
Question1 (9 points)

(a) (1 point) Find the distance from the point $(2, 1, 3)$ to the line

$$\frac{x-2}{2} = \frac{y-1}{6}; \quad z = 3$$

(b) (1 point) Find an equation for the line tangent to the curve at $t = 0$.

$$\mathbf{r}(t) = \frac{1}{t+1}\mathbf{e}_x + \frac{t}{t-1}\mathbf{e}_y + \frac{t-1}{t+1}\mathbf{e}_z$$

(c) (1 point) Find the vector equation that describes the plane that is orthogonal to

$$x + y - 2z = 1$$

and passes through the line of intersection of the planes

$$x - z = 1 \qquad \text{and} \qquad y + 2z = 3$$

(d) (1 point) Find a vector-valued function for the curve of intersection of

$$\text{the cone} \quad z = \sqrt{x^2 + y^2} \quad \text{and the plane} \quad z = 1 + y.$$

(e) (1 point) By considering two different planes whose intersection is the line

$$x = 1 + t, \qquad y = 2 - t, \qquad \text{and} \qquad z = 3 + 2t,$$

find a parametrization for all planes that contain the line

x = 1 + t, y = 2 - t, z = 3 + 2t.

(f) (2 points) Consider the line ℓ_1 that is defined by the point $(-2, 0, 2)$ and the vector

$$\mathbf{e}_x - \mathbf{e}_z$$

and the line ℓ_2 that is defined by $(-3, 2, 7)$ and the vector

$$\mathbf{e}_x + 5\mathbf{e}_y + \mathbf{e}_z$$

Find the plane that has the following two properties:

- There is a line that is perpendicular to both ℓ_1 and ℓ_2 on this plane.
- This plane makes an angle of $\frac{\pi}{4}$ with the following plane

$$x - 4y + 8z + 12 = 0$$

(g) (2 points) Let L be the line of the intersection between two planes

$$x + y - z - 1 = 0 \qquad \text{and} \qquad x - y + z + 1 = 0$$

and let π denote the plane

$$x + y + z = 0$$

Suppose M is the projection of the line L onto the plane π. Find the equation of the surface that is formed by the straight line M rotating around the z-axis.

Question2 (1 points)

Compute the determinant of $\mathbf{A}$, which is a product of the following two matrices

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -5 & 4 \\ 4 & -1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 2 \\ 3 & 7 & 1 \end{bmatrix}$$

Question3 (2 points)

Suppose there are two non-parallel lines in $\mathbb{R}^3$

$$L_i = \frac{x - x_i}{m_i} = \frac{y - y_i}{n_i} = \frac{z - z_i}{p_i} \qquad \text{for} \qquad i = 1, 2$$

Show that any point $P(x, y, z)$ on the plane which passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to both L_1 and L_2 satisfies the following equation

$$\det \begin{bmatrix} x - x_0 & y - y_0 & z - z_0 \\ m_1 & n_1 & p_1 \\ m_2 & n_2 & p_2 \end{bmatrix} = 0$$

Question4 (1 points)

Show all vectors $\mathbf{w}$ such that $\mathbf{u} \times \mathbf{w} = \mathbf{v}$ have the form $\mathbf{w} = \alpha\mathbf{u} + \dfrac{\mathbf{v} \times \mathbf{u}}{|\mathbf{u}|^2}$.

Question5 (1 points)

Let P be a point not on the plane that is defined by points Q, R and S. Derive a formula for the distance from P to the plane in terms of only $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

$$\mathbf{a} = \vec{QR}, \quad \mathbf{b} = \vec{QS}, \quad \text{and} \quad \mathbf{c} = \vec{QP}$$

Your formula shall not be in terms of any angle.

Question6 (1 points)

Use vector algebra instead of geometry to show that the three normals dropped from the vertices of a triangle perpendicular to their opposite sides intersect at the same point.

Question7 (2 points)

Consider an arbitrary tetrahedron $ABCD$. Let T_1, T_2, T_3 and T_4 denote

$$\text{the area of triangle} \, \triangle ABC, \, \triangle BCD, \, \triangle ACD \, \text{and} \, \triangle ABD$$

respectively. Let $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ and $\mathbf{n}_4$ denote the outward pointing normal vector for

$$\text{the triangular face} \, \triangle ABC, \, \triangle BCD, \, \triangle ACD \, \text{and} \, \triangle ABD$$

respectively, such that

$$T_1 = |\mathbf{n}_1|, \quad T_2 = |\mathbf{n}_2|, \quad T_3 = |\mathbf{n}_3|, \quad T_4 = |\mathbf{n}_4|$$

(a) (1 point) Use cross product to show that the sum of $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ and $\mathbf{n}_4$ is $\mathbf{0}$.

