Vv256 Lecture 24

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Higher order to Companion System

- We have seen that a $n \times n$ homogeneous system can be converted into a single linear equation of nth-order when we apply elimination method.

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 \end{aligned} \implies \ddot{x}_1 - (a_{11} + a_{22})\,\dot{x}_1 + (a_{11}a_{22} - a_{12}a_{21})\,x_1 = 0$$

- In reverse, we can express a higher order linear differential equation as a system of first-order linear equations

$$\text{Higher order equation} \longrightarrow \text{System of equations}$$

- So solving the first-order system instead of solving the higher order equation.
- Suppose we are given a higher order linear equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \tag{1}$$

where a_i are constants, and y is a function of t.

1. Define new functions,

$$
\begin{aligned}
x_1 &= y \\
x_2 &= y' \\
x_3 &= y'' \\
&\vdots \\
x_n &= y^{(n-1)}
\end{aligned}
$$

2. Differentiate each x_i and substitute all ys, we have $n-1$ equations

$$
\begin{aligned}
\dot{x}_1 &= y' \\
\dot{x}_2 &= y'' \\
&\vdots \\
\dot{x}_n &= y^{(n)}
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_{n-1} &= x_n
\end{aligned}
$$

3. We obtain the nth equation by using the original higher order equation,

$$
\dot{x}_n = -\frac{a_{n-1}}{a_n}x_n - \frac{a_{n-2}}{a_n}x_{n-1} - \cdots - \frac{a_0}{a_n}x_1
$$

- This $n \times n$ system is known as the companion system of equation (1).

Exercise

Solve the following equation,

$$\dddot{y} + 2\ddot{y} - \dot{y} - 2y = 0, \qquad \textit{where } y \textit{ is a function } t.$$

by solving the companion system of equations.

Solution

- First, notice that the characteristic equation for the equation has roots,

$$r_1 = -2, \qquad r_2 = 1, \qquad r_3 = -1$$

- In vector notation, let $\mathbf{x} = \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix}$ and differentiate $\mathbf{x}$, we have

$$\begin{aligned} &\dot{x}_1 = \dot{y} = x_2, \\ &\dot{x}_2 = \ddot{y} = x_3, \qquad \text{and} \qquad \dot{x}_3 = \dddot{y} = -2\ddot{y} + \dot{y} + 2y = -2x_3 + x_2 + 2x_1 \end{aligned}$$

Solution

- We can solve the original 3rd-order equation by solving the following system,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \qquad \text{where } \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}.$$

- Eigenvalues and eigenvectors,

$$\mathbf{D} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \mathbf{P} = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & -1 \\ 4 & 1 & 1 \end{bmatrix}$$

- Thus the solution of the system is

$$\mathbf{x} = c_1 e^{-2t} \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- So the solution of the original equation is $\quad y = x_1 = c_1 e^{2t} + c_2 e^{t} + c_3 e^{-t}.$

Coupled mass-spring system

- So far we have only consider homogeneous first-order system,

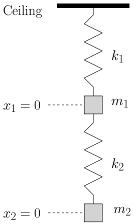
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

but often the model of a physical system is a second-order system

$$\ddot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

- Suppose that two masses m_1 and m_2 are connected to two springs A and B of negligible mass having spring constants k_1 and k_2, respectively.

- In turn the two springs are attached to a ceiling.



- Let x_1 and x_2 denote the vertical displacement from the equilibrium.
- When the system is in motion, spring B is subject to both an elongation and a compression; hence its net elongation is

$$x_2 - x_1$$

- Therefore it follows from Hooke's law that springs A and B exert forces

$-k_1 x_1$ and $k_2(x_2 - x_1)$, respectively, on m_1.

