Probabilistic Methods in Engineering

VE401

Assignment II

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Team number: 20

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Exercise 2.1

1) Since $\text{ran}X \subset \mathbb{N}$, then

$$E[X] = \sum_{x=0}^{\infty} xP[X = x] = \sum_{x=0}^{\infty} \sum_{y=1}^{x} P[X = x].$$

We change the order of summation,

$$E[X] = \sum_{x=0}^{\infty} \sum_{y=1}^{x} P[X = x] = \sum_{x=0}^{\infty} \sum_{y=x+1}^{\infty} P[X = y] = \sum_{x=0}^{\infty} P[X > x].$$

2) According to the result we obtain in 1), we perform the similar calculation to (X, f_X) where $X = 1, 2, \ldots, N - r + 1$.

$$E[X] = \sum_{x=0}^{N-r} P[X > x].$$

In fact, $P[X > x]$ is equal to the probability that the first x balls are all black.

$$\begin{aligned}
P[X > x] &= P[\text{The first x balls are all black}] \\
&= \frac{N-r}{N} \frac{N-r-1}{N-1} \cdots \frac{N-r-x+1}{N-x+1} \\
&= \frac{(N-r)!/(N-r-x)!}{N!/(N-x)!} = \frac{\binom{N-x}{r}}{\binom{N}{r}} \\
E[X] &= \frac{1}{\binom{N}{r}} \sum_{x=0}^{r'} \binom{N-x}{N-r'} = \frac{1}{\binom{N}{r}} \sum_{x=0}^{r'} \binom{N-r'+x}{N-r'} \\
&= \frac{\binom{N+1}{N-r'+1}}{\binom{N}{r}} = \frac{\binom{N+1}{r+1}}{\binom{N}{r}} = \frac{N+1}{r+1}.
\end{aligned}$$

Exercise 2.2

For $p_0(t)$, when $t = 0$ the probability of 0 failure is 1. For $p_x(t) \quad (x \geq 1)$, when $t = 0$ the probability of x failure is 0. Hence, the initial conditions are as the follows.

$$\begin{aligned}
&p_0(0) = 1 \\
&p_x(0) = 0, \quad x \geq 1.
\end{aligned}$$

Since $p_0' = -\lambda p_0$, then $p_0(t) = c \cdot e^{-\lambda t}$. Also we know that $p_0(0) = 1$, thus c=1. Hence, $p_0(t) = e^{-\lambda t}$.

Assume that $p_x(t) = (\lambda t)^x e^{-\lambda t}/x!$ for all $x \in \mathbb{N}$.
For $x = 0$, $p_0(t) = e^{-lambdat} = (\lambda t)^0 e^{-\lambda t}/0!$, which follows our assumption.
For $x = n \geq 0$, we assume the statement is true. When $x = n + 1$,

$$\begin{aligned}
p'_{n+1} + \lambda p_{n+1} &= \lambda p_n, \\
(e^{\lambda t} p_x)' &= \lambda e^{\lambda t} p_n, \\
p_{n+1} &= \frac{\int \lambda e^{\lambda t} p_n dt}{e^{\lambda t}}.
\end{aligned}$$

$$\begin{aligned}
p_{n+1} &= \frac{\lambda e^{\lambda t} p_n dt}{e^{\lambda t}} = \frac{\int \lambda e^{\lambda t} (\lambda t)^n e^{-\lambda t}/n! dt}{e^{\lambda t}} \\
&= \frac{\lambda^{n+1}/n! \int t^n dt}{e^{\lambda t}} = \lambda^{n+1}/n!/(n+1)(t^{n+1} + C)e^{-\lambda t} \\
&= (\frac{(\lambda t)^{n+1} + (\lambda)^{n+1} C}{(n+1)!})e^{-\lambda t}.
\end{aligned}$$

To satisfy that $p_{n+1}(0) = 0$, C has to be 0. Thus,

$$p_{n+1}(t) = \frac{(\lambda t)^{n+1}}{(n+1)!} e^{-\lambda t}.$$

Therefore, $p_x(t) = (\lambda t)^x e^{-\lambda t}/x!$ for all $x \in \mathbb{N}$.

