Review class

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Outline

- Determinant
- Diagonalization
- Phase Diagram
- Matrix Exponential
- Linear homogeneous systems
- Nonhomogeneous systems

Determinant

The determinant of an $n \times n$ matrix is defined recursively as

- $\det(A) = \begin{cases} a_{11} & n = 1 \\ a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} & n > 1 \end{cases}$

where C_{ij} is cofactor, and is given by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

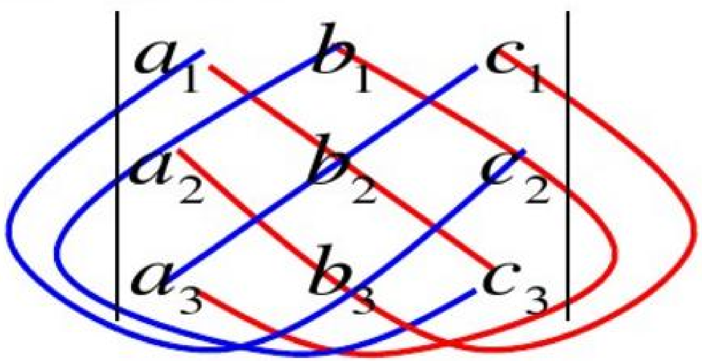
where M_{ij} is minor, and is given by

***deleting the ith row and jth column from the original matrix**

Determinant

- Determinant of $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

is $a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{33}$



Diagonalization Eigenvalues and Eigenvectors

- satisfy $\mathbf{Ax} = \lambda\mathbf{x}$ （nontrivial solution for λ and $\mathbf{x}$）

λ is the Eigenvalues and $\mathbf{x}$ is the corresponding Eigenvectors

- Way to solve

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

Expand this determinant and find a polynomial withλ

*It is called Characteristic polynomial

*The highest degree of λ is equal to the size of square matrix

- So there will n solution for the Characteristic polynomial to $n \times n$ matrix

E.g. Find the eigenvalues and eigenvectors of the following.

- A= $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

- The characteristic equation is given by

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (1 - \lambda)^3 = 0$$

$$\text{So } \lambda_1=\lambda_2=\lambda_3=1$$

$$(\mathbf{A} - \mathrm{I})\mathbf{x}=\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{x}=\mathbf{0}$$

$$\mathbf{x}=[0 \;\; 0 \;\; 1]^T$$

Diagonalization Eigenvalues and Eigenvectors

- The degree of a root λ_i of the characteristic polynomial of a matrix is called the algebraic multiplicity of the eigenvalue.

- The dimension of the eigenspace corresponding to aneigenvalue λ_i is called the geometric multiplicity of the eigenvalue.

- Geometric multiplicity is smaller or equal to algebraic multiplicity

- Geometric multiplicity is bigger or equal to one

Diagonalization Eigenvalues and Eigenvectors

properties:

When facing complex Eigenvalues the matrix and Eigenvectors are also complex

- complex Eigenvalues is in pair

λ is a Eigenvalues, then conj(λ) is also an Eigenvalue

- Transposed property

λ is a Eigenvalues to A , λ is a Eigenvalues to A^T

Diagonalization Eigenvalues and Eigenvectors

- Asqaure matrix(n×n) is said to be diagonalizable if there is an invertible matrix P and a diagonal matrix D such that:

$$\mathbf{A = PDP^{-1}}$$

- The necessary and sufficient condition for the diagonalizable is all Eigenvalues have same algebraic multiplicity and geometric multiplicity.

  Just it have n linear independent Eigenvectors

- The columns of **P** are n LI eigenvectors

- The diagonal elements of **D** are corresponding eigenvalues.

Diagonalization Eigenvalues and Eigenvectors

Diagonalize the following

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The eigenvalues and corresponding eigenvectors of **A** are

$$\lambda_1 = \lambda_2 = 0 : \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \lambda_3 = 3 : \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Thus,

$$\mathbf{A} = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & & \\ & 0 & \\ & & 3 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & & \\ & 0 & \\ & & 3 \end{bmatrix} \begin{bmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

Diagonalize the following

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The eigenvalues and corresponding eigenvectors of $\mathbf{A}$ are

$$\lambda_1 = \lambda_2 = 0 : \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \lambda_3 = 3 : \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Thus,

$$\mathbf{A} = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & & \\ & 0 & \\ & & 3 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & & \\ & 0 & \\ & & 3 \end{bmatrix} \begin{bmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

