Vv255 Lecture 5

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- Recall for a function $f\colon \mathbb{R} \to \mathbb{R}$, we had the following definition of limit.

Definition

Let f be a function defined on some open interval that contains the number a, except possibly at a itself. The value of L is the limit of $f(x)$ as x approaches a,

$$\lim_{x \to a} f(x) = L$$

if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \qquad \text{if} \qquad 0 < |x - a| < \delta$$

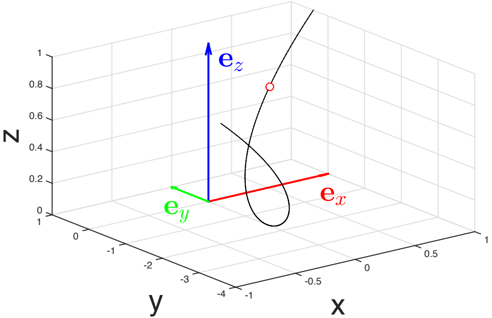
- Recall the limit L is the value that the values of $f(x)$ can be made as close as we like by taking x sufficiently close to a.

Q: Intuitively, what do you think the limit of a vector-valued function shall be

$$\mathbf{r}(t) \qquad \text{as} \qquad t \to a?$$

- Here is a vector-valued function, for which the concept of limit is useful.

$$\mathbf{r}(t) = t^3\mathbf{e}_x + \frac{2(t-0.8)^3}{(t-0.8)^2}\mathbf{e}_y + t^2\mathbf{e}_z$$



- We would like to define the limit such that the limit of $\mathbf{r}(t)$ exists and is equal to the red dot as $t \to 0.8$ despite the function being undefined there.

Q: Formally, how shall we define the limit of a vector-valued function

$$\mathbf{r}(t) \qquad \text{as} \qquad t \to a$$

Definition

Suppose

$$\mathbf{r}(t) = f(t)\mathbf{e}_x + g(t)\mathbf{e}_y + h(t)\mathbf{e}_z$$

is a vector-valued function defined on some open interval contains a.

The vector value $\boldsymbol{\ell}$ is said to be the limit of $\mathbf{r}$ as t approaches a, deonted

$$\lim_{t \to a} \mathbf{r}(t) = \boldsymbol{\ell}$$

if, for every number $\epsilon > 0$, there exists a number $\delta > 0$ such that

$$|\mathbf{r}(t) - \boldsymbol{\ell}| < \epsilon \qquad \text{whenever} \qquad 0 < |t - a| < \delta$$

Q: How shall we find the limit of a vector-valued function $\mathbf{r}(t)$ as $t \to a$?

- If $\mathbf{r}(t) = \begin{bmatrix} f(t) \\ 0 \\ 0 \end{bmatrix} = f(t)\mathbf{e}_x$, it is clear that the vector-valued function

$$\mathbf{r}(t) \to \left[\lim_{t\to a} f(t)\right]\mathbf{e}_x \qquad \text{as} \qquad t \to a$$

- Now suppose that

$$\mathbf{r}(t) = f(t)\mathbf{e}_x + g(t)\mathbf{e}_y$$

where either $f(t)$ or $g(t)$ is undefined at $t = a$, but as $t \to a$

$$f(t) \to \ell_1 \qquad \text{and} \qquad g(t) \to \ell_2$$

we expect the point $(f(t), g(t), 0)$ can be made arbitrarily close to $(\ell_1, \ell_2, 0)$, that is, for every ϵ, by taking t sufficiently close to a, that is, there exists

$$\delta = \min(\delta_1, \delta_2)$$

- It is even easier to see that the converse is true.

- Therefore, the way we calculate the limit of a vector-valued function is by computing of the limit of a scalar-valued function for each component.

Theorem

Let $\mathbf{r}(t) = f(t)\mathbf{e}_x + g(t)\mathbf{e}_y + h(t)\mathbf{e}_z$. The limit of $\mathbf{r}$ as $t \to a$ is given by

$$\lim_{t\to a} \mathbf{r}(t) = \begin{bmatrix} \lim\limits_{t\to a} f(t) \\ \\ \lim\limits_{t\to a} g(t) \\ \\ \lim\limits_{t\to a} h(t) \end{bmatrix}$$

provided all the respective limits of the component functions exist.

Q: Why the notion of the limit approaching $\pm\infty$ that we used when dealing with functions from $\mathbb{R} \to \mathbb{R}$ is not appropriate for vector-valued functions?

