Question1 (1 points)

Find the derivative function of the following function using the definition of derivative.

$$g(t) = \frac{1}{\sqrt{t}}$$

State the natural domain of $g(t)$ and the natural domain of its derivative function.

Solution:

1M By definition

$$g'(c) = \lim_{h \to 0} \frac{g(c+h) - g(c)}{h} = \lim_{h \to 0} \frac{\frac{1}{\sqrt{c+h}} - \frac{1}{\sqrt{c}}}{h}$$

Rationalizing the numerator

$$g'(c) = \lim_{h \to 0} \frac{\frac{1}{\sqrt{c+h}} - \frac{1}{\sqrt{c}}}{h} \frac{\frac{1}{\sqrt{c+h}} + \frac{1}{\sqrt{c}}}{\frac{1}{\sqrt{c+h}} + \frac{1}{\sqrt{c}}} = \lim_{h \to 0} \frac{\frac{1}{c+h} - \frac{1}{c}}{h\left(\frac{1}{\sqrt{c+h}} + \frac{1}{\sqrt{c}}\right)} = \lim_{h \to 0} \frac{\frac{-h}{c(c+h)}}{h\left(\frac{1}{\sqrt{c+h}} + \frac{1}{\sqrt{c}}\right)}$$

Simplify and get rid of the factor h in the denominator

$$g'(c) = \lim_{h \to 0} \frac{\frac{-h}{c(c+h)}}{h\left(\frac{1}{\sqrt{c+h}} + \frac{1}{\sqrt{c}}\right)} = \lim_{h \to 0} \frac{\frac{-1}{c(c+h)}}{\frac{1}{\sqrt{c+h}} + \frac{1}{\sqrt{c}}}$$

The natural domain of $g(t)$ is the set of values of t such that $g(t)$ takes real values

$$t > 0$$

which implies $c > 0$, and thus the function

$$\frac{\frac{-1}{c(c+h)}}{\frac{1}{\sqrt{c+h}} + \frac{1}{\sqrt{c}}}$$

is a composition of continuous functions in their domain, thus continuous at $h = 0$,

$$g'(c) = \lim_{h \to 0} \frac{\frac{-1}{c(c+h)}}{\frac{1}{\sqrt{c+h}} + \frac{1}{\sqrt{c}}} = \frac{\frac{-1}{c(c+0)}}{\frac{1}{\sqrt{c+0}} + \frac{1}{\sqrt{c}}} = -\frac{1}{c^2}\frac{\sqrt{c}}{2} = -\frac{1}{2}c^{-3/2}$$

Hence and the derivative function is

$$g'(t) = -\frac{1}{2}t^{-3/2}$$

the natural domain of $g'(t)$ is also

$$t > 0$$

Question2 (1 points)

Show that

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is differentiable but that f' is not continuous at $x = 0$.

Solution:

1M By definition,

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{(h)^2 \sin \frac{1}{h} - 0}{h} = \lim_{h \to 0} h \sin \frac{1}{h}$$

which we have shown to be 0 by the squeeze theorem in class, thus

$$f'(0) = 0$$

For $x \neq 0$, it follows from the product and chain rules that $f'(x)$ is given by

$$\underbrace{f'(x)}_{C} = \underbrace{2x \sin \frac{1}{x}}_{A} - \underbrace{\cos \frac{1}{x}}_{B}$$

We know $A \to 0$ as $x \to 0$, However, B is oscillating between -1 and 1,

$$\lim_{x \to 0} \cos \frac{1}{x}$$

This can be shown by considering the definition of the limit when $\epsilon = \frac{1}{2}$, it is clear that no value of L can satisfy the following,

$$\left| \cos \frac{1}{x} - L \right| < \frac{1}{2}$$

for all values of x in any deleted δ-neighbourhood of 0.

I don't expect a rigorous proof here. If you want one, it can be done by contradiction. See the proof for the Dirichlet function being nowhere continuous. It is similar.

Now if we assume the limit of C exists, then by the sum law the limit of

$$C - A$$

must exists, this leads to a contradiction of the limit of B does not exist, and thus force us to conclude that the limit of C also does not exists. Therefore f' is not continuous at $x = 0$.

Question3 (4 points)

Let f be defined on $(-\infty, \infty)$, and consider the following

$$f(c+h) = f(c) + Ah + \varepsilon(h)$$

where c is a constant and A is not a function of h.

