Vv255
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Assignment 1
Due: May 29, 2017

Question1 (2 points)

Consider the following function

$$f\colon [-1,3) \to [1,10], \qquad \text{where} \qquad f(x) = x^2 + 1.$$

(a) (1 point) Identify the domain, codomain and range of

$$f$$

by sketching them on three separate real number lines.

Solution:

- It is clearly stated the domain is

$$[-1,3) \tag{1}$$

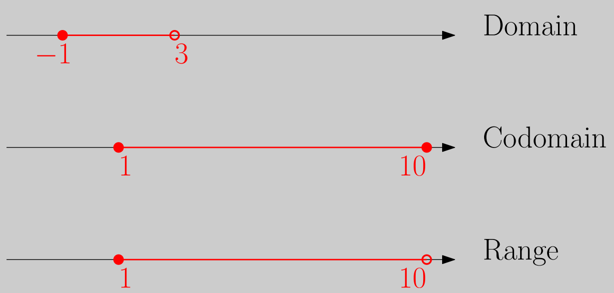
codomain is

$$[1,10] \tag{2}$$

the range is a subset of the codomain, which contains all of the values that a function can actually attain, here the range is

$$[1,10) \tag{3}$$

Hence the graphic representation of them will look like the following



To see the reason of having the concept of codomain as well as range, we go back to our definition of a function. A function f is a rule that assigns to each element in a set $\mathcal{A}$ exactly one element in a set $\mathcal{B}$, where $\mathcal{A}$ is known as the domain, and $\mathcal{B}$ the codomain. Note the definition does *not* require *every* element of the codomain has a corresponding element in the domain. So you can think the codomain is some larger set that surely captures the range, and may have elements that have no corresponding element in the domain. The range on the other hand is a set, in which every element must have a corresponding element in the domain. When the range is the same as the codomain, the function is known to be onto. When any two distinct elements in the domain map to two distinct elements in the codomain, the function is known to be one-to-one. When a function is one-to-one and onto, we say the function is a one-to-one correspondence or bijective, such functions are invertible. In fact, it is not hard to convince yourself a function is invertible if and only if there is a one-to-one correspondence between elements in its domain and elements in its codomain. The inverse function has the original codomain as its domain and vice versa. In practice, being one-to-one is good enough since we can easy turn it into a one-to-one correspondence. I will leave it to you to figure it out why.

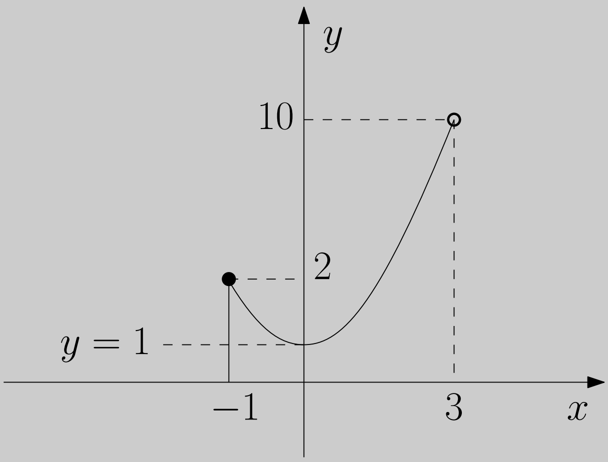
(b) (1 point) Sketch the graph of

$$y = f(x)$$

Solution:

- This is a giveaway mark. The graph of the function here is the graph of the set

$$\Big\{(x, f(x)) : x \in [-1, 3)\Big\} \tag{4}$$



Note it depicts both the domain and the range of the function, and thus the relationship between the two. The reason for asking this question will become clear next week when we talk about vector-valued function, for which we graph only the range.

