
Question1 (1 points)
The notation $\mathbb{R}^{m\times n}$ is often used to denote the set of all $m \times n$ matrices with real entries. It can be shown $\mathbb{R}^{m\times n}$ is a vector space over $\mathbb{R}$ with the usual addition and scalar multiplication. List all the axioms that will fail to hold if the usual scalar multiplication is retained but the usual addition is replaced by the following

$$\mathbf{A}+\mathbf{B}=\mathbf{A}-\mathbf{B} \qquad \text{where} \quad \mathbf{A},\mathbf{B}\in\mathbb{R}^{m\times n}$$

where the subtraction on the left is the usual subtraction between matrices in $\mathbb{R}^{m\times n}$.

Question2 (1 points)
Determine whether the set of all real skew-symmetric $n\times n$ matrices, with the usual addition and scalar multiplication, is a vector space. If not, list all of the axioms that fail to hold.

Question3 (1 points)
Determine whether the following subset of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$, where $n > 2$.

$$\{\mathbf{x} \mid \mathbf{A}\mathbf{x}=\mathbf{b}, \;\text{ where } \mathbf{A}_{m\times n}\neq\mathbf{0} \text{ and } \mathbf{b}_{m\times 1}\neq\mathbf{0}\}$$

Question4 (1 points)
Is the vector space $\mathbb{R}^{2\times 2}$ spanned by the set $\left\{\begin{bmatrix}1&1\\0&0\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}1&0\\1&1\end{bmatrix},\begin{bmatrix}0&-1\\1&0\end{bmatrix}\right\}$?

Question5 (2 points)
Let $\mathcal{V}$ be a vector space with subspaces $\mathcal{X}$ and $\mathcal{Y}$. The sum of $\mathcal{X}$ and $\mathcal{Y}$ is defined to be

$$\mathcal{X}+\mathcal{Y}=\{\mathbf{x}+\mathbf{y} \mid \mathbf{x}\in\mathcal{X} \text{ and } \mathbf{y}\in\mathcal{Y}\}$$

(a) (1 point) Show the sum $\mathcal{X}+\mathcal{Y}$ is a subspace of $\mathcal{V}$.

(b) (1 point) Let $\mathcal{V}$ be $\mathbb{R}^{n\times n}$, $\mathcal{X}$ be the subspace of all diagonal matrices of $\mathcal{V}$, and $\mathcal{Y}$ be the subspace of all lower triangular matrices with zero diagonals. What is $\mathcal{X}+\mathcal{Y}$?

Question6 (2 points)
Let $\mathcal{V}$ be a vector space with subspaces $\mathcal{X}$ and $\mathcal{Y}$.

(a) (1 point) Prove that the intersection of $\mathcal{X}\cap\mathcal{Y}$ is also a subspace of $\mathcal{V}$.

(b) (1 point) Show that the union $\mathcal{X}\cup\mathcal{Y}$ need not be a subspace of $\mathcal{V}$.

Question7 (2 points)
Let $\mathcal{V}$ denote the subset of $\mathbb{R}^{n\times n}$ such that $\operatorname{tr}(\mathbf{A})=0$, that is

$$\operatorname{tr}(\mathbf{A})=\sum_{i=1}^{n} a_{ii}=0$$

It can be shown $\mathcal{V}$ is a subspace of $\mathbb{R}^{n\times n}$.

(a) (1 point) Find a spanning set of $\mathcal{V}$ when $n=2$.

(b) (1 point) Find a spanning set of $\mathcal{V}$ when $n=9$.

Question8 (0 points)

(a) (1 point (bonus)) Prove every vector space has unique zero vector.