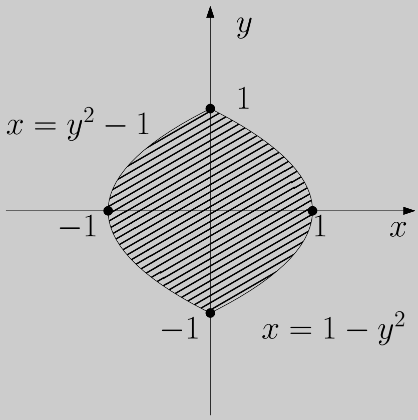

Question1 (1 points)

Find the area of the region enclosed by the curves

$$x = 1 - y^2 \qquad \text{and} \qquad x = y^2 - 1$$

Solution:

1M Area $= 2\int_{-1}^{1} (y^2 - 1)\ dy = 2\left[\frac{1}{3}y^3 - y\right]_{-1}^{1} = \frac{8}{3}$

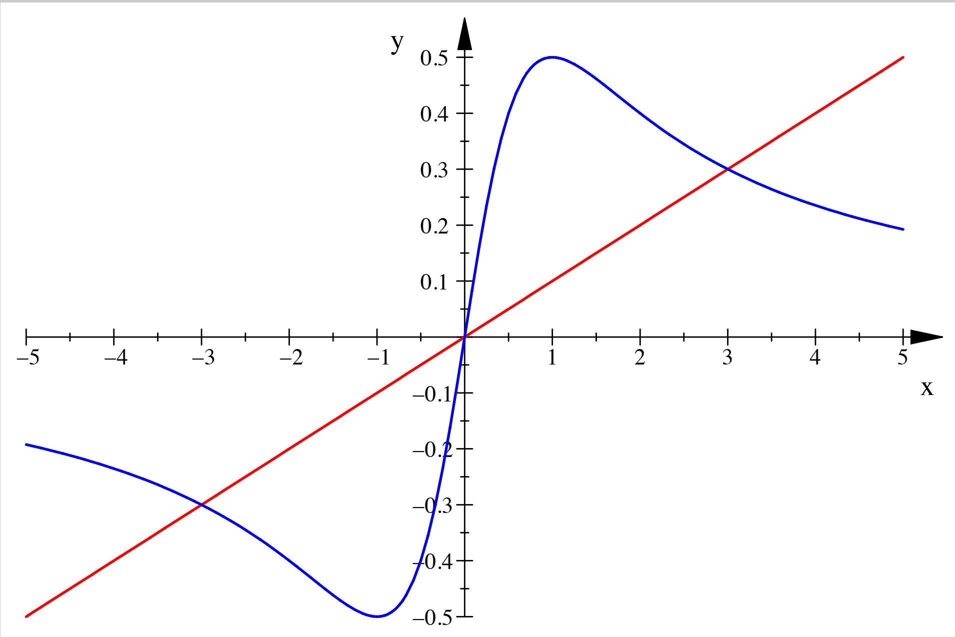


Question2 (1 points)

For what values of m do the line $y = mx$ and the curve $y = \dfrac{x}{x^2+1}$ enclose a region?

Solution:

1M Sketching a graph of the curve $y = \dfrac{x}{x^2+1}$ is essential here.



Since

$$\lim_{x\to\infty} \frac{x}{x^2+1} = 0 \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{x}{x^2+1} = 0$$

if $m \leq 0$, then there will be no region that is enclosed by the two. Note if the line $y = mx$ is as steep as the curve at $x = 0$, then there will also be no region.

