcal{S}_4} G\,dS = \frac{1}{3}$$

0M Lastly, on the slope face,

$$y + z = 1 \implies z = 1 - y \implies \mathbf{r} = u\mathbf{e}_x + v\mathbf{e}_y + (1 - v)\mathbf{e}_z \implies G(x, y, z) = 1$$
$$\implies |\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{2}$$
$$\iint_{\mathcal{S}_5} G\,dS = \iint_{\mathcal{D}} \sqrt{2}\,dA = \int_0^1 \int_0^2 \sqrt{2}\,du\,dv = 2\sqrt{2}$$

0M Therefore,

$$\iint_{\mathcal{S}} G\,dS = 1 + 1 + \frac{1}{3} + \frac{1}{3} + 2\sqrt{2} = \frac{8}{3} + 2\sqrt{2}$$

(e) (1 point) Find the outward flux of the field

$$\mathbf{F} = xz\mathbf{e}_x + yz\mathbf{e}_y + \mathbf{e}_z$$

across the surface of the upper cap cut from the solid sphere

$$x^2 + y^2 + z^2 \leq 25 \qquad \text{by} \qquad z = 3$$

Solution:

0M Use the following parametrization,

$$\mathbf{r}(u, v) = u\mathbf{e}_x + v\mathbf{e}_y + \sqrt{25 - u^2 - v^2}\mathbf{e}_z$$

for $\mathcal{D}$: $0 \le u^2 + v^2 \le 16$.

0M Find cross product of the partial derivatives, and confirm the orientation,

$$\mathbf{r}_u \times \mathbf{r}_v = \frac{u}{\sqrt{25 - u^2 - v^2}}\mathbf{e}_x + \frac{v}{\sqrt{25 - u^2 - v^2}}\mathbf{e}_y + \mathbf{e}_z$$

0M Find the component of $\mathbf{F}$ in the direction this vector in terms of u and v,

$$\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = u^2 + v^2 + 1$$

0M The surface integral of $\mathbf{F}$ over the top is

$$\iint_{\mathcal{S}_1} \mathbf{F} \cdot \mathbf{S} = \iint_{\mathcal{D}} (u^2 + v^2 + 1)\, dA = \int_0^4 \int_0^{2\pi} (r^2 + 1) r\, d\theta\, dr = 144\pi$$

0M Now for the bottom disk, we use

$$\mathbf{r}(u, v) = u\mathbf{e}_x + v\mathbf{e}_y + 3\mathbf{e}_z$$

for $\mathcal{D}$: $0 \le u^2 + v^2 \le 16$.

0M Similar computation reveals

$$\iint_{\mathcal{S}_2} \mathbf{F} \cdot \mathbf{S} = -16\pi$$

0M Therefore, we have the total outward flux of

$$144\pi - 16\pi = 128\pi$$

0M Note the divergence theorem can used here instead.

Question4 (5 points)

(a) (1 point) Let $\mathbf{F}$ be an inverse square field, show that the flux of $\mathbf{F}$ across a sphere $\mathcal{S}$ with centre the origin is independent of the radius of $\mathcal{S}$.

Solution:

0M Let $\mathcal{S}$ be a sphere of radius a centred at the origin. Then $|\mathbf{r}| = a$ and

$$\mathbf{F}(\mathbf{r}) = c\frac{\mathbf{r}}{|\mathbf{r}|^3} = \frac{c}{a^3}\left(x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z\right)$$

0M A convenient parametric representation for $\mathcal{S}$ is,

$$\mathbf{r}(u, v) = a \sin u \cos v \mathbf{e}_x + a \sin u \sin v \mathbf{e}_y + a \cos u \mathbf{e}_z, \qquad 0 \le u \le \pi, \quad 0 \le v \le 2\pi.$$

thus

$$\begin{aligned} \mathbf{r}_u &= a \cos u \cos v \mathbf{e}_x + a \cos u \sin v \mathbf{e}_y - a \sin u \mathbf{e}_z \\ \mathbf{r}_v &= -a \sin u \sin v \mathbf{e}_x + a \sin u \cos v \mathbf{e}_y + 0\mathbf{e}_z, \\ \implies \mathbf{r}_u \times \mathbf{r}_v &= a^2 \sin^2 u \cos v \mathbf{e}_x + a^2 \sin^2 u \sin v \mathbf{e}_y + a^2 \sin u \cos u \mathbf{e}_z \end{aligned}$$

0M Consider the z-component of $\mathbf{r}_u \times \mathbf{r}_v$, it is positive for $0 < u < \frac{\pi}{2}$, and negative for $\frac{\pi}{2} < u < \pi$, thus $\mathbf{r}_u \times \mathbf{r}_v$ gives the outward normal,

$$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v$$

0M The flux of $\mathbf{F}$ across $\mathcal{S}$ is

$$\begin{aligned}
\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{\mathcal{S}} \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\ dS
&= \frac{c}{a^3} \int_0^{\pi} \int_0^{2\pi} a^3 (\sin^3 u + \sin u \cos^2 v)\, dv\, du \\
&= c \int_0^{\pi} \int_0^{2\pi} \sin u\, dv\, du \\
&= 4\pi c
\end{aligned}$$

Hence the flux does not depend on the radius a.

(b) (1 point) Use Stokes' Theorem to evaluate $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F} = x^2 y \mathbf{e}_x - x y^2 \mathbf{e}_y + z^3 \mathbf{e}_z$$

and $\mathcal{C}$ is the curve of intersection of the following plane and cylinder

$$3x + 2y + z = 6; \qquad x^2 + y^2 = 4$$

oriented clockwise when viewed from above.

