Vv255 Lecture 6

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May 27, 2017

- Recall for a smooth curve $y = f(x)$ on the interval $[a, b]$, then

the arc length or simply length L

of the curve from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \underbrace{\int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}_{\text{If it has a parametrization in terms of } t}.$$

- Suppose $\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = x(t)\mathbf{e}_x + y(t)\mathbf{e}_y$, then $\dfrac{d\mathbf{r}}{dt} = \dfrac{dx}{dt}\mathbf{e}_x + \dfrac{dy}{dt}\mathbf{e}_y$ and

$$\left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \implies L = \int_\alpha^\beta \left|\frac{d\mathbf{r}}{dt}\right| dt$$

- The last formula defines the distance along a smooth plane curve.

Q: How to define and thus calculate the distance along a smooth space curve?

Definition

The arc length or simply length of a smooth curve defined by a function

$$\mathbf{r}(t) \qquad \text{for} \qquad \alpha \le t \le \beta,$$

which is traced exactly once as t increases from $t = \alpha$ to $t = \beta$, is given by

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_{\alpha}^{\beta} \left|\frac{d\mathbf{r}}{dt}\right| dt$$

- Recall if $\mathbf{r}(t)$ is the position vector of a honeybee at time t, then

$$\mathbf{v} = \dot{\mathbf{r}}$$

  is the velocity vector, and $|\mathbf{v}|$ gives the speed.

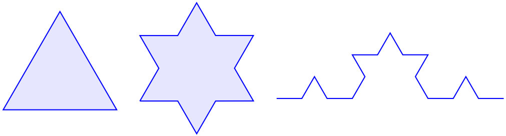
- With this interpretation, the length formula is nothing more than the familiar result that distance travelled is the integral of speed.

$$L = \int_{\alpha}^{\beta} |\dot{\mathbf{r}}|\, dt = \int_{\alpha}^{\beta} |\mathbf{v}|\, dt$$

Q: Why do we need to restrict ourselves to consider only smooth curves?

- Recall the following construction:

1. Take an equilateral triangle,



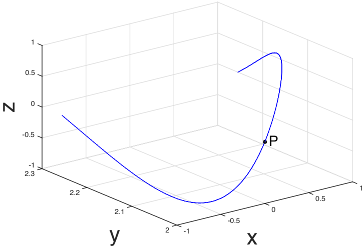
2. Divide each side into three segments of equal length.
3. Create new equilateral triangles that have the middle segment from step 1 as its base and points outward.
4. Remove the line segments that are the bases of the new triangles from step 2 continue the above three steps indefinitely for all sides.

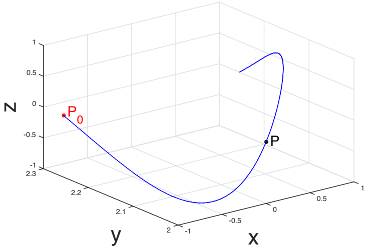
Q: What is the length of the resulting curve as the number of iterations $\to \infty$?

- Giving up smoothness and mixing with the concept of infinity can lead us to very strange objects, we want to stay away from them in this course.

Q: How can we specify a point P on a smooth curve $\mathcal{C}$ defined by $\mathbf{r}(t)$?

$$\mathbf{r}(t) = t\mathbf{e}_x + \left(e^{t/2} + e^{-t/2}\right)\mathbf{e}_y + \sin(\pi t)\mathbf{e}_z \qquad \mathbf{r}(t) = t\mathbf{e}_x + \left(e^{t/2} + e^{-t/2}\right)\mathbf{e}_y + \sin(\pi t)\mathbf{e}_z$$





- If we pick a reference point on a smooth curve $\mathcal{C}$ parametrized by t, e.g.

$$P_0 = (x(t_0), y(t_0), z(t_0))$$

- Each value of t determines a second point $P = (x(t), y(t), z(t))$ on $\mathcal{C}$, then

$$s(t) = L = \int_{t_0}^{t} \left|\frac{d\mathbf{r}}{d\tau}\right| d\tau,$$ which is known as the arc length function,

measures the "directed distance" along $\mathcal{C}$ from the reference point P_0.

