Introduction to Linear Algebra

Review Class 1

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Outline

Theorem

Every system of linear equations has either **zero, one, or infinitely many** *solutions.*

For homogeneous equations, there exists at least one solution, that is, **the trivial solution**.

Elementary Row Operations

1. *Type I: Interchange two rows.*
2. *Type II: Multiply a row by a nonzero constant.*
3. *Type III: Add a constant times one row to another.*

Key Idea: does not change the solution to the original problem.

Multiplication by A is a **"linear transformation"**. That is, if $\mathbf{w}$ is a combination of $\mathbf{u}$ and $\mathbf{v}$, then $A\mathbf{w}$ is the same combination of $A\mathbf{u}$ and $A\mathbf{v}$, namely,

$$A\mathbf{w} = cA\mathbf{u} + dA\mathbf{v}$$

This is why "Linear Algebra" is so useful and popular.

Now consider $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Combine $\mathbf{w} = c\mathbf{w} + d\mathbf{v}$, if $\mathbf{w} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$, how is $A\mathbf{w}$ connected to $A\mathbf{u}$ and $A\mathbf{v}$?

What does it mean to "multiply A times x"?

Multiplication by rows:

$$A\mathbf{x} = \begin{bmatrix} (row1) \cdot \mathbf{x} \\ (row2) \cdot \mathbf{x} \end{bmatrix}$$

Multiplication by columns:

$$A\mathbf{x} = x_1(column1) + x_2(column2) + x_3(column3)$$

Simple Question 1: For which three numbers a will elimination fail to give three pivots, i.e., $A = \begin{bmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{bmatrix}$ is singular.

Simple Question 2: Perform Gauss-Jordan Elimination to solve the equation

$$\begin{cases} 2x + 3y + z = 8 \\ 4x + 7y + 5z = 20 \\ -2y + 2z = 0 \end{cases}$$

Simple Question 3: How many individual multiplications are needed in regular matrix multiplication?

Some Useful Rules

1. If A and B are invertible matrices with the same size, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. (For any number of matrices, the product is just of the reverse order)
2. **Equivalence Theorem**. (Very Important!)
3. Inversion Algorithm.
4. Block Multiplication:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} \end{bmatrix}$$

5. Block Elimination:

$$\begin{bmatrix} I & \mathbf{0} \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ \mathbf{0} & D - CA^{-1}B \end{bmatrix}$$

Some Useful Rules (Continued)

1. A triangular matrix is invertible **if and only if** no diagonal entries are zero.
2. Diagonally dominant matrices are invertible. Diagonally dominant means, each a_{ii} on the diagonal is larger than the total sum along the rest of row i, e.g., $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$. (**Only sufficient, not necessary**)