Vv255 Lecture 24

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Definition

Suppose the following vector field is in Cartesian coordinates and is differentiable

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{e}_x + Q(x, y, z)\mathbf{e}_y + R(x, y, z)\mathbf{e}_z$$

that is, the component functions are all differentiable, then

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

is called the divergence of $\mathbf{F}$ or the divergence of the vector field defined by $\mathbf{F}$.

- Of course, this definition has its counterpart for $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$, that is,

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

  where

$$\mathbf{F} = P(x, y)\mathbf{e}_x + Q(x, y)\mathbf{e}_y$$

- A common notation for the divergence is

$$\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

where ∇ is sort of a vector in the dot product.

- However, you should not use it as anything more than a memorization technique

$$\nabla\cdot\mathbf{F} \neq \mathbf{F}\cdot\nabla$$

- Like gradients, divergences share some properties of differentiation
- Given two vector field $\mathbf{F}$ and $\mathbf{G}$ both in $\mathbb{R}^2$ or $\mathbb{R}^3$, then

$$\nabla\cdot(\alpha\mathbf{F} + \beta\mathbf{G}) = \alpha\nabla\cdot\mathbf{F} + \beta\nabla\cdot\mathbf{G}$$

where α and β are real numbers.

- Given a vector field $\mathbf{F}$ and a scalar-valued function φ, then

$$\nabla\cdot(\varphi\mathbf{F}) = \varphi\nabla\cdot\mathbf{F} + (\nabla\varphi)\cdot\mathbf{F}$$

- If $\varphi(x, y, z)$ is a continuously differentiable scalar-valued function, then

$$\mathbf{F} = \nabla\varphi = \frac{\partial\varphi}{\partial x}\mathbf{e}_x + \frac{\partial\varphi}{\partial y}\mathbf{e}_y + \frac{\partial\varphi}{\partial z}\mathbf{e}_z$$

$$\Longrightarrow \operatorname{div}\mathbf{F} = \nabla\cdot\nabla\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

which is known the Laplacian of φ, and it is often denoted as

$$\nabla^2\varphi = \nabla\cdot\nabla\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

- Of course, this definition has its counterpart for $\varphi(x, y)$ of two variables, that is,

$$\nabla^2\varphi = \nabla\cdot\nabla\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}$$

Q: What is the physical meaning of divergence of $\mathbf{F}$ and Laplacian of φ?

- Consider a sufficiently smooth velocity field of a type of fluid, a liquid or a gas

$$\mathbf{V}(x,y,z) = v_x(x,y,z)\mathbf{e}_x + v_y(x,y,z)\mathbf{e}_y + v_z(x,y,z)\mathbf{e}_z$$

in a region $\mathcal{E}$, inside which there is no source or sink, that is, no point at which the fluid is created or destroyed. Furthermore, let us assume the density

$$\rho(x,y,z,t)$$

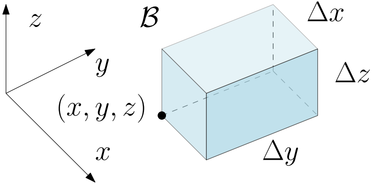
depends on (x,y,z) in space and on time t, and is continuously differentiable.

- Fluids in the restricted sense, such as water or oil, have a constant density. But fluids in the general sense, such as gas and vapour, have a non-constant density.

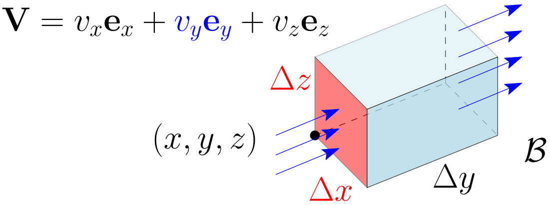
- Suppose there is a tiny box $\mathcal{B}$ in $\mathcal{E}$.

- It is clear that the box $\mathcal{B}$ has volume

$$\Delta V = \Delta x \Delta y \Delta z$$



- Let us consider the motion of the fluid in and out of the box by calculating the flux across the boundary, that is, the total mass leaving $\mathcal{B}$ per unit of time.