- Each s value is a "directed distance" from P_0, and specifies a point P on $\mathcal{C}$.
- We call s an arc length parameter for the curve, and defining the curve using s as the parameter is known as the arc length parametrization of the curve.

$$\mathbf{r}(s)$$

- We will see that the arc length parameter is particularly effective for examining the turning and twisting nature of a space curve.

Exercise

Find the arc length function for $\mathbf{r}(t) = 2t\mathbf{e}_x + 3\sin 2t\mathbf{e}_y + 3\cos 2t\mathbf{e}_z$, *and using*

$$(0, 0, 3)$$

as the reference point to find the arc length parametrization.

Q: Why the arc length parametrization is difficult to find for a given curve?

- Fortunately, however, we rarely need an exact formula $s(t)$ or its inverse $t(s)$.
- The derivatives beneath the radical are continuous (the curve is smooth), so

$$\frac{ds}{dt} = \frac{d}{dt}\int_{t_0}^{t} \left|\frac{d\mathbf{r}}{d\tau}\right| d\tau = \left|\frac{d\mathbf{r}}{d\tau}\right|$$

by the Fundamental Theorem of Calculus.

- The reference point P_0 plays a role in defining s, but it plays no role in $\dfrac{ds}{dt}$.
- Consider the honeybee again, that is, $\mathbf{r}(t)$ is the position vector at time t, then the above statement is merely stating the distance s depends on the reference point P_0, but the speed $|\mathbf{v}|$ is independent of the choice of P_0.

$$\frac{ds}{dt} = \left|\frac{d\mathbf{r}}{dt}\right| = |\mathbf{v}(t)|$$

- Note that $\dfrac{ds}{dt} = \left|\dfrac{d\mathbf{r}}{dt}\right| > 0$ since magnitude is never negative and $\mathbf{r}$ is smooth.

Hence s is a strictly increasing function of t.

Q: The velocity vector $\mathbf{v}$ is the change in the position vector $\mathbf{r}$ with respect to time t, but how does the position vector change with respect to arc length?

Q: Specifically, what does the derivative $\dfrac{d\mathbf{r}}{ds}$ represent?

Q: Why $s(t)$ is one-to-one and has an inverse that is a differentiable function?

- The derivative of the inverse is

$$\frac{dt}{ds} = \frac{1}{\frac{ds}{dt}} = \frac{1}{|\mathbf{v}|} \qquad \text{where} \quad \mathbf{v} \neq \mathbf{0}$$

- This makes $\mathbf{r}$ a differentiable function of s, using the Chain Rule, we have

$$\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

Q: What does the last formula mean?

- So $\dfrac{d\mathbf{r}}{ds}$ is the unit tangent vector in the direction of the velocity vector.

Definition

Suppose $\mathbf{r}(s)$ is a smooth curve parametrized by arc length, then the unit tangent vector, denoted by $\mathbf{T}$, is defined to be

$$\mathbf{T} = \frac{d\mathbf{r}}{ds}$$

And the curvature of the curve at a point P, denoted by κ, is defined to be

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right| \qquad \text{at the point } P.$$

Q: What does the curvature of a smooth curve $\mathcal{C}$ at point tell us? Why? How?

- Consider our honeybee again, with the position vector $\mathbf{r}(t)$ at time t, then

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

is the normalized velocity vector, which means its magnitude remains at 1.

- Only the direction of $\mathbf{T}$ might change as the honeybee traces along $\mathcal{C}$.
- Hence the rate of change of $\mathbf{T}$ tells us how much the particle is turning with respect to the parameter, here we have the arc length parameter s,

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$$

- Therefore κ measures the rate at which $\mathbf{T}$ turns per unit of length along $\mathcal{C}$.
- If κ is large, $\mathbf{T}$ turns sharply, otherwise $\mathbf{T}$ turns slowly.
- Of course, $\mathbf{T}$ can be in terms of other parameters as well, for example, time.

