Vv417 Lecture 18

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- We now consider a special linear transformation, which has a special name,

$$T\colon \mathcal{V} \to \mathcal{F}$$

where $\mathcal{V}$ is a vector space and $\mathcal{F}$ is the scalar field of $\mathcal{V}$.

Definition

A linear functional f is a linear transformation from $\mathcal{V}$ to $\mathcal{F}$, that is,

$$f\colon \mathcal{V} \to \mathcal{F}$$

Q: Have you seen linear functionals before?

$$f\colon \mathbb{R}^{n\times n} \to \mathbb{R} \qquad \text{where} \qquad f(\mathbf{A}) = \operatorname{tr}(\mathbf{A})$$

- Recall $\mathbb{R}^{n\times n}$ is is a vector space of real matrices over real.
- Notice determinant is also a transformation of a similar kind but not linear.

$$T\colon \mathbb{R}^{n\times n} \to \mathbb{R}$$

- Let $\mathcal{V}$ denote $\mathcal{C}^0[0, 2\pi]$ over $\mathbb{R}$, then for a given function $g \in \mathcal{V}$,

$$f : \mathcal{V} \to \mathbb{R} \quad \text{defined by} \quad f\left[h\right] = \frac{1}{2\pi} \int_0^{2\pi} h(t)g(t)\, dt$$

  is a linear functional on $\mathcal{V}$. In the cases when $g(t)$ is

$$\sin kt \qquad \text{or} \qquad \cos kt, \qquad \text{where } k \in \mathbb{Z},$$

  then the real value $f\left[h\right]$ is actually the kth Fourier coefficient of $h(t)$.
- Now suppose $\mathcal{V}$ is n-dimensional over $\mathbb{R}$, and $\mathcal{B}$ is a basis for $\mathcal{V}$, then

$$f_i : \mathcal{V} \to \mathbb{R} \quad \text{defined by} \quad f_i(\mathbf{v}) = \alpha_i$$

  where α_i is the ith element of $[\mathbf{v}]_{\mathcal{B}}$, is a linear functional on $\mathcal{V}$,
- The linear functional f_i is known as the

  ith coordinate function with respect to the basis $\mathcal{B}$.

Q: Is the set of all linear functionals on $\mathcal{V}$ a vector space?

Definition

For a vector space $\mathcal{V}$ over $\mathcal{F}$, the dual space of $\mathcal{V}$ is the vector space $\mathcal{L}(\mathcal{V}, \mathcal{F})$,

$$\mathcal{V}^*$$

- Notice $\mathcal{V}^*$ usee the usual addition and multiplication for transformations

$$(T + S)(\mathbf{u}) = T(\mathbf{u}) + S(\mathbf{u})$$
$$(\alpha T)(\mathbf{u}) = \alpha T(\mathbf{u}), \quad \text{where } \alpha \in \mathcal{F}.$$

- Since we consider only finite-dimensional $\mathcal{V}$, according the basis for $\mathcal{L}(\mathcal{U}, \mathcal{V})$

$$\dim(\mathcal{V}^*) = \dim(\mathcal{L}(\mathcal{V}, \mathcal{F})) = \dim(\mathcal{V}) \dim(\mathcal{F}) = \dim(\mathcal{V})$$

Q: Do you expect $\mathcal{V}$ and $\mathcal{V}^*$ are somehow related to each other?

Theorem

Suppose that $\mathcal{V}$ is a finite-dimensional vector space with the basis

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$$

Let f_i be the ith coordinate function with respect to $\mathcal{B}$, then the set

$$\mathcal{B}^* = \{f_1, f_2, \ldots, f_n\}$$

is a basis for $\mathcal{V}^*$, and for $f \in \mathcal{V}^*$, we have

$$f = \sum_{i=1}^{n} f(\mathbf{b}_i) f_i$$

Proof

- Let $f \in \mathcal{V}^*$, since $\dim(\mathcal{V}^*) = n$, we need only show $\mathcal{B}^*$ is a spanning set

