Question1 (5 points)

(a) (1 point) Evaluate

$$\iiint_E z\,dV$$

where E is the solid tetrahedron bounded by the four planes

$$x = 0, \quad y = 0, \quad z = 0, \quad \text{and} \quad x + y + z = 1$$

Solution:

1M The tetrahedron is bounded above by the plane

$$z = 1 - x - y$$

and below by the plane

$$z = 0$$

The projection of the tetrahedron onto the xy-plane is a triangle. It is bounded above by the straight line

$$y = 1 - x$$

which is the intersection of the two planes $z = 1-x-y$ and $z = 0$. It is bounded below by the straight line

$$y = 0$$

And the triangle is bounded on the left by $x = 0$ and $x = 1$. Hence

$$\iiint_E z\,dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,x = \frac{1}{24}$$

(b) (1 point) Evaluate the following integral

$$\iiint_{\mathcal{E}} (x + 2y - z)\,dV$$

where $\mathcal{E}$ is the solid region bounded between the graph of

$$z = x^2 + y^2$$

and the plane

$$3x + 5y + 2z = 12$$

Solution:

1M It is clear that the region $\mathcal{E}$ is bounded above by the plane

$$3x + 5y + 2z = 12$$

and below by the paraboloid

$$z = x^2 + y^2$$

Thus the triple integral can be converted into the following

$$I = \iiint_{\mathcal{E}} (x + 2y - z)\, dV = \iint_{\mathcal{D}} \left(\int_{x^2+y^2}^{6-3x/2-5y/2} (x + 2y - z)\ dz \right) dA$$

where the region $\mathcal{D}$ is the projection of the $\mathcal{E}$ onto the xy-plane, which is bounded by the following circle,

$$\begin{aligned} 3x + 5y + 2z &= 12 \\ x^2 + y^2 &= z \end{aligned} \implies \left(x + \frac{3}{4}\right)^2 + \left(y + \frac{5}{4}\right)^2 = \frac{65}{8}$$

This circle is the projection of the intersection of the plane and the paraboloid onto the xy-plane. It is both type-I and type-II. Thus a description of the region $\mathcal{E}$ is given by

$$\begin{aligned} -\sqrt{\frac{65}{8}} - \frac{3}{4} \leq & x \leq \sqrt{\frac{65}{8}} - \frac{3}{4} \\ -\sqrt{\frac{65}{8} - (x + \frac{3}{4})^2} - \frac{5}{4} \leq & y \leq \sqrt{\frac{65}{8} - (x + \frac{3}{4})^2} - \frac{5}{4} \\ x^2 + y^2 \leq & z \leq 6 - \frac{3}{2}x - \frac{5}{2}y \end{aligned}$$

Thus we have the following iterated integral, evaluating which, we have

$$\begin{aligned} I &= \int_{-\sqrt{\frac{65}{8}}-\frac{3}{4}}^{\sqrt{\frac{65}{8}}-\frac{3}{4}} \int_{-\sqrt{\frac{65}{8}-(x+\frac{3}{4})^2}-\frac{5}{4}}^{\sqrt{\frac{65}{8}-(x+\frac{3}{4})^2}-\frac{5}{4}} \int_{x^2+y^2}^{6-3x/2-5y/2} (x + 2y - z)\ dz\, dy\, dx \\ &= \frac{-1094275\pi}{3072} \end{aligned}$$

(c) (1 point) Find the mass and center of mass of a solid, whose density is

$$\rho(x, y, z) = 1 + xyz$$

in the shape of the cube bounded $0 \leq x \leq 1$, $0 \leq y \leq 1$ and $0 \leq z \leq 1$.

Solution:

1M The mass is given by

$$m = \int_0^1 \int_0^1 \int_0^1 (1 + xyz)\ dz\, dy\, dx = \frac{9}{8}$$

the moments are given by

$$M_y = \int_0^1 \int_0^1 \int_0^1 \left(x + x^2yz\right)\ dz\, dy\, dx = \frac{7}{12} \implies M_x = M_y = M_z = \frac{7}{12}$$

since the region and the density is symmetric in terms of x, y and z. Thus

$$\bar{x} = \bar{y} = \bar{z} = \frac{7/12}{9/8}$$

(d) (1 point) Evaluate

$$\iiint_E |xyz|\, dV$$

over the solid ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$$

Solution:

