Vv256 Lecture 2

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- Every first-order linear equation

$$\alpha(t)\dot{y} + \beta(t)y = \gamma(t)$$

can be written in the following standard form since $\alpha(t) \neq 0$,

$$\dot{y} + P(t)y = Q(t) \qquad \text{where} \qquad P(x) = \frac{\beta(t)}{\alpha(t)} \qquad \text{and} \qquad Q(x) = \frac{\gamma(t)}{\alpha(t)}$$

- For example,

$$(4+t^2)\dot{y} + 2ty = 4t \implies \dot{y} + \frac{2t}{4+t^2}y = \frac{4t}{4+t^2}$$

- Since it is a first-order equation, we could investigate it using a slope field.

$$\dot{y} = \Phi(t,y) = -\frac{2t}{4+t^2}y + \frac{4t}{4+t^2} = \frac{2t(2-y)}{4+t^2}$$

- For "simple" first-order differential equations, like this one,

$$(4+t^2)\dot{y}+2ty=4t$$

we can solve it by integrating both sides of the equation directly

$$\int \Big[(4+t^2)\dot{y}+2ty\Big]\,dt=\int 4t\,dt \qquad \text{the product/chain rule in reverse}$$

$$\int \frac{d}{dt}\Big[\,(4+t^2)\,y\Big]\,dt=2t^2+C$$

$$(4+t^2)y=2t^2+C \implies y=\frac{2t^2+C}{4+t^2}$$

- However, not every linear first-order equation can be solve in this way,

$$\alpha(t)\dot{y}+\beta(t)y=\gamma(t)$$

Q: When can we solve it in this way by directly integrating both sides?

Q: How can we solve the following equation?

$$\underbrace{(4+t^2)e^t}_{\alpha}\dot{y} + \underbrace{2te^t}_{\beta}y = 4te^t$$

- Note that β is NOT the derivative of α, however, multiplying both sides by

$$\frac{1}{e^t}$$

we obtain the previous equation without changing the underlying solutions.

$$(4+t^2)\dot{y} + 2ty = 4t$$

Q: What does the above example suggest we shall do for a general equation of

$$\alpha(t)\dot{y} + \beta(t)y = \gamma(t)$$

- Suppose there exist a function μ of t such that

$$\mu\beta = \frac{d}{dt}\Big(\mu\alpha\Big)$$

where α and β are functions of t in front of $\dot{y}$ and y, then

$$\alpha\dot{y} + \beta y = \gamma$$

$$\implies \mu\alpha\dot{y} + \mu\beta y = \mu\gamma$$

$$\Big(\mu\alpha\Big)\dot{y} + y\frac{d}{dt}\Big(\mu\alpha\Big) = \mu\gamma \qquad \text{the product/chain rule in reverse}$$

$$\frac{d}{dt}\Big[(\mu\alpha)y\Big] = \mu\gamma \implies y = \frac{\displaystyle\int \mu\gamma\, dt}{\mu\alpha}$$

Exercise

Solve the following initial value problem

$$ty' + 2y = 4t^2, \qquad y(1) = 2$$

- The function μ is known as the integrating factor, we will show it always exists and has the form given in the next theorem for any linear first-order eq.

Theorem

For a linear first-order equation

$$\alpha(t)\dot{y} + \beta(t)y = \gamma(t)$$

the integrating factor for the equation is

$$\mu = \frac{A}{\alpha} \exp\left(\int \frac{\beta}{\alpha}\, dt\right), \qquad \text{where } A \text{ is an arbitrary constant.}$$

and the general solution of the equation, which gives all possible solutions, is

$$y = \frac{\displaystyle\int \mu\gamma\, dt}{\mu\alpha}$$

Q: Why it is not a surprise to see the constant A?

