Vv256 Lecture 12

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Definition

A point x_0 is said to be an ordinary point of the differential equation

$$y'' + P(x)y' + Q(x)y = 0$$

if $P(x)$ and $Q(x)$ are analytic at x_0.

A point that is not an ordinary point is known as a singular point of the equation.

- Certain types of singular points are not too badly behaved to be studied.
- Consider the following equation, and note $x = 0$ is a singular point of it

$$2xy' - y = 0 \implies y' - \frac{1}{2x}y = 0$$

- So the theorem L11P11 will not guarantee the power series being a solution

$$y = \sum_{n=0}^{\infty} c_n x^n$$

- However, we could try to substitute the power series and see what happens

$$y = \sum_{n=0}^{\infty} c_n x^n, \qquad \text{and} \qquad y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$$

- $2xy' - y = 2x \left(\sum_{n=1}^{\infty} c_n n x^{n-1} \right) - \left(\sum_{n=0}^{\infty} c_n x^n \right)$

$$= -c_0 + \sum_{n=1}^{\infty} (2n-1)\, c_n\, x^n$$

- The identity property implies $c_0 = 0$, and

$$(2n-1)\, c_n = 0, \qquad \text{for } n = 1, 2, 3, \ldots$$

which is only possible if

$$c_n = 0, \qquad \text{for all } n.$$

- So it only leads to the trivial solution $y = 0$, which is not very useful!

- If we step back, and inspect the form of this differential equation

$$2xy' - y = 0$$

- It is a linear equation, so we could use an integrating factor, which shows

$$y = x^r, \qquad \text{is a solution for some real number } r.$$

- Alternatively, we could determine r by substituting it into the equation

$$2xy' - y = 2x\left(rx^{r-1}\right) - x^r = (2r-1)x^r = 0 \implies r = \frac{1}{2}$$

- Thus the general solution is

$$y = Cx^{1/2}, \qquad \text{for } x > 0$$

- Note when $C \neq 0$, the solution is not differentiable at $x = 0$, thus it is not analytic at $x = 0$, that is, the solution does not have a power series centred at $x = 0$ converge to it.

- Of course, not every problem with a singular point has a solution of the form

$$y = x^r$$

but some will have solutions that are related to it, namely,

$$y = x^r f(x)$$

where $f(x)$ is an analytic function.

- To investigate those cases, a singular point x_0 is further classified as either

regular or irregular.

Definition

A singular point x_0 is said to be a regular singular point of the equation

$$y'' + P(x)y' + Q(x)y = 0$$

if $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2 Q(x)$ are both analytic at x_0.

A singular point that is not regular is said to be an irregular singular point.

Exercise

Identify all singular points and classify them as regular or irregular.

$$(x^2-4)^2y''+3(x-2)y'+5y=0$$

Solution

- It is clear that $x=2$ and $x=-2$ are singular points of the equation.
- For $x=2$,

$$p(x)=(x-2)\quad P(x)=(x-2)\quad \frac{3}{(x-2)(x+2)^2}=\frac{3}{(x+2)^2}$$

$$q(x)=\ (x-2)^2Q(x)=(x-2)^2\frac{5}{(x-2)^2(x+2)^2}=\frac{5}{(x+2)^2}$$

- Both p and q are analytic at $x=2$, so $x=2$ is a regular singular point
- However, it can be found that $x=-2$ is an irregular singular point since

$$p(x)=(x+2)P(x)=\frac{3}{(x-2)(x+2)}$$ is not analytic at $x=-2$.

Theorem

If $x = x_0$ is a regular singular point of the differential equation,

$$y'' + P(x)y' + Q(x)y = 0$$

then there exist at least one solution of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}, \quad c_0 \neq 0$$

where the number r is a real number.

- Note it gives no assurance of the same sort we have from theorem L11P11
- The series solution here is known as the Frobenius-type solution

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

and the method of finding it is known as the method of Frobenius.

