Vv256 Lecture 4

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- Recall a first-order equation is linear if it has the following form

$$\alpha(t)\dot{y} + \beta(t)y = \gamma(t)$$

- Similarly, a second-order differential equation is linear if it has the form

$$\alpha(t)\ddot{y} + \beta(t)\dot{y} + \gamma(t)y = f(t) \iff \ddot{y} + P(t)\dot{y} + Q(t)y = R(t)$$

Definition

For every second-order linear equation,

$$\alpha(t)\ddot{y} + \beta(t)\dot{y} + \gamma(t)y = f(t)$$

the following homogeneous equation is called the complementary equation

$$\alpha(t)\ddot{y} + \beta(t)\dot{y} + \gamma(t)y = 0$$

or the corresponding homogeneous equation to the original equation.

- We will see later that often the corresponding homogeneous equation has to be solved first in order to solve the original equation. So we will start with

$$a\ddot{y} + b\dot{y} + cy = 0, \qquad \text{where } a,\ b \text{ and } c \text{ are constants.}$$

Q: For example, can you think of a simple solution to the following equation?

$$\ddot{y} - y = 0$$

- To solve this equation, we are basically asking ourselves to find a function

$$y(t)$$

so that the 2nd derivative of the function is the same as the function itself.

$$\phi_1 = e^t$$

Q: Is there any other function has this property?

Principle of Superposition

If ϕ_1 and ϕ_2 are two solutions of a second-order homogeneous linear equation

$$\alpha(t)\ddot{y} + \beta(t)\dot{y} + \gamma(t)y = 0$$

then the function

$$y = C_1\phi_1 + C_2\phi_2, \qquad \text{where and } C_1 \text{ and } C_2 \text{ are two arbitrary constants.}$$

is also a solution of the homogeneous equation.

Proof

$$\begin{aligned}
\alpha y'' + \beta y' + \gamma y &= \alpha\Big(C_1\phi_1 + C_2\phi_2\Big)'' + \beta\Big(C_1\phi_1 + C_2\phi_2\Big)' + \gamma\Big(C_1\phi_1 + C_2\phi_2\Big) \\
&= \alpha\Big(C_1\phi_1'' + C_2\phi_2''\Big) + \beta\Big(C_1\phi_1' + C_2\phi_2'\Big) + \gamma\Big(C_1\phi_1 + C_2\phi_2\Big) \\
&= C_1\underbrace{\left(\alpha\phi_1'' + \beta\phi_1' + \gamma\phi_1\right)}_{0} + C_2\underbrace{\left(\alpha\phi_2'' + \beta\phi_2' + \gamma\phi_2\right)}_{0} \\
&= 0
\end{aligned}$$

- Given the following IVP, and solutions ϕ_1 and ϕ_2

$$\alpha\ddot{y} + \beta\dot{y} + \gamma y = 0; \qquad y(t_0) = y_0, \quad \dot{y}(t_0) = y_1$$

we can try to determine the arbitrary constants c_1 and c_2 which must satisfy

$$C_1\phi_1(t_0) + C_2\phi_2(t_0) = y_0; \qquad C_1\phi_1'(t_0) + C_2\phi_2'(t_0) = y_1$$

Q: Can we always solve this linear system?

- Upon solving the above system,

$$C_1 = \frac{y_0\phi_2'(t_0) - y_1\phi_2(t_0)}{\phi_1(t_0)\phi_2'(t_0) - \phi_1'(t_0)\phi_2(t_0)}, \quad C_2 = \frac{-y_0\phi_1'(t_0) + y_1\phi_1(t_0)}{\phi_1(t_0)\phi_2'(t_0) - \phi_1'(t_0)\phi_2(t_0)}$$

- So C_1 and C_2 can be determined as long as the denominator is nonzero.

$$\phi_1(t_0)\phi_2'(t_0) - \phi_1'(t_0)\phi_2(t_0) \neq 0$$

- The denominator is actually a determinant of a 2×2 matrix.

Definition

The determinant is called the Wronskian determinant, or simply the Wronskian,

$$W(\phi_1, \phi_2)(t_0) = W(t_0) = \det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{bmatrix} = \phi_1(t_0)\phi_2'(t_0) - \phi_1'(t_0)\phi_2(t_0)$$

Theorem

Suppose ϕ_1 and ϕ_2 are two solutions to the following equation and $W(t_0) \neq 0$,

$$\alpha\ddot{y} + \beta\dot{y} + \gamma y = 0$$

then there is a choice of c_1 and c_2 such that the initial conditions are satisfied.

$$y(t_0) = y_0, \quad \dot{y}(t_0) = y_1$$

Exercise

Solve the initial-value problem $\quad \ddot{y} - y = 0, \quad y(0) = 1, \quad \dot{y}(0) = 1.$

- Now let us get back to the general equation with constant coefficients

$$a\ddot{y} + b\dot{y} + cy = 0$$

Q: What is the above equation actually stating?

