Question1 (3 points)
Let $\mathcal{V}$ be the space $\mathcal{C}[-1,1]$ with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$$

(a) (1 point) Find the orthogonal projection of e^x onto the subspace

$$\mathcal{H} = \operatorname{span}\left\{1, x, x^2\right\}$$

Solution:

1M In order to find the projection, we need an orthogonal basis for $\mathcal{H}$. This is similar to the exercise on L26P7, we use orthogonal project to construct an orthonormal basis for $\mathcal{H}$

$$\left\{\sqrt{\frac{1}{2}},\quad \sqrt{\frac{3}{2}}x,\quad \sqrt{\frac{45}{8}}\left(x^2-\frac{1}{3}\right)\right\}$$

where the vectors and normalization constants are found by using the given definition of inner product for $\mathcal{V}$. Clearly, such bases are not unique.

$$\langle 1, 1\rangle = \int_{-1}^{1} 1\,dx = 2$$

$$x - \left\langle \frac{1}{\sqrt{\langle 1,1\rangle}}, x\right\rangle \frac{1}{\sqrt{\langle 1,1\rangle}} = x - \frac{1}{2}\int_{-1}^{1} x\,dx = x$$

$$\langle x, x\rangle = \int_{-1}^{1} x^2\,dx = \frac{2}{3}$$

$$x^2 - \left\langle \frac{1}{\sqrt{\langle 1,1\rangle}}, x^2\right\rangle \frac{1}{\sqrt{\langle 1,1\rangle}} - \left\langle \frac{x}{\sqrt{\langle x,x\rangle}}, x^2\right\rangle \frac{x}{\sqrt{\langle x,x\rangle}} = x^2 - \frac{1}{3} - 0$$

$$\left\langle x^2-\frac{1}{3}, x^2-\frac{1}{3}\right\rangle = \frac{8}{45}$$

The projection is given by

$$\begin{aligned}
\operatorname{proj}_{\mathcal{H}} \mathrm{e}^x &= \left\langle \frac{1}{\sqrt{\langle 1,1\rangle}}, \mathrm{e}^x\right\rangle \frac{1}{\sqrt{\langle 1,1\rangle}} + \left\langle \frac{x}{\sqrt{\langle x,x\rangle}}, \mathrm{e}^x\right\rangle \frac{x}{\sqrt{\langle x,x\rangle}} \\
&\quad + \left\langle \frac{x^2-\frac{1}{3}}{\sqrt{\langle x^2-\frac{1}{3}, x^2-\frac{1}{3}\rangle}}, \mathrm{e}^x\right\rangle \frac{x^2-\frac{1}{3}}{\sqrt{\langle x^2-\frac{1}{3}, x^2-\frac{1}{3}\rangle}} \\
&= \frac{1}{2}\int_{-1}^{1} \mathrm{e}^x\,dx + \frac{3}{2}x\int_{-1}^{1} x\mathrm{e}^x\,dx + \frac{45}{8}\left(x^2-\frac{1}{3}\right)\int_{-1}^{1}\left(x^2-\frac{1}{3}\right)\mathrm{e}^x\,dx \\
&= \frac{\mathrm{e}}{2} - \frac{\mathrm{e}^{-1}}{2} + 3\,x\,\mathrm{e}^{-1} + \frac{15\,\mathrm{e}^{-1}\,\left(x^2-\frac{1}{3}\right)\,\left(\mathrm{e}^2-7\right)}{4}
\end{aligned}$$

(b) (1 point) Find the orthogonal projection of e^x onto the subspace

$$\mathcal{W} = \text{span}\{1, \cos x, \sin x\}$$

Solution:

1M This is similar to part (a), the point is that it is not the same as Fourier basis,

$$\left\{\sqrt{\frac{1}{2}}, \quad \sqrt{\frac{2}{2-\sin 2}}\sin x, \quad \sqrt{\frac{2}{2\cos 2+\sin 2}}(\cos x - \sin 1)\right\}$$

and the projection is given by

$$\text{proj}_{\mathcal{W}}\,\mathrm{e}^x = \frac{\mathrm{e}}{2} - \frac{\mathrm{e}^{-1}}{2} + \frac{2\,\sin(x)\,(\cos(1)\,\sinh(1) - \cosh(1)\,\sin(1))}{(\sin(2)-2)} + \frac{2\,(\cos(x)-\sin(1))\,(\cos(1)\,\sinh(1)+\cosh(1)\,\sin(1) - 2\,\sin(1)\,\sinh(1))}{(2\,\cos(2)+\sin(2))}$$

(c) (1 point) Explain why both of the orthogonal projections are the best.

Solution:

1M They are the best in their own subspace, that is, $\text{proj}_{\mathcal{H}}\,\mathrm{e}^x$ is the best in $\mathcal{H}$ and $\text{proj}_{\mathcal{W}}\,\mathrm{e}^x$ is the best in $\mathcal{W}$. Notice how different inner products often result difference basis and different coefficients. You shall be thankful to Fourier, for otherwise, you would have the above to do in your final.

