Vv156 Lecture 12

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- Imagine you fire a rifle straight up, assuming gravity is the only force present

Q: What can be said regarding the motion of the bullet?

- Under the usual circumstances, we expect the motion is smooth.
- And we expect the bullet goes up, and stop momentarily in the air, then comes down hitting you in the eye.

Q: Why the bullet must stop momentarily before returning?

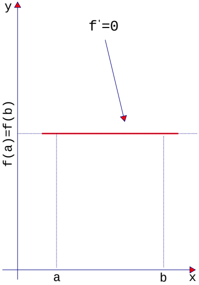
- What seems to be a trivial truth here is the essence of a principle called

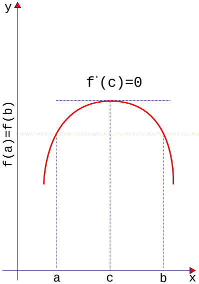
Rolle's Theorem

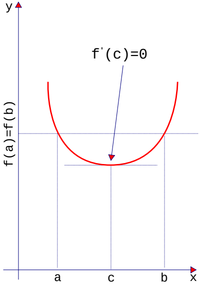
Let f be a function that satisfies the following three hypotheses:

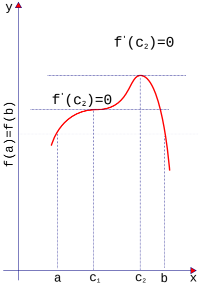
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

1. 
y
$f'=0$
$f(a)=f(b)$
a
b
x

2. 
y
$f'(c)=0$
$f(a)=f(b)$
a
c
b
x

3. 
y
$f'(c)=0$
$f(a)=f(b)$
a
c
b
x

4. 
y
$f'(c_2)=0$
$f'(c_2)=0$
$f(a)=f(b)$
a
c_1
c_2
b
x

Proof

- If $f(x) = k$, where k is a constant, then $f'(x) = 0$, so the number c can be taken to be any number in (a, b).
- If $f(x) > f(a)$ for some x in (a, b), then by EVT,

  $$\text{Hypothesis } 1 \implies f \text{ has a maximum value somewhere in } [a, b]$$

  Since $f(a) = f(b)$, so $f(x) > f(b)$, and $f(a)$ and $f(b)$ cannot be the maximum, it must attain this maximum value at a number c in the open interval (a, b).
- Therefore f has a relative maximum at c, and

  $$\text{Hypothesis } 2 \implies f'(c) = 0$$

- If $f(x) < f(a)$ for some x in (a, b), then the argument is very similar to the increasing case, the only difference is that we have a relative minimum instead of a relative maximum.

□

- Imagine that you are driving a Lykan for an hour, your average speed during this time is 51km/h. Suppose the speed limit is 50km/h.

Q: How can a policeman argue that you have been speeding and give you a ticket?

The Mean-Value theorem

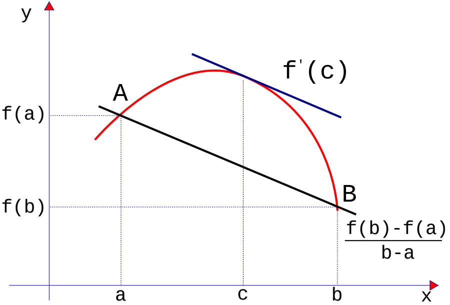
Let f be a function that satisfies the following conditions

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

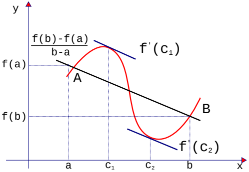
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- The Mean-Value Theorem (MVT) is stating that there is a number at which



the instantaneous rate of change is equal to the average rate of change.

- Notice it did not say the number is unique.



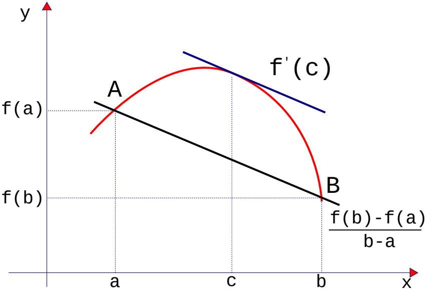
Proof

- Notice when

$$\frac{dy}{dx} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a},$$

- The equation of the segment is

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$



- The vertical distance between the curve and the segment is given by

$$h(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a)\right]$$

- This function $h(x)$ satisfies the three hypotheses of Rolle's Theorem.

Proof

- The function h is continuous on $[a, b]$ because it is the sum of f and a first-degree polynomial, both of which are continuous.
- The function h is differentiable on (a, b) because both f and the first-degree polynomial are differentiable, and the derivative is

$$h' = f'(x) - \frac{f(b) - f(a)}{b - a}$$

- lastly, we need to verify that $h(a) = h(b)$. This is clearly true since the vertical distance is zero at both ends.
- Therefore we can apply Rolle's theorem, which states that there is a number c in (a, b) such that $h'(c) = 0$, thus

$$0 = h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$
$$\implies f'(c) = \frac{f(b) - f(a)}{b - a} \quad \square$$

Theorem

- Suppose f is continuous on $[a, b]$ and differentiable on (a, b), and
  1. if $f'(x) > 0$ for every value of x in (a, b), then f is increasing on $[a, b]$.
  2. if $f'(x) < 0$ for every value of x in (a, b), then f is decreasing on $[a, b]$.

Proof

- For 1, suppose that C_1 and C_2 are points in $[a, b]$ such that $C_1 < C_2$, then we must show that $f(C_1) < f(C_2)$ if $f'(x) > 0$ for x in (a, b).
- Both requirements of MVT are satisfied, thus

$$f'(c) = \frac{f(C_2) - f(C_1)}{C_2 - C_1}, \qquad \text{where } c \text{ is a number in } (C_1, C_2) \text{ by MVT.}$$

- Therefore,

$$f'(x) > 0 \text{ for } x \text{ in } (a, b) \implies f'(c) > 0 \implies f(C_2) - f(C_1) > 0$$

- For 2 can be proved in a similar fashion. □

Exercise

(a) Prove the identity

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

(b) Prove the following inequality

$$|\sin x - \sin y| \leq |x - y|, \qquad \textit{where} \quad x, y \in \mathbb{R}$$

(c) Find all the real solutions to the equation

$$2^x + 5^x = 3^x + 4^x$$