Vv255 Lecture 2

Dr Jing Liu

UM-SJTU Joint Institute

May 17, 2017

Definition

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^2$, then the product

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$$

is called the dot product of $\mathbf{u}$ and $\mathbf{v}$. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^3$, then

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

is the corresponding dot product in $\mathbb{R}^3$.

Properties of the Dot product

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, and let α be a scalar in $\mathbb{R}$.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(\mathbf{u} \pm \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} \pm \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot (\mathbf{u} \pm \mathbf{v})$
3. $\mathbf{u} \cdot \mathbf{u} \geq 0$
4. $(\alpha \mathbf{u}) \cdot \mathbf{v} = \alpha(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (\alpha \mathbf{v})$

Q: Since a dot product is scalar, what does a dot product measure?

- If the dot product is with the vector $\mathbf{v}$ itself, say $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, then we have

$$\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + v_3^2$$

- Recall the length of $\mathbf{v}$ is

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

- Thus, the dot product of a vector $\mathbf{v}$ with itself is an indicator of its length

$$\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$$

- A vector whose length is one is called a unit vector, and the process of dividing a nonzero vector $\mathbf{v}$ by its length is known as normalizing

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{\sqrt{\mathbf{v} \cdot \mathbf{v}}}$$

- The followings are three important results involving the dot product/length.

Cauchy-Schwarz inequality

For any vectors $\mathbf{u}$ and $\mathbf{v}$, both in $\mathbb{R}^2$ or both in $\mathbb{R}^3$,

$$|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}| \, |\mathbf{v}|$$

Triangle Inequality

For any vectors $\mathbf{u}$ and $\mathbf{v}$, both in $\mathbb{R}^2$ or both in $\mathbb{R}^3$,

$$|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$$

Parallelogram Law

For any vectors $\mathbf{u}$ and $\mathbf{v}$, both in $\mathbb{R}^2$ or both in $\mathbb{R}^3$,

$$|\mathbf{u} + \mathbf{v}|^2 + |\mathbf{u} - \mathbf{v}|^2 = 2\left(|\mathbf{u}|^2 + |\mathbf{v}|^2\right).$$

Q: What is the geometric implication of Cauchy-Schwarz inequality?

Geometric formula for the dot product

If θ is the angle between the vector $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ or $\mathbb{R}^2$, then

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta$$

Proof

Consider the following three vectors in $\mathbb{R}^3$,

$$\mathbf{u}, \qquad \mathbf{v} \qquad \text{and} \qquad \mathbf{u} - \mathbf{v}$$

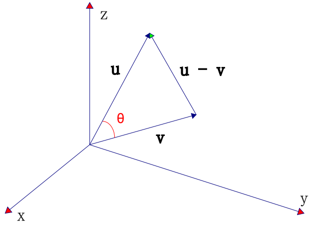
the three vectors form a triangle. Applying the Law of Cosines, we have

$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta$$

$$(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2|\mathbf{u}||\mathbf{v}|\cos\theta$$

$$\mathbf{u} \cdot \mathbf{u} - 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2|\mathbf{u}||\mathbf{v}|\cos\theta$$

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta \quad \square$$



Exercise

Find the dot product of $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix}$. What does it tell you?

- By the geometric formula for the dot product,

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| \, |\mathbf{v}| \cos\theta$$

it is clear that the dot product of two nonzero vectors is only zero if

$$\cos\theta = 0$$

- Thus the dot product of two vectors tells whether two vectors form

"a right angle triangle" in $\mathbb{R}^3$

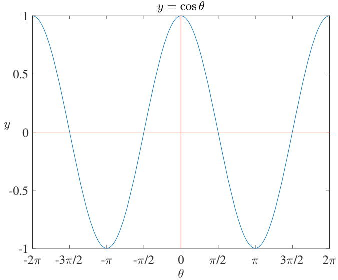
Definition

Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

$$\mathbf{u} \cdot \mathbf{v} = 0$$

They are said to be orthonormal if they are also unit vector.

