
Question1 (1 points)
Find two linearly independent power series solutions around $x = 0$ to

$$y'' - x^2y' - 3xy = 0$$

And determine the radius of convergence of the series solutions.

Question2 (1 points)
Find two linearly independent power series solutions at $x = 0$

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$$

And give a lower bound on the radius of convergence of the series solutions.

Question3 (2 points)
By using the substitution $t = x - 1$, find two linearly independent power series solution to

$$y'' + (x - 1)^2y' + (x^2 - 1)y = 0$$

in terms of t, then transform back to find the general solution in terms of $x - 1$. Show that you obtain the same result by directly finding has power series solutions around $x = 1$.

Question4 (2 points)
Find two linearly independent power series solutions about the origin to

$$e^x y'' + xy = 0$$

State the radius of convergence.

Question5 (2 points)
Find two linearly independent Frobenius series solution at the regular singular point to

$$2t^2\ddot{y} + 3t(1 + t)\dot{y} - y = 0$$

State the radius of convergence.

Question6 (2 points)
For the differential equation

$$xy'' - y = 0, \qquad x > 0$$

(a) (1 point) Show that the roots of the indicial equation are

$$r_1 = 1 \qquad \text{and} \qquad r_2 = 0$$

and determine the Frobenius series solution corresponding to $r_1 = 1$.

(b) (1 point) Find the second linearly independent solution.

Question7 (0 points)

(a) (1 point (bonus)) Let $y = \sum_{n=0}^{\infty} c_n t^n$ be the power series solution of

$$t\dot{y} + \lambda y = f(t) \qquad \text{where } \lambda \text{ is a constant and } f(t) = \sum_{n=0}^{\infty} f_n t^n.$$

Find the general solution to the differential equation.

(b) (1 point (bonus)) Find the power series solution to

$$y' = x^2 + y^2; \qquad y(0) = 1$$

(c) (1 point (bonus)) Find the power series solution to

$$y^{(3)} = (x-1)^2 + y^2 - y' - 2; \qquad y(1) = 1, \quad y'(1) = 0, \quad y''(1) = 2$$

(d) Recall a particle of mass m move along the x-axis, and bound to the equilibrium position $x = 0$ by a restoring force $-kx$, satisfies, in the absence of damping force, the equation of motion

$$m\ddot{x} = -kx$$

If we multiplying the above by $\dot{x}$, the resulting equation can be written as

$$\frac{d}{dt}\left[\frac{1}{2}m(\dot{x})^2 + \frac{1}{2}kx^2\right] = 0 \implies \frac{1}{2}m(\dot{x})^2 + \frac{1}{2}kx^2 = E$$

where $\frac{1}{2}m(\dot{x})^2$, $\frac{1}{2}kx^2$, and E are the kinetic, potential, total energies of the system, respectively. If we denote the natural frequency of system by $\omega = \sqrt{k/m}$, then

$$\frac{1}{2}m(\dot{x})^2 + \frac{1}{2}m\omega^2 x^2 = E$$

If initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$ are prescribed for the system, then the total energy can take on any nonnegative value, that is, a continuous set of values,

$$E = \frac{1}{2}m(v_0)^2 + \frac{1}{2}m\omega^2 x_0^2$$

depending on the initial conditions.

In quantum mechanics, the steady state Schrodinger wave equation corresponding to a one-dimensional problem is the ordinary differential equation

$$-\frac{\hbar^2}{2\mu}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

which I have been told is simple and practical even to freshmen. The constant $\hbar$ is Planck's constant divided by 2π, E is the total energy of the quantum system, and $V(x)$ is the potential function for the system. For example, the potential energy function for the distance between atoms in a diatomic molecule, oscillating in the neighbourhood of a stable equilibrium position, may be approximated by

$$V(x) = \frac{1}{2}\mu\omega^2 x^2$$

where ω is loosely called the classical frequency of the harmonic oscillator, and μ is the reduced mass of the system

$$\mu = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)^{-1} = \frac{m_1 m_2}{m_1 + m_2}$$

where m_1 and m_2 are the masses. When considering the vibration of a diatomic molecule, using the reduced mass assures us that we are viewing the motion from a framework that is truly stationary, and allows the two-body problem to be solved as if it were a one-body problem. Using this approximation, we have

$$-\frac{\hbar^2}{2\mu}\frac{d^2\psi}{dx^2} + \frac{1}{2}\mu\omega^2 x^2 \psi = E\psi$$

The function $\rho(x) = |\psi(x)|^2$ is interpreted as a probability density function for the position of a particle in the system. Roughly speaking, that means the product between it and the differential dx

$$\rho(x)\,dx = \rho(x) = |\psi(x)|^2\,dx$$

is the probability that upon a measurement of its position, the particle will be found in an interval of width dx about the point x. If follows that physically admissible solution

$$\psi(x)$$

known as Schrodinger wave functions, are required to satisfy

$$\psi \to 0 \quad \text{as} \quad |x| \to \infty \qquad \text{and} \qquad \int_{-\infty}^{\infty} |\psi|^2\,dx = 1$$

i. (1 point (bonus)) Show that changing the independent variable to

$$\xi = \sqrt{\mu\omega/\hbar x}$$

leads to

$$\frac{d^2\psi}{d\xi^2} + (\lambda + 1 - \xi^2)\psi = 0$$

where $\lambda + 1 = 2E/(\hbar\omega)$.

ii. (1 point (bonus)) Show that if we substitute

$$\psi = \exp\left(\frac{-\xi^2}{2}\right) y(\xi)$$

then $y(\xi)$ must satisfy

$$y'' - 2\xi y' + \lambda y = 0$$

iii. (10 points (bonus)) Show solutions of

$$-\frac{\hbar^2}{2\mu}\frac{d^2\psi}{dx^2} + \frac{1}{2}\mu\omega^2 x^2 \psi = E\psi$$

satisfying the conditions

$$\psi \to 0 \quad \text{as} \quad |x| \to \infty \qquad \text{and} \qquad \int_{-\infty}^{\infty} |\psi|^2\,dx = 1$$

exists only for a discrete set of values of E. This illustrates the well-known fact that the quantum energy states are discrete rather than continuous.
[Hint: You need power series. And believe it is simple and practical!]