Vv255 Lecture 21

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Dictionary

A Field is an area of open land.

Especially, one planted with crops or pasture: a wheat field or a field of corn.

Definition

A vector field in $\mathbb{R}^2$ is a rule that associates with each point (x, y) in the plane a *unique* vector $\mathbf{F}$ in the plane.

$$\mathbf{F}(x,y) = \begin{bmatrix} P \\ Q \end{bmatrix}, \text{ where } P \text{ and } Q \text{ are functions of } x \text{ and } y.$$

A vector field in $\mathbb{R}^3$ is a rule that associates with each point (x, y, z) in the space a *unique* vector $\mathbf{F}$ in the space.

$$\mathbf{F}(x,y,z) = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}, \quad \text{where } P,\ Q \text{ and } R \text{ are functions of } x,\ y \text{ and } z.$$

Functions P, Q and R are known as component functions of $\mathbf{F}$.

- A vector field is a vector-valued function of several variables,

$$\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$$

where the domain and the codomain have the same dimension n.

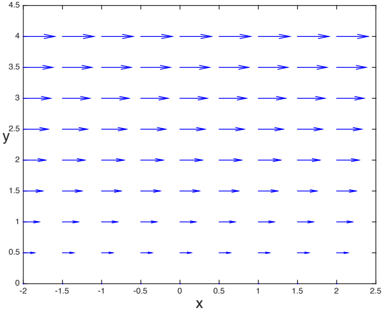
- A vector field in $\mathbb{R}^2$ can be pictured by drawing representative field vectors

$$\mathbf{F}(x, y)$$

at some well-chosen points in the xy-plane.

- For example, consider the vector field

$$\mathbf{F}(x, y) = \begin{bmatrix} \sqrt{\dfrac{y}{25}} \\ 0 \end{bmatrix}$$

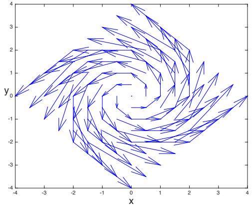


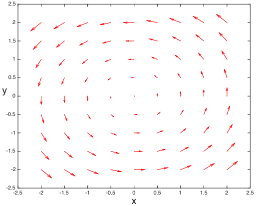
Matlab

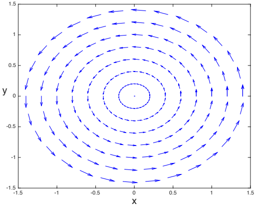
```
>> fx=@(x,y) sqrt(y)./5;
>> fy=@(x,y) 0.*x + 0.*y;

>> [X, Y] = meshgrid(-2:.5:2,0:.5:4);

>> quiver(X, Y, fx(X,Y), fy(X,Y), 0);
>> xlabel('x'); ylabel('y');
```

Q: Which of the following graphs is a plot for

$$\mathbf{F}(x,y) = -y\mathbf{e}_x + x\mathbf{e}_y$$







Matlab

```
>> fx=@(x,y) -y;
>> fy=@(x,y) x;

>> [X, Y] = meshgrid(-2:.5:2,-2:.5:2);

>> quiver(X, Y, fx(X,Y), fy(X,Y), 0);
>> xlabel('x'); ylabel('y');
```

```
>> quiver(X, Y, fx(X,Y), fy(X,Y), 0.5, '-r');
>> xlabel('x'); ylabel('y'); %Scale

>> [r, theta] = meshgrid( 0:0.2:1.5, 0:pi/15:(2*pi));

>> XX = r.*cos(theta); YY = r.*sin(theta); %Polar

>> quiver(XX, YY, fx(XX,YY), fy(XX,YY), 0.5);
>> xlabel('x'); ylabel('y');
```

Definition

If $\mathbf{r}$ is a position vector in $\mathbb{R}^2$ or $\mathbb{R}^3$, and if c is a constant, then a vector field

$$\mathbf{F} = \frac{c}{|\mathbf{r}|^3}\mathbf{r} = \frac{c}{|\mathbf{r}|^2}\frac{\mathbf{r}}{|\mathbf{r}|}$$

is called an inverse-square field.

