Vv255 Lecture 28

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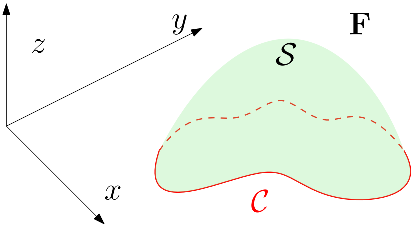
1. A line integral of $\mathbf{F}$ in $\mathbb{R}^2$ over a closed plane curve $\mathcal{C}$ to an integral of a scalar component of $\operatorname{curl}\mathbf{F}$ over the region that is enclosed by $\mathcal{C}$.

$$\oint_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\,ds = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA = \iint_{\mathcal{D}} (\operatorname{curl}\mathbf{F})\cdot\mathbf{e}_z\,dA$$

- Our next goal is to generalize this result to $\mathbb{R}^3$, that is, to relate

1. A line integral of $\mathbf{F}$ in $\mathbb{R}^3$ over a closed space curve $\mathcal{C}$ to an integral of a scalar component of $\operatorname{curl}\mathbf{F}$ over the surface $\mathcal{S}$ enclosed by $\mathcal{C}$.

$$\oint_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\,ds = \iint_{\mathcal{S}} (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\,dS$$



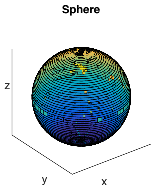
- The surface $\mathcal{S}$ is enclosed by $\mathcal{C}$ in the sense that $\mathcal{C}$ is the only boundary of $\mathcal{S}$.

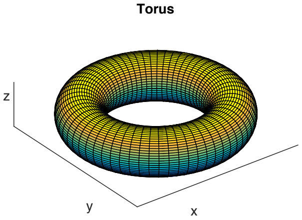
2. A line integral of the normal component of $\mathbf{F}$ in $\mathbb{R}^2$ over a closed curve $\mathcal{C}$ to a double integral of $\operatorname{div}\mathbf{F}$ over the region that is enclosed by $\mathcal{C}$.

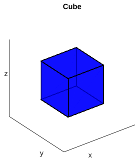
$$\oint_{\mathcal{C}} \mathbf{F}\cdot\mathbf{n}\,ds = \iint_{\mathcal{D}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\, dA = \iint_{\mathcal{D}} \operatorname{div}\mathbf{F}\,dA$$

- We wish to generalize this result as well to $\mathbb{R}^3$, that is, to relate

2. An integral of the normal component of $\mathbf{F}$ in $\mathbb{R}^3$ *over a closed surface* $\mathcal{S}$ to a triple integral of $\operatorname{div}\mathbf{F}$ over the solid $\mathcal{E}$ that is enclosed by $\mathcal{S}$.





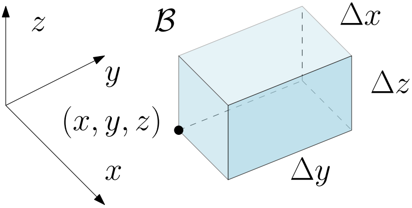


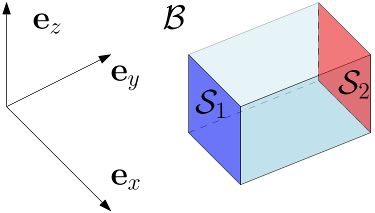
- The solid or the region $\mathcal{E}$ in $\mathbb{R}^3$ is enclosed by the closed surface $\mathcal{S}$ in the sense that $\mathcal{S}$ has no curve boundary.

