Question1 (1 points)

Find two linearly independent power series solutions around $x = 0$ to

$$y'' - x^2 y' - 3xy = 0$$

And determine the radius of convergence of the series solutions.

Solution:

1M The point $x = 0$ is an ordinary point of this equation, and it has no singularity, hence it is guaranteed by the theorem on L11P11 that we have two linearly independent power series solutions that converge for all $x \in \mathbb{R}$, that is, the radius of convergence is infinite. In general,

$$y'' + P(x)y' + Q(x)y = 0$$

where P and Q are analytic at $x = 0$, that is,

$$P(x) = \sum_{n=0}^{\infty} P_n x^n \qquad \text{and} \qquad Q(x) = \sum_{n=0}^{\infty} Q_n x^n$$

we have the following recurrence relation, which is given and derived on L11P21

$$(n+2)(n+1)c_{n+2} + \sum_{m=0}^{n} (m+1)P_{n-m}c_{m+1} + Q_{n-m}c_m = 0 \qquad \text{for} \quad n \in \mathbb{N}_0$$

In this case, P and Q are polynomials, thus the power series representation is itself,

$$\begin{array}{lllll} P_0 = 0 & P_1 = 0 & P_2 = -1 & P_k = 0 & \text{for} \quad k = 3,4,5\dots \\ Q_0 = 0 & Q_1 = -3 & Q_2 = 0 & Q_k = 0 & \text{for} \quad k = 3,4,5\dots \end{array}$$

Note there are only two nonzero terms,

$$P_2 = P_{n-m} = -1 \implies m = n-2 \qquad \text{and} \qquad Q_1 = Q_{n-m} = -3 \implies m = n-1$$

which means if $n = 0$, we have

$$2 \cdot 1 \cdot c_2 + 1 \cdot P_0 c_1 + Q_0 c_0 = 0 \implies c_2 = 0$$

for $n = 1, 2, 3, \dots$, we have

$$(n+2)(n+1)c_{n+2} - (n-1)c_{n-1} - 3c_{n-1} = 0 \implies c_{n+2} = \frac{c_{n-1}}{n+1}$$

Thus, for $k \in \mathbb{N}_1$, we have

$$c_{3k-1} = 0; \qquad c_{3k} = \frac{c_0}{\prod_{m=1}^{k}(3m-1)}; \qquad c_{3k+1} = \frac{c_1}{\prod_{m=1}^{k} 3m} = \frac{c_1}{3^k k!}$$

Therefore, two linearly independent solutions are

$$\phi_1 = 1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{\prod_{m=1}^{k}(3m-1)} \qquad \text{and} \qquad \phi_2 = x + \sum_{k=1}^{\infty} \frac{x^{3k+1}}{3^k k!}$$

Question2 (1 points)

Find two linearly independent power series solutions at $x = 0$

$$(1 - x^2)y'' - 2xy' + \lambda(\lambda + 1)y = 0$$

And give a lower bound on the radius of convergence of the series solutions.

Solution:

1M Since $x = 0$ is an ordinary point, and the closest singularity is

$$x = \pm 1$$

thus it is guaranteed by the theorem on L11P11 that we have two linearly independent power series solutions that converge for

$$-1 < x < 1$$

thus a lower bound on the radius of convergence is 1. In general, if P and Q in

$$y'' + P(x)y' + Q(x)y = 0$$

are rational functions, it is easier to use following form,

$$\alpha y'' + \beta y' + \gamma y = 0$$

in which α, β and γ are polynomials, and thus analytic everywhere, that is,

$$\alpha(x) = \sum_{n=0}^{\infty} \alpha_n x^n; \qquad \beta(x) = \sum_{n=0}^{\infty} \beta_n x^n \qquad \text{and} \qquad \gamma(x) = \sum_{n=0}^{\infty} \gamma_n x^n$$

Substituting the power series

$$\phi = \sum_{n=0}^{\infty} c_n x^n; \qquad \phi' = \sum_{n=1}^{\infty} n c_n x^{n-1} \qquad \text{and} \qquad \phi' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

