Vv255 Lecture 27

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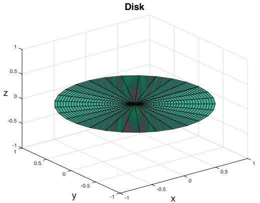
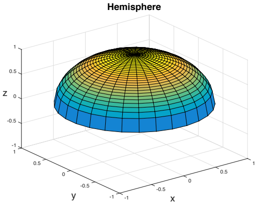
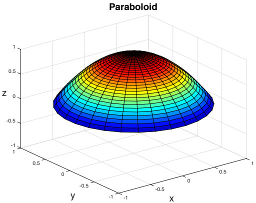
- Consider the following 3 surfaces with upwards orientation

$$\mathcal{S}_1: x^2 + y^2 \le 1 \qquad \mathcal{S}_2: z = \sqrt{1 - x^2 - y^2} \qquad \mathcal{S}_3: z = 1 - x^2 - y^2$$







- Let all three surfaces be placed in the same vector field,

$$\mathbf{F} = 2y\mathbf{e}_x + (2z - 2x)\,\mathbf{e}_y + \mathbf{e}_z$$

- Suppose we want to find the flux integral of $\mathbf{F}$ across those surfaces upwards

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dS$$

- All three surfaces can be easily parametrized using cylindrical coordinates

$$x = u\cos v \qquad y = u\sin v \qquad z = z$$

- For the disk, we have

$$\mathbf{r}(u,v) = u\cos v\mathbf{e}_x + u\sin v\mathbf{e}_y + 0\mathbf{e}_z$$

where $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$

- For the hemisphere, we have

$$\mathbf{r}(u,v) = u\cos v\mathbf{e}_x + u\sin v\mathbf{e}_y + \sqrt{1-u^2}\mathbf{e}_z$$

where $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$

- For the paraboloid, we have

$$\mathbf{r}(u,v) = u\cos v\mathbf{e}_x + u\sin v\mathbf{e}_y + (1-u^2)\mathbf{e}_z$$

where $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$

- We need the unit normal vectors,

$$\mathbf{n}$$

- We need to check the parametrization we have given to those three surfaces.

$$\mathbf{r}_u \times \mathbf{r}_v$$

- If the positive orientation of $\mathbf{r}(u, v)$ is consistent with what is required, i.e.

  the upwards direction,

  then we will use

$$\mathbf{n} = \mathbf{n}_1 = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

  if not, then we will use

$$\mathbf{n} = \mathbf{n}_2 = -\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

- Compute the cross product between partial derivatives for each surface,

$$\mathbf{r}_u \times \mathbf{r}_v = \left(u\cos^2 v + u\sin^2 v\right)\mathbf{e}_z \qquad \text{for } \mathcal{S}_1$$

$$\mathbf{r}_u \times \mathbf{r}_v = \frac{u^2\cos v}{\sqrt{1-u^2}}\mathbf{e}_x + \frac{u^2\sin v}{\sqrt{1-u^2}}\mathbf{e}_y + \left(u\cos^2 v + u\sin^2 v\right)\mathbf{e}_z \qquad \text{for } \mathcal{S}_2$$

$$\mathbf{r}_u \times \mathbf{r}_v = 2u^2\cos v\mathbf{e}_x + 2u^2\sin v\mathbf{e}_y + \left(u\cos^2 v + u\sin^2 v\right)\mathbf{e}_z \qquad \text{for } \mathcal{S}_3$$

- Since the z-component is always positive, it means the normal vector

$$\mathbf{n} = \mathbf{n}_1 = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

is always upwards, so we use the positive orientation, and the flux is given by

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = \iint_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{\mathcal{D}} \mathbf{F}\cdot\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}|\mathbf{r}_u \times \mathbf{r}_v|\,dA$$

$$= \iint_{\mathcal{D}} \mathbf{F}\cdot\left(\mathbf{r}_u \times \mathbf{r}_v\right)\,dA$$

- Compute the dot product for each surface, which forms the integrand,

$$\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = u \qquad \text{for } \mathcal{S}_1$$

$$\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 2u^2 \sin(v) + u \qquad \text{for } \mathcal{S}_2$$

$$\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 4u^2 \left(1 - u^2\right) \sin v + u \qquad \text{for } \mathcal{S}_3$$

