Probabilistic Methods in Engineering

Sample Exercises for the Second Midterm - Solutions

The following exercises have been compiled from past second midterm exams of Ve401. A Second midterm will usually consist of 4-5 such exercises to be completed in 100 minutes. In the actual exam, necessary tables of values of distributions will be provided. You may use all tables in Appendix A of the textbook to solve the sample exercises.

Exercise 1. Let X be a continuous random variable with density

$$f_\theta(x) = \begin{cases} \dfrac{\theta+1}{x^{\theta+2}} & \text{for } x > 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is a parameter. Find the method-of-moments estimator for the parameter θ. **(3 Marks)**

Solution. We calculate

$$E[X] = \int_1^\infty \frac{\theta+1}{x^{\theta+1}}\,dx = (\theta+1)\int_1^\infty x^{-\theta-1}\,dx = \frac{\theta+1}{-\theta}[x^{-\theta}]_1^\infty = 1 + \frac{1}{\theta}$$

so that

$$\theta = \frac{1}{\mathrm{E}[X]-1}.$$

Using $\widehat{\mathrm{E}[x]} = \overline{X}$ we see that the method-of-moments estmator for θ is

$$\widehat{\theta} = \frac{1}{\overline{X}-1}$$

Exercise 2. Let (X, f_X) be a continuous random variable following a uniform distribution on the interval $[0, \theta]$, $\theta > 0$, i.e.,

$$f_X(x) = \begin{cases} 1/\theta & 0 \le x \le \theta \\ 0 & \text{otherwise.} \end{cases}$$

i) Find the method-of-moments estimator for θ.

ii) Find the maximum-likelihood estimator for θ.

(2+3 Marks)

Solution. i) The mean of X is given by

$$\mathrm{E}[X] = \int_0^\theta \frac{x}{\theta}\, dx = \frac{\theta}{2}.$$

Thus,

$$\theta = 2\,\mathrm{E}[X].$$

The method-of-moments estimator for θ is then

$$\hat{\theta} = 2\overline{X}.$$

ii) The likelihood function for a random sample $X_1, \dots, X_n$ of X is given by

$$L(\theta) = \prod_{i=1}^{n} f_X(x_i) = \begin{cases} \frac{1}{\theta^n} & \text{if } 0 < x_1, \dots, x_n < \theta \\ 0 & \text{otherwise} \end{cases}$$

In order for $L(\theta) > 0$ we need $\theta \ge x_i$ for $i = 1, \dots, n$, or $\theta \ge \max_{1 \le i \le n} x_i$. The likelihood function is then maximized if $\theta = \max_{1 \le i \le n} x_i$. The maximum-likelihood estimator hence is

$$\hat{\theta} = \max_{1 \le i \le n} x_i$$

Exercise 3. A certain type of harddrive is known to have a lifetime given by an exponential distribution with (unknown) parameter $\beta > 0$. To estimate β, n identical and independent hard drives are tested and their times of failure recorded. After time $T > 0$ the test is stopped and $n - m$ hard drives are found to be still working.

Find the maximum likelihood estimator for β.
(4 Marks)

Solution. The density of the exponential distribution is given by

$$f(t) = \begin{cases} \frac{1}{\beta} e^{-t/\beta} & t > 0, \\ 0 & t \leq 0. \end{cases}$$

(1/2 Mark) This gives

$$P[t \leq T] = \int_0^T f(t)\,dt = 1 - e^{-T_k/\beta}.$$

The probability of failing after time T is

$$P[t > T] = 1 - P[t \leq T_k] = e^{-T/\beta}$$

(1/2 Mark) Suppose that the first m hard drives fail at times $T_1, \ldots, T_m < T$. The likelihood function is then

$$\begin{aligned} L(\beta) &= f_1(T_1) \cdot f_2(T_2) \ldots f_m(T_m) \cdot \underbrace{e^{-T/\beta} \cdots e^{-T/\beta}}_{n-m \text{ terms}} \\ &= \frac{1}{\beta^m} e^{-(T_1 + \cdots + T_m + (n-m)T)/\beta} \end{aligned}$$

