Vv256 Lecture 28

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- The Fourier series have many applications,

$$f = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

  we will use it is to deal with ODEs involving periodic features.

- An object of mass m on a spring of constant k are governed by the ODE,

$$m\ddot{y} + c\dot{y} + ky = f(t), \qquad \text{where } c \text{ is the damping coefficient.}$$

  and the electrical analogue, which governs RLC-circuits, is

$$Li'' + Ri' + \frac{1}{C}i = E'(t)$$

  where $f(t)$ and $E'(t)$ are known as the forcing function.

- Such equations often involve a periodic forcing function

$$f(t+2L) = f(t), \qquad E'(t+2L) = E'(t)$$

- If the forcing function is a linear combination of sine and cosine, e.g.

$$2\ddot{y} - \dot{y} + 3y = 5\cos 2t + 7\sin 2t + 4\sin t$$

we can use the method of annihilator since

$$\mathcal{L}_1\mathcal{L}_2\left(c_1\phi_1 + c_2\phi_2\right) = 0 \qquad \text{where} \qquad \mathcal{L}_1(\phi_1) = 0 \quad \text{and} \quad \mathcal{L}_2(\phi_2) = 0$$

- Recall the differential operator $\left[\mathcal{D}^2 - 2R\mathcal{D} + (R^2 + \theta^2)\right]^n$ annihilates

$$\begin{array}{ccccc} e^{Rt}\cos\theta t, & te^{Rt}\cos\theta t, & t^2e^{Rt}\cos\theta t, & \ldots, & t^{n-1}e^{Rt}\cos\theta t \\ e^{Rt}\sin\theta t, & te^{Rt}\sin\theta t, & t^2e^{Rt}\sin\theta t, & \ldots, & t^{n-1}e^{Rt}\sin\theta t \end{array}$$

- However, if the forcing function is not in terms of sine and cosine, then

$$f = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos kx + b_k\sin kx)$$

Exercise

Solve the following initial-value problem,

$$y'' + 2y = f(t) \qquad \text{where} \qquad f(t) = \begin{cases} -1 & \text{if } -\pi < t < 0, \\ 1 & \text{if } \quad 0 < t < \pi. \end{cases}$$

extended periodically. Find the steady-state solution y_p.

Solution

- Note the characteristic equation for the complementary equation is

$$\mathcal{D}^2 + 2 \implies r^2 + 2 = 0 \implies r_{1,2} = \pm\sqrt{2}\mathrm{i}$$

- Notice that the complementary solution is NOT transient,

$$y_c = c_1 \cos\sqrt{2}t + c_2 \sin\sqrt{2}$$

  however, often the steady-state solutions is defined to be the particular solution and ignore the contribution from the complementary solution.

Solution

- We consider the Fourier series representation of the function $f(t)$.

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin\Big((2k-1)\,t\Big)$$

which we have found in L26P19.

- Since the differential operator $\mathcal{D}^2 - 2\alpha\mathcal{D} + (\alpha^2+\beta^2)$ annihilates

$$e^{\alpha t}\cos\beta t, \qquad \text{and} \qquad e^{\alpha t}\sin\beta t$$

thus the annihilator for $\sin\Big((2k-1)\,t\Big)$ is

$$\mathcal{L}_k = \mathcal{D}^2 + (2k-1)^2 = \Big(\mathcal{D} + \mathrm{i}(2k-1)\Big)\Big(\mathcal{D} - \mathrm{i}(2k-1)\Big)$$

- We will need infinitely many such annihilators to annihilate an infinite sum.

Solution

- Thus the annihilator for $f(t)$ is

$$\prod_{k=1}^{\infty}\Big(\mathcal{D}+\mathrm{i}(2k-1)\Big)\Big(\mathcal{D}-\mathrm{i}(2k-1)\Big)$$

- This translates to the characteristic equation of

$$\prod_{k=1}^{\infty}\Big(\lambda+\mathrm{i}(2k-1)\Big)\Big(\lambda-\mathrm{i}(2k-1)\Big)=0$$

