
Question1 (8 points)

(a) (1 point) Determine whether the following is differentiable. Justify your answer.

$$f(x, y) = (\cos x)(\cos y)$$

Solution:

1M Since the partial derivatives

$$f_x = -\sin x \cos y \qquad \text{and} \qquad f_y = -\cos x \sin y$$

are continuous in $\mathbb{R}^2$. By the condition of differentiability, f is differentiable.

(b) (2 points) Determine whether the following is differentiable. Justify your answer.

$$g(x) = \begin{cases} 0, & \text{if} \quad x^2 < y < 2x^2, \\ 1, & \text{otherwise.} \end{cases}$$

Solution:

1M Intuitively, we don't expect this function to be differentiable on the boundaries $y = x^2$ and $y = 2x^2$. We could prove differentiability implies continuity, which I asked you to do in part (d), and use the contrapositive statement to argue that this function cannot be differentiable along the boundaries since it is not continuous there. However, I want to provide an example of using the definition to show a function is NOT differentiable at a point. We do it by constructing a contradiction. Suppose $g(x, y)$ is differentiable at $(0, 0)$, then

$$\begin{aligned} m = g_x(0,0) &= \lim_{h\to 0} \frac{g(0+h,0) - g(0,0)}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0 \\ n = g_y(0,0) &= \lim_{h\to 0} \frac{g(0,0+h) - g(0,0)}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0 \\ \varepsilon(\Delta x, \Delta y) &= g(0+\Delta x, 0+\Delta y) - g(0,0) - m\Delta x - n\Delta y \\ &= g(0+\Delta x, 0+\Delta y) - 1 \end{aligned}$$

1M However, the following

$$\begin{aligned} \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\varepsilon(\Delta x, \Delta y)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} &= \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{g(0+\Delta x, 0+\Delta y) - 1}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \\ &= \lim_{\substack{(\Delta x,\Delta y)\to(0,0) \\ \text{along } \Delta y = \frac{3}{2}(\Delta x)^2}} \frac{g(0+\Delta x, 0+\Delta y) - 1}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \\ &= \lim_{\Delta x\to 0} \frac{0-1}{\sqrt{(\Delta x)^2 + \frac{9}{4}(\Delta x)^4}} \\ &= -\infty \end{aligned}$$

contradicts the fact that the limit is 0. Hence g is not differentiable.

(c) (2 points) Determine whether the following is differentiable. Justify your answer.

$$h(x) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right), & \text{for} \quad (x,y) \neq (0,0), \\ 0, & \text{for} \quad (x,y) = (0,0). \end{cases}$$

Solution:

1M This is the example I gave as the counterexample for the condition of differentiability being necessary. Let me show it here that it is differentiable while the partial derivatives are not continuous at $(0,0)$.

$$\begin{aligned} h_x(0,0) &= \lim_{\Delta x \to 0} \frac{h(0+\Delta x, 0) - h(0,0)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(\Delta x)^2 \sin\left(\frac{1}{\sqrt{(\Delta x)^2}}\right) - 0}{\Delta x} \\ &= \lim_{\Delta x \to 0} \Delta x \sin\left(\frac{1}{|\Delta x|}\right) \\ &= \begin{cases} \lim\limits_{\Delta x \to 0^+} \Delta x \sin\left(\frac{1}{\Delta x}\right) = 0 \\ -\lim\limits_{\Delta x \to 0^-} \Delta x \sin\left(\frac{1}{\Delta x}\right) = 0 \end{cases} \end{aligned}$$

which we have shown to be zero using the squeeze theorem in vv156. However,

$$\frac{d}{dx}\left[(x^2+0^2)\sin\left(\frac{1}{\sqrt{x^2+0^2}}\right)\right] = \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x > 0 \\ \cos\left(\frac{1}{x}\right) - 2x\sin\left(\frac{1}{x}\right) & x < 0 \end{cases}$$

which lead us to conclude the following limit

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=0}} h_x(x,y)$$

does not exist, thus the following limit does not exist.

$$\lim_{(x,y)\to(0,0)} h_x(x,y)$$

Hence the partial derivative h_x is not continuous at $(0,0)$. Since h is symmetric in terms of x and y, so the partial derivative with respect to y at $(0,0)$ is

$$h_y(0,0) = 0$$

but which is also not continuous at $(0,0)$.

