Vv256 Lecture 16

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- A crucial step of the Laplace method is finding the inverse Laplace transform

$$Y(s) \longrightarrow y(t)$$

- Definition:

$$\mathcal{L}\Big[y(t)\Big] = \int_0^\infty e^{-st} y(t) = Y(s)$$

- Linear property:

$$Y(s) = c_1 Y_1(s) + c_2 Y_2(s)$$

- Transform of a derivative:

$$\mathcal{L}\big[f^{(n)}(t)\big] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s^0 f^{(n-1)}(0),$$

- Translation Theorems:

$$\mathcal{L}\left[e^{at} f(t)\right] = F(s-a) \qquad \text{and} \qquad \mathcal{L}\big[f(t-a)u(t-a)\big] = e^{-as} F(s)$$

Q: What is the inverse Laplace transform of a product?

$$H(s) = F(s)G(s) \qquad \text{where} \qquad \begin{aligned} \mathcal{L}\big[f(t)\big] &= F(s) \\ \mathcal{L}\big[g(t)\big] &= G(s) \end{aligned}$$

- If one of those function is

$$e^{-as}$$

  then we can use the second translation theorem

$$\mathcal{L}\big[f(t-a)u(t-a)\big] = e^{-as}F(s)$$

Q: What happens otherwise? For example,

$$H(s) = \frac{1}{s^2(s-1)} = \frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2} \qquad \text{or} \qquad R(s) = \frac{s}{(s^2+k^2)^2}$$

- We could use a PFD for H. However, it is not always possible. e.g. $R(s)$.

- Firstly notice

$$\mathcal{L}^{-1}\Big[H(s)\Big] \neq \mathcal{L}^{-1}\Big[F(s)\Big] \cdot \mathcal{L}^{-1}\Big[G(s)\Big]$$

- For that, consider

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s-1)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2}\right] = e^t - 1 - t$$

whereas

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] \cdot \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] = te^t$$

- Hence

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s-1)}\right] \neq \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] \cdot \mathcal{L}^{-1}\left[\frac{1}{s-1}\right]$$

Convolution Theorem

If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\int_0^\infty e^{-st}\left(\int_0^t f(\tau)g(t-\tau)\,d\tau\right)dt = \mathcal{L}\Big[f(t)\Big]\mathcal{L}\Big[g(t)\Big] = F(s)G(s)$$

Proof

- Let

$$F(s) = \int_0^\infty e^{-s\tau} f(\tau)\,d\tau, \quad \text{and} \quad G(s) = \int_0^\infty e^{-s\mu} g(\mu)\,d\mu$$

- Take the right-hand side, we have

$$\begin{aligned} F(s)G(s) &= \left(\int_0^\infty e^{-s\tau} f(\tau)\,d\tau\right)\left(\int_0^\infty e^{-s\mu} g(\mu)\,d\mu\right) \\ &= \int_0^\infty \int_0^\infty e^{-s(\tau+\mu)} f(\tau)g(\mu)\,d\mu\,d\tau \end{aligned}$$

Proof

$$F(s)G(s) = \int_0^\infty \left(\int_0^\infty e^{-s(\tau+\mu)} f(\tau) g(\mu)\, d\mu \right) d\tau$$
$$= \int_0^\infty f(\tau) \left(\int_0^\infty e^{-s(\tau+\mu)} g(\mu)\, d\mu \right) d\tau$$

- To compute the inner integral, we hold τ as fixed, and integrate w.r.t. μ,

$$\int_0^\infty e^{-s(\tau+\mu)} g(\mu)\, d\mu$$

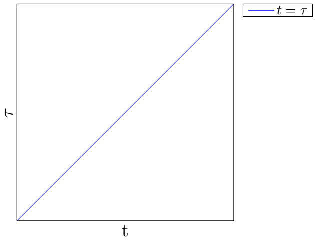
- However, we don't evaluate it directly, let $t = \tau + \mu$, and thus $\dfrac{dt}{d\mu} = 1$

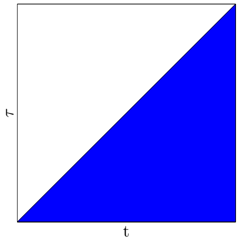
$$F(s)G(s) = \int_0^\infty f(\tau) \left(\int_0^\infty e^{-s(\tau+\mu)} g(\mu)\, d\mu \right) d\tau$$
$$= \int_0^\infty f(\tau) \left(\int_{t=\tau}^\infty e^{-st} g(t-\tau)\, dt \right) d\tau$$