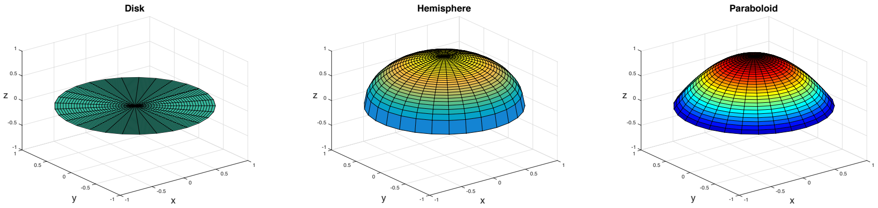
- The flux of $\mathbf{F}$ across each surface upwards is given by

$$\iint_{\mathcal{D}} \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\ dA = \int_0^1 \int_0^{2\pi} u\, dv\, du = \pi \qquad \text{for } \mathcal{S}_1$$

$$\iint_{\mathcal{D}} \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\ dA = \int_0^1 \int_0^{2\pi} \left(2u^2 \sin(v) + u\right)\ dv\, du = \pi \qquad \text{for } \mathcal{S}_2$$

$$\begin{aligned}\iint_{\mathcal{D}} \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\ dA &= \int_0^1 \int_0^{2\pi} \left(4u^2 \left(1 - u^2\right) \sin v + u\right)\ dv\, du \qquad \text{for } \mathcal{S}_3 \\ &= \pi\end{aligned}$$

- Is this a coincidence? What do those three integrals have in common?



- All three surfaces have the same boundary curve

$$\partial\mathcal{S}_1 = \partial\mathcal{S}_2 = \partial\mathcal{S}_3 = \mathcal{C}$$

where $\mathcal{C}$ is the circle

$$\mathbf{r}(t) = \cos t\,\mathbf{e}_x + \sin t\,\mathbf{e}_y + 0\mathbf{e}_z \qquad \text{for} \quad 0 \le t \le 2\pi$$

- All three surfaces are placed in the same vector field,

$$\mathbf{F} = 2y\mathbf{e}_x + (2z - 2x)\,\mathbf{e}_y + \mathbf{e}_z$$

Q: Do you notice anything special about this vector field?

Matlab

```
%Disk

>> syms u v x y z

>> assume(0 <= u <= 1);

>> assume(0 <= v <= 2*pi);

>> r = [ u*cos(v); u*sin(v); 0];

>> r_u = diff(r,u);

>> r_v = diff(r,v);

>> n = cross( r_u, r_v);

>> F = [ z^2 ; x; x^2+y^2];

>> G = curl(F, [x,y,z]);

>> G_s = subs(G,[x,y,z], ...
 r(1),r(2),r(3));

>> f = dot(G_s,n);

>> int(int(f,v,0,2*pi),u,0,1)

ans =

pi
```

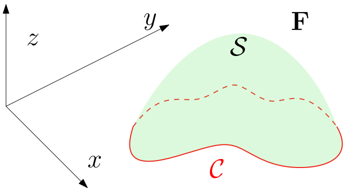
Matlab

```
%Hemisphere

>> syms u v x y z real

>> assume(0 <= u <= 1);

>> assume(0 <= v <= 2*pi);

>> r = [ u*cos(v); u*sin(v); ...
sqrt(1-u^2)];

>> r_u = diff(r,u);

>> r_v = diff(r,v);

>> n = cross( r_u, r_v);

>> F = [ z^2 ; x; x^2+y^2];

>> G = curl(F, [x,y,z]);

>> G_s = subs(G,[x,y,z], ...
r(1),r(2),r(3));

>> f = dot(G_s,n);

>> int(int(f,v,0,2*pi),u,0,1)

ans =

pi
```

Matlab

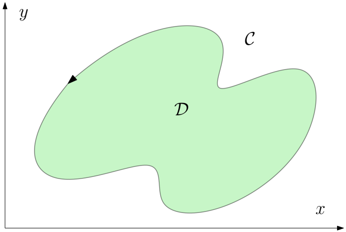
```
%Paraboloid

>> syms u v x y z real

>> assume(0 <= u <= 1);

>> assume(0 <= v <= 2*pi);

>> r = [ u*cos(v); u*sin(v);...
1-u^2];

>> r_u = diff(r,u);

>> r_v = diff(r,v);

>> n = cross( r_u, r_v);

>> F = [ z^2 ; x; x^2+y^2];

>> G = curl(F, [x,y,z]);

>> G_s = subs(G,[x,y,z],...
r(1),r(2),r(3));

>> f = dot(G_s,n);

>> int(int(f,v,0,2*pi),u,0,1)

ans =

pi
```