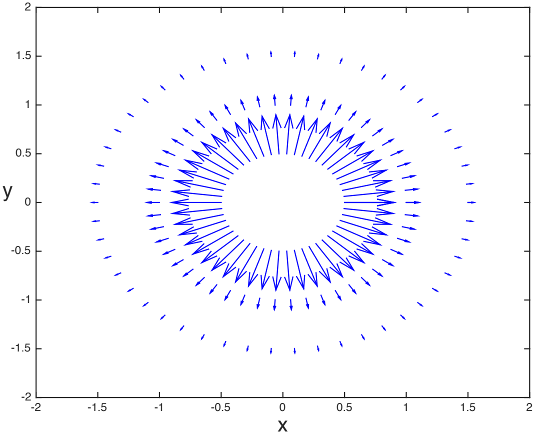
- In a plane, an inverse-square field has the form of

$$\begin{aligned}\mathbf{F} = \frac{c}{|\mathbf{r}|^3}\mathbf{r} = \frac{c}{(x^2+y^2)^{3/2}}\mathbf{r} &= \frac{c}{(x^2+y^2)^{3/2}}\left(x\mathbf{e}_x + y\mathbf{e}_y\right) \\ &= \frac{cx}{(x^2+y^2)^{3/2}}\mathbf{e}_x + \frac{cy}{(x^2+y^2)^{3/2}}\mathbf{e}_y\end{aligned}$$

- Note that if $c > 0$, then $\mathbf{F}$ has the same direction as $\mathbf{r}$, so each vector in the field is directed away from the origin; and if $c < 0$, then $\mathbf{F}$ is in the opposite direction to $\mathbf{r}$, so each vector in the field is directed towards the origin.
- In either case the magnitude of $\mathbf{F}(\mathbf{r})$ is inversely proportional to the square of the distance from the terminal point of $\mathbf{r}$ to the origin.

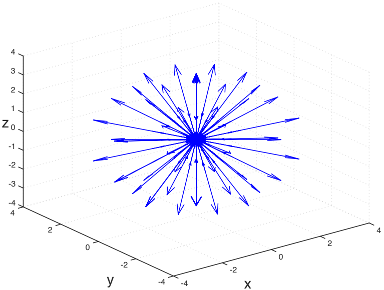
- For example, the following vector field is an inverse-square field,

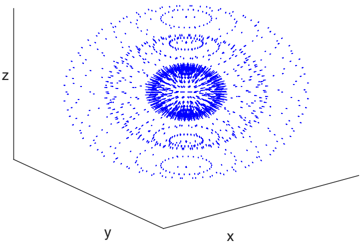
$$\mathbf{F} = \frac{x\mathbf{e}_x + y\mathbf{e}_y}{10(x^2 + y^2)^{3/2}}$$



Q: Are you surprised to see the following plot to be the graph of

$$\mathbf{F} = -\frac{x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z}{(x^2+y^2+z^2)^{3/2}}$$





Matlab

```
>> fx = @(x,y) x ./(sqrt(x.^2+y.^2)).^3 /10; fy = @(x,y) y ./(sqrt(x.^2+y.^2)).^3 /10;

>> [r, theta] = meshgrid( 0:0.5:1.5, 0:pi/25:(2*pi));
>> XX = r.*cos(theta); YY = r.*sin(theta);

>> quiver(XX, YY, fx(XX,YY), fy(XX,YY), 0)
>> xlabel('x'); ylabel('y');

>> fx = @(x,y,z) - x ./(sqrt(x.^2+y.^2+z.^2)).^3;
>> fy = @(x,y,z) - y ./(sqrt(x.^2+y.^2+z.^2)).^3;
>> fz = @(x,y,z) - z ./(sqrt(x.^2+y.^2+z.^2)).^3;

>> [rho, theta, phi] = meshgrid( 0:0.5:1.5, 0:pi/5:(2*pi), 0:pi/5:pi);

>> XX = rho.*cos(theta).*sin(phi); YY = rho.*sin(theta).*sin(phi); ZZ = rho.*cos(phi);

>> quiver3(XX,YY,ZZ,fx(XX,YY,ZZ),fy(XX,YY,ZZ),fz(XX,YY,ZZ), 0)
>> xlabel('x'); ylabel('y'); zlabel('z');

>> [rho, theta, phi] = meshgrid( 0:0.5:1.5, 0:pi/15:(2*pi), 0:pi/15:pi);

>> XX = rho.*cos(theta).*sin(phi); YY = rho.*sin(theta).*sin(phi); ZZ = rho.*cos(phi);

>> quiver3(XX,YY,ZZ,fx(XX,YY,ZZ),fy(XX,YY,ZZ),fz(XX,YY,ZZ), 0.9)
>> xlabel('x'); ylabel('y'); zlabel('z');

>> set(gca,'XTick',[]); set(gca,'YTick',[]); set(gca,'ZTick',[]);
```

