Vv156 Lecture 14

Dr Jing Liu

UM-SJTU Joint Institute

November 1, 2016

- In many problems, the properties of interest in the graph of a function are:

Properties of curves

- symmetries
- x-intercepts
- relative extrema
- intervals of increase and decrease
- asymptotes
- periodicity
- y-intercept
- concavity
- inflection points
- behavior as $x \to \infty$ or as $x \to -\infty$

- Some of these properties may not be relevant in certain cases. For example,

asymptotes are characteristic of rational functions

$$\lim_{x \to a} f(x) = \infty \qquad \text{or} \qquad \lim_{x \to a} f(x) = -\infty$$

$$\lim_{x \to \infty} f(x) = b \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = b$$

- Here we discuss how to find the features of polynomial and rational functions

$$y = P_n(x) \qquad \text{and} \qquad y = \frac{P(x)}{Q(x)}$$

however, the similar procedures can be used for other functions.

Exercise

Sketch a graph for each of the following functions and specify the locations of the intercepts, relative extrema, inflection points and asymptotes.

(a) *Polynomial function*

$$y = x^3 - 3x + 2$$

(b) *Rational function*

$$y = \frac{x^2 - 1}{x^3}$$

- Rational functions of which the degree of P did not exceed the degree of Q,

$$f(x) = \frac{P(x)}{Q(x)}$$

have either vertical asymptotes or horizontal asymptotes.

- If P of a rational function has greater degree than Q, then other "asymptotes" are possible. For example, consider the rational function

$$f(x) = \frac{x^2 + 1}{x}$$

- By division we can rewrite it as

$$f(x) = x + \frac{1}{x}$$

- The second terms approach 0 as $x \to \infty$ or as $x \to -\infty$, then

$$\Big(f(x) - x\Big) \to 0$$

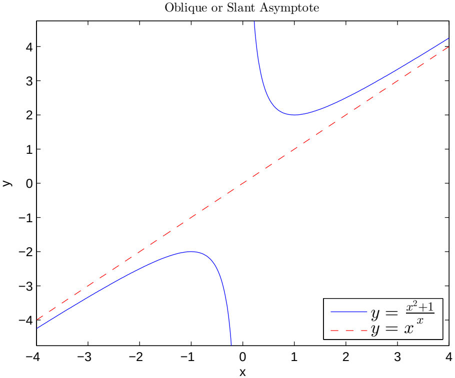
- Geometrically, this means that the graph of

$$y = f(x) \text{ and the line } y = x \text{ eventually gets closer and closer}$$

as $x \to \infty$ or as $x \to -\infty$.

Definition

The line $y = x$ is called an oblique or slant asymptote of f.



