
Question1 (2 points)

Find $\sup(\mathcal{A})$ and $\inf(\mathcal{A})$ where $\mathcal{A} = \left\{\frac{n+2}{n} \mid n \in \mathbb{N}\right\}$.

Solution:

0M The answers are very obvious. But let us see how we can convey our ideas in writing.

1M If $n \in \mathbb{N}$, then

$$n \geq 1 \implies \frac{n}{2} \geq \frac{1}{2} \implies \frac{2}{n} \leq 2 \implies 1 + \frac{2}{n} \leq 3 \implies \frac{n+2}{n} \leq 3 \implies \sup(\mathcal{A}) = 3$$

1M Since $1 \leq n < \infty$

$$\frac{n+2}{n} = 1 + \frac{2}{n} \geq 1 \implies \inf(\mathcal{A}) = 1$$

we can make a better argument after the notion of decreasing, limit of a sequence are properly introduced.

Question2 (3 points)

Suppose $\mathcal{S}$ is a nonempty subset of $\mathbb{R}$. Determine whether the following statements are true. If not, briefly explain why it is false.

(a) (1 point) Every limit point of $\mathcal{S}$ is an interior point of $\mathcal{S}$.

Solution:

1M False. A limit point might be a boundary point.

(b) (1 point) Every neighbourhood of $x \in \mathbb{R}$ intersects $\mathcal{S}$ if and only if $x \in \mathcal{S}^{\complement}$

Solution:

1M False. For sufficiently small δ, it is clear that $(x - \delta, x + \delta) \cap \mathcal{S} = \emptyset$ if $x \in \mathcal{S}^{\complement}$.

(c) (1 point) The limit points of $\mathcal{S}$ form a closed set.

Solution:

1M True.

0M Here is a proof. Let $\mathcal{A}$ be the set of all limit points of $\mathcal{S}$. The set $\mathcal{A}$ is closed if every limit of $\mathcal{A}$ is a point of $\mathcal{A}$. Let x be a limit point of $\mathcal{A}$, then every δ-neighbourhood of x contains a point $y \neq x$ such that $y \in \mathcal{A}$. If we consider a sufficiently small $\epsilon > 0$ so that $|x - y| < \epsilon$ and $(y - \epsilon, y + \epsilon) \subset (x - \delta, x + \delta)$. Since y is in $\mathcal{A}$ and thus is a limit point of $\mathcal{S}$, there must be some $z \neq y$ in $(y - \epsilon, y + \epsilon)$ such that $z \in \mathcal{S}$. This implies $z \in (x - \delta, x + \delta)$ and $z \neq x$. Hence we have shown that any neighbourhood of x contains some $z \neq x$ with $z \in \mathcal{S}$, and thus x is a limit point of $\mathcal{S}$. Therefore $\mathcal{A}$ is closed.

Question3 (2 points)

(a) (1 point) Show that the square of an odd number is odd.

Solution:

1M Let $2n+1$ denote an odd number, where n is an integer.

$$(2n+1)^2 = 4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1$$

1m The term $2n^2 + 2n$ is merely another integer, thus the square of odd number can be put in the form $2m + 1$ for some integer m, hence it is odd.

(b) (1 point) Prove that $a_n = (3-2n)(-3)^n$ is the explicit formula for

$$a_0 = 3, \quad a_1 = -3, \quad a_n = -6a_{n-1} - 9a_{n-2} \qquad \text{for} \quad n \geq 2,$$

Solution:

1M This can be proved by the principle of mathematical induction. Note the recursive formula involves two previous terms, thus in the base step, we need to show that the explicit formula is true for both $n = 0$ and $n = 1$, when $n = 0$ and $n = 1$, the given explicit formula clearly is true

$$a_0 = (3 - 2 \cdot 0)(-3)^0 = 3 \qquad a_1 = (3 - 2 \cdot 1)(-3)^1 = -3$$

Now the inductive step, let us assume that for $n \geq 2$, $a_n = (3-2n)(-3)^n$ is true to all $n \leq k$ where $k \in \mathbb{N}$. So, we need to prove the explicit formula is true for $n = k+1$. Applying the formula for $n = k$ and $n = k-1$, we have

$$\begin{aligned} a_{k+1} &= -6a_k - 9a_{k-1} \\ &= -6(3-2k)(-3)^k - 9(3 - 2(k-1))(-3)^{k-1} \\ &= 2(3-2k)(-3)^{k+1} - (5-2k)(-3)^{k+1} \\ &= (6 - 4k - 5 + 2k)(-3)^{k+1} \\ &= (3 - 2(k+1))(-3)^{k+1} \end{aligned}$$

which is the same as if we use $n = k+1$ in the explicit formula, so the formula is true for $n = k+1$. Therefore the explicit formula is true by the principle of mathematical induction.

