Vv156 Lecture 3

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- The previous definition of the limit of a sequence is very vague.
- The definition states:

  If a sequence does indeed get closer and closer to a number L, then the sequence is said to be convergent and we say it has a limit of L.

$$\lim_{n \to \infty} a_n = L$$

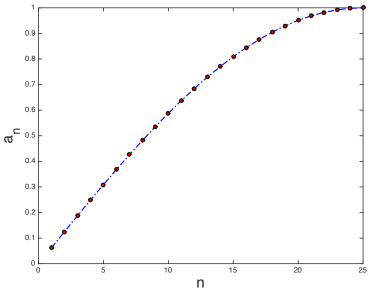
Q: Can you think of any issues with the above definition of limit?

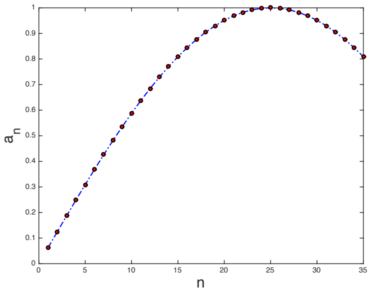
- There are two major issues with the phrase

  "get closer and closer",

- Firstly, consider the following

$$a_n = \sin\left(\frac{n\pi}{50}\right), \qquad \text{for } n \in \mathbb{N}.$$





- We certainly don't mean to define $L = 1$ as the limit of this sequence.
- In general, we don't mean to define L to be the limit of

$$\{a_n\}$$

if $\{a_n\}$ is only getting closer and closer to L for some n up to a certain integer,

$$k$$

after which it moves away from L!

- Again recall, the definition of the limit of sequence

  If a sequence does indeed get closer and closer to a number L, then the sequence is said to be convergent and we say it has a limit of L.

$$\lim_{n\to\infty} a_n = L$$

- For the second issue, consider the following sequence

$$a_n = \frac{1}{n}$$

- Intuitively it is clear that we mean to define zero to be the limit of $\{a_n\}$,

$$\lim_{n\to\infty} a_n = 0$$

  but if we stick to the current definition, what stops someone from claiming

$$\lim_{n\to\infty} a_n = -1$$

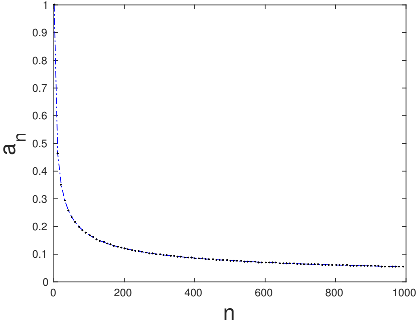
- When we say a_n approaches L, we mean regardless of how short the distance,

$$|a_n - L| < \epsilon, \qquad \text{where} \qquad \epsilon > 0$$

if a large enough n, say N, is chosen, then

$$|a_n - L| < \epsilon \qquad \text{for every } n > N.$$

- The number N tells you how far you have to go to get close to L up to ϵ.



Definition

A sequence $\{a_n\}$ has the limit L, and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \quad \text{as} \quad n \to \infty$$

if for every $\epsilon > 0$ there is a corresponding integer N such that

$$\text{if} \quad n > N \quad \text{then} \quad |a_n - L| < \epsilon$$

- In terms of δ-neighbourhoods of L,

  $$(L - \epsilon, L + \epsilon)$$

  the limit L is a value such that $\{a_n\}$ is eventually in every neighbourhood of it
- Again, if a finite limit L exists, we say the sequence converges or is convergent.
- Otherwise, we say the sequence diverges or is divergent.

Exercise

Show the sequence of reciprocals of natural numbers is convergent.

Definition

A sequence $\{a_n\}$ diverges to infinity, we write

$$\lim_{n\to\infty} a_n = \infty \qquad \text{and} \qquad a_n \to \infty \quad \text{as} \quad n \to \infty$$

if for each number $M \in \mathbb{R}$ there exists a number $N_M \in \mathbb{N}$ such that

$$a_n > M \qquad \text{for all} \quad n > N_M$$

Similarly, a sequence $\{a_n\}$ diverges to negative infinity, we write

$$\lim_{n\to\infty} a_n = -\infty \qquad \text{and} \qquad a_n \to -\infty \quad \text{as} \quad n \to \infty$$

if for each number $m \in \mathbb{R}$ there exists a number $N_m \in \mathbb{N}$ such that

$$a_n < m \qquad \text{for all} \quad n > N_m$$

Exercise

Show the sequence of even numbers diverges to infinity.

Limit Laws

Let $\{a_n\}$ and $\{b_n\}$ be two sequences such that $\lim\limits_{n\to\infty} = L_a$ and $\lim\limits_{n\to\infty} = L_b$.