- Similarly, consider the rational function

$$g(x) = \frac{x^3 - x^2 - 8}{x - 1}$$

we can rewrite it as

$$g(x) = x^2 - \frac{8}{x - 1}$$

The second terms approach 0 as $x \to \infty$ or as $x \to -\infty$, then

$$\left(g(x) - x^2\right) \to 0$$

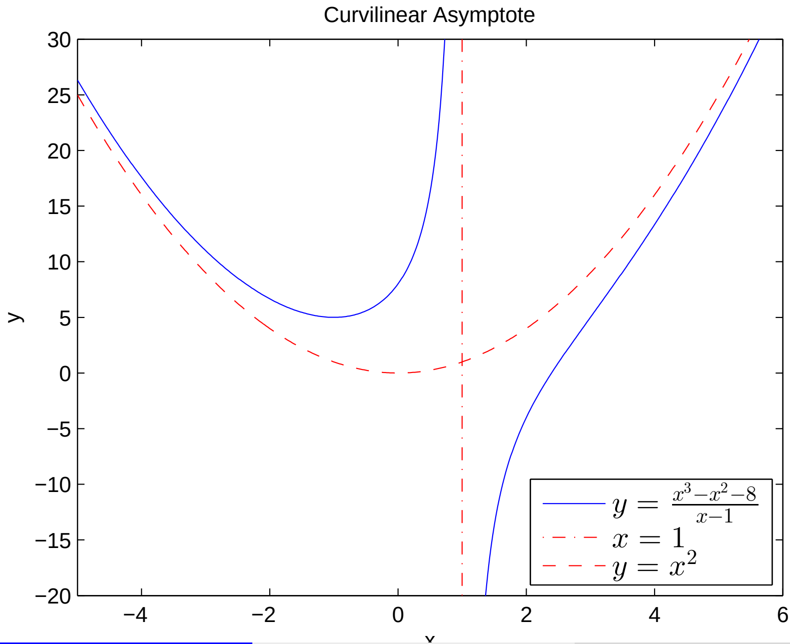
So the graph of

$y = g(x)$ eventually gets closer and closer to the parabola $y = x^2$

as $x \to \infty$ or as $x \to -\infty$.

Definition

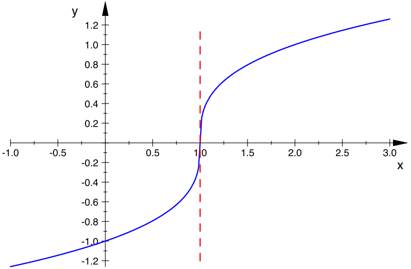
The parabola is called a curvilinear asymptote of g.


Curvilinear Asymptote
30
25
20
15
10
5
0
−5
−10
−15
−20
y
−4
−2
0
2
4
6
x
$y = \frac{x^3 - x^2 - 8}{x - 1}$
$x = 1$
$y = x^2$

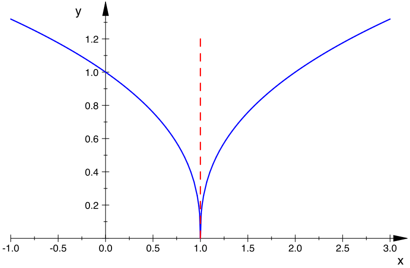
- Vertical tangents are commonly found in graphs of functions

$$f(x) = (x-a)^{p/q}$$

that involve radicals or fractional exponents.



$$y = (x-1)^{1/3}$$



$$(x-1)^{2/5}$$

- Differentiation can be used to solve various optimization problems.

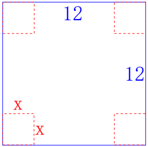
1. Maximizing or minimizing a continuous function over a finite closed interval.

Exercise

An open-top box is to be made by cutting small congruent squares from the corner of a 12cm-by-12cm sheet of tin and bending up the sides. How large should the squares cut from the corner be to make the box hold as much as possible?

Solution

- Produce a sketch,



$$V(x) = (12 - 2x)^2 x$$
$$= 144x - 48x^2 + 4x^3, \qquad 0 \le x \le 6$$

- V is continuous in the closed interval $[0, 6]$, so EVT guarantees that there is an absolute maximum value of V in $[0, 6]$.

Solution

- Find critical points

$$\begin{aligned} V'(x) &= 144 - 96x + 12x^2 \\ &= 12(2-x)(6-x) \\ &\implies x = 2; \quad x = 6 \end{aligned}$$

- Evaluate the critical points and end points

$$V(x) = (12-2x)^2 x$$

$$\implies V(0) = 0, \qquad V(2) = 128, \qquad V(6) = 0$$

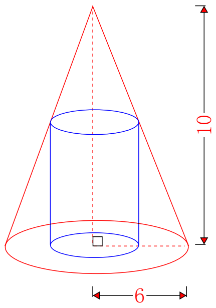
- Thus the maximum volume is 128cm^3, and the squares are 2cm-by-2cm.

Exercise

Find the radius and height of the right circular cylinder of largest volume that can be inscribed in a right circular cone with radius 6cm and height 10cm.