- An important class of vector fields arises from the gradient function.
- Recall the gradient of a function of two or three variables f is

$$\begin{aligned}\nabla f &= \frac{\partial f}{\partial x}\mathbf{e}_x + \frac{\partial f}{\partial y}\mathbf{e}_y \\ &= \underbrace{f_x(x,y)}_{P(x,y)}\mathbf{e}_x + \underbrace{f_y(x,y)}_{Q(x,y)}\mathbf{e}_y\end{aligned}$$

$$\begin{aligned}\nabla f &= \frac{\partial f}{\partial x}\mathbf{e}_x + \frac{\partial f}{\partial y}\mathbf{e}_y + \frac{\partial f}{\partial z}\mathbf{e}_z \\ &= \underbrace{f_x(x,y,z)}_{P(x,y,z)}\mathbf{e}_x + \underbrace{f_y(x,y,z)}_{Q(x,y,z)}\mathbf{e}_y + \underbrace{f_z(x,y,z)}_{R(x,y,z)}\mathbf{e}_z\end{aligned}$$

where the partial derivatives are the component functions of a field.

Definition

The gradient of a function f is a vector field, and this vector field is known as

the gradient field of f.

Exercise

Sketch the gradient field of

$$f(x, y) = x + y$$

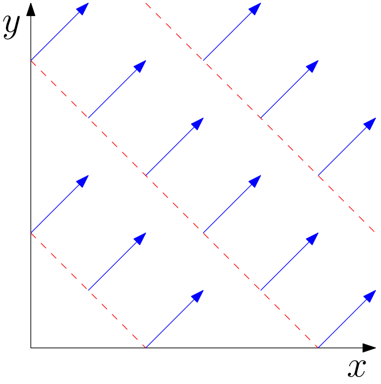
over $\mathcal{D} = \{(x, y) \mid x \geq 0, y \geq 0\}$.

Solution

- Find the gradient

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{e}_x + \frac{\partial f}{\partial y}\mathbf{e}_y$$

$$= \mathbf{e}_x + \mathbf{e}_y$$



- Consider the following function

$$f(x, y) = \frac{x}{\left(x^2 + y^2\right)^{1/2}}$$

of which the gradient field is given by

$$\mathbf{F} = \nabla f = \frac{\partial f}{\partial x}\mathbf{e}_x + \frac{\partial f}{\partial y}\mathbf{e}_y = \frac{y^2}{\left(x^2 + y^2\right)^{3/2}}\mathbf{e}_x + \frac{-xy}{\left(x^2 + y^2\right)^{3/2}}\mathbf{e}_y$$

- Notice that the original function f is greatly simplified in polar form

$$f(x, y) = \frac{x}{\left(x^2 + y^2\right)^{1/2}} \implies f\Big(x(r, \theta), y(r, \theta)\Big) = \frac{r\cos\theta}{r} = \cos\theta$$

and the gradient field of f also takes a simpler form in polar

$$\mathbf{F} = \nabla f = \frac{y}{\left(x^2 + y^2\right)^{3/2}}\Big(y\mathbf{e}_x - x\mathbf{e}_y\Big) = \frac{r\sin\theta}{r^3} - r\mathbf{e}_\theta = \frac{-\sin\theta}{r}\mathbf{e}_\theta$$

Q: How can we obtain the gradient field of a function in polar form?

- You might be tempted to suggest the following formula,

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{\partial f}{\partial \theta}\mathbf{e}_\theta$$

however, it is incorrect for a general function f!

- This can be seen from applying this incorrect formula to the last function

$$\left(\frac{\partial}{\partial r}\cos\theta\right)\mathbf{e}_r + \left(\frac{\partial}{\partial \theta}\cos\theta\right)\mathbf{e}_\theta = -\sin\theta\mathbf{e}_\theta \neq \frac{-\sin\theta}{r}\mathbf{e}_\theta$$

- In order to derive the correct formula, we need to remind ourselves

$$\nabla f$$

at a point is a vector in the direction of the maximum rate of change of f.

- Under orthonormal change of coordinate system, this property is preserved.