- Recall for a sufficiently smooth velocity field of a type of fluid,

$$\mathbf{V}(x,y,z) = v_x(x,y,z)\mathbf{e}_x + v_y(x,y,z)\mathbf{e}_y + v_z(x,y,z)\mathbf{e}_z$$

we consider the flux of $\mathbf{V}$ out of a tiny box in the xyz-coordinate system,





- Consider the flux out of such a similar box in terms of surface integrals,

$$\mathcal{B}: \qquad a \le x \le b, \qquad c \le y \le d, \qquad e \le z \le f$$

- For $\mathcal{S}_1$ and $\mathcal{S}_2$, both oriented outward, the net flux across $\mathcal{S}_1$ and $\mathcal{S}_2$

$$\iint_{\mathcal{S}_1} \mathbf{V} \cdot d\mathbf{S} + \iint_{\mathcal{S}_2} \mathbf{V} \cdot d\mathbf{S} = \iint_{\mathcal{D}_1} \mathbf{V} \cdot (-\mathbf{e}_y)\, dA + \iint_{\mathcal{D}_2} \mathbf{V} \cdot \mathbf{e}_y\, dA$$

Q: What is the region $\mathcal{D}_1$ and $\mathcal{D}_2$?

- The region $\mathcal{D}_1$ and $\mathcal{D}_2$ are the same by construction,

$$\mathcal{D}_1 = \mathcal{D}_2 : \qquad a \leq x \leq b, \qquad e \leq z \leq f$$

- If we simply use x and y to be the variable of integrations,

$$\begin{aligned}\iint_{\mathcal{S}_1} \mathbf{V} \cdot d\mathbf{S} + \iint_{\mathcal{S}_2} \mathbf{V} \cdot d\mathbf{S} &= \iint_{\mathcal{D}_1} \mathbf{V} \cdot (-\mathbf{e}_y)\, dA + \iint_{\mathcal{D}_2} \mathbf{V} \cdot \mathbf{e}_y\, dA \\ &= \int_e^f \int_a^b -v_y(x, c, z)\, dx\, dz + \int_e^f \int_a^b v_y(x, d, z)\, dx\, dz \\ &= \int_e^f \int_a^b \Big(v_y(x, d, z) - v_y(x, c, z) \Big)\, dx\, dz\end{aligned}$$

- By the fundamental theorem of Calculus,

$$v_y(x, d, z) - v_y(x, c, z) = \int_c^d \frac{\partial v_y}{\partial y}\, dy$$

- Making the substitution, and converting the iterated integral to a triple

$$v_y(x, d, z) - v_y(x, c, z) = \int_c^d \frac{\partial v_y}{\partial y}\, dy$$

$$\begin{aligned}
\implies \iint_{\mathcal{S}_1} \mathbf{V} \cdot d\mathbf{S} + \iint_{\mathcal{S}_2} \mathbf{V} \cdot d\mathbf{S} &= \int_e^f \int_a^b \Big(v_y(x, d, z) - v_y(x, c, z) \Big)\, dx\, dz \\
&= \int_e^f \int_c^d \int_a^b \frac{\partial v_y}{\partial y}\, dx\, dy\, dz \\
&= \iiint_{\mathcal{B}} \frac{\partial v_y}{\partial y}\, dV
\end{aligned}$$

- Similarly, we can obtain the flux across the remaining four faces

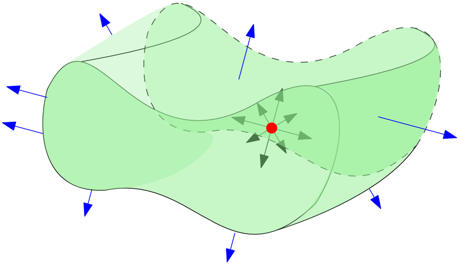
$$\iint_{\mathcal{S}_3} \mathbf{V} \cdot d\mathbf{S} + \iint_{\mathcal{S}_4} \mathbf{V} \cdot d\mathbf{S} = \iiint_{\mathcal{B}} \frac{\partial v_x}{\partial x}\, dV$$

$$\iint_{\mathcal{S}_5} \mathbf{V} \cdot d\mathbf{S} + \iint_{\mathcal{S}_6} \mathbf{V} \cdot d\mathbf{S} = \iiint_{\mathcal{B}} \frac{\partial v_z}{\partial z}\, dV$$

- Hence the net outward flux across the surface $\mathcal{S}$ of the box $\mathcal{B}$ is given by

$$\iint_{\mathcal{S}} \mathbf{V} \cdot d\mathbf{S} = \iiint_{\mathcal{B}} \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) \, dV = \iiint_{\mathcal{B}} \operatorname{div} \mathbf{V} \, dV$$

- This result is also true for any sufficiently smooth vector field $\mathbf{F}$ acting on any region $\mathcal{E}$ enclosed by a piecewise smooth oriented closed surface.