- Let $\mathcal{S}$ be an oriented piecewise smooth parametric surface $\mathbf{r}(u, v)$

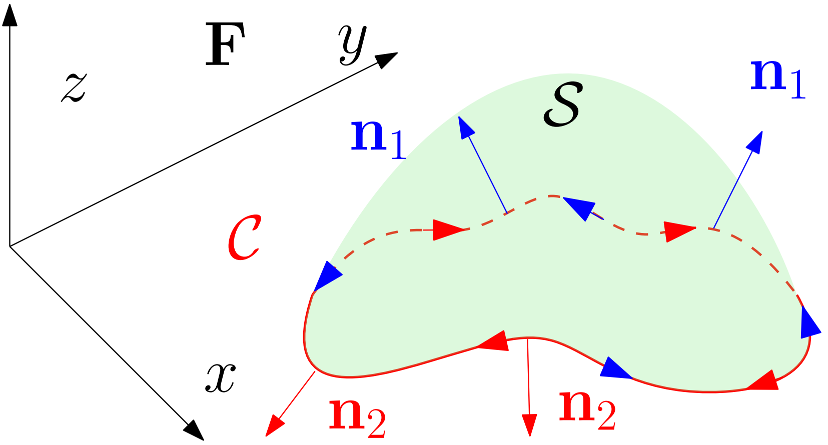


that is enclosed by positively oriented piecewise smooth simple closed curve.

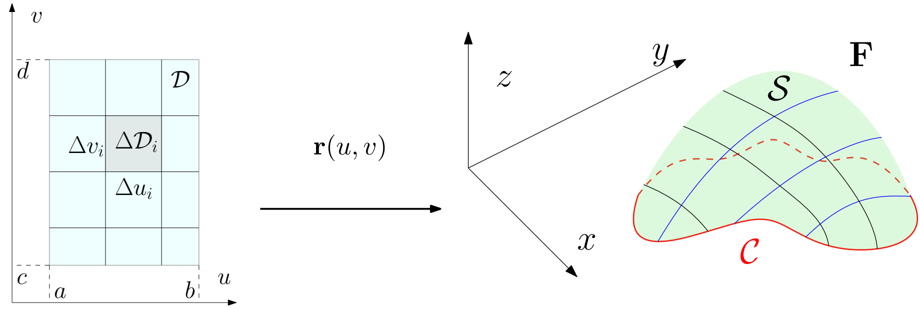
Q: Can you recall the definition of a positively oriented closed plane curve?



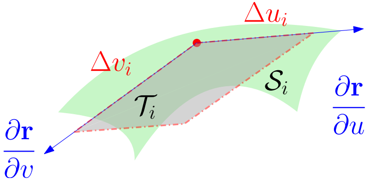
- The orientation of the boundary curve is defined by the right-handed relation to the normal of $\mathcal{S}$, if the thumb of a right hand points in the direction of $\mathbf{n}$, then the fingers curl in the positive direction of $\mathcal{C}$.



- If we partition the domain $\mathcal{D}$ of $\mathbf{r}(u, v)$, we will then have patches

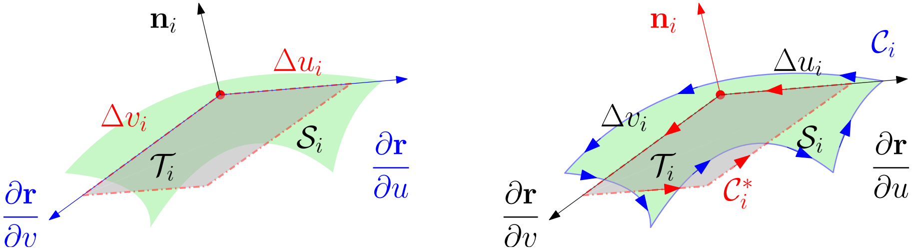


- For a sufficiently fine partition of $\mathcal{D}$, each patch is roughly a plane region $\mathcal{T}_i$



- For a sufficiently fine partition of $\mathcal{D}$, we expect the following to be reasonable

$$\iint_{\mathcal{S}_i} \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS \approx \iint_{\mathcal{T}_i} \operatorname{curl} \mathbf{F} \cdot \mathbf{n}_i\, dA$$



