Vv255 Lecture 16

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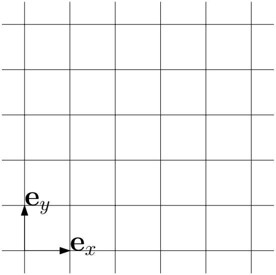
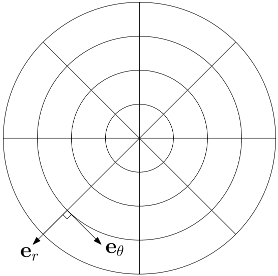
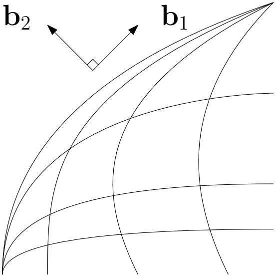
- A big difference between Cartesian, and the plane polar, cylindrical polar and spherical polar is that the coordinate lines may be curved in the later ones.







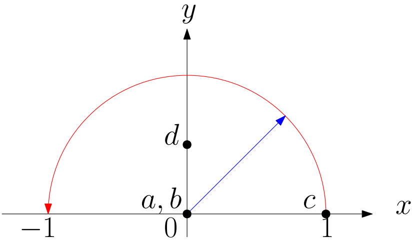
- In general, changing coordinates is a transformation from a space to another,

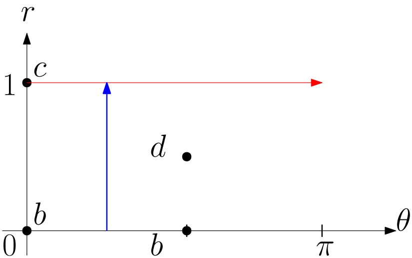
$$T\colon \mathbb{R}^2 \to \mathbb{R}^2 \qquad \text{and} \qquad T\colon \mathbb{R}^3 \to \mathbb{R}^3$$

- Specifically, it is an invertible transformation between points in the 2 spaces,

e.g. Polar $\to$ Cartesian and Cartesian $\to$ Polar

- Consider how points and vectors are transformed as we change the coordinates between Cartesian and the polar coordinates.





- Changing coordinates is similar to changing variables, we are interested in

1. changing a function of the old to be in terms of the new variables, e.g.

$$f(x, y) = F(u, v)$$

2. finding the derivatives with respect to the new variables, e.g.

$$f_u \qquad \text{and} \qquad f_v$$

- Suppose the transformation equations are given as "old in terms of new",

$$x = x(u, v) \qquad \text{and} \qquad y = y(u, v)$$

- If we actually know the function $z = f(x, y)$ explicitly, then it is easy to find

$$z = f(x, y) = f(x(u, v), y(u, v)) = F(u, v)$$

and the partial derivatives can be found directly. Even without $z = F(u, v)$,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

the chain rule provides a way of finding the partial derivatives given we know

$$\frac{\partial z}{\partial x} \qquad \text{and} \qquad \frac{\partial z}{\partial y}$$

- Now let the transformation equations be given as "new in terms of old",

$$u = u(x, y) \qquad \text{and} \qquad v = v(x, y)$$

- We might be able to solve x and y in terms of u and v, then as before

$$z = f(x, y) = f(x(u, v), y(u, v)) = F(u, v)$$

Q: What happens if we cannot solve for

$$x = x(u, v) \qquad \text{and} \qquad y = y(u, v)$$

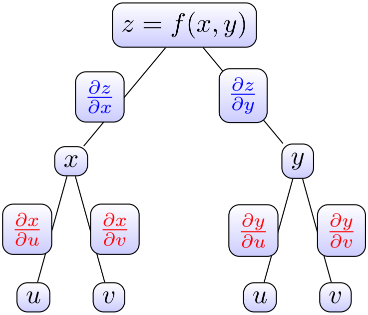
Q: Can we find $F(u, v)$ explicitly? How about F_u and F_v?

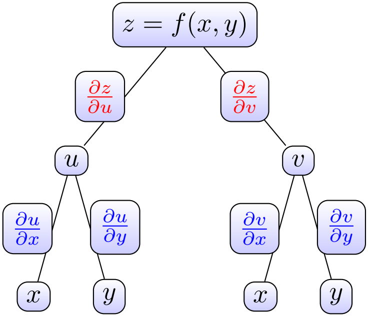
Q: Can we use the chain rule here to find the partial derivatives?