Question4 (1 points)

Give an example of two divergent sequences $\{a_n\}$ and $\{b_n\}$ such that $\{a_n + b_n\}$ is convergent.

Solution:

1M There are many examples, the following is just one of them,

$$\left\{n + \frac{1}{n}\right\}_{n=1}^{\infty} \qquad \left\{-n\right\}_{n=1}^{\infty} \qquad \left\{n + \frac{1}{n} - n\right\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

Question5 (1 points)

Consider the sequence $\{x_n\}$, where

$$x_1 = a, \quad x_2 = b, \quad x_n = \frac{x_{n-1} + x_{n-2}}{2} \qquad \text{for} \qquad n = 3, 4, \ldots$$

Determine whether $\{x_n\}$ is convergent. If so, find $\lim_{n\to\infty} x_n$. If not, prove why not.

Solution:

1M It is better to work out the explicit formula for this question,

$$\begin{aligned} x_n &= x_1 - x_1 + x_2 - x_2 + x_3 - \cdots - x_{n-1} + x_n \\ &= x_1 + (x_2 - x_1) + (x_3 - x_2) + \cdots + (x_n - x_{n-1}) \end{aligned}$$

if we consider the difference between two consecutive terms using the recurrence

$$x_n - x_{n-1} = \frac{x_{n-1} + x_{n-2}}{2} - x_{n-1} = -\frac{x_{n-1} - x_{n-2}}{2}$$

thus the difference between the difference between two consecutive terms is always a factor of $-\frac{1}{2}$. Hence the explicit formula in terms of a and b is

$$x_n = a + (b-a) - \frac{b-a}{2} + \frac{b-a}{4} - \cdots + (-1)^n \frac{b-a}{2^{n-2}} = a + \frac{2(b-a)}{3} + \frac{b-a}{3}\frac{(-1)^n}{2^{n-2}}$$

therefore the limit is

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \left(a + \frac{2(b-a)}{3} + \frac{b-a}{3}\frac{(-1)^n}{2^{n-2}} \right) = \frac{a+2b}{3}$$

Question6 (1 points)
If the sequence x_n is bounded and $\lim_{n\to\infty} y_n = 0$, show that $\lim_{n\to\infty} x_n y_n = 0$.

Solution:

1M Because the sequence $\{x_n\}$ is bounded, there is a number $M > 0$ so that,

$$|x_n| \leq M \qquad \text{for any} \qquad n \in \mathbb{N}$$

Since $y_n \to 0$ as $n \to \infty$, for any $\epsilon_1 > 0$, there is an integer N such that

$$|y_n - 0| < \epsilon_1 \qquad \text{when} \qquad n > N$$

thus for any $\epsilon_1 > 0$, there is an integer N such that, for any $n > N$,

$$|x_n| \cdot |y_n| < M\epsilon_1 \implies |x_n y_n - 0| < M\epsilon_1$$

Hence for any $\epsilon = M\epsilon_1 > 0$, there is an integer N such that,

$$|x_n y_n - 0| < M\epsilon_1 \qquad \text{when} \qquad n > N$$

Therefore the limit by definition is

$$\lim_{n\to\infty} x_n y_n = 0$$

Question7 (1 points)
Determine whether $\{\cos(n\pi)\}$ is convergent. If so, find $\lim_{n\to\infty} \cos(n\pi)$. If not, prove why not.

