Vv156 Lecture 23

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- If a constant force of magnitude F is applied in the direction of motion of an object, and if that object moves a distance d, then the work W performed by the force on the object is

$$W = F \cdot d$$

Q: How should we define the work done if a variable force instead is applied in the direction of motion along a straight line?

1. Break up the distance travelled into subintervals that are sufficiently small so that the force does not vary much inside each subinterval

$$P(x, x^*)$$

2. Approximate the work done inside each subinterval by using a constant force.

$$W_k = F_k \cdot \Delta x_k, \qquad \text{where} \quad F_k = F(x_k^*) \qquad \text{for} \quad x \in [x_{k-1}, x_k]$$

3. Add the approximations to form a Riemann sum $W \approx \sum W_k$.
4. Take the limit of the Riemann sum to find the work, hopefully, $\sum W_k \to W$.

Defintion

The work done by a variable force

$$F(x)$$

in the direction of the x-axis between $x = a$ and $x = b$ is defined to be

$$W = \lim_{\max \Delta x_k \to 0} \sum F(x_k^*) \Delta x_k = \int_a^b F(x)\, dx$$

- If the force is constant, $F(x) = F$, it reduces to the formula for a constant force,

$$\begin{aligned} W &= \int_a^b F(x)\, dx \\ &= F \int_a^b \, dx \\ &= F \cdot \Big[x\Big]_a^b = F \cdot (b - a) \end{aligned}$$

Exercise

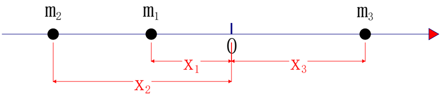
(a) *A spring exerts a force of* $5N$ *when stretched* $1m$ *beyond its natural length. How much work is required to stretch it* $1.8m$ *beyond its natural length?*

(b) *An astronaut's weight (or more precisely, earth weight) is the force exerted on the astronaut by the earth's gravity. If the earth is assumed to be a ball of radius* $6.37 \times 10^6 m$*, and its mass of* $5.98 \times 10^{24} kg$ *is concentrated at its centre. Ignore air resistance. Estimate the work required to lift an astronaut of* $100kg$ $350km$ *upward to the International Space Station.*

(c) *There is a conical container of radius* $10m$ *and height* $30m$ *on the ground. Suppose this container is filled with water to a depth of* $15m$*. How much work is required to pump all of the water out through a hole in the top of the container?*

Moments and Center of Mass: One-Dimensional System

- Consider point masses $m_1; m_2; \ldots; m_n$ located at $x_1; x_2; \ldots; x_n$. e.g.



Definition

The moment of this system of masses about the origin, denoted by M, is

$$M = m_1x_1 + m_2x_2 + m_3x_3$$

The center of mass of the system, denoted by $\bar{x}$, is

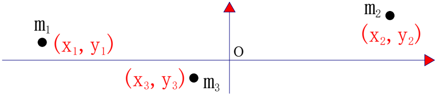
$$\bar{x} = \frac{M}{m}, \qquad \text{where} \quad m = \sum m_k$$

- The centre of mass $\bar{x}$ is the location at which the moment of the system is

$$\sum x_k m_k = \bar{x}m = \bar{x}\sum m_k = \sum \bar{x}m_k \implies \sum(x_k - \bar{x})m_k = 0$$

Moments and Center of Mass: Two-Dimensional System

- Consider point masses $m_1; \ldots; m_n$ located at $(x_1, y_1) \ldots (x_n, y_n)$, e.g.



Definition

The moment of this system of masses about the y-axis is

$$M_y = m_1x_1 + m_2x_2 + \cdots m_nx_n$$

The moment of this system of masses about the x-axis is

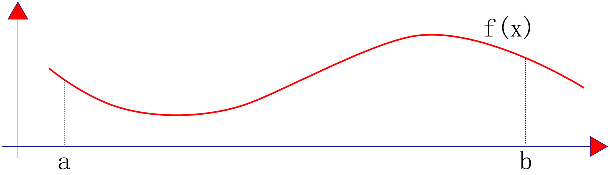
$$M_x = m_1y_1 + m_2y_2 + \cdots m_ny_n$$

The center of mass of the system is $(\bar{x} = \frac{M_y}{m}, \bar{y} = \frac{M_x}{m})$, where $m = \sum m_k$.

