Vv256 Lecture 27

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December 5, 2017

- Trigonometric polynomials are often used to approximate periodic functions

$$f(x+P) = f(x)$$

that is, we want to find a function in $\mathcal{T}_n$ to approximate f.

- To compute the approximation $\hat{f}$, we only need to compute the coefficients by evaluating the inner products between the function and the basis vectors.
- Note that

$$\left\langle f, \frac{1}{\sqrt{2\pi}} \right\rangle \frac{1}{\sqrt{2\pi}} = \langle f, 1 \rangle \frac{1}{2\pi} = \left(\frac{1}{\pi} \int_{-\pi}^{\pi} f\, dx \right) \frac{1}{2}$$

$$\left\langle f, \frac{\cos kx}{\sqrt{\pi}} \right\rangle \frac{\cos kx}{\sqrt{\pi}} = \langle f, \cos kx \rangle \frac{\cos kx}{\pi} = \left(\frac{1}{\pi} \int_{-\pi}^{\pi} f \cos kx\, dx \right) \cos kx$$

$$\left\langle f, \frac{\sin kx}{\sqrt{\pi}} \right\rangle \frac{\sin kx}{\sqrt{\pi}} = \langle f, \sin kx \rangle \frac{\sin kx}{\pi} = \left(\frac{1}{\pi} \int_{-\pi}^{\pi} f \sin kx\, dx \right) \sin kx$$

- The orthogonal projection of f onto $\mathcal{T}_n$, thus the best approximation of f is

$$f \approx \hat{f} = \mathrm{proj}_{\mathcal{T}_n} f = s_n = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx\, dx, \qquad \text{for } k = 1,\ 2,\ \ldots,\ n.$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx\, dx, \qquad \text{for } k = 1,\ 2,\ \ldots,\ n.$$

Definition

The approximation s_n is known as the nth-order Fourier approximation, which is ubiquitous in science and engineering, and the coefficients a_k's and b_k's are called the Fourier coefficients.

Exercise

Find the nth-order Fourier approximation for the 2π-periodic function

$$f(x) = |x| \qquad \text{for } x \in [-\pi, \pi].$$

Solution

- We simply need to find the coefficients

$$a_0, \qquad a_k \qquad \text{and} \qquad b_k$$

- Applying the formulas, we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{2}{\pi}\int_{0}^{\pi} x\,dx = \pi$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos kx\,dx = \frac{2}{\pi}\int_{0}^{\pi} x\cos kx\,dx$$

Solution

- Applying integration by parts, we have

$$a_k = \frac{2}{\pi}\int_0^{\pi} x\cos kx\,dx = \frac{2}{\pi}\left(\left[\frac{x\sin kx}{k}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin kx}{k}\,dx\right)$$
$$= \frac{2\Big((-1)^k - 1\Big)}{\pi k^2} = \begin{cases} 0 & \text{if } k \text{ is even,} \\ \dfrac{-4}{\pi k^2} & \text{if } k \text{ is odd.} \end{cases}$$

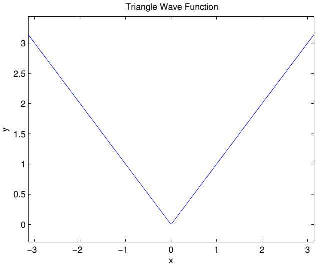
- Since the integrand is an odd function,

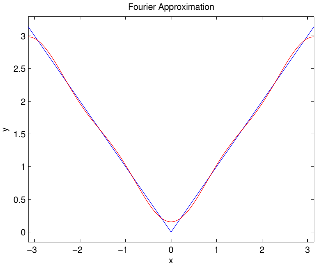
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\sin kx\,dx = 0$$

