Ve401 Probabilistic Methods in Engineering

Summer 2018 — Assigment 3

Date Due: 12:10 PM, Wednesday, the 6$^{\text{th}}$ of June 2018

JOINT INSTITUTE
交大密西根学院

This assignment has a total of **(37 Marks)**.

Exercise 3.1 Maxwell-Boltzmann Statistics

The distribution function of the speed (modulus of the velocity) V of a gas molecule is described by the Maxwell-Boltzmann law

$$f_V(v) = \begin{cases} \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{m}{kT}\right)^{3/2} v^2 e^{-\frac{m}{kT}v^2/2} & v > 0 \\ 0 & v \le 0 \end{cases}$$

where $m > 0$ is the mass of the molecule, $T > 0$ is its temperature and $k > 0$ is the Boltzmann constant.

i) Find the mean and variance of V.
(2 Marks)

ii) Find the mean of the kinetic energy $E = mV^2/2$.
(2 Marks)

iii) Find the probability density f_E of E.
(3 Marks)

Exercise 3.2 Half-Integer Values of the Gamma Function

Calculate $\Gamma((2n + 1)/2)$, $n \in \mathbb{N}$, where Γ denotes the Euler gamma function.
(3 Marks)

Exercise 3.3 Finding Probabilities with the Normal Distribution

The compressive strength of samples of cement can be modeled by a normal distribution with a mean of 6000 kilograms per square centimeter and a standard deviation of 100 kilograms per square centimeter.

i) What is the probability that a samples strength is less than 6250 kg / cm^2?
(1 Mark)

ii) What is the probability that a samples strength is between 5800 and 5900 kg / cm^2?
(1 Mark)

iii) What strength is exceeded by 95% of the samples?
(2 Marks)

(This exercise appeared in the first midterm exam in the Fall Term of 2012.)

Exercise 3.4 Continuous Uniform Distribution

A continuous random variable X is said to be *uniformly distributed* over an interval (a, b) if its density is given by

$$f(x) = \begin{cases} 1/(b-a) & \text{for } a < x < b, \\ 0 & \text{otherwise.} \end{cases}$$

i) Show that this is a density for a continuous random variable.
(1 Mark)

ii) Sketch the graph of the density and shade the area of the graph hat represents $P[X \le (a + b)/2]$.
(1 Mark)

iii) Find the probability pictured in part ii).
(1 Mark)

iv) Let (c, d) and (e, f) be subintervals of (a, b) of equal length. What is the relationship between $P[c \le X \le d]$ and $P[e \le X \le f]$?
(1 Mark)

v) Find the cumulative distribution function F for a uniformly distributed random variable.
(1 Mark)

vi) Show that $\mathrm{E}[X] = (a+b)/2$ and $\operatorname{Var} X = (b-a)^2/12$.
(2 Marks)

Exercise 3.5 A Tricky Question involving the Binomial Distribution
A mathematics textbook has 200 pages on which typographical errors in the equations could occur. Suppose there are in fact five errors randomly dispersed among these 200 pages.

i) What is the probability that a random sample of 50 pages will contain at least one error?
(2 Marks)

ii) How large must the random sample be to assure that at least three errors will be found with 90% probability? (You may use a normal approximation to the binomial distribution.)
(3 Marks)

(This exercise appeared in the first midterm exam in the Fall Term of 2012.)

Exercise 3.6 Reliability of a System
A system consists of two independent components connected in series. The life span (in hours) of the first component follows a Weibull distribution with $\alpha = 0.006$ and $\beta = 0.5$; the second has a lifespan in hours that follows the exponential distribution with $\beta = 25000$.

i) Find the reliability of the system at 2500 hours.
(2 Marks)

ii) Find the probability that the system will fail before 2000 hours.
(2 Marks)

iii) If the two components are connected in parallel, what is the system reliability at 2500 hours?
(2 Marks)

Exercise 3.7
The joint density $f_{XY} \colon \Omega \to \mathbb{R}$ for the discrete bivariate random variable $(X, Y) \colon S \to \Omega = \{(x, y) \colon 1 \le x \le y \le n\}$, where $n \in \mathbb{N}$, $n \ge 1$, is given by

$$f_{XY}(x, y) = \frac{2}{n(n+1)}.$$

i) Verify that f_{XY} is in fact a density.
(1 Mark)

ii) Find the marginal densities for X and Y.
(1 Mark)

iii) Are X and Y independent?
(1 Mark)

iv) Assume that $n = 5$. Use the joint density to find $P[X \le 3 \text{ and } Y \le 2]$. Find $P[X \le 3]$ and $P[Y \le 2]$.
(2 Marks)