Vv417 Lecture 13

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October 22, 2019

- Recall that $\mathcal{S}$ is a spanning set for a vector space $\mathcal{V}$ if and only if every vector in $\mathcal{V}$ is a linear combination of vectors in $\mathcal{S}$.
- However, spanning sets can contain redundant vectors. For example,

$$\text{row}\,(\mathbf{A}) = \text{span}\{\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_m\}$$

  where $\mathbf{r}_i$ are row vectors of an $m \times n$ matrix $\mathbf{A}$.

Q: Why we might have redundant vectors in $\text{span}\{\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_m\}$?

- Often we want a minimal spanning set, "minimal" in the sense it contains as few vectors as possible, that is, the set would no longer be a spanning set of the original vector space if any element of the set is removed.
- The notion of minimal spanning set is closely related to the following

Definition

A linearly independent spanning set for a vector space $\mathcal{V}$ is called a basis for $\mathcal{V}$.

- Just as in the case of spanning sets, a space can possess many distinct bases.

- Vectors $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, $\cdots \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ form the standard basis for $\mathbb{R}^n$.

Exercise

Do the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 9 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$ form a basis for $\mathbb{R}^3$?

Solution

- We need to check linear independence, and whether it spans $\mathbb{R}^3$.
- Since $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies \begin{array}{l} \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0}; \\ \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \beta_3\mathbf{v}_3 = \mathbf{b},\ \forall \mathbf{b} \in \mathbb{R}^3 \end{array}$
- So they are linearly independent in $\mathbb{R}^3$ and span $\mathbb{R}^3$, and form a basis for $\mathbb{R}^3$.

- The standard basis for $\mathcal{P}_n$

$$\mathbf{p}_0 = 1, \quad \mathbf{p}_1 = x, \quad \mathbf{p}_2 = x^2, \quad \ldots, \mathbf{p}_n = x^n$$

Exercise

Show that the following vectors

$$\mathbf{M}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{M}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{M}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{M}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

form a basis for the vector space $\mathcal{M}_{2\times 2}$ of 2×2 real matrices.

Solution

- We need to show the given matrices are linearly independent, that is,

$$\alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solution

- It is clear that the only solution is

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$$

- So the given matrices are linearly independent.
- We also need to show the given matrices span $\mathcal{M}_{2\times 2}$, that is, we need to show there exist β_1, β_2, β_3 and $\beta_4 \in \mathbb{R}$ for all a, b, c and $d \in \mathbb{R}$ such that

$$\begin{aligned} \beta_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \beta_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \beta_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \beta_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{bmatrix} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{aligned}$$

- It is clear that we can always let

$$\beta_1 = a, \qquad \beta_2 = b, \qquad \beta_3 = c, \qquad \text{and} \qquad \beta_4 = d$$

- Therefore the given matrices span $\mathcal{M}_{2\times 2}$, and they form a basis for $\mathcal{M}_{2\times 2}$.

Theorem

Let $\mathcal{V}$ be a vector space, and let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_1, \ldots, \mathbf{b}_n\} \subset \mathcal{V}$. Then $\mathcal{B}$ is a basis for $\mathcal{V}$ if and only if $\mathcal{B}$ is a minimal spanning set for $\mathcal{V}$.

Proof

- First let $\mathcal{B}$ be a basis for $\mathcal{V}$, then $\mathcal{B}$ spans $\mathcal{V}$ and $\mathcal{B}$ is linearly independent.
- Let $\mathcal{S}$ be the set obtained from $\mathcal{B}$ with $\mathbf{b}_i \in \mathcal{B}$ removed. If $\mathcal{S}$ spans $\mathcal{V}$, then

$$\mathbf{b}_i = \alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 + \cdots \alpha_j\mathbf{b}_j + \cdots + \alpha_n\mathbf{b}_n, \qquad \text{where} \quad \mathbf{b}_j \in \mathcal{S}$$

- It contradicts the fact $\mathcal{B} = \mathcal{S} \cup \{\mathbf{b}_i\}$ is linearly independent. So $\mathcal{B}$ is minimal.
- If $\mathcal{B}$ is a minimal spanning set, and assume $\mathcal{B}$ is not a basis, that is,

$$\mathbf{b}_i = \alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 + \cdots \alpha_j\mathbf{b}_j + \cdots + \alpha_n\mathbf{b}_n \implies \operatorname{span}(\mathcal{B}) = \operatorname{span}(\mathcal{S})$$

  where $\mathcal{S}$ is the set obtained from $\mathcal{B}$ with $\mathbf{b}_i \in \mathcal{B}$ removed.
- But this means that $\mathcal{B}$ is not minimal, contrary to our assumption. So, $\mathcal{B}$ must be linearly independent, and therefore it is a basis for $\mathcal{V}$.

Theorem

Bases of a given vector space $\mathcal{V}$ have the same number of vectors in them.