Exercise

Find the general solution of

$$3xy'' + y' - y = 0$$

Solution

- Since $x = 0$ is a regular singular point, according to the last theorem,

$$\phi = x^{r} \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r} \qquad \text{is a solution.}$$

- So the derivatives are given by

$$\phi' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} \quad \text{and} \quad \phi'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

- Substitute ϕ, ϕ' and ϕ'' into the equation and try to determine c_ns.

Solution

- We have

$$\begin{aligned}
&3x\phi'' + \phi' - \phi \\
&= 3x\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} \\
&= \sum_{n=0}^{\infty}(n+r)(3n+3r-2)c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} \\
&= r(3r-2)c_0 x^{r-1} + \sum_{n=0}^{\infty}\Big((n+r+1)\,(3n+3r+1)\,c_{n+1} - c_n\Big)x^{n+r} = 0
\end{aligned}$$

- Again the last equality leads to the following recurrence relation

$$(n+r+1)\,(3n+3r+1)\,c_{n+1} - c_n = 0 \qquad \text{for } n = 0, 1, 2, \ldots$$

and more importantly,

$$r(3r-2)c_0 = 0$$

Solution

- Notice the following conclusions

$$r(3r-2)c_0 = 0 \implies \begin{array}{l} c_0 = 0 \implies c_n = 0 \quad \text{for all } n. \\ c_0 \neq 0 \implies r(3r-2) = 0 \end{array}$$

The last equation of r is called the indicial equation.

- This particular indicial equation has roots,

$$r = 0 \qquad \text{and} \qquad r = 2/3$$

- For each value of r, we have a recurrence relation for $n = 0, 1, 2, \ldots$

$$r_1 = 0$$

$$\begin{aligned} c_{n+1} &= \frac{c_n}{(n+r+1)(3n+3r+1)} \\ &= \frac{c_n}{(n+1)(3n+1)} \end{aligned}$$

$$r_2 = 2/3$$

$$\begin{aligned} c_{n+1} &= \frac{c_n}{(n+r+1)(3n+3r+1)} \\ &= \frac{c_n}{(n+1)(3n+5)} \end{aligned}$$

Solution

$$
\begin{aligned}
r_1 &= 0 \\
c_{n+1} &= \frac{c_n}{(n+1)(3n+1)} \\
c_1 &= \frac{c_0}{1 \cdot 1} \\
c_2 &= \frac{c_1}{2 \cdot 4} = \frac{c_0}{2 \cdot 4} \\
c_3 &= \frac{c_2}{3 \cdot 7} = \frac{c_0}{3! \cdot 4 \cdot 7} \\
c_4 &= \frac{c_3}{4 \cdot 10} = \frac{c_0}{4! \cdot 4 \cdot 7 \cdot 10} \\
&\vdots \\
c_n &= \frac{c_0}{n! \cdot 4 \cdot 7 \cdots (3n-2)}
\end{aligned}
$$

$$
\begin{aligned}
r_2 &= 2/3 \\
c_{n+1} &= \frac{c_n}{(n+1)(3n+5)} \\
c_1 &= \frac{c_0}{5 \cdot 1} \\
c_2 &= \frac{c_1}{8 \cdot 2} = \frac{c_0}{2 \cdot 5 \cdot 8} \\
c_3 &= \frac{c_2}{11 \cdot 3} = \frac{c_0}{3! \cdot 5 \cdot 8 \cdot 11} \\
c_4 &= \frac{c_3}{14 \cdot 4} = \frac{c_0}{4! \cdot 5 \cdot 8 \cdot 11 \cdot 14} \\
&\vdots \\
c_n &= \frac{c_0}{n! \cdot 5 \cdot 8 \cdot 11 \cdots (3n+2)}
\end{aligned}
$$

Solution

- Hence putting r and the corresponding set of c_n into the solution, we have

$$\phi_1(x) = c_0 x^0 \left(1 + \sum_{n=1}^{\infty} \frac{1}{n! \cdot 4 \cdot 7 \cdots (3n-2)} x^n\right)$$
$$\phi_2(x) = c_0 x^{2/3} \left(1 + \sum_{n=1}^{\infty} \frac{1}{n! \cdot 5 \cdot 8 \cdot 11 \cdots (3n+2)} x^n\right)$$