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds} = \frac{\frac{d\mathbf{r}}{dt}}{\frac{ds}{dt}} = \frac{\frac{d\mathbf{r}}{dt}}{\left|\frac{d\mathbf{r}}{dt}\right|}$$

in this case, it still is the unit tangent vector but it is in terms of time.

Q: Is there any difference between $\mathbf{T}(s)$ and $\mathbf{T}(t)$? How about $\mathbf{T}'(s)$ and $\mathbf{T}'(t)$?

- If a smooth curve $\mathbf{r}(t)$ is already given in terms of some parameter t other than the arc length parameter s, the curvature is given by the Chain rule

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}}{dt}\frac{dt}{ds}\right| = \frac{1}{\left|\frac{ds}{dt}\right|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\left|\frac{d\mathbf{r}}{dt}\right|}\left|\frac{d\mathbf{T}}{dt}\right|$$

Formula for Calculating Curvature

If $\mathbf{r}(t)$ is a smooth curve, then the curvature is

$$\kappa = \frac{1}{\left|\frac{d\mathbf{r}}{dt}\right|}\left|\frac{d\mathbf{T}}{dt}\right|, \quad \text{where } \mathbf{T} = \frac{\frac{d\mathbf{r}}{dt}}{\left|\frac{d\mathbf{r}}{dt}\right|} \text{ is the unit tangent vector.}$$

Exercise

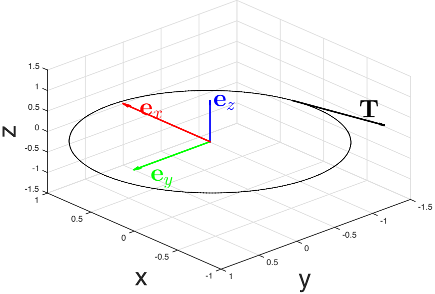
(a) *Find the curvature of a straight line.*

(b) *Find the curvature of a circle.*

- There are infinitely many vectors orthogonal to a given unit tangent vector $\mathbf{T}$

the magnitude and the direction

$$\mathbf{r}(t) = \frac{3}{4}\cos(\pi t)\mathbf{e}_x + \frac{4}{4}\cos(\pi t)\mathbf{e}_y + \frac{5}{4}\sin(\pi t)\mathbf{e}_z$$



- There is a particularly important vector among all unit vectors orthogonal to $\mathbf{T}$ because it points in the direction in which the curve is turning, denoted by

$$\mathbf{N}$$

Q: How to find this unit vector $\mathbf{N}$ that is orthogonal to $\mathbf{T}$?

Q: Why $\dfrac{d\mathbf{T}}{ds}$ is always orthogonal to $\mathbf{T}$?

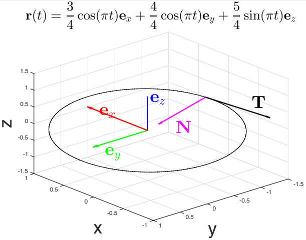
- Dividing $\dfrac{d\mathbf{T}}{ds}$ by its length κ gives us the unit vector orthogonal to $\mathbf{T}$.

Definition

At a point where $\kappa \neq 0$, the principal unit normal vector, $\mathbf{N}$, for a smooth curve is

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.$$

$\mathbf{N}$ points towards the concave side of the curve.



