Vv256 Lecture 3

Dr Jing Liu

UM-SJTU Joint Institute

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- We have discussed so far how to deal with a first-order equation that is

$$\alpha\dot{y} + \beta y = \gamma, \qquad \text{or} \qquad \dot{y} = GF$$

- Recall the key idea for linear equations is

$$\Big(\mu\alpha\Big)\dot{y} + y\frac{d}{dt}\Big(\mu\alpha\Big) = \mu\gamma \qquad \text{the product/chain rule}$$

while the key idea for separable equations is

$$\frac{d}{dy}\left(\int_{y_0}^{y}\frac{1}{G(\eta)}\,d\eta\right)\frac{dy}{dt} = \frac{d}{dt}\left(\int_{t_0}^{t}F(\tau)\,d\tau\right) \qquad \text{the chain rule}$$

- Now consider a first-order equation that has the following form

$$\frac{dy}{dx} = \Phi(x,y) = -\frac{M(x,y)}{N(x,y)} \iff M(x,y) + N(x,y)y' = 0$$

where the unknown function y is a function of x.

- For example consider the following

$$2x + y^2 + 2xyy' = 0$$

which is neither linear nor separable. However, it is special in its own way.

- Note the partial derivatives of the function

$$\Psi(x, y) = x^2 + xy^2$$

are the coefficients of the differential equation

$$\frac{\partial \Psi}{\partial x} = 2x + y^2; \qquad \frac{\partial \Psi}{\partial y} = 2xy$$

Q: How can we make use of this connection between $\Psi(x, y)$ and the equation?

$$\frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y} y' = 0$$

- In general, when a differential equation can be written as

$$\frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y} y' = 0$$

then according to the chain rule, we have

$$\frac{d\Psi}{dx} = \frac{\partial \Psi}{\partial x}\frac{dx}{dx} + \frac{\partial \Psi}{\partial y}\frac{dy}{dx} = \frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y} y'$$

which means its solutions must satisfy the following equation

$$\Longrightarrow \ \frac{d}{dx}\Big[\Psi(x,y)\Big] = 0$$

- So solutions to the differential equation must satisfy the implicit equation

$$\Psi(x,y) = C \qquad \text{where } C \text{ is an arbitrary constant.}$$

- Back to our equation,

$$2x + y^2 + 2xyy' = 0$$

$$\implies \frac{\partial}{\partial x}\left(x^2 + xy^2\right) + \frac{\partial}{\partial y}\left(x^2 + xy^2\right) \cdot \frac{dy}{dx} = 0$$

the solution $y(x)$ thus must satisfy the following relation

$$\frac{\partial}{\partial x}\left(x^2 + xy^2\right) \cdot \frac{dx}{dx} + \frac{\partial}{\partial y}\left(x^2 + xy^2\right) \cdot \frac{dy}{dx} = 0$$

$$\implies \frac{d}{dx}\left(x^2 + xy^2\right) = 0$$

- Therefore the following is an implicit solution to the differential equation.

$$x^2 + xy^2 = C \qquad \text{where } C \text{ is an arbitrary constant.}$$

Definition

In general, a differential equation

$$M(x,y) + N(x,y)y' = 0$$

is known as exact if there is a differentiable function $\Psi(x,y)$ such that

$$\frac{\partial \Psi}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial \Psi}{\partial y} = N(x,y)$$

and the following is an implicit solution to the differential equation

$$\Psi(x,y) = C$$

where Ψ is known as a potential function for the differential equation.

- To solve an exact equation, the key step is to find the potential function

$$\Psi(x,y)$$

whose partial derivatives are the coefficients of the differential equation.

The general technique of finding the potential function

1. Integrate M w.r.t x to obtain Ψ up to a function $f(y)$ of y alone.

$$\Psi(x,y) = \Psi_1(x,y) + f(y), \quad \text{where } \Psi_1(x,y) = \int M(x,y)\,dx, \text{ and } f(y)$$

   is an unknown function that plays the role of the constant of integration.

2. Differentiate $\Psi = \Psi_1 + f$ w.r.t y to obtain

$$\frac{\partial}{\partial y}\Big[\Psi_1(x,y)\Big] + f'(y) = N(x,y), \quad \text{and solve for } f'(y).$$

   compare the left with the right to find $f'(y)$.

3. Integrate $f'(y)$ w.r.t y to complete the definition of Ψ, up to a constant.

Exercise

Find the implicit solution to the differential equation $ye^{xy} + (2y + xe^{xy})y' = 0$.

Q: Have you seen a similar technique before?

- Consider the following vector field, let's say it is a force field of some kind

$$\mathbf{F}(x, y) = M(x, y)\mathbf{e}_x + N(x, y)\mathbf{e}_y$$

where M and N are the functions in the original equation,

$$M(x, y) + N(x, y)y'(x) = 0$$

Q: What do $\Psi(x, y)$ and $y = y(x)$ represent in terms of this vector field?

- The relation is even clearer if we consider the related autonomous system

$$\dot{x} = N(x, y), \qquad \dot{y} = -M(x, y)$$

a particular solution $\{x(t), y(t)\}$ of which is actually a parametric solution to

$$M(x, y) + N(x, y)y'(x) = 0$$

- The solutions of the system are trajectories, parametric curves with a velocity

$$\mathbf{v} = \dot{x}\mathbf{e}_x + \dot{y}\mathbf{e}_y = N(x, y)\mathbf{e}_x - M(x, y)\mathbf{e}_y$$

- On the other hand, the solutions of the corresponding differential equation are the same curves, the same trajectories but without any specific velocity.
- Notice the velocity $\mathbf{v}$ is always orthogonal to the vector $\mathbf{F}$

$$\mathbf{F} \cdot \mathbf{v} = \Big(M(x, y)\mathbf{e}_x + N(x, y)\mathbf{e}_y\Big) \cdot \Big(N(x, y)\mathbf{e}_x - M(x, y)\mathbf{e}_y\Big) = 0$$

- From the perspective of force fields in physics, if the potential function exists

$$\Psi(x, y)$$

then the solutions are equipotential lines in that conservative field.
- Surely not every vector field is conservative, and not every equation is exact

$$\mathbf{F}(x, y) = M(x, y)\mathbf{e}_x + N(x, y)\mathbf{e}_y$$

- However, notice there are a many autonomous systems that correspond to the same differential equation, namely all systems of the form

$$\dot{x} = \mu N(x,y), \qquad \dot{y} = -\mu M(x,y)$$

  where μ is a function of x and y.

