# Vv417 Lecture 3

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## Definition

If $\mathbf{A}$ is a square matrix, and if a matrix $\mathbf{B}$ of the same size can be found such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I},$$

then $\mathbf{A}$ is said to be invertible (or nonsingular) and $\mathbf{B}$ is called an inverse of $\mathbf{A}$.

- If no such matrix $\mathbf{B}$ can be found for $\mathbf{A}$, then $\mathbf{A}$ is said to be singular.
- Note the relationship between $\mathbf{A}$ and $\mathbf{B}$ is mutual

$$\mathbf{BA} = \mathbf{AB} = \mathbf{I}$$

## Theorem

If $\mathbf{A}$ is invertible and $\mathbf{B}$ is an inverse of $\mathbf{A}$, then $\mathbf{B}$ is invertible and $\mathbf{A}$ is its inverse

Q: Is there a square matrix with a row or a column of zeros that is invertible?

**Theorem**

Every matrix with a row or a columns of zeros is singular.

Q: Suppose $\mathbf{B}$ is an inverse of $\mathbf{A}$. Is $\mathbf{B}$ unique for $\mathbf{A}$?

**Theorem**

If $\mathbf{B}$ and $\mathbf{C}$ are both inverses of the matrix $\mathbf{A}$, then $\mathbf{B} = \mathbf{C}$.

- Therefore we can now speak of "the" inverse of an invertible matrix.
- If $\mathbf{A}$ is invertible, then its inverse will be denoted by

$$\mathbf{A}^{-1}$$

Q: Suppose $\mathbf{A}$ and $\mathbf{B}$ are invertible. Can we say anything regarding the product

$$\mathbf{AB}$$

## Theorem

If $\mathbf{A}$ and $\mathbf{B}$ are invertible matrices with the same size, then $\mathbf{AB}$ is invertible and

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

## Proof

- We can establish the formula by showing

$$\mathbf{AB}\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right) = \left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right)\mathbf{AB} = \mathbf{I}$$

- We do it by starting from the left

$$\mathbf{AB}\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right) = \mathbf{A}\left(\mathbf{BB}^{-1}\right)\mathbf{A}^{-1} = \mathbf{I}$$

- Similarly, it is clear

$$\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right)\mathbf{AB} = \mathbf{B}^{-1}\left(\mathbf{A}^{-1}\mathbf{A}\right)\mathbf{B} = \mathbf{I}$$

□

- By induction, the last result can be extended to three or more matrices:

### Theorem

A product of any number of invertible matrices is invertible, and the inverse of the product is the product of the inverses in the reverse order.

### Definition

If $\mathbf{A}$ is a square matrix, then we define the nonnegative integer powers of $\mathbf{A}$ to be

$$\mathbf{A}^0 = \mathbf{I} \quad \text{and} \quad \mathbf{A}^n = \mathbf{A}\mathbf{A}\cdots\mathbf{A} \qquad [n \text{ factors}]$$

and if $\mathbf{A}$ is invertible, then we define the negative integer powers of $\mathbf{A}$ to be

$$\mathbf{A}^{-n} = \left(\mathbf{A}^{-1}\right)^n = \mathbf{A}^{-1}\mathbf{A}^{-1}\cdots\mathbf{A}^{-1} \qquad [n \text{ factors}]$$

- The usual laws of exponents hold since the definitions parallel those for $\mathbb{R}$.

$$\mathbf{A}^r\mathbf{A}^s = \mathbf{A}^{r+s} \quad \text{and} \quad \left(\mathbf{A}^r\right)^s = \mathbf{A}^{rs}$$

- In addition, we have the following properties of negative exponents.

### Theorem

If $\mathbf{A}$ is invertible and $n$ is a nonnegative integer, then

1. $\mathbf{A}^{-1}$ is invertible and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.
2. $\mathbf{A}^n$ is invertible and $(\mathbf{A}^n)^{-1} = \mathbf{A}^{-n} = (\mathbf{A}^{-1})^n$.
3. $\alpha\mathbf{A}$ is invertible for nonzero scalar $\alpha$, and $(\alpha\mathbf{A})^{-1} = \alpha^{-1}\mathbf{A}^{-1}$.

