Vv256 Lecture 26

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Definition

A vector $\mathbf{y}$ in an inner vector space $\mathcal{V}$ is said to be orthogonal to a subspace $\mathcal{H}$ of $\mathcal{V}$ if $\mathbf{y}$ is orthogonal to all vectors in the subspace $\mathcal{H}$, meaning that

$$\langle \mathbf{y}, \mathbf{w} \rangle = 0 \qquad \text{for all vectors } \mathbf{w} \text{ in } \mathcal{H}.$$

- If $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of $\mathcal{H}$, then $\mathbf{y}$ is orthogonal to $\mathcal{H}$

  if and only if $\mathbf{y}$ is orthogonal to all the vectors in the basis $\mathcal{H}$.

Q: Why the above must be true?

Theorem

For a vector $\mathbf{y}$ in an inner space $\mathcal{V}$ and a subspace $\mathcal{H}$ of $\mathcal{V}$, we can write

$$\mathbf{y} = \mathbf{p} + \mathbf{z}, \qquad \text{where } \mathbf{p} \text{ is some vector in } \mathcal{H} \text{ and } \mathbf{z} \text{ is orthogonal to } \mathcal{H}.$$

Moreover, this representation or decomposition is unique.

Proof

- We need to show such a decomposition exists and it is unique. Suppose

$$\mathcal{Q} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$$

is an orthonormal basis of $\mathcal{H}$, and consider a vector $\mathbf{p}$ in $\mathcal{H}$,

$$\mathbf{p} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n$$

- The decomposition will exist if for every $\mathbf{y}$ in $\mathcal{V}$ there exists $\mathbf{p}$ such that

the vector $\mathbf{z} = \mathbf{y} - \mathbf{p}$ is orthogonal to $\mathcal{H}$.

- Consider the inner product for arbitrary $i = 1, \ldots, n$,

$$\begin{aligned}
\langle \mathbf{u}_i, \mathbf{z} \rangle = \langle \mathbf{u}_i, \mathbf{y} - \mathbf{p} \rangle &= \langle \mathbf{u}_i, \mathbf{y} - \alpha_1 \mathbf{u}_1 - \alpha_2 \mathbf{u}_2 - \cdots - \alpha_n \mathbf{u}_n \rangle \\
&= \langle \mathbf{u}_i, \mathbf{y} \rangle - \alpha_1 \langle \mathbf{u}_i, \mathbf{u}_1 \rangle - \alpha_2 \langle \mathbf{u}_i, \mathbf{u}_2 \rangle - \cdots - \alpha_n \langle \mathbf{u}_i, \mathbf{u}_n \rangle \\
&= \langle \mathbf{u}_i, \mathbf{y} \rangle - \alpha_i \langle \mathbf{u}_i, \mathbf{u}_i \rangle \\
&= \langle \mathbf{u}_i, \mathbf{y} \rangle - \alpha_i
\end{aligned}$$

Proof

- $\mathbf{z}$ is orthogonal to the subspace $\mathcal{H}$ if $\mathbf{z}$ is orthogonal to $\mathbf{u}_i$ for all i, that is,

$$\langle \mathbf{u}_i, \mathbf{z} \rangle = \langle \mathbf{u}_i, \mathbf{y} \rangle - \alpha_i = 0, \qquad \text{for all } i.$$

- This is achieved if and only if

$$\alpha_i = \langle \mathbf{u}_i, \mathbf{y} \rangle, \qquad \text{for all } i.$$

- Therefore if we let

$$\begin{aligned} \mathbf{p} &= \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n \\ &= \langle \mathbf{u}_1, \mathbf{y} \rangle \mathbf{u}_1 + \langle \mathbf{u}_2, \mathbf{y} \rangle \mathbf{u}_2 + \cdots + \langle \mathbf{u}_n, \mathbf{y} \rangle \mathbf{u}_n, \end{aligned}$$

  then $\mathbf{z} = \mathbf{y} - \mathbf{p}$ is orthogonal to $\mathcal{H}$.

- Since there is only one set of α_i for a given $\mathbf{y}$, the decomposition is unique. $\square$

Definition

If $\mathcal{H}$ is a subspace of $\mathcal{V}$ with an orthonormal basis $\mathbf{u}_1, \ldots, \mathbf{u}_n$, then the vector $\mathbf{p}$

$$\mathbf{p} = \langle \mathbf{u}_1, \mathbf{y} \rangle \mathbf{u}_1 + \langle \mathbf{u}_2, \mathbf{y} \rangle \mathbf{u}_2 + \cdots + \langle \mathbf{u}_n, \mathbf{y} \rangle \mathbf{u}_n, \qquad \text{for all } \mathbf{y} \text{ in } \mathcal{V},$$

is the vector such that

$$\mathbf{y} = \mathbf{p} + \mathbf{z}, \qquad \text{where } \mathbf{p} \text{ is in } \mathcal{H} \text{ and } \mathbf{z} \text{ is orthogonal to } \mathcal{H}.$$

$\mathbf{p}$ is known as the orthogonal projection of $\mathbf{y}$ onto the subspace $\mathcal{H}$, denoted by

$$\mathbf{p} = \operatorname{proj}_{\mathcal{H}} \mathbf{y}$$

Exercise

Given $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, find

$$\operatorname{proj}_{\mathcal{H}} (\mathbf{y})$$

where $\mathcal{H}$ is the subspace of $\mathbb{R}^3$ generated by the set $\mathcal{S}$.