To show h is nevertheless differentiable, consider

$$\Delta z = h(0+\Delta x, 0+\Delta y) - h(0,0) = (\Delta x^2 + \Delta y^2)\sin\left(\frac{1}{\sqrt{\Delta x^2+\Delta y^2}}\right) - 0$$

we need to show there exists constant m, n and $\varepsilon(\Delta x, \Delta y)$ such that

$$\Delta z = m\Delta x + n\Delta y + \varepsilon(\Delta x, \Delta y)$$

where

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\varepsilon(\Delta x, \Delta y)}{\sqrt{\Delta x^2 + \Delta y^2}} = 0$$

Using $m = h_x(0,0)$ and $n = h_y(0,0)$, we have

$$(\Delta x^2 + \Delta y^2)\sin\left(\frac{1}{\sqrt{\Delta x^2 + \Delta y^2}}\right) = \varepsilon(\Delta x, \Delta y)$$

Thus

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\varepsilon(\Delta x, \Delta y)}{\sqrt{\Delta x^2 + \Delta y^2}} = \lim_{(\Delta x,\Delta y)\to(0,0)} \sqrt{(\Delta x^2 + \Delta y^2)}\sin\left(\frac{1}{\sqrt{\Delta x^2 + \Delta y^2}}\right)$$

Since

$$\left|\sin\left(\frac{1}{\sqrt{\Delta x^2 + \Delta y^2}}\right)\right| \leq 1 \qquad \text{for} \qquad (\Delta x, \Delta y) \neq (0,0)$$

we have the following inequalities for $(\Delta x, \Delta y) \neq (0,0)$,

$$-\sqrt{(\Delta x^2 + \Delta y^2)} \leq \sqrt{(\Delta x^2 + \Delta y^2)}\sin\left(\frac{1}{\sqrt{\Delta x^2 + \Delta y^2}}\right) \leq \sqrt{(\Delta x^2 + \Delta y^2)}$$

It is clear that

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \sqrt{(\Delta x^2 + \Delta y^2)} = \lim_{(\Delta x,\Delta y)\to(0,0)} -\sqrt{(\Delta x^2 + \Delta y^2)} = 0$$

Therefore, by the squeeze theorem, we have

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\varepsilon(\Delta x, \Delta y)}{\sqrt{\Delta x^2 + \Delta y^2}} = 0$$

which means h is differentiable at $(0,0)$. Off the origin, the function $h(x,y)$, and its derivatives, h_x and h_y, are continuous. By the condition of differentiability, $h(x,y)$ must be differentiable.

(d) (1 point) Let $h\colon \mathbb{R}^2 \to \mathbb{R}$ be differentiable at a point P. Show h is continuous at P.

Solution:

1M We need to show

$$\lim_{(\Delta x,\Delta y)\to(0,0)} h(a + \Delta x, b + \Delta y) = h(a,b)$$

If h is differentiable, then

$$h(a+\Delta x, b+\Delta y) - h(a,b) = h_x(a,b)\Delta x + h_y(a,b)\Delta y + \varepsilon(\Delta x, \Delta y)$$

where

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\varepsilon(\Delta x,\Delta y)}{\sqrt{\Delta x^2+\Delta y^2}} = 0$$

