Vv417 Lecture 10

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- Often we are interested in a subset of a known vector space.

Definition

A non-empty subset $\mathcal{H}$ of a vector space $\mathcal{V}$ over $\mathcal{F}$ is called a subspace of $\mathcal{V}$ if $\mathcal{H}$ is itself a vector space over $\mathcal{F}$ under the same addition and scalar multiplication.

- In general, to show a set $\mathcal{H}$ with a scalar multiplication and an addition is a vector space, it is necessary to verify the 10 axioms.
- However, if $\mathcal{H}$ is a subset of $\mathcal{V}$ which is known to be a vector space, then not all axioms need to be verified. e.g. It is not necessary to verify that

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

  holds in $\mathcal{H}$ because it holds for all vectors in $\mathcal{V}$ including those in $\mathcal{H}$.

Q: Which of those 10 axioms are NOT "inherited" ?

- The next theorem shows only two of those need to be checked.

Theorem

If $\mathcal{H}$ is a non-empty subset a vector space $\mathcal{V}$ over $\mathcal{F}$, then $\mathcal{H}$ is a subspace of $\mathcal{V}$ over $\mathcal{F}$ if and only if the following conditions are satisfied.

- 1. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathcal{H}$, then $\mathbf{u} + \mathbf{v}$ is in $\mathcal{H}$.
- 2. If α is a scalar in $\mathcal{F}$ and $\mathbf{u}$ is a vector in $\mathcal{H}$, then $\alpha\mathbf{u}$ is in $\mathcal{H}$.

Proof

- Given $\mathcal{H}$ is a subspace, then those two conditions are satisfied by definition.
- It $\mathcal{H}$ is a subset of a vector space $\mathcal{V}$, then Axiom 1–6 are inherited.
- Since $\mathcal{V}$ is a vector space, property 1. and 3. of the theorem L8P16 are true.

$$1. \quad 0\mathbf{u} = \mathbf{0} \qquad 2. \quad (-1)\mathbf{u} = -\mathbf{u}$$

- So condition 2. implies $\mathbf{0}$ and $-\mathbf{u}$ are in $\mathcal{H}$, so Axiom 7–8 are satisfied.
- Hence we only need to make sure that 1. Axiom 9. is satisfied. □

- Given a vector space $\mathcal{V}$, the subset $\mathcal{Z} = \{\mathbf{0}\}$ of $\mathcal{V}$ is clearly a subspace of $\mathcal{V}$.

Q: Is the set of vectors of the following form a subspace of $\mathbb{R}^3$ over $\mathcal{F}$?

$$\mathbf{v} = \alpha \mathbf{u}, \qquad \text{where } \mathbf{u} \text{ is a vector in } \mathbb{R}^3 \text{ and } \alpha \text{ is a scalar in } \mathcal{F}.$$

- Thus straight lines through the origin in $\mathbb{R}^3$ are subspaces
- Any subset of $\mathbb{R}^3$ that does not include the origin is NOT a subspace of $\mathbb{R}^3$.

Q: Is the set of vectors of the following form a subspace of $\mathbb{R}^3$ over $\mathcal{F}$?

$$\mathbf{w} = \alpha \mathbf{u} + \beta \mathbf{v}, \qquad \text{where } \mathbf{u}, \mathbf{v} \text{ are vectors in } \mathbb{R}^3 \text{ and } \alpha, \beta \text{ are scalars in } \mathcal{F}.$$

- Thus planes through the origin in $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$ over $\mathcal{F}$.

Q: What about curves through origin, can some of them be subspaces of $\mathbb{R}^3$?

Q: What about surfaces through origin?

- We cannot easily visualize "flatness" in an arbitrary vector space, but the notion generalizes to any vector space. From now on, think of flat object passing through the origin whenever you encounter the term "subspace."

