Vv256 Lecture 21

Dr Jing Liu

UM-SJTU Joint Institute

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Exercise

Find the eigenvalues of the matrix $\mathbf{A} = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$.

Solution

- Find the characteristic equation, and solve it to find all eigenvalues,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \implies \det \begin{bmatrix} -2-\lambda & -1 \\ 5 & 2-\lambda \end{bmatrix} = 0$$

$$\implies \lambda^2 + 1 = 0 \implies \lambda_1 = \mathrm{i}, \quad \text{and} \quad \lambda_2 = -\mathrm{i}$$

- It shall be not a surprise to you that the characteristic equation of a matrix

  $\mathbf{A}$ with real entries

  can have complex solutions.

- To allow complex eigenvalues, it is necessary to allow our scalars to be complex, and consider complex entries for vectors and matrices.

Defintiion

A complex vector space is one in which the scalars are complex numbers.

- We will be concerned only with the complex generalization of $\mathbb{R}^n$.

$$\mathbb{C}^n$$

Q: Is $\mathbb{R}^n$ a subspace of $\mathbb{C}^n$?

- Every vector $\mathbf{v}$ in $\mathbb{C}^n$ can be split into real and imaginary parts as

$$\mathbf{v} = \mathrm{Re}(\mathbf{v}) + \mathrm{i}\,\mathrm{Im}(\mathbf{v})$$

  where $\mathrm{Re}(\mathbf{v})$ and $\mathrm{Im}(\mathbf{v})$ are vectors in $\mathbb{R}^n$.

- The vector

$$\overline{\mathbf{v}} = \mathrm{Re}(\mathbf{v}) - \mathrm{i}\,\mathrm{Im}(\mathbf{v})$$

  is called the complex conjugate of $\mathbf{v}$.

- It follows that the vectors in $\mathbb{R}^n$ can be viewed as those vectors in $\mathbb{C}^n$ whose imaginary part is zero; a vector $\mathbf{v}$ in $\mathbb{C}^n$ is in $\mathbb{R}^n$ if and only if

$$\overline{\mathbf{v}} = \mathbf{v}$$

- We will need to distinguish between matrices whose entries are real, called real matrices, and matrices whose entries may be either real or complex, called complex matrices.
- If $\mathbf{A}$ is a complex matrix, then

$$\mathrm{Re}(\mathbf{A}) \quad \text{and} \quad \mathrm{Im}(\mathbf{A})$$

  are the matrices formed by taking the real or imaginary part of $\mathbf{A}$.
- $\overline{\mathbf{A}}$ is the matrix formed by taking the complex conjugate of each entry in $\mathbf{A}$,

$$\overline{\mathbf{A}} = \mathrm{Re}(\mathbf{A}) - \mathrm{i}\,\mathrm{Im}(\mathbf{A})$$

- The standard operations on real matrices carry over without change to complex matrices, and all of the familiar properties continue to hold.

Theorem

- Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{C}^n$, and if α is a scalar, then:

$$1.\quad \overline{\overline{\mathbf{u}}} = \mathbf{u} \qquad 2.\quad \overline{\alpha\mathbf{u}} = \overline{\alpha}\,\overline{\mathbf{u}} \qquad 3.\quad \overline{\mathbf{u} \pm \mathbf{v}} = \overline{\mathbf{u}} \pm \overline{\mathbf{v}}$$

- Let $\mathbf{A}$ be an $m \times k$ complex matrix and $\mathbf{B}$ be a $k \times n$ complex matrix, then:

$$1.\quad \overline{\overline{\mathbf{A}}} = \mathbf{A} \qquad 2.\quad \overline{(\mathbf{A}^{\mathrm{T}})} = (\overline{\mathbf{A}})^{\mathrm{T}} \qquad 3.\quad \overline{\mathbf{AB}} = \overline{\mathbf{A}}\,\overline{\mathbf{B}}$$

Proof

- To show $\overline{\mathbf{AB}} = \overline{\mathbf{A}}\,\overline{\mathbf{B}}$, we consider the ijth element of the left-hand side

$$\left[\overline{\mathbf{AB}}\right]_{ij} = \overline{[\mathbf{AB}]_{ij}} = \overline{\sum_{p}^{k} [\mathbf{A}]_{ip} [\mathbf{B}]_{pj}} = \sum_{p}^{k} \overline{[\mathbf{A}]_{ip}}\; \overline{[\mathbf{B}]_{pj}}$$

$$= \sum_{p}^{k} \left[\overline{\mathbf{A}}\right]_{ip} \left[\overline{\mathbf{B}}\right]_{pj} = \left[\overline{\mathbf{A}}\,\overline{\mathbf{B}}\right]_{ij}$$

- Except for the use of complex scalars, the notions of linear combination, linear independence, subspace, spanning, basis, dimension, determinant, can be extended to complex vectors and matrices without much modification.

