Vv256 Lecture 25

Dr Jing Liu

UM-SJTU Joint Institute

November 30, 2017

- The definition of a vector space has only addition and scalar multiplication but no analogue of the dot product which would allow us to introduce

  length, distance, angle, and orthogonality.

- Recall the dot product in $\mathbb{R}^n$ is defined by and denoted as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \alpha, \qquad \text{where } \mathbf{u}, \mathbf{v} \in \mathbb{R}^n \text{ and } \alpha \in \mathbb{R}.$$

- In essence, the dot product of the vector space $\mathbb{R}^n$ is a way associating

  each pair of vectors to a scalar.

- While
  1. the addition for a vector space $\mathcal{V}$ is a way of associating
     each pair of vectors to another vector
  2. and the scalar multiplication for a vector space $\mathcal{V}$ is a way of associating
     each pair of a vector and a scalar to another vector.

- Recall that the dot product has the following basic properties

Properties

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$, and α be a scalar.

1. Symmetry property
$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$$
2. Distributive property
$$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$$
3. Homogeneity property
$$(\alpha \mathbf{u}) \cdot \mathbf{v} = \alpha(\mathbf{u} \cdot \mathbf{v})$$
4. Positivity property
$$\mathbf{u} \cdot \mathbf{u} \geq 0 \qquad \text{and} \quad \mathbf{u} \cdot \mathbf{u} = 0 \quad \text{if and only if} \quad \mathbf{u} = 0$$

- Those 4 properties lead to many desirable properties of $\mathbb{R}^n$.

Definition

- The length of a vector $\mathbf{v}$ in $\mathbb{R}^n$ is the nonnegative scalar

$$|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}, \qquad \text{so} \quad |\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$$

- The distance between $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is the length of $(\mathbf{u} - \mathbf{v})$. That is,

$$\mathrm{d}(\mathbf{u}, \mathbf{v}) = |\mathbf{u} - \mathbf{v}| = \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})}$$

- If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\mathbb{R}^n$,

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{\mathbf{u}}{|\mathbf{u}|} \cdot \frac{\mathbf{v}}{|\mathbf{v}|}$$

  where θ is defined as the angle between these vectors.

- Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are said to be orthogonal

$$\text{if and only if} \quad \mathbf{u} \cdot \mathbf{v} = 0$$

  and orthonormal if they are unit length as well as being orthogonal.

Definition

- The scalar projection of $\mathbf{y}$ onto $\mathbf{u}$ (the scalar component of $\mathbf{y}$ along $\mathbf{u}$)

$$\operatorname{comp}_{\mathbf{u}} \mathbf{y} = \frac{\mathbf{u} \cdot \mathbf{y}}{|\mathbf{u}|} = |\mathbf{y}| \cos \theta$$

  where θ is the angle between vectors $\mathbf{u}$ and $\mathbf{y}$.

- The vector projection of $\mathbf{y}$ onto $\mathbf{u}$ (the vector component of $\mathbf{y}$ along $\mathbf{u}$)

$$\operatorname{proj}_{\mathbf{u}} \mathbf{y} = \frac{\mathbf{u}}{|\mathbf{u}|} \operatorname{comp}_{\mathbf{u}} \mathbf{y} = \left(\frac{\mathbf{u} \cdot \mathbf{y}}{|\mathbf{u}|} \right) \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{\mathbf{u} \cdot \mathbf{y}}{|\mathbf{u}|^2} \mathbf{u}$$

- The vector component of $\mathbf{y}$ orthogonal to $\mathbf{u}$ is

$$(\mathbf{y} - \operatorname{proj}_{\mathbf{u}} \mathbf{y})$$

- A set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal, that is,

$$\mathbf{u}_i \cdot \mathbf{u}_j = 0, \qquad \text{if} \qquad i \neq j$$

- Notice all of those definitions on the last two pages, and desirable properties of them are based on the definition and the 4 properties of the dot product.

