Vv255 Lecture 11

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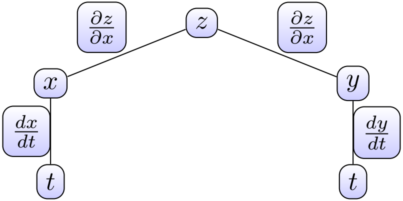
- Recall the Chain Rule for differentiable functions of a single variable $f(x)$,

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}, \qquad \text{where } x = x(t) \text{ is a differentiable function of } t.$$

- For functions of two or more variables the Chain Rule has several forms.
- If there are 1 independent variable t and 2 intermediate variables x and y.



Theorem

Let $z = f(x, y)$ be a differentiable function of x and y, $x = x(t)$ and $y = y(t)$ are differentiable functions of t, then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Proof

- We need to show that if x and y are differentiable at $t = t_0$, then

$$z = f(t) \text{ is differentiable at } t_0,$$

and

$$\left(\frac{dz}{dt}\right)_{t_0} = \left(\frac{\partial z}{\partial x}\right)_{P_0}\left(\frac{dx}{dt}\right)_{t_0} + \left(\frac{\partial z}{\partial y}\right)_{P_0}\left(\frac{dy}{dt}\right)_{t_0}, \text{where } P_0 = (x(t_0), y(t_0))$$

- The subscripts indicate where each of the derivatives is to be evaluated.
- Recall, for a differential function $z = f(x, y)$, we have found

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \gamma\Delta x + \varepsilon_x(\Delta x) + \varepsilon_y(\Delta y)$$

- Dividing Δt on both sides,

$$\frac{\Delta z}{\Delta t} = f_x(x, y)\frac{\Delta x}{\Delta t} + f_y(x, y)\frac{\Delta y}{\Delta t} + \frac{\gamma\Delta x + \varepsilon_x(\Delta x) + \varepsilon_y(\Delta y)}{\Delta t}$$

Proof

- Clearly Δx and Δy approach zero as $\Delta t \to 0$, so if we let $\Delta t \to 0$

$$\frac{\Delta z}{\Delta t} = f_x(x_0, y_0)\frac{\Delta x}{\Delta t} + f_y(x_0, y_0)\frac{\Delta y}{\Delta t} + \frac{\gamma \Delta x + \varepsilon_x(\Delta x) + \varepsilon_y(\Delta y)}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = f_x(x_0, y_0) \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + f_y(x_0, y_0) \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}$$
$$+ \lim_{\Delta t \to 0} \frac{\gamma \Delta x + \varepsilon_x(\Delta x) + \varepsilon_y(\Delta y)}{\Delta t}$$

- Since f is differentiable in terms of x and y, which in turn are differentiable functions of t, all the limits exist, thus $z = f(t)$ is differentiable at $t = t_0$.
- And the derivative is equal to

$$\frac{dz}{dt} = \left(\frac{\partial z}{\partial x}\right)_{P_0} \left(\frac{dx}{dt}\right)_{t_0} + \left(\frac{\partial z}{\partial y}\right)_{P_0} \left(\frac{dy}{dt}\right)_{t_0} + 0 \quad \square$$

Exercise

(a) Use the Chain Rule to find the derivative of w with respect to t,

$$w = f(x, y) = xy$$

along the path $x = \cos t$, $y = \sin t$. What is the derivative's value at $t = \dfrac{\pi}{2}$?

- Let $z = f(x, y)$ and $y = g(x)$, then x is the only independent variable and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\Longrightarrow \frac{dz}{dx} = \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

Exercise

(b) Find the derivative $\dfrac{dz}{dx}$ for $z = x \ln(xy) + y^3$, where $y = \cos\left(x^2 + 1\right)$.

- Suppose the temperature on earth is described by the following function

$$T = f(x, y, z), \qquad \text{where } x,\ y \text{ and } z \text{ are coordinates in space.}$$

- We might prefer to think of x, y, and z as functions of

longitudes r and latitudes s

since most of us are only interested on the earth's surface.

