Vv256 Lecture 11

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- Consider the following initial-value problem

$$\dot{y} = y, \qquad y(0) = 1$$

- If we assume it has an analytic solution, that is, it has a power series solution

$$y(t) = \phi(t) = \sum_{n=0}^{\infty} c_n t^n = c_0 + c_1 t + c_2 t^2 + \cdots$$

Q: How can we determine the coefficients?

$$c_n$$

- Substituting ϕ and $\dot{\phi}$ into the equation, we have

$$\underbrace{c_1 + 2c_2 t + 3c_3 t^2 + \cdots}_{\dot{\phi}} = \underbrace{c_0 + c_1 t + c_2 t^2 + \cdots}_{\phi}$$

- Equating the coefficients, we have the recurrence relation

$$c_n = (n+1)c_{n+1} \qquad \text{where } n \in \mathbb{N}_0.$$

- Provided that we know c_0, through this recurrence relation,

$$c_{n+1} = \frac{c_n}{n+1} \qquad \text{where } n \in \mathbb{N}_0.$$

all coefficients can be determined, thus leads us to a specific power series

$$y(t) = \sum_{n=0}^{\infty} c_n t^n = c_0 + c_1 t + c_2 t^2 + \cdots$$

to the initial-value problem

$$\dot{y} = y, \qquad y(0) = 1$$

Q: How can we determine c_0?

$$c_0 = y(0) = 1 \implies y(t) = 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 \cdots = e^t$$

- So the centre is often chosen to be 0 according to the initial condition $t_0 = 0$

Q: How can we determine the coefficients for a general solution?

Exercise

Find the general solution of the following equation.

$$y'' - xy = 0$$

Solution

- Substituting $\phi = \sum_{n=0}^{\infty} c_n x^n$ and $\phi'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}$, we have

$$\begin{aligned}
\phi'' - x\phi &= \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - x\sum_{n=0}^{\infty} c_n x^n \\
&= \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} \\
&= 2c_2 + \sum_{n=1}^{\infty} \Big[(n+1)(n+2)c_{n+2} - c_{n-1}\Big]x^n = 0
\end{aligned}$$

Solution

- Equating the coefficients, we have

$$c_2 = 0 \qquad \text{and} \qquad (n+1)(n+2)c_{n+2} - c_{n-1} = 0 \quad \text{for } n \in \mathbb{N}_1.$$

- Since $(n+1)(n+2) \neq 0$ for all values of n, we have the recurrence relation,

$$c_{n+2} = \frac{c_{n-1}}{(n+1)(n+2)}, \qquad n \in \mathbb{N}_1$$

- Because $c_2 = 0$, the following coefficients are zero

$$c_5,\ c_8,\ c_{11},\ c_{14},\ c_{17},\ \ldots,$$

- Other coefficients can be represented either in terms of c_0 or c_1,

$$c_3 = \frac{c_0}{2\cdot 3}, \quad c_6 = \frac{c_3}{5\cdot 6} = \frac{c_0}{2\cdot 3\cdot 5\cdot 6}, \quad c_9 = \frac{c_6}{8\cdot 9} = \frac{c_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}$$
$$c_4 = \frac{c_1}{3\cdot 4}, \quad c_7 = \frac{c_4}{6\cdot 7} = \frac{c_1}{3\cdot 4\cdot 6\cdot 7}, \quad c_{10} = \frac{c_7}{9\cdot 10} = \frac{c_1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}$$

Solution

- It can be shown by induction that

$$
\begin{aligned}
c_{3m} &= \frac{c_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9\cdots(3m-1)\cdot 3m}, && \text{where } m\in\mathbb{N}_1.\\
c_{3m+1} &= \frac{c_1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10\cdots 3m\cdot(3m+1)}, && \text{where } m\in\mathbb{N}_1.\\
c_{3m+2} &= 0, && \text{where } m\in\mathbb{N}_0.
\end{aligned}
$$

- Now putting back the coefficients to form the solution,

$$y=\phi(x)=\sum_{n=0}^{\infty} c_n x^n$$

- By collecting c_0 and c_1, we have

$$y=c_0\underbrace{\left(1+\sum_{n=1}^{\infty}\frac{x^{3n}}{2\cdot 3\ldots(3n-1)(3n)}\right)}_{\phi_1}+c_1\underbrace{\left(x+\sum_{n=1}^{\infty}\frac{x^{3n+1}}{3\cdot 4\ldots(3n)(3n+1)}\right)}_{\phi_2}$$

