
Question1 (1 points)

Is $\mathbf{v}$ given below in the subspace of $\mathbb{R}^4$ spanned by $\mathcal{S}$ given below? Justify your answer.

$$\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathcal{S} = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 9 \\ -5 \end{bmatrix} \right\}$$

Question2 (1 points)

Is the following set of vectors linearly independent? Justify your answer.

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 6 \end{bmatrix} \right\}$$

Question3 (4 points)

Find the smallest spanning sets for each of the four fundamental subspaces associated with

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 1 & 5 \\ -2 & -4 & 0 & 4 & -2 \\ 1 & 2 & 2 & 4 & 9 \end{bmatrix}$$

Question4 (1 points)

Let $\mathcal{S}$ be a non-empty set. Show $\mathcal{S}$ is a maximal linearly independent set of a given vector space $\mathcal{V}$ if and only if $\mathcal{S}$ is a basis of $\mathcal{V}$.

Question5 (1 points)

Let $\mathbf{A}$ be an $m \times n$ matrix. Show the following holds if $\mathbf{A}$ has linearly independent columns

$$\operatorname{null}(\mathbf{A}^{\mathrm{T}}\mathbf{A}) = \{\mathbf{0}\}$$

Question6 (1 points)

Let

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix}$$

be a square matrix such that

$$\operatorname{null}(\mathbf{A}_1) = \operatorname{col}(\mathbf{A}_2^{\mathrm{T}}),$$

prove that $\mathbf{A}$ is invertible.

Question7 (1 points)

The general solution to $\mathbf{A}\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find $\mathbf{A}$.

Question8 (1 points)

Determine $\dim\Big(\operatorname{null}(\mathbf{A}) \cap \operatorname{col}(\mathbf{B})\Big)$ for

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 1 \\ -4 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 3 & 1 & -4 \\ -1 & -3 & 1 & 0 \\ 2 & 6 & 2 & -8 \end{bmatrix}.$$

Question9 (1 points)
Are there values of r and s for which

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

has rank 1? Has rank 2? If so, find those values.

Question10 (1 points)
Give an example of square matrices $\mathbf{A}$ and $\mathbf{B}$ for which

$$\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{B})$$

but

$$\operatorname{rank}(\mathbf{A}^2) \neq \operatorname{rank}\left(\mathbf{B}^2\right)$$

Explain how you have done your construction.

Question11 (1 points)
Prove if a matrix $\mathbf{A}$ is not square, then either the row vectors or the columns vectors of $\mathbf{A}$ are linearly dependent.

Question12 (2 points)
Are the following transformations T linear? Justify your answers.

(a) (1 point) $T(\mathbf{v})$ = largest component of $\mathbf{v}$, where $\mathbf{v} \in \mathbb{R}^n$.

(b) (1 point) Let T be the transformation that maps $\mathbf{A} \in \mathbb{R}^{m\times n}$ to its transpose $\mathbf{A}^{\mathrm{T}}$.

Question13 (1 points)
Suppose $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is linear, and $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\4\end{bmatrix}$ and $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\5\end{bmatrix}$. Is T one-to-one?

Question14 (2 points)
Suppose a line L in $\mathbb{R}^3$ contains the unit vector

$$\hat{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

(a) (1 point) Show the orthogonal projection of $\mathbf{x} \in \mathbb{R}^3$ onto L,

$$T(\mathbf{x}) = \operatorname{proj}_{\mathbf{u}} \mathbf{x}$$

is a linear transformation.

(b) (1 point) Find the matrix representation of T with respect to the standard basis.

Question15 (1 points)
Find an isomorphism between the vector space of all 3×3 symmetric matrices and $\mathbb{R}^6$.

Question16 (0 points)

(a) (1 point (bonus)) Suppose $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}$ have the same fundamental subspaces. Given they are both in rref, show $\mathbf{F} = \mathbf{G}$, where $\mathbf{A} = \begin{bmatrix} \mathbf{I} & \mathbf{F} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} \mathbf{I} & \mathbf{G} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$.

(b) (1 point (bonus)) Let $\operatorname{rank}(\mathbf{A}) = r$, show there is at least one invertible submatrix of order r, that is, $r\times r$, in $\mathbf{A}$, and there is no invertible submatrix of higher order in $\mathbf{A}$.