Exercise

Recall that a polynomial is a function that can be expressed in the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \qquad \text{where } a_0,\ a_1,\ \ldots,\ a_n \text{ are constants.}$$

The degree of a polynomial is the highest power of x that occurs with a non-zero coefficient. Is the set of polynomials with a positive degree of n a subspace of

$$\mathcal{F}(-\infty, \infty)$$

Solution

- No, the subset is neither closed under addition nor scalar multiplication.
- The set of polynomial with a positive degree of n or less is a subspace of

$$\mathcal{F}(-\infty, \infty)$$

- This subspace is often denoted by

$$\mathcal{P}_n$$

Definition

If $\mathbf{u}$ is a vector in a vector space $\mathcal{V}$, then $\mathbf{u}$ is said to be a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_r$ in $\mathcal{V}$ if $\mathbf{u}$ can be expressed in the form

$$\mathbf{u} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_r \mathbf{v}_r$$

where α_1, α_2 , ..., α_r are scalars in $\mathcal{F}$. These scalars are called the coefficients.

- For example, the vector $\mathbf{u}$ is a linear combination of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ below

$$\underbrace{\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}}_{\mathbf{u}} = 3 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_1} + 2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}_2} \quad \text{or} \quad \underbrace{3x^2 + 2x + 3}_{\mathbf{u}} = 3 \underbrace{(x^2 + 1)}_{\mathbf{v}_1} + 2 \underbrace{(x)}_{\mathbf{v}_2}$$

Theorem

If $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a non-empty set of vectors in a vector space $\mathcal{V}$, then the set $\mathcal{H}$ of all possible linear combinations of the vectors in $\mathcal{S}$ is a subspace of $\mathcal{V}$.

Proof

- Let $\mathcal{H}$ be the set of all possible linear combinations of the vectors in $\mathcal{S}$.

$$\mathbf{u} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_r\mathbf{v}_r$$

- The set $\mathcal{S}$ only contains vectors in $\mathcal{V}$, and $\mathcal{V}$ is a vector space, which is closed under addition and scalar multiplication, so $\mathcal{H}$ is a subset of $\mathcal{V}$.
- So we only need to show $\mathcal{H}$ is closed under addition and scalar multiplication
- Let $\mathbf{u} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_r\mathbf{v}_r$ and $\mathbf{w} = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \cdots + \beta_r\mathbf{v}_r$, then

$$\mathbf{u} + \mathbf{w} = (\alpha_1 + \beta_1)\mathbf{v}_1 + (\alpha_2 + \beta_2)\mathbf{v}_2 + \cdots + (\alpha_r + \beta_r)\mathbf{v}_r$$

$$\gamma\mathbf{u} = (\gamma\alpha_1)\mathbf{v}_1 + (\gamma\alpha_2)\mathbf{v}_2 + \cdots + (\gamma\alpha_r)\mathbf{v}_r$$

  where α_i, β_i and γ are any scalars in $\mathcal{F}$.
- So $\mathbf{u} + \mathbf{w}$ and $\gamma\mathbf{u}$ are linear combinations of vectors in $\mathcal{S}$, and thus in $\mathcal{H}$.
- That shows $\mathcal{H}$ is closed under addition and scalar multiplication. $\square$

Defintion

The subspace $\mathcal{H}$ of all possible linear combinations of vectors in $\mathcal{S} \subset \mathcal{V}$ is called the subspace of $\mathcal{V}$ generated by $\mathcal{S}$, and we say the set $\mathcal{S}$ spans $\mathcal{H}$, or $\mathcal{H}$ is the subspace spanned by $\mathcal{S}$. We denote this subspace $\mathcal{H}$ as

$$\mathcal{H} = \mathrm{span}(\mathcal{S})$$

Alternatively, we denote it by

$$\mathcal{H} = \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} \qquad \text{where} \qquad \mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$$

The set $\mathcal{S}$ is known as the spanning set for $\mathcal{H}$.

