V_256 Assignment 4

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Q_1: $y=\sum_{n=0}^{\infty} C_n x^n$

$-3xy=-3\sum_{n=0}^{\infty} C_n x^{n+1}=-3\sum_{n=1}^{\infty} C_{n-1}x^n=-3C_0x-3\sum_{n=2}^{\infty}C_{n-1}x^n$

$y'=\sum_{n=1}^{\infty} nC_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)C_{n+1}x^n$

$-x^2y'=-\sum_{n=0}^{\infty}(n+1)C_{n+1}x^{n+2}=-\sum_{n=2}^{\infty}(n-1)C_{n-1}x^n$

$y''=\sum_{n=2}^{\infty}(n-1)nC_nx^{n-2}=\sum_{n=0}^{\infty}(n+1)(n+2)C_{n+2}x^n=2C_2+6C_3x+\sum_{n=2}^{\infty}(n+1)(n+2)C_{n+2}x^n$

$2C_2+(6C_3-3C_0)x+\sum_{n=2}^{\infty}[-3C_{n-1}-(n-1)C_{n-1}+(n+1)(n+2)C_{n+2}]x_n$

$C_2=0$

$C_0=2C_3$

$C_{n+2}=\frac{C_{n-1}}{n+1}$

$y=C_1(x+\frac{x^4}{3}+\frac{x^7}{3\times6}\dots)+C_0(1+\frac{x^3}{2}+\frac{x^6}{2\times5}\dots)$ $R=\infty$

Q_2: $1-x^2=0$

$x=\pm1\Rightarrow0$ is an ordinary point

$(1-x^2)y''-2xy'+\alpha(\alpha+1)y=0$

Assume $y=\sum_{n=0}^{\infty}C_nx^n$ then $y'=\sum_{n=1}^{\infty}nC_nx^{n-1}$ $y''=\sum_{n=2}^{\infty}n(n-1)C_nx^{n-2}$

$(1-x^2)y''=\sum_{n=2}^{\infty}n(n-1)C_nx^{n-2}-\sum_{n=2}^{\infty}n(n-1)C_nx^n=2C_2+6C_3x+\sum_{n=2}^{\infty}(n+2)(n+1)C_{n+2}x^n-(n-1)nC_nx^n$

$-2xy'=-2\sum_{n=1}^{\infty}nC_nx^n=-2C_1x-2\sum_{n=2}^{\infty}nC_nx^n$

$\alpha(\alpha+1)y=\alpha(\alpha+1)C_0+\alpha(\alpha+1)C_1x+\sum_{n=2}^{\infty}C_nx^n$

$2C_2+\alpha(\alpha+1)C_0+[6C_3-2C_1+\alpha(\alpha+1)C_1]x+\sum_{n=2}^{\infty}\{(n+2)(n+1)C_{n+2}+[\alpha(\alpha+1)-n(n+1)]C_n\}x_n=0$

$\frac{C_{n+2}}{C_n}=\frac{n(n+1)-\alpha(\alpha+1)}{(n+2)(n+1)}$

$\therefore y=C_0\{1+\frac{-\alpha(\alpha+1)}{2!}x^2+\frac{-\alpha(\alpha+1)[6-\alpha(\alpha+1)]}{4!}x^4\dots\}+C_1\{x+\frac{2-\alpha(\alpha+1)}{3!}x^3+\frac{[2-\alpha(\alpha+1)][12-\alpha(\alpha+1)]}{5!}x^5\dots\}$ $R=1$

Q_3: $y''+t^2y'+(t^2+2t)y=0$

Assume $y=\sum_{n=0}^{\infty}C_nt^n$, $y'=\sum_{n=1}^{\infty}nC_nt^{n-1}$, $y''=\sum_{n=2}^{\infty}n(n-1)C_nt^{n-2}$