$$f = \sum_{i=1}^{n} f(\mathbf{b}_i) f_i$$

Proof

- That is, we need to show, for any $\mathbf{u} \in \mathcal{V}$ and $f \in \mathcal{V}^*$, the following holds

$$f(\mathbf{u}) = \left(\sum_{i=1}^{n} f(\mathbf{b}_i) f_i\right)(\mathbf{u})$$

- Since $f \in \mathcal{V}^*$ and $f_i \in \mathcal{V}^*$ for all i, the right hand side and f must be linear,

$$f(\mathbf{u}) = \alpha_1 f\left(\mathbf{b}_1\right) + \alpha_2 f\left(\mathbf{b}_2\right) + \cdots + \alpha_n f\left(\mathbf{b}_n\right)$$

$$\left(\sum_{i=1}^{n} f(\mathbf{b}_i) f_i\right)(\mathbf{u}) = \alpha_1 \left(\sum_{i=1}^{n} f(\mathbf{b}_i) f_i\right)(\mathbf{b}_1) + \cdots + \alpha_n \left(\sum_{i=1}^{n} f(\mathbf{b}_i) f_i\right)(\mathbf{b}_n)$$

- Hence we only need to show

$$f(\mathbf{b}_j) = \left(\sum_{i=1}^{n} f(\mathbf{b}_i) f_i\right)(\mathbf{b}_j) \qquad \text{for any } 1 \geq j \geq n.$$

Proof

- For $1 \leq j \leq n$, applying the functional to $\mathbf{b}_j \in \mathcal{B}$, we have

$$\begin{aligned}\left(\sum_{i=1}^{n} f(\mathbf{b}_i) f_i\right)(\mathbf{b}_j) &= \sum_{i=1}^{n} f(\mathbf{b}_i) f_i(\mathbf{b}_j) \\ &= \sum_{i=1}^{n} f(\mathbf{b}_i)\delta_{ij}, \qquad \text{where} \quad \delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases} \\ &= f(\mathbf{b}_j)\end{aligned}$$

Definition

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis of $\mathcal{V}$, then the following basis of $\mathcal{V}^*$

$$\mathcal{B}^* = \{f_1, f_2, \ldots, f_n\} \qquad \text{where} \quad f_i(\mathbf{b}_j) = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

is the ith coordinate function with respect to $\mathcal{B}$, is known as the dual basis of $\mathcal{B}$.

- Given the connection between f_i and $\mathbf{b}_i$, the dual basis is often denoted as

$$\mathcal{B}^* = \{\mathbf{b}^1, \mathbf{b}^2, \dots, \mathbf{b}^n\} \qquad \text{where} \quad \mathbf{b}^i(\mathbf{b}_j) = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

Exercise

Find the dual basis $\mathcal{B}^*$ of $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, where $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Solution

- By definition, the dual basis consists linear functionals f_1 and f_2 such that

$$\mathbf{b}^1(\mathbf{b}_1) = 1 \implies \mathbf{b}^1(2\mathbf{e}_1 + \mathbf{e}_2) = 1 \implies 2\mathbf{b}^1(\mathbf{e}_1) + \mathbf{b}^1(\mathbf{e}_2) = 1$$
$$\mathbf{b}^1(\mathbf{b}_2) = 0 \implies \mathbf{b}^1(3\mathbf{e}_1 + \mathbf{e}_2) = 1 \implies 3\mathbf{b}^1(\mathbf{e}_1) + \mathbf{b}^1(\mathbf{e}_2) = 0$$

thus $\mathbf{b}^1(\mathbf{e}_1) = -1$ and $\mathbf{b}^1(\mathbf{e}_2) = 3,$ and the transformation matrix is

$$\mathbf{A}_1 = \begin{bmatrix} -1 & 3 \end{bmatrix} \implies \mathbf{b}^1 : \mathbb{R}^2 \to \mathbb{R} \qquad \text{defined by} \quad \mathbf{b}^1(\mathbf{u}) = \mathbf{A}_1\mathbf{u}$$

