Question1 (1 points)

Suppose $\mathbf{A}$ is a 3×3 matrix, and its eigenvalues are

$$\lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = 0,$$

and the corresponding eigenvectors are

$$\mathbf{x_1} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{x_2} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{x_3} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

respectively. Find the matrix $\mathbf{A}$.

Solution:

1M Knowing the eigenvalues and the corresponding eigenvectors, we have the following

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$$

where the diagonal matrix $\mathbf{D}$ contains the eigenvalues and $\mathbf{P}$ has the eigenvectors

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{P} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

thus we essentially need to find the inverse of $\mathbf{P}$

$$\mathbf{P}^{-1} = \frac{1}{\det(\mathbf{P})} \mathbf{C}^{\mathrm{T}} = \frac{1}{-1} \begin{bmatrix} -5 & 1 & 2 \\ -3 & 1 & 1 \\ 4 & -1 & -2 \end{bmatrix} = (-1) \begin{bmatrix} -5 & 1 & 2 \\ -3 & 1 & 1 \\ 4 & -1 & -2 \end{bmatrix}$$

and compute the product

$$\begin{aligned} \mathbf{A} &= (-1) \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -5 & 1 & 2 \\ -3 & 1 & 1 \\ 4 & -1 & -2 \end{bmatrix} \\ &= (-1) \begin{bmatrix} 1 & (-)0 & 0 \\ 2 & (-)(-2) & 0 \\ 1 & (-)1 & 0 \end{bmatrix} \begin{bmatrix} -5 & 1 & 2 \\ -3 & 1 & 1 \\ 4 & -1 & -2 \end{bmatrix} \\ &= (-1) \begin{bmatrix} -5 & 1 & 2 \\ -16 & 4 & 6 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -1 & -2 \\ 16 & -4 & -6 \\ 2 & 0 & -1 \end{bmatrix} \end{aligned}$$

Question2 (1 points)

Find the eigenvalues and the corresponding eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

Solution:

1M The eigenvalues are found by solving

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \lambda^2 - 2\lambda + 5 \implies \lambda_{1,2} = 1 \pm 2\mathrm{i}$$

Using $\lambda = 1 + 2\mathrm{i}$, the eigenvectors are found by solving

$$\begin{bmatrix} 2-2\mathrm{i} & -2 \\ 4 & -(2+2\mathrm{i}) \end{bmatrix} \mathbf{x} = \mathbf{0} \implies \begin{matrix} (2-2\mathrm{i})x_1 - 2x_2 = 0 \\ 4x_1 - (2+2\mathrm{i})x_2 = 0 \end{matrix} \implies (1-\mathrm{i})x_1 - x_2 = 0$$

Thus the eigenvector of $\mathbf{A}$ corresponding to $\lambda = 1 + 2\mathrm{i}$ is spanned by

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1-\mathrm{i} \end{bmatrix}$$

According to the theorem on L21P10, the eigenvector of $\mathbf{A}$ corresponding to

$$\lambda = 1 - 2\mathrm{i}$$

is spanned by

$$\mathbf{x}_2 = \bar{\mathbf{x}}_1 = \begin{bmatrix} 1 \\ 1+\mathrm{i} \end{bmatrix}$$

Question3 (1 points)
Prove the following theorem.

Suppose $\mathbf{A}$ is an $n \times n$ matrix, and $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_n$ are eigenvectors for $\mathbf{A}$ with distinct eigenvalues, then these vectors are linearly independent.

Solution:

1M Let r be the dimension of the subspace of $\mathbb{C}^n$ spanned by

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$$

and suppose that $r < n$, i.e.

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \quad \text{are assumed to be linearly dependent}$$

We assume that $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are linearly independent, so that

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1} \quad \text{are assumed to be linearly dependent}$$

and there exist scalar $\alpha_1, \ldots, \alpha_{r+1}$ that are not all zero such that

$$\alpha_1\mathbf{v}_1 + \cdots + \alpha_r\mathbf{v}_r + \alpha_{r+1}\mathbf{v}_{r+1} = \mathbf{0} \tag{1}$$

the coefficient $\alpha_{r+1} \neq 0$ otherwise $\mathbf{v}_1 \cdots \mathbf{v}_r$ would be linearly dependent, thus

$$\alpha_{r+1}\mathbf{v}_{r+1} \neq \mathbf{0}$$

therefore $\alpha_1 \cdots \alpha_r$ cannot be all zero. Multiply equation (1) by $\mathbf{A}$,

$$\begin{aligned} \alpha_1\mathbf{A}\mathbf{v}_1 + \cdots + \alpha_r\mathbf{A}\mathbf{v}_r + \alpha_{r+1}\mathbf{A}\mathbf{v}_{r+1} = \mathbf{0} \\ \implies \alpha_1\lambda_1\mathbf{v}_1 + \cdots + \alpha_r\lambda_r\mathbf{v}_r + \alpha_{r+1}\lambda_{r+1}\mathbf{v}_{r+1} = \mathbf{0} \end{aligned} \tag{2}$$

Subtract λ_{r+1} times equation (1) from equation (2),

$$\alpha_1(\lambda_1 - \lambda_{r+1})\mathbf{v}_1 + \cdots + \alpha_r(\lambda_r - \lambda_{r+1})\mathbf{v}_r + \alpha_{r+1}(\lambda_{r+1} - \lambda_{r+1})\mathbf{v}_{r+1} = \mathbf{0}$$

