# VE401 MID II

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I. Hypothesis testing

a. Fisher's null hypothesis testing (only to reject $H_0$)

i. Z-test

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

Use P-value to test

b. Neyman-Pearson decision theory (may reject $H_0$ and take $H_1$)

i. $H_0$ and $H_1$ are not necessarily mutual exclusive.

ii. Error

1) Type I error (reject $H_0$ thought it is true) (denoted by $\alpha$);

2) Type II error (fail to reject $H_0$ when it is false) (denoted by β)

3) Power:= 1 − β

a) Calculate $\beta$ for normal distribution

$$\beta = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-z_\beta} e^{-t^2/2} dt$$

$$n \approx \frac{\left(z_{\frac{\alpha}{2}} + z_\beta\right)^2 \sigma^2}{\delta^2}$$

4) Power is not P-value!

5) $\alpha + \beta$ may not necessarily equal 1!

iii. Acceptance sampling

Producer's risk: $\alpha$; consumer's risk: $\beta$

c. Null Hypothesis Significance Testing (NHST)

i. $H_1$ is always the logical negation of $H_0$

II. Tests

a. T-test

(test for the mean when normal distribution)

i. Statistics:

$$T_{n-1} = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$$

ii. Reject $H_0$ if ...

- $H_0: \mu = \mu_0,\qquad \left(|T_{n-1}| > t_{\frac{\alpha}{2},n-1}\right)$
- $H_0: \mu \le \mu_0,\qquad (T_{n-1} > t_{\alpha,n-1})$
- $H_0: \mu \ge \mu_0,\qquad (T_{n-1} < -t_{\alpha,n-1})$

iii. Parameter for OC curve of the test

$$d = \frac{|\mu - \mu_0|}{\sigma}$$

b. Chi-squared test

(test for the variance when normal distribution)

i. Statistic

$$\chi^2_{n-1} = \frac{(n-1)S^2}{\sigma_0^2}$$

ii. Reject $H_0$ if ...

- □ $H_0: \sigma = \sigma_0$, $\left(\chi^2_{n-1} > \chi^2_{\alpha/2,n-1} \text{ or } \chi^2_{n-1} < \chi^2_{1-\alpha/2,n-1}\right)$
- □ $H_0: \sigma \le \sigma_0$, $(\chi^2_{n-1} > \chi^2_{\alpha,n-1})$
- □ $H_0: \sigma \ge \sigma_0$, $(\chi^2_{n-1} < \chi^2_{\alpha,n-1})$

iii. Parameter for OC curve of the test

$$\lambda = \frac{\sigma}{\sigma_0}$$

c. Sign test
(test for the median when random sample)

i. Statistics

$$Q_+ = \text{number of } X_k \text{ when } X_k - M_0 > 0$$
$$Q_- = \text{number of } X_k \text{ when } X_k - M_0 < 0$$

ii. Reject $H_0$ if …

- □ $H_0$: $M \le M_0$ if $Q_-$ is too small to have occurred by chance
- □ $H_0$: $M \ge M_0$ if $Q_+$ is too small to have occurred by chance
- □ $H_0$: $M = M_0$ if $Q := \min(Q_-, Q_+)$ is too small to have occurred by chance

iii. P-value is calculated by binomial distribution. When it is a two tail test, remember to time two!

d. Wilcoxon signed rank test
(test for the median when random sample from a continuous symmetric distribution)

i. Statistics

$$W_+ = \sum_{R_i > 0} R_i$$

$$W_- = \sum_{R_i < 0} |R_i|$$

ii. Reject $H_0$ if …

- □ $H_0$: $M \le M_0$ if $W_-$ is too small to have occurred by chance
- □ $H_0$: $M \ge M_0$ if $W_+$ is too small to have occurred by chance
- □ $H_0$: $M = M_0$ if $W := \min(W_-, W_+)$ is too small to have occurred by chance

e. Test for proportion

i. Test (for large-sample)

$$H_0: p = p_0$$

ii. Statistic

$$Z = \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}}, \qquad \hat{p} = \bar{X}$$

iii. $100(1-\alpha)\%$ confidence interval for $p$:

$$\hat{p} \pm z_{\alpha/2}\sqrt{p(1-p)/n}$$

Then $d = z_{\alpha/2}\sqrt{p(1-p)/n}$

Sample size $n = \frac{z^2_{\alpha/2}}{4d^2}$ when given $d$

iv. Reject $H_0$ and in favor of $H_1$ if ...

- □ $H_1: p \neq p_0,\qquad (|Z| > z_{\alpha/2})$
- □ $H_1: p > p_0,\qquad (Z > z_\alpha)$
- □ $H_1: p < p_0,\qquad (Z < -z_\alpha)$

III. Comparing two statistics

a. Comparing two proportions (both approximately normally distributions)

i. Test

$$H_0: p_1 - p_2 = (p_1 - p_2)_0$$

ii. Statistic

$$Z = \frac{\widehat{p_1} - \widehat{p_2} - (p_1 - p_2)_0}{\sqrt{\frac{\widehat{p_1}(1-\widehat{p_1})}{n_1} + \frac{\widehat{p_2}(1-\widehat{p_2})}{n_2}}}$$

iii. Reject $H_0$ and in favor of $H_1$ if ...