Phase Diagram

- solving system of differential equations

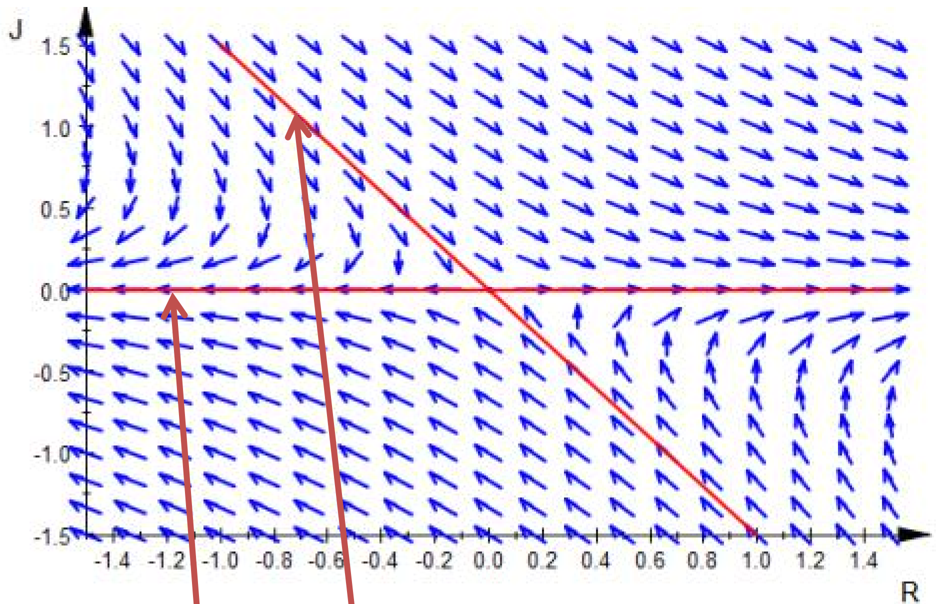
$$\begin{cases} \dfrac{dR}{dt} = a_{11}R + a_{12}J \\ \dfrac{dJ}{dt} = a_{21}R + a_{22}J \end{cases}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \qquad \mathbf{x} = \begin{bmatrix} R \\ J \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\bar{\mathbf{F}} = \frac{dR}{dt}\mathbf{e}_R + \frac{dJ}{dt}\mathbf{e}_J.$$

Phase Diagram

e.g.



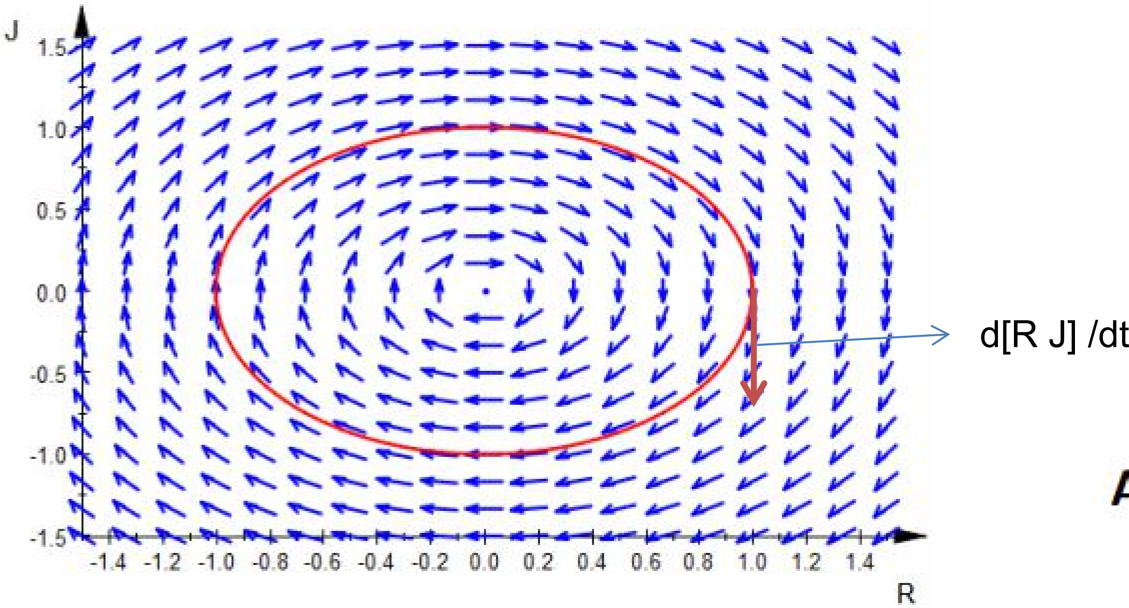
$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$$

Eigenvalues: 1, -2

Eigenvector: $[1\ 0]^T, [2\ -3]^T$

Phase Diagram

e.g.



