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**Question1** (5 points)

(a) (1 point) How many $n \times n$ matrices are both diagonal and orthogonal?

(b) (1 point) How many $n \times n$ matrices are diagonal and unitary?

(c) (1 point) Find a unitary matrix $\mathbf{U}$ that diagonalizes

$$\mathbf{A} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & -1+\mathrm{i} \\ 0 & -1-i & 0 \end{bmatrix}$$

(d) (1 point) Prove that if $\mathcal{V}$ is a complex inner product space, then

$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2 + \|\mathbf{u}+\mathrm{i}\mathbf{v}\|^2\mathrm{i} - \|\mathbf{u}-\mathrm{i}\mathbf{v}\|^2\mathrm{i}}{4}, \qquad \text{for all } \mathbf{u}, \mathbf{v} \in \mathcal{V}.$$

(e) (1 point) Show the eigenvalues of a skew-Hermitian matrix are 0 or pure imaginary.

**Question2** (5 points)

(a) (1 point) Find a symmetric matrix $\mathbf{B}$ such that

$$\mathbf{B}^2 = \begin{bmatrix} 17 & 16 & -16 \\ 16 & 41 & -32 \\ -16 & -32 & 41 \end{bmatrix}$$

(b) (1 point) Use the singular value decomposition to find the least square solution of the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ that has the smallest 2-norm, where

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 4 \\ 5 & 2 & 4 \\ 3 & 6 & 0 \\ 3 & 6 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ -1 \\ 9 \end{bmatrix}$$

(c) (1 point) A polar decomposition of an $n \times n$ matrix $\mathbf{A}$ is a factorization

$$\mathbf{A} = \mathbf{P}\mathbf{Q}$$

in which $\mathbf{P}$ is a positive semidefinite $n \times n$ matrix with the same rank as $\mathbf{A}$, and $\mathbf{Q}$ is an orthogonal $n \times n$ matrix. Show if

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$$

is the SVD of $\mathbf{A}$, then

$$\mathbf{A} = \left(\mathbf{U}\mathbf{\Sigma}\mathbf{U}^T\right)\left(\mathbf{U}\mathbf{V}^{\mathrm{T}}\right)$$

is a polar decomposition of $\mathbf{A}$.

(d) (1 point) Find a symmetric matrix $\mathbf{A}$ so that

$$f(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$$

where

$$f(\mathbf{x}) = \frac{1}{9}\left(-2x_1^2 + 7x_2^2 + 4x_3^2 + 4x_1x_2 + 16x_1x_3 + 20x_2x_3\right),$$

Is it a positive definite form?

(e) (1 point) Suppose $\mathbf{A} \in \mathbb{C}^{n\times n}$, and

$$p(\lambda) = \det\left(\mathbf{A} - \lambda\mathbf{I}\right)$$

is the characteristic polynomial of $\mathbf{A}$. Prove

$$p(\mathbf{A}) = 0$$