Vv255 Lecture 22

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1. Definite integrals: Given a *continuous* real-valued function $y = f(x)$,

$$\int_a^b f(x)\,dx$$

represents the net area between the graph of f, $y = 0$, $x = a$ and $x = b$.

- The definite integral can also be used to compute the arc length of a curve.
- Suppose a *smooth* curve $\mathcal{C}$ is defined by

$$\mathbf{r}(t) \qquad \text{for } \alpha \leq t \leq \beta.$$

then the arc length L of the curve $\mathcal{C}$ is given by

$$L = \int_\alpha^\beta \left| \frac{d\mathbf{r}}{dt} \right| \, dt$$

- The arc length function is defined to be $s(t) = \displaystyle\int_\alpha^t |\mathbf{r}'| \, d\tau$, and by FTC

$$\frac{ds}{dt} = \left| \frac{d\mathbf{r}}{dt} \right|$$

2. Double integrals: Given a *continuous* real-valued function $z = f(x, y)$,

$$\iint_{\mathcal{D}} f(x, y)\, dA$$

represents the net volume of the solid between the region $\mathcal{D}$ and the graph of

$$z = f(x, y)$$

3. Triple integrals: Given a *continuous* real-valued function $w = f(x, y, z)$,

$$\iiint_{\mathcal{E}} f(x, y, z)\, dV$$

represents the net hyper-volume of the 4-dimensional solid.

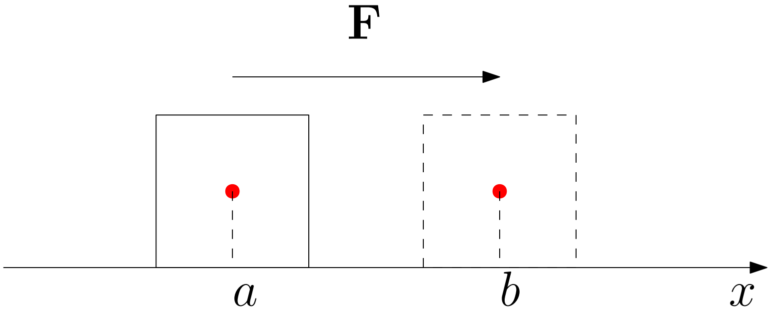
- A line integral is similar to a definite integral except instead of integrating on

$$[a, b]$$

which is really along a straight line, we integrate along a curve / path $\mathcal{C}$.

- To motivate the definition, let us recall,

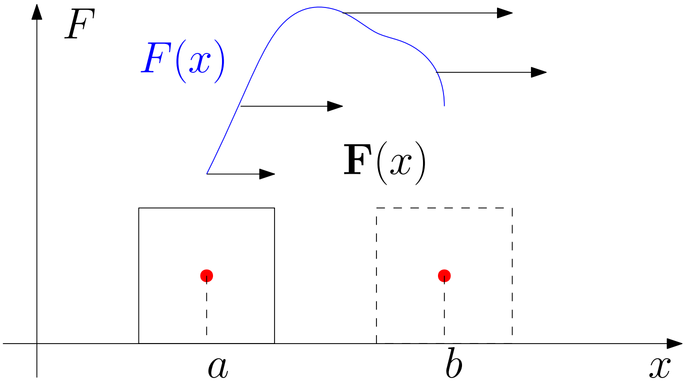
0. If a constant force F is applied to and moves an object along x-axis



then the amount of work done between $x = a$ and $x = b$ is defined to be

$$W = F \cdot (b - a).$$

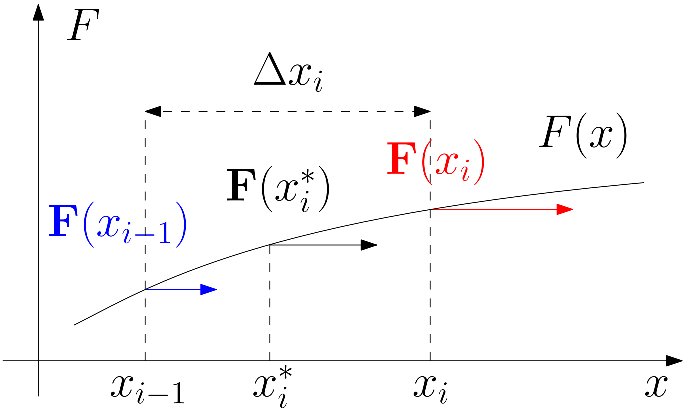
1. If the magnitude of the force is not constant, but is instead dependent on x



