
Question1 (9 points)

(a) (1 point) Find the distance from the point $(2, 1, 3)$ to the line

$$\frac{x-2}{2} = \frac{y-1}{6}; \quad z = 3$$

Solution:

1M The distance is 0 since the point $(2, 1, 3)$ is actually on the line.

(b) (1 point) Find an equation for the line tangent to the curve at $t = 0$.

$$\mathbf{r}(t) = \frac{1}{t+1}\mathbf{e}_x + \frac{t}{t-1}\mathbf{e}_y + \frac{t-1}{t+1}\mathbf{e}_z$$

Solution:

1M We need the direction $\mathbf{v} = \mathbf{r}'(0)$ of the line and a vector $\mathbf{r}(0)$ points at the line.

$$\mathbf{r}(0) = \mathbf{e}_x - \mathbf{e}_z$$
$$\mathbf{r}'(0) = \frac{-1}{(t+1)^2}\mathbf{e}_x + \frac{-1}{(t-1)^2}\mathbf{e}_y + \frac{2}{(t+1)^2}\mathbf{e}_z\bigg|_{t=0} = -\mathbf{e}_x - \mathbf{e}_y + 2\mathbf{e}_z$$

thus an equation for the tangent line is

$$\frac{x-1}{-1} = \frac{y}{-1} = \frac{z+1}{2}$$

(c) (1 point) Find the vector equation that describes the plane that is orthogonal to

$$x + y - 2z = 1$$

and passes through the line of intersection of the planes

$$x - z = 1 \qquad \text{and} \qquad y + 2z = 3$$

Solution:

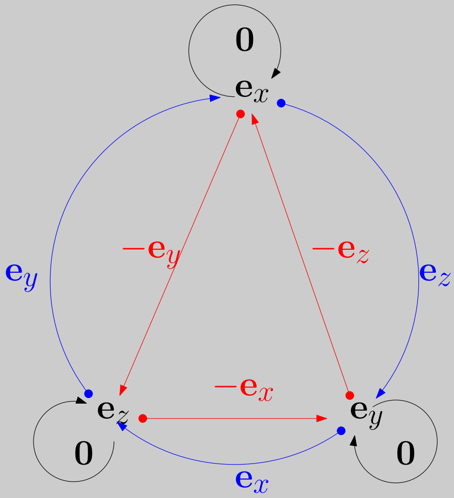
1M The normal vectors of the three planes are

$$\mathbf{n}_1 = \mathbf{e}_x + \mathbf{e}_y - 2\mathbf{e}_z$$
$$\mathbf{n}_2 = \mathbf{e}_x - \mathbf{e}_z$$
$$\mathbf{n}_1 = \mathbf{e}_y + 2\mathbf{e}_z$$

respectively, a normal vector of the plane that we are looking for is given by

$$\begin{aligned}\mathbf{n}_1 \times (\mathbf{n}_2 \times \mathbf{n}_3) &= (\mathbf{e}_x + \mathbf{e}_y - 2\mathbf{e}_z) \times (\mathbf{e}_z - 2\mathbf{e}_y + \mathbf{e}_x) \\ &= -\mathbf{e}_y - 2\mathbf{e}_z + \mathbf{e}_x - \mathbf{e}_z - 4\mathbf{e}_x - 2\mathbf{e}_y = -3\mathbf{e}_x - 3\mathbf{e}_y - 3\mathbf{e}_z\end{aligned}$$

If you haven't seen the graph on the next page, you might find it useful.



Clockwise gives the missing basis vector, Counterclockwise gives the negative miss basis vector, and with itself always gives the zero vector.

Back to the question, using

$$x - z = 1 \qquad \text{and} \qquad y + 2z = 3$$

we know the point $(2, 1, 1)$ is on the plane that we are looking for, hence

$$(\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z) \cdot (\mathbf{r}(t) - 2\mathbf{e}_x - \mathbf{e}_y - \mathbf{e}_z) = 0 \implies (\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z) \cdot \mathbf{r}(t) = 4$$

describes the plane.

(d) (1 point) Find a vector-valued function for the curve of intersection of

$$\text{the cone} \quad z = \sqrt{x^2 + y^2} \quad \text{and the plane} \quad z = 1 + y.$$

Solution:

1M We can obtain the implicit equation for the curve of intersection by solving the two given equations

$$x^2 + y^2 = y^2 + 2y + 1 \implies x^2 = 2y + 1 \implies y = \frac{x^2 - 1}{2}$$

so if we let $x = t$, then

$$\mathbf{r}(t) = t\mathbf{e}_x + \frac{t^2 - 1}{2}\mathbf{e}_y + \frac{t^2 + 1}{2}\mathbf{e}_z$$

(e) (1 point) By considering two different planes whose intersection is the line

$$x = 1 + t, \qquad y = 2 - t, \qquad \text{and} \qquad z = 3 + 2t,$$

find a parametrization for all planes that contain the line

$$x = 1 + t, \qquad y = 2 - t, \qquad z = 3 + 2t.$$

Solution:

1M The symmetric equation for the line is

$$\frac{x-1}{1}=\frac{y-2}{-1}=\frac{z-3}{2}$$

which can be thought of as two conditions

$$\frac{x-1}{1}=\frac{y-2}{-1}; \qquad \frac{y-2}{-1}=\frac{z-3}{2}$$

that must be simultaneously satisfied for (x, y, z) to be on the line, individually

$$x+y=3; \qquad 2y+z=7$$

they represent planes of which the intersection is the line, the normal vectors

$$\mathbf{n}_1=\mathbf{e}_x+\mathbf{e}_y; \qquad \mathbf{n}_2=2\mathbf{e}_y+\mathbf{e}_z$$

Consider $t=0$, we see the point $(1,2,3)$ is on the the line. Hence any plane that contain the line must take the following form

$$(\alpha\mathbf{n}_1+\beta\mathbf{n}_2)\cdot(\mathbf{r}-\mathbf{e}_x-2\mathbf{e}_y-3\mathbf{e}_z)=0$$

where α and β are real and are not simultaneously zero. The scalar equation is

$$\alpha(x-1)+(\alpha+2\beta)(y-2)+\beta(z-3)=0$$

Any parametrization satisfies the above scalar equations is valid, it is not unique.

(f) (2 points) Consider the line ℓ_1 that is defined by the point $(-2,0,2)$ and the vector

$$\mathbf{e}_x-\mathbf{e}_z$$

and the line ℓ_2 that is defined by $(-3,2,7)$ and the vector

$$\mathbf{e}_x+5\mathbf{e}_y+\mathbf{e}_z$$

Find the plane that has the following two properties:

- There is a line that is perpendicular to both ℓ_1 and ℓ_2 on this plane.
- This plane makes an angle of $\frac{\pi}{4}$ with the following plane

$$x-4y-8z+12=0$$

Solution:

1M Since $\mathbf{v}_1=\mathbf{e}_x-\mathbf{e}_z$ is clearly not a scalar multiple of $\mathbf{v}_2=\mathbf{e}_x+5\mathbf{e}_y+\mathbf{e}_z$. So ℓ_1 and ℓ_2 are not parallel. Let

$$\mathbf{r}_1=\begin{bmatrix}-2\\0\\2\end{bmatrix}+t\begin{bmatrix}1\\0\\-1\end{bmatrix} \qquad \text{and} \qquad \mathbf{r}_2=\begin{bmatrix}-3\\2\\7\end{bmatrix}+t\begin{bmatrix}1\\5\\1\end{bmatrix}$$

Since the following equation has no solution,

$$\mathbf{r}_2(t) - \mathbf{r}_1(t) = \mathbf{0} \implies \begin{bmatrix} -1 \\ 2+5t \\ 5+2t \end{bmatrix} = \mathbf{0}$$

the two lines also do not intersect. They are known as skew lines.

