Vv255 Lecture 4

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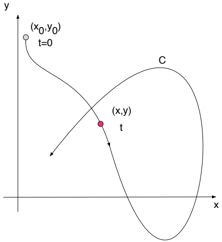
- Recall the trail of an ant on a large table can be described by



- a set of parametric equations

$$x = f(t); \qquad y = g(t)$$

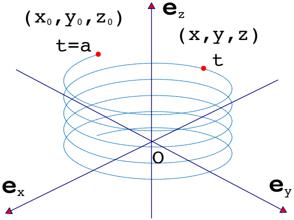
- Such a curve is called a plane curve.

- If it is a honeybee instead, then we need a set of three parametric equations

$$x = f(t), \qquad y = g(t), \qquad z = h(t),$$

$$\text{where} \qquad a \leq t \leq b$$

such a parametric curve is known as a space curve.



- Recall two distinct points define a line in $\mathbb{R}^2$

$$(x_0, y_0), \qquad \text{and} \qquad (x_1, y_1)$$

- The line can be described by the following scalar equation

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$

Q: How can we describe the line parametrically?

- Let $t = x - x_0$, then $x = t + x_0$ and

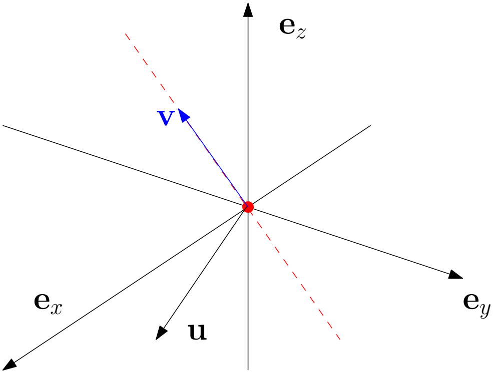
$$y = \frac{y_1 - y_0}{x_1 - x_0}t + y_0 = mt + y_0 \qquad \text{where} \quad m = \frac{y_1 - y_0}{x_1 - x_0}$$

- Thus a set of parametric equations for the line is

$$x = x_0 + t \qquad \text{and} \qquad y = y_0 + mt$$

Q: How can we find a set of parametric equations for a line in $\mathbb{R}^3$?

- First notice any line passes through the origin can be described by a vector



$$\mathbf{r} = t\mathbf{v}$$

$$= t \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} tv_1 \\ tv_2 \\ tv_3 \end{bmatrix}$$

- Thus, for a line passes through the origin,

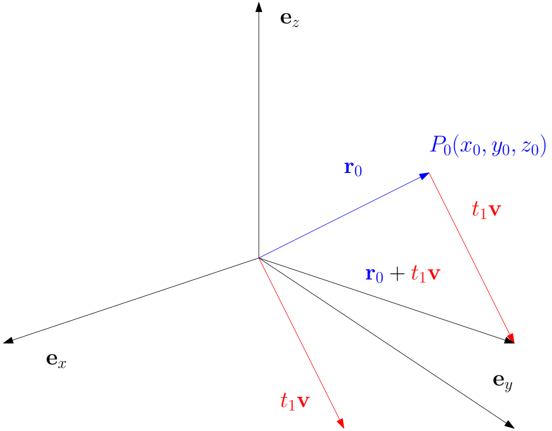
$$x = tv_1, \qquad y = tv_2, \qquad \text{and} \qquad z = tv_3$$

Q: How about lines that are off the origin?

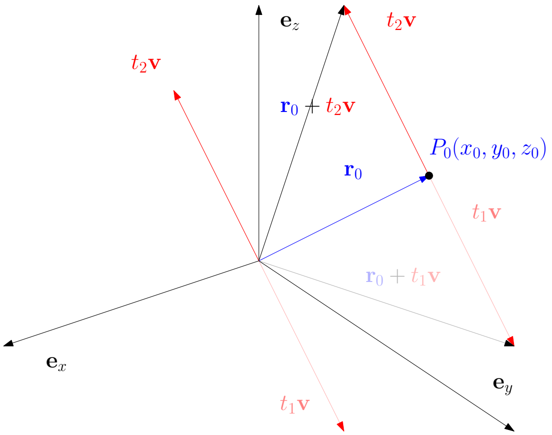
- Let $\mathbf{r}_0$ and $\mathbf{v}$ denote two constant vectors in $\mathbb{R}^3$ and t be a scalar in $\mathbb{R}$, then

$$\mathbf{r}_0 + t\mathbf{v}$$

forms a triangle with vectors $\mathbf{r}_0$ and $t\mathbf{v}$, e.g, if $t = t_1$, we have the following



Q: What happens to $\mathbf{r}_0 + t\mathbf{v}$ if we let t to vary? Where will the new vector be?



