Vv417 Lecture 6

Jing Liu

UM-SJTU Joint Institute

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- Expansions by cofactors are very useful when computing the determinant of matrices that have a row or column with several zero entries.
- However, it is also useful to establish properties of determinants.

Theorem

Suppose $\mathbf{A}$ is a square matrix.

1. If $\mathbf{A}$ has a row or column of zeros, then
$$\det(\mathbf{A}) = 0$$
2. If $\mathbf{A}$ has two identical rows or two identical columns, then
$$\det(\mathbf{A}) = 0$$

Q: How to show the second statement is true?

Q: Is the converse of each of those two statements true?

Q: What is the effect of elementary row operations on $\det(\mathbf{A})$?

Interchanging two rows of $\mathbf{A}$

Suppose $\mathbf{A}$ is an $n \times n$ matrix and

$$\mathbf{E}_{i,j} = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

is the elementary matrix corresponding to interchanging row i with row j, then

$$\det\left(\mathbf{E}_{i,j}\mathbf{A}\right) = -\det\left(\mathbf{A}\right)$$

Furthermore,

$$\det\left(\mathbf{E}_{i,j}\mathbf{A}\right) = \det\left(\mathbf{E}_{i,j}\right)\det\left(\mathbf{A}\right)$$

Multiplying a row of $\mathbf{A}$ by a nonzero constant

Suppose $\mathbf{A}$ is an $n \times n$ matrix and

$$\mathbf{E}_{(\alpha)i} = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \alpha & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

is the elementary matrix of multiplying the ith row by the nonzero scalar α, then

$$\det\left(\mathbf{E}_{(\alpha)i}\mathbf{A}\right) = \alpha \det(\mathbf{A})$$

Furthermore,

$$\det(\mathbf{E}_{(\alpha)i}\mathbf{A}) = \det(\mathbf{E}_{(\alpha)i}) \det(\mathbf{A})$$

Proof

- Notice the ith row of $\mathbf{E}_{(\alpha)i}\mathbf{A}$ is

$$\begin{bmatrix}\alpha a_{i1} & \alpha a_{i2} & \cdots & \alpha a_{in}\end{bmatrix}$$

- Expand $\det(\mathbf{E}_{(\alpha)i}\mathbf{A})$ along the ith row, we have

$$\begin{aligned}\det(\mathbf{E}_{(\alpha)i}\mathbf{A}) &= (\alpha a_{i1}C_{i1}) + (\alpha a_{i2}C_{i2}) + \cdots + (\alpha a_{in}C_{in}) \\ &= \alpha\left(a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}\right) \\ &= \alpha \det\left(\mathbf{A}\right)\end{aligned}$$

- Using the fact $\det\left(\mathbf{I}\right) = 1$ and the result above, we have

$$\begin{aligned} & \det\left(\mathbf{E}_{(\alpha)i}\mathbf{I}\right) = \alpha \det\left(\mathbf{I}\right) = \alpha \\ \Longrightarrow \quad & \det\left(\mathbf{E}_{(\alpha)i}\right) = \alpha \\ \Longrightarrow \quad & \det\left(\mathbf{E}_{(\alpha)i}\mathbf{A}\right) = \alpha \det\left(\mathbf{A}\right) = \det\left(\mathbf{E}_{(\alpha)i}\right)\det\left(\mathbf{A}\right) \quad \square\end{aligned}$$

Adding a multiple of one row to another row

Suppose $\mathbf{A}$ is an $n \times n$ matrix and

$$\mathbf{E}_{(\alpha)i,j} = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & \alpha & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

is the elementary matrix of adding α times the ith row to the jth row, then

$$\det\left(\mathbf{E}_{(\alpha)i,j}\mathbf{A}\right) = \det(\mathbf{A})$$

Furthermore,

$$\det(\mathbf{E}_{(\alpha)i,j}\mathbf{A}) = \det(\mathbf{E}_{(\alpha)i,j})\det(\mathbf{A})$$

Proof

- Note this last type of row operations only changes the jth row,

$$\begin{bmatrix} \vdots & \vdots & & \vdots \\ (a_{j1}+\alpha a_{i1}) & (a_{j2}+\alpha a_{i2}) & \cdots & (a_{jn}+\alpha a_{in}) \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

- Using the cofactor expansion along the jth row, we have

$$\begin{aligned} \det(\mathbf{E}_{(\alpha)i,j}\mathbf{A}) &= (a_{j1}+\alpha a_{i1})C_{j1} + (a_{j2}+\alpha a_{i2})C_{j2} + \\ &\qquad \vdots \\ &\qquad + (a_{jn}+\alpha a_{in})C_{jn} \\ &= (\, a_{j1}C_{j1} + a_{j2}C_{j2} \cdots + a_{jn}C_{jn} \,) \\ &\qquad + \alpha(\, a_{i1}C_{j1} + a_{i2}C_{j2} \cdots + a_{in}C_{jn} \,) \\ &= \det(\mathbf{A}) + \beta \end{aligned}$$