  Cauchy-Schwarz Inequality, The Triangle Inequality, The Parallelogram Law

- To give a vector space a similar geometric structure we need to have a 3rd operation that will generalize the concept of the dot product to general vector spaces so that basic notions such as

  length, distance, angle, and orthogonality.

  can be defined and have similar desirable properties.

- So, if we have an operation that associates with each pair of vectors in a vector space $\mathcal{V}$ to a scalar in $\mathcal{F}$ in such a way that the 4 properties hold, then we are assured that many theorems involve

  length, distance, angle, and orthogonality in $\mathbb{R}^n$

  will have their counterparts in a general vector space $\mathcal{V}$.

Definition

An inner product on a vector space $\mathcal{V}$ is an operation on $\mathcal{V}$ that assigns, to each pair of vectors $\mathbf{u}$ and $\mathbf{v} \in \mathcal{V}$, a scalar $\langle \mathbf{u}, \mathbf{v} \rangle$ in $\mathcal{F}$, satisfying the followings:

1. Symmetry property
$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$$
2. Distributive property
$$\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle, \qquad \text{where} \quad \mathbf{w} \in \mathcal{V}$$
3. Homogeneity property
$$\langle \alpha \mathbf{u}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle, \qquad \text{where } \alpha \text{ is a scalar}$$
4. Positivity property
$$\langle \mathbf{u}, \mathbf{u} \rangle \geq 0 \qquad \text{and} \quad \langle \mathbf{u}, \mathbf{u} \rangle = 0 \quad \text{if and only if} \quad \mathbf{u} = \mathbf{0}$$

- A vector space $\mathcal{V}$ with an inner product is called an inner product space.

Definitions

- The length of a vector $\mathbf{v}$ in $\mathcal{V}$ is

$$|\mathbf{v}| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

- The distance between vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{V}$ is

$$\mathrm{d}\left(\mathbf{u}, \mathbf{v}\right) = |\mathbf{u} - \mathbf{v}| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}$$

- Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{V}$ are said to be orthogonal

$$\text{if and only if} \quad \langle \mathbf{u}, \mathbf{v} \rangle = 0$$

- The angle θ between the vector $\mathbf{u}$ and $\mathbf{v}$ is

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{|\mathbf{u}||\mathbf{v}|}$$

- For two nonzero vectors $\mathbf{y}$ and $\mathbf{u}$ in V, the projection of $\mathbf{y}$ onto $\mathbf{u}$ is

$$\mathrm{proj}_{\mathbf{u}}\, \mathbf{y} = \frac{\mathbf{u}}{|\mathbf{u}|}\, \mathrm{comp}_{\mathbf{u}}\, \mathbf{y} = \left(\frac{\langle \mathbf{u}, \mathbf{y} \rangle}{|\mathbf{u}|} \right) \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{\langle \mathbf{u}, \mathbf{y} \rangle}{|\mathbf{u}|^2} \mathbf{u}$$

- Once we have defined those concepts in a general vector space $\mathcal{V}$ using an inner product, we have the some familiar theorems, three of them are given below for vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space $\mathcal{V}$.

Cauchy-Schwarz inequality

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq |\mathbf{u}||\mathbf{v}|$$

Triangle Inequality

$$|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$$

Parallelogram Law

$$|\mathbf{u} + \mathbf{v}|^2 + |\mathbf{u} - \mathbf{v}|^2 = 2\left(|\mathbf{u}|^2 + |\mathbf{v}|^2\right)$$

- The common inner product for $\mathbb{R}^n$ is the dot product.

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$$

Q: Is there any inner product on $\mathbb{R}^n$ other than the usual dot product?

