Question1 (3 points)

Find the general solution for each of the following homogeneous equations.

$$\ddot{y} - 3\dot{y} + 2y = 0 \qquad \ddot{y} - 2\dot{y} + y = 0 \qquad \ddot{y} - 2\dot{y} + 10y = 0$$

Solution:

3M Solving the characteristic equations, we have

$$\begin{aligned} r^2 - 3r + 2 = 0 &\implies r_1 = 1 \qquad \text{and} \qquad r_2 = 2 \\ r^2 - 2r + 1 = 0 &\implies r_{1,2} = 1 \\ r^2 - 2r + 10 = 0 &\implies r_1 = 1 + \mathrm{i}3 \qquad \text{and} \qquad r_2 = 1 - \mathrm{i}3 \end{aligned}$$

therefore the solutions are

$$\begin{aligned} y &= C_1 e^t + C_2 e^{2t} \\ y &= C_1 e^t + C_2 t e^t \\ y &= e^t \left(C_1 \sin 3t + C_2 \cos 3t \right) \end{aligned}$$

Question2 (1 points)

Determine whether the following two functions are linearly independent.

$$\phi_1(x) = \begin{cases} x^2 & \text{if} \quad x \geq 0, \\ 0 & \text{if} \quad x < 0. \end{cases} \qquad \phi_2(x) = \begin{cases} 0 & \text{if} \quad x \geq 0, \\ x^2 & \text{if} \quad x < 0. \end{cases}$$

Justify your answer.

Solution:

1M To show ϕ_1 and ϕ_2 are linearly independent, we have to show that

$$\alpha_1 \phi_1(x) + \alpha_2 \phi_2(x) = 0 \qquad \text{for all } x$$

is satisfied if and only if $\alpha_1 = \alpha_2 = 0$. Consider $x > 0$, then

$$\alpha_1 \phi_1(x) + \alpha_2 \phi_2(x) = \alpha_1 x^2 \neq 0$$

is clearly not identically zero unless $\alpha_1 = 0$, thus α_1 must be zero for $x > 0$. Similarly, we can conclude that α_2 must be zero for $x < 0$. Hence $\alpha_1 = \alpha_2 = 0$ is the only way for all x such that

$$\alpha_1 \phi_1(x) + \alpha_2 \phi_2(x) = 0$$

Notice when $x = 0$, α_1 and α_2 need not be zero. However, what is needed is a single set of α_1 and α_2 that works for all x. In other words, a single value of x at which α_1 and α_2 are NOT simultaneously zero leads to no conclusion. However, if there is a single value of x at which α_1 and α_2 must simultaneously zero allow us to conclude two functions are linearly independent. Also note the fact that the Wronskian of ϕ_1 and ϕ_2 is identically zero in this case.

Question3 (2 points)

(a) (1 point) First verify that

$$\phi_1(t) = t^{-1/2}\cos t$$

is one solution (for $t > 0$) of Bessel's equation

$$t^2\ddot{y} + t\dot{y} + (t^2 - \frac{1}{4})y = 0$$

Then find the second linearly independent solution.

Solution:

1M Substituting ϕ_1, we have

$$\begin{aligned}\text{LHS} &= t^2\ddot{\phi}_1 + t\dot{\phi}_1 + (t^2 - \frac{1}{4})\phi_1 \\ &= t^2\frac{3\cos t - 4t^2\cos t + 4t\sin t}{4t^{5/2}} - t\left(\frac{\cos t + 2t\sin t}{2t^{3/2}}\right) + (t^2 - \frac{1}{4})t^{-1/2}\cos t \\ &= 0 \\ &= \text{RHS}\end{aligned}$$

Using the definition of Wronskian, we have

$$\begin{aligned}W &= \phi_1\dot{\phi}_2 - \dot{\phi}_1\phi_2 \\ &= t^{-1/2}\cos t\dot{\phi}_2 + \left(\frac{\cos t + 2t\sin t}{2t^{3/2}}\right)\phi_2\end{aligned}$$