then the work done is defined to be

$$W = \int_a^b F(x)\, dx$$

- This definition is based on using the "basic" formula inside each subinterval



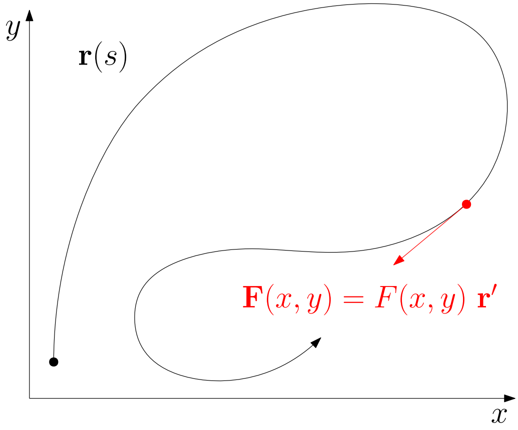
and take the limit of the sum of all those work done inside each subinterval

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i^*)\Delta x_i = \int_a^b F(x)\, dx$$

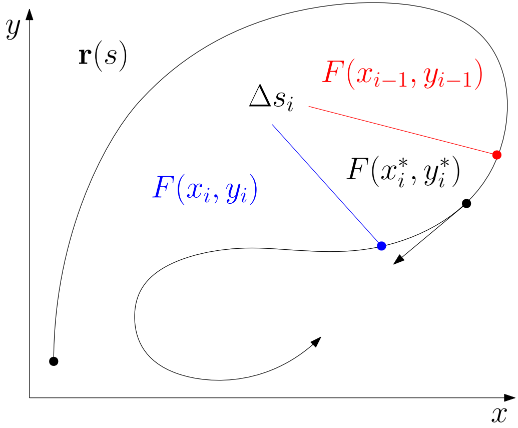
2. If a variable force is applied to and moves an object along a smooth curve $\mathcal{C}$

$$\mathbf{r} = x(s)\mathbf{e}_x + y(s)\mathbf{e}_y \qquad \text{where } s \text{ is an arc length parameter.}$$

- Suppose the force $\mathbf{F}(x, y)$ is always in the direction of motion,



- If we partition the curve $\mathcal{C}$ into sub-curves,



- We can approximate the work within each sub-curve with the work done by

$$F(x_i^*, y_i^*)\ \Delta s_i$$

- It is reasonable to expect the following definition for the work done,

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i^*, y_i^*)\ \Delta s_i$$

- If F is continuous, then we expect the limit to converge, and denoted by

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i^*, y_i^*)\ \Delta s_i = \int_{\mathcal{C}} F(x, y)\, ds$$

- The notation indicates the fact that it is integrated along the smooth

$$\mathcal{C} \colon \mathbf{r}(s) = x(s)\mathbf{e}_x + y(s)\mathbf{e}_y$$

- Although we have introduced line integrals in the context of computing work, this approach can be used to integrate any function along a smooth curve $\mathcal{C}$.

Definition

If $\mathcal{C}$ is a smooth curve in $\mathbb{R}^2$ or $\mathbb{R}^3$, then the

line integral of f with respect to s along $\mathcal{C}$

is defined to be

$$\int_{\mathcal{C}} f(x,y)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

or

$$\int_{\mathcal{C}} f(x,y,z)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta s_i$$

provided the limit exists and doesn't depend on the choice of the tagged partition.

Q: How can we evaluate such integrals?

$$\int_{\mathcal{C}} f\, ds \qquad \text{where } \mathcal{C} \text{ is defined by } \quad \mathbf{r}(t)$$

I. To evaluate a line integral along a smooth curve $\mathbf{r}(s)$ is given in terms of s

$$\int_{\mathcal{C}} f(x,y)\, ds \qquad \text{where } \mathcal{C} \text{ is defined by} \quad \mathbf{r}(s) = x(s)\mathbf{e}_x + y(s)\mathbf{e}_y$$

we only need to convert it into a ordinary single integral by substitution

$$\int_{\mathcal{C}} f(x,y)\, ds = \int f\Big(x(s), y(s)\Big)\, ds = \int F(s)\, ds$$