1 The limit of a constant sequence is the constant itself.

$$\lim_{n\to\infty} a = a$$

2 The limit of a sum/difference is the sum/difference of the limits.

$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = L_a + L_b$$

3 The limit of a product is the product of the limits.

$$\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = L_a L_b$$

4 The limit of a quotient is the quotient of the limits

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} = \frac{L_a}{L_b}, \qquad \text{provided} \quad L_b \neq 0 \quad \text{and} \quad b_n \neq 0$$

Exercise

Is the sequence

$$a_n = \frac{n}{10+n}$$

convergent or divergent? If it is convergent, find what it converges to.

Proof

- For the product law, we need to show that for $\epsilon > 0$, there exists N such that

$$|a_n b_n - L_a L_b| < \epsilon \qquad \text{for all} \quad n > N$$

- Since $\{a_n\}$ is convergent, it is bounded. Let M be the bound, i.e.

$$|a_n| < M$$

- Since $\{a_n\}$ converges to L_a, for $\epsilon_1 = \frac{\epsilon}{2(|L_b|+1)} > 0$, there exists N_1 such that

$$|a_n - L_a| < \frac{\epsilon}{2(|L_b|+1)} \qquad \text{for all} \quad n > N_1$$

Proof

- Similarly, there exists N_2 such that

$$|b_n - L_b| < \frac{\epsilon}{2(M+1)} \qquad \text{for all} \quad n > N_2$$

- Let $N = \max(N_1, N_2)$. Then, for $n > N$, consider

$$\begin{aligned} |a_n b_n - L_a L_b| &= |a_n b_n - a_n L_b + a_n L_b - L_a L_b| \\ &= |a_n(b_n - L_b) + L_b(a_n - L_a)| \\ &\leq |a_n(b_n - L_b)| + |L_b(a_n - L_a)| \\ &\leq |a_n||b_n - L_b| + |L_b||a_n - L_a| \\ &< M\frac{\epsilon}{2(M+1)} + |L_b|\frac{\epsilon}{2(|L_b|+1)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

$$|a_n b_n - L_a L_b| < \epsilon \qquad \text{for all} \quad n > N \quad \square$$

Definition

- The sequence $\{a_n\}$ is said to be increasing if

$$a_{n+1} \geq a_n \qquad \text{for all } n.$$

  and it is said to be decreasing if

$$a_{n+1} \leq a_n \qquad \text{for all } n.$$

- A sequence $\{a_n\}$ is said to be monotonic if it is one of those cases.

Monotonic Sequence Theorem

A monotonic sequence converges if and only if it is bounded.

Exercise

Suppose $a_n = \left(1 + \frac{1}{n}\right)^n$ *for* $n \in \mathbb{N}$. *Show the sequence is convergent.*

Squeeze Theorem

Suppose $\{a_n\}$ and $\{b_n\}$ are convergent, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$$

If for some $N \in \mathbb{N}$,

$$a_n \leq c_n \leq b_n \qquad \text{for all} \quad n > N$$

then the sequence $\{c_n\}$ is convergent. Moreover,

$$\lim_{n\to\infty} c_n = L$$

Exercise

Show the sequence $\left\{ \dfrac{4^n}{n!} \right\}$ *converges to zero.*

Proof

- We need to show that for each $\epsilon > 0$, there exists N such that

$$n > N \implies |c_n - L| < \epsilon$$

- The sequence $\{a_n\}$ converges to L, thus there exists N_a, when $n > N_a$, then

$$|a_n - L| < \epsilon \iff -\epsilon < a_n - L < \epsilon$$

- Similarly, for $\{b_n\}$, there exists $n > N_b$, when $n > N_b$, then

$$|b_n - L| < \epsilon \iff -\epsilon < b_n - L < \epsilon$$

- Let $N = \max(N_a, N_b)$. For $n > N$, we have

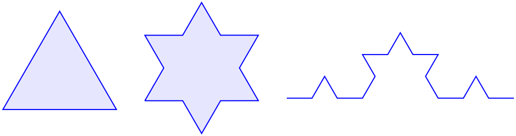
$$\begin{aligned} a_n \le c_n \le b_n &\iff a_n - L \le c_n - L \le b_n - L \\ &\iff -\epsilon < a_n - L \le c_n - L \le b_n - L < \epsilon \\ &\iff |c_n - L| < \epsilon \quad \square \end{aligned}$$

- The concept of sequence and limit of it is largely a stepping stone to

the limit of a function

- However, it is useful in terms of analysing anything to do with infinity.

- Consider an equilateral triangle,



1. Divide each side into three segments of equal length.
2. Create new equilateral triangles that have the middle segment from step 1. as its base and points outward.
3. Remove the line segments that are the bases of the new triangles from step 2.

- Continue the above three steps indefinitely for all sides.

Q: What is the perimeter of the object as the number of iterations $\to \infty$?