- Consider the flow through the left face of $\mathcal{B}$, whose area is

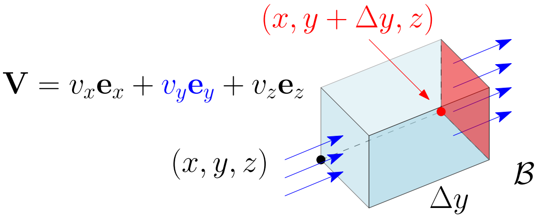
$$\Delta x \Delta z$$

- Since $v_x\mathbf{e}_x$ and $v_z\mathbf{e}_z$ are parallel to that face, the components v_x and v_y of $\mathbf{V}$ contribute nothing to this flow. Thus the mass of the fluid moving cross that face during a short time interval Δt is given approximately by

$$\rho v_y\Big|_y \Delta x \Delta z \Delta t$$

where the subscript y indicates that this expression refers to the left face.

- The mass of fluid moving through the opposite face, the right face,



during the same time interval can be approximately

$$\rho v_y\Big|_{y+\Delta y}\Delta x\Delta z\Delta t$$

where the subscript $y+\Delta y$ indicates that this expression refers to the right face

- The difference

$$\Delta u_y\Delta x\Delta z\Delta t = \frac{\Delta u_y}{\Delta y}\Delta V\Delta t, \quad \text{where} \quad \Delta u_y = \Big[\rho v_y\Big]_y^{y+\Delta y}$$

is the approximate change of mass in the direction of $\mathbf{e}_y$.

- Two similar expressions can be obtained by using the other two pairs of faces

$$\frac{\Delta u_x}{\Delta x}\Delta V \Delta t \qquad \text{and} \qquad \frac{\Delta u_z}{\Delta z}\Delta V \Delta t$$

where

$$\Delta u_x = \Big[\rho v_x\Big]_x^{x+\Delta x} \qquad \text{and} \qquad \Delta u_z = \Big[\rho v_z\Big]_z^{z+\Delta z}$$

- If we add these three expressions, we find that the total change of mass in $\mathcal{B}$ during the time interval Δt is approximately

$$\left(\frac{\Delta u_x}{\Delta x} + \frac{\Delta u_y}{\Delta y} + \frac{\Delta u_z}{\Delta z}\right)\Delta V \Delta t$$

Q: What does this change of mass mean in terms of the density inside $\mathcal{B}$?

- This change of mass in $\mathcal{B}$ is reflected by the rate of change of the density with respect to time and is thus approximately equal to

$$-\frac{\partial \rho}{\partial t}\Delta t \Delta V$$

- If we equate both expressions, divide the resulting equation by $\Delta V \Delta t$,

$$\frac{\Delta u_x}{\Delta x} + \frac{\Delta u_y}{\Delta y} + \frac{\Delta u_y}{\Delta z} \approx -\frac{\partial \rho}{\partial t}$$

and let Δx, Δy, Δz, and Δt approach zero, then we expect the error to vanish

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_x}{\partial y} + \frac{\partial u_z}{\partial z} = -\frac{\partial \rho}{\partial t}$$

- If we define a vector field as the product of ρ and $\mathbf{V}$

$$\mathbf{F} = \rho \mathbf{V} = \rho v_x \mathbf{e}_x + \rho v_y \mathbf{e}_y + \rho v_z \mathbf{e}_z = u_x \mathbf{e}_x + u_y \mathbf{e}_y + u_z \mathbf{e}_z$$

then we get an physical interpretation of the divergence,

$$\operatorname{div} \mathbf{F} = -\frac{\partial \rho}{\partial t} \implies \operatorname{div} \mathbf{u} + \frac{\partial \rho}{\partial t} = 0$$

- This important relation is called the condition for the conservation of mass or the continuity equation of a compressible fluid.

- If the flow is steady, that is, independent of time, then

$$\frac{\partial \rho}{\partial t} = 0$$

and the continuity equation becomes

$$\operatorname{div}(\rho \mathbf{V}) = 0$$

- Further, if the density ρ is constant, then the fluid is incompressible and

$$\operatorname{div} \mathbf{V} = 0$$

- It states the fact that the sum of outflow and inflow for an infinitesimal volume around a given point is 0 at any time for a steady flow of incompressible fluids.
- This relation, zero divergence, is known as the condition of incompressibility.
- A vector field which has zero divergence is also referred to as solenoidal and the corresponding fluid is referred to as incompressible.