$(t^2+2t)y=2C_0t+\sum_{n=2}^{\infty}[C_{n-2}+2C_{n-1}]t^n$ $t^2y'=\sum_{n=1}^{\infty}nC_nt^{n+1}=\sum_{n=2}^{\infty}(n-1)C_{n-1}t^n$ $y''=2C_2+6C_3t+\sum_{n=2}^{\infty}(n+1)(n+2)C_{n+2}t^n$

$2C_2+(2C_0+6C_3)t+\sum_{n=2}^{\infty}[C_{n-2}+(n+1)C_{n-1}+(n+1)(n+2)C_{n+2}]t^n$

$C_2=0$ $C_0+3C_3=0$ $(n+1)(n+2)C_{n+2}+(n+1)C_{n-1}+C_{n-2}=0$

$y=C_0+C_1t-\frac{1}{3}C_0t^3-\frac{3C_1+C_0}{12}t^4=C_0+C_1(x-1)-\frac{1}{3}C_0(x-1)^3-\frac{3C_1+C_0}{12}(x-1)^4\dots$

Assume $y=\sum_{n=0}^{\infty} C_n (x-1)^n$, $y'=\sum_{n=1}^{\infty} n C_n (x-1)^{n-1}$, $y''=\sum_{n=2}^{\infty} n(n-1) C_n (x-1)^{n-2}$

$(x-1)^2 y' = \sum_{n=2}^{\infty} (n-1) C_{n-1} (x-1)^n$ $y'' = 2C_2 + 6C_3(x-1) + \sum_{n=2}^{\infty} (n+1)(n+2) C_{n+2} (x-1)^n$

$(x^2-1)y = (x-1)(x-1+2)y = \sum_{n=2}^{\infty} C_{n-2}(x-1)^n + 2C_0(x-1) + 2\sum_{n=2}^{\infty} C_{n-1}(x-1)^n$

$\Rightarrow 2C_2 + (6C_3 + 2C_0)(x-1) + \sum_{n=2}^{\infty} [C_{n-2} + (n+1)C_{n-1} + (n+1)(n+2)C_{n+2}](x-1)^n$ It's same as using the substitution $t=x-1$

$y = C_0 + C_1(x-1) - \frac{1}{3}C_0 (x-1)^3 - \frac{3C_1+C_0}{12}(x-1)^4$

Q_4: $y=\sum_{n=0}^{\infty} C_n x^n$ $y''=\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$ $e^x y'' + xy = 0$

$xy = \sum_{n=0}^{\infty} C_n x^{n+1}$

$e^x y'' = \sum_{n=0}^{\infty} \frac{x^n}{n!} \times \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$

$(1+\frac{x}{1}+\frac{x^2}{2!}\cdots\frac{x^n}{n!})\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} + \sum_{n=0}^{\infty} C_n x^{n+1} = 0$

$C_2=0$ $C_0 + 2C_2 + 6C_3 = 0$ $C_1 + 12C_4 + 6C_3 + C_2 = 0$ $C_2 + 20C_5 + 12C_4 + 3C_3 + \frac{1}{3}C_2 = 0$

$C_3 = -\frac{1}{6}C_0$

$\therefore y = C_0 + C_1 x - \frac{C_0}{6}x^3 + \frac{C_0 - C_1}{12}x^4 + \frac{2C_1 - C_0}{40}x^5 + (\frac{C_0}{90} - \frac{C_1}{60})x^6$

$\because$ There is no singular point, so radis of convergence is ∞

$\phi_1 = C_0(1 - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{40} + \frac{1}{90}x^6 \cdots)$, $\phi_2 = C_1(x - \frac{x^4}{12} + \frac{x^5}{20} - \frac{x^6}{60} \cdots)$

Q_5: $2t^2=0$ $\frac{3t(1+t)}{2t^2} \times t = \frac{3}{2}(1+t)$ $\frac{-1}{2t^2} \times t^2 = -\frac{1}{2}$

