
Question1 (5 points)

(a) (1 point) Let $\mathcal{C}$ be the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Evaluate the line integral

$$\oint_{\mathcal{C}} y(4x^2 + y^2)\, dx + \oint_{\mathcal{C}} x(2x^2 + 3y^2)\, dy$$

(b) (1 point) Let

$$\mathbf{F} = (\sin x)\mathbf{e}_x + (x + y)\mathbf{e}_y$$

Find the line integral of $\mathbf{F}$ around the perimeter of the rectangle with corners

$$(3, 0),\ (3, 5),\ (-1, 5),\ (-1, 0),$$

traversed in that order.

(c) (1 point) Show that the line integral of

$$\mathbf{F} = x\mathbf{e}_y$$

around a positively oriented closed curve in the xy-plane measures the area of the region enclosed by the curve.

(d) (1 point) Use Green's theorem to find the area of the region enclosed by

$$\text{one arch of the cycloid } x = t - \sin t,\ y = 1 - \cos t.$$

(e) (1 point) For the following scalar-valued function and the circle

$$f(x, y) = \ln(x^2 + y^2); \qquad x^2 + y^2 = a^2$$

Evaluate the flux integral

$$\oint_{\mathcal{C}} \nabla f \cdot \mathbf{n}\, ds$$

Question2 (5 points)

(a) (1 point) Explain the meaning of the Laplacian of a scalar function φ.

(b) (1 point) Show the curl of the velocity field of a rotating rigid body has the direction of the axis of the rotation, and its magnitude equals twice the angular speed of the rotation.

(c) (1 point) For $f = z^2(x^2 + y^2 + z^2)^2$, find $\nabla \cdot (\nabla f)$ and $\nabla \times (2f\nabla f)$.

Maxwell's equations relating the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ as they vary with time in a region containing no charge and no current can be stated as follows:

$$\operatorname{div} \mathbf{E} = 0 \qquad \operatorname{div} \mathbf{H} = 0 \qquad \operatorname{curl} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \qquad \operatorname{curl} \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

where c is the speed of light. Use these equations to prove the following relationships.

(d) (1 point) $\nabla \times (\nabla \times \mathbf{E}) = -\dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{E}}{\partial t^2}$

(e) (1 point) $\nabla^2 \mathbf{H} = \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{H}}{\partial t^2}$

Question3 (5 points)

(a) (1 point) Show that the following parametric equations represent an ellipsoid.

$$x = a\sin u\cos v, \quad y = b\sin u\sin v, \quad z = c\cos u, \quad \text{where} \quad 0 \le u \le \pi, 0 \le v \le 2\pi$$

(b) (1 point) Find the area of the surface the part of the sphere

$$x^2 + y^2 + z^2 = b^2$$

that lies inside the cylinder

$$x^2 + y^2 = a^2, \qquad \text{where } 0 \le a \le b.$$

(c) (1 point) Evaluate the surface integral

$$\iint_{\mathcal{S}} \left(x^2 + y^2\right)\, dS$$

where $\mathcal{S}$ is the surface with the vector equation

$$\mathbf{r}(u,v) = 2uv\mathbf{e}_x + (u^2 - v^2)\mathbf{e}_y + (u^2 + v^2)\mathbf{e}_z, \qquad u^2 + v^2 \le 1$$

(d) (1 point) Integrate $G(x, y, z) = y + z$ over the surface of the wedge in the first octant bounded by the coordinate planes and the planes $x = 2$ and $y + z = 1$.

(e) (1 point) Find the outward flux of the field

$$\mathbf{F} = xz\mathbf{e}_x + yz\mathbf{e}_y + \mathbf{e}_z$$

across the surface of the upper cap cut from the solid sphere

$$x^2 + y^2 + z^2 \le 25 \qquad \text{by} \qquad z = 3$$

Question4 (5 points)

(a) (1 point) Let $\mathbf{F}$ be an inverse square field, show that the flux of $\mathbf{F}$ across a sphere $\mathcal{S}$ with centre the origin is independent of the radius of $\mathcal{S}$.

(b) (1 point) Use Stokes' Theorem to evaluate $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F} = x^2y\mathbf{e}_x - xy^2\mathbf{e}_y + z^3\mathbf{e}_z$$

and $\mathcal{C}$ is the curve of intersection of the following plane and cylinder

$$3x + 2y + z = 6; \qquad x^2 + y^2 = 4$$

oriented clockwise when viewed from above.

(c) (1 point) If $\mathcal{S}$ is a sphere and $\mathbf{F}$ satisfies the hypothesis of Stokes' Theorem, show

$$\iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$$

(d) (1 point) Use the Divergence Theorem to evaluate

$$\iint_{\mathcal{S}} (2x + 2y + z^2)\, dS, \qquad \text{where } \mathcal{S} \text{ is the unit sphere centred at the origin.}$$

(e) (1 point) Prove the following identity, assuming that the surface $\mathcal{S}$, and the region $\mathcal{E}$ satisfy the hypotheses of the Divergence theorem and that all necessary differentiability requirements for the functions $f(x, y, z)$ and $g(x, y, z)$ are met.

$$\iint_{\mathcal{S}} (f\nabla g - g\nabla f) \cdot \mathbf{n}\, dS = \iiint_{\mathcal{E}} (f\nabla^2 g - g\nabla^2 f)\, dV$$