- So by Newton's second law,

$$m_1 \ddot{x_1} = -k_1 x_1 + k_2(x_2 - x_1)$$

- The net force exerted on mass m_2 is due solely to the net elongation of B,

$$-k(x_2 - x_1)$$

- Similarly, by Newton's second law,

$$m_2 \ddot{x_2} = -k_2(x_2 - x_1)$$

- The equations for the coupled springs

$$m_1\ddot{x_1} = -k_1x_1 + k_2(x_2 - x_1)$$
$$m_2\ddot{x_2} = -k_2(x_2 - x_1)$$

can be written in matrix form as

$$\mathbf{M}\ddot{\mathbf{x}} = \mathbf{K}\mathbf{x}, \qquad \text{where}$$

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \qquad \mathbf{K} = \begin{bmatrix} -k_1 - k_2 & k_2 \\ k_2 & -k_2 \end{bmatrix}, \qquad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Since $\mathbf{M}$ is nonsingular,

$$\ddot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \qquad \text{where} \quad \mathbf{A} = \mathbf{M}^{-1}\mathbf{K}.$$

- We have seen that a higher order equation can be expressed as a system of first-order equations by means of substitution,

$$x_1 = y, \quad x_2 = y', \quad x_3 = y'', \ldots \quad x_n = y^{(n-1)}$$

the same idea can be used on higher order systems.

- If we introduce two more functions x_3 and x_4 of t, and let them be

$$x_3 = \dot{x}_1, \qquad \text{and} \quad x_4 = \dot{x}_2$$

then

$$\ddot{x}_1 = \dot{x}_3, \qquad \text{and} \quad \ddot{x}_2 = \dot{x}_4$$

- Thererfore

$$m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1) \implies \dot{x}_3 = -\left(\frac{k_1}{m_1} + \frac{k_2}{m_1}\right)x_1 + \frac{k_2}{m_1}x_2$$

$$m_2\ddot{x}_2 = -k_2(x_2 - x_1) \implies \dot{x}_4 = \frac{k_2}{m_2}x_1 - \frac{k_2}{m_2}x_2$$

- In matrix form,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\left(\frac{k_1}{m_1} + \frac{k_2}{m_1}\right) & \frac{k_2}{m_1} & 0 & 0\\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} \implies \dot{\mathbf{x}} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$\mathbf{A}$}} & 0 & 0\\ & & 0 & 0\end{array}\right]\mathbf{x}$$

- Thus we successfully convert a second-order system to a first-order system.

- Recall for a scalar α, the exponential function can be defined using

$$e^{\alpha} = 1 + \alpha + \frac{1}{2!}\alpha^2 + \frac{1}{3!}\alpha^3 + \cdots = \lim_{m\to\infty} \sum_{k=0}^{m} \frac{1}{k!}\alpha^k$$

Definition

For each $n \times n$ matrix $\mathbf{A}$, the exponential of $\mathbf{A}$ is defined to be matrix

$$\mathbf{I} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 + \cdots$$

which is often denoted by

$$e^{\mathbf{A}} \qquad \text{or} \qquad \exp(\mathbf{A})$$

- The sum is always convergent, and has most of the properties of

$$e^{\alpha}$$

Theorem

Suppose $\mathbf{A}$ is a matrix of $n \times n$.

1. For all integers m,

$$\mathbf{A}^m e^{\mathbf{A}} = e^{\mathbf{A}} \mathbf{A}^m$$

2. If $\mathbf{A}^{\mathrm{T}}$ denote the transpose of $\mathbf{A}$, then

$$\left(e^{\mathbf{A}}\right)^{\mathrm{T}} = e^{\mathbf{A}^{\mathrm{T}}}$$

3. If $\mathbf{AB} = \mathbf{BA}$, then

$$\mathbf{A} e^{\mathbf{B}} = e^{\mathbf{B}} \mathbf{A}$$

4. If $\mathbf{AB} = \mathbf{BA}$, then

$$e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}} e^{\mathbf{B}} = e^{\mathbf{B}} e^{\mathbf{A}}$$

