# Vv256 Lecture 20

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### Definition

If $\mathbf{u}$ is a vector in a vector space $\mathcal{V}$, then $\mathbf{u}$ is said to be a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_r$ in $\mathcal{V}$ if $\mathbf{u}$ can be expressed in the form

$$\mathbf{u} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_r \mathbf{v}_r$$

where $\alpha_1$, $\alpha_2$ , ..., $\alpha_r$ are scalars in $\mathcal{F}$. These scalars are called the coefficients.

- For example, the vector $\mathbf{u}$ is a linear combination of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ below

$$\underbrace{\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}}_{\mathbf{u}} = 3 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_1} + 2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}_2} \quad \text{or} \quad \underbrace{3x^2 + 2x + 3}_{\mathbf{u}} = 3 \underbrace{(x^2 + 1)}_{\mathbf{v}_1} + 2 \underbrace{(x)}_{\mathbf{v}_2}$$

### Theorem

If $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a non-empty set of vectors in a vector space $\mathcal{V}$, then the set $\mathcal{H}$ of all possible linear combinations of the vectors in $\mathcal{S}$ is a subspace of $\mathcal{V}$.

## Proof

- Let $\mathcal{H}$ be the set of all possible linear combinations of the vectors in $\mathcal{S}$.

$$\mathbf{u} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_r \mathbf{v}_r$$

- The set $\mathcal{S}$ only contains vectors in $\mathcal{V}$, and $\mathcal{V}$ is a vector space, which is closed under addition and scalar multiplication, so $\mathcal{H}$ is a subset of $\mathcal{V}$.
- So we only need to show $\mathcal{H}$ is closed under addition and scalar multiplication
- Let $\mathbf{u} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_r \mathbf{v}_r$ and $\mathbf{w} = \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \cdots + \beta_r \mathbf{v}_r$, then

$$\begin{aligned} \mathbf{u} + \mathbf{w} &= (\alpha_1 + \beta_1)\mathbf{v}_1 + (\alpha_2 + \beta_2)\mathbf{v}_2 + \cdots + (\alpha_r + \beta_r)\mathbf{v}_r \\ \gamma \mathbf{u} &= (\gamma\alpha_1)\mathbf{v}_1 + (\gamma\alpha_2)\mathbf{v}_2 + \cdots + (\gamma\alpha_r)\mathbf{v}_r \end{aligned}$$

  where $\alpha_i$, $\beta_i$ and $\gamma$ are any scalars in $\mathcal{F}$.
- So $\mathbf{u} + \mathbf{w}$ and $\gamma \mathbf{u}$ are linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_r$, and they are in $\mathcal{H}$
- That shows $\mathcal{H}$ is closed under addition and scalar multiplication. $\square$

Defintion

The subspace $\mathcal{H}$ of all possible linear combinations of vectors in $\mathcal{S} \subset \mathcal{V}$ is called the subspace of $\mathcal{V}$ generated by $\mathcal{S}$, and we say the set $\mathcal{S}$ spans $\mathcal{H}$, or $\mathcal{H}$ is the subspace spanned by $\mathcal{S}$. We denote this subspace $\mathcal{H}$ as

$$\mathcal{H} = \text{span}(\mathcal{S})$$

Alternatively, we denote it by

$$\mathcal{H} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} \qquad \text{where} \qquad \mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$$

The set $\mathcal{S}$ is known as the spanning set for $\mathcal{H}$.

- Let us denote that the standard unit vectors in $\mathbb{R}^n$ as

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Q: What is the geometric interpretation of

$$\mathcal{H} = \operatorname{span}\{\mathbf{e}_1\}$$

Q: What is the geometric interpretation of

$$\mathcal{H} = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2\}$$

- Of course, we can go on adding more $\mathbf{e}_i$ into the set $\mathcal{S}$, we will just end up with subspaces that are hyperplanes determined by those vectors in $\mathcal{S}$.
- If we put all $n$ of those standard unit vectors in $\mathcal{S}$, then clearly

$$\operatorname{span}(\mathcal{S}) = \mathbb{R}^n$$

- Thus the $n$ standard unit vectors span $\mathbb{R}^n$ since every vector $\mathbf{u}$ in $\mathbb{R}^n$ is a linear combination of those vectors, and the set

$$\{\mathbf{e}_1 \cdots \mathbf{e}_n\}$$

  is a spanning set for $\mathbb{R}^n$.