$$\left(\frac{x}{x^2+1}\right)'\bigg|_{x=0} = 1$$

Therefore values of m that will give a region is

$$0 < m < 1$$

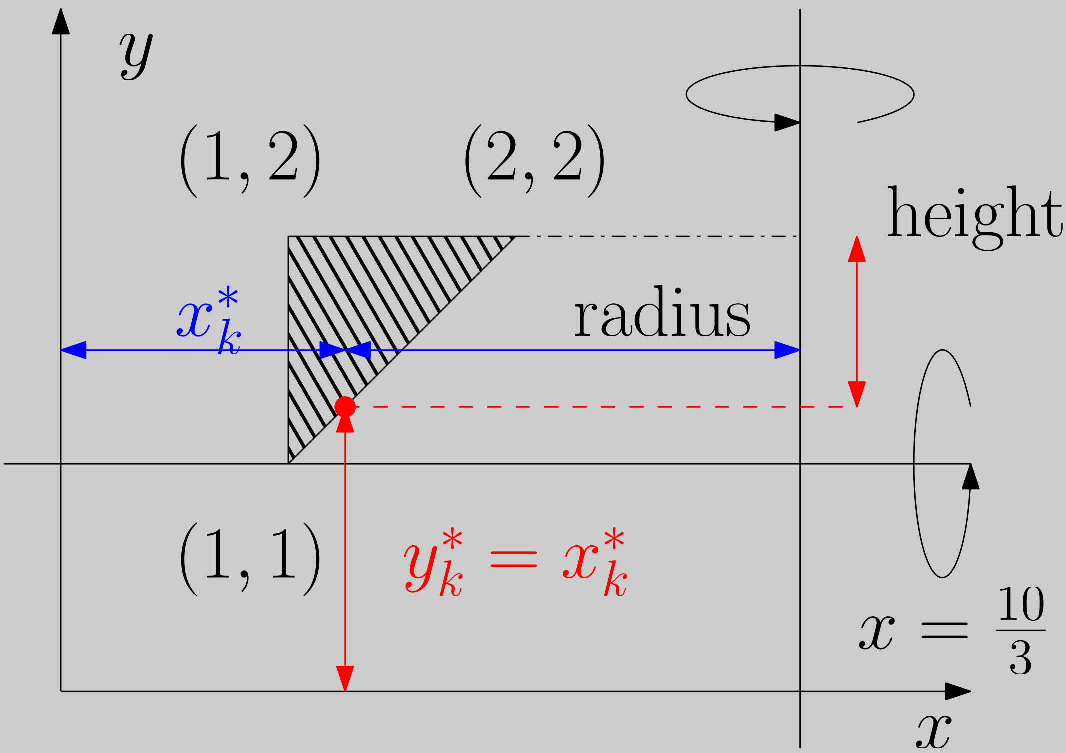
The area can actually be found in terms of m

$$\text{Area} = 2\int_0^{\sqrt{\frac{1}{m}-1}} \left(\frac{x}{x^2+1} - mx\right) dx = m - \ln m - 1$$

Question3 (1 points)
Consider the triangle with vertices $(1, 1)$, $(1, 2)$ and $(2, 2)$. Find the volume of the region generated by rotating this triangle about the line $x = \frac{10}{3}$ and the line $y = 1$.

Solution:

1M Sketch and realize the bottom line of the region is $y = x$.



About $y = 1$ is easy, it will be the difference between the cylinder and the cone

$$\text{Area} = \pi \cdot 1^2 \cdot 1 - \frac{1}{3} \cdot \pi \cdot 1 = \frac{2}{3}\pi$$

You might expect the volume to be a half of the volume of the cylinder since the triangle is a half of the square that would generate the cylinder. However, recall the Pappus theorem, it also depends only how close the region is to the axis of rotation.

About $x = \frac{10}{3}$, we use the cylindrical shell method

$$\begin{aligned}\text{Volume} &= \int_a^b 2\pi \begin{pmatrix}\text{Shell}\\ \text{Radius}\end{pmatrix}\begin{pmatrix}\text{Shell}\\ \text{Height}\end{pmatrix} dx \\ &= \int_1^2 2\pi \left(\frac{10}{3} - x\right)(2-x)\,dx = 2\pi\end{aligned}$$

Question4 (1 points)

