Vv255 Lecture 17

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- The notion of definite integral of a scale-valued function

$$f(x)$$

can be extended to scalar-valued functions of several variables.

$$f(x,y) \qquad \text{or} \qquad f(x,y,z)$$

Q: Do you remember how we motivated the study of definite integral?

Q: What is actually behind the following notation?

$$\int_a^b f(x)\,dx$$

Q: What do we mean by the term "integrable"?

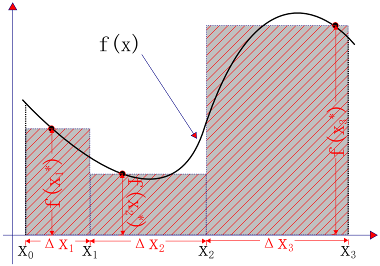
- We start with double integral, which is the extension of the definite integral to functions of two variables

$$z = f(x,y)$$

Q: How did we come up with the following limit

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

- Recall how we define and use a tagged partition of $[a,b]$ to obtain the above,



If $f(x)$ is integrable on $[a,b]$, then

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

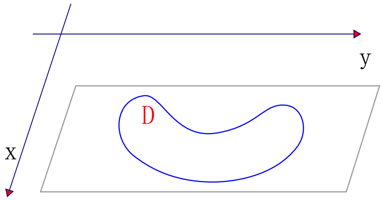
$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i,$$

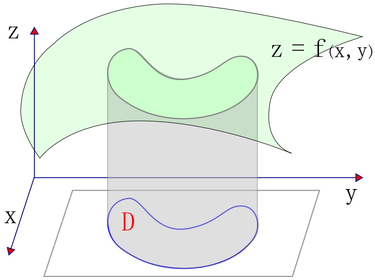
- The limit here is the process by which the number of subintervals $\to \infty$ in such a way that the norm, i.e., the maximum length of the subintervals $\to 0$

- For a functions of two variables

$$z = f(x, y)$$

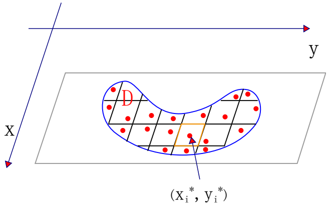
that is continuous and non-negative on a region $\mathcal{D}$ in the xy-plane, there is a well defined space between the surface $z = f(x, y)$ and the region $\mathcal{D}$.

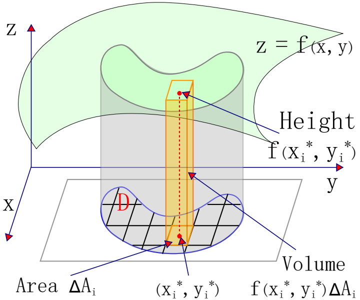




- Suppose that we are interested in associating a real number, with the name "volume", to the size of this region over $\mathcal{D}$ and below the surface.

- Consider a simple partition of the region $\mathcal{D}$ by lines parallel to the x, y-axes.





- Choose a point (x_i^*, y_i^*) in each sub-rectangle and estimate the size of it by

$$\sum_i^n f(x_i^*, y_i^*)\Delta A_i$$

Q: There are two sources of error in this computation, what are they?

Q: Do you think those errors always go away when we refine the partition?

Definition

If f is a continuous and non-negative function of two variables on a region $\mathcal{D}$ in the xy-plane, then the volume of the solid enclosed between the surface

$$z = f(x, y)$$

and the region $\mathcal{D}$ is defined to be

$$V = \lim_{n\to\infty} \sum_{i}^{n} f(x_i^*, y_i^*)\Delta A_i$$

provided that this limit exists and doesn't depend on the choices of the partition.

