Vv255 Lecture 18

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- The very first application of double integral is to compute volume and area.

Exercise

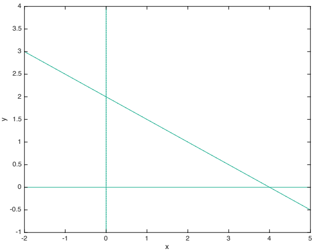
Find the volume of the solid that is below the surface

$$z = 3x + 2y$$

over the region D *on the plane* $z = 0$ *bounded by the curves* $x = 0$, $y = 0$ *and*

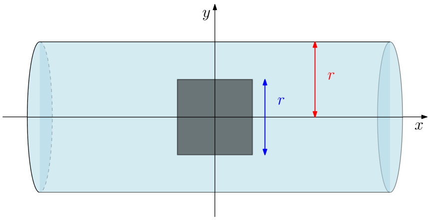
$$x + 2y = 4$$

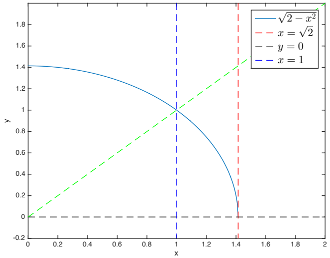
by evaluation of a double integral.



Exercise

(a) *Find the volume removed when a vertical square hole of edge length r is cut directly through the center of a long horizontal solid cylinder of radius r.*





(b) *Find the area of the region between $x = 1$, $x = \sqrt{2}$; $y = 0$, and*

$$y = \sqrt{2 - x^2}$$

by first expressing the area in terms of a double integral.

- Recall for rectangular coordinates (x, y), we have the following

Fubini's Theorem

Let $\mathcal{R}$ be the rectangle region defined by the inequalities

$$a \leq x \leq b, \qquad c \leq y \leq d$$

If $f(x, y)$ is continuous on this rectangle, then

$$\iint_{\mathcal{R}} f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

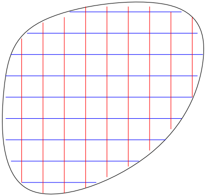
Q: Can we evaluate the double integral in polar coordinates?

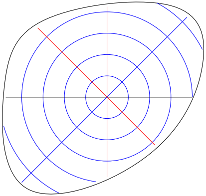
$$\iint_{\mathcal{D}} f(x, y)\, dA = \iint_{\mathcal{D}} f\Big(x(r, \theta), y(r, \theta)\Big)\, dA = \iint_{\mathcal{D}} F\big(r, \theta\big)\, dA$$

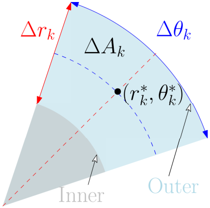
Q: Suppose $\mathcal{D} = \Big\{(r, \theta) \mid r_1 \leq r \leq r_2, \theta_1 \leq \theta \leq \theta_2\Big\}$, is the following true

$$\iint_{\mathcal{D}} f(x, y)\, dA = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f\Big(x(r, \theta), y(r, \theta)\Big)\, dr\, d\theta = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} F\big(r, \theta\big)\, dr\, d\theta$$

- Recall if the double integral exists, it will not depend on how $\mathcal{D}$ is partitioned
- Let us partition the region $\mathcal{D}$ into small patches called "polar rectangles"







- The area of a wedge-shaped sector of a circle having radius r and angle θ is

$$\pi r^2 \frac{\theta}{2\pi} = \frac{1}{2}\theta \cdot r^2$$

- So the areas of the circular sectors in the partition are

Inner radius: $\frac{1}{2}\left(r_k^* - \frac{\Delta r_k}{2}\right)^2 \Delta\theta_k,$ Outer radius: $\frac{1}{2}\left(r_k^* + \frac{\Delta r_k}{2}\right)^2 \Delta\theta_k$

- Let ΔA_k denote the area of those polar rectangles, So

$$\begin{aligned}\Delta A_k &= \text{area of large sector} - \text{area of small sector}\\ &= \frac{\Delta\theta_k}{2}\left[\left(r_k^* + \frac{\Delta r_k}{2}\right)^2 - \left(r_k^* - \frac{\Delta r_k}{2}\right)^2\right]\\ &= {r_k}^*\Delta r_k \Delta\theta_k\end{aligned}$$

- Hence, double integrals can be transformed into polar form

$$\iint_{\mathcal{D}} f(r,\theta)\,dA = \lim_{n\to\infty}\sum_{k=1}^{n} f(r_k^*,\theta_k^*) r_k^*\,\Delta r_k\,\Delta\theta_k = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta) r\,dr\,d\theta$$