- Therefore, the nth-order Fourier approximation for $f(x)$ is

$$f \approx \hat{f} = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{\ell=1}^{\lceil n/2\rceil}\frac{1}{(2\ell-1)^2}\cos(2\ell-1)x$$

```
>> syms x k
>> evalin(symengine,'assume(k,Type::Integer)');
%This tells matalb that k is an integer.

>> a = @(f,x,k) int(f*cos(k*x)/pi,x,-pi,pi);%coefficient a
>> b = @(f,x,k) int(f*sin(k*x)/pi,x,-pi,pi);%coefficient b

>> s_n = @(f,x,n) a(f,x,0)/2 + ...
    symsum(a(f,x,k)*cos(k*x) + ...
     b(f,x,k)*sin(k*x),k,1,n);%partial sum

>> f = abs(x); %The oginipal function

>> ezplot(f,[-pi,pi]); hold on
>> xlabel('x');ylabel('y');title('Triangle Wave Function')

>> pretty(s_n(f,x,6))

pi   cos(3 x) 4   4 cos(x)
-- - ---------- - --------
 2      9 pi         pi

>> pretty(simplify(a(f,x,k)))

      k + 1      k     2
  (-1)      ((-1)  - 1)
- ---------------------
               2
           pi k

>> obj = ezplot(s_n(f,x,6),[-pi,pi]);
>> set(obj, 'color','red'); set(obj, 'LineStyle', '-');clear c
>> hold off; title('Fourier Approximation')
```





Exercise

Find the nth-order Fourier approximation of the periodic function

$$f(x) = \begin{cases} -1 & \text{if} \quad -\pi < x < 0 \\ 1 & \text{if} \quad 0 < x < \pi \end{cases} \quad \text{and} \quad f(x+2\pi) = f(x).$$

Solution

- Applying the formulas, we have

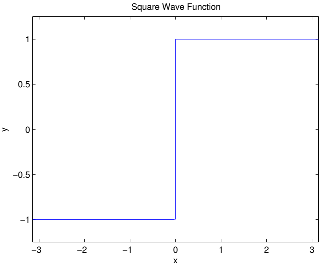
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -1\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1\,dx = -1 + 1 = 0$$

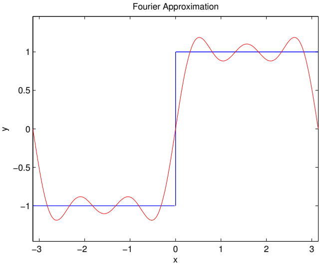
$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} -\cos kx\,dx + \int_{0}^{\pi} \cos kx\,dx\right) = 0$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} -\sin kx\,dx + \int_{0}^{\pi} \sin kx\,dx\right)$$

$$= \frac{2\,(-\cos k\pi + 1)}{k\pi} \implies \hat{f} = \frac{4}{\pi}\sum_{\ell=1}^{\lceil n/2 \rceil} \frac{\sin\Big((2\ell-1)x\Big)}{2\ell-1}$$

```
>> syms x k
>> evalin(symengine,'assume(k,Type::Integer)');
%This tells matalb that k is an integer.

>> a = @(f,x,k) int(f*cos(k*x)/pi,x,-pi,pi);%coefficient a
>> b = @(f,x,k) int(f*sin(k*x)/pi,x,-pi,pi);%coefficient b

>> s_n = @(f,x,n) a(f,x,0)/2 + ...
    symsum(a(f,x,k)*cos(k*x) + ...
     b(f,x,k)*sin(k*x),k,1,n);%partial sum

>> f = 2*heaviside(x)-1;
%This uses heaviside to define our square wave function

>> ezplot(f,[-pi,pi]); hold on
>> obj = line([0,0],[-1,1]);
>> set(obj, 'color','red'); set(obj, 'LineStyle', '-');clear obj
>> xlabel('x');ylabel('y');title('Square Wave Function')

>> pretty(s_n(f,x,6))

sin(3 x) 4   sin(5 x) 4   4 sin(x)
---------- + ---------- + --------
   3 pi         5 pi         pi

>> simplify(b(f,x,k))

ans = (4*sin((pi*k)/2)^2)/(pi*k)

>> obj = ezplot(s_n(f,x,6),[-pi,pi]);
>> set(obj, 'color','red'); set(obj, 'LineStyle', '-.');clear
>> hold off; title('Fourier Approximation')
```



