Vv255 Lecture 24

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- The line integral over a smooth closed curve $\mathcal{C}$ that is in an open region $\mathcal{D}$

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$$

provided that the vector field $\mathbf{F}$ is sufficiently smooth and conservative in $\mathcal{D}$.

- However, if the vector field is not conservative, we have to use

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt$$

where the closed smooth curve is defined by

$$\mathbf{r}(t) = x(t)\mathbf{e}_x + y(t)\mathbf{e}_y \qquad \text{for} \quad a \le t \le b$$

- Consider the following vector field

$$\mathbf{F} = -\frac{y}{2}\mathbf{e}_x + \frac{x}{2}\mathbf{e}_y$$

Q: Is the above vector field conservative?

- Suppose we need to evaluate

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F} = -\frac{y}{2}\mathbf{e}_x + \frac{x}{2}\mathbf{e}_y$ and the curve $\mathcal{C}$ is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

- We have to use the formula

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\ dt$$

- Using the the following parametrization,

$$x(t) = a\cos\theta, \qquad y(t) = b\sin\theta \qquad \text{for} \quad 0 \le \theta \le 2\pi$$

we have

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2}\int_0^{2\pi} (-b\sin\theta\mathbf{e}_x + a\cos\theta\mathbf{e}_y) \cdot (-a\sin\theta\mathbf{e}_x + b\cos\theta\mathbf{e}_y)\ d\theta$$

- If we compute the dot product, we have

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2} \int_0^{2\pi} \left(-b \sin\theta \mathbf{e}_x + a \cos\theta \mathbf{e}_y\right) \cdot \left(-a \sin\theta \mathbf{e}_x + b \cos\theta \mathbf{e}_y\right) \ d\theta$$

$$= \frac{1}{2} ab \int_0^{2\pi} \left(\sin^2\theta + \cos^2\theta\right) \ d\theta$$

$$= \frac{1}{2} ab \int_0^{2\pi} 1 \ d\theta$$

$$= \pi ab$$

Q: Is this a coincidence?

$$\iint_{\mathcal{D}} 1 \ dA = \pi ab$$

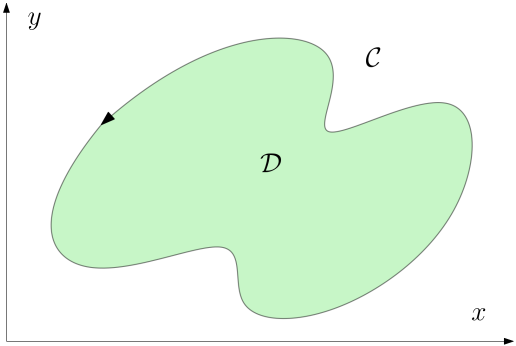
where $\mathcal{D}$ is the region enclosed by the $\mathcal{D}$, that is, the ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

- For any vector field

$$\mathbf{F} = P\mathbf{e}_x + Q\mathbf{e}_y$$

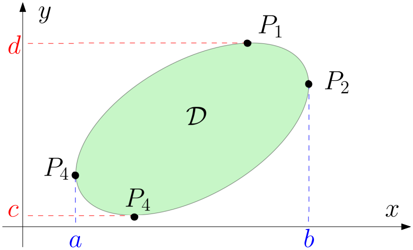
and the line integral $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ where $\mathcal{C}$ encloses a region $\mathcal{D}$



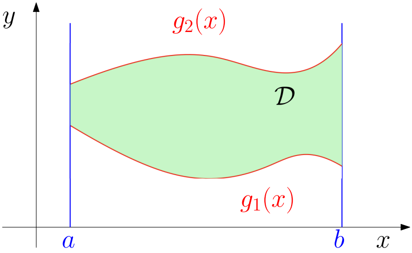
Q: Is the line integral somehow linked to a double integral over the region $\mathcal{D}$?

- To simplify the discussion, let $\mathcal{D}$ be a region that is both

Type I and Type II



Type I



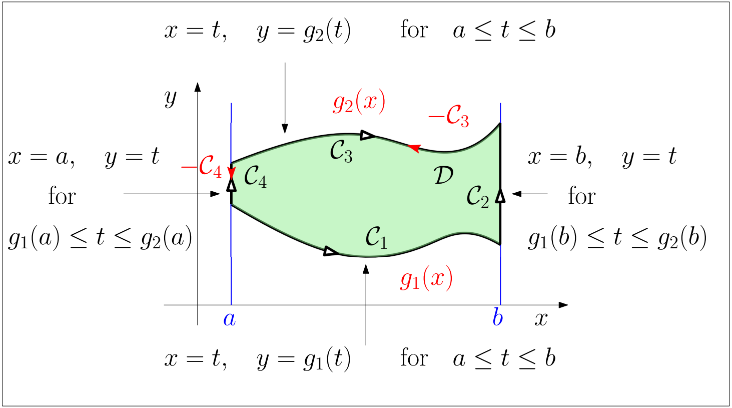
- Firstly $\mathcal{D}$ as a type I region, then the boundary of $\mathcal{D}$ is piecewise smooth $\mathcal{C}$ of