Q: What will the vector $\mathbf{r}_0 + t\mathbf{v}$ point at if t is between t_1 and t_2?

- The components of the vector

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} x_0 + tv_1 \\ y_0 + tv_2 \\ z_0 + tv_3 \end{bmatrix}$$

give the coordinates of the points on the line parallel to $\mathbf{v}$ for various t, so

$$x = x_0 + tv_1, \qquad y = y_0 + tv_2, \qquad \text{and} \qquad z = z_0 + tv_3$$

is a parametrization of a line L in $\mathbb{R}^3$.

- The vector $\mathbf{v}$ is parallel to L and $\mathbf{r}_0$ is the position vector of a point P on L.

Q: What is the relationship between a point and its position vector?

Definition

The particular representation of a vector $\vec{OP}$ from the origin O to the point P is called the position vector of the point P.

Q: How about a set of parametric equations and the vector equation?

- Both of the followings describe a line L in $\mathbb{R}^3$

$$\Big\{x = x_0 + tv_1,\, y = y_0 + tv_2,\, z = z_0 + tv_3\Big\}; \quad \mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

- If the vector $\mathbf{v}$, which gives the direction, has components v_1, v_2 and v_3, then the numbers v_1, v_2, and v_3 are called direction numbers of the line.
- If we eliminate the parameter t, we have another way of describing a line.

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}$$

- These equations are called symmetric equations of the line.
- The direction numbers a, b, and c that appear in the denominators.
- If one of v_1, v_2, or v_3 is 0, we can still eliminate t, e.g. if $a = 0$, then

$$x = x_0, \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Q: What does a 0 direction number mean? Can we have 2 of them being 0?

- Vector equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

- Parametric equations

$$x = x_0 + v_1 t \qquad y = y_0 + v_2 t \qquad z = z_0 + v_3 t$$

- Symmetric equations

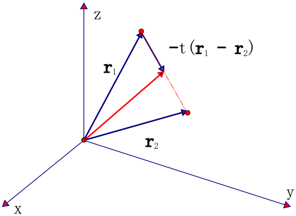
$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}$$

Q: Can we describe a line segment in $\mathbb{R}^3$?

- Line segments from $\mathbf{r}_1$ to $\mathbf{r}_2$ is given by

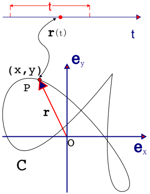
$$\mathbf{r} = (1 - t)\mathbf{r}_1 + t\mathbf{r}_2,$$

where $\quad 0 \le t \le 1$

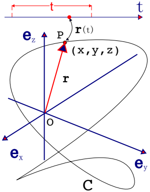


Q: What happens if components of a vector are given by other functions of t?

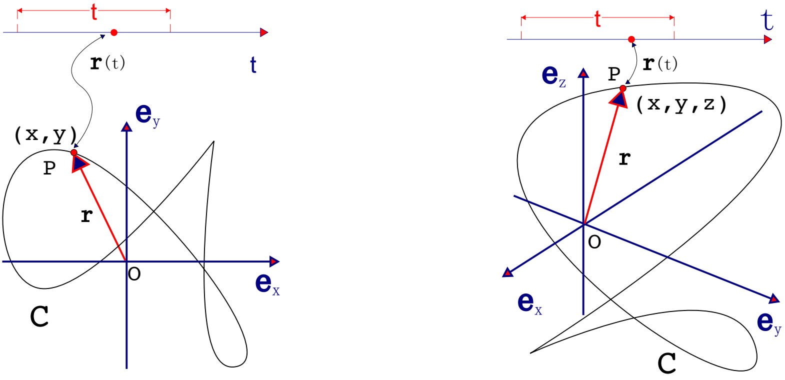
$$\mathbf{r} = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix} = f(t)\mathbf{e}_x + g(t)\mathbf{e}_y, \tag{1}$$



$$\mathbf{r} = \begin{bmatrix} f(t) \\ g(t) \\ h(t) \end{bmatrix} = f(t)\mathbf{e}_x + g(t)\mathbf{e}_y + h(t)\mathbf{e}_z, \tag{2}$$



- We have general plane and space curves, on which any point P is associated with a position vector $\mathbf{r}$, which in turn is associated with a value of t.



- The vector $\mathbf{r}$ is surely not a constant vector, it might change when t changes.

$$\mathbf{r} = \mathbf{r}(t)$$

- For every t, there is one and only one vector $\mathbf{r}$ defined by Eq (1) or Eq (2).

Q: Have you seen something similar before? What is $\mathbf{r}(t)$?

