Vv256 Lecture 18

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- Consider solving the following initial-value problem with Laplace transform

$$\ddot{y} + y = \tan t \qquad y(0) = 1 \qquad \dot{y}(0) = 2$$

- It is not clear what the Laplace transform of $\tan t$ is, however, if denote it by

$$F(s) = \int_0^\infty e^{-st} \tan t \, dt$$

- Taking the Laplace transform and use the initial conditions, we have

$$\begin{aligned}
s^2 Y(s) - s y(0) - \dot{y}(0) + Y(s) &= F(s) \\
s^2 Y(s) - s - 2 + Y(s) &= F(s) \\
\implies Y(s) &= \frac{s+2}{s^2+1} + \frac{F(s)}{s^2+1} \\
&= \frac{s}{s^2+1} + 2 \cdot \frac{1}{s^2+1} + \frac{F(s)}{s^2+1}
\end{aligned}$$

- Finding the inverse Laplace transform, we have

$$Y(s) = \frac{s+2}{s^2+1} + \frac{F(s)}{s^2+1} = \frac{s}{s^2+1} + 2\cdot\frac{1}{s^2+1} + \frac{F(s)}{s^2+1}$$

- Using the convolution theorem, we have

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s+2}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{F(s)}{s^2+1}\right\} \qquad \text{if} \quad H(s) = \frac{1}{s^2+1}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \mathcal{L}^{-1}\Big\{F(s)H(s)\Big\}$$

$$= \cos t + 2\sin t + \int_0^t \sin(t-\tau)\tan\tau\, d\tau$$

- Use the identity $\sin(A-B) = \sin A\cos B - \sin B\cos A$, we obtain

$$y(t) = \cos t + 3\sin t + \cos t \ln\left(\frac{\cos t}{1+\sin t}\right)$$

Q: What does denominator of $H(s)$ represent?

- Consider a second-order linear equation with constant coefficients

$$a\ddot{y} + b\dot{y} + cy = f(t) \qquad y(0) = y_0 \qquad \dot{y}(0) = y_1$$

- Taking the Laplace transform and applying the initial conditions, we have

$$a\left(s^2Y(s) - sy(0) - \dot{y}\right) + b\left(sY(s) - y(0)\right) + cY(s) = F(s)$$

$$\left(as^2 + bs + c\right)Y(s) - (as+b)y_0 - ay_1 = F(s)$$

- Making $Y(s)$ the subject, we have

$$Y(s) = Y_c(s) + Y_{\color{red}p}(s)$$

where

$$Y_c(s) = \frac{(as+b)y_0 + ay_1}{as^2+bs+c} \qquad \text{and} \qquad Y_{\color{red}p}(s) = \frac{F(s)}{as^2+bs+c}$$

Q: What do those two functions present in the t-domain?

- If $Y_c(s)$ is the only term present, that is,

$$Y(s) = Y_c(s) = \frac{(as+b)y_0 + ay_1}{as^2 + bs + c}$$

then $F(s)$ must be zero for all s, which means $f(t)$ must be zero, hence

$$y(t) = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}[Y_c]$$

is the solution to the corresponding homogeneous equation

$$a\ddot{y} + b\dot{y} + cy = 0$$

- Now if $Y_p(s)$ is the only term present, that is

$$Y(s) = Y_p(s) = \frac{F(s)}{as^2 + bs + c}$$

then $Y_c(s)$ must be zero for all s, which means $y_0 = 0$ and $y_1 = 0$, hence

$$y(t) = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}[Y_p]$$

is the solution to the IVP $\quad a\ddot{y} + b\dot{y} + cy = f(t), \quad y(0) = 0, \quad \dot{y}(0) = 0.$

Definition

For the initial-value problem

$$p\left(\mathcal{D}\right)[y] = f(t), \qquad y(0) = 0, \quad \dot{y}(0) = 0$$

the function

$$H(s) = \frac{1}{p(s)}$$

is called the transfer function.

- The transfer function depends only the properties of the system, e.g.

$$\underset{a}{\underset{\uparrow}{L}}\frac{d^2 i}{dt^2} + \underset{b}{\underset{\uparrow}{R}}\frac{di}{dt} + \frac{1}{\underset{c}{\underset{\uparrow}{C}}}i(t) = E'(t)$$

- The transfer function for this system is

$$H(s) = \frac{1}{Ls^2 + Rs + \frac{1}{C}}$$

Theorem

The solution to the initial-value problem,

$$a\ddot{y} + b\dot{y} + cy = f(t), \qquad y(0) = y_0, \quad \dot{y}(0) = y_1$$

is given by the following,

$$y(t) = y_c(t) + (h * f)(t)$$

where y_c is the complementary solution that satisfies the given initial conditions

$$y(0) = y_0, \quad \dot{y}(0) = y_1$$

and $h(t)$ is the inverse Laplace transform of the transfer function $H(s)$.

Exercise

Solve the following initial-value problem

$$\ddot{y} + 4y = e^{-t}, \qquad y(0) = 0 \quad \dot{y}(0) = 0$$

Solution

- Finding the characteristic polynomial, we have

$$p(r) = r^2 + 4 \implies H(s) = \frac{1}{p(s)} = \frac{1}{s^2 + 4}$$

- Finding the so-called the impulse response function

$$h(t) = \mathcal{L}^{-1}\Big[H(s)\Big] = \frac{1}{2}\sin 2t$$

- Thus the output is given by

$$y(t) = \int_0^t h(t-\tau)f(\tau)\,\tau = \int_0^t \frac{1}{2}\sin\big(2(t-\tau)\big)e^{-\tau}\,d\tau$$

$$= \frac{1}{10}\left[\sin(2t) - 2\cos(2t) + 2e^{-t}\right]$$

Q: Do you think this method is better than variation of parameters?

