Question1 (?? points)

Given that the column matrix $\mathbf{X} = (x_1, x_2, ..., x_n)^T$ satisfies $\mathbf{X^T X} = 1$, and give that

$$\mathbf{H} = \mathbf{I} - 2\mathbf{XX^T}$$

show that $\mathbf{H}$ is symmetric and $\mathbf{HH^T} = \mathbf{I}$

Solution:

$\mathbf{H^T} = (\mathbf{I} - \mathbf{2XX^T})^\mathbf{T} = \mathbf{I^T} - \mathbf{2}(\mathbf{XX^T})^\mathbf{T} = \mathbf{I} - \mathbf{2XX^T} = \mathbf{H}$, so $\mathbf{H}$ is symmetric.

And consequently

$$\begin{aligned}
\mathbf{HH^T} &= \mathbf{H}^2 \\
&= (\mathbf{I} - 2\mathbf{XX^T})^2 \\
&= \mathbf{I} - 4\mathbf{XX^T} + 4(\mathbf{XX^T})(\mathbf{XX^T}) \\
&= \mathbf{I} - 4\mathbf{XX^T} + 4\mathbf{X}(\mathbf{X^T X})\mathbf{X^T} \\
&= \mathbf{I} - 4\mathbf{XX^T} + 4\mathbf{XX^T} = \mathbf{I}
\end{aligned}$$

Question2 (?? points)

(a) (1 point) Given the permutation 32514, find the number of pairs out of natural order in it.

Solution:

There are 5 pairs.

(b) (1 point) Show that by flipping two adjacent number in a permutation, the Levi-Civita symbol will change its sign, i.e.

$$\varepsilon_{x_1 x_2 ... x_n a b y_1 y_2 ... y_m} = -\varepsilon_{x_1 x_2 ... x_n b a y_1 y_2 ... y_m}$$

Solution:

Suppose $t_i = |\{x \in \mathbb{N}^+ | x < i \wedge \ k_x > k_i\}|$ for the permutation $k_1 k_2 ... k_n$. So it can be seen that the number of pairs out of natural order is the sum of $t_i s$

It is also obvious that by interchanging the two adjacent numbers a and b, the t_i for the position of xx and ys will not change

So it will only influence that of a and b

If $a < b$, then the number for b is not changed but that of a will be added by 1

If $a > b$, then the number of a will not change but that of b will be decreased by 1

In conclusion, flipping will change the total number of out-of-natural-number pairs by 1, which will definitely change the odd-even property, thus the Levi-Civita will be negated.

Question3 (?? points)

Prove that for two 2×2 matrices $\mathbf{A}, \mathbf{B}$, $det(\mathbf{AB}) = det(\mathbf{A})det(\mathbf{B})$

Solution:

$$\begin{aligned}
det(\mathbf{A})det(\mathbf{B}) &= det(\begin{bmatrix} \mathbf{B} & -\mathbf{I} \\ \mathbf{0} & \mathbf{A} \end{bmatrix}) \\
&= det(\begin{bmatrix} b_{11} & b_{12} & -1 & 0 \\ b_{21} & b_{22} & 0 & -1 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & a_{21} & a_{22} \end{bmatrix}) \\
&= det(\begin{bmatrix} b_{11} & b_{12} & -1 & 0 \\ b_{21} & b_{22} & 0 & -1 \\ b_{11}a_{11} + b_{21}a_{12} & b_{12}a_{11} + b_{22}a_{12} & 0 & 0 \\ b_{11}a_{21} + b_{21}a_{22} & b_{12}a_{21} + b_{22}a_{22} & 0 & 0 \end{bmatrix}) \\
&= det(\begin{bmatrix} \mathbf{B} & -\mathbf{I} \\ \mathbf{AB} & \mathbf{0} \end{bmatrix}) = det(\mathbf{AB})
\end{aligned}$$

Question4 (?? points)

Suppose $\mathbf{A_n}$ is a $n \times n$ matrix which has the following property:

$$\forall i, j \in \{1, 2, ..n\}, a_{ij} = cos(i + n(j-1))$$

For example,

$$\mathbf{A_2} = \begin{bmatrix} cos1 & cos3 \\ cos2 & cos4 \end{bmatrix}$$

Now calculate $\lim_{n \to \infty} det(\mathbf{A_n})$

Hint: Perform elementary row operation $E_{(\alpha)i,j}$ will not change the determinant. All cosine functions are in radian.

Solution:

The limit will be 0. Furthermore, $det(\mathbf{A_n}) = 0$ if $n \geq 3$.

Perform the elementary row operation $E_{(1)3,1}$ to the matrix. Then consider the element on the first row. It can be seen that

$$a_{1j}^{'} = a_{1j} + a_{3j} = cos(1 + n(j-1)) + cos(3 + n(j-1)) = (2cos1)cos(2 + n(j-1))$$

Which means that $r_1^{'} = (2cos1)r_2$, then we have

$$det(\mathbf{A_n}) = det(\mathbf{A_n^{'}}) = det(\begin{bmatrix} r_1^{'} \\ r_2 \\ \vdots \end{bmatrix}) = det(\begin{bmatrix} (2cos2)r_2 \\ r_2 \\ \vdots \end{bmatrix}) = (2cos1)det(\begin{bmatrix} r_2 \\ r_2 \\ \vdots \end{bmatrix}) = 0$$