Vv417 Lecture 8

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- So far, we have considered only direct methods for solving

$$\mathbf{A}\mathbf{x} = \mathbf{b} \qquad \text{where } \mathbf{A} \text{ is } n \times n.$$

- In absence of rounding errors direct methods reach the exact solution using a finite number of arithmetic operations. However, they usually fail to take computational advantage of the sparsity.
- In an iterative method, we start with an approximation

$$\mathbf{x}^{(0)}$$

to the exact solution and then compute a sequence of

$$\left\{\mathbf{x}^{(k)}\right\}$$

such that $\mathbf{x}^{(k)}$ becomes closer and closer to the exact as k grows.
- The main advantage of iterative methods are reduced storage requirements.

- Let $\mathbf{A}$ has nonzero diagonal elements, then the diagonal matrix

$$\mathbf{D}$$

formed from the diagonal elements of $\mathbf{A}$ is invertible.

$$\begin{aligned}
\mathbf{A}\mathbf{x} &= \mathbf{b} \\
\mathbf{A}\mathbf{x} - \mathbf{D}\mathbf{x} + \mathbf{D}\mathbf{x} &= \mathbf{b} \\
\mathbf{D}\mathbf{x} &= (\mathbf{D} - \mathbf{A})\,\mathbf{x} + \mathbf{b} \\
\mathbf{x} &= \mathbf{D}^{-1}\Big(\mathbf{b} + (\mathbf{D} - \mathbf{A})\,\mathbf{x}\Big)
\end{aligned}$$

from which we have the so-called Jacobi iteration

$$\mathbf{x}^{(k+1)} = \mathbf{D}^{-1}\Big(\mathbf{b} + (\mathbf{D} - \mathbf{A})\,\mathbf{x}^{(k)}\Big)$$

Q: What does Jacobi iteration actually set to zero in parallel?

- Jacobi iteration is reasonable for a small system

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

but the convergence tends to be too slow for large systems.

Q: Intuitively, what is a clear modification to speed up the convergence?

- Instead of updating elements of $\mathbf{x}^{(k)}$ in parallel,

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right) \quad \text{for} \quad i = 1, 2, \ldots, n$$

we could update successively

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right) \quad \text{for} \quad i = 1, 2, \ldots, n$$

- Let $\mathbf{D}$, $\mathbf{L}$ and $\mathbf{U}$ be triangular matrices such that

$$d_{ij} = \begin{cases} a_{ij} & i = j \\ 0 & i \neq j \end{cases}, \quad \ell_{ij} = \begin{cases} a_{ij} & i < j \\ 0 & i \geq j \end{cases} \quad \text{and} \quad u_{ij} = \begin{cases} a_{ij} & i > j \\ 0 & i \leq j \end{cases}$$

from which it is clear that

$$\mathbf{A} = \mathbf{D} + \mathbf{L} + \mathbf{U}$$

- The modification in this notation, which is known as Gauss-Seidel iteration,

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right)$$

$$\implies \mathbf{D}\mathbf{x}^{(k+1)} = \mathbf{b} - \mathbf{L}\mathbf{x}^{(k+1)} - \mathbf{U}\mathbf{x}^{(k)}$$

$$\implies \mathbf{x}^{(k+1)} = (\mathbf{D} + \mathbf{L})^{-1} \left(\mathbf{b} - \mathbf{U}\mathbf{x}^{(k)} \right)$$

Q: Do you think the sequences produced by the Jacobi or Gauss-Seidel methods

$$\left\{\mathbf{x}^{(k)}\right\}$$

always converge to the exact solutions?

Definition

A square matrix $\mathbf{A}$ is said to be strictly diagonally dominant if

$$|a_{ii}| > \sum_{j=1}^{i-1} |a_{ij}| + \sum_{j=i+1}^{n} |a_{ij}| \qquad \text{for all} \quad i = 1, 2, \ldots n$$

Q: Is the following matrix strictly diagonally dominant?

$$\begin{bmatrix} 7 & -2 & 3 \\ 4 & 1 & -6 \\ 5 & 12 & -4 \end{bmatrix}$$

- The following theorem gives a sufficient condition for convergence.

Theorem

If $\mathbf{A}$ is strictly diagonally dominant, then

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

has a unique solution, and for any choice of the initial guess $\mathbf{x}^{(0)}$, the sequence

$$\left\{\mathbf{x}^{(k)}\right\}$$

produced by the Jacobi or Gauss-Seidel iteration converge to the exact solution.

Q: How to prove the first half of the theorem?

- We will prove it when we have the tools for defining convergence rigorously.

Q: Can you see why iterative methods save storage requirements?