we have

$$\begin{aligned}
\left(\sum_{n=0}^{\infty} \alpha_n x^n\right) \cdot \left(\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} \beta_n x^n\right) \cdot \left(\sum_{n=1}^{\infty} n c_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} \gamma_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} c_n x^n\right) &= 0 \\
\left(\sum_{n=0}^{\infty} \alpha_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n\right) + \left(\sum_{n=0}^{\infty} \beta_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} (n+1)c_{n+1} x^n\right) + \left(\sum_{n=0}^{\infty} \gamma_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} c_n x^n\right) &= 0 \\
\sum_{n=0}^{\infty} x^n \sum_{m=0}^{n} (m+2)(m+1)\alpha_{n-m} c_{m+2} + \sum_{n=0}^{\infty} x^n \sum_{m=0}^{n} (m+1)\beta_{n-m} c_{m+1} + \sum_{n=0}^{\infty} x^n \sum_{m=0}^{n} \gamma_{n-m} c_m &= 0 \\
\sum_{n=0}^{\infty} x^n \sum_{m=0}^{n} \Big((m+2)(m+1)\alpha_{n-m} c_{m+2} + (m+1)\beta_{n-m} c_{m+1} + \gamma_{n-m} c_m\Big) &= 0
\end{aligned}$$

Therefore, for $n \in \mathbb{N}_0$, the recurrent relation is

$$\sum_{m=0}^{n} \Big((m+2)(m+1)\alpha_{n-m} c_{m+2} + (m+1)\beta_{n-m} c_{m+1} + \gamma_{n-m} c_m\Big) = 0$$

In this case,

$$
\begin{array}{lllll}
\alpha_0 = 1 & \alpha_1 = 0 & \alpha_2 = -1 & \alpha_k = 0 & \text{for} \quad k = 3, 4, 5 \ldots \\
\beta_0 = 0 & \beta_1 = -2 & \beta_2 = 0 & \beta_k = 0 & \text{for} \quad k = 3, 4, 5 \ldots \\
\gamma_0 = \lambda(\lambda+1) & \gamma_1 = 0 & \gamma_2 = 0 & \gamma_k = 0 & \text{for} \quad k = 3, 4, 5 \ldots
\end{array}
$$

Thus, for $n \in \mathbb{N}_0$, we have the following recurrence relation

$$
(n+2)(n+1)c_{n+2} = \left(n^2 + n - \lambda^2 - \lambda\right) c_n \implies c_{n+2} = \frac{(n-\lambda)(n+\lambda+1)}{(n+2)(n+1)} c_n
$$

Hence, for $k \in \mathbb{N}_1$, we have

$$
\begin{aligned}
c_{2k} &= \frac{\displaystyle\prod_{m=0}^{k-1}(2m-\lambda)(2m+\lambda+1)}{\displaystyle\prod_{m=0}^{k-1}(2m+2)(2m+1)} c_0 \\
&= \frac{\displaystyle\prod_{m=0}^{k-1}(2m-\lambda)(2m+\lambda+1)}{(2k)!} c_0 \\
c_{2k+1} &= \frac{\displaystyle\prod_{m=0}^{k-1}(2m+1-\lambda)(2m+\lambda+2)}{\displaystyle\prod_{m=0}^{k-1}(2m+3)(2m+2)} c_1 \\
&= \frac{\displaystyle\prod_{m=0}^{k-1}(2m+1-\lambda)(2m+\lambda+2)}{(2m+1)!} c_1
\end{aligned}
$$

Therefore, two linearly independent solutions are

$$
\begin{aligned}
\phi_1 &= 1 + \sum_{k=1}^{\infty} \frac{\displaystyle\prod_{m=0}^{k-1}(2m-\lambda)(2m+\lambda+1)}{(2m)!} x^{2k} \\
\phi_2 &= x + \sum_{k=1}^{\infty} \frac{\displaystyle\prod_{m=0}^{k-1}(2m+1-\lambda)(2m+\lambda+2)}{(2m+1)!} x^{2k+1}
\end{aligned}
$$

Question3 (2 points)
By using the substitution $t = x - 1$, find two linearly independent power series solution to

$$
y'' + (x-1)^2 y' + (x^2-1)y = 0
$$

in terms of t, then transform back to find the general solution in terms of $x - 1$. Show that you obtain the same result by directly finding has power series solutions around $x = 1$.

Solution:

1M Make a change of variable, $t = x - 1 \implies \dfrac{dt}{dx} = 1 \implies \dfrac{d^2t}{dx^2} = 0$, we have

$$\begin{aligned}
\frac{d^2y}{dx^2} + (x-1)^2\frac{dy}{dx} + (x^2-1)y &= 0 \\
\left(\frac{d^2y}{dt^2}\left(\frac{dt}{dx}\right)^2 + \frac{dy}{dt}\frac{d^2t}{dx^2}\right) + t^2\frac{dy}{dt}\frac{dt}{dx} + (t^2+2t)y &= 0 \\
\ddot{y} + t^2\dot{y} + (t^2+2t)y &= 0
\end{aligned}$$

