
Question1 (2 points)

Consider the following differential operator

$$p(\mathcal{D}) = \mathcal{D}^2 + 2\mathcal{D} - 15$$

(a) (1 point) Find the transfer function for

$$p(\mathcal{D})\,[y] = f(t); \qquad y(0) = 0, \quad \dot{y}(0) = 0$$

Solution:

1M For the operator $p(\mathcal{D})$, the transfer function is simply

$$H(s) = \frac{1}{p(s)} = \frac{1}{s^2 + 2s - 15}$$

(b) (1 point) Find the Green's function $G(t;0)$ associated with

$$p(\mathcal{D})\,[y] = f(t); \qquad y(0) = 0, \quad \dot{y}(0) = 0$$

Solution:

1M Recall the Green's function is the solution to

$$\ddot{y} + P\dot{y} + Qy = \delta(t-a)$$

In this case, we need the solution to

$$\ddot{y} + 2\dot{y} - 15y = \delta(t)$$

Recall the Laplace transform of the Dirac delta function

$$\mathcal{L}\big[\delta(t-a)\big] = e^{-sa} = 1 \quad \text{when} \quad a = 0$$

For constant coefficients, invoking the theorem on L18P8, we have

$$\begin{aligned} G(t;0) = y_c(t) + \big(h * \delta(t)\big)(t) &= 0 + \mathcal{L}^{-1}\big[H(s)\cdot 1\big] \\ &= \mathcal{L}^{-1}\big[H(s)\big] \\ &= \mathcal{L}^{-1}\left[\frac{1}{s^2+2s-15}\right] \\ &= \frac{1}{8}\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] - \frac{1}{8}\mathcal{L}^{-1}\left[\frac{1}{s+5}\right] \\ &= \frac{1}{8}\left(e^{3t} - e^{-5t}\right) = h(t) \end{aligned}$$

Notice the Green's function is essentially the inverse Laplace transform of the transfer function in this case. So once again, Green's function is a generalisation of what we have discussed regarding transfer function, which is only applicable for constant coefficients. Whereas the Green's function is applicable to variables coefficients. However, that would lead us to G in terms of power series or

Frobenius series. It is not exactly pretty! Another thing I would like to point out here is the convolution of a function with the Dirac delta function. Because, technically speaking, we have not defined such convolution since the Dirac delta function is definitely not a piecewise continuous function! The definition takes this integral being improper into account, just like the Laplace transform of $\delta(t)$

$$\delta(t) * h(t) = \int_{0^-}^{t} \delta(\tau)h(t-\tau)\, d\tau$$

It can be shown

$$\delta(t) * h(t) = h(t)$$

by either using the definition of the Dirac delta function

$$\delta(t) = \lim_{h\to 0} \delta_h(t)$$

or property 2 of the Dirac delta function on L17P9. Last comment is that both of the transfer function and the Green's function do not depend on $f(t)$.

Question2 (1 points)

Discuss whether the set of all functions $f\colon [a,b] \to \mathbb{R}$, where $a, b \in \mathbb{R}$, such that

$$\left(\int_a^b f^2\, dx\right) < \infty$$

is a vector space under the usual addition and scalar multiplication.

Solution:

1M Nothing about the field is mentioned, so we assume the question is about real scalars. The set under consideration is the set of all real-valued functions with the domain

$$[a,b]$$

Let us denote this set by $\mathcal{V}$. Under the usual addition and scalar multiplication, Axioms 1˜5 are satisfied since they are basic properties of real number. Axiom 6 is satisfied as well, since the real number 1 serves as the multiplicative identity.

$$1\mathbf{v} = 1f(x) = f(x) = \mathbf{v} \qquad \text{for any} \quad \mathbf{v} \in \mathcal{V}$$

Axiom 7 is satisfied because the zero function can serve as the additive identity

$$\mathbf{v} + \mathbf{0} = f(x) + 0(x) = f(x) = \mathbf{v} \qquad \text{for any} \quad \mathbf{v} \in \mathcal{V}$$

Axiom 8 is satisfied because there is a negative function for every f

$$\mathbf{v} + (-\mathbf{v}) = f(x) - f(x) = 0(x) = \mathbf{0}$$

For axiom 9, we have

$$\mathbf{v} + \mathbf{u} = f(x) + g(x)$$

Consider

$$(f-g)^2 \geq 0 \implies f^2+g^2 \geq 2fg$$

By the properties of definite integral, we have

$$\int_a^b \Big(f+g\Big)^2\,dx \leq \int_a^b 2\Big(f^2+g^2\Big)\,dx \leq 2\left(\int_a^b f^2\,dx + \int_a^b g^2\,dx\right) < \infty$$

Hence axiom 9 is satisfied. The property of definite integral shows, for $\alpha \in \mathbb{R}$,

$$\alpha\mathbf{v} = \alpha f(x) \implies \int_a^b \Big(\alpha f\Big)^2\,dx = \alpha^2\int_a^b f^2\,dx < \infty$$

axiom 10 is satisfied. Therefore the set $\mathcal{V}$ under the usual addition and scalar multiplication is a vector space over $\mathbb{R}$.

