
Question1 (1 points)
A curve has the property that at *each* of its points (x, y), its slope y' is equal to twice the sum of the coordinates of the point. Write down this property by means of a differential equation for such curves.

Question2 (2 points)
Find the direction field for the following differential equation

$$\dot{y} = e^{-t} + y$$

Based on the direction field, does the behaviour of y as $t \to \infty$ depend on the initial value

$$y(0) = y_0$$

Justify your answer without explicitly solving the differential equation.

Question3 (1 points)
The general solution of a differential equation is the family of functions that satisfy a given differential equation. However, A nonlinear ODE may sometimes have an additional solution that cannot be obtained from the general solution and is then called a singular solution. The following first-order ODE is of this kind.

$$(\dot{y})^2 - t\dot{y} + y = 0$$

Verify the given equation has the general solution of

$$y = ct - c^2$$

and the singular solution of

$$y = \frac{t^2}{4}$$

Question4 (1 points)
Given $y > 0$, solve the following initial-value problem

$$\dot{y} = -\frac{ty}{\ln y}, \qquad y(0) = e^2$$

Specify the maximum open interval on which your solution is valid.

Question5 (1 points)
Solve the following initial-value problem (IVP).

$$\dot{y} - 2y = f(t), \qquad y(0) = 1, \qquad \text{where} \qquad f(t) = \begin{cases} 1 - t, & \text{if} \quad t < 1, \\ 0, & \text{if} \quad t \geq 1. \end{cases}$$

Question6 (1 points)
Test whether the following equation is exact.

$$\left(\arctan xy + \frac{xy - 2xy^2}{1 + x^2y^2}\right) + \frac{x^2 - 2x^2y}{1 + x^2y^2}\frac{dy}{dx} = 0$$

If so, find the general equations; if not, state the reason.

Question7 (1 points)

Solve the following equation using substitution.

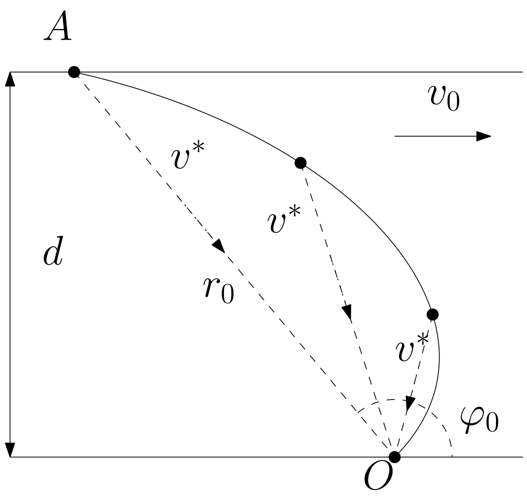
$$y' = 2\left(\frac{y+2}{x+y-1}\right)^2$$

Question8 (1 points)

Consider a large vat that contains 100 gal of sugar water. The amount flowing in is the same as the amount flowing out, so there are always 100 gallons in the vat. Sugar water with concentration of 5g/gal enters the vat through pipe A at a rate of 2 gal/min. Sugar water with concentration of 10g/gal enters the vat through pipe B at a rate of 1 gal/min. Sugar water leaves the vat through pipe C at a rate of 3 gal/min. The vat is kept well mixed, so the sugar concentration is uniform throughout the vat. Find the concentration $C(t)$ of sugar in the vat at time t measured in g/gal. Assume the concentration of sugar in the vat is 0 g/gal initially.

Question9 (1 points)

There is a small boat moving from one side of the river to the other as the figure described.



We know that the speed of the small boat v^* is bigger than the speed of the water v_0. And the boat always moving towards the destination O. The speed of the boat in stationary reference frame is the simple superposition of ship speed and water speed. Find the path of the boat using polar coordinates,

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Below are the bonus questions:

Question10 (0 points)

(a) (1 point (bonus)) Find a specific autonomous equation that admits equilibrium solutions, then find and classify the equilibrium solutions.

(b) (1 point (bonus)) Find a specific equation that is neither separable nor exact, but can be solved by converting it into an exact equation using an integrating factor.

(c) (1 point (bonus)) Find a specific Bernoulli equation, then solve it.

Question11 (0 points)

The differential equation

$$\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2$$

is known as Riccati's equation. A Riccati's equation can be solved by a succession of two substitutions provided that we know a particular solution.

(a) (1 point (bonus)) Show that the

$$y = \phi_1 + u, \qquad \text{where } \phi_1 \text{ is a solution,}$$

reduces Riccati's equation to a Bernoulli equation.

(b) (1 point (bonus)) Find a one-parameter family of solutions for the differential equation

$$\frac{dy}{dx} = -\frac{4}{x^2} - \frac{1}{x}y + y^2$$

where $\phi_1 = \dfrac{2}{x}$ is a known solution of the equation.

