Vv255 Lecture 23

Dr Jing Liu

UM-SJTU Joint Institute

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Definition

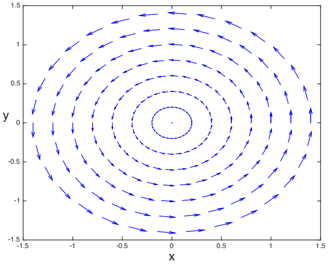
A vector field $\mathbf{F}$ is said to be conservative in a region $\mathcal{D}$ if it is the gradient field for some function f in $\mathcal{D}$, that is, if

$$\mathbf{F} = \nabla f$$

The function f is called a potential function for $\mathbf{F}$ in the region.

Q: Is every vector field conservative?

$$\mathbf{F} = -y\mathbf{e}_x + x\mathbf{e}_y$$



Q: Is the potential function of a conservative vector field unique?

The general technique of finding a potential function

For a conservative vector field

$$\mathbf{F}(x,y) = P(x,y)\mathbf{e}_x + Q(x,y)\mathbf{e}_y$$

1. Integrate $P(x,y)$ w.r.t x to obtain

   $$f(x,y) = f_1(x,y) + g(y), \quad \text{where } f_1(x,y) = \int P(x,y)\,dx, \text{ and } g(y)$$

   is an unknown function that plays the role of the constant of integration.
2. Differentiate $f = f_1 + g$ w.r.t y to obtain

   $$\frac{\partial}{\partial y}\left[f_1(x,y)\right] + g'(y) = Q(x,y), \quad \text{and solve for } g'(y).$$

3. Integrate $g'(y)$ w.r.t y to complete the definition of f, up to a constant.

A similar procedure can be used for a vector field defined on $\mathbb{R}^3$.

- Suppose $\mathcal{C}$ be a smooth curve in $\mathbb{R}^2$, and defined by

$$\mathbf{r}(t) = x(t)\mathbf{e}_x + y(t)\mathbf{e}_y \qquad \text{for} \qquad a \le t \le b$$

- If $f(x, y)$ is a continuous scalar-valued function, then

$$\int_{\mathcal{C}} f(x, y)\, ds = \int_a^b f\Big(x(t), y(t)\Big)|\dot{\mathbf{r}}(t)|\, dt$$

$$= \int_a^b f\Big(x(t), y(t)\Big)\sqrt{\dot{x}^2(t) + \dot{y}^2(t)}\, dt$$

- If $\mathbf{F}(x, y)$ is a vector field with continuous component functions P and Q,

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F} \cdot \dot{\mathbf{r}}\, dt$$

$$= \int_a^b \bigg(P\Big(x(t), y(t)\Big)\dot{x}(t) + Q\Big(x(t), y(t)\Big)\dot{y}(t)\bigg)\, dt$$

- Suppose a vector field is actually conservative in some open region $\mathcal{D}$,

$$\mathbf{F} = P\mathbf{e}_x + Q\mathbf{e}_y = \frac{\partial f}{\partial x}\mathbf{e}_x + \frac{\partial f}{\partial y}\mathbf{e}_z = \nabla f$$

where $f(x, y)$ is continuously differentiable, then the formula becomes,

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds &= \int_a^b \left(P\Big(x(t), y(t)\Big)\dot{x}(t) + Q\Big(x(t), y(t)\Big)\dot{y}(t) \right) dt \\
&= \int_a^b \left(\underbrace{\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}_{h} \right) dt \\
&= \int_a^b \frac{d}{dt}\Big[f\big(x(t), y(t)\big) \Big]\, dt \qquad f \text{ is an antiderivative of } h. \\
&= \Big[f\big(x(t), y(t)\big) \Big]_{t=a}^{t=b} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))
\end{aligned}$$

Q: What is the significance of this derivation? Have you seen a similar formula?

The Fundamental Theorem of Line Integrals

Suppose that the vector field $\mathbf{F}$ is conservative

$$\mathbf{F}(x, y) = P(x, y)\mathbf{e}_x + Q(x, y)\mathbf{e}_y = \nabla f$$

in some open region $\mathcal{D}$ containing the points A and B and that

$P(x, y)$ and $Q(x, y)$ are continuous in this region $\mathcal{D}$.

If $\mathcal{C}$ is a piecewise smooth curve given by the vector-valued function

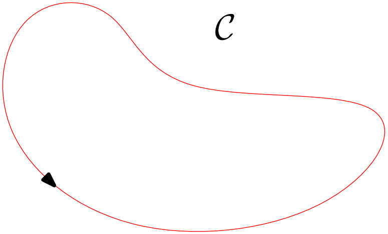
$$\mathbf{r}(t) = x(t)\mathbf{e}_x + y(t)\mathbf{e}_y, \qquad \text{for} \quad a \leq t \leq b$$

starts at $A = \mathbf{r}(a)$ and ends at $B = \mathbf{r}(b)$, and lies entirely in the region $\mathcal{D}$, then

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

- Fundamental Theorem of Line integrals, FTL, can be easily extended to $\mathbb{R}^3$.