- Given $z = f(x, y)$, $u = u(x, y)$ and $v = v(x, y)$, we can find

$$\frac{\partial z}{\partial x}, \quad \frac{\partial z}{\partial y}, \quad \frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y}$$

- Here we have two versions of the chain rule, only one of those two is useful,





- No solvable

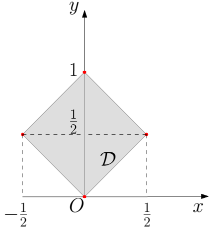
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

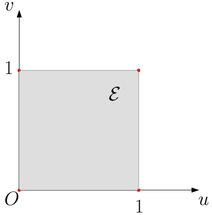
- Solvable

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

Exercise

Suppose that $T(x, y)$ is defined over the region $\mathcal{D}$, where $\mathcal{D}$ is indicated below





Now if we are interested in the rate of change of the function f along the edges, then it proves much easier to consider the transformation

$$u = y + x \qquad \text{and} \qquad v = y - x$$

Suppose that some physical quantity is defined to be

$$W = \frac{\partial^2 T}{\partial x^2} - \frac{\partial^2 T}{\partial y^2} + \frac{\partial T}{\partial x} + \frac{\partial T}{\partial y}$$

How can W be evaluated in terms of u and v?

- Therefore, for arbitrary

$$z = f(x,y), \qquad u = u(x,y) \qquad \text{and} \qquad v = v(x,y)$$

we can obtain the partial derivatives with respect to the new variables u and v by solving the linear equations, and in general, we have

$$\frac{\partial z}{\partial u} = \frac{\dfrac{\partial z}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial z}{\partial y}\dfrac{\partial v}{\partial x}}{\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}} \qquad \text{and} \qquad \frac{\partial z}{\partial v} = \frac{\dfrac{\partial z}{\partial y}\dfrac{\partial u}{\partial x} - \dfrac{\partial z}{\partial x}\dfrac{\partial u}{\partial y}}{\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}}$$

- Note both derivatives share the same denominator, and it must be non-zero for the rate of change to be defined. Recall the determinant of a 2×2 matrix

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$\Longrightarrow \det(\mathbf{J}) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}, \qquad \text{where} \quad \mathbf{J} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix}.$$

Definition

The Jacobian of transformation $u = u(x, y)$ and $v = v(x, y)$ is the determinant

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u, v)}{\partial(x, y)} = J(x, y)$$

- The matrix from which the Jacobian is defined is called the Jacobian matrix

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix}$$

- For a transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$, the Jacobian is defined in a similar way.
- The Jacobian is of special interest, because it contains the information about the transformation between one set of coordinates (x, y) and another (u, v).

- Of course, we can have Cartesian coordinates in terms of other coordinates

$$x = x(u, v) \qquad \text{and} \qquad y = y(u, v)$$

- And the Jacobian matrix and the Jacobian for the transformation are

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} \qquad \text{and} \qquad J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \det(\mathbf{J})$$

- In this case, the Jacobian matrix can be understood as

$$\mathbf{J} = \begin{bmatrix} \nabla x^{\mathrm{T}} \\ \nabla y^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \mathbf{r}}{\partial u} & \dfrac{\partial \mathbf{r}}{\partial v} \end{bmatrix}$$

Q: What does it mean in terms of the partial derivatives of $z = f(u, v)$?

Q: Will any transformation work?

Q: What kinds of transformations will not provide a useful coordinate system?

- Suppose we were considering a new set of coordinates, (u, v), given by

$$u = x^2 + y + 1 \qquad \text{and} \qquad v = x^4 + 2x^2y + y^2 + x^2 - y$$

Q: Why this transformation is not going to provide useful coordinates system ?

$$v = (u-1)^2 - (u-1) = u^2 - 3u + 2$$

- There is a functional dependence between u and v, so it is not invertible.

Theorem

If $u(x, y)$ and $v(x, y)$ are functionally dependent, then

$$\frac{\partial(u, v)}{\partial(x, y)} = 0$$

Proof

- If $u(x,y)$ and $v(x,y)$ are functionally dependent, there is an equation

$$F(u,v) = 0$$

- Apply implicit differentiation,

$$\begin{aligned} F_u u_x + F_v v_x &= 0 \\ F_u u_y + F_v v_y &= 0 \end{aligned}$$

- For consistency, we must have

$$u_x = \alpha u_y \qquad \text{and} \qquad v_x = \alpha v_y \qquad \text{where } \alpha \text{ is a constant.}$$

- That is,

$$u_x v_y - v_x u_y = \frac{\partial(u,v)}{\partial(x,y)} = 0 \quad \square$$