Vv256 Lecture 10

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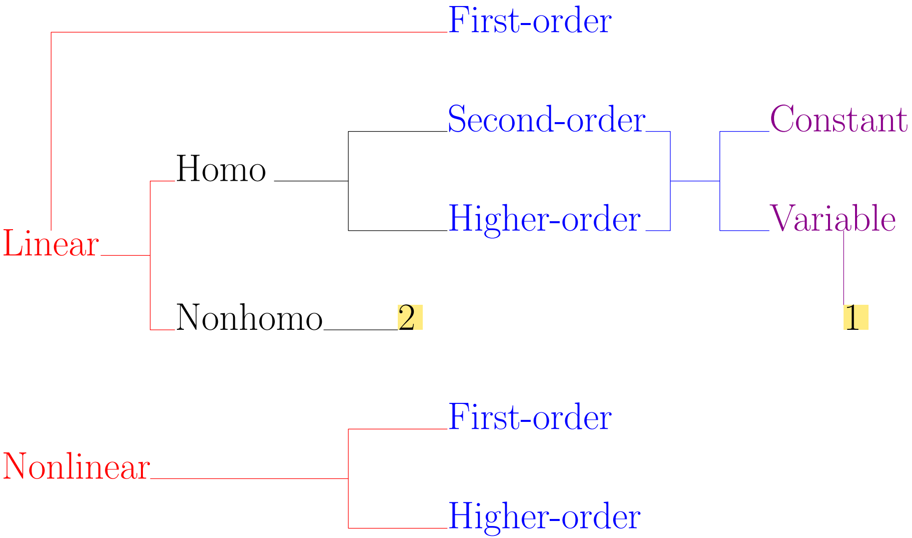
First-order
Second-order
Constant
Homo
Higher-order
Variable
Linear
Nonhomo
2
1
First-order
Nonlinear
Higher-order

- Linear

Defintion

An nth-order ordinary differential equation

$$F(x, y, y', y'', \ldots, y^{(n)}) = f(x)$$

is said to be linear if F is a linear function in terms of

$$y, y', y'', \ldots, y^{(n)}$$

Alternatively, the nth-order linear differential equation has the following form

$$\alpha_n y^{(n)} + \alpha_{n-1} y^{(n-1)} + \cdots + \alpha_1 y' + \alpha_0 y = f$$

where α_i and f are functions of x. In the standard form, it take the following form

$$y^{(n)} + P_1 y^{(n-1)} + \cdots + P_{n-1} y' + P_n y = Q$$

where P_i and Q are functions of x.

- First-order

Theorem

If P and Q are continuous on an interval $\mathcal{I}$, then the first-order linear equation

$$y' + Py = Q$$

has the following general solution, for any $x^* \in \mathcal{I}$ and $x^{**} \in \mathcal{I}$,

$$y = \frac{1}{\mu}\left(\int \mu Q\, dx\right) = C\mu^{-1} + \mu^{-1}\int_{x^{**}}^{x} \mu(\xi)Q(\xi)\, d\xi$$

where $\mu = \exp\left(\int_{x^*}^{x} P(\xi)\, d\xi\right)$. Moreover, there is one and only one solution to

$$y' + Py = Q; \qquad y(x_0) = y_0$$

where $x_0 \in \mathcal{I}$.

- Homogeneous

Defintion

An nth-order linear equation

$$y^{(n)} + P_1 y^{(n-1)} + \cdots + P_{n-1} y' + P_n y = Q$$

is said to be homogeneous if

$$Q(x) = 0$$

- Second-order

Theorem

For any $x_0, y_0, y_1 \in \mathbb{R}$, there is one and only one solution to

$$ay'' + by' + cy = 0, \qquad y(x_0) = y_0, \quad y'(x_0) = y_1$$

where a, b and c are constants.