Proof

- Consider what region we are integrating over,

$$\int_0^\infty \int_{t=\tau}^\infty dt\, d\tau = \iint dA = \int_0^\infty \int_0^t d\tau\, dt$$





- $$F(s)G(s) = \iint e^{-st} f(\tau) g(t-\tau)\, dA = \int_0^\infty e^{-st} \left(\int_0^t f(\tau) g(t-\tau)\, d\tau \right) dt$$

Definition

If $f(t)$ and $g(t)$ are piecewise continuous on the interval $[0, \infty)$, then the function

$$\int_0^t f(\tau)g(t-\tau)\,d\tau$$

is known as the convolution of f and g. This special function is denoted as

$$\Big(f * g\Big)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

- The convolution of f and g, which is a function of t, is often considered as a product between f and g since it has some properties of a typical product.

$$f * g = g * f \qquad \text{commutative}$$

$$f * (g * h) = (f * g) * h \qquad \text{associative}$$

$$f * (g + h) = f * g + f * h \qquad \text{distributive}$$

- Recall we had

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s-1)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2}\right] = e^t - 1 - t$$

- Let us apply the convolution theorem

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{s^2(s-1)}\right] &= \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] * \mathcal{L}^{-1}\left[\frac{1}{(s-1)}\right] \\
&= (t) * (e^t) = (e^t) * (t) \quad = \int_0^t f(\tau)g(t-\tau)\,d\tau \\
&= \int_0^t e^{\tau}(t-\tau)\,d\tau \\
&= e^t - t - 1
\end{aligned}$$

Exercise

Find the inverse Laplace transform of $\dfrac{1}{(s^2+k^2)^2}$.

Solution

- Let $F(s) = G(s) = \dfrac{k}{s^2+k^2}$, so

$$f(t) = g(t) = \mathcal{L}^{-1}\left[\frac{k}{s^2+k^2}\right] = \sin kt$$

- The convolution theorem states that $\mathcal{L}\Big[f * g\Big] = F(s)G(s)$

$$\begin{aligned}\mathcal{L}^{-1}\Big[F(s)G(s)\Big] = f * g &= \int_0^t \sin k\tau \sin k(t-\tau)\, d\tau \\ &= \frac{1}{2k}\sin kt - \frac{t}{2}\cos kt\end{aligned}$$

- $$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} &= \frac{1}{k^2}\mathcal{L}^{-1}\left\{\frac{k^2}{(s^2+k^2)^2}\right\} \\ &= \frac{1}{k^2}\left(\frac{1}{2k}\sin kt - \frac{t}{2}\cos kt\right) = \frac{\sin kt - kt\cos kt}{2k^3}\end{aligned}$$

Series Circuits

- In a series circuit, Kirchhoff's second law states that the sum of the voltage drops across an inductor, resistor, and capacitor is equal to the voltage

$$E(t)$$

- It is known that voltage drops across an inductor, resistor, and capacitor are,

$$\Delta V_L = L\frac{di}{dt}, \qquad \Delta V_R = Ri(t), \qquad \text{and} \qquad \Delta V_C = \frac{1}{C}\int_0^t i(\tau)\,d\tau,$$

respectively, where $i(t)$ is the current and L, R, and C are constants.

- So the current in a LRC series circuit is governed by the following equation,

$$L\frac{di}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)\,d\tau = E(t)$$

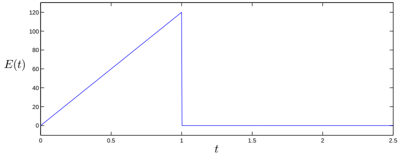
Exercise

Determine the current $i(t)$ in a series LRC circuit when

$$L = 0.1, \qquad R = 2, \qquad C = 0.1, \qquad \text{and} \quad i(0) = 0,$$

and voltage is given by

$$E(t) = 120t - 120tu(t-1).$$



Solution

- To find the current $i(t)$, we need to solve the following

$$\frac{1}{10}\frac{di}{dt} + 2i + 10\int_0^t i(\tau)\,d\tau = 120t - 120tu(t-1)$$

- This is known as integrodifferential equation.