Exercise 2.3

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} = \binom{n}{x} (\frac{k}{n})^x (1 - \frac{k}{n})^{n-x} = k^x \frac{\binom{n}{x}}{n^x} ((1 - \frac{1}{n/k})^{n/k})^k (1 - \frac{k}{n})^{-x}$$

Since

$$\begin{aligned}
&\lim_{n \to \infty} \frac{\binom{n}{x}}{n^x} = \lim_{n \to \infty} \frac{n!}{x!(n-x)! n^x} = \frac{1}{x!}, \\
&\lim_{n \to \infty} ((1 - \frac{1}{n/k})^{-n/k})^{-k} = e^{-k}, \\
&\lim_{n \to \infty} (1 - \frac{k}{n})^{-x} = 1.
\end{aligned}$$

then

$$\lim_{n \to \infty} f(x) = \frac{k^x}{x!} e^{-k}.$$

Exercise 2.4

1)

$$\begin{aligned}
E[V] &= \int_0^\infty (\frac{2}{\pi})^{1/2}(m/kT)^{3/2} v^3 e^{-\frac{m}{kT}v^2/2} dv \\
&= (\frac{2}{\pi})^{1/2}(m/kT)^{3/2} \int_0^\infty v^3 e^{-\frac{m}{kT}v^2/2} dv \quad (w = v^2) \\
&= (\frac{2}{\pi})^{1/2}(m/kT)^{3/2} \int_0^\infty w^{3/2} e^{-\frac{m}{kT}w/2} \frac{1}{2} w^{-1/2} dw \\
&= (\frac{1}{2\pi})^{1/2}(m/kT)^{3/2} \int_0^\infty w e^{-\frac{m}{kT}w/2} dw \\
&= (\frac{m}{2kT\pi})^{1/2} \int_0^\infty \frac{m}{kT} w e^{-\frac{m}{kT}w/2} dw \quad (z = -\frac{m}{kT}w/2)
\end{aligned}$$

$$E[V] = (\frac{8kT}{m\pi})^{1/2} \int_{-\infty}^0 z e^z dz = (\frac{8kT}{m\pi})^{1/2}.$$

Since $Var[V] = E[V^2] - E[V]^2$, then

$$\begin{aligned}
E[V^2] &= (\frac{2}{\pi})^{1/2}(m/kT)^{3/2} \int_0^\infty v^4 e^{-\frac{m}{kT}v^2/2} dv \\
&= -(\frac{2m}{k\pi T})^{1/2} \int_1^0 v^3 d e^{-\frac{m}{kT}v^2/2} \\
&= -(\frac{2m}{k\pi T})^{1/2} (v^3 e^{-\frac{m}{kT}v^2/2}\Big|_0^\infty - \int_0^\infty 3v^2 e^{-\frac{m}{kT}v^2/2} dv) \\
&= 3(\frac{2m}{k\pi T})^{1/2} \int_0^\infty v^2 e^{-\frac{m}{kT}v^2/2} dv \\
&= 3(\frac{2kT}{m\pi})^{1/2} \int_0^\infty e^{-\frac{m}{kT}v^2/2} dv \\
&= \frac{3kT}{m}.
\end{aligned}$$

$$Var[V] = E[V^2] - E[V]^2 = \frac{3kT}{m} - \frac{8kT}{m\pi} = \frac{kT}{m}(3 - \frac{8}{\pi}).$$

2)

$$E[E] = E[mV^2/2] = mE[V^2]/2 = \frac{3kT}{2}.$$

3) Since

$$E = \varphi(v) = \frac{mv^2}{2},$$
$$V = \varphi^{-1}(\varepsilon) = (\frac{2\varepsilon}{m})^{1/2},$$

then

$$f_E(\varepsilon) = f_V(\varphi^{-1}(\varepsilon))\left|\frac{d\varphi^{-1}(\varepsilon)}{d\varepsilon}\right| = (\frac{1}{2m\varepsilon})^{1/2}(\frac{2}{\pi})^{1/2}(\frac{m}{kT})^{3/2}\frac{2\varepsilon}{m}e^{-\frac{\varepsilon}{kT}}$$
$$= 2(\frac{\varepsilon}{\pi k^3 T^3})^{1/2}e^{-\frac{\varepsilon}{kT}}.$$

Exercise 2.5

$$\Gamma(\frac{2n+1}{2}) = \int_0^\infty t^{(2n-1)/2}e^{-t}dt = (t^{(2n-1)/2}e^{-t})\Big|_0^\infty + \frac{2n-1}{2}\int_0^\infty t^{(2n-3)/2}e^{-t}dt$$
$$= (\frac{2n-1}{2})\Gamma(\frac{2n-1}{2}).$$