- We cannot apply Green's theorem directly to each tangent plane $\mathcal{T}_i$, however,

$$\iint_{\mathcal{T}_i} \operatorname{curl} \mathbf{F} \cdot \mathbf{n}_i\, dA = \iint_{\tilde{\mathcal{D}}_i} \operatorname{curl} \tilde{\mathbf{F}} \cdot \tilde{\mathbf{n}}_i\, dA = \oint_{\tilde{\mathcal{C}}_i} \tilde{\mathbf{F}} \cdot \tilde{\mathbf{T}}\, ds = \oint_{\mathcal{C}_i^*} \mathbf{F} \cdot \mathbf{T}\, ds \approx \oint_{\mathcal{C}_i} \mathbf{F} \cdot d\mathbf{r}$$

by rotating the coordinate axes in space so that unit normal vector $\tilde{\mathbf{n}}_i$ is $\mathbf{e}_z$

Q Why can we expect such rotations will not change the integrals?

- Rotating the coordinate system is essentially an orthonormal change of basis,

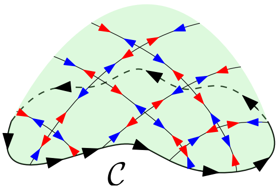
$$\mathbf{u}\cdot\mathbf{v}$$

the value of the dot product will not be altered by such change of basis.

- For a sufficiently fine partition, we expect the following to be reasonable

$$\sum_i^n \iint_{\mathcal{S}_i} \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS \approx \sum_i^n \oint_{\mathcal{C}_i} \mathbf{F}\cdot\ d\mathbf{r} = \oint_{\mathcal{C}} \mathbf{F}\cdot\ d\mathbf{r}$$

since the line integral along an interior path will vanish.



- For well-behaved $\mathbf{F}$ and $\mathcal{S}$, we expect the error to vanish if we take the limit.

Stokes' theorem

Let $\mathcal{S}$ be an oriented piecewise smooth surface that is bounded by a positively oriented, piecewise smooth, simple, closed boundary curve $\mathcal{C}$. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives in an open region in $\mathbb{R}^3$ that contains $\mathcal{S}$. Then

$$\oint_{\mathcal{C}} \mathbf{F} \cdot \, d\mathbf{r} = \oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T} \, ds = \iint_{\mathcal{S}} (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} \, dS = \iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{n}$ is the unit normal vector of $\mathcal{S}$.

Q: When Stokes' Theorem is applicable, what does it say regarding two different oriented surface $\mathcal{S}_1$ and $\mathcal{S}_2$ having the same boundary $\mathcal{C}$ placed in $\mathbf{F}$?

- The vector field

$$\operatorname{curl} \mathbf{F} \text{ is “surface independent”},$$

just like the gradient field ∇f is path independent.

Exercise

(a) *Use Stokes' theorem to evaluate*

$$\iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S},$$

where $\mathcal{S}$ is the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 5$, with upward orientation, and

$$\mathbf{F} = yz\mathbf{e}_x + xz\mathbf{e}_y + xy\mathbf{e}_z$$

(b) *Evaluate the line integral of*

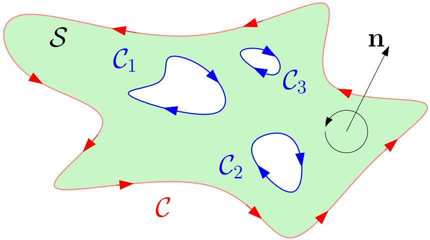
$$\mathbf{F}(x, y, z) = xy\mathbf{e}_x + 2z\mathbf{e}_y + 3y\mathbf{e}_z$$

over the curve $\mathcal{C}$ that is the intersection of

the cylinder $x^2 + y^2 = 9$ with the plane $x + z = 5$

and it is oriented counter-clockwise as viewed from above.

- Stokes' Theorem holds for surfaces that has a finite number of holes.



- The surface integral over $\mathcal{S}$ of the normal component of $\operatorname{curl}\mathbf{F}$, in this case,

$$\iint_{\mathcal{S}} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} + \oint_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{r} + \oint_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r} + \oint_{\mathcal{C}_3} \mathbf{F}\cdot d\mathbf{r}$$

equals the sum of the line integrals around all the boundary curves of the tangential component of $\mathbf{F}$, where the curves are to be traced in the direction induced by the orientation of $\mathcal{S}$.