Question3 (1 points)

Determine whether the set of solutions to the following equation is a vector space.

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0, \qquad \text{where } g \text{ and } l \text{ are two real constants.}$$

Solution:

1M Nothing regarding addition and scalar multiplication is mentioned, so we assume the usual addition and scalar multiplication. Notice this is not a linear equation, so immediately you shall suspect the set is not closed under scalar multiplication. Let $\alpha \in \mathbb{R}$, and $\theta(t)$ is a solution to the differential equation, and consider

$$\begin{aligned}\frac{d^2}{dt^2}\Big(\alpha\theta(t)\Big) + \frac{g}{l}\sin\Big(\alpha\theta(t)\Big) &= \alpha\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\Big(\alpha\theta(t)\Big)\\ &= \alpha\left(\frac{d^2\theta}{dt^2} + \frac{1}{\alpha}\frac{g}{l}\sin\Big(\alpha\theta(t)\Big)\right) \neq 0\end{aligned}$$

since, in general,

$$\frac{1}{\alpha}\sin\Big(\alpha\theta\Big) \neq \sin\theta$$

Therefore it is not closed under scalar multiplication.

Question4 (1 points)

Given a matrix $\mathbf{M}$ of $n \times n$, and let

$$\mathcal{H} = \Big\{\mathbf{A} \mid \mathbf{AM} = \mathbf{MA}\Big\}$$

Show that $\mathcal{H}$ is a subspace of the vector space of all matrices of $n \times n$.

Solution:

1M The definition of $\mathcal{H}$ restricted the matrix in it be $n \times n$, otherwise the multiplication is undefined for at least one side of

$$\mathbf{AM} = \mathbf{MA}$$

So $\mathcal{H}$ is a subset of the vector space. It is also clear that $\mathcal{H}$ is nonempty since the zero matrix of $n \times n$ is in $\mathcal{H}$,

$$\mathbf{0M} = \mathbf{0} = \mathbf{M0}$$

thus we only need to show it is closed under addition and scalar multiplication. Let

$$\mathbf{B}, \mathbf{C} \in \mathcal{H} \qquad \text{and} \qquad \alpha \in \mathbb{R}$$

Consider

$$(\mathbf{B} + \mathbf{C})\,\mathbf{M} = \mathbf{BM} + \mathbf{CM} = \mathbf{MB} + \mathbf{MC} = \mathbf{M}\,(\mathbf{B} + \mathbf{C})$$

and

$$(\alpha\mathbf{B})\,\mathbf{M} = \alpha\,(\mathbf{BM}) = \alpha\,(\mathbf{MB}) = \mathbf{M}\,(\alpha\mathbf{B})$$

So $\mathcal{H}$ is a nonempty subset of the vector space that is closed under addition and scalar multiplication, therefore $\mathcal{H}$ is subspace of the vector space of all matrices of $n \times n$.

Question5 (1 points)
Find a basis for $\operatorname{span}(\mathcal{S})$ where $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ and

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_4 = \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}$$

Solution:

1M We could remove $\mathbf{v}_4$ from $\mathcal{S}$ without changing the subspace generated from $\mathcal{S}$ since

$$\mathbf{v}_4 = -2\mathbf{v}_1$$

and the following evaluation shows

$$\det\left(\begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 3 \\ 3 & 4 & 2 \end{bmatrix}\right) = 0$$

we could remove at least one more vector. Solving for the linear independency

$$\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3 = \begin{bmatrix} \alpha - \beta + 3\gamma \\ 2\alpha + \beta + 3\gamma \\ 3\alpha + 4\beta + 2\gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{aligned} \alpha - \beta + 3\gamma &= 0 \\ 2\alpha + \beta + 3\gamma &= 0 \\ 3\alpha + 4\beta + 2\gamma &= 0 \end{aligned}$$

we have

$$\alpha + 2\gamma = 0$$
$$\beta - \gamma = 0$$

Suppose $\gamma = 1$, then

$$\beta = 1 \qquad \text{and} \qquad \alpha = -2$$

that is,

$$-2\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0} \implies \mathbf{v}_3 = 2\mathbf{v}_1 - \mathbf{v}_2$$

So we can remove $\mathbf{v}_3$, and it is clear that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent since they are not scalar multiples to each other. Therefore

$$\{\mathbf{v}_1, \mathbf{v}_2\}$$

is a linearly independent spanning set for span $(\mathcal{S})$, and thus the basis.