Hence

$$\begin{aligned}
&\lim_{(\Delta x,\Delta y)\to(0,0)} h(a+\Delta x, b+\Delta y)\\
&= \lim_{(\Delta x,\Delta y)\to(0,0)} \Big(h(a,b) + h_x(a,b)\Delta x + h_y(a,b)\Delta y + \varepsilon(\Delta x,\Delta y)\Big)\\
&= h(a,b) + 0 + \lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon(\Delta x,\Delta y)\\
&= h(a,b) + \lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon(\Delta x,\Delta y)\frac{\sqrt{\Delta x^2+\Delta y^2}}{\sqrt{\Delta x^2+\Delta y^2}}\\
&= h(a,b) + \lim_{(\Delta x,\Delta y)\to(0,0)} \sqrt{\Delta x^2+\Delta y^2}\frac{\varepsilon(\Delta x,\Delta y)}{\sqrt{\Delta x^2+\Delta y^2}}\\
&= h(a,b) + \lim_{(\Delta x,\Delta y)\to(0,0)} \sqrt{\Delta x^2+\Delta y^2} \cdot \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\varepsilon(\Delta x,\Delta y)}{\sqrt{\Delta x^2+\Delta y^2}}\\
&= h(a,b) + 0\cdot 0\\
&= h(a,b)
\end{aligned}$$

(e) (2 points) Construct a function $\phi(x,y)$ that is not differentiable at a point P, of which ϕ_x and ϕ_y exist at P, but only one of the two is continuous at P. Justify your answer.

Solution:

1M Yes, it is possible. Consider the following example

$$\phi(x,y) = \begin{cases} y, & \text{if} \quad y \geq 0 \quad \text{or} \quad x \leq 0, \\ -y, & \text{otherwise.} \end{cases}$$

which has the following partial derivatives, of which ϕ_x is clearly continuous,

$$\phi_x(x,y) = 0$$

$$\phi_y(x,y) = \begin{cases} 1, & \text{if} \quad y > 0 \quad \text{or} \quad x \leq 0, \\ -1, & \text{if} \quad y < 0 \quad \text{and} \quad x > 0, \\ 1, & \text{if} \quad x = 0 \quad \text{and} \quad y = 0, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

however, ϕ_y is clearly not continuous at $(0,0)$, at which ϕ is not differentiable.

Question2 (4 points)

(a) (1 point) If $z = f(t)$, where $t = x - y$, show that $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$.

Solution:

1M Take the left-hand side of equation and apply Chain Rule

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=\frac{dz}{dt}\frac{\partial t}{\partial x}+\frac{dz}{dt}\frac{\partial t}{\partial y}=\frac{dz}{dt}\cdot 1+\frac{dz}{dt}\cdot(-1)=0$$

which is the right-hand side of the desired equation.

(b) (1 point) Find the derivative $F'(x)$ for $F(x)=\displaystyle\int_0^{x^2} t\sin(x^2-t)dt$.

Solution:

1M Applying the Leibniz's integral rule, we have

$$\begin{aligned}F'(x)&=x^2\sin(x^2-x^2)\cdot(2x)-0\cdot\sin(x^2-0)\cdot 0+\int_0^{x^2} 2xt\sin(x^2-t)dt\\&=2x\int_0^{x^2} t\sin(x^2-t)dt\\&=2xF(x)\end{aligned}$$

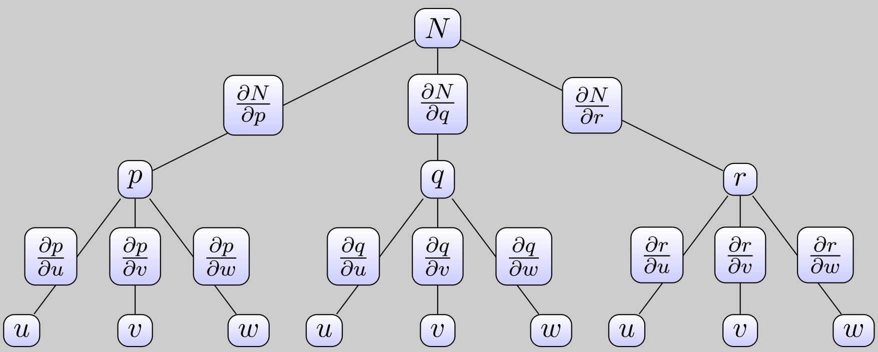
Note the integral can actually be evaluated, however, it is often not possible.