- Consider a lamina with density ρ, if it is
  1. homogeneous: $\rho = \dfrac{m}{V}$
  2. hetergeneous: $\rho(x)$ or $\rho(x, y)$
- Suppose the lamina occupies a region R in a horizontal xy-plane bounded by

$$y = f(x), \qquad y = 0, \qquad x = a, \qquad \text{and} \qquad x = b,$$

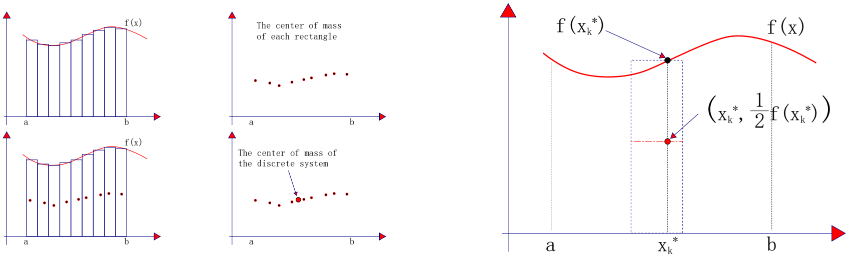
where $f(x)$ is a positive continuous function on the interval $[a, b]$.



Q: How to find the centre of mass of the lamina with its density depends only on x

$$\rho(x)$$

1. Divide the region into thin strips
2. Approximate the moment of each strip by the moment of a rectangle
3. Add the approximations to form a Riemann sum
4. Take the limit of the Riemann sum to find the centre of mass



- Let x_k^* be the x-coordinate of the centre of mass of each rectangle, then

$$\frac{1}{2} f(x_k^*)$$

is the y-coordinate of the centre of mass of each rectangle.

1. Let's work out the mathematical details,

$$\begin{aligned}\text{moment of strip about the } y\text{-axis} &\approx \text{moment of rectangle about the } y\text{-axis} \\ &= x_k^* m_k\end{aligned}$$

2. Similarly

$$\begin{aligned}\text{moment of strip about the } x\text{-axis} &\approx \text{moment of rectangle about the } x\text{-axis} \\ &= y_k^* m_k = \frac{1}{2} f(x_k^*) m_k\end{aligned}$$

3. Then centre of mass $(\bar{x}, \bar{y})$ of the discrete system is the point such that

$$\sum_{k=1}^{n} (x_k^* - \bar{x}) m_k = 0 \qquad \sum_{k=1}^{n} (y_k^* - \bar{y}) m_k = 0$$

$$\sum_{k=1}^{n} (x_k^* - \bar{x}) \, \rho(x_k^*) f(x_k^*) \Delta x_k = 0 \qquad \sum_{k=1}^{n} (\frac{1}{2} f(x_k^*) - \bar{y}) \, \rho(x_k^*) f(x_k^*) \Delta x_k = 0$$

4. As $\max \Delta x \to 0$, we expect $(\bar{x}, \bar{y})$ approaches the center of mass of the lamina.

- So we obtain following formulae for the center of mass of a lamina occupies a region $\mathcal{R}$ in a horizontal xy-plane bounded by the graphs of

$$y = f(x), \qquad y = 0, \qquad x = a, \qquad \text{and} \qquad x = b,$$

where $f(x)$ is a positive continuous function on the interval $[a, b]$.

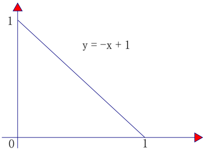
$$\int_a^b (x - \bar{x})\rho(x)f(x)\,dx = 0 \qquad \Longrightarrow \qquad \bar{x} = \frac{\displaystyle\int_a^b \rho x f\,dx}{\displaystyle\int_a^b \rho f\,dx},$$

$$\int_a^b \left(\frac{1}{2}f(x) - \bar{y}\right)\rho(x)f(x)\,dx = 0 \qquad \Longrightarrow \qquad \bar{y} = \frac{\displaystyle\int_a^b \frac{1}{2}\rho f^2\,dx}{\displaystyle\int_a^b \rho f\,dx}$$

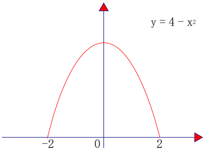
where $\rho(x)$ gives the density of the lamina at x.