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + 4 u_2 v_2$$

1. Symmetry

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + 4 u_2 v_2 = v_1 u_1 + 4 v_2 u_2 = \langle \mathbf{v}, \mathbf{u} \rangle$$

2. Distributive

$$\begin{aligned} \langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = (u_1 + v_1) w_1 + 4(u_2 + v_2) w_2 &= u_1 w_1 + 4 u_2 w_2 + v_1 w_1 + 4 v_2 w_2 \\ &= \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle \end{aligned}$$

3. Homogeneity

$$\begin{aligned} \alpha \langle \mathbf{u}, \mathbf{v} \rangle = \alpha (u_1 v_1 + 4 u_2 v_2) &= (\alpha u_1) v_1 + 4 (\alpha u_2) v_2 = \langle \alpha \mathbf{u}, \mathbf{v} \rangle \\ &= u_1 (\alpha v_1) + 4 u_2 (\alpha v_2) = \langle \mathbf{u}, \alpha \mathbf{v} \rangle \end{aligned}$$

4. Positivity

$$\langle \mathbf{u}, \mathbf{u} \rangle = u_1^2 + 4 u_2^2 \geq 0 \qquad \text{and clearly} \qquad \langle \mathbf{u}, \mathbf{u} \rangle = 0 \iff \mathbf{u} = 0$$

- Inner product of a vector space is NOT unique. e.g.

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^{n} u_i v_i w_i \qquad \text{where} \quad \mathbf{w} \in \mathbb{R}^n \quad \text{and} \quad w_i > 0 \quad \text{for} \quad \forall i.$$

- Given $\mathbf{A}$ and $\mathbf{B}$ in $\mathbb{R}^{m \times n}$, one of possible inner products is

$$\langle \mathbf{A}, \mathbf{B} \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij}$$

- For the vector space $C[a, b]$, a valid inner product is

$$\langle f, g \rangle = \int_a^b f(x) g(x) \, dx$$

Exercise

Show the above inner product on $C[a, b]$ satisfies all of the four properties.

Solution

1. Symmetry: $\langle f, g\rangle = \int_a^b f(x)g(x)\,dx = \int_a^b g(x)f(x)\,dx = \langle g, f\rangle$

2. Distributive:

$$\begin{aligned}\langle f+g, h\rangle &= \int_a^b \Big(f+g\Big)(x)h(x)\,dx \\ &= \int_a^b f(x)h(x)\,dx + \int_a^b g(x)h(x)\,dx = \langle f, h\rangle + \langle g, h\rangle\end{aligned}$$

3. Homogeneity: $\alpha\langle f, g\rangle = \alpha\int_a^b f(x)g(x)\,dx = \int_a^b \alpha f(x)g(x)\,dx = \langle \alpha f, g\rangle$
$= \langle f, \alpha g\rangle$

4. Positivity:

$$\langle 0, 0\rangle = \int_a^b 0\cdot 0\,dx = 0 \qquad \text{and} \qquad \langle f, f\rangle = \int_a^b f^2\,dx > 0$$

where f is not identically zero.

- In $\mathbb{R}^n$, it is generally more convenient to use the standard basis

$$\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$$

than to use some other basis.

- In working with an inner product space $\mathcal{V}$, it is generally desirable to have a orthonormal basis as well.

Definition

Let $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$ be nonzero vectors in an inner product space $\mathcal{V}$. If $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ whenever $i \neq j$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to be an orthogonal set of vectors.

Theorem

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space $\mathcal{V}$, then $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$ are linearly independent.

- If $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal set in an inner product space $\mathcal{V}$, then

$$\mathcal{B} \text{ is a basis for the subspace } \mathcal{S} = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k).$$

Definition

An orthonormal set of vectors is an orthogonal set of unit vectors.

$$\langle \mathbf{u}_i, \mathbf{u}_j \rangle = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

Exercise

In $C[-\pi, \pi]$ *with inner product,* $\langle f, g \rangle = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x) g(x)\, dx$*, is the set*

$$\{1, \cos x, \cos 2x, \ldots, \cos nx\}$$

an orthogonal set of vectors?

