Vv255 Lecture 25

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Definition

Suppose the following vector field is in Cartesian coordinates and is differentiable

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{e}_x + Q(x,y,z)\mathbf{e}_y + R(x,y,z)\mathbf{e}_z$$

that is, the component functions are all differentiable, then

$$\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

is called the divergence of $\mathbf{F}$ or the divergence of the vector field defined by $\mathbf{F}$.

- Of course, this definition has its counterpart for $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$, that is,

$$\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

where

$$\mathbf{F} = P(x,y)\mathbf{e}_x + Q(x,y)\mathbf{e}_y$$

- A common notation for the divergence is

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

where ∇ is sort of a vector in the dot product.

- However, it shall not be used as anything more than just a notation

$$\nabla \cdot \mathbf{F} \neq \mathbf{F} \cdot \nabla$$

- Like gradients, divergences share some properties of differentiation
- Given two vector field $\mathbf{F}$ and $\mathbf{G}$ both in $\mathbb{R}^2$ or $\mathbb{R}^3$, then

$$\nabla \cdot (\alpha\mathbf{F} + \beta\mathbf{G}) = \alpha\nabla \cdot \mathbf{F} + \beta\nabla \cdot \mathbf{G}$$

where α and β are real numbers.

- Given a vector field $\mathbf{F}$ and a scalar-valued function φ, then

$$\nabla \cdot (\varphi\mathbf{F}) = \varphi\nabla \cdot \mathbf{F} + (\nabla\varphi) \cdot \mathbf{F}$$

- If $\varphi(x, y, z)$ is a continuously differentiable scalar-valued function, then

$$\mathbf{F} = \nabla\varphi = \frac{\partial\varphi}{\partial x}\mathbf{e}_x + \frac{\partial\varphi}{\partial y}\mathbf{e}_y + \frac{\partial\varphi}{\partial z}\mathbf{e}_z$$

$$\implies \operatorname{div}\mathbf{F} = \nabla\cdot\nabla\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

  which is known the Laplacian of φ, and it is often denoted as

$$\nabla^2\varphi = \nabla\cdot\nabla\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

- Of course, this definition has its counterpart for $\varphi(x, y)$ of two variables,

$$\nabla^2\varphi = \nabla\cdot\nabla\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}$$

Q: What is the physical meaning of divergence of $\mathbf{F}$ and Laplacian of φ?

- Consider a sufficiently smooth velocity field for a fluid in a region $\mathcal{E}$

$$\mathbf{V}(x,y,z) = v_x(x,y,z)\mathbf{e}_x + v_y(x,y,z)\mathbf{e}_y + v_z(x,y,z)\mathbf{e}_z$$

inside which there is no source or sink, that is, no point at which the fluid is created or destroyed. Moreover, let the density be continuously differentiable

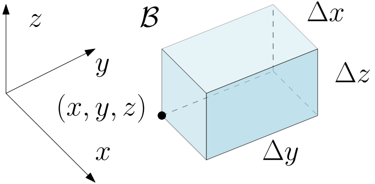
$$\rho(x,y,z,t)$$

which depends on (x,y,z) in space and on time t

- Fluids in the restricted sense, such as water or oil, have a constant density. But fluids in the general sense, such as gas or vapour, have a variable density.

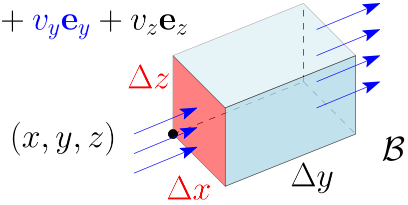
- Suppose there is a tiny box $\mathcal{B}$ in $\mathcal{E}$.
- It is clear that the box $\mathcal{B}$ has

$$\text{Volume} = \Delta V = \Delta x \Delta y \Delta z$$



- Consider the motion of the fluid in and out of the box by calculating the flux across the boundary, that is, the total mass leaving $\mathcal{B}$ per unit of time.

$$\mathbf{V} = v_x\mathbf{e}_x + v_y\mathbf{e}_y + v_z\mathbf{e}_z$$



- And consider the flow through the left face of $\mathcal{B}$, whose area is

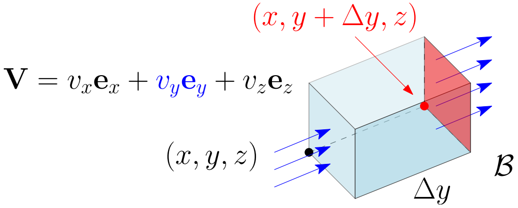
$$\Delta x \Delta z$$

- Since $v_x\mathbf{e}_x$ and $v_z\mathbf{e}_z$ are parallel to that face, the components v_x and v_y of $\mathbf{V}$ contribute nothing to this flow. Thus the mass of the fluid moving cross that face during a short time interval Δt is given approximately by

$$\rho v_y\Big|_y \Delta x \Delta z \Delta t$$

where the subscript y indicates that this expression refers to the left face.

