

Vv156 Lecture 13

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Definition

A limit of the form

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}, \quad \text{in which } f(x) \rightarrow 0 \quad \text{and} \quad g(x) \rightarrow 0, \quad \text{as } x \rightarrow a,$$

is called an **indeterminate form** of type $\frac{0}{0}$.

- Some examples encountered earlier are

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2, \quad \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

- The first limit was obtained algebraically by considering $x + 1$, and the second limits were obtained using a geometric argument and the squeeze theorem.
- However, there are many limits that have indeterminate forms for which neither algebraic nor geometric methods will simplify the limit, so we need to develop a more general method of evaluating such limits.

- Suppose that we have an indeterminate form $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ of type $\frac{0}{0}$, in which

1. $f'(x)$ and $g'(x)$ are continuous at $x = a$

2. $g'(a) \neq 0$.

- Then the following limits are equivalent,

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \rightarrow a} \frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}}$$

- Let $h = x - a$, so $x = h + a$ and clearly $h \rightarrow 0$ as $x \rightarrow a$,

$$\begin{aligned} \lim_{x \rightarrow a} \frac{f(x)}{g(x)} &= \lim_{h \rightarrow 0} \frac{\frac{f(h+a) - f(a)}{h}}{\frac{g(h+a) - g(a)}{h}} = \lim_{h \rightarrow 0} \frac{\frac{f(h+a) - f(a)}{h}}{\frac{g(a+h) - g(a)}{h}} = \frac{f'(a)}{g'(a)} = \frac{\lim_{x \rightarrow a} f'(x)}{\lim_{x \rightarrow a} g'(x)} \\ &= \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \end{aligned}$$

- Although this assumes f and g have continuous derivatives at $x = a$ and that $g'(a) \neq 0$, the result is true under less stringent conditions.

L'Hospital's rule for the form $\frac{0}{0}$

Suppose that f and g are differentiable functions on an open interval containing

$$x = a, \text{ except possibly at } x = a,$$

and that

$$\lim_{x \rightarrow a} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = 0$$

If $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ exists, or if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as

$$x \rightarrow a^- \quad x \rightarrow a^+ \quad x \rightarrow -\infty \quad x \rightarrow +\infty$$

Exercise

Find

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x^3}$$

- Applying L'Hospital's rule to limits that are not in indeterminate forms can lead to incorrect results. For example, the computation

$$\lim_{x \rightarrow 0} \frac{x+6}{x+2} = \frac{6}{2} \neq \lim_{x \rightarrow 0} \frac{(x+6)'}{(x+2)'} = \lim_{x \rightarrow 0} \frac{1}{1} = 1$$

Q: Can we apply L'Hospital's rule to

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt[3]{x}},$$

where clearly

$$\lim_{x \rightarrow \infty} \ln x = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} \sqrt[3]{x} = \infty$$

Definition

The limit of a ratio, $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$, is called an **indeterminate form** of type $\frac{\infty}{\infty}$ if

$$f(x) \rightarrow \infty \quad \text{and} \quad g(x) \rightarrow \infty \quad \text{as} \quad x \rightarrow a$$

L'Hospital's rule for the form $\frac{\infty}{\infty}$

Suppose that f and g are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$,

and that

$$\lim_{x \rightarrow a} f(x) = \infty \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = \infty$$

If $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ exists, or if this limit is $+\infty$ or $-\infty$, then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$

Moreover, this statement is also true in the case of a limit as

$$x \rightarrow a^- \quad x \rightarrow a^+ \quad x \rightarrow -\infty \quad x \rightarrow +\infty$$

Exercise

Find

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt[3]{x}}$$

- If n is any positive integer, then

$$x^n \rightarrow \infty \quad \text{as} \quad x \rightarrow \infty$$

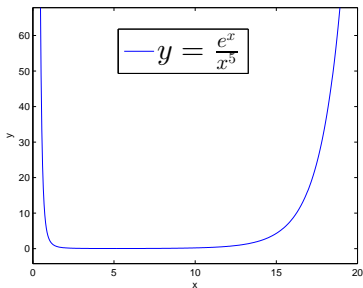
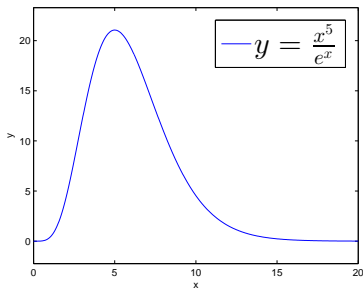
- Such integer powers of x are sometimes used as “measuring sticks” to describe how rapidly other functions grow. For example,

$$e^x \rightarrow \infty \quad \text{and} \quad x \rightarrow \infty$$

- It is reasonable to ask whether x^n grow faster or e^x grow faster.
- One way to investigate this is to examine the behaviour of the ratio

$$\frac{x^n}{e^x} \text{ or } \frac{e^x}{x^n}.$$

- For a specific n , graphically it is clear that e^x grows faster for large values of x ,



