Question1 (5 points)

(a) (1 point) Sketch the vector field

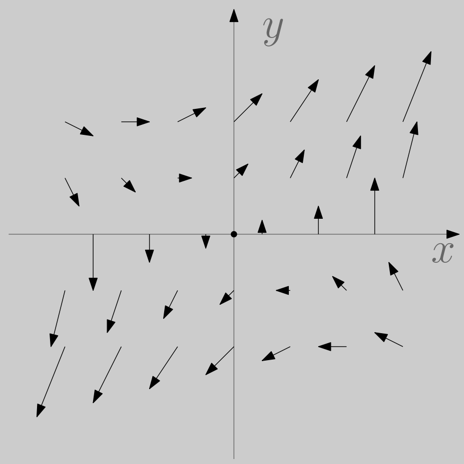
$$\mathbf{F}(x, y) = y\mathbf{e}_x + (x + y)\mathbf{e}_y$$

Solution:

1M Just like sketching $y = f(x)$, derivatives are useful.

$$\frac{\partial P}{\partial x} = 0; \quad \frac{\partial P}{\partial y} = 1; \quad \frac{\partial Q}{\partial x} = 1; \quad \frac{\partial Q}{\partial y} = 1$$

At any point (x, y), as x increases while holding y constant, the P component will not change, however, the Q component will increase linearly. When x is hold constant and y increases, both P and Q components increase linearly.



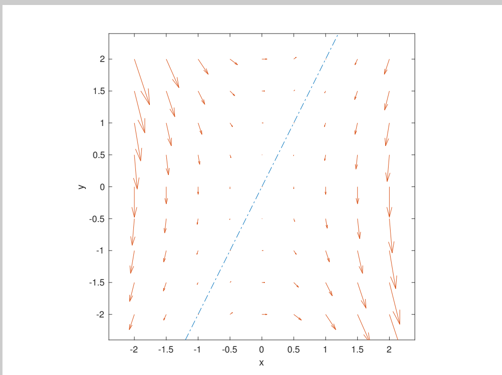
(b) (1 point) Plot the vector field in Matlab

$$\mathbf{F}(x, y) = (y^2 - 2xy)\mathbf{e}_x + (3xy - 6x^2)\mathbf{e}_y$$

And explain the appearance by finding the set of points (x, y) such that $\mathbf{F}(x, y) = \mathbf{0}$.

Solution:

1M Both components P and Q are zero on the line $y = 2x$,

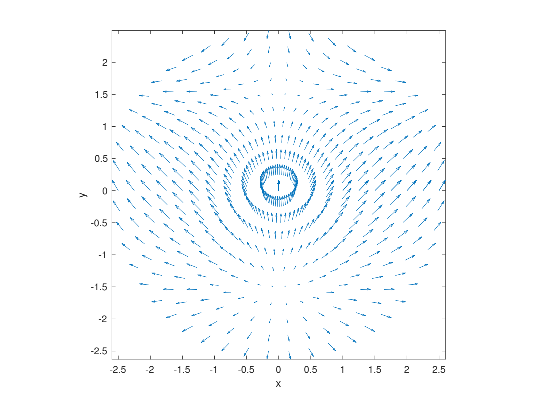


(c) (1 point) Plot the vector field using polar coordinates in Matlab for $0 \leq r \leq 2.5$ and $0 \leq \theta \leq 2\pi$, make sure you have representative non-intersecting vectors, the vectors need not be drawn to scale, but they should be in correct proportion relative to each other to make the the graph informative.

$$\mathbf{F} = (\sin x)\mathbf{e}_x + (\cos y)\mathbf{e}_y$$

Solution:

1M



(d) (1 point) Find the gradient field of

$$f(x, y, z) = z^3\sqrt{x^2 + y^2}$$

Solution:

1M In cylindrical form, we have

$$f(x, y, z) = z^3\sqrt{x^2 + y^2} = z^3 r$$

thus the gradient is given by

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z = z^3\mathbf{e}_r + 3z^2 r\mathbf{e}_z$$

(e) (1 point) A particle moves in a velocity field

$$\mathbf{V}(x, y) = x^2\mathbf{e}_x + (x + y^2)\mathbf{e}_y$$

If it is at position $(2, 1)$ at time $t = 3$, estimate its location at time $t = 3.001$.