- By the ratio test, it can be demonstrated that both series converge for

$$x \in (-\infty, \infty)$$

- Also, it should be apparent from the form of the these solutions that neither series is a constant multiple of the other, so they are linearly independent.
- So the following

$$y = C_1 \phi_1(x) + C_2 \phi_2(x)$$

is the general solution for any interval that does not contain the origin.

- It seems that the method of Frobenius involves the following steps:

1. If $x = 0$ is a regular singular point, plug Frobenius series into the equation

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

2. Collecting in terms of x^n and write everything as a single series.
3. Setting the first coefficient in the series to 0 to obtain the indicial equation

$$(r - r_1)(r - r_2) = 0$$

4. Solve the equation and use the roots to find two recurrence relations.

$$r_1 \neq r_2 \implies \text{two recurrence relations.}$$

5. Use the recurrence relations to solve for all coefficients in terms of c_0, and so

$$\phi_1(x) = x^{r_1} \sum_{n=0}^{\infty} c_n x^n \qquad \text{and} \qquad \phi_2(x) = x^{r_2} \sum_{n=0}^{\infty} c_n^* x^n$$

Summary Based on Roots of Indicial Equation

1. Two distinct real roots r_1 and r_2 such that

   $r_1 - r_2$ is NOT an integer, then two linearly independent solutions

   $$\phi_1(x) = x^{r_2} \sum_{n=0}^{\infty} a_n x^n \qquad \text{and} \qquad \phi_2(x) = x^{r_2} \sum_{n=0}^{\infty} b_n x^n$$

2. A repeated root, only 1 linearly independent Frobenius solution can be found
3. A complex root r, all the c_n are complex and linearly independent solutions

   $$\phi_1(x) = \mathrm{Re}\left(x^r \sum_{n=0}^{\infty} c_n x^n\right) \qquad \text{and} \qquad \phi_2(x) = \mathrm{Im}\left(x^r \sum_{n=0}^{\infty} c_n x^n\right)$$

4. Two distinct roots such that $r_1 - r_2$ is an integer, then

   only one linearly independent Frobenius solution may be found using

   $$r_1 \quad \text{where } \mathrm{Re}(r_1) > \mathrm{Re}(r_2)$$

Exercise

Find the series solutions to $x(x-1)y'' + 3xy' + y = 0$ about the point $x = 0$.

Solution

- It is clear that $x = 0$ is a regular singular point of the differential equation.
- Thus we shall consider Frobenius series instead of power series

$$\phi(x) = \sum_{n=0}^{\infty} c_n x^{n+r} \implies \phi'(x) = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1},$$

$$\implies \phi''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

- The last theorem guarantees a solution if we solve for c_n in the following

$$x(x-1)\phi'' + 3x\phi' + \phi = 0$$

Solution

- Collecting in terms of x^{n+r},

$$\overbrace{\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r}}^{x^2\phi''} - \overbrace{\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1}}^{x\phi''} + 3\underbrace{\sum_{n=0}^{\infty}(n+r)c_n x^{n+r}}_{x\phi'} + \underbrace{\sum_{n=0}^{\infty}c_n x^{n+r}}_{\phi} = 0$$

$$r(1-r)c_0 x^{r-1} + \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} - \sum_{n=1}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(3n+3r+1)c_n x^{n+r} = 0$$

$$r(1-r)c_0 x^{r-1} + \sum_{n=0}^{\infty}\left[\big((n+r)(n+r+2)+1\big)c_n - (n+r+1)(n+r)c_{n+1}\right]x^{n+r} = 0$$