- Suppose

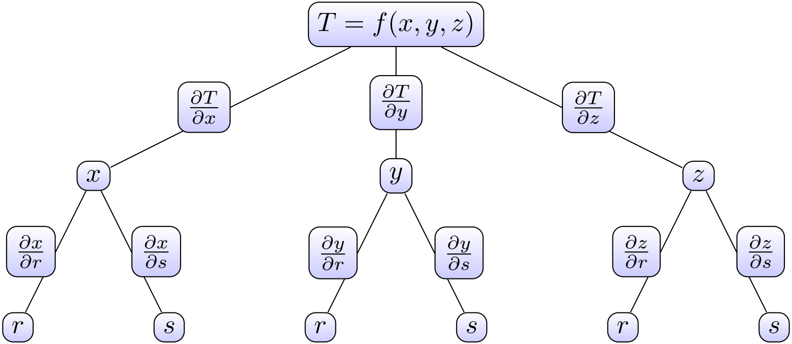
$$x = g(r, s), \qquad y = h(r, s), \qquad \text{and} \qquad z = k(r, s),$$

we then could then express the temperature T as a function of r and s

$$T = f(g(r, s), h(r, s), k(r, s)) = f(r, s)$$

and investigate rate of change of temperature with respect to r and s.

- Here we have 2 independent variables and 3 intermediate variables



Theorem

- Given $f(x, y, z)$, $x(r, s)$, $y(r, s)$, and $z(r, s)$ are differentiable functions, then the partial derivatives of f with respect to r and s exist, and are given by

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r} \quad \text{and} \quad \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

Exercise

(a) Find $\dfrac{\partial f}{\partial r}$ and $\dfrac{\partial f}{\partial s}$ in terms of r and s if

$$f = x + 2y + z^2, \qquad x = \frac{r}{s}, \qquad y = r^2 + \ln s, \qquad z = 2r.$$

(b) Let a, b, and c denote the lengths of the edges of a rectangular box. Suppose the lengths are changing with time, at the instant in question,

$$a = 4\text{m},\ b = 3\text{m},\ c = 2\text{m},$$

$$\frac{da}{dt} = 1\text{m/sec},\ \frac{db}{dt} = -2\text{m/sec, and } \frac{dc}{dt} = 1\text{m/sec}.$$

Are the interior diagonals increasing in length or decreasing at that instant?

Theorem

Suppose $h(t,x)$ has continuous partial derivatives, then

$$\frac{d}{dx}\int_a^b h(t,x)\,dt = \int_a^b \frac{\partial}{\partial x} h(t,x)\,dt$$

- Using this theorem and the chain rule, we have the Leibniz' integral rule

$$F'(x) = h(g(x),x)g'(x) - h(f(x),x)f'(x) + \int_{f(x)}^{g(x)} h_x(t,x)\,dt$$

which is a formula for finding the derivative for functions of the form

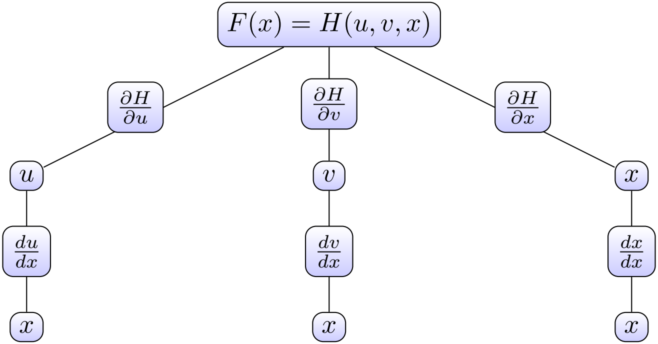
$$F(x) = \int_{f(x)}^{g(x)} h(t,x)\,dt$$

- It is particularly useful when comes to investigate convolution functions, e.g.

$$y(x) = \int_0^x e^{x-t} f(t)\,dt \qquad \text{and} \qquad y(x) = \int_0^x \sin(x-t) f(t)\,dt$$

- To understand why it works, let us introduce two middle variables u and v,

$$\underbrace{H(u, v, x)}_{F(x)} = \int_u^v h(t, x)\, dt, \qquad \text{where} \quad u = f(x), \quad \text{and} \quad v = g(x)$$