(b) (1 point) Suppose $\angle ADB = \angle BDC = \angle CDA = \dfrac{\pi}{2}$. Show that the areas satisfy

$$T_1^2 = T_2^2 + T_3^2 + T_4^2$$

Question8 (3 points)

Suppose your iPhone520s can fly like Thor's Hammer. On a very special day, your phone flies through space according to the following acceleration vector when summoned

$$\mathbf{a}(t) = \begin{bmatrix} -3\cos t \\ -3\sin t \\ 2 \end{bmatrix} \qquad \text{for } t \geq 0.$$

We also know the phone is at the point $(3, 0, 0)$ with the velocity $\mathbf{v} = 3\mathbf{e}_y$ at time $t = 0$.

The ghost named Jobs is involved in the design of iPhone520s which means it not only can fly but it is awesome and can carry you like a vehicle. However, your new iPhone automatically and instantaneously sends a "warning" message to your partner if it travels more than 5 miles away from her/him, and sends another lovely message if it comes back within this distance. Assume your partner is at the origin on the very special day, and the position vector $\mathbf{r}(t)$ is in miles.

(a) (1 point) Find the instantaneous speed at $t = 1$.

(b) (1 point) Find the number of messages your partner will receive.

(c) (1 point) Find the rate of change of the cross product between the position vector and the velocity vector with respect to time.

Question9 (2 points)

Suppose a particle P is moving along a curve $\mathcal{C}$ according to the vector-valued function

$$\mathbf{r}(t) = \begin{bmatrix} \sin\alpha \\ \alpha + \cos\alpha \\ \alpha \end{bmatrix} \qquad \text{where } \alpha \text{ is some scalar-valued function of } t.$$

The motion has a unit speed, i.e. $|\mathbf{v}| = 1$. Find the acceleration $\mathbf{a}$ in terms of only α.

Question10 (4 points)

A projectile is fired with an angle of elevation θ, and an initial velocity of $\mathbf{v}_0$. Assume the only force acting on the object is gravity.

(a) (1 point) Find the form of the vector-valued function for the position of the object.

(b) (1 point) Find an expression for the range in terms of θ.

(c) (1 point) Find the value of θ which maximises the range.

(d) (1 point) Find an expression for the distance travelled by it between t_1 and t_2 where

$$0 < t_1 \leq t_2$$

Question11 (3 points)

Suppose $\mathbf{r}(t)$ is continuous. Show the followings are true

(a) (1 point)

$$\int_a^b \mathbf{c} \cdot \mathbf{r}(t)\, dt = \mathbf{c} \cdot \int_a^b \mathbf{r}(t)\, dt, \qquad \text{where } \mathbf{c} \text{ is a constant vector.}$$

(b) (1 point)

$$\int_a^b \mathbf{c} \times \mathbf{r}(t)\, dt = \mathbf{c} \times \int_a^b \mathbf{r}(t)\, dt, \qquad \text{where } \mathbf{c} \text{ is a constant vector.}$$

(c) (1 point)

$$\frac{d}{dt}\int_a^t \mathbf{r}(\tau)\,d\tau = \mathbf{r}(t)$$

Question12 (1 points)

Suppose $\mathbf{r}(t) = (t+1)\mathbf{e}_x + (t^2+2)\mathbf{e}_y + 2t\mathbf{e}_z$. Find the osculating plane at time $t = 1$.

Question13 (4 points)

Santa Claus with his magic sleigh is travelling to see his aunt penguin as well his uncle polar bear, they are apparently separated. In order to see both of them Santa is moving along a meridian of the rotating earth with a constant speed. Let xyz be a fixed Cartesian coordinate system in space, with unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ in the directions of the axes. Let the Earth, together with a unit vector $\mathbf{b}$, be rotating about the z-axis with angular speed $\omega > 0$. Since $\mathbf{b}$ is rotating together with the Earth, it is of the form

$$\mathbf{b}(t) = \cos(\omega t)\mathbf{e}_x + \sin(\omega t)\mathbf{e}_y$$

In other words, the vector $\mathbf{b}$ defines the rotation of the Earth. Let Santa be moving on the meridian whose plane is defined by $\mathbf{b}$ and $\mathbf{e}_z$ with constant angular speed $\gamma > 0$. Then its position vector in terms of $\mathbf{b}$ and $\mathbf{e}_z$ is

$$\mathbf{r}(t) = R\cos(\gamma t)\mathbf{b} + R\sin(\gamma t)\mathbf{e}_z \qquad \text{where } R \text{ is the radius of the Earth.}$$

(a) (1 point) Find the unit tangent vector as a function of t for Santa.

(b) (1 point) Find the acceleration vector as a function of t for Santa.