- The mass of fluid moving through the opposite face, the right face,



during the same time interval can be approximately

$$\rho v_y\Big|_{y+\Delta y}\Delta x\Delta z\Delta t$$

where the subscript $y+\Delta y$ indicates this expression refers to the right face.

- The difference

$$\Delta F_y\Delta x\Delta z\Delta t = \frac{\Delta F_y}{\Delta y}\Delta V\Delta t, \quad \text{where} \quad \Delta F_y = \Big[\rho v_y\Big]_y^{y+\Delta y}$$

is the approximate change of mass in the direction of $\mathbf{e}_y$.

- Two similar expressions can be obtained by using the other two pairs of faces

$$\frac{\Delta F_x}{\Delta x}\Delta V\Delta t \qquad \text{and} \qquad \frac{\Delta F_z}{\Delta z}\Delta V\Delta t$$

where

$$\Delta F_x = \Big[\rho v_x\Big]_x^{x+\Delta x} \qquad \text{and} \qquad \Delta F_z = \Big[\rho v_z\Big]_z^{z+\Delta z}$$

- If we add these three expressions, we find that the total change of mass in $\mathcal{B}$ during the time interval Δt is approximately

$$\left(\frac{\Delta F_x}{\Delta x} + \frac{\Delta F_y}{\Delta y} + \frac{\Delta F_z}{\Delta z}\right)\Delta V\Delta t$$

Q: What does this change of mass mean in terms of the density inside $\mathcal{B}$?

- This change of mass in $\mathcal{B}$ is reflected by the rate of change of the density with respect to time and is thus approximately equal to

$$-\frac{\partial \rho}{\partial t}\Delta t\Delta V$$

- If we equate both expressions, divide the resulting equation by $\Delta V \Delta t$,

$$\frac{\Delta F_x}{\Delta x} + \frac{\Delta F_y}{\Delta y} + \frac{\Delta F_z}{\Delta z} \approx -\frac{\partial \rho}{\partial t}$$

and let Δx, Δy, Δz, and Δt approach zero, then we expect the error $\to 0$.

$$\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = -\frac{\partial \rho}{\partial t}$$

- If we define a vector field as the product of ρ and $\mathbf{V}$

$$\mathbf{F} = \rho \mathbf{V} = \rho v_x \mathbf{e}_x + \rho v_y \mathbf{e}_y + \rho v_z \mathbf{e}_z = F_x \mathbf{e}_x + F_y \mathbf{e}_y + F_z \mathbf{e}_z$$

then we get an physical interpretation of the divergence,

$$\operatorname{div} \mathbf{F} = -\frac{\partial \rho}{\partial t} \implies \operatorname{div} \mathbf{F} + \frac{\partial \rho}{\partial t} = 0$$

- This important relation is called the condition for the conservation of mass or the continuity equation of a compressible fluid.

- If the flow is steady, that is, independent of time, then

$$\frac{\partial \rho}{\partial t} = 0$$

and the continuity equation becomes

$$\operatorname{div}(\rho \mathbf{V}) = 0$$

- Further, if the density ρ is constant, then the fluid is incompressible and

$$\operatorname{div} \mathbf{V} = 0$$

- It states the fact that the sum of outflow and inflow for an infinitesimal volume around a given point is 0 at any time for.
- This relation, zero divergence, is known as the condition of incompressibility.
- A vector field which has zero divergence is also referred to as solenoidal and the corresponding fluid is referred to as incompressible.

- In general, for a velocity field $\mathbf{V}$ of a fluid, at each point within the fluid,

$$\operatorname{div} \mathbf{V}$$

measures the tendency of the fluid to diverge away from that point.

- For the vector field

$$\mathbf{F} = \rho \mathbf{V}$$

the divergence

$$\operatorname{div} \mathbf{F} = -\frac{\partial \rho}{\partial t}$$

is the negative rate of change of the density with respect to time.

- From this discussion you should conclude that, roughly speaking,

the divergence measures outflow minus inflow

- Clearly, the assumption that the flow has no source or sink in $\mathcal{B}$ is essential.