- Suppose the curve $\mathcal{C}$ is closed in $\mathcal{D}$,



that is, a curve whose the initial and terminal points of $\mathcal{C}$ are the same,

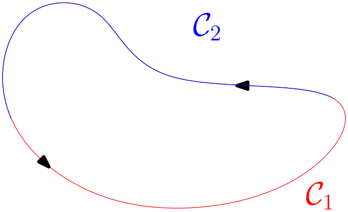
$$\mathbf{r}(b) = \mathbf{r}(a)$$

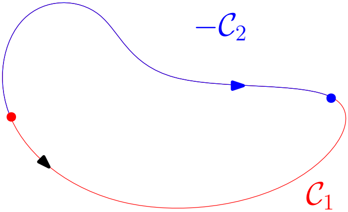
which means the line integral is zero,

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = 0$$

- This is the notation for a line integral with a closed path.

- Now if we split $\mathcal{C}$ into two curves $\mathcal{C}_1$ and $\mathcal{C}_2$, and change orientation of $\mathcal{C}_2$,





by fact that the sign of the line integral of a vector field over a curve depends on the orientation of the curve, then we have

$$0 = \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} - \int_{-\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$$

that is,

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{-\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$$

Q: What does it mean to a particle moving in a conservative force field $\mathbf{F}$?

Theorem

If $\mathbf{F}$ is conservative with continuous component functions

$$\mathbf{F}(x,y) = P(x,y)\mathbf{e}_x + Q(x,y)\mathbf{e}_y = \nabla f$$

in some open region $\mathcal{D}$, then the line integral is said to be

independent of path in $\mathcal{D}$

that is, the line integral of $\mathbf{F}$ over a piecewise smooth $\mathcal{C}$ depends only on its

initial and terminal points,

provided $\mathcal{C}$ is entirely in $\mathcal{D}$.

- The conditions are there so that the FTL of line integral can be invoked.

Q: Does independent of path in $\mathcal{D}$ implies conservativeness in $\mathcal{D}$?

Theorem

Suppose the region $\mathcal{D}$, the curve $\mathcal{C}$ and the vector field $\mathbf{F}$ satisfy the conditions of FTL and the line integral of a vector field $\mathbf{F}$ is independent of path in $\mathcal{D}$, then

$$\mathbf{F} \text{ is a conservative vector field on } \mathcal{D}.$$

Proof

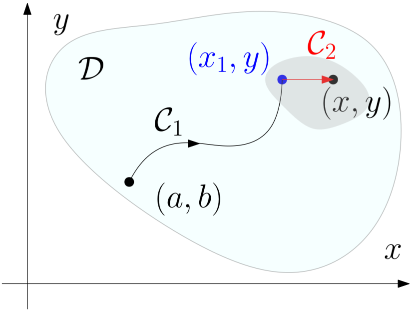
- Let $\mathbf{F} = P\mathbf{e}_x + Q\mathbf{e}_y$, we choose an arbitrary point $(a, b) \in \mathcal{D}$, and define

  $$f(x, y) = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

  where $\mathcal{C}$ is a curve between (a, b) and $(x, y) \in \mathcal{D}$.
- Since it is independent of path, any curve $\mathcal{C}$ leads to the same function f.
- Suppose $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ where $\mathcal{C}_2$ is a line segment between (x_1, y) to (x, y),

  $$f(x, y) = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$$

Proof



One possible parametrization for $\mathcal{C}_2$ is

$$\mathbf{r}_2(t) = t\mathbf{e}_x + y\mathbf{e}_y \quad \text{for } x_1 \leq t \leq x$$

$$\mathbf{r}_2'(t) = \mathbf{e}_x$$

If we can show

$$P = f_x \quad \text{and} \quad Q = f_y$$

then it is conservative.

- There always exists a curve $\mathcal{C}_1$ such that $\dfrac{\partial}{\partial x}\displaystyle\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = 0$, so we have

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} = \frac{\partial}{\partial x}\int_{x_1}^{x} \mathbf{F} \cdot \mathbf{r}_2' \, dt = \frac{\partial}{\partial x}\int_{x_1}^{x} P \, dt = P$$

- The component Q can be shown to be f_y in a similar fashion.

- In order to use the Fundamental Theorem of Line Integrals to evaluate the line integral of a conservative vector field, that is, to use the formula

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

it is necessary to find the potential function f such that

$$\nabla f = \mathbf{F}$$

Q: Can test whether a vector field is conservative or not?