Solution

- We need to check use the given inner product whether they are orthogonal.

$$\langle 1, \cos kx \rangle, \qquad \langle \cos jx, \cos kx \rangle, \qquad \text{where } j \text{ and } k \text{ are positive integers.}$$

Solution

- The inner products are zero, so it is an orthogonal set,

$$\langle 1, \cos kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos kx \, dx = 0$$

$$\langle \cos jx, \cos kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos jx \cos kx \, dx = 0 \qquad (j \neq k)$$

- The functions $\cos x$, $\cos 2x$, ..., $\cos nx$ are already unit vectors, since

$$\langle \cos kx, \cos kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 kx \, dx = 1 \qquad \text{for} \quad k = 1, 2, \dots, n$$

- To form an orthonormal set, we need only normalize in the direction of 1.

$$|1|^2 = \langle 1, 1 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} 1 \, dx = 2$$

- So the set $\{1/\sqrt{2}, \cos x, \cos 2x, \dots, \cos nx\}$ is an orthonormal set of vectors.

Parseval's Formula

If $\mathcal{S}$ is an orthonormal basis for an inner product space $\mathcal{V}$, and $\mathbf{v} \in \mathcal{V}$, then

$$|\mathbf{v}|^2 = \sum_{i=1}^{n} c_i^2 \qquad \text{where} \qquad \mathbf{v} = \sum_{i=1}^{n} c_i \mathbf{u}_i \qquad \text{and} \qquad \mathbf{u}_i \in \mathcal{S}$$

Exercise

Given $\{\frac{\sqrt{2}}{2}, \cos 2x\}$ is an orthonormal with respect to $\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx$, determine the value of $\int_{-\pi}^{\pi} \sin^4 x\,dx$ without finding any antiderivatives.

Solution

- Since $\sin^2 x = \frac{1}{2}(1 - \cos 2x) = \left(\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} + \left(-\frac{1}{2}\right)\cos 2x$, thus

$$\int_{-\pi}^{\pi} \sin^4 x\,dx = \pi\langle \sin^2 x, \sin^2 x\rangle = \pi|\sin^2 x|^2 = \pi(\frac{1}{2} + \frac{1}{4}) = \frac{3\pi}{4}$$

Definition

A function of the form

$$f(x) = c_0 + (c_1 \cos x + c_2 \cos 2x + \cdots + c_n \cos nx) \\ + (d_1 \sin x + d_2 \sin 2x + \cdots + d_n \sin nx)$$

is called a trigonometric polynomial. If c_n and d_n are not both zero, then $f(x)$ is said to have degree n.

- The set of all trigonometric polynomials of order n or less $\mathcal{T}_n$ is a subspace of
$$F(-\infty, \infty)$$
that is spanned by the functions in the set
$$\mathcal{S} = \{1, \cos x, \cos 2x, \ldots, \cos nx, \sin x, \sin 2x, \ldots, \sin nx\}$$

Exercise

Find an orthonormal basis for $\mathcal{T}_n$ with respect to $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$

Solution

- We have shown functions in red are orthogonal under the inner product, and

$$\begin{aligned}
\langle 1, \cos(kx)\rangle &= 0 \qquad k = 1, \dots, n \\
\langle 1, \sin(kx)\rangle &= 0 \qquad k = 1, \dots, n \\
\langle \sin(jx), \sin(kx)\rangle &= 0 \qquad j, k = 1, \dots, n, \quad \text{and} \quad j \neq k \\
\langle \sin(jx), \cos(kx)\rangle &= 0 \qquad j, k = 1, \dots, n,
\end{aligned}$$

- So $\mathcal{S}$ is an orthogonal basis for the subspace spanned by $\mathcal{S}$.
- If find the length of each vector,

$$|1| = \sqrt{2\pi}, \quad |\cos(kx)| = \sqrt{\pi}, \quad |\sin(kx)| = \sqrt{\pi}, \qquad \text{for } k = 1, \dots, n.$$

- So an orthonormal basis for $\mathcal{T}_n$ is

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos x, \dots, \frac{1}{\sqrt{\pi}} \cos(nx), \frac{1}{\sqrt{\pi}} \sin x, \dots, \frac{1}{\sqrt{\pi}} \sin(nx) \right\}$$