It is given that λ_i's are distinct, so the only way this equation is true is if

$$\alpha_1 = \alpha_2 = \cdots = \alpha_r = 0$$

This contradicts the independence of $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_r$, thus we reject the assumption

$$r < n$$

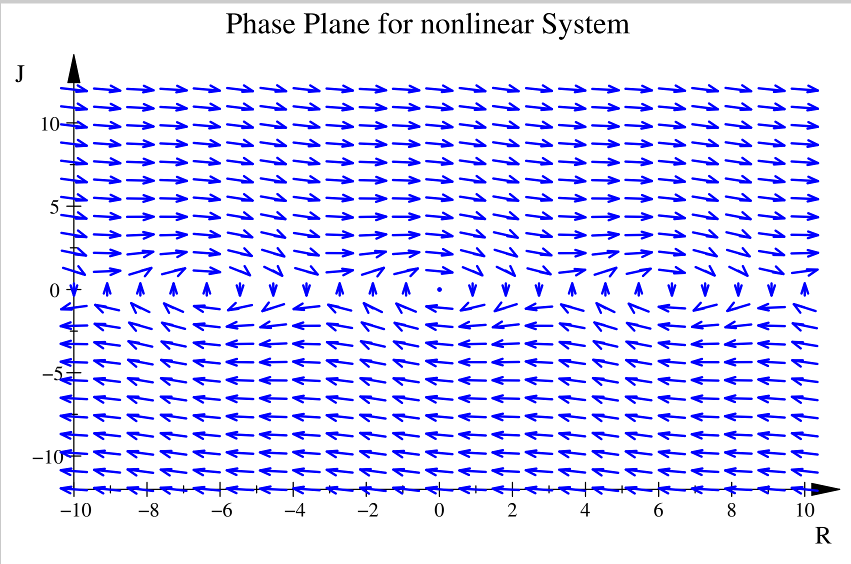
and conclude that $r = n$, therefore those eigenvectors corresponding to distinct eigenvalues are linearly independent.

Question4 (1 points)
Plot an informative phase portrait for the following system

$$\begin{aligned} \dot{R} &= J \\ \dot{J} &= -\sin R - \frac{1}{5}J \end{aligned}$$

for $-10 \leq R \leq 10$ and $-12 \leq J \leq 12$.

Solution:



1M The plot was done using Mupad in Matlab

```
>> mupad

F := matrix( [[J], [ - sin(R) - 1/5*J]])
```

$$\begin{pmatrix} J \\ -\frac{J}{5} - \sin(R) \end{pmatrix}$$

```
F := F / norm(F,Frobenius)
```

$$\begin{pmatrix} \frac{J}{\sqrt{\left|\frac{J}{5}+\sin(R)\right|^2+|J|^2}} \\ -\frac{\left(\frac{J}{5}+\sin(R)\right)}{\sqrt{\left|\frac{J}{5}+\sin(R)\right|^2+|J|^2}} \end{pmatrix}$$

```
field := plot::VectorField2d(F, R = -10..10, J = -12..12, Mesh = [ 23, 23], ArrowLength = Fixed, Axes = Frame)

plot(field, Header = "Phase Plane for nonlinear System", AxesTitles = ["R", "J"])
```

Although you are more likely to use a compute package to do a phase portrait plot, you should know what the slope and the arrow represent, and how to compute them.

Question5 (1 points)

Solve the first-order system $\dot{\mathbf{x}} = \mathbf{Ax}$ by elimination, where

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 1 \\ -3 & 1 & -2 \\ -4 & 0 & -1 \end{bmatrix}$$

Solution:

1M It can be shown that the system is not diagonalizable, using Mupad, we have

```
A:= matrix( [[3,0,1], [-3,1,-2], [-4, 0, -1]])
```

$$\begin{pmatrix} 3 & 0 & 1 \\ -3 & 1 & -2 \\ -4 & 0 & -1 \end{pmatrix}$$

```
linalg::eigenvalues(A)
```

$\{1\}$

```
v := linalg::eigenvectors(A)
```

$$\left[\left[1, 3, \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right]\right]\right]$$

Consider the general first-order system of 3×3 in operator notation, we have

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ \dot{x}_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned} \iff \begin{aligned} (D - a_{11})\, x_1 - a_{12}x_2 - a_{13}x_3 &= 0 \\ -a_{21}x_1 + (D - a_{22})\, x_2 - a_{23}x_3 &= 0 \\ -a_{31}x_1 - a_{32}x_2 + (D - a_{33})\, x_3 &= 0 \end{aligned} \quad (3)$$

From the first two equations of system (3) when eliminating x_3, we have

$$\begin{aligned} (D - a_{11})\, x_1 - a_{12}x_2 - a_{13}x_3 &= 0 \\ -a_{21} + (D - a_{22})\, x_2 - a_{23}x_3 &= 0 \end{aligned} \implies \begin{aligned} a_{23}\,(D - a_{11})\, x_1 - a_{23}a_{12}x_2 - a_{23}a_{13}x_3 &= 0 \\ -a_{13}a_{21}x_1 + a_{13}\,(D - a_{22})\, x_2 - a_{13}a_{23}x_3 &= 0 \end{aligned}$$

from the difference of the two equations above, we have

$$\big(a_{23}D-(a_{11}a_{23}-a_{13}a_{21})\big)x_1-\big(a_{13}D-(a_{13}a_{22}-a_{23}a_{12})\big)x_2=0 \tag{4}$$

