Vv255 Lecture 10

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Q: Can a function $f(x,y)$ have partial derivatives with respect to both x and y at a point without being continuous there?

- Consider the partial derivatives of the following function

$$f(x,y) = \begin{cases} 1 & \text{for} \quad x \neq 0 \quad \text{and} \quad y \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

- It is not continuous at $(0,0)$, however, both $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist at the origin.
- So the mere existence of partial derivatives does not guarantee continuity.
- Recall the existence of the derivative is equivalent to differentiability at x_0

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

and it gives the definition of the tangent line at $x = x_0$.

Q: What will be a sensible definition of tangent plane at (x_0, y_0, z_0) on the surface of $z = f(x,y)$, and thus also the definition of differentiability?

Q: Why the mere existence of partial derivatives is not enough here?

Definition

The approximation

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

is known as the linear approximation or tangent line approximation f at x_0.

- Recall another way to view differentiability is to write

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \varepsilon(h)$$

as the sum of a linear approximation of $f(x_0 + h)$ and an error term $\varepsilon(h)$.

Theorem

Suppose $f(x)$ is defined for $a \leq x \leq b$, then $f(x)$ is differentiable at $x \in (a, b)$ if and only if there exists a constant m and a function $\varepsilon(h)$ such that

$$f(x + h) = f(x) + mh + \varepsilon(h), \qquad \text{where} \quad \lim_{h \to 0} \frac{\varepsilon(h)}{h} = 0$$

- It essentially states that being differentiable is equivalent to $\lim_{h \to 0} \frac{\varepsilon(h)}{h} = 0$.

Proof

- First suppose that f is differentiable at x, and define

$$\varepsilon(h) = f(x+h) - f(x) - f'(x)h.$$

- Then

$$\lim_{h \to 0} \frac{\varepsilon(h)}{h} = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} - f'(x) \right] = 0$$

- Conversely, suppose that

$$f(x+h) = f(x) + mh + \varepsilon(h)$$

where $\frac{\varepsilon(h)}{h} \to 0$ as $h \to 0$. Then

$$\lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right] = \lim_{h \to 0} \left[m + \frac{\varepsilon(h)}{h} \right] = m$$

which proves that f is differentiable at x with $f'(x) = m$.

Definition

Suppose $f(x,y)$ is defined for $\mathcal{D} \subset \mathbb{R}^2$, then

$$z = f(x,y)$$

is differentiable at $(x_0, y_0) \in \mathcal{D}$ if and only if there exist constants m and n, and

$$\varepsilon(\Delta x, \Delta y)$$

such that

$$\Delta z = m\Delta x + n\Delta y + \varepsilon(\Delta x, \Delta y)$$

where

$$\lim_{(\Delta x, \Delta y)\to(0,0)} \frac{\varepsilon(\Delta x, \Delta y)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$$

We call f differentiable if it is differentiable at every point in its domain $\mathcal{D}$, and we say that its graph is a smooth surface.

Theorem

If $f(x, y)$ is differentiable function the constants m and n are given by

$$m = f_x(x, y) \qquad \text{and} \qquad n = f_y(x, y)$$

Proof

- Since f is differentiable, we have

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\varepsilon(\Delta x, \Delta y)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$$

$$\lim_{\substack{(\Delta x, \Delta y) \to (0,0) \\ \text{along } \Delta y = 0}} \frac{\Delta z - m\Delta x - n\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$$

$$\lim_{\Delta x \to 0^+} \frac{\Delta z - m\Delta x}{\Delta x} = 0, \qquad \text{if } \Delta x > 0,$$

$$\lim_{\Delta x \to 0^+} \frac{\Delta z}{\Delta x} = m \implies m = f_x(x, y)$$

- It clearly holds for $\Delta x < 0$, and $n = f_y(x, y)$ can be proved in a similar way.

Exercise

Is the function $f(x, y) = \sqrt{x^2 + y^2}$ differentiable at the origin?

Condition for Differentiability

If the partial derivatives, f_x and f_y, of a function $f(x, y)$ exist and are continuous in some open region $\mathcal{R}$, then $f(x, y)$ is differentiable in $\mathcal{R}$.

- So existence alone of the partial derivatives at that point is not enough, but the above condition provide a sufficient criterion for differentiability.
- It is not a necessary condition though, consider the following

$$f(x, y) = \begin{cases} (x^2 + y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & \text{for} \quad (x, y) \neq (0, 0), \\ 0 & \text{for} \quad (x, y) = (0, 0). \end{cases}$$

Theorem

If a function $f(x, y)$ is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

Exercise

Show $f(x,y) = \ln\left(x^2+y^2\right)$ is differentiable everywhere in its domain.

- Although we should be careful about thinking of ordinary derivatives in terms of fraction, they do have fraction-like behaviour, for example,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad \text{and} \qquad \frac{dy}{dx}\frac{dx}{dy} = 1$$

- However, we must be much more cautious with partial derivatives.