$$y = g_1(x) \qquad x = b \qquad y = g_2(x) \qquad x = a$$

- We need a parametrization for each piece before we can investigate

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt$$

- Let the independent variables be the parameters, we have



- If we reverse the orientation of $\mathcal{C}_3$ and $\mathcal{C}_4$, then

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup (-\mathcal{C}_3) \cup (-\mathcal{C}_4)$$

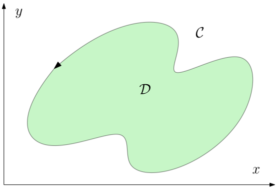
is a positively oriented simple closed curve.

Definition

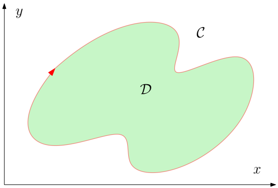
A positively oriented simple closed curve $\mathcal{C}$ is a curve with "direction" whose initial point is also the terminal point and which does not cross itself again such that the region $\mathcal{D}$ enclosed by the curve $\mathcal{C}$ is always on the left of the direction of motion.

Q: Are the following curves positively oriented? Are they simple closed curves?

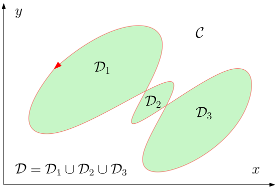
Yes



No, negatively



No, not simple



- Recall we can decompose a line integral of a vector field into

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \oint_{\mathcal{C}} P\, dx + \oint_{\mathcal{C}} Q\, dy = \int P\dot{x}\, dt + \int Q\dot{y}\, dt$$

- $$\oint_{\mathcal{C}} P\,dx = \int_{\mathcal{C}_1} P\,dx + \int_{\mathcal{C}_2} P\,dx + \int_{-\mathcal{C}_3} P\,dx + \int_{-\mathcal{C}_4} P\,dx$$
$$= \int_{\mathcal{C}_1} P\,dx + \int_{\mathcal{C}_2} P\,dx - \int_{\mathcal{C}_3} P\,dx - \int_{\mathcal{C}_4} P\,dx$$
$$= \int_a^b P\Big(t, g_1(t)\Big)\cdot(1)\,dt + \int_{g_1(b)}^{g_2(b)} P(b,t)\cdot (0)\,dt$$
$$- \int_a^b P\Big(t, g_2(t)\Big)\cdot(1)\,dt - \int_{g_1(a)}^{g_2(a)} P(a,t)\cdot (0)\,dt$$
$$= \int_a^b \Bigg(P\Big(t, g_1(t)\Big) - P\Big(t, g_2(t)\Big)\Bigg)\,dt$$
$$= \int_a^b \Big[P\Big(t,y\Big)\Big]_{y=g_2}^{y=g_1}\,dt = -\int_a^b \int_{g_1}^{g_2} \frac{\partial P}{\partial y}\,dy\,dt = -\iint_{\mathcal{D}} \frac{\partial P}{\partial y}\,dA$$
- Using a similar approach in which $\mathcal{D}$ is viewed as a region of type II,

$$\oint_{\mathcal{C}} Q\,dy = \iint_{\mathcal{D}} \frac{\partial Q}{\partial x}\,dA \implies \oint_{\mathcal{C}} P\,dx + \oint_{\mathcal{C}} Q\,dy = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA$$

- This proves the following for a region $\mathcal{D}$ that is type I as well as being type II.

Green's Theorem

If $\mathcal{C}$ is a positively oriented, piecewise smooth, simple closed curve that encloses a region $\mathcal{D}$, and P and Q have continuous first partial derivatives on some open region containing $\mathcal{D}$, then the line integral of the vector along $\mathcal{C}$

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA$$

- Another common way of stating the theorem is

$$\oint_{\partial\mathcal{D}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA$$

  where $\partial\mathcal{D}$ denotes the positively oriented boundary of $\mathcal{D}$.

- Green's Theorem can sometimes act as a bridge between

  line integrals and double integrals.

