Transfer Function

Consider the following equation

$$ay'' + by' + cy = f(t) \qquad y(0) = y_0, y'(0) = y_1$$

Using the Laplace transform, we have

$$a\left(s^2Y - sy(0) - y'(0)\right) + b\left(sY - y(0)\right) + cY = F$$

$$(as^2 + bs + c)Y - (as + b)y_0 - ay_1 = F$$

$$Y(s) = \underbrace{\boxed{\frac{(as+b)y_0 + ay_1}{as^2 + bs + c}}}_{Y_c} + \underbrace{\boxed{\frac{F(s)}{as^2 + bs + c}}}_{Y_p}$$

Transfer Function

- Definition

For the initial-value problem $p(D)\, y = f(t)$,

the function $H(s) = \frac{1}{P(s)}$ is called the transfer function.

- For the equation

$$ay'' + by' + cy = f(t) \qquad y(0) = y_0, y'(0) = y_1$$

the transfer function is $\frac{1}{as^2+bs+c}$.

Transfer Function

- The solution for the following equation

$$ay'' + by' + cy = f(t) \qquad y(0) = y_0, y'(0) = y_1$$

is $y(t) = y_c(t) + (h * f)(t)$.

Example 1:

$$y'' + y = \tan(t) \qquad y(0) = 0, y'(0) = 0$$

Solution: $H(s) = \frac{1}{s^2+1}$, so $h(t) = \sin(t)$.

Therefore, $y(t) = \int_0^t \sin(t-\tau) \tan\tau \, d\tau$

Green Function

- Definition
  The green function is the function satisfying

$$G'' + PG' + QG = \delta(x - a)$$

- If P and Q are continuous function of x, then the solution of the differential equation

$$y'' + Py' + Qy = f$$

is

$$\phi = \int_{-\infty}^{\infty} f(a)G(x; a)\, da$$

Green Function

Two properties:

1. G is continuous at x=a
2. G' has a finite jump discontinuity of magnitude 1 at x=a

Reasons:

1. NO $\delta'(x-a)$ on the left side.
2. $$\int_{a-\epsilon}^{a+\epsilon} G''\,dx + \int_{a-\epsilon}^{a+\epsilon} PG'\,dx + \int_{a-\epsilon}^{a+\epsilon} QG\,dx = \int_{a-\epsilon}^{a+\epsilon} \delta(x-a)\,dx \quad \text{as} \quad \epsilon \to 0$$
$$\lim_{x\to a^+} G' - \lim_{x\to a^-} G' = 1$$

Green Function

$$G(x;a) = \begin{cases} A_1\phi_1 + A_2\phi_2 & x < a \\ B_1\phi_1 + B_2\phi_2 & x > a \end{cases}$$

A_1, A_2, B_1, B_2 are constants to be determined.

ϕ_1 and ϕ_2 are two linearly independent solutions of the complementary equation.

1. G is continuous at x=a

   $\rightarrow A_1\emptyset_1 + A_2\emptyset_2 = B_1\emptyset_1 + B_2\emptyset_2$

2. G' has a finite jump discontinuity of magnitude 1 at x=a

   $\rightarrow A_1\emptyset'_1 + A_2\emptyset'_2 + 1 = B_1\emptyset'_1 + B_2\emptyset'_2$

Green Function

$$B_1 - A_1 = -\frac{\phi_2(a)}{W[\phi_1(a), \phi_2(a)]},$$
$$B_2 - A_2 = -\frac{\phi_1(a)}{W[\phi_1(a), \phi_2(a)]}.$$

If we let $A_1 = A_2 = 0$, then

$$G(x;a) = \begin{cases} 0 & x < a \\ \dfrac{-\phi_2(a)\phi_1(x) + \phi_1(a)\phi_2(x)}{W[\phi_1(a), \phi_2(a)]} & x \geq a \end{cases}$$

So $y = \int_{-\infty}^{\infty} f(a)G(x;a)da$

$$= -\phi_1(x)\int_{-\infty}^{x} \frac{f(a)\phi_2(a)}{W[\phi_2(a)\phi_1(x)]}da + \phi_2(x)\int_{-\infty}^{x} \frac{f(a)\phi_1(a)}{W[\phi_2(a)\phi_1(x)]}da$$

Fourier

- We can find a best approximation $\hat{f}$ in a space H for a function $\boldsymbol{f}$ by the method of projection.

$$\hat{f} = \langle f, f_1 \rangle f_1 + \langle f, f_2 \rangle f_2 + \cdots + \langle f, f_n \rangle f_n, \quad \text{where } \langle f, f_k \rangle = \int_a^b f(x) f_k(x)\, dx.$$

- The subspace T_n with the orthonormal basis

$$\mathcal{S} = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \ldots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \ldots, \frac{\sin nx}{\sqrt{\pi}} \right\}$$

can be used to approximate periodic functions.

