Honors Calculus III

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- Multivariable calculus /Multivariate calculus is the extension of calculus

  in one variable to calculus in more than one variable.

- This extension is done by simply considering a broader class of functions.

Q: What is a function?

Definition

A function f is a rule that assigns to each element in a set $\mathcal{A}$ exactly one element in a set $\mathcal{B}$, where $\mathcal{A}$ is known as the domain, and $\mathcal{B}$ the codomain.

- Formally, we use the following notation when we want to emphasise the sets

$$f : \mathbb{R} \to \mathbb{R} \qquad \text{defined by} \qquad f(x) = x^2$$

  where the domain $\mathcal{A}$ and the codomain $\mathcal{B}$ are $\mathbb{R}$, and the range of f is $\mathbb{R}_0^+$.

- Recall the domain and codomain are often not specified in practice, e.g.

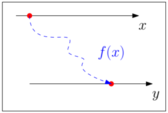
$$g(x) = \sqrt{x}$$

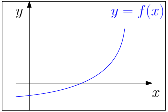
  in which case the domain is assumed to be the natural domain, that is, the largest set of real numbers such that the range is a subset of $\mathbb{R}$.

Q: What kind of functions did we consider last year ?

1. Explicitly e.g.

$$y = e^x - 3$$



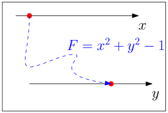


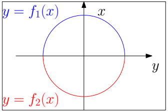
- $f : \mathbb{R} \to \mathbb{R}$

New types :

2. Implicitly e.g.

$$x^2 + y^2 = 1$$

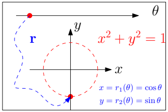


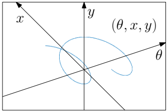


- $f_1 : \mathcal{D} \to \mathbb{R}$
  $f_2 : \mathcal{D} \to \mathbb{R}$
- $F : \mathbb{R}^2 \to \mathbb{R}$

3. Parametrically e.g.

$$x = \cos\theta; \quad y = \sin\theta$$

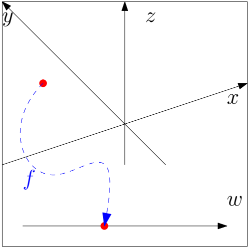




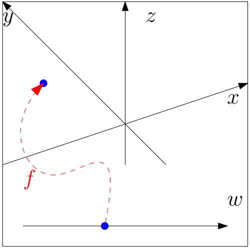
- $r_1 : \mathcal{D} \to \mathbb{R}$
  $r_2 : \mathcal{D} \to \mathbb{R}$
- $\mathbf{r} : \mathcal{D} \to \mathbb{R}^2$

- In addition to functions like F and $\mathbf{r}$, we will consider functions where the domain or codomain are $\mathbb{R}^3$, that is,

$$f : \mathbb{R}^3 \to \mathbb{R} \qquad \text{or} \qquad f : \mathbb{R} \to \mathbb{R}^3$$



4-dimensional plot



4-dimensional plot

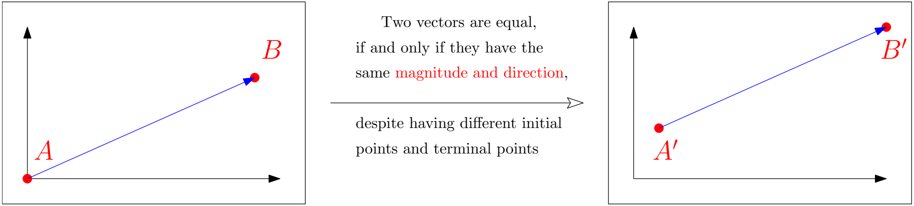
- Towards the end of this course, we will consider functions where the domain and codomain are high-dimensional at the same time, that is,

$$f : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \text{or} \qquad f : \mathbb{R}^3 \to \mathbb{R}^3$$

- Recall a vector is used to describe a physical quantity that has both

  magnitude and direction.