(a) (1 point) Find the simplest expression for the following limit.

$$\lim_{h\to 0} \varepsilon(h)$$

Solution:

1M This is not a trick question. You simply use the given expression and simplify by using the fact that $Ah \to 0$ as $h \to 0$ since A is given to be a constant, not a function h.

$$\lim_{h\to 0} \varepsilon(h) = \lim_{h\to 0} \Big(f(c+h) - f(c) - Ah\Big) = \lim_{h\to 0} f(c+h) - f(c)$$

(b) (1 point) Show that $f(x)$ is continuous at c if and only if

$$\lim_{h\to 0} \varepsilon(h) = 0$$

Solution:

1M Let $x = c + h$, it is clear that $x \to c$ as $h \to 0$, then

$$\lim_{h\to 0} \varepsilon(h) = \lim_{x\to c} f(x) - f(c)$$

Suppose $f(x)$ is continuous, then

$$\lim_{x\to c} f(x) = f(c)$$

and that implies

$$\lim_{h\to 0} \varepsilon(h) = 0$$

Conversely, suppose

$$\lim_{h\to 0} \varepsilon(h) = 0$$

then

$$\lim_{x\to c} f(x) - f(c) = 0 \implies \lim_{x\to c} f(x) = f(c)$$

thus continuous at any point c.

(c) (1 point) Suppose $f(x) = x^2$. Show that $f(x)$ is differentiable and $f'(c) = A = 2c$ if

$$\lim_{h\to 0} \frac{\varepsilon(h)}{h} = 0$$

Solution:

1M This is essentially the half of the proof that you have shown in the lecture slides. The only difference is this is about the specific case $f(x) = x^2$, we have

$$(c+h)^2 = c^2 + Ah + \varepsilon(h) \implies \frac{(c+h)^2 - c^2}{h} - A = \frac{\varepsilon(h)}{h}$$

Therefore

$$\lim_{h\to 0}\frac{\varepsilon(h)}{h}=0 \implies \lim_{h\to 0}\frac{(c+h)^2-c^2}{h}-\lim_{h\to 0}A=0 \implies A=\lim_{h\to 0}\frac{(c+h)^2-c^2}{h}$$

Notice the right-hand side is the definition of the derivative of f at c

$$f'(c)=\lim_{h\to 0}\frac{2ch+h^2}{h}=2c=A$$

(d) (1 point) Suppose $f(x)$ is differentiable and we choose A to be $f'(c)$. Show that

$$\lim_{h\to 0}\frac{\varepsilon(h)}{h}=0$$

What happens to the values of the two limits if an "incorrect" $A\neq f'(c)$ is used ?

$$\lim_{h\to 0}\frac{\varepsilon(h)}{h} \qquad \text{and} \qquad \lim_{h\to 0}\varepsilon(h)$$

Explain your answers using the tangent line approximation.

Solution:

1M The first part is the other half of the proof that you have shown you in the lecture slides. So I will not duplicate the proof here. For the second part,

$$A\neq f'(c)$$

is used, then $\lim_{h\to 0}\frac{\varepsilon(h)}{h}$ is not going to be zero. Since

$$\frac{\varepsilon}{h}$$

represents the difference between the slope of the curve and the slope of the tangent line approximation. Having the wrong A means the line will not be tangent to the curve at $h=0$, thus the difference between the two slopes will not vanish at $h=0$. However, $\lim_{h\to 0}\varepsilon(h)$ is going to be zero, since

$$\varepsilon$$

represents the error of using the tangent line approximation instead of evaluating the function $f(x)$. Since the line intersects the curve at $h=0$, the error will vanish at $h=0$.

Question4 (4 points)

Find the derivative y', show all your workings.

