Vv255 Lecture 15

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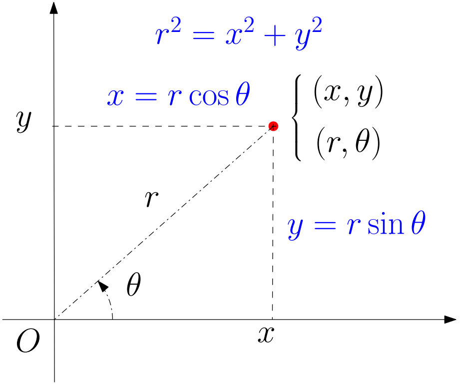
- Recall the polar coordinates in relation to Cartesian coordinates.



- For the same curve in the same space $\mathbb{R}^2$, we might have 2 representations

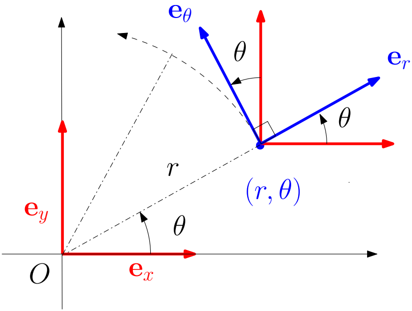
$$\begin{aligned} y &= y(x) \\ r &= r(\theta) \end{aligned} \qquad \text{e.g.} \quad y = \frac{3}{2}x - 1 \iff r = \frac{2}{3\cos\theta - 2\sin\theta}$$

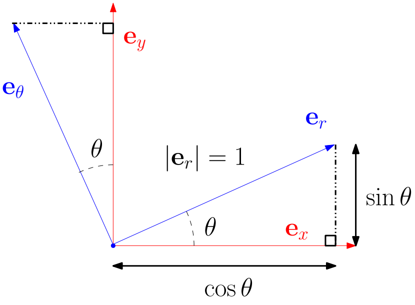
- In terms of the standard basis $\mathbf{e}_x$ and $\mathbf{e}_y$, we have

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}_{\mathcal{C}} = x\mathbf{e}_x + y\mathbf{e}_y, \qquad \text{where} \quad \mathcal{C} = \{\mathbf{e}_x, \mathbf{e}_y\}$$

- We want to have an orthonormal basis $\mathcal{P} = \{\mathbf{e}_r, \mathbf{e}_\theta\}$,

$$\begin{bmatrix} r \\ \theta \end{bmatrix}_{\mathcal{P}} = r\mathbf{e}_r + \theta\mathbf{e}_\theta, \qquad \text{where} \quad \begin{array}{ll} \mathbf{e}_r & \text{is the direction of increasing } r. \\ \mathbf{e}_\theta & \text{is the direction of increasing } \theta. \end{array}$$





- Therefore, we expect the unit vector in the direction of increasing r to be

$$\mathbf{e}_r = \frac{(\cos\theta)\,\mathbf{e}_x + (\sin\theta)\,\mathbf{e}_y}{\sqrt{(\cos\theta)^2 + (\sin\theta)^2}} = (\cos\theta)\,\mathbf{e}_x + (\sin\theta)\,\mathbf{e}_y$$

- Similarly, the unit vector in the direction of increasing θ,

$$\mathbf{e}_\theta = \frac{(-\sin\theta)\,\mathbf{e}_x + (\cos\theta)\,\mathbf{e}_y}{\sqrt{(-\sin\theta)^2 + (\cos\theta)^2}} = (-\sin\theta)\,\mathbf{e}_x + (\cos\theta)\,\mathbf{e}_y$$

- We can verify the orthogonality of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ by computing

$$\mathbf{e}_r \cdot \mathbf{e}_\theta = 0$$

- Note the Cartesian position vector $\mathbf{r}$ in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ is

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}_{\mathcal{C}} = r\mathbf{e}_r \neq r\mathbf{e}_r + \theta\mathbf{e}_\theta = \begin{bmatrix} r \\ \theta \end{bmatrix}_{\mathcal{P}}$$