(1 Mark) To find the maximum of L, we take the logarithm,

$$\ln(L(\beta)) = -(T_1 + \cdots + T_m + (n-m)T)/\beta - m\ln(\beta)$$

(1/2 Mark) and set the derivative equal to zero,

$$\frac{T_1 + \cdots + T_m + (n-m)T}{\beta^2} - \frac{m}{\beta} = 0.$$

(1/2 Mark) Solving for β gives

$$\widehat{\beta} = \frac{T_1 + \cdots + T_m + (n-m)T}{m.}$$

(1 Mark)

Exercise 4. The diameter of steel rods manufactured on two different extrusion machines is being investigated. Two random samples of $n_1 = 15$ and $n_2 = 18$ are selected, and the sample means and sample variances are $\bar{x}_1 = 8.73$, $s_1^2 = 0.34$, $\bar{x}_2 = 8.68$, and $s_2^2 = 0.30$, respectively.

i) Test $\sigma_1^2 = \sigma_2^2$ at $\alpha = 0.10$.

ii) Assuming that $\sigma_1^2 = \sigma_2^2$, test $\mu_1 = \mu_2$ using a pooled T-test.

iii) Assuming that $\sigma_1^2 = \sigma_2^2$, construct a 95% two-sided confidence interval on the difference in mean rod diameter.

(2+2+2 Marks)

Solution.

i) The acceptance region for the test is given by

$$[1/f_{0.05,14,17}, f_{0.05,17,14}] = [1/2.43, 2.33] = [0.412, 2.33].$$

The test statistic is

$$F = \frac{s_1^2}{s_2^2} = \frac{0.34}{0.30} = 1.13.$$

Since the value of the test statistic is within this region we do not have enough evidence to reject $H_0\colon \sigma_1^2 = \sigma_2^2$.

ii) Assuming equal variances, we have the pooled sample variation

$$S_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2} = \frac{9.86}{31} = 0.318$$

The T-test statistic is

$$T = \frac{\overline{x}_1 - \overline{x}_2}{\sqrt{S_p^2(1/n_1 + 1/n_2)}} = \frac{0.05}{0.197} = 0.254.$$

Since $t_{0.1,n_1+n_2-2} = t_{0.1,31} > t_{0.05,32} = 1.309$, the P-value of this test is greater than 0.2. Therefore, there is not enough evidence to reject $H_0\colon \mu_1 = \mu_2$.

iii) A 95% two-sided confidence interval is given by

$$(\overline{X}_1 - \overline{X}_2) \pm t_{\alpha/2,n_1+n_2-2}\sqrt{S_p^2(1/n_1 + 1/n_2)} = 0.05 \pm 2.0395 \cdot 0.197 = 0.05 \pm 0.402$$

where we have interpolated between $T_{0.025,30} = 2.042$ and $T_{0.025,32} = 2.039$.

Exercise 5. A soft drink bottler is studying the internal pressure strength of 1-liter glass non-returnable bottles. A random sample of 16 bottles is tested and the pressure strengths obtained. The data (in units of psi) are shown below.

226.16	202.20	219.54	193.73	208.15	195.45	193.71	200.81
211.14	203.62	188.12	224.39	221.31	204.55	202.21	201.63

i) Draw a histogram for these data.

ii) Create a boxplot, clearly identifying any near and far outliers.

iii) Do you believe the data is normally distributed? Why or why not?

iv) Assuming normality, find 95% confidence intervals for the mean pressure strength μ and the variance σ^2.