(a) (1 point) $y=x^8-3\sqrt{x}+5x^{-3}$

(b) (1 point) $y=\sin x+2\cos^3 x$

(c) (1 point) $y=\dfrac{\sin x\cos x}{\sqrt{x}}$

(d) (1 point) $y=x^x$

Solution:

(a) Apply the sum and power rules of differentiation, we have

$$y'=8x^7-\frac{3}{2}x^{-1/2}-15x^{-4}$$

(b) Apply the sum and chain rules, we have

$$y' = \cos x - 6\cos^2 x \sin x$$

(c) Apply the product and quotient rules, we have

$$y' = \frac{x^{1/2}\left(\cos^2 x - \sin^2 x\right) - \frac{1}{2}x^{-1/2}\sin x\cos x}{x}$$

(d) By the properties of logarithmic function and exponential function, we have

$$y' = (x^x)' = \left(e^{\ln x^x}\right)' = \left(e^{x\ln x}\right)'$$

by the chain rule and the product rule, we have

$$y' = e^{x\ln x}\cdot(\ln x + 1) = x^x(\ln x + 1)$$

We could Log both sides

$$\ln y = \ln x^x \implies \ln y = x\ln x$$

Differentiate implicitly, we have

$$\frac{1}{y}\frac{dy}{dx} = \ln x + 1 \implies y' = y(\ln x + 1) = x^x(\ln x + 1)$$

However, this needs the assumption that the function is differentiable, which is a correct assumption but I didn't give you. If you are using this assumption, you need to state it for the very least.

Question5 (1 points)
Suppose that $F(x) = (f\circ g)(x)$ and $g(3) = m$, $g'(3) = 4$, $f'(3) = 2$, and $f'(6) = 7$. If your instructor insists that he has given you enough information to determine the value of $F'(3)$, find the value/s that m must take and thus its corresponding value of $F'(3)$.

Solution:

1M Since

$$F(x) = f(g(x))) \implies F'(x) = f'(u)\cdot g'(x). \qquad \text{where } u = g(x).$$

At $x = 3$,

$$F'(3) = f'(u)\cdot g'(3) = 4f'(m)$$

We only have the value of $f'(3)$ and $f'(6)$, so m can either be 3 or 6, which leads to

$$F'(3) = 4\cdot 2 = 8 \qquad \text{or} \qquad F'(3) = 4\cdot 7 = 28$$

Question6 (1 points)

Consider the function

$$f(x) = (x^{156} - 1)g(x)$$

where $g(x)$ is continuous at $x = 1$, and $g(1) = 1$. Find the derivative of $f(x)$ at $x = 1$.

Solution:

1M By definition

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1}$$
$$= \lim_{x\to 1} \frac{(x^{156} - 1)g(x) - (1^{156} - 1)g(1)}{x - 1}$$
$$= \lim_{x\to 1} \frac{(x^{156} - 1)}{x - 1} g(x)$$

By the product rule of limits,

$$f'(1) = \lim_{x\to 1} \frac{(x^{156} - 1)}{x - 1} g(x) = \underbrace{\lim_{x\to 1} \frac{(x^{156} - 1)}{x - 1}}_{A} \underbrace{\lim_{x\to 1} g(x)}_{B}$$

The factor A is essentially the derivative of x^{156} at $x = 1$, thus

$$A = 156x^{155} = 156 \qquad \text{at x = 1,}$$

and B is 1 by the continuity of $g(x)$, hence

$$f'(1) = 156 \cdot 1 = 156$$

Question7 (1 points)

For a constant a, evaluate $\lim_{x\to 0} \dfrac{\sin(a + 2x) - 2\sin(a + x) + \sin(a)}{x^2}$ in terms of a.

Solution:

1M This is actually an alternative formula for the definition of the second derivative function of $\sin x$. Let us see why this is the case in general.

By definition,

$$f''(a) = \lim_{h\to 0} \frac{f'(a+h) - f'(a)}{h} = \lim_{h\to 0} \frac{\lim_{v\to 0} \frac{f(a+h+v)-f(a+h)}{v} - \lim_{u\to 0} \frac{f(a+u)-f(a)}{u}}{h}$$

Note there are three limiting processes. Let us consider the two inner limits first.

$$\lim_{v\to 0} \frac{f(a+h+v) - f(a+h)}{v} \qquad \text{and} \qquad \lim_{u\to 0} \frac{f(a+u) - f(a)}{u}$$

If f is differentiable near a, then both limits must exist, and u and v are two dummy variables, we could use the same dummy variable v and apply the sum law

$$\lim_{u\to 0}\frac{f(a+u)-f(a)}{u}=\lim_{v\to 0}\frac{f(a+v)-f(a)}{v}$$
$$\implies f''(a)=\lim_{h\to 0}\frac{\lim_{v\to 0}\frac{f(a+h+v)-f(a+h)-f(a+v)+f(a)}{v}}{h}$$

Note we need to be more careful with the two remaining limiting processes, since unlike the last case, they are "dependent".

$$\begin{aligned}f''(a)&=\lim_{h\to 0}\frac{\lim_{v\to 0}\frac{f(a+h+v)-f(a+h)-f(a+v)+f(a)}{v}}{h}\\&=\lim_{h\to 0}\lim_{v\to 0}\frac{f(a+h+v)-f(a+h)-f(a+v)+f(a)}{vh}\end{aligned}$$

that is, how quick $h\to 0$ relative to $v\to 0$ may affect the value of the final result. In other word, the relationship between h and v may affect the answer. In general, for example, we don't expect the same value when $h=v$ from when $h=v^2$.