Proof

- So we need to show the following is true

$$\beta = a_{i1}C_{j1} + a_{i2}C_{j2} \cdots + a_{in}C_{jn} = 0$$

- Consider the matrix $\mathbf{A}^*$ which differs from $\mathbf{A}$ only in row j, specifically,

$$[\mathbf{A}^*]_{pq} = \begin{cases} [\mathbf{A}]_{pq} & \text{if } p \neq j, \\ [\mathbf{A}]_{iq} & \text{if } p = j. \end{cases}$$

- Notice, by construction, we have

$$a^*_{jq} = a^*_{iq} = a_{iq} \qquad \text{and} \qquad C^*_{jq} = C_{jq}$$

- Since there are two identical rows in $\mathbf{A}^*$, namely, the ith and the jth row

$$\det(\mathbf{A}^*) = 0$$

Proof

- However, the following cofactor expansion of $\mathbf{A}^*$ along the jth row reveals

$$0 = \det\left(\mathbf{A}^*\right) = a^*_{j1}C^*_{j1} + a^*_{j2}C^*_{j2} + \cdots + a^*_{jn}C^*_{jn}$$

$$= a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} = \beta$$

- This completes the main result,

$$\det\left(\mathbf{E}_{(\alpha)i,j}\mathbf{A}\right) = \det(\mathbf{A}) + \beta = \det(\mathbf{A})$$

- Using the fact $\det\left(\mathbf{I}\right) = 1$ and the result above, we have

$$\det\left(\mathbf{E}_{(\alpha)i,j}\mathbf{I}\right) = \det\left(\mathbf{I}\right) = 1$$

$$\Longrightarrow \quad \det\left(\mathbf{E}_{(\alpha)i,j}\right) = 1$$

$$\Longrightarrow \quad \det\left(\mathbf{E}_{(\alpha)i,j}\mathbf{A}\right) = 1 \cdot \det\left(\mathbf{A}\right) = \det\left(\mathbf{E}_{(\alpha)i,j}\right)\det\left(\mathbf{A}\right) \quad \square$$

Q: Why is the following theorem obviously true given what we have discussed?

Theorem

Suppose $\mathbf{A}$ is a square matrix.

1. If $\mathbf{A}$ has a row or column of zeros, then
$$\det(\mathbf{A}) = 0$$
2. If $\mathbf{A}$ has two identical rows or two identical columns, then
$$\det(\mathbf{A}) = 0$$
3. If $\mathbf{A}$ has a row that is a multiple of another row of $\mathbf{A}$, then
$$\det(\mathbf{A}) = 0$$
4. If $\mathbf{A}$ has a column that is a multiple of another column of $\mathbf{A}$, then
$$\det(\mathbf{A}) = 0$$

- Interchanging rows

$$\det\left(\mathbf{E}_{i,j}\right) = -1$$
$$\det\left(\mathbf{E}_{i,j}\mathbf{A}\right) = \det\left(\mathbf{E}_{i,j}\right)\det\left(\mathbf{A}\right)$$

- Multiplying a row

$$\det\left(\mathbf{E}_{(\alpha)i}\right) = \alpha$$
$$\det\left(\mathbf{E}_{(\alpha)i}\mathbf{A}\right) = \det\left(\mathbf{E}_{(\alpha)i}\right)\det\left(\mathbf{A}\right)$$

- Adding a multiple of a row to another

$$\det\left(\mathbf{E}_{(\alpha)i,j}\right) = 1$$
$$\det\left(\mathbf{E}_{(\alpha)i,j}\mathbf{A}\right) = \det\left(\mathbf{E}_{(\alpha)i,j}\right)\det\left(\mathbf{A}\right)$$

- Hence we have essentially shown that

$$\det(\mathbf{EA}) = \det(\mathbf{E})\det(\mathbf{A}), \qquad \text{where } \mathbf{E} \text{ is an elementary matrix.}$$

- A natural question to ask next is what if $\mathbf{E}$ is not an elementary matrix.

$$\det(\mathbf{AB})$$

- In order to show

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$$

we need the following result, which is surely not a surprise!