Question2 (3 points)

Consider the following two vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ x+2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 3x \\ 2x \end{bmatrix}$$

(a) (1 point) Find the linear combination of **u** and **v** that is equal to

$$\mathbf{w} = \begin{bmatrix} 7+3x \\ 14+9x \end{bmatrix}$$

that is, to find α and β such that

$$\alpha \mathbf{u} + \beta \mathbf{v} = \mathbf{w}$$

Solution:

- We simply convert the vector equation into a system of linear equations

$$\alpha \mathbf{u} + \beta \mathbf{v} = \mathbf{w}$$
$$\Longrightarrow \begin{bmatrix} \alpha \\ \alpha(x+2) \end{bmatrix} + \begin{bmatrix} 3\beta x \\ 2\beta x \end{bmatrix} = \begin{bmatrix} 7+3x \\ 14+9x \end{bmatrix} \Longrightarrow \begin{aligned} \alpha + 3x\beta &= 7+3x \\ (x+2)\alpha + 2x\beta &= 14+9x \end{aligned}$$

Solving it the system of linear equations in terms of x, we have

$$\alpha = 7 \qquad \text{and} \qquad \beta = 1$$

(b) (1 point) For which x are $\mathbf{u}$ and $\mathbf{v}$ linearly independent.

Solution:

- They are linearly independent if the only solution to the following is $\alpha = \beta = 0$.

$$\alpha\mathbf{u} + \beta\mathbf{v} = \mathbf{0} \tag{5}$$

- Convert into linear system, we have

$$\begin{aligned} \alpha + 3x\beta &= 0 \\ (x+2)\alpha + 2x\beta &= 0 \end{aligned}$$

- Multiply the first equation by $x+2$, we have

$$\begin{aligned} (x+2)\alpha + 3x(x+2)\beta &= 0 \\ (x+2)\alpha + 2x\beta \quad &= 0 \end{aligned}$$

- We will have two equations for two unknowns, thus a unique solution if

$$3x(x+2) \neq 2x$$

- So $\mathbf{u}$ and $\mathbf{v}$ are linearly independent as long as

$$x \neq 0 \qquad \text{and} \qquad x \neq -\frac{4}{3}$$

- We will discuss linear independence further when we talk about the cross product and determinant. It will give us a more elegant approach. But this is how we do it for now.

(c) (1 point) Show that the vector component of $\mathbf{u}$ along $\mathbf{v}$ is orthogonal to

$$(\mathbf{u} - \mathrm{proj}_{\mathbf{v}}\,\mathbf{u})$$

Solution:

- Apply the definition,

$$(\mathrm{proj}_{\mathbf{v}}\,\mathbf{u}) \cdot (\mathbf{u} - \mathrm{proj}_{\mathbf{v}}\,\mathbf{u}) = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} \cdot \left(\mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}\right) = \frac{(\mathbf{u}\cdot\mathbf{v})^2}{\mathbf{v}\cdot\mathbf{v}} - \frac{(\mathbf{u}\cdot\mathbf{v})^2}{\mathbf{v}\cdot\mathbf{v}} = 0$$

Question3 (3 points)

Given the following matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

consider the matrix multiplication between each of the four matrices with the followings

$$\mathbf{e}_x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For example, two of such multiplications are

$$\mathbf{A}\mathbf{e}_x \qquad \text{and} \qquad \mathbf{A}\mathbf{e}_y$$

(a) (1 point) State the geometric effect of multiplying each of the four matrices to a nonzero vector $\mathbf{v}$ in $\mathbb{R}^2$.

Solution:

- Multiplying matrix $\mathbf{A}$ causes $\mathbf{e}_y$ to become $-\mathbf{e}_y$ while having not effect on $\mathbf{e}_x$, and since any vector $\mathbf{v}$ in $\mathbb{R}^2$ can be represented by

$$\mathbf{v} = v_x\mathbf{e}_x + v_y\mathbf{e}_y$$

$\mathbf{A}$ reflect any vector about the line $y = 0$.

- Similarly, we see

matrix $\mathbf{B}$ reflects any vector about the line $x = 0$.

matrix $\mathbf{C}$ reflect any vector about the origin.

matrix $\mathbf{D}$ reflect any vector about the line $y = x$

(b) (1 point) Let $\mathbf{R}$ be a 2×2 matrix. Find the elements of $\mathbf{R}$ such that the vector

$$\mathbf{v}' = \mathbf{R}\mathbf{v}$$

is the image of $\mathbf{v}$ after rotating θ degree counterclockwise about the origin.

