
Question1 (5 points)

(a) (1 point) How many $n \times n$ matrices are both diagonal and orthogonal?

Solution:

0M 2^n

$$\begin{bmatrix} \pm 1 & 0 & \cdots & 0 \\ 0 & \pm 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \pm 1 \end{bmatrix}$$

(b) (1 point) How many $n \times n$ matrices are diagonal and unitary?

Solution:

0M There are infinitely many because each diagonal entry can be any point on the unit circle in the complex plane.

(c) (1 point) Find a unitary matrix $\mathbf{U}$ that diagonalizes

$$\mathbf{A} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & -1+\mathrm{i} \\ 0 & -1-i & 0 \end{bmatrix}$$

Solution:

0M Find the eigenvalues

$$\mathbf{D} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

0M Find and normalize the eigenvectors corresponding to those eigenvalues,

$$\mathbf{U} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1-\mathrm{i}}{\sqrt{3}} & \frac{-1+\mathrm{i}}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \end{bmatrix}$$

(d) (1 point) Prove that if $\mathcal{V}$ is a complex inner product space, then

$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2 + \|\mathbf{u}+\mathrm{i}\mathbf{v}\|^2\mathrm{i} - \|\mathbf{u}-\mathrm{i}\mathbf{v}\|^2\mathrm{i}}{4}, \qquad \text{for all } \mathbf{u}, \mathbf{v} \in \mathcal{V}.$$

Solution:

0M Expanding each of those top terms

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= \|\mathbf{u}\|^2 + \langle \mathbf{u},\mathbf{v}\rangle + \overline{\langle \mathbf{u},\mathbf{v}\rangle} + \|\mathbf{v}\|^2 \\
\|\mathbf{u}-\mathbf{v}\|^2 &= \|\mathbf{u}\|^2 - \langle \mathbf{u},\mathbf{v}\rangle - \overline{\langle \mathbf{u},\mathbf{v}\rangle} + \|\mathbf{v}\|^2 \\
\|\mathbf{u}+\mathrm{i}\mathbf{v}\|^2 &= \|\mathbf{u}\|^2 - \mathrm{i}\langle \mathbf{u},\mathbf{v}\rangle + \mathrm{i}\overline{\langle \mathbf{u},\mathbf{v}\rangle} + \|\mathbf{v}\|^2 \\
\|\mathbf{u}-\mathrm{i}\mathbf{v}\|^2 &= \|\mathbf{u}\|^2 + \mathrm{i}\langle \mathbf{u},\mathbf{v}\rangle - \mathrm{i}\overline{\langle \mathbf{u},\mathbf{v}\rangle} + \|\mathbf{v}\|^2
\end{aligned}$$

0M Putting those terms together,

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2 &= 2\left(\langle \mathbf{u},\mathbf{v}\rangle + \overline{\langle \mathbf{u},\mathbf{v}\rangle}\right), \\
\mathrm{i}\left(\|\mathbf{u}+\mathrm{i}\mathbf{v}\|^2 - \|\mathbf{u}-\mathrm{i}\mathbf{v}\|^2\right) &= 2\left(\langle \mathbf{u},\mathbf{v}\rangle - \overline{\langle \mathbf{u},\mathbf{v}\rangle}\right)
\end{aligned}$$

0M Thus together these show that

$$\langle \mathbf{u},\mathbf{v}\rangle = \frac{\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2 + \|\mathbf{u}+\mathrm{i}\mathbf{v}\|^2\mathrm{i} - \|\mathbf{u}-\mathrm{i}\mathbf{v}\|^2\mathrm{i}}{4}, \qquad \text{for all } \mathbf{u},\mathbf{v}\in\mathcal{V}.$$

(e) (1 point) Show the eigenvalues of a skew-Hermitian matrix are 0 or pure imaginary.

Solution:

0M Let $\mathbf{A}$ be skew-Hermitian, and $\alpha = \mathbf{x}^{\mathrm{H}}\mathbf{A}\mathbf{x}$, then

$$\overline{\alpha} = \alpha^{\mathrm{H}} = \left(\mathbf{x}^{\mathrm{H}}\mathbf{A}\mathbf{x}\right)^{\mathrm{H}} = \mathbf{x}^{\mathrm{H}}\mathbf{A}^{\mathrm{H}}\mathbf{x} = -\mathbf{x}^{\mathrm{H}}A\mathbf{x} = -\alpha$$

therefore $\alpha = \mathbf{x}^{\mathrm{H}}\mathbf{A}\mathbf{x}$ is zero or pure imaginary.