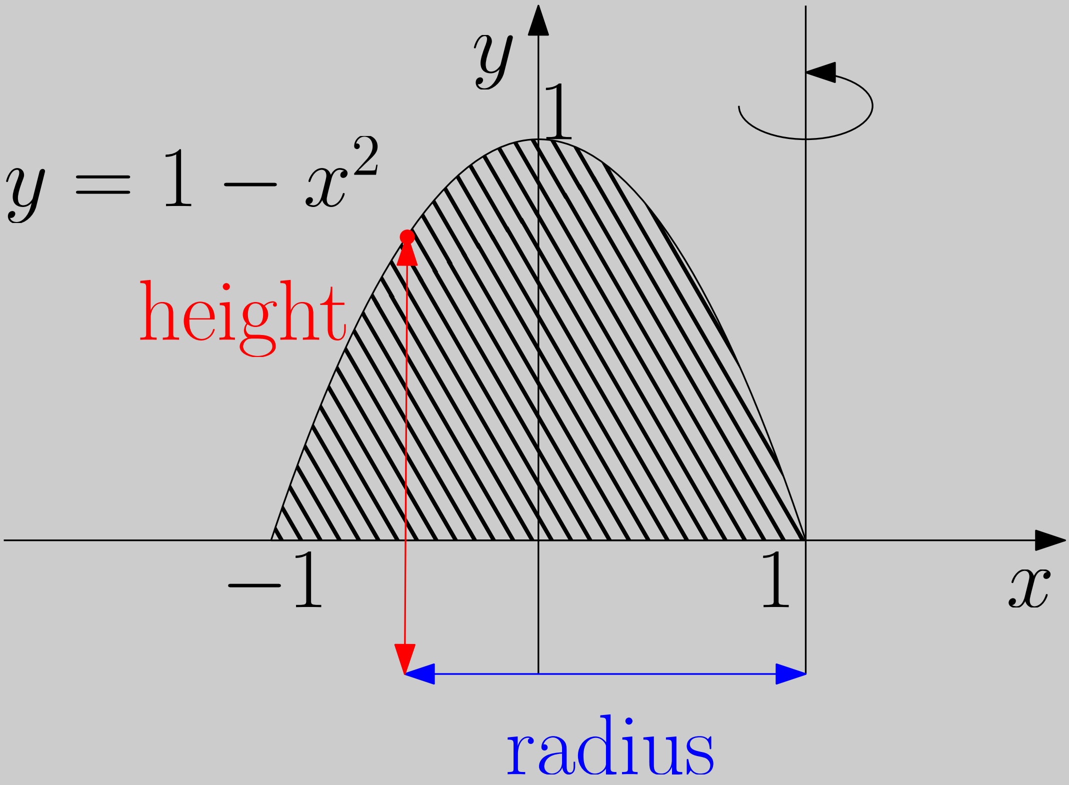
Let $\mathcal{A}$ be the region that is enclosed by the x-axis and the curve

$$y = 1 - x^2$$

Find the volume of the solid generated when the region $\mathcal{A}$ is rotated around the line $x = 1$.

Solution:

1M Sketch and realize the bottom line of the region is $y = x$.



Using the cylindrical shell method

$$\begin{aligned}\text{Volume} &= \int_a^b 2\pi \begin{pmatrix}\text{Shell}\\ \text{Radius}\end{pmatrix}\begin{pmatrix}\text{Shell}\\ \text{Height}\end{pmatrix} dx \\ &= \int_{-1}^1 2\pi(1-x)(1-x^2)\,dx = \frac{8\pi}{3}\end{aligned}$$

Question5 (1 points)

Find the length of the curve

$$y = \ln(\sec x), \qquad \text{for} \qquad 0 \le x \le \pi/4$$

Solution:

1M Using the definition of the arc length of a curve, we have

$$
\begin{aligned}
y' = \tan x \implies \text{Arc length} &= \int_0^{\pi/4} \sqrt{1 + (y')^2}\, dx \\
&= \int_0^{\pi/4} \sec x \, dx \\
&= \Big[\ln|\sec x + \tan x| \Big]_0^{\pi/4} \\
&= \ln\left(\sqrt{2} + 1\right)
\end{aligned}
$$

Question6 (1 points)

Find the area of the surface generated by revolving the following curve about x-axis.

$$y = \cos x, \qquad \text{for} \qquad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

Solution:

1M Using the definition of the surface area of an object recreated by rotation, we have

$$
\begin{aligned}
\text{Surface Area} &= \int_{-\pi/2}^{\pi/2} 2\pi \cos x \sqrt{1 + \sin^2 x}\, dx \\
&= 2\pi \int_{-1}^{1} \sqrt{1 + u^2}\, du \\
&= 2\pi \int_{-\pi/4}^{\pi/4} \sec^3 \theta \, d\theta \\
&= 2\pi \left(\left[\frac{\tan\theta \sec\theta}{2} \right]_{-\pi/4}^{\pi/4} + \frac{1}{2} \int_{-\pi/4}^{\pi/4} \sec\theta \, d\theta \right) \\
&= 2\sqrt{2}\pi + 2\pi \ln(1 + \sqrt{2})
\end{aligned}
$$