Solution:

- Let α be the angle between $\mathbf{e}_x$ and the vector

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

then

$$\sin\alpha = \frac{v_2}{\sqrt{v_1^2 + v_2^2}} \qquad \text{and} \qquad \cos\alpha = \frac{v_1}{\sqrt{v_1^2 + v_2^2}}$$

The angle between $\mathbf{e}_x$ and $\mathbf{v}'$ must be

$$\alpha + \theta$$

If we let

$$\mathbf{v}' = \begin{bmatrix} v_1' \\ v_2' \end{bmatrix}$$

then

$$v_1' = \sqrt{v_1^2 + v_2^2}\cos(\alpha+\theta) = v_1 \cdot \cos\theta - v_2 \cdot \sin\theta$$
$$v_2' = \sqrt{v_1^2 + v_2^2}\sin(\alpha+\theta) = v_1 \cdot \sin\theta + v_2 \cdot \cos\theta$$

When we multiply $\mathbf{R}$ and $\mathbf{v}$, we have

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \cdot r_{11} + v_2 \cdot r_{12} \\ v_1 \cdot r_{21} + v_2 \cdot r_{22} \end{bmatrix}$$

Therefore

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

(c) (1 point) Based on your answer to part (a) and (b), speculate what matrix and matrix multiplication represent in general.

Solution:

- From part (a) and (b), it is reasonable to expect matrix multiplication in general transforms vector one way or another, but it offers more possibilities than simple vector addition and scalar multiplication can. It plays a similar role as a function $y = f(x)$, where a real number becomes another according to some rule. In fact, going back to our definition. A function f is a rule that assigns to each element in a set $\mathcal{A}$ exactly one element in a set $\mathcal{B}$. Of course, $\mathcal{A}$ and $\mathcal{B}$ can be sets of vectors. So any matrix can be considered as a precise representation of some function between two sets of vectors $\mathcal{A}$ and $\mathcal{B}$. Moreover, if you ponder long enough, you will see it does not represent every type of functions between sets of vectors, the type of functions it can represent must be linear. A detailed discussion of matrices is typically included in a linear algebra course, so I will not go any further here.

Question4 (3 points)

(a) (1 point) Construct an example of non-zero matrices such that

$$\mathbf{AB} = \mathbf{AC}, \qquad \text{where} \quad \mathbf{B} \neq \mathbf{C}$$

Solution:

- Let me remind you that column and row vectors are matrices, and the dot product is a special form of matrix multiplication, by given the following example,

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

so essentially, we cannot expect two vectors to be equal when the dot products between each of them with a third vector are equal.

(b) (1 point) Determine the unknown quantities x, y and z in the following expression.

$$2\begin{bmatrix} x & 3 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 2y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ y & z \end{bmatrix}^{\mathrm{T}}$$

Solution:

- Two matrices of the same size are equal if and only if all the corresponding elements in the two matrices are equal.

$$\begin{bmatrix} 2x+4 & 2y+6 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} 3 & y \\ 6 & z \end{bmatrix}^{\mathrm{T}} \implies x = -\frac{1}{2}, \quad y = -6, \quad z = 0$$

(c) (1 point) Determine whether the set $\mathcal{S}$ is a basis for $\mathbb{R}^3$.

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 11 \\ 17 \\ 19 \end{bmatrix}$$

If so, find the coordinates of $\mathbf{v}$ with respect to $\mathcal{S}$. If not, justify your answer.

Solution:

- We simply solve the following

$$\alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 17 \\ 19 \end{bmatrix} \implies \begin{aligned} \alpha + \beta + \gamma &= 11 \\ \alpha + 2\beta + \gamma &= 17 \\ \alpha + \beta + 3\gamma &= 19 \end{aligned}$$

If the solution is unique, then $\mathcal{S}$ is a basis for $\mathbb{R}^3$. If not, that is, no solution or finitely many solution, then $\mathcal{S}$ is not a basis for $\mathbb{R}^3$. In this case, we actually have a unique solution, thus $\mathcal{S}$ is a basis for $\mathbb{R}^3$.

$$\alpha = 1, \qquad \beta = 6, \qquad \gamma = 4$$

Question5 (3 points)
Let $\mathbf{A}$ be an $n \times n$ matrix. The matrix $\mathbf{A}$ is known to be symmetric if

$$\mathbf{A} = \mathbf{A}^{\mathrm{T}}$$

or skew-symmetric matrix if

$$\mathbf{A} = -\mathbf{A}^{\mathrm{T}}$$

(a) (1 point) Suppose $\mathbf{A}$ and $\mathbf{B}$ are symmetric. Determine which of the following two matrices is symmetric and which is skew-symmetric. Justify your answer.