- Graphically a vector is an arrow from an initial point A to a terminal point B



- In typesetting, we denote a vector by a lower case letter in boldface, e.g.

$$\mathbf{v}$$

- In handwriting, arrows, tildes or straight lines below or above are common.

- We specify vectors by enclosing the components in square brackets, i.e.

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2 \qquad \text{and} \qquad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$$

- Recall the following definitions and terminologies for vectors

Definition

- Addition

If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ are vectors in $\mathbb{R}^2$, then the sum is given by

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

Definition

- Scalar Multiplication

  Let $\alpha \in \mathbb{R}$, which is known as a scalar, then the scalar multiple is given by

$$\alpha \mathbf{u} = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \end{bmatrix}$$

- Length or magnitude

  The magnitude or length of the vector $\mathbf{v} \in \mathbb{R}^2$ is, denoted by $|\mathbf{v}|$,

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}$$

- Addition, scalar multiplication and the length of vectors in $\mathbb{R}^3$ are defined in a similar fashion, the only difference is that there is a third component. e.g.

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} \qquad \text{where} \qquad \mathbf{v} \in \mathbb{R}^3$$

Properties of addition and scalar mutlipliсation

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ and α and β are scalars in $\mathbb{R}$, then

1. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
2. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
3. $\alpha(\mathbf{u}+\mathbf{v})=\alpha\mathbf{u}+\alpha\mathbf{v}$
4. $(\alpha+\beta)\mathbf{u}=\alpha\mathbf{u}+\beta\mathbf{u}$
5. $(\alpha\beta)\mathbf{u}=\alpha(\beta\mathbf{u})$
6. $1\mathbf{u}=\mathbf{u}$
7. $\mathbf{u}+\mathbf{0}=\mathbf{u}$
8. $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$

Definition

Combining the two operations gives a linear combination of the given vectors, e.g.

$$\alpha\mathbf{u}+\beta\mathbf{v} \qquad \text{and} \qquad \alpha\mathbf{u}+\beta\mathbf{v}+\gamma\mathbf{w}$$

where $\alpha,\beta,\gamma\in\mathbb{R}$.

Q: What is the difference between points and vectors?

- A *scalar* is a physical quantity that keeps track of 'one piece of information'.

  e.g. temperature

- A *vector* is a physical quantity that keeps track of 'two pieces of information'

  e.g. displacement

  where magnitude and direction are the "two pieces of information".

Q: What happens when we need to keep track of 'three pieces of information'?

- A tensor is needed, for example, a stress tensor has 9 components, it keeps track of the magnitude, the direction and the plane on which it is relevant.
- In its simplest form, a tensor can be represented as a matrix.

$$\mathbf{A} = \begin{bmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{bmatrix}$$

- In general, a matrix is a rectangular array of numbers, for example,

$$\underbrace{\begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix}}_{\mathbf{A}} \quad \text{and} \quad \underbrace{\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}}_{\mathbf{B}}$$

  of which will denote by upper-case letters in boldface in general.
- The numbers in the brackets are called entries or elements of a matrix,

$$a_{ij}$$

  note it has two indices, identifying their location within the matrix.
- The first index identifies the row it is in, while the second index identifies the column, so that together the entry's position is uniquely identified.
- When we say "an $m \times n$ matrix", we mean a matrix with m rows and n columns, together $m \times n$ is called the size of the matrix.

- The second matrix

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

is a square matrix, which means that it has as many rows as columns.

- For a square matrix, entries on the diagonal, that is,

$$a_{11}, a_{22}, \ldots, a_{nn}$$

is called the main diagonal of the square matrix.

- Vectors can be treated as matrices of the size $m \times 1$ or $1 \times n$, e.g.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

which are known as column vectors and row vectors, respectively.

Definition

- Addition

  The sum of two matrices $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ of the same size

$$\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$$

  is obtained by adding the corresponding entries of $\mathbf{A}$ and $\mathbf{B}$ together.

- Scalar Multiplication

  The product of any matrix $\mathbf{A} = [a_{ij}]$ and any scalar α is written as

$$\alpha\mathbf{A} = [\alpha a_{ij}]$$

  and is obtained by multiplying each entry of $\mathbf{A}$ by α.