- Note both $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are functions of θ.

$$\mathbf{e}_r(\theta) \qquad \text{and} \qquad \mathbf{e}_\theta(\theta)$$

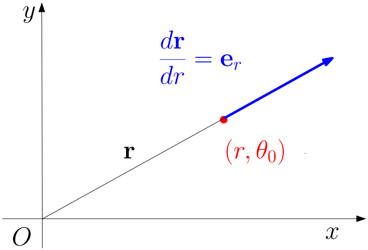
- Let $\theta = \theta_0$, then the Cartesian position vector is a vector-valued function,

$$\mathbf{r}(r) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos\theta_0 \\ r\sin\theta_0 \end{bmatrix} = r\begin{bmatrix} \cos\theta_0 \\ \sin\theta_0 \end{bmatrix}$$

- The rate of change of $\mathbf{r}$ with respect to r is

$$\frac{d\mathbf{r}}{dr} = \begin{bmatrix} \cos\theta_0 \\ \sin\theta_0 \end{bmatrix} = \mathbf{e}_r\Big|_{\theta=\theta_0}$$

which gives the tangential direction of the curve defined by $\mathbf{r}$ at r as usual.



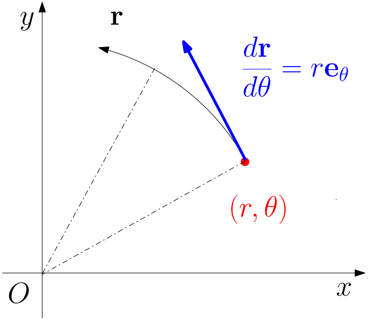
- If $r = r_0$, then the Cartesian position vector is a vector-valued function of θ,

$$\mathbf{r}(r) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r_0 \cos\theta \\ r_0 \sin\theta \end{bmatrix}$$

- The rate of change of $\mathbf{r}$ with respect to θ is

$$\frac{d\mathbf{r}}{d\theta} = r_0 \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = r_0 \mathbf{e}_\theta = r\mathbf{e}_\theta \bigg|_{r=r_0}$$

which gives the tangential direction of the curve defined by $\mathbf{r}$ at r as usual.



- We have two derivatives of

$$\mathbf{r}(r, \theta)$$

1. The rate of change of $\mathbf{r}$ with respect to r while holding θ fixed.

$$\frac{d\mathbf{r}}{dr}$$

2. The rate of change of $\mathbf{r}$ with respect to θ while holding r fixed.

$$\frac{d\mathbf{r}}{d\theta}$$

Q: Have you seen derivatives that are similar to those?

$$\frac{\partial \mathbf{r}}{\partial r} \qquad \text{and} \qquad \frac{\partial \mathbf{r}}{\partial \theta}$$

- The partial derivatives give the change of the function, here the vector-valued function, with respect to one independent variable, holding other independent variables constant.

- Now consider the scalar-valued function of two variables,

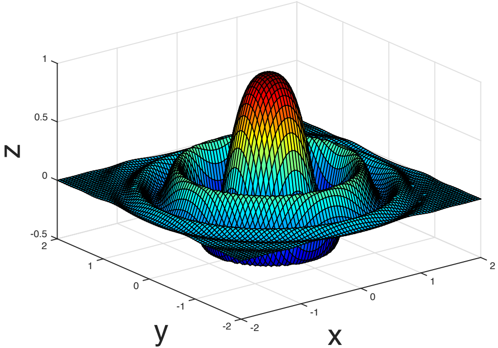
$$z = f(x, y)$$

- Recall a partial derivative of z is a directional derivative in the direction of