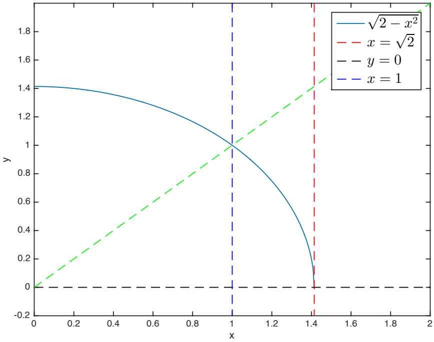
- Consider again the area of the region between $x=1$, $x=\sqrt{2}$; $y=0$, and

$$y=\sqrt{2-x^2}$$

Q: Can we convert the double integral into an iterated integral in polar form?

Q: What are the upper and lower limits, r_1 and r_2, for this region?

Q: What are the upper and lower limits, θ_1 and θ_2, for this region?



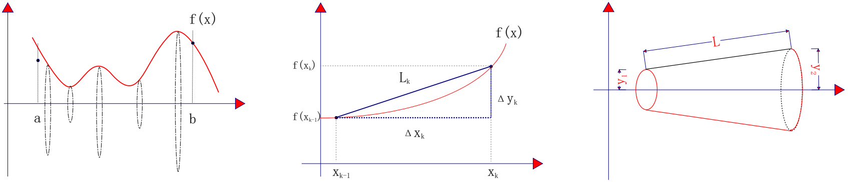
- Therefore the area is

$$\text{Area} = \iint_{\mathcal{D}} dA = \int_0^{\pi/4} \int_{\sec\theta}^{\sqrt{2}} r \, dr \, d\theta = \frac{\pi}{4} - \frac{1}{2}$$

- Recall how we define the area of the surface, A.K.A surface area, created by revolving about the x-axis the graph of a nonnegative smooth function

$$y = f(x), \qquad a \leq x \leq b,$$

1. Divide the curve into small curve segments.



2. Approximate the area using a line segment instead of a curve segment
3. Add the approximations to form a Riemann sum.
4. Take the limit of the Riemann sum to find the area when the limit exists.

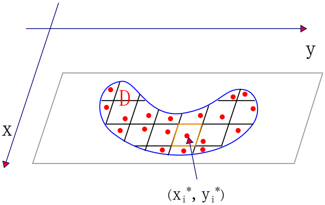
- Area of a surface S created by revolving the curve $y = f(x)$ between a and b

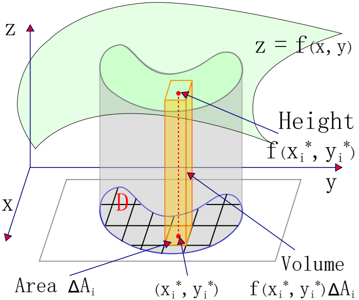
$$S = \int_a^b 2\pi f(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

- Now let us consider the surface defined by

$$z = f(x, y)$$

the size of the region over a region $\mathcal{D}$ in the domain of $z = f(x, y)$.





- Recall what we have done for volume.

Q: What should we use to approximate each small surface area?

1. We divide $\mathcal{D}$ into rectangles $\mathcal{R}_i$ with area ΔA_i and sample points (x_i^*, y_i^*).

- There is a small portion of the surface over each sub-rectangle,

$$\Delta S_i$$

- Consider a tangent plane T_i to the surface directly above each sub-rectangle.

$$z = f(x_i^*, y_i^*) + f_x(x_i^*, y_i^*)(x - x_i^*) + f_y(x_i^*, y_i^*)(y - y_i^*)$$

2. The tangent planes over the rectangle $\mathcal{R}_i$ has a shape of a parallelogram.

$$\Delta S_i \approx \Delta T_i \qquad \text{if our sub-rectangles are small.}$$

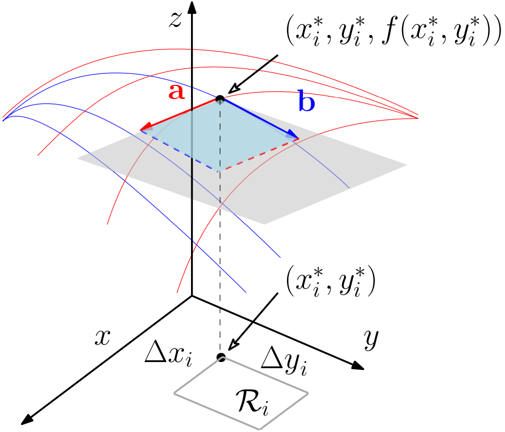
where ΔT_i is the area of the parallelogram over the sub-rectangle $\mathcal{R}_i$.