Solution

- Assume the usual addition, scalar multiplication and inner product for $\mathbb{R}^3$.
- Notice that

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 1 - 1 + 0 = 0$$

which means $\mathcal{Q} = \{\frac{\mathbf{v}_1}{|\mathbf{v}_1|}, \frac{\mathbf{v}_2}{|\mathbf{v}_2|}\}$ forms an orthonormal basis for $\mathcal{H} = \mathrm{span}\,(\mathcal{S})$.
- Thus

$$\begin{aligned}
\mathrm{proj}_{\mathcal{H}} &= \langle \frac{\mathbf{v}_1}{|\mathbf{v}_1|}, \mathbf{y}\rangle \frac{\mathbf{v}_1}{|\mathbf{v}_1|} + \langle \frac{\mathbf{v}_2}{|\mathbf{v}_2|}, \mathbf{y}\rangle \frac{\mathbf{v}_2}{|\mathbf{v}_2|} \\
&= \frac{1}{|\mathbf{v}_1|^2}\langle \mathbf{v}_1, \mathbf{y}\rangle \mathbf{v}_1 + \frac{1}{|\mathbf{v}_2|^2}\langle \mathbf{v}_2, \mathbf{y}\rangle \mathbf{v}_2 = \frac{\langle \mathbf{v}_1, \mathbf{y}\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\mathbf{v}_1 + \frac{\langle \mathbf{v}_2, \mathbf{y}\rangle}{\langle \mathbf{v}_2, \mathbf{v}_2\rangle}\mathbf{v}_2 \\
&= \frac{5}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{-2}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 \\ 8 \\ 5 \end{bmatrix}
\end{aligned}$$

Exercise

For function space $\mathcal{C}[0,1]$ with the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 f(x)g(x)\,dx$$

Find $\text{proj}_{\mathcal{W}}(\mathbf{v})$ for $\mathbf{v} = e^x$ and $\mathcal{W}$ is the subspace of $\mathcal{C}[0,1]$ spanned by $\{1, x\}$.

Solution

- We need an orthonormal basis, but $\mathcal{S}$ is orthogonal since

$$\int_0^1 1 \cdot x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2} \neq 0$$

- To construct an orthogonal set, we use theorem on $\boxed{2}$ on $\mathbf{y} = x$,

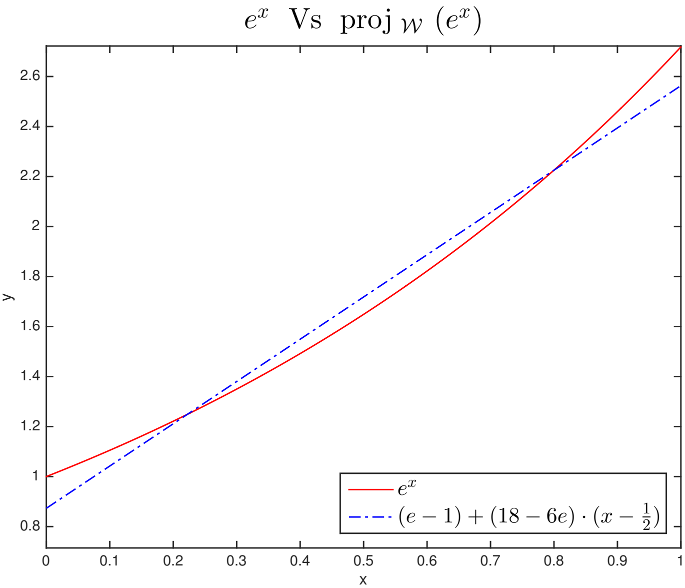
$$\mathbf{y} = \mathbf{p} + \mathbf{z} \implies \mathbf{z} = \mathbf{y} - \mathbf{p} \implies \mathbf{z} = x - \text{proj}_{\mathcal{H}}\, x, \qquad \text{where} \quad \mathcal{H} = \text{span}(1)$$

$$= x - \langle 1, x \rangle 1 = x - \left(\int_0^1 x\,dx\right) 1 = x - \frac{1}{2}$$