$$\mathbf{AB} + \mathbf{BA} \qquad \text{and} \qquad \mathbf{AB} - \mathbf{BA}$$

Solution:

- It is given that

$$\mathbf{A} = \mathbf{A}^{\mathrm{T}} \qquad \text{and} \qquad \mathbf{B} = \mathbf{B}^{\mathrm{T}}$$

If we consider

$$\begin{aligned} (\mathbf{AB} + \mathbf{BA})^{\mathrm{T}} &= (\mathbf{AB})^{\mathrm{T}} + (\mathbf{BA})^{\mathrm{T}} \\ &= \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} + \mathbf{A}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}} \\ &= \mathbf{BA} + \mathbf{AB} \\ &= \mathbf{AB} + \mathbf{BA} \end{aligned}$$

we see $\mathbf{AB} + \mathbf{BA}$ is symmetric and $(\mathbf{AB} - \mathbf{BA})$ is skew-symmetric

$$\begin{aligned} (\mathbf{AB} - \mathbf{BA})^{\mathrm{T}} &= (\mathbf{AB})^{\mathrm{T}} - (\mathbf{BA})^{\mathrm{T}} \\ &= \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} - \mathbf{A}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}} \\ &= \mathbf{BA} - \mathbf{AB} \\ &= -(\mathbf{AB} - \mathbf{BA}) \end{aligned}$$

(b) (1 point) Suppose $\mathbf{A}$ is a symmetric and $\mathbf{B}$ is a skew-symmetric. Show

$$\mathbf{AB} \text{ is skew-symmetric if and only if } \mathbf{AB} = \mathbf{BA}.$$

Solution:

- If $\mathbf{AB}$ is skew-symmetic matrix, then

$$\mathbf{AB} = -(\mathbf{AB})^{\mathrm{T}} = -\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$$

Since $\mathbf{A}$ is a symmetric and $\mathbf{B}$ is a skew-symmetric

$$\mathbf{A} = \mathbf{A}^{\mathrm{T}}, \qquad \text{and} \qquad \mathbf{B} = -\mathbf{B}^{\mathrm{T}}$$

we have

$$\mathbf{AB} = -(\mathbf{AB})^{\mathrm{T}} = -\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} \implies \mathbf{AB} = \mathbf{BA}$$

- If $\mathbf{AB} = \mathbf{BA}$, we have

$$\mathbf{AB} = \mathbf{BA} = -\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = -(\mathbf{AB})^{\mathrm{T}}$$

since $\mathbf{A}$ is a symmetric and $\mathbf{B}$ is a skew-symmetric. Therefore, $\mathbf{AB}$ is a skew-symmetric matrix. □

(c) (1 point) Show any $n \times n$ matrix $\mathbf{A}$ can be decomposed into a sum of a symmetric matrix and a skew-symmetric matrix.

Solution:

- For any element, a_{ij}, of the matrix $\mathbf{A}$, we consider another element a_{ji} and let

$$b_{ij} = \frac{a_{ij} + a_{ji}}{2} \qquad c_{ij} = \frac{a_{ij} - a_{ji}}{2}$$

It is clear that

$$a_{ij} = b_{ij} + c_{ij}$$

and

$$b_{ij} = \frac{a_{ij} + a_{ji}}{2} = \frac{a_{ji} + a_{ij}}{2} = b_{ji}; \quad c_{ij} = \frac{a_{ij} - a_{ji}}{2} = -\frac{a_{ji} - a_{ij}}{2} = -c_{ji}$$

Therefore, $\mathbf{B}$ is a symmetric matrix and $\mathbf{C}$ is a Skew-symmetric matrix.

Question6 (1 points)
Suppose an object with $10\sqrt{2}$ N is traveling due east at a constant speed of 10 m/s. A wind is blowing towards the northwestern direction with a constant force of 2 N. There also exists a friction with friction coefficient $\mu = 0.25$ between the object and the ground. Specify all the forces acting on the object using vectors in $\mathbb{R}^3$. Use dot product and projection to find how much work is done by the wind over a span of 5 s. How about the friction and gravity? Assume the acceleration due to gravity is $g = 10$ m/s^2.