- In general, it is very expensive to compute the matrix exponential $e^{\mathbf{A}}$
- For a diagonal matrix $\mathbf{D} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$,

$$e^{\mathbf{D}} = \lim_{m\to\infty} \left(\mathbf{I} + \mathbf{D} + \frac{1}{2!}\mathbf{D}^2 + \frac{1}{3!}\mathbf{D}^3 + \cdots + \frac{1}{m!}\mathbf{D}^m \right)$$

$$= \lim_{m\to\infty} \begin{bmatrix} \sum_{k=0}^{m} \frac{1}{k!}\lambda_1^k & & \\ & \ddots & \\ & & \sum_{k=0}^{m} \frac{1}{k!}\lambda_n^k \end{bmatrix} = \begin{bmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{bmatrix}$$

- If $\mathbf{A}$ is diagonalizable, then

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$$

- So the square of $\mathbf{A}$ can be written as

$$\mathbf{A}^2 = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}\mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\mathbf{D}^2\mathbf{P}^{-1}$$

- In general, powers of $\mathbf{A}$, if $\mathbf{A}$ is diagonalizable, can be computed as

$$\mathbf{A}^k = \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1} = \mathbf{P}\begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix}\mathbf{P}^{-1}$$

- Therefore the powers and the exponential of $\mathbf{A}$ are given by

$$\mathbf{A}^k = \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1} \qquad \text{for} \quad k = 1, 2, \ldots$$

$$e^{\mathbf{A}} = \mathbf{P}\left(\mathbf{I} + \mathbf{D} + \frac{1}{2!}\mathbf{D}^2 + \frac{1}{3!}\mathbf{D}^3 + \cdots\right)\mathbf{P}^{-1} = \mathbf{P}e^{\mathbf{D}}\mathbf{P}^{-1}$$

Exercise

Compute the exponential function $e^{\mathbf{A}}$ for $\mathbf{A} = \begin{bmatrix} -2 & -6 \\ 1 & 3 \end{bmatrix}$

Solution

The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 1$ and $\lambda_2 = 0$, with the corresponding eigenvectors

$$\mathbf{x}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

so $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$

$$\begin{aligned} \Longrightarrow \ e^{\mathbf{A}} &= \mathbf{P}e^{\mathbf{D}}\mathbf{P}^{-1} \\ &= \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^1 & 0 \\ 0 & e^0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 3-2e & 6-6e \\ e-1 & 3e-2 \end{bmatrix} \end{aligned}$$

Mupad

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A := matrix( [[-2, -6], [1, 3]]):
exp(A)
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$$\begin{pmatrix} 3 - 2\,\mathrm{e} & 6 - 6\,\mathrm{e} \\ \mathrm{e} - 1 & 3\,\mathrm{e} - 2 \end{pmatrix}$$

- Since for any $n \times n$ matrix $\mathbf{A}$, the matrix exponential $e^{\mathbf{A}}$ is

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 + \cdots + \frac{1}{k!}\mathbf{A}^k + \cdots$$

  and thus the matrix exponential function of t is given by

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}(\mathbf{A}t)^2 + \frac{1}{3!}(\mathbf{A}t)^3 + \cdots + \frac{1}{k!}(\mathbf{A}t)^k + \cdots$$

- Now consider the derivative of the matrix exponential function of t, like the derivative of a vector-valued function, it is a component-wise derivative

$$\begin{aligned}
\frac{d}{dt}\left(e^{\mathbf{A}t}\right) &= \frac{d}{dt}\left(\mathbf{I} + \mathbf{A}t + \frac{1}{2!}(\mathbf{A}t)^2 + \frac{1}{3!}(\mathbf{A}t)^3 + \cdots + \frac{1}{k!}(\mathbf{A}t)^k + \cdots\right) \\
&= \mathbf{0} + \mathbf{A} + \mathbf{A}^2 t + \frac{1}{2!}(\mathbf{A})^3 t^2 + \cdots + \frac{1}{(k-1)!}\mathbf{A}^k t^{k-1} + \cdots \\
&= \mathbf{A}\left(\mathbf{I} + \mathbf{A}t + \frac{1}{2!}(\mathbf{A})^2 t^2 + \cdots + \frac{1}{(k-1)!}\mathbf{A}^{k-1} t^{k-1} + \cdots\right) = \mathbf{A}\, e^{\mathbf{A}t}
\end{aligned}$$