Solution

- Similarly, solving the system based on $\mathbf{b}^2(\mathbf{b}_1) = 0$ and $\mathbf{b}^2(\mathbf{b}_2) = 1$, we have

$$\begin{aligned} \mathbf{b}^2(\mathbf{e}_1) &= 1 \\ \mathbf{b}^2(\mathbf{e}_2) &= -2 \end{aligned} \implies \boxed{\mathbf{A}_2 = \begin{bmatrix} 1 & 2 \end{bmatrix}}$$

$$\implies \mathbf{b}^2 \colon \mathbb{R}^2 \to \mathbb{R} \qquad \text{defined by} \quad \mathbf{b}^2(\mathbf{u}) = \mathbf{A}_2\mathbf{u}$$

- The dual basis of $\mathcal{B}$ is $\mathcal{B}^* = \{\mathbf{b}^1, \mathbf{b}^2\}$, where $\mathbf{b}^1$ and $\mathbf{b}^2$ are defined above.

- Consider the coordinate matrix of $T \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ with respect to $(\mathcal{B}_\mathcal{U}, \mathcal{B}_\mathcal{V})$,

$$[T(\mathbf{u})]_{\mathcal{B}_\mathcal{V}} = [T]_{\mathcal{B}_\mathcal{U}\mathcal{B}_\mathcal{V}} [\mathbf{u}]_{\mathcal{B}_\mathcal{U}}$$

- It is clear which transformation is associated with $[T]^{-1}_{\mathcal{B}_\mathcal{U}\mathcal{B}_\mathcal{V}}$ when it exits.

Q: How about the transformation associated with the matrix $[T]^{\mathrm{T}}_{\mathcal{B}_\mathcal{U}\mathcal{B}_\mathcal{V}}$?

Q: Suppose there is a linear transformation S such that the coordinate matrix

$$[S] = [T]^{\mathrm{T}}_{\mathcal{B}_\mathcal{U}\mathcal{B}_\mathcal{V}}$$

is there any relationship between S and T and under which bases this holds?

Theorem

Let $\mathcal{U}$ and $\mathcal{V}$ be vector spaces over $\mathcal{F}$ with bases $\mathcal{B}_{\mathcal{U}}$ and $\mathcal{B}_{\mathcal{V}}$, respectively, and let

$$f \in \mathcal{V}^* \qquad \text{and} \qquad T \in \mathcal{L}(\mathcal{U}, \mathcal{V})$$

that is, both $f : \mathcal{V} \to \mathbb{R}$ and $T : \mathcal{U} \to \mathcal{V}$ are linear, then the following

$$S : \mathcal{V}^* \to \mathcal{U}^* \qquad \text{defined by} \quad S[f] = f \circ T$$

is a linear transformation between the dual space of $\mathcal{U}$ to $\mathcal{V}$ such that

$$[S]_{\mathcal{B}_{\mathcal{V}}^* \mathcal{B}_{\mathcal{U}}^*} = [T]_{\mathcal{B}_{\mathcal{U}} \mathcal{B}_{\mathcal{V}}}^{\mathrm{T}}$$

where $\mathcal{B}_{\mathcal{V}}^*$ and $\mathcal{B}_{\mathcal{U}}^*$ are dual bases of $\mathcal{B}_{\mathcal{U}}$ and $\mathcal{B}_{\mathcal{V}}$, respectively,

Proof

- Since any $f \in \mathcal{V}^*$ and $T \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ are linear, for $\alpha, \beta \in \mathcal{F}$, $\mathbf{x}, \mathbf{y} \in \mathcal{U}$,

$$(f \circ T)(\alpha \mathbf{x} + \beta \mathbf{y}) = f \circ (\alpha T(\mathbf{x}) + \beta T(\mathbf{y})) = \alpha (f \circ T)(\mathbf{x}) + \beta (f \circ T)(\mathbf{y})$$