However, according to Abel, the Wronskian is given by

$$W = C\exp\left(-\int\frac{t}{t^2}\,dt\right) = \frac{C}{t}$$

from which, we can find the second linearly independent solution by solving

$$\frac{\cos t}{t^{1/2}}\dot{\phi}_2 + \left(\frac{\cos t + 2t\sin t}{2t^{3/2}}\right)\phi_2 = \frac{C_2}{t} \qquad \text{for} \quad C \neq 0$$

Note if $C_2 = 0$, we will just obtain a scalar multiple of ϕ_1. This is first-order linear equation. Using the following integrating factor,

$$\mu = \frac{1}{\alpha}\exp\left(\int\frac{\beta}{\alpha}\,dt\right) = \frac{t^{1/2}}{\cos t}\exp\int\left(\frac{1}{2t} + \tan t\right)dt = \frac{t}{\cos^2 t}$$

we have the second linearly independent solution

$$\begin{aligned}y &= \frac{1}{\mu\alpha}\int\mu\gamma\,dt \\ &= t^{-1/2}\cos t\int\frac{C_2}{\cos^2 t}\,dt \\ &= C_2 t^{-1/2}\cos t\left(\tan t + \frac{C_1}{C_2}\right) = C_2 t^{-1/2}\sin t\end{aligned}$$

By setting $\frac{C_1}{C_2} = 0$, we have a second linearly independent solution that is "simple", but actually for any nonzero C_2, the following is a second linearly independent solution

$$y = C_1 t^{-1/2}\cos t + C_2 t^{-1/2}\sin t$$

(b) (1 point) Find the general solution to Legendre's equation

$$(1-x^2)y'' - 2xy' + 2y = 0 \qquad \text{for} \quad -1 < x < 1$$

Solution:

1M In this case, our guess that a power function $y = x^r$ being a solution

$$\begin{aligned}(1-x^2)r(r-1)x^{r-2} - 2xrx^{r-1} + 2x^r &= 0\\ r(r-1)(1-x^2)x^{r-2} - 2x(r-1) &= 0\\ \implies r &= 1\end{aligned}$$

will not fail us, that is,

$$\phi_1(x) = x$$

is a solution to the given Legendre equation. From Abel's formula, we have

$$x\phi_2' - \phi_2 = \frac{C}{1-x^2}$$

from which, we have

$$\phi_2 = C\left(\frac{x}{2}\ln\frac{1+x}{1-x} - 1\right)$$

Hence the general solution is

$$y = C_1 + C_2\left(\frac{x}{2}\ln\frac{1+x}{1-x} - 1\right)$$

Question4 (2 points)

Find the general solution by using the method of undetermined coefficients.

(a) (1 point) $\ddot{y} - 2\dot{y} + 10y = 20t^2 + 2t - 8$

Solution:

1M We have found the complementary solution in question 1

$$y_c = e^t\left(C_1 \sin 3t + C_2 \cos 3t\right)$$

so a particular solution shall be in the following family

$$\phi(t) = A_2t^2 + A_1t + A_0$$

Substituting $\phi(t)$ into the equation, and equating the coefficients, we have

$$A_2 = 2; \quad A_1 = 1; \quad A_0 = -1$$

Thus the general solution is given by

$$y = y_c + y_p = e^t\left(C_1 \sin 3t + C_2 \cos 3t\right) + 2t^2 + t - 1$$

(b) (1 point) $\ddot{y} - 4\dot{y} + 4y = e^{2t}$

Solution:

1M Solve the complementary equation. The characteristic equation is:

$$r^2 - 4r + 4 = (r-2)^2$$

So the general solution to the corresponding homogeneous equation is

$$y_c(t) = (C_1 + C_2 t)e^{2t}$$

For the particular solution. Since the complementary solution takes the form of

$$y_c(t) = (C_1 + C_2 t)e^{2t}$$

both $\phi(t) = Ae^{2t}$ and $\phi(t) = Ate^{2t}$ will give 0 instead of e^{2t} on the left-hand side the equation, we need a different form for the second linearly independent solution. We assume the particular solution has the following form:

$$\begin{aligned}
\phi &= At^2e^{2t} \\
\dot{\phi} &= A(2te^{2t} + 2t^2e^{2t}) \\
\ddot{\phi} &= A(2e^{2t} + 8te^{2t} + 4t^2e^{2t})
\end{aligned}$$