II. If the smooth curve $\mathbf{r}(t)$ is given in terms of other parameter t

$$\int_{\mathcal{C}} f(x,y)\, ds \qquad \text{where } \mathcal{C} \text{ is defined by} \quad \mathbf{r}(t) = x(t)\mathbf{e}_x + y(t)\mathbf{e}_y$$

we have to use the u-substitution formula to change the variable $s = s(t)$

$$\begin{aligned}
\int_{\mathcal{C}} f(x,y)\, ds &= \int f\Big(x(s), y(s)\Big)\, ds \\
&= \int f\Big(x(t), y(t)\Big) s'(t)\, dt \qquad s = s(t) = \int_{\alpha}^{t} |\mathbf{r}'(\tau)|\, d\tau \\
&= \int f\Big(x(t), y(t)\Big) |\mathbf{r}'(t)|\, dt \qquad s'(t) = |\mathbf{r}'(t)|
\end{aligned}$$

Exercise

(a) *Evaluate the following line integral*

$$\int_{\mathcal{C}} x^2 z \, ds$$

where $\mathcal{C}$ is the line segment between

$$(0, 6, -1) \text{ and } (4, 1, 5).$$

(b) *Evaluate the following line integral*

$$\int_{\mathcal{C}} 2x + 9z \, ds$$

where $\mathcal{C}$ is defined by the parametric equations

$$x = t, \qquad y = t^2, \qquad z = t^3$$

for $0 \leq t \leq 1$.

- It is usually impractical to evaluate line integrals directly from the definition.
- However, the definition is important in the interpretation of line integrals.

Q: If $\mathcal{C}$ is a smooth plane curve and $f(x, y)$ is a non-negative continuous on $\mathcal{C}$, what does the following line integral represent geometrically?

$$\int_{\mathcal{C}} f(x, y)\, ds$$

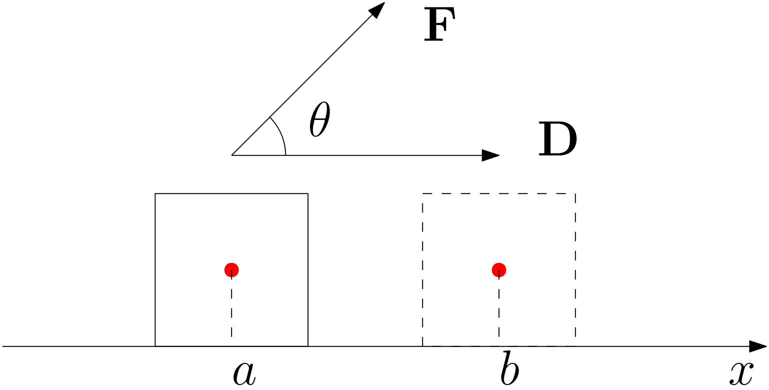
- If $\mathcal{C}$ is a curve in $\mathbb{R}^3$ that models a thin wire, and if $f(x, y, z)$ gives the density function of the wire, then the mass m of the wire is given by

$$m = \int_{\mathcal{C}} f(x, y, z)\, ds$$

- If $\mathcal{C}$ is a smooth curve of arc length L, and f is identically 1, then

$$\int_{\mathcal{C}} ds = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta s_i = \lim_{n \to \infty} L = L$$

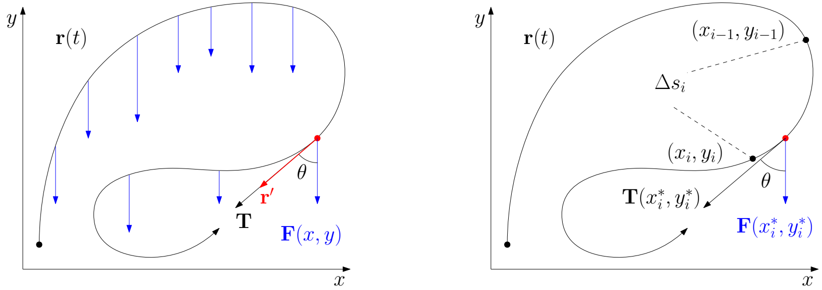
3. If a constant force $\mathbf{F}$ is applied to an object NOT in the direction of motion,



but rather makes an angle θ with a constant displacement vector $\mathbf{D}$,

$$W = |\mathbf{F}| \cos\theta |\mathbf{D}| = \mathbf{F} \cdot \mathbf{D}$$

4. Suppose an object moving along a curve $\mathcal{C}$ in a force field $\mathbf{F}(x,y)$, so subject to a variable force, both the direction and the magnitude might be changing



