

18.650
Statistics for Applications

Chapter 2: Parametric Inference

The rationale behind statistical modeling

- ▶ Let X_1, \dots, X_n be n independent copies of X .
- ▶ The goal of statistics is to learn the distribution of X .
- ▶ If $X \in \{0, 1\}$, easy! It's $\text{Ber}(p)$ and we only have to learn the parameter p of the Bernoulli distribution.
- ▶ Can be more complicated. For example, here is a (partial) dataset with number of siblings (including self) that were collected from college students a few years back: 2, 3, 2, 4, 1, 3, 1, 1, 1, 1, 1, 2, 2, 3, 2, 2, 2, 3, 2, 1, 3, 1, 2, 3, ...
- ▶ We could make no assumption and try to learn the pmf:

x	1	2	3	4	5	6	≥ 7
$\mathbb{P}(X = x)$	p_1	p_2	p_3	p_4	p_5	p_6	$\sum_{i \geq 7} p_i$

That's 7 parameters to learn.

- ▶ Or we could assume that $X \sim \text{Poiss}(\lambda)$. That's 1 parameter to learn!

Statistical model (1)

Formal definition

Let the observed outcome of a statistical experiment be a sample X_1, \dots, X_n of n i.i.d. random variables in some measurable space E (usually $E \subseteq \mathbb{R}$) and denote by \mathbb{P} their common distribution. A *statistical model* associated to that statistical experiment is a pair

$$(E, (\mathbb{P}_\theta)_{\theta \in \Theta}),$$

where:

- ▶ E is *sample space*;
- ▶ $(\mathbb{P}_\theta)_{\theta \in \Theta}$ is a family of probability measures on E ;
- ▶ Θ is any set, called *parameter set*.



Statistical model (2)

- ▶ Usually, we will assume that the statistical model is *well specified*, i.e., defined such that $\mathbb{P} = \mathbb{P}_\theta$, for some $\theta \in \Theta$.
- ▶ This particular θ is called the true parameter, and is unknown: The aim of the statistical experiment **is to estimate θ** , or check it's properties when they have a special meaning ($\theta > 2?$, $\theta \neq 1/2?$, ...)
- ▶ For now, we will always assume that $\Theta \subseteq \mathbb{R}^d$ for some $d \geq 1$: The model is called *parametric*.

Statistical model (3)

Examples

1. For n Bernoulli trials:



$$\left(\{0, 1\}, (\text{Ber}(p))_{p \in (0,1)} \right).$$

2. If $X_1, \dots, X_n \stackrel{iid}{\sim} \text{Exp}(\lambda)$, for some unknown $\lambda > 0$:

$$(\mathbb{R}_+^*, (\text{Exp}(\lambda))_{\lambda > 0}).$$

3. If $X_1, \dots, X_n \stackrel{iid}{\sim} \text{Pois}(\lambda)$, for some unknown $\lambda > 0$:

$$(\mathbb{N}, (\text{Pois}(\lambda))_{\lambda > 0}).$$

4. If $X_1, \dots, X_n \stackrel{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$, for some unknown $\mu \in \mathbb{R}$ and $\sigma^2 > 0$:

$$\left(\mathbb{R}, (\mathcal{N}(\mu, \sigma^2))_{(\mu, \sigma^2) \in \mathbb{R} \times \mathbb{R}_+^*} \right).$$

Identification

The parameter θ is called *identified* iff the map $\theta \in \Theta \mapsto \mathbb{P}_\theta$ is injective, i.e.,

$$\theta = \theta' \Rightarrow \mathbb{P}_\theta = \mathbb{P}_{\theta'}.$$

Examples



1. In all four previous examples, the parameter was **identified**.



2. If $X_i = \mathbb{I}_{Y_i \geq 0}$, where $Y_1, \dots, Y_n \stackrel{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$, for some unknown $\mu \in \mathbb{R}$ and $\sigma^2 > 0$, are unobserved: μ and σ^2 are not identified (but $\theta = \mu/\sigma$ is).

