Vv417 Lecture 20

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- Recall the dot product of $\mathbf{u} = u_1\mathbf{e}_1 + v_2\mathbf{e}_2$ and $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2$

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2$$

Dot product

In general, the dot product of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is defined and denoted by

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n = \mathbf{u}^{\mathrm{T}}\mathbf{v}$$

where $\mathbf{u}$ and $\mathbf{v}$ are column vectors.

- The dot product is a scalar quantity, and is different from the outer product

$$\mathbf{u}^{\mathrm{T}}\mathbf{v} \neq \mathbf{u}\mathbf{v}^{\mathrm{T}}$$

Properties of the Dot product

- Suppose $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are column vectors in $\mathbb{R}^n$, and let α be a scalar.

1. $\mathbf{u}^{\mathrm{T}}\mathbf{v} = \mathbf{v}^{\mathrm{T}}\mathbf{u}$
2. $(\mathbf{u}\pm\mathbf{v})^{\mathrm{T}}\mathbf{w} = \mathbf{u}^{\mathrm{T}}\mathbf{w} \pm \mathbf{v}^{\mathrm{T}}\mathbf{w} = \mathbf{w}^{\mathrm{T}}(\mathbf{u}\pm\mathbf{v})$
3. $\mathbf{u}^{\mathrm{T}}\mathbf{u} \geq 0$
4. $(\alpha\mathbf{u})^{\mathrm{T}}\mathbf{v} = \alpha(\mathbf{u}^{\mathrm{T}}\mathbf{v}) = \mathbf{u}^{\mathrm{T}}(\alpha\mathbf{v})$

- Recall many key concepts in geometry can be defined using this product.

Definition

The length or magnitude of a vector $\mathbf{v}$ in $\mathbb{R}^n$ is the non-negative scalar,

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = \sqrt{\mathbf{v}^{\mathrm{T}}\mathbf{v}} \implies |\mathbf{v}|^2 = \mathbf{v}^{\mathrm{T}}\mathbf{v}$$

The distance d between vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is defined by

$$d = \mathrm{d}\left(\mathbf{u}, \mathbf{v}\right) = |\mathbf{u} - \mathbf{v}|$$

The angle θ between vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is defined by

$$\mathbf{u}^{\mathrm{T}}\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta \iff \cos\theta = \frac{\mathbf{u}^{\mathrm{T}}\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$$

Two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

$$\mathbf{u}^{\mathrm{T}}\mathbf{v} = 0$$

They are said to be orthonormal if they are also unit length.

Cauchy-Schwarz inequality

For any vectors $\mathbf{u}$ and $\mathbf{v}$, both in $\mathbb{R}^n$,

$$|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}|\,|\mathbf{v}|$$

Triangle Inequality

For any vectors $\mathbf{u}$ and $\mathbf{v}$, both in $\mathbb{R}^n$,

$$|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$$

Parallelogram Law

For any vectors $\mathbf{u}$ and $\mathbf{v}$, both in $\mathbb{R}^n$,

$$|\mathbf{u} + \mathbf{v}|^2 + |\mathbf{u} - \mathbf{v}|^2 = 2\left(|\mathbf{u}|^2 + |\mathbf{v}|^2\right)$$

- In a normed vector space, we have the notion of length and distance.
- However, to have the notion of angle and thus orthogonality, we have to have something similar to dot product for an arbitrary vector space.

Q: What actually is a dot product?

- Recall the dot product associate each pair of vectors in $\mathbb{R}^n$ with a scalar.

$$\mathbf{u} \cdot \mathbf{v} = \alpha$$

- If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$, and α be a scalar.
  1. Symmetry property
  $$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$$
  2. Distributive property
  $$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$$
  3. Homogeneity property
  $$(\alpha\mathbf{u}) \cdot \mathbf{v} = \alpha(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (\alpha\mathbf{v})$$
  4. Positivity property
  $$\mathbf{u} \cdot \mathbf{u} \geq 0 \qquad \text{and} \quad \mathbf{u} \cdot \mathbf{u} = 0 \quad \text{if and only if} \quad \mathbf{u} = 0$$

- Recall many properties regarding distance, angle, etc. are due properties 1–4.

- In order to have the same properties in a general vector space $\mathcal{V}$, we define:

Definition

An inner product on a vector space $\mathcal{V}$ is an operation on $\mathcal{V}$ that assigns, to each pair of vectors $\mathbf{u}$ and $\mathbf{v} \in \mathcal{V}$, a scalar $\langle \mathbf{u}, \mathbf{v} \rangle$ in $\mathbb{R}$, satisfying the followings:

1. Symmetry property
$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$$
2. Distributive property
$$\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle, \qquad \text{where} \quad \mathbf{w} \in \mathcal{V}$$
3. Homogeneity property
$$\langle \alpha \mathbf{u}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \alpha \mathbf{v} \rangle, \qquad \text{where } \alpha \text{ is a scalar}$$
4. Positivity property
$$\langle \mathbf{u}, \mathbf{u} \rangle \geq 0 \qquad \text{and} \quad \langle \mathbf{u}, \mathbf{u} \rangle = 0 \quad \text{if and only if} \quad \mathbf{u} = \mathbf{0}$$

- A vector space $\mathcal{V}$ with an inner product is called an inner product space.
- The standard inner product for $\mathbb{R}^n$ is the dot product.