1M It is easier to consider the following change of variable

$$x = au; \qquad y = bv; \qquad z = cw$$

The corresponding region is

$$u^2 + v^2 + w^2 \leq 1$$

According to the Jacobian theorem, the triple integral can be converted to

$$\iiint_E |xyz|\, dV = 8\int_0^1 \int_0^{\sqrt{1-u^2}} \int_0^{\sqrt{1-u^2-v^2}} |abc|\, uvw \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dw\, dv\, du$$

The Jacobian is given by

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = abc$$

Thus

$$\iiint_E |xyz|\, dV = 8a^2b^2c^2 \int_0^1 \int_0^{\sqrt{1-u^2}} \int_0^{\sqrt{1-u^2-v^2}} uvw\, dw\, dv\, du = \frac{1}{6}a^2b^2c^2$$

(e) (1 point) Suppose $f(x)$ is continuous, show that

$$\int_0^t \int_0^z \int_0^y f(x)\, dx\, dy\, dz = \frac{1}{2}\int_0^t (t-x)^2 f(x)\, dx$$

Solution:

1M The left-hand side can be converted into a triple integral,

$$\int_0^t \int_0^z \int_0^y f(x)\, dx\, dy\, dz = \iiint_{\mathcal{E}} f(x)\, dV$$

where $\mathcal{E} = \Big\{(x,y,z) \mid 0 \leq x \leq y, 0 \leq y \leq z, 0 \leq z \leq t\Big\}$. The given iterated integral treats the region $\mathcal{E}$ as a region bounded by the surfaces

$$x = 0 \qquad \text{and} \qquad x = y$$

However, this region $\mathcal{E}$ can also be treated as a region bounded by the surface $y = z$ as its upper bound and $y = x$ as it lower bound. The projection of the $\mathcal{E}$ onto the xz-plane is bounded above by the line $z = t$ and below by the line $z = x$, the intersection of $z = y$ and $y = x$, and bounded on the left by the vertical line $x = 0$, on the right by the vertical line $x = t$. Hence

$$\begin{aligned}
\int_0^t \int_0^z \int_0^y f(x)\,dx\,dy\,dz &= \iiint_{\mathcal{E}} f(x)\,dV \\
&= \int_0^t \int_x^t \int_x^z f(x)\,dy\,dz\,dx \\
&= \int_0^t \int_x^t (z-x)f(x)\,dz\,dx \\
&= \int_0^t f(x)\left[\frac{1}{2}z^2 - xz\right]_x^t dx \\
&= \int_0^t f(x)\left[\frac{1}{2}t^2 - xt + \frac{1}{2}x^2\right] dx \\
&= \frac{1}{2}\int_0^t (t-x)^2 f(x)\,dx
\end{aligned}$$

Question2 (5 points)

(a) (1 point) Find a double integral for the area of $\mathcal{D}$ in the first quadrant formed by

$$xy = 4; \qquad xy = 8; \qquad xy^3 = 5; \qquad xy^3 = 15$$

Then convert the double integral into an iterated integral with constant limits.

Solution:

1M The area is simply given by

$$\iint_{\mathcal{D}} dA$$

To have an iterated integral over a rectangular region for this area, it is clear that we shall use the substitution

$$u = xy; \qquad v = xy^3$$

which leads to

$$x = \sqrt{\frac{u^3}{v}}; \qquad y = \sqrt{\frac{v}{u}}$$

The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{3}{2}\sqrt{\frac{u}{v}} & -\frac{1}{2}\sqrt{\frac{u^3}{v^3}} \\ -\frac{1}{2}\sqrt{\frac{v}{u^3}} & \frac{1}{2}\sqrt{\frac{1}{uv}} \end{bmatrix} = \frac{1}{2v}$$

Thus

$$\iint_{\mathcal{D}} dA = \frac{1}{2}\int_5^{15}\int_4^8 \frac{1}{v}\,du\,dv$$

(b) (1 point) Evaluate

$$\iint_{\mathcal{S}} (x^4 - y^4)e^{xy}\, dA,$$

where $\mathcal{S}$ is the region in the first quadrant enclosed by the hyperbolas

$$xy = 1, \quad xy = 3, \quad x^2 - y^2 = 3, \quad x^2 - y^2 = 4.$$

Solution:

1M Consider the following substitution,

$$u = xy; \qquad v = x^2 + y^2$$

then

$$v + 2u = (x+y)^2 \implies x + y = \sqrt{v + 2u}$$

since $\mathcal{S}$ is in the first quadrant. And

$$x - y = \sqrt{v - 2u} \implies x^2 - y^2 = \sqrt{v^2 - 4u^2} \implies x^4 - y^4 = v\sqrt{v^2 - 4u^2}$$

