
Question1 (3 points)

Find the general solution for each of the following homogeneous equations.

$$\ddot{y} - 3\dot{y} + 2y = 0 \qquad \ddot{y} - 2\dot{y} + y = 0 \qquad \ddot{y} - 2\dot{y} + 10y = 0$$

Question2 (1 points)

Determine whether the following two functions are linearly independent.

$$\phi_1(x) = \begin{cases} x^2 & \text{if} \quad x \geq 0, \\ 0 & \text{if} \quad x < 0. \end{cases} \qquad \phi_2(x) = \begin{cases} 0 & \text{if} \quad x \geq 0, \\ x^2 & \text{if} \quad x < 0. \end{cases}$$

Justify your answer.

Question3 (2 points)

(a) (1 point) First verify that

$$\phi_1(t) = t^{-1/2} \cos t$$

is one solution (for $t > 0$) of Bessel's equation

$$t^2 \ddot{y} + t\dot{y} + (t^2 - \frac{1}{4})y = 0$$

Then find the second linearly independent solution.

(b) (1 point) Find the general solution to Legendre's equation

$$(1 - x^2)y'' - 2xy' + 2y = 0 \qquad \text{for} \quad -1 < x < 1$$

Question4 (2 points)

Find the general solution by using the method of undetermined coefficients.

(a) (1 point) $\ddot{y} - 2\dot{y} + 10y = 20t^2 + 2t - 8$

(b) (1 point) $\ddot{y} - 4\dot{y} + 4y = e^{2t}$

Question5 (2 points)

Find the general solution by using the method of variation of parameters.

(a) (1 point) $\ddot{y} + y = \sec(t) \csc(t)$

(b) (1 point) $\ddot{y} - 3\dot{y} + 2y = \cos(e^{-t})$

Question6 (2 points)

Suppose we found by variation of parameters, $y_p(x) = u_1(x)\phi_1(x) + u_2(x)\phi_2(x)$, where ϕ_1 and ϕ_2 form a fundamental set of solutions for the complementary equation, is a particular solution for the nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = g(x)$$

on an interval I for which p, q, and g are continuous.

(a) (1 point) Show that y_p can be written as

$$y_p(x) = \int_{x_0}^{x} G(x, t)g(t)\, dt,$$

where x and x_0 are in I, $G(x, t) = \dfrac{\phi_1(t)\phi_2(x) - \phi_1(x)\phi_2(t)}{W(t)}$, and $W(t) = W(\phi_1(t), \phi_2(t))$ is the Wronskian. The function $G(x, t)$ is called the Green's function for the differential equation. It is fundamental to advanced theories on differential equations.

(b) (1 point) Use Green's function to find a solution of the following initial-value problem

$$y'' - y = e^{2x}, \qquad y(0) = 0,\, y'(0) = 0.$$

Question7 (1 points)

Recall the curvature of a curve defined by a function $y = f(x)$ is

$$\kappa = \frac{y''}{[1 + (y')^2]^{\frac{3}{2}}}.$$

Find $y = f(x)$ for which $\kappa = 1$, show all your workings.

[Hint: For simplicity, assume constants of integration is zero.]

Question8 (3 points)

Sometimes a differential equation with variable coefficients.

$$\ddot{y} + P(t)\dot{y} + Q(t)y = 0$$

can be put in a more suitable form for finding a solution by making a change of independent variable. In this equation we determine conditions on P and Q such that the above equation can be transformed into an equation with constant coefficients. Let the new variable be

$$x = u(t)$$

(a) (1 point) Show, with the new variable, that the given equation becomes

$$(\dot{x})^2 \frac{d^2y}{dx^2} + (\ddot{x} + P(t)\dot{x}) \frac{dy}{dx} + Q(t)y = 0$$

(b) (1 point) In order to for the equation in part (a) to have constant coefficients, the coefficients of $\frac{d^2y}{dx^2}$ and of y must be proportional. If $Q(t) > 0$ and the constant of proportionality to be 1, show that we need the following substitution

$$x = u(t) = \int \Big(Q(t)\Big)^{1/2} dt$$

(c) (1 point) When $Q(t) > 0$, find the condition under which the original equation can be transformed into one with constant coefficients by the above substitution

$$x = u(t) = \int \Big(Q(t)\Big)^{1/2} dt$$

Below are the bonus questions:

Question9 (0 points)

(a) (1 point (bonus)) Find out about linear second-order exact equation and solve one.

(b) (1 point (bonus)) Let $f(x)$ and $g(x)$ be infinitely many times differentiable for all $x \in \mathbb{R}$. Suppose the Wronskian of f and g is identically zero. Can we determine whether f and g are linearly independent? Justify your answer.