Fourier

- The approximation of a periodic function f is

$$\hat{f} = \operatorname{proj}_{\mathcal{T}_n} f = s_n = \frac{a_0}{2} + \sum^{n}(a_k \cos kx + b_k \sin kx)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx\, dx, \qquad \text{for } k = 1,\ 2,\ \ldots,\ n.$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx\, dx, \qquad \text{for } k = 1,\ 2,\ \ldots,\ n.$$

- The series on the right is called the Fourier series.

The proof of convergence of Fourier Series

If $f(x)$ and $f'(x)$ are piecewise continuous on $[a,b]$, then

$$\lim_{\lambda\to\infty}\int_a^b f(x)\sin(x\lambda)dx = 0, \qquad \text{and} \qquad \lim_{\lambda\to\infty}\int_a^b f(x)\cos(x\lambda)dx = 0,$$

- $\{x_1 < x_2 < \cdots < x_n\}$ are points where f, f' are defined.

$$\int_a^b f(x)\sin(x\lambda)dx = \sum_{i=1}^{n-1}\int_{x_i}^{x_{i+1}} f(x)\sin(x\lambda)\,dx,$$

$$\lim_{\lambda\to\infty}\int_{x_i}^{x_{i+1}} f(x)\sin(x\lambda)\,dx = \lim_{\lambda\to\infty}\left[\frac{-f(x)\cos(x\lambda)}{\lambda}\right]_{x_i}^{x_{i+1}} + \frac{1}{\lambda}\int_{x_i}^{x_{i+1}} f'(x)\cos(x\lambda)\,dx.$$

$$= 0$$

- Same for $\lim_{\lambda\to\infty}\int_a^b f(x)\cos(x\lambda)dx = 0.$

The proof of convergence of Fourier Series

- We can rewrite the Fourier partial sum

$$s_n = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

in the form of

$$s_n = \frac{\displaystyle\int_{-\pi}^{\pi} f(t) \left[\frac{1}{2} + \sum_{k=1}^{n} \cos(k(t-x)) \right] dt}{\pi} = \frac{\displaystyle\int_{-\pi}^{\pi} f(t) \frac{\sin\left(\left(n+\frac{1}{2}\right)(t-x)\right)}{2\sin\left(\frac{t-x}{2}\right)} dt}{\pi}$$

use the angle difference identity and the identity

$$\frac{1}{\pi}\left(\frac{1}{2} + \sum_{k=1}^{n} \cos(k\alpha)\right) = \frac{\sin((n+\frac{1}{2})\alpha)}{2\pi \sin(\frac{\alpha}{2})}.$$

The proof of convergence of Fourier Series

- We introduce a function $D_n(x)$ according to the new form of s_n.

$$s_n = \frac{\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{k=1}^{n} \cos(k(t-x))\right] dt}{\pi} = \frac{\int_{-\pi}^{\pi} f(t) \frac{\sin\left(\left(n+\frac{1}{2}\right)(t-x)\right)}{2\sin\left(\frac{t-x}{2}\right)} dt}{\pi}$$

The Dirichlet kernel is the collection of functions defined by

Even function

$$D_n(x) = \begin{cases} \dfrac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{2\pi \sin\left(\dfrac{x}{2}\right)} & x \neq 0, \pm 2\pi, \ldots \\ \dfrac{2n+1}{2\pi} & x = 0, \pm 2\pi, \ldots \end{cases}$$

$$\int_{0}^{\pi} D_n(u)\, du = \frac{1}{2}$$

The proof of convergence of Fourier Series

- With the definition of Dirichlet kernel, we can rewrite s_n.

$$s_n = \int_{-\pi}^{\pi} f(t) D_n(t-x)\,dt \overset{u=t-x}{=} \int_{-\pi-x}^{\pi-x} f(u+x) D_n(u)\,du = \int_{-\pi}^{\pi} f(u+x) D_n(u)\,du$$

$$\overset{D_n \text{ is even}}{=} \int_{-\pi}^{\pi} f(t) D_n(x-t)\,dt = \int_{-\pi}^{\pi} f(x-u) D_n(u)\,du$$

$$= \int_{-\pi}^{\pi} \frac{f(x+u)+f(x-u)}{2} D_n(u)\,du$$

$$= \int_{0}^{\pi} (f(x+u)+f(x-u)) D_n(u)\,du$$

The proof of convergence of Fourier Series

- We want to prove that

$$S_f(x) = \begin{cases} f(x) & \text{if } f \text{ is continuous at } x \\ \dfrac{f(x^+) + f(x^-)}{2} & \text{if } f \text{ is discontinuous at } x. \end{cases}$$

- With $\displaystyle\int_0^\pi D_n(u)\,du = \frac{1}{2}$

$$S_f = \frac{f(x^+) + f(x^-)}{2} = \Big(f(x^+) + f(x^-)\Big) \int_0^\pi D_n(u)\,du$$

$$s_n - S_f = \int_0^\pi [f(x+u) + f(x-u)]\, D_n(u)\,du$$

$$- \int_0^\pi [f(x^+) + f(x^-)]\, D_n(u)\,du = \int_0^\pi \Big(\phi_1(u,x) + \phi_2(u,x)\Big) D_n(u)\,du$$

where $\phi_1(u,x) = f(x+u) - f(x^+)$ and $\phi_2(u,x) = f(x-u) - f(x^-)$.