Similarly, from the last two equations of system (3) when eliminating x_3, we have

$$\begin{aligned}-a_{21}\left(D-a_{33}\right)x_1+\left(D-a_{33}\right)\left(D-a_{22}\right)x_2-a_{23}\left(D-a_{33}\right)x_3&=0\\ -a_{23}a_{31}x_1-a_{23}a_{32}x_2+a_{23}\left(D-a_{33}\right)x_3&=0\end{aligned}$$

from which we obtain a second equation involving only x_1 and x_2,

$$\big(-a_{21}D-(a_{23}a_{31}-a_{21}a_{33})\big)x_1+\big(D^2-(a_{22}+a_{33})D-(a_{23}a_{32}-a_{22}a_{33})\big)x_2=0$$

Eliminating x_2, we have

$$\begin{aligned}&\Big(\big(a_{23}D-(a_{11}a_{23}-a_{13}a_{21})\big)\big(D^2-(a_{22}+a_{33})D-(a_{23}a_{32}-a_{22}a_{33})\big)\\ &\quad+\big(a_{13}D-(a_{13}a_{22}-a_{23}a_{12})\big)\big(-a_{21}D-(a_{23}a_{31}-a_{21}a_{33})\big)\Big)x_1=0\\ &\implies\left(D^3+\alpha D^2+\beta D+\gamma\right)x_1=0\end{aligned}$$

where

$$\begin{aligned}&\alpha=-\left(a_{11}+a_{22}+a_{33}\right)\\ &\beta=a_{11}a_{22}-a_{12}a_{21}+a_{11}a_{33}-a_{13}a_{31}+a_{22}a_{33}-a_{23}a_{32}\\ &\gamma=-\left(a_{11}a_{22}a_{33}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}\right)\end{aligned}$$

Thus by solving the characteristic equation

$$\lambda^3+\alpha\lambda^2+\beta\lambda+\gamma=0$$

we willing have the general solution for

$$x_1=C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}+C_3e^{\lambda_3 t}$$

from which we obtain x_2 by using equation (4) and x_3 in turn by using any one of the original equations. Notice it can shown by direction computation that

$$\lambda^3+\alpha\lambda^2+\beta\lambda+\gamma=0$$

is actually $\det\left(\mathbf{A}-\lambda\mathbf{I}\right)=0$, that is, roots to the characteristic polynomial of

$$\left(D^3+\alpha D^2+\beta D+\gamma\right)x_1=0$$

are actually the eigenvalues of $\mathbf{A}$. The same can be said to x_2 or x_3. So in practice, instead of manipulating every time, we can use eigenvalues to find x_1, then use the equations obtained from elimination to obtain x_2 and x_3 for a 3×3 system.

For this particular question, since $\lambda_1=\lambda_2=\lambda_3=1$,

$$x_1=\left(C_1+C_2t+C_3t^2\right)e^t$$

In this case, it is easier to work out x_3 first. Using the first equation of system (3),

$$(D-3)x_1 - x_3 = 0 \implies x_3 = (D-3)x_1$$

Using the exponential shift law, we have

$$\begin{aligned} x_3 &= (D-3)\left(C_1 + C_2 t + C_3 t^2\right) e^t \\ &= e^t (D-2)\left(C_1 + C_2 t + C_3 t^2\right) \\ &= \left[(C_2 - 2C_1) + 2(C_3 - C_2)t - 2C_3 t^2\right] e^t \end{aligned}$$

From equation (4) and exponential shift law, we have

$$\begin{aligned} \big(-2D - (3\cdot(-2) - 1\cdot(-3))\big)x_1 - \big(1D - (1\cdot 1 - (-2)\cdot 0)\big)x_2 &= 0 \\ (-2D+3)x_1 - (D-1)x_2 &= 0 \\ -2\left(D - \frac{3}{2}\right)\left(C_1 + C_2 t + C_3 t^2\right) e^t &= (D-1)x_2 \\ -2e^t\left(D - \frac{1}{2}\right)\left(C_1 + C_2 t + C_3 t^2\right) &= (D-1)x_2 \\ -2e^t\left[\left(C_2 - \frac{1}{2}C_1\right) + \left(2C_3 - \frac{1}{2}C_2\right) - \frac{1}{2}C_3 t^2\right] &= (D-1)x_2 \end{aligned}$$