Solution

- By the ratio test, we can reach the conclusion that both series

$$\phi_1(x) = \left(1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2\cdot 3 \ldots (3n-1)(3n)}\right)$$
$$\phi_2(x) = \left(x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{3\cdot 4 \ldots (3n)(3n+1)}\right)$$

are convergent. For example, if we denote $\phi_1(x) = 1 + \sum_{n=1}^{\infty} a_n$, then we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{3n+3}}{x^{3n}} \cdot \frac{2\cdot 3\cdot 5\cdot 6\cdots(3n-1)(3n)}{2\cdot 3\cdot 5\cdot 6\cdots(3n-1)(3n)(3n+2)(3n+3)}\right|$$
$$= \frac{|x|^3}{(3n+2)(3n+3)} \to 0 \qquad \text{as} \qquad n \to \infty$$

thus we can conclude ϕ_1 is convergent for all $x \in \mathbb{R}$ by the ratio test.

Solution

- We have shown, for any arbitrary constants c_0 and c_1, the function

$$y = c_0 \underbrace{\left(1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2 \cdot 3 \ldots (3n-1)(3n)}\right)}_{\phi_1} + c_1 \underbrace{\left(x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{3 \cdot 4 \ldots (3n)(3n+1)}\right)}_{\phi_2}$$

  is a solution to

$$y'' - xy = 0$$

- However, in order to claim it is the general solution, we have to show no other solution exists outside of this form. This can be done by showing they are linearly independent. Checking the Wronskian, we have

$$W(\phi_1, \phi_2)(0) = \phi_1(0)\phi_2'(0) - \phi_1'(0)\phi_2(0) = 1 \cdot 1 - 0 \cdot 0 = 1 \neq 0$$

  from which, we can conclude ϕ_1 and ϕ_2 are linearly independent, and

$$y = c_0\phi_1 + c_1\phi_2$$

  is the general solution to the equation for all $x \in \mathbb{R}$.

- Recall a function $f(x)$ is said to be analytic at a point

$$x = a$$

if it can represented by a power series

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

with a positive radius of convergence.

Definition

A point x_0 is said to be an ordinary point of the differential equation

$$y'' + P(x)y' + Q(x)y = 0$$

if

$$P(x) \text{ and } Q(x) \text{ are analytic at } x_0.$$

A point that is not an ordinary point is known as a singular point of the equation.

- For example, every value of x is an ordinary point of the differential equation

$$y'' + (e^x)\, y' + (\sin x)\, y = 0$$

since we know their power series representation of

$$e^x \qquad \text{and} \qquad \sin x$$

converges everywhere.

Q: How can we recognize functions that are not analytic?

- If one of $P(x)$ or $Q(x)$ fails to be analytic at x_0, then x_0 is a singular point.

$$y'' + \frac{1}{x^2-4} y' + \frac{1}{x+1} y = 0$$

has the following singular points since $P(x)$ or $Q(x)$ are not continuous at

$$x = \pm 2 \qquad \text{and} \qquad x = -1$$

Theorem

If x_0 is an ordinary point of a homogeneous linear second-order equation, we can always find two linearly independent solutions in the form of a power series at x_0

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

and it converges at least for all x such that

$$|x - x_0| < R,$$

where R is the distance from x_0 to the closest singular point.

- The above is known as the series solution about the ordinary point x_0
- The distance R is the lower bound for the radius of convergence of power series solutions of the differential equation about x_0.
- In other words, its actual radius of convergence, which can be found by ratio test, might be bigger than R.

- Consider the following differential equation,

$$(x^2 - 2x + 5)y'' + xy' - y = 0$$

$$\implies y'' + \frac{x}{x^2 - 2x + 5} y' - \frac{1}{x^2 - 2x + 5} y = 0$$

- The point $x = 0$ is an ordinary point of the homogeneous differential equation since a quotient of analytic functions are analytic when the denominator is nonzero.

- According to the last theorem, we have two linearly independent solutions

$$\phi_1(x) = \sum_{n=0}^{\infty} c_n x^n \qquad \text{and} \qquad \phi_2(x) = \sum_{n=0}^{\infty} c_n^* x^n$$

- The complex numbers

$$x = 1 \pm 2\mathrm{i}$$

are singular points of the equation since both P and Q are not continuous.

