VV255 Final Review

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Vector Field

1. Vector Field is a vector-valued function of several variable,

$$\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$$

2. $$\mathbf{F} = \frac{c}{|\mathbf{r}^3|}\mathbf{r}$$

   is called inverse-square field where $\mathbf{r}$ is a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ and c is a constant.

3. Draw vector field in Cartesian coordinate and Polar coordinate. Select the point in different way and draw the vector of that point.

Vector Field

1. The gradient of a function f is the gradient field.

$$\nabla f = f_x \mathbf{e}_x + f_y \mathbf{e}_y + f_z \mathbf{e}_z$$

2. The gradient field in polar

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta$$

3. The gradient field in cylindrical

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z$$

4. The gradient field in spherical

$$\nabla f = \frac{\partial f}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho \sin\phi}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi$$

Vector Field

The algebraic rule

1. $$\nabla(\alpha f \pm \beta g) = \alpha \nabla f \pm \beta g$$

2. $$\nabla(fg) = g\nabla f + f\nabla g$$

3. $$\nabla(\frac{f}{g}) = \frac{g\nabla f - f\nabla g}{g^2}$$

4. $$\nabla f^n = nf^{n-1}\nabla f$$

Line Integral

Line integral of scalar function:

$$\int_C f(x,y)ds = \int f(x(t),y(t))|\mathbf{r}'(t)|dt$$

This kind of integral is independent of the orientation of the path, which means $\int_C f(x,y)ds = \int_{-C} f(x,y)ds$.

In general, to calculate this kind of line integral, we need to find a suitable parametrization. When we do the integral using the parameter t, we always integrate from the smaller t to the larger t.

1. Line Segment from point **a** to **b**

$$\mathbf{r}(t) = t\mathbf{b} + (1-t)\mathbf{a}$$

2. Curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\mathbf{r}(t) = a\cos t\mathbf{e}_x + b\sin t\mathbf{e}_y$$

Line Integral

Line integral of vector-valued function:

$$\int_C \mathbf{F}(x,y)d\mathbf{r} = \int_a^b \mathbf{F}\cdot\mathbf{r}'(t)dt = \int_C Pdx + \int_C Qdy$$

In this case, the orientation of the path matters.

$$\int_C \mathbf{F}(x,y)d\mathbf{r} = -\int_{-C} \mathbf{F}(x,y)d\mathbf{r}$$

Line Integral

Some Physical meaning of the line integral

1. For the Line integral of scalar function:
   - The area of the vertical surface of $f(x, y)$ along the curve C.
   - The mass of the line (if f is the density).
2. For the Line integral of vector-valued function:
   - The work done by the force F along the path C.
   - The flow of the curve.

For the following vector field, determine whether the line integral $\int_C \mathbf{F} d\mathbf{r}$ is positive, negative or zero

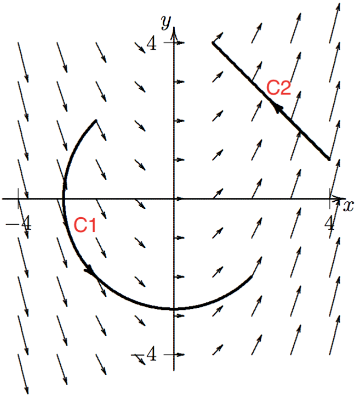

Figure: The vector field

Line Integral

- Find the line integral
$$\int_C y\mathbf{e}_x + x\mathbf{e}_y\, d\mathbf{r}$$
where C is the quarter-circle centered at the origin starting at $(0,2)$ and proceeding clockwise to $(2,0)$

Conservative Field

- A vector field $\mathbf{F}$ is said to be conservative in a region D if it is the gradient field for some function f in D, that is, if $\mathbf{F} = \nabla f$. f is called the potential function of the $\mathbf{F}$.
- - The vector field $\mathbf{F}$ is conservative in some open region D containing the points A and B.
  - Component functions are continuous in this region D.
  - C is a piecewise smooth curve starts at A and ends at B.
  - C lies entirely in the region D.