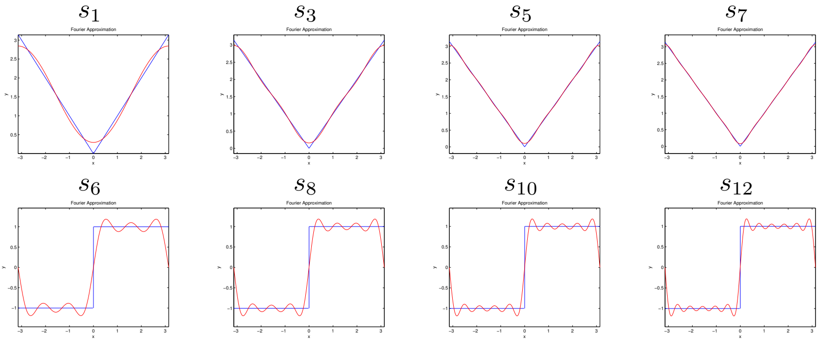


- A central question in Fourier analysis is whether or not

  the approximation s_n converge to f, that is, whether

$$f = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

the formula holds. The series on the right is called the Fourier series.



Definition

The function f defined on $[a, b]$, is said to be piecewise continuous if and only if there exits a partition $\{x_1, x_2, .., x_n\}$ of $[a, b]$ such that

1. $f(x)$ is continuous on $[a, b]$ except may be for the points x_i,
2. the right-limit and left-limit of $f(x)$ at the points x_i exist.

We say that $f(x)$ is piecewise smooth if and only if $f(x)$ as well as its derivatives are piecewise continuous .

Convergence and Sum of a Fourier Series

Let f be periodic with period 2π and piecewise continuous on the interval $[-\pi, \pi]$. Furthermore, suppose $f(x)$ have a left-hand derivative and a right-hand derivative at each point of that interval. Then the Fourier series of $f(x)$ converges

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

to $f(x)$ for all x, except at point x_0 where $f(x)$ is discontinuous, there the series converges to the average of the left- and right-hand limits of $f(x)$ at x_0.

- Before we can prove the main result on convergence of Fourier series, we need some intermediary results. First of all, FTC needs to be modified,

Theorem

For a piecewise smooth function $f(x)$ on the open interval (a, b), if we denote the left-limit and right-limit of $f(x)$ at a point x_0 by

$$\lim_{x \to x_0^-} f(x) = f(x_0^-) \qquad \text{and} \qquad \lim_{x \to x_0^+} f(x) = f(x_0^+),$$

then the fundamental theorem of calculus has the form

$$f(b^-) - f(a^+) = \int_a^b f'(x)\, dx.$$

- We need this version of FTC, because there is no reason for $f(x)$ and $f'(x)$ to be defined at the end-points a and b when $f(x)$ is only piecewise smooth on the open interval (a, b).

Theorem

If $f(x)$ and $f'(x)$ are piecewise continuous on $[a, b]$, then

$$\lim_{\lambda \to \infty} \int_a^b f(x) \sin(x\lambda) dx = 0, \qquad \text{and} \qquad \lim_{\lambda \to \infty} \int_a^b f(x) \cos(x\lambda) dx = 0,$$

Proof

- Let the points

  $$\{x_1 < x_2 < \ldots < x_n\}$$

  be where the functions $f(x)$ and $f'(x)$ are not continuous. Since

  $$\int_a^b f(x) \sin(x\lambda) dx = \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} f(x) \sin(x\lambda)\, dx,$$

  it is enough to show that

  $$\lim_{\lambda \to \infty} \int_{x_i}^{x_{i+1}} f(x) \sin(x\lambda)\, dx = 0.$$

Proof

- Integration by parts gives

$$\int_{x_i}^{x_{i+1}} f(x)\sin(x\lambda)\,dx = \left[\frac{-f(x)\cos(x\lambda)}{\lambda}\right]_{x_i}^{x_{i+1}} + \frac{1}{\lambda}\int_{x_i}^{x_{i+1}} f'(x)\cos(x\lambda)\,dx.$$

- Since $\cos(x\lambda)$ is bounded and $f'(x)$ are bounded on the interval $[x_i, x_{i+1}]$,

$$\lim_{\lambda\to\infty}\frac{\cos(x\lambda)}{\lambda} = 0, \qquad \text{and} \qquad \lim_{\lambda\to\infty}\frac{1}{\lambda}\int_{x_i}^{x_{i+1}} f'(x)\cos(x\lambda)\,dx = 0,$$

- Therefore $\displaystyle\lim_{\lambda\to\infty}\int_{x_i}^{x_{i+1}} f(x)\sin(x\lambda)\,dx = 0.$

