Vv256 Lecture 7

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Definition

When $f(t)$ is not identically zero, but a, b, and c are constants, then equation

$$a\ddot{y} + b\dot{y} + cy = f(t) \tag{1}$$

is known as nonhomogeneous with constant coefficients.

The corresponding homogeneous equation

$$a\ddot{y} + b\dot{y} + cy = 0 \tag{2}$$

is called the complementary equation of equation (1).

Theorem

The general solution of equation (1) can be written as

$$y = y_p + y_c,$$

where y_p is a particular solution to (1) and y_c is the general solution to (2).

Proof

- Consider

$$y^* = y - y_p$$

where y and y_p are the general and particular solution of equation (1).

- Consider

$$\begin{aligned} a(y-y_p)'' + b(y-y_p)' + c(y-y_p) &= (ay'' + by' + cy) - \left(ay_p'' + by_p' + cy_p\right) \\ &= f(t) - f(t) \\ &= 0 \end{aligned}$$

- Therefore $y^* = y - y_p$ is the general solution of equation (2), hence

$$y = y_p + y_c$$

is the general solution of equation (1).

- This suggests the following steps for solving equation (1).

Steps for finding the general solution for nonhomogeneous equations

1. Find the complementary solution, which is the general solution

$$y_c = C_1\phi_1(t) + C_2\phi_2(t)$$

   to the corresponding homogeneous equation.

2. Find a particular solution, which is any solution y_p to equation (1).
3. Add the complementary and the particular solutions together

$$y = y_c + y_p$$

- So the only problem left is how to find the particular solution.

undetermined coefficients and variation of parameters

- The method of undetermined coefficients is simple but works only if $f(t)$ is
  1. Exponential
  2. Polynomial
  3. Sines and cosines
  4. Sums and products of those types of functions
- The basic idea is this, consider the nonhomogeneous equation

$$a\ddot{y} + b\dot{y} + cy = f(t)$$

- If $f(t)$ is a relatively simple function, and since the coefficients a, b and c are constants, we might be able to make a good guess as to what sort of function y_p produces $f(t)$ after being plugged into the left side of the above equation.
- For example,

$$\ddot{y} - 2\dot{y} - 3y = 36e^{5t}$$

- Since all derivatives of e^{5t} equal some constant multiple of e^{5t}, if we let

$$y = \text{some constant multiple of } e^{5t}$$

then

$$\begin{aligned} \ddot{y} - 2\dot{y} - 3y &= \text{some other multiple of } e^{5t} \\ &= 36\, e^{5t} \end{aligned}$$

- So we let A be some constant "to be determined", and try

$$y_p = Ae^{5t}$$

as a particular solution to our differential equation:

$$y'' - 2y' - 3y = \left(Ae^{5t}\right)'' - 2\left(Ae^{5t}\right)' - 3\left(Ae^{5t}\right) = 36e^{5t}$$

$$\implies 12Ae^{5t} = 36e^{5t} \implies A = 3$$

- So $y_p(t) = Ae^{5t}$ satisfies the differential equation only if $A = 3$.

$$y_p(t) = 3e^{5t}$$

is a particular solution to our nonhomogeneous differential equation.

- To obtain the general solution, we need to solve complementary equation

$$\ddot{y} - 2\dot{y} - 3y = 0$$

- The characteristic equation for it is

$$r^2 - 2r - 3 = 0 \iff (r+1)(r-3) = 0$$

- So the general solution to the corresponding homogeneous equation is

$$y_c(t) = c_1 e^{-t} + c_2 e^{3t}$$

- Thus the general solution to the nonhomogeneous equation is

$$y(t) = y_c(t) + y_p(t) = c_1 e^{-t} + c_2 e^{3t} + 3e^{5t}$$

- To solve the initial-value problem involves the equation, we proceed as usual

$$y'' - 2y' - 3y = 36e^{5t}, \qquad y(0) = 9, \quad y'(0) = 25$$

- We know the general solution to the equation is

$$\begin{aligned} y(t) = y_c(t) + y_p(t) &= c_1e^{-t} + c_2e^{3t} + 3e^{5t} \\ \implies y'(t) &= -c_1e^{-t} + 3c_2e^{3t} + 15e^{5t} \end{aligned}$$