- If a smooth curve $\mathbf{r}(t)$ is already given in terms of some parameter t other than the arc length parameter s, then $\mathbf{N}$ can be found using the Chain rule

$$\mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{\left|\frac{d\mathbf{T}}{ds}\right|} = \frac{\frac{d\mathbf{T}}{dt}\frac{dt}{ds}}{\left|\frac{d\mathbf{T}}{dt}\frac{dt}{ds}\right|} = \frac{\frac{d\mathbf{T}}{dt}}{\left|\frac{d\mathbf{T}}{dt}\right|}, \qquad \text{since } \frac{dt}{ds} = \frac{1}{ds/dt} > 0.$$

Formula for Calculating $\mathbf{N}$

If $\mathbf{r}(t)$ is a smooth curve, then the principal unit normal is

$$\mathbf{N} = \frac{\frac{d\mathbf{T}}{dt}}{\left|\frac{d\mathbf{T}}{dt}\right|}, \quad \text{where } \mathbf{T} = \frac{\frac{d\mathbf{r}}{dt}}{\left|\frac{d\mathbf{r}}{dt}\right|} \text{ is the unit tangent vector.}$$

- The above works with any parameter, including the arc length parameter s.

Q: Does $\ddot{\mathbf{r}}$ share the same direction as $\mathbf{N}$?

- If you are Tony Stark in his suit, the xyz-coordinate system for representing the vectors and describing your motion is not truly relevant to you.

- What is meaningful are the vectors representative of

1. The forward direction $\mathbf{T}$.
2. The extent to which the path is turning $\mathbf{N}$.
3. The tendency of the motion to "twist" out of the plane defined by $\mathbf{T}$ and $\mathbf{N}$

Definition

Suppose $\mathbf{T}$ and $\mathbf{N}$ are the unit tangent vector and the principal unit normal for a smooth curve respectively, then the unit vector $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ is called binormal.

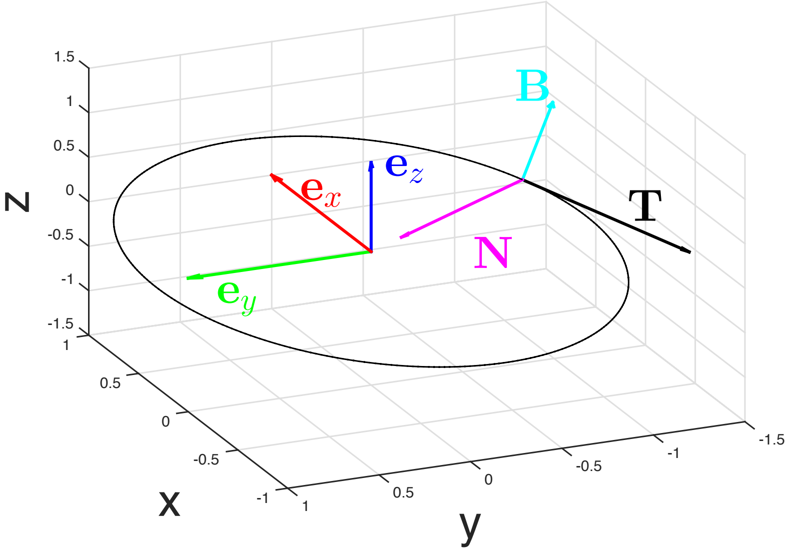
- $\mathbf{B}$ is a unit vector and is orthogonal to both $\mathbf{T}$ and $\mathbf{N}$ by construction.

Q: Why is $\mathbf{B}$ always a unit vector ?

$$|\mathbf{T} \times \mathbf{N}| = |\mathbf{T}||\mathbf{N}||\sin\theta|, \qquad \text{where } \theta \text{ is the angle between } \mathbf{T} \text{ and } \mathbf{N}.$$

- Expressing the acceleration vector along the curve as a linear combination of this $\mathbf{TNB}$ frame of mutually orthogonal unit vectors travelling with the motion is particularly revealing of the nature of the path and motion along it.

$$\mathbf{r}(t) = \frac{3}{4}\cos(\pi t)\mathbf{e}_x + \frac{4}{4}\cos(\pi t)\mathbf{e}_y + \frac{5}{4}\sin(\pi t)\mathbf{e}_z$$



- When an object is accelerated by external forces, we want to know how much of the acceleration acts in the direction of motion, that is, in the direction of

$$\mathbf{T}$$

- We can calculate this using the Chain Rule to rewrite the velocity vector $\mathbf{v}$ as