Proof

- Let $\mathcal{U} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{W} = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ be bases of $\mathcal{V}$, where

$$m = n + 1$$

  that is, basis $\mathcal{W}$ has one more vector than basis $\mathcal{U}$.

- Since $\mathcal{W} \subset \mathcal{V}$, everything vector in $\mathcal{W}$ is linear combination of vectors in $\mathcal{U}$.
- Consider n out of those $n+1$ vectors in $\mathcal{W}$, which can be written as

$$\mathbf{w}_i = \sum_{j=1}^{n} a_{ij}\mathbf{u}_j \qquad \text{for} \quad i = 1, 2, \ldots, n$$

- To avoid contradicting the set of $\mathbf{w}_j$'s being linearly independent, the $\mathbf{A}_{n\times n}$ matrix of a_{ij} must have n linearly independent rows.

Proof

- Since the row space is not altered by elementary row operations, thus

$$\text{row}\,(\mathbf{A}) = \text{row}\,(\mathbf{I}) \iff \mathbf{A} \sim \mathbf{I}$$

  from which, we see the system of vector equations

$$\mathbf{w}_i = \sum_{j=1}^{n} a_{ij}\mathbf{u}_j \qquad \text{for} \quad i = 1, 2, \ldots, n$$

  can be manipulated by adding and scaling rows into the following system

$$\sum_{i=1}^{n} a^*_{ij}\mathbf{w}_i = \mathbf{u}_j \qquad \text{for} \quad j = 1, 2, \ldots, n$$

- This leads to the contradiction that the remaining vector $\mathbf{w}_{n+1}$, which must be a linear combination of $\mathbf{u}_i$'s, must be a linear combination of $\mathbf{w}_1, \ldots \mathbf{w}_n$.
- Therefore, to avoid contradicting $\mathcal{W}$ being linearly independent, $m = n$. $\square$

- A vector space $\mathcal{V}$ can have many different bases, but the preceding 2 results:
  - all bases of $\mathcal{V}$ contain the same number of vectors.
  - all minimal spanning sets of $\mathcal{V}$ contain the same number of vectors.
- If $\mathcal{B}_1$ and $\mathcal{B}_2$ are each a basis for $\mathcal{V}$, then each is a minimal spanning set, thus they must contain the same number of vectors. It can be shown that it is also the number of vectors in a maximal linearly independent set.

Definition

The dimension of a vector space $\mathcal{V}$ is denoted by $\dim(\mathcal{V})$ and is defined to be the number of vectors in a basis for $\mathcal{V}$.

- Engineers often use the term degrees of freedom as a synonym for dimension
- Dimensions of some familiar vector spaces

$$\dim(\mathbb{R}^n) = n \qquad \dim(\mathcal{P}_n) = n + 1 \qquad \dim(\mathcal{M}_{m\times n}) = mn$$

Exercise

Find a basis for the solution space of $\mathbf{Ax} = \mathbf{0}$ if the general solution of it is

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0$$

What is the dimension of the solution space?

Solution

- It is more clear in the vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \underbrace{\begin{bmatrix} -3r - 4s - 2t \\ r \\ -2s \\ s \\ t \\ 0 \end{bmatrix}}_{\text{zero only if } r=s=t=0} = r \underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{v}_1} + s \underbrace{\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{v}_2} + t \underbrace{\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}_3}$$

- Thus, $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for the null space and the dimension is 3.

Exercise

Show that the vector space

$$\mathcal{P}_\infty$$

the space of all polynomials with real coefficients is infinite-dimensional.

Solution

- Assume $\mathcal{P}_\infty$ is finite-dimensional with a basis $\mathcal{S}$ of size n for some integer n.
- Since $\mathcal{S}$ is finite, there are vectors in $\mathcal{S}$ that have the highest degree, say m.
- It is clear that x^{m+1} is in $\mathcal{P}_\infty$, however,

  $$x^{m+1}$$

  is not a linear combination of vectors in $\mathcal{S}$.
- So the finite basis does not span $\mathcal{P}_\infty$, contradiction to the assumption.
- Hence the vector space is infinite-dimensional.

- The simplest of all vector spaces is the zero vector space $\mathcal{V} = \{\mathbf{0}\}$.

Q: What do you think the dimension of the zero vector space should be?

Q: What is the spanning set for the zero vector space?

Q: What is then the dimension of the zero vector space?

Q: Is the following set linearly independent?

$$\{\mathbf{0}, \mathbf{u}\}, \qquad \text{where } \mathbf{u} \text{ is any vector in } \mathcal{V}.$$

- However, the empty set $\emptyset$ is defined to be the basis for $\mathcal{V} = \{\mathbf{0}\}$.
- Recall engineers use the term degrees of freedom instead of dimension.
- In the zero space case, there are no degrees of freedom; you can go nowhere.
- It is important not to confuse the dimension of a vector space $\mathcal{V} \subset \mathbb{R}^n$ with the number of components contained in the individual vectors from $\mathcal{V}$. e.g.,

A plane through the origin in $\mathbb{R}^3$.