Solution

- The indicial equation is

$$r(1-r)=0 \implies r_2=0 \quad \text{and} \quad r_1=1$$

- According to Frobenius, when the indicial equation has two distinct roots, but differ by an integer, then there may be only one solution in the form

$$\phi = x^r \sum_{n=0}^{\infty} c_n x^n.$$

and this solution can be found using the larger root

$$r=1$$

- When $r=1$, the recurrence relation simplifies to the following for $n \in \mathbb{N}_0$,

$$\big((n+r)(n+r+2)+1\big)c_n - (n+r+1)(n+r)c_{n+1} = 0$$
$$\big((n+1)(n+3)+1\big)c_n - (n+2)(n+1)c_{n+1} = 0$$

Solution

- Simplifying the recurrence relation, we have

$$c_{n+1} = \frac{n+2}{n+1} c_n \qquad \text{for} \quad n \in \mathbb{N}_0$$

- Notice that

$$\begin{aligned}
c_{0+1} &= \frac{0+2}{0+1} c_0 \implies c_1 = 2c_0 \\
c_{1+1} &= \frac{1+2}{1+1} c_1 \implies c_2 = \frac{3}{2} c_1 = 3c_0 \\
&\vdots \\
c_n = \frac{n-1+2}{n-1+1} \cdot \frac{n-2+2}{n-2+1} \cdots \frac{0+2}{0+1} c_0 &\implies c_n = (n+1) c_0
\end{aligned}$$

- So the solution is $\phi_1 = x \sum_{n=0}^{\infty} (n+1) x^n = x(1 + 2x + 3x^2 + \cdots)$

Solution

- If we attempt using $r = 0$, the smaller root, the recurrence relation is

$$\Big((n+0)(n+0+2)+1\Big)c_n - (n+0+1)(n+0)c_{n+1} = 0$$
$$\implies c_{n+1} = \frac{n+1}{n} c_n$$

- So unless c_0 is zero c_1 is undefined, which means all other c_n are undefined.
- However, if $c_0 = 0$, all we got is the trivial solution. So the method fails.
- To find the second linearly independent solution, we could theoretically use

1. Abel's theorem: we know one solution to the homogeneous linear equation,

$$\phi_2(x) = \phi_1(x) \int \frac{W(x)}{\phi_1^2(x)}\, dx$$

where $W(x) = A \exp\left(-\int P(x)\, dx\right)$.

- However, it is easier to use the following formulas to find the second solution.

The form of the Second Linearly Independent Solution

2. A repeated roots, the second linearly independent solution is of the form

$$\phi_2 = \phi_1 \ln x + x^{r_1} \sum_{n=0}^{\infty} c_n^* x^n,$$

   where $r_1 = r_2$ are the roots of the indicial equation.

4. Two distinct roots such that $r_1 - r_2 > 0$ is an integer, then

$$\phi_2 = C\phi_1 \ln x + x^{r_2} \sum_{n=0}^{\infty} c_n^* x^n$$

   where r_1 and r_2 are the roots of the indicial equation.

- Constants C and c_n^* can be found by substituting ϕ_2 into the equation

Solution

- To find a second linearly independent solution near $x = 0$ for the equation

$$x(x-1)y'' + 3xy' + y = 0 \quad \text{in addition to} \quad \phi_1 = x\sum_{n=0}^{\infty}(n+1)x^n$$

- We substitute $\phi_2 = C\phi_1 \ln x + x^{r_2}\sum_{n=0}^{\infty} c_n^* x^n$ into the equation.
- The roots of the corresponding indicial equation is $r_1 = 1$ and $r_2 = 0$. So

$$\phi_2 = C\phi_1 \ln x + \sum_{n=0}^{\infty} c_n^* x^n \implies \phi_2' = C\left(\phi_1' \ln x + \frac{1}{x}\phi_1\right) + \sum_{n=0}^{\infty} n c_n^* x^{n-1}$$