Substituting into the differential equation, we have

$$\begin{aligned}
\ddot{\phi} - 4\dot{\phi} + 4y &= A(2e^{2t} + 8te^{2t} + 4t^2e^{2t} - 8te^{2t} - 8t^2e^{2t} + 4t^2e^{2t}) \\
&= 2Ae^{2t} \\
&= e^{2t}
\end{aligned}$$

from which, we have

$$A = \frac{1}{2}$$

Thus the particular solution is

$$y_p(t) = \frac{1}{2}t^2e^{2t}$$

The general solution is

$$y = y_p + y_C\frac{1}{2}t^2e^{2t} + (C_1 + C_2 t)e^{2t}$$

Question5 (2 points)

Find the general solution by using the method of variation of parameters.

(a) (1 point) $\ddot{y} + y = \sec(t)\csc(t)$

Solution:

1M Solve the complementary equation, we have

$$y_c = C_1 \sin t + C_2 \cos t$$

assuming the particular solution

$$y_p = u_1(t)\sin t + u_2(t)\cos t$$

we have the following system

$$\begin{aligned}\dot{u}_1 \sin t + \dot{u}_2 \cos t &= 0\\ \dot{u}_1 \cos t - \dot{u}_2 \sin t &= \sec t \csc t\end{aligned}$$

solving which, we have

$$\dot{u}_1 = \csc t \qquad \text{and} \qquad \dot{u}_2 = -\sec t$$

Integrating both, we have

$$u_1 = \ln(\csc t - \cot t) \qquad \text{and} \qquad u_2 = -\ln(\sec t + \tan t)$$

Hence,

$$y_p = \ln(\csc t - \cot t)\sin t - \ln(\sec t + \tan t)\cos t$$

and the general solution is

$$y = y_c + y_p$$

where y_c and y_p are given above.

(b) (1 point) $\ddot{y} - 3\dot{y} + 2y = \cos(e^{-t})$

Solution:

1M We have found the complementary solution in question 1

$$y_c = C_1 e^t + C_2 e^{2t}$$

solving the following system,

$$\begin{aligned}\dot{u}_1 e^t + \dot{u}_2 e^{2t} &= 0\\ \dot{u}_1 e^t + 2\dot{u}_2 e^{2t} &= \cos(e^{-t})\end{aligned}$$

we have

$$\dot{u}_1 = -e^{-t}\cos(e^{-t}) \qquad \text{and} \qquad \dot{u}_2 = e^{-2t}\cos(e^{-t})$$

Integrating both, we have

$$u_1 = \sin(e^{-t}) \qquad \text{and} \qquad u_2 = -\cos(e^{-t}) - e^{-t}\sin(e^{-t})$$

Hence

$$y_p = u_1 e^t + u_2 e^{2t} = -\cos(e^{-t})e^{2t}$$

and the general solution is

$$y = C_1 e^t + C_2 e^{2t} - e^{2t}\cos(e^{-t})$$

Question6 (2 points)

Suppose we found by variation of parameters, $y_p(x) = u_1(x)\phi_1(x) + u_2(x)\phi_2(x)$, where ϕ_1 and ϕ_2 form a fundamental set of solutions for the complementary equation, is a particular solution for the nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = g(x)$$

on an interval I for which p, q, and g are continuous.

(a) (1 point) Show that y_p can be written as

$$y_p(x) = \int_{x_0}^{x} G(x,t)g(t)\,dt,$$

where x and x_0 are in I, $G(x,t) = \dfrac{\phi_1(t)\phi_2(x) - \phi_1(x)\phi_2(t)}{W(t)}$, and $W(t) = W(\phi_1(t), \phi_2(t))$ is the Wronskian. The function $G(x,t)$ is called the Green's function for the differential equation. It is fundamental to advanced theories on differential equations.

