VV256 Assignment 6

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Q1.

(a) $H(s) = \frac{1}{p(s)} = \frac{1}{s^2+2s-15}$

(b) $G'' + 2G' - 15G = \delta(x)$

$s^2F(s) + 2sF(s) - 15F(s) = e^{-as}$

$$F(s) = \frac{e^{-as}}{(s+5)(s-3)} = \frac{e^{-as}}{8}\left(\frac{1}{s-3} - \frac{1}{s+5}\right) = \frac{1}{s-3} - \frac{1}{s+5}$$

$$G(s) = L^{-1}[F(s)] = \frac{1}{8}e^{3t} - \frac{1}{8}e^{-5t}$$

Q2 It's clear $\int_a^b f^2 dx \in R$ and R is a vector space, so if $(\int_a^b f^2 dx) < \infty$ is a vector space, then it's a subspace.

$\forall \alpha \in R$ $\int_a^b (\alpha f)^2 dx = \alpha^2 \int_a^b f^2 dx < \infty$

$\int_a^b (f+g)^2 dx = \int_a^b f^2 + g^2 + 2fg\, dx = \int_a^b f^2 + g^2 dx + \int_a^b 2fg\, dx$

According to the definition of Riemann Sum, if $\int_a^b f^2 dx < \infty$ then $\int_a^b f dx < \infty$, $\int_a^b fg\, dx < \infty$

$\therefore \int_a^b (f+g)^2 dx < \infty$ $\therefore \int_a^b f^2 dx$ is a subspace and it's a vector space

Q3:

$$\frac{d^2\alpha\theta}{dt^2} + \frac{g}{l}\sin(\alpha\theta)$$

$$= \alpha\theta'' + \frac{g}{l}\sin\alpha\theta$$

$$= -\frac{\alpha g}{l}\sin\theta + \frac{g}{l}\sin\alpha\theta$$ which can't always be 0 for any θ

$\therefore$ It's not a vector space.

Q4:

$(\alpha A)M$

$= \alpha(AM)$

$= (MA)\alpha$

$= M(\alpha A)$

$\therefore \alpha A \in H$

Assume $A, B \in H$

$(A+B)M = AM + BM = MA + MB = M(A+B)$

$\therefore A+B \in H$

$\therefore H$ is a subspace

Q5 $V_4 = -2V_1$

$V_3 = 2V_1 - V_2$

$\alpha_1 V_1 + \alpha_2 V_2 = 0$

$$\begin{cases} \alpha_1 - \alpha_2 = 0 \\ 2\alpha_1 + \alpha_2 = 0 \end{cases} \Rightarrow \begin{cases} \alpha_1 = 0 \\ \alpha_2 = 0 \end{cases}$$

$\therefore$ V_1, V_2 are linearly independent

$\therefore$ The basis can be $\{V_1, V_2\}$.

Q6: $(a-\lambda)(d-\lambda) - bc = 0$

$\lambda^2 - (a+d)\lambda + ad - bc = 0$

Q7: $$\det(A) = 1 \times \begin{vmatrix} 3 & -7 & 9 & 6 \\ 0 & 2 & 7 & 3 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} - 5 \times \begin{vmatrix} 3 & -7 & 8 & 6 \\ 0 & 2 & -5 & 3 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$$= 3 \times \begin{vmatrix} 2 & 7 & 3 \\ 0 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} = 6 \times \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} = -12$$

Q8 $$A - \lambda I = \begin{vmatrix} 7-\lambda & 0 & -3 & 0 \\ -9 & -2-\lambda & 3 & 0 \\ 18 & 0 & -8-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = -2-\lambda \begin{vmatrix} 7-\lambda & -3 & 0 \\ 18 & -8-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (\lambda-1)(\lambda+2) \begin{vmatrix} 7-\lambda & -3 \\ 18 & -8-\lambda \end{vmatrix}$$

$$= (\lambda-1)(\lambda+2)[(\lambda-7)(\lambda+8)+54] = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = -2, \lambda_3 = 1, \lambda_4 = -2$$

For $\lambda = 1$

$$\begin{vmatrix} 6 & 0 & -3 & 0 \\ -9 & -3 & 3 & 0 \\ 18 & 0 & -9 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} x = 0 \Rightarrow \begin{cases} 2x_1 - x_3 = 0 \\ -3x_1 - x_2 + x_3 = 0 \\ 0 = 0 \end{cases} \Rightarrow \text{the eigenvector is span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}$$

For $\lambda = -2$

$$\begin{vmatrix} 9 & 0 & -3 & 0 \\ -9 & 0 & 3 & 0 \\ 18 & 0 & -6 & 0 \\ 0 & 0 & 0 & 3 \end{vmatrix} x = 0 \Rightarrow \begin{cases} 3x_1 - x_3 = 0 \\ \\ 3x_4 = 0 \end{cases} \Rightarrow \text{the eigenvector is span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix} \right\}$$