Q: For what values of θ is $\cos\theta$ positive?

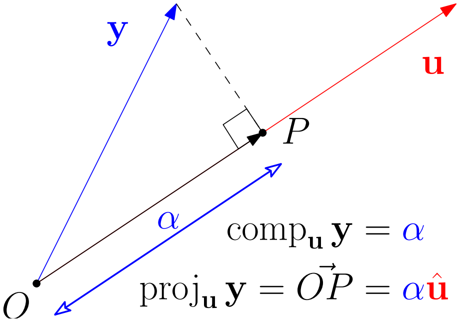


$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta \implies \hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = \cos\theta$$

- Hence the dot product of two vectors measures the extent to which the two vectors point in the same general direction.

- The dot product of two vectors is directly related to projections, consider

$$\operatorname{comp}_{\mathbf{u}} \mathbf{y} = \alpha$$
$$= |\mathbf{y}| \cos\theta$$



$$\operatorname{proj}_{\mathbf{u}} \mathbf{y} = \alpha\hat{\mathbf{u}} \quad \text{for some } \alpha \in \mathbb{R}.$$
$$= \alpha \frac{\mathbf{u}}{|\mathbf{u}|}$$
$$= |\mathbf{y}| \cos\theta \frac{\mathbf{u}}{|\mathbf{u}|}$$
$$= |\mathbf{y}| \frac{\mathbf{y} \cdot \mathbf{u}}{|\mathbf{y}|\,|\mathbf{u}|} \frac{\mathbf{u}}{|\mathbf{u}|}$$
$$= \frac{\mathbf{y} \cdot \mathbf{u}}{|\mathbf{u}|\,|\mathbf{u}|}\mathbf{u}$$
$$= \frac{\mathbf{u} \cdot \mathbf{y}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u}$$

Definition

Scalar projection of $\mathbf{y}$ onto $\mathbf{u}$,

$$\operatorname{comp}_{\mathbf{u}} \mathbf{y} = \frac{\mathbf{u} \cdot \mathbf{y}}{|\mathbf{u}|} = |\mathbf{y}| \cos \theta$$

where θ is the angle between vectors $\mathbf{u}$ and $\mathbf{y}$.

- It also known as the scalar component of $\mathbf{y}$ along $\mathbf{u}$.

Definition

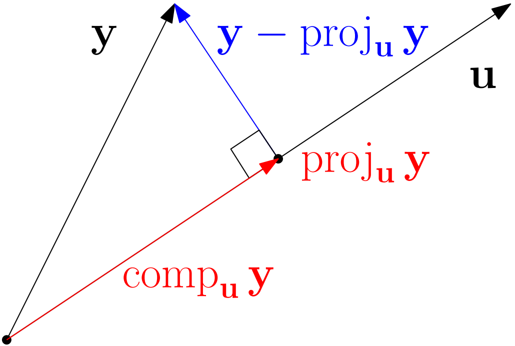
Vector projection of $\mathbf{y}$ onto $\mathbf{u}$

$$\operatorname{proj}_{\mathbf{u}} \mathbf{y} = \frac{\mathbf{u}}{|\mathbf{u}|} \operatorname{comp}_{\mathbf{u}} \mathbf{y} = \frac{\mathbf{u}}{|\mathbf{u}|} \left(\frac{\mathbf{u} \cdot \mathbf{y}}{|\mathbf{u}|} \right) = \frac{\mathbf{u} \cdot \mathbf{y}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u}$$

- It is also called the vector component of $\mathbf{y}$ along $\mathbf{u}$.

- We often want to decompose a vector $\mathbf{y}$ into two vector components,

  parallel and perpendicular to a vector $\mathbf{u}$.



- The vector $\mathbf{y} - \mathrm{proj}_{\mathbf{u}}\,\mathbf{y}$ is called the vector component of $\mathbf{y}$ orthogonal to $\mathbf{u}$.

