Vv417 Lecture 1

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UM-SJTU Joint Institute

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- Instructor:

  Jing Liu

- Lectures:

Monday	(16:00 – 17:40)	in CRQ-107	Odd weeks only
Tuesday	(16:00 – 17:40)	in CRQ-107	
Thursday	(16:00 – 17:40)	in CRQ-108	

- Office Hours:

Tuesay	(09:00am – 11:00am)	in JI-Building 441A
Thursday	(09:00am – 11:00am)	in JI-Building 441A

- Email:

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  When sending an email related to this course please include the tag [vv417] in the subject e.g. Subject: [vv417] Special Request

- Teaching Assistant/s:

  See Canvas for his/her contact information

Grading Policy

- Assignment:

  30% There will be 8 assignments in the form of problem sets, and may require extra reading and the use of Matlab.

- Exam:

  70% There will be three exams:

Midterm I	Midterm II	Final
20%	20%	30%

- Quiz (Optional):

  20% Quizzes will be given frequently in class.

- For those who attempt all quizzes, their grade is whichever is the higher of:

  0. 30% Assignment + 0% Quiz + 70% Exam
  1. 30% Assignment + 20% Quiz + 50% Exam

- For this course, the grade will be curved to achieve a median grade of "B".

Honour Code

- Assignments/projects are to be solved by each student individually. You are encouraged to discuss problems with other students, but you are advised not to show your written work to others. Copying someone else's work is a very serious violation of the honour code.
- Students may read resources on the Internet, such as articles on Wikipedia, Wolfram MathWorld or any other forums, but you are not allowed to post the original assignment question online and ask for answers. It is regarded as a violation of the honour code.
- Since it is impossible to list all conceivable instance of honour code violations, the students has the responsibility to always act in a professional manner and to seek clarification from appropriate sources if their or another student's conduct is suspected to be in conflict with the intended spirit of the honour code.

Teaching Schedule

Week	Topics
1	Introduction Gaussian Elimination Matrix Multiplication
2	Invertible Matrices Diagonal, Triangular, Symmetric and Block Matrices
3	Determinant The LU Factorisation More determinant and LU
4	**National Holiday**
5	Sparse System Vector Spaces, Subspaces and Spanning sets (Review) Linear Independence (Review)
6	**First Midterm Exam** Fundamental Subspaces

7	Basis and Dimension Rank and Nullity Homomorphism
8	Isomorphism Change of Basis
9	Dual spaces (Optional) Metric, Normed and inner Product Spaces (Review) **Second Midterm Exam**
10	Orthogonality Orthogonalization
11	Applications: Least Squares (Optional) Eigenvalues and eigenvectors Similarity
12	Applications: Page Ranking (Optional) Complex Scalars
13	Hermitian, Unitary, and Normal Matrices Single Value Decomposition Positive Definite Matrices
14	**Final Exam**

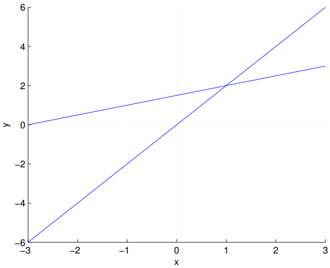
Textbook

- Gilbert STRANG, Introduction to Linear Algebra (4th edition)
- Some Additional Material:
  - Otto BRETSCHER, Linear Algebra with Applications (5th edition)
  - David LAY, Linear Algebra and its Applications (3rd edition)
  - Jim HEFFERON, Linear Algebra (3rd edition)
- More Advanced Material:
  - Stephen FRIEDBERG, Linear Algebra (4th edition)
  - Sheldon AXLER, Linear Algebra Done Right (2nd edition)
  - Kenneth HOFFMAN and Ray KUNZE, Linear Algebra (2nd edition)

- Recall solving a system of two linear equations of two unknown variables

  - Approach 1

$$\begin{aligned} 2x - \quad y &= 0 \\ -x + 2y &= 3 \end{aligned}$$



gives the point of intersection of the two lines if it exits.

Q: How many intersections can two arbitrary straight lines in $\mathbb{R}^2$ have?

