Vv255 Lecture 3

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Q: Given two nonzero vectors in $\mathbb{R}^3$,

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

is there a vector

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

that is orthogonal to *both* $\mathbf{u}$ and $\mathbf{v}$? If so, how to find $\mathbf{w}$? Is it unique?

- Let us assume one such vector exists, then $\mathbf{w}$ must satisfy

$$\mathbf{w} \cdot \mathbf{u} = 0$$
$$\mathbf{w} \cdot \mathbf{v} = 0$$

- In scalar form, we have

$$\begin{aligned} \mathbf{w}\cdot\mathbf{u} &= 0 \\ \mathbf{w}\cdot\mathbf{v} &= 0 \end{aligned} \Longrightarrow \begin{aligned} w_1u_1 + w_2u_2 + w_3u_3 &= 0 \\ w_1v_1 + w_2v_2 + w_3v_3 &= 0 \end{aligned}$$

- Eliminating one of the component from the equations, say w_3,

$$v_3\left(w_1u_1 + w_2u_2 + w_3u_3\right) = v_3 0 \qquad (1)$$

$$-u_3\left(w_1v_1 + w_2v_2 + w_3v_3\right) = -u_3 0 \qquad (2)$$

- If we add equations (1) to (2), and collect w_1 and w_2, we have

$$(u_1v_3 - u_3v_1)\, w_1 + (u_2v_3 - u_3v_2)\, w_2 = 0$$

by which we can conclude there are infinitely many $\mathbf{w}$ in general.

- One such $\mathbf{w}$ has the following form,

$$w_1 = u_2v_3 - v_3u_2 \qquad w_2 = u_3v_1 - u_1v_3 \qquad w_3 = u_1v_2 - u_2v_1$$

Definition

If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, then the cross product of $\mathbf{u}$ and $\mathbf{v}$, denoted as $\mathbf{u} \times \mathbf{v}$,

is the vector $$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

Theorem

The vector $\mathbf{u} \times \mathbf{v}$ is orthogonal to both nonzero vectors $\mathbf{u}$ and $\mathbf{v}$.

- Notice that the cross product $\mathbf{u} \times \mathbf{v}$ only gives one of infinitely many vectors

$$\alpha\left(\mathbf{u} \times \mathbf{v}\right) \qquad \text{where } \alpha \in \mathbb{R}$$

that are orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

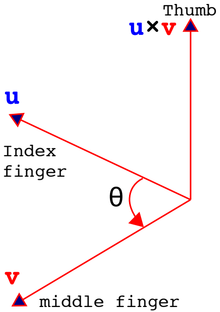
Matlab

```
>> syms u_1 u_2 u_3 v_1 v_2 v_3 real
>> u = [ u_1; u_2; u_3];
>> v = [ v_1; v_2; v_3];
>> dot( cross(u,v), u)

ans =  0

>> dot( cross(u,v), v)

ans =  0

>> cross

ans =

u_2*v_3 - u_3*v_2
u_3*v_1 - u_1*v_3
u_1*v_2 - u_2*v_1
```

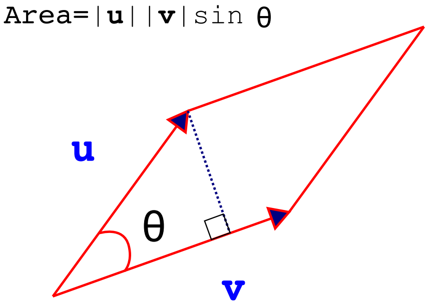
- Notice that the cross product $\mathbf{u} \times \mathbf{v}$, unlike the dot product, is a vector having both magnitude and direction.

Q: What is the geometric meaning of $\mathbf{u} \times \mathbf{v}$ in terms of $\mathbf{u}$ and $\mathbf{v}$?

- The direction determined by the right-hand rule.



- The magnitude is equal to the area of the parallelogram with sides $\mathbf{u}$ and $\mathbf{v}$.



Theorem

If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, $0 \leq \theta \leq \pi$, then

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta$$

Proof

If we square the LHS, we have

$$\begin{aligned}
|\mathbf{u} \times \mathbf{v}|^2 &= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \\
&= u_2^2v_3^2 - 2u_2u_3v_2v_3 + u_3^2v_2^2 + u_3^2v_1^2 - 2u_1u_3v_1v_3 + u_1^2v_3^2 \\
&\quad + u_1^2v_2^2 - 2u_1u_2v_1v_2 + u_2^2v_1^2 + u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2 - u_1^2v_1^2 - u_2^2v_2^2 - u_3^2v_3^2 \\
&= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \\
&= |\mathbf{u}|^2\,|\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2 \\
&= |\mathbf{u}|^2\,|\mathbf{v}|^2 - |\mathbf{u}|^2\,|\mathbf{v}|^2 \cos^2\theta \\
&= |\mathbf{u}|^2\,|\mathbf{v}|^2 (1 - \cos^2\theta) \\
&= |\mathbf{u}|^2\,|\mathbf{v}|^2 \sin^2\theta
\end{aligned}$$

Taking the square root and observing that $\sqrt{\sin^2\theta} = \sin\theta$ because $\sin\theta \geq 0$,

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta \quad \square$$

Q: What is the cross product of two vectors, of which one is parallel another?