- Therefore the particular solution must have the form

$$\begin{aligned} y_p=\sum_{k=1}^{\infty} c_i e^{\lambda_k t}+d_i e^{-\lambda_k t} &= \sum_{k=1}^{\infty} c_i e^{(2k-1)\mathrm{i}t}+d_i e^{-(2k-1)\mathrm{i}t} \\ &= \sum_{k=1}^{\infty} e^{0t}\Big[A_k\cos\left[\pm\left(2k-1\right)t\right]+B_k\sin\left[\pm\left(2k-1\right)t\right]\Big] \\ &= \sum_{k=1}^{\infty}\Big(A_k\cos\left(2k-1\right)t+B_k\sin\left(2k-1\right)t\Big) \end{aligned}$$

Solution

- To determine the coeffcents A_k and B_k, we substitute y_p and y_p'' into,

$$y'' + 2y = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin\Big((2k-1)\, t \Big)$$

- Since

$$\begin{aligned} y'' + 2y = & \sum_{k=1}^{\infty} \Big(-(2k-1)^2 A_k \cos(2k-1)\, t - (2k-1)^2 B_k \sin(2k-1)\, t \Big) \\ & + \quad 2 \sum_{k=1}^{\infty} \Big(A_k \cos(2k-1)\, t + B_k \sin(2k-1)\, t \Big) \\ = & \sum_{k=1}^{\infty} \left(2 - (2k-1)^2\right) A_k \cos(2k-1)\, t \\ & + \sum_{k=1}^{\infty} \left(2 - (2k-1)^2\right) B_k \sin(2k-1)\, t \end{aligned}$$

Solution

- Equating the coefficents, we see that

$$A_k = 0, \qquad \text{and} \qquad B_k = \frac{4}{\pi\,(2k-1)\,(2-(2k-1)^2)}$$

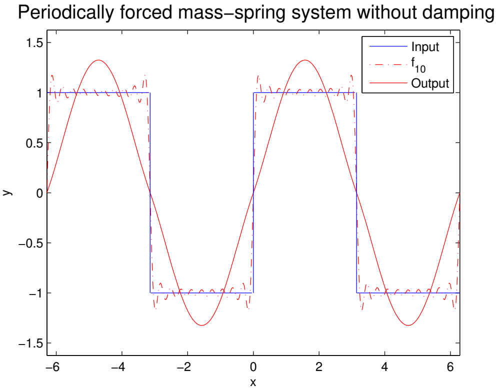
- Therefore the steady periodic solution has the Fourier series representation of

$$y_p = \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k-1)\,(2-(2k-1)^2)} \sin\,(2k-1)\,t$$

- The steady-state solution is periodic with the same period as $f(t)$ itself.

$$f(t) = \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{2k-1} \sin\Big((2k-1)\,t\Big)$$

```
>> syms x k
>> evalin(symengine,'assume(k,Type::Integer)');
%This tells matalb that k is an integer.

>> f = sign(sin(x));
%This uses sign function
%to define our square wave function

>> f_n = @(x,n) 4/pi * symsum(...
1/(2*k-1)*sin((2*k-1)*x),k,1,n);%partial sum

>> obj = ezplot(f,[-2*pi,2*pi]);
>> set(obj, 'color','blue');
>> clear obj; hold on

>> obj = ezplot(f_n(x,10),[-2*pi,2*pi]);
>> set(obj, 'color','red');
>> set(obj, 'LineStyle', '-.');clear obj

>> y_n = @(x,n) symsum(...
4/(pi*(2*k-1)*(2-(2*k-1)^2))...
    *sin((2*k-1)*x),k,1,n);%partial sum

>> obj = ezplot(y_n(x,10),[-2*pi,2*pi]);
>> set(obj, 'color','red'); clear obj
```




```
>> legend({'Input','f_{10}','Output'}, 'Location','northeast','FontSize',10);

>> line([-pi,-pi],[-1,1]); line([0,0],[-1,1]); line([pi,pi],[-1,1]); hold off

>> xlabel('x'); ylabel('y');
>> title(' Periodically forced mass-spring system without damping','FontSize',15)
```

- Clearly, periodic functions in applications may have any period, not just 2π.
- We need to consider the Fourier series of periodic functions with period $2L$,

$$f(t+2L) = f(t)$$

where L is any finite real number.

