Vv256 Lecture 9

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- In general, the nth-order linear differential equation is

$$\alpha_n y^{(n)} + \alpha_{n-1} y^{(n-1)} + \cdots + \alpha_1 y' + \alpha_0 y = f \tag{1}$$

where α_i and f are functions of x. In the standard form, we have

$$y^{(n)} + P_1 y^{(n-1)} + \cdots + P_{n-1} y' + P_n y = Q \tag{2}$$

where P_i and Q are functions of x.

- In general, for IVPs of an nth-order linear equation, we have the following

Existence and Uniqueness

If P_1, P_2, ..., P_n, and Q are continuous on the open interval $\mathcal{I}$, then there exists exactly one solution y on the open interval $\mathcal{I}$ for

$$y^{(n)} + P_1 y^{(n-1)} + \cdots + P_{n-1} y' + P_n y = Q$$

that also satisfies the initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}$$

- In developing a general theory of linear equations, it is helpful to introduce

Linear differential operator

- In calculus, Leibniz and Euler's notation for differentiation is more dominant,

$$\frac{d}{dx}, \qquad \mathcal{D}$$

for the notations emphasize differentiation being an operation, just like

$$\sqrt{\quad}$$

- In addition to $\dfrac{d}{dx}$ or $\mathcal{D}$, there are many differential operators, e.g. ∇.
- Other differential operator can be "built" from the simple operator $\mathcal{D}$:

$$y'' = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2} = \mathcal{D}(\mathcal{D}y) = \mathcal{D}^2 y, \qquad \text{and} \qquad \frac{d^n}{dt^n} = \mathcal{D}^n y$$

where y represents a sufficiently differentiable function.

- Not only powers of $\mathcal{D}$ denote an operator,

$$\mathcal{D}^n$$

polynomial expressions of $\mathcal{D}$ are used to denote differential operators, e.g.

$$\mathcal{D}+3, \quad \mathcal{D}^2+3\mathcal{D}-4, \quad \text{and} \quad 5x^3\mathcal{D}^3-6x^2\mathcal{D}^2+4x\mathcal{D}+9$$

are valid notations for differential operators.

Definition

In general, we define an nth-order linear differential or polynomial operator to be

$$\mathcal{L}=\alpha_n\mathcal{D}^n+\alpha_{n-1}\mathcal{D}^{n-1}+\cdots+\alpha_1\mathcal{D}+\alpha_0 \tag{3}$$

where α_0, α_1, ..., α_n are functions of x.

- Any linear differential equation can be expressed in terms of $\mathcal{D}$, for example,

$$\begin{aligned} y''+5y'+6y=5x-3 &\iff \mathcal{D}^2y+5\mathcal{D}y+6y=5x-3 \\ &\iff \left(\mathcal{D}^2+5\mathcal{D}+6\right)y=5x-3 \end{aligned}$$

- Once we have defined a specific differential operator $\mathcal{L}$, for example

$$\mathcal{L} = \mathcal{D}^2 + 5\mathcal{D} + 6$$

then the following defines a differential equation

$$\mathcal{L}(y) = 5x - 3$$

- Homogeneous linear equations with constant coefficients is simply written as

$$\mathcal{L}(y) = 0$$

where $\mathcal{L} = a_n \mathcal{D}^n + a_{n-1}\mathcal{D}^{n-1} + \cdots + a_1 \mathcal{D} + a_0$, and a_i are constants.

- As a consequence of two basic properties of differentiation,

1. $\mathcal{D}[cf] = c\mathcal{D}[f]$ 2. $\mathcal{D}[f+g] = \mathcal{D}[f] + \mathcal{D}[g]$

the polynomial operator $\mathcal{L}$ a linear operator.

Superposition Principle

Let ϕ_1, ..., ϕ_k be solutions of the homogeneous nth-order linear equation, then

$$y = c_1\phi_1 + c_2\phi_2 + \ldots + c_k\phi_k, \quad \text{where the } c_i\text{'s are arbitrary constants.}$$

is also a solution.

