Vv255 Lecture 12

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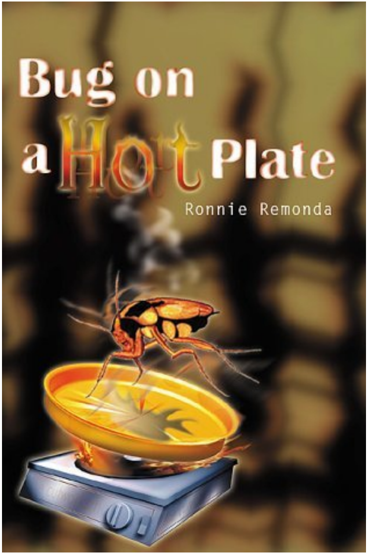
- What would you do if you were a happy heat-loving bug on a hot plate?



- Let $z = f(x, y)$ be differentiable, and $\mathcal{C}$ be a smooth curve defined by

$$x = x(t) \qquad \text{and} \qquad y = y(t)$$

then the rate at which f changes with respect to t along $\mathcal{C}$ is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

and is known as the total derivative of f with respect to t on $\mathcal{C}$.

Q: What does $\dfrac{df}{dt}$ represent if t is actually some time parameter?

Q: What does $\dfrac{df}{dt}$ represent if t is actually the arc length s?

- To address the 2nd question, let $\mathcal{C}$ be a straight line in the direction of

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

which passes through a point $P_0(x_0, y_0)$.

- For a given $\mathbf{v} = v_1\mathbf{e}_x + v_2\mathbf{e}_y$ and a point $P_0(x_0, y_0)$, we have the following

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} = (x_0 + tv_1)\mathbf{e}_x + (y_0 + tv_2)\mathbf{e}_y$$

- A simple computation reveals

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = v_1\frac{\partial f}{\partial x} + v_2\frac{\partial f}{\partial y}$$

- Now suppose $\mathbf{v} = 2\mathbf{e}_x + 0\mathbf{e}_y$, then

$$\frac{df}{dt} = 2\frac{\partial f}{\partial x}$$

Q: What does it tell us? How can we tell whether t is the arc length parameter?

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

Q: How can we extend the definition of partial derivatives so that we have a rate of change of f in any direction in the domain of f, not just along the axes?

Definition

The rate of change of f at (x_0, y_0) in the direction of $\mathbf{v}$ is defined as

$$\lim_{h\to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h},$$

where $\mathbf{u} = u_1\mathbf{e}_x + u_2\mathbf{e}_y$ is the unit vector in the direction of $\mathbf{v}$, that is,

$$\mathbf{u} = \hat{\mathbf{v}}$$

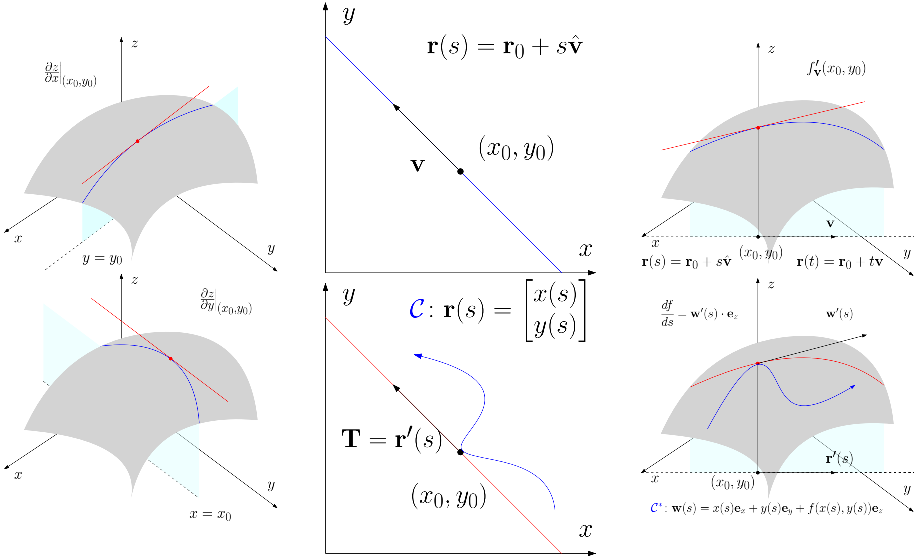
This rate of change is known as a directional derivative, and often denoted by

$$\mathrm{D}_{\mathbf{v}}\, f(x_0, y_0) = f'_{\mathbf{v}}(x_0, y_0)$$

Q: Why $f'_{\mathbf{v}}$ is essentially a special case of the total derivative $\dfrac{df}{ds}$?

