Vv156 Lecture 5

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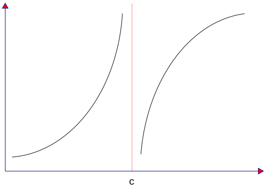
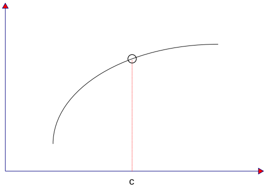
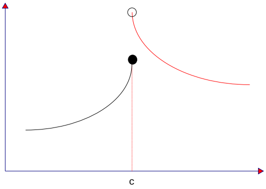
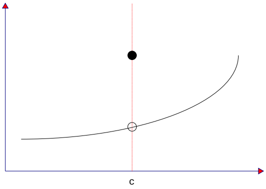
Q: What does a curve, that is not continuous, look like?

1. 

1. 

2. 

3. 

Definition

Let f be a function defined on an *open* interval that contains the number c, then f is said to be continuous at $x = c$ if the following conditions are satisfied:

1. $f(c)$ is defined; 2. $\lim_{x \to c} f(x)$ exists ; 3. $\lim_{x \to c} f(x) = f(c)$

Exercise

(a) *Find K which makes the following function continuous at $x = 1$.*

$$f(x) = \begin{cases} x^2 - 2 & \text{if} \quad x < 1, \\ Kx - 4 & \text{if} \quad 1 \leq x. \end{cases}$$

(b) *Show the sine function is continuous at every point $c \in (-\infty, \infty)$.*

(c) *Determine whether the following function is continuous at every point $c \in \mathbb{R}$.*

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

(d) *Show the function* $f(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q}, \\ 0 & \text{if} \quad x \notin \mathbb{Q} \end{cases}$ *is nowhere continuous.*

- Discontinuities can be further classified according to their nature.

Defintion

- Removable discontinuity: Both $f(c)$ and $\lim_{x \to c} f(x) = L$ exist, but

$$f(c) \neq L$$

in which case we can make f continuous at c by redefining $f(c) = L$.

- Jump discontinuity: Both of the one-sided limits exist, but

$$\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$$

- Essential discontinuity: At least one of the one-sided limits does not exist.

Q: Is it possible to have an essential discontinuity where the function f is bounded?

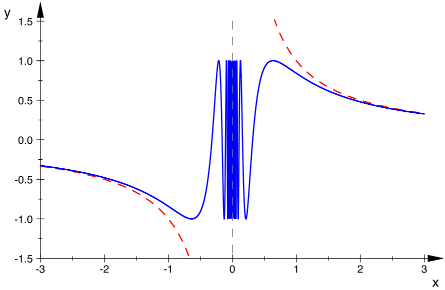
$$f(x) = \sin\left(\frac{1}{x}\right) \qquad \text{at} \qquad x = 0$$

- The graph on the right is a plot of

$$y = \sin\left(\frac{1}{x}\right)$$

together with

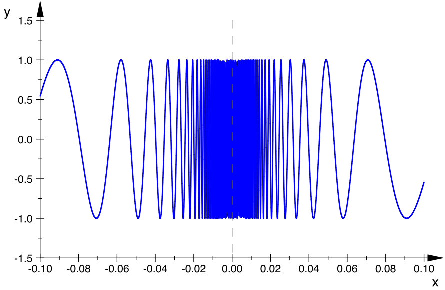
$$y = \frac{1}{x}$$



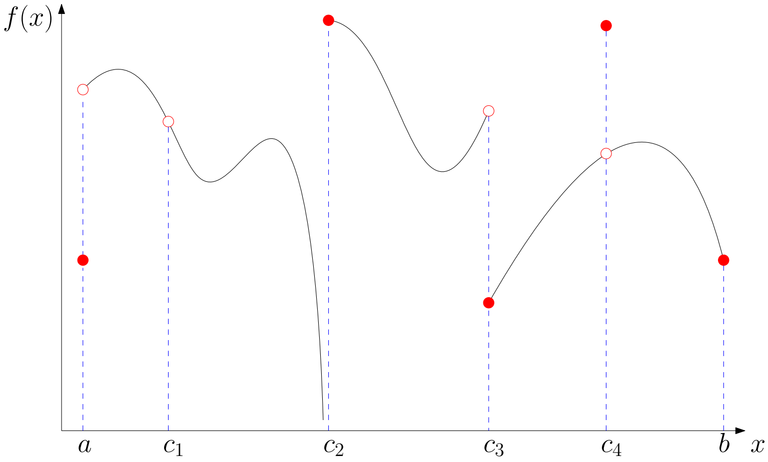
- The second is a closer look at

$$y = \sin\left(\frac{1}{x}\right)$$

near $x = 0$.