Solution

- First notice that, in general,

$$f * g = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t f(\tau)\,d\tau \quad \text{for} \quad g(t-\tau)=1$$

- According to the convolution theorem,

$$\mathcal{L}\Big[f*g\Big] = \mathcal{L}\Big[f\Big]\cdot\mathcal{L}\Big[g\Big] = \frac{F(s)}{s}, \quad \text{where} \quad \mathcal{L}\Big[f\Big] = F(s)$$

  for the case when $g(t-\tau)=1$.

- Back to the current $i(t)$,

$$\mathcal{L}\left[\int_0^t f(\tau)\,d\tau\right] = \frac{F(s)}{s} \implies \mathcal{L}\left[\int_0^t i(\tau)\,d\tau\right] = \frac{I(s)}{s}$$

  where $I(s) = \mathcal{L}\Big[i(t)\Big]$.

Solution

- Thus the Laplace transform of the integrodifferential equation is

$$\mathcal{L}\left[\frac{1}{10}\frac{di}{dt} + 2i + 10\int_0^t i(\tau)\,d\tau\right] = \mathcal{L}\left[120t - 120tu(t-1)\right]$$

$$\Longrightarrow\ \frac{1}{10}sI(s) + 2I(s) + 10\frac{I(s)}{s} = 120\left(\frac{1}{s^2} - e^{-s}\mathcal{L}\left[t+1\right]\right)$$

$$\Longrightarrow\ \frac{1}{10}sI(s) + 2I(s) + 10\frac{I(s)}{s} = 120\left(\frac{1}{s^2} - e^{-s}\frac{1}{s^2} - e^{-s}\frac{1}{s}\right)$$

- Rearranging and collecting $I(s)$,

$$I(s) = 1200\left[\frac{1}{s(s+10)^2} - \frac{1}{s(s+10)^2}e^{-s} - \frac{1}{(s+10)^2}e^{-s}\right]$$

Solution

- By partial fractions, $I(s) = 1200\left[\overbrace{\frac{1/100}{s}}^{(1)} - \overbrace{\frac{1/100}{s+10}}^{(2)} - \overbrace{\frac{1/10}{(s+10)^2}}^{(3)} - \overbrace{\frac{1/100}{s}e^{-s}}^{(4)}\right.$

$$\left. + \underbrace{\frac{1/100}{s+10}e^{-s}}_{(5)} - \underbrace{\frac{9/10}{(s+10)^2}e^{-s}}_{(6)}\right]$$

- Transforming back to the time domain,

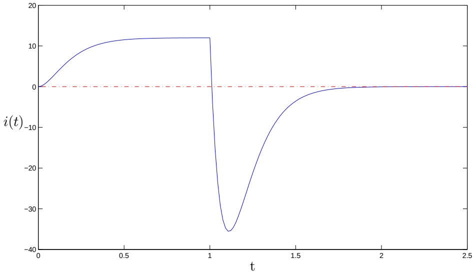
$$i(t) = 1200\left[\overbrace{\frac{1}{100}}^{(1)} - \overbrace{\frac{1}{100}e^{-10t}}^{(2)} - \overbrace{\frac{1}{10}te^{-10t}}^{(3)} - \overbrace{\frac{1}{100}u(t-1)}^{(4)}\right.$$

$$\left. + \underbrace{\frac{1}{100}e^{-10(t-1)}u(t-1)}_{(5)} - \underbrace{\frac{9}{10}(t-1)e^{-10(t-1)}u(t-1)}_{(6)}\right]$$

Solution

- Written as a piecewise-defined function,

$$i(t) = \begin{cases} 12 - 12e^{-10t} - 120te^{-10t}, & 0 \le t < 1 \\ -12e^{-10t} + 12e^{-10(t-1)} - 120te^{-10t} - 1080(t-1)e^{-10(t-1)} & 1 \le t. \end{cases}$$



- The input $E(t)$ is discontinuous, however, the output $i(t)$ is a continuous

- Consider a periodic function $\mathcal{L}\Big[f(t)\Big]$ where $f(t+T) = f(t)$.