(c) (2 points) Find the second order partial derivative $\dfrac{\partial^2 N}{\partial v^2}$ and $\dfrac{\partial^2 N}{\partial v\partial u}$ for

$$N=\frac{p+q}{p+r},\qquad \text{where } p=u+vw,\ q=v+uw,\ r=w+uv.$$

Solution:

1M From the way it is given, we take p, q and r as the intermediate variables, while u, v and w are independent variables. From a tree diagram,



we expect the chain rule takes the following form

$$\begin{aligned}\frac{\partial N}{\partial v}&=\frac{\partial N}{\partial p}\frac{\partial p}{\partial v}+\frac{\partial N}{\partial q}\frac{\partial q}{\partial v}+\frac{\partial N}{\partial r}\frac{\partial r}{\partial v}\\&=\frac{p+r-p-q}{(p+r)^2}\cdot w+\frac{1}{p+r}\cdot 1+\frac{-p-q}{(p+r)^2}\cdot u\\&=\frac{(r-q)w+(p+r)-(p+q)u}{(p+r)^2}\end{aligned}$$

However, I hope this is enough to convince you that blindly using the chain rule when comes to many variables can be deadly. Just imagine how many terms you would have for the second order chain rule formula in this form! So I would stop once I saw this partial derivative, and consider an alternative approach. Eliminating p, q and r, the variable N in terms of u, v and w is given by

$$N = \frac{u + vw + v + uw}{u + vw + w + uv} = \frac{(w+1)(u+v)}{(v+1)(w+u)}$$

Finding some symmetry and isolating the sources of dependency can greatly simplify a problem. Let $a = w+1$, $b = u+v$, $c = (v+1)^{-1}$, and $d = (w+u)^{-1}$.

$$N = abcd$$

Now applying the chain rule, we have

$$\frac{\partial N}{\partial v} = acd + abd\frac{-1}{(v+1)^2} = acd - abc^2d$$

which is easy to compute for a and d ain't functions of v, thus contribute nothing

$$\begin{aligned}\frac{\partial^2 N}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial N}{\partial v}\right) &= \frac{\partial}{\partial v}\left(acd - abc^2d\right)\\ &= ad\frac{-1}{(v+1)^2} - ac^2d - 2abcd\frac{-1}{(v+1)^2}\\ &= 2(abc^3d - ac^2d)\end{aligned}$$

Similarly, the mixed derivative is

$$\begin{aligned}\frac{\partial^2 N}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial N}{\partial v}\right) &= \frac{\partial}{\partial u}\left(acd - abc^2d\right)\\ &= ac\frac{-1}{(w+u)^2} - ac^2d - abc^2\frac{-1}{(w+u)^2}\\ &= -acd^2 - ac^2d + abc^2d^2\end{aligned}$$

where $a = w+1, \quad b = u+v, \quad c = \dfrac{1}{v+1}, \quad$ and $\quad d = \dfrac{1}{w+u}.$

Question3 (1 points)

Suppose that

$$z = x(e^x + e^{-y})$$

where x and y are found to be 2 and $\ln 2$ with maximum possible errors of

$$|\Delta x| = |dx| = 0.1 \text{ and } |\Delta y| = |dy| = 0.02.$$

Estimate the maximum possible error in the computed value of z using differentials.