Q: How many solutions can an arbitrary system of the following form have?

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

where a_{ij} and b_i are arbitrary constants.

Definition

In general, we say a system of linear equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_2 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

is consistent if it has at least one solution and inconsistent if it has no solution.

- Clearly a system of linear equations is either consistent or inconsistent.

Q: Is it possible to have two and only two distinct solutions for the following?

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 &= b_1 \\
a_{21}x_1 + a_{22}x_2 &= b_2
\end{aligned}
$$

Q: How about a system of three equations of three unknown variables?

Eight possibilities of linear systems of three equations in three unknowns

1. Three parallel planes; no common intersection
2. Two parallel planes; no common intersection
3. No parallel planes and no coincident planes but no common intersection
4. Two coincident planes parallel to the third; no common intersection
5. Intersection is a point
6. Intersection is a line
7. All three planes are all coincident; intersection is a plane
8. Two coincident planes; intersection is a line

- It is difficult to argue the following is true by using the same approach.

Theorem

Every system of linear equations has either zero, one, or infinitely many solutions.

- In general, the study of linear algebra is to reverse this line of thinking.
- It is one of our objectives before midterm I to prove the last theorem, hence confirm our expectation about the number of solutions of a system of linear equations: unique, exactly one, and infinitely many.

Q: If the system is homogeneous, will we still have those three possibilities?

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_2 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{aligned}
$$

Recall the above is known as homogenous.

- Every homogeneous system of linear equations has the trivial solution

$$x_1 = x_2 = \cdots = x_n = 0$$

- Recall a matrix is a rectangular array of numbers, e.g.

$$\begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

which are often denoted by upper-case letters in boldface. i.e.

$$\mathbf{A} = \begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

- The numbers in the brackets are called entries or elements of a matrix,

$$a_{ij} = \big[\mathbf{A}\big]_{ij} \qquad \text{or} \qquad \mathbf{A} = [a_{ij}]$$

where the index i identifies the row it is in, while j identifies the column.

- When we say "an $m \times n$ matrix", we mean a matrix with m rows and n columns, together $m \times n$ is called the size of the matrix.

- The second matrix is said to a square matrix,

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

since it has as many rows as columns.

- For a square matrix, entries on the diagonal collectively, that is,

$$b_{11}, b_{22}, \ldots, b_{nn}$$

is called the main diagonal of the square matrix, other entries as off-diagonal.

- Scalars, row or column vectors are simply considered as special matrices. e.g.

$$\begin{bmatrix} 7 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

- Recall a system of linear equations can be recorded or defined using

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3 \end{aligned} \implies \mathbf{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

where $\mathbf{A}$ is known as the coefficient matrix.

- We can also include the vector $\mathbf{b}$ by augmenting $\mathbf{A}$ with $\mathbf{b}$,

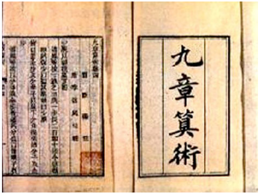
$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 3 \end{bmatrix} = \left[\begin{array}{cc|c} 2 & -1 & 0 \\ -1 & 2 & 3 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{b} \end{array}\right]$$

where $\left[\begin{array}{c|c} \mathbf{A} & \mathbf{b} \end{array}\right]$ is known as the augmented matrix of the system.

- Since we can always go back and recapture the system of linear equations from the augmented matrix $\left[\begin{array}{c|c} \mathbf{A} & \mathbf{b} \end{array}\right]$, it contains all the information of the system, so the augmented matrix is all we need to solve the linear system.

Q: What should we do to $\left[\begin{array}{c|c} \mathbf{A} & \mathbf{b} \end{array}\right]$ to "extract" the solution out of it?

- The most important treatise in the history of Chinese mathematics

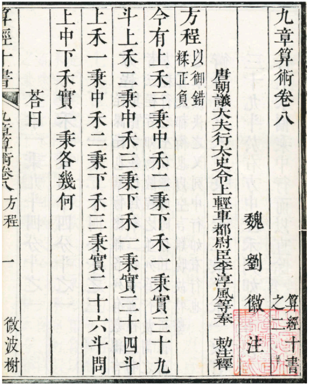


is the *Nine Chapters of the Mathematical Art*.

