

Vv156 Lecture 9

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- In terms of computing derivatives, the definition of derivative is not very friendly

$$\frac{d}{dx} \sqrt{2x^2 + 1}$$

- Based on the formal definition of derivative, we want to derive a set of laws.

Theorem

The derivative of a constant function is 0; that is, if c is any real number, then

$$\frac{d}{dx}(c) = 0$$

Proof

- By definition, $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} = \lim_{h \rightarrow 0} 0 = 0 \quad \square$

- Geometrically, the graph of $f(x) = c$ is a horizontal line, which has a slope of zero everywhere, so the tangent line must have a slope of 0 everywhere as well.

The Power Rule

If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Proof

$$\begin{aligned}\frac{d}{dx}(x^n) &= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} \\&= \lim_{h \rightarrow 0} \frac{\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n \right] - x^n}{h} \\&= \lim_{h \rightarrow 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h} \\&= \lim_{h \rightarrow 0} \left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1} \right] = nx^{n-1}\end{aligned}$$

□

- Although our proof of the power rule applies only to positive integer powers of x , it is not difficult to show that the same formula holds for all real number r .

The General Power Rule

If r is any real number, then

$$\frac{d}{dx}(x^r) = rx^{r-1}$$

- In words, to differentiate a power function,

$$y = x^r$$

1. We take the exponent, and multiply it into the coefficient,
2. then reduce the exponent by 1.

Theorem

If f is differentiable at x and c is any real number, then

cf is also differentiable at x and

$$\frac{d}{dx}(cf) = c \frac{d}{dx}(f)$$

Proof

- By definition,

$$\frac{d}{dx}(cf) = \lim_{h \rightarrow 0} \frac{cf(x+h) - cf(x)}{h} = c \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = c \frac{df}{dx}$$



- In words, a constant factor can be moved through a differential operator.

Theorem

If f and g are differentiable at x , then so are $f + g$ and $f - g$, and

$$\frac{d}{dx}(f \pm g) = \frac{d}{dx}(f) \pm \frac{d}{dx}(g)$$

Proof

$$\begin{aligned}\frac{d}{dx}(f \pm g) &= \lim_{h \rightarrow 0} \frac{[f(x+h) \pm g(x+h)] - [f(x) \pm g(x)]}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \pm \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &= \frac{d}{dx}(f) \pm \frac{d}{dx}(g)\end{aligned}$$

- In words, the derivative of a sum equals the sum of the derivatives, and the derivative of a difference equals the difference of the derivatives.
- Although this theorem is stated for for sums and differences of two functions, they can be easily extended to any finite number of functions.

Exercise

(a) Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}\bigg|_{x=1}$ where

$$y = 3x^8 - 2x^5 + 6x + 1$$

(b) At what points, if any, does the graph of

$$y = x^3 - 3x + 4$$

have a horizontal tangent line?

(c) Evaluate the limit by first converting it to a derivative at a particular x -value.

$$\lim_{x \rightarrow 1} \frac{x^{50} - 1}{x - 1}$$

- You might be tempted to conjecture that the derivative of a product of two functions is the product of their derivatives. However, consider a simple example

$$\begin{array}{ll} f(x) = x & g(x) = x^2 \\ \implies f'(x) = 1 & \implies g'(x) = 2x \\ f'(x) \cdot g'(x) = 1 \cdot 2x = 2x & \left(f(x) \cdot g(x)\right)' = (x^3)' = 3x^2 \end{array}$$

$$f'(x) \cdot g'(x) \neq \left(f(x) \cdot g(x)\right)'$$

The product rule

If f and g are differentiable at x , then so is the product $f \cdot g$, and

$$\frac{d}{dx}(f \cdot g) = \frac{df}{dx} \cdot g + f \cdot \frac{dg}{dx}$$

Proof

- By definition,

$$\begin{aligned}\frac{d}{dx} [f \cdot g] &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\&= \lim_{h \rightarrow 0} f(x+h) \frac{g(x+h) - g(x)}{h} + \lim_{h \rightarrow 0} g(x) \frac{f(x+h) - f(x)}{h} \\&= f' \cdot g + f \cdot g'\end{aligned}$$

Exercise

Find $f'(x)$ for

$$f(x) = 3\sqrt[3]{x} \cdot \left(\frac{1}{2}x^2 + x\right)$$

- Just as the derivative of a product is **not** the product of the derivatives, so the derivative of a quotient is **not** generally the quotient of the derivatives.

The Quotient Rule

If f and g are differentiable at x and if $g(x) \neq 0$, then $\frac{f}{g}$ is differentiable at x and

$$\frac{d}{dx} \left(\frac{f}{g} \right) = \frac{\frac{df}{dx} \cdot g - f \cdot \frac{dg}{dx}}{(g)^2}$$

