Vv417 Lecture 16

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- Recall we defined a matrix transformation

$$T_{\mathbf{A}} \colon \mathbb{R}^n \to \mathbb{R}^m$$

to be a transformation of the form

$$T_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\mathbf{x}$$

in which $\mathbf{A}$ is an $m \times n$ matrix. We established later that every matrix transformation is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ and vice versa.

- We continue the study of linear transformations by turning our attention to arbitrary vector spaces instead of focusing on Euclidean spaces.
- A transformation $T \colon \mathcal{U} \to \mathcal{V}$ is linear if

$$T(\alpha \mathbf{u}_1 + \beta \mathbf{u}_2) = \alpha T(\mathbf{u}_1) + \beta T(\mathbf{u}_2)$$

which simply combines the two linearity properties.

Theorem

If $T \colon \mathcal{U} \to \mathcal{V}$ is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0}$$

Proof

- Let $\mathbf{u}$ be any vector in $\mathcal{U}$, then

$$T(\mathbf{0}) = T(0\mathbf{u})$$

since $\mathbf{0} = 0\mathbf{u}$ for any vector space.

- Use the fact that T is linear, thus

$$T(\mathbf{0}) = T(0\mathbf{u}) = 0T(\mathbf{u}) = \mathbf{0}$$

- This states the zero vector is in the kernel of every linear transformation, e.g.

$$\mathbf{0} \in \mathrm{null}(\mathbf{A})$$

Q: Can you think of any linear transformations between non-Euclidean spaces?

- The zero transformation and the identity operator are linear.

$$T(\mathbf{u}) = \mathbf{0} \qquad \text{and} \qquad T(\mathbf{u}) = \mathbf{u} \qquad \text{for all} \quad \mathbf{u}$$

Q: Is the following transformation $T\colon \mathcal{P}_n \to \mathcal{P}_{n+1}$ linear?

$$T(p(x)) = xp(x), \qquad \text{where} \quad p(x) \in \mathcal{P}_n$$

Q: How about the following transformation $T\colon \mathcal{M}_{n\times n} \to \mathbb{R}$?

$$T(\mathbf{A}) = \det(\mathbf{A})$$

- Let $\mathcal{V}$ be a subspace of $\mathcal{F}(-\infty, \infty)$, and

$$x_1, x_2, \dots, x_n$$

be a sequence of real numbers, and let $T\colon \mathcal{V} \to \mathbb{R}^n$ be

$$T(f) = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix}$$

Q: Is this transformation linear?

- For matrix transformation $T_{\mathbf{A}}: \mathbb{R}^n \to \mathbb{R}^m$, we have

$$T(\mathbf{x}) = x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) + \cdots + x_n T(\mathbf{e}_n) = \mathbf{A}\mathbf{x}$$

Theorem

Suppose $T: \mathcal{U} \to \mathcal{V}$ is a linear transformation, where $\mathcal{U}$ is finite-dimensional. Let

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$$

be a basis for $\mathcal{U}$, then the image of any vector $\mathbf{u}$ in $\mathcal{U}$ can be expressed as

$$T(\mathbf{x}) = \beta_1 T(\mathbf{b}_1) + \beta_2 T(\mathbf{b}_2) + \cdots + \beta_n T(\mathbf{b}_n)$$

where β_1, β_2, ..., β_n are the components of the coordinate vector $[\mathbf{u}]_{\mathcal{B}}$.

- It simply states a result of T being linear: the image of any vector in the domain under T is a linear combination of images a basis of the domain.

Theorem

Suppose $T\colon \mathcal{U} \to \mathcal{V}$ is a linear transformation, then

1. the kernel of T is a subspace of $\mathcal{U}$.
2. the range of T is a subspace of $\mathcal{V}$.

If the vector space $\mathcal{U}$ is finite-dimensional, then

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(\mathcal{U})$$

where $\operatorname{rank}(T) = \dim(\operatorname{range}(T))$ and $\operatorname{nullity}(T) = \dim(\operatorname{kernel}(T))$.

Q: How to prove the above general version of rank and nullity?

Definition

Suppose $T\colon \mathcal{U} \to \mathcal{V}$ is a linear transformation, then T is said to be

1. one-to-one if it maps distinct vectors in $\mathcal{U}$ into distinct vectors in $\mathcal{V}$.
2. onto $\mathcal{V}$ if every vector in $\mathcal{V}$ is the image of at least one vector in $\mathcal{U}$.

Theorem

Let $\mathcal{U}$ and $\mathcal{V}$ are finite-dimensional vector spaces with the same dimension, and if

$$T : \mathcal{U} \to \mathcal{V}$$

is a linear transformation, then the following statements are equivalent:

(a) T is one-to-one. (b) $\text{kernel}(T) = \{\mathbf{0}\}$. (c) T is onto, $\text{range}(T) = \mathcal{V}$.

- If $\mathcal{U}$ and $\mathcal{V}$ have different dimensions, then only the first two are equivalent.

Proof

- (a) $\implies$ (b)
- Since T is linear, by the theorem on $\boxed{2}$,

$$T(\mathbf{0}) = \mathbf{0} \implies \text{kernel}(T) = \{\mathbf{0}\}$$

for there can be no other vector in $\mathcal{U}$ that maps into $\mathbf{0}$ if it is one-to-one.