- Let us denote that the standard unit vectors in $\mathbb{R}^n$ as

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Q: What is the geometric interpretation of

$$\mathcal{H} = \text{span}\{\mathbf{e}_1\}$$

Q: What is the geometric interpretation of

$$\mathcal{H} = \text{span}\{\mathbf{e}_1, \mathbf{e}_2\}$$

- Of course, we can go on adding more $\mathbf{e}_i$ into the set $\mathcal{S}$, we will just end up with subspaces that are hyperplanes determined by those vectors in $\mathcal{S}$.
- If we put all n of those standard unit vectors in the $\mathcal{S}$, then clearly

$$\text{span}(\mathcal{S}) = \mathbb{R}^n$$

- Thus the n standard unit vectors span $\mathbb{R}^n$ since every vector $\mathbf{u}$ in $\mathbb{R}^n$ is a linear combination of those vectors, and the set

$$\{\mathbf{e}_1 \cdots \mathbf{e}_n\}$$

is a spanning set for $\mathbb{R}^n$.

Exericise

Show the following set $\mathcal{S}$ is a spanning set for $\mathcal{P}_2$.

$$\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$$

where $\mathbf{v}_1 = x^2 + 3x - 2,\ \mathbf{v}_2 = 2x^2 + 5x - 3,$ and $\mathbf{v}_3 = -x^2 - 4x + 4.$

Solution

- We need to show $\mathcal{S}$ spans $\mathcal{P}_2$,

$$\mathcal{P}_2 = \mathrm{span}(\mathcal{S})$$

that is, every polynomial in $\mathcal{P}_2$ is a linear combination of vectors in $\mathcal{S}$.

$$\begin{aligned}(-8\beta_1 + 5\beta_2 + 3\beta_3)(x^2 + 3x - 2) + (4\beta_1 - 2\beta_2 - \beta_3)(2x^2 + 5x - 3)\\ + (-\beta_1 + \beta_2 + \beta_3)(-x^2 - 4x + 4) = \beta_1 x^2 + \beta_2 x + \beta_3 \quad \square\end{aligned}$$

- Every flat space, that is every vector space, can be generated by some set $\mathcal{S}$.

- Sometimes we will want to find the "smallest" subspace of a vector space $\mathcal{V}$ that contains all of the vectors in some set of interest.

Theorem

The subspace $\mathcal{H} = \operatorname{span}(\mathcal{S})$, where $\mathcal{S}$ consists of vectors in a vector space $\mathcal{V}$, is the "smallest" subspace of $\mathcal{V}$ that contains all of the vectors in $\mathcal{S}$ in the sense that any other subspace of $\mathcal{V}$ that contains those vectors contains $\mathcal{H}$.

Proof

- Let $\mathcal{H}^*$ be any subspace of $\mathcal{V}$ that contains all the vectors in $\mathcal{S}$. $\mathcal{H}^*$ is vector space, so it is closed under addition and scalar multiplication, hence it must contain all linear combination of the vectors in $\mathcal{S}$ and so $\mathcal{H}^*$ must contain $\mathcal{H}$.

- So the "smallest" flat surface, which contains vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$, is

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$$

- Of course, we are not restricting ourselves in $\mathbb{R}^n$. It is also true for any $\mathcal{V}$.

- The subspace $\operatorname{span}\{\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3\}$ is the smallest subspace of $\mathbb{R}^{2\times 2}$ that has

$$\mathbf{M}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{M}_2 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{M}_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Exericise

Let $\mathbf{a}_i \in \mathcal{V}$, where $\mathcal{V}$ is a subspace of $\mathbb{R}^m$, are columns of a matrix $\mathbf{A}_{m\times n}$, and

$$\mathcal{S} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$$

Show the set $\mathcal{S}$ spans $\mathcal{V}$ if and only if $\mathbf{A}\boldsymbol{\alpha} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathcal{V}$.