- Notice what we have shown here

$$\dot{\mathbf{M}} = \mathbf{A}\mathbf{M}, \qquad \text{where} \qquad \mathbf{M} = e^{\mathbf{A}t} \qquad \text{is a } n \times n \text{ matrix.}$$

- Consider any linear combination of the columns of the LHS and the RHS,

$$\begin{aligned}
\dot{\mathbf{M}}\mathbf{c} &= \mathbf{A}\mathbf{M}\mathbf{c}, && \text{where } \mathbf{c} \text{ is any constant vector in } \mathbb{R}^n. \\
c_1\dot{\mathbf{m}}_1 + c_2\dot{\mathbf{m}}_2 + \cdots + c_n\dot{\mathbf{m}}_n &= \mathbf{A}\mathbf{M}\mathbf{c} && \text{where } \dot{\mathbf{m}}_i\text{s are columns of } \dot{\mathbf{M}} \\
c_1\frac{d}{dt}\mathbf{m}_1 + c_2\frac{d}{dt}\mathbf{m}_2 + \cdots + c_n\frac{d}{dt}\mathbf{m}_n + &= \mathbf{A}\mathbf{M}\mathbf{c} \\
\frac{d}{dt}(c_1\mathbf{m}_1) + \frac{d}{dt}(c_2\mathbf{m}_2) + \cdots + \frac{d}{dt}(c_n\mathbf{m}_n) &= \mathbf{A}\mathbf{M}\mathbf{c} \\
\frac{d}{dt}(c_1\mathbf{m}_1 + c_2\mathbf{m}_2 + \cdots + c_n\mathbf{m}_n) &= \mathbf{A}\mathbf{M}\mathbf{c} \\
\frac{d}{dt}(\mathbf{M}\mathbf{c}) &= \mathbf{A}\mathbf{M}\mathbf{c} \\
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} && \text{where } \mathbf{x} = \mathbf{M}\mathbf{c}.
\end{aligned}$$

Q: What does this mean in terms of the solution of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$?

- So $\mathbf{x} = e^{\mathbf{A}t}\mathbf{c}$ is a solution of the first-order system for any constant $\mathbf{c} \in \mathbb{R}^n$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x},$$

- In general, the initial value problem

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0$$

has the solution

$$\mathbf{x} = e^{\mathbf{A}t}\mathbf{c} \qquad \text{if we can find } \mathbf{c} \text{ such that} \qquad \mathbf{x}_0 = \mathbf{e}^{0\mathbf{A}}\mathbf{c}$$

- Consider $t = 0$, we have

$$\mathbf{x}_0 = e^{0\mathbf{A}}\mathbf{c} = \left(\mathbf{I} + \mathbf{A}\cdot 0 + \frac{1}{2!}\mathbf{A}^2\cdot 0 + \cdots + \frac{1}{k!}\mathbf{A}^k\cdot 0 + \cdots\right)\mathbf{c} = \mathbf{I}\mathbf{c} = \mathbf{c}$$

- So such $\mathbf{c}$ always exists and the following is a particular solution for the IVP

$$\mathbf{x} = e^{\mathbf{A}t}\mathbf{x}_0$$

Q: This seems easier than decoupling or elimination, why we are not using this?

