MID2 REVIEW L7-11

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L7 Function of Two Variables

- A function of two variables, denoted by $f(x, y)$, is a rule that assigns each point (x, y) in some $D \subset R^2$ to a unique number in R.
- If f is a function of two variables with domain D, then the graph of f is the set of all points in such that $z = f(x, y)$ and (x,y) is in D.

L7 Level Curve

- The level curves or contour lines of a function $f(x, y)$ are the curves defined by $f(x, y) = k$, where k is a constant in the range of f.

L7 Different kind of curves

- elliptic paraboloid
  - $F(x, y) = 4x^2 + y^2$
- hyperbolic paraboloid
  - $F(x, y) = y^2 - x^2$

Typical Question L7

- Sketch the domain of $f(x, y)$
  - Figure out the domain: $\ln(\mathrm{x}), \frac{1}{x}, x!$
  - Notice sketch detail: open&close, special points, asymptotic lines
- Give you a surface, write down its equation
  - Make sure that $F(x, y) \Longleftrightarrow$ Surface

L8 Limit along a curve

Suppose C is a smooth curve that is defined by

$\mathbf{r}(\mathrm{t}) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and $\mathbf{r}(t_0) = \begin{bmatrix} x(t_0) \\ y(t_0) \end{bmatrix}$

then the limit of f(x,y) along a curve C at (x_0, y_0) is defined to be

$$\lim_{\substack{(x,y)\to(x_0,y_0) \\ along C}} f(x,y) = \lim_{t\to t_0} f\big(x(t), y(t)\big) = \lim_{t\to t_0} f(t)$$

Transform limit of function of two variables to limit of function of one variable, which you have learned!

L8 Limit of functions in R^2

- Suppose $f(x, y)$ is a function of two variables defined at all points of some open disk centred at (a, b), except possibly at (a, b). Then the value of L is the limit of f as (x, y) approaches (a, b) and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ } if \text{ } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

This means that along any smooth curve!

L8 Continuity of function in R^2

- A function f(x,y) is said to be continuous at (a,b) if f(a,b) is defined and if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

In addition, if f is continuous at every point in an open set D, then we say that f is continuous on D, and if f is continuous at every point in the xy-plane, then we say that f is continuous everywhere.

Typical Questions L8

- Evaluae $\lim_{(x,y)\to(a,b)} f(x,y)$ if exists, if not, explain why the limit doesn't exist.
  - 1. If $f(x,y)$ is continuous at (a,b) (obviously stuff, such as polynomials, rational, trigonometric, logarithm or composition of above functions), $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$
  - 2. Try squeeze theorem if there exists $\frac{x^2}{x^2+y^2} g(x,y), \sin(x)\, g(x,y)$
  - 3. If 1 is not satisfied, use original definition! $\varepsilon - \delta$ definition
  - 4. If you cannot prove the limit or you think the limit doesn't exist when you see this equation, use limit laws to prove by contradiction! Refer to L8 P11

Typical Questions L8

- Evaluate whether $f(x, y)$ is continuous at (a,b)
  - Just check $\lim_{(x,y)\to(a,b)} f(x, y)\ and\ f(a, b)$

L9 Partial Derivatives

- The partial derivatives of $f(x, y)$ with respect to x at the point (x_0, y_0) is

$$\frac{\partial f}{\partial x}|_{(x_0,y_0)} = \lim_{h \longrightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

provided the limit exist.

$$\frac{\partial f}{\partial x}|_{(x_0,y_0)} = \frac{df(x,y_0)}{dx}|_{x=x_0}$$

With respect to y: $\frac{\partial f}{\partial y}|_{(x_0,y_0)} = \lim_{h \longrightarrow 0} \frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}$

Implicit differentiation works the same way! For example, $x^2 + cosy + z^3 = 1$

L9 Mixed Derivative

- If $f(x, y)$ and its partial derivatives $fx, fy, fxy\ , fyx$ are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$fxy(a, b) = fyx(a, b)$$

We usually use this theorem when $f(x, y)$ obviously satisfies fx, fy, fxy, fyx are continuous at(a,b), so this theorem can help us simplify the calculation.

