Vv255 Lecture 8

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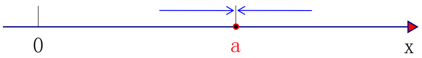
June 7, 2017

- Recall a function of one variable has two one-sided limits at a point $x = a$,

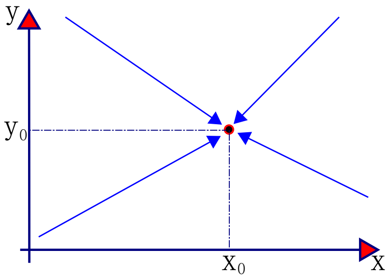
$$\lim_{x \to a^+} f(x) \qquad \text{and} \qquad \lim_{x \to a^-} f(x)$$

reflecting the fact that there are two "ways" by which x can approach a.

$\mathcal{D} \subset \mathbb{R}$



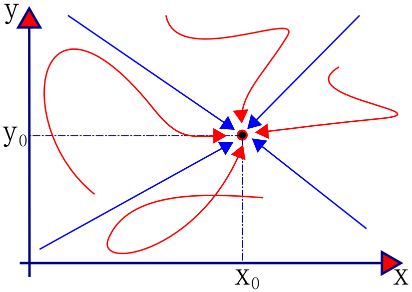
$\mathcal{D} \subset \mathbb{R}^2$



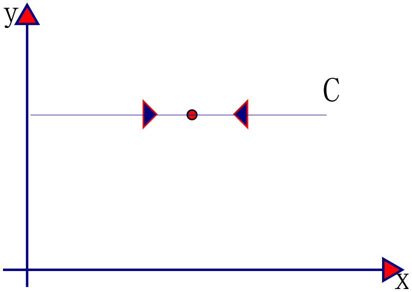
Q: Why is the situation more complicated for functions of two variables $f(x, y)$?

- Not only there are infinitely many directions, there also exist infinitely many different paths along which one point can be approached.

$\mathcal{D} \subset \mathbb{R}^2$



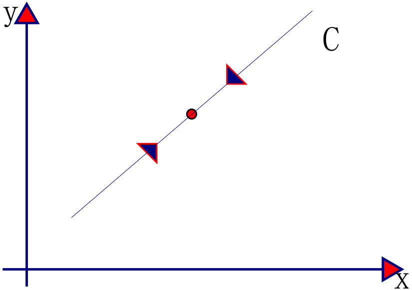
$(x, y) \to (x_0, y_0)$ horizontally



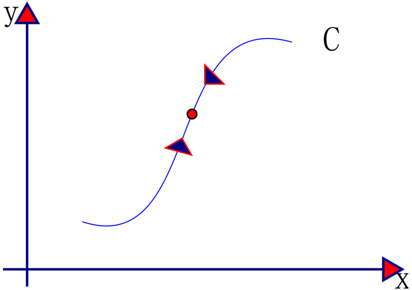
- Let us take on the limit of f along a path before the limit of f in general.

Q: What is the simplest path? how should we deal with this path?

$(x, y) \to (x_0, y_0)$ along a line.



$(x, y) \to (x_0, y_0)$ along a curve.



Q: How can we compute the limit of $f(x, y)$ at (x_0, y_0) along a path defined by

$$\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

where $x(t)$ and $y(t)$ are the component functions of $\mathbf{r}(t)$.

Definition

Suppose C is a smooth curve that is defined by

$$\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \qquad \text{and} \qquad \mathbf{r}(t_0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

then the limit of $f(x, y)$ along a curve C at (x_0, y_0) is defined to be

$$\lim_{\substack{(x,y)\to(x_0,y_0) \\ \text{along } C}} f(x, y) = \lim_{t\to t_0} f(x(t), y(t)) = \lim_{t\to t_0} f(t)$$

- By considering the limit along a curve C, we reduces the problem back to the limit of a function of just one variable.

Exercise

(a) Find the limit of $f(x, y) = \dfrac{3x^2y}{x^2 + y^2}$ as $(x, y) \to (0, 0)$ along the x-axis.

(b) Find the limit of $f(x, y)$ along the parabola $y = x^2$.

- Limits along specific curves are useful for many purposes, but they do not tell the whole story about the limiting behaviour of a function at a point.
- Recall the definition of a continuous function $f(x)$ requires

$$\lim_{x\to a} f(x) = f(a)$$

Q: What is a sensible definition of continuity for functions of several variable?

$$\underbrace{\lim_{\substack{(x,y)\to(x_0,y_0)\\ \text{along } C}} f(x,y) = f(x_0,y_0)}_{\text{not good enough}}$$

Q: Consider the following function, do you think it is continuous instinctively?

$$f(x,y) = \begin{cases} 1 & \text{for} \quad x \neq 0 \quad \text{and} \quad y \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

- So we need a limit concept that accounts for the behaviour of the function in an entire vicinity of a point, not just along curves passing through the point.