0M And, since norm is always real,

$$\alpha = \mathbf{x}^{\mathrm{H}}\mathbf{A}\mathbf{x} = \mathbf{x}^{\mathrm{H}}\lambda\mathbf{x} = \lambda\|\mathbf{x}\|^2 \implies \lambda \text{ is zero or pure imaginary as well.}$$

Question2 (5 points)

(a) (1 point) Find a symmetric matrix $\mathbf{B}$ such that

$$\mathbf{B}^2 = \begin{bmatrix} 17 & 16 & -16 \\ 16 & 41 & -32 \\ -16 & -32 & 41 \end{bmatrix}$$

Solution:

0M Note $\mathbf{B}^2$ is symmetric, so we can find the orthogonal diagonalization for $\mathbf{B}^2$.

$$\mathbf{B}^2 = \mathbf{QDQ}^{\mathrm{T}} = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-2}{3\sqrt{5}} & \frac{-1}{3} \\ 0 & \frac{5}{3\sqrt{5}} & \frac{-2}{3} \\ \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 81 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ \frac{-2}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{-1}{3} & \frac{-2}{3} & \frac{2}{3} \end{bmatrix}$$

1M Therefore

$$\mathbf{B} = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-2}{3\sqrt{5}} & \frac{-1}{3} \\ 0 & \frac{5}{3\sqrt{5}} & \frac{-2}{3} \\ \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ \frac{-2}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{-1}{3} & \frac{-2}{3} & \frac{2}{3} \end{bmatrix}$$

(b) (1 point) Use the singular value decomposition to find the least square solution of the system $\mathbf{Ax} = \mathbf{b}$ that has the smallest 2-norm, where

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 4 \\ 5 & 2 & 4 \\ 3 & 6 & 0 \\ 3 & 6 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ -1 \\ 9 \end{bmatrix}$$

Solution:

0M Suppose $\mathbf{A}$ has the following SVD,

$$\mathbf{A} = \mathbf{U\Sigma V}^{\mathrm{T}}$$

then the first k columns of $\mathbf{U}$ form an orthonormal basis for col($\mathbf{A}$)

0M Using the singular value decomposition, we know rank $\mathbf{A} = 2$

$$\begin{aligned} \mathbf{A} &= \mathbf{U\Sigma V}^{\mathrm{T}} \\ &= \begin{bmatrix} -1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 12 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2/3 & -1/3 & -2/3 \\ -2/3 & 2/3 & 1/3 \\ -1/3 & -2/3 & 2/3 \end{bmatrix}^{\mathrm{T}} \end{aligned}$$

0M It can be shown by the definition of SVD that

$$\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)\hat{\mathbf{x}} = \mathbf{A}^{\mathrm{T}}\mathbf{b}$$
$$\mathbf{\Sigma}^{\mathrm{T}}\mathbf{\Sigma V}^{\mathrm{T}}\hat{\mathbf{x}} = \mathbf{\Sigma}^{\mathrm{T}}\mathbf{U}^{\mathrm{T}}\mathbf{b}$$
$$\begin{bmatrix} 144 & 0 & 0 \\ 0 & 36 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{V}^{\mathrm{T}}\hat{\mathbf{x}} = \begin{bmatrix} 12 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{U}^{\mathrm{T}}\mathbf{b}$$
$$\begin{bmatrix} 1/144 & 0 & 0 \\ 0 & 1/36 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 144 & 0 & 0 \\ 0 & 36 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{V}^{\mathrm{T}}\hat{\mathbf{x}} = \begin{bmatrix} 1/144 & 0 & 0 \\ 0 & 1/36 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 12 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{U}^{\mathrm{T}}\mathbf{b}$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{V}^{\mathrm{T}}\hat{\mathbf{x}} = \begin{bmatrix} 1/12 & 0 & 0 & 0 \\ 0 & 1/6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{U}^{\mathrm{T}}\mathbf{b}$$

0M Note the last row of the first matrix on both sides is redundant

$$\hat{\mathbf{x}} = \mathbf{V\Sigma}^{*}\mathbf{U}^{\mathrm{T}}\mathbf{b} = \begin{bmatrix} 1/3 \\ 1/2 \\ 1/12 \end{bmatrix}$$

where

$$\mathbf{\Sigma}^* = \begin{bmatrix} 1/12 & 0 & 0 & 0 \\ 0 & 1/6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) (1 point) A polar decomposition of an $n \times n$ matrix $\mathbf{A}$ is a factorization

$$\mathbf{A} = \mathbf{PQ}$$

in which $\mathbf{P}$ is a positive semidefinite $n \times n$ matrix with the same rank as $\mathbf{A}$, and $\mathbf{Q}$ is an orthogonal $n \times n$ matrix. Show if

$$\mathbf{A} = \mathbf{U\Sigma V}^T$$

is the SVD of $\mathbf{A}$, then

$$\mathbf{A} = \left(\mathbf{U\Sigma U}^T\right)\left(\mathbf{UV}^{\mathrm{T}}\right)$$

is a polar decomposition of $\mathbf{A}$.