Here we have to assume the line ℓ_3 that is perpendicular to ℓ_1 and ℓ_2 actually intersects with ℓ_1 and ℓ_2. We clearly cannot determine the location of ℓ_3 if ℓ_3 is skew with respect to ℓ_1 or ℓ_2 as well. Under this assumption, ℓ_3 can be uniquely determined. The distance between ℓ_1 and ℓ_2 is given by

$$\begin{aligned} D &= \frac{1}{|\mathbf{v}_2|}\left|\Big(\mathbf{r}_1(t) - (-3\mathbf{e}_x + 2\mathbf{e}_y + 7\mathbf{e}_z)\Big) \times \mathbf{v}_2\right| \\ &= \sqrt{\frac{(2t+6)^2 + (5t+7)^2 + (5t+23)^2}{27}} \end{aligned}$$

which is minimised at $t = -3$. Hence we know the line ℓ_3 that is perpendicular to both ℓ_1 and ℓ_2 pass through the point $(-1, 0, 5)$. The direction of ℓ_3 is given by the cross product

$$\mathbf{v}_3 = \mathbf{v}_1 \times \mathbf{v_2} = (\mathbf{e}_x - \mathbf{e}_z) \times (\mathbf{e}_x + 5\mathbf{e}_y + \mathbf{e}_z) = 5\mathbf{e}_x - 2\mathbf{e}_y + 5\mathbf{e}_z$$

since ℓ_3 is orthogonal to both ℓ_1 and ℓ_2. Any plane contains ℓ_3 has

$$\mathbf{n} = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 = (\alpha+\beta)\mathbf{e}_x + 5\beta\mathbf{e}_y + (\beta-\alpha)\mathbf{e}_z$$

as its normal vector.

1M To satisfy the second property, we must have

$$\begin{gathered} \frac{\mathbf{n}\cdot\mathbf{m}}{|\mathbf{n}|\,|\mathbf{m}|} = \frac{\sqrt{2}}{2} \quad \text{where } \mathbf{m} = \mathbf{e}_x - 4\mathbf{e}_y - 8\mathbf{e}_z. \\ -486\beta(4\alpha+3\beta) = 0 \implies \alpha = 1; \beta = 0 \quad \text{or} \quad \alpha = -3; \beta = 4 \end{gathered}$$

This suggests the following two normal vectors up to a scalar

$$\mathbf{n} = \mathbf{e}_x - \mathbf{e}_z \qquad \text{or} \qquad \mathbf{n} = \mathbf{e}_x + 20\mathbf{e}_y + 7\mathbf{e}_z$$

So there are two planes that have the two given properties.

$$x - z + 6 = 0 \qquad \text{and} \qquad x + 20y + 7z - 34 = 0$$

The first property creates a constraint that is satisfied by every vector on a plane for the normal vector, while the second property creates a constraint that is satisfied by every vector on the surface of a cone for the normal vector. The number of intersections, which need to be a line, between a plane and a cone can be 0, 1 or 2. That is why I had to correct the typo, otherwise there may not be any solution.

(g) (2 points) Let L be the line of the intersection between two planes

$$x + y - z - 1 = 0 \qquad \text{and} \qquad x - y + z + 1 = 0$$

and let π denote the plane

$$x + y + z = 0$$

Suppose M is the projection of the line L onto the plane π. Find the equation of the surface that is formed by the straight line M rotating around the z-axis.

Solution:

1M The direction of L is given by

$$\mathbf{v} = (\mathbf{e}_x + \mathbf{e}_y - \mathbf{e}_z) \times (\mathbf{e}_x - \mathbf{e}_y + \mathbf{e}_z) = -2\mathbf{e}_y - 2\mathbf{e}_z$$

Set $x = 0$, the first two equations reduce to

$$y - z - 1 = 0$$

So we known point $P(0, 0, -1)$ is on L. So, an equation of L is given by

$$x = 0, \quad \frac{y}{-2} = \frac{z+1}{-2}$$

One of the normal vectors of plane π is $\mathbf{n} = \mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z$, the direction vector of M is any scalar multiple of the difference between $\mathbf{v}$ and its projection onto $\mathbf{n}$.

$$\mathbf{u} = \mathbf{v} - \operatorname{proj}_{\mathbf{n}} \mathbf{v} = \frac{2\sqrt{3}}{3} (2\mathbf{e}_x - \mathbf{e}_y - \mathbf{e}_z)$$

thus let us use the direction vector

$$2\mathbf{e}_x - \mathbf{e}_y - \mathbf{e}_z$$

Solving the intersection between L and π gives a point Q on M,

$$Q(0, \frac{1}{2}, -\frac{1}{2})$$

Thus a set of parametric equation for M is

$$x(t) = 2t, \quad y(t) = \frac{1}{2} - t, \quad z(t) = -\frac{1}{2} - t$$

1M Since the surface is formed by rotation about the z-axis, any point on the surface shall satisfy the following relation

$$x^2 + y^2 = (2t)^2 + \left(\frac{1}{2} - t\right)^2$$

which says the squared distance from the z-axis is set by M. The z-value is

$$z(t) = -\frac{1}{2} - t \implies 2t = -1 - 2z \implies \frac{1}{2} - t = z + 1$$

Thus eliminating t gives a scalar equation of the surface.

$$x^2 + y^2 - 5z^2 - 6z - 2 = 0$$

Question2 (1 points)

Compute the determinant of $\mathbf{A}$, which is a product of the following two matrices

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -5 & 4 \\ 4 & -1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 2 \\ 3 & 7 & 1 \end{bmatrix}$$

Solution:

1M Matrix multiplication followed by definition of the determinant of a matrix

$$\mathbf{A} = \begin{bmatrix} 10 & 15 & 0 \\ 12 & 8 & -6 \\ 28 & 60 & 10 \end{bmatrix} \implies \det \mathbf{A} = 80$$

Question3 (2 points)

Suppose there are two non-parallel lines in $\mathbb{R}^3$

$$L_i = \frac{x - x_i}{m_i} = \frac{y - y_i}{n_i} = \frac{z - z_i}{p_i} \qquad \text{for} \qquad i = 1, 2$$

Show that any point $P(x, y, z)$ on the plane which passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to both L_1 and L_2 satisfies the following equation

$$\det \begin{bmatrix} x - x_0 & y - y_0 & z - z_0 \\ m_1 & n_1 & p_1 \\ m_2 & n_2 & p_2 \end{bmatrix} = 0$$

Solution:

1M Let $\mathbf{r}_0$ denote the position vector of the point P_0, then the plane passes through P_0 can described by

$$(\mathbf{r} - \mathbf{r_0}) \cdot \mathbf{n} = 0$$

where $\mathbf{n}$ is the normal vector, and can be determined by L_1 and L_2. The direction vectors of L_1 and L_2, denoted by $\mathbf{v}_1$, $\mathbf{v}_2$, is

$$\mathbf{v}_1 = m_1\mathbf{e}_x + n_1\mathbf{e}_y + p_1\mathbf{e}_z, \quad \mathbf{v}_2 = m_2\mathbf{e}_x + n_2\mathbf{e}_y + p_2\mathbf{e}_z$$

Since the plane is parallel to L_1 and L_2, the normal vector of plane is is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$, thus

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$$

hence

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{v}_1 \times \mathbf{v}_2 = 0$$

Since the left hand side is a scalar triple product which is equal to the determinant

$$\det \begin{bmatrix} x - x_0 & y - y_0 & z - z_0 \\ m_1 & n_1 & p_1 \\ m_2 & n_2 & p_2 \end{bmatrix} = 0$$

Question4 (1 points)

Show all vectors $\mathbf{w}$ such that $\mathbf{u} \times \mathbf{w} = \mathbf{v}$ have the form $\mathbf{w} = \alpha \mathbf{u} + \dfrac{\mathbf{v} \times \mathbf{u}}{|\mathbf{u}|^2}$.

