
Question1 (5 points)

(a) (1 point) Evaluate

$$\iint_{\mathcal{R}} \frac{x}{1+xy}\, dA,$$

where $\mathcal{R}$ is the rectangular region $\mathcal{R} = [0,1] \times [0,1]$.

(b) (1 point) Find the volume of the solid that lies under the paraboloid

$$z = x^2 + y^2$$

and above the region $\mathcal{D}$ in the xy-plane bounded by the line

$$y = 2x$$

and the parabola

$$y = x^2$$

(c) (1 point) Find the volume of the tetrahedron bounded by the planes

$$x + 2y + z = 2, \quad x = 2y, \quad x = 0, \quad \text{and } z = 0$$

(d) (1 point) Evaluate

$$\iint_{\mathcal{D}} \frac{\sin x}{x}\, dA,$$

where $\mathcal{D}$ is the triangular region

$$\{(x,y) \mid 0 \le y \le x,\ 0 \le x \le \pi\}$$

(e) (1 point) If $f(x,y)$ is continuous on $[a,b] \times [c,d]$ and

$$g(x,y) = \int_a^x \int_c^y f(s,t)\, dt\, ds, \qquad \text{for } a < x < b,\ c < y < d.$$

Show that

$$g_{xy} = g_{yx} = f(x,y)$$

Question2 (5 points)

(a) (1 point) Let $f(x,y)$ be continuous. Find a single iterated integral that is equal to

$$\int_0^1 \int_0^{x^2} f(x,y)\, dy\, dx + \int_1^3 \int_0^{\frac{3-x}{2}} f(x,y)\, dy\, dx$$

by changing the order of integration.

(b) (1 point) Evaluate

$$\iint_{\mathcal{D}} \frac{1}{(1+x^2+y^2)^2}\, dA$$

where $\mathcal{D}$ is the region in the xy-plane outside the circle of radius 1 centred at the origin.

(c) (1 point) Let $f(x)$ and $g(x)$ be continuous on $[a, b]$, and monotonically increasing.

$$\left[\int_a^b f(x)\,dx\right]\left[\int_a^b g(x)\,dx\right] \le (b-a)\int_a^b f(x)g(x)\,dx$$

Show the above inequality is true.

(d) (2 points) Find the value of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

at $s = 2$ by considering the values of

$$\int_0^{\pi/2}\int_0^1 \frac{\sin\theta}{\sqrt{1 - x^2\sin^2\theta}}\,dx\,d\theta \qquad \text{and} \qquad \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

Question3 (5 points)

(a) (1 point) Find the surface area of the part of the surface

$$z = x^2 + 2y$$

that lies above the triangle in the xy-plane with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$.

(b) (1 point) Find the area of the part of the sphere

$$x^2 + y^2 + z^2 = 4z$$

that lies inside the paraboloid

$$z = x^2 + y^2$$

(c) (1 point) Find the area of the portion of the paraboloid

$$\mathbf{r}(u, v) = \begin{bmatrix} u\cos v \\ u\sin v \\ u^2 \end{bmatrix},$$

for which $1 \le u \le 2$ and $0 \le v \le 2\pi$.

(d) (2 points) Let $\mathcal{D}$ be the half-annulus

$$9 \le x^2 + y^2 \le 16 \qquad \text{where} \quad y \ge 0.$$

Suppose we have a lamina whose shape is $\mathcal{D}$ and has uniform density. Find the centroid.

Question4 (0 points)

(a) (1 point (bonus)) The function e^{-x^2} has no elementary antiderivative. So you cannot use the fundamental theorem of calculus to evaluate the definite integral

$$\int_a^b e^{-x^2}\,dx$$

However, the integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx$$

is especially important in many applications. Show

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

by using the theory of double integrals.