Vv256 Lecture 6

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- Suppose we have two continuously differentiable functions

$$\phi_1 \qquad \text{and} \qquad \phi_2$$

Q: Can we conclude ϕ_1 and ϕ_2 are linearly dependent if

$$W(\phi_1, \phi_2) = 0 \qquad \text{for all } t.$$

Q: When ϕ_1 and ϕ_2 are linearly independent, can we conclude

$$W(\phi_1, \phi_2) \neq 0 \qquad \text{for some } t.$$

- Now suppose ϕ_1 and ϕ_2 are two linearly independent solutions to

$$\alpha\ddot{y} + \beta\dot{y} + \gamma y = 0$$

Q: Can we conclude that

$$W(\phi_1, \phi_2) \neq 0 \qquad \text{for some } t.$$

- When ϕ_1 and ϕ_2 are solutions of the same second-order equation, then the next theorem leads to a necessary and sufficient condition for linear independence in terms of the Wronskian of ϕ_1 and ϕ_2.

Abel's theorem

If ϕ_1 and ϕ_2 are solutions of the differential equation

$$\ddot{y} + P\dot{y} + Qy = 0$$

where P and Q are continuous on an open interval $\mathcal{I}$, then the Wronskian is

$$W(\phi_1, \phi_2)(t) = C \exp\left[-\int P(t)\, dt\right],$$

where C is a constant that depends on ϕ_1 and ϕ_2, but not on t. Consequently,

When C is zero, $W(\phi_1, \phi_2)(t) = 0$, for all t in $\mathcal{I}$.
When C is nonzero, $W(\phi_1, \phi_2)(t) \neq 0$, for all t in $\mathcal{I}$.

Proof

- Since ϕ_1 is a solution,

$$\phi_1'' + P\phi_1' + Q\phi_1 = 0 \implies -\phi_2\left(\phi_1'' + P\phi_1' + Q\phi_1\right) = 0 \tag{1}$$

Proof

- Similarly, ϕ_2 is also a solution,

$$\phi_2'' + P\phi_2' + Q\phi_2 = 0 \implies \phi_1\left(\phi_2'' + P\phi_2' + Q\phi_2\right) = 0 \tag{2}$$

- Now consider (1)+(2),

$$\phi_1\left(\phi_2'' + P\phi_2' + Q\phi_2\right) - \phi_2\left(\phi_1'' + P\phi_1' + Q\phi_1\right) = 0$$

$$\left(\phi_1\phi_2'' - \phi_2\phi_1''\right) + P\left(\phi_1\phi_2' - \phi_2\phi_1'\right) = 0 \tag{3}$$

- The term is actually the Wronskian of ϕ_1 and ϕ_2,

$$W(\phi_1, \phi_1) = \phi_1\phi_2' - \phi_2\phi_1'$$

- Differentiate W, we have $W' = \phi_1'\phi_2' + \phi_1\phi_2'' - \phi_2'\phi_1' - \phi_2\phi_1''$

$$= \left(\phi_1\phi_2'' - \phi_2\phi_1''\right)$$

Proof

- Therefore, (3) is a first-order linear equation

$$\boxed{W'} + P\,\boxed{W} = 0$$

- Solve it, we have

$$W = C\exp\left[-\int P\,dt\right]$$

- Note C is an arbitrary constant, it will be determined by initial condition

$$W(\phi_1, \phi_2)(t_0) = W_0$$

- Since the exponential function is never zero, so if W_0 is zero, then C is zero, and Wronskian is identically zero. If not, then C is not zero and so does W.
- So in this way Wronskian here depends only on the functions ϕ_1 and ϕ_2, not the actual values of t_0. That completes the second part of the theorem $\square$

Theorem

Suppose $y = u(t)$ and $y = v(t)$ are solutions of the following initial-value problem

$$\ddot{y} + P\dot{y} + Qy = R, \qquad y(t_0) = y_0, \qquad \dot{y}(t_0) = y_1$$

where P, Q and R are continuous on an open interval $\mathcal{I}$ which contains t_0, then

$$u(t) = v(t) \qquad \text{for all } t \in \mathcal{I}.$$

Proof

- Suppose $y = w(t) = u(t) - v(t)$, then w must satisfy the following IVP

$$\ddot{y} + P\dot{y} + Qy = 0, \qquad y(t_0) = \dot{y}(t_0) = 0$$

- Consider the following function

$$z(t) = (\dot{w})^2 + w^2 \implies z(t) \geq 0 \qquad \text{and} \qquad z(t_0) = 0$$
$$\dot{z}(t) = 2\dot{w}\ddot{w} + 2w\dot{w} = 2\dot{w}\left(-P\dot{w} - Qw\right) + 2w\dot{w} = 2w\dot{w}\left(1 - Q\right) - 2P\left(\dot{w}\right)^2$$

Proof

- Since P and Q are continuous on an open interval $\mathcal{I}$, for any finite closed subinterval $[t_1, t_2]$ of $\mathcal{I}$, there exists a constant M such that

$$|P| \leq M \qquad \text{and} \qquad |Q| \leq M \qquad \text{for all } t \in [t_1, t_2].$$