- The second derivative is

$$\phi_2'' = C\left(\phi_1'' \ln x + \frac{1}{x}\phi_1' - \frac{1}{x^2}\phi_1 + \frac{1}{x}\phi_1'\right) + \sum_{n=0}^{\infty} n(n-1)c_n^* x^{n-2}$$

$$= C\left(\phi_1'' \ln x + \frac{2}{x}\phi_1' - \frac{1}{x^2}\phi_1\right) + \sum_{n=0}^{\infty} n(n-1)c_n^* x^{n-2}$$

Solution

- If we substitute ϕ_2, ϕ_2' and ϕ_2'' into the original equation, and simplify

$$Cx(x-1)\left(\phi_1''\ln x+\frac{2}{x}\phi_1'-\frac{1}{x^2}\phi_1\right)+x(x-1)\sum_{n=0}^{\infty}n(n-1)c_n^*x^{n-2}$$

$$+C3x\left(\phi_1'\ln x+\frac{1}{x}\phi_1\right)+3x\sum_{n=0}^{\infty}nc_n^*x^{n-1}+C\phi_1\ln x+\sum_{n=0}^{\infty}c_n^*x^n$$

$$=C\Big(x(x-1)\phi_1''+3x\phi_1'+\phi_1\Big)\ln x+C\left(x(x-1)\left(\frac{2}{x}\phi_1'-\frac{1}{x^2}\phi_1\right)+3\phi_1\right)$$

$$+\sum_{n=0}^{\infty}\left(n(n-1)c_n^*\left(x^n-x^{n-1}\right)+3nc_n^*x^n+c_n^*x^n\right)$$

$$=C\left(x(x-1)\left(\frac{2}{x}\sum_{n=0}^{\infty}(n+1)^2x^n-\frac{1}{x^2}\sum_{n=0}^{\infty}(n+1)x^{n+1}\right)+3\sum_{n=0}^{\infty}(n+1)x^{n+1}\right)$$

$$+\sum_{n=0}^{\infty}(n+1)^2c_n^*x^n-\sum_{n=0}^{\infty}(n+1)nc_{n+1}^*x^n$$

Solution

- Simplify further, with lots of algebra, we have

$$
\begin{aligned}
2C\sum_{n=0}^{\infty}(n+1)^2x^{n+1} - C\sum_{n=0}^{\infty}(n+1)x^{n+1} - 2C\sum_{n=0}^{\infty}(n+1)^2x^n + C\sum_{n=0}^{\infty}(n+1)x^n \\
+ 3C\sum_{n=0}^{\infty}(n+1)x^{n+1} + \sum_{n=0}^{\infty}(n+1)^2c_n^*x^n - \sum_{n=0}^{\infty}(n+1)nc_{n+1}^*x^n \\
= c_0^* - C + \sum_{n=1}^{\infty}(n+1)\left(C - c_n^* - nc_n^* + nc_{n+1}^*\right)x^n
\end{aligned}
$$

- Apply the identity property to

$$
c_0^* - C + \sum_{n=0}^{\infty}(n+2)\left(C - (n+2)c_{n+1}^* + (n+1)c_{n+2}^*\right)x^{n+1} = 0
$$

$$
\implies C = c_0^*, \quad \text{and} \quad c_{n+2}^* = \frac{(n+2)c_{n+1}^* - c_0^*}{n+1} \quad \text{for all } n \in \mathbb{N}_0.
$$

Solution

- Using the recurrence relation,

$$c^*_{n+2} = \frac{(n+2)c^*_{n+1} - c^*_0}{n+1}$$

by inspection and induction, we have

$$\begin{aligned}
c^*_{0+2} &= \frac{(0+2)c^*_{0+1} - c^*_0}{0+1} = 2c^*_1 - c^*_0 \\
c^*_{1+2} &= \frac{(1+2)c^*_{1+1} - c^*_0}{1+1} = \frac{3}{2}c^*_2 - \frac{1}{2}c^*_0 = 3c^*_1 - 2c^*_0 \\
&\vdots \\
\Longrightarrow \ c^*_n &= nc^*_1 - (n-1)c^*_0
\end{aligned}$$