- You can prove this rule by adding and subtracting $f \cdot g$ in the right place.
- Sometimes it is better to simplify a function first than to apply the quotient rule immediately. For example, it is easier to differentiate

$$f(x) = \frac{x^{3/2} + x}{\sqrt{x}} = x + x^{1/2}$$

- If we assume that the variable x in the trigonometric functions $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\csc x$ is measured in radians, we derive formulae of trigonometric function using the definition of derivative. For example,

$$\begin{aligned}
 \frac{d}{dx}(\sin x) &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h} \\
 &= \lim_{h \rightarrow 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \frac{\sin h}{h} \right] \\
 &= \underbrace{\left[\lim_{h \rightarrow 0} \sin x \left(\frac{\cos h - 1}{h} \right) \right]}_{\left[\lim_{h \rightarrow 0} \sin x \right] \left[\lim_{h \rightarrow 0} \frac{\cos h - 1}{h} \right]} + \left[\lim_{h \rightarrow 0} \cos x \right] \left[\lim_{h \rightarrow 0} \frac{\sin h}{h} \right] \\
 &= \cos x. \qquad \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1; \qquad \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} = 0
 \end{aligned}$$

Exercise

Find the derivatives: (a) $\frac{d}{dx} \left(\cos \left(x + \frac{\pi}{2} \right) \right)$ (b) $\frac{d}{dx} (\sin 2x)$

- Other derivatives of trigonometric function can be found in a similar fashion, or using known derivatives or rules of differentiation, especially the next rule.

$$\frac{d}{dx} c = 0$$

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} \ln|x| = \frac{1}{x}$$

$$\frac{d}{dx} x^n = nx^{n-1}$$

$$\frac{d}{dx} a^x = a^x \ln a$$

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$$

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \csc x = -\csc x \cot x$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\frac{d}{dx} \cot x = -\csc^2 x$$

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$$

$$\frac{d}{dx} \csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx} \cot^{-1} x = -\frac{1}{1+x^2}$$

The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then
the composition $f \circ g$ is differentiable at x .

Moreover, if

$$y = f(g(x)) \quad \text{and} \quad u = g(x)$$

then $y = f(u)$ and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Exercise

Find the derivatives using the chain rule:

$$(a) \frac{d}{dx} \left(\cos \left(x + \frac{\pi}{2} \right) \right) \qquad (b) \frac{d}{dx} (\sin 2x)$$

Proof

- We want to compute

$$\lim_{h \rightarrow 0} \frac{f(g(x+h)) - f(g(x))}{h} \quad (1)$$

- Let $v = \frac{g(x+h) - g(x)}{h} - g'$, notice

$$v \rightarrow 0 \quad \text{as} \quad h \rightarrow 0$$

make $g(x+h)$ the subject

$$g(x+h) = \underbrace{g(x)}_u + \underbrace{[g'(x) + v]h}_k$$

- Notice

$$k \rightarrow 0 \quad \text{as} \quad h \rightarrow 0$$

Proof

- Let $w = \frac{f(u+k) - f(u)}{k} - f'(u)$, then $w \rightarrow 0$ as $k \rightarrow 0$.
- Make $f(u+k)$ the subject, we have

$$f(u+k) = f(u) + [f'(u) + w]k$$

$$f(g(x+h)) = f(g(x)) + [f'(g(x)) + w][g'(x) + v]h$$

Collect terms in equation (1),

$$\begin{aligned}\Rightarrow \lim_{h \rightarrow 0} \frac{f(g(x+h)) - f(g(x))}{h} &= \lim_{h \rightarrow 0} [f'(g(x)) + w][g'(x) + v] \\ &= f'(g(x)) \cdot g'(x) \\ &= \frac{dy}{du} \cdot \frac{du}{dx} \quad \square\end{aligned}$$

A common flawed proof for the chain rule

- Use the alternative definition of derivative

$$\begin{aligned}\frac{dy}{dx} &= \lim_{x \rightarrow a} \frac{f(g(x)) - f(g(a))}{x - a} \\&= \lim_{x \rightarrow a} \frac{f(g(x)) - f(g(a))}{x - a} \frac{g(x) - g(a)}{g(x) - g(a)} \\&= \lim_{x \rightarrow a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} \\&= \lim_{x \rightarrow a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \lim_{x \rightarrow a} \frac{g(x) - g(a)}{x - a} \\&= \frac{dy}{du} \cdot \frac{du}{dx}\end{aligned}$$

- The major problem with this proof is that we cannot be sure that we didn't multiply and divide by zero.
- The definition of limit guarantees that (as $x \rightarrow a$) x will not equal a ; but the same cannot be said about $g(x)$ and $g(a)$.

Exercise

Find the derivative $\frac{d}{dx} \left(\sin^2 \left(\sqrt{2x^2 + 1} \right) \right)$.

Q: Is the derivative of

$\sin(u)$, where u is measured in degrees,

equal to the derivative of $\sin(x)$ where x is the same angle but in radians?

- In radians,

$$\frac{d}{dx} (\sin x) = \cos x$$

- In degrees,

$$\begin{aligned} \frac{d}{du} (\sin u) &= \frac{d}{du} \left(\sin \frac{180^\circ}{\pi} x \right) = \frac{d}{dx} \left(\sin \frac{180^\circ}{\pi} x \right) \frac{dx}{du} \\ &= \frac{180^\circ}{\pi} \left(\cos \frac{180^\circ}{\pi} x \right) \frac{\pi}{180^\circ} = \cos u \end{aligned}$$

- However, $\frac{d}{dx} \sin u = \frac{d}{du} (\sin u) \frac{du}{dx} = \frac{180^\circ}{\pi} \cos u = \frac{180^\circ}{\pi} \cos \left(\frac{180^\circ}{\pi} x \right)$