Properties of the limit

Let $\mathbf{u}(t)$, $\mathbf{v}(t)$ be two vector-valued functions and $\phi(t)$ is a scalar-valued function.

1. $$\lim_{t\to a}\Big(\mathbf{u}(t)+\mathbf{v}(t)\Big)=\lim_{t\to a}\mathbf{u}(t)+\lim_{t\to a}\mathbf{v}(t)$$

2. $$\lim_{t\to a}\Big(\phi(t)\mathbf{u}(t)\Big)=\lim_{t\to a}\phi(t)\lim_{t\to a}\mathbf{v}(t)$$

3. $$\lim_{t\to a}\Big(\mathbf{u}(t)\cdot\mathbf{v}(t)\Big)=\lim_{t\to a}\mathbf{u}(t)\cdot\lim_{t\to a}\mathbf{v}(t)$$

4. $$\lim_{t\to a}\Big(\mathbf{u}(t)\times\mathbf{v}(t)\Big)=\lim_{t\to a}\mathbf{u}(t)\times\lim_{t\to a}\mathbf{v}(t)$$

Q: Why intuitively those laws are clearly true?

- Recall how we defined a function $y = f(x)$ being continuous at a point a,

$$3.\ \lim_{x \to a} f(x) = f(a)$$

Definition

A vector-valued function $\mathbf{r}(t)$ is continuous at a point $t = a$ in its domain if

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$$

The function is continuous on a set if it is continuous at every point in it.

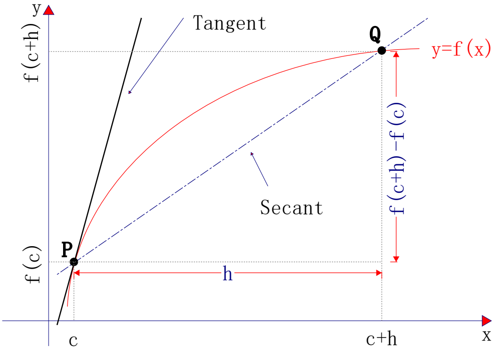
- Therefore a vector-valued function $\mathbf{r}(t)$ is continuous at $t = a$ if and only if all component functions are continuous at $t = a$.

Q: Is the following function continuous at $t = 0$,

$$\mathbf{r}(t) = \begin{cases} t\mathbf{e}_x + (2t+1)\mathbf{e}_y + 2t\mathbf{e}_z & \text{if} \quad t \leq 0, \\ t\mathbf{e}_x + 2t\mathbf{e}_y - t\mathbf{e}_z & \text{if} \quad t > 0. \end{cases}$$

- Recall the definition of the derivative of a scalar-valued function $y = f(x)$ is:

$$f'(c) = \lim_{h \to 0} \left[\frac{f(c+h) - f(c)}{h} \right]$$

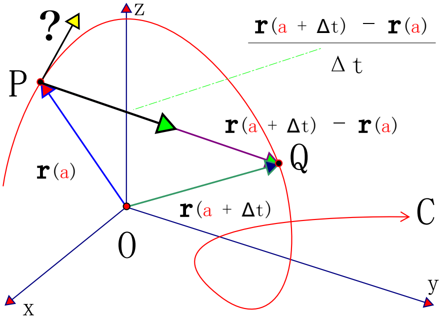


- And we say $f(x)$ is differentiable at $x = c$, when the above limit exists.

Q: How shall we define the derivative for a vector-valued function?

Q: Can we still form a secant? Can we still think in terms of the motion of Q?

- Let $\mathbf{r}(t) = f(t)\mathbf{e}_x + g(t)\mathbf{e}_y + h(t)\mathbf{e}_z$ be the position vector of a particle moving along a curve $\mathcal{C}$ in space and that f, g, and h are differentiable.



- The difference between the positions at time $t = a$ and time $c + \Delta t$ is

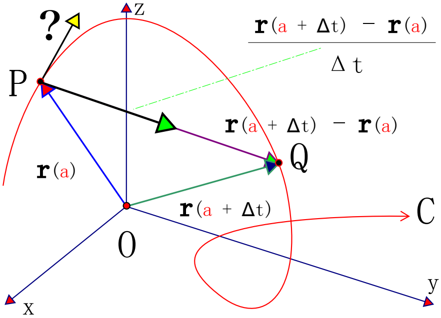
$$\mathbf{r}(a + \Delta t) - \mathbf{r}(a)$$

- Note the difference is the change in position, thus it is the displacement.