(2+2+1+2 Marks)

Solution.

i)

```
Data = {226.16, 202.20, 219.54, 193.73, 208.15, 195.45, 193.71,
  200.81, 211.14, 203.62, 188.12, 224.39, 221.31, 204.55, 202.21, 201.63}

{226.16, 202.2, 219.54, 193.73, 208.15, 195.45, 193.71, 200.81,
 211.14, 203.62, 188.12, 224.39, 221.31, 204.55, 202.21, 201.63}

n = 5;
CatLength = Round[(Max[Data] - Min[Data]) / n, 0.01]

7.61

categories = Table[Min[Data] - 0.005 + i CatLength, {i, 0, n}]

{188.115, 195.725, 203.335, 210.945, 218.555, 226.165}

Histogram[Data, {Min[Data] - 0.005, Min[Data] - 0.005 + n * CatLength,
  CatLength}, AxesOrigin → Min[Data] - 3, Ticks → {categories, {1, 2, 3, 4}}]
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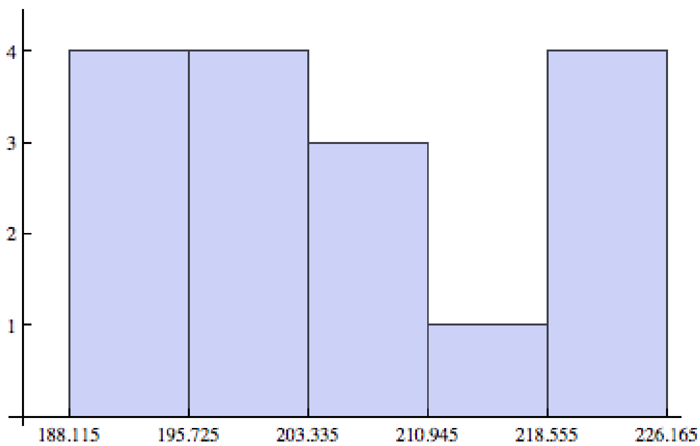
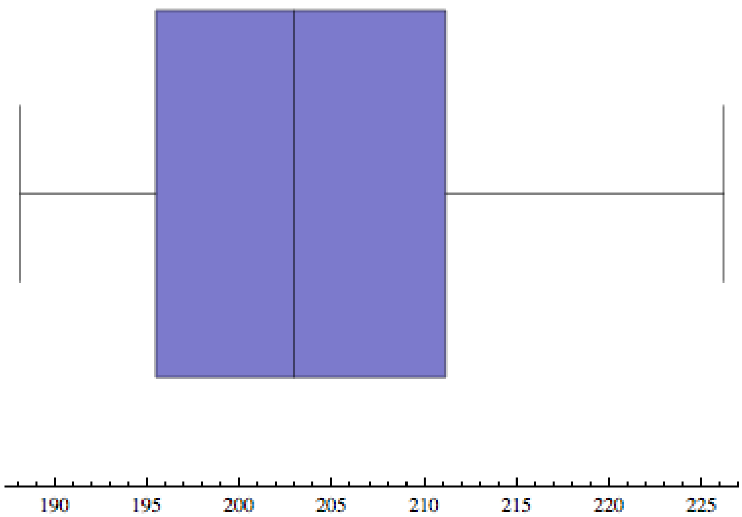


ii)



iii) The histogram does not have the characteristic shape of a normal distribution, but also shows no significant skewing. The boxplot confirms that the values are distributed symmetrically. The lack of "Gauss-curve" shape may be due to the small sample size. While there is no evidence for a normal distribution, there is also non against it.

iv) We calculate the sample mean and standard deviation: $\overline{x} = 206.045$, $s = 11.571$. Using the value $t_{0.025,15} = 2.131$ we obtain the confidence interval for the mean

$$\overline{x} \pm 2.131 s/\sqrt{15} = 206.045 \pm 6.164 = (199.88, 212.21).$$

Similarly, using $\chi^2_{0.025,15} = 27.5$, $\chi^2_{0.975,15} = 6.26$ we have

$$\left[15s^2/\chi^2_{0.025,15},\ 15s^2/\chi^2_{0.975,15}\right] = [73.03, 320.82]$$

for the variance.

Exercise 6. The manufacturer of a power supply is interested in the variability of output voltage. He has tested 16 units, chosen at random, with the following results:

5.34	5.00	5.07	5.25	5.65	5.55	5.35	5.35
4.96	5.54	5.54	4.61	5.28	5.93	5.38	5.47

i) It is thought that the output voltage is normally distributed. Create a stem-and-leaf diagram for the data. Does it support this assumption?

ii) Create a histogram for the data.

iii) Create a boxplot for the data.

iv) Assuming that $\sigma^2 = 0.1$, give a 95% confidence interval for the mean.

v) Using the sample variance, use the T-distribution to give a 95% confidence interval for the mean.

vi) Test the hypothesis that $\sigma^2 = 0.1$ at the $\alpha = 1\%$ level of significance (you should formulate H_0 and H_1 explicitly).

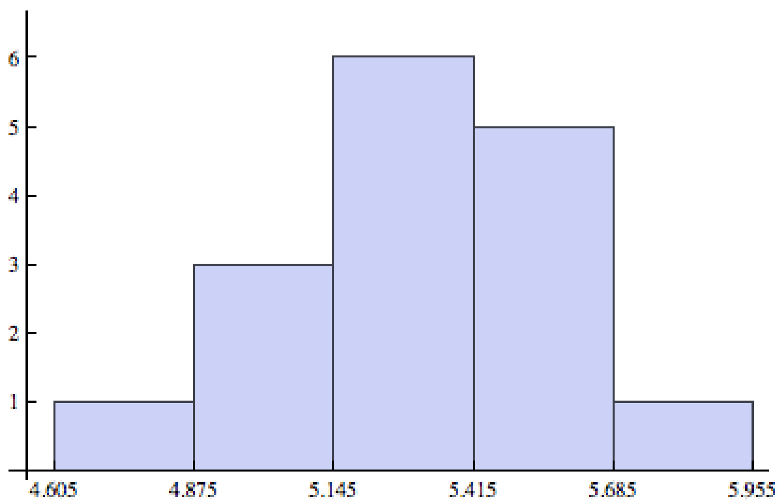
((2 + 1) + 2 + 2 + 3 + 2 + 2 Marks)

Solution.

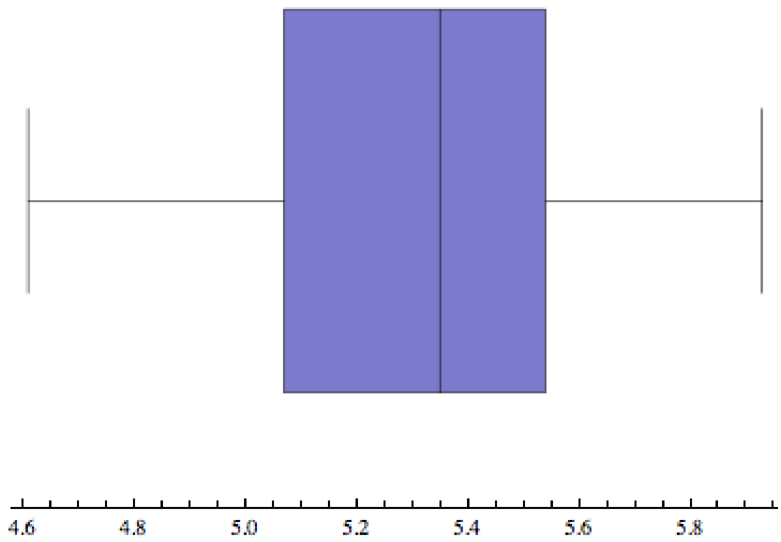
i)

Stem	Leaves
46	1
49	6
50	07
52	58
53	4558
54	7
55	445
56	5
59	3

ii)



iii)



iv) A 95% confidence interval using the standard normal distribution is given by