Question2 (1 points)
Using the Fourier series to find the steady-state current $i(t)$ in a simple RLC-circuit,

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{dE}{dt}$$

where $R = 10\Omega$, $L = 1H$, $C = 10^{-1}F$ and

$$E(t) = \begin{cases} -50t^2 & \text{if} \quad -\pi < t < 0 \\ 50t^2 & \text{if} \quad 0 < t < \pi \end{cases} \qquad E(t+2\pi) = E(t)$$

Solution:

1M Again the steady-state solution here is taken to be the particular solution

$$\frac{dE}{dt} = \begin{cases} -100t & \text{if} \quad -\pi < t < 0 \\ 100t & \text{if} \quad 0 < t < \pi \end{cases}$$

Find the Fourier series,

$$a_0 = 100\pi \qquad a_n = -\frac{100\,(-1)^{n+1}\,((-1)^n - 1)^2}{\pi\,n^2} \qquad \text{and} \qquad b_n = 0$$

since the $\dfrac{dE}{dt}$ is even.

$$\frac{d^2i}{dt^2} + 10\frac{di}{dt} + 10i = 50\pi + \frac{-400}{\pi}\cos t + \frac{-400}{\pi 3^2}\cos 3t + \frac{-400}{\pi 5^2}\cos 5t + \cdots$$

Identify the annihilator

$$\mathcal{D}\prod_{n=1}^{\infty}\left(\mathcal{D}+\mathrm{j}(2n-1)\right)\left(\mathcal{D}-\mathrm{j}(2n-1)\right) \qquad \text{where j is the imaginary unit.}$$

Hence the steady-state solution takes the following form

$$i_p = A_0 + A_1\cos t + B_1\sin t + A_3\cos 3t + B_3\sin 3t + \ldots,$$

Substitute $\dfrac{d^2i_p}{dt^2}$, $\dfrac{di_p}{dt}$ and i_p into

$$\frac{d^2i}{dt^2} + 10\frac{di}{dt} + 10i = 50\pi + \frac{-400}{\pi}\cos t + \frac{-400}{\pi 3^2}\cos 3t + \frac{-400}{\pi 5^2}\cos 5t + \cdots$$

We have

$$i_p = 5\pi + A_1\cos t + B_1\sin t + A_3\cos 3t + B_3\sin 3t + \ldots,$$

where

$$A_n = \frac{-400(10-n^2)}{n^2\pi\big((n^2-10)^2+100n^2\big)} \quad \text{and} \quad B_n = \frac{-400(10n)}{n^2\pi\big((n^2-10)^2+100n^2\big)}$$

Question3 (1 points)
Using the Fourier series to find the steady-state current $i(t)$ in a simple RLC-circuit,

$$L\frac{di}{dt} + Ri = E(t), \quad \text{where} \quad E(t) = \begin{cases} 1 & \text{if} \quad 0<t<1 \\ 0 & \text{if} \quad 1<t<2 \end{cases}, \qquad E(t+2) = E(t)$$

Solution:

1M Note this problem is not from $-\pi$ to π. Finding the Fourier series for $E(t)$, we have

$$\begin{aligned}
a_0 &= \frac{1}{1}\int_{-1}^{1} E(t)\,dt &&= \int_0^1 1\,dt &&= 1 \\
a_k &= \frac{1}{1}\int_{-1}^{1} E(t)\cos\frac{k\pi}{1}t\,dt &&= \int_0^1 \cos k\pi t\,dt &&= 0 \\
b_k &= \frac{1}{1}\int_{-1}^{1} E(t)\sin\frac{k\pi}{1}t\,dt &&= \int_0^1 \sin k\pi t\,dt &&= \begin{cases} 0 & \text{if } k \text{ is even,} \\ 2/(k\pi) & \text{if } k \text{ is odd.} \end{cases}
\end{aligned}$$

thus

$$E(t) = \frac{1}{2} + \sum_{\ell=1}^{\infty}\frac{2}{(2\ell-1)\pi}\sin(2\ell-1)\pi t$$

The annihilator for $E(t)$ is given by

$$\mathcal{D}\prod_{\ell=1}^{\infty}\left(\mathcal{D}+\mathrm{j}(2\ell-1)\pi\right)\left(\mathcal{D}-\mathrm{j}(2\ell-1)\pi\right)$$

Thus the steady-state solution is given by

$$i_p = A_0+\sum_{\ell=1}^{\infty}\left(A_{2\ell-1}\cos(2\ell-1)\pi t+B_{2\ell-1}\sin(2\ell-1)\pi t\right)$$

Applying the differential operator $(L\mathcal{D}+R)$ to i_p to determine those coefficients.

$$\begin{aligned} RA_0 &= \frac{1}{2} \\ B_{2\ell-1}R - A_{2\ell-1}L(2\ell-1)\pi &= \frac{2}{(2\ell-1)\pi} \\ A_{2\ell-1}R - B_{2\ell-1}L(2\ell-1)\pi &= 0 \end{aligned}$$

Therefore

$$A_0=\frac{1}{2R};\quad A_{2\ell-1}=\frac{-2L}{D_{2\ell-1}};\quad B_{2\ell-1}=\frac{-2R}{(2\ell-1)\pi D_{2\ell-1}}$$

where

$$D_{2\ell-1}=L^2(2\ell-1)^2\pi^2-R^2$$

Note this is the example we covered in class when we were talking about using Laplace transform on periodic forcing functions. It isn't something you do in your head, however, it can be done and easier this way. This is the end of Vv256! Thanks for reading this. Good luck with your exam!