Q: How can we show for any positive integer n

$$\lim_{x \rightarrow \infty} \frac{x^n}{e^x} = 0$$

- We have discussed indeterminate forms of type

$$\frac{0}{0} \quad \text{and} \quad \frac{\infty}{\infty}$$

- However, these two forms are not the only possibilities; in general, the limit of an expression that has one of the forms

$$\frac{f(x)}{g(x)}, \quad f(x)g(x), \quad f(x)^{g(x)}, \quad f(x) \pm g(x)$$

is called an indeterminate form if the limits of $f(x)$ and $g(x)$ individually exert **conflicting influences** on the limit of the entire expression. For example,

$$\lim_{x \rightarrow 0^+} x \ln x$$

is an indeterminate form of type $0 \cdot \infty$ because the limit of the first factor is 0, the limit of the second factor is $-\infty$, so they have conflicting influences on the value of the limit, therefore the limit in this form is indeterminate

Exercise

Evaluate

$$(a) \quad \lim_{x \rightarrow 0^+} x \ln x$$

- A limit problem that leads to one of the expressions

$$\infty - \infty, \quad -\infty - (-\infty), \quad \infty + (-\infty), \quad -\infty + \infty$$

is an **indeterminate form of type $\infty - \infty$** .

- However, limit problems that lead to one of the expressions

$$\infty + \infty, \quad \infty - (-\infty), \quad -\infty + (-\infty), \quad -\infty - \infty$$

is **not** an indeterminate form.

- Indeterminate forms of $\infty - \infty$ can sometimes be evaluated by combining the terms and manipulating the result to produce an indeterminate form of type

$$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}$$

Exercise

Evaluate

$$(b) \quad \lim_{x \rightarrow 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$$

- Limits of the form

$$\lim_{x \rightarrow a} \left(f(x) \right)^{g(x)}$$

can give rise to indeterminate forms of the types ∞^0 , 0^0 , 1^∞ . For example,

$$\lim_{x \rightarrow 0} (1 + \sin x)^{\frac{1}{x}}$$

- Such indeterminate forms can be evaluated by first introducing a variable y

$$y = f(x)^{g(x)}$$

$$\implies \ln y = g(x) \ln f(x)$$

and then compute the limit of $\ln y$.

Exercise

Find

$$\lim_{x \rightarrow 0} (1 + \sin x)^{1/x}$$

No.	Form	Indeterminate	Technique
1	$(0/0)$	Yes	Direct
2	(∞/∞)		
3	$0 \cdot \infty$		$(0/(1/\infty))$ or $(\infty/(1/0))$ factorize or exponential transformation
4	$\infty - \infty$		
5	0^0		
6	∞^0		logarithmic transformation
7	1^∞		
8	0^∞	No	$0 \cdot (1/\infty)$
9	$(0/\infty)$		
10	$(\infty/0)$		$\infty \cdot (1/0)$
11	$\infty \cdot \infty$		∞
12	$+\infty + (+\infty)$		∞
13	$+\infty - (-\infty)$		∞
14	$-\infty + (-\infty)$		$-\infty$
15	$-\infty - (+\infty)$		$-\infty$

Table: Table of Indeterminate forms

When NOT to use L'Hospital's rule

1. When it is making the problem worse. For example,

$$\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{x}{(\ln x)^{-1}} = \lim_{x \rightarrow 0^+} \frac{1}{-\frac{1/x}{(\ln x)^2}} = \lim_{x \rightarrow 0^+} [-x(\ln x)^2]$$

which is more complicated than the original problem

$$\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} = \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} (-x) = 0$$

2. When there is a better way to get the answer.

$$\lim_{x \rightarrow +\infty} \frac{x^2 - 1}{2x^2 + 1} = \lim_{x \rightarrow +\infty} \frac{\frac{x^2-1}{x^2}}{\frac{2x^2+1}{x^2}} = \lim_{x \rightarrow +\infty} \frac{1 - \frac{1}{x^2}}{2 + \frac{1}{x^2}} = \frac{1}{2}$$

3. It isn't an indeterminate form!

$$\lim_{x \rightarrow 0^+} \frac{\cos x}{x} = +\infty \neq \lim_{x \rightarrow 0^+} \frac{(\cos x)'}{(x)'} = \lim_{x \rightarrow 0^+} \frac{-\sin x}{1} = 0$$