- In general, for a velocity field $\mathbf{V}$ of a fluid, then, at each point within the fluid,

$$\operatorname{div}\mathbf{V}$$

measures the tendency of the fluid to diverge away from that point.

- For the vector field

$$\mathbf{F} = \rho\mathbf{V}$$

the divergence

$$\operatorname{div}\mathbf{F} = -\frac{\partial\rho}{\partial t}$$

is the negative rate of change of the density with respect to time.

- From this discussion you should conclude and remember that, roughly speaking,

the divergence measures outflow minus inflow

- Clearly, the assumption that the flow has no source or sink in $\mathcal{B}$ is essential.

Q: What is the physical meaning of the Laplacian of φ?

Definition

Suppose the following vector field in Cartesian coordinates is differentiable,

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{e}_x + Q(x, y, z)\mathbf{e}_y + R(x, y, z)\mathbf{e}_z$$

that is, the component functions are all differentiable, then

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{e}_z$$

is called the curl of $\mathbf{F}$ or the curl of the vector field defined by $\mathbf{F}$.

- Other common notations for the divergence are

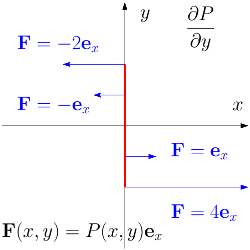
$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \operatorname{rot}\mathbf{F}$$

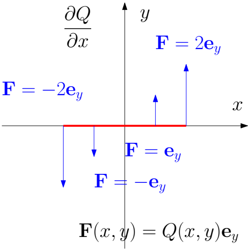
- The notation $\operatorname{rot}\mathbf{F}$ is used because of the connection between rotation and curl.
- Note that the curl of a vector field is a vector while the divergence is a scalar.

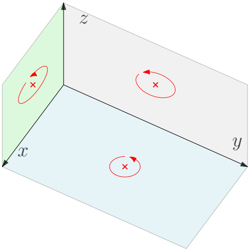
- Recall in the xy-plane, the following scalar evaluated at (x, y)

$$\boxed{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}$$

indicates whether there is a tendency to rotate at (x, y).







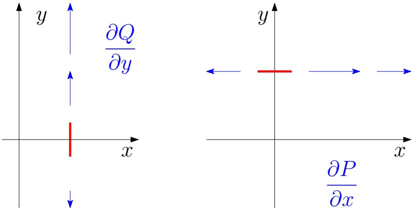
- Now the curl, a vector of three component, is a generalization of it into $\mathbb{R}^3$,

$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{e}_y + \boxed{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}\mathbf{e}_z$$

- Recall in the xy-plane, the following two scalar evaluated at (x, y)

$$\frac{\partial P}{\partial x} \quad \text{and} \quad \frac{\partial Q}{\partial y}$$

are not relevant to the tendency of rotation at (x, y).



- Now notice the sum is actually the divergence of the vector field in $\mathbb{R}^2$,

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

Q: Based on our discussion of divergence, do you see the difference between

$$\operatorname{div} \mathbf{F} \quad \text{and} \quad \operatorname{curl} \mathbf{F}$$

- Curls share some properties of differentiation as well.

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{e}_z$$

- Given two vector field $\mathbf{F}$ and $\mathbf{G}$ both in $\mathbb{R}^2$ or $\mathbb{R}^3$, then

$$\nabla \times (\alpha\mathbf{F} + \beta\mathbf{G}) = \alpha(\nabla \times \mathbf{F}) + \beta(\nabla \times \mathbf{G})$$

  where α and β are real numbers.

- Given a vector field $\mathbf{F}$ and a scalar-valued function φ, then

$$\nabla \times (\varphi\mathbf{F}) = \varphi(\nabla \times \mathbf{F}) + (\nabla\varphi) \times \mathbf{F}$$

Theorem

The curl of the velocity field of a rotating rigid body has the direction of the axis of the rotation, and its magnitude equals twice the angular speed of the rotation.