- In a similar way, we can show

$$\lim_{\lambda\to\infty}\int_a^b f(x)\cos(x\lambda)\,dx = 0. \quad \square$$

- Recall that our initial problem is to approximate $f(x)$ by Fouier partial sum,

$$s_n = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

- If we substitute a_k and b_k values, we have

$$s_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \sum_{k=1}^{n} \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\Big[\cos(kx)\cos(kt) + \sin(kx)\sin(kt))\Big]\,dt,$$

which can be written in the following form using angle difference identity

$$s_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{k=1}^{n}\cos(k(t-x))\right]\,dt.$$

- Next we use the following identity

$$\frac{1}{\pi}\left(\frac{1}{2} + \sum_{k=1}^{n}\cos(k\alpha)\right) = \frac{\sin((n+\frac{1}{2})\alpha)}{2\pi\sin(\frac{\alpha}{2})}. \qquad (1)$$

- To prove identity (1), we multiply the LHS by $\sin\alpha$,

$$\sin(\alpha)\frac{1}{\pi}\left(\frac{1}{2}+\sum_{k=1}^{n}\cos(k\alpha)\right)=\frac{1}{2\pi}\left(\sin\alpha+2\sum_{k=1}^{n}\cos(k\alpha)\sin\alpha\right)$$

- From the product to sum identity

$$\sin A\cos B=\frac{1}{2}\left(\sin\left(A+B\right)+\sin\left(A-B\right)\right)\ =\frac{1}{2}\left(\sin\left(A+B\right)-\sin\left(B-A\right)\right)$$

- Apply the above result,

$$\begin{aligned}\frac{1}{2\pi}\left[\sin\alpha+2\sum_{k=1}^{n}\cos(k\alpha)\sin\alpha\right]&=\frac{1}{2\pi}\Big[\sin\alpha+\sin(\alpha+\alpha)-\sin(\alpha-\alpha)\\&\qquad+\sin(\alpha+2\alpha)-\sin(2\alpha-\alpha)\\&\qquad+\sin(\alpha+3\alpha)-\sin(3\alpha-\alpha)\cdots\Big]\\&=\frac{1}{2\pi}\left[\sin(n\alpha)+\sin\Big((n+1)\alpha\Big)\right]\end{aligned}$$

- Now we use the sum to product identity

$$\frac{1}{\pi}\left(\frac{1}{2}\sin\alpha+\sum_{k=1}^{n}\cos(k\alpha)\sin\alpha\right)=\frac{1}{2\pi}\left[\sin(n\alpha)+\sin\Big((n+1)\alpha\Big)\right]$$

$$=\frac{1}{\pi}\sin\Big((n+\frac{1}{2})\alpha\Big)\cos\frac{1}{2}\alpha$$

- Divide this by $\sin\alpha$, and then apply double angle identity

$$\text{LHS}=\frac{1}{\pi}\frac{\sin\Big((n+\frac{1}{2})\alpha\Big)\cos\frac{1}{2}\alpha}{\sin\alpha}=\frac{\sin\Big((n+\frac{1}{2})\alpha\Big)\cos\frac{1}{2}\alpha}{2\pi\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha}$$

$$=\frac{\sin\Big((n+\frac{1}{2})\alpha\Big)}{2\pi\sin\frac{1}{2}\alpha}=\text{RHS}$$

- This proves identity (1).

$$\frac{1}{\pi}\left(\frac{1}{2}+\sum_{k=1}^{n}\cos(k\alpha)\right)=\frac{\sin((n+\frac{1}{2})\alpha)}{2\pi\sin(\frac{\alpha}{2})}.$$

- Using identity (1), we get

$$s_n = \frac{\displaystyle\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{k=1}^{n} \cos(k(t-x))\right] dt}{\pi} = \frac{\displaystyle\int_{-\pi}^{\pi} f(t) \frac{\sin\left(\left(n+\frac{1}{2}\right)(t-x)\right)}{2\sin\left(\frac{t-x}{2}\right)} dt}{\pi}$$

- This gives the Fourier approximation of $f(x)$ without the coefficients.