Corollary

Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

Exercise

Determine whether the following points are collinear, if not, find the area of the triangle formed by these points and find a vector orthogonal to the triangle.

$$P(1,4,6), \quad Q(-2,5,-1), \quad \text{and} \quad R(1,-1,1)$$

Properties of cross products

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^3$ and α is a scalar, then

1. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
2. $(\alpha\mathbf{u}) \times \mathbf{v} = \alpha(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (\alpha\mathbf{v})$
3. $\mathbf{w} \times (\mathbf{u} + \mathbf{v}) = \mathbf{w} \times \mathbf{u} + \mathbf{w} \times \mathbf{v}$
4. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$

- Recall for vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, then we have $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^{\mathrm{T}}\mathbf{v}$ and $\mathbf{u} \otimes \mathbf{v} = \mathbf{u}\mathbf{v}^{\mathrm{T}}$.

Q: Can the cross product also be treated as a matrix product of some kind?

- The cross product can be written as a matrix product,

$$\mathbf{u}\times\mathbf{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \left[\mathbf{u}\right]_\times \mathbf{v}$$

$$= \begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \left[\mathbf{v}\right]_\times^{\mathrm{T}} \mathbf{u}$$

  where the matrix $\left[\mathbf{u}\right]_\times$ is called the cross product matrix associated to $\mathbf{u}$, which is also called the cross product tensor associated to $\mathbf{u}$, alternatively

$$[\mathbf{u}]_\times = \mathbf{C_u}$$

- Using the properties of matrix multiplication, we can prove 3. and 4. easily.

  e.g. $\mathbf{w}\times(\mathbf{u}+\mathbf{v}) = \left[\mathbf{w}\right]_\times(\mathbf{u}+\mathbf{v}) = \left[\mathbf{w}\right]_\times\mathbf{u} + \left[\mathbf{w}\right]_\times\mathbf{v} = \mathbf{w}\times\mathbf{u} + \mathbf{w}\times\mathbf{v}$

- Recall the dot product tells whether two vectors are orthogonal,

$$\mathbf{u} \cdot \mathbf{v} = 0$$

and the cross product tells whether $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ are parallel/collinear

$$\mathbf{u} \times \mathbf{v} = \mathbf{0}$$

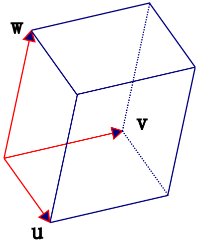
Q: How can we tell whether three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ are coplanar?

- Recall Volume = Base $\times$ Hight

$$= |\mathbf{u} \times \mathbf{v}| \left|\mathrm{comp}_{(\mathbf{u}\times\mathbf{v})}\, \mathbf{w}\right|$$

$$= |\mathbf{u} \times \mathbf{v}| \left|\frac{\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})}{\mathbf{u} \times \mathbf{v}}\right|$$

$$= |\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|$$



Theorem

The volume of the parallelepiped determined by the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is

$$\text{Volume} = |\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|$$

Definition

The scalar $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$ is known as the scalar triple product, and

$$\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

Exercise

Show the vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 0 \\ -9 \\ 18 \end{bmatrix}$ are coplanar.

- Since both the dot product and the cross product are special forms of matrix multiplications, we expect some connection between the scalar triple product and matrices.

Definition

- The determinant is a scalar associated with every square matrix,

$$\det(\mathbf{A}), \quad \text{or} \quad |\mathbf{A}|$$

- A determinant of order 2 is defined by $\det\left(\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right) = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$.

$$= u_1 v_2 - u_2 v_1$$

- A higher-order determinant can be defined in terms of lower-order dets, e.g.

$$\begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = w_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + w_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

Exercise

Find the determinant of $\mathbf{A} = \begin{bmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{bmatrix}$.

- The determinant $\det(\mathbf{A}_{3\times 3})$ is closely related to the scalar triple product.

$$
\begin{aligned}
\begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} &= w_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + w_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \\
&= w_1(u_2v_3 - u_3v_2) - w_2(u_1v_3 - u_3v_1) + w_3(u_1v_2 - u_2v_1) \\
&= w_1(u_2v_3 - u_3v_2) + w_2(u_3v_1 - u_1v_3) + w_3(u_1v_2 - u_2v_1) \\
&= \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})
\end{aligned}
$$

Theorem

The volume of the parallelepiped determined by the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is equal to the absolute value of the determinant of the corresponding matrix,

$$
\text{Volume} = |\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})| = \left| \det \left(\begin{bmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \right) \right|
$$

- So to find the volume of the parallelepiped determined by vectors, e.g.