- Hence, for $t \in [t_1, t_2]$, we have the following

$$\dot{z} = 2w\dot{w}\left(1 - Q\right) - 2P\left(\dot{w}\right)^2$$

$$\begin{aligned}
|\dot{z}| &\leq \left|2w\dot{w}\left(1 - Q\right)\right| + \left|-2P\left(\dot{w}\right)^2\right| \\
&\leq 2\left|w\right|\left|\dot{w}\right|\left(1 + M\right) + 2M\left(\dot{w}\right)^2 \\
&\leq \left(1 + M\right)\left(\left(\dot{w}\right)^2 + w^2\right) + 2M\left(\left(\dot{w}\right)^2 + w^2\right) \\
&= \left(1 + 3M\right)z
\end{aligned}$$

Proof

- Thus, we have the following inequality for all $t \in [t_1, t_2]$

$$-kz \leq \dot{z} \leq kz \qquad \text{where} \quad k = 1 + 3M$$

- Considering the following inequality for $t \in [t_0, t_2]$, we have

$$\begin{aligned} \dot{z} \leq kz \implies & e^{-kt}\dot{z} - ke^{-kt}z \leq 0 \\ \implies & \quad \frac{d}{dt}\left(e^{-kt}z\right) \leq 0 \end{aligned}$$

which implies

$$e^{-kt}z \leq e^{-kt_0}z(t_0) = 0 \implies z(t) \leq 0 \qquad \text{for} \quad t_0 \leq t \leq t_2$$

- Since $z(t) \geq 0$ for all $t \in [t_1, t_2]$,

$$z(t) = 0 \qquad \text{for} \quad t_0 \leq t \leq t_2.$$

Proof

- Similarly, considering the following inequality for $t \in [t_1, t_0]$, we obtain

$$\dot{z} \geq -kz \implies z(t) = 0 \qquad \text{for} \quad t_1 \leq t \leq t_0$$

which implies

$$z(t) = 0 \qquad \text{for all } t \in \mathcal{I}.$$

since t_1 and t_2 are arbitrary.

- Solving the following differential equation, we can conclude

$$0 = (\dot{w})^2 + w^2 \implies \dot{w} = \pm \mathrm{i} w \implies w = Ce^{\pm \mathrm{i} t} \implies w(t) = 0$$

for all $t \in \mathcal{I}$, is the only real solution, which means

$$u(t) = v(t) \qquad \text{for all } t \in \mathcal{I}. \quad \square$$

Theorem

Suppose ϕ_1 and ϕ_2 are two solutions to the homogeneous equation

$$\ddot{y} + P\dot{y} + Qy = 0$$

where P and Q are continuous on an open interval $\mathcal{I}$, then the Wronskian is

1. identically zero if and only if ϕ_1 and ϕ_2 are linearly dependent.
2. never zero if and only if ϕ_1 and ϕ_2 are linearly independent.

Proof

- Abel's theorem helped us to understand that $W(\phi_1, \phi_2)$ is either identically zero or nowhere zero in $\mathcal{I}$ if ϕ_1 and ϕ_2 are solutions of the same equation.
- We will prove statement 1., since statement 2. will follow immediately.
- Assume the two solutions ϕ_1 and ϕ_2 are linearly dependent, we can invoke the theorem L5P4 , which states W is identically zero.
- If assume $W(\phi_1, \phi_2)$ is identically zero, then the corresponding system has nontrivial solution for any $t_0 \in \mathcal{I}$.

Proof

- That is, we have a nontrivial solution α_1 and α_2 for a given $t = t_0$.

$$\begin{bmatrix} \phi_1 & \phi_2 \\ \dot{\phi}_1 & \dot{\phi}_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{aligned} \alpha_1\phi_1 + \alpha_2\phi_2 &= 0 \\ \alpha_1\dot{\phi}_1 + \alpha_2\dot{\phi}_2 &= 0 \end{aligned}$$

- Consider the following IVP,

$$\ddot{y} + P\dot{y} + Qy = 0; \quad \dot{y}(t_0) = 0, \quad y(t_0) = 0$$

- The last theorem says $y = 0$ for all $t \in \mathcal{I}$ is the only solution to this IVP.
- However, setting $C_1 = \alpha_1$ and $C_2 = \alpha_2$, the following is a solution

$$y = C_1\phi_1 + C_2\phi_2 = \alpha_1\phi_1 + \alpha_2\phi_2$$

for all $t \in \mathcal{I}$. Thus for some nontrivial α_1 and α_2, the following is true

$$\alpha_1\phi_1 + \alpha_2\phi_2 = 0 \qquad \text{for all } t \in \mathcal{I}.$$

- Therefore ϕ_1 and ϕ_2 are linearly dependent. □

- There are two main virtues of Abel's theorem,

1. It was used to prove the last theorem, and reach the fact that we need two linearly independent solutions ϕ_1 and ϕ_2 for our general solution, that is

$$y = c_1\phi_1 + c_2\phi_2$$

   is the general solution if and only if ϕ_1 and ϕ_2 are linearly independent.

2. It gives us a second way to compute the Wronskian.

- A general rule in mathematics is that whenever you can compute something in two different ways, something good will happen. In this case, we known

$$W = C\exp\left[-\int P(t)\,dt\right],$$

  On the other hand, by definition,

$$W = \det\begin{bmatrix}\phi_1 & \phi_2\\ \phi_1' & \phi_2'\end{bmatrix} = \phi_1\phi_2' - \phi_1'\phi_2.$$

Exercise

(a) *Find the general solution to the equation using Abel's theorem.*

$$\ddot{y} - 2r\dot{y} + r^2 y = 0, \qquad \textit{where } r \textit{ is a constant.}$$

(b) *Solve the Euler differential equation using Abel's theorem.*

$$t^2\ddot{y} - 7t\dot{y} + 16y = 0$$