- This treatise, which is a collection of 246 problems and their solutions, was assembled in its final form in 263 AD by *Liu Hui*.

- Its contents, however, go back to at least the beginning of the Han dynasty.

- In the treatise, it describes a procedure for solving linear systems, the first example in the eighth chapter is the following,



九章算術卷八

魏 劉徽 注

唐朝議大夫行太史令上輕車都尉臣李淳風等奉 勑注釋

方程 以御錯糅正負

今有上禾三秉中禾二秉下禾一秉實三十九斗上禾二秉中禾三秉下禾一秉實三十四斗上禾一秉中禾二秉下禾三秉實二十六斗問上中下禾實一秉各幾何

答曰

算經十書 九章算術卷八 方程 一 微波榭

There are three classes of corn, of which three bundles of the first class, two of the second, and one of the third make 39 measures.

Two of the first, three of the second, and one of the third make 34 measures.

And one of the first, two of the second, and three of the third make 26 measures.

How many measures of grain are contained in one bundle of each class?

- The approach is almost identical to the Gaussian elimination, which is named after Gauss as a result of confusion over the history of the subject.

Q: Can you solve the above problem? Can you do it by manipulating a matrix?

- Let x, y, and z be the measures of the first, second, and third classes of corn
- The conditions of the problem lead to the following linear system

$$
\begin{aligned}
3x + 2y + z &= 39 \\
2x + 3y + z &= 34 \\
x + 2y + 3z &= 26
\end{aligned}
$$

- The solution in the treatise represented the coefficients of each equation by an appropriate number placed within squares on a counting table.
- The counting table was set up so that the coefficients of each equation appear in columns with the first equation in the rightmost column:

1	2	3
2	3	2
3	1	1
26	34	39

- Recall the augmented matrix of a system, it is closely related to the array.

- Next, the numbers within the squares were adjusted following two steps:

1. Two times the numbers of the 3rd column were subtracted from three times the numbers in the 2nd column
2. The numbers in the 3rd column were subtracted from three times the numbers in the 1st column.

- The result was the following array:

1	2	3
2	3	2
3	1	1
26	34	39

$\longrightarrow$

		3
4	5	2
8	1	1
39	24	39

$\longrightarrow$

		3
	5	2
36	1	1
99	24	39

$\longrightarrow$

$$\begin{aligned} 3x + 2y + z &= 39 \\ 5y + z &= 24 \\ 36z &= 99 \end{aligned}$$

- After the first two steps, in the resulting array, four times the numbers in the 2nd column were subtracted from five times the numbers in the 1st column.
- This last array is equivalent to the triangular linear system, which was solved by a method known as back substitution nowadays to obtain

$$x = \frac{37}{4}, \qquad y = \frac{17}{4}, \qquad z = \frac{11}{4}$$

The Ancient Chinese Idea

The fundamental idea for solving a linear system is to perform

- a succession of algebraic operations on the system,

  which do not alter the solution set,

  so that the system becomes increasingly simple.

It continues to a point at which it is clear whether the system is consistent, if so, what the solutions are.

Q: What types of operations can we perform without altering the solution set?

- In terms of scalar equations, we can perform following algebraic operations:

1. Interchange two equations.
2. Multiply an equation through by a nonzero constant.
3. Add a constant times one equation to another.

- Recall we use an augmented matrix to store equations in a row-wise fashion

$$\begin{aligned} 3x + 2y + z &= 39 \\ 2x + 3y + z &= 34 \\ x + 2y + 3z &= 26 \end{aligned} \Longrightarrow \left[\begin{array}{ccc|c} 3 & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{array}\right]$$

- Hence we expect applying the following three operations

Type I	:	Interchange two rows	$E_{i,j}$
Type II	:	Multiply a row by a nonzero constant	$E_{(\alpha)i}$
Type III	:	Add a constant times one row to another	$E_{(\alpha)i,j}$

on an augmented matrix will not alter the underlying solution

$$x = \frac{37}{4}, \qquad y = \frac{17}{4}, \qquad z = \frac{11}{4}$$

- Those three operations are called elementary row operations of a matrix.