Solution:

1M It is a velocity field, if we use linear approximation, the direction is given by

$$\mathbf{V}(2, 1) = 4\mathbf{e}_x + 3\mathbf{e}_y$$

the position of the particle is roughly

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0.001\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2.004 \\ 1.003 \end{bmatrix} \implies (2.004, 1.003)$$

Question2 (5 points)

(a) (1 point) Evaluate $\int_{\mathcal{C}} z + y^2\, ds$, where $\mathcal{C}$ is the line segment from $(3,4,0)$ to $(1,4,2)$.

Solution:

1M Find a parametrisation for the line segment

$$\mathbf{r}(t) = (1-t)\begin{bmatrix}3\\4\\0\end{bmatrix} + t\begin{bmatrix}1\\4\\2\end{bmatrix} = \begin{bmatrix}3\\4\\0\end{bmatrix} + t\begin{bmatrix}-2\\0\\2\end{bmatrix} = (3-2t)\mathbf{e}_x + 4\mathbf{e}_y + 2t\mathbf{e}_z$$

The line integral can be converted into a definite integral

$$\int_{\mathcal{C}} z + y^2\, ds = \int_0^1 \left(z(t) + y(t)^2\right)\left|\mathbf{r}'\right|\, dt = \int_0^1 (2t+16)\sqrt{4+4}\, dt = 34\sqrt{2}$$

(b) (1 point) Evaluate $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$, where $\mathcal{C}$ is the curve $y = e^x$ from $(2, e^2)$ to $(0,1)$ and

$$\mathbf{F} = x^2\mathbf{e}_x - y\mathbf{e}_y$$

Solution:

1M We can use the following parametrisation for the curve,

$$\mathbf{r}(t) = t\mathbf{e}_x + e^t\mathbf{e}_y$$

The line integral can be converted into a definite integral,

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \int_2^0 \mathbf{F}\cdot\mathbf{r}'\, dt = \int_2^0 \left(t^2 - e^t e^t\right)\, dt = \frac{e^4}{2} - \frac{19}{6}$$

(c) (1 point) Find the work done by the force field

$$\mathbf{F} = xy\mathbf{e}_x + yz\mathbf{e}_y + xz\mathbf{e}_z,$$

on a particle that moves along the curve $\mathcal{C}$.

$$\mathbf{r}(t) = t\mathbf{e}_x + t^2\mathbf{e}_y + t^3\mathbf{e}_z \qquad 0 \le t \le 1$$

Solution:

1M The work done by the force field is given by the line integral

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\, ds = \int_0^1 \mathbf{F}\cdot\mathbf{r}'\, dt &= \int_0^1 \left(t^3\mathbf{e}_x + t^5\mathbf{e}_y + t^4\mathbf{e}_z\right)\cdot\left(\mathbf{e}_x + 2t\mathbf{e}_y + 3t^2\mathbf{e}_z\right)\, dt\\ &= \int_0^1 \left(t^3 + 5t^6\right)\, dt = \frac{27}{28}\end{aligned}$$

(d) (1 point) Suppose that $\mathbf{F}$ represents the velocity field of a fluid flowing through a region in space. If $\mathbf{r}(t)$ parametrizes a smooth curve $\mathcal{C}$ in the domain of the continuous velocity field $\mathbf{F}$, the flow along the curve from $A = \mathbf{r}(a)$ to $B = \mathbf{r}(b)$ is defined to be

$$\text{Flow} = \int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\, ds$$

The line integral in this case is known as a flow integral. If the curve starts and ends at the same point, so that $A = B$, the flow is called the circulation around the curve. Consider the velocity field

$$\mathbf{F} = xy\mathbf{e}_x + y\mathbf{e}_y - yz\mathbf{e}_z$$

Find the flow from $(0,0,0)$ to $(1,1,1)$ along the curve of intersection of

$$y = x^2 \text{ and } z = x.$$

Solution:

