Introduction to Linear Algebra

Midterm 2 Review Class

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Outline

Content

20' True / False
- look carefully at your quiz and questions on slides

4' Vector space & linear independence
- check about the **abstract** definition of vector space, not just specific one we get used to

5' Fundamental subspaces
- know clearly how to find out those subspaces in general

6' Linear transformation
- relate the terms involved to the properties of matrix, specifically **Equivalence Theorem**

5' Homomorphism & isomorphism & coordinate
- apply these concepts to real problems, *e.g.*, solving tricky integrals

10' Proof
- look at those short but nontrivial proofs on slides, especially vector space, linear independence, spanning set and linear transformation

Vector Space

Vector space is a concept that brings various objects in Mathematics into same scenario. It consists of four parts:

1. Scalar field ($\mathcal{F}$)
2. Set of vectors ($\mathcal{V}$)
3. Vector addition
4. Multiplication between scalar and vectors

Here the word "vector" means not only those ordinary vectors in **Euclidean Space** ($\mathbb{R}^n$), but arbitrary set of mathematical objects following some rules.

Scalar Field

A scalar field can be either **real** or **complex**. For this field there should be also two operations:

1. Scalar addition
2. Scalar multiplication

The two operations are different things compared with the two operations in the previous page (3 and 4).

There are 9 axioms that a scalar field should follow. In particular, you should pay attention to (all of them are **unique**)

- Additive identity: $\mathbf{0} + \alpha = \alpha$
- Multiplicative identity: $\mathbf{1} \cdot \alpha = \alpha$
- Additive inverse: $\alpha + (-\alpha) = 0$
- Multiplicative inverse: $\alpha \cdot \alpha^{-1} = 1$

Set of Vectors

There are 10 axioms for the set of vectors $\mathcal{V}$ and the scalar field $\mathcal{F}$. If all of them are satisfied, then we say $\mathcal{V}$ is a vector space over $\mathcal{F}$.

In practice we check the vector space in the following order:

1. Axiom 9,10 (closure of addition and scalar multiplication)
2. Axiom 6,7,8 (existence of additive identity, additive inverse and multiplicative identity)
3. Axiom 1-5 (commutive, associative and distributive law)

Note: A field is not equivalent to a vector space.

Scalar Field and Vector Space

Question 1: Show the following are scalar fields

1. Galois field: $\{0,1\}$ using **xor** as addition and **and** as multiplication.
2. $\{\begin{bmatrix} x & -y \\ y & x \end{bmatrix} | x, y \in \mathbb{R}\}$ using normal matrix addition and multiplication.

Question 2: Use the properties of vector space to show that if $u + v = u + w$, then $v = w$.

Question 3: Check whether $\{\mathbf{A} \in \mathcal{M}_{2\times 2} \mid \det(\mathbf{A}) = 0\}$ is a subspace of $\mathcal{M}_{2\times 2}$.

Subspace

Some small observations:

- When checking whether $\mathcal{H}$ is a subspace of $\mathcal{V}$, only the **closure of addition** and **scalar multiplication** needs to be checked;
- Any subset of $\mathbb{R}^3$ that does not include the origin is **NOT** a subspace of $\mathbb{R}^3$;
- The set of all possible linear combinations of the vectors in $\mathcal{S}$ is a subspace of $\mathcal{V}$.

Spanning Set

Two important facts:

- Spanning set of $\mathcal{S}$ is the smallest subspace of $\mathcal{V}$ that contains $\mathcal{S}$;
- Two sets of vectors in space $\mathcal{V}$ span to the same set **if and only if** all vectors in one set are contained in the spanning set of the other. (This directly gives us the fact that $\text{span}(\text{span}(\mathcal{S})) = \text{span}(\mathcal{S})$)

Two important types of problems:

- Given a nonempty set $\mathcal{S}$ of vectors in $\mathbb{R}^n$ and a vector $\mathbf{v}$ in $\mathbb{R}^n$, determine if $\mathbf{v}$ is a linear combination of the vectors in $\mathcal{S}$;
- Given a nonempty set $\mathcal{S}$ of vectors in $\mathbb{R}^n$, determine whether $\mathcal{S}$ span $\mathbb{R}^n$.

Linear Independence

1. Determine linear independence (**basic**): consider vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, and then consider whether $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\mathbf{x} = 0$ has only the trivial solution;
2. The **Vandemonde matrix V** is invertible if it is from using n distinct x;
3. If the **Wronskian** of a set of functions is **not identically zero**, then the functions form a linearly independent set (its negative and inverse statements are not true);

Fundamental Subspaces

There are basically four fundamental subspaces of a given matrix **A**:

- Null space ($\mathbf{Ax} = 0$)
- Left-hand null space ($\mathbf{A}^{\mathrm{T}}\mathbf{y} = 0$)
- Row space (span$\{\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_m\}$)
- Column space (span$\{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n\}$)

Note: Elementary row operation will change column space of a matrix, but not the row space and null space, and it does not change the linear dependency of columns.

