# 1 Question 1

Find the line integral

$$\int_C y\mathbf{e}_x + x\mathbf{e}_y d\mathbf{r}$$

where $C$ is the quarter-circle centered at the origin starting at $(0, 2)$ and proceeding clockwise to $(2, 0)$

## 1.1 solution

For this kind of the curve, we need to use the polar parameterization of the curve.

$$\mathbf{r}(t) = 2\cos t\mathbf{e}_x + 2\sin t\mathbf{e}_y$$

The derivative is

$$\mathbf{r}(t) = -2\sin t\mathbf{e}_x + 2\cos t\mathbf{e}_y$$

$$\int_C y\mathbf{e}_x + x\mathbf{e}_y d\mathbf{r} = \int_0^{\pi/2} 4\cos^2 t - 4\sin^2 t dt = 4\int_0^{\pi/2} \cos 2t dt = 0$$

# 2 Question 2

Find

$$\int_L \frac{(x-y)dx + (x+y)dy}{x^2+y^2}$$

where $L$ is

$$\begin{cases} x = t - \sin t - \pi \\ y = 1 - \cos t \end{cases}$$

from $t = 0$ to $t = 2\pi$

## 2.1 solution

First, we need to verify whether the vector field is conservative to not. In the open simply connected region $D$ defined by $\{(x, y)|y > 0\}$, the partial derivative of component function is continuous and equal.

$$\frac{\partial Q}{\partial y} = \frac{\partial P}{\partial x}$$

Thus, the vector field is conservative in this region.

The consequence of this is that the line integral is independent of the path and we need to change the parameter so that we can calculate it easier. One way is to use polar

$$\mathbf{r} = \pi\cos\theta\mathbf{e}_x + \pi\sin\theta\mathbf{e}_y$$

Then the integral becomes

$$\int_{\pi}^{0} \frac{(\pi\cos\theta - \pi\sin\theta)(-\pi\sin\theta) + (\pi\cos\theta + \pi\sin\theta)(\pi\cos\theta)}{\pi^2} d\theta = -\pi$$

Notice here, we can only use the curve above the x-axis to satisfy the condition of the conservative field test.

# 3 Question 3

Find the integral

$$\int_L (e^x \sin y - my)dx + (e^x \cos y - m)dy$$

where $L$ is the upper half of the curve $x^2 + y^2 = ax$ from $A(a, 0)$ to $O(0, 0)$.

## 3.1 solution

The vector field is not conservative, but we can treat the part of $F$ as a conservative field. Let $f$ to be

$$e^x \sin y - my$$

Then the integral becomes

$$(e^x \sin y - my)\Big|_{(a,0)}^{(0,0)} - m\int_a^0 ydx = \frac{m\pi}{8}a^2$$

# 4 Question 4

Find the area of the region enclosed by the parameterized curve

$$\mathbf{r}(t) = (t - t^2)\mathbf{e}_x + (t - t^3)\mathbf{e}_y \quad 0 \le t \le 1$$

## 4.1 solution

To calculate the area, we have three common vector field $\mathbf{F}$ which is $\mathbf{F} = -y\mathbf{e}_x$, $\mathbf{F} = x\mathbf{e}_y$ and $\mathbf{F} = -\frac{1}{2}y\mathbf{e}_x + \frac{1}{2}x\mathbf{e}_y$

For this question, we can choose either vector field that satisfies the condition

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

I choose $\mathbf{F} = x\mathbf{e}_y$ and the line integral becomes

$$A = \int_0^1 (t - t^2)(1 - 3t^2)dt = \frac{1}{60}$$

# 5 Question 5

Find the curve that maximize the line integral

$$\oint_C \left(x + \frac{y^3}{3}\right) dx + \left(y + x - \frac{2}{3}x^3\right) dy$$

## 5.1 solution

Use the green theorem to convert the line integral to the double integral, we can get

$$\oint_C \left(x + \frac{y^3}{3}\right) dx + \left(y + x - \frac{2}{3}x^3\right) dy = \iint_D (1 - 2x^2 - y^2) dxdy$$

To maximize the line integral means we also need to maximize the double integral. Thus, we need

$$1 - 2x^2 - y^2 \geq 0$$

The area is enclosed by the curve

$$1 - 2x^2 - y^2 = 0$$

and this is curve we need

# 6 Question 6

Use Green's Theorem to prove the change of variables formula for a double integral for the case where $f(x, y) = 1$.

$$\iint_R dxdy = \iint_S \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv$$

Here $R$ is the region in the xy-plane that corresponds to the region $S$ in the uv-plane under the transformation given by $x = g(u, v)$ and $y = h(u, v)$.

## 6.1 solution

Convert the double integral to the line integral.

$$dy = \frac{\partial h}{\partial u} du + \frac{\partial h}{\partial v} dv$$

$$
\begin{aligned}
\int_{\partial R} x dy &= \int_{\partial S} g(u,v)\frac{\partial h}{\partial u}du + h(u,v)\frac{\partial h}{\partial v}dv \\
&= \iint_S \Big| \frac{\partial}{\partial u}\left(h(u,v)\frac{\partial h}{\partial v}\right) - \frac{\partial}{\partial v}\left(g(u,v)\frac{\partial h}{\partial u}\right)\Big| dudv \\
&= \iint_S \Big| \frac{\partial g}{\partial u}\frac{\partial h}{\partial v} + g(u,v)\frac{\partial^2 h}{\partial u \partial v} - \frac{\partial g}{\partial v}\frac{\partial h}{\partial u} - g(u,v)\frac{\partial^2 h}{\partial u \partial v}\Big| dudv \\
&= \iint_S \Big| \frac{\partial g}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial g}{\partial v}\frac{\partial h}{\partial u}\Big| dudv \\
&= \iint_S \Big| \frac{\partial(x,y)}{\partial(u,v)}\Big| dudv
\end{aligned}
$$