- Apply the initial conditions, we have

$$9 = 3 + c_1 + c_2 \qquad \text{and} \qquad 25 = 15 - c_1 + 3c_2$$

- Solve this system, we have $c_1 = 2$ and $c_2 = 4$, so the particular solution to the given differential equation that satisfies the given initial conditions is

$$y(x) = 2e^{-t} + 4e^{3t} + 3e^{5t}$$

Q: Can you think of a trial solution for the differential equation

$$y'' - 2y' - 3y = 65\cos 2t$$

- A naive first guess for a particular solution might be

$$y_p(t) = A\cos 2t, \qquad \text{where } A \text{ is some constant to be determined.}$$

- Unfortunately,

$$\begin{aligned} y_p'' - 2y_p' - 3y_p &= -4A\cos 2t + 4A\sin 2t - 3A\cos 2t \\ &= A\left(-7\cos 2t + 4\sin 2t\right) \end{aligned}$$

- We try again,

$$y_p(t) = A\cos 2t + B\sin 2t$$

where A and B are constants to be determined.

- Plugging this y_p, y_p' and y_p'' into the equation and simplify, we have

$$y_p'' - 2y_p' - 3y_p = (-7A - 4B)\cos 2t + (4A - 7B)\sin 2t$$

- Equating the two expressions of $y_p'' - 2y_p' - 3y_p$

$$(-7A - 4B)\cos 2t + (4A - 7B)\sin 2t = 65\cos 2t$$

- We obtain a linear system,

$$-7A - 4B = 65; \qquad 4A - 7B = 0$$

- solving which, we have

$$A = -7 \qquad \text{and} \qquad B = -4$$

- Thus we have the following particular solution,

$$y_p = -7\cos 2t - 4\sin 2t$$

Exercise

Solve the differential equation $\ddot{y} + \dot{y} - 2y = t^2$.

Summary

- Exponential

  $$f = Ke^{\alpha t} \implies y_p(t) = Ae^{\alpha t}$$

  where A is a constant to be determined.

- Sine and Cosine

  $$f = K_c \cos \omega t + K_s \sin \omega t \implies y_p(t) = A \cos \omega t + B \sin \omega t$$

  where A and B are constants to be determined.

- Polynomials

  $$f = P_n(t) \implies y_p(t) = A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0$$

  where A_k's are constants to be determined.

- If $f(t)$ is a sum of the functions above,

$$f(t) = f_1(t) + f_2(t)$$

then we can find the right y_p by using the principle of superposition,

$$ay''_{p_1} + by'_{p_1} + cy_{p_1} = f_1 \qquad \text{and} \qquad ay''_{p_2} + by'_{p_2} + cy_{p_2} = f_2$$

then $y_{p_1} + y_{p_2}$ is a solution of

$$ay'' + by' + cy = f_1 + f_2$$

- If $f(t)$ is a product of the functions above, in general

$$f(t) = P(t)e^{\alpha t}\cos\omega t + Q(x)e^{\alpha t}\sin\omega t$$

where P and Q are polynomials.

- In this case, everything need to be included, that is, a particular solution is

$$y_p = \left(A_0 t^K + A_1 t^{K-1} + \cdots + A_{k-1} x + A_k\right) e^{\alpha t} \cos \omega t$$

$$\left(B_0 t^K + B_1 t^{K-1} + \cdots + B_{k-1} t + B_k\right) e^{\alpha t} \sin \omega t$$

where the A_ks' and B_ks' are constants to be determined

- The integer K is the highest degree of the polynomials $P(t)$ or $Q(t)$

$$f(t) = P(t) e^{\alpha t} \cos \omega t + Q(x) e^{\alpha t} \sin \omega t$$

Exercise

Find a particular solution to

$$y'' - 2y' - 3y = 65t \cos 2t$$

- Consider

$$\ddot{y} - 2\dot{y} - 3y = 28e^{3t}$$

- Our first guess is going to be

$$y_p(t) = Ae^{3t}$$

- Plugging it into the left-hand side of the above differential equation:

$$\left(Ae^{3t}\right)'' - 2\left(Ae^{3t}\right)' - 3Ae^{3t} = 9Ae^{3t} - 6Ae^{3t} - 3Ae^{3t} = 0 \neq 28e^{3t}$$

- We see no value of A can make $y_p(t) = Ae^{3t}$ a particular solution.