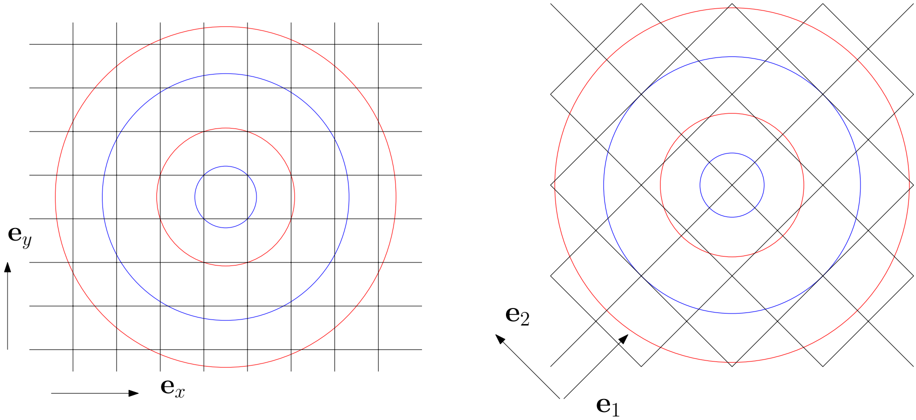
$$\mathbf{e}_x \qquad \text{or} \qquad \mathbf{e}_y$$

Q: Are these always the best direction to consider the rate of z?

$$f(x, y) = e^{-(x^2+y^2)} \cos\left(4(x^2 + y^2)\right)$$



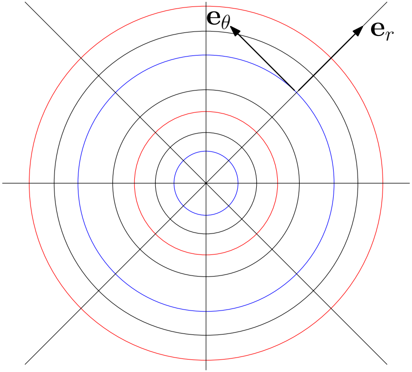
- Clearly $\mathbf{e}_x$ and $\mathbf{e}_y$ are no better than any linear orthogonal directions.



Q: Why using polar coordinates is a better choice here?

- Since it shows the radial symmetry of the function

$$
\begin{aligned}
f(x,y) &= e^{-(x^2+y^2)}\cos\Big(4\left(x^2+y^2\right)\Big) \\
&= e^{-r^2}\cos\left(4r^2\right)
\end{aligned}
$$

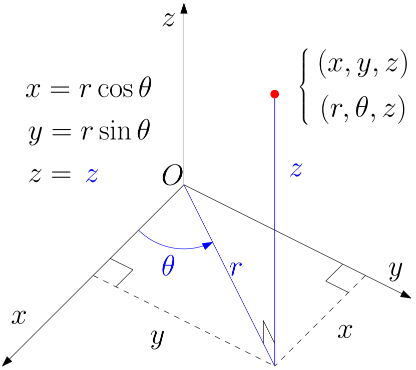


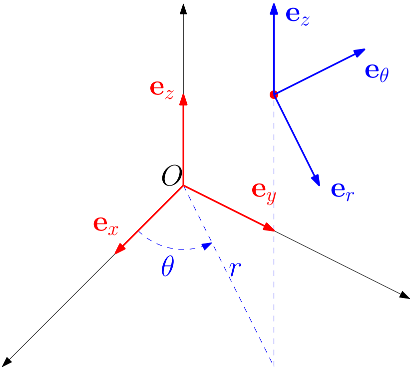
- It is often the symmetry of a given physical problem that points to the most convenient choice of basis or coordinates.
- If we add the usual z coordinate to the plane polar coordinates, then we will have a cylindrical coordinate system.

Definition

Cylindrical coordinates represent a point P in $\mathbb{R}^3$ by (r, θ, z) in which

1. r and θ are the polar coordinates for the projection of P onto the xy-plane
2. z is the Cartesian vertical coordinate.





- Similar to the plane polar basis, the cylindrical polar basis can be found by differentiating the Cartesian position vector

$$\mathbf{r}(r,\theta,z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix}$$

- As may be directly verified, the following basis is orthonormal everywhere,

$$\mathbf{e}_r = \frac{\frac{\partial \mathbf{r}}{\partial r}}{\left|\frac{\partial \mathbf{r}}{\partial r}\right|} = \frac{\partial \mathbf{r}}{\partial r} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix},$$ where $\mathbf{e}_r$ gives the direction of increasing r.