The proof of convergence of Fourier Series

- Then we need to show that

$$\lim_{n\to\infty} \int_0^{\pi} \phi_1(u,x) D_n(u)\, du = 0, \quad \text{and} \quad \lim_{n\to\infty} \int_0^{\pi} \phi_2(u,x) D_n(u)\, du = 0$$

- Let $g(u) = \dfrac{\phi_1(u)}{u}$

g and g' are piecewise continuous on $[0, \pi]$

$$U(u) = \begin{cases} \dfrac{u}{2\pi \sin\left(\frac{u}{2}\right)}, & \text{if } u \neq 0; \\ 1/\pi, & \text{if } u = 0. \end{cases}$$

U and U' are continuous on $[-\pi, \pi]$

$$\lim_{n\to\infty} \int_0^{\pi} g(u) U(u) \sin(n + \frac{1}{2}) u\, du = 0 \quad = \quad \lim_{n\to\infty} \int_0^{\pi} \phi_2(u,x) D_n(u)\, du$$

Recall the theorem proved before

Fourier

- For Fourier series of periodic functions with period $2L$,

$$f(t) = f\left(\frac{L}{\pi}x\right) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

$$= \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos \frac{k\pi}{L}t + b_k \sin \frac{k\pi}{L}t) \qquad \text{where}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}x\right)\, dx = \frac{1}{L}\int_{-L}^{L} f\,(t)\, dt$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}x\right) \cos kx\, dx = \frac{1}{L}\int_{-L}^{L} f\,(t) \cos \frac{k\pi}{L}t\, dt, \qquad \text{for } k = 1,\ 2,\ \ldots,\ n.$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}x\right) \sin kx\, dx = \frac{1}{L}\int_{-L}^{L} f\,(t) \sin \frac{k\pi}{L}t\, dt, \qquad \text{for } k = 1,\ 2,\ \ldots,\ n.$$

Fourier

- For even function

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi t}{L}$$

$$a_0 = \frac{2}{L}\int_0^L f(t)dt \qquad a_k = \frac{2}{L}\int_0^L f(t)\cos\frac{k\pi t}{L}dt$$

- For odd function

$$f(t) = \sum_{k=1}^{\infty} b_k \sin\frac{k\pi t}{L}$$

$$b_k = \frac{2}{L}\int_0^L f(t)\sin\frac{k\pi t}{L}dt$$

$f(x) = -x \quad for \quad -2 \leq x < 0$
$f(x) = x \quad for \quad 0 \leq x < 2$
$f(x+4) = f(x)$ Determine the coeficients in this Fourier series.

- Even function➔$b_k = 0$

$a_0 = 0.5 \times \int_{-2}^{0} -x dx + 0.5 \times \int_0^2 x dx = 2$
$a_k = 0.5 \times \int_{-2}^{0} -x cos(\frac{k\pi x}{2}) dx + 0.5 \times \int_0^2 x \cos(\frac{k\pi x}{2}) dx = -\frac{8}{(k\pi)^2} (k = odd)$

Find $\sum\limits_{k=1}^{\infty} \frac{1}{(2k-1)^2}$, from previous result.

- $s_n = 1 + \sum_{k=1}^{\infty} a_{2k-1} \cos kx = 1 + \sum_{k=1}^{\infty} a_{2k-1} (x = 0)$
- $s_n = 1 - \sum_{k=1}^{\infty} \frac{8}{(2k-1)^2 \pi^2} = 0$
- $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$

Find the Fourier series expression for the following function.

$$f(t) = t, -L < t < L$$

$$f(-L) = f(L)$$

whose period is $2L$.

- Odd function➔$a_0 = a_k = 0$

$$f = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi t}{L}$$

Using the Fourier series to find the stead-state response of the following system.

$$L\frac{\mathrm{d}^2 i}{\mathrm{d}t^2} + R\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{C}i = f(t)$$

where $R = 100\Omega$, $L = 1\mathrm{H}$, $C = 10^{-1}\mathrm{F}$, $f(t)$ is calculated above.

- Steady-state response ➔ find the particular solution
- $f = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi t}{L}$ ➔ the annihilate is $\prod_{n=1}^{\infty}(D + i\frac{n\pi}{L})(D - i\frac{n\pi}{L})$
- Then the particular solution must take the form of

$$y_p = \sum_{n=1}^{\infty}(A_k\cos\frac{n\pi}{L}t + B_k sin\frac{n\pi}{L}t)$$

- Substitute y_p and y_p' into the original equation to determine the coefficients.