Solution:

1M Notice vectors

$$\mathbf{u} \qquad \text{and} \qquad \mathbf{v} \times \mathbf{u}$$

are orthogonal. The expression is essentially an orthogonal decomposition of $\mathbf{w}$,

$$\mathbf{w} = \alpha \mathbf{u} + \beta \left(\mathbf{v} \times \mathbf{u}\right)$$

Crossing with $\mathbf{u}$ and apply Lagrange's formula, we have

$$\begin{aligned} \mathbf{u} \times \mathbf{w} &= \alpha \mathbf{u} \times \mathbf{u} + \beta \left[\mathbf{u} \times \left(\mathbf{v} \times \mathbf{u}\right)\right] \\ \mathbf{v} &= \alpha \mathbf{0} + \beta \left[\mathbf{v}\left(\mathbf{u} \cdot \mathbf{u}\right) - \mathbf{u}\left(\mathbf{u} \cdot \mathbf{v}\right)\right] \end{aligned}$$

since $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, so

$$\mathbf{v} = \beta \left|\mathbf{u}\right|^2 \mathbf{v}$$

which implies

$$\beta = \frac{1}{\left|\mathbf{u}\right|^2}$$

Hence

$$\mathbf{w} = \alpha \mathbf{u} + \frac{\mathbf{v} \times \mathbf{u}}{\left|\mathbf{u}\right|^2}$$

Question5 (1 points)

Let P be a point not on the plane that is defined by points Q, R and S. Derive a formula for the distance from P to the plane in terms of only $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

$$\mathbf{a} = \vec{QR}, \quad \mathbf{b} = \vec{QS}, \quad \text{and} \quad \mathbf{c} = \vec{QP}$$

Your formula shall not be in terms of any angle.

Solution:

1M The plane is defined by the point Q and the normal vector

$$\mathbf{a} \times \mathbf{b}$$

the distance is simply the scalar component of $\mathbf{c}$ onto $\mathbf{a} \times \mathbf{b}$

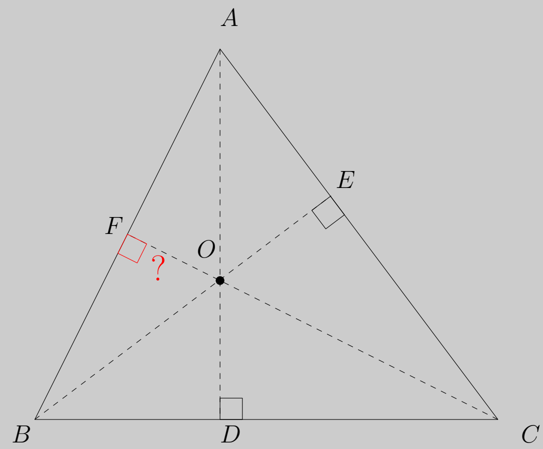
$$\operatorname{comp}_{\mathbf{a} \times \mathbf{b}} \mathbf{c} = \frac{\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})}{|\mathbf{a} \times \mathbf{b}|}$$

Question6 (1 points)

Use vector algebra instead of geometry to show that the three normals dropped from the vertices of a triangle perpendicular to their opposite sides intersect at the same point.

Solution:

1M Let us use the notation indicated in the diagram below



Suppose

$$\vec{AD} \cdot \vec{BC} = 0 \qquad \vec{BE} \cdot \vec{AC} = 0$$

and O is the intersection of AD and BE, and

$$\vec{CO} \times \vec{OF} = \mathbf{0}$$

that is, there are parallel and thus lying on the same line. We need to show $\vec{OF}$ is orthogonal to $\vec{AB}$, that is

$$\vec{OF} \cdot \vec{AB} = 0$$

The vector $\vec{CE}$ is the projection of $\vec{OC}$ onto $\vec{AC}$, that is,

$$\vec{CE} = \text{proj}_{\vec{AC}}\, \vec{OC} = \frac{\vec{OC} \cdot \vec{AC}}{\vec{AC} \cdot \vec{AC}} \vec{AC}$$

However, $\vec{CE}$ is also the projection of $\vec{BC}$ onto $\vec{AC}$, that is,

$$\vec{CE} = \text{proj}_{\vec{AC}}\, \vec{BC} = \frac{\vec{BC} \cdot \vec{AC}}{\vec{AC} \cdot \vec{AC}} \vec{AC}$$

Thus, we can conclude

$$\vec{OC} \cdot \vec{AC} = \vec{BC} \cdot \vec{AC}$$

Similarly,

$$\left.\begin{aligned} \vec{DC} &= \frac{\vec{OC} \cdot \vec{BC}}{\vec{BC} \cdot \vec{BC}} \vec{BC} \\ \vec{DC} &= \frac{\vec{AC} \cdot \vec{BC}}{\vec{BC} \cdot \vec{BC}} \vec{BC} \end{aligned}\right\} \implies \vec{OC} \cdot \vec{BC} = \vec{AC} \cdot \vec{BC}$$

which means

$$\vec{OC} \cdot \vec{AC} = \vec{BC} \cdot \vec{AC} = \vec{OC} \cdot \vec{BC}$$

Now if we consider the dot product

$$\vec{OC} \cdot \vec{AB} = \vec{OC} \cdot \left(\vec{AC} - \vec{BC}\right) = \vec{OC} \cdot \vec{AC} - \vec{OC} \cdot \vec{BC} = 0$$

which means

$$\vec{OF} \cdot \vec{AB} = 0$$

since $\vec{OF}$ and $\vec{OC}$ are on the same line. Therefore, we can conclude they intersect at the same point O.

Question7 (2 points)

Consider an arbitrary tetrahedron $ABCD$. Let T_1, T_2, T_3 and T_4 denote

$$\text{the area of triangle} \triangle ABC,\ \triangle BCD,\ \triangle ACD \text{ and} \triangle ABD$$

respectively. Let $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ and $\mathbf{n}_4$ denote the outward pointing normal vector for

$$\text{the triangular face} \triangle ABC,\ \triangle BCD,\ \triangle ACD \text{ and} \triangle ABD$$

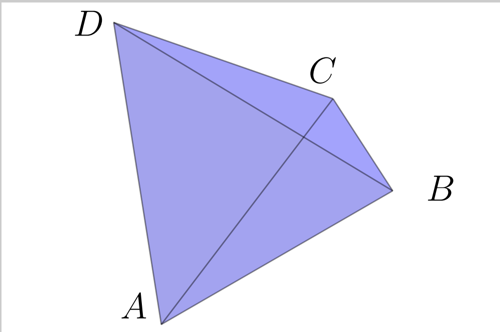
respectively, such that

$$T_1 = |\mathbf{n}_1|, \quad T_2 = |\mathbf{n}_2|, \quad T_3 = |\mathbf{n}_3|, \quad T_4 = |\mathbf{n}_4|$$

(a) (1 point) Use cross product to show that the sum of $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ and $\mathbf{n}_4$ is $\mathbf{0}$.

Solution:

1M Consider the following tetrahedron,



Because the cross product of two vectors gives a vector that is orthogonal to both of the vectors. A normal vector of any triangular face can be represented as the cross product of the two vectors that define the triangular face. For example, to obtain $\mathbf{n}_1$, we simply need to scale the cross product $\vec{BA} \times \vec{BC}$ by half to reflect the area of $\triangle ABC$ is half of the parallelogram that is defined by the two vectors. So

$$\mathbf{n}_1 = \frac{\vec{BA} \times \vec{BC}}{2}$$

Similarly,

$$\mathbf{n}_2 = \frac{\vec{DB} \times \vec{DC}}{2} \qquad \mathbf{n}_3 = \frac{\vec{CA} \times \vec{CD}}{2} \qquad \mathbf{n}_4 = \frac{\vec{DA} \times \vec{DB}}{2}$$

The sum of these four vectors can be represented as

$$\mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3 + \mathbf{n}_4 = \frac{1}{2}\left(\vec{BA} \times \vec{BC} + \vec{DB} \times \vec{DC} + \vec{CA} \times \vec{CD} + \vec{DA} \times \vec{DB}\right)$$

We have been given that the sum is zero, so we simply need to do the addition, and everything will varnish. This is a perfect example to illustrate that sometimes orthonormal basis is not the best basis to use. Since it is not clear why those cross products shall add to zero in the standard basis. Let us use another basis to show the sum is indeed zero.

Hopefully, you have managed to prove in assignment 1 that all bases for n-dimensional space consist of n linearly independent vectors. And we have discussed in class, when we were talking about cross and scalar triple product, linearly independent vectors in $\mathbb{R}^3$ are vectors that are not coplanar.