In this case,

$$P(t) = t^2 \qquad Q(t) = t^2 + 2t$$

thus the coefficients for their power series representation are

$$\begin{array}{llllll}
P_0 = 0 & P_1 = 0 & P_2 = 1 & P_k = 0 & \text{for} & k = 3, 4, 5\ldots \\
Q_0 = 0 & Q_1 = 2 & Q_2 = 1 & Q_k = 0 & \text{for} & k = 3, 4, 5\ldots
\end{array}$$

According to the recurrence relation in general, we have

$$\begin{aligned}
(n+2)(n+1)c_{n+2} + \sum_{m=0}^{n}(m+1)P_{n-m}c_{m+1} + Q_{n-m}c_m &= 0 \qquad \text{for} \quad n \in \mathbb{N}_0 \\
\implies c_2 = 0 \quad \text{and} \quad 6c_3 + 2c_0 = 0 \implies c_3 = -\frac{1}{3}c_0 & \\
\implies (n+2)(n+1)c_{n+2} + (n-1)c_{n-1} + c_{n-2} + 2c_{n-1} &= 0 \qquad \text{for} \quad n = 2, 3, 4, \ldots \\
\implies (n+2)(n+1)c_{n+2} + (n+1)c_{n-1} + c_{n-2} &= 0 \qquad \text{for} \quad n = 2, 3, 4, \ldots
\end{aligned}$$

this recurrence relation cannot be solved easily, but can obtain as many terms as we desire from this recurrence relation. So we illustrate the answer we list a few nonzero terms in both linearly independent solutions. In the exam, I will say explicitly to list at least three nonzero terms for difficult recurrence relations,

$$\begin{aligned}
\phi_1 &= 1 - \frac{1}{3}t^3 - \frac{1}{12}t^4 + \frac{1}{18}t^6 + \cdots \\
&= 1 - \frac{1}{3}(x-1)^3 - \frac{1}{12}(x-1)^4 + \frac{1}{18}(x-1)^6 + \cdots \\
\phi_2 &= t - \frac{1}{4}t^4 - \frac{1}{20}t^5 + \frac{1}{28}t^7 + \cdots \\
&= (x-1) - \frac{1}{4}(x-1)^4 - \frac{1}{20}(x-1)^5 + \frac{1}{28}(x-1)^7 + \cdots
\end{aligned}$$

Now if we were to expand everything around $x = 1$, we would use

$$\phi = \sum_{n=0}^{\infty} c_n(x-1)^n$$

and

$$P(x) = \sum_{n=0}^{\infty} P_n(x-1)^n = (x-1)^2$$
$$Q(x) = \sum_{n=0}^{\infty} Q_n(x-1)^n = (x-1+2)(x-1) = (x-1)^2 + 2(x-1)$$

Notice we would have the same P_n and Q_n, thus we would reach the same recurrence relation, which leads to the same solutions.

Question4 (2 points)
Find two linearly independent power series solutions about the origin to

$$e^x y'' + xy = 0$$

State the radius of convergence.

Solution:

1M Consider the power series of e^x around $x = 0$, and the power series solution, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=0}^{\infty} c_n x^n &= 0 \\
\sum_{n=2}^{\infty} \sum_{m=0}^{n} (n-m)(n-m-1)c_{n-m}x^{n-m-2}\frac{x^m}{m!} + \sum_{n=0}^{\infty} c_n x^{n+1} &= 0 \\
\sum_{n=2}^{\infty} x^{n-2} \sum_{m=0}^{n} \frac{(n-m)(n-m-1)c_{n-m}}{m!} + \sum_{n=0}^{\infty} c_n x^{n+1} &= 0 \\
2c_2 + \sum_{n=0}^{\infty} \left[c_n + \sum_{m=0}^{n+3} \frac{(n-m+3)(n-m+2)c_{n-m+3}}{m!} \right] x^{n+1} &= 0
\end{aligned}$$