Since the third-order equation that leads to x_2 has the same eigenvalues, we know

$$x_2 = \left(C_1^* + C_2^* t + C_3^* t^2\right) e^t$$

using which, we have

$$\begin{aligned} -2e^t\left[\left(C_2 - \frac{1}{2}C_1\right) + \left(2C_3 - \frac{1}{2}C_2\right)t - \frac{1}{2}C_3 t^2\right] &= (D-1)\left(C_1^* + C_2^* t + C_3^* t^2\right) e^t \\ &= e^t D\left(C_1^* + C_2^* t + C_3^* t^2\right) \\ &= e^t\left(C_2^* + 2C_3^* t\right) \end{aligned}$$

from which we conclude $C_3 = 0$ to avoid a contradiction and

$$C_2^* = -2\left(C_2 - \frac{1}{2}C_1\right) = -2C_2 + C_1 \qquad C_3^* = -\left(2C_3 - \frac{1}{2}C_2\right) = \frac{1}{2}C_2$$

while C_1^* is arbitrary. Therefore, solution is given by

$$\begin{aligned} \mathbf{x} &= e^t \begin{bmatrix} C_1 + C_2 t \\ C_1^* + (C_1 - 2C_2)t + \frac{1}{2}C_2 t^2 \\ C_2 - 2C_1 - 2C_2 t \end{bmatrix} \\ &= C_1 e^t \begin{bmatrix} 1 \\ t \\ -2 \end{bmatrix} + \frac{1}{2}C_2 e^t \begin{bmatrix} 2t \\ t^2 - 4t \\ 2 - 4t \end{bmatrix} + C_1^* e^t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

where C_1, C_2 and C_1^* are arbitrary constants. This case shows, when $\mathbf{A}$ is not diagonalizable, we have complications in the form of the general solution.

Question6 (1 points)

Prove the following theorem.

Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices, then the matrix product

$$\mathbf{AB} \quad \text{and} \quad \mathbf{BA}$$

have the same eigenvalues.

Solution:

1M If λ is an eigenvalue of $\mathbf{AB}$ and $\mathbf{x}$ is the corresponding eigenvector,

$$\mathbf{ABx} = \lambda \mathbf{x},$$

then

$$\mathbf{BABx} = \lambda \mathbf{Bx} \implies \mathbf{BAy} = \lambda \mathbf{y}, \quad \text{where } \mathbf{y} = \mathbf{Bx}.$$

Thus, as long as $\mathbf{y} \neq \mathbf{0}$ for nonzero $\mathbf{x}$, $\mathbf{y}$ is an eigenvector of $\mathbf{BA}$ with eigenvalue λ. However, if $\mathbf{y} = \mathbf{0}$ for a nonzero $\mathbf{x}$, then $\mathbf{Bx} = \mathbf{0} \implies \mathbf{ABx} = \mathbf{0}$ and

$$\mathbf{0} = \lambda \mathbf{x}, \qquad \text{for nonzero } \mathbf{x}$$

thus $\lambda = 0$ must be the eigenvalue. If $\lambda = 0$ is an eigenvalue of $\mathbf{AB}$, then $\mathbf{AB}$ must be singular. If $\mathbf{AB}$ is singular, then $\mathbf{BA}$ must be singular as well, which means $\lambda = 0$ is also an eigenvalue of $\mathbf{BA}$.

Question7 (1 points)

Determine whether $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{bmatrix}$ is singular. If $\mathbf{A}$ is not singular, find $\mathbf{A}^{-1}$.

Solution:

1M The determinant of $\mathbf{A}$ tells whether it is singular.

$$\begin{aligned} \det(\mathbf{A}) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 1 \cdot (-1)^{1+1}(-2-1) + (-1) \cdot (-1)^{1+2}(-1+2) + 0 \cdot (-1)^{1+3}(1+4) \\ &= -2 \end{aligned}$$

thus it is invertible/nonsingular, and

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = -\frac{1}{2}\begin{bmatrix} -3 & -1 & -1 \\ -1 & -1 & -1 \\ 5 & 1 & 3 \end{bmatrix}$$

Question8 (2 points)

Prove the following theorem.

Let $\mathbf{A}$ be an $n \times n$ matrix, then

1. λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$ if $\mathbf{A}$ is invertible and λ is an eigenvalue of $\mathbf{A}$.
2. $\mathbf{A}$ is singular if and only if $\lambda = 0$ is an eigenvalue of $\mathbf{A}$.

Solution:

1M Let $\mathbf{x}$ be an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ, then

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \implies \mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}^{-1}\lambda\mathbf{x} \implies \mathbf{x} = \lambda\mathbf{A}^{-1}\mathbf{x} \implies \lambda^{-1}\mathbf{x} = \mathbf{A}^{-1}\mathbf{x}$$

thus λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$. Suppose $\mathbf{A}$ is singular, then

$$\det(\mathbf{A}) = 0$$

which means there is a choice of x_1, x_2, $\ldots$, x_n, not simultaneously zero, such that

$$\begin{aligned} x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n &= \mathbf{0} \\ \mathbf{A}\mathbf{x} &= 0\mathbf{x} \end{aligned}$$

where $\mathbf{a}_i$ are the ith column of $\mathbf{A}$ and x_i are the ith element of $\mathbf{x}$. Since $\mathbf{x} \neq \mathbf{0}$, $\lambda = 0$ is an eigenvalue of $\mathbf{A}$. Now suppose $\lambda = 0$ is an eigenvalue of $\mathbf{A}$, then there is a nonzero vector $\mathbf{x}$ such that

$$\begin{aligned} \mathbf{A}\mathbf{x} &= 0\mathbf{x} \\ x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n &= \mathbf{0} \end{aligned}$$

which means the columns of $\mathbf{A}$ are linearly dependent, and $\det(\mathbf{A}) = 0$.