- Recall coordinates are really just the scalars α, β, and γ to represent a vector

$$\mathbf{v} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$$

with respect to some basis $\mathcal{B} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$.

- So in a Cartesian coordinate system, the gradient is really just a vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{e}_x + \frac{\partial f}{\partial y}\mathbf{e}_y$$

having coordinates $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ with respect to $\{\mathbf{e}_x, \mathbf{e}_y\}$.

- Recall for the coordinates of a vector with respect to a orthonormal basis are found by taking scalar projection of the vector onto the basis vectors, that is,

$$\mathbf{v} = (\mathbf{v}\cdot\mathbf{a})\,\mathbf{a} + (\mathbf{v}\cdot\mathbf{b})\,\mathbf{b} + (\mathbf{v}\cdot\mathbf{c})\,\mathbf{c}$$

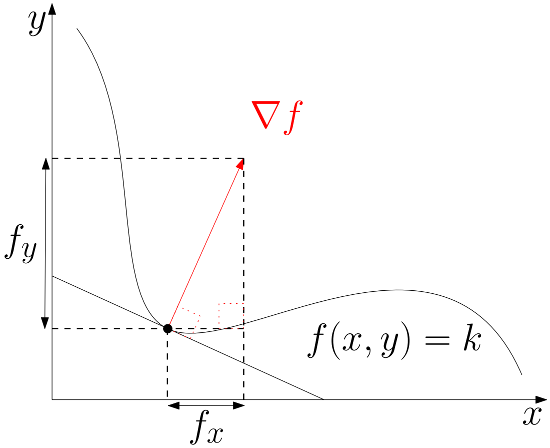
- At a particular point, ∇f is a constant vector that can be understood as

$$\nabla f = (\nabla f\cdot\mathbf{e}_x)\,\mathbf{e}_x + (\nabla f\cdot\mathbf{e}_y)\,\mathbf{e}_y$$

- For any orthonormal basis $\{\mathbf{a}, \mathbf{b}\}$, the gradient field

$$\mathbf{F} = (\mathbf{F} \cdot \mathbf{a})\,\mathbf{a} + (\mathbf{F} \cdot \mathbf{b})\,\mathbf{b} = (\nabla f \cdot \mathbf{e}_x)\,\mathbf{e}_x + (\nabla f \cdot \mathbf{e}_y)\,\mathbf{e}_y = \frac{\partial f}{\partial x}\mathbf{e}_x + \frac{\partial f}{\partial y}\mathbf{e}_y$$

so the standard basis is just a particular representation of the gradient field.



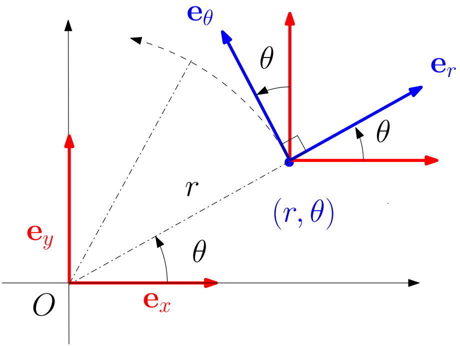
- Recall, $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y = r\cos\theta\mathbf{e}_x + r\sin\theta\mathbf{e}_y$, the polar basis consists of

$$\begin{aligned}\mathbf{e}_r &= \frac{\partial \mathbf{r}}{\partial r} \\ &= \quad \cos\theta\mathbf{e}_x + \sin\theta\mathbf{e}_y\end{aligned}$$

is the direction of increasing r.

$$\begin{aligned}\mathbf{e}_\theta &= \frac{1}{r}\frac{\partial \mathbf{r}}{\partial \theta} \\ &= -\sin\theta\mathbf{e}_x + \cos\theta\mathbf{e}_y\end{aligned}$$

is the direction of increasing θ.