(c) (1 point) What is the centripetal acceleration due to the rotation of the Earth?

(d) (1 point) What is the centripetal acceleration due to the motion of Santa on the meridian of the rotating Earth.

Question14 (6 points)

(a) (1 point) Find an expression for the torsion of C which is defined by

$$\mathbf{r} = t\mathbf{e}_x + t^2\mathbf{e}_y + t^3\mathbf{e}_z.$$

(b) (1 point) The smaller the curvature of a bend in a road, the faster a car can travel. Assume that the maximum speed around a turn is inversely proportional to the square root of the curvature. A car moving on the path $y = \frac{1}{3}x^3$ (x and y are measured in miles) can safely go 30 miles per hour at $\left(1, \frac{1}{3}\right)$. How fast can it go at $\left(\frac{3}{2}, \frac{9}{8}\right)$?

(c) (1 point) A curve C is given by the polar equation $r = f(\theta)$. Show that the curvature κ at the point (r, θ) is

$$\kappa = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}}$$

(d) (1 point) For a smooth curve given by the parametric equations $x = f(t)$ and $y = g(t)$, show that the curvature is given by

$$\kappa = \frac{|f'g'' - g'f''|}{\left\{[f']^2 + [g']^2\right\}^{3/2}}$$

(e) (1 point) For a smooth space curve defined by $\mathbf{r}(t)$, show the curvature is given by

$$\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

(f) (1 point) A sufficiently differentiable curve with zero torsion lies in a plane is a special case of the fact that a particle whose velocity $\mathbf{v}$ remains perpendicular to a fixed vector $\mathbf{c}$ moves in a plane perpendicular to $\mathbf{c}$. This, in turn, can be viewed as the following result. Suppose

$$\mathbf{r}(t) = f(t)\mathbf{e}_x + g(t)\mathbf{e}_y + h(t)\mathbf{e}_z$$

is twice differentiable for all t in an interval $[a, b]$, that $\mathbf{r} = \mathbf{0}$ when $t = a$, and that

$$\mathbf{v} \cdot \mathbf{e}_z = 0$$

for all t in $[a, b]$. Show that $h(t) = 0$ for all t in $[a, b]$.

Question15 (0 points)

(a) (1 point (bonus)) Suppose $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are unit vectors in $\mathbb{R}^3$. What is the geometric interpretation of the vector triple product? Justify your answer.

(b) (1 point (bonus)) A particle was at point $P_1 = (x_1, y_1, z_1)$ at time t_1 and is moving at the constant velocity

$$\mathbf{v} = v_1\mathbf{e}_x + v_2\mathbf{e}_y + v_3\mathbf{e}_z$$

Another particle was at $P_2 = (x_2, y_2, z_2)$ at t_2 and is moving at the constant velocity

$$\mathbf{u} = u_1\mathbf{e}_x + u_2\mathbf{e}_y + u_3\mathbf{e}_z$$

How close did the particles get to each other and at what time? What conditions are needed for a collision?

(c) (1 point (bonus)) Consider whether the following two functions/curves are smooth.

$$\mathbf{r}_1 = t^3\mathbf{e}_x + t^6\mathbf{e}_y + 0\mathbf{e}_z \qquad \text{and} \qquad \mathbf{r}_2 = t\mathbf{e}_x + t^2\mathbf{e}_y + 0\mathbf{e}_z$$

What is the seemingly unresolvable dilemma? What do those two examples provide regarding smoothness?

(d) (1 point (bonus)) Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vector-valued functions of t in $\mathbb{R}^3$, and

$$\begin{aligned} f(t) &= f(\mathbf{u}, \mathbf{v}, \mathbf{w}) \\ &= \det\left(\begin{bmatrix} u_1(t) & u_2(t) & u_3(t) \\ v_1(t) & v_2(t) & v_3(t) \\ w_1(t) & w_2(t) & w_3(t) \end{bmatrix}\right) \end{aligned}$$

where $u_i(t)$, $v_i(t)$ and $w_i(t)$ are component functions of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Show that

$$\begin{aligned} \frac{df}{dt} &= \frac{d}{dt}\left(f(\mathbf{u}(t), \mathbf{v}(t), \mathbf{w}(t))\right) \\ &= f(\mathbf{u}'(t), \mathbf{v}(t), \mathbf{w}(t)) + f(\mathbf{u}(t), \mathbf{v}'(t), \mathbf{w}(t)) + f(\mathbf{u}(t), \mathbf{v}(t), \mathbf{w}'(t)) \end{aligned}$$

(e) (1 point (bonus)) A cable of radius r and length L is wound around a cylinder of radius R without overlapping. What is the shortest length along the cylinder that is covered by the cable?