Question1 (10 points)

(a) Derive $\mathcal{L}[f(t)]$ from the definition and state the interval of convergence.

i. (1 point) $f(t) = te^{4t}$.

ii. (1 point) $f(t) = \begin{cases} \sin t & \text{if } 0 \le t < \pi \\ 0 & \text{if } t \ge \pi. \end{cases}$

iii. (1 point) $f(t) = \sinh t$.

(b) One definition of the gamma function is given by the improper integral

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt, \qquad \alpha > 0$$

i. (1 point) Show $\Gamma(\alpha)$ converges for $0 < \alpha < 1$.

ii. (1 point) Show that

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha).$$

iii. (1 point) Show that

$$\mathcal{L}\left[t^\alpha\right] = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \qquad \alpha > -1.$$

iv. (1 point) Show that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

v. (1 point) Show that

$$\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

(c) (1 point) Show that $f(t) = t\cos t$ is of exponential order.

(d) (1 point) Solve by Laplace's method the initial value problem

$$\dot{y} = 5 - 2t, \qquad y(0) = 1$$

Question2 (4 points)

(a) (1 point) Write $f(t) = \begin{cases} 2\sin t, & 0 \le t < 3 \\ 7\cos t, & 3 \le t < 5 \\ 3\sin t, & t \ge 5. \end{cases}$ in terms of unit step functions.

(b) (1 point) Find the convolution t^2 and $e^{2t}\sin 2t$.

(c) (1 point) Show the operation of convolution is associative.

(d) (1 point) Find $\lim_{t\to\infty} f(t)$, where $\mathcal{L}\left[f(t)\right] = \dfrac{1}{s(s+a)(s+b)}$.

Question3 (3 points)

Consider the equation

$$\phi(t) + \int_0^t h(t-\xi)\phi(\xi)\, d\xi = f(t),$$

in which f and h are known functions, and ϕ is to be determined. Since the unknown function ϕ appears under an integral sign, the given equation is called an integral equation; in particular, it belongs to a class of integral equations known as Volterra integral equations.

The inverse Laplace transform of $\mathcal{L}\{\phi(t)\}$ is the solution of the original integral equation.

Consider the following Volterra integral equation

$$\phi(t) + \int_0^t (t-\xi)\phi(\xi)\,d\xi = \sin 2t. \tag{1}$$

That is, $h(t-\xi) = t - \xi$ and $f(t) = \sin 2t$.

(a) (1 point) Solve Equation (1) by using the Laplace method.

(b) (1 point) Show that if u is a function such that $u''(t) = \phi(t)$, then

$$u''(t) + u(t) - tu'(0) - u(0) = \sin 2t.$$

(c) (1 point) Show that Equation (1) is equivalent to the initial value problem

$$u''(t) + u(t) = \sin 2t; \qquad u(0) = 0, u'(0) = 0.$$

Question4 (4 points)

Solve the following initial-value problem using the Laplace method.

(a) (1 point) $y' + 2y = u(t-\pi)\sin 2t, \quad y(0) = 3$

(b) (1 point) $y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0,$ where $f(t) = \begin{cases} 4, & 0 \le t < 2 \\ t+2, & 2 \le t \end{cases}$

(c) (1 point) $y'' + 2y' + y = \delta(t-1), \quad y(0) = 0, \quad y'(0) = 0$

(d) (1 point) $ty'' + 2(t-1)y' - 2y = 0, \quad y(0) = 0$

Question5 (1 points)

Suppose a tank holds 10 gallon of pure water. There are two sources of brine solution: the first source has concentration of 0.5g of salt per gallon while the second source has a concentration of 2.5g of salt per gallon. The first source pours into the tank at a rate of 1 gallon per minute for 5 minutes after which it is turned off. In the meanwhile, the second source is turned on at a rate of 1 gallon per minute. The well-mixed solution pours out of the tank at a rate of 1 gallon per minute. Find the amount of salt in the tank at time t.

Question6 (0 points)

(a) (1 point (bonus)) Show that the following

$$x^3y'' + x(x+1)y' - y = 0$$

has an irregular singular point at $x = 0$. Find its solution in the form

$$\phi = \sum_{n=0}^{\infty} c_n x^{-n}$$

(b) (1 point (bonus)) The beta function is defined by $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$. Show

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

(c) (1 point (bonus)) Show

$$\mathcal{L}\left[\frac{\sin t}{t}\right] = \arctan\frac{1}{s}$$

(d) (1 point (bonus)) Solve the partial differential equation using Laplace transform

$$u_x = 2u_t + u, u(x,0) = 6e^{-3x}$$

which is bounded for all $x > 0$, $t > 0$.