- If divide the displacement over the change in time,

$$\frac{\Delta \mathbf{r}}{\Delta t} = \frac{\mathbf{r}(a + \Delta t) - \mathbf{r}(a)}{\Delta t}$$

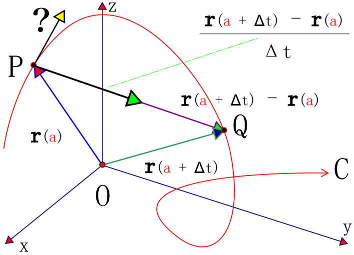
we obtain the average velocity.



Q: What happens to our average velocity as Δt approaches zero?

- As Δt approaches zero, three things seem to happen simultaneously.

1. Q approaches P along the curve C.
2. The vector $\vec{PQ}$ approaches its limiting position on the curve at P.
3. The average velocity $\dfrac{\Delta \mathbf{r}}{\Delta t}$ approaches the instantaneous velocity at P.



Q: Clearly the average velocity is 0 if $\Delta t = 0$, how about the instantaneous one?

Q: Is the limit of the average velocity zero as $\Delta t \to 0$?

- It is clear that the instantaneous velocity is the vector that consists of the derivatives of the component functions of $\mathbf{r}$.

$$\lim_{\Delta \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \lim_{\Delta \to 0} \frac{\mathbf{r}(a+\Delta t) - \mathbf{r}(a)}{\Delta t} = \begin{bmatrix} \lim_{\Delta t \to 0} \dfrac{f(a+\Delta t) - f(a)}{\Delta t} \\ \lim_{\Delta t \to 0} \dfrac{g(a+\Delta t) - g(a)}{\Delta t} \\ \lim_{\Delta t \to 0} \dfrac{h(a+\Delta t) - h(a)}{\Delta t} \end{bmatrix} = \begin{bmatrix} f'(a) \\ g'(a) \\ h'(a) \end{bmatrix}$$

Definition

In general, the vector-valued function $\mathbf{r}(t) = f(t)\mathbf{e}_x + g(t)\mathbf{e}_y + h(t)\mathbf{e}_z$ is said to be differentiable at $t = a$ if and only if f, g, and h are differentiable at $t = a$.

The derivative function, or simply derivative, of $\mathbf{r}(t)$ is the vector-value function

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = f'(t)\mathbf{e}_x + g'(t)\mathbf{e}_y + h'(t)\mathbf{e}_z$$

$\mathbf{r}$ is said to be differentiable if it is differentiable at every point of its domain.

- The derivative, when differs from $\mathbf{0}$, gives the direction along the curve at t,

$$\mathbf{r}'(t)$$

and is defined to be the vector tangent to the curve.

- The tangent line to the curve at a point $(f(a), g(t_0), h(a))$ is defined to be the line through the point parallel to $\mathbf{r}'(a)$.

Definition

The curve traced by $\mathbf{r}(t)$ is defined to be smooth if $\mathbf{r}'$ is continuous and never $\mathbf{0}$, that is, if the component functions have continuous first derivatives

$$\lim_{t \to a} \mathbf{r}'(t) = \mathbf{r}'(a)$$

and are not simultaneously 0 at any t.

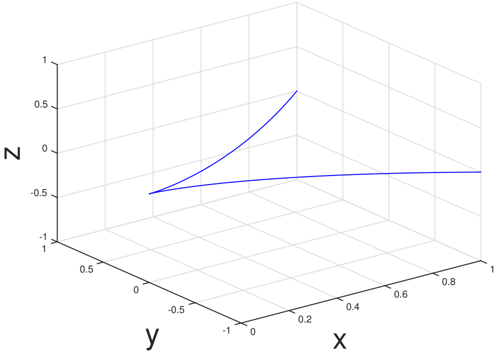
$$\mathbf{r}'(t) \neq \mathbf{0}$$

A curve that is made up of a finite number of smooth curves pieced together in a continuous fashion is called piecewise smooth.

- Why the definition requires the extra condition?

$$\mathbf{r}'(t) \neq \mathbf{0}$$

$$\mathbf{r}(t) = t^2\mathbf{e}_x + t^3\mathbf{e}_y + 0\mathbf{e}_z$$



Matlab

```
>> han1 = ezplot3('t^2','t^3','0', [-1,1]); set(han1,'LineStyle','-', 'Color','blue', 'LineWidth',1);
>> grid on; xlabel('x', 'fontsize', 30); ylabel('y', 'fontsize', 30); zlabel('z', 'fontsize', 30)
```

Motion

If $\mathbf{r}$ is the position vector of a particle moving along a smooth curve in space, then

$$\mathbf{v}(t) = \dot{\mathbf{r}}$$

is the particle's velocity vector, tangent to the curve. At any time t, the direction of $\mathbf{v}$ is the direction of motion, the magnitude of $\mathbf{v}$ is the particle's speed, and the derivative $\mathbf{a} = \dot{\mathbf{v}}$, when it exists, is the particle's acceleration vector.