Solution:

1M Intuitively, the limit $\lim_{n\to\infty} \cos n\pi$ clearly doesn't exist. Let us make a more concrete argument by contradiction. Suppose that the limit exists and takes value L

$$\lim_{n\to\infty} \cos n\pi = L$$

Let us consider $m = n+1$, then the sequence $\{\cos m\pi\}$ is essentially the same sequence without the very first term of $\{a_n\}$, that is,

$$\{a_n\}_{n=1}^{\infty} = \{a_m\}_{m=2}^{\infty}$$

Hence they must have the same limit

$$\lim_{m\to\infty} \cos(m\pi) = \lim_{n\to\infty} \cos(n\pi) = L$$

However, the following trigonometric identity

$$\cos m\pi = \cos\Big((n+1)\pi\Big) = -\cos n\pi$$

reveals the following value for the limit of $\{a_m\}$ by properties of limits,

$$\lim_{m\to\infty} \cos(m\pi) = \lim_{m\to\infty} -\cos(n\pi) = -L$$

from which we are forced to conclude that

$$L = 0$$

If there is any $\epsilon > 0$ such that NO $N \in \mathbb{N}$ exists for the following to be true

$$|a_n - 0| < \epsilon \qquad \text{when} \qquad n > N$$

then $L = 0$ is NOT the limit. Now if we consider $\epsilon = \frac{1}{2}$, then clearly the inequality

$$|\cos(n\pi)| < \frac{1}{2}$$

is never satisfied by any natural number n since cosine is either 1 or -1 for any integer copies of π. So indeed no such N exists, and we can conclude $L \neq 0$. Therefore we reach a contradiction regarding the limit of $\{a_n\}$ if we assume the limit exists, which forces us to conclude that the limit does not exist and $\{a_n\}$ is divergent.

Question8 (3 points)

Consider the sequence

$$a_1 = \sqrt{6}$$
$$a_2 = \sqrt{6+\sqrt{6}}$$
$$a_3 = \sqrt{6+\sqrt{6+\sqrt{6}}}$$
$$a_4 = \sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6}}}}$$
$$\vdots$$

(a) (1 point) Find a recursion formula for the sequence.

Solution:

1M It is that

$$a_1 = \sqrt{6}, \qquad a_n = \sqrt{6 + a_{n-1}}$$

(b) (1 point) Show this sequence converges.

Solution:

1M By the monotonic sequence theorem, all need to be shown is the sequence is monotonic and bounded. Use induction to show it is increasing, firstly the base step

$$a_2 - a_1 = \sqrt{6+\sqrt{6}} - \sqrt{6} > 0$$

then the inductive step, let us assume that $a_n > a_{n-1}$ for $n = k$, then

$$a_{k+1} - a_k = \sqrt{6+a_k} - \sqrt{6+a_{k-1}} \implies a_{k+1} - a_k > 0$$

Thus it is monotonically increasing by the principle of mathematical induction. And $\sqrt{6}$ follows immediately being a lower bound for this sequence. Now use induction to show it is bounded above, firstly the base step,

$$\sqrt{6} < \sqrt{9} = 3$$

then the inductive step, let us assume that $a_n < 3$ for $n = k$, then

$$a_k + 6 < 9 \implies \sqrt{a_k + 6} < \sqrt{9} \implies a_{k+1} < 3$$

So it is bounded above by the principle of mathematical induction. Thus it is bounded as well as being monotonic, and is convergent by the monotonic sequence theorem.

(c) (1 point) Find the limit of this sequence.

Solution:

1M To work out the value of limit we need the following theorem,

If $\{a_n\}$ converges to L with $a_n \geq 0$, then $\{\sqrt{a_n}\}$ converges to $\sqrt{L}$.

Let us see why the theorem is true. There are two possibilities since $a_n \geq 0$.

$$L = 0 \qquad \text{and} \qquad L > 0$$

First suppose $L = 0$, then for every $\epsilon^* > 0$, there exists one N such that

$$|a_n - 0| < \epsilon^* \qquad \text{when} \qquad n > N$$

which implies for every $\epsilon^* > 0$, there exists one N such that

$$\sqrt{a_n} < \sqrt{\epsilon^*} \qquad \text{when} \qquad n > N$$

thus for every $\epsilon = \sqrt{\epsilon^*} > 0$, there exist one N such that

$$\left|\sqrt{a_n} - \sqrt{0}\right| < \epsilon \qquad \text{when} \qquad n > N$$