Q: What types of discontinuities does the following function on $[a, b]$ has?



Q: Can we say anything regarding the continuity of the above function at a and b?

Definition

Let f be a function defined on a *closed* interval $[a, b]$, then f is

continuous at $x = a$

if the following conditions are satisfied:

1. $f(a)$ is defined; 2. $\lim_{x \to a^+} f(x)$ exists ; 3. $\lim_{x \to a^+} f(x) = f(a)$

continuous at $x = b$

if the following conditions are satisfied:

1. $f(b)$ is defined; 2. $\lim_{x \to b^-} f(x)$ exists ; 3. $\lim_{x \to b^-} f(x) = f(b)$

Definition

A function is said to be continuous on a *set* if it is continuous at every point in it, and simply continuous if it is continuous at everywhere in its domain.

Definition

Suppose f is defined on (a, b), then f is continuous on (a, b) if and only if

$$\lim_{x \to c} f(x) = f(c) \qquad \text{for every} \quad a < c < b$$

Definition

Suppose f is defined on $[a, b]$, then f is continuous on $[a, b]$ if and only if

$$\lim_{x \to c} f(x) = f(c); \qquad \lim_{x \to a^+} f(x) = f(a); \qquad \lim_{x \to b^-} f(x) = f(b)$$

for every $a < c < b$.

- We can define continuity in a broad sense without the limit concept.

Definition

A function f is said to be continuous at $x = c$ if and only if c is in the domain $\mathcal{A}$ of a function f and for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(c)| < \epsilon \qquad \text{when} \qquad |x - c| < \delta \qquad \text{and} \qquad x \in \mathcal{A}$$

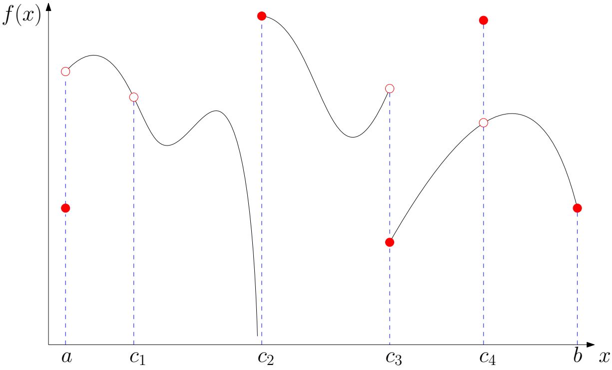
- In terms of neighbourhoods, the function f is

$$\text{continuous at } c \in \mathcal{A}$$

if for every neighbourhood $\mathcal{V}$ of $f(c)$ there is a neighbourhood $\mathcal{U}$ of c such that

$$x \in \mathcal{U} \cap \mathcal{A} \implies f(x) \in \mathcal{V}$$

- Note that c must belong to the domain in order to have continuity at c.



Theorem

If the functions f and g have the same domain and are continuous at c, then

1. The sum is also continuous at c.

$$f \pm g$$

2. The product is also continuous at c.

$$fg$$

3. The quotient is continuous at c if $g(c) \neq 0$.

$$\frac{f}{g}$$

- This is a direct result of various limit laws, and together gives the following

Theorem

Every polynomial function or rational function is continuous on its domain.

Defintion

Suppose $\mathcal{A}$ is the domain of f and $\mathcal{B}$ is the domain of g, where

$$g(\mathcal{B}) \subset \mathcal{A}$$

that is, the domain of f contains the range of g, then the composition

$$(f \circ g)(x) = f\Big(g(x)\Big)$$

is a function of x.

- The next theorem states the composition of continuous functions is continuous.

Theorem

Let f and g be two continuous functions in their domains. If c is in the domain of g and $g(c)$ is in the domain of f, then the composite function

$$f \circ g$$

is also continuous at c.

Proof

- For $\epsilon > 0$, since f is continuous at $g(c)$, there exists δ_1 such that

$$|y - g(c)| < \delta_1 \implies |f(y) - f(g(c))| < \epsilon.$$

- Next, since g is continuous at c, there exists $\delta > 0$ such that

$$|x - c| < \delta \implies |g(x) - g(c)| < \delta_1.$$

- Combine these inequalities shows that the composite function is continuous.

$$|x - c| < \delta \implies |f(g(x)) - f(g(c))| < \epsilon \quad \square$$

Q: A Tibetan monk leaves the monastery at 7:00 AM and takes his usual path to the top of the mountain, arriving at 7:00 PM. The following morning, he starts at 7:00 AM at the top and takes the same path back, arriving at the monastery at 7:00 PM. Is there always a point on the path that the monk will cross at exactly the same time of day on both days?