Solution

- $\mathcal{S}$ spans $\mathcal{V}$ if and only if for each $\mathbf{b} \in \mathcal{V}$ there exist scalars α_i such that

$$\mathbf{b} = \alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2 + \cdots + \alpha_n\mathbf{a}_n$$

- We need to show the existence of α_i's for every $\mathbf{b} \in \mathcal{V}$.

Solution

- It is equivalent to having a solution to the following system for every $\mathbf{b} \in \mathcal{V}$,

$$\mathbf{b} = \alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \cdots + \alpha_n \mathbf{a}_n = \left[\begin{array}{c|c|c|c} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{array}\right] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \mathbf{A}\boldsymbol{\alpha}$$

- Hence $\mathbf{A}\boldsymbol{\alpha} = \mathbf{b}$ being consistent for every $\mathbf{b} \in \mathcal{V}$ is equivalent to $\mathcal{S}$ spans $\mathcal{V}$.

- The next 3 exercises are concerned with two important types of problems:

1. Given a nonempty set $\mathcal{S}$ of vectors in $\mathbb{R}^n$ and a vector $\mathbf{v}$ in $\mathbb{R}^n$, determine if $\mathbf{v}$ is a linear combination of the vectors in S.
2. Given a nonempty set $\mathcal{S}$ of vectors in $\mathbb{R}^n$, determine whether $\mathcal{S}$ span $\mathbb{R}^n$.

Exericise

Show that $\mathbf{w} = \begin{bmatrix} 9 \\ 2 \\ 7 \end{bmatrix}$ is a linear combination of $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$.

Solution

- The vector $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ if there exist α_1 and α_2 s.t

$$\alpha_1\mathbf{u} + \alpha_2\mathbf{v} = \mathbf{w} \iff \mathbf{A}\mathbf{x} = \mathbf{w}, \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} 1 & 6 \\ 2 & 4 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

- Apply row operations, we see

$$\begin{bmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

- Thus $\mathbf{w}$ is indeed a linear combination of $\mathbf{u}$ and $\mathbf{v}$ with $\alpha_1 = -3$ and $\alpha_2 = 2$.

Exericise

Show that $\mathbf{z} = \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix}$ is *not* a linear combination of $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$.

Solution

- The vector $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ if there exist α_1 and α_2 s.t.

$$\alpha_1\mathbf{u} + \alpha_2\mathbf{v} = \mathbf{z} \iff \mathbf{A}\mathbf{x} = \mathbf{z}, \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} 1 & 6 \\ 2 & 4 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

- Apply row operations, we see the system is inconsistent,

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- So no such α_1 and α_2 exist, and $\mathbf{z}$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Exericise

Determine whether the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ span $\mathbb{R}^3$.

Solution

- To determine whether they form a spanning set for $\mathbb{R}^3$, we apply the result we obtain the exercise on page $\boxed{12}$ and invoking the equivalence theorem.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \neq \mathbf{I}$$

- Together, it means the vectors span $\mathbb{R}^3$ if and only if $\mathbf{A}$ is row equivalent to $\mathbf{I}$
- Thus $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ do not span $\mathbb{R}^3$.

- It is important to recognize that spanning sets are not unique.
- For example, if $\mathcal{H}$ is a line through the origin, then any non-zero vector $\mathbf{v}$ on the line forms a spanning set of $\mathcal{H}$.

$$\mathcal{H} = \operatorname{span}\{\mathbf{v}\}$$

- If $\mathcal{W}$ is a plane in $\mathbb{R}^3$ through the origin, then any two non-collinear vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ in the plane forms a spanning set of $\mathcal{W}$.

$$\mathcal{W} = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2\}$$

Theorem

If $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ and $\mathcal{S}^\star = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ are non-empty sets of vectors in a vector space $\mathcal{V}$, then

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$$

if and only if each vector in $\mathcal{S}$ is a linear combination of those in $\mathcal{S}^\star$, and each vector in $\mathcal{S}^\star$ is a linear combination of those in $\mathcal{S}$.