Question9 (1 points)
Suppose $\lambda_1 = 4$ and $\lambda_2 = -3$ are the eigenvalues of an unknown 2×2 matrix

$$\mathbf{A}$$

and the corresponding eigenvectors, respectively, are

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Find the eigenvalues and and the corresponding eigenspaces for the matrix

$$\mathbf{A}^3$$

Solution:

1M Considering the powers of $\mathbf{A}$,

$$\mathbf{A}^3\mathbf{x} = \mathbf{A}^2\mathbf{A}\mathbf{x} = \mathbf{A}^2\lambda\mathbf{x} = \lambda\mathbf{A}\mathbf{A}\mathbf{x} = \lambda\mathbf{A}\lambda\mathbf{x} = \lambda^2\mathbf{A}\mathbf{x} = \lambda^3\mathbf{x}$$

So the eigenvalues are

$$4^3 = 64 \qquad \text{and} \qquad (-3)^3 = -27$$

and corresponding eigenvectors are,

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Question10 (1 points)

Verify that the following matrix is diagonalizable

$$\mathbf{A} = \begin{bmatrix} 3 & -2 & -2 \\ -3 & -2 & -6 \\ 3 & 6 & 10 \end{bmatrix}$$

and find an invertible matrix $\mathbf{P}$ such that the following is a diagonal matrix

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

Solution:

1M To determine whether it is diagonalizable, we have to solve the eigenvalue problem, and see whether there are 3 linearly independent eigenvectors.

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \implies -(\lambda-4)^2(\lambda-3) = 0 \implies \lambda_{1,2} = 4 \quad \lambda_3 = 3$$

Solving $(\mathbf{A} - 4\mathbf{I})\,\mathbf{x} = \mathbf{0}$, we have

$$\begin{bmatrix} -1 & -2 & -2 \\ -3 & -6 & -6 \\ 3 & 6 & 6 \end{bmatrix} \mathbf{x} = \mathbf{0} \implies x_1 + 2x_2 + 2x_3 = 0 \implies \mathbf{x} = \alpha \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Solving $(\mathbf{A} - 3\mathbf{I})\,\mathbf{x} = \mathbf{0}$, we have

$$\begin{bmatrix} 0 & -2 & -2 \\ -3 & -5 & -6 \\ 3 & 6 & 7 \end{bmatrix} \mathbf{x} = \mathbf{0} \implies \begin{aligned} 3x_1 + x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned} \implies \mathbf{x} = \gamma \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix}$$

Since there are 3 linearly independent eigenvectors, $\mathbf{A}$ is diagonalizable, and

$$\mathbf{P} = \begin{bmatrix} -2 & -2 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & 3 \end{bmatrix}$$

is a matrix that diagonalizes $\mathbf{A}$.

Question11 (1 points)

Find the general solution of the following system,

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & 2+\mathrm{i} \\ -1 & -1-\mathrm{i} \end{bmatrix} \mathbf{x}$$

Solution:

1M Solving the eigenvalue problem, we have

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \implies (2-\lambda)(-1-\mathrm{i}-\lambda) + (2+\mathrm{i}) &= 0 \\ \lambda^2 + (\mathrm{i}-1)\lambda - \mathrm{i} &= 0 \\ (\lambda-1)(\lambda+\mathrm{i}) &= 0 \\ \implies \lambda_1 = 1 \quad \lambda_2 &= -\mathrm{i} \end{aligned}$$

Solving $(\mathbf{A} - \mathbf{I})\,\mathbf{x} = \mathbf{0}$, we have

$$\begin{bmatrix} 1 & 2+\mathrm{i} \\ -1 & -2-\mathrm{i} \end{bmatrix} \mathbf{x} = \mathbf{0} \implies x_1 + (2+\mathrm{i})x_2 = 0 \implies \mathbf{x} = \begin{bmatrix} 2+\mathrm{i} \\ -1 \end{bmatrix}$$

Solving $(\mathbf{A} + \mathrm{i}\mathbf{I})\,\mathbf{x} = \mathbf{0}$, we have

$$\begin{bmatrix} 2+\mathrm{i} & 2+\mathrm{i} \\ -1 & -1 \end{bmatrix} \mathbf{x} = \mathbf{0} \implies x_1 + x_2 = 0 \implies \mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Note the derivation on L23P16 is applicable to $\mathbb{C}^n$ as well since we didn't assume anything regarding $\mathbf{A}$, thus the general solution is given by

$$\mathbf{x} = C_1 e^{t} \begin{bmatrix} 2+\mathrm{i} \\ -1 \end{bmatrix} + C_2 e^{-\mathrm{i}t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Question12 (1 points)

Solve the given initial value problem, then describe the behaviour of the solution as $t \to \infty$.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 7 \\ 5 \\ 5 \end{bmatrix}$$

Solution:

1M This system is diagonalizable. So the easiest way is to use the formula

$$\mathbf{x} = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + C_n e^{\lambda_n t}\mathbf{v}_n$$

where λ_i are the eigenvalues and $\mathbf{v}_i$ are the corresponding eigenvectors. However, let me show you one more method. Don't blink!