Any three edges from the same vertex of the tetrahedron $ABCD$ are clearly not coplanar, thus the corresponding vectors for the three edges are linearly independent and hence form a basis for $\mathbb{R}^3$. For example, let us use the vertex D, then the following is a basis for $\mathbb{R}^3$.

$$\mathcal{B} = \{\vec{DA}, \vec{DB}, \vec{DC}\}$$

Every vector $\mathbf{v} \in \mathbb{R}^3$, including those on the left-hand side of the following

$$\mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3 + \mathbf{n}_4 = \frac{1}{2}\left(\vec{BA} \times \vec{BC} + \vec{DB} \times \vec{DC} + \vec{CA} \times \vec{CD} + \vec{DA} \times \vec{DB}\right)$$

can be represented as a linear combination of those three basis vectors.

$$\mathbf{v} = \alpha\vec{DA} + \beta\vec{DB} + \gamma\vec{DC}$$

Here we don't need to solve any linear system despite $\mathcal{B}$ not being orthogonal.

$$\begin{aligned}
&\vec{BA} = \vec{DA} - \vec{DB}, \qquad &&\vec{BC} = \vec{DC} - \vec{DB}, \qquad &&\vec{DB} = \vec{DB}, \qquad &&\vec{DC} = \vec{DC} \\
&\vec{CA} = \vec{DA} - \vec{DC}, \qquad &&\vec{CD} = -\vec{DC}, \qquad &&\vec{DA} = \vec{DA}, \qquad &&\vec{DB} = \vec{DB}
\end{aligned}$$

So we have

$$\begin{aligned}
2\mathbf{n}_1 &= \vec{BA} \times \vec{BC} = (\vec{DA} - \vec{DB}) \times (\vec{DC} - \vec{DB}) \\
&\qquad\qquad\quad = \vec{DA} \times \vec{DC} - \vec{DA} \times \vec{DB} - \vec{DB} \times \vec{DC} \\
2\mathbf{n}_2 &= \vec{DB} \times \vec{DC} = \vec{DB} \times \vec{DC} \\
2\mathbf{n}_3 &= \vec{CA} \times \vec{CD} = (\vec{DA} - \vec{DC}) \times (-\vec{DC}) = -\vec{DA} \times \vec{DC} \\
2\mathbf{n}_4 &= \vec{DA} \times \vec{DB} = \vec{DA} \times \vec{DB}
\end{aligned}$$

With this basis, we can see clearly that everything vanishes.

$$\mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3 + \mathbf{n}_4 = \mathbf{0}$$

(b) (1 point) Suppose $\angle ADB = \angle BDC = \angle CDA = \dfrac{\pi}{2}$. Show that the areas satisfy

$$T_1^2 = T_2^2 + T_3^2 + T_4^2$$

Solution:

1M Since the three angles at D are all $\dfrac{\pi}{2}$, we have

$$\vec{DA}\cdot\vec{DB} = 0, \qquad \vec{DB}\cdot\vec{DC} = 0, \qquad \vec{DC}\cdot\vec{DA} = 0$$

The dot product

$$\begin{aligned}\mathbf{n}_2\cdot\mathbf{n}_3 &= -\frac{1}{4}\left(\vec{DB}\times\vec{DC}\right)\cdot\left(\vec{DA}\times\vec{DC}\right)\\ &= -\frac{1}{4}\left[(\vec{DB}\cdot\vec{DA})(\vec{DC}\cdot\vec{DC}) - (\vec{DB}\cdot\vec{DC})(\vec{DC}\cdot\vec{DA})\right] = 0\end{aligned}$$

similarly, we can easily show

$$\mathbf{n}_2\cdot\mathbf{n}_4 = 0 \qquad \mathbf{n}_3\cdot\mathbf{n}_4 = 0$$

So the three normal vectors $\mathbf{n}_2$, $\mathbf{n}_3$, $\mathbf{n}_4$ are orthogonal to each other. We have shown that $\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3+\mathbf{n}_4=\mathbf{0}$, thus the squared length of $\mathbf{n}_1$ is

$$\begin{aligned}\mathbf{n}_1 = -\left(\mathbf{n}_2+\mathbf{n}_3+\mathbf{n}_4\right) \implies |\mathbf{n}_1|^2 &= |\mathbf{n}_2+\mathbf{n}_3+\mathbf{n}_4|^2\\ &= (\mathbf{n}_2+\mathbf{n}_3+\mathbf{n}_4)\cdot(\mathbf{n}_2+\mathbf{n}_3+\mathbf{n}_4)\\ &= \mathbf{n}_2\cdot\mathbf{n}_2+\mathbf{n}_3\cdot\mathbf{n}_3+\mathbf{n}_4\cdot\mathbf{n}_4\\ &= |\mathbf{n}_2|^2+|\mathbf{n}_3|^2+|\mathbf{n}_4|^2\\ T_1^2 &= T_2^2+T_3^2+T_4^2\end{aligned}$$

This is like the Pythagoras's theorem but in $\mathbb{R}^3$.

Question8 (3 points)

Suppose your iPhone520s can fly like Thor's Hammer. On a very special day, your phone flies through space according to the following acceleration vector when summoned

$$\mathbf{a}(t) = \begin{bmatrix}-3\cos t\\ -3\sin t\\ 2\end{bmatrix} \qquad \text{for } t \geq 0.$$

We also know the phone is at the point $(3,0,0)$ with the velocity $\mathbf{v} = 3\mathbf{e}_y$ at time $t=0$.

The ghost named Jobs is involved in the design of iPhone520s which means it not only can fly but it is awesome and can carry you like a vehicle. However, your new iPhone automatically and instantaneously sends a "warning" message to your partner if it travels more than 5 miles away from her/him, and sends another lovely message if it comes back within this distance. Assume your partner is at the origin on the very special day, and the position vector $\mathbf{r}(t)$ is in miles.

(a) (1 point) Find the instantaneous speed at $t=1$.

Solution:

1M The velocity function is given by

$$\begin{aligned}\mathbf{v} = \mathbf{v}(0) + \int_0^t \mathbf{a}(\tau)\,d\tau &= \mathbf{e}_x \int_0^t -3\cos\tau\,d\tau + \mathbf{e}_y \int_0^t -3\sin\tau\,d\tau + \mathbf{e}_z \int_0^t 2\,d\tau \\ &= -3\sin t\mathbf{e}_x + 3\cos t\mathbf{e}_y + 2t\mathbf{e}_z \\ \implies |\mathbf{v}| &= \sqrt{9+4t^2} \\ &= \sqrt{13} \qquad \text{at} \quad t = 1\end{aligned}$$

(b) (1 point) Find the number of messages your partner will receive.

Solution:

1M The position vector is given by

$$\begin{aligned}\mathbf{r} = \mathbf{r}(0) + \int_0^t \mathbf{v}(\tau)\,d\tau &= \mathbf{e}_x \int_0^t -3\sin\tau\,d\tau + \mathbf{e}_y \int_0^t +3\cos\tau\,d\tau + \mathbf{e}_z \int_0^t 2\tau\,d\tau \\ &= 3\cos t\mathbf{e}_x + 3\sin t\mathbf{e}_y + t^2\mathbf{e}_z\end{aligned}$$

The distance from the origin is given by

$$|\mathbf{r}| = \sqrt{9+t^4}$$

which is strictly increasing for $t > 0$, so only one message will be received.

(c) (1 point) Find the rate of change of the cross product between the position vector and the velocity vector with respect to time.

Solution:

1M The rate is

$$\begin{aligned}\frac{d}{dt} = \frac{d}{dt}\mathbf{r}\times\mathbf{v} &= \dot{\mathbf{r}}\times\mathbf{v} + \mathbf{r}\times\dot{\mathbf{v}} \\ &= \mathbf{v}\times\mathbf{v} + \mathbf{r}\times\mathbf{a} \\ &= \mathbf{r}\times\mathbf{a} \\ &= (3\cos t\mathbf{e}_x + 3\sin t\mathbf{e}_y + t^2\mathbf{e}_z)\times(-3\cos t\mathbf{e}_x - 3\sin t\mathbf{e}_y + 2\mathbf{e}_z) \\ &= -9\cos t\sin t\mathbf{e}_z - 6\cos t\mathbf{e}_y + 9\sin t\cos t\mathbf{e}_z + 6\sin t\mathbf{e}_x \\ &\quad - 3t^2\cos t\mathbf{e}_y + 3t^2\sin t\mathbf{e}_x \\ &= (3t^2+6)\sin t\mathbf{e}_x - 3(2+t^2)\cos t\mathbf{e}_y\end{aligned}$$

Question9 (2 points)

Suppose a particle P is moving along a curve $\mathcal{C}$ according to the vector-valued function

$$\mathbf{r}(t) = \begin{bmatrix} \sin\alpha \\ \alpha + \cos\alpha \\ \alpha \end{bmatrix} \quad \text{where } \alpha \text{ is some scalar-valued function of } t.$$

The motion has a unit speed, i.e. $|\mathbf{v}| = 1$. Find the acceleration $\mathbf{a}$ in terms of only α.