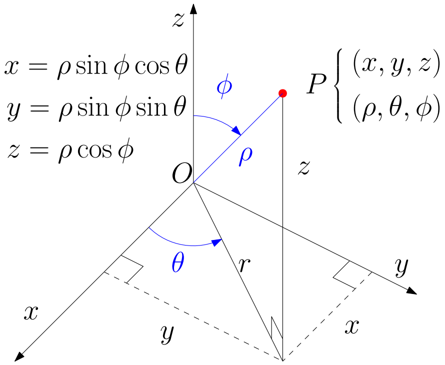
$$\mathbf{e}_\theta = \frac{\frac{\partial \mathbf{r}}{\partial \theta}}{\left|\frac{\partial \mathbf{r}}{\partial \theta}\right|} = \frac{1}{r}\frac{\partial \mathbf{r}}{\partial \theta} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix},$$ where $\mathbf{e}_\theta$ gives the direction of increasing θ.

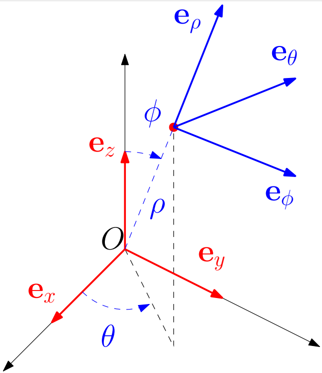
$$\mathbf{e}_z = \frac{\frac{\partial \mathbf{r}}{\partial z}}{\left|\frac{\partial \mathbf{r}}{\partial z}\right|} = \frac{\partial \mathbf{r}}{\partial z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$ where $\mathbf{e}_z$ gives the direction of increasing z.

Definition

Spherical coordinates represent a point P in $\mathbb{R}^3$ by (ρ, θ, ϕ) in which

1. ρ is the distance between the point P to the origin O.
2. θ is the angular coordinate for the projection of P on the xy-plane.
3. ϕ is the angle $\vec{OP}$ makes with the positive z-axis.





- Again the spherical polar basis can be found by differentiating

$$\mathbf{r}(\rho,\theta,\phi)=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}\rho\sin\phi\cos\theta\\\rho\sin\phi\sin\theta\\\rho\cos\phi\end{bmatrix}$$

- Again it can be directly verified, the following basis is orthonormal in $\mathbb{R}^3$,

$$\mathbf{e}_\rho=\frac{\frac{\partial\mathbf{r}}{\partial\rho}}{\left|\frac{\partial\mathbf{r}}{\partial\rho}\right|}=\frac{\partial\mathbf{r}}{\partial\rho}=\begin{bmatrix}\sin\phi\cos\theta\\\sin\phi\sin\theta\\\cos\phi\end{bmatrix},$$ where $\mathbf{e}_\rho$ is the direction of increasing ρ.

$$\mathbf{e}_\theta=\frac{\frac{\partial\mathbf{r}}{\partial\theta}}{\left|\frac{\partial\mathbf{r}}{\partial\theta}\right|}=\frac{1}{\rho\sin\phi}\frac{\partial\mathbf{r}}{\partial\theta}=\begin{bmatrix}-\sin\theta\\\cos\theta\\0\end{bmatrix},$$ where $\mathbf{e}_\theta$ is the direction of increasing θ.

$$\mathbf{e}_\phi=\frac{\frac{\partial\mathbf{r}}{\partial\phi}}{\left|\frac{\partial\mathbf{r}}{\partial\phi}\right|}=\frac{1}{\rho}\frac{\partial\mathbf{r}}{\partial\phi}=\begin{bmatrix}\cos\phi\cos\theta\\\cos\phi\sin\theta\\-\sin\phi\end{bmatrix},$$ where $\mathbf{e}_\phi$ is the direction of increasing ϕ.