Exercise

Evaluate the following line integral

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

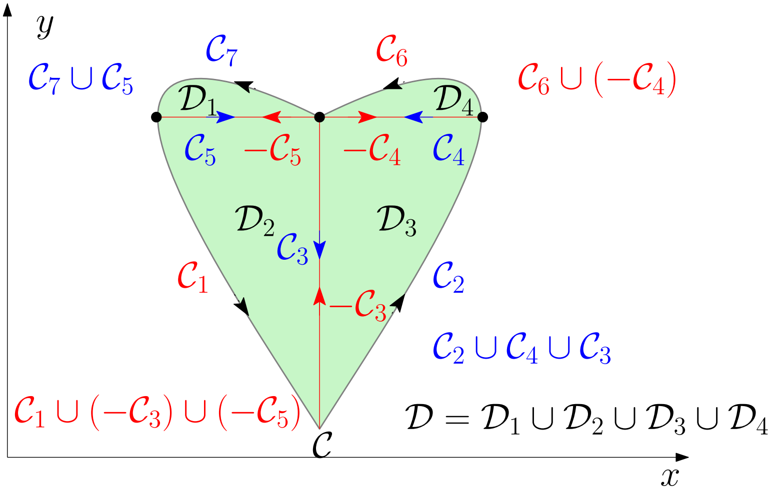
where $\mathcal{C}$ *is the boundary of the following square with positive orientation*

$$\mathcal{R} = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

and the vector field is

$$\mathbf{F} = y \cos x \mathbf{e}_x + x^2 \mathbf{e}_y$$

- For more general regions than those that are of both type I and type II,



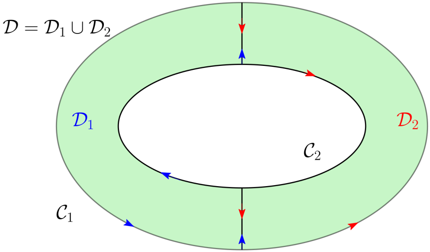
we consider the union of regions that are of both type I and type II.

- It follows that for a vector field that satisfy the assumptions of Green's theorem on $\mathcal{D}$, we can apply the theorem to $\mathcal{D}_i$ individually for $i = 1, \ldots, 4$.

$$\begin{aligned}
\iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA &= \sum_{i=1}^{4} \iint_{\mathcal{D}_i} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\
&= \oint_{\mathcal{C}_7 \cup \mathcal{C}_5} \mathbf{F} \cdot d\mathbf{r} + \oint_{\mathcal{C}_1 \cup (-\mathcal{C}_3) \cup (-\mathcal{C}_5)} \mathbf{F} \cdot d\mathbf{r} \\
&\quad + \oint_{\mathcal{C}_2 \cup \mathcal{C}_4 \cup \mathcal{C}_3} \mathbf{F} \cdot d\mathbf{r} + \oint_{\mathcal{C}_6 \cup (-\mathcal{C}_4)} \mathbf{F} \cdot d\mathbf{r} \\
&= \oint_{\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{D}_6 \cup \mathcal{D}_7} \mathbf{F} \cdot d\mathbf{r} = \oint_{C} \mathbf{F} \cdot d\mathbf{r}
\end{aligned}$$

- It can be concluded that Green's theorem holds on this $\mathcal{D}$.
- The same argument can be used to easily show that Green's theorem is valid on any finite union of regions that are regions of both type I and type II.

- Green's theorem can also be applied to regions with "holes", that is, regions that are not simply connected. To see this, let $\mathcal{D}$ be a region enclosed by two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ that are both positively oriented with respect to $\mathcal{D}$, that is, $\mathcal{D}$ is on the left as either $\mathcal{C}_1$ or $\mathcal{C}_2$ is traversed.



- Suppose curve $\mathcal{C}_2$ is contained within the region enclosed by curve $\mathcal{C}_1$; that is, curve $\mathcal{C}_2$ is the boundary of the "hole" in $\mathcal{D}$.
- Then partition the region $\mathcal{D}$ into two simply connected regions $\mathcal{D}_1$ and $\mathcal{D}_2$ by connecting $\mathcal{C}_2$ to $\mathcal{C}_1$ along two separate curves that lie in the region $\mathcal{D}$.

- Applying Green's theorem to regions $\mathcal{D}_1$ and $\mathcal{D}_2$ individually, we find that the line integrals along the common boundaries of $\mathcal{D}_1$ and $\mathcal{D}_2$ cancel, since they have opposite orientations with respect to these regions. Hence, we have

$$\begin{aligned}\iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \, dA &= \iint_{\mathcal{D}_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \, dA + \iint_{\mathcal{D}_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \, dA \\ &= \oint_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} \\ &= \oint_{\mathcal{C}_1 \cup \mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}\end{aligned}$$

- Hence Green's theorem applies to a non-simply connected region $\mathcal{D}$ as well.