Question10 (5 points)

A discrete model for change in price of a stock $S(t)$ over a time interval $[0, T]$ is

$$S_{n+1} = S_n + \mu S_n \Delta t + \sigma S_n \epsilon_{n+1} \sqrt{\Delta t}, \qquad S_0 = s \tag{1}$$

where the constant μ is the annual growth rate of the stock, and σ is a measure of the stock's annual price volatility or tendency to fluctuate. Volatile stocks have large σ.

$$S_n = S(t_n)$$

is the stock price at time $t_n = n\Delta t$, and n takes $0, 1, 2, \ldots, N-1$, and

$$\Delta t = \frac{T}{N}$$

Each term in the sequence $\epsilon_1, \epsilon_2, \ldots$ takes on the value 1 or -1 with equal probability.

$$\epsilon_n = \begin{cases} 1 & \text{with probability} = 0.5, \\ -1 & \text{with probability} = 0.5. \end{cases}$$

A sequence of such numbers can easily be generated by Matlab.

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>> 2 * randi([0,1],1,10) - 1
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Given such a sequence, the difference equation can then be used to simulate a sample path or trajectory of stock prices, $\{s, S_1, S_2, \ldots, S_N\}$. The random terms $\sigma S_n \epsilon_{n+1} \sqrt{\Delta t}$ on the right hand side of the equation can be thought of as "shocks" or "disturbances" that model fluctuations in the stock price. By repeatedly simulating stock price trajectories and computing appropriate averages, it is possible to obtain estimates of the price of a European call option, a type of financial derivative. A statistical simulation algorithm of this type is call a Monte Carlo method.

A European call option is a contract between two parties, a holder and a writer, whereby, for a premium paid to the writer, the holder acquires the right (but not the obligation) to purchase the stock at a future date T (the expiration date) at a price K (the strike price) agree upon in the contract. If the buyer elects to exercise the option on the expiration date, the writer is obligated to sell the underlying stock to the buyer at the price K. Thus the option has, associated with it, a payoff function

$$f(S) = \max(S - K, 0)$$

where $S = S(T)$ is the price of the underlying stock at the time T when the option expires. The payoff function gives the value of the option at time T since, if $S(T) > K$, the holder can purchase, at price K, stock with market value $S(T)$ and thereby make a profit equal to $S(T) - K$ not counting the option premium. If $S(T) < K$, the holder will simply let the option expire since it would be irrational to purchase stock at a price that exceeds the market value.

Back to our simple Monte Carlo method, to estimate the price of a call option, a set

$$\left\{ S_N^{(k)} = S^{(k)}(T), \qquad k = 1, \ldots, M \right\}$$

of M stock prices at expiration is generated using the difference equation

$$S_{n+1}^{(k)} = S_n^{(k)} + rS_n^{(k)}\Delta t + \sigma S_n^{(k)}\epsilon_{n+1}^{(k)}\sqrt{\Delta t}, \qquad S_0 = s \tag{2}$$

For each $k = 1, \dots, M$, difference equation (2) is identical to the difference equation (1) except that the growth rate μ is replaced by the annual rate of interest r that it costs the writer to borrow money. Option pricing theory requires that the average value of the payoffs

$$\left\{f(S_N^{(k)}), \qquad k = 1, \dots, M\right\}$$

be equal to the compounded total return obtained by investing the option premium,

$$C(s)$$

at rate r over the life of the option, that is,

$$\frac{1}{M}\sum_{k=1}^{M} f(S_N^{(k)}) = (1 + r\Delta t)^N C(s)$$

Thus the Monte Carlo estimate is

$$\hat{C}(s) = (1 + r\Delta t)^{-N}\left\{\frac{1}{M}\sum_{k=1}^{M} f(S_N^{(k)})\right\}$$

(a) (1 point) Show that Euler's method applied to the differential equation

$$\frac{dS}{dt} = \mu S$$

yields the equation (1) in the absence of random disturbances, that is, when $\sigma = 0$.

(b) (1 point) Use Matlab to simulate five sample trajectories of equation (1) for the following parameter values and plot the trajectories in a single graph.

$$\mu = 0.12, \quad \sigma = 0.1, \quad T = 1, \quad s = 40, \quad N = 254$$

(c) (1 point) Repeat the simulation for $\sigma = 0.25$ for the volatility. Do the sample trajectories generated in this case appear to exhibit a greater degree of variability?

(d) (1 point) Use the difference equation (2) to generate a set of stock prices $S_N^{(k)} = S^{(k)}(N\Delta t)$, $k = 1, \dots, M$ where $T = N\Delta t$ and then use the given formula to compute a Monte Carlo estimate of the value of a five month call option ($T = \frac{5}{12}$) for the following parameter values: $r = 0.06$, $\sigma = 0.2$, and $K = 50$. Find the estimates corresponding to current stock prices of $S(0) = s = 45$ and $S(0) = s = 55$. Use $N = 200$ time steps for each trajectory and $M = 10,000$ sample trajectories for each Monte Carlo estimate.

(e) (1 point) Check the accuracy of your results above by making a comparison between your approximations with the value computed from the exact Black-Scholes formula

$$C(s) = \frac{s}{2}\operatorname{erfc}\left(-\frac{d_1}{\sqrt{2}}\right) - \frac{K}{2}e^{-rT}\operatorname{erfc}\left(-\frac{d_2}{\sqrt{2}}\right)$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{s}{K}\right) + \left(r + \frac{\sigma^2}{s}\right)T\right], \qquad d_2 = d_1 - \sigma\sqrt{T}$$

and $\operatorname{erfc}(x)$ is the complementary error function,

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2}\,dt$$