Theorem

Gradient fields are irrotational. That is, if a continuously differentiable vector field is the gradient of scalar function f, then its curl is the zero vector,

$$\operatorname{curl}(\operatorname{grad} f) = \mathbf{0}$$

- This is equivalent to say that the curl of a conservative vector field is zero.
- This theorem can be proved from the definitions of curl and gradient, and using

  Clairaut's theorem.

- The converse, that is

  a vector field $\mathbf{F}$ for which $\operatorname{curl} \mathbf{F} = \mathbf{0}$ is conservative,

  is also true if

  the curl of the vector field is zero within a simply connected domain.

- The curl and divergence can be used to restate

Green's Theorem

in forms that are more directly generalizable to surfaces and solids in $\mathbb{R}^3$.

- Suppose $\mathbf{F} = \begin{bmatrix} P \\ Q \\ 0 \end{bmatrix}$, so it is a two-dimensional vector field embedded in $\mathbb{R}^3$, then

$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{e}_z$$

- It follows that

$$\operatorname{curl} \mathbf{F} \cdot \mathbf{e}_z = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{e}_z \cdot \mathbf{e}_z = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

- This expression is called the scalar curl of the two-dimensional vector field

$$\mathbf{F} = P\mathbf{e}_x + Q\mathbf{e}_y$$

Green's Theorem

If $\mathcal{C}$ is a positively oriented, piecewise smooth, simple closed curve that encloses a region $\mathcal{D}$, and P and Q have continuous first partial derivatives on some open region containing $\mathcal{D}$, then the line integral of the vector along $\mathcal{C}$

Tangential form $$\oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_{\mathcal{D}} \operatorname{curl} \mathbf{F}\, dA$$

Normal form $$\oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{n}\, ds = \iint_{\mathcal{D}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA = \iint_{\mathcal{D}} \operatorname{div} \mathbf{F}\, dA$$

where $\mathbf{F} = P\mathbf{e}_x + Q\mathbf{e}_y$, $\quad \mathbf{T} = \frac{dx}{ds}\mathbf{e}_x + \frac{dy}{ds}\mathbf{e}_y$, and $\quad \mathbf{n} = \frac{dy}{ds}\mathbf{e}_x - \frac{dx}{ds}\mathbf{e}_y$.

- And if $\mathbf{F}$ is a conservative vector field with the potential function f, then

$$\iint_{\mathcal{D}} \nabla^2 f\, dA = \oint_{\mathcal{C}} \nabla f \cdot \mathbf{n}\, ds$$

that is, the divergence is replaced by the Laplacian of the potential funciton.

- For some problems it is convenient to work in polar rather than Cartesians. Just like gradient, divergence and curl can all be found in terms of derivatives using other coordinates systems, and in particular we will often want
- Cylindrical coordinate system

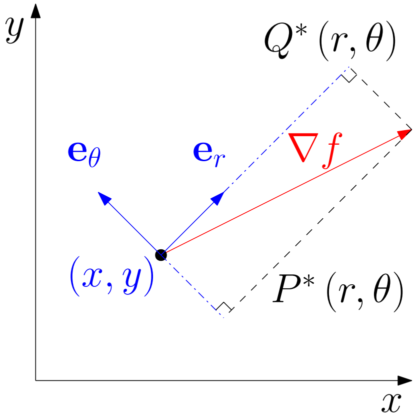
$$\mathbf{e}_r = \frac{1}{\left|\frac{\partial \mathbf{r}}{\partial r}\right|}\frac{\partial \mathbf{r}}{\partial r} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \quad \mathbf{e}_\theta = \frac{1}{\left|\frac{\partial \mathbf{r}}{\partial \theta}\right|}\frac{\partial \mathbf{r}}{\partial \theta} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}, \quad \mathbf{e}_z = \frac{1}{\left|\frac{\partial \mathbf{r}}{\partial z}\right|}\frac{\partial \mathbf{r}}{\partial z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Spherical coordinate system