Solution:

1M Let us break the position vector into two components,

$$\mathbf{r} = \alpha \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix} \implies \mathbf{v} = \dot{\mathbf{r}} = \dot{\alpha} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \cos\alpha \\ -\sin\alpha \\ 0 \end{bmatrix} \dot{\alpha} = \dot{\alpha} \underbrace{\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \cos\alpha \\ -\sin\alpha \\ 0 \end{bmatrix} \right)}_{\mathbf{b}}$$

The speed is always 1, so

$$|\mathbf{v}| = |\dot{\alpha}|\, |\mathbf{b}| = |\dot{\alpha}| \sqrt{\cos^2\alpha + (1-\sin\alpha)^2 + 1} = 1$$

From this we can obtain the expression of $\dot{\alpha}$ and $\ddot{\alpha}$ in terms of α

$$|\dot{\alpha}| = \frac{1}{\sqrt{\cos^2\alpha + (1-\sin\alpha)^2 + 1}} = (3 - 2\sin\alpha)^{-\frac{1}{2}}$$
$$\ddot{\alpha} = -\frac{1}{2}(3 - 2\sin\alpha)^{-\frac{3}{2}} \cdot (-2\cos\alpha) \cdot \alpha' = \frac{\pm\cos\alpha}{3 - 2\sin\alpha}$$

The acceleration vector is

$$\mathbf{a} = \ddot{\mathbf{r}} = \ddot{\alpha}\mathbf{b} + \dot{\alpha}\dot{\mathbf{b}} = \ddot{\alpha} \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \cos\alpha \\ -\sin\alpha \\ 0 \end{bmatrix} \right) + \dot{\alpha}^2 \begin{bmatrix} -\sin\alpha \\ -\cos\alpha \\ 0 \end{bmatrix}$$
$$= \frac{\pm\cos\alpha}{3 - 2\sin\alpha} \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \cos\alpha \\ -\sin\alpha \\ 0 \end{bmatrix} \right) - \frac{1}{3 - 2\sin\alpha} \begin{bmatrix} \sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix}$$

Question10 (4 points)
A projectile is fired with an angle of elevation θ, and an initial velocity of $\mathbf{v}_0$. Assume the only force acting on the object is gravity.

(a) (1 point) Find the form of the vector-valued function for the position of the object.

Solution:

1M Let us set the coordinate system such that the upward direction is in the direction of $\mathbf{e}_y$ and the positive horizontal direction is in the direction of $\mathbf{e}_x$. Additionally, suppose the motion of the projectile begins at the origin. Since the only force is the downward gravity, we have

$$\mathbf{a}(t) = -g\mathbf{e}_y \implies \mathbf{v}(t) = \int \mathbf{a}(t)\, dt = c_1\mathbf{e}_x + (c_2 - gt)\mathbf{e}_y$$

Applying the initial condition $\mathbf{v}(0) = \mathbf{v}_0$, we have

$$\mathbf{v}(t) = |\mathbf{v}_0| \cos\theta \mathbf{e}_x + (|\mathbf{v}_0| \sin\theta - gt)\mathbf{e}_y$$
$$\implies \mathbf{r}(t) = \int \mathbf{v}(t)\, dt = (c_3 + t\,|\mathbf{v}_0| \cos\theta)\, \mathbf{e}_x + (c_4 + t\,|\mathbf{v}_0| \sin\theta - \frac{1}{2}gt^2)\mathbf{e}_y$$

Applying the initial condition $\mathbf{r}(0) = \mathbf{0}$, we have

$$\mathbf{r}(t) = (t\,|\mathbf{v}_0| \cos\theta)\,\mathbf{e}_x + (t\,|\mathbf{v}_0| \sin\theta - \frac{1}{2}gt^2)\mathbf{e}_y$$

(b) (1 point) Find an expression for the range in terms of θ.

Solution:

1M The range is the horizontal distance when the projectile return to the ground

$$t\left(|\mathbf{v}_0| \sin\theta - \frac{1}{2}gt\right) = 0 \implies t = 0 \quad \text{or} \quad t = \frac{2\,|\mathbf{v}_0| \sin\theta}{g}$$

$$\implies \text{Range} = |\mathbf{v}_0| \cos\theta \frac{2\,|\mathbf{v}_0| \sin\theta}{g} = \frac{|\mathbf{v}_0|^2 \sin 2\theta}{g}$$

(c) (1 point) Find the value of θ which maximises the range.

Solution:

1M This happens when $\sin 2\theta = 1$, that is, $\theta = \dfrac{\pi}{4}$.

(d) (1 point) Find an expression for the distance travelled by it between t_1 and t_2 where

$$0 < t_1 \le t_2$$

Solution:

1M This of course simply is the arc length between t_1 and t_2

$$\int_{t_1}^{t_2} |\mathbf{v}(t)|\, dt$$

Question11 (3 points)

Suppose $\mathbf{r}(t)$ is continuous. Show the followings are true

(a) (1 point)

$$\int_a^b \mathbf{c} \cdot \mathbf{r}(t)\, dt = \mathbf{c} \cdot \int_a^b \mathbf{r}(t)\, dt, \qquad \text{where } \mathbf{c} \text{ is a constant vector.}$$

Solution:

1M Note it is an integral of a scalar-valued function, so it is just a matter of writing everything out, let $\mathbf{c} = c_x\mathbf{e}_x + c_y\mathbf{e}_y + c_z\mathbf{e}_z$ and $\mathbf{r} = r_x\mathbf{e}_x + r_y\mathbf{e}_y + t_z\mathbf{e}_z$, then

$$\begin{aligned}
\int_a^b \mathbf{c} \cdot \mathbf{r}(t)\, dt &= \int_a^b (c_x r_x + c_y r_y + c_z r_z)\, dt \\
&= \int_a^b c_x r_x\, dt + \int_a^b c_y r_y\, dt + \int_a^b c_z r_z\, dt \\
&= c_x \int_a^b r_x\, dt + c_y \int_a^b r_y\, dt + c_z \int_a^b r_z\, dt \\
&= \mathbf{c} \cdot \int_a^b \mathbf{r}(t)\, dt
\end{aligned}$$

(b) (1 point)

$$\int_a^b \mathbf{c} \times \mathbf{r}(t)\, dt = \mathbf{c} \times \int_a^b \mathbf{r}(t)\, dt, \qquad \text{where } \mathbf{c} \text{ is a constant vector.}$$

Solution:

1M Using the same notation and the result in part (a), and consider

$$\begin{aligned} \mathbf{e}_x \cdot \int_a^b \mathbf{c} \times \mathbf{r}(t)\, dt = \int_a^b \mathbf{e}_x \cdot \Big(\mathbf{c} \times \mathbf{r}(t)\Big)\, dt &= \int_a^b 1(c_y r_z - c_z r_y)\, dt \\ &= c_y \int_a^b r_z\, dt - c_z \int_a^b r_y\, dt \\ &= \mathbf{e}_x \cdot \left(\mathbf{c} \times \int_a^b \mathbf{r}(t)\, dt\right) \end{aligned}$$

which shows the first component on the left is equal to the first component on the right. Since we have complete symmetry in the equation, the second and the third components must also hold.

(c) (1 point)

$$\frac{d}{dt} \int_a^t \mathbf{r}(\tau)\, d\tau = \mathbf{r}(t)$$

Solution:

1M This is just a matter of breaking into components and applying FTC to each component. FTC is applicable here because $\mathbf{r}(t)$ is continuous, thus each component must be continuous.

Question12 (1 points)
Suppose $\mathbf{r}(t) = (t+1)\mathbf{e}_x + (t^2+2)\mathbf{e}_y + 2t\mathbf{e}_z$. Find the osculating plane at time $t = 1$.