- Thus the second linearly independent solution is $\phi_2 = c^*_0\phi_1 \ln x + \sum_{n=0}^{\infty} c^*_n x^n$

Solution

- For any arbitrary c_0^* and c_1^*, the following is a solution

$$\phi_2 = c_0^* \phi_1 \ln x + \sum_{n=0}^{\infty} \left[n c_1^* - (n-1) c_0^* \right] x^n$$

- In this case, if we choose $c_0^* = 0$, and set c_1^* to be the arbitrary constant.

$$\phi_2 = c_0^* \phi_1 \ln x + \sum_{n=0}^{\infty} \left[n c_1^* - (n-1) c_0^* \right] x^n = \sum_{n=0}^{\infty} n c_1^* x^n = c_1^* \phi_1$$

- So the easiest way to have two linearly independent solutions is to set $c_1^* = 0$

$$\begin{aligned} y &= c_0 \phi_1 + c_0^* \phi_2 \\ &= c_0 \sum_{n=0}^{\infty} (n+1) x^{n+1} + c_0^* \left(\ln x \sum_{n=0}^{\infty} (n+1) x^{n+1} - \sum_{n=0}^{\infty} (n-1) x^n \right) \end{aligned}$$

- Let us now derive the formula for the second linearly independent solution for

$$r = r_1 = r_2$$

near the regular singular point $x = 0$ for the following equation,

$$x^2 y'' + xp(x)y' + q(x)y = 0$$

- Note it is the only type of equations that has zero as a regular singular point.
- In order to have $x = 0$ as a regular singular point, the explicit form must be

$$x^2 y'' + xp(x)y' + q(x)y = 0 \implies y'' + \underbrace{\frac{p(x)}{x}}_{P} y' + \underbrace{\frac{q(x)}{x^2}}_{Q} y = 0$$

where P and Q are not analytic while p and q are analytic at $x = 0$,

- So both p and q must have power series at $x = 0$

$$p(x) = xP(x) = \sum_{n=0}^{\infty} p_n x^n \qquad \text{and} \qquad q(x) = x^2 Q(x) = \sum_{n=0}^{\infty} q_n x^n$$

- If we substitute

$$y = \phi_1 = \sum_{n=0}^{\infty} c_n x^{n+r} \qquad p(x) = \sum_{n=0}^{\infty} p_n x^n \qquad q(x) = \sum_{n=0}^{\infty} q_n x^n$$

into the original equation

$$x^2 y'' + xp(x)y' + q(x) = 0 \iff \mathcal{L}\left[y\right] = 0$$

where $\mathcal{L}$ is the corresponding differential operator for the equation.

$$\mathcal{L}\left[\phi\right] = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} + \left(\sum_{n=0}^{\infty} p_n x^n\right)\left(\sum_{n=0}^{\infty}(n+r)c_n x^{n+r}\right)$$

$$+ \left(\sum_{n=0}^{\infty} q_n x^n\right)\left(\sum_{n=0}^{\infty} c_n x^{n+r}\right) = 0$$

- Multiplying the power series together,

$$\left(\sum_{n=0}^{\infty} p_n x^n\right)\left(\sum_{n=0}^{\infty}(n+r)c_n x^{n+r}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} p_{n-k}x^{n-k}(k+r)c_k x^{k+r}$$

$$\left(\sum_{n=0}^{\infty} q_n x^n\right)\left(\sum_{n=0}^{\infty} c_n x^{n+r}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} q_{n-k}x^{n-k}c_k x^{k+r}$$

- Thus the equation becomes the following

$$\mathcal{L}[\phi] = \sum_{n=0}^{\infty}\left\{(n+r)(n+r-1)c_n + \sum_{k=0}^{n}\Big[p_{n-k}(k+r) + q_{n-k}\Big]c_k\right\}x^{n+r} = 0$$

- Equating the coefficients, we have the recurrence relation for $n \geq 1$.