- The transition from 2π to $2L$ can be obtained by a suitable change of scale.
- Suppose $f(t)$ is a periodic function of period $2L$, and consider the following

$$t = \frac{L}{\pi}x \implies x = \frac{\pi}{L}t$$

then

$x = \pm\pi$ corresponds to $t = \pm L$.

- So f, as a function of x, is a periodic function of period 2π.

- So the Fourier series of this form is,

$$f(t) = f\left(\frac{L}{\pi}x\right) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}x\right)\,dx$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}x\right)\cos kx\,dx, \qquad \text{for } k = 1,\ 2,\ \ldots,\ n.$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}x\right)\sin kx\,dx, \qquad \text{for } k = 1,\ 2,\ \ldots,\ n.$$

- Substitute x back to t, we have

$$x = \frac{\pi}{L} t \implies \frac{dx}{dt} = \frac{\pi}{L}$$

- Thus

$$\begin{aligned} f(t) = f\left(\frac{L}{\pi} x\right) &= \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \\ &= \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos \frac{k\pi}{L} t + b_k \sin \frac{k\pi}{L} t) \qquad \text{where} \end{aligned}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi} x\right) \, dx = \frac{1}{L} \int_{-L}^{L} f\,(t) \, dt$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi} x\right) \cos kx \, dx = \frac{1}{L} \int_{-L}^{L} f\,(t) \cos \frac{k\pi}{L} t \, dt, \qquad \text{for } k = 1,\, 2,\, \ldots,\, n.$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi} x\right) \sin kx \, dx = \frac{1}{L} \int_{-L}^{L} f\,(t) \sin \frac{k\pi}{L} t \, dt, \qquad \text{for } k = 1,\, 2,\, \ldots,\, n.$$

Exercise

Find the Fourier series of the function

$$f(t) = \begin{cases} 0 & \text{if} \quad -2 < t < -1 \\ \gamma & \text{if} \quad -1 < t < \ \ 1 \\ 0 & \text{if} \quad \ \ 1 < t < \ \ 2 \end{cases} \qquad \text{and} \qquad f(t+4) = f(t)$$

Solution

- Note this function has a period of $2L = 4$, thus applying the formulas,

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(t)\,dt = \frac{1}{2}\int_{-1}^{1} \gamma\,dt = \gamma$$

$$a_k = \frac{1}{2}\int_{-2}^{2} f(t)\cos\left(\frac{k\pi t}{2}\right)\,dt = \frac{1}{2}\int_{-1}^{1} \gamma\cos\left(\frac{k\pi t}{2}\right)\,dt$$

$$= \frac{2\gamma}{k\pi}\sin\frac{k\pi}{2} = \begin{cases} 0 & \text{if } k \text{ is even} \\ \frac{2\gamma}{k\pi}\sin\frac{k\pi}{2} & \text{if } k \text{ is odd} \end{cases}$$

Solution

- Hence

$$a_k = \begin{cases} 0 & \text{if } k \text{ is even} \\ \frac{2\gamma}{k\pi} \sin \frac{k\pi}{2} & \text{if } k \text{ is odd} \end{cases}$$

$$= \begin{cases} 0 & \text{if } k \text{ is even} \\ -\frac{2\gamma}{k\pi} & \text{if } (k+1)/2 \text{ is even} \\ \frac{2\gamma}{k\pi} & \text{if } (k+1)/2 \text{ is odd} \end{cases}$$

- Compute the coefficients
$$b_n = \frac{1}{2} \int_{-2}^{2} f(t) \sin \frac{k\pi t}{2} \, dt = \frac{1}{2} \int_{-1}^{1} \gamma \sin \frac{k\pi t}{2} \, dt = 0$$
- Therefore the Fourier series for the given function is

$$f(t) = \frac{\gamma}{2} + \frac{2\gamma}{\pi} \sum_{\ell=1}^{\infty} (-1)^{\ell+1} \frac{1}{2\ell - 1} \cos \frac{(2\ell-1)\pi}{2} t$$