$$\mathcal{L}[\dot{\mathbf{x}}] = \begin{bmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{bmatrix} \mathcal{L}[\mathbf{x}]$$

where $\mathcal{L}$ denotes the Laplace transform operator, that is,

$$\begin{aligned} &\mathcal{L}[\dot{x}_1] = -\mathcal{L}[x_3] & & -7 + sX_1(s) = -X_3(s) \\ &\mathcal{L}[\dot{x}_2] = 2\mathcal{L}[x_1] & \implies & -5 + sX_2(s) = 2X_1(s) \\ &\mathcal{L}[\dot{x}_3] = -\mathcal{L}[x_1] + 2\mathcal{L}[x_2] + 4\mathcal{L}[x_3] & & -5 + sX_3(s) = -X_1(s) + 2X_2(s) + 4X_3(s) \end{aligned}$$

Solving this algebraic system, we have

$$X_1(s) = -\frac{-7s^2 + 33s + 10}{(s^2-1)(s-4)}$$
$$X_2(s) = -\frac{-5s^2 + 6s + 71}{(s^2-1)(s-4)}$$
$$X_2(s) = \frac{5s^2 + 3s + 28}{(s^2-1)(s-4)}$$

Using the final-value theorem, we have

$$\lim_{t\to\infty} \mathbf{x}(t) = \lim_{s\to 0} s\mathbf{X}(s) = \lim_{s\to 0} \begin{bmatrix} 7 - X_3(s) \\ 5 + 2X_1(s) \\ 5 - X_1(s) + 2X_2(s) + 4X_3(s) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -8 \end{bmatrix}$$

Don't panic or roll your eyes! This method is not examinable.

Question13 (1 points)
Find the solution of

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

where the coefficient matrix is $\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 & 0 & 1 \\ -5 & 1 & -4 & 0 & 0 \\ 0 & -2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ which can be diagonalized by

$$\mathbf{P} = \begin{bmatrix} 2 & -1 & -2 & -8 & 0 \\ -10 & 1 & 1 & 15 & 0 \\ -5 & 1 & 2 & 10 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution:

1M Since the coefficient matrix is given to be diagonalizable, the solution must be

$$\mathbf{x} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + c_3 e^{\lambda_3 t}\mathbf{v}_3 + c_4 e^{\lambda_4 t}\mathbf{v}_4 + c_5 e^{\lambda_5 t}\mathbf{v}_5$$

where c_i are arbitrary constants, λ_i are eigenvalues of $\mathbf{A}$ and $\mathbf{v}_i$ are columns of $\mathbf{P}$, thus only the eigenvalues need to be determined. Since $\mathbf{P}$ as well as $\mathbf{A}$ are given

$$\mathbf{PD} = \mathbf{AP} \implies \mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$$

We can use a row of $\mathbf{A}$ and columns of $\mathbf{P}$ to find eigenvalues, for example,

$$\begin{bmatrix} 1 & 2 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathit{2} \\ -1 \\ -5 \\ 0 \\ 3 \end{bmatrix} = \lambda_1 \cdot \mathit{2} \implies \lambda_1 = 0$$

Similarly, we obtain

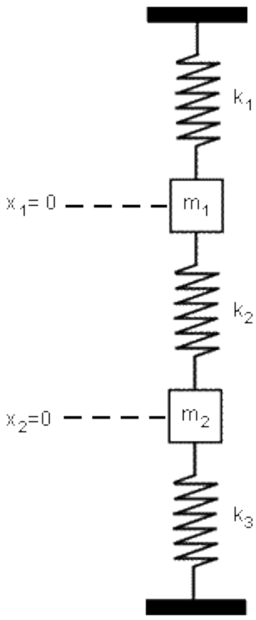
$$\lambda_2 = 2, \qquad \lambda_3 = 3, \qquad \lambda_4 = 1$$

and note λ_5 can be found using the fourth row of $\mathbf{A}$ and the fifth column of $\mathbf{P}$

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathit{1} \\ 0 \end{bmatrix} = \lambda_5 \cdot \mathit{1} \implies \lambda_5 = 1$$

Question14 (1 points)

Consider the spring–mass system beside. The two masses, $m_1 = m_2 = 1$, are constrained by the three springs whose constants are $k_1 = k_2 = k_3 = 1$. If no external force is on the system and no damping force is present, By Newton's second law we can write the following equations for the coordinates x_1 and x_2 of the two masses:



$$\begin{aligned} m_1 \frac{d^2 x_1}{dt^2} &= k_2(x_2 - x_1) - k_1 x_1 \\ &= -(k_1 + k_2)x_1 + k_2 x_2 \end{aligned}$$

$$\begin{aligned} m_2 \frac{d^2 x_2}{dt^2} &= -k_3 x_2 - k_2(x_2 - x_1) \\ &= k_2 x_1 - (k_2 + k_3)x_2. \end{aligned}$$

Solve the system with the initial conditions

$$x_1(0) = 0, \qquad \dot{x_1}(0) = -1, \qquad x_2(0) = 0, \qquad \dot{x_2}(0) = 1$$

Solution:

1M This is very similar to the example in Lecture 23.