Solution

- Produce a sketch,



$r=$radius of the cylinder

$h=$height of the cylinder

$V=$volume of the cylinder

- Find V as a function of only one variable,

$$V = \pi r^2 h$$

- Similar triangles implies

$$\frac{10-h}{r} = \frac{10}{6} \implies h = 10 - \frac{5}{3}r$$

$$\implies V = \pi r^2 (10 - \frac{5}{3}r), \qquad 0 \le r \le 6$$

Solution

- V is continuous in the closed interval $[0,6]$, so EVT guarantees that there is an absolute maximum value of V in $[0,6]$.
- Find critical points

$$\begin{aligned} V' &= 20\pi r - 5\pi r^2 \\ &= 5\pi r(4-r) \end{aligned}$$

$$\implies r = 0; \quad r = 4$$

- Evaluate the critical points and end points

$$V(r) = \pi r^2(10 - \frac{5}{3}r)$$
$$\implies V(0) = 0, \qquad V(4) = \frac{160}{3}\pi, \qquad V(6) = 0$$

- So the maximum volume is $\frac{160}{3}\pi \text{cm}^3$, and this happens when $r = 4$.

2. Maximizing or minimizing a continuous function over a set that is not compact.

Exercise

A cylindrical can is to be made to hold 1L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

Solution

- Let h, r, S be the height, the radius and the surface area of the can respectively.
- Assume there is no waste or overlap, we need to minimise the surface area

$$S = 2\pi r^2 + 2\pi rh$$

- The volume of the can needs to be $1\text{L} = 1000\ \text{cm}^3$, so h in terms of r is

$$1000 = \pi r^2 h \implies h = \frac{1000}{\pi r^2} \implies S = 2\pi r^2 + \frac{1000}{r}$$

- Thus we have reduced the problem to finding a value of r in the interval $[0, \infty)$ for which S is a minimum. So EVT is NOT applicable here, however

$$S' = 4\pi r - 2000r^{-2} = 2r^{-2}(2\pi r^3 - 1000)$$

Solution

- The critical points are at $r = 0$ and $r = \dfrac{10}{\sqrt[3]{2\pi}}$, by the first derivative test,

$$\begin{array}{ccc} r < 0 & S' < 0 & \text{decreasing} \\ 0 < r < \dfrac{10}{\sqrt[3]{2\pi}} & S' < 0 & \text{decreasing} \\ \dfrac{10}{\sqrt[3]{2\pi}} < r & S' > 0 & \text{increasing} \end{array}$$

- Hence

$$r = \frac{10}{\sqrt[3]{2\pi}}$$

gives a local minimum.

- Therefore

$$h = \frac{1000}{\pi r^2} = \frac{20}{\sqrt[3]{2\pi}} \qquad \text{and} \qquad r = \frac{10}{\sqrt[3]{2\pi}}$$

is the dimension of the can that minimises the surface area, and thus the cost.

- The speed of light depends on the medium through which it travels and tends to be slower in denser media.
- In a vacuum, it travels at the famous speed $c = 3x10^8$m/sec, but in the earth's atmosphere it travels slightly slower than that, and even slower in glass.

Fermat's principle of least time

light travels from one point to another along a path for which the time of travel is a minimum.

Exercise

Find the path that a ray of light will follow in going from a point A in a medium where the speed of light is c_1 across a straight boundary to a point B in another medium where the speed of light is c_2.