Thus

$$x = \frac{\sqrt{v+2u} + \sqrt{v-2u}}{2}; \qquad y = \frac{\sqrt{v+2u} - \sqrt{v-2u}}{2}$$

and the new boundaries are

$$1 \le u \le 3; \qquad \sqrt{9 + 4u^2} \le v \le \sqrt{16 + 4u^2}$$

The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2\sqrt{v^2 - 4u^2}}$$

Therefore,

$$\begin{aligned}\iint_{\mathcal{S}} (x^4 - y^4)e^{xy}\, dA &= \int_1^3 \int_{\sqrt{9+4u^2}}^{\sqrt{16+4u^2}} v\sqrt{v^2 - 4u^2}e^u \frac{1}{2\sqrt{v^2 - 4u^2}}\, dv\, du \\ &= \frac{7}{4}\left(e^3 - e^1\right)\end{aligned}$$

(c) (1 point) Find the volume of the solid region lying below the surface

$$f(x,y) = \frac{xy}{1 + x^2y^2}$$

and above the plane region bounded by $xy = 1$, $xy = 4$, $x = 1$, and $x = 4$.

Solution:

1M The volume is given by

$$\int_1^4 \int_{1/x}^{4/x} \frac{xy}{1 + x^2y^2}\, dy\, dx$$

Considering the following substitution $u = x$ and $v = xy$, we have

$$x = u \qquad \text{and} \qquad y = \frac{v}{u}$$

and the Jacobian is given by

$$J(u,v) = \det \begin{bmatrix} 1 & 0 \\ \frac{-v}{u^2} & \frac{1}{u} \end{bmatrix} = \frac{1}{u}$$

Hence the volume can be found by evaluating the following integral instead

$$\begin{aligned} \int_1^4 \int_1^4 \frac{v}{1+v^2} \left| \frac{1}{u} \right| du\, dv &= \int_1^4 \int_1^4 \frac{v}{1+v^2} \frac{1}{u}\, du\, dv \\ &= \ln 4 \int_1^4 \frac{v}{1+v^2}\, dv \\ &= (\ln 2) \left(\ln \frac{17}{2} \right) \end{aligned}$$

(d) (1 point) Evaluate the following integral

$$\iiint_{\mathcal{E}} xyz\, dV$$

where $\mathcal{E}$ is a region formed by the following surfaces:

$$m = \frac{x^2+y^2}{z}; \quad n = \frac{x^2+y^2}{z}; \quad a^2 = xy; \quad b^2 = xy; \quad \alpha = \frac{y}{x}; \quad \beta = \frac{y}{x}$$

in the first octant, that is,

$$x > 0; \qquad y > 0; \qquad z > 0$$

and $0 < a < b$, $0 < \alpha < \beta$ and $0 < m < n$.

Solution:

1M The boundary of this problem suggests the following substitution:

$$u = \frac{z}{x^2+y^2}; \qquad v = xy; \qquad w = \frac{y}{x}$$

which leads to

$$x = \sqrt{\frac{v}{w}}; \qquad y = \sqrt{vw}; \qquad z = uv\left(w + \frac{1}{w}\right)$$

With this substitution, we end up with constant bounds

$$\frac{1}{n} \le u \le \frac{1}{m}; \qquad a^2 \le v \le b^2; \qquad \alpha \le w \le \beta$$

The Jacobian is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{bmatrix} 0 & \frac{1}{2\,w\sqrt{\frac{v}{w}}} & -\frac{v}{2\,w^2\sqrt{\frac{v}{w}}} \\ 0 & \frac{w}{2\sqrt{v\,w}} & \frac{v}{2\sqrt{v\,w}} \\ v\,\left(w+\frac{1}{w}\right) & u\,\left(w+\frac{1}{w}\right) & -u\,v\,\left(\frac{1}{w^2}-1\right) \end{bmatrix} = \frac{v(w^2+1)}{2w^2}$$