Q: Consider the perfect gas law

$$pV = RT$$

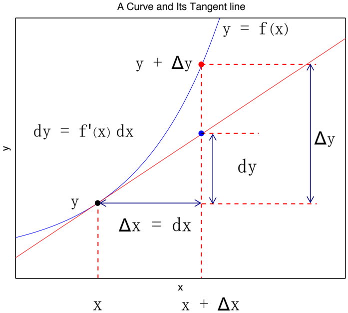
what is $\dfrac{\partial p}{\partial V}\dfrac{\partial V}{\partial T}\dfrac{\partial T}{\partial p}$?

- In fact, we will be able to show if $f(x,y,z) = 0$, then

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1$$

- Recall for a differentiable function of a single variable $y = f(x)$,

  the differential dy and the change Δy are generally different,



- However, the differential dy will nonetheless be a good approximation of Δy

$$\Delta y \approx dy = f' dx, \qquad \text{where } f' \text{ is the ordinary derivative of } x.$$

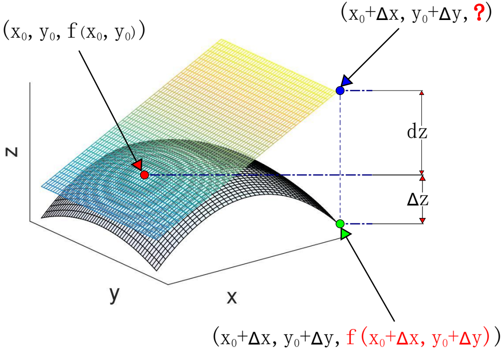
provided the differential $dx = \Delta x$ is small.

- The differential dy gives an estimate of the change in $y = f(x)$ near $x = x_0$

$$\Delta y \approx dy = f'(x_0)dx$$

without knowing the function f, provided we know f' at $x = x_0$.

- Essentially, we are using the tangent approximation near $x = x_0$.
- For a differentiable function $z = f(x, y)$, we want to do something similar.



Definition

Suppose the function

$$z = f(x, y)$$

is differentiable, then

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

is called the total differential of f

Q: Why the total differential dz is a good approximation for Δz near (x_0, y_0)?

- Given small changes Δx and Δy, we have some change in z

$$
\begin{aligned}
\Delta z &= f(x+\Delta x, y+\Delta y) - f(x,y) \\
&= \Big(\underbrace{f(x+\Delta x, y+\Delta y)}_{a} - \underbrace{f(x, y+\Delta y)}_{b}\Big) + \Big(\underbrace{f(x, y+\Delta y)}_{c} - \underbrace{f(x,y)}_{d}\Big)
\end{aligned}
$$

- Recall if $z = f(x,y)$ is differentiable in some open region $\mathcal{D}$ containing (x_0, y_0) , then the partial derivatives f_x and f_y exist for values in $\mathcal{D}$, thus

$$
\begin{aligned}
\underbrace{f(x+\Delta x, y+\Delta y)}_{a} &= \underbrace{f(x, y+\Delta y)}_{b} + m\Delta x + \varepsilon_x(\Delta x), \\
\underbrace{f(x, y+\Delta y)}_{c} &= \underbrace{f(x,y)}_{d} + n\Delta y + \varepsilon_y(\Delta y),
\end{aligned}
$$

where $\displaystyle\lim_{\Delta x \to 0} \frac{\varepsilon_x(\Delta x)}{\Delta x} = 0$ and $\displaystyle\lim_{\Delta y \to 0} \frac{\varepsilon_y(\Delta y)}{\Delta y} = 0$ by the theorem on $\boxed{3}$.

- Therefore,

$$
\begin{aligned}
\Delta z &= \underbrace{f_x(x, y+\Delta y)}_{m}\Delta x + \varepsilon_x(\Delta x) + \underbrace{f_y(x,y)}_{n}\Delta y + \varepsilon_y(\Delta y) \\
&= \Big(f_x(x,y) + \gamma\Big)\Delta x + \varepsilon_x(\Delta x) + f_y(x,y)\Delta y + \varepsilon_y(\Delta y), \qquad \lim_{\Delta y \to 0} \gamma = 0 \\
&= \underbrace{f_x(x,y)\Delta x + f_y(x,y)\Delta y}_{dz} + \underbrace{\gamma\Delta x + \varepsilon_x(\Delta x) + \varepsilon_y(\Delta y)}_{\to 0 \quad \text{as} \quad \Delta x, \Delta y \to 0}
\end{aligned}
$$

- The error $\to 0$ faster than Δx and Δy, we expect the approximation

$$\Delta z \approx dz$$

  becomes better and better as we approaches (x_0, y_0).
- The tangent plane is a linear approximation in the same way that the tangent line is a linear approximation to a function of a single variable.
- Of course, something very similar can be done for a function of 3 variables.

Exercise

(a) Estimate how much the value of

$$f(x, y, z) = y \sin x + 2yz$$

will change if move 0.1 unit from $(0, 1, 0)$ straight toward $(2, 2, -2)$.

(b) The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum possible error in calculating the volume of the cone using those measurements.

(c) Consider the formula

$$S = x \cos \theta$$

for $x = (2.0 \pm 0.2)$ meters and $\theta = (0.9250 \pm 0.0035)$ radians. Estimate the resulting possible error in the calculation of S.