$$\overline{X} \pm \frac{z_{0.025} \cdot \sigma}{\sqrt{n}} = 5.33 \pm 1.96 \cdot 0.317/4 = 5.33 \pm 0.16.$$

v) The sample variance is $s^2 = 0.09751$. A 95% confidence interval using the T-distribution is given by

$$\overline{X} \pm \frac{t_{0.025,15} \cdot S}{\sqrt{n}} = 5.33 \pm 2.131 \cdot 0.312/4 = 5.33 \pm 0.17.$$

vi) We test

$$H_0\colon \sigma^2 = 0.1, \qquad H_1\colon \sigma^2 \neq 0.1.$$

The statistic $(n-1)S^2/\sigma^2$ follows a chi-squared distribution with 15 degrees of freedom. The critical values are $\chi^2_{0.005,15} = 4.6$ and $\chi^2_{0.995,15} = 32.8$ (two-tailed test!), so the critical region is given by

$$(n-1)S^2/\sigma_0^2 < 4.6 \qquad \text{or} \qquad (n-1)S^2/\sigma_0^2 > 32.8$$

where $\sigma_0^2 = 0.1$ is the null value of the statistic. The sample variance is $s^2 = 0.09751$, so we have

$$(n-1)S^2/\sigma_0^2 = 14.6265.$$

This is not within the critical region, so we have no evidence to reject H_0.

Exercise 7. A manufacturer of precision measuring instruments claims that the standard deviation in the use of an instrument is not more than 0.00002 inch. An analyst, who is unaware if the claim, uses the instrument eight times and obtains a sample standard deviation of 0.00005 inch.

i) Using $\alpha = 0.01$, is the manufacturer's claim justified?

ii) What is the power of the test if the true standard deviation equals 0.00004 inch?

iii) What is the smallest sample size that can be used to detect a true standard deviation of 0.00004 inch or more with a probability of at least 0.95? Use $\alpha = 0.01$.

(2+1+1 Marks)

Solution. i) We test the hypotheses

$$H_0 : \sigma \le 0.00002, \qquad\qquad H_1 : \sigma > 0.00002.$$

at $\alpha = 0.01$. If H_0 is true, the statistic

$$X^2_{n-1} = (n-1)\frac{S^2}{\sigma_0^2}$$

follows a chi-squared distribution with $n - 1 = 7$ degrees of freedom. **(1/2 Mark)** The critical value is $\chi^2_{0.01,7} = 18.5$. **(1/2 Mark)** The value of the statistic is

$$x_7^2 = 7 \cdot \frac{25 \cdot 10^{-8}}{4 \cdot 10^{-8}} = 43.75.$$

(1/2 Mark) Since this exceeds the critical value, we can reject H_0 at the 1% level of significance. **(1/2 Mark)** There is evidence that the manufacturer's claim is not justified.

ii) We use the OC curve for the right-tailed chi-squared test. The abscissa parameetr is

$$\lambda = \frac{S}{\sigma_0} = 2$$

and the sample size is $n = 8$. We read off $\beta \approx 0.34$, so the power is approximately $1 - \beta = 0.66$.

iii) Again, we use the OC chart with $\lambda = 2$ and $\beta = 1 - 0.95 = 0.05$. A sample size of $n = 20$ is sufficient to achieve the power stated.