### Proof

- 1. and 2. are trivial given the definition of inverse and integer powers of $\mathbf{A}$.
- Statement 3. is true if we can show

$$(\alpha\mathbf{A})(\alpha^{-1}\mathbf{A}^{-1}) = (\alpha^{-1}\mathbf{A}^{-1})(\alpha\mathbf{A}) = \mathbf{I}$$

- $\begin{matrix} (\alpha\mathbf{A})(\alpha^{-1}\mathbf{A}^{-1}) = \alpha\alpha^{-1}\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \\ (\alpha^{-1}\mathbf{A}^{-1})(\alpha\mathbf{A}) = \alpha\alpha^{-1}\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \end{matrix} \implies (\alpha\mathbf{A})^{-1} = \alpha^{-1}\mathbf{A}^{-1}$

### Definition

Matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be row equivalent if either $\mathbf{A}$ or $\mathbf{B}$ can be obtained from the other by a sequence of elementary row operations, which is denoted by

$$\mathbf{A} \sim \mathbf{B}$$

Q: Are elementary matrices row equivalent to each other?

Q: Is every elementary matrix invertible?

### Equivalence Theorem

If $\mathbf{A}$ is an $n \times n$ matrix, then the following statements are equivalent.

(1.) $\mathbf{A}$ is invertible.

(2.) $\mathbf{A}\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(3.) The reduced echelon form of $\mathbf{A}$ is $\mathbf{I}_n$.

(4.) $\mathbf{A}$ is expressible as a product of elementary matrices.

## Proof

- We will prove the equivalence by establishing the chain of implications
- (1.) $\implies$ (2.): Let $\mathbf{x}_0$ be any solution of

$$\mathbf{A}\mathbf{x}_0 = \mathbf{0} \implies \mathbf{A}^{-1}\mathbf{A}\mathbf{x}_0 = \mathbf{A}^{-1}\mathbf{0} \implies \mathbf{x}_0 = \mathbf{0}$$

- Any solution of it must be trivial, so the trivial solution is the only solution.
- (2.) $\implies$ (3.): Having only the trivial solution, we must have the following

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{array}\right]$$

- The reduced echelon form of $\mathbf{A}$ is the left part of the above matrix

$$\operatorname{rref}(\mathbf{A}) = \mathbf{I}_n$$

## Proof

- (3.) $\implies$ (4.): $\mathrm{rref}(\mathbf{A}) = \mathbf{I}_n$ implies there is a sequence of row operations

$$\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{I}$$

- Since elementary matrices are invertible,

$$\mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\cdots\mathbf{E}_{k-1}^{-1}\mathbf{E}_k^{-1}\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\cdots\mathbf{E}_{k-1}^{-1}\mathbf{E}_k^{-1}\mathbf{I}$$

$$\mathbf{A} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\cdots\mathbf{E}_{k-1}^{-1}\mathbf{E}_k^{-1}$$

- (4.) $\implies$ (1.): If $\mathbf{A}$ is a product of elementary matrices,

$$\mathbf{A} = \mathbf{E}_1^*\mathbf{E}_2^*\cdots\mathbf{E}_{k-1}^*\mathbf{E}_k^*$$

then $\mathbf{A}$ must be invertible for it is a product of invertible matrices. $\square$

Q: What do the equivalence theorem and its proof give us?

1. The first application of the last theorem is that it gives us a way to determine whether a given matrix is invertible.

Check if the $\mathrm{rref}(\mathbf{A})$ is $\mathbf{I}$.

2. Secondly, the theorem gives a way to find the inverse of an invertible matrix.

$$\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{I}$$

$$\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}\mathbf{A}^{-1}$$

$$\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{I} = \mathbf{A}^{-1}$$

### Inversion Algorithm

To find the inverse of an invertible matrix $\mathbf{A}$, find a sequence of elementary row operations that reduces $\mathbf{A}$ to the identity and then perform that same sequence of operations on $\mathbf{I}$ to obtain $\mathbf{A}^{-1}$.

## Exercise

Find the inverse of the following matrix if it is invertible $\mathbf{A} = \begin{bmatrix} 0 & 4 & 1 \\ 3 & 8 & 1 \\ 1 & 2 & 1 \end{bmatrix}$.