Hence,

$$\Gamma(\frac{2n+1}{2}) = \frac{(2n-1)(2n-3)\ldots(1)}{2\cdot 2\ldots 2}\Gamma(\frac{1}{2}) = \frac{(2n-1)!!}{2^n}\Gamma(\frac{1}{2}).$$

Also we obtain

$$\Gamma(\frac{1}{2}) = \int_0^\infty t^{-1/2}e^{-t}dt \quad (t = w^2)$$
$$= \int_0^\infty w^{-1}e^{-w^2}2wdw = 2\int_0^\infty e^{-w^2}dw = \sqrt{\pi}$$

Therefore,

$$\Gamma(\frac{2n+1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}.$$

Exercise 2.6

1) Denote that

$$Z = \frac{X-\mu}{\sigma} = \frac{X-6000}{100}$$

so that $Z \sim N(0,1)$. Then,

$$P[\text{A sample strength is less than 6250 kg/cm}^2]$$
$$= P[X < 6250] = P[Z < 2.50] = \Phi(2.50) = 0.9938.$$

2)

$$
\begin{aligned}
P[5800 \leq X \leq 5900] &= P[-2.00 \leq Z \leq -1.00] \\
&= \Phi(-1.00) - \Phi(-2.00) \\
&= 0.1587 - 0.0228 \\
&= 0.1359.
\end{aligned}
$$

3) According to the standard normal distribution table, we know that

$$\Phi(-1.64) = 0.0505, \quad \Phi(-1.65) = 0.0495.$$

When $Z = -1.65$, $X = -1.65 \times 100 + 6000 = 5835\text{kg/cm}^2$.

Hence, the strength of 5835 kg/cm^2 exceeds 5% samples, saying that is exceeded by 95% samples.

Exercise 2.7

1) The random variable is a map $X : S \to \mathbb{R}$ together with a function $f_X : \mathbb{R} \to \mathbb{R}$. Then, the two properties of a continuous random variable will be showed.

For $a < x < b$ where $b > a$, $f(x) = 1/(b-a) > 0$. For other conditions, $f(x) = 0 >= 0$. Hence, $f(x) \geq 0$.

Also,

$$\int_{-\infty}^{\infty} f(x)dx = \int_{a}^{b} 1/(b-a)dx = (b-a)/(b-a) = 1.$$

Hence, this is a density for a continuous random variable.

2) The graph is as the following.

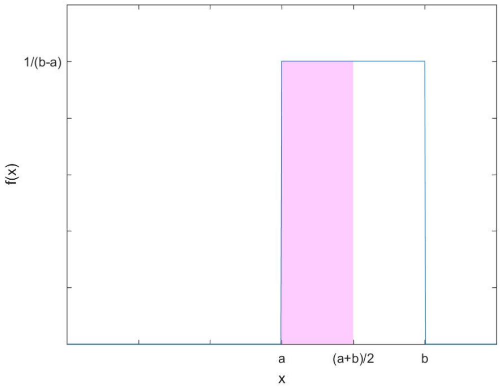

Figure 1: Graph for Exercise 2.7 (2)

3)

$$P[X <= (a+b)/2] = \int_{-\infty}^{(a+b)/2} f(x)dx = \int_{a}^{(a+b)/2} 1/(b-a)dx = (b-a)/2/(b-a) = \frac{1}{2}.$$

4) Since [c,d] and [e,f] are the subintervals of [a,b], then

$$P[c \le X \le d] = \int_{c}^{d} 1/(b-a)dx = (d-c)/(b-a).$$
$$P[e \le X \le f] = \int_{e}^{f} 1/(b-a)dx = (f-e)/(b-a).$$

Since $d-c = f-e$, then $P[c \le X \le d] = P[e \le X \le f]$.

5)

$$F(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \le x \le b \\ 1, & x > b \end{cases}$$

6)

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} xf(x)dx = \int_{a}^{b} \frac{x}{b-a}dx \\ &= \frac{x^2}{2(b-a)}\bigg|_{a}^{b} = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}. \end{aligned}$$

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 f(x)dx = \int_{a}^{b} \frac{x^2}{b-a}dx \\ &= \frac{x^3}{3(b-a)}\bigg|_{a}^{b} = \frac{b^3-a^3}{3(b-a)} = \frac{a^2+ab+b^2}{3}. \end{aligned}$$

Hence,

$$
\begin{aligned}
Var[X] = E[X^2] - E[X]^2 &= \frac{a^2 + ab + b^2}{3} - (\frac{a+b}{2})^2 \\
&= \frac{(b-a)^2}{12}.
\end{aligned}
$$