Theorem

The general solution to

$$ay'' + by' + cy = 0$$

is given by any two linearly independent solutions of the above equation

$$y(x) = C_1\phi_1(x) + C_2\phi_2(x)$$

$$= \begin{cases} C_1 e^{r_1 t} + C_2 e^{r_2 t}, & p(r) = (r - r_1)(r - r_2) \\ \big(C_1 + C_2 t\big) e^{r_1 t}, & p(r) = (r - r_1)^2 \\ e^{Rt}\Big(C_1 \cos(\theta t) + C_2 (\sin\theta t)\Big), & p(r) = (r - R - \mathrm{i}\theta)(r - R + \mathrm{i}\theta) \end{cases}$$

where $p(r) = ar^2 + br + c$ is the corresponding characteristic polynomial.

- Higher-order

Theorem

For any $x_0, y_0, y_1, \ldots, y_{n-1} \in \mathbb{R}$, there is one and only one solution to

$$p\left(\mathcal{D}\right)\Big(y\Big) = 0, \qquad y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots\ y^{(n-1)}(x_0) = y_{n-1}$$

where $p\left(\mathcal{D}\right)$ is an nth-order linear differential operator with constant coefficients.

$$y = C_1\phi_1 + C_2\phi_2 + \cdots + C_n\phi_n = \sum_{i=1} \sum_{j=0}^{k_i} C_{ij}\phi_{ij}$$

is the general solution to the differential equation, where

$$\phi_{ij} = x^j e^{r_i x}$$

and r_i is a root of multiplicity of k_i of the corresponding characteristic polynomial

$$p(r)$$

- Nonhomogeneous

Defintion

Given an nth-order linear equation

$$y^{(n)} + P_1 y^{(n-1)} + \cdots + P_{n-1} y' + P_n y = Q$$

the following equation is called the complementary equation

$$y^{(n)} + P_1 y^{(n-1)} + \cdots + P_{n-1} y' + P_n y = 0$$

or the corresponding homogeneous to the nonhomogeneous equation.

Theorem

The general solution of a nonhomogeneous equation is given by

$$y = y_c + y_p$$

where y_p is a particular solution to the nonhomogeneous equation and y_c is the general solution to the complementary solution.

- Nonhomogeneous with Constant coefficients

Theorem

For any $x_0, y_0, y_1, \ldots, y_{n-1} \in \mathbb{R}$, there is one and only one solution to

$$p(\mathcal{D})\left(y\right) = Q, \qquad y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots\ y^{(n-1)}(x_0) = y_{n-1}$$

where $p(\mathcal{D})$ is an nth-order linear differential operator with constant coefficients.

- Undetermined coefficients/Annihilator can be used to find y_p for

$$p(\mathcal{D}) = Q = \sum_i c_i x^{k_i} e^{r_i x}$$

where $p(\mathcal{D})$ is a linear differential operator with constant coefficients,

$$Q = \sum_i c_i x^{k_i} e^{r_i x}$$

and $c_i \in \mathbb{R}$, $k_i \in \mathbb{N}_0$ and $r_i \in \mathbb{C}$ are constants.

- Variation of parameters can be used to find y_p for any *continuous* Q.

- Nonhomogeneous with Variable coefficients

Theorem

If $p(\mathcal{D})$ is an nth-order linear differential operator with variable coefficients that are continuous on the open interval $\mathcal{I}$, then there is one and only one solution to

$$p(\mathcal{D})\Big(y\Big) = Q, \qquad y(x_0) = y_0,\ y'(x_0) = y_1,\ \dots\ y^{(n-1)}(x_0) = y_{n-1}$$

1. The complementary solution y_c can be occasionally found by

   Abel's theorem or Substitution

- But we lack...

2. The particular solution y_p can be found by

   Undetermined coefficients/Annihilator or Variation of parameters

- But we cannot deal with...
- We will address those two questions using second-order equations in part II.

- Nonlinear first-order

Theorem

If $\Phi(x, y)$ and $\Phi_y(x, y)$ are continuous in a rectangle $\mathcal{R} = \{a \le x \le b, c \le y \le d\}$ that contains the point (x_0, y_0), then there is one and only one solution to

$$y' = \Phi(x, y), \qquad y(x_0) = y_0$$

over some interval $\mathcal{I}$ inside the interval $[a, b]$.

- A certain types of such equations can be readily solved

separable, exact or by substitution

- Nonlinear higher-order

hardly know anything in general!