Theorem

An $n \times n$ matrix $\mathbf{A}$ is singular if and only if

$$\det(\mathbf{A}) = 0$$

Proof

- Consider a sequence of elementary matrices such that

$$\mathbf{E}_k\mathbf{E}_{k-1}\mathbf{E}_{k-2}\ldots\mathbf{E}_1\mathbf{A} = \mathrm{rref}(\mathbf{A}) = \mathbf{R}$$

Proof

- Since $\mathbf{E}_i$ for $i = 1, \ldots k$ are elementary matrices, we have,

$$\det(\mathbf{E}_k)\det(\mathbf{E}_{k-1})\det(\mathbf{E}_{k-2})\ldots\det(\mathbf{E}_1)\det(\mathbf{A}) = \det(\mathbf{R})$$

- Since the determinant of any elementary matrix is non-zero

$$\det(\mathbf{A}) = 0 \iff \det(\mathbf{R}) = 0$$

- Recall the rref of a square matrix can either be $\mathbf{I}$ or have a row of zeros.
- If $\mathbf{A}$ is singular, then $\mathbf{R} = \mathrm{rref}(\mathbf{A})$ has an entire row of 0, which leads us to

$$\det(\mathbf{R}) = 0 \implies \det(\mathbf{A}) = 0$$

- If $\det(\mathbf{A}) = 0$, then

$$\det(\mathbf{R}) = 0 \implies \mathbf{R} \neq \mathbf{I}$$

which means $\mathbf{A}$ is singular. □

- We can now formally add one extra statement to our equivalent theorem.

Equivalence Theorem

If $\mathbf{A}$ is an $n \times n$ matrix, then the following statements are equivalent,

1. $\mathbf{A}$ is invertible.
2. $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.
3. The reduced echelon form of $\mathbf{A}$ is $\mathbf{I}_n$.
4. $\mathbf{A}$ is expressible as a product of elementary matrices.
5. $\mathbf{Ax} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
6. $\mathbf{Ax} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
7. $\det(\mathbf{A}) \neq 0$.

Theorem

Suppose $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices, then

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$$

Proof

- If $\mathbf{A}$ is singular, then $\mathbf{AB}$ is singular.

$$\begin{aligned}\det(\mathbf{AB}) &= 0 \\ &= \det(\mathbf{A})\det(\mathbf{B})\end{aligned}$$

- If $\mathbf{A}$ is invertible, then $\mathbf{A}$ can be written as a product of elementary matrices

$$\mathbf{A} = \mathbf{E}_k\mathbf{E}_{k-1}\mathbf{E}_{k-2}\ldots\mathbf{E}_1$$

- Therefore,

$$\begin{aligned}\det(\mathbf{AB}) &= \det(\mathbf{E}_k\mathbf{E}_{k-1}\mathbf{E}_{k-2}\ldots\mathbf{E}_1\mathbf{B}) \\ &= \det(\mathbf{E}_k)\det(\mathbf{E}_{k-1})\det(\mathbf{E}_{k-2})\ldots\det(\mathbf{E}_1)\det(\mathbf{B}) \\ &= \det(\mathbf{E}_k\mathbf{E}_{k-1}\mathbf{E}_{k-2}\ldots\mathbf{E}_1)\det(\mathbf{B}) \\ &= \det(\mathbf{A})\det(\mathbf{B})\end{aligned}$$

Q: Is there any further connection between determinants and the solution to

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

besides giving us some idea on the number of solutions?

Definition

If every element in an $n \times n$ matrix $\mathbf{A}$ is replaced by its cofactor, the resulting matrix is called the matrix of cofactors and is denoted $\mathbf{C}$. The transpose of the matrix of cofactors, is called the adjoint of $\mathbf{A}$ and is denoted

$$\operatorname{adj}(\mathbf{A}) = \mathbf{C}^{\mathrm{T}}$$

Theorem

Suppose $\mathbf{A}$ is a square matrix. If $\det(\mathbf{A}) \neq 0$, then

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \operatorname{adj}(\mathbf{A})$$

Proof

- Recall we have show the following result in the cofactor proof,

$$\sum_{k=1}^{n} a_{ik} C_{jk} = \begin{cases} \det(\mathbf{A}) & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

- Consider the elements of the following product

$$\left[\mathbf{A} \frac{1}{\det(\mathbf{A})} \operatorname{adj}(\mathbf{A})\right]_{ij} = \frac{1}{\det(\mathbf{A})} \left[\mathbf{A}\mathbf{C}^{\mathrm{T}}\right]_{ij} = \frac{1}{\det(\mathbf{A})} \sum_{k=1}^{n} a_{ik} C_{jk}$$

$$= \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

- Therefore,

$$\mathbf{A} \frac{1}{\det(\mathbf{A})} \operatorname{adj}(\mathbf{A}) = \mathbf{I} \implies \mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \operatorname{adj}(\mathbf{A}) \quad \square$$

Cramer's rule

Let $\mathbf{A}$ be an $n \times n$ matrix. If $\det(\mathbf{A}) \neq 0$, then the unique solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} \det(\mathbf{B}_1) \\ \det(\mathbf{B}_2) \\ \vdots \\ \det(\mathbf{B}_k) \\ \vdots \\ \det(\mathbf{B}_n) \end{bmatrix}$$

where $\mathbf{B}_k$ denotes the matrix obtained by replacing the kth column of $\mathbf{A}$ by $\mathbf{b}$.