$$\mathbf{e}_\rho = \frac{1}{\left|\frac{\partial \mathbf{r}}{\partial \rho}\right|}\frac{\partial \mathbf{r}}{\partial \rho} = \frac{\partial \mathbf{r}}{\partial \rho}, \quad \mathbf{e}_\theta = \frac{1}{\left|\frac{\partial \mathbf{r}}{\partial \theta}\right|}\frac{\partial \mathbf{r}}{\partial \theta} = \frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}, \quad \mathbf{e}_\phi = \frac{1}{\left|\frac{\partial \mathbf{r}}{\partial \phi}\right|}\frac{\partial \mathbf{r}}{\partial \phi} = \frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi}$$

$$= \begin{bmatrix} \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{bmatrix} \qquad = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \qquad = \begin{bmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ -\sin\phi \end{bmatrix}$$

- Recall how we find the gradient in polar coordinates



- Note P^* and Q^* are directional derivatives in the direction of $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

$$\nabla f = f_x\mathbf{e}_x + f_y\mathbf{e}_y = P^*(r,\theta)\,\mathbf{e}_r + Q^*(r,\theta)\,\mathbf{e}_\theta = (\mathrm{D}_{\mathbf{e}_r} f)\,\mathbf{e}_r + (\mathrm{D}_{\mathbf{e}_\theta} f)\,\mathbf{e}_\theta$$

$$= (\nabla f\cdot\mathbf{e}_r)\,\mathbf{e}_r + (\nabla f\cdot\mathbf{e}_\theta)\,\mathbf{e}_\theta$$

- Now let us illustrate how it can be done by considering divergence in polar.

$$\nabla \cdot \mathbf{F} \qquad \text{where} \quad \mathbf{F} = \begin{cases} P(x,y)\mathbf{e}_x + Q(x,y)\mathbf{e}_y \\ P^*(r,\theta)\mathbf{e}_x + Q^*(r,\theta)\mathbf{e}_y \\ F_r(r,\theta)\mathbf{e}_r + F_\theta(r,\theta)\mathbf{e}_\theta \end{cases}$$

- First note the divergence can be understood as the sum of scalar projections

$$\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \mathbf{e}_x \cdot \left(\frac{\partial P}{\partial x}\mathbf{e}_x + \frac{\partial Q}{\partial x}\mathbf{e}_y\right) + \mathbf{e}_y \cdot \left(\frac{\partial P}{\partial y}\mathbf{e}_x + \frac{\partial Q}{\partial y}\mathbf{e}_y\right)$$

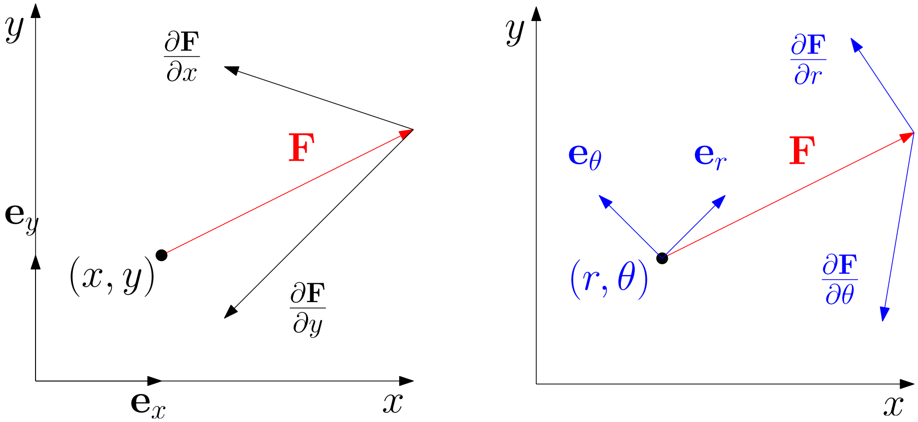
$$= \mathbf{e}_x \cdot \frac{\partial \mathbf{F}}{\partial x} + \mathbf{e}_y \cdot \frac{\partial \mathbf{F}}{\partial y}$$

$$= \mathbf{e}_x \cdot \mathrm{D}_{\mathbf{e}_x} \mathbf{F} + \mathbf{e}_y \cdot \mathrm{D}_{\mathbf{e}_y} \mathbf{F}$$

- The directional derivatives of a vector field is defined to be

$$\mathrm{D}_{\hat{\mathbf{u}}} \mathbf{F} = \begin{bmatrix} \mathrm{D}_{\hat{\mathbf{u}}} P \\ \mathrm{D}_{\hat{\mathbf{u}}} Q \end{bmatrix}$$

- The rate of change of **F** in polar and in Cartesian are going to be quite different.