- The limit process here is the one by which the number of sub-rectangles in $\mathcal{D}$ in such a way that both the lengths and the widths $\to 0$.
- We use the following notation for this limit when it exists,

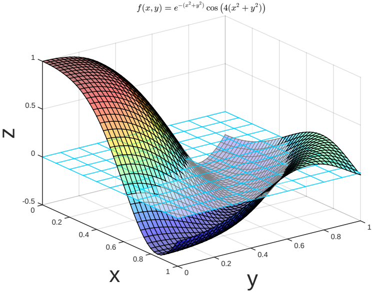
$$\iint\limits_{\mathcal{D}} f(x, y)\, dA = \lim_{n\to\infty} \sum_{i}^{n} f(x_i^*, y_i^*)\Delta A_i$$

- If f is continuous but has both positive and negative values in $\mathcal{D}$, then

$$\iint\limits_{\mathcal{D}} f(x,y)\, dA$$

no longer represents the volume between $\mathcal{D}$ and the surface; rather, it gives the difference between the volume above the xy-plane and the volume below

- We call this the net signed volume between the region $\mathcal{D}$ and the surface z.
- A positive value for the double integral of f over $\mathcal{D}$ means that there is more volume above $\mathcal{D}$ than below, and vice versa.



Matlab

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>> f = @(x,y) exp(-(x.^2+y.^2)).*cos(4.*(x.^2+y.^2));
>> [X, Y] = meshgrid((0:0.025:1),(0:0.025:1)); Z = f(X,Y);

>> surf(X,Y,Z); xlim([0,1]);ylim([0,1]); hold on;
>> [X, Y] = meshgrid((0:0.1:1),(0:0.1:1)); Z = 0*X;
>> h = mesh(X,Y,Z); alpha(h, 0.5); hold off;

>> q = integral2(f,0,1,0,1)

q =

0.0109
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- It is impractical to obtain the value of a double integral from its definition.
- Recall the evaluation of a single integral using the definition is difficult as well, but the fundamental theorem of calculus provides a much easier way.

$$\int_a^b f(x)\,dx = F(b) - F(a), \qquad \text{where } F \text{ is any antiderivative of } f.$$

- Recall the partial derivatives of a function $f(x, y)$ are calculated by holding one of the variables fixed and differentiating with respect to the other one.
- Let us consider the reverse of this process, partial integration,

$$\int f(x, y)\,dx \quad \text{and} \quad \int f(x, y)\,dy$$

- Partial definite integral with respect to x

$$\int_0^2 xy^2\,dx = y^2 \int_0^2 x\,dx = \left.\frac{y^2x^2}{2}\right|_{x=0}^{x=2} = 2y^2$$

- Partial definite integral with respect to y

$$\int_0^3 xy^2\,dy = x \int_0^3 y^2\,dy = \left.\frac{xy^3}{3}\right|_{y=0}^{y=3} = 9x$$

- A partial definite integral with respect to x is a function of y and hence can be in turn integrated with respect to y;

$$\int_0^3 \int_0^2 xy^2 \, dx \, dy = \int_0^3 2y^2 \, dy = \left. \frac{2y^3}{3} \right|_{y=0}^{y=3} = 18$$

- There is no reason for not doing it the other way around,

$$\int_0^2 \int_0^3 xy^2 \, dy \, dx = \int_0^2 9x \, dx = \left. \frac{9x^2}{2} \right|_{x=0}^{x=2} = 18$$

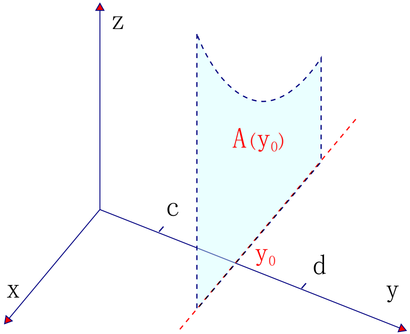
- This two-stage integration process is called iterated, or repeated, integration.
- It is no accident that the two iterated integrals produced the same answer.