This shows the result is true for

$$L = 0$$

Now suppose $L > 0$, then for every $\epsilon^* > 0$, there exists one N such that

$$|a_n - L| < \epsilon^* \qquad \text{when} \qquad n > N$$

which implies every $\epsilon^* > 0$, there exists one N such that

$$\frac{|a_n - L|}{\sqrt{L}} < \frac{\epsilon^*}{\sqrt{L}} \qquad \text{when} \qquad n > N$$

Therefore the following inequality

$$\left|\sqrt{a_n} - \sqrt{L}\right| = \frac{|a_n - L|}{\sqrt{a_n} + \sqrt{L}} < \frac{|a_n - L|}{\sqrt{L}}$$

implies for every $\epsilon = \frac{\epsilon^*}{\sqrt{L}} > 0$, there exists one N such that

$$\left|\sqrt{a_n} - \sqrt{L}\right| < \epsilon \qquad \text{when} \qquad n > N$$

This shows the theorem is true. Applying it with some basic properties of limits,

$$\lim_{n\to\infty} \sqrt{6 + a_n} = \sqrt{6 + L}$$

However, the sequence $\left\{\sqrt{6 + a_n}\right\}$ is essentially $\{a_n\}$ without the first term, hence surely converges to the same L, thus we have an equation of L

$$\sqrt{6 + L} = L \implies L^2 = 6 + L \implies L = 3$$

since L is positive.

Question9 (1 points)

Suppose $\{a_n\}$ is monotonic and bounded. Use the definition of the supremum and the infimum to argue $\{a_n\}$ must be convergent by the definition of convergence.

Solution:

1M Since $\{a_n\}$ is bounded, then both the supremum and the infimum of the set $\mathcal{S}$ of all terms of $\{a_n\}$ must exist. First let us suppose $\{a_n\}$ is increasing, and let $L = \sup(\mathcal{S})$. If we can show this L is the limit of the $\{a_n\}$, then $\{a_n\}$ is convergent by the definition of convergence.

Since we have assumed $\{a_n\}$ is increasing and L is the supremum. For any $\epsilon > 0$, there exists a natural number N such that

$$L - \epsilon < a_n \qquad \text{for all } n > N.$$

which means the following must be true for any $\epsilon > 0$,

$$L - a_n < \epsilon \qquad \text{for all } n > N. \tag{1}$$

By the definition of L,

$$a_n \leq L \qquad \text{for all } n.$$

Hence, for any $\epsilon > 0$,

$$a_n < L + \epsilon \qquad \text{for all } n.$$

which means the following must be true for any $\epsilon > 0$,

$$-\epsilon < L - a_n \qquad \text{for all } n > N. \tag{2}$$

If we combine (1) and (2), we conclude the following statement is true for any $\epsilon > 0$,

$$|a_n - L| < \epsilon \qquad \text{when} \qquad n > N$$

Question10 (1 points)

In class, we have introduced the definitions of *interior point*, *limit point*, *boundary point* and *open set* under $\mathbb{R}$. These definitions can be easily extended to a higher dimension, so do many other mathematical concepts. Give your **OWN** reasonable definitions of *interior point*, *limit point boundary point* and *open set* for two-dimensional space.

Solution:

1M They are various ways of stating the same thing, here are the possible definitions:

interior point:

For a 2-dimensional set $\mathcal{S}$, a point z is an interior point of $\mathcal{S}$ if there is an open disk about z containing only points of $\mathcal{S}$.

limit point:

A point z is a limit point of a set $\mathcal{S}$ if every open disk about z contains at least one point of $\mathcal{S}$ different from z.

boundary point:

A point z is a boundary point of $\mathcal{S}$ if every open disk about z contains at least one point in $\mathcal{S}$ and at least one point not in $\mathcal{S}$.

a open set:

A set $\mathcal{S} \subset \mathbb{R}^2$ is open in $\mathbb{R}^2$,if for every point $z \in \mathcal{S}$ there is an open disk about z containing only points of $\mathcal{S}$.

