Vv256 Lecture 5

Dr Jing Liu

UM-SJTU Joint Institute

September 21, 2017

Definition

Vectors $\mathbf{v}_1, \mathbf{v}_2$ are said to be linearly independent if the only way to satisfy

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 = \mathbf{0}$$

is to have both α_1 and α_2 being zero simultaneously,

$$\alpha_1 = \alpha_2 = 0$$

- The notion of linear independence can be extended to functions.

Definition

Functions $f(t)$ and $g(t)$ are said to be linearly independent if the only way to have

$$\alpha_1 f(t) + \alpha_2 g(t) = 0, \qquad \text{for all } t.$$

is to have both constants α_1 and α_2 being zero simultaneously,

$$\alpha_1 = \alpha_2 = 0$$

Exercise

(a) Determine whether the following two functions are linearly independent.

$$f(x) = 9\cos 2x, \qquad g(x) = 2\cos^2 x - 2\sin^2 x$$

(b) Determine whether the following two functions are linearly independent.

$$f(x) = 2x^2, \qquad g(x) = x^4$$

- In general, it is not an easy job to determine whether two arbitrary functions

$$f(x) \qquad \text{and} \qquad g(x)$$

  are linearly independent.
- However, there is a systematic approach for differentiable functions.

Theorem

If f and g are differentiable functions on an open interval $\mathcal{I}$ and if the Wronskian

$$W(f, g) \text{ is not identically zero in } \mathcal{I},$$

then f and g are linearly independent. The contrapositive statement is often used,

$$f \text{ and } g \text{ are linearly dependent, then } W(f, g) \text{ is identically zero in } \mathcal{I}.$$

Proof

- It is given that W is not identically zero in $\mathcal{I}$,

$$W(x_0) = \det \begin{bmatrix} f & g \\ f' & g' \end{bmatrix} \neq 0 \qquad \text{for some } x_0.$$

- That tells us something about the following linear system,

$$\begin{aligned} f\alpha_1 + g\alpha_2 &= 0 \\ f'\alpha_1 + g'\alpha_2 &= 0 \end{aligned} \iff \begin{bmatrix} f & g \\ f' & g' \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \alpha_1 \begin{bmatrix} f \\ f' \end{bmatrix} + \alpha_2 \begin{bmatrix} g \\ g' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Proof

- The column vectors of the matrix

$$\begin{bmatrix} f \\ f' \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} g \\ g' \end{bmatrix} \quad \text{are not collinear for some } x_0.$$

- So at least for $x = x_0$, $\alpha_1 = \alpha_2 = 0$ is the only way the following is true

$$\alpha_1 \begin{bmatrix} f(x_0) \\ f'(x_0) \end{bmatrix} + \alpha_2 \begin{bmatrix} g(x_0) \\ g'(x_0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Therefore $\alpha_1 = \alpha_2 = 0$ is the only way that the following is true for all x.

$$\alpha_1 f + \alpha_2 g = 0$$

- Hence by definition f and g are linearly independent.

Q: Is the last statement true in reverse?

$$\phi_1 = |x|\, x^2 \qquad \phi_2 = x^3$$

Theorem

If $\phi_1(t)$ and $\phi_2(t)$ are linearly independent solutions to

$$a\ddot{y} + b\dot{y} + cy = 0, \qquad \text{where } a, b \text{ and } c \text{ are constants.}$$

then the general solution to this homogeneous equation of constant coefficients is

$$y(t) = C_1\phi_1(t) + C_2\phi_2(t)$$

where C_1 and C_2 are arbitrary constants.

- For a second-order equation, the set of linearly independent solutions

$$\{\phi_1, \phi_2\}$$

is known as a fundamental set of solutions.

Q: What does this theorem mean in terms of the Wronskian of

$$\phi_1 \qquad \text{and} \qquad \phi_2$$

Q: Have we proved this theorem? If not, how can we prove it?

Theorem

If the characteristic equation to

$$a\ddot{y} + b\dot{y} + cy = 0, \qquad \text{where } a, b \text{ and } c \text{ are constants.}$$

has two distinct real roots r_1 and r_2, then the general solution is

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

If the characteristic equation has a repeated root r, then the general solution is

$$y = (C_1 + C_2 t)\, e^{rt}$$

If it has complex roots $R \pm \mathrm{i}\theta$, where $\theta > 0$, then the general solution is

$$y = e^{Rt}\Big(C_1 \cos\theta t + C_2 \sin\theta t\Big)$$

Q: What is the critical step that we used to reach those three solutions?