Question12 (0 points)
An important application of differential equations outside physics involves population growth. Consider a population $P(t)$ of animals. As likelihood of reproduction depends on the number of animals present, it is natural to assume that the rate of change of $P(t)$ is directly proportional to $P(t)$. Phrased in terms of the derivative, this assumption means that

$$\frac{dP}{dt} = kP$$

where k is some positive constant. It can be shown easily that the general solution is

$$P(t) = P_0 e^{kt}$$

This model exhibits unbounded growth over time, so it is not realistic beyond a relatively short period of time. A related, but more sophisticated, model of population growth is the logistic differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{A}\right)$$

where the constant k is considered the reproductive rate of the population and the constant A is the surrounding environment's carrying capacity.

Suppose a population exhibits logistic growth which has $k = 0.05$ and $A = 75$.

(a) (1 point (bonus)) Use Matlab to produce an informative graph of a slope field of the equation, and determine the value(s) of P for which P is increasing most rapidly.

(b) (1 point (bonus)) Solve the IVP *explicitly* for P with the initial condition $P(0) = 10$, then write down the Matlab code that can be used to check your explicit solution.

Question13 (0 points)
One of the famous problems in the history of mathematics is the Brachistochrone problem. Below are some interesting histories regarding the birth of Brachistochrone written by James Ferguson from University of Victoria.

In June of 1696, Johann Bernoulli (1667—1748), a member of the most famous mathematical family in history, issued an open challenge to the mathematical world with the following problem:

> Given two points P and Q in a vertical plane, find the path which a movable particle will traverse in shortest time, assuming that its acceleration is due only to gravity.

The problem is in fact based on a similar problem considered by Galileo Galilei (1564—1642) in 1638. Galileo did not solve the problem explicitly and did not use methods based on the calculus. Due to the incomplete nature of Galileo's work on the subject, Johann was fully justified in bringing the matter to the attention of the world. After stating the problem, Johann assured his readers that the solution to the problem was very useful in mechanics and that it was not a straight line but rather a curve familiar to geometers. He gave the world until the end of 1696 to solve problem, at which time he promised to publish his own solution. At the end of the year, he published the challenge a second time, adding an additional problem (one of a geometrical nature), and extending his deadline until Easter of 1697.

At the time of the initial challenge to the world, Johann Bernoulli had also sent the problem privately to one of the most gifted minds of the day, Gottfried Wilhelm Leibniz (1646-1716), in a letter dated 9 June 1696. A short time later, he received a complete solution in reply, dated 16 June 1696! In our modern society, which has become obsessed doing everything as soon as possible, focusing so much on speed that we often sacrifice quality, it is refreshing to see that technology is not a prerequisite for timeliness. It also gives us an indication of Leibniz's genius. It was in correspondence between Leibniz and Johann Bernoulli that the name Brachistochrone was born. Leibniz had originally suggested the name Tachistoptotam (from the Greek tachistos, swiftest, and piptein, to fall). However, Bernoulli overruled him and christened the problem under the name Brachistochrone (from the Greek brachistos, shortest, and chronos, time).

The other great mathematical mind of the day, Newton, was also able to solve the problem posed by Johann Bernoulli. As legend has it, on the afternoon of 29 January 1697, Newton found a copy of Johann Bernoulli's challenge waiting for him as he returned home after a long day at work. At this time, Newton was Warden of the London Mint. He had resigned the Lucasian Professorship at Cambridge in April 1696. By four o'clock that morning, after roughly twelve hours of continuous work, Newton had succeeded in solving both of the problems found in Bernoulli's challenge! That same day, Newton communicated his solution anonymously to the Royal Society. While it is quite a feat, comparable to that of Leibniz's rapid response to Bernoulli, one should note that Bernoulli himself claimed that neither problem should take "a man capable of it more than half an hour's careful thought." As Ball slyly notes, since it actually took Newton twelve hours, it is "a warning from the past of how administration dulls the mind." Indeed, it is rather surprising that it took Newton so long, considering the similarities that the Brachistochrone problem has with Newton's previously solved problem of bodies in a resisting medium. According to Newtonian scholar Tom Whiteside, he said that Newton would have solved it in a few minutes in his younger days. This story gives some idea of Newton's power, since Johann Bernoulli himself took two weeks to solve it.

When Johann originally posed the problem, it is likely that his main motivation was to fuel the fire of his bitter feud with elder brother, Jacob Bernoulli (1654-1705). Johann had publicly described his brother Jacob as incompetent and was probably using the Brachistochrone problem, which he has already solved, as a means of publicly triumphing over his brother. Such an attitude towards one's contemporaries prompted one scholar to remark that it must have been Johann Bernoulli who first said the words, "It is not enough for you to succeed; your colleagues must also fail."