Exercise 8. The diameters of bolts are known to have a standard deviation of 0.0001 inch. A random sampe of 10 bolts yields an average diameter of 0.2546 inch.

i) Test the hypothesis that the true mean diameter of bolts equals 0.255 inch, using $\alpha = 0.05$.

ii) What size sample would be necessary to detect a true mean bolt diameter of 0.2552 inch or more with a probability of at least 0.90, assuming $\alpha = 0.05$?

(2+2 Marks)

Solution. i) We test $H_0\colon \mu = 0.255$ at $\alpha = 0.05$. We will use the statistic

$$Z = \frac{\overline{X} - 0.255}{\sigma/\sqrt{n}},$$

which follows a standard normal distribution if H_0 is true. For $\alpha = 5\%$, we will reject H_0 if $|Z| > z_{0.025} = 1.96$. Now

$$z = \frac{0.2546 - 0.255}{0.0001/\sqrt{10}} = -12.65.$$

Since $|-4| > 1.96$, we reject H_0, i.e., the true mean diamater is different from 0.255 inch.

ii) Since in our case

$$d = \frac{\mu - \mu_0}{\sigma} = \frac{0.2552 - 0.255}{0.0001} = 2,$$

we can see from the OC curve that in our case $n = 3$ is sufficient.

Exercise 9. A company wants to test whether a new assembly line procedure increases the physical stress on its workers. It selects eleven workers to work for one day using each of the assembly line procedures. At the end of each day, their pulse frequency is measured:

Procedure 1	X	63	65	71	75	72	75	68	74	62	73	72
Procedure 2	Y	80	78	96	87	88	96	82	83	77	79	71

It is thought that the median pulse frequency is higher in Procedure 2 than in Procedure 1.

i) Formulate H_0 and H_1.

ii) Use the Wilcoxon signed rank test at the 5% level of significance to determine whether you can reject H_0.

iii) Use a paired T-test (formally; the sample size is actually to small for it to give meaningful results) at the 5% level of significance to determine whether you can reject H_0.

(2 + 2 + 2 Marks)

Solution.

i) Denote by M_X the median pulse frequency in procedure 1 and by M_Y the median pulse frequency in procedure 2. Then we have

$$H_0\colon M_Y \leq M_X, \qquad\qquad H_1\colon M_Y > M_X.$$

(We are trying to find evidence to support the hypothesis that the median pulse frequency is higher in Procedure 2 than in Procedure 1.)

ii) We calculate $Y - X$:

$Y - X$	17	13	25	12	16	21	14	9	15	6	-1

(1/2 Mark) We can see from the table of $Y - X$ that there is only a single negative value of $Y - X$, which has rank 1. Therefore,

$$W_+ = 2 + 3 + \ldots + 11 = \frac{11 \cdot 12}{2} - 1 = 65, \qquad\qquad |W_-| = 1.$$

(1/2+1/2 Mark) Therefore, $W = \min(W_+, |W|_-) = 1$. **(1/2 Mark)** According to the table for the Wilcoxon signed-rank test, we reject H_0 at the 5% level of significance if $W < 14$, so we can here reject H_0. **(1/2 Mark)**