- There are two big tasks left for linear equations

$$\underbrace{\underset{\uparrow}{\alpha_1}(t)\ddot{y} + \underset{\uparrow}{\alpha_2}(t)\dot{y} + \underset{\uparrow}{\alpha_3}(t)y}_{1.} = \underbrace{f(t)}_{2.}$$

1. Variable Coefficients $\alpha_i(t)$ 2. Non-elementary $f(t)$

- The next task is to derive a systematic method for variable coefficients,

$$\alpha_i$$

which we assume to be continuous at first, but will be relaxed afterwards.

- The reason that linear equations with variable coefficients deserve a separate treatment is the fact that there is often no elementary solution to them.
- So we must expand the types of functions that we consider.
- A more general way to define a function is by using an infinite series, e.g.

$$y = y(t) = \sum_{n=0}^{\infty} c_n(t-a)^n = c_0 + c_1(t-a) + \cdots + c_n(t-a)^n + \cdots$$

Q: How should we proceed if functions that are defined by power series

$$y = \sum_{n=0}^{\infty} c_n (t-a)^n = c_0 + c_1(t-a) + \cdots + c_n(t-a)^n + \cdots$$

are allowed to be solutions to

$$\alpha_1(t)\ddot{y} + \alpha_2(t)\dot{y} + \alpha_3(t)y = f(t)$$

- It is similar to the method of undetermined coefficients, we try to determine

$$c_n$$

  by substituting the function

$$y = \sum_{n=0}^{\infty} c_n (t-a)^n \qquad \text{about a particular point } t = a$$

  into the differential equation and equating the coefficients.
- Only linear second-order differential equations with variable coefficients that possess solutions in the form of power series will be considered, but the idea can be extended to higher order easily.

Power series about a

A power series is said to be about $x = a$ if the series has the following form,

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n + \cdots$$

where a is a constant called the center, and c_n's are coefficients.

- Recall it is the choice of c_n and a that defines a power series. For example, taking all the coefficients to be 1, and let it be about $x = 0$, then

$$\sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

  which is the geometric series with first term of 1 and ratio of x.

Q: Why a power series may or may not be a function of x?

- A power series may converge for some values of x but diverge for others

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for} \quad -1 < x < 1$$

the above power series only represents the function $\dfrac{1}{1-x}$ for $-1 < x < 1$.

- Recall a power series is convergent at x_0 if its partial sums $\{s_n\}$ converges,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=0}^{n} c_k (x_0 - a)^k$$

- If the above limit does not exist at x, then the series is said to be divergent.
- Every power series has an interval of convergence,

$$(a-R, a+R), \quad [a-R, a+R], \quad (a-R, a+R], \quad [a-R, a+R).$$

which is the set of all values of x for which the series converges.

- Every power series has a radius of convergence R.

1. If $R > 0$, then

   the power series converges for $|x-a| < R$, and diverges for $|x-a| > R$.

2. If $R = 0$, then

   the series converges only at its centre a.

3. If $R = \infty$, then

   the series converges for all x.

- The interval of convergence is therefore

$$(a-R, a+R), \quad [a-R, a+R], \quad (a-R, a+R], \quad [a-R, a+R).$$

- The radius of convergence tells nothing about the endpoints,

$$a-R \qquad \text{and} \qquad a+R$$

Testing for Convergence

1. Use the Ratio Test to find the interval where the series converges absolutely.

$$|x-a|<R \qquad \text{or} \qquad a-R<x<a+R.$$

2. If the interval of absolute convergence is $a-R<x<a+R$, then

the series diverges for $|x-a|>R$.

3. If the interval of absolute convergence is finite, then we need to

test for convergence or divergence at each endpoint.

Definition

A series $\displaystyle\sum_{k=0}^{n} a_k$ is called absolutely convergent if the series $\displaystyle\sum_{k=0}^{n} |a_k|$ is convergent.

- Recall the ratio test is useful in determining the radius of convergence.

The Ratio Test

Suppose a_n represent the nth terms of a series.

$$\text{If} \quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$$
$$\implies \quad \text{Absolutely convergent}$$
$$\implies \quad \text{Convergent}$$
$$\text{If} \quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$$
$$\implies \quad \text{divergent}$$
$$\text{If} \quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$$
$$\implies \quad \text{Inconclusive}$$

Integral test

Suppose $\{a_n\}$ is a sequence such that $a_n = f(n)$, where $f(x)$ is a

1. continuous
2. positive
3. decreasing function on $[1, \infty)$.