Parameter estimation (1)

Idea: Given an observed sample X_1, \dots, X_n and a statistical model $(E, (\mathbb{P}_\theta)_{\theta \in \Theta})$, one wants to *estimate* the parameter θ .



Definitions

- ▶ **Statistic:** Any measurable¹ function of the sample, e.g., $\bar{X}_n, \max_i X_i, X_1 + \log(1 + |X_n|)$, sample variance, etc...
- ▶ **Estimator of θ :** Any statistic whose expression does not depend on θ .
- ▶ An estimator $\hat{\theta}_n$ of θ is *weakly* (resp. *strongly*) **consistent** iff

$$\hat{\theta}_n \xrightarrow[n \rightarrow \infty]{\mathbb{P} \text{ (resp. a.s.)}} \theta \quad (\text{w.r.t. } \mathbb{P}_\theta).$$



¹Rule of thumb: if you can compute it exactly once given data, it is measurable. You may have some issues with things that are implicitly defined such as sup or inf but not in this class

Parameter estimation (2)

- *Bias* of an estimator $\hat{\theta}_n$ of θ :

$$\mathbb{E} \left[\hat{\theta}_n \right] - \theta.$$



- *Risk* (or *quadratic risk*) of an estimator $\hat{\theta}_n$:

$$\mathbb{E} \left[|\hat{\theta}_n - \theta|^2 \right].$$



Remark: If $\Theta \subseteq \mathbb{R}$,

"Quadratic risk = bias² + variance".

Confidence intervals (1)

Let $(E, (\mathbb{P}_\theta)_{\theta \in \Theta})$ be a statistical model based on observations X_1, \dots, X_n , and assume $\Theta \subseteq \mathbb{R}$.

Definition

Let $\alpha \in (0, 1)$.

- ▶ *Confidence interval (C.I.) of level $1 - \alpha$ for θ* : Any random (i.e., depending on X_1, \dots, X_n) interval \mathcal{I} whose boundaries do not depend on θ and such that:

$$\mathbb{P}_\theta [\mathcal{I} \ni \theta] \geq 1 - \alpha, \quad \forall \theta \in \Theta.$$

- ▶ *C.I. of asymptotic level $1 - \alpha$ for θ* : Any random interval \mathcal{I} whose boundaries do not depend on θ and such that:

$$\lim_{n \rightarrow \infty} \mathbb{P}_\theta [\mathcal{I} \ni \theta] \geq 1 - \alpha, \quad \forall \theta \in \Theta.$$

Confidence intervals (2)

Example: Let $X_1, \dots, X_n \stackrel{iid}{\sim} \text{Ber}(p)$, for some unknown $p \in (0, 1)$.

- ▶ LLN: The sample average \bar{X}_n is a strongly consistent estimator of p .
- ▶ Let $q_{\alpha/2}$ be the $(1 - \frac{\alpha}{2})$ -quantile of $\mathcal{N}(0, 1)$ and

$$\mathcal{I} = \left[\bar{X}_n - \frac{q_{\alpha/2} \sqrt{p(1-p)}}{\sqrt{n}}, \bar{X}_n + \frac{q_{\alpha/2} \sqrt{p(1-p)}}{\sqrt{n}} \right].$$

- ▶ CLT: $\lim_{n \rightarrow \infty} \mathbb{P}_p [\mathcal{I} \ni p] = 1 - \alpha, \quad \forall p \in (0, 1).$
- ▶ Problem: \mathcal{I} depends on p !

Confidence intervals (3)

Two solutions:

- ▶ Replace $p(1 - p)$ with $1/4$ in \mathcal{I} (since $p(1 - p) \leq 1/4$).
- ▶ Replace p with \bar{X}_n in \mathcal{I} and use Slutsky's theorem.

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