If for a $m \times n$ matrix **A** of rank r , another nonsingular matrix **E** transform A into row-echelon form, then the last $(m - r)$ rows of **E** spans null($\mathbf{A}^{\mathrm{T}}$).

Fundamental Subspaces

Ways to find the fundamental subspaces:

- Null space & left-hand null space:
  1. Reduce the matrix to its **rref**. Solve the homogeneous equation and obtain a parametric representation of solution. Rewrite the solution as a linear combination of vectors.
  2. Consider $\mathbf{EA} = \mathbf{U}$ and $\mathbf{E} = \begin{bmatrix} \mathbf{E_1} \\ \mathbf{E_2} \end{bmatrix}$ can find the left-hand null space and null space.
- Column space & row space:
  1. Reduce the matrix to its **rref** and obtain a basis for the row space easily. For the column space, just mark the **pivot columns** in rref and index the corresponding columns in the **original matrix**.

Basis and Dimension

Some important facts:

- Basis is equivalent to a minimal spanning set of a vector space, as well as the maximal linearly independent subset;
- **Dimension** of a vector space is the number of vectors in the basis of that space;
- The dimension (degrees of freedom) of a subspace of a vector space $\mathcal{V}$ is always less than or equal to $\dim(\mathcal{V})$ and if the equality is satisfied, the subspace is just $\mathcal{V}$ itself;
- The dimension of a vector space $\mathcal{V} \subset \mathbb{R}^n$ is different from the number of components contained in the individual vectors from $\mathcal{V}$;
- A set of vectors that contain the zero vector is **linearly dependent**;
- Find the extension set for a given linearly independent set in $\mathbb{R}^n$.

Back to Rank

We can define the rank of any matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ as:

- the number of pivots when performing Gauss elimination;
- the maximum number of linearly independent columns of $\mathbf{A}$, which also can be denoted as $\dim(\mathrm{col}(\mathbf{A}))$;
- the dimension of row space, denoted as $\dim(\mathrm{row}(\mathbf{A}))$.

Sometimes it's quite useful to consider the fact that rank of any submatrix $\mathbf{A}_i$ is **less than or equal to** the rank of the matrix $\mathbf{A}$.

Sum of Vector Spaces

If $\mathcal{X}$ and $\mathcal{Y}$ are subspaces of a vector space $\mathcal{V}$,then the sum of $\mathcal{X}$ and $\mathcal{Y}$ is defined to be the set of all possible sums of vectors from $\mathcal{X}$ with vectors from $\mathcal{Y}$. That is, $\mathcal{X} + \mathcal{Y} = \{x + y | x \in \mathcal{X}, y \in \mathcal{Y}\}$.

- dim$(\mathcal{X} + \mathcal{Y}) =$ dim$\mathcal{X}$ $+$ dim$\mathcal{Y}$ - dim$(\mathcal{X} \cap \mathcal{Y})$;
- rank$(\mathbf{A} + \mathbf{B}) \leq$ rank$(\mathbf{A})$ $+$ rank$(\mathbf{B})$ for any matrices $\mathbf{A}, \mathbf{B} \in \mathcal{M}_{m \times n}$.

Rank of Matrix Multiplication

1. If $\mathbf{A}$ is a matrix of $m \times n$ and $\mathbf{B}$ is a matrix of $n \times r$, then

$$\mathrm{rank}(\mathbf{AB}) = \mathrm{rank}(\mathbf{B}) - \dim(\mathrm{null}(\mathbf{A}) \cap \mathrm{col}(\mathbf{B}))$$

2. $\mathrm{rank}(\mathbf{AB}) \leq \min(\mathrm{rank}(\mathbf{A}), \mathrm{rank}(\mathbf{B}))$ [consider the subspace];
3. If $\mathbf{B}$ is invertible, then $\mathrm{rank}(\mathbf{AB}) = \mathrm{rank}(\mathbf{A})$ [$\mathrm{rank}((AB)B^{-1}) \leq \mathrm{rank}(AB)$]; If $\mathbf{A}$ is invertible, then $\mathrm{rank}(\mathbf{AB}) = \mathrm{rank}(\mathbf{B})$;
4. $\mathrm{rank}(\mathbf{AB})$ not necessarily equal to $\mathrm{rank}(\mathbf{BA})$, *e.g.*, $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ where column vectors of $\mathbf{B}$ lie in the null space of $\mathbf{A}$.

