Vv156 Lecture 2

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Dictionary

"A set of related events, movements, or things
that follow each other in a particular order."

- We will be interested in sequences of a more mathematical nature;

  Mostly numbers, occasionally points, and functions in the end.

- John picks colored marbles from a bag, first he picks a red marble, then a blue one, another blue one, a violet one, a red one and finally a blue one.
- The sequence of marbles he has chosen could be represented by

$$1, 2, 2, 3, 1, 2$$

  where 1, 2, 3 stand for red, blue and violet respectively.

- Infinite sequence, such as the Fibonnaci sequence, is our primary interest

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

Notation I

- The notation for representing a general infinite sequence is:

$$a_1, a_2, a_3, \ldots$$

where a_1 represents the first element in the sequence, and a_2 the second, etc.

- If we wish to discuss a term in a sequence without being specific, we write

$$a_i \qquad \text{or} \qquad a_n$$

- To represent a finite sequence, we would write

$$a_1, a_2, a_3, \ldots, a_{29} \qquad \text{or} \qquad a_1, a_2, a_3, \ldots, a_N$$

where the later represent a series that is finite but of some unknown length N.

Notation II

- A more compact way of representing the general infinite sequence is

$$\{a_i\}_{i=1}^{\infty}.$$

- The finite sequence $a_1, a_2, \ldots, a_N$ is similarly represented by:

$$\{a_i\}_{i=1}^{N}.$$

- Since we study mostly infinite sequences, we will often abbreviate further

$$\{a_i\}_{i=1}^{\infty} \quad \text{with simply} \quad \{a_i\} \quad \text{or} \quad \{a_n\}.$$

- It looks like set notation, but you should be careful not to confuse a sequence with the set whose elements are the entries of the sequence.

1. A set has no particular ordering of its elements but a sequence certainly does.
2. Each element of a set must be unique, but terms of a sequence need not be.

Specification

- There are three ways to specify a particular sequence:

    1. For a (short) finite sequence, one can simply list the terms in order.

$$1, 2, 4, 8, 16, 32, 64$$

    2. A better method is to define a sequence with an explicit formula.

$$a_n = 2^{n-1}$$

- However, an explicit formula for many sequences are hard to obtain.

    3. A third way of describing a sequence is through a recursive formula.

$$a_1 = 1, \qquad a_{n+1} = 2a_n$$

it describes the nth term of the sequence in terms of previous terms.

Mathematical Induction

- The principle of Mathematical Induction

  It is often used to prove a given statement for all natural numbers

1. The Base Case:

   Prove the desired result for a certain n.

2. The Inductive Step:

   Prove that if the result is true for n, then it is also true for $n+1$.

Exercise

Let $\{a_n\}$ be the sequence defined recursively by

$$a_{n+1} = a_n + (n+1), \qquad \textit{and} \qquad a_1 = 1$$

Prove that in general the explicit formula for the sequence is given by

$$a_n = \frac{n(n+1)}{2}$$

Harry and Hermione playing a game

- Two people sit facing each other in a room, Harry and Hermione.
- Two consecutive natural numbers are written on their foreheads, one on each.
- Harry and Hermione both know the number that's not their own.
- They also both know that the two numbers are consecutive.
- The game proceeds with one player, say Harry, asking Hermione if she knows what her number is. If she does, she says so and the game ends.
- If not, Harry's turn ends and Hermione gets her chance to ask him the same question. As before, if he does then he says so and the game ends.
- Otherwise, back to Harry asking the same question to Hermione.
- This back and forth questioning continues until someone says "Yes", if ever.
- We assume that the two players are honest, and they are "perfect reasoners", so that if there was some way for either of them at any point to deduce their own number then they would do it.

Q: Does this game ever end?

- Some sequences of real numbers get closer and closer to a single number

$$L$$

while other sequences exhibit no such behaviour.