- Note that this is different from how we were solving first-order equations.

- So we have shown that those three functions are indeed solutions, however, we haven't shown that every solution actually has one of those three forms.
- To prove the last 2 theorems, we must show all possible solutions are in this form since the general solution of a linear equation captures all solutions.
- We can obtain the solutions to the first two cases by consider the following

$$v = a\dot{\varphi} + (ar_1 + b)\varphi$$

  where φ is a continuously differentiable function of t.
- The derivative of v is given by

$$\dot{v} = a\ddot{\varphi} + (ar_1 + b)\dot{\varphi}$$

- Now consider the following

$$\dot{v} - r_1 v = a\ddot{\varphi} + (ar_1 + b)\dot{\varphi} - r_1 a\dot{\varphi} - r_1(ar_1 + b)\varphi = a\ddot{\varphi} + b\dot{\varphi} - (ar_1^2 + br_1)\varphi$$

- From the fact that r_1 is a solution to the characteristic equation, we know

$$ar_1^2 + br_1 + c = 0 \implies ar_1^2 + br_1 = -c \implies \dot{v} - r_1 v = a\ddot{\varphi} + b\dot{\varphi} + c\varphi = 0$$

Q: What happens if we set it to zero?

- This shows if we use the solutions to the first-order linear equation

$$\dot{v} - r_1 v = 0$$

as the constant term for the first-order linear differential equation of φ

$$a\dot{\varphi} + (ar_1 + b)\varphi = v$$

then a function φ is a solution to the above first-order equation if and only if it is a solution to the original second-order equation

$$a\dot{\varphi} + (ar_1 + b)\varphi = v \iff a\ddot{\varphi} + b\dot{\varphi} + c\varphi = 0 \iff a\ddot{y} + b\dot{y} + cy = 0$$

- Now if we solve the first-order linear equation of v, we have

$$\dot{v} - r_1 v = 0 \implies v = d_1 e^{r_1 t} \qquad \text{where } d_1 \text{ is an arbitrary constant.}$$

thus the first-order linear equation of φ is

$$a\dot{\varphi} + (ar_1 + b)\varphi = d_1 e^{r_1 t} \iff \dot{\varphi} + (r_1 + \frac{b}{a})\varphi = \frac{d_1}{a} e^{r_1 t}$$

- Since r_1 and r_2 are solutions to the characteristic equation,

$$\begin{aligned} ar^2 + br + c = 0 \implies ar^2 + br + c &= a(r - r_1)(r - r_2) \\ &= ar^2 - a(r_1 + r_2)r + ar_1 r_2 \\ \implies b &= -a(r_1 + r_2) \\ \implies -r_2 &= r_1 + \frac{b}{a} \end{aligned}$$

- Thus the first-order linear equation of φ can be written as

$$\dot{\varphi} + (r_1 + \frac{b}{a})\varphi = \frac{d_1}{a} e^{r_1 t} \iff \dot{\varphi} - r_2 \varphi = \frac{d_1}{a} e^{r_1 t}$$

- Using the integrating factor $\mu = \exp(-r_2 t)$ and solve for φ, we have

$$\varphi = \frac{d_1}{a} \exp(r_2 t) \int \exp\Big((r_1 - r_2)\, t \Big)\, dt$$

- If $r_1 \neq r_2$, then $\int \exp\Big((r_1 - r_2)\,t\Big)\,dt = \dfrac{\exp\Big((r_1 - r_2)\,t\Big)}{r_1 - r_2} + d_2$

$$\implies \varphi = \frac{d_1}{a}\exp(r_2 t)\left(\frac{\exp\Big((r_1 - r_2)\,t\Big)}{r_1 - r_2} + d_2\right)$$

$$= C_1 e^{r_1 t} + C_2 e^{r_2 t} \qquad \text{where } C_1 = \frac{d_1}{a(r_1 - r_2)} \text{ and } C_2 = \frac{d_1 d_2}{a}.$$

- If $r = r_1 = r_2$, then $\int \exp\Big((r_1 - r_2)\,t\Big)\,dt = t + d_3$

$$\implies \varphi = \frac{d_1}{a}\exp(rt)\,(t + d_3) = C_1 e^{rt} + C_2 t e^{rt}$$

where $C_1 = \dfrac{d_1 d_3}{a}$ and $C_2 = \dfrac{d_1}{a}$.