Solution:

1M Variation of Parameters involves soling the following system

$$\begin{aligned} u_1'\phi_1 + u_2'\phi_2 &= 0 \\ u_1'\phi_1' + u_2'\phi_2' &= g \end{aligned}$$

which has the following solution in general

$$\begin{aligned} u_1' &= \frac{-g\phi_2}{\phi_1\phi_2' - \phi_1'\phi_2} \\ u_2' &= \frac{g\phi_1}{\phi_1\phi_2' - \phi_1'\phi_2} \end{aligned}$$

The particular solution is thus

$$\begin{aligned} y_p &= \left(\int_{x_0}^{x} u_1'(t)\,dt\right)\phi_1(x) + \left(\int_{x_0}^{x} u_2'(t)\,dt\right)\phi_2(x) \\ &= \left(\int_{x_0}^{x} \frac{-g(t)\phi_2(t)}{\phi_1(t)\phi_2'(t) - \phi_1'(t)\phi_2(t)}\,dt\right)\phi_1(x) \\ &\quad + \left(\int_{x_0}^{x} \frac{g(t)\phi_1(t)}{\phi_1(t)\phi_2'(t) - \phi_1'(t)\phi_2(t)}\,dt\right)\phi_2(x) \\ &= \int_{x_0}^{x} \frac{\phi_1(t)\phi_2(x) - \phi_1(x)\phi_2(t)}{\phi_1(t)\phi_2'(t) - \phi_1'(t)\phi_2(t)} g(t)\,dt \end{aligned}$$

which is the form required to show. Also notice

$$y_p(x_0) = \int_{x_0}^{x_0} G(x,t), g(t)\,dt = 0$$

and the derivative

$$\begin{aligned}
y_p'(x_0) &= \frac{d}{dx}\left(\int_{x_0}^{x} u_1'(t)\,dt\right)\Bigg|_{x=x_0} \phi_1(x_0) + \phi_1'(x_0)\left(\int_{x_0}^{x_0} u_1'(t)\,dt\right) \\
&\quad + \frac{d}{dx}\left(\int_{x_0}^{x} u_2'(t)\,dt\right)\Bigg|_{x=x_0} \phi_2(x_0) + \phi_2'(x_0)\left(\int_{x_0}^{x_0} u_2'(t)\,dt\right) \\
&= \frac{-g(x_0)\phi_1(x_0)\phi_2(x_0)}{\phi_1(x_0)\phi_2'(x_0) - \phi_1'(x_0)\phi_2(x_0)} + \frac{g(x_0)\phi_1(x_0)\phi_2(x_0)}{\phi_1(x_0)\phi_2'(x_0) - \phi_1'(x_0)\phi_2(x_0)} \\
&= 0
\end{aligned}$$

which shows the particular solution obtained from using Green's function satisfies the initial conditons

$$y(x_0) = y'(x_0) = 0$$

(b) (1 point) Use Green's function to find a solution of the following initial-value problem

$$y'' - y = e^{2x}, \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution:

1M Green's function in this context is merely an explicit formula for the solution from variation of parameters,

$$y = \int_0^x \frac{\phi_1(t)\phi_2(x) - \phi_1(x)\phi_2(t)}{\phi_1(t)\phi_2'(t) - \phi_1'(t)\phi_2(t)} e^{2t}\,dt$$

where $\phi_1 = e^t$ and $\phi_2 = e^{-t}$ are two linearly independent solutions to the complementary equation. Computing the integral, we obtain the solution

$$y = \frac{1}{6}\left(e^{-x}\left(e^x - 1\right)^2\left(2e^x + 1\right)\right)$$

Question7 (1 points)

Recall the curvature of a curve defined by a function $y = f(x)$ is

$$\kappa = \frac{y''}{[1 + (y')^2]^{\frac{3}{2}}}.$$

Find $y = f(x)$ for which $\kappa = 1$, show all your workings.

[Hint: For simplicity, assume constants of integration is zero.]

