
Question1 (2 points)

Let $\mathbf{A}$ be an $n \times n$ upper triangular matrix with nonzero diagonal elements.

(a) (1 point) Show the product of two upper triangular matrices is upper triangular.

(b) (1 point) Show $\mathbf{A}$ is invertible and the inverse is upper triangular.

Question2 (1 points)

Suppose $\mathbf{A}$ is a skew-symmetric. Show

$$\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = 0$$

Recall $\mathbf{x}$ denotes a column vector unless otherwise specified in this course.

Question3 (1 points)

Consider two commuting matrices $\mathbf{A}$ and $\mathbf{B}$, that is,

$$\mathbf{AB} = \mathbf{BA}$$

Given $\mathbf{A}$ and $\mathbf{B}$ are skew-symmetric, is $\mathbf{AB}$ skew-symmetric? Justify your answer.

Question4 (1 points)

Determine whether the product can be computed using the given partitioning.

$$\left[\begin{array}{cc:c} 2 & 1 & -1 \\ 3 & 0 & 5 \\ \hdashline 4 & 2 & 1 \end{array}\right] \left[\begin{array}{cc:c} 1 & -2 & -1 \\ 3 & 0 & -2 \\ \hdashline -1 & 3 & 2 \end{array}\right]$$

If so, use it to compute the product.

Question5 (1 points)

State the values of the indices p and q such that the following is a term in the determinant of a 4×4 matrix $\mathbf{A}$ with a nonzero Levi-Civita, ε_{1q23}. Determine the value of ε_{1q23}.

$$a_{21}a_{3q}a_{p2}a_{43}$$

Question6 (1 points)

Evaluate the determinant of the following matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \end{bmatrix}$$

Question7 (1 points)

Evaluate the determinant of the matrix by first reducing the matrix to row echelon form and then using some combination of row operations and cofactor expansion.

$$\begin{bmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Question8 (1 points)

Frank and Krystor are playing a game that concerns a 3×3 matrix of zeros and ones,

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

The locations of zeros and ones are decided by Frank and Krystor in turn. Frank can only put 1 while Krystor can only put 0 into the matrix. No player can skip a turn, that is, each player must decide one entry in each turn of the game. The game ends when all 9 entries are decided. If the determinant of the matrix is not zero, then Frank wins. If the determinant is zero, then Krystor wins. Assuming Frank goes first, and both of them are "perfect players" of this game, who will always be the winner? Justify your answer.

Question9 (1 points)

Use the fact that 21375, 38798, 34162, 40223, 79154 are all divisible by 19 to show that

$$\det\left(\begin{bmatrix} 2 & 1 & 3 & 7 & 5 \\ 3 & 8 & 7 & 9 & 8 \\ 3 & 4 & 1 & 6 & 2 \\ 4 & 0 & 2 & 2 & 3 \\ 7 & 9 & 1 & 5 & 4 \end{bmatrix}\right)$$

is divisible by 19 without directly evaluating the determinant.

Question10 (0 points)

Consider the following matrix

$$\mathbf{A} = \begin{bmatrix} e^t & e^{-t} \\ \cos t & \sin t \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} e^t & -e^{-t} \\ \cos t & \sin t \end{bmatrix} \qquad \text{and} \qquad \mathbf{C} = \begin{bmatrix} e^t & e^{-t} \\ -\sin t & \cos t \end{bmatrix}$$

where $t \in \mathbb{R}$.

(a) (1 point (bonus)) Find the derivative

$$\frac{d}{dt}(\det \mathbf{A})$$

(b) (1 point (bonus)) Find

$$\det(\mathbf{B}) + \det(\mathbf{C})$$

What does it suggest to you?

(c) (1 point (bonus)) Based on part (a) and part(b), propose a theorem regarding the derivative of a determinant in general, then prove it.

(d) (1 point (bonus)) What does it mean in terms of Wronskian of a set of functions?

$$\mathcal{S} = \{f_1(t), f_2(t), f_3(t), \cdots, f_n(t)\}$$

where f_i is $n-1$ times differentiable function of t.