Vv256 Lecture 22

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Love instead of hammer or circuit

- The role of eigenvalue and eigenvector in differential equations.

- These examples are taken from,

  Steven H. Strogatz, Nonlinear Dynamics and Chaos, 1994.

Modeling love

- Let $R(t)$ denote Romeo's affection for Juliet at time t, and
  - $R > 0$ corresponds to positive affection, i.e.

    Love.
  - $R < 0$ corresponds to negative affection, i.e.

    Hate.
  - $R = 0$ corresponds to zero affection, i.e.

    No feeling.
- Similarly, $J(t)$ Juliet's affection for Romeo at time t, then the derivatives

$$\frac{dR}{dt} \quad \text{and} \quad \frac{dJ}{dt}$$

give how Romeo and Juliet's feeling for each other are changing at time t.

The secret love model

- Suppose how Romeo and Juliet's feeling for each other changes depending only on themselves and not on how the other feels about them:

  1. The derivative $\dot{R}$ is a function of R, not a function of J.
  2. The derivative $\dot{J}$ is a function of J, not a function of R.

- Specifically, let the instantaneous rate of change in Romeo's love for Juliet be proportional to how much feeling he already has for her, and vice versa

$$\frac{dR}{dt} = \lambda_1 R \qquad \text{and} \qquad \frac{dJ}{dt} = \lambda_2 J$$

- Althought this is a system of linear differential equations, the two equations are totally independent of one another, so we can solve them individually,

$$R = R_0 e^{\lambda_1 t} \qquad \text{and} \qquad J = J_0 e^{\lambda_2 t}$$

  where R_0 and J_0 are two constants representing the initial states.

Exercise

Suppose the initial conditions are given by

$$(R_0, J_0) = (0.5, 1),$$

which corresponds to Romeo having a slight interest in Juliet and Juliet having a moderate interest in Romeo, and God told me that proportionality constants are

$$\lambda_1 = 2 \qquad \textit{and} \qquad \lambda_2 = -1$$

According to this secret love model, what is going to happen in the end?

Solution

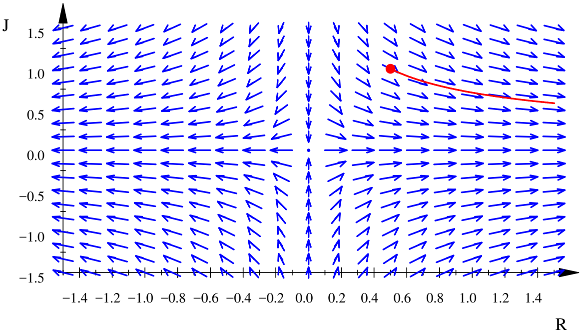
- We could use the exact solution in this case, however, such solutions may not be available for other cases. So let us investigate qualitatively.

$$\begin{aligned} \dot{R} &= \lambda_1 R \\ \dot{J} &= \lambda_2 J \end{aligned} \implies \frac{dJ}{dR} = \frac{\dot{J}}{\dot{R}} = \frac{\lambda_2 J}{\lambda_1 R}$$

Solution

- We can plot what is known as a "phase diagram" or "phase plane".

Phase Plane for the secret love model



- Following the arrows, we can see Juliet's love for Romeo decays over time, and Romeo's love for Juliet grows stronger.

Exercise

What are some of the outcomes of their love under this secret love model?

$$\frac{dR}{dt} = \lambda_1 R \qquad \textit{and} \qquad \frac{dJ}{dt} = \lambda_2 J$$

Solution

- Let's write the original system in matrix form

$$\begin{bmatrix} \dot{R} \\ \dot{J} \end{bmatrix} = \begin{bmatrix} \lambda_1 R \\ \lambda_2 J \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix}$$

- If $\lambda_1 > 0$, and
  - Romeo starts out with some love for Juliet, $R_0 > 0$, then
    Romeo's love for Juliet feeds off itself and grows exponentially.
  - If $R_0 < 0$, then
    Romeo's hate for Juliet feeds off itself and grows exponentially.
- Note λ_1 and λ_2 are the eigenvalues of the matrix representing the system.