Q: How can we compute ? $\cancel{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$

Matrices of different sizes cannot be added.

Properties of Matrix addition and scalar multiplication

If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are matrices of the same size, and α and β are scalars,

1. $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$
2. $\mathbf{A}+(\mathbf{B}+\mathbf{C})=(\mathbf{A}+\mathbf{B})+\mathbf{C}$
3. $\alpha(\mathbf{A}+\mathbf{B})=\alpha\mathbf{A}+\alpha\mathbf{B}$
4. $(\alpha+\beta)\mathbf{A}=\alpha\mathbf{A}+\beta\mathbf{A}$
5. $(\alpha\beta)\mathbf{A}=\alpha(\beta\mathbf{A})$
6. $1\mathbf{A}=\mathbf{A}$
7. $\mathbf{A}+\mathbf{0}=\mathbf{A}$
8. $\mathbf{A}+(-\mathbf{A})=\mathbf{0}$

- Here $\mathbf{0}$ denote the zero matrix of the right size, that is,

$$\mathbf{0}=[a_{ij}] \qquad \text{where } a_{ij}=0 \text{ for all } i \text{ and } j. \qquad \text{e.g.}$$

$$[0], \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Note matrix shares all of the 8 properties with its vector counterpart.

Multiplication of a Matrix by a Matrix

The product $\mathbf{C} = \mathbf{AB}$ of a matrix $\mathbf{A}_{m\times r} = [a_{ij}]$ and $\mathbf{B}_{p\times n} = [b_{ij}]$ is defined,

if and only if $r = p$,

to be the $m \times n$ matrix $\mathbf{C} = [c_{ij}]$ with entries

$$c_{ij} = \sum_{k=1}^{r} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj} \quad \text{for} \quad \begin{cases} i = 1, \cdots, m \\ j = 1, \cdots, n \end{cases}$$

- For instance,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \\ c_{41} & c_{42} \end{bmatrix}$$

- This is done by multiplying rows into columns,

$$c_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}$$

Properties of matrix multiplication

Suppose $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are matrices of the right size for which the indicated sums and products are defined, and α is a scalar, then

1. $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
2. $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$
3. $(\mathbf{B}+\mathbf{C})\mathbf{A} = \mathbf{BA}+\mathbf{CA}$
4. $\alpha(\mathbf{AB}) = (\alpha\mathbf{A})\mathbf{B} = \mathbf{A}(\alpha\mathbf{B})$
5. $\mathbf{I}_m\mathbf{A} = \mathbf{A} = \mathbf{A}\mathbf{I}_n$

1. $2\times(3\times 5) = (2\times 3)\times 5$
2. $2\times(3+5) = 2\times 3+2\times 5$
3. $(3+5)\times 2 = 3\times 2+5\times 2$
4. $\alpha\times(2\times 3) = \alpha 2\times 3 = 2\times\alpha 3$
5. $1\times 2 = 2 = 2\times 1$

where $\mathbf{I}_k$ is known as the identity matrix, they are square matrices of $k\times k$ with ones on the main diagonal and zeros everywhere else. For example,

$$\mathbf{I}_1 = [1], \quad \mathbf{I}_2 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad \mathbf{I}_3 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \cdots \quad \mathbf{I}_k = \begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{bmatrix}$$

Exercises

Evaluate the followings, and explain what can be learnt from the computations?

(a) $\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix}$

- Common mistakes regarding matrix multiplication.
  1. In general, you cannot change the order
  $$\mathbf{AB} \neq \mathbf{BA}$$
  2. In general, the cancellation laws do not hold
  $$\mathbf{AB} = \mathbf{AC} \qquad \not\Rightarrow \qquad \mathbf{B} = \mathbf{C}$$
  3. In general, you cannot conclude that
  $$\mathbf{AB} = \mathbf{0} \qquad \not\Rightarrow \qquad \mathbf{A} = \mathbf{0} \quad \text{or} \quad \mathbf{B} = \mathbf{0}$$