Vv256 Lecture 14

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- We consider nonhomogeneous linear differential equations again such as

$$a\ddot{y} + b\dot{y} + cy = f(t)$$

where a, b and c are constants.

- While we have discussed using methods of
  1. Undetermined coefficients
  2. Variation of parameters

  to solve this problem, there are reasons to consider a different method.

- The most important reason is that most of examples to date,

  $f(t)$ is continuous and differentiable.

- In many applications, however, it is possible for

  $f(t)$ to be piecewise defined, discontinuous or worse.

- Electrical circuits with a voltage source provide a common situation where the forcing function, that is $f(t)$, is not continuous.
- Let R, L and C be resistance, inductance and capacitance, respectively.
- Recall an RLC circuit can be modelled by

$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

  where E denote an external voltage source, and Q be the electric charge
- If we flip a switch to turn the power on, the input, that is, the voltage

$$E(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t < 4, \\ 100 & \text{if} \quad t \geq 4. \end{cases}$$

  is a step function that leaps from zero to a constant value.

- To motivate the definition of the Laplace transform, consider power series

$$\sum_{n=0}^{\infty} a_n z^n = A(z) \qquad |z| < R \quad \text{or} \quad z \in \mathcal{I}$$

- The coefficients can be thought of as a function

$$a_n = a(n) \qquad n = 0, 1, 2, \ldots$$

- In the interval of convergence R, there seems to be an 1-to-1 correspondence

$$\text{between functions} \qquad a \qquad \text{and} \qquad A$$

- $A(z)$ seems to contain all the "information" about $a(n)$, perhaps surprising that $a(n)$ also contains all the "information" about $A(z)$ for $z \in \mathcal{I}$.
- So power series can be thought as a transformation between $a(n)$ and $A(z)$.

$$a(n) \longrightarrow \boxed{\text{Power series}} \longrightarrow A(z)$$

- For example,

$$a(n) = 1 \longrightarrow A(z) = \frac{1}{1-z} \qquad \text{for} \qquad |z| < 1$$

$$a(n) = \frac{1}{n!} \longrightarrow A(z) = e^z$$

- Now suppose we extend power series to its continuous analogue, that is,

$$r \in \mathbb{R} \qquad \text{instead of} \qquad n = 0, 1, 2, 3, \ldots$$

Q: What will you do in order to sum over the continuous variable r?

$$\sum_{n=0}^{\infty} a_n z^n \xrightarrow{\text{Continuous Analogue}} \int_0^{\infty} a(r) z^r \, dr$$

- Note z^r is the exponential function of base z, which can be changed to

$$z^r = e^{\ln z^r} = e^{r \ln z}$$

- Now let us consider the convergence of this improper integral,

$$\int_0^\infty a(r)z^r\,dr = \int_0^\infty a(r)e^{r\ln z}\,dr$$

Q: Can we allow any z value for an arbitrary $a(r)$?

$$0 < z < 1 \implies \ln z < 0$$

- If we introduce a new variable s to replace $\ln z$

$$s = -\ln z \implies -s = \ln z$$

then we have

$$A(s) = \int_0^\infty a(r)e^{-sr}\,dr \qquad F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

Q: How can we interpret this integral?

Definition

Suppose $f(t)$ is a function defined on the interval $[0, \infty)$. The Laplace transform of $f(t)$, denoted by $F(s)$ or $\mathcal{L}[f]$, is the function defined by

$$F(s) = \mathcal{L}[f] = \int_0^\infty e^{-st} f(t)\,dt$$

provided that the integral converges.

- The Laplace transform is an integral transform,

$$F(s) = \int_a^b K(s,t) f(t)\,dt$$

  where $K(s,t) = e^{-st}$ is called the kernel of the transform.

- The two-sided Laplace transform

$$F(s) = \mathcal{B}[f] = \int_{-\infty}^\infty e^{-st} f(t)\,dt$$

- If $K(s,t) = e^{-2\pi \mathrm{i} ts}$, then we have the Fourier transform of $f(t)$,

$$F(s) = \int_{-\infty}^{\infty} e^{-2\pi \mathrm{i} ts} f(t)\, dt$$

- Notice the sine/cosine series in complex form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=-\infty}^{\infty} c_n e^{\mathrm{i}nx}$$

- The Fourier and Laplace are related. Both transform a function of time into some other domain, in which the problem may be greatly simplified.
- The Fourier is used primarily for steady state analysis, while Laplace is used for transient state analysis.
- The Laplace transform is used to looking for the response to input functions that are pulses, step, delta, while Fourier is used for continuous functions.

Exercise

(a) *Determine whether the Laplace transform exist for* $f(t) = 1$ *for* $t \geq 0$.

