
Question1 (2 points)

Find $\sup(\mathcal{A})$ and $\inf(\mathcal{A})$ where $\mathcal{A} = \left\{ \frac{n+2}{n} \mid n \in \mathbb{N} \right\}$.

Question2 (3 points)

Suppose $\mathcal{S}$ is a nonempty subset of $\mathbb{R}$. Determine whether the following statements are true. If not, briefly explain why it is false.

(a) (1 point) Every limit point of $\mathcal{S}$ is an interior point of $\mathcal{S}$.

(b) (1 point) Every neighbourhood of $x \in \mathbb{R}$ intersects $\mathcal{S}$ if and only if $x \in \mathcal{S}^{\complement}$

(c) (1 point) The limit points of $\mathcal{S}$ form a closed set.

Question3 (2 points)

(a) (1 point) Show that the square of an odd number is odd.

(b) (1 point) Prove that $a_n = (3-2n)(-3)^n$ is the explicit formula for

$$a_0 = 3, \quad a_1 = -3, \quad a_n = -6a_{n-1} - 9a_{n-2} \qquad \text{for} \quad n \geq 2,$$

Question4 (1 points)

Give an example of two divergent sequences $\{a_n\}$ and $\{b_n\}$ such that $\{a_n + b_n\}$ is convergent.

Question5 (1 points)

Consider the sequence $\{x_n\}$, where

$$x_1 = a, \quad x_2 = b, \quad x_n = \frac{x_{n-1} + x_{n-2}}{2} \qquad \text{for} \qquad n = 3, 4, \ldots$$

Determine whether $\{x_n\}$ is convergent. If so, find $\lim_{n\to\infty} x_n$. If not, prove why not.

Question6 (1 points)

If the sequence x_n is bounded and $\lim_{n\to\infty} y_n = 0$, show that $\lim_{n\to\infty} x_n y_n = 0$.

Question7 (1 points)

Determine whether $\{\cos(n\pi)\}$ is convergent. If so, find $\lim_{n\to\infty} \cos(n\pi)$. If not, prove why not.

Question8 (3 points)

Consider the sequence

$$\begin{aligned}
a_1 &= \sqrt{6} \\
a_2 &= \sqrt{6 + \sqrt{6}} \\
a_3 &= \sqrt{6 + \sqrt{6 + \sqrt{6}}} \\
a_4 &= \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6}}}} \\
&\vdots
\end{aligned}$$

(a) (1 point) Find a recursion formula for the sequence.

(b) (1 point) Show this sequence converges.

(c) (1 point) Find the limit of this sequence.

Question9 (1 points)
Suppose $\{a_n\}$ is monotonic and bounded. Use the definition of the supremum and the infimum to argue $\{a_n\}$ must be convergent by the definition of convergence.

Question10 (1 points)
In class, we have introduced the definitions of *interior point*, *limit point*, *boundary point* and *open set* under $\mathbb{R}$. These definitions can be easily extended to a higher dimension, so do many other mathematical concepts. Give your **OWN** reasonable definitions of *interior point*, *limit point* *boundary point* and *open set* for two-dimensional space.

Question11 (4 points)
In high school, if it is required to find the explicit formula for a sequence involves

$$a_{n+1} = ca_n + d, \qquad \text{where} \quad c \neq 0,$$

You would properly consider of the alternative sequence $\left\{a_n + \dfrac{d}{c-1}\right\}$, which is geometric, in order to solve the explicit formula for the original sequence. A similar idea can be used to find the explicit formula for a sequence with a second-order linear recurrence relation

$$a_{n+1} = pa_n + qa_{n-1} \qquad \text{where} \qquad p^2 + 4q \geq 0$$

We can try to find the explicit formula by considering the alternative sequence

$$\{a_{n+1} + ta_n\}$$

For simplicity, let us assume that

$$a_{n+1} + ta_n = s(a_n + ta_{n-1})$$

(a) (1 point) Assume that there are two distinct sets of real solutions for the unknowns s and t, denoted as (s_1, t_1) and (s_2, t_2). Express the explicit formula for a_n in terms of a_1, a_2, t_1, t_2, s_1 and s_2.

(b) (1 point) If the solutions of s and t are repeated, that is, $s_1 = s_2, t_1 = t_2$, what is the explicit formula for a_n in this case?

(c) (1 point) Fibonacci sequence is a world famous sequence. It is defined as:

$$F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-1} + F_{n-2}, \qquad \text{for} \quad n > 2$$

We can use the method that is described in the last two parts to solve the explicit formula for $\{F_n\}$. However, there exists a much easier method. For a sequence with the recurrence relation

$$a_1, \quad a_2, \qquad a_{n+1} = pa_n + qa_{n-1}, \qquad \text{where} \quad p^2 + 4q > 0$$

we can solve the equation $r^2 - pr - q = 0$ to obtain two distinct solutions $r_1 = s_1, r_2 = s_2$. Then the explicit formula for a_n can be expressed in the form of

$$a_n = k_1 s_1^n + k_2 s_2^n$$

Use this method to find out the explicit formula for the Fibonacci sequence.

(d) (1 point) Explain why the method used in part (c) is valid by using results in part (a).

Question12 (0 points)

A set of real number $\mathcal{S} \subset \mathbb{R}$ is disconnected if there are open sets $\mathcal{U}, \mathcal{V} \subset \mathbb{R}$ such that $\mathcal{U} \cap \mathcal{V}$ are empty, and $\mathcal{S} \cap \mathcal{U}$ and $\mathcal{S} \cap \mathcal{V}$ are nonempty and

$$\mathcal{S} = (\mathcal{S} \cap \mathcal{U}) \cup (\mathcal{S} \cap \mathcal{V})$$

A set is connected if it is not disconnected.

(a) (1 point (bonus)) Show the set $\{0, 1\}$ is disconnected.

(b) (1 point (bonus)) Let $\mathcal{S} \subset \mathbb{R}$, show $\mathcal{S}$ is connected if and only if it is an interval.

Question13 (0 points)

Let a be an odd natural number and let

$$b = \left\lfloor \frac{a^2}{2} \right\rfloor \qquad \text{and} \qquad c = \left\lceil \frac{a^2}{2} \right\rceil$$

be the integer floor and ceiling for $a^2/2$ respectively.

(a) (1 point (bonus)) Show that

$$a^2 + b^2 = c^2, \qquad \text{for all } a.$$

(b) (1 point (bonus)) Find

$$\lim_{a \to \infty} \frac{b}{c}$$