Vv417 Lecture 11

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October 17, 2019

Q: Recall the concept of linear independence, how did we define it?

Defintion

Suppose

$$\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$$

is a set of two or more vectors in a vector space $\mathcal{V}$, then $\mathcal{S}$ is said to be

linearly independent

if no vector in $\mathcal{S}$ can be expressed as a linear combination of the others.

Theorem

Let $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a set of two or more vectors in $\mathcal{V}$. Then $\mathcal{S}$ is linearly independent if and only if the only solution to the following equation

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_r\mathbf{v}_r = \mathbf{0}$$

is $\alpha_1 = \alpha_2 = \cdots = \alpha_r = 0$.

Proof

- Given the set $\mathcal{S}$ is linearly independent, and assume α_i is non-zero, then

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_r\mathbf{v}_r = \mathbf{0}$$
$$\implies \mathbf{v}_i = \left(-\frac{\alpha_1}{\alpha_i}\right)\mathbf{v}_1 + \left(-\frac{\alpha_2}{\alpha_i}\right)\mathbf{v}_2 + \cdots + \left(-\frac{\alpha_r}{\alpha_i}\right)\mathbf{v}_r$$

- This contradicts the fact of linear independence, so α_i's must all be zero.
- Given

$$\alpha_1 = \alpha_2 = \cdots = \alpha_r = 0$$

is the only solution, and assume $\mathcal{S}$ is linearly dependent, then

$$\mathbf{v}_i = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \cdots + \beta_r\mathbf{v}_r$$
$$\implies \beta_1\mathbf{v}_1 + \cdots - \mathbf{v}_i + \cdots + \beta_r\mathbf{v}_r = \mathbf{0}$$

- But this contradicts the fact that all α_i's being zero is the only solution, so $\mathcal{S}$ must be linearly independent.

Exercise

Determine whether the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 6 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ are linearly independent or linearly dependent in $\mathbb{R}^4$.

Solution

- We have to find whether the trivial solution is the only solution to

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0} \iff \begin{bmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{x} = \mathbf{0} \quad \text{where} \quad \mathbf{x} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

- Since $\begin{bmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 0.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, they are not linearly independent.

Theorem

For a set $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ of two or more vectors in a vector space $\mathcal{V}$, then

1. If $\mathcal{S}$ has a linearly dependent subset, then $\mathcal{S}$ must be linearly dependent.
2. If $\mathcal{S}$ is linearly independent, then every subset of $\mathcal{S}$ is linearly independent
3. If $\mathcal{S}$ is linearly independent and if $\mathbf{u} \in \mathcal{V}$, then the extension set

$$\mathcal{S}_{\text{ext}} = \mathcal{S} \cup \{\mathbf{u}\}, \quad \text{is linearly independent if and only if } \mathbf{u} \notin \operatorname{span}(\mathcal{S}).$$

Proof

1. Suppose that $\mathcal{S}$ contains a linearly dependent subset, and let the vectors in $\mathcal{S}$ be listed so that this dependent set is $\mathcal{S}_{\text{dep}} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

- By definition of dependence, there are scalars α_1, α_2, ..., α_k not all of which are zero such that $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k = \mathbf{0}$. This means, we have

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k + 0\mathbf{v}_{k+1} + \cdots + 0\mathbf{v}_n = \mathbf{0}$$

where not all of the scalars are zero, and hence $\mathcal{S}$ is linearly dependent. □

Q: What is a Vandemonde matrix $\mathbf{V} = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{n-1} \end{bmatrix}$?

Q: Why is the Vandemonde matrix $\mathbf{V}$ invertible if it is from using n distinct x?

- $\mathbf{V}$ is invertible if and only if the homogeneous system has only trivial solution

$$\mathbf{Va} = \mathbf{0}$$

$$a_0\mathbf{v}_0 + a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_{n-1}\mathbf{v}_{n-1} = \mathbf{0}, \quad \text{where } \mathbf{v}_i \text{ are columns of } \mathbf{V}.$$

- That is, for each $i = 1, 2, \ldots, n$,

$$a_0 + a_1 x_i + a_2 x_i^2 + \cdots + a_{n-1} x_i^{n-1} = 0$$

- Since x_i are distinct, the corresponding polynomial has n distinct roots.

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

- But the fundamental theorem of algebra guarantees that if $P(x)$ is not the zero polynomial, then $P(x)$ of degree $n-1$ can have at most $n-1$ roots.
- Therefore the only solution to

$$\mathbf{Va} = \mathbf{0} \qquad \text{is the trivial solution.}$$

- Hence any Vandemonde matrix is invertible for n distinct x, and since

$$\mathbf{Va} = a_0\mathbf{v}_0 + a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_{n-1}\mathbf{v}_{n-1} = \mathbf{0},$$

  the columns of $\mathbf{V}$ forms a linearly independent set.
- Using the same argument, we can show that

$$1, \quad x, \quad x^2, \ldots, x^n$$

  form a linearly independent set in $\mathcal{P}_n$.