- If $f(t)$ is an even function, that is,

$$f(-t) = f(t)$$

then its Fourier series reduces to a Fourier cosine series

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi t}{L}, \qquad \text{where}$$

$$a_0 = \frac{2}{L}\int_0^L f(t)\,dt \qquad \text{and} \qquad a_k = \frac{2}{L}\int_0^L f(t)\cos\frac{k\pi t}{L}\,dt$$

- Notice that the integration is from 0 to L only!
- If $f(t)$ is an odd function, that is,

$$f(-t) = -f(t)$$

then its Fourier series reduces to a Fourier sine series

$$f(t) = \sum_{k=1}^{\infty} b_k \sin\frac{k\pi t}{L} \qquad \text{where} \quad b_k = \frac{2}{L}\int_0^L f(t)\sin\frac{k\pi t}{L}\,dt$$

Exercise

Find the Fourier series of the function

$$f(x) = x + \frac{\pi}{2} \qquad \text{if} \quad -\pi < x < \pi \qquad and \quad f(x+2\pi) = f(x).$$

Solution

- Note the given function can be slipped into

$$f(x) = \underbrace{x}_{odd} + \underbrace{\frac{\pi}{2}}_{constant}$$

- We can leave out the constant,

$$\frac{\pi}{2}$$

  find the Fourier series for $g(x) = x$, which will also be the Fourier series for

$$f(x) - \frac{\pi}{2}$$

Solution

- Since $g(x) = x$ is odd, we know

$$a_0 = 0 \qquad \text{and} \qquad a_k = 0$$

and using integration by parts, we have

$$\begin{aligned} b_k &= \frac{2}{\pi} \int_0^\pi x \sin kx \, dx \\ &= \frac{2}{\pi} \left[\frac{-x \cos kx}{k} \Bigg|_0^\pi + \frac{1}{k} \int_0^\pi \cos kx \, dx \right] = \frac{2}{\pi} \left(\frac{-\pi \cos k\pi}{k} + 0 \right) \\ &= -\frac{2}{k} \cos k\pi \end{aligned}$$

- Therefore, the Fourier series is

$$f(x) = \frac{\pi}{2} + x = \frac{\pi}{2} + 2 \sum_{k=1}^{\infty} b_k \sin kx = \frac{\pi}{2} + 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin kx$$

Exercise

Consider the equation

$$I'' + 0.05I' + 25I = E'(t)$$

Find the steady-state solution y_p, where $f(t)$ is given by

$$E'(t) = \begin{cases} t + \frac{\pi}{2} & \text{if} \quad -\pi < t < 0 \\ -t + \frac{\pi}{2} & \text{if} \quad 0 < t < \pi \end{cases} \quad \text{and} \quad E'(t + 2\pi) = E'(t)$$

Solution

- The forcing function is essentially

$$E'(t) = \underbrace{-|t|}_{\text{even}} + \underbrace{\frac{\pi}{2}}_{\text{constant}}$$

- Thus we only need to find the Fourier series for the even function

$$-|t|$$

Solution

- According to the formulas for an even function, we have

$$a_0 = \frac{2}{\pi}\int_0^\pi -t\,dt = \frac{2}{\pi}\left[\frac{-t^2}{2}\right]_0^\pi = -\pi$$

$$\begin{aligned} a_k &= \frac{2}{\pi}\int_0^\pi -t\cos kt\,dt \\ &= -\frac{2}{\pi}\left(\left[\frac{t\sin kt}{k}\right]_0^\pi - \int_0^\pi \frac{\sin kt}{k}\,dt\right) = -\frac{2\Big((-1)^k - 1\Big)}{\pi k^2} \\ &= \begin{cases} 0 & \text{if } k \text{ is even,} \\ \dfrac{4}{\pi k^2} & \text{if } k \text{ is odd.} \end{cases} \end{aligned}$$

$$b_k = 0$$

- So the Fourier series for the forcing function is $E' = \dfrac{4}{\pi}\displaystyle\sum_{\ell=1}^{\infty}\frac{\cos(2\ell-1)t}{(2\ell-1)^2}$

Solution

- Back to the equation

$$I'' + 0.05I' + 25I = E'(t)$$

$$\left(\mathcal{D}^2 + 0.05\mathcal{D} + 25\right)I = \frac{4}{\pi}\sum_{\ell=1}^{\infty}\frac{\cos(2\ell-1)t}{(2\ell-1)^2}$$