Solution:

- Let us set up the coordinate system such that the positive x, y and z directions correspond to due east, due north and upward respectively, and the object is at the origin initially. Since the object is traveling in a straight line at a constant speed. Thus all the forces acting on the object must be completely balanced, that is,

$$\mathbf{F} = \mathbf{0}$$

It is clear that there are at least four forces on the object, namely, the force of gravity, the normal force, resistive force of friction, and the force due to wind, respectively,

$$\mathbf{G}, \quad \mathbf{N}, \quad \mathbf{F}_f, \quad \mathbf{F}_w$$

For the gravity, it is clear that the representation of it in this coordinate system or in terms of the standard basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ is given by

$$\mathbf{G} = \begin{bmatrix} 0 \\ 0 \\ -10\sqrt{2} \end{bmatrix} = 0\mathbf{e}_x + 0\mathbf{e}_y - 10\sqrt{2}\mathbf{e}_z = -10\sqrt{2}\mathbf{e}_z$$

For the support, it is clear that it is equal in magnitude but opposite to $\mathbf{G}$

$$\mathbf{N} = 10\sqrt{2}\mathbf{e}_z$$

For the friction, it is against the direction of motion, so it must opposite of $\mathbf{e}_x$,

$$\mathbf{F}_f = -\left|\mathbf{F}_f\right|\mathbf{e}_x$$

where the magnitude of this force is proportional to the magnitude of the normal force that pushes the object and ground together, so

$$\left|\mathbf{F}_f\right| = \mu\left|\mathbf{N}\right| = \frac{5\sqrt{2}}{2} \implies \mathbf{F}_f = -\frac{5\sqrt{2}}{2}\mathbf{e}_x$$

For the wind, its magnitude is 2, and it is in direction of $-\mathbf{e}_x + \mathbf{e}_y$, so

$$\mathbf{F}_w = \left|\mathbf{F}_w\right|\hat{\mathbf{F}}_w = 2\frac{-\mathbf{e}_x + \mathbf{e}_y}{\left|-\mathbf{e}_x + \mathbf{e}_y\right|} = 2\frac{-\mathbf{e}_x + \mathbf{e}_y}{\sqrt{(-\mathbf{e}_x + \mathbf{e}_y)\cdot(-\mathbf{e}_x + \mathbf{e}_y)}} = -\sqrt{2}\mathbf{e}_x + \sqrt{2}\mathbf{e}_y$$

Thus resultant force of those four fources is given by

$$\mathbf{G} + \mathbf{N} + \mathbf{F}_f + \mathbf{F}_w = -\sqrt{2}\mathbf{e}_x + \sqrt{2}\mathbf{e}_y - \frac{5\sqrt{2}}{2}\mathbf{e}_x = -\frac{7\sqrt{2}}{2}\mathbf{e}_x + \sqrt{2}\mathbf{e}_y$$

So there must be a fifth force equal in magnitude but opposite to the force above

$$\mathbf{F}_m = \frac{7\sqrt{2}}{2}\mathbf{e}_x - \sqrt{2}\mathbf{e}_y$$

From the gravitational force, we can work out the mass of the object,

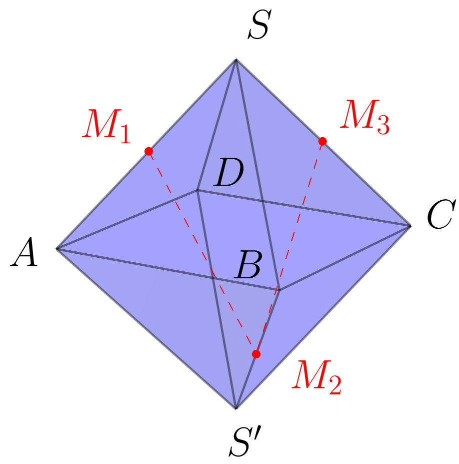
$$m = \sqrt{2}\ \text{kg}$$

So perhaps the object is something from Rowling's world or it has a miniaturised internal combustion engine designed by Stark. Either way, we have

$$\begin{aligned} W_w &= \mathbf{F}_w \cdot \mathbf{D} = \mathbf{F}_w \cdot (5 \times 10\mathbf{e}_x) = -50\sqrt{2}\ \text{J} \\ W_f &= \mathbf{F}_f \cdot \mathbf{D} = \mathbf{F}_f \cdot (5 \times 10\mathbf{e}_x) = -125\sqrt{2}\ \text{J} \\ W_g &= \mathbf{G} \cdot \mathbf{D} = \mathbf{G} \cdot (5 \times 10\mathbf{e}_x) = 0\ \text{J} \end{aligned}$$