- Now the partial derivative of F with respect to x is

$$\frac{dF}{dx} = \frac{d}{dx}\Big(H(u, v, x)\Big) = H_u \frac{du}{dx} + H_v \frac{dv}{dx} + H_x \frac{dx}{dx}$$

Q: Why H_u and H_v are readily available?

$$\frac{d}{dx}\Big(H(u,v,x)\Big) = H_u\frac{du}{dx} + H_v\frac{dv}{dx} + H_x\frac{dx}{dx}, \quad \text{where } H = \int_u^v h(t,x)\, dt.$$

- Split the integral into two,

$$H(u,v,x) = \int_u^a h(t,x)\, dt + \int_a^v h(t,x)\, dt, \qquad \text{where } a \in \mathbb{R}.$$

- It is clear that

$$H_u = -h(u,x) \qquad \text{and} \qquad H_v = h(v,x)$$

- Together with the theorem on $\boxed{9}$, we have,

$$\begin{aligned} F'(x) &= -h(u,x)\frac{du}{dx} + h(v,x)\frac{dv}{dx} + \left(\int_u^v \frac{\partial}{\partial x}h(t,x)\, dt\right)\left(\frac{dx}{dx}\right) \\ &= h(g(x),x)g'(x) - h(f(x),x)f'(x) + \int_{f(x)}^{g(x)} h_x(t,x)\, dt \quad \square \end{aligned}$$

Exercise

(a) Find $\dfrac{dF}{dx}$ where $F(x) = \int_0^x \sin(x^2+t^2)\,dt$.

(b) Find $\dfrac{dy}{dx}$ where $y^2 - x^2 = \sin xy$.

- We can treat

$$y^2 - x^2 = \sin xy \iff y^2 - x^2 - \sin xy = 0$$

  as a function of x and y taking the value 0.
- That is, suppose $F(x,y) = y^2 - x^2 - \sin xy$, then

$$F(x,y) = 0 \iff y^2 - x^2 = \sin xy$$

- Since $F(x,y) = 0$ when $y = y(x)$ defined by the implicit equation, then

$$\frac{dF}{dx} = 0 \quad \text{when } y \text{ is defined by} \quad y^2 - x^2 - \sin xy = 0$$

Q: Notice $F'(x)$ is different from F_x, what does each of them represent?

- We can also find the derivative by the chain rule formula we found on $\boxed{5}$

$$\frac{dF}{dx} = F_x \frac{dx}{dx} + F_y \frac{dy}{dx} = F_x \cdot 1 + F_y \frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{F_x}{F_y}$$

Theorem

Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

- This result can be easily extended to three variables.
- If $F(x, y, z) = 0$ defines z implicitly as a function of x and y, then

$$F(x, y, f(x, y)) = 0$$

for all (x, y) in the domain of f

- Suppose that F and f are differentiable functions, we can use the chain rule to differentiate the equation $F(x, y, z) = 0$ with respect to x, we have

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$F_x \cdot 1 + F_y \cdot 0 + F_z \cdot \frac{\partial z}{\partial x} = 0$$

$$\Longrightarrow \; F_x + F_z \frac{\partial z}{\partial x} = 0 \; \Longrightarrow \; \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

- A similar calculation for differentiating with respect to y gives

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

- But there still is an important question that we have not addressed, that is

  When does an implicit equation actually define a differentiable function?

- The following theorem, which we cannot prove yet, states when it is valid,

Implicit Function Theorem

- Suppose F is a function of x, y and z, if the following conditions are satisfied

1. The partial derivatives F_x, F_y, and F_z are continuous throughout an open region R in space containing the point (x_0, y_0, z_0).
2. For some constant c, $F(x_0, y_0, z_0) = c$ and $F_z(x_0, y_0, z_0) \neq 0$, then

$$F(x, y, z) = c$$

defines z implicitly as a differentiable function of x and y near (x_0, y_0, z_0), and the partial derivatives of z are given by

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Exercise

Is z a differentiable function of x and y at $(0, 0, 0)$ if $x^3 + z^2 + ye^{xz} + z\cos y = 0$?