Vv156 Lecture 27

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- For the harmonic series, Bernoulli noticed that the subsequence diverges

$$s_2, \quad s_4, \quad s_8, \quad s_{16}, \quad s_{32}, \quad \ldots$$

$$\begin{aligned}
s_2 &= 1 + \frac{1}{2} \\
s_4 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + 1 \\
s_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{3}{2} \\
&\vdots
\end{aligned}$$

- Hence he concluded that the harmonic series diverges; however, we might not be able to do the same to every other series, for example,

Q: Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$$

- There is no closed formula for the partial sum of this series, so we cannot check

$$\lim_{n\to\infty} s_n$$

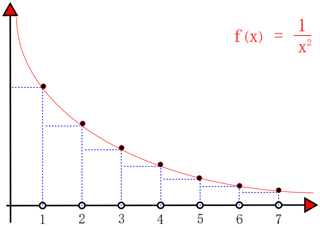
- And the test for divergence is inconclusive,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$$

- We can determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ by comparing it with $\displaystyle\int_1^{\infty} \frac{1}{x^2}\, dx$.

- Let $f(x) = \dfrac{1}{x^2}$, then the partial sum is

$$\begin{aligned} s_n &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{1^n} \\ &= f(1) + f(2) + f(3) + \cdots + f(n) \\ &< f(1) + \int_1^n f(x)\, dx \end{aligned}$$



- Thus the limit of the partial sum is less than the following sum

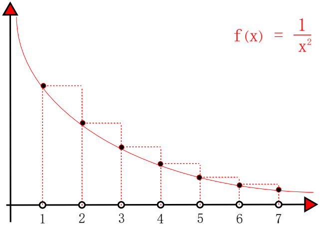
$$s_n < f(1) + \int_1^n f(x)\,dx \implies \lim_{n\to\infty} s_n < f(1) + \lim_{n\to\infty}\int_1^n f(x)\,dx$$

- However, we have the following alternative inequality

$$s_n = f(1) + f(2) + \cdots + f(n) > \int_1^n f(x)\,dx \implies \lim_{n\to\infty} s_n > \lim_{n\to\infty}\int_1^n f(x)\,dx$$

- So the series will be convergent if the improper integral is convergent, and will diverge if the improper integral diverges.

$$\begin{aligned}\lim_{n\to\infty}\int_1^n f(x)\,dx &= \lim_{n\to\infty}\int_1^n \frac{1}{x^2}\,dx \\ &= \lim_{n\to\infty}\left[-\frac{1}{x}\right]_1^n = 1\end{aligned}$$



- Hence the series is convergent.

Integral test

Suppose $\left\{a_n\right\}$ is a sequence such that $a_n = f(n)$, where $f(x)$ is a

1. continuous, 2. positive, 3. decreasing function on $[1, \infty)$.

Then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\int_1^{\infty} f(x)\,dx$ both converge or both diverge.

Exercise

Test the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

for convergence or divergence.

The Comparison Test

- Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

  If

  1. $\sum_{n=1}^{\infty} b_n$ is convergent and
  2. $a_n \leq b_n$ for all n.

  then the series $\sum a_n$ is convergent.

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The limit Comparison Test

- Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c, \qquad \text{where}$$

1. if c is a finite and $c > 0$, then either both series converge or both diverge.
2. if $c = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
3. if $c = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Exercise

For what values of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

Alternating Series test

If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$, where $b_n > 0$, satisfies

$$1. b_{n+1} \leq b_n \qquad \text{and} \qquad 2. \lim_{n \to \infty} b_n = 0$$

then the series is convergent.

Proof

- We consider the even-numbered partial sum $s_2, s_4, s_6, \ldots, s_{2n}$,

$$s_{2n} = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots + b_{2n-1} - b_{2n}$$

- Because $b_n - b_{n+1} \geq 0$ for all n, the sequence of partial sums $\{s_{2n}\}$ is increasing.
- Also, because, for all n,

$$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n} \leq b_1$$

Proof

- Therefore the sequence $\left\{ s_{2n} \right\}$ is bounded above as well as being increasing.
- Clearly an increasing sequence is bounded below, and thus the limit exists

$$s = \lim_{n\to\infty} s_{2n}$$

  by the monotonic sequence theorem.
- It remains to show the odd-numbered partial sums s_{2n+1} also converges to s.