Solution:

1M Osculating plane is the plane spanned by $\mathbf{T}$ and $\mathbf{N}$, and thus $\mathbf{B}$ is its normal vector.

$$\dot{\mathbf{r}} = \mathbf{e}_x + 2t\mathbf{e}_y + 2\mathbf{e}_z \implies |\dot{\mathbf{r}}| = \sqrt{5 + 4t^2}$$

Thus the unit tangent vector is given by

$$\begin{aligned} \mathbf{T} &= \frac{1}{\sqrt{5+4t^2}} (\mathbf{e}_x + 2t\mathbf{e}_y + 2\mathbf{e}_z) \\ \dot{\mathbf{T}} &= \frac{1}{\sqrt{(5+4t^2)^3}} (-4t\mathbf{e}_x + 10\mathbf{e}_y - 8t\mathbf{e}_z) \\ \implies \left|\dot{\mathbf{T}}\right| &= \frac{\sqrt{80t^2+100}}{\sqrt{(5+4t^2)^3}} \\ \implies \mathbf{N} &= \frac{1}{\sqrt{80t^2+100}} (-4t\mathbf{e}_x + 10\mathbf{e}_y - 8t\mathbf{e}_z) \\ \implies \mathbf{B} &= \mathbf{T} \times \mathbf{N} \\ &= \frac{1}{\sqrt{(5+4t^2)^3}} \frac{1}{\sqrt{80t^2+100}} (\mathbf{e}_x + 2t\mathbf{e}_y + 2\mathbf{e}_z) \times (-4t\mathbf{e}_x + 10\mathbf{e}_y - 8t\mathbf{e}_z) \end{aligned}$$

At $t = 1$, we have

$$\mathbf{r} = 2\mathbf{e}_x + 3\mathbf{e}_y + 4\mathbf{e}_z \qquad \mathbf{B} = -\frac{2}{\sqrt{5}}\mathbf{e}_x + \frac{1}{\sqrt{5}}\mathbf{e}_z$$

So the equation can be described by

$$-\frac{2}{\sqrt{5}}(x-2) + \frac{1}{\sqrt{5}}(z-4) = 0$$

Question13 (4 points)

Santa Claus with his magic sleigh is travelling to see his aunt penguin as well his uncle polar bear, they are apparently separated. In order to see both of them Santa is moving along a meridian of the rotating earth with a constant speed. Let xyz be a fixed Cartesian coordinate system in space, with unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ in the directions of the axes. Let the Earth, together with a unit vector $\mathbf{b}$, be rotating about the z-axis with angular speed $\omega > 0$. Since $\mathbf{b}$ is rotating together with the Earth, it is of the form

$$\mathbf{b}(t) = \cos(\omega t)\mathbf{e}_x + \sin(\omega t)\mathbf{e}_y$$

In other words, the vector $\mathbf{b}$ defines the rotation of the Earth. Let Santa be moving on the meridian whose plane is defined by $\mathbf{b}$ and $\mathbf{e}_z$ with constant angular speed $\gamma > 0$. Then its position vector in terms of $\mathbf{b}$ and $\mathbf{e}_z$ is

$$\mathbf{r}(t) = R\cos(\gamma t)\mathbf{b} + R\sin(\gamma t)\mathbf{e}_z \qquad \text{where } R \text{ is the radius of the Earth.}$$

(a) (1 point) Find the unit tangent vector as a function of t for Santa.

Solution:

1M Consider the rate of change of $\mathbf{b}$,

$$\mathbf{b}'(t) = -\omega\sin(\omega t)\mathbf{e}_x + \omega\cos(\omega t)\mathbf{e}_y$$

$$\mathbf{b}''(t) = -\omega^2\cos(\omega t)\mathbf{e}_x - \omega^2\sin(\omega t)\mathbf{e}_y = -\omega^2\mathbf{b}(t)$$

Notice $\mathbf{b}$ is unit length, and

$$|\mathbf{b}'| = \omega \qquad \text{and} \qquad |\mathbf{b}''| = \omega^2$$

Since $|\mathbf{b}| = 1$, $\mathbf{b}'$ is orthogonal to $\mathbf{b}$.

$$\mathbf{b}' \cdot \mathbf{b} = 0$$

Note both $\mathbf{b}'$ and $\mathbf{b}$ are orthogonal to $\mathbf{e}_z$, so we have an orthonormal basis

$$\left\{\frac{\mathbf{b}'}{\omega}, \mathbf{b}, \mathbf{e}_z\right\}$$

and

$$\frac{\mathbf{b}'}{\omega} \times \mathbf{b} = -\mathbf{e}_z, \qquad \frac{\mathbf{b}'}{\omega} \times \mathbf{e}_z = \mathbf{b} \qquad \text{and} \qquad \mathbf{b} \times \mathbf{e}_z = -\frac{\mathbf{b}'}{\omega}$$

The velocity is given by

$$\begin{aligned}\mathbf{v}(t) &= R\cos(\gamma t)\mathbf{b}' - \gamma R\sin(\gamma t)\mathbf{b} + \gamma R\cos(\gamma t)\mathbf{e}_z \\ &= \omega R\cos(\gamma t)\frac{\mathbf{b}'}{\omega} - \gamma R\sin(\gamma t)\mathbf{b} + \gamma R\cos(\gamma t)\mathbf{e}_z\end{aligned}$$

Thus

$$\begin{aligned}|\mathbf{v}| &= \sqrt{\omega^2 R^2\cos^2(\gamma t) + \gamma^2 R^2\sin^2(\gamma t) + \gamma^2 R^2\cos^2(\gamma t)} \\ &= R\sqrt{\omega^2\cos^2(\gamma t) + \gamma^2}\end{aligned}$$

The unit tangent is

$$\begin{aligned}\mathbf{T}(t) &= \frac{\mathbf{v}}{|\mathbf{v}|} \\ &= \frac{\omega\cos(\gamma t)\frac{\mathbf{b}'}{\omega} - \gamma\sin(\gamma t)\mathbf{b} + \gamma\cos(\gamma t)\mathbf{e}_z}{\sqrt{\omega^2\cos^2(\gamma t) + \gamma^2}}\end{aligned}$$

(b) (1 point) Find the acceleration vector as a function of t for Santa.

Solution:

1M The acceleration is given by

$$\begin{aligned}\mathbf{a}(t) &= R\cos(\gamma t)\mathbf{b}'' - 2\gamma R\sin(\gamma t)\mathbf{b}' - \gamma^2 R\cos(\gamma t)\mathbf{b} - \gamma^2 R\sin(\gamma t)\mathbf{e}_z \\ &= \underbrace{R\cos(\gamma t)\mathbf{b}''}_{\text{Earth}}\underbrace{-2\gamma R\sin(\gamma t)\mathbf{b}'}_{\text{Coriolis}}\underbrace{-\gamma^2\mathbf{r}}_{\text{Santa}}\end{aligned}$$

(c) (1 point) What is the centripetal acceleration due to the rotation of the Earth?

Solution:

1M See part (b)

(d) (1 point) What is the centripetal acceleration due to the motion of Santa on the meridian of the rotating Earth.

Solution:

1M See part (b)

Question14 (6 points)

(a) (1 point) Find an expression for the torsion of C which is defined by

$$\mathbf{r} = t\mathbf{e}_x + t^2\mathbf{e}_y + t^3\mathbf{e}_z.$$

Solution:

1M When none of **TNB** is available, using the formula with the jerk vector in it is usually the best option

$$\begin{aligned}\mathbf{v} &= \mathbf{e}_x + 2t\mathbf{e}_y + 3t^2\mathbf{e}_z \\ \mathbf{a} &= 2\mathbf{e}_y + 6t\mathbf{e}_z \\ \mathbf{j} &= 6\mathbf{e}_z \\ \implies \tau &= \frac{\mathbf{j}\cdot(\mathbf{v}\times\mathbf{a}))}{|\mathbf{v}\times\mathbf{a})|^2} = \frac{3}{9t^4+9t^2+1}\end{aligned}$$

(b) (1 point) The smaller the curvature of a bend in a road, the faster a car can travel. Assume that the maximum speed around a turn is inversely proportional to the square root of the curvature. A car moving on the path $y = \frac{1}{3}x^3$ (x and y are measured in miles) can safely go 30 miles per hour at $\left(1, \frac{1}{3}\right)$. How fast can it go at $\left(\frac{3}{2}, \frac{9}{8}\right)$?