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \text{ and } \mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix},$$

we can ask Matlab for either the scalar triple product or the determinant.

Matlab

```
>> u = [ 1 2 3 ];
>> v = [ 1 1 2 ];
>> w = [ 2 1 4 ];
>> A = [ u; v; w ]
A =
1      2      3
1      1      2
2      1      4
>> det(A)
ans =  -1
>> abs(ans)
ans =   1
```

```
>> dot(u, cross(v,w))
ans =   -1
>> dot(w, cross(u,v))
ans =   -1
>> dot(v, cross(w,u))
ans =   -1
>> dot(u, cross(w,v))
ans =   1
>> dot(w, cross(v,u))
ans =   1
>> dot(v, cross(u,w))
ans =   1
```

Q: What is the geometric interpretation of the determinant of a 2×2 matrix?

Theorem

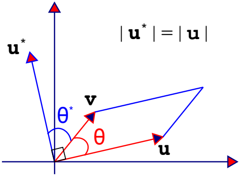
The area of the parallelogram determined by two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ is

$$\text{Area} = |\det(\mathbf{A})| = \left|\det\left(\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right)\right|, \qquad \text{where} \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix}$$

Proof

- Now consider a vector $\mathbf{u}^*$, which is $\mathbf{u}$ after a counter-clockwise rotation of 90°, what are the components of $\mathbf{u}^*$?

$$\begin{aligned}
\text{Area} = |\mathbf{u} \times \mathbf{v}| &= |\mathbf{u}|\,|\mathbf{v}|\,|\sin\theta| \\
&= |\mathbf{u}^*|\,|\mathbf{v}|\,\left|\cos\left(\frac{\pi}{2} - \theta\right)\right| \\
&= |\mathbf{u}^*|\,|\mathbf{v}|\,|\cos\theta^*| \\
&= |\mathbf{u}^* \cdot \mathbf{v}| = |u_1 v_2 - u_2 v_1| = |\det(\mathbf{A})| \quad \square
\end{aligned}$$



Q: What does it mean to have a determinant equal to zero in general?

Matlab

```
>> u = [ 1 4 -7 ];
>> v = [ 2 -1 4 ];
>> w = [ 0 -9 18 ];
>> A = [ u; v; w ];
A =
1      4     -7
2     -1      4
0     -9     18
>> det(A)
ans =  0
```

```
>> A_T = transpose(A)
A_T =
1      2      0
4     -1     -9
-7     4     18
>> det(A_T)
ans =  0
>> mldivide(A_T(:,1:2),A_T(:,3))
ans =
-2.0000
1.0000
```

Exercise

Express $-\left(\mathbf{v}\mathbf{u}^{\mathrm{T}} - \mathbf{u}\mathbf{v}^{\mathrm{T}}\right)\mathbf{w}$ using cross products.

Definition

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^3$, then the vector

$$\mathbf{w} \times (\mathbf{u} \times \mathbf{v})$$

is known as the vector triple product.

Lagrange's formula

If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are vectors in $\mathbb{R}^3$, then

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

- Engineers often call it the "bac-cab" identity.
- It states that any vector can be written as a linear combination of two mutually orthogonal vectors according arbitrary unit vector $\mathbf{e}$:

$$\begin{aligned} \mathbf{e} \times (\mathbf{u} \times \mathbf{e}) = \mathbf{u}(\mathbf{e} \cdot \mathbf{e}) - \mathbf{e}(\mathbf{e} \cdot \mathbf{u}) &\implies \mathbf{u} = \mathbf{e} \times (\mathbf{u} \times \mathbf{e}) + \mathbf{e}(\mathbf{e} \cdot \mathbf{u}) \\ &\implies \mathbf{u} = \mathbf{e} \times (\mathbf{u} \times \mathbf{e}) + \operatorname{proj}_{\mathbf{e}} \mathbf{u} \end{aligned}$$

Jacobi's identity

If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are vectors in $\mathbb{R}^3$, then

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$$

- The sum of all the cyclic permutations of the vector triple product is zero.

Dot of crosses and Cross of crosses

If $\mathbf{a}$, $\mathbf{b}$. $\mathbf{c}$ and $\mathbf{d}$ are vectors in $\mathbb{R}^3$, then

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) &= (\mathbf{a} \cdot (\mathbf{c} \times \mathbf{d}))\,\mathbf{b} - (\mathbf{b} \cdot (\mathbf{c} \times \mathbf{d}))\,\mathbf{a} \\ &= (\mathbf{a} \cdot (\mathbf{b} \times \mathbf{d}))\,\mathbf{c} - (\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}))\,\mathbf{d}\end{aligned}$$