$$(n+r)(n+r-1)c_n + \sum_{k=0}^{n}\Big[(k+r)p_{n-k} + q_{n-k}\Big]c_k = 0$$

- If $n=0$, we have a simple equation of c_0 instead of a relation between c_n

$$\Big[r(r-1) + p_0 r + q_0\Big]c_0 = 0 \implies r(r-1) + p_0 r + q_0 = 0 \quad \text{since } c_0 \neq 0.$$

- Notice $\mathcal{L}[\phi]$ consists two parts,

$$\mathcal{L}[\phi] = \sum_{n=0}^{\infty} \left\{ (n+r)(n+r-1)c_n + \sum_{k=0}^{n} \Big[p_{n-k}(k+r) + q_{n-k} \Big] c_k \right\} x^{n+r}$$

$$= \overbrace{\Big[r(r-1) + p_0 r + q_0 \Big] c_0 x^r}^{F}$$

$$+ \sum_{n=1}^{\infty} \underbrace{\left\{ (n+r)(n+r-1)c_n + \sum_{k=0}^{n} \Big[(k+r)p_{n-k} + q_{n-k} \Big] c_k \right\} x^{n+r}}_{G_n}$$

- So far, we have always solved the indicial equation given by

$$F(r) = \Big[r(r-1) + p_0 r + q_0 \Big] c_0 x^r = 0 \iff (r-r_1)(r-r_2) = 0$$

to obtain the roots before solving the recurrence relation given by

$$G_n(r; c_0, c_1, \cdots, c_n) = 0$$

to obtain the "correct" c_n, for $n \in \mathbb{N}_1$.

- We could find the "correct" c_n first by solving the following in terms of r

$$G_n(r; c_0, c_1, \cdots, c_n) = 0 \qquad \text{for all } n \geq 1.$$

- Let us denote the function $\phi(x)$ with the "correct" c_n for all $n \geq 1$ to be

$$\varphi(x, r) = x^r \sum_{n=0}^{\infty} c_n x^n$$

which is not a solution to the differential equation

$$\mathcal{L}\left[y\right] = x^2 y'' + xp(x)y' + q(x)y = 0$$

unless r satisfies the indicial equation, that is, $r = r_1$ or $r = r_2$,

$$\begin{aligned} F(r) = \Big[r(r-1) + p_0 r + q_0\Big] c_0 x^r &= 0 \\ \iff (r - r_1)(r - r_2) &= 0 \end{aligned}$$

where r_1 and r_2 are the solutions to the indicial equation.

- Notice we are treating the function

$$\varphi(x,r) = x^{r}\sum_{n=0}^{\infty} c_n x^n$$

as a function of both x and r.

- Since φ has the "correct" c_n, that is, $G_n(r; c_0, c_1, \cdots, c_n) = 0$ for $n \in \mathbb{N}_1$,

$$\mathcal{L}\left[\varphi\right] = F(r) + \sum_{n=1}^{\infty} G_n(r; c_0, c_1, \cdots, c_n) = c_0(r-r_1)(r-r_2)x^r$$

- Recall we are considering the case with repeated solutions roots, that is,

$$r_1 = r_2$$

so the resulting function after applying the operator $\mathcal{L}$ is identically zero

$$f(x,r) = \mathcal{L}\left[\varphi\right] = c_0(r-r_1)^2 x^r = 0 \qquad \text{if and only if} \qquad r = r_1$$

- If we consider the partial derivative of $f(x,r)$ with respect to r

$$f_r(x,r) = \frac{\partial}{\partial r}\Big(c_0(r-r_1)^2 x^r\Big) = 2c_0(r-r_1)x^r + c_0(r-r_1)^2 x^r \ln x$$
$$= c_0(r-r_1)x^r\Big(2 + (r-r_1)\ln x\Big)$$