1M This line integral takes the same form as the work integral

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\,ds &= \int_{\mathcal{C}} xy\,dx + \int_{\mathcal{C}} y\,dy - \int_{\mathcal{C}} yz\,dz \\ &= \int_0^1 x^3\,dx + \int_0^1 x^2\frac{dy}{dx}\,dx - \int_0^1 x^3\frac{dz}{dx}\,dx \\ &= \int_0^1 \left(x^3 + 2x^3 - x^3\right)\,dx = \frac{1}{2}\end{aligned}$$

(e) (1 point) A curve in the xy-plane is simple if it does not cross itself. When a curve starts and ends at the same point, it is a closed curve. If $\mathcal{C}$ is a smooth simple closed curve in the domain of a continuous vector field $\mathbf{F}$ in the plane, and if $\mathbf{n}$ is the outward-pointing unit normal vector on $\mathcal{C}$, the Flux of $\mathbf{F}$ across $\mathcal{C}$ is defined to be

$$\text{Flux of } \mathbf{F} \text{ across } \mathcal{C} = \int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{n}\,ds$$

The flux is the rate of a fluid leaving or entering the region defined by $\mathcal{C}$. Flux is Latin for flow, but many flux calculations involve no motion at all. If $\mathbf{F}$ were an electric field or a magnetic field, for instance, the integral of $\mathbf{F}\cdot\mathbf{n}$ would still be called the flux of the field across $\mathcal{C}$. Notice the difference between flux and circulation. Flux is the integral of the normal component of $\mathbf{F}$; circulation is the integral of the tangential component of $\mathbf{F}$. Consider the velocity field

$$\mathbf{F} = -y\mathbf{e}_x + x\mathbf{e}_y \quad \text{and the circle} \quad \mathbf{r}(t) = (\cos t)\mathbf{e}_x + (\sin t)\mathbf{e}_y, \quad 0 \le t \le 2\pi$$

Find the flux of the field across the circle.

Solution:

1M Since $\mathcal{C}$ is a circle, the unit normal vector for $\mathcal{C}$ is given by

$$\mathbf{n} = \mathbf{r}(t) = \cos t\mathbf{e}_x + \sin t\mathbf{e}_y$$

The line integral is given by

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi} \mathbf{F}\cdot\mathbf{r}\left|\mathbf{r}'\right|\,dt &= \int_0^{2\pi} \left(-\sin t\mathbf{e}_x + \cos t\mathbf{e}_y\right)\cdot\left(\cos t\mathbf{e}_x + \sin t\mathbf{e}_y\right)\,dt \\ &= \int_0^{2\pi} 0\,dt = 0\end{aligned}$$

Question3 (5 points)

(a) (1 point) Evaluate the line integral

$$\int_{\mathcal{C}} \sin y \, dx + \int_{\mathcal{C}} (x \cos y - \sin y) \, dy,$$

where $\mathcal{C}$ is a smooth curve from $(2, 0)$ to $(1, \pi)$.

Solution:

1M The line integral corresponds to the following vector field

$$\mathbf{F} = \sin y \mathbf{e}_x + (x \cos y - \sin y) \, \mathbf{e}_y$$

If we integrate $\sin y$ partially with respect to x, we have

$$f(x, y) = \int \sin y \, dx = x \sin y + g(y)$$

differentiate with respect to y, we have

$$\frac{\partial f}{\partial y} = x \cos y + g'(y)$$

by equating with the second component, we conclude

$$g' = -\sin y$$

which implies the vector field is conservative in $\mathbb{R}^2$, and the potential function

$$f(x, y) = x \sin y + \cos y + c$$

thus by the FTL the line integral is equal to

$$\int_{\mathcal{C}} \sin y \, dx + \int_{\mathcal{C}} (x \cos y - \sin y) \, dy, = f(1, \pi) - f(2, 0) = -2$$