Question7 (1 points)

Consider a regular octahedron $SABCDS'$,where S and S' are the tips of the octahedron



where points M_1,M_2 and M_3 are the midpoints of sides SA, $S'B$ and SC. Find the angle

$$\angle M_1M_2M_3$$

Solution:

- Since scaling the size of the octahedron will not change the angle, so let us consider one that has unit length edges. Suppose it is placed in a rectangular coordinate system such that the rectangle $ABCD$ is in the xy-plane with the following coordinates

$$A\left(\frac{1}{2},-\frac{1}{2},0\right),\quad B\left(\frac{1}{2},\frac{1}{2},0\right),\quad C\left(-\frac{1}{2},\frac{1}{2},0\right),\quad D\left(-\frac{1}{2},-\frac{1}{2},0\right),$$

then, if we let $S(s_x, s_y, s_z)$, the definition of a regular octahedron implies

$$\begin{array}{l}\left|\vec{AS}\right|=1\\ \left|\vec{BS}\right|=1\\ \left|\vec{CS}\right|=1\\ \left|\vec{DS}\right|=1\end{array}\implies \begin{array}{l}\left(s_x-\frac{1}{2}\right)^2+\left(s_y+\frac{1}{2}\right)^2+s_z^2=1\\ \left(s_x-\frac{1}{2}\right)^2+\left(s_y-\frac{1}{2}\right)^2+s_z^2=1\\ \left(s_x+\frac{1}{2}\right)^2+\left(s_y-\frac{1}{2}\right)^2+s_z^2=1\\ \left(s_x+\frac{1}{2}\right)^2+\left(s_y+\frac{1}{2}\right)^2+s_z^2=1\end{array}\implies \begin{array}{l}s_x=0\\ s_y=0\\ s_z=\pm\frac{1}{\sqrt{2}}\end{array}$$

which leads us to the coordinates for both S and S'

$$S\left(0,0,\frac{1}{\sqrt{2}}\right),\quad S'\left(0,0,-\frac{1}{\sqrt{2}}\right)$$

Let the coordinates of M_1 be (m_x, m_y, m_z), then

$$\frac{1}{2}\vec{AS} = \vec{AM} \implies \frac{1}{2}\begin{bmatrix} 0-\frac{1}{2} \\ 0+\frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} m_x - \frac{1}{2} \\ m_y + \frac{1}{2} \\ m_z \end{bmatrix} \implies \begin{aligned} m_x &= \frac{1}{4} \\ m_y &= -\frac{1}{4} \\ m_z &= \frac{\sqrt{2}}{4} \end{aligned}$$

In a similar fashion, we obtain

$$M_1\left(\frac{1}{4}, -\frac{1}{4}, \frac{\sqrt{2}}{4}\right), \quad M_2\left(\frac{1}{4}, \frac{1}{4}, -\frac{\sqrt{2}}{4}\right), \quad M_3\left(-\frac{1}{4}, \frac{1}{4}, \frac{\sqrt{2}}{4}\right)$$

So

$$\vec{M_1M_2} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \quad \text{and} \quad \vec{M_3M_2} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

Therefore, the angle $\angle M_1M_2M_3$ equals

$$\angle M_1M_2M_3 = \arccos\left(\frac{\vec{M_1M_2}\cdot\vec{M_3M_2}}{\left|\vec{M_1M_2}\right|\left|\vec{M_3M_2}\right|}\right) = \arccos\left(\frac{2}{3}\right) = 0.8411$$

Question8 (2 points)
Show that all bases for $\mathbb{R}^n$, which is called the n-dimensional Euclidean space, consist of n linearly independent vectors in $\mathbb{R}^n$. You may assume $\mathbb{R}^n$ is n-dimensional.