3. Add the approximations to form a Riemann sum and 4. take the limit,

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta T_i$$

Q: How to determine the area ΔT_i?

- Let (x_i^*, y_i^*) be the point of tangency, then



$$\Delta T_i = |\mathbf{a} \times \mathbf{b}|$$

$$\mathbf{a} = \begin{bmatrix} \Delta x_i \\ 0 \\ \Delta z_x \end{bmatrix} = \begin{bmatrix} \Delta x_i \\ 0 \\ f_x(x_i^*, y_i^*)\Delta x_i \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 0 \\ \Delta y_i \\ \Delta z_y \end{bmatrix} = \begin{bmatrix} 0 \\ \Delta y_i \\ f_y(x_i^*, y_i^*)\Delta y_i \end{bmatrix}$$

Q: How can we find $\mathbf{a}$ and $\mathbf{b}$?

- Compute the the cross product, we have $\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 - f_x(x_i^*, y_i^*)\Delta x_i \Delta y_i \\ 0 - f_y(x_i^*, y_i^*)\Delta x_i \Delta y_i \\ \Delta x_i \Delta y_i - 0 \end{bmatrix}$.
- The length of the cross product gives the area of the parallelogram over $\mathcal{R}_i$

$$\Delta T_i = |\mathbf{a} \times \mathbf{b}| = \Delta x_i \Delta y_i \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{f_x^2 + f_y^2 + 1}\Delta A_i$$

- Therefore

$$S = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta T_i = \iint_{\mathcal{D}} \sqrt{f_x^2 + f_y^2 + 1}\, dA$$

Definition

Area of a smooth surface with equation $z = f(x, y)$ over a region $\mathcal{D}$,

$$S = \iint_{D} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Exercise

(a) *Find the area of the part of the paraboloid*

$$z = x^2 + y^2$$

that lies under the plane

$$z = 9$$

(b) *Find the area of the surface represented by*

$$x = u, \quad y = u\cos v, \quad z = u\sin v$$

for $0 \le u \le 2$ *and* $0 \le v \le 2\pi$.

- The area and volume might have specific meanings in different applications.
- For example, suppose the population density of a city can be modeled by

$$f(x,y) = \frac{5000xe^y}{1+2x^2}$$

where the density is in people per square kilometre, and x, y are in kms.

Q: What does the volume given by the following double integral represent?

$$\int_0^4 \int_{-2}^0 \frac{5000xe^y}{1+2x^2}\, dy\, dx$$

- The population inside the rectangular area defined by the vertices

$$(0,0), \quad (4,0), \quad (0,-2), \quad \text{and} \quad (4,-2)$$

Definition

If $f(x,y)$ is integrable over the region in the plane $\mathcal{D}$ which has an area A, then its average value over $\mathcal{D}$ is given by

$$\text{Average value} = \frac{1}{A}\iint\limits_{\mathcal{D}} f(x,y)\,dA$$

- A firm's weekly profit in marketing two products is given by

$$P(x,y) = 192x + 576y - x^2 - 5y^2 - 2xy - 5000,$$

where x and y represent the numbers of units of each product sold weekly.

Exercise

Estimate the average weekly profit if x is given to be between 40 and 50 units, and y is given to be between 45 and 50 units.

$$\textit{Average profit} = \frac{1}{50}\int_{40}^{50}\int_{45}^{50} P(x,y)\,dy\,dx$$

Definition

An idealized flat object that is thin enough to be viewed as a two-dimensional plane region is called a lamina

- A lamina is called homogeneous if its composition is uniform throughout,

  then the density: $\rho = \dfrac{m}{A}$

  where m is the mass, and A the area, of the homogeneous lamina.

- If it is not uniform, then the lamina is called inhomogeneous, and its density vary from point to point, and defined to be the following limit

$$\rho(x, y) = \lim_{\Delta A \to 0} \frac{\Delta m}{\Delta A}$$

- From this relationship we obtain the approximation

$$\Delta m \approx \rho(x, y)\Delta A$$

Definition

If a lamina with a continuous density function $\rho(x, y)$ occupies a region $\mathcal{D}$ in the xy-plane, then its total mass m is given by

$$m = \iint_{\mathcal{D}} \rho(x, y)\, dA$$

- Physicists also consider other types of density in the same manner.
- For example, consider the following exercise.