Exercise

Consider an n-sided polygon with vertices

$$(x_1, y_1), \quad (x_2, y_2), \quad \ldots, \quad (x_n, y_n)$$

Find the area of the polygon in terms of the coordinates of vertices.

- Green's theorem can be thought as a generalization of FTC or FTL.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\int_{\mathcal{C}} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

$$\iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\, dA = \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

- However, the meaning of Green's theorem can be studied in terms of velocity field of water flow $\mathbf{F}$. When $\mathcal{C}$ is an oriented simple closed curve, the integral

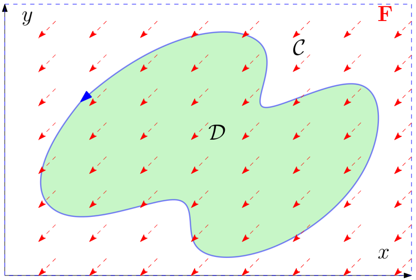
$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\,ds$$

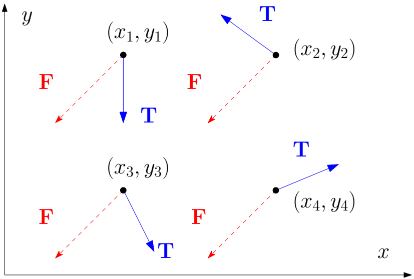
would indicate how much the water tends to circulate around the path in the direction of its orientation, the tendency to circulate or rotate.

Q: Why does it indicate the amount of water circulating the the region?

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds$$

- Of course the velocity $\mathbf{F}$ is in general not in the direction of $\mathbf{T}$.





- The integrand

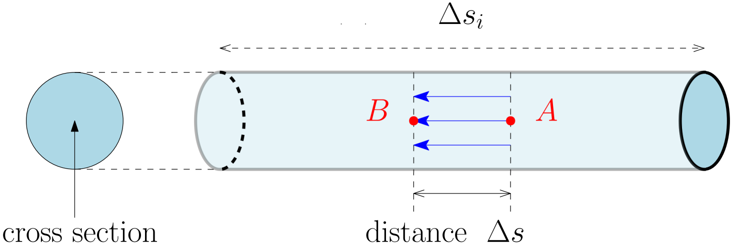
$$\alpha = \mathbf{F} \cdot \mathbf{T}$$

gives the scalar component of the velocity $\mathbf{F}$ of water in the direction of $\mathbf{T}$.

Q: What does the following integral represent?

$$\int_a^b \mathbf{F} \cdot \mathbf{T}\, dt$$

- The distance travelled by water with a specification of the cross section gives



- Now back to the line integral of a velocity field,

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F} \cdot \mathbf{T}\ |\mathbf{r}'(t)|\ dt$$

- So the line integral, which is also known as flow integral, is an indication of

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

the amount of water circulating when $\mathcal{C}$ is a closed curve, thus the circulation

$$\mathbf{r} = x(t)\mathbf{e}_x + y(t)\mathbf{e}_y \qquad \text{for} \quad a \leq t \leq b$$

- Now let us consider the other side of the formula in Green's theorem

$$\oint_{\partial\mathcal{D}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA$$

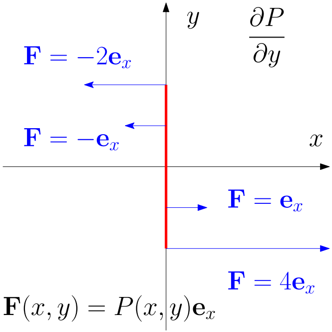
Q: Given a velocity field

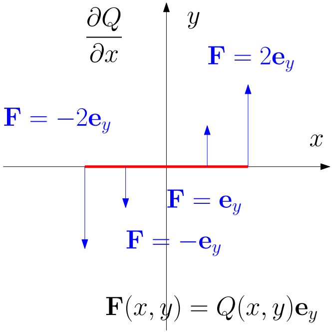
$$\mathbf{F} = P(x, y)\mathbf{e}_x + Q(x, y)\mathbf{e}_y$$

How can we determine whether $\mathbf{F}$ would cause a tiny object to rotate?

Q: Does it depend on the magnitude or the direction of $\mathbf{F}$ at (x, y)?

- It depends on the rate of change of $\mathbf{F}$ at (x, y).