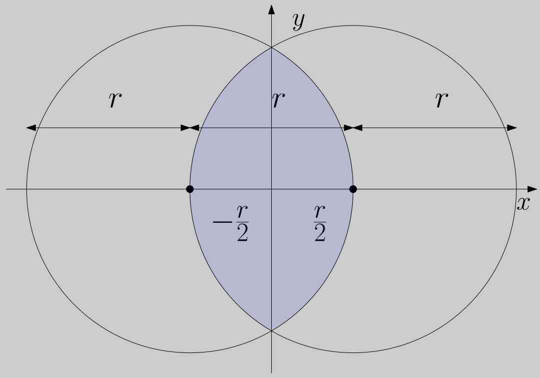
Question7 (1 points)

Find the volume common to two spheres, each with radius r, if the centre of each sphere lies on the surface of the other sphere.

Solution:

1M Sketch and set up a coordinate system

$$\left(x \pm \frac{r}{2}\right)^2 + y^2 = r^2$$



If we partition the x-axis, then cross-sections are circles of radius y

$$A(x) = 2\pi y^2 = 2\pi\left(r^2 - \left(x + \frac{r}{2}\right)^2\right)$$

Therefore

$$\text{Volume} = 2\int_0^{r/2} \pi\left(r^2 - \left(x + \frac{r}{2}\right)^2\right)\,dx = \frac{5}{12}\pi r^3$$

Question8 (3 points)

(a) (1 point) A 20 meter long steel cable has density 2 kilograms per meter, and is hanging straight down. How much work is required to lift the entire cable to the height of its top end?

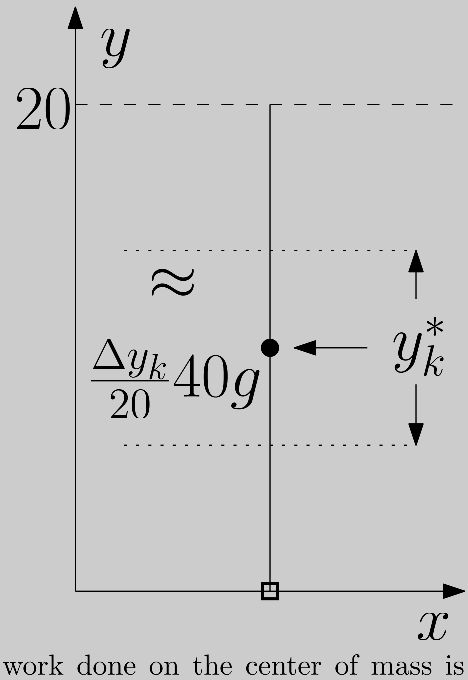
Solution:

1M If we set up the problem so the cable is aligned with y-axis, then

$$\text{work} = \lim_{\|P\|\to 0}\sum_{k=1}^{n} 20\cdot 2\cdot g\frac{\Delta y_k}{20}\cdot(20 - y_k^*)$$

$$= 2g\int_0^{20}(20 - y)\,dy$$

$$= 400 * 9.8 = 3920\text{J}$$



Note we could use

Note we will get the same answer if the work done on the center of mass is considered.

(b) (1 point) Now suppose the cable in the previous question has a 100kg bucket of water attached to its lower end. How much work is required to lift the entire cable and bucket to the height of its top end?

Solution:

1M This is simply the sum of work done on the cable and the bucket

$$\text{work} = 3920 + 100 \cdot g \cdot 20 = 23520\text{J}$$

(c) (1 point) Consider again the cable and bucket of the previous question. How much work is required to lift the bucket 10 meters by raising the cable 10 meters? i.e. The top half of the cable ends up at the height of the top end of the cable, while the bottom half of the cable is lifted 10 meters.