Theorem

Let $f(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order, and periodic with period T, then

$$\mathcal{L}\Big[f(t)\Big] = \frac{1}{1-e^{-sT}} \int_0^{T} e^{-st} f(t)\, dt$$

Proof

- Break the interval into two parts,

$$\mathcal{L}\Big[f(t)\Big] = \int_0^{T} e^{-st} f(t)\, dt + \int_T^{\infty} e^{-st} f(t)\, dt$$

- When we let $t = u + T$, then the second integral becomes,

$$\int_0^{\infty} e^{-s(u+T)} f(u+T)\, du = e^{-sT} \int_0^{\infty} e^{-su} f(u)\, du = e^{-sT} \mathcal{L}\Big[f(t)\Big]$$

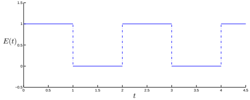
Exercise

The current $i(t)$ in a single-loop LR series circuit is governed by

$$L\frac{di}{dt} + Ri = E(t)$$

Find the current $i(t)$ when $i(0) = 0$ and $E(t)$ is a square wave function

$$E(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & 1 \le t \le 2 \end{cases}, \quad E(t) = E(t+2)$$



Solution

- Find the Laplace transform of input function $E(t)$,

$$\mathcal{L}\Big[E(t)\Big] = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}E(t)\,dt$$
$$= \frac{1}{1-e^{-2s}}\int_0^1 e^{-st}\cdot 1\,dt = \frac{1}{1-e^{-2s}}\frac{1-e^{-s}}{s} = \frac{1}{s(1+e^{-s})}$$

Solution

- Applying the Laplace transform to the equation, we have

$$LsI(s) + RI(s) = \frac{1}{s(1+e^{-s})}$$

- Rearranging and collecting $I(s)$, we have

$$I(s) = \frac{1/L}{s(s+R/L)} \cdot \frac{1}{1+e^{-s}}$$

- To find the current in the t-domain, we first make use of power series,

$$1 - r + r^2 - r^3 \cdots = \frac{1}{1-(-r)} \implies \frac{1}{1+e^{-s}} = 1 - e^{-s} + e^{-2s} - e^{-3s} \cdots$$

- By partial fraction,

$$\frac{1/L}{s(s+R/L)} = \frac{1}{L}\left(\frac{L/R}{s} - \frac{L/R}{s+R/L}\right)$$

Solution

- Together, we have

$$I(s) = \frac{1}{R}\left(\frac{1}{s} - \frac{1}{(s+R/L)}\right)\left(1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots\right)$$

$$= \frac{1}{R}\left(\frac{1}{s} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} + \cdots\right)$$

$$- \frac{1}{R}\left(\frac{1}{s+R/L} - \frac{e^{-s}}{s+R/L} + \frac{e^{-2s}}{s+R/L} - \frac{e^{-3s}}{s+R/L} + \cdots\right)$$

- Now we are ready to back transform!!!

$$i(s) = \frac{1}{R}\Big(1 - u(t-1) + u(t-2) - u(t-3) + \cdots\Big)$$

$$- \frac{1}{R}\left(e^{-Rt/L} - e^{-R(t-1)/L}u(t-1) + e^{-R(t-2)/L}u(t-2) - \cdots\right)$$

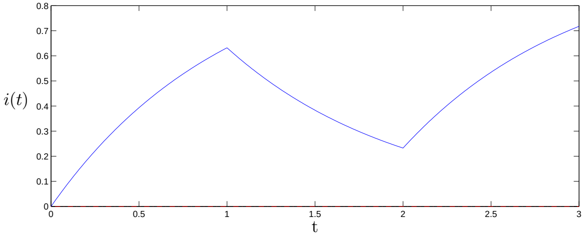
Solution

- Tidy it up, we have

$$i(t) = \frac{1}{R}\left(1 - e^{-Rt/L}\right) + \frac{1}{R}\sum_{k=1}^{\infty}(-1)^k\left(1 - e^{-R(t-k)/L}\right)u(t-k)$$

- For $R = 1$, $L = 1$, and let's consider $0 \le t \le 3$, then

$$i(t) = 1 - e^{-t} - (1 - e^{-(t-1)})u(t-1) + (1 - e^{-(t-2)})u(t-2)$$



- Recall we had our theorem on the Laplace transform of a derivative function

$$\mathcal{L}\left[\frac{df}{dt}\right] = s\mathcal{L}\Big[f(t)\Big] - f(0) = sF(s) - f(0)$$