Solution:

0M Let $\mathcal{S}$ be the part of the plane

$$3x + 2y + z = 6$$

that lies inside the cylinder

$$x^2 + y^2 = 1$$

oriented downward. Then

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}, \qquad \text{where} \qquad \operatorname{curl} \mathbf{F} = (-y^2 - x^2)\mathbf{e}_z$$

0M A vector-valued function for $\mathcal{S}$ is, where $u^2 + v^2 \le 4$,

$$\mathbf{r}(u, v) = \begin{bmatrix} u \\ v \\ 6 - 3u - 2v \end{bmatrix} \implies \begin{aligned} \mathbf{r}_u &= (1)\mathbf{e}_x + 0\mathbf{e}_y + (-3)\mathbf{e}_z \\ \mathbf{r}_v &= 0\mathbf{e}_x + (1)\mathbf{e}_y + (-2)\mathbf{e}_z \end{aligned} \implies \mathbf{r}_u \times \mathbf{r}_v = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

thus $\mathbf{r}_u \times \mathbf{r}_v$ is upward.

0M So

$$\iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{D}} \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, du\, dv = \iint_{\mathcal{D}} \operatorname{curl} \mathbf{F} \cdot (-\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv$$
$$= \iint_{\mathcal{D}} (x^2 + y^2)\, dudv$$
$$= \int_0^{2\pi} \int_0^2 r^3\, dr\, d\theta$$
$$= 8\pi$$

(c) (1 point) If $\mathcal{S}$ is a sphere and $\mathbf{F}$ satisfies the hypothesis of Stokes' Theorem, show

$$\iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$$

Solution:

0M Assume $\mathcal{S}$ is centred at the the origin with radius a and let $\mathcal{H}_u$ and $\mathcal{H}_l$ be the upper and lower hemispheres, respectively, of S. Then

$$\iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{H}_u} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} + \iint_{\mathcal{H}_l} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_{\mathcal{C}_u} \mathbf{F} \cdot d\mathbf{r} + \oint_{\mathcal{C}_l} \mathbf{F} \cdot d\mathbf{r}$$

0M However, $\mathcal{C}_u$ and $\mathcal{C}_l$ represent the same circle with opposite orientation. Hence

$$\oint_{\mathcal{C}_u} \mathbf{F} \cdot d\mathbf{r} = -\oint_{\mathcal{C}_l} \mathbf{F} \cdot d\mathbf{r} \implies \oint_{\mathcal{C}_u} \mathbf{F} \cdot d\mathbf{r} + \oint_{\mathcal{C}_l} \mathbf{F} \cdot d\mathbf{r} = 0 \implies \iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$$

(d) (1 point) Use the Divergence Theorem to evaluate

$$\iint_{\mathcal{S}} (2x + 2y + z^2)\, dS, \qquad \text{where } \mathcal{S} \text{ is the unit sphere centred at the origin.}$$

Solution:

0M Treat the unit sphere as a level surface, thus the unit normal vector for the sphere is

$$\mathbf{n} = \frac{\nabla\left(x^2 + y^2 + z^2\right)}{|\nabla\left(x^2 + y^2 + z^2\right)|} = \frac{2x\mathbf{e}_x + 2y\mathbf{e}_y + 2z\mathbf{e}_z}{|2x\mathbf{e}_x + 2y\mathbf{e}_y + 2z\mathbf{e}_z|} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$$

0M Thus applying the Divergence Theorem, we have

$$\iint_{\mathcal{S}} (2x + 2y + z^2)\, dS = \iint_{\mathcal{S}} (2\mathbf{e}_x + 2\mathbf{e}_y + z\mathbf{e}_z) \cdot (x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z)\, dS$$
$$= \iiint_{\mathcal{E}} \operatorname{div}(2\mathbf{e}_x + 2\mathbf{e}_y + z\mathbf{e}_z)\, dV$$
$$= \iiint_{\mathcal{E}} 1\, dV = \frac{4\pi}{3}$$

(e) (1 point) Prove the following identity, assuming that the surface $\mathcal{S}$, and the region $\mathcal{E}$ satisfy the hypotheses of the Divergence theorem and that all necessary differentiability requirements for the functions $f(x,y,z)$ and $g(x,y,z)$ are met.

$$\iint_{\mathcal{S}} (f\nabla g - g\nabla f)\cdot \mathbf{n}\, dS = \iiint_{\mathcal{E}} (f\nabla^2 g - g\nabla^2 f)\, dV$$

Solution:

0M Use the Divergence Theorem,

$$\iint_{\mathcal{S}} (f\nabla g)\cdot \mathbf{n}\, dS = \iiint_{E} \operatorname{div}(f\nabla g)\, dV = \iiint_{E} \left(f\nabla^2 g + \nabla g\cdot\nabla f\right)\, dV$$

0M Similarly,

$$\iint_{\mathcal{S}} (g\nabla f)\cdot \mathbf{n}\, dS = \iiint_{E} \operatorname{div}(g\nabla f)\, dV = \iiint_{E} \left(g\nabla^2 f + \nabla g\cdot\nabla f\right)\, dV$$

0M Thus,

$$\begin{aligned}\iint_{\mathcal{S}} (f\nabla g - g\nabla f)\cdot \mathbf{n}\, dS &= \iiint_{\mathcal{E}} \left(f\nabla^2 g + \nabla g\cdot\nabla f\right)\, dV \\ &\quad - \iiint_{\mathcal{E}} \left(g\nabla^2 f + \nabla g\cdot\nabla f\right)\, dV \\ &= \iiint_{\mathcal{E}} (f\nabla^2 g - g\nabla^2 f)\, dV\end{aligned}$$