Solution:

1M Using a definite integral, we have

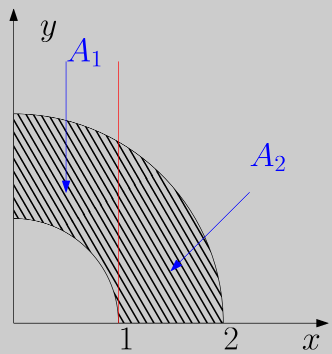
$$\text{work} = 2g \int_{10}^{20} (20 - y)\, dy + 2g \int_0^{10} 10\, dy + 100 \cdot g \cdot 10 = 12740\text{J}$$

Question9 (1 points)

A lamina occupies the region between the circle $x^2 + y^2 = 4$ and the circle $x^2 + y^2 = 1$ in the first quadrant. Find the centroid.

Solution:

1M Divide the region into two parts, the first region from $x = 0$ to $x = 1$,



the second region from $x = 1$ to $x = 2$. The whole region is symmetric, so $\bar{x} = \bar{y}$ and

$$\bar{x} = \bar{y} = \frac{\bar{x}_1 m_1 + \bar{x}_2 m_2}{m_1 + m_2}$$

where m_1 and m_2 are the masses of region 1 and region 2. Here the density is uniform, so we simply compute the areas of region 1 and region 2.

$$\begin{aligned} m_1 + m_2 &= \int_0^1 \left(\sqrt{2^2 - x^2} - \sqrt{1^2 - x^2} \right) dx + \int_1^2 \sqrt{2^2 - x^2}\, dx \\ &= \frac{\pi}{12} + \frac{\sqrt{3}}{2} + \frac{2\pi}{3} - \frac{\sqrt{3}}{2} = \frac{3\pi}{4} \end{aligned}$$

Therefore

$$\bar{x} = \bar{y} = \frac{\bar{x}_1 m_1 + \bar{x}_2 m_2}{m_1 + m_2} = \frac{\int_0^1 x\left(\sqrt{4-x^2} - \sqrt{1-x^2}\right)\,dx + \int_1^2 x\sqrt{4-x^2}\,dx}{\frac{3\pi}{4}} = \frac{28}{9\pi}$$

Question10 (1 points)

Find the centroid of a thin wire shaped like an open semicircle of radius r.

Solution:

1M Let us consider the following circle

$$x^2 + y^2 = r^2$$

Differentiate implicitly, we have

$$\frac{dy}{dx} = -\frac{x}{y}$$

By symmetric it is clear that

$$\bar{x} = 0$$

Using the formula for the centroid, we have

$$\bar{y} = \frac{\int_{-r}^{r} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}{\int_{-r}^{r}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx} = \frac{\int_{-r}^{r} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}{\text{Arc length}}$$

$$= \frac{\int_{-r}^{r}\sqrt{x^2+y^2}\,dx}{\pi r} = \frac{1}{\pi r}\int_{-r}^{r} r\,dx = \frac{2r}{\pi}$$

Question11 (2 points)

We can model the voltage in a typical home appliance with the sine function

$$V = V_{\max}\sin 120\pi t,$$

which expresses the voltage V in volts as a function of time t in seconds. The function runs through 60 cycles each second (its frequency is 60 Hz). The positive constant $V_{\max}$ is known as the peak voltage. The average value of V over the half-cycle from 0s to 1/120s is

$$\frac{2V_{\max}}{\pi}$$

(a) (1 point) If we measured the voltage with a standard moving-coil galvanometer, the meter would read zero. Explain this observation.

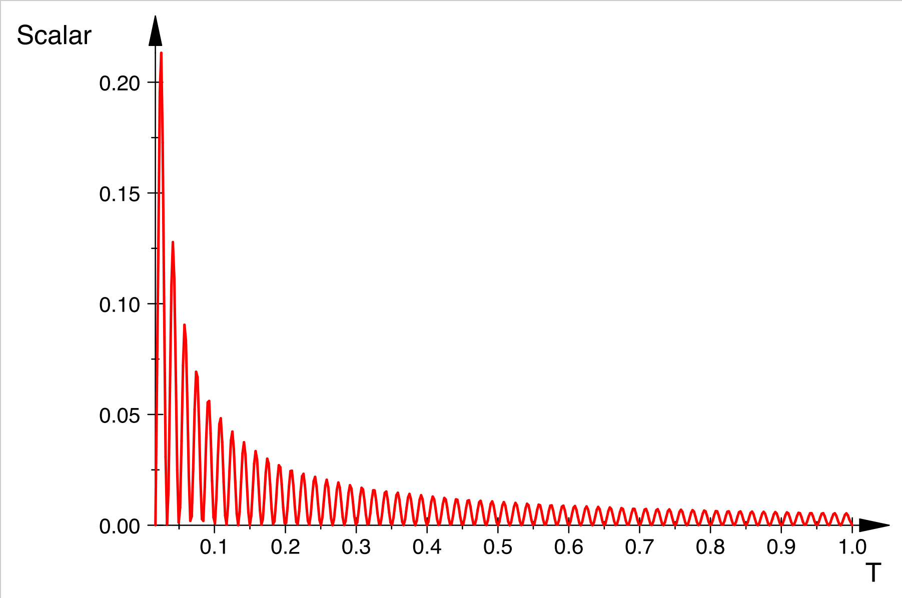
Solution:

1M Note the average voltage is zero over a full-cycle,

$$V_{av} = \frac{1}{1/60 - 0} \int_0^{1/60} V_{\max} \sin 120\pi t \, dt = 60 V_{\max} \left[-\frac{1}{120\pi} \cos 120\pi t \right]_0^{1/60} = 0$$

and roughly zero over a time much bigger than $\frac{1}{60}$s

$$V_{av} = \frac{V_{\max}}{T - 0} \int_0^T \sin 120\pi t \, dt = V_{\max} \underbrace{\frac{\sin(60\pi T)^2}{60\pi T}}_{\text{Scalar}\approx 0}$$



since the response time of a standard moving-coil galvanometer is bigger than 1s. So it essentially measures the average voltage, which is roughly zero.

To measure the voltage effectively, we use an instrument that measures the square root of the average value of the square of the voltage, namely

$$V_{rms} = \sqrt{(V^2)_{av}}$$

where the subscript "rms" stands for "root mean square", and $(V^2)_{av}$ is for the average value of the square of the voltage.

(b) (1 point) The values given for household currents and voltages are always rms values. The circuit that runs your microwave oven is rated 240 volts. What is the peak value of the allowable voltage?

Solution:

1M You must establish the relationship between the rms voltage and the peak voltage using the concept of average value of the square of the voltage.

$$(V^2)_{av} = \frac{1}{1/60 - 0} \int_0^{1/60} (V_{\max})^2 \sin^2 120\pi t \, dt = \frac{(V_{\max})^2}{2}$$

So the rms voltage is

$$V_{rms} = \sqrt{\frac{(V_{\max})^2}{2}} = \frac{V_{\max}}{\sqrt{2}}$$

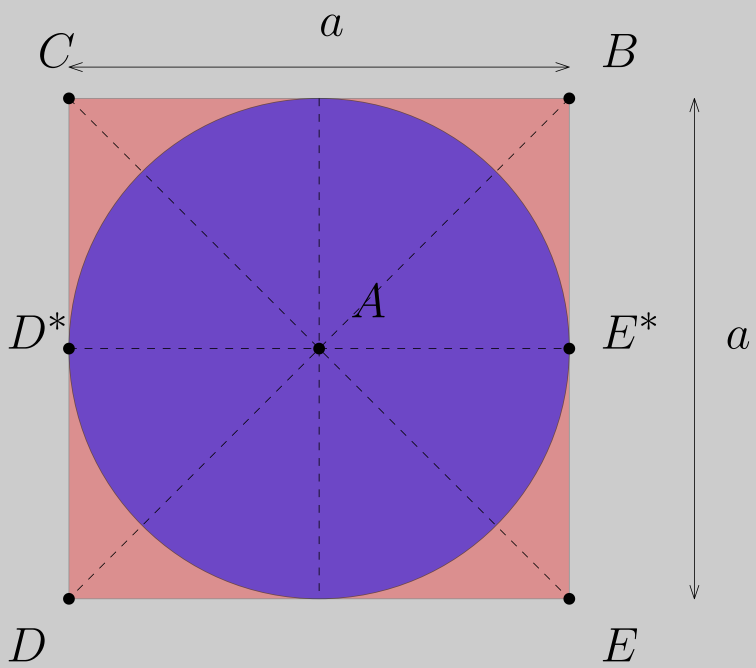
1M The peak voltage is therefore

$$V_{\max} = \sqrt{2}V_{rms} = 240\sqrt{2} = 339\text{volts}$$

Question12 (1 points)
Consider a pyramid with a square base of side a and a sphere with its centre on the base of the pyramid. Suppose the sphere is tangent to all eight sides of the pyramid. Find the height of the pyramid and then find the volume of the intersection of the pyramid and the sphere.

