
Question1 (8 points)

(a) (1 point) Determine whether the following is differentiable. Justify your answer.

$$f(x,y) = (\cos x)(\cos y)$$

(b) (2 points) Determine whether the following is differentiable. Justify your answer.

$$g(x) = \begin{cases} 0, & \text{if} \quad x^2 < y < 2x^2, \\ 1, & \text{otherwise.} \end{cases}$$

(c) (2 points) Determine whether the following is differentiable. Justify your answer.

$$h(x) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right), & \text{for} \quad (x,y) \neq (0,0), \\ 0, & \text{for} \quad (x,y) = (0,0). \end{cases}$$

(d) (1 point) Let $h\colon \mathbb{R}^2 \to \mathbb{R}$ be differentiable at a point P. Show h is continuous at P.

(e) (2 points) Construct a function $\phi(x,y)$ that is not differentiable at a point P, of which ϕ_x and ϕ_y exist at P, but only one of the two is continuous at P. Justify your answer.

Question2 (4 points)

(a) (1 point) If $z = f(t)$, where $t = x - y$, show that $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$.

(b) (1 point) Find the derivative $F'(x)$ for $F(x) = \displaystyle\int_0^{x^2} t\sin(x^2 - t)dt$.

(c) (2 points) Find the second order partial derivative $\dfrac{\partial^2 N}{\partial v^2}$ and $\dfrac{\partial^2 N}{\partial v \partial u}$ for

$$N = \frac{p+q}{p+r}, \qquad \text{where } p = u + vw,\ q = v + uw,\ r = w + uv.$$

Question3 (1 points)

Suppose that

$$z = x(e^x + e^{-y})$$

where x and y are found to be 2 and $\ln 2$ with maximum possible errors of

$$|\Delta x| = |dx| = 0.1 \text{ and } |\Delta y| = |dy| = 0.02.$$

Estimate the maximum possible error in the computed value of z using differentials.

Question4 (1 points)

Find an equation of the tangent plane to the following ellipsoid at $(2, 1, -3)$.

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

Question5 (1 points)

A multiplication problem of two numbers is altered by taking a tiny amount from a factor and adding it to the other. How can you tell using tangent plane approximation whether the product increases or decreases for this tiny change?

Question6 (2 points)

Let $f(x, y)$ be differentiable everywhere. Suppose $f(a, a) = a$, $f_x(a, a) = b$, $f_y(a, a) = c$, where a, b and c are constants. Find $g(a)$ and $h'(a)$ in terms of a, b and c, where

$$g(x) = f\Big(x, f\big(x, f(x, x)\big)\Big) \qquad \text{and} \qquad h(x) = [g(x)]^2$$

Question7 (3 points)

Let $\mathbf{x}$ and $\mathbf{a}$ be the position vectors of points $P(x_1, x_2, x_3)$ and $A(a_1, a_2, a_3)$ respectively, where a_1, a_2 and a_3 are constants. Suppose $f\colon \mathbb{R}^3 \to \mathbb{R}$ is differentiable in an open ball $\mathcal{B}$ containing A, and denote

$$f(x_1, x_2, x_3) = f(\mathbf{x})$$

(a) (1 point) Show there exists some $\mathbf{b}$ on the line segment defined by $\mathbf{x}$ and $\mathbf{a}$ such that

$$f(\mathbf{x}) = f(\mathbf{a}) + \sum_{k=1}^{3} (x_k - a_k) \frac{\partial f}{\partial x_k}(\mathbf{b}) \qquad \text{where} \qquad \mathbf{x} \in \mathcal{B}$$

(b) (1 point) Suppose f is twice differentiable in $\mathcal{B}$, that is, all the first order partial derivatives are differentiable in $\mathcal{B}$. Show there exists some $\mathbf{b}$ on the line segment defined by $\mathbf{x}$ and $\mathbf{a}$ such that

$$f(\mathbf{x}) = f(\mathbf{a}) + \sum_{k=1}^{3} (x_k - a_k) \frac{\partial f}{\partial x_k}(\mathbf{a}) + \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} (x_i - a_i)(x_j - a_j) \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{b})$$

where $\mathbf{x} \in \mathcal{B}$.

(c) (1 point) Find a similar formula for $f(\mathbf{x})$ involving third order partial derivatives when f is three times differentiable in $\mathcal{B}$.

Question8 (0 points)

(a) (1 point (bonus)) Let us use the notation above, and define a function $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$,

$$\mathbf{F}(\mathbf{x}) = F_x(\mathbf{x})\mathbf{e}_x + F_y(\mathbf{x})\mathbf{e}_y + F_z(\mathbf{x})\mathbf{e}_z$$

to be differentiable at $\mathbf{a}$ if and only if its scalar-valued component functions F_x, F_y and F_z are differentiable at $\mathbf{a}$. What does the following partial derivative at $\mathbf{a}$ give us?

$$\frac{\partial \mathbf{F}}{\partial x} = \lim_{h \to 0} \frac{\mathbf{F}(\mathbf{a} + h\mathbf{e}_x) - \mathbf{F}(\mathbf{a})}{h}$$

(b) (1 point (bonus)) Show we can uniquely solve for u and v as a function of x and y

$$xu + yvu^2 = 2$$
$$xu^3 + y^2v^4 = 2$$

near the point $(x, y, u, v) = (1, 1, 1, 1)$. Then compute $\dfrac{\partial u}{\partial x}$ at the point $(1, 1, 1, 1)$.