Probabilistic Methods in Engineering

Sample Exercises for the First Midterm

The following exercises have been compiled from past first midterm exams of Ve401. A first midterm will usually consist of 4-5 such exercises to be completed in 100 minutes. In the actual exam, necessary tables of values of distributions will be provided. You may use all tables in Appendix A of the textbook to solve the sample exercises.

Exercise 1. A company produces widgets in three factories, A, B and C. Factory A produces 20% of the widgets, factory B produces 45% of the widgets and factory C produces the remaining 35%. Of all widgets produced, 5% fail tolerance. Of those that fail tolerance, 25% were produced in factory A, 35% were produced in factory B and 40% were produced in factory C. In each factory, what percentage of the widgets produced fails tolerance?
(3 Marks)

Exercise 2. A coin is tossed 28800 times. Assuming that the coin is fair (i.e., heads and tails each have a 50% chance of coming up), use the normal approximation to the binomial distribution to give the probability that between 14380 and 14399 heads come up.
(4 Marks)

Exercise 3. Consider an experiment with three possible outcomes P, Q, R. The probability of outcome P is $p \in (0,1)$, that of Q is $q \in (0, 1-p)$ and that of R is $r = 1-p-q$. We perform $n \in \mathbb{N} \setminus \{0\}$ independent and identical experments of this type. Let X be the random variable denoting the number of outcomes P and Y the random variable denoting the number of outcomes Q.

i) What is the probability density function f_{XY} of the joint random variable (X,Y)? Explain how you obtain f_{XY} and verify that it is indeed a density function.

ii) From the definition of a marginal density function, calculate $f_X(x)$. Could you have obtained this result more easily?

iii) Are X and Y independent?

iv) Calculate $\mathrm{E}[X]$.

v) In quality control, a widget is either approved, rejected or marked for further testing. Of all widgets, 70% are accepted, 20% are rejected and 10% are marked for further testing. 10 widgets are tested and of those, 5 are approved. What is the probability that of the remaining 5, 3 are rejected?

(3+(2+1)+1+1+2 Marks)

Exercise 4. A fair coin is tossed repeatedly.

i) What is the probability that the 5^{th} head occurs on the 10^{th} toss?

ii) Suppose that we know that the 10^{th} head occurs on the 25^{th} toss. Find the probability density function of the toss number of the 5^{th} head.

(1+3+2 Marks)

Exercise 5. Suppose that a hotel has 200 equally-sized rooms. From experience, it is known that 10% of travellers who have reserved a room do not turn up. Use the normal approximation to the binomial distribution to answer the following questions.

i) Suppose that the hotel has received reservations for 215 rooms. What is the probability that there will enough rooms for all travelers who turn up?

ii) Suppose that the hotel has received reservations for 215 rooms. What is the probability that more than 190 rooms (but of course not more than 200) will be occupied?

iii) At what point should the hotel stop accepting reservations if management wants to be at least 99% certain to have enough rooms for all travelers who show up?

(2+2+2 Marks)

Exercise 6. The target value for the thickness of a machined cylinder is 8 cm. The upper specification limit is 8.2 cm and the lower specification limit is 7.9 cm. A machine produces cylinders that have a mean thickness of 8.1 cm and a standard deviation of 0.1 cm. Assume that the thickness of the cylinders produced by this machine follows a normal distribution.

i) What is the probability that the thickness of a randomly selected cylinder is within specification?

ii) What is the probability that the thickness of a randomly selected cylinder exceeds the target value?

iii) Find a bound $L > 0$ such that the thickness of 90% of all cylinders lies within $(8.1 \pm L)$ cm.

(2+2+2 Marks)

Exercise 7. Consider the continuous random variable X with density

$$f_X(x) = \frac{c}{e^{-x} + e^{x}} \qquad \text{for } x \in \mathbb{R}.$$

i) Determine the constant $c \in \mathbb{R}$.

ii) Find $P[X \leq 1]$.

iii) Find the density of the random variable X^2.

(1+1+2 Marks)

Exercise 8. The distribution function of the speed (modulus of the velocity) V of a gas molecule is described by the Maxwell-Boltzmann law

$$f_V(v) = \begin{cases} \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{m}{kT}\right)^{3/2} v^2 e^{-\frac{m}{kT}v^2/2} & v > 0 \\ 0 & v \leq 0 \end{cases}$$

where $m > 0$ is the mass of the molecule, $T > 0$ is its temperature and $k > 0$ is the Boltzmann constant.

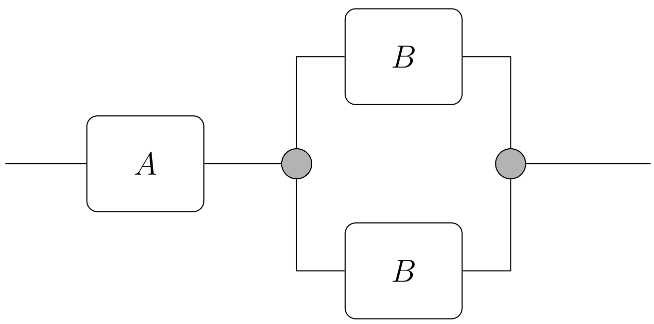
i) Find the mean and variance of V.

ii) Find the mean of the kinetic energy $E = mV^2/2$.

iii) Find the probability density f_E of E.

(3 + 2 + 2 Marks)

Exercise 9. Consider the following system of components:



The system will fail if either component A or both components marked B fail. The components A and B have failure densities

$$f_A(t) = \frac{1}{100}e^{-t/100}, \qquad f_B(t) = \frac{1}{50}e^{-t/50}, \qquad t \geq 0,$$

respectively.

i) Find the reliability functions $R_A(t)$ and $R_B(t)$ of component A and each component B.

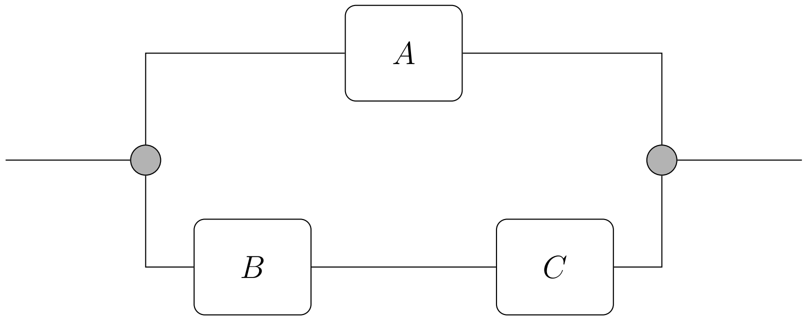
ii) Find the reliability function $R(t)$ of the system.

iii) What is the expected time of failure of each individual component?

iv) What is the expected time of failure of the system?

(2+2+2+2 Marks)

Exercise 10. Consider the following system of components:



The components A, B and C have failure densities

$$f_A(t) = \frac{t}{50}e^{-t^2/100}, \qquad f_B(t) = \frac{1}{40}e^{-t/40}, \qquad f_C(t) = \frac{1}{25}e^{-t/25}, \qquad t \geq 0,$$

respectively.

i) Find the reliability functions $R_A(t)$, $R_B(t)$ and $R_C(t)$.

ii) Find the reliability function $R(t)$ of the system.

iii) What is the expected time of failure of each component? Evaluate all integrals explicitly; there should be no gamma functions or other integrals in your answer.

iv) What is the expected time of failure of the system? Evaluate all integrals explicitly; there should be no gamma functions or other integrals in your answer.

(2+2+2+3 Marks)