1M Therefore, inspecting a few n, we have

$$\begin{aligned}
c_2 &= 0 \\
c_0 + 6c_3 = 0 &\implies c_3 = -\frac{c_0}{6} \\
c_1 + 12c_4 + 6c_3 = 0 &\implies c_4 = \frac{c_0 - c_1}{12} \\
c_2 + 20c_5 + 12c_4 + 3c_3 = 0 &\implies c_5 = -\frac{1}{40}c_0 + \frac{1}{20}c_1 \\
c_3 + 30c_6 + 20c_5 + 6c_4 + c_3 = 0 &\implies c_6 = \frac{1}{90}c_0 - \frac{1}{60}c_1 \\
&\vdots
\end{aligned}$$

from those, we can write the first few terms of the solution

$$\begin{aligned}y &= c_0 + c_1 x - \frac{c_0}{6}x^3 + \left(\frac{1}{12}c_0 - \frac{1}{12}c_1\right)x^4 \\ &\quad + \left(-\frac{1}{40}c_0 + \frac{1}{20}c_1\right)x^5 + \left(\frac{1}{90}c_0 - \frac{1}{60}c_1\right)x^6 + \cdots \\ \phi_1 &= 1 - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{40}x^5 + \cdots \\ \phi_2 &= x - \frac{1}{12}x^4 + \frac{1}{20}x^5 - \frac{1}{60}x^6 + \cdots\end{aligned}$$

Since this equation has no singular point, the radius of convergence is infinite. Again, I will state explicitly how many nonzero terms you need to work out in your midterm.

Question5 (2 points)
Find two linearly independent Frobenius series solution at the regular singular point to

$$2t^2\ddot{y} + 3t(1+t)\dot{y} - y = 0$$

State the radius of convergence.

Solution:

1M Putting the equation into the standard form, we have

$$\ddot{y} + P(t)\dot{y} + \frac{-1}{2t^2}y = 0 \qquad \text{where} \quad P(t) = \frac{3(1+t)}{2t} \quad \text{and} \quad Q(t) = \frac{-1}{2t^2}$$

which shows P and Q are not analytic at $x = 0$. If we consider the following, which shows both p and q are polynomial, thus analytic at $x = 0$,

$$p(t) = tP(t) = \frac{3(1+t)}{2} = \frac{3}{2} + \frac{3}{2}t \qquad \text{and} \qquad q(t) = t^2 Q(t) = -\frac{1}{2}$$

we can reach the conclusion that $x = 0$ is a regular singular point of this equation.

$$\begin{array}{lllll} p_0 = \frac{3}{2} & p_1 = \frac{3}{2} & p_2 = 0 & p_k = 0 & \text{for} \quad k = 3,4,5\ldots \\ q_0 = -\frac{1}{2} & q_1 = 0 & q_2 = 0 & q_k = 0 & \text{for} \quad k = 3,4,5\ldots \end{array}$$

In general, we have derived for the following equation on L12P29,

$$t^2\ddot{y} + tp\dot{y} + qy = 0$$

we need to satisfy the following conditions:

$$\begin{aligned} F &= [r(r-1) + p_0 r + q_0]\, c_0 x^r = \rho(r) c_0 x^r = 0 \\ G_n &= (n+r)(n+r-1)c_n + \sum_{k=0}^{n}\Big[(k+r)p_{n-k} + q_{n-k}\Big]c_k = 0 \quad \text{for } n \in \mathbb{N}_1. \end{aligned}$$

The first condition is satisfied by solving the indicial equation

$$\begin{aligned}\rho(r) &= 0\\ r(r-1)+p_0r+q_0 &= 0\\ r(r-1)+\frac{3}{2}r-\frac{1}{2} &= 0\\ (2r-1)(r+1) &= 0\\ \implies r_1 = \frac{1}{2}; \qquad r_2 &= -1\end{aligned}$$

Using p_n and q_n, since many of them are zeros, we can simplify G_n and set it to zero

$$\begin{aligned}(n+r)(n+r-1)c_n+\left(\frac{3}{2}(n+r)-\frac{1}{2}\right)c_n+\frac{3}{2}(n+r-1)c_{n-1} &= 0\\ (2n+2r-1)(n+r+1)c_n+3(n+r-1)c_{n-1} &= 0\\ \implies c_n = -\frac{3(n+r-1)}{(2n+2r-1)(n+r+1)}c_{n-1}\end{aligned}$$

Using the two distinct solutions to the indicial equation, we have

$$\begin{aligned}c_n &= -\frac{3(2n-1)}{2n(2n+3)}c_{n-1} && \text{or} \qquad c_n = -\frac{3(n-2)}{n(2n-3)}c_{n-1}\\ &= \left(-\frac{3}{2}\right)^n\frac{3}{n!(2n+3)(2n+1)}c_0 && \implies c_1 = -3c_0 \quad \text{and} \quad c_k = 0\end{aligned}$$

for $k = 2, 3, 4, \ldots$. Therefore two linearly independent solutions are

$$\begin{aligned}\phi_1 &= t^{1/2}\sum_{n=0}^{\infty}\left(-\frac{3}{2}\right)^n\frac{3}{n!(2n+3)(2n+1)}t^n\\ \phi_2 &= t^{-1}(1-3t)\end{aligned}$$

Using ratio test, we have the following for ϕ_1

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3}{2}\frac{2n+1}{(n+1)(2n+1)}|t| \to 0 \qquad \text{as} \quad n\to\infty$$

and ϕ_2 clearly converges for all $t \neq 0$, thus the radius of convergence is infinite.