iii) For the purposes of the t-test, we assume that the medians are equal to the means, i.e., $\mu_X = M_X$, $\mu_Y = M_Y$. If H_0 is true, $\mu_{Y-X} = 0$ and

$$\frac{\hat{\mu}_{Y-X} - \mu_{Y-X}}{\sqrt{S^2_{Y-X}/n}} = \frac{\hat{\mu}_{Y-X}}{\sqrt{S^2_{Y-X}/n}}$$

satisfies the T-distribution with $\gamma = 10$ degrees of freedom. **(1/2 Mark)** By the table for the T-distribution, at $\alpha = 5\%$ level of significance and $\gamma = 10$, the critical value is 1.812.

In our case, the sample mean of $Y - X$ is

$$\hat{\mu}_{Y-X} = \overline{Y - X} = \frac{1}{11}(17 + 13 + 25 + 12 + 16 + 21 + 14 + 9 + 15 + 6 - 1) = 13.36.$$

(1/2 Mark) The sample variance is

$$S^2_{Y-X} = \frac{1}{10}\sum_{i=1}^{11}(Y_i - X_i - \overline{Y - X})^2 = 49.85.$$

(1/2 Mark) We obtain

$$\frac{\hat{\mu}_{Y-X}}{\sqrt{S^2_{Y-X}/n}} = 6.28.$$

Since $6.28 > 1.812$, we can reject H_0.**(1/2 Mark)**

Exercise 10. In a hardness test, a steel ball is pressed into the material being tested at a standard load. The diameter of the indentation is measured, which is related to the hardness. Two types of steel balls are available, and their performance is compared on 10 randomly selected specimens. The hypothesis that the two steel balls give the same expected hardness measurement is to be tested at a significance level of $\alpha = 0.05$. Each specimen is tested twice, once with each ball. The results are given below:

Specimen	1	2	3	4	5	6	7	8	9	10
Ball x	75	46	57	43	58	38	61	56	64	65
Ball y	52	41	43	47	32	49	52	44	57	60

Use each of the following methods to test the hypothesis

i) A pooled T-test (assume that the variances are unequal).

ii) A Wilcoxon signed rank test.

iii) A paired T-test.

Compare the results obtained by each of the above tests. What assumptions are necessary for the validity of each test? What is your final conclusion regarding the hypothesis?
(2+2+2+3 Marks)

Solution.

i) We first compute the sample means and variances:

$$\overline{x} = 56.3, \qquad \overline{y} = 47.7$$
$$s_X^2 = 125.344, \qquad s_Y^2 = 67.122$$

(1/2 Mark) The value of the pooled test statistic is

$$T_\gamma = \frac{(\overline{x} - \overline{y})}{\sqrt{s_x^2/10 + s_y^2/10}} = 1.96.$$

(1/2 Mark) The degrees of freedom for this test are

$$\gamma = \frac{\left(s_x^2/10 + s_y^2/10\right)^2}{\frac{(s_x^2/10)^2}{9} + \frac{(s_x^2/10)^2}{10}} = 16.49$$

rounded down to 16. **(1/2 Mark)** Since $t_{0.025,16} = 2.120 > 1.96$, we do not have enough evidence to reject H_0. **(1/2 Mark)**

Assumptions: X, Y both follow a normal distribution. (The sample size is too small for the Central Limit theorem to be applicable.) **(1/2 Mark)**

ii) The Wilcoxon statistics are