- The proof can be extended to non-rectangular region $\mathcal{E}$ by considering

$$x = x(u, v, w), \qquad y = y(u, v, w), \qquad z = z(u, v, w)$$

The Divergence Theorem

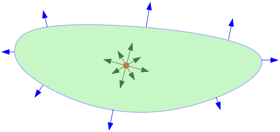
Let $\mathcal{S}$ be a piecewise smooth closed surface with outward orientation and $\mathcal{E}$ be a region in $\mathbb{R}^3$ that is enclosed by $\mathcal{S}$. Suppose $\mathbf{F}$ is a vector field with a continuous partial derivatives in an open region containing $\mathcal{E}$, then

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_{\mathcal{E}} \operatorname{div} \mathbf{F} \, dV$$

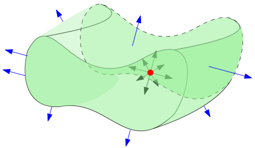
where $\mathbf{n}$ is the outward unit normal vector of $\mathcal{S}$.

- Roughly speaking, it relates the flux integral of a vector field across a closed surface $\mathcal{S}$ to an integral of its divergence in the enclosed region $\partial\mathcal{S}$

The normal form of Green's theorem



The divergence theorem



- The theorem can serve as a bridge between surface and triple integral.

Exercise

(a) *Use the divergence theorem to calculate the surface integral*

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S},$$

where $\mathcal{S}$ *is the surface of the box with vertices* $(\pm 1, \pm 2, \pm 3)$ *and*

$$\mathbf{F}(x,y,z) = x^2 z^3 \mathbf{e}_x + 2xyz^3 \mathbf{e}_y + xz^4 \mathbf{e}_z$$

(b) *Use the divergence theorem to evaluate*

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S},$$

where $\mathcal{S}$ *is the top half of the sphere* $x^2 + y^2 + z^2 = 1$, *and*

$$\mathbf{F}(x,y,z) = z^2 x \mathbf{e}_x + \left(y^3/3 + \tan z\right) \mathbf{e}_y + (x^2 z + y^2) \mathbf{e}_z$$

- If we bring the density at t and position (x, y, z) into consideration,

$$\rho(x, y, z, t)$$

- Applying the divergence theorem to the vector field,

$$\mathbf{F} = \rho \mathbf{V}$$

- The total mass flux out of the region $\mathcal{E}$ at t is given by the triple integral

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_{\mathcal{E}} \operatorname{div} \mathbf{F} \, dV$$

- However, this flux must equal the rate of change of mass in the region $\mathcal{E}$, i.e.

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n} \, dS = -\frac{\partial}{\partial t} \iiint_{\mathcal{E}} \rho \, dV = -\iiint_{\mathcal{E}} \frac{\partial \rho}{\partial t} \, dV$$

the minus sign is for the fact we are measuring net mass loss due to outflow.

- Thus equating the two expressions of the total mass flux,

$$\iiint_{\mathcal{E}} \operatorname{div} \mathbf{F} \, dV = - \iiint_{\mathcal{E}} \frac{\partial \rho}{\partial t} \, dV \implies \iiint_{\mathcal{E}} \left(\operatorname{div} \mathbf{F} + \frac{\partial \rho}{\partial t} \right) \, dV$$

- The domain is arbitrary, so this can only happen if the integrand vanishes,

$$\operatorname{div} \mathbf{F} + \frac{\partial \rho}{\partial t} = 0$$

- For a steady state fluid flow,

$$\operatorname{div}(\rho \mathbf{V}) + \frac{\partial \rho}{\partial t} = 0 \implies \rho \operatorname{div} \mathbf{V} = 0 \implies \operatorname{div} \mathbf{V} = 0 \implies \text{Incompressible}$$

  and the net fluid flux through $\mathcal{S}$ must be zero,

$$\iint_{\mathcal{S}} \mathbf{V} \cdot d\mathbf{S} = \iiint_{\mathcal{E}} \operatorname{div} \mathbf{V} \, dV = 0$$