For example, the problem on L9 P9

Typical Questions L9

- Show whether partial derivatives of $f(x, y)$ exist (a,b)
  - Use definition $\frac{\partial f}{\partial x}|_{(x_0,y_0)} = \lim_{h \longrightarrow 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$

If the question just asks to evaluate the partial derivatives of $f(x, y)$ at (a,b)

1. If the partial derivatives are continuous (obviously), evaluate

$$\frac{\partial f}{\partial x}|_{(x_0,y_0)}=\frac{df(x,y_0)}{dx}|_{x=x_0}$$

2. Else, use definition

$$\frac{\partial f}{\partial x}|_{(x_0,y_0)} = \lim_{h \longrightarrow 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$$

L10 Linear Approximation

- The approximation
$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$
is known as the linear approximation or tangent line approximation of f at x_0

From approximation we can understand the 'new' definition of differentiability (both one variable and two variables)

L10 Differentiability

- Suppose $f(x)$ is defined for $a \leq x \leq b$, then $f(x)$ is differentiable at $x \in (a, b)$ if and only if there exists a constant m and a function $\varepsilon(h)$ such that

$$f(x+h) = f(x) + mh + \varepsilon(h), where \lim_{h \longrightarrow 0} \frac{\varepsilon(h)}{h} = 0$$

And $m = f'(x)$

Refer to proof on L10 P4

L10 Differentiability

- Suppose $f(x, y)$ is defined for $D \subset R2$, then $z = f(x, y)$ is differentiable at (x0 , y0) ∈ D if and only if there exist constants m and n, and ε(Δx, Δy) such that

$$\Delta z = m\Delta x + n\Delta y + \varepsilon(\Delta x, \Delta y)$$

where

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\varepsilon(\Delta x, \Delta y)}{\sqrt{\Delta x, \Delta y}} = 0$$

- $m = fx(x, y), n = fy(x, y)$

  Refer to L10 P4-6 for proof (at least get the big picture)

L10 Condition for Differentiability

- If the partial derivatives, fx and fy, of a function $f(x,y)$ exist and are continuous in some open region R, then $f(x,y)$ is differentiable in R.
- fx, fy continuous $\Longrightarrow$ $f(x,y)$ differentiable

$$\not\Longleftarrow$$

L10 Total Differential

- Suppose the function $z = f(x, y)$ is differentiable, then

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

We can use this equation to have a good approximation of Δz

Refer to L10 P12 for explanation.

Typical Questions L10

- Evaluate whether $f(x, y)$ is differentiable at (a,b).
  - 1. If fx, fy are continuous at (a,b), then f(x,y) is differentiable at (a,b)
  - 2. Else, use definition to prove that $\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\varepsilon(\Delta x,\Delta y)}{\sqrt{\Delta x,\Delta y}} = 0$

A loooooooooot of T/F questions

Solution: Life Unexpected

L11 Differentiable functions

Let $z = f(x, y)$ be a differentiable functions of x and y, $x = x(t), y = y(t)$ are differentiable functions of t, then z is a differentiable functions of t, and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Refer to proof on L11 P3-4 (at least get a big picture)
What will happen if x is the only independent variable?
I.e., $y = g(x)$ is a differentiable function

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

L11 General Chain Rules

- Given $f(x,y,z), x(r,s), y(r,s)\ and\ z(r,s)$ are differentiable functions, then the partial derivatives of f with respect to r and s exist, and

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$
$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

- Whenever you encounter any functions, figure out the independent variables and dependent variables!

L11 Leibniz' Integral Rule

- A very import theorem:

$$\frac{d}{dx}\int_a^b h(t,x)dt = \int_a^b \frac{\partial}{\partial x}h(t,x)dt$$

- This theorem tells us the arithmetic rule of integration and derivative of functions of several variables.
- And we can have the Leibniz Integral Rule(Recite it)

$$F'(x) = h(g(x),x)g'(x) - h(f(x),x)f'(x) + \int_{f(x)}^{g(x)} h_x(t,x)dt$$

if

$$F(x) = \int_{f(x)}^{g(x)} h(t,x)dt$$

L11 Implicit Function Theorem

- Suppose that F(x,y) is differentiable and that the equation F(x,y)=0 defines y as a differentiable function of x, then at any point where $F_y \neq 0$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

In most cases, you can just use this theorem when calculating the derivative of a implicit function, but you should be care that $F_y \neq 0$

L11 Implicit Function Theorem

- Suppose F is a function of x, y and z, if the following conditions are satisfied

1. The partial derivatives F_x, F_y, and F_z are continuous throughout an open region R in space containing the point (x_0, y_0, z_0).
2. For some constant c, $F(x_0, y_0, z_0) = c$ and $F_z(x_0, y_0, z_0) \neq 0$, then

$$F(x, y, z) = c$$

defines z implicitly as a differentiable function of x and y near (x_0, y_0, z_0), and the partial derivatives of z are given by