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1} = s \quad \square$$

Exercise

Test the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ *for convergence or divergence.*

Q: It is sufficient but not necessary. Can you think of a counterexample?

Definition

A series $\sum_{n}^{\infty} a_n$ is called absolutely convergent if the series $\sum_{n}^{\infty} |a_n|$ is convergent.

Q: Is alternating harmonic series absolutely convergent?

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

- It is not absolutely convergent because the corresponding series of absolute values is the harmonic series and is therefore divergent.

Q: Is it convergent?

- However, it can be shown that it is convergent by the alternating series test.

Definition

A series $\sum_{n}^{\infty} a_n$ is known as conditionally convergent if the series is convergent but not absolutely convergent.

Exercise

(a) Prove the following using mathematical induction.

$$\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\cdots+\frac{1}{2n-1}-\frac{1}{2n}=\frac{1}{n}\left[\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\cdots+\frac{1}{1+\frac{n}{n}}\right]$$

(b) Find the value to which the alternating harmonic series converges to.

Riemann series theorem

If series $\sum_{n}^{\infty} a_n$ is a conditionally convergent and r is any real number, then there is a rearrangement of $\sum_{n}^{\infty} a_n$ that has a sum equal to r.

- $$\begin{aligned} 0 &= (1-1)+(1-1)+(1-1)+\cdots \\ &\neq 1-1+1-1+1-1+\cdots \neq 1+(-1+1)+(-1+1)+(-1+1)+\cdots \\ &= 1 \end{aligned}$$

- If a series is an absolutely convergent with sum s, then any rearrangement of it has the same sum s. However, any series that is only conditionally convergent can be rearranged to give a different sum.
- If we halve the alternating harmonic series,

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\ln 2 \implies \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots=\frac{1}{2}\ln 2$$

- Now if we introduce some zeros

$$\begin{array}{cccccc}
\frac{1}{2} & -\frac{1}{4} & +\frac{1}{6} & -\frac{1}{8} & +\cdots & =\frac{1}{2}\ln 2 \\
0+\frac{1}{2}+0 & -\frac{1}{4}+0 & +\frac{1}{6}+0 & -\frac{1}{8}+0 & +\cdots & =\frac{1}{2}\ln 2
\end{array}$$

- If we add the alternating harmonic series and the last series together,

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots=\frac{3}{2}\ln 2$$

- The last series contains the same terms as the original alternating harmonics series, but rearranged so that one negative term occurs after two positive terms.

Theorem

If a series is absolutely convergent, then it is convergent.

Proof

- Notice

$$0 \leq a_n + |a_n| \leq 2|a_n|$$

- Absolutely convergence means the series $\sum_{n=1}^{\infty} |a_n|$ is convergent, thus the series $\sum_{n=1}^{\infty} 2|a_n|$ converges, so by the comparison test the following series converges

$$\sum_{n=1}^{\infty} (a_n + |a_n|)$$

- So $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|$ converges since both series converge.

The Ratio Test

This test is useful in determining whether a given series is absolutely convergent.

$$
\begin{aligned}
\text{If} \quad & \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1 \\
& \implies \text{Absolutely convergent} \\
& \implies \text{Convergent} \\
\text{If} \quad & \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1 \\
& \implies \text{divergent} \\
\text{If} \quad & \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1 \\
& \implies \text{Inconclusive}
\end{aligned}
$$

Exercise

Test the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n}$ *for absolute convergence.*

- If we have integer powers, it is more convenient to use the following test, e.g.

$$\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2} \right)^n$$

The Root Test

$$\text{If} \qquad \lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$$

$$\implies \quad \text{Absolutely convergent}$$

$$\implies \quad \text{Convergent}$$

$$\text{If} \qquad \lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$$

$$\implies \quad \text{divergent}$$

$$\text{If} \qquad \lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$$

$$\implies \quad \text{Inconclusive}$$