Solution

- If $\lambda_1 < 0$, on the other hand, then R decays exponentially, that is,

  Romeo's feeling, either positive or negative, for Juliet gradually wear off.

- Juliet's feelings are similarly determined by λ_2.
- Notice what the eigenspaces of the matrix representing this system,

Mupad

```
A := matrix([[ 2, 0],[0, -1]])
```

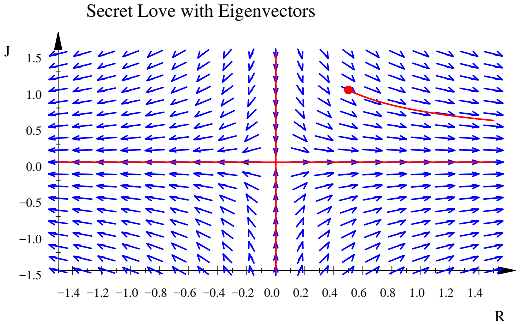
$$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

```
linalg::eigenvalues(A)
```

$\{-1, 2\}$

```
linalg::eigenvectors(A)
```

$$\left[\left[-1, 1, \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]\right], \left[2, 1, \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]\right]\right]$$



A sensitive couple model

- Now let's look at a different model. This time, instead of Romeo and Juliet having a secret love for each other, suppose that they actually talk to each other about their feelings, and they are very easily moved, thus their feelings change entirely based on how the other feels toward them. In other words,

$$\frac{dR}{dt} = aJ \qquad \frac{dJ}{dt} = bR$$

where a and b are positive constants.

- Notice, unlike the last model, we cannot solve the system directly.
- According to this model, Romeo grows fonder of Juliet the more she likes him, and Juliet grows fonder of Romeo the more he likes her. This system can be written in matrix notation as

$$\begin{bmatrix} \dot{R} \\ \dot{J} \end{bmatrix} = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix}$$

- Suppose we want to investigate the long-run behaviour of R and J when
$$a = 4 \qquad \text{and} \qquad b = 9$$
by plotting the phase plane, and working out the eigenvectors of the matrix

Mupad

```
A := matrix([ [ 0, a], [b, 0]])
```

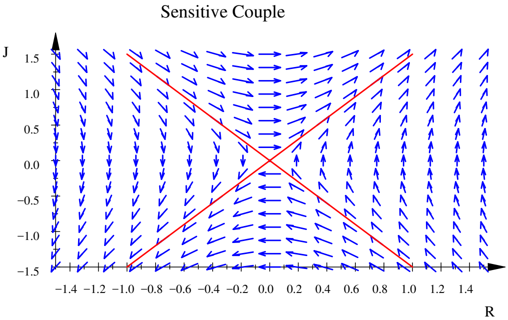
$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$

```
linalg::eigenvalues(A)
```

$$\left\{\sqrt{a}\,\sqrt{b}, -\sqrt{a}\,\sqrt{b}\right\}$$

```
linalg::eigenvectors(A)
```

$$\left[\left[\sqrt{a\,b}, 1, \left[\begin{pmatrix} \frac{\sqrt{a\,b}}{b} \\ 1 \end{pmatrix}\right]\right], \left[-\sqrt{a\,b}, 1, \left[\begin{pmatrix} -\frac{\sqrt{a\,b}}{b} \\ 1 \end{pmatrix}\right]\right]\right]$$



Two equally cautious lovers

- Let us now consider the system

$$\begin{bmatrix} \dot{R} \\ \dot{J} \end{bmatrix} = \begin{bmatrix} -a & b \\ b & -a \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix}, \qquad \text{where } a > 0 \text{ and } b > 0.$$

Q: What does this model capture in terms of Romeo and Juliet's feelings?