$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

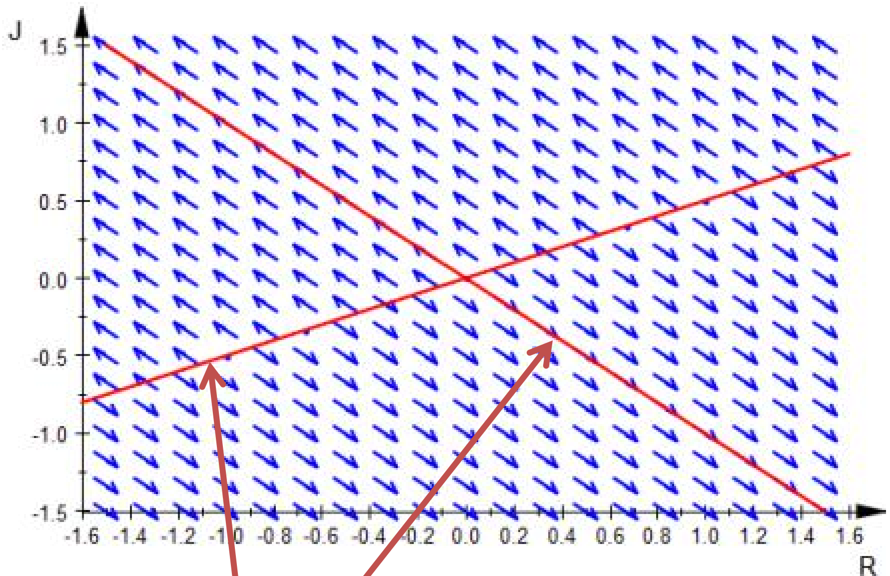
Eigenvalues: i, -i

Real part is 0

find d[R J]/dt for[1 0] is A$[1\ 0]^T$ is $[0\ -1]^T$ so the arrow shall at the direction of $[0\ -1]^T$ just down side so it is clockwise

Phase Diagram

e.g.



Eigenvalues: 0, 1

Phase Diagram

Eigenvalues		Stability	Origin case
Real	λ1 ≥ λ2 > 0	Unstable	source
	λ1 ≤ λ2 < 0	stable	sink
	λ1 < 0 < λ2	Unstable	Saddle point
	λ1 = 0	e.g.	
Complex	Re(λ1) = 0	Stable	Center of circle
	Re(λ1) > 0	Unstable	Spiral source
	Re(λ1) < 0	stable	Spiral sink

*caution: do not find dJ/dR to define whether clockwise or conter-clockwise

Matrix Exponential

- We define the exponential of a matrix same as exponential of function:

$e^x = 1 + x + x^2/2 + x^3/6 + \ldots + x^n/n! + \ldots$

$e^{\mathbf{A}} = 1 + \mathbf{A} + \mathbf{A}^2/2 + \mathbf{A}^3/6 + \ldots + \mathbf{A}^n/n! + \ldots$

- If **D** is diagonal, then

$$\mathbf{D}^k = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}^k = \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix}$$

- Thus $e^{\mathbf{D}} = \mathrm{diag}(e^{D_{11}}, e^{D_{22}}, e^{D_{33}}, \ldots)$

Matrix Exponential

- For the reason that we have the formula $\mathbf{A}^k = \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1}$

Prove: e.g for k=3

$$A^3 = AAA = \mathbf{PD(P^{-1}P)D(P^{-1}P)DP^{-1}} = \mathbf{PDIDIDP^{-1}} = \mathbf{PD^3P^{-1}}$$

- $e^{\mathbf{A}} = 1 + \mathbf{A} + \mathbf{A}^2/2 + \mathbf{A}^3/6 + \ldots + \mathbf{A}^n/n! + \ldots$

