PROBABILISTIC METHODS IN ENGINEERING

VE401

Assignment V

Due: April 12, 2018

Team number: 20

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April 11, 2018

Exercise 5.1

We know that $\bar{x} = 26.035$, $s = 4.78476$. Here the null hypothesis is

$$H_0: \ \mu \leq 25.$$

Hence,

$$\begin{aligned}
&P[\overline{X} \geq \overline{x}\big|\mu \leq 25] \leq P[\overline{X} \geq \overline{x}\big|\mu = 25]\\
=&P[\frac{\bar{X}-25}{s/\sqrt{n}} \geq \frac{26.035-25}{s/\sqrt{n}}]\\
=&P[T_{19} \geq \frac{26.035-25}{4.78476}]\\
=&P[T_{19} \geq 0.97]\\
\approx&0.17 > 0.05.
\end{aligned}$$

If we set $\alpha = 0.05$, then $t_{0.05}(19) = 1.729$.

We can reject H_0 if $t = 0.97 > t_{0.05}(19)$ but we fail, which means we say that we have no significant evidence to reject that $\mu \leq 25$. Hence, we support the claim. The P-value is approximately 0.17.

Exercise 5.2

1) We denote that $Z = \frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}}$.

$$\begin{aligned}
\alpha &= P[\text{reject } H_0\big|H_0 \text{ true}]\\
&= P[\overline{X} \geq \overline{x}\big|\mu \leq 4]\\
&\leq P[\overline{X} \geq \overline{x}\big|\mu = 4]\\
&= P[Z \geq \frac{\overline{x}-4}{0.2/\sqrt{50}}]\\
&= P[Z \geq z_\alpha]\\
&\overset{!}{=} 0.05.
\end{aligned}$$

Since $z_{0.05} = 1.645$, then

$$\begin{aligned}
\frac{\overline{x}-4}{0.2/\sqrt{50}} &= 1.645\\
\overline{x} &= 4.047.
\end{aligned}$$

Thus the critical region is

$$\overline{X} \geq 4.047.$$

2)

$$\begin{aligned} P &= P[\text{reject } H_0 \big| H_1 \text{ true}] \\ &= P[Z \geq z_{\alpha=0.05} \big| \mu \geq 4.5] \\ &\geq P[\frac{\overline{X} - 4}{0.2/\sqrt{50}} \geq 1.645 \big| \mu = 4.5] \\ &= P[\overline{X} \geq 4.047 \big| \mu = 4.5] \\ &= 1 - \Phi(\frac{4.047 - 4.5}{0.2/\sqrt{50}}) \\ &= 1. \end{aligned}$$

Hence, the power of this test is 1.

3) Since $z_{1-0.97} = -1.885$, then $z \leq -1.885$, which means

$$\frac{\overline{X} - 4.5}{0.2/\sqrt{n}} = \frac{(z_{0.05} \times 0.2/\sqrt{n} + 4) - 4.5}{0.2/\sqrt{n}} \leq -1.885,$$

which gives $n \geq 1.99$.

Hence, the sample size is at least 2.

4) Since $\bar{x} = 4.05 > 4.047$, we conclude that we can reject H_0 at $p < 0.05$.

A 95% confidence interval for μ is $4.05 \pm \frac{z_{0.025} \cdot 0.2}{\sqrt{50}} = 4.05 \pm 0.06$

Exercise 5.3

1)

$$n = 8, \quad \sigma = 4, \quad \mu_0 = 100.$$

Thus the normalized statistic is

$$z = \frac{\bar{x} - 100}{4/\sqrt{8}} = 1.556$$

Since $z_{0.05} = 1.645 < 1.556$, then we conclude that we reject H_0 at $p < 0.05$.

