Question1 (1 points)
Let $\mathcal{V}$ be a vector space, and $\mathbf{v} \in \mathcal{V}$. Show $\mathbf{v} = \mathbf{0}$ if $L(\mathbf{v}) = 0$ for all $L \in \mathcal{V}^*$.

Question2 (1 points)
Show the second dual $\mathcal{V}^{**}$ is isomorphic to the original vector space $\mathcal{V}$.

Question3 (1 points)
Let $\mathcal{M}$ be a metric space with d being the metric. Show the following is a metric on $\mathcal{M}$.

$$d^*(x, y) = \ln\Big(1 + d(x, y)\Big) \qquad \text{for } x, y \in \mathcal{M}$$

Question4 (1 points)
Show the following defines a valid norm on $\mathbb{R}^n$, and is the limiting case of the ℓ_p norm

$$\|\mathbf{x}\|_\infty = \max_{1 \le i \le n} \{|x_i|\}$$

This is known as the infinity norm or maximum norm.

Question5 (2 points)

(a) (1 point) If $\|\cdot\|$ is an operator norm on $\mathbb{R}^{n\times n}$, show $\|\mathbf{I}\| = 1$, where $\mathbf{I}$ is the identity.

(b) (1 point) Is there a vector norm that induces the Frobenius norm as an operator norm?

Question6 (4 points)

(a) (1 point) Show the following is a valid inner product for the vector space $\mathbb{R}^{m\times n}$.

$$\langle \mathbf{A}, \mathbf{B} \rangle = \operatorname{trace}\left(\mathbf{A}^{\mathrm{T}}\mathbf{B}\right)$$

(b) (1 point) Show the following is true for all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}$.

$$\left(\operatorname{trace}\left(\mathbf{A}^{\mathrm{T}}\mathbf{B}\right)\right)^2 \le \operatorname{trace}\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)\operatorname{trace}\left(\mathbf{B}^{\mathrm{T}}\mathbf{B}\right)$$

(c) (1 point) Suppose $\mathcal{B}$ is a basis for a finite-dimensional inner product space $\mathcal{V}$. Show if

$\langle \mathbf{u}, \mathbf{v} \rangle = 0$ for all $\mathbf{v} \in \mathcal{B}$, then $\mathbf{u} = \mathbf{0}$.

(d) (1 point) Show that if $\mathcal{V}$ is an inner product space, then

$$|\langle \mathbf{u}, \mathbf{v} \rangle| = \|\mathbf{u}\|\,\|\mathbf{v}\| \qquad \text{where } \mathbf{u}, \mathbf{v} \in \mathcal{V}.$$

if and only if the vectors $\mathbf{u}$ and $\mathbf{v}$ are scalar multiple of each other.

Question7 (1 points)
Let $\mathbf{A}$ be an $m \times n$ matrix with linearly independent row vectors. Find a matrix representation for the orthogonal projection of $\mathbb{R}^n$ onto the row space of $\mathbf{A}$.

Question8 (2 points)
Let $\mathbf{A} = \begin{bmatrix} 3 & 1 & 1 \end{bmatrix}$, and let $\mathbf{V}$ be the nullspace of $\mathbf{A}$.

(a) (1 point) Find a matrix representation for the orthogonal projection of $\mathbb{R}^3$ onto $\mathbf{V}^\perp$.

(b) (1 point) Find a matrix representation for the orthogonal projection of $\mathbb{R}^3$ onto $\mathbf{V}$.

Question9 (1 points)

Show $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is invertible when $\mathbf{A}$ is a rectangular matrix with linearly independent columns.

Question10 (1 points)

Consider solving a rectangular system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}^{-1}$ does not exist. It is clear that

$$\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}$$

is a left inverse of $\mathbf{A}$ but not a right-inverse. Discuss what does the following represent

$$\mathbf{A}\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{b}$$

where $\mathbf{A}$ has linearly independent columns.

Question11 (1 points)

Find the QR factorization of

$$\begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Question12 (4 points)

(a) (1 point) State the geometric and algebraic multiplicity of each eigenvalue of

$$\mathbf{A} = \begin{bmatrix} 19 & -9 & -6 \\ 25 & -11 & -9 \\ 17 & -9 & -4 \end{bmatrix}$$

Determine whether $\mathbf{A}$ is diagonalizable. If $\mathbf{A}$ is diagonalizable, find a matrix $\mathbf{P}$ s.t.

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$$

(b) (1 point) For a scalar t, determine the matrix exponential $e^{\mathbf{A}t}$, where

$$\mathbf{A} = \begin{bmatrix} -\alpha & \beta \\ \alpha & -\beta \end{bmatrix} \qquad \text{with} \qquad \alpha + \beta \neq 0.$$

(c) (1 point) Show $\mathbf{A}$ and $\mathbf{A}^{\mathrm{T}}$ have the same eigenvalues.

(d) (1 point) Show that if $\mathbf{A}$ is a real symmetric matrix, then $\mathbf{A}$ has only real eigenvalues.

Question13 (0 points)

(a) (1 point (bonus)) Norms are basic tools for defining and analysing limiting behaviour in a vector space $\mathcal{V}$. Recall a sequence of vectors $\{\mathbf{u}_k\} \subset \mathcal{V}$ is said to converge to $\mathbf{u}$ if

$$\|\mathbf{u}_k - \mathbf{u}\| \to 0$$

This depends on the choice of the norm, that is, $\mathbf{u}_k$ might approach $\mathbf{u}$ with one norm but not with another. Fortunately, this is impossible in finite-dimensional spaces. Given two valid norms $\|\cdot\|$ and $\|\cdot\|_\star$ for a finite-dimensional space $\mathcal{V}$, show there are positive constants α and β such that

$$\alpha \leq \frac{\|\mathbf{v}\|}{\|\mathbf{v}\|_\star} \leq \beta \qquad \text{for all non-zero vector } \mathbf{v} \in \mathcal{V}.$$