- Romeo and Juliet are both afraid of their own feelings toward the other, thus the more Romeo loves Juliet the more he pulls back, and similarly for Juliet.
- But they respond positively to other's affection.
- If we solve eigenvalue/eigenvector problem for this system, we find

$$\begin{aligned} \lambda_1 &= b - a & \qquad & \lambda_2 = -a - b \\ \mathbf{x}_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \text{and} \qquad & \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

- If $b > a$, which corresponds to Romeo and Juliet being more sensitive to each other's feelings than to their own, then

We have one positive eigenvalue and one negative eigenvalue

Mupad

```
a := 4: b := 9:
A := matrix([ [ -a, b], [b, -a]])
```

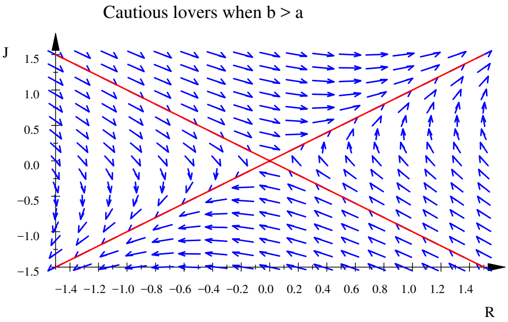
$$\begin{pmatrix} -4 & 9 \\ 9 & -4 \end{pmatrix}$$

```
linalg::eigenvalues(A)
```

$$\{-13, 5\}$$

```
linalg::eigenvectors(A)
```

$$\left[\left[-13, 1, \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right]\right], \left[5, 1, \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]\right]\right]$$



- So they will either end up loving each other or hating each other in this case.

- If $a > b$, which means that Romeo and Juliet are both more sensitive to their own feelings than to each others, then

We have two negative eigenvalue, the origin is the attractor

Mupad

```
a := 9: b := 4:
A := matrix([ [ -a, b], [b, -a]])
```

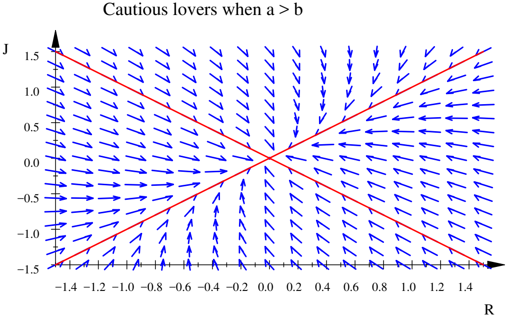
$$\begin{pmatrix} -9 & 4 \\ 4 & -9 \end{pmatrix}$$

```
linalg::eigenvalues(A)
```

$\{-13, -5\}$

```
linalg::eigenvectors(A)
```

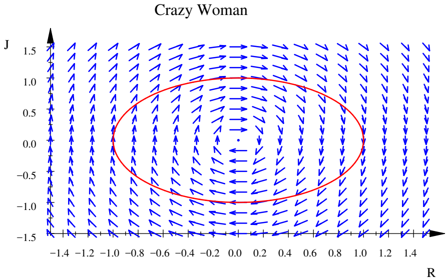
$$\left[\left[-13, 1, \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right]\right], \left[-5, 1, \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]\right]\right]$$



- It is very very very sad that, in this case, time kills any feeling between Romeo and Juliet, no matter where they start. Don't let yourself dominate!

A crazy woman model

- Now consider the system of equations, $\begin{aligned} \dot{R} &= aJ \\ \dot{J} &= -bR \end{aligned}$ where $a > 0$ and $b > 0$.



- In this case, Romeo responds positively to Juliet's affection for him, but Juliet, a crazy woman, likes Romeo more when Romeo dislikes her, and conversely, she likes Romeo less when Romeo likes her more.

- Writing the system that led to this phase portrait in matrix form,

$$\begin{bmatrix} \dot{R} \\ \dot{J} \end{bmatrix} = \begin{bmatrix} 0 & a \\ -b & 0 \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix}.$$

- This leads to pure imaginary eigenvalues,

$$\lambda = \pm i\sqrt{ab}$$

this helps to explain why there is no asymptote in our phase diagram.