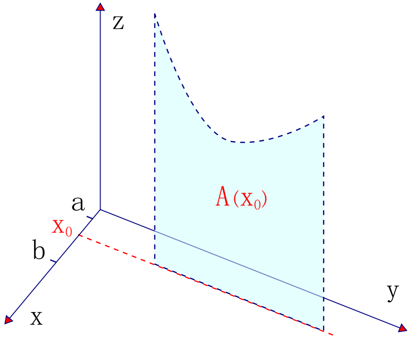
Q: What does this two stage integration process represent geometrically?

$$z = f(x, y) = xy^2$$

- Recall to evaluate the volume of a region by the method of cross-sections

$$\int_a^b A(x)\,dx, \qquad \text{where } A(x) \text{ gives the area of the cross-section at } x.$$





Fubini's Theorem

Let $\mathcal{R}$ be the rectangle region defined by the inequalities

$$a \leq x \leq b, \qquad c \leq y \leq d$$

If $f(x, y)$ is continuous on this rectangle, then

$$\iint\limits_{\mathcal{R}} f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

- A double integral over a rectangle can be computed by iterated integration.

Exercise

Find the volume of the solid S that is bounded by the elliptic paraboloid

$$x^2 + 2y^2 + z = 16,$$

the planes $x = 2$ and $y = 2$, and the three coordinate planes.

- When $f(x,y)$ can be factored as the product of a function of x only and a function of y only, the double integral of f can be written in a simpler form.
- Suppose $f(x,y) = g(x)h(y)$ and $\mathcal{R} = [a,b] \times [c,d]$, then

$$\begin{aligned}\iint\limits_{\mathcal{R}} f(x,y)\, dA &= \int_c^d \int_a^b g(x)h(y)\, dx\, dy \\ &= \int_c^d \left[\int_a^b g(x)h(y)\, dx\right] dy \\ &= \int_c^d \left[h(y)\left(\int_a^b g(x)\, dx\right)\right] dy = \int_a^b g(x)\, dx \int_c^d h(y)\, dy\end{aligned}$$

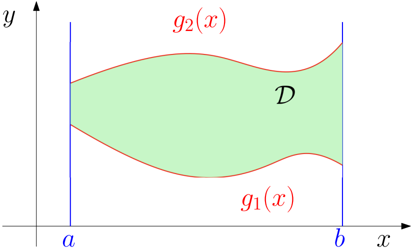
- For example, if $\mathcal{R} = \left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right]$, we have

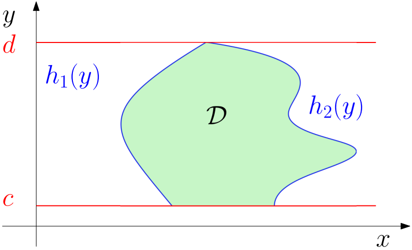
$$\iint\limits_{\mathcal{R}} \sin x \cos y\, dA = \int_0^{\pi/2} \sin x\, dx \int_0^{\pi/2} \cos y\, dy = [-\cos x]_0^{\pi/2}\, [\sin y]_0^{\pi/2} = 1$$

- Computing a double integrals over a very general region is not a small task.
- We will limit our study of double integrals to two basic types of regions,

Definition

A type I region is bounded on the left and right by vertical lines $x = a$ and $x = b$ and is bounded below and above by continuous curves $y = g_1(x)$ and $y = g_2(x)$, where $g_1(x) \leq g_2(x)$ for $a \leq x \leq b$.





A type II region is bounded below and above by horizontal lines $y = c$ and $y = d$ and is bounded on the left and right by continuous curves $x = h_1(y)$, $x = h_2(y)$ satisfying $h_1(y) \leq h_2(y)$ for $c \leq y \leq d$.

