
Question1 (5 points)

Consider the following initial-value problem

$$y'' - 3y' + 2y = 2xe^{3x} + 3\sin(x)$$

(a) (1 point) Find the complementary solution y_c of the given nonhomogeneous equation.

(b) (1 point) State an annihilator for

$$2xe^{3x} + 3\sin(x)$$

(c) (1 point) Find the general solution

$$y_h$$

from which a particular solution y_p of the given nonhomogeneous equation belongs to.

(d) (1 point) Find a particular solution by determining all the coefficients of

$$y_p$$

(e) (1 point) State the general solution of the given nonhomogeneous equation

$$y = y_c + y_p$$

then find the particular solution that satisfies the following IVP

$$y'' - 3y' + 2y = 2xe^{3x} + 3\sin(x), \qquad y(0) = 1, \quad y'(0) = 1$$

Question2 (1 points)

Suppose $\mathcal{L}(y) = 0$ is a homogeneous 4th-order linear equation with constant coefficient and

$$x^3 e^{-x}$$

is a solution to the equation. Find the general solution y and the differential operator $\mathcal{L}$ of

$$\mathcal{L}(y) = 0$$

Question3 (1 points)

Based on the method of variation of parameters for second-order equations, show how to use variation of parameters for solving the given third-order differential equation.

$$y''' + y' = \tan x$$

Question4 (3 points)

Find the radius of convergence for each of the following series

(a) (1 point) $\displaystyle\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n}$

(b) (1 point) $\displaystyle\sum_{n=0}^{\infty} (\ln x)^n$

(c) (1 point) $\displaystyle\sum_{n=0}^{\infty} c_n x^n$, where c_n takes the value 1 if n is odd, and 2 if n is even.

Question5 (0 points)

(a) (1 point (bonus)) Suppose $\lambda = R + \mathrm{i}\theta \neq 0$, where R and θ are real numbers, and

$$f(t) = \exp\left(\lambda t\right),$$

Use the power series definition of exp to show the derivative of f with respect to t is

$$\dot{f} = \lambda f$$

Briefly discuss the significance of this in terms of nonhomogeneous equation.

(b) (1 point (bonus)) If r is not a root of the polynomial p, then show that

$$y(x) = a\frac{e^{rt}}{p(r)}$$

is a particular solution to the differential equation

$$p(\mathcal{D})y = ae^{rt}$$

(c) (1 point (bonus)) Find the first four nonzero terms of the Taylor series at $t = 0$ of

$$f(t) = \frac{e^t}{1+t}$$

(d) (1 point (bonus)) Find the first four nonzero terms of the Taylor series at $t = 0$ of

$$g(t) = \int_0^t \frac{\sin\tau}{\tau}\, d\tau$$

(e) (1 point (bonus)) Use Matlab to plot the above two functions against the Taylor polynomials you found. Discuss how to find the radius of convergence of a Taylor series.

(f) (1 point (bonus)) Discuss whether the following function

$$f(x) = \begin{cases} \exp(-1/x^2) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

is infinitely differentiable and whether it is analytic.

(g) (1 point (bonus)) Discuss how to use what we have done for first-order nonlinear equations, and second-order linear equation with variable coefficients to extend the method of annihilator to equations

$$\mathcal{L}(y) = Q$$

where Q is NOT a solution to any linear equation with constant coefficients, and

$$\mathcal{L}$$

is a n-order linear differential operator with constant coefficients.