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i$$

- Inner product of a vector space is NOT unique. e.g. for $\mathbb{R}^n$,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^{n} u_i v_i w_i \qquad \text{where} \quad \mathbf{w} \in \mathbb{R}^n \quad \text{and} \quad w_i > 0 \quad \text{for} \quad \forall i.$$

- Given $\mathbf{A}$ and $\mathbf{B}$ in $\mathbb{R}^{m \times n}$, we can define an inner product by

$$\langle \mathbf{A}, \mathbf{B} \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij}$$

- For the vector space $\mathcal{C}[a, b]$, we often use following inner product

$$\langle f, g \rangle = \int_a^b f(x) g(x)\, dx$$

Defintions

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in an inner product space $\mathcal{V}$, then

- The length of a vector $\mathbf{v}$ in $\mathcal{V}$ is defined and denoted by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

- The distance between vectors $\mathbf{u}$ and $\mathbf{v}$ is

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}$$

- Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{V}$ are said to be orthogonal

$$\text{if and only if} \quad \langle \mathbf{v}, \mathbf{v} \rangle = 0$$

- The angle θ between the vector $\mathbf{u}$ and $\mathbf{v}$ is

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \, \|\mathbf{v}\|}$$

- Notice the norm notation was used, in fact, we have the following theorem

Theorem

If $\mathcal{V}$ is an inner product space, then the following is a valid norm on $\mathcal{V}$

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}, \qquad \text{for all } \mathbf{v} \in \mathcal{V}$$

- We refer to this norm $\|\cdot\|$ as the norm induced by $\langle \cdot, \cdot \rangle$, from which we have

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

  which is the induced metric on $\mathcal{V}$.
- Every inner product space is a metric space as well as a normed space.
- It may be possible to place other norms on an inner product space $\mathcal{V}$, but unless it is stated otherwise, we assume all norm-related statements on an inner product space are taken with respect to the induced norm.

Proof

- Nonnegativity is clearly satisfied by the positivity of any inner product.
- Use the homogeneity property of an inner product space

$$\|\alpha\mathbf{v}\| = \sqrt{\langle \alpha\mathbf{v}, \alpha\mathbf{v}\rangle} = \sqrt{\alpha^2\langle \mathbf{v}, \mathbf{v}\rangle} = |\alpha|\|\mathbf{v}\|$$

- Consider vectors $\mathbf{u}$ and $\mathbf{v}$

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle \\
&= \langle \mathbf{u}, \mathbf{u}\rangle + 2\langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle \\
&= \|\mathbf{u}\|^2 \cos 0 + 2\|\mathbf{u}\|\|\mathbf{v}\| \cos\theta + \|\mathbf{v}\|^2 \cos 0 \\
&\leq \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 \\
&= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2
\end{aligned}$$

$$\Longrightarrow \quad \|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

- Not only the properties but all results based on dot products in $\mathbb{R}^n$ hold e.g.

$$\langle 1, x\rangle = \int_{-1}^{1} x\,dx = 0 \implies \|1\|^2 + \|x\|^2 = \|x+1\|^2$$

since $1, x \in \mathcal{C}[-1,1]$ and the following is a inner product for $\mathcal{C}[-1,1]$

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx$$

- We can verify it explicitly, which helps us to understand what Pythagora says

$$\|1\| = \sqrt{\langle 1, 1\rangle} = \sqrt{\int_{-1}^{1} 1\,dx} = \sqrt{2};$$

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{\int_{-1}^{1} x^2\,dx} = \sqrt{\frac{2}{3}};$$

$$\|x+1\| = \sqrt{\langle x+1, x+1\rangle} = \sqrt{\int_{-1}^{1} (x+1)^2\,dx} = \sqrt{\frac{8}{3}};$$

- Notice ℓ_p-norm, for $p \neq 2$, does not correspond to any inner product.
- In the case of a norm that is not derived from any inner product. e.g. $\|\cdot\|_4$
- The Pythagorean law will not hold, consider

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}_2 = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

then the lengths are

$$\begin{aligned} \|\mathbf{x}_1\|_4 &= \left(1^4 + 2^4\right)^{1/4} = \sqrt[4]{17} \\ \|\mathbf{x}_2\|_4 &= \left((-4)^4 + 2^4\right)^{1/4} = \sqrt[4]{272} \\ \|\mathbf{x}_1 + \mathbf{x}_2\|_4 &= \left((-3)^4 + 4^4\right)^{1/4} = \sqrt[4]{337} \end{aligned}$$

- Thus

$$\|\mathbf{x}_1\|_4^2 + \|\mathbf{x}_2\|_4^2 \neq \|\mathbf{x}_1 + \mathbf{x}_2\|_4^2$$

Cauchy-Schwarz inequality

For any vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space $\mathcal{V}$, then

$$\langle \mathbf{u}, \mathbf{v} \rangle^2 \leq \langle \mathbf{u}, \mathbf{u} \rangle \langle \mathbf{v}, \mathbf{v} \rangle$$

Triangle Inequality

For any vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space $\mathcal{V}$, then

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

Parallelogram Law

For any vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space $\mathcal{V}$, then

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right)$$

Definition

An inner product space $\mathcal{H}$ is called a Hilbert space if every Cauchy sequence in $\mathcal{H}$ converges to an element of $\mathcal{H}$ with respect to the induced norm.