Exercise

Find the solution of the initial-value problem,

$$\cos(t)\dot{y} + \sin(t)y = 2\cos^3(t)\sin(t) - 1, \qquad y\left(\frac{\pi}{4}\right) = 3\sqrt{2} \quad \text{for} \quad 0 \le t < \frac{\pi}{2}$$

- To prove the last theorem, we need to find μ for arbitrary α and β such that

$$(\mu\beta) = \frac{d}{dt}\Big(\mu\alpha\Big) = \dot{\mu}\alpha + \dot{\alpha}\mu$$

- Therefore the integrating factor is a solution to the differential equation

$$\alpha\dot{\mu} = (\beta - \dot{\alpha})\,\mu$$

- This equation is a homogeneous linear first-order equation of μ.

$$\alpha\dot{\mu} - (\beta - \dot{\alpha})\,\mu = 0$$

which is always separable and can be solved using the next theorem.

Definition

A first-order differential equation is called separable if it can be written in the form

$$\dot{y} = GF$$

where G is only a function of y and F is only a function of t.

- Note a separable equation is linear if and only if

$$G(y) = Cy \qquad \text{or} \qquad G(y) = C$$

- For any α and β, the function $\mu = 0$ is always a solution to the equation

$$\alpha\dot{\mu} = (\beta - \dot{\alpha})\,\mu$$

- But we are interested in non-trivial solutions, that is, not identically zero.

Theorem

If $G(y)$ and $F(x)$ are continuous, then a separable equation has the solution,

$$\int \frac{1}{G}\,dy = \int F\,dt$$

Proof

Given a separable equation $\dot{y} = GF$, we rearrange to obtain

$$\frac{1}{G(y)} \frac{dy}{dt} = F(t)$$

Write $\dfrac{1}{G(y)}$ and $F(t)$ as the derivative of their antiderivative using FTC,

$$\frac{d}{dy} \left(\int_{y_0}^{y} \frac{1}{G(\eta)} \, d\eta \right) \frac{dy}{dt} = \frac{d}{dt} \left(\int_{t_0}^{t} F(\tau) \, d\tau \right)$$

Use the chain rule in reverse for the left-hand side

$$\frac{d}{dt} \left(\int_{y_0}^{y} \frac{1}{G(\eta)} \, d\eta \right) = \frac{d}{dt} \left(\int_{t_0}^{t} F(\tau) \, d\tau \right)$$

Two functions have the same derivative must only differ by an additive constant

$$\int_{y_0}^{y} \frac{1}{G(\eta)} \, d\eta = \int_{t_0}^{t} F(\tau) \, d\tau + C \iff \int \frac{1}{G} \, dy = \int F \, dt$$

- Let us see how this theorem leads to the formula for the integrating factor

$$\begin{aligned} \alpha\dot{\mu} = (\beta - \dot{\alpha})\,\mu \implies \int \frac{1}{\mu}\,d\mu &= \int \frac{\beta - \dot{\alpha}}{\alpha}\,dt \\ \implies \quad \ln|\mu| &= \int \frac{\beta}{\alpha}\,dt - \ln|\alpha| \end{aligned}$$

- Exponentiating both sides of the last equation

$$\begin{aligned} |\mu| = \frac{1}{|\alpha|}\exp\left(\int \frac{\beta}{\alpha}\,dt\right) &= \frac{1}{|\alpha|}\exp\left(\int \frac{\beta}{\alpha}\,dt + A_1\right) \\ &= \frac{A_2}{|\alpha|}\exp\left(\int \frac{\beta}{\alpha}\,dt\right) \\ \implies \mu &= \frac{A}{\alpha}\exp\left(\int \frac{\beta}{\alpha}\,dt\right) \end{aligned}$$

- In the standard form a linear first-order equation has the general solution of

$$y = \frac{1}{\mu}\left(\int \mu Q\,dt + C\right) \qquad \text{where} \quad \mu = A\exp\left(\int P\,dt\right)$$

Exercise

(a) Solve the following differential equation

$$\frac{dy}{dt} = -2ty$$

(b) Solve the following initial value problem

$$(1 - y^2)\dot{y} = t^2, \qquad y(1) = 3$$

(c) Find all solutions of the following differential equation.

$$\dot{y} = y(1 - y)$$

(d) Suppose an object of mass m is falling from rest near sea level. Assume the air resistance is proportional to the velocity of the object, and the drag coefficient is K. Derive a model for the motion of the object using Newton's second law, then solve it to find the velocity function.