Question6 (1 points)

Find the characteristics equation for an arbitrary 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Solution:

1M By definition, we have

$$\begin{aligned} p(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \det\left(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right) &= \lambda^2 - (a+d)\lambda + (ad - bc) = 0 \\ &= \lambda^2 - \lambda \operatorname{tr} \mathbf{A} + \det \mathbf{A} \end{aligned}$$

Question7 (1 points)

Compute $\det(\mathbf{A})$, where $\mathbf{A} = \begin{bmatrix} 3 & -7 & 8 & 9 & 6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$.

Solution:

1M Definition of determinant that I gave in class is known as the cofactor expansion along the the first row. In fact, the definition can be generalised to any row or any column, for example, in this case, it is most convenient to expand down the first column since there is only one nonzero element

$$\det \mathbf{A} = a_1 1 C_1 1 = 3 \cdot (-1)^{1+1} \cdot \det\left(\begin{bmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{bmatrix}\right)$$

Again expand down the first column of the minor

$$\det \mathbf{A} = 3 \cdot 2 \cdot (-1)^{1+1} \det \left(\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \right)$$

This time expand cross the last row of the minor

$$\det \mathbf{A} = 6 \cdot (-2) \cdot (-1)^{3+2} \det \left(\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \right) = 12 \cdot (-1 - 0) = -12$$

I will leave it to you to confirm the same value of $\det(\mathbf{A})$ will be obtained if the first row is used instead. The proof for this equivalence is provided in Vv214. You only need to know this result for Vv256.

Question8 (2 points)

Find the eigenvalues and the corresponding eigenvectors of the matrix $\mathbf{A} = \begin{bmatrix} 7 & 0 & -3 & 0 \\ -9 & -2 & 3 & 0 \\ 18 & 0 & -8 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Solution:

1M Finding the characteristic equation, we have

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= \det \left(\begin{bmatrix} 7-\lambda & 0 & -3 & 0 \\ -9 & -2-\lambda & 3 & 0 \\ 18 & 0 & -8-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix} \right) \\ &= (1-\lambda) \cdot (-1)^{4+4} \det \left(\begin{bmatrix} 7-\lambda & 0 & -3 \\ -9 & -2-\lambda & 3 \\ 18 & 0 & -8-\lambda \end{bmatrix} \right) \\ &= (1-\lambda)(-2-\lambda) \cdot (-1)^{2+2} \det \left(\begin{bmatrix} 7-\lambda & -3 \\ 18 & -8-\lambda \end{bmatrix} \right) \\ &= (\lambda - 1)^2 (\lambda + 2)^2 \end{aligned}$$

the eigenvalues are $\lambda_{1,2} = 1$ and $\lambda_{3,4} = -2$. Corresponding to $\lambda = 1$, we have

$$\mathbf{B} = \mathbf{A} - \mathbf{I} = \begin{bmatrix} 6 & 0 & -3 & 0 \\ -9 & -3 & 3 & 0 \\ 18 & 0 & -9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{B}\mathbf{x} = \mathbf{0} \implies \begin{aligned} 6x_1 - 3x_3 &= 0 \\ -9x_1 - 3x_2 + 3x_3 &= 0 \\ 18x_1 - 9x_3 &= 0 \end{aligned} \implies \begin{aligned} 2x_1 - x_3 &= 0 \\ -3x_1 - x_2 + x_3 &= 0 \end{aligned}$$

Suppose $x_1 = \alpha$, then $x_3 = 2\alpha$ and $x_2 = -\alpha$, and x_4 can be any real value, thus let it be β. Hence the eigenvector for $\mathbf{A}$ corresponding to $\lambda = 1$ must be in the form of

$$\mathbf{x} = \begin{bmatrix} \alpha \\ -\alpha \\ 2\alpha \\ \beta \end{bmatrix} = \alpha \underbrace{\begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}}_{\mathbf{v}_1} + \beta \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_2} \implies \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$$

Similarly, eigenvectors of $\mathbf{A}$ corresponding to $\lambda = 2$ can be any nonzero vector in

$$\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\} \qquad \text{where} \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Notice what I have given above, $\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_2\}$, for eigenvectors are not the only solution. They are merely two bases for the eigenspaces of $\mathbf{A}$ corresponding to $\lambda = 1$ and $\lambda = 2$. There are infinitely many bases for any given vector space.