Proof

- Suppose $\mathcal{L}$ is the differential operator defined by equation (3), and if

$$\mathcal{L}(\phi_1) = 0, \quad \ldots, \quad \mathcal{L}(\phi_k) = 0, \qquad \text{and} \qquad y = c_1\phi_1 + c_2\phi_2 + \ldots + c_k\phi_k$$

then

$$\begin{aligned}\mathcal{L}(y) &= \mathcal{L}\left(c_1\phi_1 + c_2\phi_2 \ldots + c_k\phi_k\right)\\ &= c_1\mathcal{L}(\phi_1) + c_2\mathcal{L}(\phi_2) + \cdots + c_k\mathcal{L}(\phi_k)\\ &= c_1 \cdot 0 + c_2 \cdot 0 + \cdots + c_k \cdot 0 = 0\end{aligned}$$

Theorem

Let y_p be any particular solution of the nonhomogeneous equation (1), and y_c be the complementary solution, then the general solution of equation (1) is

$$y = y_c + y_p$$

Proof

- Suppose $\mathcal{L}$ is the corresponding differential operator for equation (1), and let

$$\mathcal{L}(y_p) = f$$

- If we let $y^* = y - y_p$, where y is the general solution of $\mathcal{L}(y) = f$, then

$$\mathcal{L}(y^*) = \mathcal{L}(y - y_p) = \mathcal{L}(y) - \mathcal{L}(y_p) = f - f = 0$$

  so y^* is the general solution of $\mathcal{L}(y) = 0$, hence the complementary solution.

Q: Can you guess the form of the complementary solution y_c?

- Suppose $y = e^{rx}$, where $r \in \mathbb{C}$, is a solution of the homogeneous equation

$$\mathcal{L}(y) = 0, \qquad \text{where } \mathcal{L} \text{ is defined by equation (3).}$$

- Since $\mathcal{D}^k(e^{rx}) = r^k e^{rx}$, then

$$\begin{aligned}\mathcal{L}(e^{rx}) &= \alpha_n \mathcal{D}^n e^{rx} + \alpha_{n-1}\mathcal{D}^{n-1}e^{rx} + \cdots + \alpha_1 \mathcal{D}e^{rx} + \alpha_0 e^{rx} \\ &= \alpha_n r^n e^{rx} + \alpha_{n-1} r^{n-1} e^{rx} + \cdots + \alpha_1 r e^{rx} + \alpha_0 e^{rx} \\ &= \left(\alpha_n r^n + \alpha_{n-1} r^{n-1} + \cdots + \alpha_1 r + \alpha_0\right) e^{rx}\end{aligned}$$

- Hence $\mathcal{L}(e^{rx}) = 0$ if and only if r is a root of the characteristic polynomial

$$p(r) = \alpha_n r^n + \alpha_{n-1} r^{n-1} + \cdots + \alpha_1 r + \alpha_0$$

- You should be able to extend theorems for second-order equations, without too much effort, to higher-order equations if the characteristic polynomial leads to distinct roots.

Q: What is the form of the general solution for

$$y^{(4)} - y = 0$$

- Solve the corresponding characteristic equation

$$r^4 = 1 \implies r_{1,2} = \pm 1 \quad \text{and} \quad r_{3,4} = \pm \mathrm{i}$$

- Thus we expect the general solution to be

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + d_1 e^{r_3 x} + d_2 e^{r_4 x} = c_1 e^{x} + c_2 e^{-x} + d_1 e^{\mathrm{i}x} + d_2 e^{-\mathrm{i}x}$$

- Eliminating complex numbers,

$$\begin{aligned} y &= c_1 e^{x} + c_2 e^{-x} + e^{0 \cdot x} \left(c_3 \cos 1 \cdot x + c_4 \sin 1 \cdot x \right) \\ &= c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x \end{aligned}$$

Q: Can we always solve the initial-value problem involving the above equation?