$$m_1 \frac{d^2 x_1}{dt^2} = -(k_1 + k_2)x_1 + k_2 x_2$$

$$m_2 \frac{d^2 x_2}{dt^2} = k_2 x_1 - (k_2 + k_3)x_2$$

$$\implies \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -(k_1+k_2) & k_2 \\ k_2 & -(k_2+k_3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\implies \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} 1/m_1 & 0 \\ 0 & 1/m_2 \end{bmatrix} \begin{bmatrix} -(k_1+k_2) & k_2 \\ k_2 & -(k_2+k_3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\implies \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -(k_1+k_2)/m_1 & k_2/m_1 \\ k_2/m_2 & -(k_2+k_3)/m_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Define two more functions

$$x_3 = \dot{x}_1 \qquad \text{and} \qquad x_4 = \dot{x}_2$$

This converts this second-order system intro a first-order system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(k_1+k_2)/m_1 & k_2/m_1 & 0 & 0 \\ k_2/m_2 & -(k_2+k_3)/m_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Eigenvalues are

$$\lambda_{1,2} = \pm\mathrm{i} \qquad \text{and} \qquad \lambda_{3,4} = \pm\mathrm{i}\sqrt{3}$$

and the corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} -\mathrm{i} \\ -\mathrm{i} \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} \mathrm{i} \\ \mathrm{i} \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} \sqrt{3}\mathrm{i} \\ -\sqrt{3}\mathrm{i} \\ -3 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_4 = \begin{bmatrix} -\sqrt{3}\mathrm{i} \\ \sqrt{3}\mathrm{i} \\ -3 \\ 3 \end{bmatrix}$$

Note they appear in complex conjugacy, so we only need to solve two cases.

To find the particular solution that satisfies the initial condition, we solve for c_i

$$\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} = c_1 \mathrm{e}^{\lambda_1 0}\mathbf{v}_1 + c_2 \mathrm{e}^{\lambda_2 0}\mathbf{v}_2 + c_3 \mathrm{e}^{\lambda_3 0}\mathbf{v}_3 + c_4 \mathrm{e}^{\lambda_4 0}\mathbf{v}_4$$

$$\Longrightarrow \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\mathrm{i} & \mathrm{i} & \sqrt{3}\,\mathrm{i} & -\sqrt{3}\,\mathrm{i} \\ -\mathrm{i} & \mathrm{i} & -\sqrt{3}\,\mathrm{i} & \sqrt{3}\,\mathrm{i} \\ 1 & 1 & -3 & -3 \\ 1 & 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \Longrightarrow c_1 = c_2 = 0, \qquad c_3 = c_4 = \frac{1}{6}$$

Note multiplying the first and second equations by i, we can avoid complex algebra. Therefore

$$x_1 = \frac{\sqrt{3}\mathrm{i}}{6}\mathrm{e}^{\sqrt{3}\mathrm{i}t} - \frac{\sqrt{3}\mathrm{i}}{6}\mathrm{e}^{-\sqrt{3}\mathrm{i}t} = -\frac{\sqrt{3}\sin\sqrt{3}t}{3} \qquad \text{and} \qquad x_2 = -\frac{\sqrt{3}\mathrm{i}}{6}\mathrm{e}^{\sqrt{3}\mathrm{i}t} + \frac{\sqrt{3}\mathrm{i}}{6}\mathrm{e}^{-\sqrt{3}\mathrm{i}t} = \frac{\sqrt{3}\sin\sqrt{3}t}{3}$$

Question15 (1 points)

Given the following

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{4t} & 0 \\ te^{2t} & 0 & e^{2t} \end{bmatrix} \qquad \text{for} \qquad \mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Solve the following initial-value problem

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Solution:

1M Note the matrix **A** is not diagonalizable, however, I have given you **A**t, so it is just a matter of writing down the solution and compute a matrix product

$$\mathbf{x} = \mathrm{e}^{\mathbf{A}t}\mathbf{x}_0 = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{4t} & 0 \\ te^{2t} & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{2t} \\ 2\mathrm{e}^{4t} \\ 3\mathrm{e}^{2t} + te^{2t} \end{bmatrix}$$

Note the solution can be written as the following

$$\mathbf{x} = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{4t} & 0 \\ te^{2t} & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \mathrm{e}^{2t}\left(t\underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_2} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{w}_2} \right) + 2\mathrm{e}^{4t}\underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}_1} + 3\mathrm{e}^{2t}\underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_2}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors corresponding to $\lambda_1 = 4$ and $\lambda_2 = 2$, respectively,

$$(\mathbf{A} - 4\mathbf{I})\,\mathbf{v}_1 = \mathbf{0} \qquad \text{and} \qquad (\mathbf{A} - 2\mathbf{I})\,\mathbf{v}_2 = \mathbf{0}$$

while $\mathbf{w}_2$ is known as the generalized eigenvector corresponding to $\lambda_2 = 2$,

$$(\mathbf{A} - 2\mathbf{I})\,\mathbf{w}_2 = \mathbf{v}_2$$

and the decomposition

$$\mathbf{A} = \mathbf{P}\mathbf{J}\mathbf{P}^{-1}$$

is known to be the Jordan form of $\mathbf{A}$, where $\mathbf{P}$ contains eigenvectors and generalized eigenvectors as its columns, and $\mathbf{J}$ is a block diagonal matrix, e.g.

$$\mathbf{A} = \mathbf{P}\mathbf{J}\mathbf{P}^{-1}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Question16 (1 points)
Find the matrix exponential $e^{\mathbf{B}t}$ for

$$\mathbf{B} = \begin{bmatrix} 9 & -5 \\ 0 & 1 \end{bmatrix}$$

Solution:

1M Solving the eigenvalue/eigenvector problem, we have

$$\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \qquad \text{and} \qquad \mathbf{P} = \begin{bmatrix} 5 & 1 \\ 8 & 0 \end{bmatrix}$$

Thus the matrix can be diagonalized and the matrix exponential is given by

$$\mathrm{e}^{\mathbf{B}t} = \mathbf{P}\mathrm{e}^{\mathbf{D}t}\mathbf{P}^{-1} = -\frac{1}{8}\begin{bmatrix} 5 & 1 \\ 8 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{e}^{t} & 0 \\ 0 & \mathrm{e}^{9t} \end{bmatrix} \begin{bmatrix} 0 & -8 \\ -1 & 5 \end{bmatrix} = -\frac{1}{8}\begin{bmatrix} -8\mathrm{e}^{9t} & -5\mathrm{e}^{t} + 5\mathrm{e}^{9t} \\ 0 & -8\mathrm{e}^{t} \end{bmatrix}$$

Question17 (3 points)
Solve the following initial value problems

(a) (1 point)

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t \\ 0 \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solution:

1M This is not diagonalizable, however, it can be solved easily by elimination.

$$x_1 = \mathrm{e}^{t} - t\mathrm{e}^{t} - (t+1) \qquad \text{and} \qquad x_2 = \mathrm{e}^{t}$$

(b) (1 point)

$$\dot{\mathbf{x}} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 4e^{2t} \\ 4e^{4t} \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solution:

1M This is diagonalizable,

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \left(\frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \right)$$

Let us use decoupling on this nonhomogeneous system,

$$\begin{aligned} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \boldsymbol{\beta} \implies \dot{\mathbf{x}} &= \mathbf{P}\mathbf{D}\mathbf{P}^{-1}\mathbf{x} + \boldsymbol{\beta} \\ \implies \mathbf{P}^{-1}\dot{\mathbf{x}} &= \mathbf{D}\mathbf{P}^{-1}\mathbf{x} + \mathbf{P}^{-1}\boldsymbol{\beta} \\ \implies \dot{\mathbf{y}} &= \mathbf{D}\mathbf{y} + \mathbf{P}^{-1}\boldsymbol{\beta} \\ \implies \dot{\mathbf{y}} &= \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \mathbf{y} + \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 4e^{2t} \\ 4e^{4t} \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2\,\mathrm{e}^{2\,t} + 2\,\mathrm{e}^{4\,t} \\ 2\,\mathrm{e}^{4\,t} - 2\,\mathrm{e}^{2\,t} \end{bmatrix} \end{aligned}$$

$$\implies \dot{y_1} = 2y_1 + 2\,\mathrm{e}^{2\,t} + 2\,\mathrm{e}^{4\,t} \qquad \dot{y_2} = 4y_2 + 2\,\mathrm{e}^{4\,t} - 2\,\mathrm{e}^{2\,t}$$

Solve y_1 and y_2 individually, then back transform, we have

$$\begin{aligned} x_1(t) &= \mathrm{e}^{2\,t}\left(2t + \mathrm{e}^{2t}\right) - \mathrm{e}^{4t}\left(2t + \mathrm{e}^{-2t} - 1\right) \\ x_2(t) &= \mathrm{e}^{2t}\left(2t + \mathrm{e}^{2t}\right) + \mathrm{e}^{4t}\left(2t + \mathrm{e}^{-2t} - 1\right) \end{aligned}$$

(c) (1 point)

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solution:

1M This is diagonalizable, however, it involves complex eigenvalues and eigenvectors

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \begin{bmatrix} 2-\mathrm{i} & 2+\mathrm{i} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{bmatrix} \left(\frac{1}{2} \begin{bmatrix} \mathrm{i} & 1-2\mathrm{i} \\ -\mathrm{i} & 1+2\mathrm{i} \end{bmatrix} \right)$$

Since $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, we can simplify the formula,

$$\begin{aligned} \mathbf{x} &= \mathrm{e}^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t \mathrm{e}^{(t-\tau)\mathbf{A}} \boldsymbol{\beta}(\tau)\, d\tau \\ &= \int_0^t \mathrm{e}^{(t-\tau)\mathbf{A}} \boldsymbol{\beta}(\tau)\, d\tau \\ &= \frac{1}{4} \int_0^t \begin{bmatrix} 2-\mathrm{i} & 2+\mathrm{i} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{e}^{-\mathrm{i}(t-\tau)} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}(t-\tau)} \end{bmatrix} \begin{bmatrix} \mathrm{i} & 1-2\mathrm{i} \\ -\mathrm{i} & 1+2\mathrm{i} \end{bmatrix} \begin{bmatrix} -\mathrm{e}^{-\tau\mathrm{i}} - \mathrm{e}^{\tau\mathrm{i}} \\ \mathrm{i}\mathrm{e}^{-\tau\mathrm{i}} - \mathrm{i}\mathrm{e}^{\tau\mathrm{i}} \end{bmatrix} d\tau \end{aligned}$$

After some complex algebra and integration! We have

$$\mathbf{x} = \begin{bmatrix} 2\,t\,\cos(t) - 3\,\sin(t) - t\,\sin(t) \\ t\,\cos(t) - \sin(t) \end{bmatrix}$$

Note we could use the steps in part (b) to solve this system as well.