Question11 (4 points)

In high school, if it is required to find the explicit formula for a sequence involves

$$a_{n+1} = ca_n + d, \qquad \text{where} \quad c \neq 0,$$

You would properly consider of the alternative sequence $\left\{a_n + \dfrac{d}{c-1}\right\}$, which is geometric, in order to solve the explicit formula for the original sequence. A similar idea can be used to find the explicit formula for a sequence with a second-order linear recurrence relation

$$a_{n+1} = pa_n + qa_{n-1} \qquad \text{where} \qquad p^2 + 4q \geq 0$$

We can try to find the explicit formula by considering the alternative sequence

$$\{a_{n+1} + ta_n\}$$

For simplicity, let us assume that

$$a_{n+1} + ta_n = s(a_n + ta_{n-1})$$

(a) (1 point) Assume that there are two distinct sets of real solutions for the unknowns s and t, denoted as (s_1, t_1) and (s_2, t_2). Express the explicit formula for a_n in terms of a_1, a_2, t_1, t_2, s_1 and s_2.

Solution:

1M Under the assumption

$$a_{n+1} + t_1 a_n = s_1(a_n + t_1 a_{n-1})$$
$$a_{n+1} + t_2 a_n = s_2(a_n + t_2 a_{n-1})$$

It is clear that $\{a_{n+1} + t_1 a_n\}$ and $\{a_{n+1} + t_2 a_n\}$ are geometric progressions with common ratio of s_1 and s_2 respectively. So can be expressed in the form of:

$$a_{n+1} + t_1 a_n = (a_2 + t_1 a_1)s_1^{n-1}$$
$$a_{n+1} + t_2 a_n = (a_2 + t_2 a_1)s_2^{n-1}$$

Take the difference between the above two equations, we have

$$a_n = \frac{a_2 + t_1 a_1}{s_1(t_1 - t_2)} s_1^n - \frac{a_2 + t_2 a_1}{s_2(t_1 - t_2)} s_2^n$$

(b) (1 point) If the solutions of s and t are repeated, that is, $s_1 = s_2, t_1 = t_2$, what is the explicit formula for a_n in this case?

Solution:

1M Repeated solutions occur if

$$p^2 - 4q = 0$$

Solving it for $p = s - t$ and $q = st$, we have

$$t_1 = -s_1$$

then the recurrence $a_{n+1} + t_1 a_n = s_1(a_n + t_1 a_{n-1}) = \cdots = s_1^{n-1}(a_2 + t_1 a_1)$

$$\Longrightarrow \frac{a_{n+1}}{s_1^{n+1}} + \frac{a_1}{s_1} = \frac{a_n}{s_1^n} + \frac{a_2}{s_1^2}$$

So, $\left\{\frac{a_n}{s_1^n}\right\}$ is a arithmetic progression. Finally, we can get the expression

$$a_n = s_1^n \left(\frac{a_1}{s_1} + \frac{(a_2 + t_1 a_1)(n-1)}{s_1^2} \right)$$

(c) (1 point) Fibonacci sequence is a world famous sequence. It is defined as:

$$F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-1} + F_{n-2}, \qquad \text{for} \quad n > 2$$

We can use the method that is described in the last two parts to solve the explicit formula for $\{F_n\}$. However, there exists a much easier method. For a sequence with the recurrence relation

$$a_1, \quad a_2, \qquad a_{n+1} = pa_n + qa_{n-1}, \qquad \text{where} \quad p^2 + 4q > 0$$

we can solve the equation $r^2 - pr - q = 0$ to obtain two distinct solutions $r_1 = s_1, r_2 = s_2$. Then the explicit formula for a_n can be expressed in the form of

$$a_n = k_1 s_1^n + k_2 s_2^n$$

Use this method to find out the explicit formula for the Fibonacci sequence.

Solution:

1M Apply the method directly

$$a_n = \frac{1}{\sqrt{5}} \left((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n \right)$$

(d) (1 point) Explain why the method used in part (c) is valid by using results in part (a).

Solution:

1M Recall the equations $s - t = p$ and $st = q$. Eliminating t, we can have

$$s^2 - ps - q = 0$$

which is equivalent to the equation

$$r^2 - pr - q = 0$$

Observing the expression of a_n in part (a), we can find that two roots of the equation are those two numbers in exponential functions. And the coefficients before the exponential functions are constants, which are expressed in k_1 and k_2 in question 3. Hence this kind of method is valid.