Q: Is operation 3. still valid if it is the other way around?

Exercise

Solve the following system by using elementary row operations

$$\begin{array}{rcrcrcr} x & + & y & + & 2z & = & 9 \\ 2x & + & 4y & - & 3z & = & 1 \\ 3x & + & 6y & - & 5z & = & 0 \end{array}$$

followed by back substitution on the corresponding augmented matrix.

Solution

- Writing down the corresponding augmented matrix, we have

$$\begin{array}{rcrcrcr} x & + & y & + & 2z & = & 9 \\ 2x & + & 4y & - & 3z & = & 1 \\ 3x & + & 6y & - & 5z & = & 0 \end{array} \qquad \left[\ \mathbf{A}\ \middle|\ \mathbf{b}\ \right] = \left[\begin{array}{rrr|r} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

- Type III: Adding (-2) times the first row to the second row.

Solution

- Type III: Adding (-2) times the first row to the second row, we have

$$\begin{array}{rcrcrcr} x & + & y & + & 2z & = & 9 \\ & & 2y & - & 7z & = & -17 \\ 3x & + & 6y & - & 5z & = & 0 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

- Type III: Adding (-3) times the first row to the third row, we have

$$\begin{array}{rcrcrcr} x & + & y & + & 2z & = & 9 \\ & & 2y & - & 7z & = & -17 \\ & & 3y & - & 11z & = & -27 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right]$$

- Type III: Adding $(-\frac{3}{2})$ times the second row to the third row, we have

$$\begin{array}{rcrcrcr} x & + & y & + & 2z & = & 9 \\ & & 2y & - & 7z & = & -17 \\ & & & - & \frac{1}{2}z & = & -\frac{3}{2} \end{array} \qquad \left[\begin{array}{rrr|r} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right]$$

- Applying back substitution.

Solution

- Applying back substitution.

$$\begin{array}{rcrcrcr} x & + & y & + & 2z & = & 9 \\ & & 2y & - & 7z & = & -17 \\ & & & - & \frac{1}{2}z & = & -\frac{3}{2} \end{array} \qquad \left[\begin{array}{rrr|r} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right]$$

- The third row gives,

$$z = \frac{b_3}{a_{33}} = 3$$

- Given $z = 3$, the second row gives,

$$y = \frac{b_2 - a_{23}z}{a_{22}} = 2$$

- Given $z = 3$ and $y = 2$, the first row gives,

$$x = \frac{b_1 - a_{12}y - a_{13}z}{a_{11}} = 1$$

- In the West, this approach, which you have known for years, is known as

  Gaussian Elimination with back substitution

- Notice we could, less efficient though, avoid back substitution.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right] \xrightarrow{E_{(-14)3,2}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right]$$

$$\xrightarrow{E_{(4)3,1}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right]$$

$$\xrightarrow{E_{(-\frac{1}{2})2,1}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right]$$

$$\xrightarrow{E_{(\frac{1}{2})2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right] \xrightarrow{E_{(-2)3}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

- Note the initial augmented matrix is reduced to the following

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{b} \end{array}\right] = \left[\begin{array}{rrr|r} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right] \xrightarrow{E} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

which is an example of what is known as reduced row echelon form.

Q: Note only $E_{(\alpha)i}$ and $E_{(\alpha)i,j}$ were used in this case, when do we need $E_{i,j}$?

- To obtain a similarly "informative" form for the following, we need $E_{i,j}$,

$$\begin{bmatrix} 1 & 1 & 3 & -1 \\ 2 & 2 & 6 & -2 \\ 0 & 1 & -4 & 2 \end{bmatrix} \xrightarrow{E} \begin{bmatrix} 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 2 \end{bmatrix}$$

$$\xrightarrow{E} \begin{bmatrix} 1 & 1 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{E} \begin{bmatrix} 1 & 0 & 7 & -3 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

the final matrix is another matrix in reduced row echelon form.