Definition

Given three vectors in $\mathbb{R}^3$,

$$\mathbf{a}, \quad \mathbf{b}, \quad \mathbf{c}$$

if there exist three unique scalars

$$\alpha, \quad \beta, \quad \gamma$$

such that

$$\mathbf{v} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$$

for any arbitrary vector $\mathbf{v} \in \mathbb{R}^3$, then we say that $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ form a basis for $\mathbb{R}^3$.

The scalars

$$\alpha, \quad \beta, \quad \text{and} \quad \gamma$$

are called the components/coordinates of $\mathbf{v}$ with respect to the basis $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

- Because every vector in $\mathbb{R}^3$ can be represented as a linear combination of

$$\mathbf{e}_x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

  they form a basis for $\mathbb{R}^3$.

- The set

$$\mathcal{S} = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$$

  is known as the standard basis for $\mathbb{R}^3$.

- Note vectors in $\mathcal{S}$ are orthogonal to each other and of unit length. Such a set is known as an orthonormal basis, however, a basis need not be orthonormal.

Q: Can you think of another basis for $\mathbb{R}^3$? How many bases are there for $\mathbb{R}^3$?

$$\mathbf{a} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Linear Independence

The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent if the only way to satisfy

$$\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = \mathbf{0}$$

is for all the scalars α, β and γ to be simultaneously zero,

$$\alpha = \beta = \gamma = 0.$$

Dimension

The dimension of a space $\mathbb{R}^n$ is the largest number of linearly independent vectors

n

in that space. For example, the dimension of $\mathbb{R}^3$ is 3.

- A basis for that space consists of n linearly independent vectors.
- A vector $\mathbf{v}$ in that space has n components (some of them possibly zero) with respect to any basis in that space.

Q: Given an arbitrary vector $\mathbf{v}$ in $\mathbb{R}^3$, and a basis for this space. How to determine the components of $\mathbf{v}$ with respect to the basis?

Q: Given an arbitrary vector $\mathbf{v}$ in $\mathbb{R}^3$, and a orthogonal basis for this space. How to determine the components of $\mathbf{v}$ with respect to the basis?

- Finding those components is much simpler if the basis is orthogonal, i.e.

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = 0.$$

- In that case, take the dot product of both sides of the equation ealier

$$\mathbf{v} = \alpha \mathbf{a} + \beta \mathbf{b} + \gamma \mathbf{c}$$

with each of the 3 basis vectors and show that

$$\alpha = \frac{\mathbf{a} \cdot \mathbf{v}}{\mathbf{a} \cdot \mathbf{a}}, \quad \beta = \frac{\mathbf{b} \cdot \mathbf{v}}{\mathbf{b} \cdot \mathbf{b}}, \quad \gamma = \frac{\mathbf{c} \cdot \mathbf{v}}{\mathbf{c} \cdot \mathbf{c}}.$$

Q: Given an arbitrary vector $\mathbf{v}$ in $\mathbb{R}^3$, and a orthonormal basis for this space. How to determine the components of $\mathbf{v}$ with respect to the basis?

Exercise

If two vectors $\mathbf{u}$ and $\mathbf{v}$ are defined in terms of an orthonormal basis $\mathcal{B} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$,

$$\mathbf{u} = \alpha_u \mathbf{a} + \beta_u \mathbf{b} + \gamma_u \mathbf{c}, \qquad \mathbf{v} = \alpha_v \mathbf{a} + \beta_v \mathbf{b} + \gamma_v \mathbf{c}$$

Find the dot product of $\mathbf{u}$ and $\mathbf{v}$. What can you conclude from your answer?

Q: Is there another orthonormal basis for $\mathbb{R}^3$?

Matlab

```
>> a = [ 1; 1; 1]; b = [-1/3; 2/3; -1/3]; c = [ -2; 0; 2];
>> a = a/sqrt(3); b = b*sqrt(3/2); c = c/sqrt(8);
>> norm(a), norm(b), norm(c) %magnitude of a vector
ans = 1
ans = 1
ans = 1
>> dot(a,b), dot(a,c), dot(b,c)
ans = 0
ans = 0
ans = 0
```

Dictionary

"Transpose cause (two or more things) to change places with each other ."