$\therefore t_0=0$ is a regular singular point

$y = t^r \sum_{n=0}^{\infty} C_n t^n = \sum_{n=0}^{\infty} C_n t^{n+r}$ $y' = \sum_{n=0}^{\infty} (n+r) C_n t^{n+r-1}$ $y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) C_n t^{n+r-2}$

$2t^2 y'' = \sum_{n=1}^{\infty} 2(n+r)(n+r-1) C_n t^{n+r} + 2r(r-1)C_0 t^r$ $3t(1+t)y' = 3rC_0 t^r + \sum_{n=1}^{\infty} 3[(n+r)C_n + (n+r-1)C_{n-1}]t^{n+r}$ $-y = -C_0 t^r - \sum_{n=1}^{\infty} C_n t^{n+r}$

$(2r^2+r-1)C_0 t^r + \sum_{n=1}^{\infty} \{[(n+r)(2n+2r+1)-1]C_n + 3(n+r-1)C_{n-1}\} t^{n+r}$

$2r^2+r-1=0$

$\begin{cases} r_1 = -1 \Rightarrow C_n = -\frac{3(n-2)}{n(2n-3)} C_{n-1} \\ r_2 = \frac{1}{2} \Rightarrow C_n = -\frac{3(n-\frac{1}{2})}{2n^2+3n} C_{n-1} \end{cases}$ $\Rightarrow y = C_0 t^{-1}(1 + \sum_{n=1}^{\infty} \frac{(-3)^n \times (-1) \times 0 \times (n-2)!}{n! \times (-1) \times 1 \times (2n-3)!} t^n) + C_0 t^{\frac{1}{2}}(1 + \sum_{n=1}^{\infty} \frac{(-3)^n \times (\frac{1}{2}) \times \frac{3}{2} \times \cdots (n-\frac{1}{2})}{n! \times 5 \times 7 \cdots (2n+3)} t^n)$

According to ratio test, $R = \frac{4?}{?}$

(a) $\frac{-y}{x} \times x^2 = -xy$ $\quad xy'' - y = 0$

$x=0$ is a regular singular point

Assume $y = x^r \sum_{n=0}^{\infty} C_n x^n$ then $y = \sum_{n=0}^{\infty} C_n x^{n+r}$, $y'' = (n+r)(n+r-1) \sum_{n=0}^{\infty} C_n x^{n+r-2}$

$$r(r-1) C_0 x^{r-1} + \sum_{n=0}^{\infty} [(n+r)(n+r+1) C_{n+1} - C_n] x^{n+r}$$

$r(r-1)=0$ $\quad r_1 = 0$ $\quad r_2 = 1$

$$(n+1)(n+2) C_{n+1} - C_n = 0$$

$$C_{n+1} = \frac{C_n}{(n+1)(n+2)}$$

$$y_1 = C_0 x \left(1 + \frac{1}{1\times2} x + \frac{1}{1\times2\times2\times3} x^2 + \frac{1}{1\times2\times2\times3\times3\times4} x^3 \cdots\right) = C_0 \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} x^{n+1}$$

(b)

$$y_2 = C \ln x \, C_0 \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} x^{n+1} + \sum_{n=0}^{\infty} C_n^* x^n$$

$$y_2' = C\left(y_1' \ln x + \frac{1}{x} y_1\right) + \sum_{n=0}^{\infty} n C_n^* x^{n-1}$$

$$y_2'' = C\left(y_1'' \ln x + \frac{2}{x} y_1' - \frac{1}{x^2} y_1\right) + \sum_{n=0}^{\infty} n(n-1) C_n^* x^{n-2}$$

$$(x y_1'' - y_1) C \ln x + C\left(2y_1' - \frac{1}{x} y_1\right) + \sum_{n=0}^{\infty} [n(n-1) C_n^* x^{n-1} - C_n^* x^n] = 0$$