Q: Why did it fail?

- The complementary solution is

$$y_c = c_1 e^{-t} + c_2 e^{3t}$$

- So Ae^{3t} gives you zero, we need another linearly independent solution.

- In general, whenever our first guess contains a term that is also a solution to the corresponding homogeneous differential equation, then we shall

multiply the first guess by t as the second guess

- If, however, the second guess also fails, for it also contains a term satisfying the corresponding homogeneous equation, then we shall

multiply the second guess by t again as the third guess

- The second guess is used only if the first fails, that is, the first has a term that satisfies the homogeneous equation. Likewise, if the second works, the third is not only unnecessary, it will not work.
- So for our example,

$$\ddot{y} - 2\dot{y} - 3y = 28e^{3t}$$

- We look at our second guess

$$y_p = Ate^{3t}$$

- Plugging into the equation, we will find that $A = 7$, hence $y_p(t) = 7te^{3t}$.

- The method of undetermined coefficients works only for restricted functions f, on the other hand, the method of variation of parameters works for every function $f(t)$ but it is usually more difficult to apply in practice.
- Suppose the complementary equation has the solution

$$y = c_1\phi_1 + c_2\phi_2$$

where c_1 and c_2 are two arbitrary constants.

- The basic idea is to look for a particular solution to equation (1) of the form

$$y_p = u_1(t)\phi_1(t) + u_2(t)\phi_2(t)$$

that is, we let the parameter ot be a function of t

$$y_p' = u_1'(t)\phi_1(t) + u_1(t)\phi_1'(t) + u_2'(t)\phi_2(t) + u_2(t)\phi_2'(t)$$

$$= (u_1'\phi_1 + u_2'\phi_2) + (u_1\phi_1' + u_2\phi_2')$$

- Since u_1 and u_2 are arbitrary functions, we can impose two conditions:
    1. One condition is that y_p is a solution of the differential equation.
    2. We can choose the other condition so as to simplify our calculations.
- If we choose

$$(u_1'\phi_1 + u_2'\phi_2) = 0 \qquad \text{to be condition 2.}$$

then

$$y_p'' = (u_1'\phi_1 + u_2'\phi_2)' + (u_1\phi_1' + u_2\phi_2')'$$

$$= 0 + u_1'\phi_1' + u_1\phi_1'' + u_2'\phi_2' + u_2\phi_2''$$

- Substituting into the nonhomogeneous equation $ay_p'' + by_p' + cy_p = f$,

$$a(u_1'\phi_1' + u_1\phi_1'' + u_2'\phi_2' + u_2\phi_2'') + b(u_1\phi_1' + u_2\phi_2') + c(u_1\phi_1 + u_2\phi_2) = f$$

$$u_1(a\phi_1'' + b\phi_1' + c\phi_1) + u_2(a\phi_2'' + b\phi_2' + c\phi_2) + a(u_1'\phi_1' + u_2'\phi_2') = f$$

- But ϕ_1 and ϕ_2 are solutions of of the complementary equation, so we have

$$a(u_1'\phi_1' + u_2'\phi_2') = f$$

- So u_1 and u_2 are to be chosen so that the following system is satisfied

$$\begin{aligned} u_1'\phi_1 + u_2'\phi_2 &= 0 \\ a\left(u_1'\phi_1' + u_2'\phi_2'\right) &= f \end{aligned}$$

- Once we have u_1' and u_2', the parameters are

$$u_1 = \int u'\,dt \qquad \text{and} \qquad u_2 = \int u_2'\,dt$$

Exercise

Use the method of variation of parameters to solve the equation

(a) $\ddot{y} - 4y = te^t + \cos 2t$

(b) $t^2\ddot{y} - 2y = 3t^{-1}$ for $t > 0$