Solution:

1M Using the linear approximation, we have

$$\begin{aligned} dz &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \\ &= \left(e^x + e^{-y} + xe^x\right)dx + \left(-xe^{-y}\right)dy \end{aligned}$$

Thus the maximum possible error is

$$\left|e^x + e^{-y} + xe^x\right||dx| + \left|-xe^{-y}\right||dy| = \left|e^2 + \frac{1}{2} + 2e^2\right||0.1| + \left|2\frac{1}{2}\right||0.02| = 2.287$$

Question4 (1 points)
Find an equation of the tangent plane to the following ellipsoid at $(2, 1, -3)$.

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

Solution:

1M Recall your total differential

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

gives the change in z on the tangent plane, so an equation of the tangent plane with the point of tangency being $P(x_0, y_0, z_0)$ is

$$z - z_0 = \left.\frac{\partial z}{\partial x}\right|_P (x - x_0) + \left.\frac{\partial z}{\partial y}\right|_P (y - y_0)$$

It is clear that the ellipsoid is smooth, we expect the equation implicitly defines z as a differentiable function of x and y, and we can differentiate implicitly when $z \neq 0$,

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{x^2}{4} + y^2 + \frac{z^2}{9}\right) &= \frac{\partial}{\partial x}(3) \implies \frac{x}{2} + \frac{2z}{9}\frac{\partial z}{\partial x} = 0 \implies \frac{\partial z}{\partial x} = -\frac{9x}{4z} \\ \frac{\partial}{\partial y}\left(\frac{x^2}{4} + y^2 + \frac{z^2}{9}\right) &= \frac{\partial}{\partial y}(3) \implies 2y + \frac{2z}{9}\frac{\partial z}{\partial y} = 0 \implies \frac{\partial z}{\partial y} = -\frac{9y}{z} \end{aligned}$$

Hence an equation for the tangent plane is

$$z + 3 = -\frac{9 \cdot 2}{4 \cdot (-3)}(x - 2) - \frac{9 \cdot 1}{-3}(y - 1) \implies -\frac{3}{2}(x - 2) - 3(y - 1) + (z + 3) = 0$$

Question5 (1 points)
A multiplication problem of two numbers is altered by taking a tiny amount from a factor and adding it to the other. How can you tell using tangent plane approximation whether the product increases or decreases for this tiny change?

Solution:

1M Let $f(x,y) = xy$. Then $f(x+\Delta h, y-\Delta h) - f(x,y)$ can be approximated by

$$f_x(x,y)\Delta h - f_y(x,y)\Delta h$$

and the partial derivatives are $f_x = y$ and $f_y = x$.

Hence the change in the product is $(y-x)\Delta h$. Therefore the product increases when the increment h is taken from the larger factor, and vice versa.

Question6 (2 points)
Let $f(x,y)$ be differentiable everywhere. Suppose $f(a,a) = a$, $f_x(a,a) = b$, $f_y(a,a) = c$, where a, b and c are constants. Find $g(a)$ and $h'(a)$ in terms of a, b and c, where

$$g(x) = f\Big(x, f\big(x, f(x,x)\big)\Big) \qquad \text{and} \qquad h(x) = [g(x)]^2$$

Solution:

1M It is clear that

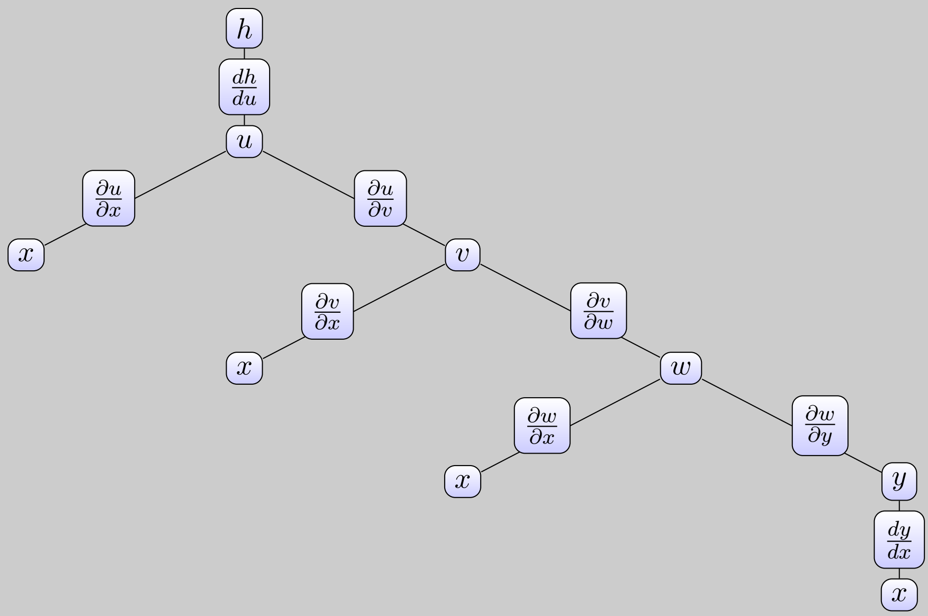
$$g(a) = a$$

In fact, any such composition is always a. We will use this fact in the next part.

1M We rely on the chain rule to find $h'(a)$, let

$$u = g(x) = f(x,v) \quad \text{where} \quad v = f\big(x, f(x,x)\big) = f(x,w) \quad \text{and} \quad w = f(x,x)$$

The tree diagram for the dependency



this suggests the following chain rule

$$
\begin{aligned}
h'(x) &= \frac{dh}{dx} \\
&= \frac{dh}{du}\frac{du}{dx} \\
&= \frac{dh}{du}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial v}\frac{dv}{dx}\right) \\
&= \frac{dh}{du}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial w}\frac{dw}{dx}\right)\right) \\
&= \frac{dh}{du}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial w}\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\frac{dy}{dx}\right)\right)\right) \\
&= 2g(a)\left(f_x(a,v) + f_y(a,v)\left(f_x(a,w) + f_y(a,w)\left(f_x(a,a) + f_y(a,a)\cdot 1\right)\right)\right) \\
&= 2g(a)\left(f_x(a,a) + f_y(a,a)\left(f_x(a,a) + f_y(a,a)\left(f_x(a,a) + f_y(a,a)\cdot 1\right)\right)\right) \\
&= 2a\left(b + c\left(b + c\left(b + c\right)\right)\right)
\end{aligned}
$$

Question7 (3 points)

Let $\mathbf{x}$ and $\mathbf{a}$ be the position vectors of points $P(x_1, x_2, x_3)$ and $A(a_1, a_2, a_3)$ respectively, where a_1, a_2 and a_3 are constants. Suppose $f\colon \mathbb{R}^3 \to \mathbb{R}$ is differentiable in an open ball $\mathcal{B}$ containing A, and denote

$$f(x_1, x_2, x_3) = f(\mathbf{x})$$

(a) (1 point) Show there exists some $\mathbf{b}$ on the line segment defined by $\mathbf{x}$ and $\mathbf{a}$ such that

$$f(\mathbf{x}) = f(\mathbf{a}) + \sum_{k=1}^{3}(x_k - a_k)\frac{\partial f}{\partial x_k}(\mathbf{b}) \qquad \text{where} \qquad \mathbf{x} \in \mathcal{B}$$

Solution:

1M Consider the function

$$g(t) = f\big(\mathbf{a} + t(\mathbf{x}^* - \mathbf{a})\big) \qquad \text{where} \qquad \mathbf{x}^* \in \mathcal{B}$$

Since f is differentiable, $g(t)$ is differentiable by the chain rule.