- Instead of decomposing the change into $\mathbf{e}_x$ and $\mathbf{e}_y$, we now use $\mathbf{e}_r$ and $\mathbf{e}_\theta$,

$$\nabla \cdot \mathbf{F} = \mathbf{e}_x \cdot \mathrm{D}_{\mathbf{e}_x} \mathbf{F} + \mathbf{e}_y \cdot \mathrm{D}_{\mathbf{e}_y} \mathbf{F} = \underbrace{\mathbf{e}_r \cdot \mathrm{D}_{\mathbf{e}_r} \mathbf{F}}_{r} + \underbrace{\mathbf{e}_\theta \cdot \mathrm{D}_{\mathbf{e}_\theta} \mathbf{F}}_{\theta}$$

- The basis vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are not constant, they are functions of θ

$$\nabla \cdot \mathbf{F} = \mathbf{e}_r \cdot \mathrm{D}_{\mathbf{e}_r} \mathbf{F} + \mathbf{e}_\theta \cdot \mathrm{D}_{\mathbf{e}_\theta} \mathbf{F}$$

so the basis must be differentiated along with the coefficients when computing the directional derivatives, and we must do so before taking the dot product

$$\mathrm{D}_{\mathbf{e}_r} \mathbf{F} = \begin{bmatrix} \mathbf{e}_r \cdot \nabla P^* \\ \mathbf{e}_r \cdot \nabla Q^* \end{bmatrix} = \begin{bmatrix} \mathbf{e}_r \cdot \left(\dfrac{\partial P^*}{\partial r}\mathbf{e}_r + \dfrac{1}{r}\dfrac{\partial P^*}{\partial \theta}\mathbf{e}_\theta \right) \\ \mathbf{e}_r \cdot \left(\dfrac{\partial Q^*}{\partial r}\mathbf{e}_r + \dfrac{1}{r}\dfrac{\partial Q^*}{\partial \theta}\mathbf{e}_\theta \right) \end{bmatrix} = \frac{\partial \mathbf{F}}{\partial r}$$

$$\mathrm{D}_{\mathbf{e}_\theta} \mathbf{F} = \begin{bmatrix} \mathbf{e}_\theta \cdot \nabla P^* \\ \mathbf{e}_\theta \cdot \nabla Q^* \end{bmatrix} = \begin{bmatrix} \mathbf{e}_\theta \cdot \left(\dfrac{\partial P^*}{\partial r}\mathbf{e}_r + \dfrac{1}{r}\dfrac{\partial P^*}{\partial \theta}\mathbf{e}_\theta \right) \\ \mathbf{e}_\theta \cdot \left(\dfrac{\partial Q^*}{\partial r}\mathbf{e}_r + \dfrac{1}{r}\dfrac{\partial Q^*}{\partial \theta}\mathbf{e}_\theta \right) \end{bmatrix} = \frac{1}{r}\frac{\partial \mathbf{F}}{\partial \theta}$$

- Therefore, divergence in polar coordinates is given by

$$\nabla \cdot \mathbf{F} = \mathbf{e}_r \cdot \frac{\partial \mathbf{F}}{\partial r} + \frac{1}{r}\mathbf{e}_\theta \cdot \frac{\partial \mathbf{F}}{\partial \theta} \qquad \text{where} \quad \mathbf{F} = F_r(r,\theta)\mathbf{e}_r + F_\theta(r,\theta)\mathbf{e}_\theta$$

Cylindrical

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z$$

$$\nabla \cdot \mathbf{F} = \frac{\partial F_r}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z} + \frac{F_r}{r}$$

$$\nabla \times \mathbf{F} = \left(\frac{1}{r}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)\mathbf{e}_\theta + \left(\frac{\partial F_\theta}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial \theta} + \frac{F_\theta}{r}\right)\mathbf{e}_z$$