Q: Intuitively, what is the geometric interpretation of a vector space in general?

## Definition

Given three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^3$, if there exist three unique scalars

$$\alpha, \quad \beta, \quad \text{and} \quad \gamma$$

such that

$$\mathbf{v} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} \qquad \text{for any arbitrary vector } \mathbf{v} \text{ in } \mathbb{R}^3.$$

then we say that $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ form a basis for the space $\mathbb{R}^3$. The scalars

$$\alpha, \quad \beta, \quad \text{and} \quad \gamma$$

are called the components/coordinates of $\mathbf{v}$ with respect to the basis $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

- The notions of

  "basis vectors" and "coordinate systems"

  can be extended naturally to a general vector space $\mathcal{V}$.

### Definition

The vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are said to be linearly independent if the only way to have

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n = \mathbf{0}$$

is for all the $\alpha$'s to be zero,

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

### Defintion

The dimension of a vector space $\mathcal{V}$ is defined to be the largest number of linearly independent vectors in $\mathcal{V}$, often denoted by

$$\dim \mathcal{V}$$

### Definition

In general, a basis for a vector space $\mathcal{V}$ is a linearly independent spanning set of $\mathcal{V}$

$$\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

- The idea of Wronskian can be easily extended to more than two functions.

### Definition

If $\mathbf{f}_1 = f_1(x)$, $\mathbf{f}_2 = f_2(x)$, $\cdots$, $\mathbf{f}_n = f_n(x)$ are functions that are $n-1$ times continuously differentiable on the interval $(-\infty, \infty)$, then the determinant

$$W(x) = \det\left(\begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix}\right)$$

is called the Wronskian of $f_1$, $f_2$, ..., $f_n$.

### Theorem

If the functions $\mathbf{f}_1, \mathbf{f}_2, \cdots, \mathbf{f}_n$ have $n-1$ continuous derivatives on $(-\infty, \infty)$, and if the Wronskian of these functions is not identically zero on $(-\infty, \infty)$, then these functions form a linearly independent set of vectors in $\mathcal{C}^{n-1}(-\infty, \infty)$.

### Exercise

*Find a basis and the dimension of the solution space of*

$$\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)y = 0$$

### Solution

- The solution space is the set of all solutions of the equation, and it is easy to show that it is a subspace of $F(-\infty, \infty)$ by considering the general solution.
- We know the general solution of the equation is

$$y = c_1 e^{0t} + c_2 e^{1t} + c_3 e^{2t}$$

- Notice that the set of all solutions is simply,

$$\mathcal{H} = \operatorname{span}(\mathcal{S}), \qquad \text{where} \quad \mathcal{S} = \{1, e^t, e^{2t}\}$$

thus must be a vector space, and hence the subspace of $F(-\infty, \infty)$.

## Solution

- To find a basis for $\mathcal{H}$, we need to find a linearly independent spanning set of $\mathcal{H}$, the set $\mathcal{S}$ is a spanning set of $\mathcal{H}$ by definition, so if $\mathcal{S}$ is linearly independent, then $\mathcal{S}$ is a basis for $\mathcal{H}$ and the dimension of $\mathcal{H}$ is 3.
- Consider the Wronskian $W(t)$,

$$W = \det\left(\begin{bmatrix} 1 & e^t & e^{2t} \\ 0 & e^t & 2e^{2t} \\ 0 & e^t & 4e^{2t} \end{bmatrix}\right) = 1 \det\left(\begin{bmatrix} e^t & 2e^{2t} \\ e^t & 4e^{2t} \end{bmatrix}\right)$$
$$= 4e^{3t} - 2e^{3t} = 2e^{3t} \neq 0, \qquad \text{for all } t.$$

- By theorem $\boxed{8}$, the set $\mathcal{S}$ is linearly independent as well as being the span.
- Essentially we have been looking for a basis for the solution space when we are solving a differential equation,
- Note $\mathcal{S}$ is the fundamental set of solutions. In general, the fundamental set of solutions is a basis for the solution space by definition.