Definition

The rule $\mathbf{r}(t)$ is known as a vector-valued or vector function,

$$\mathbf{r}(t) = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix} \qquad \text{or} \qquad \mathbf{r}(t) = \begin{bmatrix} f(t) \\ g(t) \\ h(t) \end{bmatrix}$$

$$= f(t)\mathbf{e}_x + g(t)\mathbf{e}_y \qquad \text{or} \qquad = f(t)\mathbf{e}_x + g(t)\mathbf{e}_y + h(t)\mathbf{e}_z$$

and $f(t)$, $g(t)$, and $h(t)$ are known as component functions of $\mathbf{r}(t)$.

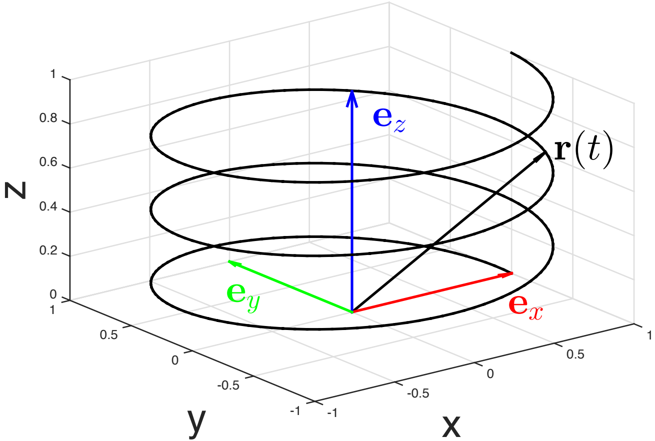
- The corresponding parametric curve of a vector-valued function

  $$x = f(t), \qquad y = g(t), \qquad \text{and/or} \qquad z = h(t)$$

  is defined to be the graph of the vector-valued function.
- The graph, the range of the function, is traced by the vector $\mathbf{r}$ as t changes.

- Here is a vector-valued function and its graph for $0 \le t \le 6\pi$.

$$\mathbf{r}(t) = \cos(t)\mathbf{e}_x + \sin(t)\mathbf{e}_y + \frac{t}{6\pi}\mathbf{e}_z$$



Q: For a nonzero vector in $\mathbb{R}^3$, how many plane can a vector orthogonal to?

Q: For a nonzero vector and a point in $\mathbb{R}^3$, is there always a plane of which the vector is orthogonal to and the point is on that plane?

Q: Is there a second plane of which the vector is orthogonal to and the point is on that plane?

Vector equation

A plane in three-dimensional space is determined by a point $P_0(x_0, y_0, z_0)$ on the plane and a vector $\mathbf{n}$ that is orthogonal to the plane. Let $\mathbf{r}_0$ denote the position vector of the point P_0, then

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

where $\mathbf{r}$ is the position vector of any points on the plane, and the vector

$$\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

is called a normal vector of the plane.

- To obtain a scalar equation for the plane, simply expand the dot product.

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

Exercise

Find a scalar equation of the plane that passes through the points

$$(1, 4, 6), \quad (-2, 5, -1) \quad \text{and} \quad (1, -1, 1)$$

Matlab

```
>> syms x y z real

>> r = [ x y z ];
>> r_1 = [ 1 4 6 ];
>> r_2 = [ -2 5 -1 ];
>> r_3 = [ 1 -1 1 ];
>> A = [ r - r_1; r_2 - r_1 ; r_3 - r_1 ];
```

```
>> det(A)

ans = 15*z - 15*y - 40*x + 10

>> dot (r-r_1, cross(r_2-r_1, r_3-r_1))

ans = 15*z - 15*y - 40*x + 10
```

Definition

Two planes are parallel if their normal vectors are parallel. If two planes are not parallel, they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors $0 < \theta < \frac{\pi}{2}$.

Exercise

(a) Find the angle between the planes

$$x + y + z = 1$$

$$x - 2y + 3z = 1$$

(b) Find symmetric equations for the line of intersection of the two planes.

(c) Find a formula for the distance D from a point $P_1 = (x_1, y_1, z_1)$ to a line that passes through a point $P_0 = (x_0, y_0, z_0)$ and is parallel to a vector $\mathbf{v}$.

(d) Find a formula for the distance D from a point $P_1 = (x_1, y_1, z_1)$ to a plane that is defined by a point $P_0(x_0, y_0, z_0)$ and a normal vector $\mathbf{n}$.

- Vector equation

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

- Scalar equation

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

- Linear equation

$$ax + by + cz + d = 0 \qquad \text{where} \quad d = -ax_0 - by_0 - cz_0$$

- Distance D from a Point to a Line in Space

$$D = \frac{\left|\vec{P_0 P_1} \times \mathbf{v}\right|}{|\mathbf{v}|}$$

- Distance D from a Point to a Plane

$$D = \frac{\left|\vec{P_0 P_1} \cdot \mathbf{n}\right|}{|\mathbf{n}|}$$