
Question1 (5 points)

Consider the matrices $\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 & 4 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 6 \\ 1 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 4 & 5 & 0 \\ 1 & 2 & 3 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 & 6 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 8 & 1 & 7 & -8 & 2 & 16 \\ 8 & 6 & 0 & -8 & 12 & 14 \\ 5 & 1 & 4 & -5 & 2 & 10 \\ 12 & 0 & 7 & -12 & 0 & 19 \\ 6 & 1 & 3 & -6 & 2 & 10 \\ 10 & 7 & 3 & -10 & 14 & 20 \end{bmatrix}$.

(a) (1 point) Find an upper triangular matrix $\mathbf{U}^*$ by elimination such that

$$\mathbf{A} \sim \mathbf{U}^*$$

(b) (1 point) Explain why $\mathbf{A}$ possesses an LU decomposition.

(c) (1 point) Find the LU decomposition for

$$\mathbf{A} = \mathbf{LU} \qquad \text{where } \mathbf{L} \text{ has unit diagonals.}$$

(d) (1 point) Given $\mathbf{A} = \mathbf{LU}$, solve $\mathbf{AX} = \mathbf{B}$ by using as few multiplications as possible.

(e) (1 point) Given $\mathbf{A} = \mathbf{LU}$, find $\mathbf{A}^{-1}$ by using as few multiplications as possible.

Question2 (5 points)

Consider the matrix $\mathbf{A} = \begin{bmatrix} 9 & 0 & 3 & 4 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 & 6 \\ 1 & 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 3 & 4 & 9 & 0 \\ 1 & 2 & 9 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 & 9 \end{bmatrix}$ and the vector $\mathbf{b} = \begin{bmatrix} 16 \\ 15 \\ 10 \\ 16 \\ 12 \\ 13 \end{bmatrix}$.

(a) (1 point) Explain why $\mathbf{A}$ is not strictly diagonally dominant.

(b) (1 point) Construct a matrix $\mathbf{A}^* \sim \mathbf{A}$ that is is strictly diagonally dominant.

(c) (1 point) Implement the Jacobi iteration in Matlab to solve

$$\mathbf{Ax} = \mathbf{b}$$

and obtain $\mathbf{x}^{(3)}$ by computing three iterations with the initial guess $\mathbf{x}^{(0)} = \mathbf{0}$.

(d) (1 point) Implement the Gauss-Seidel iteration in Matlab to solve

$$\mathbf{Ax} = \mathbf{b}$$

and obtain $\mathbf{x}^{(3)}$ by computing three iterations with the initial guess $\mathbf{x}^{(0)} = \mathbf{0}$.

(e) (1 point) The true solution to $\mathbf{Ax} = \mathbf{b}$ is

$$\mathbf{x} = \mathbf{1}$$

where $\mathbf{1}$ denote a vector of unit elements. Which of the two iterative methods seems to converge faster based on the squared common Euclidean distance

$$\left(\mathbf{x}^{(3)} - \mathbf{1}\right)^{\mathrm{T}} \left(\mathbf{x}^{(3)} - \mathbf{1}\right)$$