Solution:

1M Let us denote corners of the pyramid by A, B, C, D and E as indicated below.



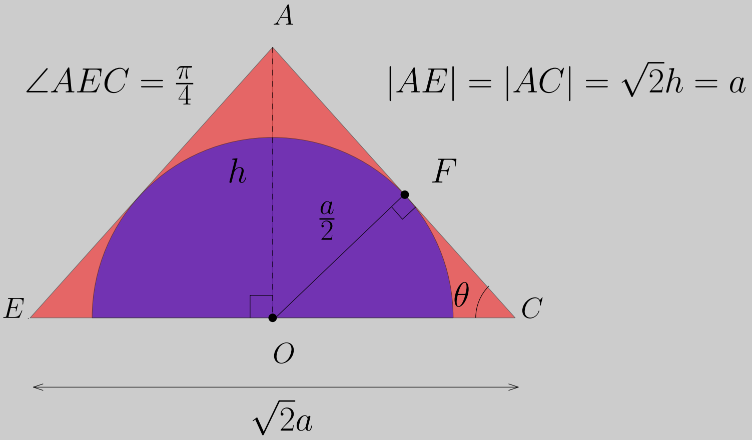
This is a sketch of the pyramid and the sphere when looking down from above. Since the sphere is tangent to the opposite sides in the base of the pyramid, the radius of the sphere must be

$$\frac{a}{2}$$

And the diagonal of the square base must have a length of

$$|CE| = |BD| = \sqrt{2}a$$

If we look at the pyramid in the direction of BD, then we have the following sketch of the pyramid and the sphere, which shows the sphere being tangent to two slant sides of the pyramid, thus $OF \perp AC$ and $|OF|$ is equal to the radius of the sphere.



Considering the sine of θ, which is equal to $\angle AEC$ since $\triangle AEC$ is isosceles, we have

$$\sin\theta = \frac{|OF|}{|OC|} = \frac{\frac{a}{2}}{\frac{1}{2}\sqrt{2}a} \implies \angle AEC = \theta = \frac{\pi}{4}$$

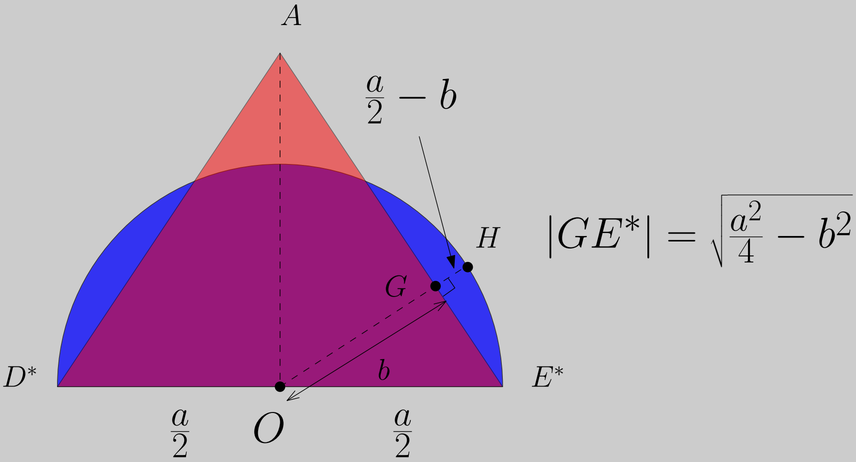
This means

$$\angle FOC = \angle AOF = \angle OAF = \frac{\pi}{4} \implies |AF| = |OF| = \frac{a}{2}$$

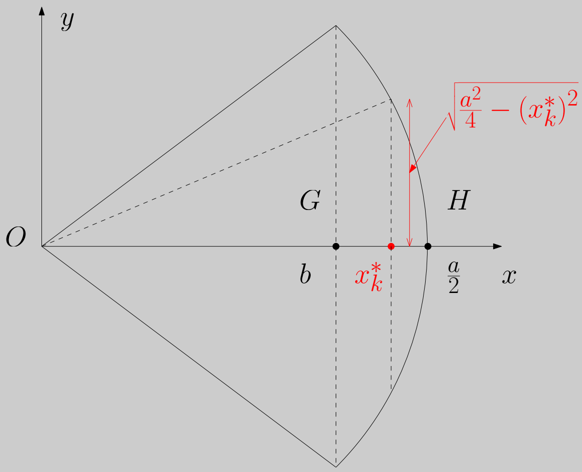
And thus the height of the pyramid is

$$h = \frac{a}{\sqrt{2}}$$

If we look in the direction that is perpendicular to DE, then we have the following sketch of the pyramid and the sphere. It shows some caps of the sphere is outside the pyramid. There are four identical caps of this kind outside the pyramid in total.