- Without actually finding these solutions, we know that

$$\phi_1 \qquad \text{and} \qquad \phi_2$$

must converge at for all x such that

$$|x| < \sqrt{5}$$

because $R = \sqrt{5}$ is the distance between 0 and $1 \pm 2\mathrm{i}$.

- Alternatively, if we seek solutions about a different ordinary point, say,

$$x = -1$$

- We shall, for the sake of simplicity, find solutions only about the ordinary point

$$x = 0$$

- If it is necessary to find a power series solution about

$$x_0 \neq 0$$

we can make the change of variable $t = x - x_0$ and translates $x = x_0$ to $t = 0$, and solve the new equation at $t = 0$, and then back transform.

- Consider a series solution about $x = 0$,

$$y = f(x) = \sum_{n=0}^{\infty} c_n x^n$$

- The first and second derivatives of y with respect to x are

$$y' = \sum_{n=0}^{\infty} c_n n x^{n-1} \qquad \text{and} \qquad y'' = \sum_{n=0}^{\infty} c_n n(n-1) x^{n-2}$$

- Notice that the first term in y' and the first two terms in y'' are zero, thus

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} \qquad \text{and} \qquad y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$$

Identity property

Suppose

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = 0$$

for all x in the interval of convergence, where $R > 0$, then $c_n = 0$ for all n.

Exercise

Solve the equation $(x^2+1)y''+xy'-y=0$ using power series about $x=0$.

Solution

- The given differential equation has singular points at $x=\pm\mathrm{i}$, and so a power series solution centred at 0 will converge at least for $|x|<1$.
- Substitute the series into the equation and regroup to determine c_n

$$y=\sum_{n=0}^{\infty}c_nx^n,\qquad y'=\sum_{n=1}^{\infty}c_nnx^{n-1}\qquad\text{and}\qquad y''=\sum_{n=2}^{\infty}c_nn(n-1)x^{n-2}$$

- We have

$$(x^2+1)\sum_{n=2}^{\infty}c_nn(n-1)x^{n-2}+x\sum_{n=1}^{\infty}c_nnx^{n-1}-\sum_{n=0}^{\infty}c_nx^n=0$$

$$2c_2-c_0+6c_3x+\sum_{n=2}^{\infty}\Big[(n+1)(n-1)c_n+(n+2)(n+1)c_{n+2}\Big]x^n=0$$

Solution

- Again the identity property implies

$$c_2 = \frac{1}{2}c_0, \quad c_3 = 0, \quad \text{and} \quad c_{n+2} = \frac{1-n}{2+n}c_n, \quad n = 2, 3, 4, \dots$$

- Collect and put c_n either in terms of c_0 or c_1, we have

$$y = c_0\left(1 + \frac{1}{2}x^2 - \frac{1}{2^2 2!}x^4 + \frac{1\cdot 3}{2^3 3!}x^6 - \frac{1\cdot 3\cdot 5}{2^4 4!}x^8 + \cdots\right) + c_1 x \quad |x| < 1$$

- Thus the two linearly independent solutions are

$$\phi_1 = 1 + \frac{1}{2}x^2 + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n n!}x^{2n}$$

and

$$\phi_2 = x$$

Exercise

Find the general solution of the equation $y'' + (\cos x)\, y = 0$ using power series.

Solution

- Use the same procedure, we have

$$y'' + (\cos x)\, y = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)\sum_{n=0}^{\infty} c_n x^n = 0$$

- This leads to a recurrence relation, the first few are

$$2c_2 + c_0 = 0, \quad 6c_3 + c_1 = 0, \quad 12c_4 + c_2 - \frac{1}{2}c_0 = 0, \quad 20c_5 + c_3 - \frac{1}{2}c_1 = 0$$

- This gives

$$c_2 = -\frac{1}{2}c_0, \quad c_3 = -\frac{1}{6}c_1, \quad c_4 = \frac{1}{12}c_0, \quad c_5 = \frac{1}{30}c_1, \qquad \ldots$$
$$\Longrightarrow\ y = c_0\left(1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \cdots\right) + c_1\left(x - \frac{1}{6}x^3 + \frac{1}{30}x^5 - \cdots\right)$$