# Vv156 Lecture 25

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- Up to now, we have implicitly required definite integrals to have two properties:

1. The domain of integration $\mathcal{I} = [a, b]$ be finite, that is, $a$ and $b$ are finite.

Q: Do you know what the other is?

2. The integrand has only a finite number of removable and jump discontinuities.

- In practice, we have problems that fail to meet one or both requirements. e.g.

$$\int_1^\infty \frac{1}{x^2}\,dx \neq \left[\frac{-1}{x}\right]_1^\infty \qquad \int_{-1}^1 \frac{1}{x^2}\,dx \neq \left[\frac{-1}{x}\right]_{-1}^1 \qquad \int_{-\infty}^\infty \frac{1}{x^2}\,dx \neq \left[\frac{-1}{x}\right]_{-\infty}^\infty$$

- Such integrals are said to be improper, and are evaluated as the limit of a limit

Q: Why we have been avoiding those definite integrals.

**The Fundamental Theorem of Calculus Part-I Evaluation**

If $f$ is piecewise continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a), \qquad \text{where } F \text{ is any antiderivative of } f.$$

# Type I — Infinite interval

- The improper integral of $f$ over the interval $[a, \infty)$ is defined to be

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

- The improper integral of $f$ over the interval $(-\infty, b]$ is defined to be

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

- The improper integral of $f$ over the interval $(-\infty, \infty)$ is defined to be

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^\infty f(x)\,dx$$

  where $c$ is any real number.

- In each case, the improper integral is said to be convergent if the corresponding limit exists and divergent if the limit does not exist.

## Exercise

(a) *Evaluate*

$$\int_1^\infty \frac{1}{x^2}\, dx$$

(b) *Find the area of the region between the curves*

$$y = \frac{1}{x^2}, \qquad x = 1, \qquad y = 0$$

(c) *Evaluate*

$$\int_{-\infty}^\infty e^{-|x|}\, dx$$

(d) *Evaluate*

$$\int_{-\infty}^\infty \cos x\, dx$$

# Type II — Infinite discontinuity

- If $f$ is continuous on the interval $[a, b)$, and have an infinite discontinuity at $b$,

$$\int_a^b f(x)\,dx = \lim_{k \to b^-} \int_a^k f(x)\,dx$$

- If $f$ is continuous on the interval $(a, b]$, and have an infinite discontinuity at $a$,

$$\int_a^b f(x)\,dx = \lim_{k \to a^+} \int_k^b f(x)\,dx$$

- If $f$ is continuous on the interval $[a, b]$, except for an infinite discontinuity at a point $c$ in (a, b), then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

- In each case, the improper integral is said to be convergent if the corresponding limit exists and divergent if the limit does not exist.

## Exercise

*(a) Evaluate*

$$\int_{-1}^{1} \frac{1}{x^2}\, dx$$

*(b) Evaluate*

$$\int_{1}^{4} \frac{dx}{(x-2)^{2/3}}$$

*(c) For what values of $p$ does the integral* $\displaystyle\int_{1}^{\infty} \frac{dx}{x^p}$ *converges?*

*(d) Evaluate*

$$\int_{-1}^{1} \frac{1}{x}\, dx$$

Q: What are the regions the following integrals corresponding to

$$\int_{-\infty}^{\infty} \cos x\, dx \qquad \text{and} \qquad \int_{-1}^{1} \frac{1}{x}\, dx$$

- The area of the regions corresponding to the following integrals are undefined

$$\int_{-\infty}^{\infty} \cos x \, dx \qquad \text{and} \qquad \int_{-1}^{1} \frac{1}{x} \, dx$$

since both integrals are divergent.

- The same can be said to the area of any region with a divergent integral.
- The improper integral of $f$ over the interval $(-\infty, \infty)$ is convergent

$$\int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{c} f(x) \, dx + \int_{c}^{\infty} f(x) \, dx$$

if and only if both integrals on the right-hand side are convergent.

- The same can be said about the case of infinite discontinuity. If $f$ is continuous on the interval $[a, b]$, except for an essential discontinuity at a point $c$ in (a, b),

$$\int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx$$

converges if and only if both integrals on the right-hand side are convergent.

- Sometimes it is impossible to find the exact value of an improper integral and yet it is important to know whether it is convergent or divergent.

### Comparison Test

Suppose that $f$ and $g$ are continuous functions with the fact

$$f(x) \geq g(x) \geq 0 \qquad \text{for} \quad x \geq 0$$

then

$$\int_a^\infty f(x)\,dx \text{ being convergent implies that } \int_a^\infty g(x)\,dx \text{ is convergent.}$$

and

$$\int_a^\infty g(x)\,dx \text{ being divergent implies that } \int_a^\infty f(x)\,dx \text{ is divergent.}$$

### Exercise

*Show that* $\int_0^\infty e^{-x^2}\,dx$ *is convergent.*

- Recall the question on the work required to send an astronaut from the surface of the earth to the International Space Station.
- We have the following formula for the work done,

$$W = \int_{r_0}^{r_1} \frac{k}{r^2}\, dr = \left[ -\frac{k}{r} \right]_{r_0}^{r_1} = -\frac{k}{r_1} + \frac{k}{r_0}$$

- Notice that as $r_1$ increases indefinitely,

$$W = \lim_{r_1 \to \infty} \left( -\frac{k}{r_1} + \frac{k}{r_0} \right) = \frac{k}{r_0}$$

Q: What does this improper integral represent?

Q: Are you surprised that this improper integral is convergent?

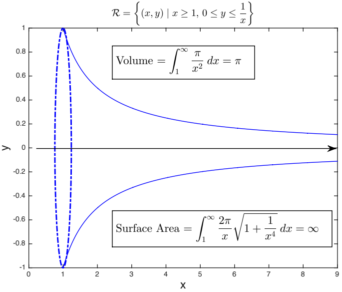
- Suppose the region

$$\mathcal{R} = \left\{ (x, y) \mid x \geq 1,\, 0 \leq y \leq \frac{1}{x} \right\}$$

is rotated about the $x$-axis. It can be shown that

the volume of the resulting solid is finite while the surface area is infinite.

Q: What is the apparent contradiction?




Q: Why there is no contradiction mathematically?

- The surface is shown in the figure and is known as Gabriel's horn.