Definition

If a sequence does indeed get closer and closer to a number L, then the sequence is said to be convergent and we say it has a limit of L. We will write

$$\lim_{n \to \infty} a_n = L$$

Alternatively, we denote $\{a_n\}$ converges to L using the following notation

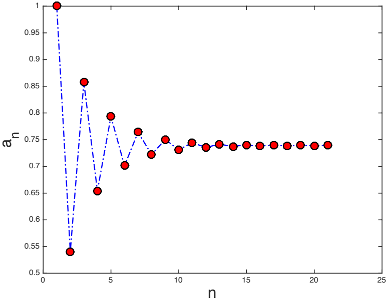
$$a_n \to L \quad \text{as} \quad n \to \infty$$

If a sequence is not convergent, it is called divergent.

- The behaviour of a sequence can be investigated graphically. For example,

$$a_{n+1} = \cos(a_n), \qquad a_1 = 1$$

Q: Does it grow indefinitely? Is it periodic or does it have anything special feature?



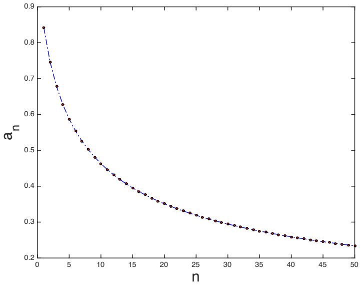
Matlab

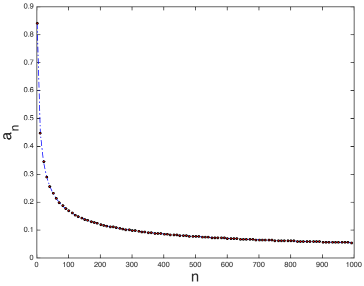
```
>> a(1) = 1;
>> for n = 1:20
>>  a(n+1) = cos(a(n));
>> end
>> plot(a, '-.bo', ...
>> 'LineWidth', 1.5, ...
>> 'MarkerFaceColor', 'r', ...
>> 'MarkerEdgeColor', 'black', ...
>> 'MarkerSize', 10);
>> xlabel('n','fontsize', 20);
>> ylabel('a_n','fontsize', 20);
```

- It seems the terms are getting closer and closer to a single number.

Q: How about the following sequence?

$$a_{n+1} = \sin(a_n), \qquad a_1 = \sin(1)$$



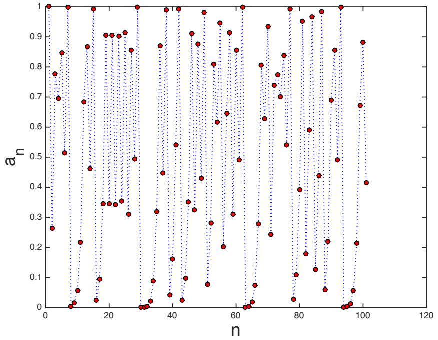


Matlab

```
>> N = 1000; step = 10; a(1) = sin(1);
>> for n = 1:N
>>  a(n+1) = sin(a(n));
>> end;
>> plot(1:step:N, a(1:step:N), '-.bo', 'LineWidth', 1, 'MarkerFaceColor', 'r', ...
>> 'MarkerEdgeColor', 'black', 'MarkerSize', 3); xlabel('n', 'fontsize', 20); ylabel('a_n', 'fontsize', 20);
```

- The following plot is for the sequence with the explicit formula

$$a_n = \sin^2\left(2^n \pi \sqrt{2}\right)$$



Matlab

```
>> a(1) = 1;
>> for n = 1:100
>>  a(n+1) = sin(sqrt(2)*pi*2^n)^2 ;
>> end
>> plot(a, ':bo', ...
>> 'LineWidth', 1, ...
>> 'MarkerFaceColor', 'r', ...
>> 'MarkerEdgeColor', 'black', ...
>> 'MarkerSize', 5);
>> xlabel('n','fontsize', 20);
>> ylabel('a_n','fontsize', 20);
```

- It seems there is no number that the terms are getting closer and closer.