Rank of Matrix Multiplication

Question: Show that if matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ satisfy the following multiplication

$$\mathbf{D} = \mathbf{ABC}$$

while $\mathbf{A}$ and $\mathbf{C}$ are two invertible square matrices, then $\text{rank}(\mathbf{D}) = \text{rank}(\mathbf{B})$.

Solution: $\text{rank}(\mathbf{D}) = \text{rank}(\mathbf{ABC}) \leq \text{rank}(\mathbf{BC}) \leq \text{rank}(\mathbf{B})$. As $\mathbf{A}$ and $\mathbf{C}$ are invertible, we have $\mathbf{B} = \mathbf{A}^{-1}\mathbf{D}\mathbf{C}^{-1}$. To this end, we have $\text{rank}(\mathbf{B}) \leq \text{rank}(\mathbf{D})$. Combining the two inequalities, we conclude that $\text{rank}(\mathbf{B}) = \text{rank}(\mathbf{D})$.

This essentially reveals very interesting facts regarding several decomposition methods (*i.e.*, eigenvalue, SVD) that will be covered later. Intuitively the middle matrix $\mathbf{B}$ already contains the major information of original matrix $\mathbf{D}$.

Nullity

1. If $\mathbf{A}$ is a matrix with n columns, then

$$\mathrm{rank}(\mathbf{A}) + \mathrm{nullity}(\mathbf{A}) = n$$

which can be understood by the concept of leading variables and free variables;

2. $\mathrm{nullity}(\mathbf{A}) = 0$ can be added into equivalence theorem;
3. For system of linear equations $\mathbf{A}\mathbf{x} = \mathbf{b}$, we have
   - it has solution **if and only if** $\mathbf{b} \in \mathrm{col}(\mathbf{A})$;
   - if $\mathbf{x}_p$ is a particular solution, then the general solution can be represented as $\{\mathbf{x}_p + \mathbf{x}_c | \mathbf{x}_c \in \mathrm{null}(\mathbf{A})\}$.

Transformation

Matrix multiplication $\mathbf{y} = \mathbf{A}\mathbf{x}$ defines a **matrix transformation** $T_{\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{A}$ is the **transformation matrix**. Specifically, if $m = n$, then $T_{\mathbf{A}}$ is known as a **matrix operator**. The **codomain** $\mathbb{R}^m$ of $T_{\mathbf{A}}$ is larger or equal than the **range** of it.

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation **if and only if** for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and α in $\mathcal{F}$:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$;
2. $T(\alpha\mathbf{u}) = \alpha T(\mathbf{u})$.

Two matrix transformations are the same **if and only if** their transformation matrices are the same (one-to-one correspondence).

Standard Matrix

The matrix with the image vectors of the standard vectors as its columns

$$[T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ \cdots\ T(\mathbf{e}_n)]$$

is called the **standard matrix** for the transformation.

A matrix transformation $T_{\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if $T_{\mathbf{A}}$ maps distinct vectors in $\mathbb{R}^n$ into distinct vectors in $\mathbb{R}^m$.

A matrix transformation is said to be **onto** if every vector in $\mathbb{R}^m$ is the image of at least one vector in $\mathbb{R}^n$.

One-to-One

Quiz 3 Question 3:
Let $\mathcal{U}$ and $\mathcal{V}$ be finite dimensional vector spaces over a scalar field $\mathcal{F}$. Consider $T : \mathcal{U} \to \mathcal{V}$. Show if $\dim(\mathcal{U}) > \dim(\mathcal{V})$, then T cannot be one-to-one.

Solution: Recall T being one-to-one is equivalent to $\text{kernel}(T) = \{0\}$. Therefore we need to show $\text{kernel}(T)$ cannot be $\{0\}$. As T is not necessarily onto, we can state that $\dim(\text{range}(T)) \leq \dim(\mathcal{V})$. Thus $\text{rank}(T) \leq \dim(\mathcal{V}) < \dim(\mathcal{U})$ holds. In the meantime, T can be represented as a matrix with $\dim(\mathcal{V}) \times \dim(\mathcal{U})$. Recall the relationship $\text{rank}(T) + \text{nullity}(T) = n = \dim(\mathcal{U})$, we can conclude that $\text{nullity}(T) \geq 1$ and hence $\text{kernel}(T)$ cannot be $\{0\}$.

Intuitively, this says that some vectors in $\mathcal{U}$ have to be mapped to "finished" vectors in $\mathcal{V}$ as the capacity of $\mathcal{V}$ is not large enough.