- Recall for a given linear system, e.g.

$$\begin{aligned} 3x + 2y + \phantom{3}z &= 39 \\ 2x + 3y + \phantom{3}z &= 34 \\ x + 2y + 3z &= 26 \end{aligned} \implies \mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

- We can form a coefficient matrix, $\mathbf{A}$, by listing the coefficients of the unknowns in the position in which they appear in the linear equations.

  and can also include the right-hand side by augmenting $\mathbf{A}$ with $\mathbf{B}$.

$$\begin{bmatrix} 3 & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{bmatrix} = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{array}\right] = \mathbf{A}|\mathbf{B},$$

  where $\mathbf{A}|\mathbf{B}$ is known as the augmented matrix of the system.

- Since we can always go back and recapture the system of linear equations from the augmented matrix $\mathbf{A}|\mathbf{B}$, it contains all the information of the system and can thus be used to solve the linear system.

# Solving $n$ linear equations with $n$ unknowns View 1

Q: What question are we actually addressing when we solve a linear system?

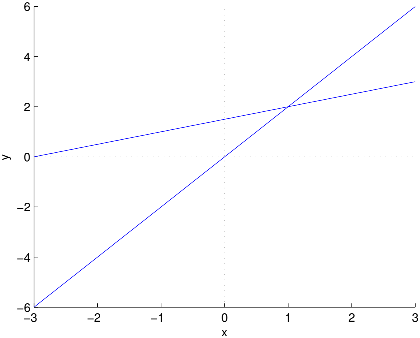
$$\begin{aligned} 2x - \phantom{2}y &= 0 \\ -x + 2y &= 3 \end{aligned}$$

- The first equation gives

$$y = 2x,$$

substitute into the second,

$$\begin{aligned} -x + 4x = 3 &\implies x = 1 \\ &\implies y = 2 \end{aligned}$$




- Asking for the point, if any, satisfies the first and the second equation, i.e.

the intersection of the two lines

## Solving $n$ linear equations with $n$ unknowns View 2

Q: How to solve the following?

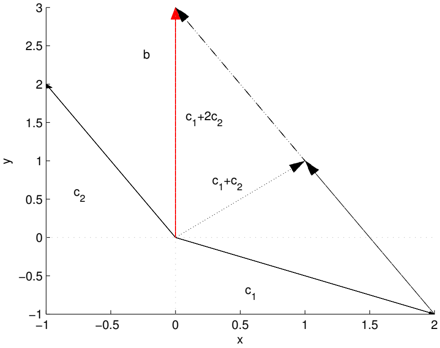
$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

- Try one copy of each vector

$$1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Try one copy of the first and two copies of the second

$$1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$




- Asking for, if possible, the right amount of the first and the second vector such that their linear combination equal to the third.

# Solving $n$ linear equations with $n$ unknows View 3

Q: How can we solve $\mathbf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

- By the definition of matrix multiplication and addition, we have

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \implies \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \implies \begin{bmatrix} 2x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} -x_2 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\implies x_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

- This means the matrix equation is equivalent to perspective 2, which in term can be understood and solved using perspective 1.

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \iff x_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \iff \begin{aligned} 2x_1 - x_2 &= 0 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$

# Three ways of looking at a linear system of equations

- Three different ways of asking the same question

Row (Intersection) Column (Combination) Matrix (Inverse image)

- It becomes harder to see as the dimension increase, but the same idea applies

$$\begin{aligned} 4y + z &= 5 \\ 3x + 8y + z &= 9 \\ x + 2y + z &= 3 \end{aligned} \qquad x\begin{bmatrix}0\\3\\1\end{bmatrix} + y\begin{bmatrix}4\\8\\2\end{bmatrix} + z\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}5\\9\\3\end{bmatrix} \qquad \begin{bmatrix}0 & 4 & 1\\3 & 8 & 1\\1 & 2 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}5\\9\\3\end{bmatrix}$$

- We will primarily try to solve a matrix equation using the first method.

$$\mathbf{Ax} = \mathbf{b}$$

Q: What question does a matrix equation try to address? Specifically

$$\mathbf{Ax} = \mathbf{0}$$

- Recall the definition of the determinant of a $2 \times 2$ or $3 \times 3$ matrix.

**Definition**

The determinant is a scalar associated with every square matrix,

$$\det(\mathbf{A}), \quad \text{or} \quad |\mathbf{A}|$$

A determinant of order 2 is defined by

$$\det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = u_1 v_2 - u_2 v_1$$

A third-order can be defined in terms of second-order determinants.

$$\begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = w_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + w_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

Q: How shall we define determinant of an $n \times n$ matrix ?