Exercise

Solve the system of differential equations,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \textit{where } \mathbf{A} = \begin{bmatrix} -7 & -9 & 9 \\ 3 & 5 & -3 \\ -3 & -3 & 5 \end{bmatrix}.$$

Solution

- We know the solution is

$$\mathbf{x} = e^{\mathbf{A}t}\mathbf{x}_0$$

- However, we need to determine $e^{\mathbf{A}t}$ before we can make use of this solution.
- For diagonalizable $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, then

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{I} + \mathbf{A}t + \frac{1}{2!}(\mathbf{A}t)^2 + \frac{1}{3!}(\mathbf{A}t)^3 + \cdots + \frac{1}{k!}(\mathbf{A}t)^k + \cdots \\ &= \mathbf{P}\left(\mathbf{I} + \mathbf{D}t + \frac{1}{2!}\mathbf{D}^2t^2 + \frac{1}{3!}\mathbf{D}^3t^3 + \cdots + \frac{1}{k!}\mathbf{D}^kt^k + \cdots\right)\mathbf{P}^{-1} \\ &= \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1} \end{aligned}$$

Solution

- For the given matrix $\mathbf{A}$, the eigenvalues and the eigenvectors are

$$\mathbf{D} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \mathbf{P} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

- Thus the particular solution is

$$\begin{aligned} \mathbf{x} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}\mathbf{x}_0 &= \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \\ &= 0e^{-t} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + 2e^{2t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + 3e^{2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = e^{2t} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \end{aligned}$$

- Now let us consider the nonhomogeneous linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \boldsymbol{\beta} \tag{2}$$

  where $\mathbf{A}$ is a constant matrix, and $\boldsymbol{\beta}$ is some vector-valued function of t.
- In order to solve (2) we have to solve the homogeneous system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

  which is known as the corresponding homogeneous system.

Q: What do you think the general solution of the linear system (2) is made of?

$$\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p$$

where $\mathbf{x}_p$ is any particular solution to the nonhomogeneous system (2) and $\mathbf{x}_c$ is the general solution to the corresponding homogeneous system,

$$\mathbf{x}_c = e^{\mathbf{A}t}\mathbf{c}$$

which is known as the complementary solution as before.

- So we basically need to derive a method for finding a particular solution

$$\mathbf{x}_p$$

- Suppose a particular solution can be written as

$$\mathbf{x}_p = e^{\mathbf{A}t}\mathbf{u}$$

where $\mathbf{u}$ is a vector-valued function of t.

- The rest is reminiscent of the variation of parameters, we consider

$$\begin{aligned}
\dot{\mathbf{x}}_p &= \mathbf{A}\mathbf{x}_p + \boldsymbol{\beta} \\
\implies \left(e^{\mathbf{A}t}\mathbf{u}\right)' &= \mathbf{A}e^{\mathbf{A}t}\mathbf{u} + \boldsymbol{\beta} \\
\implies \left(\mathbf{A}e^{\mathbf{A}t}\right)\mathbf{u} + e^{\mathbf{A}t}\dot{\mathbf{u}} &= \left(\mathbf{A}e^{\mathbf{A}t}\right)\mathbf{u} + \boldsymbol{\beta} \\
\implies e^{\mathbf{A}t}\dot{\mathbf{u}} &= \boldsymbol{\beta}
\end{aligned}$$

$$\implies e^{-\mathbf{A}t}e^{\mathbf{A}t}\dot{\mathbf{u}} = e^{-\mathbf{A}t}\boldsymbol{\beta} \implies \dot{\mathbf{u}} = e^{-\mathbf{A}t}\boldsymbol{\beta} \implies \mathbf{x}_p = e^{\mathbf{A}t}\int e^{-\mathbf{A}t}\boldsymbol{\beta}\,dt$$

- Recall the general solution of the single linear first-order differential equation

$$y' = ay + f, \qquad \text{where } a \text{ is a constant and } f \text{ a function of } t,$$

can be obtained by an integrating factor, and has the form

$$x = x_c + x_p = ce^{at} + e^{at}\int_{t_0}^{t} e^{-a\tau}\beta(\tau)\,d\tau, \qquad \text{where } c \text{ is a constant.}$$

- Note the general solution of a nonhomogeneous linear first-order system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \boldsymbol{\beta}$$

has a very similar form

$$\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p = e^{\mathbf{A}t}\mathbf{c} + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}\tau}\boldsymbol{\beta}(\tau)\,d\tau, \qquad \text{where } \mathbf{c} \text{ is a constant vector.}$$

where t_0 is an initial value.