Proof

- Since $\det(\mathbf{A}) \neq 0$, using the theorem on page $\boxed{16}$, we have

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{1}{\det(\mathbf{A})} \operatorname{adj}(\mathbf{A})\,\mathbf{b} = \frac{1}{\det(\mathbf{A})} \mathbf{C}^{\mathrm{T}}\mathbf{b}$$

where $\mathbf{C}$ is the cofactor matrix.

Proof

- Considering the kth term, we have

$$x_k = \frac{1}{\det(\mathbf{A})} \sum_{j=1}^{n} b_j C_{jk} = \frac{\det(\mathbf{B}_k)}{\det(\mathbf{A})}$$

where the sum is the cofactor expansion along the k column of

$$\det(\mathbf{B}_k) = \begin{bmatrix} a_{11} & a_{12} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n & \cdots & a_{nn} \end{bmatrix} \quad \square$$

Exercise

Find $\det(\mathbf{A})$, where $\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{bmatrix}$, first by using the definition, then by the cofactor expansion. Count the number of additions and multiplications used.

Solution

- In general, by definition, we have $n!$ nonzero terms

$$\det(\mathbf{A}) = \sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\sum_{k_3=1}^{3} \varepsilon_{k_1k_2k_3} a_{1k_1}a_{2k_2}a_{3k_3}$$

$$= 2\cdot 2\cdot 4 - 2\cdot 1\cdot(-3)$$

$$- 1\cdot 4\cdot 4 + 1\cdot 1\cdot 6$$

$$3\cdot 4\cdot(-3) - 3\cdot 2\cdot 6 = -60$$

- Notice 5 additions and 12 multiplications are used for this 3×3 matrix.
- According to the expansion by cofactors along the first row, we have

$$\det(\mathbf{A}) = 2\cdot(2\cdot 4 + 1\cdot 3) - 1\cdot(4\cdot 4 - 1\cdot 6) + 3\cdot(-4\cdot 3 - 2\cdot 6) = -60$$

- Notice 5 additions and 9 multiplications are used in this approach.

- That illustrates why we don't use the definition to compute determinants.

Q: Is there a better way than the cofactor expansion if we are forced to compute

$$\det(\mathbf{A})$$

Q: What do we know regarding the determinant of an upper triangular matrix?

Q: How can we exploit the fact that we only need 2 multiplications to compute

$$\det(\mathbf{U}) \qquad \text{where } \mathbf{U} \text{ is upper triangular and } \mathbf{U} \sim \mathbf{A},$$

and its link to $\det(\mathbf{A})$ when we need to compute the determinant of

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{bmatrix}$$

Q: How many additions and multiplications do we need in the elimination step?

- The elimination step consists of finding the necessary row operations

$$\mathbf{A} \sim \mathbf{E}_{(-3)1,3}\mathbf{E}_{(-2)1,2}\begin{bmatrix}2 & 1 & 3\\ 4 & 2 & 1\\ 6 & -3 & 4\end{bmatrix} \sim \mathbf{E}_{2,3}\begin{bmatrix}2 & 1 & 3\\ 0 & 0 & -5\\ 0 & -6 & -5\end{bmatrix}$$

$$\sim \begin{bmatrix}2 & 1 & 3\\ 0 & -6 & -5\\ 0 & 0 & -5\end{bmatrix} = \mathbf{U}$$

$$\Longrightarrow \mathbf{A} = \mathbf{E}_{(2)1,2}\mathbf{E}_{(3)1,3}\mathbf{E}_{3,2}\mathbf{U}$$

in which 4 additions and 6 multiplications are needed for this particular case.

- The next step involves finding $\det(\mathbf{U})$, and making "corrections"

$$\det(\mathbf{A}) = -\det(\mathbf{U}) = -2 \cdot (-6) \cdot (-5) = -60$$

- In this case, only 4 additions and 8 multiplications are needed in total.

Q: How many multiplications do we need in total for an arbitrary 3×3 matrix?

- Many theorems to do with determinants can be proved using of the cofactor expansion. But it is also not used to find the determinant of a big matrix.
- Below gives the number of arithmetic operations involved in each method.

	Leibiniz		Laplace		Gauss	
n	$+$	$\times$	$+$	$\times$	$+$	$\times$
2	1	2	1	2	1	3
3	5	12	5	9	5	10
4	23	72	23	40	14	23
5	119	480	119	205	30	44
10	3,628,799	32,659,200	3,628,799	6,235,300	285	339

- Note computing the determinant of a 50×50 matrix directly would take a computer 10^{40} years! that is, more than 10^{30} times the age of the universe!
- So don't blindly do it on your laptop! Often it can be avoided altogether.

Q: How about finding the solution of a large square nonsingular linear system?

$$\mathbf{Ax} = \mathbf{b}$$