Proof

- Let $\mathbf{A} = [T]^{\mathrm{T}}_{\mathcal{B}_{\mathcal{U}}\mathcal{B}_{\mathcal{V}}}$ and the followings be bases for the corresponding spaces

$$\begin{aligned} \mathcal{B}_{\mathcal{U}} &= \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\} \qquad \text{and} \qquad \mathcal{B}_{\mathcal{V}} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \\ \mathcal{B}^*_{\mathcal{U}} &= \{\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^n\} \qquad \text{and} \qquad \mathcal{B}^*_{\mathcal{V}} = \{\mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^m\} \end{aligned}$$

- The jth column of $[S]_{\mathcal{B}^*_{\mathcal{V}}\mathcal{B}^*_{\mathcal{U}}}$ by the definition of coordinate matrix is given by

$$[S(\mathbf{v}^j)]_{\mathcal{B}^*_{\mathcal{U}}} = [\mathbf{v}^j \circ T]_{\mathcal{B}^*_{\mathcal{U}}} = \begin{bmatrix} \left(\mathbf{v}^j \circ T\right)(\mathbf{u}_1) \\ \left(\mathbf{v}^j \circ T\right)(\mathbf{u}_2) \\ \vdots \\ \left(\mathbf{v}^j \circ T\right)(\mathbf{u}_n) \end{bmatrix}$$

according to the definition and properties of dual basis, that is,

$$f = \sum_{i=1}^{n} f(\mathbf{b}_i)\mathbf{b}^i$$

Proof

- Hence we need to consider

$$\left(\mathbf{v}^j \circ T\right)\left(\mathbf{u}_k\right)$$

- Note $\mathbf{v}^j \circ T$ is a linear functional on $\mathcal{U}$, so acting it on $\mathbf{u}_k$, we have

$$\left(\mathbf{v}^j \circ T\right)\left(\mathbf{u}_k\right) = \mathbf{v}^j\Big(T\left(\mathbf{u}_k\right)\Big)$$

- Recall the definition of coordinate matrix

$$[T]_{\mathcal{B}_\mathcal{U}\mathcal{B}_\mathcal{V}} = \begin{bmatrix} [T(\mathbf{u}_1)]_{\mathcal{B}_\mathcal{V}} & [T(\mathbf{u}_2)]_{\mathcal{B}_\mathcal{V}} & \cdots & [T(\mathbf{u}_n)]_{\mathcal{B}_\mathcal{V}} \end{bmatrix} = \mathbf{A}^{\mathrm{T}}$$

- Thus putting the coefficients and basis vectors together, we have

$$\left(\mathbf{v}^j \circ T\right)\left(\mathbf{u}_k\right) = \mathbf{v}^j\left(\sum_{\ell=1}^{m} a_{k\ell}\mathbf{v}_\ell\right)$$

Proof

- Since $\mathbf{v}^j$ is the jth coordinate function with respect to $\mathcal{B}_\mathcal{V}$, we have

$$\mathbf{v}^j\left(\mathbf{v}_i\right)=\delta_{ij} \implies \left(\mathbf{v}^j \circ T\right)\left(\mathbf{u}_k\right)=\mathbf{v}^j\left(\sum_{\ell=1}^{m} a_{k\ell}\mathbf{v}_\ell\right)=a_{kj}$$

which means $[S(\mathbf{v}^j)]_{\mathcal{B}_\mathcal{U}^*}=[\mathbf{v}^j \circ T]_{\mathcal{B}_\mathcal{U}^*}=\begin{bmatrix}\left(\mathbf{v}^j \circ T\right)\left(\mathbf{u}_1\right)\\ \left(\mathbf{v}^j \circ T\right)\left(\mathbf{u}_2\right)\\ \vdots \\ \left(\mathbf{v}^j \circ T\right)\left(\mathbf{u}_n\right)\end{bmatrix}=\begin{bmatrix}a_{1j}\\ a_{2j}\\ \vdots\\ a_{mj}\end{bmatrix}$.