Kernel and Range

1. kernel$(T_{\mathbf{A}}) = \text{null}(\mathbf{A})$;
2. range$(T_{\mathbf{A}}) = \text{col}(\mathbf{A})$;
3. Now three extra statements can be added into equivalence theorem:
    - kernel$(T_{\mathbf{A}}) = \{\mathbf{0}\}$;
    - $T_{\mathbf{A}}$ is one-to-one;
    - range$(T_{\mathbf{A}}) = \mathbb{R}^n$;
4. if $\mathbf{m} = \mathbf{n}$, then the following statements are equivalent:
    - $T_{\mathbf{A}}$ is one-to-one;
    - kernel$(T_{\mathbf{A}}) = \{\mathbf{0}\}$;
    - T is onto, range$(T_{\mathbf{A}}) = n$.

Linear Transformation

A linear transformation T is invertible **if and only if** it is **one-to-one** and **onto**. Do not get confused with what the equivalence theorem states, which only holds for the linear transformation with the same dimension.

Isomorphism between spaces $\mathcal{U}$ and $\mathcal{V}$ is a linear transformation from $\mathcal{U}$ to $\mathcal{V}$ which is one-to-one and onto. And in this case $\mathcal{U}$ and $\mathcal{V}$ are isomorphic to each other, denoted as $\mathcal{U} \cong \mathcal{V}$.

1. Every real n-dimensional vector space is isomorphic to $\mathbb{R}^n$;
2. One specific category of isomorphism above is $\mathbf{u} \to [\mathbf{u}]_{\mathcal{S}}$, where $\mathcal{S}$ is a basis for a vector space $\mathcal{V}$.

Transition Matrix

A transition matrix is a matrix to transform a coordinate vector on a basis $\mathcal{B}$ into the coordinate vector on another basis $\mathcal{B}'$ for a vector space $\mathcal{V}$. It can be presented as

$$\mathbf{P}_{\mathcal{B}\to\mathcal{B}'} = [\ [\mathbf{u}_1]_{\mathcal{B}'}\ \ [\mathbf{u}_2]_{\mathcal{B}'}\ \ \cdots\ \ [\mathbf{u}_n]_{\mathcal{B}'}\ ]$$

1. The inverse of this matrix is the transform matrix from $\mathcal{B}'$ to $\mathcal{B}$;
2. Transition matrix from standard basis to another basis is just an alignment of the vectors in that basis.

Set of Linear Transformations

For each pair of vector spaces $\mathcal{U}$ and $\mathcal{V}$ over $\mathcal{F}$, the set $\mathcal{L}(\mathcal{U}, \mathcal{V})$ of all linear transformations from $\mathcal{U}$ and $\mathcal{V}$ is a vector space over $\mathcal{F}$.

Suppose $\mathcal{B}_{\mathcal{U}} = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ and $\mathcal{B}_{\mathcal{V}} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m\}$ are bases for $\mathcal{U}$ and $\mathcal{V}$, respectively, and let B_{ji} be the linear transformation from $\mathcal{U}$ into $\mathcal{V}$ defined by

$$B_{ji}(\mathbf{u}) = \gamma_j \mathbf{v}_i, \text{ where } \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix} = [\mathbf{u}]_{\mathcal{B}_{\mathcal{U}}},$$

then $\mathcal{B}_{\mathcal{L}} = \{B_{ji}\}_{j=1\dots n}^{i=1\dots m}$ is a basis for $\mathcal{L}(\mathcal{U}, \mathcal{V})$.

Coordinate

As an extension of original **standard matrix**, we want to evaluate the change of coordinate for any linear transformation,

$$\begin{tikzcd}
\mathbf{u} \in \mathcal{U} \arrow[r, "\text{Direct}", "\text{Computation}"'] \arrow[d] & T(\mathbf{u}) \in \mathcal{V} \\
{[\mathbf{u}]_{\mathcal{B}_\mathcal{U}} \in \mathbb{R}^n} \arrow[r, "\text{Indirect}", "\text{Matrix transformation}"'] & {[T(\mathbf{u})]_{\mathcal{B}_\mathcal{V}} \in \mathbb{R}^m} \arrow[u]
\end{tikzcd}$$

and the matrix transformation is just

$$[T]_{\mathcal{B}_\mathcal{U}\mathcal{B}_\mathcal{V}} = \begin{bmatrix} [T(\mathbf{u}_1)]_{\mathcal{B}_\mathcal{V}} & [T(\mathbf{u}_2)]_{\mathcal{B}_\mathcal{V}} & \cdots & [T(\mathbf{u}_n)]_{\mathcal{B}_\mathcal{V}} \end{bmatrix}$$

Thanks !
Good luck for your exam !