- The term we might not have been expecting, comes from the fact that basis vectors are functions of the coordinates as well, and are constantly changing.
- Consider the divergence

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \mathbf{e}_r \cdot \frac{\partial \mathbf{F}}{\partial r} + \frac{1}{r}\mathbf{e}_\theta \cdot \frac{\partial \mathbf{F}}{\partial \theta} + \mathbf{e}_z \cdot \frac{\partial \mathbf{F}}{\partial z} \\ &= \mathbf{e}_r \cdot \frac{\partial}{\partial r}\left(F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_z\mathbf{e}_z\right) + \frac{1}{r}\mathbf{e}_\theta \cdot \frac{\partial}{\partial \theta}\left(F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_z\mathbf{e}_z\right) \\ &\quad + \mathbf{e}_z \cdot \frac{\partial}{\partial z}\left(F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_z\mathbf{e}_z\right)\end{aligned}$$

- The basis vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are functions of θ, and $\mathbf{e}_z$ is a constant vector,

$$
\begin{aligned}
\nabla \cdot \mathbf{F} = \mathbf{e}_r \cdot \frac{\partial}{\partial r}\left(F_r \mathbf{e}_r + F_\theta \mathbf{e}_\theta + F_z \mathbf{e}_z\right) \\
+ \frac{1}{r}\mathbf{e}_\theta \cdot \frac{\partial}{\partial \theta}\left(F_r \mathbf{e}_r + F_\theta \mathbf{e}_\theta + F_z \mathbf{e}_z\right) \\
+ \mathbf{e}_z \cdot \frac{\partial}{\partial z}\left(F_r \mathbf{e}_r + F_\theta \mathbf{e}_\theta + F_z \mathbf{e}_z\right)
\end{aligned}
$$

$$
= \frac{\partial F_r}{\partial r} + \frac{1}{r}\mathbf{e}_\theta \cdot \left(\frac{\partial F_r}{\partial \theta}\mathbf{e}_r + F_r \mathbf{e}_\theta + \frac{\partial F_\theta}{\partial \theta}\mathbf{e}_\theta - F_\theta \mathbf{e}_r + \frac{\partial F_z}{\partial \theta}\mathbf{e}_z\right) + \frac{\partial F_z}{\partial z}\mathbf{e}_z
$$

$$
= \frac{\partial F_r}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{F_r}{r} + \frac{\partial F_z}{\partial z}\mathbf{e}_z
$$

- The additional term is from the fact that $\mathbf{e}_r$ is also a function of θ.

Spherical

$$\nabla f = \frac{\partial f}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho \sin\phi}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi$$

$$\nabla \cdot \mathbf{F} = \frac{\partial F_\rho}{\partial \rho} + \frac{1}{\rho \sin\phi}\frac{\partial F_\theta}{\partial \theta} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{2F_\rho}{\rho} + \frac{F_\phi}{\rho \tan\phi}$$

$$\nabla \times \mathbf{F} = \left(\frac{1}{\rho}\frac{\partial F_\theta}{\partial \phi} - \frac{1}{\rho \sin\phi}\frac{\partial F_\phi}{\partial \theta} + \frac{F_\theta}{\rho \tan\phi}\right)\mathbf{e}_\rho + \left(\frac{\partial F_\phi}{\partial \rho} - \frac{1}{\rho}\frac{\partial F_\rho}{\partial \phi} + \frac{F_\phi}{\rho}\right)\mathbf{e}_\theta + \left(\frac{1}{\rho \sin\phi}\frac{\partial F_\rho}{\partial \theta} - \frac{\partial F_\theta}{\partial \rho} - \frac{F_\theta}{\rho}\right)\mathbf{e}_\phi$$

Exercise

For the two dimensional vortex embedded in $\mathbb{R}^3$,

$$\mathbf{v} = \frac{k}{x^2+y^2}\left(-y\mathbf{e}_x + x\mathbf{e}_y + 0\mathbf{e}_z\right)$$

where k *is a constant. Find the divergence and the curl.*