Exercise

(a) *Find the centre of mass of the triangular lamina with vertices* $(0,0)$*,* $(0,1)$*, and* $(1,0)$ *and density* $\rho = 3$*.*



(b) *Find the centre of mass of a lamina covering the region bounded above by the parabola* $y = 4 - x^2$ *and below by the* x*-axis.*

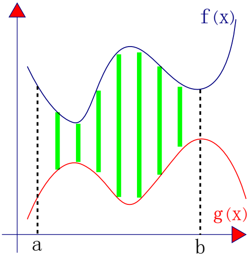


The density of the lamina at (x, y) *is given by* $\rho = 2x^2$*.*

- We can reply on the same sort of argument to obtain following formulae for the centre of mass of a lamina occupies the region R between continuous curves, $f(x)$, $g(x)$, $x = a$, and $x = b$, where $f(x) \geq g(x)$ and $b \geq a$.

$$\bar{x} = \frac{\displaystyle\int_a^b \rho x(f-g)\,dx}{\displaystyle\int_a^b \rho(f-g)\,dx},$$

$$\bar{y} = \frac{\displaystyle\int_a^b \frac{1}{2}\rho(f^2-g^2)\,dx}{\displaystyle\int_a^b \rho(f-g)\,dx}$$



where $\rho(x)$ gives the density of the lamina.

Exercise

Find the centre of mass for a lamina bounded above by $f(x) = \sqrt{x}$ and below by $g(x) = \frac{x}{2}$ between $x = 0$ and $x = 1$. The density of the lamina is given by $\rho = x^2$.

Centroid

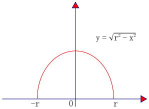
- When the density function ρ is constant, it cancels out of the numerator and denominator of the formulas for $\bar{x}$ and $\bar{y}$.

$$\bar{x} = \frac{\int_a^b \cancel{\rho} x f \, dx}{\int_a^b \cancel{\rho} f \, dx}, \qquad \bar{y} = \frac{\int_a^b \frac{1}{2} \cancel{\rho} f^2 \, dx}{\int_a^b \cancel{\rho} f \, dx} \qquad\qquad \bar{x} = \frac{\int_a^b \cancel{\rho} x (f - g) \, dx}{\int_a^b \cancel{\rho} (f - g) \, dx}, \qquad \bar{y} = \frac{\int_a^b \frac{1}{2} \cancel{\rho} (f^2 - g^2) \, dx}{\int_a^b \cancel{\rho} (f - g) \, dx}$$

- Therefore, when the density is constant, the location of the centre of mass is a feature of the geometry of the object and not of the material from which it is made. In such cases, we often call the centre of mass the centroid of the shape.

Exercise

(a) *Find the centroid of a semicircular lamina of radius r.*



(b) *Find the centroid of a thin wire shaped like an open semicircle of radius r.*

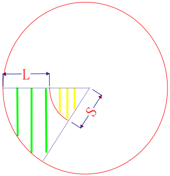
Center of mass of an arc

- Let a piece of thin wire with a uniform density $\rho(x)$, whose shape is given by

$$y = f(x),$$

joins two points, P and Q, at $x = a$ and $x = b$, respectively.

- To find the centroid of the wire, we can



1. Break up the arc into small curve segments
2. Approx. the moment of each curve segment by the moment of a line segment
3. Add the approximations to form a Riemann sum
4. Take the limit of the Riemann sum to find the center of mass

1. The moments of each curve segment is roughly the moments of a line segment

$$(M_y)_k \approx x_k^* m_k = x_k^* \rho(x_k^*) \sqrt{1 + [f'(x_k^*)]^2} \Delta x_k$$
$$(M_x)_k \approx y_k^* m_k = f(x_k^*) \rho(x_k^*) \sqrt{1 + [f'(x_k^*)]^2} \Delta x_k$$

2. Use $\sum (x_k - \bar{x}) m_k = 0$, we have

$$\sum (x_k^* - \bar{x}) \rho(x_k^*) \sqrt{1 + [f'(x_k^*)]^2} \Delta x_k \approx 0$$