Solution:

1M Let us denote the maximum speed by

$$|\mathbf{v}| = \frac{\lambda}{\sqrt{\kappa}}$$

Consider the following parametrisation

$$\mathbf{r} = t\mathbf{e}_x + \frac{1}{3}t^3\mathbf{e}_y$$

For the curvature, finding it through the rate of change of $\mathbf{T}$ is not too tedious.

$$\begin{aligned}\mathbf{v} = \mathbf{e}_x + t^2\mathbf{e}_y \implies \mathbf{T} = \hat{\mathbf{v}} = \frac{1}{\sqrt{1+t^4}}\left(\mathbf{e}_x + t^2\mathbf{e}_y\right) \\ \implies \kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sqrt{1+t^4}}\frac{1}{\sqrt{(1+t^4)^3}}\left|-2t^3\mathbf{e}_x + 2t\mathbf{e}_y\right| = \frac{|2t|}{(1+t^4)^{3/2}}\end{aligned}$$

Using 30 as the maximum speed at $(1, 1/3)$ when $t = 1$, we have

$$30 = \frac{\lambda}{\sqrt{2/2^{3/2}}} \implies \lambda = 15\sqrt[4]{2^3}$$

At $(3/2, 9/8)$ when $t = 3/2$, we have

$$|\mathbf{v}| = \frac{15\sqrt[4]{2^3}}{\sqrt{\kappa(3/2)}} = \frac{5\sqrt[4]{65712456}}{8} = 56.27\text{miles/hours}$$

(c) (1 point) A curve C is given by the polar equation $r = f(\theta)$. Show that the curvature κ at the point (r, θ) is

$$\kappa = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}}$$

Solution:

1M Let us use the following notation,

$$\hat{\boldsymbol{\rho}} = \cos\theta \mathbf{e}_x + \sin\theta \mathbf{e}_y$$

then the the curve C can be described by the following vector-valued function

$$\mathbf{r} = r(\theta)\cos\theta \mathbf{e}_x + r(\theta)\sin\theta \mathbf{e}_y = r(\theta)\hat{\boldsymbol{\rho}}$$

So first derivative can be represented as

$$\frac{d\mathbf{r}}{d\theta} = r'\hat{\boldsymbol{\rho}} + r\frac{d\hat{\boldsymbol{\rho}}}{d\theta} = r'\hat{\boldsymbol{\rho}} + r\underbrace{\left(-\sin\theta \mathbf{e}_x + \cos\theta \mathbf{e}_y\right)}_{\hat{\boldsymbol{\phi}}} = r'\hat{\boldsymbol{\rho}} + r\hat{\boldsymbol{\phi}}$$

Note $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ form a orthonormal basis for $\mathbb{R}^2$. And the unit tangent vector is

$$\mathbf{T}(\theta) = \frac{\frac{d\mathbf{r}}{d\theta}}{\left|\frac{d\mathbf{r}}{d\theta}\right|} = \frac{r'\hat{\boldsymbol{\rho}} + r\hat{\boldsymbol{\phi}}}{\sqrt{(r'\hat{\boldsymbol{\rho}} + r\hat{\boldsymbol{\phi}})\cdot(r'\hat{\boldsymbol{\rho}} + r\hat{\boldsymbol{\phi}})}} = \left[(r')^2 + r^2\right]^{-1/2}\left(r'\hat{\boldsymbol{\rho}} + r\hat{\boldsymbol{\phi}}\right)$$

The curvature is defined as the length of the the derivative of the unit tangent vector with respect to arc length, according to the chain rule

$$\begin{aligned}
\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{d\theta}\frac{d\theta}{ds} = \frac{d\theta}{ds}\frac{d}{d\theta}\left(\frac{d\mathbf{r}}{ds}\right) &= \frac{d\theta}{ds}\frac{d}{d\theta}\left(\frac{d\mathbf{r}}{d\theta}\frac{d\theta}{ds}\right) \\
&= \frac{d\theta}{ds}\left(\frac{d^2\mathbf{r}}{d\theta^2}\frac{d\theta}{ds} + \frac{d\mathbf{r}}{d\theta}\frac{d}{d\theta}\left(\frac{d\theta}{ds}\right)\right) \\
&= \frac{d\theta}{ds}\left(\frac{d^2\mathbf{r}}{d\theta^2}\frac{d\theta}{ds} + \frac{d\mathbf{r}}{d\theta}\frac{d}{d\theta}\left(\frac{1}{\left|\frac{d\mathbf{r}}{d\theta}\right|}\right)\right) \\
&= \frac{d^2\mathbf{r}}{d\theta^2}\left(\frac{d\theta}{ds}\right)^2 + \frac{d\mathbf{r}}{d\theta}\frac{d\theta}{ds}\frac{d}{d\theta}\left(\frac{1}{\left|\frac{d\mathbf{r}}{d\theta}\right|}\right) \\
&= \frac{d^2\mathbf{r}}{d\theta^2}\left(\frac{d\theta}{ds}\right)^2 + \mathbf{T}\left|\frac{d\mathbf{r}}{d\theta}\right|\frac{d\theta}{ds}\frac{d}{d\theta}\left(\frac{1}{\left|\frac{d\mathbf{r}}{d\theta}\right|}\right)
\end{aligned}$$

Note $\mathbf{T}$ and $\mathbf{N}$ form another orthonormal basis for $\mathbb{R}^2$, so

$$\frac{d\mathbf{T}}{ds} = \alpha\mathbf{T} + \beta\mathbf{N} \implies \left|\frac{d\mathbf{T}}{ds}\right| = \sqrt{\alpha^2 + \beta^2}$$

where

$$\alpha = \frac{d\mathbf{T}}{ds}\cdot\mathbf{T} \qquad \beta = \frac{d\mathbf{T}}{ds}\cdot\mathbf{N}$$

Since $\mathbf{T}$ is a unit vector, the dot product $\frac{d\mathbf{T}}{ds}\cdot\mathbf{T} = 0$,

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = |\beta| = \left|\frac{d\mathbf{T}}{ds}\cdot\mathbf{N}\right| = \left|\left(\frac{d\theta}{ds}\right)^2\left(\frac{d^2\mathbf{r}}{d\theta^2}\cdot\mathbf{N}\right)\right|$$

Next we need to figure out $\mathbf{N}$, which is unit length and orthogonal to $\mathbf{T}$, that is

$$\mathbf{T}\cdot\mathbf{N}=0$$

So there are only two possibilities for $\mathbf{N}$

$$\mathbf{T}=\left[(r')^2+r^2\right]^{-1/2}\left(r'\hat{\boldsymbol{\rho}}+r\hat{\boldsymbol{\phi}}\right)\implies\mathbf{N}=\pm\left[(r')^2+r^2\right]^{-1/2}\left(r\hat{\boldsymbol{\rho}}-r'\hat{\boldsymbol{\phi}}\right)$$

Either way, we have

$$\begin{aligned}\kappa&=\left(\frac{d\theta}{ds}\right)^2\left[(r')^2+r^2\right]^{-1/2}\left|\left(\frac{d^2\mathbf{r}}{d\theta^2}\cdot\left(r\hat{\boldsymbol{\rho}}-r'\hat{\boldsymbol{\phi}}\right)\right)\right|\\&=\left(\frac{1}{\left|\frac{d\mathbf{r}}{d\theta}\right|}\right)^2\left[(r')^2+r^2\right]^{-1/2}\left|\left(\left((r''-r)\hat{\boldsymbol{\rho}}+2r'\hat{\boldsymbol{\phi}}\right)\cdot\left(r\hat{\boldsymbol{\rho}}-r'\hat{\boldsymbol{\phi}}\right)\right)\right|\\&=\left[(r')^2+r^2\right]^{-3/2}\left|r(r''-r)-2r'r'\right|\\&=\frac{|2(r')^2-rr''+r^2|}{[(r')^2+r^2]^{3/2}}\end{aligned}$$