Solution:

- Let us break it into two parts, firstly, we show vectors in every basis must be linearly independent. Then complete the argument by showing every basis for $\mathbb{R}^n$, which is given to be n-dimensional, contains exactly n vectors.
- We prove the first part by contradiction, let $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ be a basis for $\mathbb{R}^n$, and suppose the vectors in $\mathcal{B}$ are *not* linearly independent. Since vectors in $\mathcal{B}$ are not linearly independent, then

$$\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$$

is not the unique solution to

$$\mathbf{0} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_m\mathbf{u}_m$$

This presents a contradiction, since $\mathcal{B}$ is a basis, any vector in $\mathbb{R}^n$, including the zero vector, can be *uniquely* represented as a linear combination of vectors in $\mathcal{B}$

$$\mathbf{0} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_m\mathbf{u}_m$$

Hence vectors in any basis must be linearly independent.

- For the second part, since we have shown all vectors in a basis must be linearly independent, it is clear that the number of vectors in a basis for $\mathbb{R}^n$ cannot exceed n, because the dimension of a space by definition is the maximum number of linearly independent vectors in the space. So we only need to consider whether it is possible for m, the number of vectors in $\mathcal{B}$, to be less than n. Suppose the following is a maximal set of linearly independent vectors for $\mathbb{R}^n$,

$$\mathcal{S} = \{\underbrace{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m}_{\mathcal{B}}, \underbrace{\mathbf{u}_{m+1}, \mathbf{u}_{m+2}, \ldots, \mathbf{u}_n}_{\mathcal{A}}\}$$

  which consists of vectors in $\mathcal{B}$ and an additional $n-m$ vectors in $\mathbb{R}^n$. Consider a vector that is a linear combination of vectors in $\mathcal{A}$, that is, in $\mathcal{S}$ but not in $\mathcal{B}$,

$$\mathbf{v} = \beta_1 \mathbf{u}_{m+1} + \beta_2 \mathbf{u}_{m+2} + \ldots + \beta_{n-m} \mathbf{u}_n$$

  where β_i are not simultaneously zero. So we are considering a nonzero vector. Hopefully, you can convince yourself $\mathbf{v}$ is a vector in $\mathbb{R}^n$ because vectors in $\mathcal{A}$ are themselves in $\mathbb{R}^n$. For $\mathcal{B}$ to be a basis for $\mathbb{R}^n$, every vector in $\mathbb{R}^n$, including $\mathbf{v}$, can be uniquely represented as a linear combination of vectors in $\mathcal{B}$

$$\begin{aligned} \mathbf{v} &= \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_m \mathbf{u}_m \\ \beta_1 \mathbf{u}_{m+1} + \beta_2 \mathbf{u}_{m+2} + \ldots + \beta_{n-m} \mathbf{u}_n &= \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_m \mathbf{u}_m \end{aligned}$$

  Rearrange everything to the left, we have

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_m \mathbf{u}_m - \beta_1 \mathbf{u}_{m+1} - \beta_2 \mathbf{u}_{m+2} - \ldots - \beta_{n-m} \mathbf{u}_n = \mathbf{0}$$

  Unless $m = n$, otherwise we have reached a contradiction, since all vectors in $\mathcal{S}$ are linearly independent, but we know at least one of β_i is nonzero by construction. Hence we conclude $m = n$, and that completes the argument. □

- So this exercise tells/confirms that the dimension of a space is equal to the number of vectors in any basis for that space.

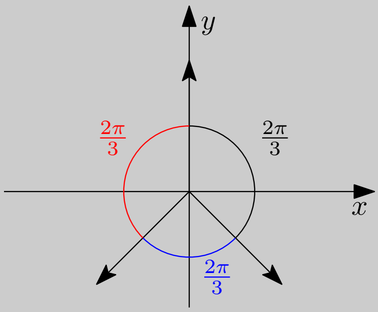
Question9 (2 points)
Suppose $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are some vectors in $\mathbb{R}^2$. Is it possible that all of the following pairwise dot products are less than zero?

$$\mathbf{u} \cdot \mathbf{v} < 0, \qquad \mathbf{u} \cdot \mathbf{w} < 0, \qquad \mathbf{v} \cdot \mathbf{w} < 0$$

Construct an example or provide a contradiction to support your answer. What is the maximum number of vectors in $\mathbb{R}^3$ can we have such that all of the pairwise dot products of those vectors are less than zero? Justify your answer.