- Consider a partition of the curve $\mathcal{C}$, in each sub-curve, we treat $\mathbf{F}$ as a constant force $\mathbf{F}(x_i^*, y_i^*)$ and the displacement as $\Delta s_i \mathbf{T}(x_i^*, y_i^*)$

$$W_i \approx \mathbf{F}(x_i^*, y_i^*) \cdot (\Delta s_i \mathbf{T}(x_i^*, y_i^*)) = \mathbf{F}(x_i^*, y_i^*) \cdot \mathbf{T}(x_i^*, y_i^*) \Delta s_i$$

- It is reasonable to expect the following definition, when the limit converges,

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{F}(x_i^*, y_i^*) \cdot \mathbf{T}(x_i^*, y_i^*) \Delta s_i = \int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds$$

Definition

If $\mathcal{C}$ is a smooth curve in $\mathbb{R}^2$ or $\mathbb{R}^3$, then the

line integral of $\mathbf{F}$ with respect to s along $\mathcal{C}$

is defined to be

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds$$

Q: Is this integral so different from the first type? How to evaluated this type?

- Recall the unit tangent vector is given by $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, and $s' = |\mathbf{r}'(t)|$

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\; ds = \int_a^b \mathbf{F}(x(t), y(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} |\mathbf{r}'(t)|\; dt = \int_a^b \mathbf{F} \cdot \mathbf{r}'\; dt = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

- The last form of the line integral is merely an abbreviation.

- For a given $\mathbf{F}$ with component functions P and Q, and a curve $\mathcal{C}$ defined by

$$\mathbf{r}(t) = x(t)\mathbf{e}_x + y(t)\mathbf{e}_y, \quad \text{where } a \le t \le b$$

- The line integral of $\mathbf{F}$ can be evaluated using line integral of P and Q,

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt \\
&= \int_a^b \Big[P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t) \Big]\, dt \\
&= \int_a^b P(x(t), y(t))x'(t)\, dt + \int_a^b Q(x(t), y(t))y'(t)\, dt \\
&= \int_{\mathcal{C}} P(x, y)\, dx + \int_{\mathcal{C}} Q(x, y)\, dy
\end{aligned}$$

Q: Why using the line integral notation in the last line?

- So the line integral of the vector field $\mathbf{F}$ over $\mathcal{C}$ can be obtained by summing the line integrals of its component functions w.r.t. x and y individually.

- However, it is important to note that unlike line integrals with respect to s,

$$\int_{\mathcal{C}} f(x,y)\,ds = \int_a^b f(x(t),y(t))|\mathbf{r}'(t)|\,dt$$

line integrals with respect to x or y depends on the orientation of $\mathcal{C}$.

- For example,

$$\int_{\mathcal{C}} P(x,y)\,dx = \int_a^b P(x(t),y(t))x'(t)\,dt$$

- If the curve is traced in reverse (that is, from the terminal point to the initial point), then the sign of the line integral is reversed as well. We denote by $-\mathcal{C}$ the curve $\mathcal{C}$ with its orientation reversed. We then have

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = -\int_{-\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$$

since the components $\displaystyle\int_{\mathcal{C}} P\,dx = -\int_{-\mathcal{C}} P\,dx,$ and $\displaystyle\int_{\mathcal{C}} Q\,dy = -\int_{-\mathcal{C}} Q\,dy$.

- All of this discussion can be generalized to space curves in a similar way.

Exercise

(a) *Evaluate the line integral*

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F} = z\mathbf{e}_x + y\mathbf{e}_y - x\mathbf{e}_z$ *and* $\mathcal{C}$ *is the curve defined by*

$$\mathbf{r}(t) = t\mathbf{e}_x + (\sin t)\mathbf{e}_y + (\cos t)\mathbf{e}_z \qquad \text{for} \quad 0 \le t \le \pi.$$

(b) *Evaluate the line integral*

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \quad \text{where} \quad \mathbf{F}(x, y) = (\sin x)\mathbf{e}_x + (\cos y)\mathbf{e}_y$$

and $\mathcal{C}$ *is the top half of the circle*

$$x^2 + y^2 = 1$$

from $(1, 0)$ *to* $(-1, 0)$, *then the line segment from* $(-1, 0)$ *to* $(-2, 3)$.