Vv156 Lecture 8

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Definition

A function $f(x)$ is said to be differentiable on an open interval $\mathcal{I}$ if the function is differentiable at every point $x \in \mathcal{I}$. And it is simply said to be differentiable if it is differentiable at every point inside its domain.

- We could also define a function that is differentiable on a closed interval:

Definition

A function $f(x)$ is differentiable on a closed interval $[a, b]$ if it is differentiable on (a, b) and both of the one-sided derivatives $f'(a^+)$ and $f'(b^-)$ exist.

- However, it is often sufficient to use the assumption that

$$f \text{ is continuous on } [a, b] \text{ and differentiable on } (a, b).$$

- So far we have been using only one notation for our derivative function of $f(x)$

$$f'(x)$$

- There are other common notations for the derivative function of $y = f(x)$.

1. Lagrange's notation: $f'(x) = y'$
2. Leibniz's notation: $\dfrac{dy}{dx} = \dfrac{df}{dx} = \dfrac{d}{dx} f(x)$
3. Euler's notation: $\mathcal{D} f = \mathcal{D}_x f$
4. Newton's notation: $\dot{y}$

- The symbols $\mathcal{D}$ and $\dfrac{d}{dx}$ are called differentiation operators because they indicate the operation of differentiation, which is the process of calculating a derivative.

Exercise

Find the derivative function $f'(x)$, where $f = -\dfrac{1}{x^2}$ for $x \neq 0$.

Q: What is the connection between $f_1(x) = \dfrac{1}{x}$, $f_2(x) = -\dfrac{1}{x^2}$ and $f_3(x) = \dfrac{2}{x^3}$?

Definition

Suppose f is differentiable on an interval, and f' is itself a differentiable function, then the derivative of f' is known to be the second derivative of f.

1. Lagrange's notation: $f''(x) = y''$
2. Leibniz's notation: $\dfrac{d^2y}{dx^2} = \dfrac{d^2f}{dx^2} = \dfrac{d^2}{dx^2}f(x)$
3. Euler's notation: $\mathcal{D}^2 f = \mathcal{D}_x^2 f$
4. Newton's notation: $\ddot{y}$

- Continuing in this manner, we denote derivatives as

$$f, \qquad f', \qquad f'', \qquad f^{(3)}, \ldots, f^{(n)}$$

  each of which is the first derivative of the proceeding one.
- The derivative $f^{(n)}$ is called the nth derivative, or the derivative of order n, of f.

Defintion

A function f is continuously differentiable on (a, b), written as

$$f \in \mathcal{C}^1(a, b)$$

if f is differentiable and f' is continuous on (a, b). In general, $f \in \mathcal{C}^k$ denotes

$$f', f'', \ldots, f^{(k)} \qquad \text{exist and are continuous.}$$

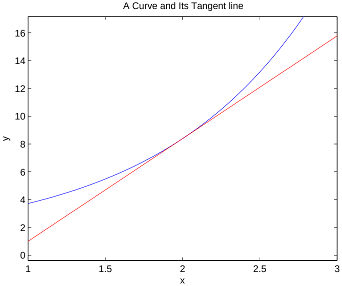
A function f is smooth if it has continuous derivatives up to some desired order. The number of continuous derivatives necessary for a function to be considered smooth depends on the problem at hand, and may vary from two to infinity.

Exercise

Does the following piecewise function belong to $\mathcal{C}^2$?

$$f(x) = \begin{cases} x^2 & \textit{if} \quad x > 0, \\ 0 & \textit{if} \quad x \leq 0. \end{cases}$$

- If you look at the graph very closely near the point of tangency.



- Note how the curve gets closer and closer to its tangent as we keep zooming in
- This is the basis for finding approximations within a small neighbourhood.

Matlab

```
>> syms x
>> ezplot('exp(x)+1',[1:0.00001:3]); hold on
>> obj = ezplot('exp(2) * ( x- 2) + exp(2) +1',[1:0.00001:3]); set(obj, 'color','red'); clear obj
>> hold off; xlabel('x'); ylabel('y'); title('A Curve and Its Tangent line')
```

- The equation of the tangent line at $x = a$ for $y = f(x)$ is given by

$$y - f(a) = f'(a)(x-a) \implies y = \boxed{f(a) + f'(a)(x-a)}$$

- Based on our observation, we expect near the point of tangency

$$f(x) \approx \boxed{f(a) + f'(a)(x-a)}$$

Definition

The approximation

$$f(x) \approx f(a) + f'(a)(x-a)$$

is known as the linear approximation or tangent line approximation f at a.

The linear function $L(x)$ whose graph is the tangent line,

$$L(x) = f(a) + f'(a)(x-a)$$

is called the linearization of f at a.

- Another way to view differentiability is to write

$$f(c+h) = f(c) + f'(c)h + \varepsilon(h)$$

as the sum of a linear approximation of $f(c+h)$ and an error term $\varepsilon(h)$.

- In general, the error ε also depends on c, so the error $\epsilon(h)$ will alter if c alters.

Theorem

Suppose $f(x)$ is defined for $a \leq x \leq b$, then f is differentiable at $c \in (a,b)$ if and only if there exists a constant A and a function $\varepsilon(h)$ such that

$$f(c+h) = f(c) + Ah + \epsilon(h), \qquad \text{where} \quad \lim_{h \to 0} \frac{\varepsilon(h)}{h} = 0$$

- This theorem essentially states that differentiability is equivalent to

$$\lim_{h \to 0} \frac{\varepsilon(h)}{h} = 0$$

- And you will see in the proof that $A = f'(c)$ if it is differentiable at c.

Proof

- First suppose that f is differentiable at c, and define

$$\varepsilon(h) = f(c+h) - f(c) - f'(c)h.$$

Then

$$\lim_{h \to 0} \frac{\varepsilon(h)}{h} = \lim_{h \to 0} \left[\frac{f(c+h) - f(c)}{h} - f'(c) \right] = 0$$

- Conversely, suppose that

$$f(c+h) = f(c) + Ah + \varepsilon(h)$$

where $\frac{\varepsilon(h)}{h} \to 0$ as $h \to 0$. Then

$$\lim_{h \to 0} \left[\frac{f(c+h) - f(c)}{h} \right] = \lim_{h \to 0} \left[A + \frac{\varepsilon(h)}{h} \right] = A$$

which proves that f is differentiable at c with $f'(c) = A$.