Q: What will be the inverse Laplace transform of

$$\frac{d}{ds}F(s) \qquad \text{where} \qquad F(s) = \mathcal{L}\Big[f(t)\Big]$$

- Suppose $f(t)$ is well-behaved, thus we can interchange the order of

integration and differentiation

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty \frac{d}{ds}\Big(e^{-st} f(t)\Big)\,dt$$

$$= \int_0^\infty -te^{-st} f(t)\,dt = -\mathcal{L}\Big[tf(t)\Big]$$

Exercise

Find the Laplace transform of te^{3t} using the last theorem.

Solution

- We can treat it as a translation

$$\mathcal{L}\Big[e^{at}f(t)\Big] = F(s-a)$$

- Then

$$\begin{aligned}\mathcal{L}[te^{3t}] &= \frac{n!}{(s-a)^{n+1}} \\ &= \frac{1}{(s-3)^2}\end{aligned}$$

for $s > 3$.

- We can treat it as a derivative

$$\mathcal{L}\Big[tg(t)\Big] = -\frac{d}{ds}F(s)$$

- Then

$$\begin{aligned}\mathcal{L}[te^{3t}] &= -\frac{d}{ds}\left(\frac{1}{s-a}\right) \\ &= -\frac{d}{ds}\left(\frac{1}{s-3}\right) \\ &= \frac{1}{(s-3)^2}\end{aligned}$$

for $s > 3$.

- We can use the last result to find the Laplace transform of

$$\mathcal{L}\Big[t^2 f(t)\Big] = \mathcal{L}\Big[t \cdot t f(t)\Big] = -\frac{d}{ds}\mathcal{L}\Big[t f(t)\Big] = -\frac{d}{ds}\left(-\frac{d}{ds}\mathcal{L}\Big[f(t)\Big]\right)$$

$$= \frac{d^2}{ds^2}\left(\mathcal{L}\Big[f(t)\Big]\right)$$

- The preceding two cases suggest the following general result:

Theorem

If $F(s) = \mathcal{L}\Big[f(t)\Big]$ and $n = 1, 2, 3, \ldots$, then

$$\mathcal{L}\Big[t^n f(t)\Big] = (-1)^n \frac{d^n}{ds^n} F(s)$$

Q: When is this theorem useful?

Exercise

Solve $t\ddot{y} - t\dot{y} + y = 2, \quad y(0) = 2, \quad \dot{y}(0) = -4$ using the method of Laplace.

Solution

- Take the Laplace transform and apply the last theorem, we have

$$\begin{aligned} -\frac{d}{ds}\mathcal{L}\Big[\ddot{y}\Big] + \frac{d}{ds}\mathcal{L}\Big[\dot{y}\Big] + \mathcal{L}\Big[y\Big] &= \mathcal{L}\Big[2\Big] \\ -\frac{d}{ds}\Big(s^2Y(s) - sy(0) - y'(0)\Big) + \frac{d}{ds}\Big(sY(s) - y(0)\Big) + Y(s) &= \frac{2}{s} \\ -2sY(s) - s^2Y'(s) + y(0) + Y(s) + sY'(s) + Y(s) &= \frac{2}{s} \\ Y'(s) + \frac{2}{s}Y(s) &= \frac{2}{s^2} \end{aligned}$$

- From this first order linear equation, and the initial condition $\dot{y}(0) = -4$

$$Y(s) = 2s^{-1} + Cs^{-2} \implies y(t) = 2 + ct \implies y(t) = 2 - 4t$$

- A linear equation with variable coefficients can be solved by using the method of Laplace transform, but it is not a universal method.

Using series is still the standard method for variable coefficients

- However, a second approach offers a way to compute various things, e.g.

the Laplace transform of special functions

Q: How can we find the closed-form for the Laplace transform of J_0.

$$\mathcal{L}\Big[J_0\Big] = \mathcal{L}\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\lambda+n)} \left(\frac{t}{2}\right)^{2n+\lambda}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+n)} \left(\frac{1}{2}\right)^{2n} \mathcal{L}\Big[t^{2n}\Big]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+n)} \left(\frac{1}{2}\right)^{2n} \frac{(2n)!}{s^{2n+1}}$$

Q: The closed form is not obvious, what shall we do next?