$$g'(t) = (x_1^* - a_1)\frac{\partial f}{\partial x_1} + (x_2^* - a_2)\frac{\partial f}{\partial x_2} + (x_3^* - a_3)\frac{\partial f}{\partial x_3}$$

where the partial derivatives are evaluated at $\big(\mathbf{a} + t(\mathbf{x}^* - \mathbf{a})\big)$. By MVT, there exists $c \in (0, 1)$ such that

$$g'(c) = g(1) - g(0)$$

Note that $g(0) = f(\mathbf{a})$ and $g(1) = f(\mathbf{x}^*)$, thus we have the desired result,

$$
\begin{aligned}
f(\mathbf{x}^*) - f(\mathbf{a}) &= (x_1^* - a_1)\frac{\partial f}{\partial x_1} + (x_2^* - a_2)\frac{\partial f}{\partial x_2} + (x_3^* - a_3)\frac{\partial f}{\partial x_3} \\
f(\mathbf{x}^*) &= f(\mathbf{a}) + \sum_{k=1}^{3}(x_k^* - a_k)\frac{\partial f}{\partial x_k}(\mathbf{b})
\end{aligned}
$$

where $\mathbf{b} = \mathbf{a} + c(\mathbf{x}^* - \mathbf{a})$. Using the gradient notation, we have

$$f(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{b}) \cdot (\mathbf{x} - \mathbf{a})$$

(b) (1 point) Suppose f is twice differentiable in $\mathcal{B}$, that is, all the first order partial derivatives are differentiable in $\mathcal{B}$. Show there exists some $\mathbf{b}$ on the line segment defined by $\mathbf{x}$ and $\mathbf{a}$ such that

$$f(\mathbf{x}) = f(\mathbf{a}) + \sum_{k=1}^{3}(x_k - a_k)\frac{\partial f}{\partial x_k}(\mathbf{a}) + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}(x_i - a_i)(x_j - a_j)\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{b})$$

where $\mathbf{x} \in \mathcal{B}$.

Solution:

1M It looks complicated, but it is just an extension of the second order case of Taylor's theorem, which states there exists $c \in (0, 1)$ such that

$$g(1) = g(0) + g'(0)(1-0) + \frac{1}{2}g''(c)(1-0)^2 = g(0) + g'(0) + \frac{1}{2}g''(c)$$

By the chain rule, we have

$$g'(t) = \sum_{k=1}^{3}(x_k^* - a_k)\frac{\partial f}{\partial x_k}\big(\mathbf{a} + t(\mathbf{x}^* - \mathbf{a})\big)$$
$$g''(t) = \sum_{i=1}^{3}\sum_{j=1}^{3}(x_i^* - a_i)(x_j^* - a_j)\frac{\partial^2 f}{\partial x_i \partial x_j}\big(\mathbf{a} + t(\mathbf{x}^* - \mathbf{a})\big)$$

Again use the fact that $g(0) = f(\mathbf{a})$ and $g(1) = f(\mathbf{x}^*)$, and

$$g'(0) = \sum_{k=1}^{3}(x_k^* - a_k)\frac{\partial f}{\partial x_k}(\mathbf{a})$$

we have the desired result,

$$f(\mathbf{x}^*) = f(\mathbf{a}) + \sum_{k=1}^{3}(x_k^* - a_k)\frac{\partial f}{\partial x_k}(\mathbf{a}) + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}(x_i^* - a_i)(x_j^* - a_j)\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{b})$$

where $\mathbf{b} = \big(\mathbf{a} + c(\mathbf{x}^* - \mathbf{a})\big)$.

(c) (1 point) Find a similar formula for $f(\mathbf{x})$ involving third order partial derivatives when f is three times differentiable in $\mathcal{B}$.

Solution:

1M The third order Taylor's theorem

$$f(\mathbf{x}) = f(\mathbf{a}) + \sum_{k=1}^{3}(x_k - a_k)\frac{\partial f}{\partial x_k}(\mathbf{a}) + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}(x_i - a_i)(x_j - a_j)\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a})$$
$$+ \frac{1}{6}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}(x_i - a_i)(x_j - a_j)(x_k - a_k)\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(\mathbf{b})$$

where $\mathbf{b}$ is on the line segment defined by $\mathbf{x}$ and $\mathbf{a}$.