### Definition

The determinant of an $n \times n$ matrix $\mathbf{A}$, denoted

$$\det(\mathbf{A}),$$

is a scalar associated with the matrix $\mathbf{A}$ that is defined successively as

$$\det(\mathbf{A}) = \begin{cases} a_{11} & \text{if} \quad n = 1 \\ a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} & \text{if} \quad n > 1 \end{cases}$$

where $C_{ij}$ is known as the cofactor for $a_{ij}$,

$$C_{ij} = (-1)^{i+j} \det(\mathbf{M}_{ij})$$

where $\mathbf{M}_{ij}$ is known as the minor of $a_{ij}$, which

is the submatrix formed by deleting the $i$-th row and $j$-th column

Q: What is the geometric significance of the determinant of an $n \times n$ matrix?

### The eigenvalue/eigenvector problem

Suppose $\mathbf{A}$ is an $n \times n$ matrix. A scalar $\lambda$ is said to be an eigenvalue of $\mathbf{A}$ if there exists a nonzero vector $\mathbf{x}$ such that

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x} \tag{1}$$

The vector $\mathbf{x}$ is said to be an eigenvector corresponding to $\lambda$.

- Recall we have discussed how to solve an eigenvalue problem,

$$\mathbf{A}\mathbf{x} - \lambda \mathbf{x} = \mathbf{0} \implies (\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0} \tag{2}$$

  which requires $\mathbf{x}$ to be orthogonal to every row of the matrix $(\mathbf{A} - \lambda \mathbf{I})$,

$$\mathbf{r}_i^{\mathrm{T}} \mathbf{x} = \mathbf{r}_i \cdot \mathbf{x} = 0 \qquad \text{for } i = 1, \ldots, n.$$

  where $\mathbf{r}_i^{\mathrm{T}}$s are the rows of $(\mathbf{A} - \lambda \mathbf{I})$.
- So the matrix equation (2) asks $\mathbf{r}_i$ and $\mathbf{x}$ to be orthogonal for $i = 1, \ldots, n$.

- Rows of $(\mathbf{A}-\lambda\mathbf{I})$ cannnot be linearly independent, so the determinant is 0,

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

Q: Why can we not allow rows of $(\mathbf{A}-\lambda\mathbf{I})$ be linearly independent?

**Definition**

For a given $n\times n$ matrix $\mathbf{A}$, the polynomial $p(\lambda)$ defined by

$$p(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})$$

is called the characteristic polynomial of $\mathbf{A}$,  and the equation

$$p(\lambda)=0$$

is called the characteristic equation of $\mathbf{A}$.

**The Fundamental Theorem of Algebra**

Every polynomial of degree $n$ has exactly $n$ roots in $\mathbb{C}$ if repeated are included.

Q: How many eigenvectors $\mathbf{x}$ will we have for a particular eigenvalue $\lambda_i$.

$$\mathbf{A}\mathbf{x} = \lambda_i \mathbf{x}$$

### Definition

The subspace of all eigenvectors $\mathbf{x}$ is called the eigenspace corresponding to $\lambda_i$.

### Procedures to solve the eigenvalue/eigenvector problem

1. Find all scalars $\lambda$ such that $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. These are the eigenvalues of $\mathbf{A}$

2. If $\lambda_1$, $\lambda_2$, ..., $\lambda_k$ are the distinct eigenvalues obtained in step 1, then solve the $k$ systems of linear equations

$$(\mathbf{A} - \lambda_i\mathbf{I})\,\mathbf{x} = \mathbf{0}$$

to find all eigenvectors $\mathbf{x}$ corresponding to each eigenvalue $\lambda_i$.

- Complications may occur when we have repeated eigenvalues.

## Exercise

Find the eigenvalues and the corresponding eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$$

## Solution

1. Find the characteristic equation, and solve it to find all eigenvalues,

$$\det\left(\mathbf{A} - \lambda \mathbf{I}\right) = 0 \implies \det \begin{bmatrix} 3-\lambda & 2 \\ 3 & -2-\lambda \end{bmatrix} \implies \lambda^2 - \lambda - 12 = 0$$

   Thus the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 4$ and $\lambda_2 = -3$.