$$y_1' = C_0 \sum_{n=0}^{\infty} \frac{1}{(n!)^2} x^n$$

$$C C_0 \sum_{n=0}^{\infty} \left[\frac{2}{(n!)^2} x^n - \frac{1}{n!(n+1)!} x^n\right] + \sum_{n=0}^{\infty} C_n^* [n(n-1) x^{n-1} - x^n] = 0$$

$$\sum_{n=0}^{\infty} \left[C\left(\frac{2}{(n!)^2} - \frac{1}{n!(n+1)!}\right) + C_{n+1}^* (n+1) n - C_n^*\right] x^n = 0$$

when $n=0$ $\quad C - C_0^* = 0$ $\quad C = C_0^*$ $\quad C_1^* = 0$

$$C_{n+1}^* = \frac{1}{(n+1)n}\left[C_n^* - \frac{C_0}{(n!)^2}\left(2 - \frac{1}{n+1}\right)\right]$$

$$\therefore y_2 = C_0^* y_1 \ln x + \sum_{n=0}^{\infty} C_n^* x^n = C_0^* \times C_0 \ln x \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} x^{n+1} + C_0^* \left(1 - \frac{3}{4} x^2 - \frac{7}{36} x^3 \cdots\right)$$

(a) $ty' + \lambda y = f(t)$

$y' = \sum_{n=1}^{\infty} n C_n t^{n-1}$

$\sum_{n=1}^{\infty} n C_n t^n + \lambda \sum_{n=0}^{\infty} C_n t^n = \sum_{n=0}^{\infty} f_n t^n$

$\lambda C_0 = f_0$

$(n+\lambda) C_n = f_n \quad (n \geq 1)$ $\qquad y = \sum_{n=0}^{\infty} \frac{f_n}{n+\lambda} t^n$

(b) Assume $y = \sum_{n=0}^{\infty} C_n x^n$

$\sum_{n=1}^{\infty} n C_n x^{n-1} = x^2 + \sum_{n=0}^{\infty} C_n x^n \cdot \sum_{n=0}^{\infty} C_n x^n$

$C_0 = 1 \quad C_1 = C_0^2 \quad 2C_2 = 2C_1C_0 \quad 3C_3 = 1 + C_1^2 + 2C_0C_2 \quad 4C_4 = 2C_0C_3 + 2C_2C_1$

$C_0 = 1 \quad C_1 = 1 \quad C_2 = 1 \quad C_3 = \frac{4}{3} \quad C_4 = \frac{7}{8}$

$y = 1 + x + x^2 + \frac{4}{3}x^2 + \frac{7}{8}x^3 \cdots$

(c) Assume $y = \sum_{n=0}^{\infty} C_n (x-1)^n \qquad y' = \sum_{n=1}^{\infty} n C_n (x-1)^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) C_n (x-1)^{n-2}$

$C_0 = 1 \quad C_1 = 0 \quad C_2 = 1$

$\sum_{n=3}^{\infty} n(n-1)(n-2) C_n (x-1)^{n-3} = (x-1)^2 + \sum_{n=0}^{\infty} C_n (x-1)^n \times \sum_{n=0}^{\infty} C_n (x-1)^n - \sum_{n=1}^{\infty} n C_n (x-1)^{n-1} - 2$

Assume $x-1 = t$

$6C_3 + 24C_4 t + 60C_5 t^2 + \sum_{n=3}^{\infty} (n+3)(n+2)(n+1) C_{n+3} t^n = (C_0^2 - 2 - C_1) + 2(C_0C_1 - C_2)t + (1 + 2C_0C_2 + C_1 - 3C_3)t^2$

$C_0 = 1 \quad C_1 = 0 \quad C_2 = 1 \quad C_3 = -\frac{1}{6} \quad C_4 = -\frac{1}{12} \quad C_5 = \frac{1}{24}$

$\therefore y = 1 + (x-1)^2 - \frac{1}{6}(x-1)^3 - \frac{1}{12}(x-1)^4 + \frac{1}{24}(x-1)^5 \cdots$