- Suppose that $\mathbf{F}(x, y) = P\mathbf{e}_x + Q\mathbf{e}_y$ is a conservative vector field, then

$$\frac{\partial f}{\partial x} = P, \quad \frac{\partial f}{\partial y} = Q \qquad \text{for some function } f.$$

- If the conditions of the mixed derivative theorem are met, that is,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial P}{\partial y}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

are continuous in the open region $\mathcal{D}$, then we have $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

Theorem

Suppose the vector field

$$\mathbf{F} = P\mathbf{e}_x + Q\mathbf{e}_y = \nabla f$$

is a conservative vector field in some open region $\mathcal{D}$, and the component functions

$$P(x, y) \quad \text{and} \quad Q(x, y)$$

have continuous first-order partial derivatives in $\mathcal{D}$, then throughout $\mathcal{D}$ we have

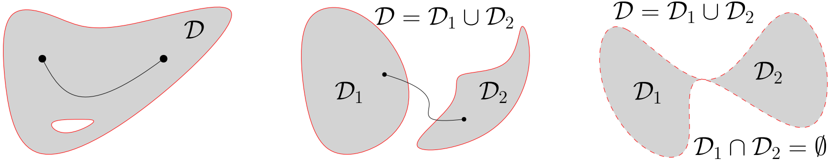
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

- This is a necessary condition for conservativeness, is it sufficient?

Definition

A region $\mathcal{D}$ is known to be connected if any two points in $\mathcal{D}$ can be connected by a path $\mathcal{C}$ entirely within $\mathcal{D}$.

Q: Consider the following regions, are they open and connected?



Definition

A subset of $\mathcal{S}$ of $\mathbb{R}^n$ is called simply connected if it is connected and every close curve in $\mathcal{S}$ can be contracted to a point in $\mathcal{S}$.

- Intuitively, if the domain $\mathcal{D}$ is simply connected, then
  1. one piece
  2. does not have any "holes" that go all the way through

Q: Are the above regions open simply connected? How about

$$\{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0)\} \qquad \{(x, y, z) \in \mathbb{R}^2 \mid (x, y, z) \neq (0, 0, 0)\}$$

Conservative Field Test

Suppose the vector field

$$\mathbf{F} = P\mathbf{e}_x + Q\mathbf{e}_y$$

is defined on an open simply connected region $\mathcal{D}$, and the partial derivatives

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

are equal and continuous throughout the region $\mathcal{D}$, then

$\mathbf{F}$ is conservative.

Similarly, in an open simply connected region $\mathcal{E} \subset \mathbb{R}^3$,

$$\mathbf{F} = P\mathbf{e}_x + Q\mathbf{e}_y + R\mathbf{e}_z$$

is conservative if P, Q and R have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \text{and} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

Q: Are inverse-square fields in $\mathbb{R}^2$ conservative?

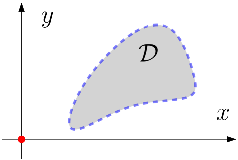
$$\mathbf{F} = \frac{c}{|\mathbf{r}|^3}\mathbf{r} = c\left(x^2+y^2\right)^{-3/2}(x\mathbf{e}_x + y\mathbf{e}_y)$$

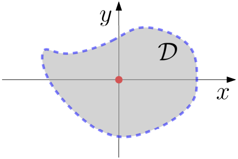
Q: Can we judge using the conservative field test?

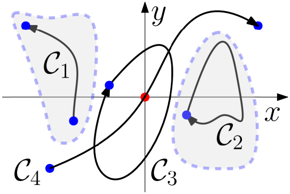
$$\frac{\partial P}{\partial y} = -3cxy(x^2+y^2)^{-5/2} \qquad \frac{\partial Q}{\partial x} = -3cxy(x^2+y^2)^{-5/2}$$

- Consider the gradient of the following scalar-valued function

$$f(x,y) = \frac{-c}{\sqrt{x^2+y^2}} = \frac{-c}{r} \implies \nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r = \frac{c}{r^3}r\mathbf{e}_r = \frac{c}{|\mathbf{r}|^3}\mathbf{r}$$







Exercise

(a) *Suppose* $\mathbf{F} = (2xz + y^2)\mathbf{e}_x + 2xy\mathbf{e}_y + (x^2 + 3z^2)\mathbf{e}_z$. *Evaluate the following*

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

where $\mathcal{C}$ *is the curve defined by*

$$\mathbf{r}(t) = t^2\mathbf{e}_x + (t+1)\mathbf{e}_y + (2t-1)\mathbf{e}_z \qquad \text{for} \quad 0 \le t \le 1.$$

(b) *Suppose* $\mathbf{F} = \dfrac{-y}{x^2+y^2}\mathbf{e}_x + \dfrac{x}{x^2+y^2}\mathbf{e}_y$. *Evaluate the line integral*

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

where $\mathcal{C}$ *is a circle of radius 1 centred at the point* $(2,2)$.

(c) *What if the circle is centred at the origin instead?*

Characterization of conservative vector fields

Suppose $P(x,y)$ and $Q(x,y)$ are continuously differentiable on some open simply connected region $\mathcal{D}$, then then the following statements are equivalent :

1. $\mathbf{F} = P\mathbf{e}_x + Q\mathbf{e}_y$ is a conservative vector field on the region $\mathcal{D}$.
2. $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ at every point in $\mathcal{D}$.
3. $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$ for every piecewise smooth *closed* curve $\mathcal{C}$ in $\mathcal{D}$.
4. $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is independent of the path.

- All of the results above can be easily extended to $\mathbb{R}^3$.