- Given the values of a, b and c, there might be an exponential function

$$y = e^{rt}$$

  may satisfy the equation for some value of r

$$\dot{y} = re^{rt} \implies \ddot{y} = r^2 e^{rt}$$

- Substitute the function and its derivatives into the equation,

$$a(r^2 e^{rt}) + b(re^{rt}) + c(e^{rt}) = 0 \implies (ar^2 + br + c)e^{rt} = 0$$

$$\implies ar^2 + br + c = 0$$

- The quadratic equation

$$ar^2 + br + c = 0$$

is known as the characteristic equation of the differential equation

$$a\ddot{y} + b\dot{y} + cy = 0$$

- The form of the solution to the original equation depends on the discriminant

$$\Delta = b^2 - 4ac$$

- If $\Delta > 0$, then the following is a solution to the original differential equation

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

- If $\Delta = 0$, then the following is a solution to the original differential equation

$$y = (C_1 + C_2 t)\, e^{rt}$$

- If $\Delta < 0$, then the following is a solution to the original differential equation

$$y = e^{Rt}\Big(C_1 \cos\theta t + C_2 \sin\theta t\Big)$$

- Case: $b^2 - 4ac > 0$
- The solutions r_1 and r_2 of the characteristic equation are real and distinct,

$$r_1 \neq r_2$$

- Thus the two solutions to the original differential equation are simply

$$\phi_1 = e^{r_1 t} \quad \text{and} \quad \phi_2 = e^{r_2 t}$$

- The Wronskian of ϕ_1 and ϕ_2 is

$$\begin{aligned} W = \phi_1\phi_2' - \phi_1'\phi_2 = e^{r_1 t} r_2 e^{r_2 t} - e^{r_2 t} r_1 e^{r_1 t} &= (r_2 - r_1)e^{(r_1+r_2)t} \\ &\neq 0 \end{aligned}$$

- Hence we can determine a solution to any IVP of the given equation

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

- Case: $b^2 - 4ac = 0$
- The solutions r_1 and r_2 of the characteristic equation are real and equal.

$$r_1 = r_2 = r$$

- So we obtain only one instead of two solutions to the differential equation

$$\phi_1 = e^{rt}$$

- The method of finding ϕ_2 will covered next week, nevertheless we can verify

$$\phi_2 = te^{rt}$$

is another solution by simply substituting it back to the original equation

$$\begin{aligned} a\phi_2'' + b\phi_2' + c\phi_2 &= a(2re^{rt} + r^2te^{rt}) + b(e^{rt} + rte^{rt}) + cte^{rt} \\ &= \underbrace{(2ar + b)}_{0} e^{rt} + \underbrace{(ar^2 + br + c)}_{0} te^{rt} = 0 \end{aligned}$$

- The Wronskian is $W = e^{2rt} \neq 0$, so any IVP of the equation can be solved

$$y = C_1e^{rt} + C_2te^{rt} = (C_1 + C_2t)\, e^{rt}$$

- Case: $b^2 - 4ac < 0$
- The solutions r_1 and r_2 of the characteristic equation are conjugates

$$r_1 = R + \mathrm{i}\theta, \qquad r_2 = R - \mathrm{i}\theta, \qquad \text{where} \quad \theta > 0$$

- By the definition of the complex exponential function,

$$e^{x+\mathrm{i}y} = e^x \left(\cos y + \mathrm{i} \sin y\right)$$

we can write a solution of the original differential equation

$$\begin{aligned} y = d_1 e^{r_1 t} + d_2 e^{r_2 t} &= d_1 e^{(R+\mathrm{i}\theta)t} + d_2 e^{(R-\mathrm{i}\theta)t} \\ &= d_1 e^{Rt} \left(\cos \theta t + \mathrm{i} \sin \theta t\right) + d_2 e^{Rt} \left(\cos \theta t - \mathrm{i} \sin \theta t\right) \\ &= e^{Rt} \Big[(d_1 + d_2) \cos \theta t + \mathrm{i}(d_1 - d_2) \sin \theta t \Big] \\ &= e^{Rt} \Big(C_1 \cos \theta t + C_2 \sin \theta t \Big) \end{aligned}$$

where $C_1 = d_1 + d_2$ and $C_2 = \mathrm{i}(d_1 - d_2)$.

- Therefore, we have two solutions ϕ_1 and ϕ_2 in trigonometric form,

$$\phi_1 = e^{Rt}\cos\theta t, \qquad \phi_2 = e^{Rt}\sin\theta t$$

  where R is the real part, and θ is the imaginary part which is positive.
- The Wronskian is given by

$$W = \theta e^{2Rt} \neq 0$$

- Hence any IVP of the given equation can be solved

$$y = e^{Rt}\Big(C_1\cos\theta t + C_2\sin\theta t\Big)$$

Exercise

Solve the initial-value problem

$$\ddot{y} + y = 0, \qquad y(0) = 2, \qquad y'(0) = 3$$