```
Diff = SortBy[X - Y, Abs]

{-4, 5, 5, 7, 9, -11, 12, 14, 23, 26}

W+ = 0;
For[i = 1, i ≤ Length[Diff], i++, If[Positive[Diff[[i]]], W+ = W+ + i,]];
W+

48

W- = 0;
For[i = 1, i ≤ Length[Diff], i++, If[Negative[Diff[[i]]], W- = W- + i,]];
W-

7
```

(1/2 Mark) so the test statistic is $W = \min(W_+, |W_-|) = 7$. **(1/2 Mark)** For a two-sided test at $\alpha = 0.05$ the critical value is 8, **(1/2 Mark)** so there is enough evidence to reject H_0. **(1/2 Mark)**

Assumptions: $X - Y$ follows a symmetric distribution so that the mean is equal to the median. **(1/2 Mark)**

iii) For the paired T-test we calculate the sample mean and variance of $D = X - Y$:

$$\overline{d} = 8.6, \qquad\qquad s_d^2 = 124.7.$$

(1/2 Mark) We test $H_0\colon D = 0$. The statistic used is

$$T_9 = \frac{\overline{d}}{\sqrt{s_d^2/10}} = 2.435$$

(1/2 Mark) The critical value is $t_{0.025,9} = 2.262$. **(1/2 Mark)** Since the value of the test statistic exceeds this, we reject H_0. **(1/2 Mark)**

Assumptions: X, Y both follow a normal distribution. (The sample size is too small for the Central Limit theorem to be applicable.) **(1/2 Mark)**

Based on the test results, we can reject the null hypothesis. **(1/2 Mark)** While the pooled test was not powerful enough to do so, by the elimination of extraneous factor through pairing we were able to collect enough evidence to reject H_0. **(1 Mark)**

Exercise 11. A product developer is interested in reducing the drying time of a primer paint. Two formulations of the paint are tested; formulation 1 is the standard chemistry, and formulation 2 has a new drying ingredient that should reduce the drying time. From experience, it is known that the standard deviation of drying time is 8 minutes, and this inherent variability should be unaffected by the addition of the new ingredient. Ten specimens are painted with formulation 1, and another 10 specimens are painted with formulation 2; the 20 specimens are painted in random order.