1. Velocity is the derivative of position:

$$\mathbf{v}(t) = \dot{\mathbf{r}}$$

2. Speed is the magnitude of velocity:

$$\text{Speed} = |\mathbf{v}|$$

3. Acceleration is the derivative of velocity:

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}$$

4. The unit vector $\hat{\mathbf{v}}$ is the direction of motion at time t.

Differentiation rules for vector-valued function

Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions of t.

1. Addition
$$\frac{d}{dt}[\mathbf{u}+\mathbf{v}] = \mathbf{u}' + \mathbf{v}'$$

2. Scalar multiplication
$$\frac{d}{dt}[f\mathbf{u}] = f'\mathbf{u} + f\mathbf{u}', \quad \text{where } f \text{ is a real-valued function of } t.$$

3. Dot product
$$\frac{d}{dt}[\mathbf{u}\cdot\mathbf{v}] = \mathbf{u}'\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v}'$$

4. Cross product
$$\frac{d}{dt}[\mathbf{u}\times\mathbf{v}] = \mathbf{u}'\times\mathbf{v} + \mathbf{u}\times\mathbf{v}'$$

5. Chain rule
$$\frac{d}{dt}\left[\mathbf{u}\left(f(t)\right)\right] = f'(t)\mathbf{u}'(f(t)), \quad \text{where } f \text{ is a real-valued function of } t.$$

Exercise

(a) Use the product rule to find the derivative of the cross product between

$$\mathbf{u}(t) = \begin{bmatrix} e^{-t} \\ 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}(t) = \begin{bmatrix} t^2 \\ \sin t \\ 0 \end{bmatrix}$$

(b) Show that

$$|\mathbf{r}(t)| = c, \qquad \text{where } c \text{ a constant.}$$

then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

- A vector-valued function $\mathbf{R}(t)$ is an antiderivative of $\mathbf{r}(t)$ on an interval $\mathcal{I}$ if

$$\frac{d\mathbf{R}}{dt} = \mathbf{r} \quad \text{at each point of } \mathcal{I}.$$

- If $\mathbf{R}$ is an antiderivative of $\mathbf{r}$ on $\mathcal{I}$, then it can be shown that every antiderivative of $\mathbf{r}$ on $\mathcal{I}$ has the form $\mathbf{R} + \mathbf{C}$ for some constant vector $\mathbf{C}$.

Definition

The indefinite integral of $\mathbf{r}$ with respect to t is the set of all antiderivatives of $\mathbf{r}$, denoted by $\int \mathbf{r}(t)\,dt$. If $\mathbf{R}$ is any antiderivative of $\mathbf{r}$, then

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C}$$

- Definite integrals of vector-valued functions are defined component-wise.

Definition

If the components functions, $f(t)$, $g(t)$ and $h(t)$, of $\mathbf{r}(t)$ are integrable over $[a, b]$, then $\mathbf{r}$ is also integrable, and the definite integral of $\mathbf{r}$ from a to b is defined to be

$$\int_a^b \mathbf{r}(t)\,dt = \begin{bmatrix} \int_a^b f(t)\,dt \\ \int_a^b g(t)\,dt \\ \int_a^b h(t)\,dt \end{bmatrix}$$

Definition

If the components of $\mathbf{r}(t) = \begin{bmatrix} f(t) \\ g(t) \\ h(t) \end{bmatrix}$ are integrable over $[a, b]$, then so is $\mathbf{r}$, and the definite integral of $\mathbf{r}$ from a to b is defined to be $\displaystyle\int_a^b \mathbf{r}(t)\,dt = \begin{bmatrix} \displaystyle\int_a^b f(t)\,dt \\ \displaystyle\int_a^b g(t)\,dt \\ \displaystyle\int_a^b h(t)\,dt \end{bmatrix}$.

Fundamental Theorem of Calculus

Suppose $\mathbf{r}(t)$ is continuous, then

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big|_a^b = \mathbf{R}(b) - \mathbf{R}(a) \qquad \text{where } \mathbf{R} \text{ is any antiderivative of } \mathbf{r}.$$