- Therefore,

$$[S]_{\mathcal{B}_\mathcal{V}^*\mathcal{B}_\mathcal{U}^*}=\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \vdots\\ a_{m1} & \cdots & \cdots & a_{mn}\end{bmatrix}=\mathbf{A}=[T]_{\mathcal{B}_\mathcal{U}\mathcal{B}_\mathcal{V}}^{\mathrm{T}}$$

Definition

Let $\mathcal{U}$ and $\mathcal{V}$ be vector spaces over $\mathcal{F}$ with bases $\mathcal{B}_{\mathcal{U}}$ and $\mathcal{B}_{\mathcal{V}}$, respectively, and

$$f \in \mathcal{V}^* \qquad \text{and} \qquad T \in \mathcal{L}\left(\mathcal{U}, \mathcal{V}\right)$$

then the following is known as the transpose/dual transformation of T,

$$T^* \colon \mathcal{V}^* \to \mathcal{U}^* \qquad \text{defined by} \quad T^*\left[f\right] = f \circ T$$

Exercise

Verify the last theorem regarding the transpose of a linear transformation using

$$T \colon \mathcal{P}_1 \to \mathbb{R}^2 \quad \text{by} \quad T(p) = \begin{bmatrix} p(0) \\ p(2) \end{bmatrix}$$

Solution

- Let $[T]_{\mathcal{B}_{\mathcal{P}_1}\mathcal{B}_{\mathbb{R}^2}}$ be the coordinate matrix with

$$\mathcal{B}_{\mathcal{P}_1} = \{1, x\} \qquad \text{and} \qquad \mathcal{B}_{\mathbb{R}^2} = \{\mathbf{e}_1, \mathbf{e}_2\}$$

Solution

- The coordinate matrix is given by

$$[T]_{\mathcal{B}_{\mathcal{P}_1}\mathcal{B}_{\mathbb{R}^2}} = \begin{bmatrix} [T(1)]_{\mathcal{B}_{\mathbb{R}^2}} & [T(x)]_{\mathcal{B}_{\mathbb{R}^2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

- We can compute $[T^*]_{\mathcal{B}_{\mathbb{R}^2}\mathcal{B}_{\mathcal{P}_1}}$ directly from the definition. Let

$$\mathcal{B}^*_{\mathcal{P}_1} = \{\mathbf{p}^1, \mathbf{p}^2\} \qquad \text{and} \qquad \mathcal{B}^*_{\mathbb{R}^2} = \{\mathbf{e}^1, \mathbf{e}^2\}$$

- Suppose $[T^*]_{\mathcal{B}^*_{\mathbb{R}^2}\mathcal{B}^*_{\mathcal{P}_1}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then it implies $T^*\left[\mathbf{e}^1\right] = a\mathbf{p}^1 + c\mathbf{p}^2$.
- Apply this functional in $\mathcal{P}_1^*$ to both elements in $\mathcal{B}_{\mathcal{P}_1} = \{\mathbf{p}_1, \mathbf{p}_2\} = \{1, x\}$.

$$\begin{aligned} \left(T^*\left[\mathbf{e}^1\right]\right)(1) &= \left(a\mathbf{p}^1 + c\mathbf{p}^2\right)(\mathbf{p}_1) \\ \mathbf{e}^1\left(T(1)\right) &= a\mathbf{p}^1(\mathbf{p}_1) + c\mathbf{p}^2(\mathbf{p}_1) \\ \mathbf{e}^1\left(\mathbf{e}_1 + \mathbf{e}_2\right) &= a \implies a = 1 \end{aligned}$$

$$\begin{aligned} \left(T^*\left[\mathbf{e}^1\right]\right)(x) &= \left(a\mathbf{p}^1 + c\mathbf{p}^2\right)(\mathbf{p}_2) \\ \mathbf{e}^1\left(T(x)\right) &= a\mathbf{p}^1(\mathbf{p}_2) + c\mathbf{p}^2(\mathbf{p}_2) \\ \mathbf{e}^1\left(0\mathbf{e}_1 + 2\mathbf{e}_2\right) &= c \implies c = 0 \end{aligned}$$

Solution

- By similar computations, we obtain $b = 1$ and $d = 2$, hence

$$[T^*]_{\mathcal{S}^*_{\mathbb{R}^2}\mathcal{S}^*_{\mathcal{P}_1}} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

which confirms $[T^*]_{\mathcal{S}^*_{\mathbb{R}^2}\mathcal{S}^*_{\mathcal{P}_1}} = [T]^{\mathrm{T}}_{\mathcal{S}_{\mathcal{P}_1}\mathcal{S}_{\mathbb{R}^2}}$ as stated by the last theorem.