## Solution

1. $\mathbf{E}_{1,3}$ $\left[\begin{array}{cccccc} 0 & 4 & 1 & 1 & 0 & 0 \\ 3 & 8 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$

2. $\mathbf{E}_{(-3)1,2}$ $\left[\begin{array}{cccccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 3 & 8 & 1 & 0 & 1 & 0 \\ 0 & 4 & 1 & 1 & 0 & 0 \end{array}\right]$

3. $\mathbf{E}_{(\frac{1}{2})2}$ $\left[\begin{array}{cccccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 2 & -2 & 0 & 1 & -3 \\ 0 & 4 & 1 & 1 & 0 & 0 \end{array}\right]$

4. $\mathbf{E}_{(-4)2,3}$ $\left[\begin{array}{cccccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 4 & 1 & 1 & 0 & 0 \end{array}\right]$

5. $\mathbf{E}_{(\frac{1}{5})3}$ $\left[\begin{array}{cccccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 5 & 1 & -2 & 6 \end{array}\right]$

6. $\mathbf{E}_{(1)3,2}$ $\left[\begin{array}{cccccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{2}{5} & \frac{6}{5} \end{array}\right]$

7. $\mathbf{E}_{(-1)3,1}$ $\left[\begin{array}{cccccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{5} & \frac{1}{10} & -\frac{3}{10} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{2}{5} & \frac{6}{5} \end{array}\right]$

8. $\mathbf{E}_{(-2)2,1}$ $\left[\begin{array}{cccccc} 1 & 2 & 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & \frac{1}{5} & \frac{1}{10} & -\frac{3}{10} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{2}{5} & \frac{6}{5} \end{array}\right]$

$$\mathbf{A}^{-1} = \begin{bmatrix} -3/5 & 1/5 & 2/5 \\ 1/5 & 1/10 & -3/10 \\ 1/5 & -2/5 & 6/5 \end{bmatrix}$$

Q: Notice the above algorithm of computing $\mathbf{A}^{-1}$ is essentially Gauss-Jordan elimination. Can we use gaussian elimination with back substitution?

## Matlab

Command Window

```
>>
>> format rational
>> A = [ 0 4 1; 3 8 1; 1 2 1]

A =

       0              4              1
       3              8              1
       1              2              1

>> inv(A)

ans =

      -3/5            1/5            2/5
       1/5            1/10          -3/10
       1/5           -2/5            6/5

>> B = eye(3)

B =

       1              0              0
       0              1              0
       0              0              1

>>
```

Command Window

```
>>
>> AB = GaussianElimination(A,B)

AB =

       3              8              1              0              1              0
       0              4              1              1              0              0
       0              0              5/6            1/6           -1/3            1

>> Ainv = BackSubstitution(AB)

Ainv =

      -3/5            1/5            2/5
       1/5            1/10          -3/10
       1/5           -2/5            6/5

>>
```

- We made the following assertion in our very first lecture.

**Theorem**

A system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.

- We are now in a position to prove this fundamental result.

**Proof**

- There are many examples where we have no or a unique solution, thus we only need to show systems that have more than one solution actually have infinitely many solutions.
- Consider an arbitrary system
$$\mathbf{Ax} = \mathbf{b}$$
- Assume that $\mathbf{Ax} = \mathbf{b}$ has more than one solution, say,
$$\mathbf{x}_1 \qquad \text{and} \qquad \mathbf{x}_2$$

## Proof

- Let $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2$, as $\mathbf{x}_1$ and $\mathbf{x}_2$ are distinct, we can conclude $\mathbf{x}_0$ is non-zero

$$\mathbf{A}\mathbf{x}_0 = \mathbf{A}\left(\mathbf{x}_1 - \mathbf{x}_2\right) = \mathbf{A}\mathbf{x}_1 - \mathbf{A}\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

  thus $\mathbf{x}_0$ is a solution to the corresponding homogeneous system.

- Now consider $\mathbf{x}_1 + \alpha\mathbf{x}_0$ where $\alpha \in \mathbb{R}$, and see whether it is a solution,

$$\mathbf{A}\left(\mathbf{x}_1 + \alpha\mathbf{x}_0\right) = \mathbf{A}\mathbf{x}_1 + \alpha\mathbf{A}\mathbf{x}_0 = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

  thus the vector $\mathbf{x}_1 + \alpha\mathbf{x}_0$ is also a solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$ for all $\alpha \in \mathbb{R}$.