Solution

- Thus $\mathcal{Q} = \left\{ \frac{1}{\sqrt{\langle 1,1 \rangle}}, \frac{x-\frac{1}{2}}{\sqrt{\langle x-\frac{1}{2}, x-\frac{1}{2} \rangle}} \right\}$ is an orthonormal basis for $\operatorname{span}(\mathcal{S})$.
- Use the definition of a orthogonal projection of a vector onto a subspace,

$$
\begin{aligned}
\operatorname{proj}_{\mathcal{W}}(\mathbf{v}) &= \left\langle \frac{1}{\sqrt{\langle 1,1 \rangle}}, e^x \right\rangle \frac{1}{\sqrt{\langle 1,1 \rangle}} \\
&\quad + \left\langle \frac{x-\frac{1}{2}}{\sqrt{\langle x-\frac{1}{2}, x-\frac{1}{2} \rangle}}, e^x \right\rangle \frac{x-\frac{1}{2}}{\sqrt{\langle x-\frac{1}{2}, x-\frac{1}{2} \rangle}} \\
&= \frac{\langle 1, e^x \rangle}{\langle 1, 1 \rangle} 1 + \frac{\langle x-\frac{1}{2}, e^x \rangle}{\langle x-\frac{1}{2}, x-\frac{1}{2} \rangle} (x - \frac{1}{2}) \\
&= \frac{\int_0^1 e^x \, dx}{\int_0^1 1 \, dx} + \frac{\int_0^1 (x-\frac{1}{2}) e^x \, dx}{\int_0^1 (x-\frac{1}{2})^2 \, dx} (x - \frac{1}{2}) = e - 1 + (18 - 6e)(x - \frac{1}{2})
\end{aligned}
$$


e^x Vs $\text{proj}_{\mathcal{W}}\,(e^x)$
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0.8
y
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x
e^x
$(e-1)+(18-6e)\cdot(x-\frac{1}{2})$

Definition

For a subspace $\mathcal{H}$ of an inner product space $\mathcal{V}$, the orthogonal complement $\mathcal{H}^\perp$ of $\mathcal{H}$ is the set of all vectors $\mathbf{y}$ in $\mathcal{V}$ that are orthogonal to $\mathcal{H}$:

$$\mathcal{H}^\perp = \{\mathbf{y} \text{ in } \mathcal{V} : \langle \mathbf{v}, \mathbf{y} \rangle = 0, \text{ for all } \mathbf{v} \text{ in } \mathcal{H}\}$$

- Clearly, combine this notion with previous theorems, we have the following,

Theorem

Let $\mathcal{H}$ is a subspace of an inner product space $\mathcal{V}$, if $\mathbf{p}$ is the orthogonal projection of any vector $\mathbf{y}$ in $\mathcal{V}$ onto $\mathcal{H}$, then

$$\mathbf{p} - \mathbf{y} \quad \text{is in the orthogonal complement } \mathcal{H}^\perp$$

Moreover, $\mathbf{p}$ is the vector in $\mathcal{H}$ that is closest to $\mathbf{y}$, that is,

$$|\mathbf{w} - \mathbf{y}| > |\mathbf{p} - \mathbf{y}|$$

for any other vector $\mathbf{w}$ in $\mathcal{H}$ that is distinct from $\mathbf{p}$.

Proof

- Since $\mathbf{w}$ and $\mathbf{p}$ are both in $\mathcal{H}$, then

$$\mathbf{w} - \mathbf{p} \in \mathcal{H}$$

and by theorem on page $\boxed{2}$,

$$\mathbf{p} - \mathbf{y} \in \mathcal{H}^{\perp}$$

- Thus the Pythagoras' theorem states,

$$|\mathbf{w} - \mathbf{y}|^2 = |\mathbf{w} - \mathbf{p}|^2 + |\mathbf{p} - \mathbf{y}|^2$$

- Since $\mathbf{w}$ and $\mathbf{p}$ are distinct, it must be true that $|\mathbf{w} - \mathbf{p}|^2 > 0$

$$|\mathbf{w} - \mathbf{y}|^2 > |\mathbf{p} - \mathbf{y}|^2$$

- Hence $\mathbf{p}$ is the closest vector to $\mathbf{y}$ in $\mathcal{H}$.

- The last theorem provided a way of solving the following problem

Minimum distance problem for functions

Suppose that $\mathcal{C}[a, b]$ has the integral inner product,

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x)g(x)\,dx$$

Given a subspace $\mathcal{H}$ of $\mathcal{C}[a, b]$ and a function f that is continuous on the interval $[a, b]$, find a function $\hat{f}$ in $\mathcal{H}$ that is closest to f in the sense that

$$|\hat{f} - f| < |g - f|$$

for every function g in $\mathcal{H}$ that is distinct from $\hat{f}$.

- Such a function $\hat{f}$ is known as a best approximation to f from $\mathcal{H}$.

Theorem

If $\mathcal{H}$ is a finite-dimensional subspace of $\mathcal{C}[a,b]$, and if

$$\{f_1, f_2, \dots, f_n\}$$

is an orthonormal basis for $\mathcal{H}$, then

each function f in $\mathcal{C}[a,b]$ has a unique best approximation $\hat{f}$ in $\mathcal{H}$,

and that approximation is given by

$$\hat{f} = \langle f, f_1\rangle f_1 + \langle f, f_2\rangle f_2 + \cdots + \langle f, f_n\rangle f_n, \quad \text{where } \langle f, f_k\rangle = \int_a^b f(x) f_k(x)\,dx.$$

- We have an orthonormal basis for the subspace $\mathcal{T}_n$,

$$\mathcal{S} = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \dots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \dots, \frac{\sin nx}{\sqrt{\pi}} \right\}$$

- Therefore we can easily compute the best approximation to some f from $\mathcal{T}_n$.