- In operational calculus, we have the following theorem,

Theorem

Suppose a_is and b_js are constant, and

$$\mathcal{P} = \sum_{i=0}^{k} a_i \mathcal{D}^i \qquad \text{and} \qquad \mathcal{Q} = \sum_{j=0}^{m} b_j \mathcal{D}^j$$

are two linear differential operators, then

$$\mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} = \sum_{i=0}^{k} \sum_{j=0}^{m} a_i b_j \mathcal{D}^{i+j}$$

- So we can, without altering the result, factor and change the orders e.g.

$$y'' + 5y' + 6y = \left(\mathcal{D}^2 + 5\mathcal{D} + 6\right) y = (\mathcal{D}+2)(\mathcal{D}+3)\, y = (\mathcal{D}+3)(\mathcal{D}+2)\, y$$

- Notice the above is NOT true for variable coefficients.

- The characteristic polynomial completely determines the operator, we write

$$p(\mathcal{D}) = a_n\mathcal{D}^n + a_{n-1}\mathcal{D}^{n-1} + \cdots + a_1\mathcal{D} + a_0$$

where p is a polynomial. So p takes differential operator $\mathcal{D}$ as a "variable".

The Exponential Shift Law

If p is a polynomial and r is a constant $\in \mathbb{C}$, then

$$p(\mathcal{D})\Big(e^{rx}h\Big) = e^{rx}p(\mathcal{D}+r)\Big(h\Big)$$

where h is a general function of x.

Proof

By the product rule,

$$\mathcal{D}\Big(e^{rx}h\Big) = e^{rx}\mathcal{D}\Big(h\Big) + re^{rx}h = e^{rx}\mathcal{D}\Big(h\Big) + re^{rx}\mathcal{I}\Big(h\Big) = e^{rx}\left(\mathcal{D}+r\right)\Big(h\Big)$$

where $\mathcal{I}$ is the identity operator.

Proof

Now subtracting $se^{rx}h$ from both sides, where $s \in \mathbb{C}$,

$$\mathcal{D}\Big(e^{rx}h\Big) - se^{rx}h = e^{rx}\left(\mathcal{D}+r\right)\Big(h\Big) - se^{rx}h$$

$$\mathcal{D}\Big(e^{rx}h\Big) - s\mathcal{I}\Big(e^{rx}h\Big) = e^{rx}\Big[\left(\mathcal{D}+r\right)\Big(h\Big) - s\mathcal{I}\Big(h\Big)\Big]$$

$$\left(\mathcal{D}-s\mathcal{I}\right)\Big(e^{rx}h\Big) = e^{rx}\left(\mathcal{D}+r-s\mathcal{I}\right)\Big(h\Big)$$

Dropping the identity operator,

$$\left(\mathcal{D}-s\right)\Big(e^{rx}h\Big) = e^{rx}\left(\mathcal{D}+r-s\right)\Big(h\Big) \tag{4}$$

Now with the equation (4) proved, we next need to show that

$$\left(\mathcal{D}-s\right)^{k}\Big(e^{rx}h\Big) = e^{rx}\left(\mathcal{D}+r-s\right)^{k}\Big(h\Big), \qquad \text{where } k \text{ is an integer.}$$

This can be done by mathematical induction.

Proof

We have just proved that it is true for $k=1$, let us assume it is true for $k=n$, where n is some integer. We need to show it is true for $k=n+1$.

$$(\mathcal{D}-s)^{n+1}\left(e^{rx}h\right)$$

Using the last theorem, we can factor $(\mathcal{D}-s)^{n}$,

$$(\mathcal{D}-s)^{n+1}\left(e^{rx}h\right)=(\mathcal{D}-s)^{n}(\mathcal{D}-s)\left(e^{rx}h\right)$$

Applying the base case, that is equation (4),

$$(\mathcal{D}-s)^{n+1}\left(e^{rx}h\right)=(\mathcal{D}-s)^{n}e^{rx}(\mathcal{D}+r-s)\left(h\right)$$