- Recall for $f : \mathbb{R} \to \mathbb{R}$, we had the following definition of limit.

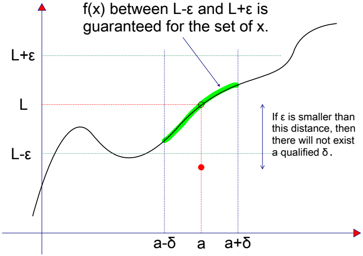
Definition

Let f be a function defined on some open interval of $\mathbb{R}$ that contains the number a, except possibly at a. The value of L is the limit of $f(x)$ as x approaches a,

$$\lim_{x \to a} f(x) = L$$

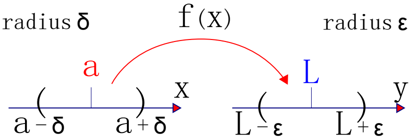
if for every $\epsilon > 0$ there is a $\delta > 0$ such that

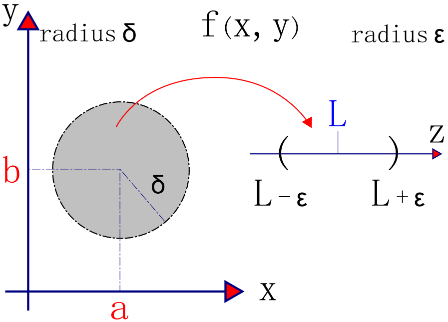
$$|f(x) - L| < \epsilon \qquad \text{if} \qquad 0 < |x - a| < \delta$$



Q: What do the last two inequalities provide?

- If L is indeed the limit of $f(x)$ at a, then for every neighbourhood of L, there is a neighbourhood of a such that x chosen from it ensures $y = f(x)$ is in the neighbourhood of L.





- Similarly, for the function $f(x, y)$, if L is indeed the limit of $f(x, y)$ at (a, b), then for every neighbourhood of L, there is a neighbourhood of (a, b) such that (x, y) chosen from it ensures $z = f(x, y)$ is in the neighbourhood of L.

Definition

Suppose $f(x, y)$ is a function of two variables defined at all points of some open disk centred at (a, b), except possibly at (a, b).

Then the value of L is the limit of f as (x, y) approaches (a, b) and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if, for any number $\epsilon > 0$, there is a number $\delta > 0$ such that $f(x, y)$ satisfies

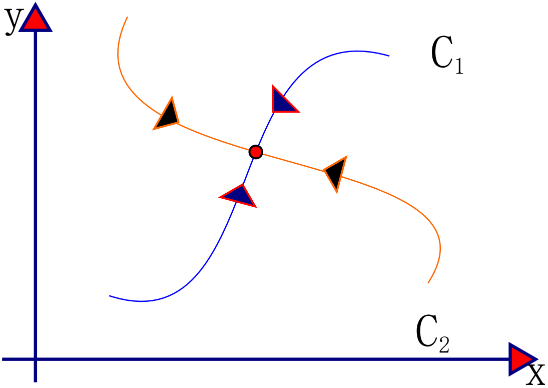
$$|f(x, y) - L| < \epsilon$$

whenever the distance between (x, y) and (a, b) satisfies

$$0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$$

- Loosely speaking, if $f(x, y)$ has a limit L at (a, b), then the value of $f(x, y)$ can be made as close to L as we please by having (x, y) sufficiently close to (a, b), but not being (a, b).

Q: Why is the existence of limit of a function $f(x, y)$ as (x, y) approaches (a, b) stronger than the existence of limit the function $f(x, y)$ as (x, y) approaches (a, b) along any smooth curve C?



Theorem

If $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = L$, then $\displaystyle\lim_{\substack{(x,y)\to(a,b)\\ \text{along } \mathcal{C}}} f(x,y) = L$ for any smooth $\mathcal{C}$.

If $\displaystyle\lim_{\substack{(x,y)\to(a,b)\\ \text{along } C}} f(x,y)$ fails to exist, then $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

If $\displaystyle\lim_{\substack{(x,y)\to(a,b)\\ \text{along } \mathcal{C}_1}} f(x,y) \neq \lim_{\substack{(x,y)\to(a,b)\\ \text{along } \mathcal{C}_2}} f(x,y)$, then $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

- This gives a way of checking whether a function of two variables has a limit.

Exercise

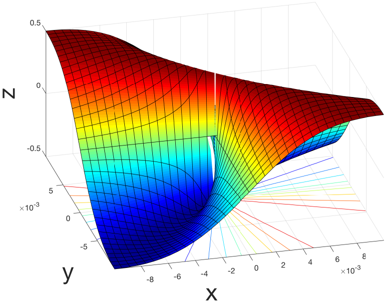
(a) Find the limit of $f(x,y)$ as $(x,y)\to(0,0)$ along the line $y=x$.