In the end, Jacob Bernoulli was able to solve the problem set to him by his brother, joining Leibniz, Newton, and L'Hôpital as the only people to correctly solve the problem. Johann's solution was elementary and clever: the path of quickest descent is the same as the light ray passing through a fluid of variable density. However, it was later suggested that Johann

Bernoulli, "found an incorrect proof that the curve is a cycloid. After Jacob Bernoulli correctly proved the curve sought is a cycloid, Johann tried to substitute his brother's proof for his own. "

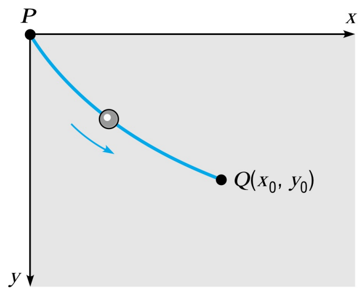
In an attempt to outdo his brother, Jacob Bernoulli created a harder version of the Brachistochrone problem. In solving it, he developed new methods that were refined by Leonhard Euler into what the latter called (in 1766) the calculus of variations. Joseph-Louis Lagrange did further work that resulted in modern infinitesimal calculus.

It is also interesting to note that even though Newton sent in his result anonymously, Johann Bernoulli was not fooled. He later wrote to Leibniz that Newton's unmistakable style was easy to spot and that "he knew the lion from his touch." Far from being gracious, however, Johann was quick to proclaim his superiority over others when summarising the results of his challenge:

> I have with one blow solved two fundamental problems, one optical and the other mechanical and have accomplished more than I have asked of others: I have shown that the two problems, which arose from totally different fields of mathematics, nevertheless possess the same nature.

Bernoulli refers to the fact that he was the first to publicly demonstrate that the least time principle of Fermat and the least time nature of the Brachistochrone are two manifestations of the same phenomenon.

Below are actual mathematical questions instead of history regarding Brachistochrone.



In solving this problem it is convenient to take the origin as the upper point P and Q to orient the axes as shown in the Figure. The lower point Q has coordinates (x_0, y_0). It is then possible to show that the curve of minimum time is given by a function $y = \phi(x)$ that satisfies the differential equation

$$(1 + {y'}^2)y = k^2 \tag{1}$$

where k^2 is a certain positive constant to be determined later. Solve equation (1) for y',

$$y' = \pm\sqrt{\frac{k^2}{y} - 1} \tag{2}$$

(a) (1 point (bonus)) Why is it necessary to choose the positive square root?

(b) (1 point (bonus)) Introduce the new variable t by the relation

$$y = k^2 \sin^2 t$$

Show that the equation (2) then takes the form

$$2k^2 \sin^2 t = \frac{dx}{dt} \tag{3}$$

(c) (1 point (bonus)) Letting $\theta = 2t$, show that the solution of equation (3) for which $x = 0$ when $y = 0$ is given by

$$x = \frac{k^2(\theta - \sin\theta)}{2}, \qquad y = \frac{k^2(1 - \cos\theta)}{2} \tag{4}$$

Equations (4) are parametric equations of the solution of equation (1) that passes through $(0, 0)$. The graph of equations (4) is called a cycloid. If we make a proper choice of the constant k, then the cycloid also passes through the point (x_0, y_0) and is the solution of the brachistochrone problem.

(d) (1 point (bonus)) Find k for

$$x_0 = 1 \qquad \text{and} \qquad y_0 = 2$$

Question14 (0 points)

Textbooks on Differential equations often give the impression that most differential equations can be solved in closed form, but experience does not bear this out. It remains true that solutions of the vast majority of first order initial value problems cannot be found by analytical means. Therefore, it is important to be able to approach the problem in other ways. Today there are numerous methods that produce numerical approximations to solutions of differential equations. Here we introduce the oldest and simplest such method, originated by Euler about 1768. It is called the tangent line method or the Euler's method.

There are various modifications to the Euler's method. The original one step Euler's method is undoubtedly the simplest method for approximating the solution to an ordinary differential equation. It goes something like this: Given a first order initial value problem

$$x' = f(t, x),\ t \in (a, b) \qquad\qquad x(a) = x_a$$

we observe that defining $t_j = a + h_j$, for $j = 0, 1, 2, \ldots, N$ where $h = (b - a)/N$ we have

$$\frac{x(t_{j+1}) - x(t_j)}{h} \approx f(t_j, x(t_j)).$$

Therefore we can write

$$x(t_{j+1}) \approx x(t_j) + hf(t_j, x(t_j)).$$

With this we can define an iterative scheme for computing approximations x_j for $x(t_j)$ using the iterative scheme