- □ $H_1: p_1 - p_2 \neq (p_1 - p_2)_0\,,\qquad (|Z| > z_{\alpha/2})$
- □ $H_1: p_1 - p_2 > (p_1 - p_2)_0,\qquad (Z > z_\alpha)$
- □ $H_1: p_1 - p_2 < (p_1 - p_2)_0,\qquad (Z < -z_\alpha)$

b. Pooled large sample test for equality of proportions (both Bernoulli distributions)

i. Test

$$H_0: p_1 - p_2 = 0$$

ii. Statistics

$$Z = \frac{\widehat{p_1} - \widehat{p_2}}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n_1} + \frac{\hat{p}(1-\hat{p})}{n_2}}}, \qquad \hat{p} = \frac{n_1\widehat{p_1} + n_2\widehat{p_2}}{n_1 + n_2}$$

c. Comparing two means (variance known)

i. For OC curves

$$\mathrm{d} = \frac{|\mu_1 - \mu_2|}{\sqrt{\sigma_1^2 + \sigma_2^2}}, \qquad \text{when } n_1 = n_2 = n$$

$$n = \frac{\sigma_1^2 + \sigma_2^2}{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}, \qquad \text{when } n_1 \neq n_2$$

ii. Pooled estimator: (equal variances)

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

iii. $100(1-\alpha)\%$ confidence interval for $\mu_1 - \mu_2$:

$$(\overline{X_1} - \overline{X_2}) \pm t_{\frac{\alpha}{2}, n_1+n_2-2}\sqrt{S_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

iv. Pooled $T$-test

(when variances are independent and equal)

1) Test

$$H_0:\ \mu_1 - \mu_2 = (\mu_1 - \mu_2)_0$$

2) Statistic

$$T_{n_1+n_2-2} = \frac{(\overline{X_1} - \overline{X_2}) - (\mu_1 - \mu_2)_0}{\sqrt{S_p^2(\frac{1}{n_1} + \frac{1}{n_2})}}$$

3) Reject $H_0$ and in favor of $H_1$ if ...

- $H_1: \mu_1 - \mu_2 \neq (\mu_1 - \mu_2)_0$ , $\left(|T_{n_1+n_2-2}| > t_{\frac{\alpha}{2}, n_1+n_2-2}\right)$
- $H_1: \mu_1 - \mu_2 > (\mu_1 - \mu_2)_0$ , $(T_{n_1+n_2-2} > t_{\alpha, n_1+n_2-2})$
- $H_1: \mu_1 - \mu_2 < (\mu_1 - \mu_2)_0$ , $(T_{n_1+n_2-2} < -t_{\alpha, n_1+n_2-2})$

4) OC curve

$$\mathrm{d} = \frac{|\mu_1 - \mu_2|}{2\sigma}$$

v. Pooled *T*-test
(when variances are independent but unequal)

$$\gamma = \frac{\left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}\right)^2}{\frac{(S_1^2/n_1)^2}{n_1 - 1} + \frac{(S_2^2/n_2)^2}{n_2 - 1}}$$

$$T_\gamma = \frac{(\overline{X_1} - \overline{X_2}) - (\mu_1 - \mu_2)_0}{\sqrt{S_1^2/n_1 + S_2^2/n_2}}$$

Compared with $t_{\alpha,\gamma}$

vi. Paired *T*-test when variances are dependent

$$\mathrm{D} = \mathrm{X} - \mathrm{Y}$$

$$\mathrm{T} = \frac{\overline{D}}{s_D/\sqrt{n}}$$

vii. Comparison between Paired and Pooled T-Tests

$$T_{\text{pooled}} = \frac{(\overline{X_1} - \overline{X_2})}{\sqrt{S_p^2(\frac{1}{n_1} + \frac{1}{n_2})}}$$

$$T_{\text{paired}} = \frac{\overline{D}}{S_D/\sqrt{n}}$$

viii. *F*-test (comparing two variances) (related to *F*-distribution)

1) Test

$$H_0: \sigma_1 = \sigma_2$$

2) Statistic

$$F_{n_1-1, n_2-1} = S_1^2/S_2^2$$

3) Reject $H_0$ and in favor of $H_1$ if ...

- $H_1: \sigma_1 > \sigma_2$ , $\left(\frac{S_1^2}{S_2^2} > f_{\alpha, n_1-1, n_2-1}\right)$
- $H_1: \sigma_1 < \sigma_2$ , $\left(\frac{S_2^2}{S_1^2} > f_{\alpha, n_2-1, n_1-1}\right)$
- $H_1: \sigma_1 \neq \sigma_2$ , $\left(\frac{S_1^2}{S_2^2} > f_{\alpha/2, n_1-1, n_2-1} \cup \frac{S_2^2}{S_1^2} > f_{\alpha/2, n_2-1, n_1-1}\right)$

4) OC curve for *F*-test

When $n_1 = n_2 = n$, $\lambda = \frac{\sigma_1}{\sigma_2}$.

d. Wilcoxon Rank-Sum Test

(for medians of same kind of distribution)

i. Test

$$H_0: M_1 = M_2$$

IV. *F*-distribution with $\gamma_1$ and $\gamma_2$ degrees of freedom

$$F_{\gamma_1,\gamma_2} = \frac{\chi^2_{\gamma_1}/\gamma_1}{\chi^2_{\gamma_2}/\gamma_2}$$

$$f_{\gamma_1,\gamma_2} = \gamma_1^{\gamma_1/2}\gamma_2^{\gamma_2/2}\frac{\Gamma(\frac{\gamma_1+\gamma_2}{2})}{\Gamma(\frac{\gamma_1}{2})\Gamma(\frac{\gamma_2}{2})}\frac{x^{\frac{\gamma_1}{2}-1}}{(\gamma_1 x + \gamma_2)^{(\gamma_1+\gamma_2)/2}}$$

$$f_{1-\alpha,\gamma_1,\gamma_2} = \frac{1}{f_{\alpha,\gamma_1,\gamma_2}}$$