Exercise

Determine whether the polynomials

$$\mathbf{p}_1 = 1 - x, \quad \mathbf{p}_2 = 5 + 3x - 2x^2, \quad \mathbf{p}_3 = 1 + 3x - x^2$$

are linearly dependent or linearly independent in $\mathcal{P}_2$.

Solution

- Linearly independent only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$ is the only way for all $x \in \mathbb{R}$

$$\mathbf{0} = \alpha_1 \mathbf{p}_1 + \alpha_2 \mathbf{p}_2 + \alpha_3 \mathbf{p}_3$$
$$0 = \underbrace{(\alpha_1 + 5\alpha_2 + \alpha_3)}_{0} + \underbrace{(-\alpha_1 + 3\alpha_2 + 3\alpha_3)}_{0} x + \underbrace{(-2\alpha_2 - \alpha_3)}_{0} x^2$$

- Only the zero polynomial satisfies it, see whether all $\alpha = 0$ is the only way

$$\begin{bmatrix} 1 & 5 & 1 \\ -1 & 3 & 3 \\ 0 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \implies$$

Hence $\alpha_1 = \alpha_2 = \alpha_2 = 0$ is not the only solution, so $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are linearly dependent.

Exercise

Determine whether the following vectors

$$\mathbf{f}_1 = \sin^2 x, \quad \mathbf{f}_2 = \cos^2 x, \quad \mathbf{f}_3 = 5$$

are linearly dependent or linearly independent.

Solution

- Using the trigonometric identity

$$\sin^2 x + \cos^2 x = 1$$

- We have

$$5\mathbf{f}_1 + 5\mathbf{f}_2 - \mathbf{f}_3 = \mathbf{0}$$

- Therefore they are linearly dependent.

- It is rare that linear independence or dependence of arbitrary functions can be determined by algebraic methods.
- There is a theorem that can be useful when the functions are differentiable. It uses a special function known as the Wronskian.

Definition

Let

$$\mathbf{f}_1 = f_1(x), \qquad \mathbf{f}_2 = f_2(x), \qquad \cdots, \mathbf{f}_n = f_n(x)$$

be functions that are $n-1$ times continuously differentiable for $x \in \mathbb{R}$, then

$$W(x) = \det\left(\begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix}\right)$$

is called the Wronskian of f_1, f_2, ..., f_n.

- Suppose that $\mathbf{f}_1, \mathbf{f}_2, \cdots \mathbf{f}_n$ are linearly dependent vectors in $\mathcal{C}^{n-1}(-\infty, \infty)$.
- This implies that the vector equation

$$\alpha_1 \mathbf{f}_1 + \alpha_2 \mathbf{f}_2 + \cdots + \alpha_n \mathbf{f}_n = \mathbf{0}$$

  is satisfied by values of the coefficients α_1, α_2, $\ldots$, α_n that are not all zero.
- This also implies that for these coefficients α_i's the equation

$$\alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_n f_n(x) = 0$$

  is satisfied for all $x \in \mathbb{R}$.
- If the functions are $n-1$ times differentiable, then we have the system

$$\begin{array}{ccccccccc}
\alpha_1 f_1(x) & + & \alpha_2 f_2(x) & + & \cdots & + & \alpha_n f_n(x) & = & 0 \\
\alpha_1 f_1'(x) & + & \alpha_2 f_2'(x) & + & \cdots & + & \alpha_n f_n'(x) & = & 0 \\
 & & \vdots & & & \vdots & & & \\
\alpha_1 f_1^{(n-1)}(x) & + & \alpha_2 f_2^{(n-1)}(x) & + & \cdots & + & \alpha_n f_n^{(n-1)}(x) & = & 0
\end{array}$$

- Thus, the linear dependence of $\mathbf{f}_1, \mathbf{f}_2, \cdots \mathbf{f}_n$ implies that the linear system

$$\begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

  has a non-trivial solution for every x in the interval $(-\infty, \infty)$.
- This in turn implies that the determinant of the coefficient matrix $W(x)$, the Wronskian, is zero for every such x.

Theorem

If the functions $\mathbf{f}_1, \mathbf{f}_2, \cdots, \mathbf{f}_n$ have $n-1$ continuous derivatives on $(-\infty, \infty)$, and if the Wronskian of these functions is not identically zero on $(-\infty, \infty)$, then these functions form a linearly independent set of vectors in $\mathcal{C}^{n-1}(-\infty, \infty)$.

Q: Can we conclude anything regarding the set of differentiable functions if

$$W(x) = 0 \qquad \text{for all} \quad x$$

Exercise

Use the Wronskian to show that

$$1, \quad e^x, \quad \text{and} \quad e^{2x}$$

are linearly independent vectors.

Solution

- Compute the Wronskian,

$$W(x) = \det \begin{bmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{bmatrix} = 2e^{3x}$$

- Since exponential function is never zero,

$$1, e^x, e^{2x}$$

  are linearly independent.