$$x_{j+1} = x_j + f(t_j, x_j), \quad j = 0, 1, \ldots, (N - 1).$$

As an example let us consider the IVP

$$x'(t) = -tx(t), \quad x(0) = 1 \qquad \text{on} \quad 0 < t < 5.$$

The exact solution is given by

$$x(t) = e^{-1/2t^2}$$

We can put the following lines of code into an m-file *euler.m* and save it to disk.

```
% Euler's method 1
>> clear all;
>> defn=inline('-t*x','t','x');
>> a=0; b=5; N=20; xa=1;
>> h=(b-a)/N; t=a+h*(0:N); LT=length(t);
>> x(1)=xa;
>>     for j=2:LT
>>         x(j)=x(j-1)+h*defn(t(j-1),x(j-1));
>>     end
>> xact_sol=exp(-t.^2/2);
>> plot(t,x,'b',t,xact_sol,'r--','LineWidth',2)
>> grid on
```

Execute the file in the Workspace by typing the first name of the file without the .m extension.

(a) (1 point (bonus)) Run the given Matlab code, write down the error at $t = 3.75$.

There are many variations on this simple Euler's method. Here let us consider *Modified Euler's methods*. Once again we consider the following first order initial value problem

$$x' = f(t, x),\ t \in (a, b) \qquad x(a) = x_a$$

Apply the fundamental theorem of calculus to get

$$\int_{t_1}^{t_2} f(t, x(t))\, dt = \int_{t_1}^{t_2} x'(t)\, dt = x(t_2) - x(t_1).$$

This implies that

$$x(t_2) = x(t_1) + \int_{t_1}^{t_2} f(t, x(t))\, dt.$$

Now we consider two possible ways to approximate the integral on the right above.

- The Midpoint Rule
- The Trapezoid Rule

The **midpoint** method uses the original Euler's method to step halfway across the interval, evaluates the function at this intermediate point, then uses that slope to take the actual step.

$$\begin{aligned} s_1 &= f(t_k, x_k) \\ s_2 &= f\left(t_k + \frac{h}{2}, x_k + \frac{h}{2} s_1\right) \\ x_{k+1} &= x_k + h s_2 \\ t_{k+1} &= t_k + h \end{aligned}$$

This formula is obtained as follows:
The key to deriving Euler's method is the approximate equality

$$x(t + h) \approx x(t) + h f(t, x(t))$$

which is obtained from the slope formula,

$$x'(t) \approx \frac{x(t+h)-x(t)}{h}$$

First we replace the approximation for the slope formula by a more accurate approximation,

$$x'\left(t+\frac{h}{2}\right) \approx \frac{x(t+h)-x(t)}{h} \quad \text{instead of} \quad x'(t) \approx \frac{x(t+h)-x(t)}{h}$$

Using the more accurate approximation, we have

$$x(t+h) \approx x(t) + hf(t+\frac{h}{2}, x(t+\frac{h}{2})) \quad \text{instead of} \quad x(t+h) \approx x(t) + hf(t, x(t))$$

Then we apply Taylor's formula of order one

$$g(\xi) = g(a) + (\xi - a)g'(a) + \dots$$

with $\xi = t + h/2$, $a = t$ and $g = x$, to obtain

$$x\left(t+\frac{h}{2}\right) \approx x(t) + \frac{h}{2}x'(t) = x(t) + \frac{h}{2}f(t, x(t))$$

So substitute $x\left(t+\frac{h}{2}\right)$ into the more accurate approximate equality, we arrive at

$$x(t+h) \approx x(t) + hf(t+\frac{h}{2}, x(t) + \frac{h}{2}f(t, x(t))).$$

For the **trapezoid** rule we approximate the integral with step size $h = t_2 - t_1$ using

$$x(t_2) \approx x(t_1) + \frac{h}{2}\left[f(t_1, x(t_1)) + f(t_2, x(t_2))\right]$$

Unfortunately, we do not know $x(t_2)$ on the right side, so we use, for example, Euler's method to approximate it:

$$x(t_2) \approx x(t_1) + hf(t_1, x(t_1))$$

We thus obtain a numerical procedure

$$x_k^* = x_k + hf(t_k, x_k), \quad t_{k+1} = t_k + h,$$

$$x_{k+1} = x_k + \frac{h}{2}\left[f(t_k, x_k) + f(t_{k+1}, x_k^*)\right].$$

The rest of this problem is concerned with comparing the accuracy of the two variations on the Euler's method discussed in the text.

- Euler
- Midpoint
- Trapezoid

For this comparison use the following differential equation:

$$x' = x - tx^3e^{-2t},\ 0 \le t \le 1,\ x(0) = 1, \quad \text{exact solution: } x = (t^2+1)^{-1/2}e^t$$

Write a Matlab script file that will:

(b) (1 point (bonus)) Compare Euler, Midpoint and Trapezoidal methods using $N = 50$.