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{T}\frac{ds}{dt}$$

- Then we differentiate both ends of this string of equalities to get

$$\mathbf{a} = \frac{d}{dt}\left[\frac{ds}{dt}\mathbf{T}\right] = \frac{d^2s}{dt^2}\mathbf{T} + \left[\frac{ds}{dt}\right]\frac{d}{dt}\mathbf{T} = \frac{d^2s}{dt^2}\mathbf{T} + \left[\frac{ds}{dt}\right]\frac{d\mathbf{T}}{ds}\frac{ds}{dt}$$

- By definition $\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$ and $\mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{\left|\frac{d\mathbf{T}}{ds}\right|}$, we have

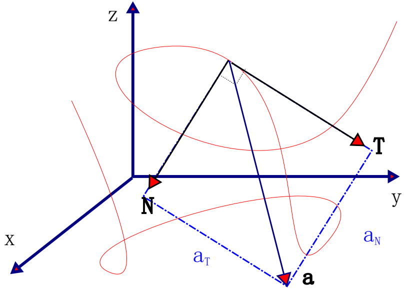
$$\mathbf{r}'' = \mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)^2\left[\frac{\kappa}{\left|\frac{d\mathbf{T}}{ds}\right|}\frac{d\mathbf{T}}{ds}\right] = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}$$

Definition

An acceleration vector can be decomposed as the following

$$\mathbf{a} = a_{\mathrm{T}}\mathbf{T} + a_{\mathrm{N}}\mathbf{N}, \quad \text{where} \quad \begin{aligned} a_{\mathrm{T}} &= \frac{d^2 s}{dt^2} = \frac{d}{dt}|\mathbf{v}| \\ a_{\mathrm{N}} &= \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\mathbf{v}|^2 \end{aligned}, \quad \text{which are known as}$$

the tangential and the normal scalar components of $\mathbf{a}$, respectively.



Q: Do you expect a third component in the decomposition of $\mathbf{a}$?

$$\mathbf{a} = a_{\mathrm{T}}\mathbf{T} + a_{\mathrm{N}}\mathbf{N} \cancel{+ a_{\mathrm{B}}\mathbf{B}}$$

- No matter how the path of the moving object we are watching may appear to twist in space, the acceleration $\mathbf{a}$ always is in the plane of $\mathbf{T}$ and $\mathbf{N}$.
- Acceleration $\mathbf{a}$ is the rate of change of velocity $\mathbf{v}$ with respect to time t.

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$

Q: What does each of the scalar components measure?

$$\mathbf{a} = a_{\mathrm{T}}\mathbf{T} + a_{\mathrm{N}}\mathbf{N} = \left(\tfrac{d}{dt}|\mathbf{v}|\right)\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}$$

- The tangential comp. measures the rate of change of the length of $\mathbf{v}$.
- The normal comp. measures the rate of change of the direction of $\mathbf{v}$.

- To obtain a formula for a_{N} without κ, consider

$$|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a} = a_{\mathrm{T}}^2 + a_{\mathrm{N}}^2$$

since the dot product is invariant under orthonormal change of basis.

Formula for calculating the normal component of acceleration

$$a_{\mathrm{N}} = \sqrt{|\mathbf{a}|^2 - {a_{\mathrm{T}}}^2}$$

- With this formula, we can find a_{N} without having to calculate κ first.
- Recall $\dfrac{d\mathbf{r}}{ds} = \mathbf{T}$, and $\dfrac{d\mathbf{T}}{ds}$ gives the direction of $\mathbf{N}$, and its magnitude gives κ.

Q: How do $\dfrac{d\mathbf{N}}{ds}$ and $\dfrac{d\mathbf{B}}{ds}$ behave in relation to $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$?