i) The two sample average drying times are $\overline{x}_1 = 121$ minutes and $\overline{x}_2 = 112$ minutes. Perform a significance test to judge the effectiveness of the new ingredient. What is the P-value of the test? What conclusions can you draw about the effectiveness of the new ingredient?

ii) If the true difference in mean drying times is as much as 10 minutes, find the sample sizes required to detect this difference with probability at least 0.90, assuming the hypothesis test is conducted with $\alpha = 0.01$.

(3+3 Marks)

Solution.

i) We test $H_1\colon \mu_2 < \mu_1$, $H_0\colon \mu_2 \geq \mu_1$. **(1/2 Mark)** The test statistic

$$Z = \frac{\overline{X}_1 - \overline{X}_2}{\sqrt{\sigma^2/n_1 + \sigma^2/n_1}}$$

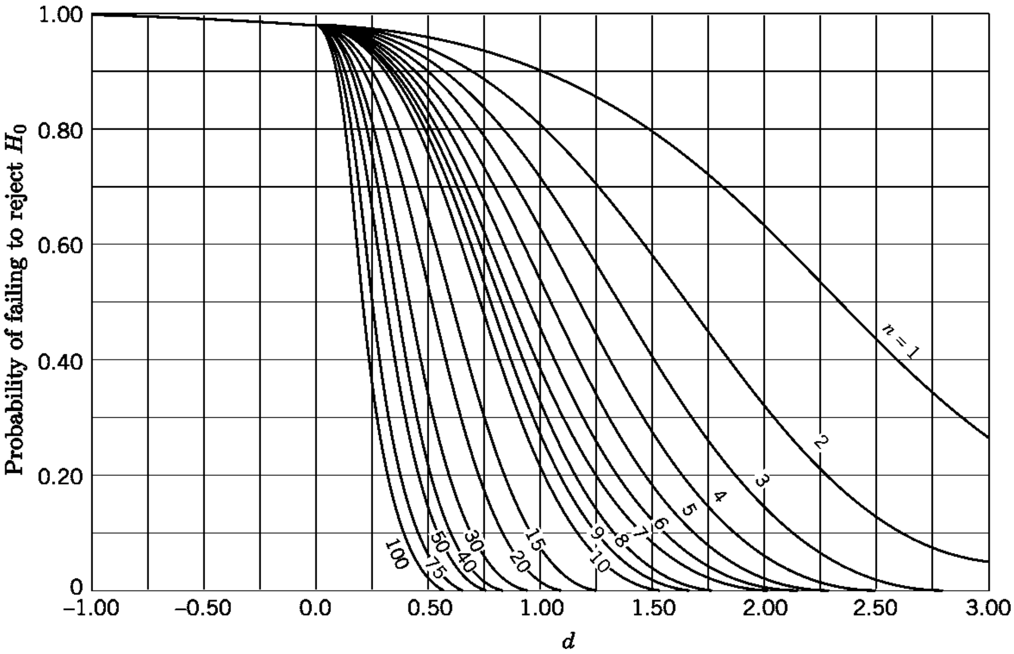
follows (at least approximately) a standard normal distribution. We will reject H_0 if Z is too large to have occurred by chance if H_0 is true. **(1/2 Mark)**

The observed value of Z is

$$z = \frac{121 - 112}{8\sqrt{1/10 + 1/10}} = \frac{9\sqrt{10}}{8\sqrt{2}} = 2.52.$$

(1/2 Mark) The probability of observing this large or a larger result if $\mu_1 = \mu_2$ is $0.5 - 0.4941 = 0.0059$. This is the P-value of the test. **(1 Mark)** Since the P-value is significantly less than 1%, we can conclude that there is evidence that the new drying ingredient reduces the drying time. **(1/2 Mark)**

ii) We use the OC chart for a one-sided test based on the normal distribution:



with $d = (\mu_1 - \mu_2)/\sqrt{\sigma_1^2 + \sigma_2^2} = 10/\sqrt{128} = 10/(8\sqrt{2}) = 0.88$ **(1/2 Mark)** and $\beta = 0.1$. **(1/2 Mark)** This gives a minimum sample size of about $n_1 = n_2 = 17$. **(2 Marks for any number greater than 15 and not greater than 20)**.