(b) (1 point) Show that the following vector field $\mathbf{Q}$ is conservative in any open region $\mathcal{E}$

$$\mathbf{Q} = \begin{bmatrix} 3x^2(y+z) + y^3 + z^3 \\ 3y^2(z+x) + z^3 + x^3 \\ 3z^2(x+y) + x^3 + y^3 \end{bmatrix}$$

Solution:

1M The partial derivatives of the component functions are equal in any open region,

$$\frac{\partial P}{\partial y} = 3x^2 + 3y^2 = \frac{\partial Q}{\partial x}; \quad \frac{\partial P}{\partial z} = 3x^2 + 3z^2 = \frac{\partial R}{\partial x}; \quad \frac{\partial R}{\partial y} = 3z^2 + 3y^2 = \frac{\partial Q}{\partial z}$$

and the vector field is defined and the partial derivatives are continuous in any open region, so the vector field is conservative by the conservative field test.

(c) (1 point) Construct the potential function for the vector field $\mathbf{Q}$ above.

Solution:

1M This is similar to the case in $\mathbb{R}^2$,

$$\begin{aligned} f(x,y,z) &= \int \left(3x^2(y+z)+y^3+z^3\right)\,dx \\ &= x^3(y+z)+x(y^3+z^3)+g(y,z) \\ \implies f_y(x,y,z) &= x^3+3xy^2+g_y(y,z) \\ \implies f_z(x,y,z) &= x^3+3xz^2+g_z(y,z) \end{aligned}$$

which implies

$$\begin{aligned} 3y^2z+z^3 &= g_y \\ \int \left(3y^2z+z^3\right)\,dy &= g(y,z) \\ y^3z+z^3y+h(z) &= g(y,z) \end{aligned} \qquad\qquad \begin{aligned} 3yz^2+y^3 &= g_z \\ \int \left(3yz^2+y^3\right)\,dz &= g(y,z) \\ yz^3+y^3z+w(y) &= g(y,z) \end{aligned}$$

to avoid a contradiction,

$$h(z)=w(y)=C$$

Therefore, the potential function must take the following form

$$f(x,y,z)=x^3(y+z)+x(y^3+z^3)+y^3z+yz^3+C$$

(d) (1 point) Evaluate the following integral

$$\int_{\mathcal{C}} \mathbf{Q}\cdot d\mathbf{r}$$

where the vector field $\mathbf{Q}$ is defined above and $\mathcal{C}$ is a smooth curve from

$$(1,-1,1) \text{ to } (2,1,2).$$

Solution:

1M We simply use the above potential function by the FTL

$$\int_{\mathcal{C}} \mathbf{Q}\cdot d\mathbf{r} = f(2,1,2)-f(1,-1,1)=54$$

(e) (1 point) Let $\mathbf{r}=x\mathbf{e}_x+y\mathbf{e}_y+z\mathbf{e}_z$ and the following vector to be a constant vector

$$\mathbf{a}=a_1\mathbf{e}_x+a_2\mathbf{e}_y+a_3\mathbf{e}_z$$

Let $\mathcal{C}$ be a curve from the origin to the point with the position vector

$$\mathbf{r}_0=x_0\mathbf{e}_x+y_0\mathbf{e}_y+z_0\mathbf{e}_z, \qquad \text{where} \quad |\mathbf{r}_0|=10$$

what is the maximum possible value of

$$\int_{\mathcal{C}} \nabla(\mathbf{r}\cdot\mathbf{a})\cdot d\mathbf{r}$$

Solution:

1M Since it is a line integral of a gradient field, it is equal to

$$\int_{\mathcal{C}} \nabla(\mathbf{r}\cdot\mathbf{a})\cdot d\mathbf{r} = \mathbf{r}\cdot\mathbf{a}\Big|_{(x_0,y_0,z_0)} - 0 = \mathbf{r}_0\cdot\mathbf{a}$$

the maximum of a dot occurs when the two vectors are in the same direction,

$$\int_{\mathcal{C}} \nabla(\mathbf{r}\cdot\mathbf{a})\cdot d\mathbf{r} = |\mathbf{r}_0|\,|\mathbf{a}| = 10\sqrt{a_1^2+a_2^2+a_3^2}$$