Exercise

Solve the initial value problem,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \boldsymbol{\beta}, \quad \mathbf{x}(0) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \quad \textit{where } \mathbf{A} = \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix} \textit{ and } \boldsymbol{\beta} = \begin{bmatrix} 4e^{2t} \\ 8e^{-t} \end{bmatrix}.$$

Solution

- Through diagonalization we can find $e^{\mathbf{A}t}$,

$$e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1/4 & 1/4 \\ 3/4 & -1/4 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{4}e^{-t} + \frac{3}{4}e^{3t} & \frac{1}{4}e^{-t} - \frac{1}{4}e^{3t} \\ \frac{3}{4}e^{-t} - \frac{3}{4}e^{3t} & \frac{3}{4}e^{-t} + \frac{1}{4}e^{3t} \end{bmatrix}$$

- Thus the complementary solution is

$$\mathbf{x}_c = e^{\mathbf{A}t}\mathbf{x}_0 = \begin{bmatrix} \frac{1}{4}e^{-t} + \frac{3}{4}e^{3t} & \frac{1}{4}e^{-t} - \frac{1}{4}e^{3t} \\ \frac{3}{4}e^{-t} - \frac{3}{4}e^{3t} & \frac{3}{4}e^{-t} + \frac{1}{4}e^{3t} \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-t} + 3e^{3t} \\ 3e^{-t} - 3e^{3t} \end{bmatrix}$$

Solution

- The solution of the initial value problem is given by

$$
\begin{aligned}
\mathbf{x} &= e^{\mathbf{A}t}\mathbf{c} + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}\tau}\boldsymbol{\beta}(\tau)\,d\tau \\
&= e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{(t-\tau)\mathbf{A}}\boldsymbol{\beta}(\tau)\,d\tau \\
&= \begin{bmatrix} e^{-t}+3e^{3t} \\ 3e^{-t}-3e^{3t} \end{bmatrix} + \int_0^t \begin{bmatrix} \frac{1}{4}e^{-(t-\tau)}+\frac{3}{4}e^{3(t-\tau)} & \frac{1}{4}e^{-(t-\tau)}-\frac{1}{4}e^{3(t-\tau)} \\ \frac{3}{4}e^{-(t-\tau)}-\frac{3}{4}e^{3(t-\tau)} & \frac{3}{4}e^{-(t-\tau)}+\frac{1}{4}e^{3(t-\tau)} \end{bmatrix} \begin{bmatrix} 4e^{2\tau} \\ 8e^{-\tau} \end{bmatrix} d\tau \\
&= \begin{bmatrix} e^{-t}+3e^{3t} \\ 3e^{-t}-3e^{3t} \end{bmatrix} + \int_0^t \begin{bmatrix} 2\,e^{-t}+3\,e^{-\tau}\,e^{3\,t}+e^{3\,\tau}\,e^{-t}-2\,e^{-4\,\tau}\,e^{3\,t} \\ 6\,e^{-t}-3\,e^{-\tau}\,e^{3\,t}+3\,e^{3\,\tau}\,e^{-t}+2\,e^{-4\,\tau}\,e^{3\,t} \end{bmatrix} d\tau \\
&= \begin{bmatrix} e^{-t}+3e^{3t} \\ 3e^{-t}-3e^{3t} \end{bmatrix} + \frac{1}{6}\begin{bmatrix} e^{-t}\,\left(12\,t-16\,e^{3\,t}+15\,e^{4\,t}+1\right) \\ 3e^{-t}\,\left(12\,t+8\,e^{3\,t}-5\,e^{4\,t}-3\right) \end{bmatrix} \\
&= \frac{1}{36}\begin{bmatrix} e^{-t}\,\left(12\,t-16\,e^{3\,t}+33\,e^{4\,t}+7\right) \\ 3e^{-t}\,\left(12\,t+8\,e^{3\,t}-11\,e^{4\,t}+3\right) \end{bmatrix}
\end{aligned}
$$