Question6 (2 points)

For the differential equation

$$xy''-y=0, \qquad x>0$$

(a) (1 point) Show that the roots of the indicial equation are

$$r_1 = 1 \qquad \text{and} \qquad r_2 = 0$$

and determine the Frobenius series solution corresponding to $r_1 = 1$.

Solution:

1M Again using the following

$$F = [r(r-1) + p_0 r + q_0]\, c_0 x^r = \rho(r) c_0 x^r = 0$$
$$G_n = (n+r)(n+r-1)c_n + \sum_{k=0}^{n} \Big[(k+r)p_{n-k} + q_{n-k}\Big] c_k = 0 \quad \text{for } n \in \mathbb{N}_1.$$

and $p_0 = 0$ and $q_0 = 0$, we have

$$\rho(r) = r(r-1) = 0 \implies r = 0 \qquad \text{or} \qquad r = 1$$

Using $q_1 = -1$ and all other p_n and q_n are zero, we have

$$G_n = (n+r)(n+r-1)c_n - c_{n-1}$$

The first solution is guaranteed to be found by using the larger root $r = 1$,

$$G_n = 0 \quad \text{for } n \in \mathbb{N}_1.$$
$$n(n+1)c_n - c_{n-1} = 0$$
$$\implies c_n = \frac{c_{n-1}}{n(n+1)} = \frac{c_0}{(n!)^2\,(n+1)}$$

Thus the first solution is

$$\phi_1 = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n!)^2\,(n+1)}$$

(b) (1 point) Find the second linearly independent solution.

Solution:

1M As shown in class, in general, we have

$$\mathcal{L}[\varphi(x,r)] = f(x,r) = c_0\rho(r)x^r = c_0(r-r_1)(r-r_2)x^r$$
$$\frac{\partial}{\partial r}\mathcal{L}[\varphi(x,r)] = \frac{\partial}{\partial r}[c_0(r-r_1)(r-r_2)x^r]$$
$$= c_0\left[(2r - r_1 - r_2)\,x^r + r(r-r_1)(r-r_2)x^{r-1}\right]$$

which shows the partial derivative is NOT a solution when we have distinct roots to $\rho(r)$, however, in general the following is a solution, and the reason is given below

$$\mathcal{L}\left[\frac{\partial}{\partial r}\Big((r-r_2)\varphi(x,r)\Big)\right] = \mathcal{L}\left[\varphi + (r-r_2)\frac{\partial \varphi}{\partial r}\right]$$
$$= \mathcal{L}[\varphi] + (r-r_2)\mathcal{L}\left[\frac{\partial \varphi}{\partial r}\right]$$
$$= c_0(r-r_1)(r-r_2)x^r + (r-r_2)\mathcal{L}\left[\frac{\partial \varphi}{\partial r}\right]$$

When $r = r_2$, we have

$$\mathcal{L}\left[\frac{\partial}{\partial r}\Big((r-r_2)\varphi(x,r)\Big)\right]\Bigg|_{r=r_2} = 0 + 0 = 0$$

thus it is a solution to the differential equation since $\mathcal{L}$ is with respect to x only. However, in this case, $\varphi(x, r)$ is not easy to derive,

$$G_n = 0 \implies c_n = \frac{1}{(n+r)(n+r-1)} c_{n-1} \qquad \text{for} \quad n \in \mathbb{N}_1$$
$$\implies \varphi(x, r) = c_0 x^r \left(1 + \sum_{n=1}^{\infty} \frac{n+r}{\Gamma(n+r+1)^2} x^n \right)$$

and working out the derivative function of a gamma function requires a lot of additional work. So in such cases, substitution is actually an easy option. Using

$$\phi_2 = C\phi_1 \ln x + x^{r_2} \sum_{n=0}^{\infty} c_n^* x^n$$

we can workout the following to be the second linearly independent solution,

$$\phi_2 = \ln x \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n!)^2 (n+1)} + \left(1 - \frac{3}{4} x^2 - \frac{7}{36} x^3 - \frac{35}{1728} x^4 + \cdots \right)$$

where the arbitrary constant $C = c_0^*$, and have been set to one. I omit the details here, see L12P21 for a similar example.