- We consider the following linear equation again

$$y'' + Py' + Qy = f$$

where P and Q are continuous functions of x.

Q: Can we write down the solution y as an integral of some kind?

Definition

The Green's function

$$G(x; a)$$

associated with the equation is the function such that

$$G'' + PG' + QG = \delta(x - a)$$

where $\delta(x - a)$ is the Dirac delta function.

Q: Is it surprising that the solution to the original equation is related to $G(x; a)$?

Theorem

Suppose P and Q are continuous functions of x, then

$$\phi = \int_{-\infty}^{\infty} f(a) G(x;a)\, da$$

is the solution to the following equation

$$y'' + Py' + Qy = f$$

Proof

- Let $\mathcal{L} = \mathcal{D}^2 + P\mathcal{D} + Q$ be the differential operator for the given equation,

$$\mathcal{L}\Big[\phi\Big] = \mathcal{L}\int_{-\infty}^{\infty} f(a)G(x;a)\, da = \int_{-\infty}^{\infty} f(a)\mathcal{L}\Big[G(x;a)\Big]\, da = \int_{-\infty}^{\infty} f(a)\delta(x-a)\, da$$

- Use substitution $u = x - a$, we have

$$\mathcal{L}\Big[\phi\Big] = \int_{-\infty}^{\infty} f(a)\delta(x-a)\, da = -\int_{\infty}^{-\infty} f(x-u)\delta(u)\, du = f(x-0) = f(x)$$

- Since by definition,

$$G'' + PG' + QG = \delta(x-a)$$

- For $x \neq a$, the Dirac delta function is zero

$$\delta(x-a) = 0$$

- So the Green's function is given by the complementary solutions for $x \neq a$

$$G(x;a) = \begin{cases} A_1\phi_1(x) + A_2\phi_2(x) & x < a \\ B_1\phi_1(x) + B_2\phi_2(x) & x > a \end{cases}$$ where A_i and B_i are constants,

and ϕ_1 and ϕ_2 are two linearly independent solutions

$$y'' + Py' + Qy = 0$$

- There are two properties of $G(x;a)$ we can use to determine the constants.

1. The continuity of $G(x;a)$ at $x=a$.
2. The finite jump discontinuity of $G'(x;a)$ of magnitude 1 at $x=a$.

Q: Why property 1. is true?

- Let us suppose $G(x;a)$ is not continuous, say a jump discontinuity, then

$$G' \propto \delta(x-a) \qquad \text{and} \qquad G'' \propto \delta'(x-a)$$

which contradicts to the fact that, after multiplying continuous P and Q,

$$G'' + PG' + QG = \delta(x-a)$$

Roughly speaking, it would be "more singular" than $\delta(x-a)$.

Q: Why property 2. is true?

$$\int_{a-\epsilon}^{a+\epsilon} G''\,dx + \int_{a-\epsilon}^{a+\epsilon} PG'\,dx + \int_{a-\epsilon}^{a+\epsilon} QG\,dx = \int_{a-\epsilon}^{a+\epsilon} \delta(x-a)\,dx \quad \text{as} \quad \epsilon \to 0$$

$$\lim_{x\to a^+} G' - \lim_{x\to a^-} G' = 1$$

- Apply properties 1. and 2. to

$$G(x;a) = \begin{cases} A_1\phi_1 + A_2\phi_2 & x < a \\ B_1\phi_1 + B_2\phi_2 & x > a \end{cases}$$

1. From the continuity of G at $x = a$, we have

$$A_1\phi_1(a) + A_2\phi_2(a) = B_1\phi_1(a) + B_2\phi_2(a)$$

2. From the jump of G' at $x = a$, we have

$$B_1\phi_1'(a) + B_2\phi_2'(a) - A_1\phi_1'(a) - A_2\phi_2'(a) = 1$$

- Solving these equation, we have

$$B_1 - A_1 = -\frac{\phi_2(a)}{W\left[\phi_1(a), \phi_2(a)\right]} \qquad \text{and} \qquad B_2 - A_2 = \frac{\phi_1(a)}{W\left[\phi_1(a), \phi_2(a)\right]}$$

- This means Green's function is not unique, we have two free parameters.

- If we let $A_1 = A_2 = 0$, then

$$G(x;a) = \begin{cases} 0 & x < a \\ \dfrac{-\phi_2(a)\phi_1(x) + \phi_1(a)\phi_2(x)}{W\left[\phi_1(a), \phi_2(a)\right]} & x \geq a \end{cases}$$

- Therefore the solution to the original equation is given by

$$y = \int_{-\infty}^{\infty} f(a) G(x;a)\, da$$

$$= -\phi_1(x) \int_{-\infty}^{x} \frac{f(a)\phi_2(a)}{W\left[\phi_1(a), \phi_2(a)\right]}\, da + \phi_2(x) \int_{-\infty}^{x} \frac{f(a)\phi_1(a)}{W\left[\phi_1(a), \phi_2(a)\right]}\, da$$

- This is identical to the solution obtained using variation of parameters.
- Note the choice of $A_1 = A_2 = 0$ was arbitrary, so variation of parameters is a special case of general Greens function.

Q: Can we use the method of Laplace transform to find the Green's function?