# Vv255 Lecture 9

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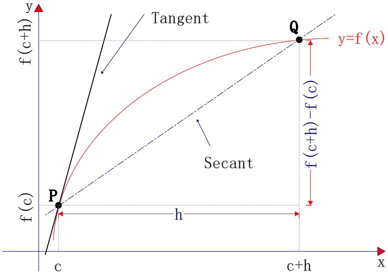
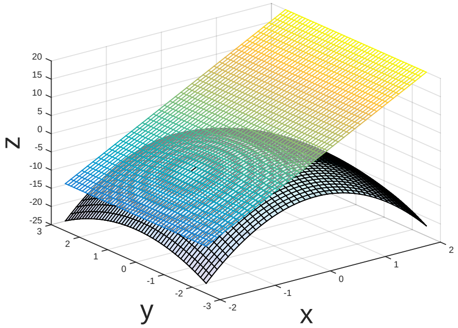
June 8, 2017

- Recall the definition of the derivative of a scalar-valued function $y = f(x)$ is:




$$f(x,y) = -4x^2 - y^2$$

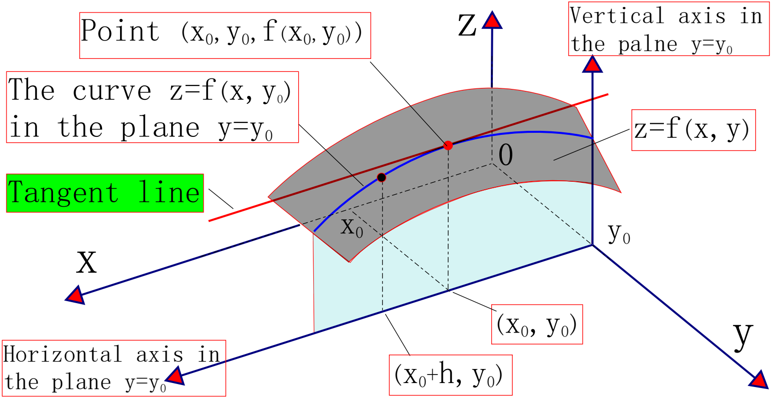



- With the function $f(x,y)$, we have infinitely many tangent lines at a point

$$(x_0, y_0)$$

- Here we need a definition of the tangent plane

- Similar to what we did for the limit and the continuity of $f(x, y)$, let us begin by considering the derivative of $f(x, y)$ along a horizontal path.



- Consider the surface $z = f(x, y)$.
- Suppose $(x_0, y_0)$ is a point in the domain of a function $f(x, y)$.
- The vertical plane $y = y_0$ cuts the surface $z = f(x, y)$ in half.
- The curve $z = f(x, y_0)$ is the intersection of $z = f(x, y)$ and $y = y_0$, and

  $(x_0, y_0, f(x_0, y_0))$ is on this intersection.
- In the plane $y = y_0$, the only independent variable is $x$, having an increment $h$ in $x$ defines a second point $(x_0 + h, y_0, f(x_0 + h, y_0))$ on the curve of

$$z = f(x, y_0)$$

- Consider the secant line that passes through both points

  $(x_0 + h, y_0, f(x_0 + h, y_0))$ and $(x_0, y_0, f(x_0, y_0))$.
- As $h \to 0$, the secant becomes the tangent to the curve $z = f(x, y_0)$ at $(x_0, y_0, f(x_0, y_0))$ in the $y = y_0$ plane, and

$$\frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \to f(x_0; y_0) \qquad \text{as} \qquad h \to 0$$

Q: So what is the geometric interpretation of

$$\lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$$

### Definition

The partial derivative of $f(x,y)$ with respective to $x$ at the point $(x_0,y_0)$ is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}=\lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h},\qquad \text{provided the limit exists.}$$

- We define the partial derivative of $f$ with respect to $x$ at the point $(x_0,y_0)$ as the ordinary derivative of $f(x,y_0)$ with respect to $x$ at the point $x=x_0$.
- An equivalent expression for the partial derivative is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}=\left.\frac{d}{dx}f(x,y_0)\right|_{x=x_0}$$

- The partial derivative of $f$ with respect to $x$ at $(x_0, y_0)$ gives the slope of the curve $z = f(x, y_0)$ at the point $(x_0, y_0, f(x_0, y_0))$ in the plane $y = y_0$.
- In other words, the partial derivative $\dfrac{\partial f}{\partial x}$ at $(x_0, y_0)$ gives the rate of change of $f$ with respect to $x$ when $y$ is held fixed at the value $y_0$ .
- There are two more common notations for the partial derivative:

$$f_x \qquad \partial_x f$$

- The definition of the partial derivative with respect to $y$ at $(x_0, y_0)$ is similar to the definition of the partial derivative of $f$ with respect to $x$. We hold $x$ fixed at the value $x_0$ and differentiate $f(x_0, y)$ with respect to $y$ at $y_0$.

### Definition

The partial derivative of $f(x, y)$ with respective to $y$ at the point $(x_0, y_0)$ is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \left.\frac{d}{dy} f(x_0, y)\right|_{y=y_0} = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

provided the limit exists.

## Exercise

(a) Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1$$

- Implicit differentiation works for partial derivatives in the same way.

## Exercise

(b) Given the following equation defines $z$ as a function of the two independent variables $x$ and $y$ and the partial derivative exists, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

$$x^2 + \cos y + z^3 = 1$$

(c) The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at $(1, 2, 5)$.

- If we differentiate a function $f(x, y)$ twice, we obtain its second-order partial derivatives. These derivatives are usually denoted by

$$\frac{\partial^2 f}{\partial x^2} \quad \text{or} \quad f_{xx}, \qquad \frac{\partial^2 f}{\partial y^2} \quad \text{or} \quad f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} \quad \text{or} \quad f_{yx}, \qquad \frac{\partial^2 f}{\partial y \partial x} \quad \text{or} \quad f_{xy}$$

- The defining equations are

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \quad \text{and so on.}$$

- Notice the order in which the mixed partial derivatives are taken:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx} = (f_y)_x$$

**Exercise**

Suppose $f(x, y) = x \cos y + y e^x$, find the second-order derivatives.

Q: Are the mixed partial derivatives $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ always equal?

- Consider the mixed partial derivatives of the following function

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{for} \quad (x,y) \neq (0,0), \\ 0 & \text{for} \quad (x,y) = (0,0). \end{cases}$$

### The Mixed Derivative Theorem

If $f(x,y)$ and its partial derivatives $f_x$, $f_y$, $f_{xy}$ and $f_{yx}$ are defined throughout an open region containing a point $(a,b)$ and are all continuous at $(a,b)$, then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

### Exercise

Find $\dfrac{\partial^2 w}{\partial x \partial y}$ where $w = xy + \dfrac{e^y}{y^2+1}$