- We again ignore any possible contribution from the complementary solution,

$$\prod_{\ell=1}^{\infty}\Big(\mathcal{D} + \mathrm{i}(2\ell-1)\Big)\Big(\mathcal{D} - \mathrm{i}(2\ell-1)\Big) \implies \lambda = \pm\mathrm{i}(2\ell-1)$$

- Therefore the steady-state solution is given by

$$I_p = \sum_{\ell=1}^{\infty} I_\ell \qquad \text{where} \qquad I_\ell = A_{2\ell-1}\cos(2\ell-1)t + B_{2\ell-1}\sin(2\ell-1)t$$

$$= A_1\cos t + B_1\sin t + A_3\cos 3t + B_3\sin 3t + \cdots$$

Solution

- To determine the coefficients

$$A_{2\ell-1} \qquad \text{and} \qquad B_{2\ell-1}$$

we need to substitute this particular solution and its derivatives

$$I_p = \underbrace{A_1 \cos t + B_1 \sin t}_{k=1} + \underbrace{A_3 \cos 3t + B_3 \sin 3t}_{k=3} + \cdots$$

into the equation

$$\left(\mathcal{D}^2 + 0.05\mathcal{D} + 25\right) I_p = \frac{4}{\pi}\left(\underbrace{\cos t}_{k=1} + \frac{1}{9}\underbrace{\cos 3t}_{k=3} + \frac{1}{25}\underbrace{\cos 5t}_{k=5} + \cdots\right)$$

- Note the kth term on the left can only be obtained from the kth term of I_p.

$$\left(\mathcal{D}^2 + 0.05\mathcal{D} + 25\right)\left(A_k \cos kt + B_k \sin kt\right) = \frac{4}{\pi k^2} \cos kt \qquad k = 1, 3, 5, \ldots$$

Solution

- So we only need consider one such equation

$$\left(\mathcal{D}^2 + 0.05\mathcal{D} + 25\right)(A_k \cos kt + B_k \sin kt) = \frac{4}{\pi k^2} \cos kt$$

$$\begin{aligned} -k^2 A_k \cos kt - k^2 B_k \sin kt + 0.05\left(-kA_k \sin kt + kB_k \cos kt\right) & \\ + 25\left(A_k \cos kt + B_k \sin kt\right) &= \frac{4}{\pi k^2} \cos kt \end{aligned}$$

$$\Longrightarrow \quad \begin{aligned} -k^2 A_k + 0.05kB_k + 25A_k &= \frac{4}{\pi k^2} \\ -k^2 B_k - 0.05kA_k + 25B_k &= 0 \end{aligned}$$

- Solving the system, we have the following for all $k = 1, 3, 5, \ldots$,

$$A_k = \frac{4(25 - k^2)}{k^2\pi\left[(25 - k^2)^2 + (0.05k)^2\right]} \quad \text{and} \quad B_k = \frac{0.2}{k\pi\left[(25 - k^2)^2 + (0.05k)^2\right]}$$

- It is useful to express the trigonometric function in exponential form. We can define complex Fourier coefficients c_k in terms of the real coefficients:

$$c_k = \frac{1}{2}(a_k - \mathrm{i}b_k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos kx - \mathrm{i}\sin kx)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-\mathrm{i}kx}\,dx$$

- If we denote the complex conjugate of c_k to be c_{-k},

$$c_{-k} = \frac{1}{2}(a_k + \mathrm{i}b_k)$$

then we can solve for a_k and b_k , then

$$a_k = c_k + c_{-k} \qquad \text{and} \qquad b_k = \mathrm{i}(c_k - c_{-k})$$

- It follows that

$$c_k e^{\mathrm{i}kx} + c_{-k}e^{-\mathrm{i}kx} = (c_k + c_{-k})\cos kx + \mathrm{i}(c_k - c_{-k})\sin kx = a_k \cos kx + b_k \sin kx$$

- Hence the Fourier series has complex form

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) = \sum_{k=-\infty}^{\infty} c_k e^{\mathrm{i}kx}, \text{ where } c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-\mathrm{i}kx}\,dx$$