- Thus the position vector in polar is

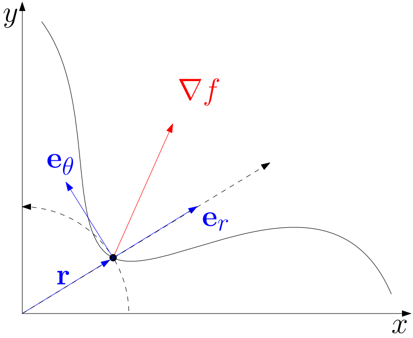
$$\mathbf{r} = r\mathbf{e}_r$$

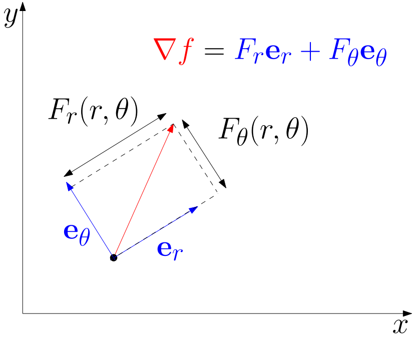
Q: Why do we need the basis vector

$$\mathbf{e}_\theta$$

- So using the basis $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ to represent ∇f will give us the polar form of it

$$\mathbf{F} = (\mathbf{F}\cdot\mathbf{a})\,\mathbf{a} + (\mathbf{F}\cdot\mathbf{b})\,\mathbf{b} = \underbrace{(\nabla f\cdot\mathbf{e}_r)}_{F_r}\mathbf{e}_r + \underbrace{(\nabla f\cdot\mathbf{e}_\theta)}_{F_\theta}\mathbf{e}_\theta$$





Q: How can we find the component F_r and F_θ without first finding

$$\frac{\partial f}{\partial x} \qquad \text{and} \qquad \frac{\partial f}{\partial y}$$

- The basis vectors are in terms of the partial derivatives,

$$\begin{aligned}\nabla f &= (\nabla f \cdot \mathbf{e}_r)\,\mathbf{e}_r + (\nabla f \cdot \mathbf{e}_\theta)\,\mathbf{e}_\theta \\ &= \left(\nabla f \cdot \frac{\partial \mathbf{r}}{\partial r}\right)\mathbf{e}_r + \left(\nabla f \cdot \frac{1}{r}\frac{\partial \mathbf{r}}{\partial \theta}\right)\mathbf{e}_\theta \\ &= \underbrace{\left(\nabla f \cdot \frac{\partial \mathbf{r}}{\partial r}\right)}_{\text{Chain Rule}}\mathbf{e}_r + \frac{1}{r}\underbrace{\left(\nabla f \cdot \frac{\partial \mathbf{r}}{\partial \theta}\right)}_{\text{Chain Rule}}\mathbf{e}_\theta = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta\end{aligned}$$

- Therefore, the gradient field of f in polar has component functions

$$\mathbf{F} = \nabla f = F_r(r,\theta)\mathbf{e}_r + F_\theta(r,\theta)\mathbf{e}_\theta = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta$$

Exercise

Verify the formulae for the components functions of ∇f in polar using

$$f(x,y) = \frac{1}{(x^2+y^2)^{1/2}} = \frac{1}{r}$$

- Similarly, for functions of three variables,

$$w = f(x, y, z)$$

we can derive the gradient field of f in cylindrical and spherical coordinates

$$\mathbf{F}(r,\theta,z) = \nabla f = P(r,\theta,z)\mathbf{e}_r + Q(r,\theta,z)\mathbf{e}_\theta + R(r,\theta,z)\mathbf{e}_z$$

$$= \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z$$

$$\mathbf{F}(\rho,\theta,\phi) = \nabla f = P^*(\rho,\theta,\phi)\mathbf{e}_\rho + Q^*(\rho,\theta,\phi)\mathbf{e}_\theta + R^*(\rho,\theta,\phi)\mathbf{e}_\phi$$

$$= \frac{\partial f}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho\sin\phi}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi \qquad \text{where}$$

$$\mathbf{e}_\rho = \frac{\partial \mathbf{r}}{\partial \rho} = \begin{bmatrix} \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{bmatrix}; \mathbf{e}_\theta = \frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}; \mathbf{e}_\phi = \frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi} = \begin{bmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ -\sin\phi \end{bmatrix}$$

Exercise

Find the gradient field in terms of x, y and z for the following function

$$f(x,y,z) = \left(\ln\sqrt{x^2+y^2+z^2}\right)^2$$

Algebra Rules for Gradients

- Linear:

$$\nabla(\alpha f \pm \beta g) = \alpha \nabla f \pm \beta \nabla g \qquad \text{for any real number } \alpha \text{ and } \beta.$$

- Product:

$$\nabla(fg) = f\nabla g + g\nabla f$$

- Quotient:

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$$

- Power:

$$\nabla f^n = n f^{n-1}\nabla f$$