Q: Back to arbitrary $\mathcal{C}$, how can we interpret the total derivative geometrically

$$\frac{df}{dt} = \frac{df}{ds}\frac{ds}{dt}$$


z
$\frac{\partial z}{\partial x}\Big|_{(x_0,y_0)}$
x
y
$y = y_0$
z
$\frac{\partial z}{\partial y}\Big|_{(x_0,y_0)}$
x
y
$x = x_0$
y
$\mathbf{r}(s) = \mathbf{r}_0 + s\hat{\mathbf{v}}$
(x_0, y_0)
$\mathbf{v}$
x
y
$\mathcal{C}: \mathbf{r}(s) = \begin{bmatrix} x(s) \\ y(s) \end{bmatrix}$
$\mathbf{T} = \mathbf{r}'(s)$
(x_0, y_0)
x
z
$f'_{\mathbf{v}}(x_0, y_0)$
$\mathbf{v}$
x
(x_0, y_0)
y
$\mathbf{r}(s) = \mathbf{r}_0 + s\hat{\mathbf{v}}$
$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$
z
$\frac{df}{ds} = \mathbf{w}'(s) \cdot \mathbf{e}_z$
$\mathbf{w}'(s)$
$\mathbf{r}'(s)$
x
(x_0, y_0)
y
$\mathcal{C}^*: \mathbf{w}(s) = x(s)\mathbf{e}_x + y(s)\mathbf{e}_y + f(x(s), y(s))\mathbf{e}_z$

Exercise

Use the definition to find the rate of change of the following function at $(1, 2)$

$$f(x, y) = x^2 + xy$$

in the direction of $\mathbf{v} = 1/\sqrt{2}\mathbf{e}_x + 1/\sqrt{2}\mathbf{e}_y$.

- The definitions is not user-friendly, in practice, we use the chain rule instead
- Since the following equations parametrize the line through (x_0, y_0)

  $$x = x_0 + su_1, \qquad y = y_0 + su_2$$

  with the arc length parameter s in the direction of the unit vector

  $$\hat{\mathbf{v}} = \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

- The directional derivative is the total derivative along a straight line

  $$f'_{\mathbf{v}} = \frac{df}{ds}$$

- By the chain rule of one independent variable and two intermediate variables,

$$
\begin{aligned}
f'_{\mathbf{v}}(x_0, y_0) &= \frac{df}{ds} \\
&= \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} \\
&= \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 \qquad \text{If we use the standard unit vectors } \mathbf{e}_i, \\
&= \left(\frac{\partial f}{\partial x}\mathbf{e}_x + \frac{\partial f}{\partial y}\mathbf{e}_y\right) \cdot (u_1\mathbf{e}_x + u_2\mathbf{e}_y) \\
&= \mathbf{w} \cdot \mathbf{u} \qquad\qquad \text{where} \quad \mathbf{w} = \frac{\partial f}{\partial x}\mathbf{e}_x + \frac{\partial f}{\partial y}\mathbf{e}_y
\end{aligned}
$$

- So the directional derivative of f in the direction of $\mathbf{v}$ is the dot product of $\mathbf{u} = \hat{\mathbf{v}}$ with a vector of partial derivatives of f.

Q: What does a dot between any vector and a unit vector represent?

Definition

The gradient vector (or simply gradient) of $f(x, y)$ at a point (x_0, y_0) is the vector

$$\nabla f = \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \\ \dfrac{\partial f}{\partial y} \end{bmatrix} = \frac{\partial f}{\partial x}\mathbf{e}_x + \frac{\partial f}{\partial y}\mathbf{e}_y$$

obtained by evaluating the partial derivatives of f at (x_0, y_0).

In general, the gradient of a function of several variables is

$$\nabla f = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} \\ \vdots \\ \dfrac{\partial f}{\partial x_n} \end{bmatrix} = \frac{\partial f}{\partial x_1}\mathbf{e}_1 + \cdots + \frac{\partial f}{\partial x_n}\mathbf{e}_n$$

Q: What does a gradient vector tell us about f?

Theorem

If f is differentiable in an open region, then the derivative f in the direction of $\mathbf{v}$,

$$f'_{\mathbf{v}} = \hat{\mathbf{v}} \cdot \nabla f$$

where ∇f is the gradient of f.

Exercise

(a) Find the derivative of

$$g(x, y) = x^2 + xy$$

at $(1, 2)$ in the direction of the unit vector $\mathbf{u} = 1/\sqrt{2}\mathbf{e}_x + 1/\sqrt{2}\mathbf{e}_y$.

