Question1 (5 points)

(a) (1 point) Sketch the vector field

$$\mathbf{F}(x,y) = y\mathbf{e}_x + (x+y)\mathbf{e}_y$$

(b) (1 point) Plot the vector field in Matlab

$$\mathbf{F}(x,y) = (y^2 - 2xy)\mathbf{e}_x + (3xy - 6x^2)\mathbf{e}_y$$

And explain the appearance by finding the set of points (x,y) such that $\mathbf{F}(x,y) = \mathbf{0}$.

(c) (1 point) Plot the vector field using polar coordinates in Matlab for $0 \le r \le 2.5$ and $0 \le \theta \le 2\pi$, make sure you have representative non-intersecting vectors, the vectors need not be drawn to scale, but they should be in correct proportion relative to each other to make the the graph informative.

$$\mathbf{F} = (\sin x)\mathbf{e}_x + (\cos y)\mathbf{e}_y$$

(d) (1 point) Find the gradient field of

$$f(x,y,z) = z^3\sqrt{x^2+y^2}$$

(e) (1 point) A particle moves in a velocity field

$$\mathbf{V}(x,y) = x^2\mathbf{e}_x + (x+y^2)\mathbf{e}_y$$

If it is at position $(2,1)$ at time $t=3$, estimate its location at time $t=3.001$.

Question2 (5 points)

(a) (1 point) Evaluate $\displaystyle\int_{\mathcal{C}} z + y^2\, ds$, where $\mathcal{C}$ is the line segment from $(3,4,0)$ to $(1,4,2)$.

(b) (1 point) Evaluate $\displaystyle\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$, where $\mathcal{C}$ is the curve $y = e^x$ from $(2,e^2)$ to $(0,1)$ and

$$\mathbf{F} = x^2\mathbf{e}_x - y\mathbf{e}_y$$

(c) (1 point) Find the work done by the force field

$$\mathbf{F} = xy\mathbf{e}_x + yz\mathbf{e}_y + xz\mathbf{e}_z,$$

on a particle that moves along the curve $\mathcal{C}$.

$$\mathbf{r}(t) = t\mathbf{e}_x + t^2\mathbf{e}_y + t^3\mathbf{e}_z \qquad 0 \le t \le 1$$

(d) (1 point) Suppose that $\mathbf{F}$ represents the velocity field of a fluid flowing through a region in space. If $\mathbf{r}(t)$ parametrizes a smooth curve $\mathcal{C}$ in the domain of the continuous velocity field $\mathbf{F}$, the flow along the curve from $A = \mathbf{r}(a)$ to $B = \mathbf{r}(b)$ is defined to be

$$\text{Flow} = \int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\, ds$$

The line integral in this case is known as a flow integral. If the curve starts and ends at the same point, so that $A = B$, the flow is called the circulation around the curve. Consider the velocity field

$$\mathbf{F} = xy\mathbf{e}_x + y\mathbf{e}_y - yz\mathbf{e}_z$$

Find the flow from $(0,0,0)$ to $(1,1,1)$ along the curve of intersection of

$$y = x^2 \text{ and } z = x.$$

(e) (1 point) A curve in the xy-plane is simple if it does not cross itself. When a curve starts and ends at the same point, it is a closed curve. If $\mathcal{C}$ is a smooth simple closed curve in the domain of a continuous vector field $\mathbf{F}$ in the plane, and if $\mathbf{n}$ is the outward-pointing unit normal vector on $\mathcal{C}$, the Flux of $\mathbf{F}$ across $\mathcal{C}$ is defined to be

$$\text{Flux of } \mathbf{F} \text{ across } \mathcal{C} = \int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{n}\, ds$$

The flux is the rate of a fluid leaving or entering the region defined by $\mathcal{C}$. Flux is Latin for flow, but many flux calculations involve no motion at all. If $\mathbf{F}$ were an electric field or a magnetic field, for instance, the integral of $\mathbf{F} \cdot \mathbf{n}$ would still be called the flux of the field across $\mathcal{C}$. Notice the difference between flux and circulation. Flux is the integral of the normal component of $\mathbf{F}$; circulation is the integral of the tangential component of $\mathbf{F}$. Consider the velocity field

$$\mathbf{F} = -y\mathbf{e}_x + x\mathbf{e}_y \quad \text{and the circle} \quad \mathbf{r}(t) = (\cos t)\mathbf{e}_x + (\sin t)\mathbf{e}_y, \quad 0 \le t \le 2\pi$$

Find the flux of the field across the circle.

Question3 (5 points)

(a) (1 point) Evaluate the line integral

$$\int_{\mathcal{C}} \sin y\, dx + \int_{\mathcal{C}} (x \cos y - \sin y)\, dy,$$

where $\mathcal{C}$ is a smooth curve from $(2, 0)$ to $(1, \pi)$.

(b) (1 point) Show that the following vector field $\mathbf{Q}$ is conservative in any open region $\mathcal{E}$

$$\mathbf{Q} = \begin{bmatrix} 3x^2(y+z) + y^3 + z^3 \\ 3y^2(z+x) + z^3 + x^3 \\ 3z^2(x+y) + x^3 + y^3 \end{bmatrix}$$

(c) (1 point) Construct the potential function for the vector field $\mathbf{Q}$ above.

(d) (1 point) Evaluate the following integral

$$\int_{\mathcal{C}} \mathbf{Q} \cdot d\mathbf{r}$$

where the vector field $\mathbf{Q}$ is defined above and $\mathcal{C}$ is a smooth curve from

$$(1, -1, 1) \text{ to } (2, 1, 2).$$

(e) (1 point) Let $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$ and the following vector to be a constant vector

$$\mathbf{a} = a_1\mathbf{e}_x + a_2\mathbf{e}_y + a_3\mathbf{e}_z$$

Let $\mathcal{C}$ be a curve from the origin to the point with the position vector

$$\mathbf{r}_0 = x_0\mathbf{e}_x + y_0\mathbf{e}_y + z_0\mathbf{e}_z, \qquad \text{where} \quad |\mathbf{r}_0| = 10$$

what is the maximum possible value of

$$\int_{\mathcal{C}} \nabla(\mathbf{r} \cdot \mathbf{a}) \cdot d\mathbf{r}$$

Question4 (0 points)

(a) (1 point (bonus)) Suppose $f(z)$ is a smooth function, and there is a vector field

$$\mathbf{F} = \frac{1}{y}\Big(1 + y^2 f(xy)\Big)\mathbf{e}_x + \frac{x}{y^2}\Big(y^2 f(xy) - 1\Big)\mathbf{e}_y$$

and a smooth curve $\mathcal{C}$ that is entirely above the x-axis. Let (a, b) and (c, d) be the initial point and terminal point of $\mathcal{C}$, respectively. Given $ab = cd$, find

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

(b) (1 point (bonus)) Suppose $f(z)$ is continuous. Constructing a function to show

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$$

where $\mathcal{C}$ is a piecewise smooth and

$$\mathbf{F} = f(x^2 + y^2)\,(x\mathbf{e}_x + y\mathbf{e}_y)$$

(c) (1 point (bonus)) Find an upper bound for

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

where $\mathcal{C}$ is some circle of radius R centred at the origin in the xy-plane and

$$\mathbf{F} = \frac{y\mathbf{e}_x}{(x^2 + y^2 + xy)^2} + \frac{-x\mathbf{e}_y}{(x^2 + y^2 + xy)^2}$$