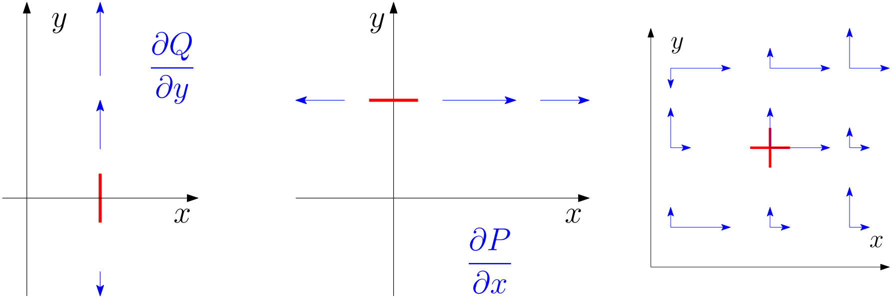
- If only the horizontal component of $\mathbf{F}$ is relevant, then it depends on whether

$$\frac{\partial P}{\partial y} < 0$$

- If only the vertical component of $\mathbf{F}$ is relevant, it depends on whether

$$\frac{\partial Q}{\partial x} > 0$$

- However, the other two partial derivatives have nothing to do with rotation.



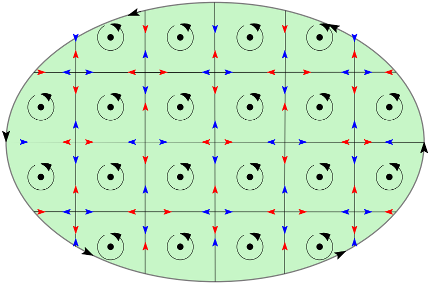
- Therefore we see the difference in the partial derivatives tells whether

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} > 0 \qquad \text{at} \quad (x, y)$$

there is a general tendency to rotate counterclockwise near (x, y).

- We expect Green's theorem as the partition becomes finer and finer

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \underbrace{\oint_{\mathcal{C}} P\, dx + \oint_{\mathcal{C}} Q\, dy}_{\text{Macroscopic circulation}} = \underbrace{\iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA}_{\text{Microscopic circulation}}$$



- Green's theorem says that if you add up all the microscopic circulation inside $\mathcal{C}$, then the sum is exactly the same as the macroscopic circulation around $\mathcal{C}$.

- Now consider a related vector field $\mathbf{G}$, which is orthogonal to $\mathbf{F}$ everywhere

$$\mathbf{G}(x, y) = -Q(x, y)\mathbf{e}_x + P(x, y)\mathbf{e}_y$$

- Applying Green's theorem,

$$\iint\limits_{\mathcal{D}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA = \oint\limits_{\mathcal{C}} -Q\, dx + \oint\limits_{\mathcal{C}} P\, dy = \oint\limits_{\mathcal{C}} P\dot{y}\, dt + \oint\limits_{\mathcal{C}} Q(-\dot{x})\, dt$$

- In terms of the original vector field,

$$\mathbf{F}(x, y) = P(x, y)\mathbf{e}_x + Q(x, y)\mathbf{e}_y$$

- we can rewrite the above formula using the direction normal to the $\mathbf{r}'(s)$,

$$\iint_{\mathcal{D}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA = \oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{n}\, ds \qquad \text{where} \quad \mathbf{n} = \begin{bmatrix} y'(s) \\ -x'(s) \end{bmatrix}$$

Q: How to interpret the above formula if $\mathbf{F}$ is a velocity field of water flow?

Green's Theorem

If $\mathcal{C}$ is a positively oriented, piecewise smooth, simple closed curve that encloses a region $\mathcal{D}$, and P and Q have continuous first partial derivatives on some open region containing $\mathcal{D}$, then the line integral of the vector along $\mathcal{C}$

Tangential form $$\oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Normal form $$\oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{n}\, ds = \iint_{\mathcal{D}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA$$

where $\mathbf{F} = P\mathbf{e}_x + Q\mathbf{e}_y$, $\quad \mathbf{T} = \frac{dx}{ds}\mathbf{e}_x + \frac{dy}{ds}\mathbf{e}_y$, $\quad$ and $\quad \mathbf{n} = \frac{dy}{ds}\mathbf{e}_x - \frac{dx}{ds}\mathbf{e}_y$.

- If $\mathbf{F}$ is a conservative vector field with the potential function f, then

$$\iint_{\mathcal{D}} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) dA = \oint_{\mathcal{C}} \nabla f \cdot \mathbf{n}\, ds$$

where the directional derivative $\nabla f \cdot \mathbf{n}$ is called the normal derivative of f.