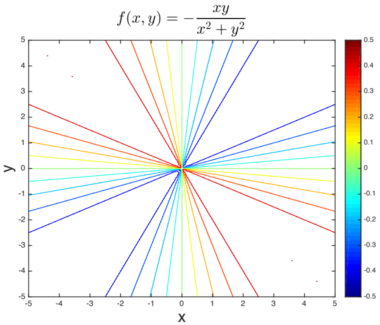
$$f(x,y) = -\frac{xy}{x^2+y^2}$$

(b) Find the limit of $f(x,y)$ as $(x,y)\to(0,0)$ along the parabola $y=x^2$.

- The limit of the function does not exist as $(x, y) \to (0, 0)$ despite the limits of the function along smooth curves exist.
- This is very much like the limit of a function of one variable exist if and only if the two one-sided limits exist and are equal.

Q: Can you imagine the shape of the graph of $f(x, y) = -\dfrac{xy}{x^2 + y^2}$?





Limit Laws

Let $\lim\limits_{(x,y)\to(a,b)} f(x,y) = K$ and $\lim\limits_{(x,y)\to(a,b)} g(x,y) = L$, and c be a constant,

The limit of a constant is the constant itself.

$$\lim_{(x,y)\to(a,b)} c = c$$

The limit of a sum/difference is the sum/difference of the limits.

$$\lim_{(x,y)\to(a,b)} \big[f(x,y) \pm g(x,y)\big] = \lim_{(x,y)\to(a,b)} f(x,y) \pm \lim_{(x,y)\to(a,b)} g(x,y) = K \pm L$$

The limit of a product is the product of the limits.

$$\lim_{(x,y)\to(a,b)} \big[f(x,y)g(x,y)\big] = \Big[\lim_{(x,y)\to(a,b)} f(x,y)\Big]\Big[\lim_{(x,y)\to(a,b)} g(x,y)\Big] = KL$$

The limit of a quotient is the quotient of the limits.

$$\lim_{(x,y)\to(a,b)} \left[\frac{f(x,y)}{g(x,y)}\right] = \frac{\lim\limits_{(x,y)\to(a,b)} f(x,y)}{\lim\limits_{(x,y)\to(a,b)} g(x,y)} = \frac{K}{L}, \quad \text{provided} \lim_{(x,y)\to(a,b)} g \neq 0$$

Limit Laws

Let $\lim_{(x,y)\to(a,b)} f(x,y) = K$ and $\lim_{(x,y)\to(a,b)} g(x,y) = L$, and c be a constant,

If f is a polynomial or a rational function and (a,b) is in the domain of f, then

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

If $f(x,y) = g(x,y)$ for all (x,y) near (a,b), possibly except at (a,b), then

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} g(x,y), \qquad \text{provided the limits exist}$$

The Squeeze Theorem

If $g(x,y) \le f(x,y) \le h(x,y)$ when (x,y) is near (a,b), except possibly at (a,b),

and if $$\lim_{(x,y)\to(a,b)} g(x,y) = L = \lim_{(x,y)\to(a,b)} h(x,y),$$

then $$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

- Intuitively, function $f(x, y)$ is continuous if the surface has no tears or holes.
- The precise definition of continuity at a point for functions of two variables is similar to that for functions of one variable. We require the limit of the function and the value of the function to be the same at the point.

Definition

A function $f(x, y)$ is said to be continuous at (a, b) if $f(a, b)$ is defined and if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

In addition, if f is continuous at every point in an open set $\mathcal{D}$, then we say that f is continuous on $\mathcal{D}$, and if f is continuous at every point in the xy-plane, then we say that f is continuous everywhere.

Exercise

Show that $f(x, y) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2} & if \quad (x, y) \neq (0, 0) \\ 0 & if \quad (x, y) = (0, 0) \end{cases}$ is continuous at $(0, 0)$.

Theorem

1. If $g(t)$ and $h(t)$ are continuous at d with $g(d) = a$ and $h(d) = b$, and $f(x, y)$ is continuous at (a, b), then the composition $f(g(t), h(t))$ is continuous at d.
2. A sum, difference, or product of continuous functions is continuous.
3. A quotient of continuous functions is continuous if the denominator is not 0.
4. If $h(x, y)$ is continuous at (a, b) and $g(u)$ is continuous at $u = h(a, b)$, then the composition $f(x, y) = g(h(x, y))$ is continuous at (a, b).

Exercise

Evaluate

$$\lim_{(x,y)\to(-1,2)} -\frac{xy}{x^2 + y^2}$$

Q: Does continuity in every linear direction implies continuity?