- For all $\mathbf{A}^k = \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1}$

we can extract $\mathbf{P}$ and $\mathbf{P}^{-1}$ and get $e^A = \mathbf{P}e^{\mathbf{D}}\mathbf{P}^{-1}$

Linear homogeneous systems

- Elimination Method

For a homogeneous system of first-order diff equ.

$$\begin{cases} \dot{R} = a_{11}R + a_{12}J \\ \dot{J} = a_{21}R + a_{22}J \end{cases}$$

$$\begin{aligned} \ddot{R} &= a_{11}\dot{R} + a_{12}\dot{J} \\ &= a_{11}\dot{R} + a_{12}a_{21}R + a_{12}a_{22}J \\ &= a_{11}\dot{R} + a_{12}a_{21}R + a_{22}\dot{R} - a_{22}a_{11}R \end{aligned}$$

$$\ddot{R} - (a_{11} + a_{22})\dot{R} + (a_{11}a_{22} - a_{12}a_{21})R = 0$$

Linear homogeneous systems

- For higher order it is too difficult to eliminate!

For a linear homogeneous system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$

$$d\mathbf{x}/dt = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}\mathbf{x}$$

Thus: $$\mathbf{P}^{-1}d\mathbf{x}/dt = \mathbf{D}\mathbf{P}^{-1}\mathbf{x}$$

we assume $\mathbf{P}^{-1}\mathbf{x}=\mathbf{y}$ then $d\mathbf{y}/dt = \mathbf{P}^{-1}d\mathbf{x}/dt$

Thus $d\mathbf{y}/dt=\mathbf{D}y$

we get $\mathbf{y}_i=C_i e^{\lambda_i t}$

Thus $\mathbf{y}=e^{Dt}\mathbf{C}$ where $\mathbf{C}$ is Arbitrary vector

Linear homogeneous systems

- e.g. $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$$

- The solution is $e^{\mathbf{A}t}\mathbf{C}$ where $\mathbf{C}=\mathbf{x}(0)$

$$e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1} = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{5t} \end{bmatrix} \begin{bmatrix} -2/5 & 1/5 \\ 1/5 & 2/5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^{5t}}{5} + \frac{4}{5} & \frac{2e^{5t}}{5} - \frac{2}{5} \\ \frac{2e^{5t}}{5} - \frac{2}{5} & \frac{4e^{5t}}{5} + \frac{1}{5} \end{bmatrix}$$

Thus solution is $\begin{bmatrix} \frac{e^{5t}}{5} + \frac{4}{5} & \frac{2e^{5t}}{5} - \frac{2}{5} \\ \frac{2e^{5t}}{5} - \frac{2}{5} & \frac{4e^{5t}}{5} + \frac{1}{5} \end{bmatrix} \mathbf{x}(0)$

Nonhomogeneous systems

- Formula:

$$d\mathbf{x}/dt = A\mathbf{x} + \boldsymbol{\beta}$$

- Solution is $\mathbf{x}=e^{At}\mathbf{x}(to)+\mathbf{e}^{At}\int_{to}^{t}e^{-A\tau}\boldsymbol{\beta}(\tau)d\tau$

- The formula is very difficult to calculate!
- The method is same as previous one

$$d\mathbf{x}/dt= \mathbf{PDP}^{-1}\mathbf{x} + \boldsymbol{\beta}$$

$$\mathbf{P}^{-1}d\mathbf{x}/dt= \mathbf{DP}^{-1}\mathbf{x} + \mathbf{P}^{-1}\boldsymbol{\beta}$$

$$\mathbf{y}=\mathbf{P}^{-1}\mathbf{x}$$

$$d\mathbf{y}/dt=\mathbf{D}y+\mathbf{P}^{-1}\boldsymbol{\beta}$$

$$y=e^{\mathbf{D}t}\mathbf{C}\text{-}\mathbf{D}^{-1}\mathbf{P}^{-1}\boldsymbol{\beta}$$ <u>**(maybe not right you can check it)**</u>

This is more simple than formula


ngiyabonga
danke
謝謝
teşekkür ederim
спасибо
dank je
tapadh leat
thank you
gracias
maururu
hvala
bedankt
mochchakkeram
dziękuję
go raibh maith agat
sagolun
sukriya
kop khun krap
arigatô
takk
dakujem
obrigado
grazie
мерси
terima kasih
감사합니다
ευχαριστώ
merci