Definition

The Dirichlet kernel is the collection of functions defined by

$$D_n(x) = \begin{cases} \dfrac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{2\pi \sin\left(\dfrac{x}{2}\right)} & x \neq 0, \pm 2\pi, \ldots \\ \dfrac{2n+1}{2\pi} & x = 0, \pm 2\pi, \ldots \end{cases}$$

- The function $D_n(x)$ is even, continuous and periodic, with $\dfrac{2\pi}{n}$ as its period.

- We are now ready to prove the main result, which was done by Dirichlet.
- We essentially need to show $\hat{f} = s_n$ converges to

$$S_f(x) = \begin{cases} f(x) & \text{if } f \text{ is continuous at } x \\ \dfrac{f(x^+) + f(x^-)}{2} & \text{if } f \text{ is discontinuous at } x. \end{cases}$$

- Without loss of generality, we may assume $f(x)$ is defined and 2π-periodic on the entire real line $\mathbb{R}$. So the function $D_n(t-x)f(t)$ is 2π-periodic in t.
- Using the definition of Dirichlet kernel, we obtain

$$s_n = \int_{-\pi}^{\pi} f(t) D_n(t-x)\, dt$$

- If we make the substitution $u = t - x$, then

$$s_n = \int_{-\pi-x}^{\pi-x} f(u+x) D_n(u)\, du = \int_{-\pi}^{\pi} f(u+x) D_n(u)\, du$$

- Since D_n is an even function, we deduce

$$s_n = \int_{-\pi}^{\pi} f(t) D_n(t-x)\, dt = \int_{-\pi}^{\pi} f(t) D_n(x-t)\, dt = \int_{-\pi}^{\pi} f(x-u) D_n(u)\, du$$

- Hence

$$s_n = \int_{-\pi}^{\pi} \frac{f(x+u)+f(x-u)}{2} D_n(u)\, du$$

- Now since the integrand is even, we have

$$s_n = \int_{0}^{\pi} (f(x+u)+f(x-u)) D_n(u)\, du$$

- On the other hand, using identity (1), we have

$$\int_{0}^{\pi} D_n(u)\, du = \frac{1}{2}$$

  which means

$$S_f = \frac{f(x^+)+f(x^-)}{2} = \Big(f(x^+)+f(x^-)\Big) \int_{0}^{\pi} D_n(u)\, du$$

$$s_n - S_f = \int_0^{\pi} [f(x+u) + f(x-u)]\, D_n(u)\, du$$
$$- \int_0^{\pi} \left[f(x^+) + f(x^-)\right] D_n(u)\, du = \int_0^{\pi} \Big(\phi_1(u,x) + \phi_2(u,x)\Big) D_n(u)\, du$$

where $\phi_1(u,x) = f(x+u) - f(x^+)$ and $\phi_2(u,x) = f(x-u) - f(x^-)$.

- To complete the proof, let us show that

$$\lim_{n\to\infty} \int_0^{\pi} \phi_1(u,x) D_n(u)\, du = 0, \quad \text{and} \quad \lim_{n\to\infty} \int_0^{\pi} \phi_2(u,x) D_n(u)\, du = 0$$

- Set $g(u) = \dfrac{\phi_1(u)}{u}$, and

$$U(u) = \begin{cases} \dfrac{u}{2\pi \sin\left(\frac{u}{2}\right)}, & \text{if } u \neq 0; \\ 1/\pi, & \text{if } u = 0. \end{cases}$$

- It can be shown U is continuous and has a continuous derivative on $[0, \pi]$.

- It can be shown $g(u)$ and its derivative are piecewise continuous on $[0, \pi]$

$$\lim_{u \to 0^+} g(u) = f'(x^+) \qquad \text{and} \qquad \lim_{u \to 0^+} g'(u) = f''(x^+)$$

- Thus, $g(u)U(u)$ and its derivative are piecewise continuous on $[0, \pi]$, and

$$\lim_{n \to \infty} \int_0^\pi g(u)U(u) \sin(n + \frac{1}{2})u \, du = 0$$

  by the theorem on $\boxed{12}$.

- And this shows the limit is zero since

$$\int_0^\pi g(u)U(u) \sin(n + \frac{1}{2})u \, du = \int_0^\pi \phi_1(u, x) D_n(u) \, du$$

- That completes the proof since we can show in a similar way,

$$\lim_{n \to \infty} \int_0^\pi \phi_2(u, x) D_n(u) \, du = 0$$

$\square$