(b) Find the derivative of

$$f(x, y) = xe^y + \cos(xy)$$

at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{e}_x - 4\mathbf{e}_y$.

```
>> v = [ 3 -4];
% Since v is not unit.
>> u = v/norm(v)

% Define x, y and f(x,y) symbolically
>> syms x y real
>> f_sym = x*exp(y) + cos(x*y);

% Find the symbolic gradient of f
>> gradf=jacobian(f_sym,[x,y]);

% Evaluation at it at (2,0)
>> subs(gradf, [x,y] ,[2,0])

>> dot(u, ans)

ans =      -1
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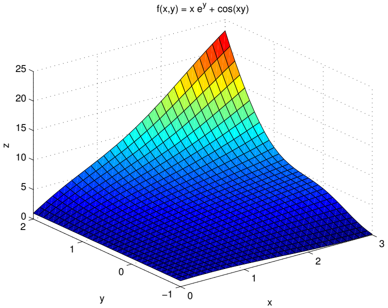

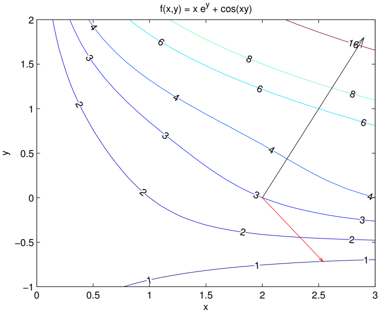


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>> f = inline('x.*exp(y) + cos(x.*y)','x','y');

>> [x, y] = meshgrid((0:0.1:3),(-1:0.1:2)); z = f(x,y);

>> surf(x,y,z); xlabel('x'); ylabel('y'); zlabel('z'); title('f(x,y) = x e^y + cos(xy)');

>> [k,h]=contour(x,y,z,[0,1,2,3,4,6,8,16,32]);
>> clabel(k,h)

>> xlabel('x'); ylabel('y'); title('f(x,y) = x e^y + cos(xy)');

>> fx=@(x,y) eval(vectorize(gradf(1)));
>> fy=@(x,y) eval(vectorize(gradf(2)));

>> hold on; quiver(2, 0, fx(2,0), fy(2,0) ,'color','black'); quiver(2, 0, 3/5, -4/5 ,'color','r'); hold off
```

- The dot product in terms of the following formula again reveals some insight

$$f'_{\mathbf{v}} = \hat{\mathbf{v}} \cdot \nabla f = |\nabla f||\hat{\mathbf{v}}| \cos\theta = |\nabla f| \cos\theta$$

  where θ is the angle between the vectors $\mathbf{v}$ and ∇f.

- The function f increases most rapidly when $\cos\theta = 1$, that is,

  when $\mathbf{v}$ is in the direction of the gradient vector ∇f.

- And the maximum rate of change of f at any point in the domain is

$$f'_{\nabla f} = |\nabla f|$$

- Similarly, f decreases most rapidly in the direction of $-\nabla f$,

$$f'_{-\nabla f} = -|\nabla f|$$

Q: In which direction is the function f neither increasing nor decreasing?

Exercise

Find the direction of zero change in $f(x, y) = \dfrac{x^2}{2} + \dfrac{y^2}{2}$ at $(1, 1)$?

- If a differentiable function $f(x, y)$ takes a constant value c along a smooth

$$\mathcal{C}\colon \mathbf{r} = x(t)\mathbf{e}_x + y(t)\mathbf{e}_y$$

then the graph of $\mathbf{r}$ is a level curve of the function f, and

$$f(x(t), y(t)) = c$$

- Differentiating both sides of this equation with respect to t,

$$\begin{aligned}
\frac{d}{dt}\Big[f(x(t), y(t))\Big] &= \frac{dc}{dt} \\
\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} &= 0 \\
\underbrace{\left(\frac{\partial f}{\partial x}\mathbf{e}_1 + \frac{\partial f}{\partial y}\mathbf{e}_2\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dx}{dt}\mathbf{e}_x + \frac{dy}{dt}\mathbf{e}_y\right)}_{\mathbf{r}'} &= 0
\end{aligned}$$

Q: It means ∇f is orthogonal to the tangent vector $\mathbf{r}'$, but is the same as $\mathbf{N}$?

Theorem

At any point P_0 in the domain of a differentiable function $f(x, y)$, the gradient of f is normal to the level curve through P_0.

- This enables us to find an equation for the tangent line to level curves.
- Every vector $\vec{P_0P}$ on the tangent line through a point P_0 is orthogonal to

$$\text{the gradient } \nabla f \text{ evaluated at } P_0.$$

- Thus the dot product must be zero,

$$\vec{P_0P} \cdot (\nabla f)_{P_0} = 0 \qquad \text{where} \quad \vec{P_0P} = \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} \quad \text{is on the line.}$$

- The scalar form,

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0$$

Exercise

(a) Find an equation for the tangent to the ellipse at the point $(-2, 1)$.

$$\frac{x^2}{4} + y^2 = 2$$

(b) Find the derivative of

$$f(x, y, z) = x^2 - xy^2 - z$$

at $(1, 1, 0)$ in the direction of $\mathbf{v} = 2\mathbf{e}_x - 3\mathbf{e}_y + 6\mathbf{e}_z$.

(c) Find the equations of the tangent plane and normal line at $(2, 1, -3)$ to

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

- If we know the gradients of two functions f and g, we automatically know the gradient of the sum, difference, constant multiple, product, and quotient.

Algebra Rules for Gradients

- Sum:

$$\nabla(f \pm g) = \nabla f \pm \nabla g$$

- Constant Multiple:

$$\nabla(\alpha f) = \alpha \nabla f \qquad \text{for any real number } \alpha.$$

- Product:

$$\nabla(fg) = f\nabla g + g\nabla f$$

- Quotient:

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$$