Vv156 Lecture 20

Dr Jing Liu

UM-SJTU Joint Institute

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- The idea behind integrating rational functions

$$\int \frac{P(x)}{Q(x)}\,dx$$

is to expand the rational function as a sum of simpler fractions, called partial fractions, which are easily integrated.

- For example, consider putting the following two fractions together

$$\frac{2}{x-4}+\frac{3}{x+1}$$

$$=\frac{2(x+1)+3(x-4)}{(x-4)(x+1)}$$

$$=\frac{5x-10}{x^2-3x-4}$$

$$\int \frac{5x-10}{x^2-3x-4}\,dx$$

$$=\int\left(\frac{2}{x-4}+\frac{3}{x+1}\right)dx$$

$$=\int \frac{2}{x-4}\,dx+\int\frac{3}{x+1}\,dx$$

$$=2\ln|x-4|+3\ln|x+1|+C$$

$$\int \frac{1}{ax+b}\,dx$$

$$u=g(x)=ax+b$$

$$\implies g'(x)=a$$

$$=\int \frac{1}{ax+b}\frac{1}{a}a\,dx$$

$$=\int u^{-1}\frac{1}{a}\,du$$

$$=\frac{1}{a}\ln|ax+b|+C$$

Definition

This method is called integration by partial fraction, and the expansion is known as partial fraction decomposition (P.F.D.).

- Every proper rational function, $\deg(P(x)) \leq \deg(Q(x))$, can be expanded as

$$\frac{P(x)}{Q(x)} = F_1(x) + F_2(x) + \cdots + F_n(x)$$

where F_1, F_2, ..., F_n are terms of partial fractions of $\dfrac{P(x)}{Q(x)}$.

- The form of F_i depends on factors in $Q(x)$. For each simple factor in $Q(x)$,

Factor in $Q(x)$	Term in P.F.D.
$ax + b$	$\dfrac{A}{ax+b}$

Exercise

Find $\displaystyle\int \frac{x+1}{(2x+1)(x+2)}\,dx$ *by partial fraction.*

- If a particular simple factor in $Q(x)$ is repeated, which is known as a simple repeated factor, then

Factor in $Q(x)$	Term in P.F.D.
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b}+\cdots+\dfrac{A_k}{(ax+b)^k}$

where k is a positive integer.

Exercise

Find the partial fraction decomposition for $\dfrac{x^2+1}{x(x-1)^3}$.

- If a particular quadratic factor cannot be further factored into real terms, then it is known as irreducible quadratic factor,

Factor in $Q(x)$	Term in P.F.D.
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b}+\cdots+\dfrac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$

where k is an integer.

Exercise

Expand $\dfrac{5x-4}{(x+2)(x^2+3)}$ *using partial fraction.*

Q: Is x^2 an irreducible quadratic factor or a repeated linear factor?

$$\frac{Ax+B}{x^2} \quad \text{or} \quad \frac{A}{x}+\frac{B}{x^2}$$

Q: How about x^3, x^4, $x^5 \ldots$?

Factor in $Q(x)$	Term in P.F.D.
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b}+\cdots+\dfrac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\cdots+\dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$

Exercise

Expand using partial fraction

$$\frac{4x^5 - 7x^4 + 8x^3 - 2x^2 + 4x - 7}{x^2\left(x^2+1\right)^2}$$

You don't need to evaluate the coefficients.

- Improper fraction is when the degree of the top is greater than the degree of the bottom, for example,

$$\frac{x^5 - 2x^4 + x^3 + x + 5}{x^3 - 2x^2 + x - 2} = x^2 + \frac{2x^2 + x + 5}{x^3 - 2x^2 + x - 2}$$

- Long division (Polynomial division)

$$\begin{array}{rrl} & x^2 & \\ \cline{2-2} x^3 - 2x^2 + x - 2 \big) & x^5 - 2x^4 + x^3 & + x + 5 \\ & -x^5 + 2x^4 - x^3 + 2x^2 & \\ \cline{2-2} & 2x^2 & \end{array}$$

Exercise

Find the following integrals.

(a)

$$\int \frac{3x^2 + 2}{x^3 + 2x - 8}\, dx$$

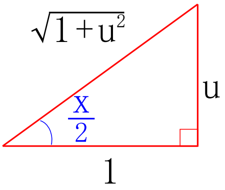
(b)

$$\int \frac{2x - 1}{x^2 + 1}\, dx$$

- In essence, the partial fraction decomposition helps in terms of identifying the antiderivative of a rational function by breaking down the rational function into a sum of smaller and simpler rational functions, thus PFD is just as applicable for definite integrals as it is for indefinite integral.

Q: How would you integrate a composite function of trigonometric function(s)?

- Consider a right angled triangle, and define x and u as indicated,



- The sine, cosine and tangent of $\frac{x}{2}$ are

$$\sin\frac{x}{2} = \boxed{\frac{u}{\sqrt{1+u^2}}}, \quad \cos\frac{x}{2} = \boxed{\frac{1}{\sqrt{1+u^2}}}, \quad \tan\frac{x}{2} = \frac{u}{1} = u$$

- Use double angle identity, we have $\sin x$ and $\cos x$ in terms of u,

$$\sin x = 2\boxed{\sin\frac{x}{2}}\boxed{\cos\frac{x}{2}} = \frac{2u}{1+u^2}, \quad \cos x = 1 - 2\sin^2\frac{x}{2} = \frac{1-u^2}{1+u^2}$$

- Lastly, the derivative of $u = g(x)$ with respect to x in terms of u

$$u = g(x) = \tan \frac{x}{2}$$

$$\implies g'(x) = \frac{1}{2}\sec^2 \frac{x}{2} = \frac{1}{2\cos^2 \frac{x}{2}} = \frac{1+u^2}{2}$$

Exercise

Find the following integrals.

(a)

$$\int \frac{dx}{3 - 5\sin x}$$

(b)

$$\int_{\pi/3}^{\pi/2} \frac{dx}{1 + \sin x - \cos x}$$