- The third case, in which the characteristic equation has two complex roots

$$R \pm \mathrm{i}\theta$$

things are a bit more complicated.

- Now consider the following,

$$v = e^{-Rt}\varphi \implies \dot{v} = -Re^{-Rt}\varphi + e^{-Rt}\dot{\varphi}$$
$$\implies \ddot{v} = e^{-Rt}\left(\ddot{\varphi} - 2R\dot{\varphi} + R^2\varphi\right)$$

where φ is a continuously differentiable function of t.

- Now if we combine $\ddot{v}$ and $\dot{v}$ in the following way

$$a\ddot{v} + \theta^2 av = e^{-Rt}\Big(a\ddot{\varphi} - 2aR\dot{\varphi} + a(R^2 + \theta^2)\varphi\Big)$$

- We will see what those coefficients in red are if we factor

$$ar^2 + br + c = a\Big(r - (R + \mathrm{i}\theta)\Big)\Big(r - (R - \mathrm{i}\theta)\Big)$$

$$= a\Big(r^2 - 2Rr + \left(R^2 + \theta^2\right)\Big)$$

- Thus the relationship between R, θ and the coefficients are

$$b = -2aR \qquad c = a(R^2 + \theta^2)$$

- Therefore we have

$$a\ddot{v} + \theta^2 av = e^{-Rt}\Big(a\ddot{\varphi} - 2aR\dot{\varphi} + a(R^2 + \theta^2)\varphi\Big)$$

$$= e^{-Rt}\left(a\ddot{\varphi} + b\dot{\varphi} + c\varphi\right) = 0$$

Q: What happen if we set it to zero?

- This shows φ is a solution to the original differential equation if and only if

$$v = e^{-Rt}\varphi$$

satisfies the following

$$a\ddot{v} + \theta^2 av = 0 \iff \ddot{v} + \theta^2 v = 0 \qquad \text{since } a \neq 0.$$

- To solve this equation, let $u = \theta t$, according to the chain rule

$$\dot{v} = \frac{dv}{dt} = \frac{dv}{du}\frac{du}{dt} = \theta\frac{dv}{du} \implies \ddot{v} = \frac{d}{dt}\left(\theta\frac{dv}{du}\right) = \theta\frac{d^2v}{du^2}\frac{du}{dt} = \theta^2\frac{d^2v}{du^2} = \theta^2 v''(u)$$

- Therefore

$$\ddot{v} + \theta^2 v = 0 \iff \theta^2 v''(u) + \theta^2 v(u) = 0 \iff v''(u) + v(u) = 0$$

since the roots are known to be not real.

- If we perform one last substitution

$$w = v'(u) + v \tan u$$

- and consider one last equation

$$\begin{aligned}
\frac{dw}{du} - w \tan u &= \frac{d}{du}\left(\frac{dv}{du} + v \tan u\right) - \left(\frac{dv}{du} + v \tan u\right) \tan u \\
&= \frac{d^2v}{du^2} + \frac{dv}{du} \tan u + v \sec^2 u - \frac{dv}{du} \tan u - v \tan^2 u \\
&= \frac{d^2v}{du^2} + v(\sec^2 u - \tan^2 u) = v''(u) + v(u) = 0
\end{aligned}$$

Q: What happens if we set it to zero?

- This shows $w(u)$ is a solution to the following linear first-order equation

$$\frac{dw}{du} - w \tan u = 0$$

if and only if the corresponding $v(u)$ is a solution to the following equation

$$v''(u) + v(u) = 0$$

- Using the integrating factor

$$\mu = \cos u$$

we can find the following general solution

$$w = C_1 \sec u, \qquad \text{where } C_1 \text{ is an arbitrary constant.}$$

- The function $v(u)$ can be found in turn by solving the following equation

$$v'(u) + v \tan u = C_1 \sec u$$

- The integrating factor $\nu = \sec u$ allows us to obtain

$$v = C_1 \sin u + C_2 \cos u, \qquad \text{where } C_2 \text{ is also an arbitrary constant.}$$

- Lastly, since $u = \theta t$ and $v = e^{-Rt}\varphi$, back substitutions lead us to

$$\varphi = e^{Rt}\Big(C_1 \cos \theta t + C_2 \sin \theta t\Big) \quad \square$$