Definition

We obtain the transpose of a matrix,

$$\mathbf{A}^{\mathrm{T}}$$

by writing its rows as columns or vice versa.

- For instance,

$$\text{Suppose} \qquad \mathbf{A} = \begin{bmatrix} 0 & 4 & 1 \\ 3 & 8 & 1 \\ 1 & 2 & 1 \end{bmatrix}, \quad \text{then} \quad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 0 & 3 & 1 \\ 4 & 8 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

Q: What will the transpose of a column vector $\mathbf{u}$ be ?

- If $\mathbf{u}$ and $\mathbf{v}$ are column vectors, then the dot product is equivalent to

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^{\mathrm{T}}\mathbf{v} = \mathbf{v}^{\mathrm{T}}\mathbf{u}$$

the later two are matrix products.

Properties of Transpose

Let $\mathbf{A}$ and $\mathbf{B}$ be matrices, and let α be a scalar.

1. $\left(\mathbf{A}^{\mathrm{T}}\right)^{\mathrm{T}} = \mathbf{A}$
2. $(\alpha\mathbf{A})^{\mathrm{T}} = \alpha\mathbf{A}^{\mathrm{T}}$
3. $(\mathbf{A}+\mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}$
4. $(\mathbf{A}\mathbf{B})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$

Properties of the Dot product

- Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be column vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, and let α be a scalar in $\mathbb{R}$.

1. $\mathbf{u}^{\mathrm{T}}\mathbf{v} = \mathbf{v}^{\mathrm{T}}\mathbf{u}$
2. $(\mathbf{u} \pm \mathbf{v})^{\mathrm{T}}\mathbf{w} = \mathbf{u}^{\mathrm{T}}\mathbf{w} \pm \mathbf{v}^{\mathrm{T}}\mathbf{w} = \mathbf{w}^{\mathrm{T}}(\mathbf{u} \pm \mathbf{v})$
3. $\mathbf{u}^{\mathrm{T}}\mathbf{u} \geq 0$
4. $(\alpha\mathbf{u})^{\mathrm{T}}\mathbf{v} = \alpha(\mathbf{u}^{\mathrm{T}}\mathbf{v}) = \mathbf{u}^{\mathrm{T}}(\alpha\mathbf{v})$

- Dot product is a scalar quantity, not a matrix.

$$\mathbf{u}^{\mathrm{T}}\mathbf{v} = \mathbf{v}^{\mathrm{T}}\mathbf{u} \neq \mathbf{u}\mathbf{v}^{\mathrm{T}} = \left(\mathbf{v}\mathbf{u}^{\mathrm{T}}\right)^{\mathrm{T}}$$

- The product in blue is known as the tensor product, the resulting square matrix is actually a tensor. Commonly, it is also denoted as

$$\mathbf{u}\mathbf{v}^{\mathrm{T}} = \mathbf{u} \otimes \mathbf{v} = \mathbf{A}$$

Properties of Tensor product

- Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be column vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$.

1. $\mathbf{u}\mathbf{v}^{\mathrm{T}} = \left(\mathbf{v}\mathbf{u}^{\mathrm{T}}\right)^{\mathrm{T}}$
2. $\left(\mathbf{u}\mathbf{v}^{\mathrm{T}}\right)\mathbf{w} = \mathbf{u}\left(\mathbf{v}^{\mathrm{T}}\mathbf{w}\right)$
3. $\mathbf{u}\left(\mathbf{v} + \mathbf{w}\right)^{\mathrm{T}} = \mathbf{u}\mathbf{v}^{\mathrm{T}} + \mathbf{u}\mathbf{w}^{\mathrm{T}}$
4. $\mathbf{u}^{\mathrm{T}}\left(\mathbf{v}\mathbf{w}^{\mathrm{T}}\right) = \left(\mathbf{u}^{\mathrm{T}}\mathbf{v}\right)\mathbf{w}^{\mathrm{T}}$