Solution:

- Recall the dot product measures the extend to which two vectors are pointing in the same general direction. Having a negative value for the dot product means the two vectors are not pointing in the same general direction. To construct three such vectors, we need to make sure the angle between any two of the three vectors is big enough, specifically, bigger than $\frac{\pi}{2}$. It is clearly possible to construct three such vectors, for example,



It's not difficult to see that it is impossible to have four such vectors in $\mathbb{R}^2$, because the sum of the angles between such four vectors would be greater than 2π.

- In $\mathbb{R}^3$, the maximum number of vectors we can have is four. Intuitively, the trick is to make sure no more than two vectors are ever on the same plane, thus we can use the "extra room" in the 3-dimensional space to force each angle to be bigger than

$$\frac{\pi}{2}$$

By geometry, we can reach the following example,

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \sqrt{3}/2 \\ 0 \\ -1/2 \end{bmatrix}, \quad \begin{bmatrix} -\sqrt{3}/4 \\ 3/4 \\ -1/2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \sqrt{3}/4 \\ -3/4 \\ -1/2 \end{bmatrix}$$

It turns out four is the maximum number. To prove there cannot be five vectors that have this property, suppose we have five unit vectors

$$\mathbf{v}_1 = \hat{\mathbf{v}}_1, \quad \mathbf{v}_2 = \hat{\mathbf{v}}_2, \quad \mathbf{v}_3 = \hat{\mathbf{v}}_3, \quad \mathbf{v}_4 = \hat{\mathbf{v}}_4 \quad \text{and} \quad \mathbf{v}_5 = \hat{\mathbf{v}}_5$$

where the dot product for any two of them is negative. Since the length of the vectors will not affect the sign of the dot product, there is no loss of generality in assuming they are unit. Consider the vector component of $\mathbf{v}_i$ orthogonal to $\mathbf{v}_1$ for $2 \leq i \leq 5$, and let us denote these vectors by

$$\mathbf{u}_i = \mathbf{v}_i - (\mathbf{v}_i \cdot \mathbf{v}_1)\mathbf{v}_1$$

Since $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$ and $\mathbf{u}_5$ are orthogonal to the same vector $\mathbf{v}_1$, they must be in the same plane. Now if we consider the dot product of any two of those four vectors

$$\begin{aligned} \mathbf{u}_i \cdot \mathbf{u}_j &= (\mathbf{v}_i - (\mathbf{v}_i \cdot \mathbf{v}_1)\mathbf{v}_1) \cdot (\mathbf{v}_j - (\mathbf{v}_j \cdot \mathbf{v}_1)\mathbf{v}_1) \quad \text{where} \qquad i \neq j \\ &= (\mathbf{v}_i \cdot \mathbf{v}_j) - (\mathbf{v}_i \cdot \mathbf{v}_1)\mathbf{v}_1 \cdot \mathbf{v}_j - (\mathbf{v}_j \cdot \mathbf{v}_1)\mathbf{v}_1 \cdot \mathbf{v}_i + (\mathbf{v}_i \cdot \mathbf{v}_1)\mathbf{v}_1 \cdot (\mathbf{v}_j \cdot \mathbf{v}_1)\mathbf{v}_1 \\ &= (\mathbf{v}_i \cdot \mathbf{v}_j) + (\mathbf{v}_i \cdot \mathbf{v}_1)(\mathbf{v}_1 \cdot \mathbf{v}_j)(\mathbf{v}_1 \cdot \mathbf{v}_1 - 2) \end{aligned}$$

Because the vector $\mathbf{v}_i$ is a unit vector and the dot product between $\mathbf{v}_i$ and $\mathbf{v}_j$ is negative, we conclude

$$\mathbf{u}_i \cdot \mathbf{u}_j < 0$$

which means there are four vectors in the same plane that have all the pairwise dot products being less than 0. It contradicts the fact that there cannot be more than three such vectors in a plane. Therefore there cannot be five such vectors in $\mathbb{R}^3$. $\square$