Vv156 Lecture 19

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November 17, 2016

Reduction formula for powers of sine and cosine

- For positive integers n, we have

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

Proof

- Split a copy of $\cos x$,

$$\int \cos^n x\,dx = \int \cos^{n-1} x\cos x\,dx,$$

- Apply integartion by parts

$$\int f(x)g(x)\,dx = f(x)G(x) - \int f'(x)G(x)\,dx$$

Proof

- If we let,

$$f(x) = \cos^{n-1} x; \qquad g(x) = \cos x$$
$$f'(x) = -(n-1)\cos^{n-2} x \sin x; \qquad G(x) = \sin x$$

- then

$$\int \cos^n x\, dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin x \sin x\, dx$$

- With the identity $\sin^2\theta + \cos^2\theta = 1$, we have

$$\boxed{\int \cos^n x\, dx} = \cos^{n-1} x \sin x + (n-1)\left[\int \cos^{n-2} x\, dx - \boxed{\int \cos^n x\, dx}\right]$$
$$\Longrightarrow \boxed{\int \cos^n x\, dx} = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\, dx \quad \square$$

Odd Power

If both m and n are positive integers, and suppose either m or n is odd, then

$$\int \sin^m x \cos^n x \, dx$$

can be evaluated by

1. splitting off a factor of sine or cosine whichever has the odd power
2. using substitution with the other factor being g, and use

$$\sin^2\theta + \cos^2\theta = 1$$

Exercise

Evaluate

$$\int \sin^4 x \cos^5 x \, dx$$

Even Power

If both m and n are even positive integers, then

$$\int \sin^m x \cos^n x \, dx$$

can be evaluated by using

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

1. to eliminate sine
2. then apply the reduction formula for powers of sine and cosine

Exercise

Evaluate

$$\int \sin^2 x \cos^4 x \, dx$$

- Integrals of products of sines and cosines of the form

$$\int \sin mx \cos nx \, dx \qquad \int \sin mx \sin nx \, dx \qquad \int \cos mx \cos nx \, dx$$

can be found by using the trigonometric identities

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$$
$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$
$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$$

Exercise

Evaluate

$$\int \sin 3x \cos 5x \, dx$$

- Integrating tangent and secant closely parallel those for sine and cosine.

Reduction formulae for tangent and secant function

For integer powers $n \geq 2$ of tangent and secant, we use the reduction formulae

$$\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx,$$
$$\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$

- When n is even, it is solved by successive applications of the formula.
- When n is odd, the exponent can be reduced to 1, and we then need to solve

$$\int \tan x\,dx \qquad \text{or} \qquad \int \sec x\,dx$$

- Neither $\left(\int \tan x\, dx\right)$ nor $\left(\int \sec x\, dx\right)$ are usually in the derivative table,

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$$

$$u = g(x) = \cos x \implies g' = -\sin x$$

$$\int \tan x\, dx = -\int u^{-1}\, du$$

$$= -\ln|\cos x| + C$$

$$= \ln|\sec x| + C$$

$$\int \sec x\, dx = \int \sec x \frac{\tan x + \sec x}{\tan x + \sec x}\, dx$$

$$= \int \frac{\sec x \tan x + \sec^2 x}{\tan x + \sec x}\, dx$$

$$u = g(x) = \tan x + \sec x$$

$$\implies g' = (\sec^2 x + \sec x \tan x)\, dx$$

$$\int \sec x\, dx = \int u^{-1}\, du$$

$$= \ln|\tan x + \sec x| + C$$

- The following integrals are common, and worth to commit to memory

$$\int \tan^2 x\, dx = \tan x - x + C; \qquad \int \sec^2 x\, dx = \tan x + C$$

Exercise

Find the following integrals

(a)

$$\int \tan^2 x \sec^4 x \, dx$$

(b)

$$\int \tan^3 x \sec^3 x \, dx$$

(c)

$$\int \tan^2 x \sec x \, dx$$

Integrating products of tangents and secents

- If both m and n are even positive integers, then the integral

$$\int \tan^m x \sec^n x \, dx$$

can be evaluated by using the following identity and the reduction formulae

$$\sec^2 x = \tan^2 x + 1$$

- Alternatively,

1. If we have even powers of $\sec x$, then consider the substitution $u = \tan x$
2. If we have odd powers of $\tan x$, then consider the substitution $u = \sec x$
3. If we have even powers of $\tan x$ and odd powers of $\sec x$, then

    use the above identity to reduce the integrand to powers of $\sec x$ alone.

- Trigonometric substitutions is a method in which we replace the variable of integration by a trigonometric function. If x is the variable of integration, then

$$x = g(\theta), \qquad \text{where } g(\theta) \text{ is some trigonometric function.}$$

- Notice it is different from the usual substitution

$$u = g(x) \qquad \text{where } g(x) \text{ is a part of the integrand.}$$

- Trig substitution is useful when one of the followings is a part of the integrand.

$$1. \quad \sqrt{a^2 - x^2}; \qquad 2. \quad \sqrt{a^2 + x^2}; \qquad 3. \quad \sqrt{x^2 - a^2}$$

  where a is a positive constant.

- The corresponding substitutions are

$$1. \quad x = a\sin\theta; \qquad 2. \quad x = a\tan\theta; \qquad 3. \quad x = a\sec\theta$$

- The basic idea for making such a substitution is to eliminate the radical.

Exercise

(a) *Find*

$$\int \sqrt{4-x^2}\,dx$$

(b) *Find*

$$\int \frac{dx}{\sqrt{4+x^2}}$$

(c) *Evaluate*

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx$$

(d) *Find the area of the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$