Let $h^*=(\mathcal{D}+r-s)\left(h\right)$, use the assumption that it is true $k=n$,

$$(\mathcal{D}-s)^{n+1}\left(e^{rx}h\right)=(\mathcal{D}-s)^{n}\left(e^{rx}h^*\right)=e^{rx}(\mathcal{D}+r-s)^{n}h^*$$

Proof

We complete the induction by plugging $h^* = (\mathcal{D}+r-s)^k \Big(h\Big)$ back,

$$(\mathcal{D}-s)^{n+1}\Big(e^{rx}h\Big) = e^{rx}(\mathcal{D}+r-s)^{n+1}\Big(h\Big)$$

Therefore the following equation is true for all integer k,

$$(\mathcal{D}-s)^{k}\Big(e^{rx}h\Big) = e^{rx}(\mathcal{D}+r-s)^{k}\Big(h\Big) \tag{5}$$

Then successively apply the above equation for different operator $(\mathcal{D}-s_i)^{k_i}$,

$$(\mathcal{D}-s_1)^{k_1}\cdots(\mathcal{D}-s_m)^{k_m}\Big(e^{rx}h\Big) = e^{rx}(\mathcal{D}-s_1+r)^{k_1}\cdots(\mathcal{D}-s_m+r)^{k_m}\Big(h\Big)$$

Lastly, by the fundamental theorem of algebra, every polynomial of degree n has exactly n roots, thus can be factored into the following form

$$p(\mathcal{D}) = (\mathcal{D}-s_1)^{k_1}(\mathcal{D}-s_2)^{k_2}\cdots(\mathcal{D}-s_m)^{k_m}$$

where $s_i \in \mathbb{C}$, are the roots of the polynomial p. $\square$

Theorem

If λ_i is a root of multiplicity k_i of the characteristic polynomial

$$p(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

of the linear differential operator $p(\mathcal{D})$ with constant coefficients, then

$$x^\gamma e^{\lambda_i x}, \qquad \text{where } \gamma = 0,\ 1,\ \ldots,\ (k_i - 1).$$

are solution of

$$p(\mathcal{D})\Big(y\Big) = 0$$

Proof

- By the exponential shift law,

$$p(\mathcal{D})\Big(e^{rx}h\Big) = e^{rx}p(\mathcal{D}+r)\Big(h\Big)$$

- it follows that

$$\left(\mathcal{D}-\lambda_i\right)^{k_i}\left(x^\gamma e^{\lambda_i x}\right) = e^{\lambda_i x}\left(\mathcal{D}-\lambda_i+\lambda_i\right)^{k_i}\left(x^\gamma\right) = e^{\lambda_i x}D^{k_i}x^\gamma$$

Proof

- For $\gamma = 0,\ 1,\ 2,\ \ldots,\ (k_i - 1)$.

$$(\mathcal{D} - \lambda_i)^{k_i} \left(x^\gamma e^{\lambda_i x} \right) = e^{\lambda_i x} D^{k_i} x^\gamma = 0$$

- On the other hand, $p(\mathcal{D})$ must contain the factor

$$(\mathcal{D} - \lambda_i)^{k_i}$$

- We can factor $p(\mathcal{D}) = (\mathcal{D} - \lambda_i)^{k_i} q(\mathcal{D})$, where $q(\mathcal{D}) = \prod_{\lambda_j \neq \lambda_i} (\mathcal{D} - \lambda_j)^{k_j}$
- Now apply the differential operator $p(\mathcal{D})$ to

$$x^\gamma e^{\lambda_i x}$$

- By the theorem on L910 , we can interchange the order

$$p(\mathcal{D})\left(x^\gamma e^{\lambda_i x} \right) = q(\mathcal{D}) \left[(\mathcal{D} - \lambda_i)^{k_i} \left(x^\gamma e^{\lambda_i x} \right) \right] = 0 \quad \text{for } \gamma = 0,\ \ldots\ (k_i - 1).$$

Exercise

(a) *Find the general solution of*

$$y^{(5)} - 3y^{(4)} + 3y''' - y'' = 0$$

(b) *What is the general solution for*

$$\mathcal{L}(y) = 0$$

where $\mathcal{L}$ has a characteristic polynomial of

$$(\lambda^2 + 1)^3 (\lambda + 1)^2$$

Definition

If f is a sufficiently differentiable function and $\mathcal{L}$ is a linear differential operator with constant coefficients such that

$$\mathcal{L}(f) = 0$$

then $\mathcal{L}$ is said to be an annihilator of the function.