- Because $\mathbf{x}_0$ is non-zero and $\alpha$ is any scalar, we can conclude $\mathbf{A}\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

## Theorem

Let $\mathbf{A}$ be an $n \times n$ matrix, then either $\mathrm{rref}\left(\mathbf{A}\right)$ has a row of zeros or $\mathrm{rref}\left(\mathbf{A}\right) = \mathbf{I}$.

- Two procedures for solving linear systems

  Gauss–Jordan elimination and Gaussian elimination.

- The following theorem provides an formula for the solution of a linear system of $n$ equations in $n$ unknowns when the coefficient matrix is invertible.

### Theorem

If $\mathbf{A}$ is an invertible square matrix of size $n$, then for each $\mathbf{b} \in \mathbb{R}^n$, the system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$ has exactly one solution, namely,

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

### Proof

- It is clear that $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ is a solution, $\mathbf{A}\left(\mathbf{A}^{-1}\mathbf{b}\right) = \mathbf{b}$.
- To show it is the only solution, consider any solution $\mathbf{x}_0$,

$$\mathbf{A}\mathbf{x}_0 = \mathbf{b} \implies \mathbf{x}_0 = \mathbf{A}^{-1}\mathbf{b} \quad \square$$

## Exercise

Under what conditions would the following system be consistent?

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & 2x_3 & = & b_1 \\ x_1 & + & & + & x_3 & = & b_2 \\ 2x_1 & + & x_2 & + & 3x_3 & = & b_3 \end{array}$$

## Solution

- Form the augmented matrix and apply Gauss Elimination

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}$$

- Thus the coefficient matrix is not invertible.
- It has a solution if and only if

$$b_3 - b_2 - b_1 = 0$$

## Exercise

Under what conditions would the following system be consistent?

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & = & b_1 \\ 2x_1 & + & 5x_2 & + & 3x_3 & = & b_2 \\ x_1 & + & & + & 8x_3 & = & b_3 \end{array}$$

## Solution

- Form the augmented matrix and apply Gauss Elimination

$$\begin{bmatrix} 1 & 2 & 3 & b_1 \\ 2 & 5 & 3 & b_2 \\ 1 & 0 & 8 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -40b_1 + 16b_2 + 9b_3 \\ 0 & 1 & 0 & 13b_1 - 5b_2 - 3b_3 \\ 0 & 0 & 1 & 5b_1 - 2b_2 - b_3 \end{bmatrix}$$

- Invertible coefficient matrix, so it is consistent for any set of $b_1$, $b_2$ and $b_3$.
- And the solution is unique.

- Up to now, to show that an $n \times n$ matrix $\mathbf{A}$ is invertible, it has been necessary to find an $\mathbf{B}$ such that

$$\mathbf{AB} = \mathbf{I} \qquad \text{and} \qquad \mathbf{BA} = \mathbf{I}$$

- The next theorem shows that if we produce an $n \times n$ matrix $\mathbf{B}$ satisfying either condition, then the other condition will hold automatically.

**Theorem**

Let $\mathbf{A}$ be a square matrix.

1. If $\mathbf{B}$ is a square matrix satisfying $\mathbf{BA} = \mathbf{I}$, then $\mathbf{B} = \mathbf{A}^{-1}$.
2. If $\mathbf{B}$ is a square matrix satisfying $\mathbf{AB} = \mathbf{I}$, then $\mathbf{B} = \mathbf{A}^{-1}$.

**Proof**

- If we can show that $\mathbf{A}$ is invertible, then we can multiply $\mathbf{BA} = \mathbf{I}$ on both sides by $\mathbf{A}^{-1}$, and obtain what we need $\mathbf{BAA}^{-1} = \mathbf{IA}^{-1} \implies \mathbf{B} = \mathbf{A}^{-1}$.
- To show $\mathbf{A}$ is invertible, we only need to show $\mathbf{Ax} = \mathbf{0}$ has only trivial sol.
- Let $\mathbf{x}_0$ be any solution of $\mathbf{Ax} = \mathbf{0}$, then $\mathbf{BAx}_0 = \mathbf{B0} \implies \mathbf{x}_0 = \mathbf{0}$.