- The notion of matrix transformation $T_{\mathbf{A}} = \mathbf{A}\mathbf{x}$ and linear transformation

$$T : \mathcal{U} \to \mathcal{V}$$

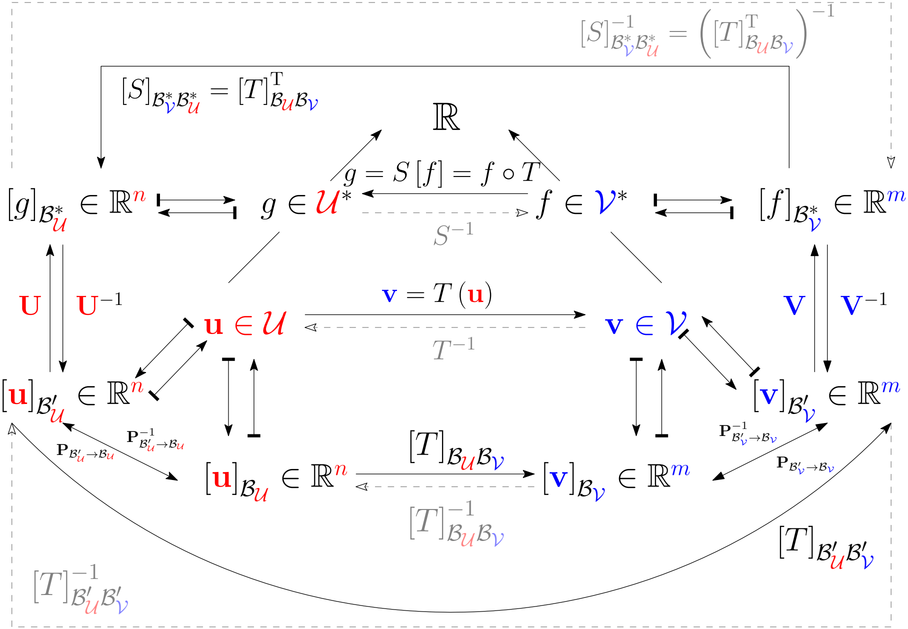
and the notion of basis $\mathcal{B}$, isomorphism, and coordination transformation

$$\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$$

put different vector spaces into the same picture, while dual spaces/bases

$$\mathcal{B}^*$$

give us new understanding regarding the transpose of a matrix and more.