3. Use $\sum (y_k - \bar{y}) m_k = 0$, we have

$$\sum (f(x_k^*) - \bar{y}) \rho(x_k^*) \sqrt{1 + [f'(x_k^*)]^2} \Delta x_k \approx 0$$

4. Take the limit, and rearrange

- The centroid of an arc of a smooth curve with equation,

$$y = f(x),$$

between $x = a$ and $x = b$, are given by

$$\bar{x} = \frac{\int_a^b x\sqrt{1+(f')^2}\,dx}{\int_a^b \sqrt{1+(f')^2}\,dx} = \frac{\int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}{\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$$

$$\bar{y} = \frac{\int_a^b f\sqrt{1+(f')^2}\,dx}{\int_a^b \sqrt{1+(f')^2}\,dx} = \frac{\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}{\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$$

Exercise

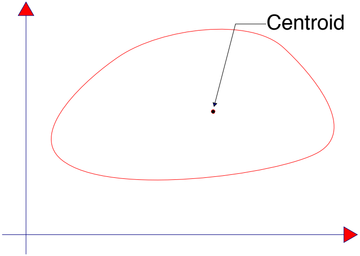
Find the centroid of the arc of the circle $x^2 + y^2 = r^2$ lying above the x-axis.

The theorems of Pappus I

Pappus's Theorem for Volumes

If a plane region is revolved once about a line in the plane that does not cut through the region's interior, then the volume of the solid it generates is equal to the region's area A times the distance traveled by the region's centroid during the revolution. If D is the distance from the axis of revolution to the centroid,

then we have $\quad V = 2\pi D A$



Proof

1. Suppose y-axis is the axis of rotation, and $x = a$ and $x = b$ are the end points of the region on the left and on the right respectively. Let $H(x)$ denote the length of the region perpendicular to the x-axis. We assume that $H(x)$ is a continuous function.

2. Use cylindrical shells,

$$\text{Volume} = \int_a^b 2\pi \begin{pmatrix}\text{Shell}\\ \text{Radius}\end{pmatrix} \begin{pmatrix}\text{Shell}\\ \text{Height}\end{pmatrix} \, dx = \int_a^b 2\pi x H(x)\, dx = 2\pi \int_a^b xH(x)\, dx$$

3. Note $\displaystyle\int_a^b H(x)\, dx = \int_a^b \Big[f(x) - g(x)\Big]\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx = \text{Area}$

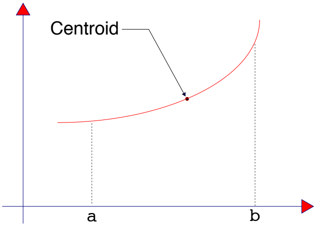
4. And $\displaystyle\bar{x} = \frac{\int_a^b xH(x)\, dx}{\text{Area}} = \frac{\int_a^b x\Big[f(x) - g(x)\Big]\, dx}{A} \implies \int_a^b xH(x)\, dx = \bar{x}A$

5. So Volume $= 2\pi\bar{x}A$, since $\bar{x}$ is actually D the perpendicular distance from the centroid to the y-axis, so $V = 2\pi DA$. □

Pappus's Theorem for Surface Areas

If an arc of a smooth plane curve is revolved once about a line in the plane that does not cut through the arc's interior, then the area of the surface generated by the arc equals the length L of the arc times the distance travelled by the arc's centroid during the revolution. If D is the distance from the axis of revolution to the centroid,

then we have $S = 2\pi D L$



Proof

1. Surface area is given by $S = \int_a^b 2\pi\, \boxed{y\sqrt{1+[y'(x)]}}\, dx$
2. Since

$$\bar{y} = \frac{\int_a^b \boxed{y\sqrt{1+[y'(x)]^2}}\, dx}{\int_a^b \sqrt{1+[y'(x)]^2}\, dx} = \frac{\int_a^b \boxed{y\sqrt{1+[y'(x)]^2}}\, dx}{L} \implies S = 2\pi DL \quad \square$$

Exercise

(a) Find the volume of the torus generated by revolving a circular disk of radius a about an axis in its plane at a distance $b \geq a$ from its centre.

(b) Locate the centroid of a semicircular region of radius r.

(c) Use Pappus's area theorem to find the surface area of the torus.