- It can be shown that this system has the following analytic solutions

$$e^{i\sqrt{ab}t} \qquad \text{and} \qquad e^{-i\sqrt{ab}t}$$

- Using Euler's formula, we can rewrite our solution in terms of

$$\cos(\sqrt{ab}t) \qquad \text{and} \qquad \sin(\sqrt{ab}t).$$

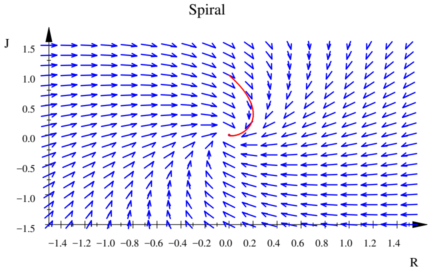
A spiral

- To conclude, consider the system

$$\begin{bmatrix} \dot{R} \\ \dot{J} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix}.$$

- In this case

$$\lambda = -\frac{3}{2} \pm i\frac{\sqrt{3}}{2}.$$



both of which contain a non-zero real part and non-zero imaginary part.

- It can be shown that this system has the following analytic solutions

$$e^{-3t/2}\cos(t\sqrt{3}/2) \qquad \text{and} \qquad e^{-3t/2}\sin(t\sqrt{3}/2)$$

- The fact that the real part is negative corresponds to exponential decay. If the real part had been positive, the spirals would move outward as time goes

- First-order linear system of differential equations with constant coefficients is

$$\frac{dx_i}{dt} = \dot{x}_i = \sum_{j=1}^{n} a_{ij}x_j + \beta_i, \qquad i = 1, 2, \ldots, n.$$

where x_i are unknown functions of the variable t, which often denote time, and a_{ij} are certain constant coefficients, and β_i are given functions of t.

- We assume that all these functions are continuous on an interval $[a, b]$ of t.
- The system of differential equations can be written in matrix form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \boldsymbol{\beta}, \qquad \text{where}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}, \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{11} & a_{12} & \cdots & a_{nn} \end{bmatrix}.$$

- If the vector $\boldsymbol{\beta}$ is identically zero, then the system is said to be homogeneous

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

- Recall for the secret love model is a linear homogeneous system of equations,

$$\begin{bmatrix} \dot{R} \\ \dot{J} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix}$$

we can solve R and J since the two functions do not depend on each other.

$$\dot{R} = \lambda_1 R \implies R = R_0 e^{\lambda_1 t} \qquad \text{and} \qquad \dot{J} = \lambda_2 J \implies J = J_0 e^{\lambda_2 t}$$

- However, for the sensitive couple and the equally cautious lovers,

$$\begin{bmatrix} \dot{R} \\ \dot{J} \end{bmatrix} = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} \dot{R} \\ \dot{J} \end{bmatrix} = \begin{bmatrix} -a & b \\ b & -a \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix}$$

where a and b are both positive constants.

Q: How can we solve linear homogeneous systems of equations in general?

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

Elimination method

- For a simple homogeneous system, often for a 2×2 system, we can reduce the system to a single linear equation of nth order. For example,

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 \quad (1)$$
$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 \quad (2)$$

where x_1 and x_2 are functions of the variable t.

- Differentiate eq (1) and substitute the derivative $\dot{x}_2$ from eq (2)

$$\ddot{x}_1 = a_{11}\dot{x}_1 + a_{12}\dot{x}_2 \implies \ddot{x}_1 = a_{11}\dot{x}_1 + a_{12}\left(a_{21}x_1 + a_{22}x_2\right)$$

- Now we substitute $a_{12}x_2$ from equation (1).

$$\ddot{x}_1 = a_{11}\dot{x}_1 + a_{12}a_{21}x_1 + a_{22}\left(a_{12}x_2\right) = a_{11}\dot{x}_1 + a_{12}a_{21}x_1 + a_{22}\left(\dot{x}_1 - a_{11}x_1\right)$$

- As a result we obtain a second-order linear homogeneous equation:

$$\ddot{x}_1 - \left(a_{11} + a_{22}\right)\dot{x}_1 + \left(a_{11}a_{22} - a_{12}a_{21}\right)x_1 = 0$$

- Once x_1 is determined, another function x_2 can be found from eq (1).