Proof

- (b) $\implies$ (a)
- If $\mathbf{u}_1$ and $\mathbf{u}_2$ are distinct vectors in $\mathcal{U}$, then

$$\mathbf{u}_1 - \mathbf{u}_2 \neq \mathbf{0}$$

- Since $\text{kernel} = \{\mathbf{0}\}$,

$$T(\mathbf{u}_1 - \mathbf{u}_2) \neq \mathbf{0}$$

- Because T is linear, it follows

$$T(\mathbf{u}_1) - T(\mathbf{u}_2) = T(\mathbf{u}_1 - \mathbf{u}_2) \neq \mathbf{0}$$

- So T maps distinct vectors in $\mathcal{U}$ into distinct vectors in $\mathcal{V}$, thus is one-to-one.
- Invoke the theorem on $\boxed{6}$, it is clear that

(b) $\iff$ (c)

Q: Is the following transformation $T\colon \mathcal{P}_3 \to \mathbb{R}^4$ one-to-one and onto?

$$T(a + bx + cx^2 + dx^3) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Q: How about $T\colon \mathcal{M}_{2\times 2} \to \mathbb{R}^4$

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Q: How about $T\colon \mathcal{P}_n \to \mathcal{P}_{n+1}$?

$$T(p(x)) = xp(x), \qquad \text{where} \quad p(x) \in \mathcal{P}_n$$

Q: Is this contradicting the last theorem?

Q: Under what conditions will we have the inverse of a linear $T : \mathcal{U} \to \mathcal{V}$, that is,

$$\left(T^{-1} \circ T\right)(\mathbf{u}) = \mathbf{u} \quad \text{and} \quad \left(T \circ T^{-1}\right)(\mathbf{v}) = \mathbf{v} \qquad \text{for all} \quad \mathbf{u} \in \mathcal{U}, \mathbf{v} \in \mathcal{V}$$

Theorem

A linear transformation T is invertible if and only if it is one-to-one and onto.

Defintion

Let $T : \mathcal{U} \to \mathcal{V}$ be a linear transformation that is one-to-one and onto, then

T is said to be an isomorphism between $\mathcal{U}$ and $\mathcal{V}$,

and we say $\mathcal{V}$ is isomorphic to $\mathcal{U}$, and vice versa

$$\mathcal{U} \cong \mathcal{V}$$

- This terminology is used since T connects two different vector spaces of the same "form", even though they may consist of different kinds of objects.

- For example,

$$\alpha_0 + \alpha_1 x + \alpha_2 x^2 \xleftrightarrow[T^{-1}]{T} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix}$$

- The following theorem reveals the fundamental importance of the space $\mathbb{R}^n$.

Theorem

Every real n-dimensional vector space is isomorphic to $\mathbb{R}^n$.

Proof

- Let $\mathcal{V}$ be a real n-dimensional vector space. To show that $\mathcal{V}$ is isomorphic to $\mathbb{R}^n$, we must find a linear transformation

$$T\colon \mathcal{V} \to \mathbb{R}^n$$

that is one-to-one and onto.

Proof

- Let $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be any basis for $\mathcal{V}$, and

$$\mathbf{u} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n \qquad \text{and} \qquad \mathbf{w} = \gamma_1\mathbf{v}_1 + \gamma_2\mathbf{v}_2 + \cdots + \gamma_n\mathbf{v}_n$$

be the representation of a $\mathbf{u}$ in $\mathcal{V}$ and let $T\colon \mathcal{V} \to \mathbb{R}^n$ be the coordinate map

$$T(\mathbf{u}) = [\mathbf{u}]_{\mathcal{S}}$$

- We need to show T is isomorphism, that is, linear, one-to-one, and onto.

$$T(\beta\mathbf{u}) = T(\beta\alpha_1\mathbf{v}_1 + \beta\alpha_2\mathbf{v}_2 + \cdots + \beta\alpha_n\mathbf{v}_n) = \begin{bmatrix} \beta\alpha_1 \\ \beta\alpha_1 \\ \vdots \\ \beta\alpha_n \end{bmatrix} = \beta \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_3 \end{bmatrix} = \beta T(\mathbf{u})$$

- Similarly,

$$T(\mathbf{u} + \mathbf{w}) = [\mathbf{u} + \mathbf{w}]_{\mathcal{S}} = [\mathbf{u}]_{\mathcal{S}} + [\mathbf{w}]_{\mathcal{S}} = T(\mathbf{u}) + T(\mathbf{w})$$

Proof

- To show that T is one-to-one, we must show that if $\mathbf{u}$ and $\mathbf{w}$ are distinct in $\mathcal{V}$, then so are their images in $\mathbb{R}^n$. If $\mathbf{u} \neq \mathbf{w}$, then

$$\alpha_i \neq \gamma_i \qquad \text{for at least one } i.$$

  hence

$$T(\mathbf{u}) = [\mathbf{u}]_{\mathcal{S}} \neq [\mathbf{w}]_{\mathcal{S}} = T(\mathbf{w})$$

  which shows that $\mathbf{u}$ and $\mathbf{w}$ have distinct images under T.

- Finally, the coordinate transformation T is onto, for if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$,

  then $\mathbf{x}$ is the image under T of the vector

$$\mathbf{y} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n \quad \square$$

Theorem

Suppose $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space $\mathcal{V}$, then

$$\mathbf{u} \longrightarrow [\mathbf{u}]_{\mathcal{S}}$$

is an isomorphism between $\mathcal{V}$ and $\mathbb{R}^n$.

- Recall the coordinate maps depend on the order in which the basis vectors are listed. Thus it describes $n!$ possible isomorphisms.
- $T\colon \mathcal{P}_n \to \mathbb{R}^{n+1}$ is an isomorphism of between $\mathcal{P}_n$ and $\mathbb{R}^{n+1}$.

  $$\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n \xrightarrow{T} \begin{bmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_n \end{bmatrix}$$

  and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{T} \begin{bmatrix} a & b & c & d \end{bmatrix}$ is an isomorphism between $\mathcal{M}_{2\times 2}$ and $\mathbb{R}^4$.
- Both are known as the natural isomorphisms for the vector spaces involved.