- The equation governs the motion of a mass m falling near sea level is

$$m\dot{v} = mg - Kv$$

where g is the gravitational constant and K is the drag coefficient.

- It is a special/simple equation since the derivative in which can be expressed without any explicit reference to time

$$\dot{v} = \Phi(t, v) = \frac{mg - Kv}{m}$$

Definition

An autonomous first-order equation is any equation of the form

$$\dot{y} = G(y)$$

- Since there is no time t, all autonomous equations are separable.

$$\int \frac{1}{G(y)}\, dy = t + C$$

- For our autonomous equation

$$\dot{v} = \frac{mg - Kv}{m}$$

there is one solution that has particular physical importance.

Definition

An equilibrium/steady solution is any constant function

$$y(t) = C$$

that is a solution to the differential equation.

Q: For the above equation of a falling mass, what does the equilibrium solution

$$v = \frac{mg}{K}$$

represent?

- Notice the derivative of a constant function is always zero,

$$y(t) = C \implies \dot{y} = 0$$

thus we find equilibrium solutions by solving for y in the equation

$$\dot{y} = \Phi(t, y) = 0$$

- So the equilibrium/steady velocity is

$$0 = \frac{mg - Kv}{m} \implies v = \frac{mg}{K}$$

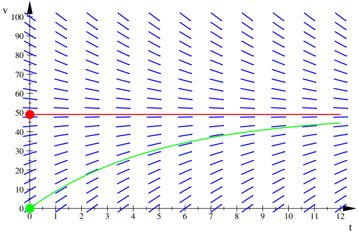
- For a particular mass, e.g., an object of $m = 10$kg and $K = 2$kg/s , we have

$$\dot{v} = 9.8 - \frac{v}{5}$$

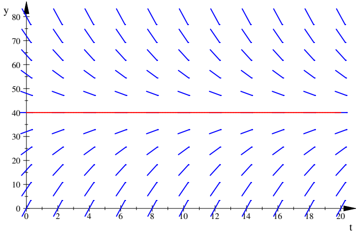
the slope filed can be used to see the effect of having different initial velocity and the role that equilibrium solution plays.

- Solutions with different initial velocity converge to the equilibrium velocity.

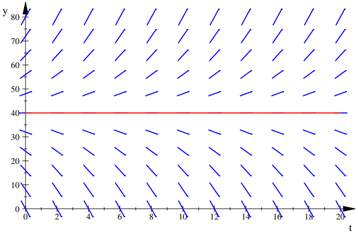
(a) Equilibrium Velocity



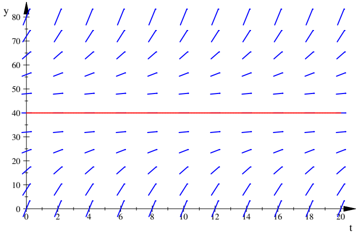
(b) Stable Equilibrium



(c) Unstable Equilibrium



(d) Semi-stable Equilibrium



- We classify equilibrium solutions according to their behaviour as $t \to \infty$.

Definition

The equilibrium solution $y(t) = c$ is

stable if all solutions with initial conditions y_0 near $y = c$

approach c as $t \to \infty$.

unstable if all solutions with initial conditions y_0 near $y = c$

diverge away from c as $t \to \infty$.

semi-stable if initial conditions y_0 on one side of c lead to solutions $y(t)$ that approach c as $t \to \infty$, while on the other side of c diverge away from c.

Exercise

Find and classify the equilibrium solutions of

$$\dot{y} = (1 - y)(3 - y)$$