2)

$$
\begin{aligned}
p &= P[\overline{X} \geq \bar{x} \big| \mu \leq 100] \\
&\leq P[Z \geq 1.556] \\
&= (0.0606 + 0.5904)/2 \\
&= 0.06.
\end{aligned}
$$

3)

$$
\begin{aligned}
\text{Power} &= P[\text{reject } H_0 \big| \mu = 105] = P[Z \geq z_{0.05} \big| \mu = 105] \\
&= P[\frac{\overline{X} - 100}{4/\sqrt{8}} \geq z_{0.05} \big| \mu = 105] \\
&= P[\overline{X} \geq 102.3 \big| \mu = 105] \\
&= 1 - \Phi(\frac{102.3 - 105}{4/\sqrt{8}}) \\
&= 0.9706.
\end{aligned}
$$

4) Since $z_{1-0.85} = -1.035$, then $z \leq -1.035$, which means

$$
\frac{\overline{X} - 105}{4/\sqrt{n}} = \frac{(z_{0.05} \times 4/\sqrt{n} + 100) - 105}{4/\sqrt{n}} \leq -1.035,
$$

which gives $n \geq 4.60$.

Hence, the sample size is at least 5.

5) If we construct a 95% lower confidence bound for μ, which is

$$
\mu \geq \overline{X} - \frac{z_{0.05} \cdot 4}{\sqrt{8}} = 102.2 - 2.3 = 99.9.
$$

Hence, we can say $\mu \geq 99.9$ with the confidence of 95%. It's equivalent to the test in part(i). The results are both rejecting H_0.

Exercise 5.4

Denote that

$$
\begin{aligned}
G &= \sum_{k=1}^{n} X_k, \\
K &= \frac{2}{\beta} \sum_{k=1}^{n} X_k.
\end{aligned}
$$

$$F_K(y) = P[K \leq y] = P[G \leq y \cdot \beta/2],$$
$$f_K(y) = F_K'(y) = \frac{\beta}{2} f_G(y \cdot \frac{\beta}{2}).$$

Then when $y > 0$,

$$\begin{aligned}
f_K(y) &= \frac{\beta}{2} \frac{1}{\Gamma(n)\beta^n} \left(y \cdot \frac{\beta}{2} \right)^{n-1} e^{-\frac{y\beta}{\beta \cdot 2}} \\
&= \left(\frac{\beta}{2} \right)^n \frac{1}{\Gamma(n)\beta^n} y^{n-1} e^{-\frac{y}{2}} \\
&= \frac{1}{\Gamma(n)2^n} y^{n-1} e^{-\frac{y}{2}} \\
&= \frac{1}{\Gamma(n'/2)2^{n'/2}} y^{n'/2-1} e^{-\frac{y}{2}} \\
&= f_{\chi^2_{n'}}(y).
\end{aligned}$$

where $n' = 2n$.

Hence, K follows a chi-squared distribution with $2n$ degrees of freedom.

For $H_0: \ \beta = \beta_0$,

$$\begin{aligned}
\chi^2_{1-\alpha/2,2n} &\leq K \leq \chi^2_{\alpha/2,2n} \\
\chi^2_{1-\alpha/2,2n} &\leq \frac{2}{\beta} \sum_{k=1}^{n} \overline{X}_k \leq \chi^2_{\alpha/2,2n} \\
\frac{\chi^2_{1-\alpha/2,2n}}{2\sum_{k=1}^{n} \overline{X}_k} &\leq \frac{1}{\beta} \leq \frac{\chi^2_{\alpha/2,2n}}{2\sum_{k=1}^{n} \overline{X}_k} \\
\frac{2\sum_{k=1}^{n} \overline{X}_k}{\chi^2_{\alpha/2,2n}} &\leq \beta \leq \frac{2\sum_{k=1}^{n} \overline{X}_k}{\chi^2_{1-\alpha/2,2n}}.
\end{aligned}$$

Hence the critical region is

$$\beta \leq \frac{2\sum_{k=1}^{n} \overline{X}_k}{\chi^2_{\alpha/2,2n}} \text{ or } \beta \geq \frac{2\sum_{k=1}^{n} \overline{X}_k}{\chi^2_{1-\alpha/2,2n}}.$$

For $H_0: \ \beta \leq \beta_0$,

$$\begin{aligned}
K &\geq \chi^2_{\alpha,2n} \\
\beta &\leq \frac{2\sum_{k=1}^{n} \overline{X}_k}{\chi^2_{\alpha,2n}}.
\end{aligned}$$

Hence the critical region is

$$\beta \geq \frac{2\sum_{k=1}^{n} \overline{X}_k}{\chi^2_{\alpha,2n}}.$$