(d) (1 point) For a smooth curve given by the parametric equations $x=f(t)$ and $y=g(t)$, show that the curvature is given by

$$\kappa=\frac{|f'g''-g'f''|}{\left\{[f']^2+[g']^2\right\}^{3/2}}$$

Solution:

1M Let $\varphi(t)$ be the angle that the tangent vector $\dot{\mathbf{r}}(t)$ makes with $\mathbf{e}_x$,

$$\dot{\mathbf{r}}(t)=\dot f(t)\mathbf{e}_x+\dot g(t)\mathbf{e}_y=|\dot{\mathbf{r}}|\left(\cos\varphi\mathbf{e}_x+\sin\varphi\mathbf{e}_y\right)$$

In the last part, we have derived that

$$\begin{aligned}\kappa=\left|\frac{d\mathbf{T}}{ds}\cdot\mathbf{N}\right|&=\left|\left(\lim_{\Delta s\to0}\frac{\mathbf{T}(s+\Delta s)-\mathbf{T}(s)}{\Delta s}\right)\cdot\mathbf{N}(s)\right|\\&=\left|\lim_{\Delta s\to0}\frac{\mathbf{T}(s+\Delta s)\cdot\mathbf{N}(s)-\mathbf{T}(s)\cdot\mathbf{N}(s)}{\Delta s}\right|\\&=\left|\lim_{\Delta s\to0}\frac{\mathbf{T}(s+\Delta s)\cdot\mathbf{N}(s)}{\Delta s}\right|\\&=\left|\lim_{\Delta s\to0}\frac{\cos\left(\frac{\pi}{2}+\Delta\varphi\right)}{\Delta s}\right|\end{aligned}$$

where $\Delta\varphi$ is the angle between $\mathbf{T}(s)$ and $\mathbf{T}(s+\Delta s)$. Since $\mathbf{T}$ is always unit,

$$\frac{\Delta s}{1}=\Delta\varphi\implies\Delta s=\Delta\varphi\implies\Delta\varphi\to0\quad\text{as}\quad\Delta s\to0$$

thus we have

$$\begin{aligned}
\kappa = \left| \lim_{\Delta s \to 0} \frac{\cos\left(\frac{\pi}{2} + \Delta\varphi\right)}{\Delta s} \right| &= \left| \lim_{\Delta s \to 0} \frac{\lim\limits_{\Delta\varphi \to 0} \cos\left(\frac{\pi}{2} + \Delta\varphi\right)}{\Delta s} \right| \\
&= \left| \lim_{\Delta s \to 0} \frac{\lim\limits_{\Delta\varphi \to 0} -\Delta\varphi}{\Delta s} \right| \\
&= \left| \lim_{\Delta s \to 0} \frac{\Delta\varphi}{\Delta s} \right| \\
&= \left| \frac{d\varphi}{ds} \right| \qquad \text{since} \quad \lim_{\Delta\theta \to 0} \frac{\cos\left(\frac{\pi}{2} + \Delta\theta\right)}{\Delta\theta} = -1
\end{aligned}$$

Hopefully, this makes sense to you, the absolute value of the rate of change of φ, the angle between the tangent and $\mathbf{e}_x$, tells how sharp the curve is bending. Since the curve is smooth, we can apply implicit differentiation,

$$\tan\varphi = \frac{\dot{y}}{\dot{x}} = \frac{\dot{g}}{\dot{f}}$$
$$\implies \frac{d}{dt}\tan\varphi = \frac{d}{dt}\frac{\dot{g}}{\dot{f}} \implies \frac{d\varphi}{dt}\sec^2\varphi = \frac{\ddot{g}\dot{f} - \dot{g}\ddot{f}}{(\dot{f})^2}$$

and use the following identity, we have

$$\sec^2\varphi = \tan^2\varphi + 1 = \left(\frac{\dot{g}}{\dot{f}}\right)^2 + 1 = \frac{(\dot{f})^2 + (\dot{g})^2}{(\dot{f})^2}$$

So we can solve for

$$\frac{d\varphi}{dt} = \frac{\ddot{g}\dot{f} - \dot{g}\ddot{f}}{(\dot{f})^2 + (\dot{g})^2}$$

The rest is just a matter of substitution and simplification

$$\kappa = \left|\frac{d\varphi}{ds}\right| = \left|\frac{d\varphi}{dt}\frac{dt}{ds}\right| = \frac{|\dot{f}\ddot{g} - \dot{g}\ddot{f}|}{\left(\left(\dot{f}\right)^2 + (\dot{g})^2\right)^{3/2}}$$

(e) (1 point) For a smooth space curve defined by $\mathbf{r}(t)$, show the curvature is given by

$$\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Solution:

1M We have derived the following in class

$$\mathbf{a}(t) = \frac{d^2s}{dt^2}\mathbf{T}(t) + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}(t)$$

Taking the cross product of both sides of this equation with $\mathbf{T}(t)$, that is,

$$\begin{aligned}\mathbf{T}(t) \times \mathbf{a}(t) &= \frac{d^2 s}{dt^2}\mathbf{T}(t) \times \mathbf{T}(t) + \kappa \left(\frac{ds}{dt}\right)^2 \mathbf{T}(t) \times \mathbf{N}(t) \\ &= \kappa \left(\frac{ds}{dt}\right)^2 \mathbf{T}(t) \times \mathbf{N}(t)\end{aligned}$$

Taking the magnitude of both sides and noticing that

$$\mathbf{T}(t) \times \mathbf{N}(t) = \mathbf{B}(t)$$

we have

$$|\mathbf{T}(t) \times \mathbf{a}(t)| = \kappa \left(\frac{ds}{dt}\right)^2 |\mathbf{T}(t) \times \mathbf{N}(t)| = \kappa \left(\frac{ds}{dt}\right)^2 |\mathbf{B}(t)|$$

Since $|\mathbf{B}(t)| = 1$, we have

$$|\mathbf{T}(t) \times \mathbf{a}(t)| = \kappa \left(\frac{ds}{dt}\right)^2$$

Substitute $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, $\mathbf{a}(t) = \mathbf{r}''(t)$ and $\dfrac{ds}{dt} = |\mathbf{r}'(t)|$, we have

$$\frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \kappa |\mathbf{r}'(t)|^2 \implies \kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

(f) (1 point) A sufficiently differentiable curve with zero torsion lies in a plane is a special case of the fact that a particle whose velocity $\mathbf{v}$ remains perpendicular to a fixed vector $\mathbf{c}$ moves in a plane perpendicular to $\mathbf{c}$. This, in turn, can be viewed as the following result. Suppose

$$\mathbf{r}(t) = f(t)\mathbf{e}_x + g(t)\mathbf{e}_y + h(t)\mathbf{e}_z$$

is twice differentiable for all t in an interval $[a, b]$, that $\mathbf{r} = \mathbf{0}$ when $t = a$, and that

$$\mathbf{v} \cdot \mathbf{e}_z = 0$$

for all t in $[a, b]$. Show that $h(t) = 0$ for all t in $[a, b]$.

Solution:

1M Since

$$\begin{aligned}\mathbf{v} \cdot \mathbf{e}_z &= 0 \\ \left(\dot{f}\mathbf{e}_x + \dot{g}\mathbf{e}_y + \dot{h}\mathbf{e}_z\right) \cdot \mathbf{e}_z &= 0 \\ \dot{h} &= 0\end{aligned}$$

the component function $h(t)$ must be a constant for all $t \in [a, b]$,

$$h(t) = c$$

Using the initial condition, we have

$$h(a) = 0$$

thus

$$h(t) = 0$$

for all t. The torsion is defined as

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\left(\mathbf{T} \times \frac{d\mathbf{N}}{ds}\right) \cdot \mathbf{N} = -\mathbf{N} \cdot \left(\mathbf{T} \times \frac{d\mathbf{N}}{ds}\right)$$

which shows the torsion is essentially a scalar triple product. Having one component of $\mathbf{r}$ as a constant, there is no way

$$\frac{d\mathbf{N}}{ds}$$

is outside the plane defined by $\mathbf{T}$ and $\mathbf{N}$. All three vectors being coplanar means the scalar triple product, thus the torsion, must be zero.