$[S]^{-1}_{\mathcal{B}^*_{\mathcal{V}}\mathcal{B}^*_{\mathcal{U}}} = \left([T]^{\mathrm{T}}_{\mathcal{B}_{\mathcal{U}}\mathcal{B}_{\mathcal{V}}}\right)^{-1}$
$[S]_{\mathcal{B}^*_{\mathcal{V}}\mathcal{B}^*_{\mathcal{U}}} = [T]^{\mathrm{T}}_{\mathcal{B}_{\mathcal{U}}\mathcal{B}_{\mathcal{V}}}$
$\mathbb{R}$
$g = S[f] = f \circ T$
$[g]_{\mathcal{B}^*_{\mathcal{U}}} \in \mathbb{R}^n$
$g \in \mathcal{U}^*$
$f \in \mathcal{V}^*$
$[f]_{\mathcal{B}^*_{\mathcal{V}}} \in \mathbb{R}^m$
S^{-1}
$\mathbf{U}$
$\mathbf{U}^{-1}$
$\mathbf{v} = T(\mathbf{u})$
$\mathbf{V}$
$\mathbf{V}^{-1}$
$\mathbf{u} \in \mathcal{U}$
$\mathbf{v} \in \mathcal{V}$
T^{-1}
$[\mathbf{u}]_{\mathcal{B}'_{\mathcal{U}}} \in \mathbb{R}^n$
$[\mathbf{v}]_{\mathcal{B}'_{\mathcal{V}}} \in \mathbb{R}^m$
$\mathbf{P}_{\mathcal{B}'_{\mathcal{U}}\to\mathcal{B}_{\mathcal{U}}}$
$\mathbf{P}^{-1}_{\mathcal{B}'_{\mathcal{U}}\to\mathcal{B}_{\mathcal{U}}}$
$[T]_{\mathcal{B}_{\mathcal{U}}\mathcal{B}_{\mathcal{V}}}$
$\mathbf{P}^{-1}_{\mathcal{B}'_{\mathcal{V}}\to\mathcal{B}_{\mathcal{V}}}$
$\mathbf{P}_{\mathcal{B}'_{\mathcal{V}}\to\mathcal{B}_{\mathcal{V}}}$
$[\mathbf{u}]_{\mathcal{B}_{\mathcal{U}}} \in \mathbb{R}^n$
$[\mathbf{v}]_{\mathcal{B}_{\mathcal{V}}} \in \mathbb{R}^m$
$[T]^{-1}_{\mathcal{B}_{\mathcal{U}}\mathcal{B}_{\mathcal{V}}}$
$[T]_{\mathcal{B}'_{\mathcal{U}}\mathcal{B}'_{\mathcal{V}}}$
$[T]^{-1}_{\mathcal{B}'_{\mathcal{U}}\mathcal{B}'_{\mathcal{V}}}$

- With this new understanding regarding various spaces, let us consider

$$\mathcal{P}_n \quad \text{over} \quad \mathbb{R}$$

the polynomial space of degree n or less in general, and the standard basis

$$\mathcal{B}_{\mathcal{P}_n} = \{\mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_n\} \ , \qquad \text{where} \quad \mathbf{b}_k = x^k$$

Q: Can you think of a linear functional on $\mathcal{P}_n$, thus understand $\mathcal{P}_n^*$?

$$f(p) = p(a) \qquad \text{where} \quad p \in \mathcal{P}_n, a \in \mathbb{R}$$

Q: Can you work out the dual basis of $\mathcal{B}_{\mathcal{P}_n}$?

$$\mathcal{B}^* = \{\mathbf{b}^0, \mathbf{b}^1, \mathbf{b}^2, \dots, \mathbf{b}^n\} \qquad \text{where} \quad \mathbf{b}^j(\mathbf{b}_i) = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

- Check the following for any polynomial $p(x) = \gamma_0 + \gamma_1 x + \cdots + \gamma_n x^n \in \mathcal{P}_n$,

$$\mathbf{b}^k(p) = \frac{1}{k!}\frac{d^k}{dx^k}p(x)\bigg|_{x=0} = \frac{p^{(k)}(0)}{k!} = \gamma_k$$

Q: What does this give us? Have you seen this before?

- Let x_0, x_1, ..., x_n be distinct points, and consider the following functionals

$$f_i(p) = p(x_i) \qquad \text{for} \quad p \in \mathcal{P}_n \quad \text{and} \quad i = 0, 1, \ldots, n$$

Q: Why is the set linearly independent? And what does it mean?

$$\{f_0, f_1, f_2, \ldots, f_n\}$$

- Consider applying the functionals to the following polynomials,

$$\begin{aligned} \ell_j(x) &= \prod_{\substack{0 \le m \le n \\ m \ne j}} \frac{x - x_m}{x_j - x_m} \qquad \text{for } j = 0, 1, 2, \ldots, n, \\ &= \frac{x - x_0}{x_j - x_0} \cdots \frac{x - x_{j-1}}{x_j - x_{j-1}} \frac{x - x_{j+1}}{x_j - x_{j+1}} \cdots \frac{x - x_n}{x_j - x_n} \end{aligned}$$

- It is clear that we have a pair of dual bases,

$$f_i(\ell_j) = \delta_{ij}$$

Q: What does this give us? Have you seen this before?