
Question1 (2 points)

Consider the following function

$$f\colon [-1,3) \to [1,10], \qquad \text{where} \qquad f(x) = x^2 + 1.$$

(a) (1 point) Identify the domain, codomain and range of

$$f$$

by sketching them on three separate real number lines.

(b) (1 point) Sketch the graph of

$$y = f(x)$$

Question2 (3 points)

Consider the following two vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ x+2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 3x \\ 2x \end{bmatrix}$$

(a) (1 point) Find the linear combination of **u** and **v** that is equal to

$$\mathbf{w} = \begin{bmatrix} 7+3x \\ 14+9x \end{bmatrix}$$

that is, to find α and β such that

$$\alpha\mathbf{u} + \beta\mathbf{v} = \mathbf{w}$$

(b) (1 point) For which x are **u** and **v** linearly independent.

(c) (1 point) Show that the vector component of **u** along **v** is orthogonal to

$$(\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u})$$

Question3 (3 points)

Given the following matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

consider the matrix multiplication between each of the four matrices with the followings

$$\mathbf{e}_x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For example, two of such multiplications are

$$\mathbf{A}\mathbf{e}_x \qquad \text{and} \qquad \mathbf{A}\mathbf{e}_y$$

(a) (1 point) State the geometric effect of multiplying each of the four matrices to a nonzero vector **v** in $\mathbb{R}^2$.

(b) (1 point) Let $\mathbf{R}$ be a 2×2 matrix. Find the elements of $\mathbf{R}$ such that the vector

$$\mathbf{v}' = \mathbf{R}\mathbf{v}$$

is the image of $\mathbf{v}$ after rotating θ degree counterclockwise about the origin.

(c) (1 point) Based on your answer to part (a) and (b), speculate what matrix and matrix multiplication represent in general.

Question4 (3 points)

(a) (1 point) Construct an example of non-zero matrices such that

$$\mathbf{AB} = \mathbf{AC}, \qquad \text{where} \quad \mathbf{B} \neq \mathbf{C}$$

(b) (1 point) Determine the unknown quantities x, y and z in the following expression.

$$2\begin{bmatrix} x & 3 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 2y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ y & z \end{bmatrix}^{\mathrm{T}}$$

(c) (1 point) Determine whether the set $\mathcal{S}$ is a basis for $\mathbb{R}^3$.

$$\mathcal{S} = \{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 11 \\ 17 \\ 19 \end{bmatrix}$$

If so, find the coordinates of $\mathbf{v}$ with respect to $\mathcal{S}$. If not, justify your answer.

Question5 (3 points)

Let $\mathbf{A}$ be an $n \times n$ matrix. The matrix $\mathbf{A}$ is known to be symmetric if

$$\mathbf{A} = \mathbf{A}^{\mathrm{T}}$$

or skew-symmetric matrix if

$$\mathbf{A} = -\mathbf{A}^{\mathrm{T}}$$

(a) (1 point) Suppose $\mathbf{A}$ and $\mathbf{B}$ are symmetric. Determine which of the following two matrices is symmetric and which is skew-symmetric. Justify your answer.

$$\mathbf{AB} + \mathbf{BA} \qquad \text{and} \qquad \mathbf{AB} - \mathbf{BA}$$

(b) (1 point) Suppose $\mathbf{A}$ is a symmetric and $\mathbf{B}$ is a skew-symmetric. Show

$$\mathbf{AB} \text{ is skew-symmetric if and only if } \mathbf{AB} = \mathbf{BA}.$$

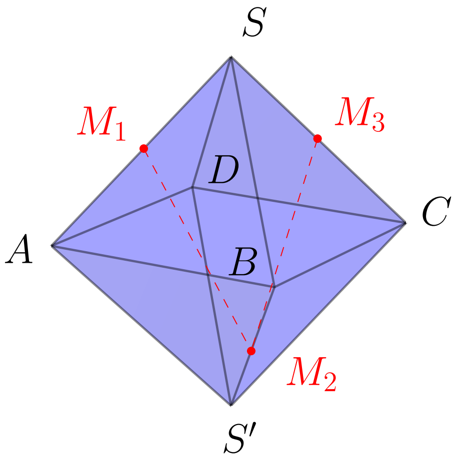
(c) (1 point) Show any $n \times n$ matrix $\mathbf{A}$ can be decomposed into a sum of a symmetric matrix and a skew-symmetric matrix.

Question6 (1 points)

Suppose an object with $10\sqrt{2}$ N is traveling due east at a constant speed of 10 m/s. A wind is blowing towards the northwestern direction with a constant force of 2 N. There also exists a friction with friction coefficient $\mu = 0.25$ between the object and the ground. Specify all the forces acting on the object using vectors in $\mathbb{R}^3$. Use dot product and projection to find how much work is done by the wind over a span of 5 s. How about the friction and gravity? Assume the acceleration due to gravity is $g = 10$ m/s^2.

Question7 (1 points)

Consider a regular octahedron $SABCDS'$,where S and S' are the tips of the octahedron



where points M_1,M_2 and M_3 are the midpoints of sides SA, $S'B$ and SC. Find the angle

$$\angle M_1M_2M_3$$

Question8 (2 points)

Show that all bases for $\mathbb{R}^n$, which is called the n-dimensional Euclidean space, consist of n linearly independent vectors in $\mathbb{R}^n$. You may assume $\mathbb{R}^n$ is n-dimensional.

Question9 (2 points)

Suppose $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are some vectors in $\mathbb{R}^2$. Is it possible that all of the following pairwise dot products are less than zero?

$$\mathbf{u}\cdot\mathbf{v}<0, \qquad \mathbf{u}\cdot\mathbf{w}<0, \qquad \mathbf{v}\cdot\mathbf{w}<0$$

Construct an example or provide a contradiction to support your answer. What is the maximum number of vectors in $\mathbb{R}^3$ can we have such that all of the pairwise dot products of those vectors are less than zero? Justify your answer.

Question10 (0 points)

(a) (1 point (bonus)) If $\mathbf{A}$ is a square matrix, then the trace of $\mathbf{A}$, denoted by

$$\operatorname{tr}(\mathbf{A})$$

is defined to be the sum of the entries on the main diagonal of $\mathbf{A}$, The trace of $\mathbf{A}$ is undefined if $\mathbf{A}$ is not a square matrix. Prove

$$\operatorname{tr}(\mathbf{u}\otimes\mathbf{v})=\mathbf{u}\cdot\mathbf{v}$$

where $\mathbf{u}$ and $\mathbf{v}$ are vectors from the same space.

(b) (1 point (bonus)) Find a geometric interpretation of the tensor product

$$\mathbf{u}\otimes\mathbf{v}$$

in terms of projection.

[Hint: you may find your answers to other questions in this assignment useful.]