However, if f is twice differentiable, then it doesn't matter how h relates to v, as long as they are approaching zero, we will have the same value. So let $h=v$, that is, h is approaching zero at the same rate as v, then

$$\begin{aligned}f''(a)&=\lim_{h\to 0}\lim_{v\to 0}\frac{f(a+h+v)-f(a+h)-f(a+v)+f(a)}{vh}\\&=\lim_{v\to 0}\frac{f(a+v+v)-f(a+v)-f(a+v)+f(a)}{v^2}\\&=\lim_{v\to 0}\frac{f(a+v+v)-2f(a+v)+f(a)}{v^2}\end{aligned}$$

Compare this with the given limit, and we know $\sin x$ is twice differentiable

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(a+2x)-2\sin(a+x)+\sin(a)}{x^2}&=\lim_{v\to 0}\frac{f(a+2v)-2f(a+v)+f(a)}{v^2}\\&=f''(a)=\frac{d^2}{dx^2}\Big(\sin(x)\Big)\Big|_{x=a}=-\sin(a)\end{aligned}$$

We will discuss another way of evaluating this limit next week, which is based on the fact that this limit is in an indeterminate form of zero over zero. We will discuss why the choice of the relationship between h and v will not affect this limit in summer next year. You can try to evaluate the limit by using $h=v^2$ when $f(x)=\sin x$, and convince yourself that I am not lying!

Question8 (1 points)

For a positive integer n, consider

$$f(x)=a_1\sin(x)+a_2\sin(2x)+\cdots+a_n\sin(nx)$$

where $a_1,a_2,...,a_n$ are real numbers such that $|f(x)|\leq|\sin(x)|$ for all $x\in\mathbb{R}$. Show that

$$|a_1+2a_2+3a_3+...+na_n|\leq 1$$

Solution:

1M Find the derivative function of f by differentiation, we have

$$f'(x) = a_1 \cos(x) + 2a_2 \cos(2x) + ... + na_n \cos(nx)$$

At $x = 0$, we have

$$f'(0) = a_1 + 2a_2 + ... + na_n$$

Now consider the definition of the derivative of f at $x = 0$, we have

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{f(x)}{x}$$

If we take the absolute value of it, and apply a basic limit law, we have

$$\left|f'(0)\right| = \left|\lim_{x\to 0} \frac{f(x)}{x}\right| = \lim_{x\to 0} \left|\frac{f(x)}{x}\right|$$

For all x values, we have

$$|f(x)| \le |\sin(x)| \implies \left|\frac{f(x)}{x}\right| \le \left|\frac{\sin(x)}{x}\right| \implies \lim_{x\to 0} \left|\frac{f(x)}{x}\right| \le \lim_{x\to 0} \left|\frac{\sin(x)}{x}\right| = 1$$

Therefore

$$\left|f'(0)\right| \le 1 \implies |a_1 + 2a_2 + ... + na_n| \le 1$$

Question9 (1 points)
Suppose there is a bowl in the shape of a hemisphere of radius a meters. Water is pouring into the bowl with a constant rate of $5\pi a^3$ cubic meters per second. Find the rate at which the water level is rising in the bowl.

Solution:

1M The volume is given by

$$V = \frac{1}{3}\pi h^2(3a - h)$$

The instantaneous rate change of the volume with respect to h is given by

$$\frac{dV}{dh} = \pi(2ah - h^2)$$

It is clear that the volume V is a differentiable function of h, and it is 1-to-1, thus

$$\frac{dh}{dV} = \frac{1}{\pi(2ah - h^2)}$$

The rate of change V with respect to time is given, thus by the chain rule, we have

$$\frac{dV}{dt} = 5\pi a^3 \implies \frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt} = \frac{5a^3}{(2ah - h^2)}$$