(b) *Find* $\mathcal{L}[f(t)]$, *where* $f(t) = t$ *for* $t \geq 0$.

Solution

Basically need to check whether it converges

$$\mathcal{L}[1] = \int_0^\infty e^{-st} \cdot 1\, dt = \lim_{b\to\infty} \left[\frac{-e^{-st}}{s}\right]_0^b$$

$$= \lim_{b\to\infty} \left(\frac{-e^{-sb}}{s} + \frac{1}{s}\right) = \begin{cases} \frac{1}{s} & s > 0 \\ \infty & \text{otherwise} \end{cases}$$

Again we use the definition, and check the convergence

$$\mathcal{L}[t] = \lim_{b\to\infty} \int_0^b te^{-st}\, dt = \lim_{b\to\infty} \left(-\frac{b}{s}e^{-bs} - \frac{1}{s^2}e^{-bs} + \frac{1}{s^2}\right) = \begin{cases} \frac{1}{s^2} & s > 0 \\ \infty & \text{otherwise} \end{cases}$$

- The basic idea behind using the Laplace transform to solve

$$a\ddot{y} + b\dot{y} + cy = f(t)$$

is similar to that of integrating factors for solving first-order linear equations.

- Before, instead of solving the original equation, we go around and solve

$$\alpha\dot{y} + \beta y = \gamma \implies \mu\alpha\dot{y} + \mu\beta y = \mu\gamma$$

- This time, we transform it to an algebraic equation,

$$\begin{tikzcd}
\text{ODE} \arrow[r, "\mathcal{L}"'] \arrow[d] & \text{Algebraic Equation} \arrow[d] \\
y(t) & Y(s) \arrow[l, "\mathcal{L}^{-1}"']
\end{tikzcd}$$

- Recall that the big reason for using the Laplace transform is to deal with

$$a\ddot{y} + b\dot{y} + cy = f(t)$$

where $f(t)$ is not continuous.

- The two functions that we have considered are continuous.

$$\mathcal{L}[1] = \frac{1}{s} \qquad \text{and} \qquad \mathcal{L}[t] = \frac{1}{s^2} \qquad \text{for} \qquad s > 0$$

- However, the real power of the Laplace transform comes from the fact that

discontinuous functions may also be transformed

- So we want address the question

What types of functions have their Laplace transform?

- We will discuss a sufficient condition for the existence the Laplace transform.

- Recall there are various types of discontinuities:

Defintion

- Removable discontinuity: Both $f(c)$ and $\lim_{t \to c} f(t) = L$ exist, but

$$f(c) \neq L$$

in which case we can make f continuous at c by redefining $f(c) = L$.

- Jump discontinuity: Both of the one-sided limits exist, but

$$\lim_{t \to c^-} f(t) \neq \lim_{t \to c^+} f(t)$$

- Essential discontinuity: At least one of the one-sided limits does not exist.

Q: Which type of discontinuity do you expect to be problematic?

Definition

The function f defined on $[0, \infty)$, is said to be piecewise continuous if and only if there exits a finite partition $\{t_1, t_2, \dots, t_i, \dots, t_n\}$ of $[0, \infty)$ such that

1. $f(t)$ is continuous on $[0, \infty)$ except may be for the points t_i,
2. The two one-sided limits of $f(t)$ at the points t_i exist.

Q: Are the following functions piecewise continuous on $[0, 3]$?

$$f(t) = \begin{cases} 2 & t = 1 \\ t & t \neq 1 \end{cases} \quad g(t) = \begin{cases} t^2 + 1, & 0 \leq t \leq 1 \\ 2 - t & 1 < t \leq 2 \\ 1 & 2 < t \leq 3 \end{cases} \quad h(t) = \begin{cases} \dfrac{1}{1 - t} & 0 \leq t < 1 \\ t & 1 < t \leq 3 \end{cases}$$

Exercise

Find the Laplace transform of the piecewise continuous function

$$E(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t < 4, \\ 100 & \text{if} \quad t \geq 4. \end{cases}$$

Solution

By the definition,

$$
\begin{aligned}
\mathcal{L}\left[E\right] = \int_0^\infty e^{-st} E(t)\, dt &= \int_0^4 e^{-st} \cdot 0 \, dt + \int_4^\infty e^{-st} \cdot 100 \, dt \\
&= 0 + \lim_{b\to\infty} \int_4^b e^{-st} \cdot 100 \, dt \\
&= 100 \lim_{b\to\infty} \left[\frac{-e^{-st}}{s}\right]_4^b = 100 \lim_{b\to\infty} \left(\frac{-e^{-sb}}{s} + \frac{e^{-4s}}{s}\right) \\
&= \begin{cases} \dfrac{100e^{-4s}}{s} & s > 0 \\ \infty & \text{Otherwise} \end{cases}
\end{aligned}
$$

Q: Does every piecewise continuous function possess the Laplace transform?

$$f(t) = e^{t^2}$$

- To avoid such functions, which grow too fast, we introduce the following:

Definition

A function f is said to be of exponential order c if there exist constants

1. c, 2. $M > 0$, and 3. $T > 0$

such that

$$|f(t)| \leq Me^{ct} \qquad \text{for all} \qquad t > T$$

Theorem

If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order c, then

$$\mathcal{L}[f(t)]$$

exists for $s > c$.