Exercise

Suppose

$$\mathbf{v} = \begin{bmatrix} 3+\mathrm{i} \\ -2\mathrm{i} \end{bmatrix} \quad \textit{and} \quad \mathbf{A} = \begin{bmatrix} 1+\mathrm{i} & -\mathrm{i} \\ 4 & 6-2\mathrm{i} \end{bmatrix}$$

Find $\det(\mathbf{A})$ *and determine whether* $\mathbf{Av}$ *and* $\mathbf{v}$ *are linearly independent.*

Solution

- Determinant

$$\det(\mathbf{A}) = (1+\mathrm{i})(6-2\mathrm{i}) + 4\mathrm{i} = 8 + 8\mathrm{i}$$

- Compute $\mathbf{Av} = (3+\mathrm{i}) \begin{bmatrix} 1+\mathrm{i} \\ 4 \end{bmatrix} + (-2\mathrm{i}) \begin{bmatrix} -\mathrm{i} \\ 6-2\mathrm{i} \end{bmatrix} = \begin{bmatrix} 4\mathrm{i} \\ 8-8\mathrm{i} \end{bmatrix}$

Solution

- As before, we need to determine whether there is non-zero scalar α_1 and α_2

$$\alpha_1 \begin{bmatrix} 3+\mathrm{i} \\ -2\mathrm{i} \end{bmatrix} + \alpha_2 \begin{bmatrix} 4\mathrm{i} \\ 8-8\mathrm{i} \end{bmatrix} = \mathbf{0}$$

- Note the scalar field is $\mathbb{C}$, so α_1 and α_2 need not be real, and $\mathbf{0} \in \mathbb{C}^2$.
- However, the equivalence theorem is applicable, so we can simply consider

$$\underbrace{\begin{bmatrix} 3+\mathrm{i} & 4\mathrm{i} \\ -2\mathrm{i} & 8-8\mathrm{i} \end{bmatrix}}_{\mathbf{B}} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and conclude $\mathbf{Av}$ and $\mathbf{v}$ are linearly independent if and only if $\det(\mathbf{B}) \neq 0$

$$\det(\mathbf{B}) = (3+\mathrm{i})(8-8\mathrm{i}) - 8 = 24 - 16\mathrm{i} \neq 0$$

- Therefore $\mathbf{v}$ and $\mathbf{Av}$ are linearly independent.

- As in the real case, λ is a complex eigenvalue of $\mathbf{A}$ if and only if

  there exists a nonzero vector $\mathbf{x}$ in $\mathbb{C}^n$ such that $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$.

- Each such $\mathbf{x}$ in $\mathbb{C}^n$ is called a complex eigenvector of $\mathbf{A}$ corresponding to λ.

Exercise

Find the eigenspace of the matrix corresponding to the eigenvalue $\lambda = \mathrm{i}$.

$$\mathbf{A} = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$$

Solution

- Compute the matrix

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} -2-\mathrm{i} & -1 \\ 5 & 2-\mathrm{i} \end{bmatrix}$$

- We know $\mathbf{A} - \lambda\mathbf{I}$ must be singular, thus we only need one of the two rows.

Solution

- Let us use the second equation,

$$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \begin{bmatrix} -2-\mathrm{i} & -1 \\ 5 & 2-\mathrm{i} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies 5x_1 + (2-i)x_2 = 0$$

$$\implies \mathbf{x} = \alpha \begin{bmatrix} 2-\mathrm{i} \\ -5 \end{bmatrix} \qquad \text{for any } \alpha \neq 0.$$

- Thus the eigenspace corresponding to $\lambda = \mathrm{i}$ is the span of

$$\begin{bmatrix} 2-\mathrm{i} \\ -5 \end{bmatrix}$$

Theorem

If λ is an eigenvalue of a real $n \times n$ matrix $\mathbf{A}$, and if $\mathbf{x}$ is a eigenvector belonging to λ, then $\overline{\lambda}$ is also an eigenvalue of $\mathbf{A}$, and $\overline{\mathbf{x}}$ is a corresponding eigenvector.

Proof

- Since λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{x}$ is a corresponding eigenvector, we have

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \implies \overline{\mathbf{A}\mathbf{x}} = \overline{\lambda\mathbf{x}} \implies \overline{\mathbf{A}}\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}$$

- However, $\mathbf{A} = \overline{\mathbf{A}}$, since $\mathbf{A}$ is real, we have

$$\mathbf{A}\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}$$

which shows $\overline{\lambda}$ is an eigenvalue of $\mathbf{A}$, and $\overline{\mathbf{x}}$ is a corresponding eigenvector.

Theorem

The eigenvalues of the following real matrix are $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$.

$$\mathbf{C} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \qquad \text{where} \quad b \geq 0$$

If a and b are *not both zero*, then this matrix can be factored as

$$\mathbf{C} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where θ is the argument of λ_1 and r is the modulus of λ_1,

$$\theta = \arg(\lambda_1), \qquad r = \operatorname{mod}(\lambda_1)$$

- For every vector $\mathbf{x} \in \mathbb{R}^2$, consider the matrix multiplication between $\mathbf{C}$ and $\mathbf{x}$

$$\mathbf{C}\mathbf{x} = r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \mathbf{x}$$

- It essentially states that multiplication by a matrix of the given form is a rotation through the angle θ followed by a scaling with factor r.

Proof

- The first part is trivial,

$$(a-\lambda)^2 + b^2 = 0 \implies \lambda = a \pm b\mathrm{i}$$

- To prove the second part,

$$\mathbf{C} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \frac{a}{r} & -\frac{b}{r} \\ \frac{b}{r} & \frac{a}{r} \end{bmatrix}$$

- Consider the polar form,

$$a = \mathrm{Re}(\lambda_1) = r\cos\theta, \qquad b = \mathrm{Im}(\lambda_1) = r\sin\theta$$

- Thus

$$\mathbf{C} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \square$$