- Hence it is possible in theory to introduce an integrating factor μ to convert a non-conservative field into a conservative one

$$\mathbf{G} = \mu\mathbf{F} = \mu M(x,y)\mathbf{e}_x + \mu N(x,y)\mathbf{e}_y = \Psi^*_x\mathbf{e}_x + \Psi^*_y\mathbf{e}_y$$

  thus converting a non-exact first-order differential equation into an exact one

$$M + Ny' = 0 \implies \mu M + \mu N y' = 0$$

- However, this often runs into partial differential equations which are difficult to solve and will be left to more advanced courses.

- It is possible to determine whether a first-order equation is exact or not without first finding the potential function.

Theorem

Let the functions M, N be continuous with continuous partial derivatives in some simply-connected region $\mathcal{D}$. Then there exists a potential function Ψ such that

$$M = \frac{\partial \Psi}{\partial x} \qquad \text{and} \qquad N = \frac{\partial \Psi}{\partial y}$$

if and only if

$$M_y = N_x \qquad \text{for all points in } \mathcal{D}$$

Q: Are linear equations always exact?

$$\alpha(x)y' + \beta(x)y = \gamma(x)$$

Q: How about separable?

- Sometimes we can solve differential equations by changing the variables.

$$v = h(t, y)$$

- For instance, the differential equation

$$\dot{y} = (t + y + 3)^2$$

practically demands the substitution

$$v = t + y + 3$$

which will reduce the equation into simpler one where we can to solve.

$$\dot{y} = v^2$$

Q: What do we need to do next?

Exercise

Solve the differential equation $\dot{y} = (t + y + 3)^2$.

- In general, if the substitution relation $v = h(t, y)$ can be solved for

$$y = \eta(t, v)$$

then by the chain rule, regarding v as an intermediate variable, we have

$$\dot{y} = \frac{dy}{dt} = \frac{\partial \eta}{\partial t}\frac{dt}{dt} + \frac{\partial \eta}{\partial v}\frac{dv}{dt} = \eta_t + \eta_v \dot{v}$$

- Thus the following new equation from the original first-order equation

$$\dot{y} = \Phi(t, y) \implies \dot{v} = \frac{\Phi(t, y) - \eta_t}{\eta_v} = \Psi(t, v)$$

- Once this new equation is solved, we then use use

$$y = \eta\big(t, v(t)\big)$$

to determine the function form of y.

Definition

A function $f(t)$ is said to be homogeneous of degree n if,

$$f(\lambda t) = \lambda^n f(t), \qquad \text{for all nonzero } \lambda.$$

Similarly, we have homogeneous functions of two variables, $f(t, y)$, of degree n if,

$$f(\lambda t, \lambda y) = \lambda^n f(t, y)$$

A first-order differential equation in which,

$$\dot{y} = \Phi(t, y) = \frac{M(t, y)}{N(t, y)},$$

where M and N are homogeneous functions of the same degree n, then

$$\dot{y} = \Phi(t, y)$$

is known as a homogeneous first-order differential equation.

Q: Are the following differential equations homogeneous?

$$1. \quad \dot{y} = \frac{-1-3t^2}{3y^2-4t} \qquad 2. \quad \dot{y} = \frac{-3t^2-y^2}{ty^2} \qquad 3. \quad \dot{y} = \frac{y^2+ty}{t^2}$$

- Homogeneous equations can be solved by using the substitution

$$v = \frac{y}{t}$$

- For a homogeneous equation, we can always rewrite it as the following

$$\dot{y} = \frac{M(t,y)}{N(t,y)} = \frac{\left(\frac{1}{t}\right)^n M(t,y)}{\left(\frac{1}{t}\right)^n N(t,y)} = \frac{M\left(1, \frac{y}{t}\right)}{N\left(1, \frac{y}{t}\right)}$$

which is separable/autonomous in terms of t and v.

- Let the homogeneous equation be denoted as

$$\dot{y} = \frac{M\left(1, \dfrac{y}{t}\right)}{N\left(1, \dfrac{y}{t}\right)} = \Psi\left(\frac{y}{t}\right)$$

- If we make the substitution, $v = h(t, y) = \dfrac{y}{t}$, then

$$y = \eta(t, v) = vt \implies \dot{y} = v + t\dot{v}$$

thus we have a separable equation of the following form

$$\dot{v} = \Big(\Psi(v) - v\Big)\frac{1}{t}$$

Exercise

Solve the differential equation $2ty\dot{y} = 4t^2 + 3y^2$.

Definition

A Bernoulli equation is a first-order differential equation of the form

$$\dot{y} + Py = Qy^n$$

where P and Q are functions of t.

- The following substitution can be used to solve a Bernoulli equation,

$$v = y^{1-n}$$

Exercise

(a) Solve the differential equation by treating it as a Bernoulli equation.

$$2ty\dot{y} = 4t^2 + 3y^2$$

(b) Solve the differential equation

$$\dot{y} = \sin(t - y)$$