Then the series $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\, dx$ both converge or both diverge.

Alternating Series test

If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$, where $b_n > 0$, satisfies

$$1. b_{n+1} \leq b_n \qquad \text{and} \qquad 2. \lim_{n\to\infty} b_n = 0$$

then the series is convergent.

Exercise

For what values of x do the power series $\displaystyle\sum_{n=0}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$ *converge?*

Solution

- Ratio Test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{2n+1}}{2n+1}\frac{2n-1}{x^{2n-1}}\right| = \frac{2n-1}{2n+1}x^2 \to x^2 \qquad \text{as} \quad n \to \infty$$

- So the power series converges absolutely for $-1 < x < 1$ and diverges for

$$x < -1 \qquad \text{or} \qquad x > 1$$

- At $x = 1$ or $x = -1$, it converges by the Alternating Series test.

$$b_n = \frac{1}{2n-1}$$

- Therefore the series converges for $-1 \leq x \leq 1$ and diverges elsewhere.

- If $R > 0$, then f is continuous, differentiable and integrable on the interval

$$(a - R, a + R)$$

- Moreover,

$$f'(x) \text{ and } \int f(x)\,dx$$

  can be found by term-by-term differentiation and integration.

- Convergence of a power series

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

  at endpoints may be either lost by differentiation or gained by integration.

- Consider the domains of $f(x)$ and $g(x)$, and of $f'(x)$ and $\int g(x)\,dx$,

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} \qquad \text{and} \qquad g(x) = \sum_{n=0}^{\infty} (-1)^n x^n \qquad \text{for } |x| < 1.$$

- The ratio test implies the radius of convergence for the first series is 1, and the p-series test implies that the series converges absolutely at both ends.
- However, the derivative of the first series is not convergent at $x = 1$

$$f'(x) = \frac{d}{dx} \sum_{n=1}^{\infty} \frac{x^n}{n^2} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}$$

  since it becomes the harmonic series at $x = 1$.
- It can be shown that

$$g(x) = \sum_{n=0}^{\infty} (-1)^n x^n$$

  has a radius of convergence $R = 1$, but is divergent at both endpoints.
- However, the integral of the second series is convergent at $x = 1$

$$\int g(x)\, dx = \int \sum_{n=0}^{\infty} (-1)^n x^n \, dx = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} + c$$

  since it becomes the alternative harmonic series at $x = 1$.

- Notice differentiation may shift the indices

$$\frac{d}{dx}\sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

- Sometimes it is desirable to shift the indices after differentiation, e.g.

$$\begin{aligned}\sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n &= \sum_{n=0}^{\infty}(n+1)c_{n+1}x^n + \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{n=0}^{\infty}\left((n+1)c_{n+1}x^n + c_n x^n\right) \\ &= \sum_{n=0}^{\infty}\left((n+1)c_{n+1} + c_n\right)x^n\end{aligned}$$

- In general, for $\displaystyle\sum_{n=k}^{\infty} c_n(x-a)^n$, make the substitution $m = n-k$, then

$$\sum_{n=k}^{\infty} c_n(x-a)^n = \sum_{m=0}^{\infty} c_{m+k}(x-a)^{m+k} = \sum_{n=0}^{\infty} c_{n+k}(x-a)^{n+k}$$

Analytic at a point

A function f is analytic at a point $x = a$ if it can represented by a power series in terms of $x - a$ with a nonzero radius of convergence.

- For example, the following two functions are analytic

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{and} \quad f(x) = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1}$$

- In fact, any elementary functions are analytic in their domain,

$$x^2 + 3x, \quad \frac{x^2+1}{x-5}, \quad x^{3/4}, \quad \ln(x), \quad e^x, \quad \sin x, \quad \text{and} \quad \cos x \quad \text{etc.}$$

- Note those special functions you have taken for granted

$$\exp x, \quad \sin x, \quad \text{and} \quad \cos x$$

are symbols that are given to their power series representations for simplicity.