Proof

- By the additive interval property of integral

$$\mathcal{L}\Big[f(t)\Big] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt = I_1 + I_2$$

- Since $f(t)$ is piecewise continuous, I_1 must exist.
- Because f is of exponential order, there exists c, $M > 0$, and $T > 0$ so that

$$|f(t)| \leq Me^{ct} \qquad \text{for} \quad t > T.$$

which means $|I_2| \leq \displaystyle\int_T^\infty e^{-st}|f(t)|\,dt$

$$\leq M\int_T^\infty e^{-st}e^{ct}\,dt = M\int_T^\infty e^{-(s-c)t}\,dt \leq M\frac{e^{-(s-c)T}}{s-c}$$

- For $s > c$, the right hand side is convergent, so I_2 is convergent for $s > c$ and thus the Laplace transform exists.

- By the last theorem, the following functions have their Laplace transforms

1. Polynomial $P(t)$
2. Sine $\sin kt$ and Cosine $\cos kt$
3. Exponential e^{kt}
4. Sums and products of these functions
5. Piecewise functions with finitely many finite discontinuities made of 1–4.

Q: Can we use the last theorem to determine whether $\mathcal{L}[f(t)]$ exist for

$$f(t) = t^{-1/2}$$

- Recall by using the substitution $u = sx$, we have shown

$$\int_0^\infty x^{z-1}e^{-sx}\,dx = \frac{\Gamma(z)}{s^z} \implies \mathcal{L}\left[t^{-1/2}\right] = \int_0^\infty t^{1/2-1}e^{-sx}\,dt = \frac{\Gamma(\frac{1}{2})}{s^{1/2}}$$

converges for $s > 0$ and diverges for all other s.

Q: How can we compute

$$\mathcal{L}\Big[2t+2\Big] \qquad \text{for} \qquad t \geq 0$$

- Consider the Laplace transform of $c_1 f_1 + c_2 f_2$

$$F(s) = \mathcal{L}[c_1 f_1 + c_2 f_2]$$

where f_1 and f_2 have Laplace transforms for $s > a_1$ and $s > a_2$, respectively.

- Then, for s bigger than the maximum of a_1 and a_2, that is, $s > \max(a_1, a_2)$

$$\begin{aligned} F(s) = \mathcal{L}\Big[c_1 f_1 + c_2 f_2\Big] &= \int_0^\infty e^{-st}(c_1 f_1 + c_2 f_2)\,dt \\ &= \int_0^\infty e^{-st} c_1 f_1\,dt + \int_0^\infty e^{-st} c_2 f_2\,dt \\ &= c_1\mathcal{L}[f_1] + c_2\mathcal{L}[f_2] = c_1 F_1(s) + c_2 F_2(s) \end{aligned}$$

- Therefore the Laplace transform is linear.

Exercise

Suppose $f(t) = 3t + 2$ for $t \geq 0$. Find

$$\mathcal{L}[f(t)]$$

Solution

- Since the Laplace transform is a linear operator

$$\mathcal{L}\Big[3t+2\Big] = 3\mathcal{L}\Big[t\Big] + 2\mathcal{L}\Big[1\Big]$$

$$= \begin{cases} \dfrac{3}{s^2} + \dfrac{2}{s} & \text{if} \quad s > 0 \\ \infty & \text{otherwise} \end{cases}$$

- Often the Laplace transform is only written for values of s that converges, so

$$\mathcal{L}\Big[3t+2\Big] = \frac{3}{s^2} + \frac{2}{s}, \qquad \text{for} \quad s > 0$$

- Recall when we were talking about the Gamma function, we had

$$\int_0^\infty x^{z-1}e^{-sx}\,dx = \frac{\Gamma(z)}{s^z} \implies \int_0^\infty (2-3x+5x^2)e^{-sx}\,dx = \frac{2}{s} - \frac{3}{s^2} + \frac{10}{s^3}$$

and suppose $f(x)$ is a continuous function on $[0,\infty)$, such that

$$\int_0^\infty f(x)e^{-sx}\,dx = -\frac{3}{s} + \frac{10}{s^2} = F(s)$$

Q: What is $f(x)$? Is it related to

$$2-3x+5x^2$$

- Consider the Laplace transform of the derivative of a function $f(t)$ for $t \geq 0$,

$$\mathcal{L}\big[f'(t)\big] = \int_0^\infty e^{-st}f'(t)\,dt = \lim_{b\to\infty}\int_0^b e^{-st}f'(t)\,dt$$