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Typical Questions L11

- Evaluate Leibniz' integral
- Evaluate derivatives of implicit functions

$\exists f_x, f_y$

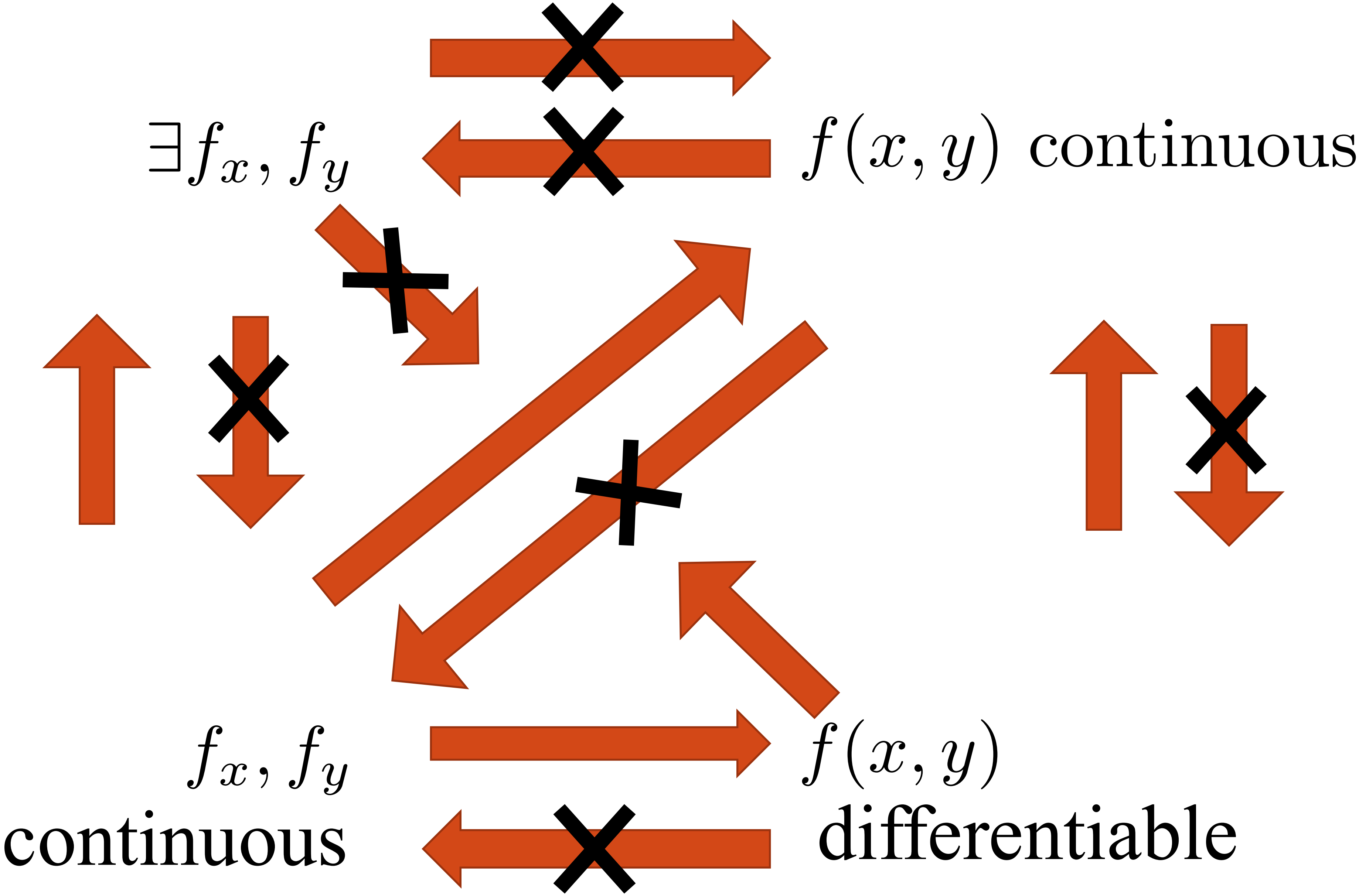
$f(x, y)$ continuous

f_x, f_y

continuous

$f(x, y)$

differentiable


$\exists f_x, f_y$
$f(x, y)$ continuous
f_x, f_y continuous
$f(x, y)$ differentiable

Some Questions

1. Let $w = f(x+y+z, xyz)$, and f has second order continuous partial derivatives, find $\frac{\partial w}{\partial x}$ and $\frac{\partial^2 w}{\partial x \partial z}$

2. We all know that the gravity can be measured by simple pendulum motion, and $g = \frac{4\pi^2 l}{T^2}$. We know that $l = 100 \pm 0.1\text{cm}$, $T = 2 \pm 0.004\text{cm}$. Calculate the uncertainty of g.

3. $x^2 + y^2 + z^2 - 4z = 0$, please calculate $\frac{\partial^2 z}{\partial x^2}$

4. $PV = RT$(R is const),please calculate $\frac{\partial p}{\partial V} \cdot \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial p}$

5. Please prove that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0$