Q9 Assume $-\frac{c}{2m}=\beta$

(i) $G''+\frac{c}{m}G'+\frac{k}{m}G=\delta(x-a)$

Use Laplace transform

$\therefore s^2F(s)+\frac{c}{m}sF(s)+\frac{k}{m}F(s)=e^{-sa}$

$$F(s)=\frac{me^{-sa}}{s^2m+cs+k}=\frac{e^{-sa}}{[s-(\beta+\mu)][s-(\beta-\mu)]}=e^{-as}L[g(t+a)]$$

$$\because G(t+a;a)=\frac{1}{\mu}e^{\beta t}\sin(\mu t)\Rightarrow L[G(t+a);a)]=\frac{1}{\mu}\times\frac{\mu}{(s-\beta)^2+\mu^2}=\frac{1}{s^2-2\beta s+\beta^2+\mu^2}=\frac{1}{[s-(\beta+\mu)][s-(\beta-\mu)]}$$

$\therefore G(t;a)=\frac{1}{\mu}e^{-\frac{c(t-a)}{2m}}\sin((t-a)\mu)$ is the corresponding Green Function, and the solution $\phi=\int_{-\infty}^{\infty}f(a)\sin[(t-a)\mu]\frac{1}{\mu}e^{-\frac{c(t-a)}{2m}}da$

(ii) Similarly to last question

$$F(s)=\frac{e^{-sa}}{(s-\beta)^2}=e^{-sa}\times\frac{1}{(s-\beta)^2}=L[(t-a)e^{\beta(t-a)}u(t-a)]=e^{-as}L[g(t+a)]$$

$$L[g(t+a)]=\frac{1}{(s-\beta)^2}$$

$g(t+a)=\frac{1}{(s-\beta)^2}=e^{\beta t}t$; $\Rightarrow g(t)=e^{-\frac{c}{2m}(t-a)}(t-a)=G(t;a)$

$\therefore G(t;a)$ is the solution

$\therefore \phi=\int_{-\infty}^{\infty}f(a)(t-a)e^{-\frac{c}{2m}(t-a)}da$

(iii) It's similar to the first question

$$G(t+a;a)=\frac{1}{v}e^{\beta t}\sinh(vt)\Rightarrow L[G(t+a);a]=\frac{1}{v}\times\frac{v}{(s-\beta)^2-v^2}=\frac{1}{(s-\beta-v)(s-\beta+v)}$$

$$F(s)=\frac{e^{-sa}}{(s-\beta-v)(s-\beta+v)}=e^{-as}L[g(t+a)]$$

$\therefore g(t+a)=G[(t+a;a]\Rightarrow G(t;a)$ is the corresponding Green Function

$\phi=\int_{-\infty}^{\infty}f(a)\frac{1}{v}\sinh[(t-a)v)e^{-\frac{c(t-a)}{2m}}da$

(b) Since $\sin\ln t'=\frac{1}{t}\cos\ln t$, $\sin\ln t''=-\frac{\cos\ln t}{t^2}-\frac{\sin\ln t}{t^2}$

$\therefore$ Assume $y=A\sin\ln t\Rightarrow t\dot{y}=A\cos\ln t$, $t^2\ddot{y}=-A(\sin\ln t+\cos\ln t)$

$-A[\sin\ln t+\cos\ln t]+A\cos\ln t+4\sin\ln t=\sin\ln t$

$(4-A)=1$

$A=3$

$\therefore y=3\sin\ln t$

(c) Assume $k \in$ the vector space, then there only exists one pair (α_1, α_2) such that $k = \alpha_1 u + \alpha_2 v$

$$\begin{cases} u = \frac{1}{2}(u+v) + i \times \frac{1}{2i}(u-v) \\ v = \frac{1}{2}(u+v) - i \times \frac{1}{2i}(u-v) \end{cases}$$

$\therefore k = (\alpha_1 + \alpha_2)\frac{1}{2}(u+v) + (\alpha_1 i - \alpha_2 i)\frac{1}{2i}(u-v)$

$\therefore \alpha_1' = \alpha_1 + \alpha_2$, $\alpha_2' = \alpha_1 i - \alpha_2 i$ and there only one pair (α_1', α_2') such that $k = \alpha_1' \times \frac{1}{2}(u+v) + (\alpha_1 - \alpha_2) i \times \frac{1}{2i}(u-v)$

$\therefore \{\frac{1}{2}(u+v), \frac{1}{2i}(u-v)\}$ is also a basis for a vector space over C