- Apply integration by parts,

$$\begin{aligned}
\mathcal{L}\big[f'(t)\big] &= \lim_{b\to\infty}\int_0^b e^{-st} f'(t)\,dt \\
&= \lim_{b\to\infty}\left(\Big[e^{-st}f(t)\Big]_{t=0}^{t=b} + \int_0^b s e^{-st} f(t)\,dt\right) \\
&= \lim_{b\to\infty}\Big(e^{-sb}f(b) - f(0)\Big) + s\,\lim_{b\to\infty}\int_0^b e^{-st} f(t)\,dt \\
&= -f(0) + s\,\mathcal{L}\big[f(t)\big]
\end{aligned}$$

for $s > c$ when f is exponential order of c.

Q: How about the second derivative

$$f''$$

- Similarly, for higher-order derivatives

$$\begin{aligned}
\mathcal{L}\big[f''(t)\big] = \int_0^\infty e^{-st} f''(t)\,dt = \Big[e^{-st} f'(t)\Big]_0^\infty + s\int_0^\infty e^{-st} f'(t)\,dt \\
= \lim_{r\to\infty}\Big(e^{-sr} f'(r) - e^0 f'(0)\Big) + s\mathcal{L}\big[f'(t)\big] \\
= s\mathcal{L}\big[f'\big] - f'(0) \\
= s^2\mathcal{L}\big[f\big] - sf(0) - f'(0) \qquad \text{for} \quad s > c
\end{aligned}$$

Transform of a derivative

If f, f', ..., $f^{(n-1)}$ are continuous on $[0, \infty)$ and are of exponential order of c and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathcal{L}\big[f^{(n)}(t)\big] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s^0 f^{(n-1)}(0),$$

where $F(s) = \mathcal{L}[f(t)]$ for $s > c$.

Exercise

Solve the following initial-value problem using the Laplace transform

$$\ddot{y} - 3\dot{y} + 2y = e^{-4t}, \qquad y(0) = 1, \qquad \dot{y}(0) = 5$$

Solution

- If two functions are equivalent,

$$f(t) = g(t)$$

  then we expect their Laplace transforms to be the same, in our case here

$$\mathcal{L}\Big[y'' - 3y' + 2y\Big] = \mathcal{L}\Big[e^{-4t}\Big]$$

- Apply the formula for the Laplace transform of the derivatives

$$\begin{aligned}\mathcal{L}\Big[e^{-4t}\Big] &= \mathcal{L}\Big[y''\Big] - 3\mathcal{L}\Big[y'\Big] + 3\mathcal{L}\Big[y\Big] \\ &= s^2Y(s) - sy(0) - y'(0) - 3\Bigg(sY(s) - y(0)\Bigg) + 2Y(s)\end{aligned}$$

Solution

- Now let us find

$$
\begin{aligned}
\mathcal{L}\left[e^{-4t}\right] &= \lim_{b\to\infty}\int_0^b e^{-st}e^{-4t}\,dt \\
&= \lim_{b\to\infty}\int_0^b e^{-(s+4)t}\,dt \\
&= \lim_{b\to\infty}\left[\frac{e^{-(s+4)t}}{-(s+4)}\right]_{t=0}^{t=b} \\
&= \lim_{b\to\infty}\left(\frac{e^{-(s+4)b}}{-(s+4)} - \frac{e^{0}}{-(s+4)}\right) = \frac{1}{s+4} \qquad \text{for} \quad s > -4
\end{aligned}
$$

- So the Laplace transform of $y(t)$, $Y(s)$, must satisfy the algebraic equation

$$
s^2Y(s) - sy(0) - y'(0) - 3\Big(sY(s) - y(0)\Big) + 2Y(s) = \frac{1}{s+4}, \quad s > -4
$$

Solution

- Use the initial condition,

$$y(0) = 1, \qquad y'(0) = 5$$

- We have

$$s^2Y(s) - s - 5 - 3(sY(s) - 1) + 2Y(s) = \frac{1}{s+4} \qquad \text{for} \quad s > -4$$

- Simplify and make $Y(s)$ the subject,

$$Y(s) = \frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)} \qquad \text{for} \quad s > -4$$

- In order to obtain the solution

$$y(t)$$

  we need to find the function whose Laplace transform matches

$$Y(s)$$