- To evaluate the double integral of a function $f(x, y)$ over a type I region $\mathcal{D}$,

$$\iint_{\mathcal{D}} f(x, y)\, dA$$

- Consider a rectangular region $\mathcal{R}$ that contains $\mathcal{D}$, and the following function

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } \mathcal{D} \\ 0 & \text{if } (x, y) \text{ is in } \mathcal{R} \text{ but not in } \mathcal{D} \end{cases}$$

- If F is integrable over $\mathcal{R}$, then by the definition of the double integral,

$$\begin{aligned} \iint_{\mathcal{D}} f(x, y)\, dA &= \iint_{\mathcal{R}} F(x, y)\, dA \\ &= \int_a^b \left[\int_c^d F(x, y)\, dy \right] dx \qquad \text{by Fubini's Theorem} \\ &= \int_a^b \left[\int_{g_1(x)}^{g_2(x)} F(x, y)\, dy \right] dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy \right] dx \end{aligned}$$

Theorem

- If $\mathcal{D}$ is a type I region on which $f(x, y)$ is continuous, then

$$\iint\limits_{\mathcal{D}} f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

where

$$\mathcal{D} = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

- If $\mathcal{D}$ is a type II region on which $f(x, y)$ is continuous, then

$$\iint\limits_{\mathcal{D}} f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

where

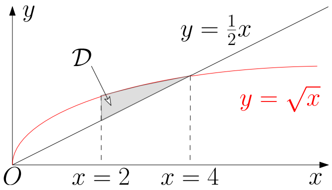
$$\mathcal{D} = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

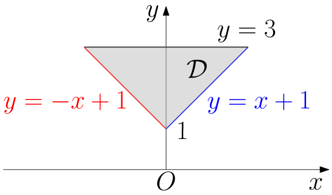
Exercise

(a) *Evaluate*

$$\iint_{\mathcal{D}} xy\, dA$$

over the region $\mathcal{D}$ *enclosed between* $y = \dfrac{1}{2}x$, $y = \sqrt{x}$, $x = 2$, *and* $x = 4$.





(b) *Evaluate*

$$\iint_{\mathcal{D}} (2x - y^2)\, dA$$

over the triangular region $\mathcal{D}$ *enclosed between* $y = -x + 1$, $y = x + 1$, $y = 3$.

Definition

A function $f(x, y)$ is called integrable if the limit actually exists and that its value does not depend on the choice of the partition.

- Having continuity is sufficient for a functions f to be integrable, but it is not a necessary condition.
- Recall we have the following for functions of one variable:

If $y = f(x)$ is continuous on $[a, b]$, or
if f has finitely many discontinuities but is bounded on $[a, b]$, then
f is integrable on $[a, b]$.

- Something similar can be said to functions of two or more variables.

Theorem

If $z = f(x, y)$ is continuous in $\mathcal{D}$, except on a finite number of smooth curves on which $f(x, y)$ is bounded, then f is integrable over $\mathcal{D}$, where $\mathcal{D}$ is some union of type I-II regions.

Properties of double integrals

Assume that all of the following integrals exist.

- Let c be a constant, then

$$\iint_{\mathcal{D}} cf(x,y)\,dA = c\iint_{\mathcal{D}} f(x,y)\,dA$$

$$\iint_{\mathcal{D}} [f(x,y)+g(x,y)]\,\,dA = \iint_{\mathcal{D}} f(x,y)\,dA + \iint_{\mathcal{D}} g(x,y)\,dA$$

- If $f(x,y) \geq g(x,y)$ for all (x,y) in $\mathcal{D}$, then

$$\iint_{\mathcal{D}} f(x,y)\,dA \geq \iint_{\mathcal{D}} g(x,y)\,dA$$

More Properties of double integrals

- If $\mathcal{D}$ is partitioned into $\mathcal{D}_1$ and $\mathcal{D}_2$, then

$$\iint_{\mathcal{D}} f(x,y)\,dA = \iint_{\mathcal{D}_1} f(x,y)\,dA + \iint_{\mathcal{D}_2} f(x,y)\,dA$$

- If we integrate the constant function $f(x,y) = h$ over a region $\mathcal{D}$, we have

$$\iint_{\mathcal{D}} h\,dA = h\iint_{\mathcal{D}} 1\,dA = V = h \cdot A$$

  where A is the area of the region $\mathcal{D}$.

- If f is bounded, that is, $m \le f(x,y) \le M$ for all (x,y) in $\mathcal{D}$, then

$$mA \le \iint_{\mathcal{D}} f(x,y)\,dA \le MA$$