- Let us consider the values of J_0 and J_0' at $t = 0$.

$$J_0(0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+n)} \left(\frac{t}{2}\right)^{2n} \Bigg|_{t=0} = 1 + 0 + 0 + \cdots = 1$$

and

$$J'(0) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!\Gamma(1+n)} \frac{2n}{2} \left(\frac{t}{2}\right)^{2n-1} \Bigg|_{t=0} = 0$$

- Thus the following initial-value problem defines J_0.

$$t^2\ddot{y} + t\dot{y} + (t^2 - \lambda^2)y = 0 \qquad y(0) = 1 \qquad \dot{y}(0) = 0$$

where $\lambda = 0$.

Q: Why is this useful? What shall we do next?

- Take the Laplace transform, we have

$$\mathcal{L}\Big[t\ddot{y} + \dot{y} + ty\Big] = \mathcal{L}\Big[0\Big]$$

- Apply the last theorem, we have

$$-\frac{d}{ds}\mathcal{L}\Big[\ddot{y}\Big] + \mathcal{L}\Big[\dot{y}\Big] - \frac{d}{ds}\mathcal{L}\Big[y\Big] = \mathcal{L}\Big[0\Big]$$

- Apply the theorem of the Laplace transform of a derivative,

$$\begin{aligned}
-\frac{d}{ds}\Big(s^2Y(s) - sy(t_0) - y'(t_0)\Big) + sY(s) - y(t_0) - \frac{dY}{ds} &= 0 \\
-2sY(s) - s^2Y'(s) + y(t_0) + sY(s) - y(t_0) - Y'(s) &= 0 \\
-(s^2+1)Y'(s) - sY(s) &= 0
\end{aligned}$$

- Solve this first order linear equation, we have

$$Y(s) = \frac{C}{\sqrt{s^2+1}}$$ where C is an arbitrary constant.

Q: How can we determine the arbitrary constant C?

$$Y(s) = \frac{C}{\sqrt{s^2+1}}$$

- We need some connection between initial values in t-domain and s-domain.

Theorem

If $f(t)$ and $f'(t)$ are continuous, and of exponential order, then

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) \qquad \text{where} \qquad F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Furthermore,

$$\lim_{t\to \infty} f(t) = \lim_{s\to 0} sF(s) \qquad \text{where} \qquad F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

provided the limit of $f(t)$ exists when $t \to \infty$.

- This is often known as the initial-value/final-value theorem.

- Since J_0 converges for $t \in \mathbb{R}$, and it is a solution to a second-order equation

$$\lim_{t \to 0} J_0 = J_0(0) = 1$$

- So the initial-value theorem implies

$$1 = \lim_{s \to \infty} s\mathcal{L}\left[J_0\right] = \lim_{s \to \infty} s \frac{C}{\sqrt{s^2+1}} = \lim_{s \to \infty} \frac{C}{\sqrt{1 + \frac{1}{s^2}}} = C$$

- Hence the Laplace transform of Bessel function of the first kind of order 0 is

$$\mathcal{L}\left[J_0\right] = \frac{1}{\sqrt{s^2+1}}$$

Exercise

Solve $t\ddot{y} + \dot{y} + 4ty = 0, \quad y(0) = 3, \quad \dot{y}(0) = 0$ using the method of Laplace.

Solution

- Taking the Laplace transform, we obtain

$$-\frac{d}{ds}\Big(s^2Y(s) - sy(0) - y'(0)\Big) + sY(s) - y(0) - 4\frac{dY}{ds} = 0$$

- Use the initial conditions and simplify, we have

$$(s^2+4)Y'(s) + sY(s) = 0$$

- Solve this first-order linear equation, and use the following theorem, we have

$$Y = \frac{C}{\sqrt{s^2+4}} = \frac{C}{2\sqrt{\left(\frac{s}{2}\right)^2+1}} \implies y(t) = CJ_0(2t) \implies y(t) = 3J_0(2t)$$

Theorem

Let $F(s) = \int_0^\infty e^{-st} f(t)\,dt$, then $\frac{1}{a}F\left(\frac{s}{a}\right) = \int_0^\infty e^{-st} f(at)\,dt$, where $a > 0$.