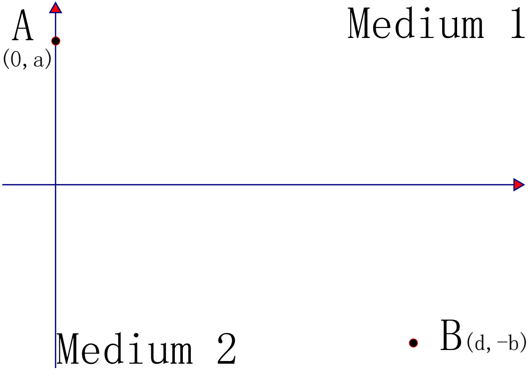
Solution

- According to Fermat's principle, we should minimise the time of travel,

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

Solution

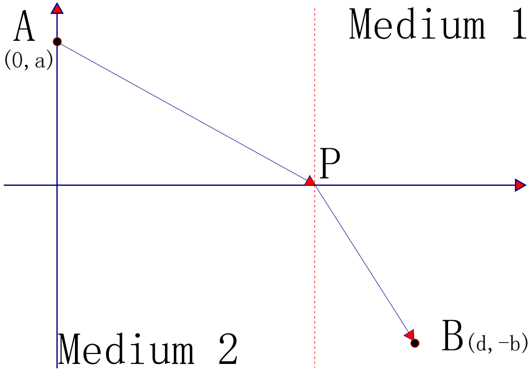
- Suppose that A and B lie in the xy-plane and that the line separating the two media is x-axis



Solution

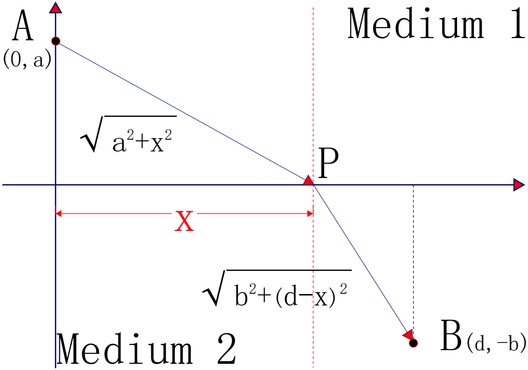
- In a uniform medium, where the speed of light remain constant, "shortest time" means "shortest path", and the ray of light will follow a straight line. Hence

> *the path from A to B will consist of a line segment from A to a boundary point P, followed by another line segment from P to B.*



Solution

- Let x be the x-coordinate of P, then



- The times required for light from A to P and from P to B, respectively, are

$$t_1 = \frac{AP}{c_1} = \frac{\sqrt{a^2+x^2}}{c_1}, \qquad \text{and} \qquad t_2 = \frac{PB}{c_2} = \frac{\sqrt{b^2+(d-x)^2}}{c_2}$$

Solution

- So from A and to B is

$$t = t_1 + t_2 = \frac{\sqrt{a^2 + x^2}}{c_1} + \frac{\sqrt{b^2 + (d-x)^2}}{c_2}$$

- This expresses t as a differentiable function of x for $0 \le x \le d$, and

$$\frac{dt}{dx} = \frac{x}{c_1\sqrt{a^2 + x^2}} - \frac{d-x}{c_2\sqrt{b^2 + (d-x)^2}}$$

  is continuous, and is negative at $x = 0$ and is positive $x = d$.

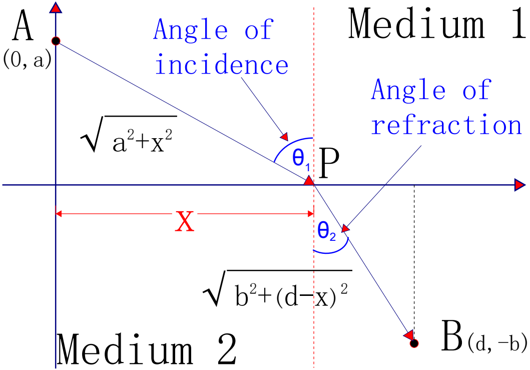
- Therefore IVT guarantees there is a point between 0 and d such that

$$0 = \frac{x}{c_1\sqrt{a^2 + x^2}} - \frac{d-x}{c_2\sqrt{b^2 + (d-x)^2}}$$

- There is only one such point since $\dfrac{d^2t}{dx^2} > 0$ for $0 < x < d$.

Solution

- In terms of angles, θ_1 and θ_2



- we have

$$\frac{\sin\theta_1}{c_1} - \frac{\sin\theta_2}{c_2} = 0$$

which is known as the Snell's law or the law of refraction.