Thus the volume of the intersection is equal to half of the volume of the sphere, minus the volume of the four caps. If set up a x-axis in the direction of OH with O being the origin, then the following is sketch of the coordinate system



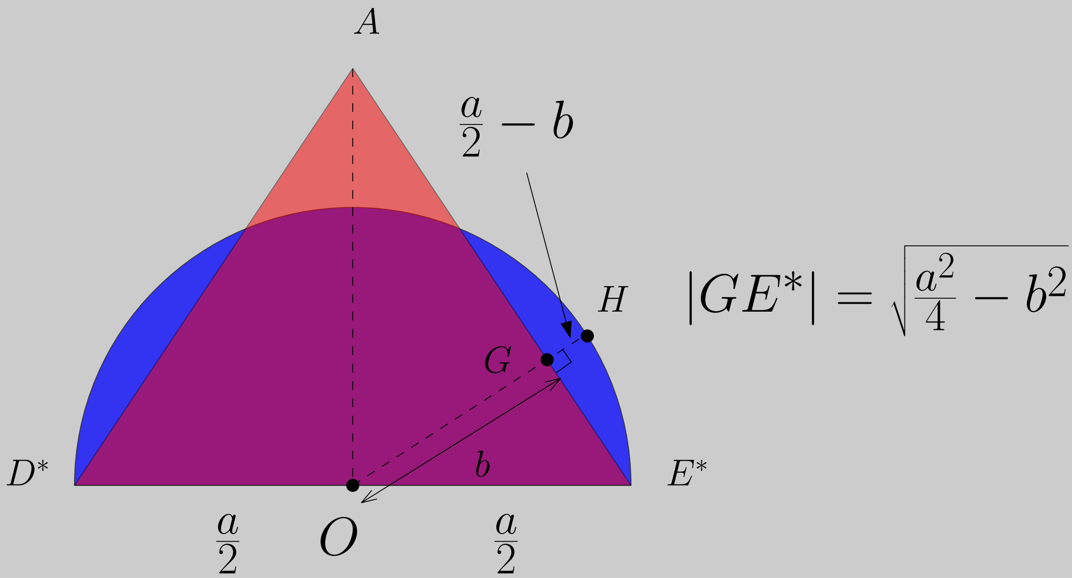
Therefore the volume of each cap is

$$
\begin{aligned}
V_c &= \int_b^{a/2} (\text{cross-section area})\, dx \\
&= \int_b^{a/2} \pi\, (\text{radius})^2\, dx \\
&= \int_b^{a/2} \pi \left(\frac{a^2}{4} - x^2 \right) dx \\
&= \frac{(a+b)(a-2b)^2}{12} \pi
\end{aligned}
$$

To find b in terms of a, we use the fact

$$|AO| = h = \frac{a}{\sqrt{2}}$$

and since $\triangle AOE^*$ and $\triangle OGE^*$ are similar,



thus consider the following ratio and applying the Pythagorean theorem on $\triangle AOE^*$,

$$\frac{|OG|}{|OE*|} = \frac{|AO|}{|AE^*|} \implies b = \frac{\frac{a}{\sqrt{2}} \frac{a}{2}}{\sqrt{\frac{a^2}{4} + \frac{a^2}{2}}} \implies b = \frac{a}{\sqrt{6}}$$

Therefore the volume of the intersection is

$$
\begin{aligned}
V &= \frac{1}{2} \frac{4}{3} \pi \left(\frac{a}{2} \right)^3 - 4 \cdot \frac{(a+b)(a-2b)^2}{12} \pi \\
&= \frac{\pi\, a^3\, \left(14\sqrt{6} - 27\right)}{108}
\end{aligned}
$$