## Equivalence Theorem

If $\mathbf{A}$ is an $n \times n$ matrix, then the following statements are equivalent,

(1.) $\mathbf{A}$ is invertible.

(2.) $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.

(3.) The reduced echelon form of $\mathbf{A}$ is $\mathbf{I}_n$.

(4.) $\mathbf{A}$ is expressible as a product of elementary matrices.

(5.) $\mathbf{Ax} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^n$.

(6.) $\mathbf{Ax} = \mathbf{b}$ has exactly one solution for every $\mathbf{b} \in \mathbb{R}^n$.

## Proof

- We have proved (1.), (2.), (3.), (4.) are equivalent, it is sufficient to show

$$(1.) \implies (6.) \implies (5.) \implies (1.)$$

- (1.) $\implies$ (6.) This is essentially identical to the theorem on page $\boxed{15}$.

## Proof

- (6.) $\implies$ (5.) This is almost self-evident, for if $\mathbf{Ax} = \mathbf{b}$ has exactly one solution for every $\mathbf{b} \in \mathbb{R}^n$, then $\mathbf{Ax} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^n$.
- (5.) $\implies$ (1.) It the system $\mathbf{Ax} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^n$, then

$$\mathbf{Ax}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{Ax}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \qquad \cdots \qquad \mathbf{Ax}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

have at least one solution each. Consider the product

$$\mathbf{AC} = \mathbf{A}\left[\begin{array}{cccc} \mathbf{x}_1^* & \mathbf{x}_2^* & \cdots & \mathbf{x}_n^* \end{array}\right] = \left[\begin{array}{cccc} \mathbf{Ax}_1^* & \mathbf{Ax}_2^* & \cdots & \mathbf{Ax}_n^* \end{array}\right] = \mathbf{I}$$

where $\mathbf{x}_1^*$, $\mathbf{x}_2^*$, ..., $\mathbf{x}_n^*$ are solutions of the respective systems.

- We then invoke the theorem on page $\boxed{18}$, and conclude $\mathbf{A}$ is invertible. $\square$

- We have shown the product of invertible matrices is invertible. Next we have the converse.

### Theorem

Suppose $\mathbf{A}$ and $\mathbf{B}$ are matrices of $n \times n$. If $\mathbf{AB}$ is invertible, then $\mathbf{A}$ and $\mathbf{B}$ must also be invertible.

### Proof

- We will use statements (1.) and (2.) of the equivalence theorem,

$$\mathbf{Bx} = \mathbf{0} \quad \text{having only the trivial solution} \iff \mathbf{B} \quad \text{is invertible}$$

- So we will first need to show $\mathbf{Bx} = \mathbf{0}$ has only the trivial solution.
- Let $\mathbf{x}_0$ be any solution of $\mathbf{Bx} = \mathbf{0}$

$$\mathbf{Bx}_0 = \mathbf{0} \implies \mathbf{ABx}_0 = \mathbf{A0} \implies (\mathbf{AB})\,\mathbf{x}_0 = \mathbf{0}$$

so $\mathbf{x}_0$ is also a solution of $(\mathbf{AB})\,\mathbf{x}_0 = \mathbf{0}$.

## Proof

- However, $\mathbf{AB}$ is known to be invertible, so the backward implication states

$$\mathbf{x}_0 = \mathbf{0}$$

  is the only solution.
- The vector $\mathbf{x}_0$ is defined as a solution of

$$\mathbf{Bx} = \mathbf{0}$$

  thus $\mathbf{B}$ is invertible by the forward implication of the equivalence theorem.
- Notice $\mathbf{A}$ is a product of

$$\mathbf{A} = \mathbf{AI} = \mathbf{ABB}^{-1} = (\mathbf{AB})\,\mathbf{B}^{-1}$$

- Since $\mathbf{B}$ is invertible, $\mathbf{B}^{-1}$ is invertible. Together with the given fact $\mathbf{AB}$ is invertible, $\mathbf{A}$ must be invertible because the product of invertible matrices is invertible. $\square$