Exercise

Suppose $\mathcal{V}$ is an n-dimensional vector space and

$$\mathcal{S}_r = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}, \qquad \text{where } r < n,$$

is a linearly independent subset of $\mathcal{V}$. Show it is possible to find an extension set

$$\mathcal{S}_{\text{ext}} = \{\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\} \subset \mathcal{V}$$

such that

$$\mathcal{S}_n = \mathcal{S}_r \cup \mathcal{S}_{\text{ext}} = \{\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$$

is a basis for $\mathcal{V}$.

Solution

- Since

$$r < n \implies \operatorname{span}(\mathcal{S}_r) \neq \mathcal{V}$$

Solution

- Hence there is a vector

$$\mathbf{v}_{r+1} \in \mathcal{V}$$

such that

$$\mathbf{v}_{r+1} \notin \operatorname{span}(\mathcal{S}_r)$$

- The extension set below is a linearly independent subset of $\mathcal{V}$.

$$\mathcal{S}_{r+1} = \mathcal{S}_r \cup \{\mathbf{v}_{r+1}\}$$

- This can be repeated to generate linear independent subsets

$$\mathcal{S}_{r+2}, \quad \mathcal{S}_{r+2}, \ldots$$

- Eventually

$$\dim(\mathcal{S}_r \cup \mathcal{S}_{\text{ext}}) = n$$

Theorem

For vector spaces $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M} \subset \mathcal{N}$, then

$$\dim \mathcal{M} \leq \dim \mathcal{N}$$

If $\dim \mathcal{M} = \dim \mathcal{N}$, then

$$\mathcal{M} = \mathcal{N}$$

Proof

- Let $\mathcal{B}_{\mathcal{M}}$ and $\mathcal{B}_{\mathcal{N}}$ be minimal spanning sets of $\mathcal{M}$ and $\mathcal{N}$, respectively.
- Suppose

  $$\dim \mathcal{M} > \dim \mathcal{N}$$

  that is, there are more vectors in $\mathcal{B}_{\mathcal{M}}$ than in $\mathcal{B}_{\mathcal{N}}$.
- Although $\mathcal{B}_{\mathcal{N}}$ is not necessarily a minimal spanning set of $\mathcal{M}$, it must be a spanning set of $\mathcal{M}$ since $\mathcal{M} \subset \mathcal{N}$, which means a minimal spanning of $\mathcal{M}$ formed from $\mathcal{B}_{\mathcal{N}}$ must have equal or less number of vectors than $\mathcal{B}_{\mathcal{N}}$.
- This contradicts the fact that all minimal spanning set of $\mathcal{M}$ should have the same number of vectors because $\mathcal{B}_{\mathcal{M}}$ already has more vectors than $\mathcal{B}_{\mathcal{N}}$.

Proof

- Suppose $\dim \mathcal{M} = \dim \mathcal{N}$ but $\mathcal{M} \neq \mathcal{N}$, then there is an extension set

$$\mathcal{S}_{\text{ext}}$$

such that

$$\mathcal{B}_{\mathcal{M}} \cup \mathcal{S}_{\text{ext}}$$

is minimal spanning set of $\mathcal{N}$, which means this minimal spanning set of $\mathcal{N}$,

$$\mathcal{B}_{\mathcal{M}} \cup \mathcal{S}_{\text{ext}}$$

has more vectors than the minimal spanning set $\mathcal{B}_{\mathcal{N}}$.

- This contradicts the fact that the number vectors in minimal spanning sets of the vector space is the same. Therefore, no $\mathcal{S}_{\text{ext}}$ should be found, that is

$$\mathcal{M} = \mathcal{N} \quad \square$$

Q: How can we find the extension set for a given linearly independent set in $\mathbb{R}^n$?

$$\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$$

- Let the following set be any basis for $\mathbb{R}^n$,

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$$

- Now if we place vectors in $\mathcal{S}$ along with vectors in $\mathbb{B}$ as columns in a matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_r & \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}$$

- Clearly

$$\operatorname{col}(\mathbf{A}) = \mathbb{R}^n$$

- The set of pivot columns from $\mathbf{A}$ is a basis for $\operatorname{col}(\mathbf{A})$. Notice vectors

$$\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$$

are the first r pivot columns in $\mathbf{A}$ for $\mathcal{S}$ is linearly independent, the rest in $\mathcal{B}$.

Exercise

Find the extension set to the following

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}$$

to form basis for $\mathbb{R}^4$.

Solution

- Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

- Hence the extension set of $\mathcal{S}_{\text{ext}} = \{\mathbf{e}_2, \mathbf{e}_3\}$ will make $\mathcal{S} \cup \mathcal{S}_{\text{ext}}$ a basis for $\mathbb{R}^4$.