- The exponential function $\exp\colon \mathbb{R} \to \mathbb{R}$ can be defined using power series

$$\exp x = \sum_{k=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

  which can be shown to be convergent for any $x \in \mathbb{R}$ using the ratio test.

- Too see why the exponential function $\exp\colon \mathbb{C} \to \mathbb{C}$ has the following form

$$\exp z = \exp(x + \mathrm{i}y) = \exp(x)\,(\cos y + \mathrm{i}\sin y)$$

  we consider the following power series definition

$$\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

  which is a power series with terms in $\mathbb{C}$ when z takes complex numbers.

- Similar to the real case, a series with terms in $\mathbb{C}$

$$\sum_{n=0}^{\infty} a_n$$

is defined to be convergent if the sequence of its partial sums

$$\{s_n\} \qquad \text{where} \quad s_n = \sum_{k=0}^{n} a_k$$

is convergent, that is, for every $\epsilon > 0$, there is a $N \in \mathbb{N}$ such that

$$|s_n - L| < \epsilon \qquad \text{whenever} \quad n > N$$

- The number L is the limit of $\{s_n\}$ or what the series $\sum_{n=0}^{\infty} a_n$ converges to

$$\lim_{n\to\infty} s_n = L; \qquad \text{or} \qquad \sum_{n=0}^{\infty} a_n = L$$

- Note $|\cdot|$ denote the modulus of a complex nubmer, and $L \in \mathbb{C}$.

- It can be shown the ratio test is also applicable to series with complex terms

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

using which we can conclude $\displaystyle\sum_{n=0}^{\infty}\frac{z^n}{n!}$ is convergent for all $z \in \mathbb{C}$ since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{z^{n+1}}{(n+1)!}\cdot\frac{n!}{z^n}\right| = \lim_{n\to\infty}\frac{|z|}{n+1} = 0 < 1$$

- There are two properties of $\exp z$ left to be shown before we establish

$$\exp z = \exp(x + \mathrm{i}y) = \exp(x)\,(\cos y + \mathrm{i}\sin y)$$

namely,

$$\exp(z_1 + z_2) = \exp z_1 \exp z_2 \qquad \text{and} \qquad \exp \mathrm{i}y = \cos y + \mathrm{i}\sin y$$

- Multiplying $\exp z_1$ and $\exp z_2$ term-by-term and rearranging, we have

$$\begin{aligned}
\exp z_1 \exp z_2 &= \left(\sum_{n=0}^{\infty} \frac{z_1^n}{n!}\right) \cdot \left(\sum_{m=0}^{\infty} \frac{z_2^m}{m!}\right) \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{z_1^n z_2^m}{n!m!} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{z_1^{n-m} z_2^m}{(n-m)!m!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \frac{n!}{(n-m)!m!} z_1^{n-m} z_2^m \\
&= \sum_{n=0}^{\infty} \frac{(z_1+z_2)^n}{n!} = \exp(z_1+z_2)
\end{aligned}$$

- Separating the odd terms from the even terms, we have

$$
\begin{aligned}
\exp \mathrm{i}y &= \sum_{n=0}^{\infty} \frac{(\mathrm{i}y)^n}{n!} \\
&= \underbrace{1 + \frac{(\mathrm{i}y)^2}{2!} + \frac{(\mathrm{i}y)^4}{4!} + \cdots}_{\text{even}} + \underbrace{\mathrm{i}y + \frac{(\mathrm{i}y)^3}{3!} + \frac{(\mathrm{i}y)^5}{5!} + \cdots}_{\text{odd}} \\
&= 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots + \mathrm{i}\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \cdots\right) \\
&= \cos y + \mathrm{i} \sin y
\end{aligned}
$$

from which, we have the desired result and the following identities

$$
\exp(-\mathrm{i}y) = \cos y - \mathrm{i} \sin y \implies \begin{aligned} \cos y &= \frac{e^{\mathrm{i}y} + e^{-\mathrm{i}y}}{2} \\ \sin y &= \frac{e^{\mathrm{i}y} - e^{-\mathrm{i}y}}{2\mathrm{i}} \end{aligned}
$$