Exercise

Solve the system of differential equations $\begin{aligned}\dot{x_1} &= 2x_1 + 3x_2 \\ \dot{x_2} &= 4x_1 - 2x_2\end{aligned}$ by elimination.

Solution

- Differentiate the first and make a substitution using the second,

$$\ddot{x_1} = 2\dot{x_1} + 3\dot{x_2} = 2\dot{x_1} + 3\left(4x_1 - 2x_2\right) = 2\dot{x_1} + 12x_1 - 2 \cdot 3x_2$$

- Rearrange the first equation, and substitute into the last equation,

$$\ddot{x_1} - 16x_1 = 0 \implies r^2 = 16 \implies r = \pm 4 \implies x_1 = c_1 e^{4t} + c_2 e^{-4t}$$

  where c_1 and c_2 are arbitrary constants.

- Now compute $\dot{x}_1$, and plug both x_1 and $\dot{x}_1$ into the first equation, we have

$$x_2 = \frac{2}{3} c_1 e^{4t} - 2c_2 e^{-4t}.$$

- Recall there seems to exist some kind of connection between the solutions to

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

and eigenvalue problem of the coefficient matrix

$$\mathbf{A}$$

Mupad

```
A := matrix([[ 2, 0],[0, -1]])
```

$$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

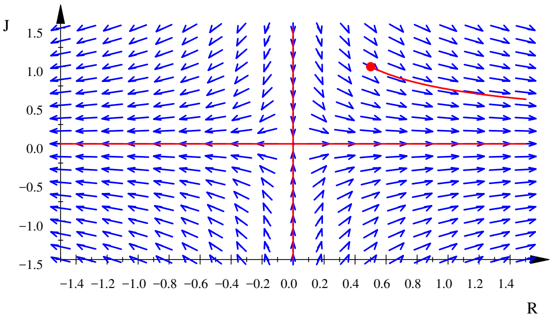
```
linalg::eigenvalues(A)
```

$$\{-1, 2\}$$

```
linalg::eigenvectors(A)
```

$$\left[\left[-1, 1, \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]\right], \left[2, 1, \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]\right]\right]$$

Secret Love with Eigenvectors



Mupad

```
a := 9: b := 4:
```

```
A := matrix([ [ -a, b], [b, -a]])
```

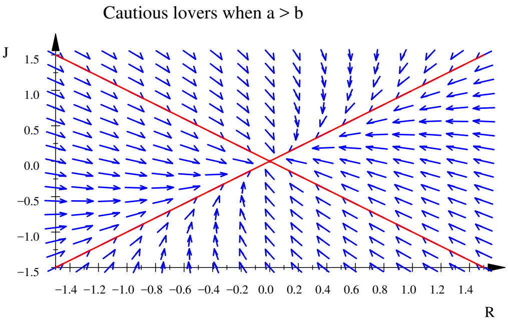
$$\begin{pmatrix} -9 & 4 \\ 4 & -9 \end{pmatrix}$$

```
linalg::eigenvalues(A)
```

$$\{-13, -5\}$$

```
linalg::eigenvectors(A)
```

$$\left[\left[-13, 1, \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right]\right], \left[-5, 1, \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]\right]\right]$$



- This is even more striking when we consider the system we have just solved

$$\begin{aligned} \dot{x_1} &= 2x_1 + 3x_2 \\ \dot{x_2} &= 4x_1 - 2x_2 \end{aligned} \implies \dot{\mathbf{x}} = \begin{bmatrix} 2 & 3 \\ 4 & -2 \end{bmatrix} \mathbf{x}$$

Mupad

```
A := matrix([ [ 2, 3], [4, -2]])
```

$$\begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix}$$

```
linalg::eigenvalues(A)
```

$$\{-4, 4\}$$

```
linalg::eigenvectors(A)
```

$$\left[\left[4, 1, \left[\begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}\right]\right], \left[-4, 1, \left[\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}\right]\right]\right]$$

- If we rewrite our solution, we have

$$\begin{aligned} x_1 &= c_1 e^{4t} + c_2 e^{-4t} \\ x_2 &= \frac{2}{3} c_1 e^{4t} - 2 c_2 e^{-4t} \\ \implies \mathbf{x} &= c_1 e^{4t} \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ &= c_1 e^{4t} \mathbf{x}_1 + c_2 e^{-4t} \mathbf{x}_2 \end{aligned}$$

Q: Why the solution is in terms of eigenvalues and eigenvectors?