Solution:

1M We can solve the following to obtain the curve that has a constant curvature of 1.

$$y'' = \left(1 + (y')^2\right)^{3/2}$$

We solve by using a technique that we used in class, but I did not spell out its name.

$$v = y' \implies v' = \left(1 + v^2\right)^{3/2}$$

which is a first-order separable equation, applying the formula, we have

$$\frac{v}{(v^2+1)^{1/2}} = x + C \implies y' = v = \pm\frac{x}{\sqrt{1-x^2}}$$

where C was set to be zero. Integrating both, we have

$$y = \pm\sqrt{1-x^2}$$

Question8 (3 points)

Sometimes a differential equation with variable coefficients.

$$\ddot{y} + P(t)\dot{y} + Q(t)y = 0$$

can be put in a more suitable form for finding a solution by making a change of independent variable. In this equation we determine conditions on P and Q such that the above equation can be transformed into an equation with constant coefficients. Let the new variable be

$$x = u(t)$$

(a) (1 point) Show, with the new variable, that the given equation becomes

$$(\dot{x})^2\frac{d^2y}{dx^2} + (\ddot{x} + P(t)\dot{x})\frac{dy}{dx} + Q(t)y = 0$$

Solution:

1M According the chain rule, we have

$$\dot{y} = \frac{dy}{dx}\frac{dx}{dt} \implies \ddot{y} = \frac{d}{dx}\left(\frac{dy}{dx}\frac{dx}{dt}\right)\frac{dx}{dt} = \left(\frac{d^2y}{dx^2}\frac{dx}{dt} + \frac{d^2x}{dt^2}\frac{dt}{dx}\cdot\frac{dy}{dx}\right)\frac{dx}{dt}$$

Making the substitution, we have

$$\frac{d^2y}{dx^2}(\dot{x})^2 + \frac{dy}{dx}\ddot{x} + P(t)\dot{x}\frac{dy}{dx} + Qy = (\dot{x})^2\frac{d^2y}{dx^2} + (\ddot{x} + P(t)\dot{x})\frac{dy}{dx} + Q(t)y = 0$$

as required.

(b) (1 point) In order for the equation in part (a) to have constant coefficients, the coefficients of $\dfrac{d^2y}{dx^2}$ and of y must be proportional. If $Q(t) > 0$ and the constant of proportionality to be 1, show that we need the following substitution

$$x = u(t) = \int\left(Q(t)\right)^{1/2}dt$$

Solution:

1M In order to satisfying the condition that the constant of proportionality is 1, x must satisfy

$$\frac{Q}{(\dot{x})^2} = 1 \iff Q^{1/2} = \dot{x}$$

direction integration shows, we need

$$x = u(t) = \int \Big(Q(t)\Big)^{1/2}\, dt$$

(c) (1 point) When $Q(t) > 0$, find the condition under which the original equation can be transformed into one with constant coefficients by the above substitution

$$x = u(t) = \int \Big(Q(t)\Big)^{1/2}\, dt$$

Solution:

1M In general, let

$$\frac{Q(t)}{(\dot{x})^2} = k > 0$$

then the equation can be written as

$$\frac{d^2y}{dx^2} + \frac{\ddot{x} + P(t)\dot{x}}{(\dot{x})^2}\frac{dy}{dx} + ky = 0$$

We will have an equation with constant coefficients only if $x = u(t)$ satisfies the following equation

$$\frac{\ddot{x} + P(t)\dot{x}}{(\dot{x})^2} = \ell$$

for some constant ℓ. Substituting

$$\dot{x} = \frac{d}{dt}\int \Big(Q(t)\Big)^{1/2}\, dt = \Big(Q(t)\Big)^{1/2}$$
$$\ddot{x} = \frac{1}{2}\Big(Q(t)\Big)^{-1/2}\dot{Q}$$

into the equation, we have

$$\frac{1}{2}Q^{-3/2}\dot{Q} + PQ^{-1/2} = \ell$$

which gives the condition, that P and Q need to satisfy, such that

$$x = u(t) = \int \Big(Q(t)\Big)^{1/2}\, dt$$

can be used to converted the original equation into one with only constant coefficients.