2. To find the eigenvectors, we must solve the two linear systems

$$\left(\mathbf{A} - \lambda_1 \mathbf{I}\right)\mathbf{x} = \mathbf{0}, \qquad \text{and} \qquad \left(\mathbf{A} - \lambda_2 \mathbf{I}\right)\mathbf{x} = \mathbf{0}$$

## Solution

- For $\lambda_1 = 4$,

$$\mathbf{A} - 4\mathbf{I} = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$$

- Find a basis for the eigenspace

$$\begin{aligned} -x_1 + 2x_2 &= 0 \\ 3x_1 - 6x_2 &= 0 \end{aligned} \implies x_1 = 2x_2$$

- So the eigenspace is spanned by

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

- Any nonzero multiple of $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda_1$.

- For $\lambda_2 = -3$,

$$\mathbf{A} + 3\mathbf{I} = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$$

- Find a basis for the eigenspace

$$\begin{aligned} 6x_1 + 2x_2 &= 0 \\ 3x_1 + 1x_2 &= 0 \end{aligned} \implies -3x_1 = x_2$$

- The eigenspace is spanned by

$$\mathbf{x}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix},$$

- Any nonzero multiple of $\mathbf{x}_2$ is an eigenvector corresponding to $\lambda_2$.

## Exercise

Find the eigenspaces of $\mathbf{A} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$.

## Solution

1. Construct the characteristic equation, and solve it to find all eigenvalues,

$$\det \begin{bmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{bmatrix} = -\lambda(\lambda-1)^2 \implies \lambda_1 = 0, \quad \lambda_2 = \lambda_3 = 1$$

2. To find the eigenvectors, we must solve the two linear systems

$$(\mathbf{A} - \lambda_1 \mathbf{I})\,\mathbf{x} = \mathbf{0}, \qquad \text{and} \qquad (\mathbf{A} - \lambda_2 \mathbf{I})\,\mathbf{x} = \mathbf{0}$$

- The eigenspace corresponding to $\lambda = 0$ is given by

$$(\mathbf{A} - 0\mathbf{I})\,\mathbf{x} = \mathbf{0} \implies \mathbf{A}\mathbf{x} = \mathbf{0} \implies x_1 = x_2 = x_3$$

- The eigenspace corresponding to $\lambda_1 = 0$ is the span of $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$.

## Solution

- To find the eigenspace corresponding to $\lambda = 1$, we must solve $(\mathbf{A} - \mathbf{I})\mathbf{x} = \mathbf{0}$,

$$\begin{bmatrix} 1 & -3 & 1 & 0 \\ 1 & -3 & 1 & 0 \\ 1 & -3 & 1 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0} \implies \begin{aligned} x_1 - 3x_2 + x_3 &= 0 \\ x_1 - 3x_2 + x_3 &= 0 \\ x_1 - 3x_2 + x_3 &= 0 \end{aligned} \implies \begin{aligned} x_1 - 3x_2 + x_3 &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

- Setting $x_2 = \alpha$ and $x_3 = \beta$, $x_1 = 3\alpha - \beta$,

$$\begin{bmatrix} 3\alpha - \beta \\ \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

- Thus, the eigenspace corresponding to $\lambda = 1$ is

$$\text{span}\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \}$$

- Hence, we have three eigenvalues and three eigenvectors for a $3 \times 3$ matrix.

### Definition

The degree of a root (eigenvalue) of the characteristic polynomial of a matrix, that is the number of times the root is repeated, is called the

algebraic multiplicity of the eigenvalue.

The dimension of the eigenspace corresponding to a given $\lambda$, that is the number of linearly independent eigenvectors corresponding to the eigenvalue, is called the

geometric multiplicity of the eigenvalue.

- Consider $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, the characteristic polynomial of $\mathbf{A}$ is $(1-\lambda)^2$, so

$$\lambda_1 = \lambda_2 = 1$$

it follows, $\quad (\mathbf{A} - 1\mathbf{I})\,\mathbf{x} = \mathbf{0}$

$$\Longrightarrow \begin{aligned} 0x_1 + x_2 &= 0 \\ 0x_1 + 0x_2 &= 0 \end{aligned} \Longrightarrow x_2 = 0, \text{ so eigenspace is } \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}.$$

- So the geometric multiplicity is 1 but the algebraic multiplicity is 2.