
Question1 (1 points)

Find the derivative function of the following function using the definition of derivative.

$$g(t) = \frac{1}{\sqrt{t}}$$

State the natural domain of $g(t)$ and the natural domain of its derivative function.

Question2 (1 points)

Show that

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is differentiable but that f' is not continuous at $x = 0$.

Question3 (4 points)

Let f be defined on $(-\infty, \infty)$, and consider the following

$$f(c+h) = f(c) + Ah + \varepsilon(h)$$

where c is a constant and A is not a function of h.

(a) (1 point) Find the simplest expression for the following limit.

$$\lim_{h \to 0} \varepsilon(h)$$

(b) (1 point) Show that $f(x)$ is continuous at c if and only if

$$\lim_{h \to 0} \varepsilon(h) = 0$$

(c) (1 point) Suppose $f(x) = x^2$. Show that $f(x)$ is differentiable and $f'(c) = A = 2c$ if

$$\lim_{h \to 0} \frac{\varepsilon(h)}{h} = 0$$

(d) (1 point) Suppose $f(x)$ is differentiable and we choose A to be $f'(c)$. Show that

$$\lim_{h \to 0} \frac{\varepsilon(h)}{h} = 0$$

What happens to the values of the two limits if an "incorrect" $A \neq f'(c)$ is used ?

$$\lim_{h \to 0} \frac{\varepsilon(h)}{h} \quad \text{and} \quad \lim_{h \to 0} \varepsilon(h)$$

Explain your answers using the tangent line approximation.

Question4 (4 points)

Find the derivative y', show all your workings.

(a) (1 point) $y = x^8 - 3\sqrt{x} + 5x^{-3}$

(b) (1 point) $y = \sin x + 2\cos^3 x$

(c) (1 point) $y = \dfrac{\sin x \cos x}{\sqrt{x}}$

(d) (1 point) $y = x^x$

Question5 (1 points)
Suppose that $F(x) = (f \circ g)(x)$ and $g(3) = m$, $g'(3) = 4$, $f'(3) = 2$, and $f'(6) = 7$. If your instructor insists that he has given you enough information to determine the value of $F'(3)$, find the value/s that m must take and thus its corresponding value of $F'(3)$.

Question6 (1 points)
Consider the function

$$f(x) = (x^{156} - 1)g(x)$$

where $g(x)$ is continuous at $x = 1$, and $g(1) = 1$. Find the derivative of $f(x)$ at $x = 1$.

Question7 (1 points)
For a constant a, evaluate $\lim_{x \to 0} \dfrac{\sin(a+2x) - 2\sin(a+x) + \sin(a)}{x^2}$ in terms of a.

Question8 (1 points)
For a positive integer n, consider

$$f(x) = a_1 \sin(x) + a_2 \sin(2x) + \cdots + a_n \sin(nx)$$

where $a_1, a_2, ..., a_n$ are real numbers such that $|f(x)| \le |\sin(x)|$ for all $x \in \mathbb{R}$. Show that

$$|a_1 + 2a_2 + 3a_3 + ... + na_n| \le 1$$

Question9 (1 points)
Suppose there is a bowl in the shape of a hemisphere of radius a meters. Water is pouring into the bowl with a constant rate of $5\pi a^3$ cubic meters per second. Find the rate at which the water level is rising in the bowl.

Question10 (0 points)
In economics, a demand function f is a function between the demand in a market and the price of a product. It is often denoted as

$$Q = f(P)$$

where P is the market price and Q is the total number of the product the market desires.

The price Elasticity of demand is defined as follows:

$$E = \frac{P}{Q}\frac{dQ}{dP}$$

Usually the demand of a product will decrease if the price increases, thus E is often negative.

Suppose there are two companies Sumsang and Banana competing in the same market on the same good. The current demand for the good is saturated based on Sumsang's and Banana's current pricing policy. The demand function for Sumsang is given by

$$Q_S = 400 - \frac{1}{2}P_A$$

while the demand function for Banana is given by

$$Q_B = 300 - \frac{1}{5}P_B$$

Currently, Sumsang is charging $P_S = \$400$ for its good while Banana is charging $P_B = \$500$.

(a) (1 point (bonus)) Find and compare the elasticities for the two companies. Discuss the significance of your answers.