- However, in general, we have

$$\begin{aligned} f_r(x,r) &= \frac{\partial}{\partial r}\Big(\mathcal{L}\left[\varphi\right]\Big) \\ &= \frac{\partial}{\partial r}\left(x^2\frac{\partial^2\varphi}{\partial x^2}\right) + \frac{\partial}{\partial r}\left(xp(x)\frac{\partial\varphi}{\partial x}\right) + \frac{\partial}{\partial r}\Big(q(x)\varphi(x,r)\Big) \\ &= x^2\frac{\partial^2}{\partial x^2}\left(\frac{\partial\varphi}{\partial r}\right) + xp(x)\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial r}\right) + q(x)\left(\frac{\partial\varphi}{\partial r}\right) = \mathcal{L}\left[\frac{\partial}{\partial r}\varphi(x,r)\right] \end{aligned}$$

- Therefore

$$\mathcal{L}\left[\frac{\partial}{\partial r}\varphi(x,r)\right] = c_0(r-r_1)x^r\Big(2 + (r-r_1)\ln x\Big)$$

Q: What does this equation imply?

- Thus if we differentiate

$$\varphi(x,r) = x^r \sum_{n=0}^{\infty} c_n(r) x^n$$

with respect to r and evaluate it at the repeated root

$$r = r_1 = r_2$$

we will have the second linearly independent solution

$$\begin{aligned}
\phi_2 = \left.\frac{\partial \varphi(x,r)}{\partial r}\right|_{r=r_1} &= \frac{\partial}{\partial r}\left(x^r \sum_{n=0}^{\infty} c_n(r) x^n\right)\Bigg|_{r=r_1} \\
&= x^{r_1} \ln x \sum_{n=0}^{\infty} c_n(r_1) x^n + x^{r_1} \sum_{n=0}^{\infty} c_n'(r_1) x^n \\
&= \phi_1 \ln x + x^{r_1} \sum_{n=0}^{\infty} c_n^* x^n
\end{aligned}$$

- This completes the derivation for the formula for having a repeated root.

- In the light of the above derivative, it gives a better way to find the second linearly independent, for example, consider the following equation again

$$x(x-1)y'' + 3xy' + y = 0$$

- Recall we have the following indicial equation

$$r(1-r) = 0 \implies r_2 = 0 \quad \text{and} \quad r_1 = 1$$

- According to the above derivation, we have

$$\begin{aligned} \mathcal{L}\Big[\varphi(x,r)\Big] &= F(r) = c_0(r-r_1)(r-r_2)x^r \\ \implies \frac{\partial}{\partial r}\Big(\mathcal{L}\Big[\varphi\Big]\Big) &= \mathcal{L}\left[\frac{\partial \varphi}{\partial r}\right] \\ &= c_0\left[(2r-r_1-r_2)x^r + r(r-r_1)(r-r_2)x^{r-1}\right] \end{aligned}$$

Q: How can we avoid solving coefficients algebraically?

- Recall we have the following recurrence relation for $n \in \mathbb{N}_0$,

$$0 = \big((n+r)(n+r+2)+1\big)c_n - (n+r+1)(n+r)c_{n+1}$$

$$\begin{aligned}
\Longrightarrow \quad c_{n+1} &= \frac{\big((n+r)(n+r+2)+1\big)}{(n+r+1)(n+r)}c_n \Longrightarrow c_n = \frac{n+r}{r}c_0 \\
\Longrightarrow \varphi(x,r) &= c_0 x^r \sum_{n=0}^{\infty} \frac{n+r}{r} x^n \\
\Longrightarrow \quad \phi_1(x) &= x \sum_{n=0}^{\infty} (n+1)x^n \\
\Longrightarrow \quad \phi_2(x) &= \frac{1}{c_0}\frac{\partial}{\partial r}\Big((r-r_2)\varphi(x,r)\Big)\Bigg|_{r=r_2} \\
&= \frac{\partial}{\partial r}\left(x^r \sum_{n=0}^{\infty}(n+r)x^n\right)\Bigg|_{r=0} = \ln x \sum_{n=0}^{\infty} n x^n + \sum_{n=0}^{\infty} x^n
\end{aligned}$$

which is essentially the same as the solution we found by substitution.