- Clearly, the differential operator $\mathcal{D}^n$ annihilates each of the function

$$x^k, \qquad \text{for} \quad k = 0,\ 1,\ \ldots,\ (n-1)$$

- The functions that are annihilated by a polynomial operator $\mathcal{L}$ are simply those functions that can be obtained from the general solution of

$$\mathcal{L}(y) = 0$$

Exercise

Find a differential operator that annihilates each of the functions

$$e^{\lambda t}, \quad te^{\lambda t}, \quad t^2 e^{\lambda t}, \quad \ldots, \quad t^{n-1} e^{\lambda t}$$

- When R and θ, $\theta > 0$ are real numbers, the quadratic formula reveals

$$\lambda^2 - 2R\lambda + (R^2 + \theta^2) = 0$$

has complex roots $\lambda = R \pm \mathrm{i}\theta$, and clearly

$$\left[\lambda^2 - 2R\lambda + (R^2 + \theta^2)\right]^n = 0$$

has $\lambda = R \pm \mathrm{i}\theta$ of multiplicity of n.

- Hence the differential operator $\left[\mathcal{D}^2 - 2R\mathcal{D} + (R^2+\theta^2)\right]^n$ annihilates

$$e^{Rt}\cos\theta t, \quad te^{Rt}\cos\theta t, \quad t^2e^{Rt}\cos\theta t, \quad \ldots, \quad t^{n-1}e^{Rt}\cos\theta t$$

$$e^{Rt}\sin\theta t, \quad te^{Rt}\sin\theta t, \quad t^2e^{Rt}\sin\theta t, \quad \ldots, \quad t^{n-1}e^{Rt}\sin\theta t$$

Exercise

Find a differential operator that annihilates $f(t) = 5e^{-t}\cos 2t - 9e^{-t}\sin 2t$.

Theorem

Suppose $\mathcal{L}$ is a linear differential operator with constant coefficients, then

$$\mathcal{L}(c_1\phi_1 + c_2\phi_2) = 0 \qquad \text{where} \qquad \mathcal{L}(\phi_1) = 0 \quad \text{and} \quad \mathcal{L}(\phi_2) = 0$$

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be linear differential operators with constant coefficients, then

$$\mathcal{L}_1\mathcal{L}_2\left(c_1\phi_1 + c_2\phi_2\right) = 0 \qquad \text{where} \qquad \mathcal{L}_1(\phi_1) = 0 \quad \text{and} \quad \mathcal{L}_2(\phi_2) = 0$$

Proof

- Note $\mathcal{L}_1$ annihilates ϕ_1 and $\mathcal{L}_2$ annihilates ϕ_2, but they do not necessarily

$$\mathcal{L}_1(\phi_2) \neq 0 \qquad \text{and} \qquad \mathcal{L}_2(\phi_1) \neq 0$$

- However, for constant coefficients, we can change the order of $\mathcal{L}_1$ and $\mathcal{L}_2$,

$$\begin{aligned} \mathcal{L}_1\mathcal{L}_2\left(c_1\phi_1 + c_2\phi_2\right) &= c_1\mathcal{L}_1\mathcal{L}_2\left(\phi_1\right) + c_2\mathcal{L}_1\mathcal{L}_2\left(\phi_2\right) \\ &= 0 + 0 = 0 \end{aligned}$$

Exercise

(a) Solve the following nonhomogeneous equation

$$y'' + 3y' + 2y = 4t^2$$

(b) Solve the following nonhomogeneous equation

$$y''' - 4y'' + 4y' = 5t^2 - 6t + 4t^2e^{2t} + 3e^{5t}$$