Therefore,

$$\begin{aligned}\iiint_{\mathcal{E}} xyz\,dV &= \int_{1/n}^{1/m}\int_{a^2}^{b^2}\int_{\alpha}^{\beta} uv^2\left(w+\frac{1}{w}\right)\frac{v(w^2+1)}{2w^2}\,dw\,dv\,du\\ &= \int_{1/n}^{1/m}\frac{u}{2}\,du\int_{a^2}^{b^2} v^3\,dv\int_{\alpha}^{\beta}\frac{(w^2+1)^2}{w^3}\,dw\\ &= \frac{1}{32}\left(\frac{1}{m^2}-\frac{1}{n^2}\right)(b^8-a^8)\left[(\beta^2-\alpha^2)\left(1+\frac{1}{\alpha^2\beta^2}\right)+4\ln\frac{\beta}{\alpha}\right]\end{aligned}$$

(e) (1 point) Suppose partial derivatives of $f(x,y,z)$ are continuous and satisfy

$$\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2\le 1$$

Show that the following is true

$$f(x_0,y_0,z_0)-\frac{3}{4}R\le\bar{f}\le f(x_0,y_0,z_0)+\frac{3}{4}R$$

where $\bar{f}$ is the average value

$$\bar{f}=\frac{1}{V}\iiint_{\mathcal{S}} f(x,y,z)\,dV$$

over the spherical region $\mathcal{S}$ defined by

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\le R^2$$

and V is the volume of the region $\mathcal{S}$.

Solution:

1M Recall what we have derived in the assignment 4. In an ball $\mathcal{B}$, we can always find some $\mathbf{b}$ in the line segment of $\mathbf{a}$ and $\mathbf{x}$ such that

$$f(\mathbf{x})=f(\mathbf{a})+\sum_{k=1}^{3}(x_k-a_k)\frac{\partial f}{\partial x_k}(\mathbf{b})$$

with the notion of gradient of a function, we have the following representation,

$$f(\mathbf{x})=f(\mathbf{a})+\nabla f(\mathbf{b})\cdot(\mathbf{x}-\mathbf{a})$$

Hence we have the following formula in general,

$$\iiint_{\mathcal{S}}\Big(f(\mathbf{x})-f(\mathbf{a})\Big)\,dV=\iiint_{\mathcal{S}}\nabla f(\mathbf{b})\cdot(\mathbf{x}-\mathbf{a})\,dV$$

Since

$$|\nabla f(\mathbf{b})\cdot(\mathbf{x}-\mathbf{a})|\le|\nabla f(\mathbf{b})||(\mathbf{x}-\mathbf{a})|$$

and it is given that

$$|\nabla f(\mathbf{b})|=\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2}\le 1$$

we have

$$
\begin{aligned}
|\nabla f(\mathbf{b}) \cdot (\mathbf{x}-\mathbf{a})| &\leq |(\mathbf{x}-\mathbf{a})| \\
&\leq \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}
\end{aligned}
$$

which allows to reach the following conclusion,

$$
\begin{aligned}
\left|\iiint_{\mathcal{S}} \Big(f(\mathbf{x})-f(\mathbf{a})\Big)\, dV\right| &\leq \iiint_{\mathcal{S}} |(f(\mathbf{x})-f(\mathbf{a}))|\, dV \\
&= \iiint_{\mathcal{S}} |\nabla f(\mathbf{b}) \cdot (\mathbf{x}-\mathbf{a})|\, dV \\
&\leq \iiint_{\mathcal{S}} \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}\, dV
\end{aligned}
$$

The average value of f is defined as

$$
\bar{f} = \left(\frac{4\pi}{3}R^3\right)^{-1} \iiint_{\mathcal{S}} f(\mathbf{x})\, dV \implies \iiint_{\mathcal{S}} f(\mathbf{x})\, dV = \frac{4\pi}{3}R^3 \bar{f}
$$

Notice $f(\mathbf{a})$ is a constant, thus

$$
\iiint_{\mathcal{S}} f(\mathbf{a}) dV = \frac{4\pi}{3} R^3 f(\mathbf{a})
$$

Therefore

$$
\frac{4\pi}{3} R^3 |\overline{f} - f(\mathbf{a})| \leq \iiint_{\mathcal{S}} \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} dV
$$

Converting the integral on the left-hand side into spherical coordinates, we have

$$
\begin{aligned}
\iiint_{\mathcal{S}} \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} dV &= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^3 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \pi R^4
\end{aligned}
$$

Thus, we have

$$
|\overline{f} - f(\mathbf{a})| \leq \frac{3}{4} R
$$

which is

$$
f(x_0, y_0, z_0) - \frac{3}{4}R \leq \overline{f} \leq f(x_0, y_0, z_0) + \frac{3}{4}R
$$