Solution:

0M Since $\mathbf{U}^{\mathrm{T}}\mathbf{U}$ is the identity matrix $\mathbf{I}$,

$$\mathbf{A} = \mathbf{U\Sigma V}^T = \mathbf{U\Sigma}\left(\mathbf{U}^{\mathrm{T}}\mathbf{U}\right)\mathbf{V}^T = \left(\mathbf{U\Sigma U}^T\right)\left(\mathbf{UV}^{\mathrm{T}}\right)$$

0M Since $\mathbf{\Sigma}$ can in general have zero diagonal elements, $\mathbf{P} = \mathbf{U\Sigma U}^T$, which shares the same eigenvalues with $\mathbf{\Sigma}$ for they are similar, is positive semidefinite. And they share the same rank for the same reason.

0M It is trivial to verify that $\mathbf{UV}^{\mathrm{T}}$ is orthogonal

$$\left(\mathbf{UV}^{\mathrm{T}}\right)^{\mathrm{T}}\left(\mathbf{UV}^{\mathrm{T}}\right) = \mathbf{VU}^{\mathrm{T}}\mathbf{UV}^{\mathrm{T}} = \mathbf{I}$$

(d) (1 point) Find a symmetric matrix $\mathbf{A}$ so that

$$f(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$$

where

$$f(\mathbf{x}) = \frac{1}{9}\left(-2x_1^2 + 7x_2^2 + 4x_3^2 + 4x_1x_2 + 16x_1x_3 + 20x_2x_3\right),$$

Is it a positive definite form?

Solution:

0M When $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$ is expanded, the coefficient of x_ix_j is given by $(a_{ij} + a_{ji})/2$.

0m Therefore,

$$\mathbf{A} = \frac{1}{9}\begin{bmatrix} -2 & 2 & 8 \\ 2 & 7 & 10 \\ 8 & 10 & 4 \end{bmatrix}$$

0M No, the form is indefinite.

(e) (1 point) Suppose $\mathbf{A} \in \mathbb{C}^{n\times n}$, and

$$p(\lambda) = \det\left(\mathbf{A} - \lambda \mathbf{I}\right)$$

is the characteristic polynomial of $\mathbf{A}$. Prove

$$p(\mathbf{A}) = 0$$

Solution:

0M If $\mathbf{A}$ is diagonalizable, then we can find an invertible matrix $\mathbf{P}$ such that

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \qquad \text{where } \mathbf{D} \text{ is diagonal.}$$

since the power of a diagonal matrix is

$$\mathbf{D}^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n^k \end{bmatrix}$$

then a polynomial of diagonal matrix is

$$p(\mathbf{D}) = \begin{bmatrix} p(\lambda_1) & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & p(\lambda_n) \end{bmatrix}$$

0M Since $p(\lambda_i) = 0$ for all i, thus

$$p(\mathbf{D}) = \mathbf{0}$$

and $\mathbf{A}^k = \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1}$ for all k. we have

$$p(\mathbf{A}) = \mathbf{P}p(\mathbf{D})\mathbf{P}^{-1} \implies p(\mathbf{A}) = \mathbf{0}$$

This completes the proof of the Cayley-Hamilton theorem in this special case.

0M To show the Cayley-Hamilton theorem in general, we use the fact that any matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ can be approximated by diagonalizable matrices. More precisely, given any matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$, we can find a sequence of matrices $\{\mathbf{A}_k, k \in \mathbb{N}\}$ such that $\mathbf{A}_k \to \mathbf{A}$ as $k \to \infty$ and each matrix $\mathbf{A}_k$ has n distinct eigenvalues.

0M This can be proved by the Schur's theorem,

$$\mathbf{A} = \mathbf{U}\mathbf{R}\mathbf{U}^{\mathrm{H}} \implies \mathbf{A} \approx \mathbf{U}\mathbf{R}^*\mathbf{U}^{\mathrm{H}}$$

by changing some of the diagonal entries of $\mathbf{R}$ by less than ϵ so that all of those diagonal elements become distinct, thus diagonalizable.

0M Since $p(\mathbf{A})$ can be written as a polynomial of $\mathbf{A}$. With

$$\lim_{k\to\infty} \mathbf{A}_k = \mathbf{A}$$

we conclude that

$$\lim_{k\to\infty} p_k(\mathbf{A}_k) = p(\mathbf{A})$$

Since $p_k(\mathbf{A}_k) = \mathbf{0}$ for every $k \in \mathbb{N}$, we must have $p(\mathbf{A}) = \mathbf{0}$.