Exercise

Charge is distributed over the triangular region $\mathcal{D}$ with vertices $(0, 0)$, $(0, 1)$, and $(1, 0)$ so that the charge density at (x, y) is

$$\sigma(x, y) = xy,$$

measured in Coulombs per square meter. Find the total charge in the region $\mathcal{D}$.

Definition

The center of gravity or the center of mass of an object is the point such that the effect of gravity is "equivalent" to that of a single force acting at the point.

- Recall we obtained the formulae for the centre of mass of a lamina occupies the region $\mathcal{D}$ between continuous curves,

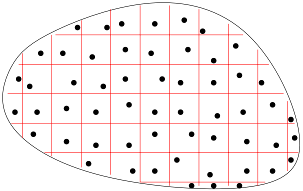
$$f(x), \quad g(x), \quad x = a, \quad \text{and} \quad x = b, \quad \text{where } f(x) \geq g(x) \text{ and } b \geq a$$

$$\bar{x} = \frac{\displaystyle\int_a^b \rho x(f-g)\,dx}{\displaystyle\int_a^b \rho(f-g)\,dx}, \qquad \bar{y} = \frac{\displaystyle\int_a^b \frac{1}{2}\rho(f^2-g^2)\,dx}{\displaystyle\int_a^b \rho(f-g)\,dx}$$

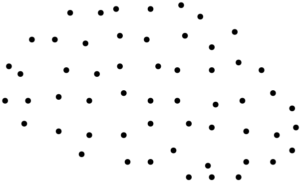
where $\rho(x)$ gives the density of the lamina.

- With the notion of double integral, we can consider cases, where the density is function of y as well as x, and of more general shapes.

- Now consider a lamina with a continuous density function $\rho(x, y)$ occupies a region $\mathcal{D}$ in the xy-plane. Let $(\bar{x}, \bar{y})$ be coordinates of the center of mass.
- Imagine the lamina is subdivided into rectangular pieces, then



- Converting into a discrete approximation,



- If the lamina is subdivided into rectangular pieces, then

$$\text{the mass of the } k\text{th piece:} \qquad \Delta m_k \approx \rho(x_k^*, y_k^*)\Delta A_k$$

- The lamina balances at $(\bar{x}, \bar{y})$, so the sum of the moments of the rectangular pieces about y-axis can be approximated by $\bar{x}$ times the total mass,

$$\sum_{k=1}^{n} x_k^* \Delta m_k \approx \bar{x} \sum_{k=1}^{n} \Delta m_k$$

$$\sum_{k=1}^{n} x_k^* \rho(x_k^*, y_k^*)\Delta A_k \approx \bar{x} \sum_{k=1}^{n} \rho(x_k^*, y_k^*)\Delta A_k$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} x_k^* \rho(x_k^*, y_k^*)\Delta A_k = \lim_{n\to\infty} \bar{x} \sum_{k=1}^{n} \rho(x_k^*, y_k^*)\Delta A_k$$

$$\iint_{\mathcal{D}} x\rho(x,y)\, dA = \bar{x} \iint_{\mathcal{D}} \rho(x,y)\, dA$$

Theorem

The coordinates $(\bar{x}, \bar{y})$ of the center mass of a lamina occupying the region $\mathcal{D}$ with density function $\rho(x, y)$ are

$$\bar{x} = \frac{\displaystyle\iint_{\mathcal{D}} x\rho(x,y)\,dA}{\displaystyle\iint_{\mathcal{D}} \rho(x,y)\,dA}, \qquad \bar{y} = \frac{\displaystyle\iint_{\mathcal{D}} y\rho(x,y)\,dA}{\displaystyle\iint_{\mathcal{D}} \rho(x,y)\,dA}.$$

the center of mass of a homogeneous lamina is called the centroid of the lamina,

$$\bar{x} = \frac{\displaystyle\iint_{\mathcal{D}} x\,dA}{\displaystyle\iint_{\mathcal{D}} dA} = \frac{1}{\text{Area}} \iint_{\mathcal{D}} x\,dA, \qquad \bar{y} = \frac{\displaystyle\iint_{\mathcal{D}} y\,dA}{\displaystyle\iint_{\mathcal{D}} dA} = \frac{1}{\text{Area}} \iint_{\mathcal{D}} y\,dA.$$

Exercise

Find the centroid of the semicircular region.