$$\int_C \mathbf{F} d\mathbf{r} = f(\mathbf{r}(a)) - f(\mathbf{r}(b))$$

Conservative Field

Suppose $\mathbf{F} = P\mathbf{e}_x + Q\mathbf{e}_y$ is defined on an open simply connected region D, and the partial derivatives

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

are equal and continuous throughout the region D, then $\mathbf{F}$ is conservative. For vector field in $\mathbb{R}^3$, we need

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \text{ and } \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

The open simply connected region is only needed when we use the conservative field test. If we can find the potential function directly in the region, we can say that this vector field is conservative no matter it is an open simply connected region or not.

Conservative Field

Suppose we have a continuous vector field $\mathbf{F}$ on a domain D in $\mathbb{R}^2$. Then the following conditions are equivalent:

- $\mathbf{F}$ is a conservative field
- $\mathbf{F}$ is a gradient field
- The line integral $\int_C \mathbf{F} d\mathbf{r}$ is independent of path
- For any close loop C, $\oint_C \mathbf{F} d\mathbf{r} = 0$

Furthermore, if D is open and simply connected, then they are equivalent to

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Conservative Field

- Is inverse-square field a conservative field in its domain?
- Is the vector field $\mathbf{F} = \frac{-y}{x^2+y^2}\mathbf{e}_x + \frac{x}{x^2+y^2}\mathbf{e}_y$ a conservative field in its domain?

Conservative Field

- Find

$$\int_L \frac{(x-y)dx + (x+y)dy}{x^2+y^2}$$

where L is

$$\begin{cases} x = t - \sin t - \pi \\ y = 1 - \cos t \end{cases}$$

from $t = 0$ to $t = 2\pi$

Conservative Field

- Find the integral

$$\int_L (e^x \sin y - my)dx + (e^x \cos y - m)dy$$

where L is the upper half of the curve $x^2 + y^2 = ax$ from $A(a, 0)$ to $O(0, 0)$.

Green Theorem

A positively oriented simple closed curve C has the properties

- initial point is also the terminal point
- does not cross itself again
- the region D enclosed by the curve C is always on the left of the direction of motion.

Green Theorem

If C is a positively oriented, piecewise smooth, simple closed curve that encloses a region D, and P and Q have continuous first partial derivatives on some open region containing D, then the line integral of the vector along C

$$\text{Tangential Form} \quad \oint_C \mathbf{F} \cdot \mathbf{T} ds = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\text{Normal Form} \quad \oint_C \mathbf{F} \cdot \mathbf{n} ds = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA$$

where $\mathbf{T} = x'(s)\mathbf{e}_x + y'(s)\mathbf{e}_y$ and $\mathbf{n} = \mathbf{T} \times \mathbf{k} = y'(s)\mathbf{e}_x - x'(s)\mathbf{e}_y$

Green Theorem

Tangential Form: Flow and circulation

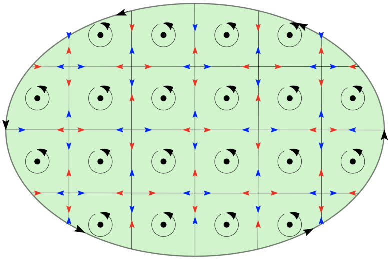

Figure: Flow and circulation

Normal Form: Flux

Green Theorem

Find the area of the region enclosed by the parameterized curve

$$\mathbf{r}(t) = (t - t^2)\mathbf{e}_x + (t - t^3)\mathbf{e}_y \quad 0 \leq t \leq 1$$

Green Theorem

Find the curve that maximize the line integral

$$\oint_C \left(x + \frac{y^3}{3}\right) dx + \left(y + x - \frac{2}{3}x^3\right) dy$$

Green Theorem

Use Green's Theorem to prove the change of variables formula for a double integral for the case where $f(x, y) = 1$.

$$\iint_R dxdy = \iint_S \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv$$

Here R is the region in the xy-plane that corresponds to the region S in the uv-plane under the transformation given by $x = g(u, v)$ and $y = h(u, v)$.