- From the rule for differentiating a cross product, we have

$$\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T} \times \mathbf{N}) = \frac{d\mathbf{T}}{ds} \times \mathbf{N} + \mathbf{T} \times \frac{d\mathbf{N}}{ds}$$

- Thus the rate of change of binormal with respect to arc length is

$$\frac{d\mathbf{B}}{ds} = \frac{d\mathbf{T}}{ds} \times \mathbf{N} + \mathbf{T} \times \frac{d\mathbf{N}}{ds}$$

- Since $\mathbf{N}$ is the direction of $\frac{d\mathbf{T}}{ds}$, $\quad \frac{d\mathbf{T}}{ds} \times \mathbf{N} = \mathbf{0}$ and

$$\frac{d\mathbf{B}}{ds} = \mathbf{0} + \mathbf{T} \times \frac{d\mathbf{N}}{ds} = \mathbf{T} \times \frac{d\mathbf{N}}{ds}$$

- From this we see that $d\mathbf{B}/ds$ is orthogonal to $\mathbf{T}$

Q: Is $d\mathbf{B}/ds$ orthogonal to $\mathbf{B}$?

- It follows that $\frac{d\mathbf{B}}{ds}$ is orthogonal to the plane of $\mathbf{B}$ and $\mathbf{T}$. In other words, $\frac{d\mathbf{B}}{ds}$ is parallel to $\mathbf{N}$, so $\frac{d\mathbf{B}}{ds}$ is scalar multiple of $\mathbf{N}$, that is, $\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$.
- The scalar τ is called the torsion along the curve.

$$\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\tau\mathbf{N} \cdot \mathbf{N} = -\tau(1) = -\tau$$

Definition

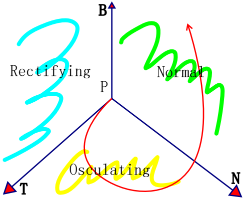
Let $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. The torsion function of a smooth curve is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$$

Q: What is the difference between curvature and torsion?

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| \implies \mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} \implies \kappa = \frac{d\mathbf{T}}{ds} \cdot \mathbf{N}$$

- The curvature can be thought of as the rate at which the normal plane turns as P moves along its path.
- Similarly, the torsion is the rate at which the osculating plane turns about as P moves along the curve.



- Intuitively, curvature measures the failure of a curve to be a straight line, while torsion measures the failure of a curve to be planar.

Computation Formulas for Curves in Space

Velocity vector:	$\mathbf{v} = \frac{d\mathbf{r}}{dt}$
Acceleration vector:	$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$
Jerk vector:	$\mathbf{j} = \frac{d\mathbf{a}}{dt} = \frac{d^2\mathbf{v}}{dt^2} = \frac{d^3\mathbf{r}}{dt^3}$
Unit tangent vector:	$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{\lvert\mathbf{v}\rvert}$
Curvature:	$\kappa = \left\lvert\frac{d\mathbf{T}}{ds}\right\rvert = \frac{1}{\lvert\mathbf{v}\rvert}\left\lvert\frac{d\mathbf{T}}{dt}\right\rvert = \frac{\lvert\mathbf{v}\times\mathbf{a}\rvert}{\lvert\mathbf{v}\rvert^3}$
Principal unit normal vector:	$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{\lvert d\mathbf{T}/dt\rvert}$
Binormal vector:	$\mathbf{B} = \mathbf{T}\times\mathbf{N}$
Torsion:	$\tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N} = \frac{\mathbf{j}\cdot(\mathbf{v}\times\mathbf{a})}{\lvert\mathbf{v}\times\mathbf{a}\rvert^2}$
	$\mathbf{a} = a_{\mathrm{T}}\mathbf{T} + a_{\mathrm{N}}\mathbf{N}$
Tangential component:	$a_{\mathrm{T}} = \frac{d}{dt}\lvert\mathbf{v}\rvert$
Normal component:	$a_{\mathrm{N}} = \kappa\lvert\mathbf{v}\rvert^2 = \sqrt{\lvert\mathbf{a}\rvert^2 - a_{\mathrm{T}}^2}$
Frenet–Serret theorem:	$\begin{bmatrix} d\mathbf{T}/ds \\ d\mathbf{N}/ds \\ d\mathbf{B}/ds \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}$