
Question1 (3 points)

Consider the following function.

$$f(x,y) = \frac{\sqrt{y-x^2}}{1-x} + \ln(2+x^3-y)$$

(a) (1 point) Sketch the domain of the function.

(b) (1 point) Find the minimum of $f(x,y)$ along the path $y = x^2$ for $0 < x < 1$.

Question2 (5 points)

Evaluate each of the following limits, if it exists.

(a) (1 point)

$$\lim_{(x,y,z)\to(1,-1,-1)} \arcsin\left(\frac{2xy+yz}{x^2+z^2}\right)$$

(b) (1 point)

$$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x-y}$$

(c) (1 point)

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

(d) (1 point)

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$$

(e) (1 point)

$$\lim_{(x,y)\to(1,0)} \frac{(x-1)^2\ln x}{(x-1)^2+y^2}$$

Question3 (5 points)

(a) (2 points) Consider the following function,

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}.$$

Shown f is NOT continuous despite the fact that it is continuous in both x-direction and y-direction, and then find all directions that f is not continuous.

(b) (3 points) It is even possible to construct examples which are continuous in every linear direction at a particular point but not continuous at the point. Consider the following function,

$$g(x,y) = \begin{cases} \dfrac{y}{x^2}\left(1-\dfrac{y}{x^2}\right) & \text{if} \quad 0 < y < x^2 \\ 0 & \text{if} \quad y \le 0 \text{ or } y \ge x^2 \end{cases}$$

Explain why g is continuous at all points other than $(0,0)$. Show that g is continuous in every linear direction at $(0,0)$. Then show that g is NOT continuous at $(0,0)$ by finding a curve approaching the origin along which the limit at the origin is not zero.

Question4 (2 points)

(a) (1 point) Verify that the function $z = \ln(e^x + e^y)$ satisfies the following

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$$

and

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$$

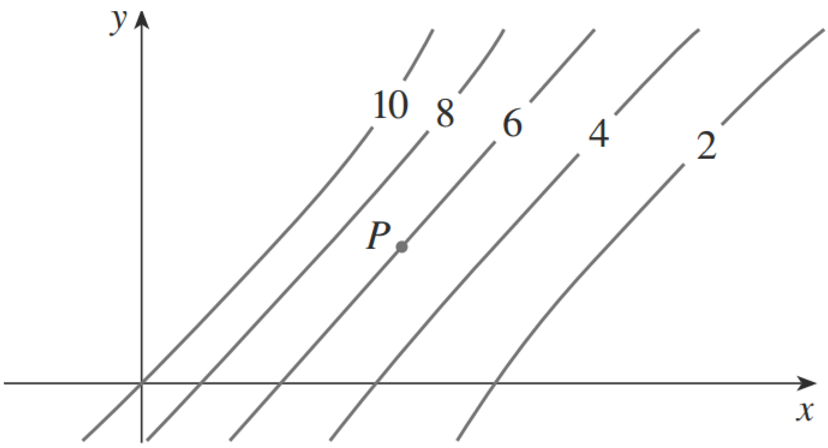
(b) (1 point) You are told that there is a function f whose partial derivatives are

$$f_x(x, y) = x + 4y \qquad \text{and } f_y(x, y) = 3x - y$$

Should you believe it? Justify your answer.

Question5 (1 points)

Some level curves of a functions $f(x, y)$ are shown below. Determine whether each of the partial derivatives $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ is positive or negative at P. Justify your answers.



Question6 (4 points)

Consider the function

$$f(x, y) = \sqrt{|xy|}$$

(a) (1 point) Show $f_x(0, 0)$ and $f_y(0, 0)$ exist.

(b) (1 point) Is $f(x, y)$ continuous at $(0, 0)$?

(c) (1 point) Is there an arbitrary open region containing $(0, 0)$ in which f_x is continuous?

(d) (1 point) Is $f(x, y)$ differentiable at (0,0)?

Question7 (0 points)

(a) (1 point (bonus)) Let p, q be positive real, and m, n be even numbers. Show the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^p y^q}{x^m + y^n}$$

exists if and only if the constants satisfy the inequality $\frac{p}{m} + \frac{q}{n} > 1$.