- We have been solving the vector and matrix equations using the 1st way

$$\begin{aligned} 4y + z &= 5 \\ 3x + 8y + z &= 9 \\ x + 2y + z &= 3 \end{aligned} \quad x\begin{bmatrix}0\\3\\1\end{bmatrix} + y\begin{bmatrix}4\\8\\2\end{bmatrix} + z\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}5\\9\\3\end{bmatrix} \quad \begin{bmatrix}0&4&1\\3&8&1\\1&2&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}5\\9\\3\end{bmatrix}$$

Q: Is there any way to solve the matrix equation directly?

$$\mathbf{Ax} = \mathbf{b} \implies \mathbf{BAx} = \mathbf{Bb} \implies \mathbf{Ix} = \mathbf{Bb} \implies \mathbf{x} = \mathbf{Bb}$$

Definition

If $\mathbf{A}$ is a square matrix, and if a matrix $\mathbf{B}$ of the same size can be found such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I},$$

then $\mathbf{A}$ is said to be invertible (or nonsingular) and $\mathbf{B}$ is called an inverse of $\mathbf{A}$.

If no such matrix $\mathbf{B}$ can be found, then $\mathbf{A}$ is said to be singular.

Q: Can an invertible matrix have more than one inverse?

- Suppose $\mathbf{B}$ and $\mathbf{C}$ are both inverse of $\mathbf{A}$.

$$\mathbf{BA} = \mathbf{I} \implies \mathbf{BAC} = \mathbf{IC} = \mathbf{C} \implies \mathbf{B}\left(\mathbf{AC}\right) = \mathbf{C} \implies \mathbf{BI} = \mathbf{C} \implies \mathbf{B} = \mathbf{C}$$

- So we can now speak of "the" inverse of an invertible matrix.

Theorem

If $\mathbf{B}$ and $\mathbf{C}$ are both inverses of the matrix $\mathbf{A}$, then $\mathbf{B} = \mathbf{C}$.

- If $\mathbf{A}$ is invertible, then its inverse will be denoted by

$$\mathbf{A}^{-1}$$

- And by definition,

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

Q: How can we determine whether $\mathbf{A}$ is invertible, and how can we find $\mathbf{A}^{-1}$?

Theorem

If $\mathbf{A}$ is an invertible matrix, then

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}}$$

where $\mathbf{C}^{\mathrm{T}}$ is the transpose of the matrix of cofactors from $\mathbf{A}$,

Exercise

Find the inverse of the coefficient matrix, and thus solve the linear system,

$$\begin{bmatrix} 0 & 4 & 1 \\ 3 & 8 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 3 \end{bmatrix}$$

Solution

- Find the cofactor matrix $\mathbf{C}$,

$$C_{ij} = (-1)^{i+j} \det(\mathbf{M}_{ij})$$

Solution

- Recall $\det(\mathbf{M}_{ij})$ is the determinant of the submatrix of $\mathbf{A}$ after ith row and jth column are removed. Thus the cofactor matrix here is given by

$$\mathbf{C} = \begin{bmatrix} 6 & -2 & -2 \\ -2 & -1 & 4 \\ -4 & 3 & -12 \end{bmatrix}$$

- Compute the determinant,

$$\det(\mathbf{A}) = 0 \cdot 6 + 3 \cdot (-2) + 1 \cdot (-4) = -10$$

- Having a zero determinant is the only way that the inverse of a matrix is undefined, so the determinant tells whether the inverse exists or not.

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{-10}\begin{bmatrix} 6 & -2 & -4 \\ -2 & -1 & 3 \\ -2 & 4 & -12 \end{bmatrix}$$