Vv417 Lecture 12

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- Given a matrix $\mathbf{A}$ of $m \times n$, the solutions of the homogeneous linear system

$$\mathbf{Ax} = \mathbf{0}, \qquad \text{is a subset of } \mathbb{R}^n.$$

Q: Is the solution set a subspace?

- If so, we have a useful insight into the geometric structure of the solution set

The solution set is "flat".

Theorem

Given a matrix $\mathbf{A}$ of $m \times n$, the solutions of the homogeneous linear system

$$\mathbf{Ax} = \mathbf{0}$$

is a subspace of $\mathbb{R}^n$, and it is called the null space of $\mathbf{A}$, denoted by

$$\text{null}(\mathbf{A})$$

Proof

- Let $\mathcal{H}$ be the solution set of the system. $\mathcal{H}$ is not empty since it contains $\mathbf{0}$.
- $\mathcal{H}$ is a subset of $\mathbb{R}^n$, to show that $\mathcal{H}$ is a subspace of $\mathbb{R}^n$, we must show that it is closed under addition and scalar multiplication.
- Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be any vectors in $\mathcal{H}$, so $\mathbf{A}\mathbf{x}_1 = \mathbf{0}$ and $\mathbf{A}\mathbf{x}_2 = \mathbf{0}$. Consider

$$\begin{aligned} \mathbf{A}(\mathbf{x}_1 + \mathbf{x}_2) &= \mathbf{A}\mathbf{x}_1 + \mathbf{A}\mathbf{x}_2 \\ &= \mathbf{0} + \mathbf{0} = \mathbf{0} \end{aligned} \qquad \begin{aligned} \mathbf{A}(\alpha\mathbf{x}_1) &= \alpha(\mathbf{A}\mathbf{x}_1) \\ &= \alpha(\mathbf{0}) = \mathbf{0} \quad \square \end{aligned}$$

Theorem

Given a matrix $\mathbf{A}$ of $m \times n$, the solutions of the homogeneous linear system

$\mathbf{A}^{\mathrm{T}}\mathbf{y} = \mathbf{0}$, is a subspace of $\mathbb{R}^m$, denoted, $\mathrm{null}(\mathbf{A}^{\mathrm{T}})$.

- The vector space $\mathrm{null}(\mathbf{A}^{\mathrm{T}})$ is sometime called the left-hand null space of $\mathbf{A}$ for it is the set of all solutions to the left-hand homogeneous system

$$\mathbf{y}^{\mathrm{T}}\mathbf{A} = \mathbf{0}^{\mathrm{T}}$$

Exercise

Find a spanning set for $\mathrm{null}(\mathbf{A})$, where $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$.

Solution

- Since $\mathrm{null}\,(\mathbf{A})$ is simply the general solution of

$$\mathbf{A}\mathbf{x} = \mathbf{0}$$

- Applying Gaussian elimination, we have $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$, so

$$x_1 = -2x_2 - 3x_3$$

- In the vector form, we have $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}_1} + x_3 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_2}$

- Therefore, the subspace $\mathrm{null}\,(\mathbf{A})$ is spanned by $\{\mathbf{v}_1, \mathbf{v}_2\}$.

Definition

For a given matrix $\mathbf{A}$ of $m \times n$, we have the followings

1. The column space is the set of all linear combinations of the columns of $\mathbf{A}$,

$$\mathrm{col}(\mathbf{A}) = \mathrm{span}\{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n\}$$

2. The row space is the set of all linear combinations of the rows of $\mathbf{A}$, denoted,

$$\mathrm{row}(\mathbf{A}) = \mathrm{span}\{\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_m\}$$

where $\mathbf{c}_j$ and $\mathbf{r}_i$ are columns and rows of $\mathbf{A}$ respectively.

Q: Why is the following theorem clearly true.

Theorem

The column space and the row space of an $m \times n$ matrix $\mathbf{A}$ are subspaces.

- There are six important vector spaces associated with a matrix $\mathbf{A}$ and $\mathbf{A}^{\mathrm{T}}$
  - row space of $\mathbf{A}$
  - column space of $\mathbf{A}$
  - null space of $\mathbf{A}$
  - row space of $\mathbf{A}^{\mathrm{T}}$
  - column space of $\mathbf{A}^{\mathrm{T}}$
  - null space of $\mathbf{A}^{\mathrm{T}}$
- However, transposing a matrix converts row vectors into column vectors.
- So the row space of $\mathbf{A}^{\mathrm{T}}$ is the same as the column space of $\mathbf{A}$,

$$\mathrm{row}(\mathbf{A}^{\mathrm{T}}) = \mathrm{col}(\mathbf{A})$$

  and the column space of $\mathbf{A}^{\mathrm{T}}$ is the same as the row space of $\mathbf{A}$.

$$\mathrm{col}(\mathbf{A}^{\mathrm{T}}) = \mathrm{row}(\mathbf{A})$$

- The remaining spaces $\mathrm{null}(\mathbf{A})$, $\mathrm{null}(\mathbf{A}^{\mathrm{T}})$, $\mathrm{col}(\mathbf{A})$ and $\mathrm{row}(\mathbf{A})$ are called the

  four fundamental subspaces of $\mathbf{A}$.

Q: How to determine whether two matrices $\mathbf{A}$ and $\mathbf{B}$ have the same row space?

Theorem

For matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size, $\mathrm{row}(\mathbf{A}) = \mathrm{row}(\mathbf{B})$ if and only if $\mathbf{A} \sim \mathbf{B}$.

Proof

- First let $\mathbf{A} \sim \mathbf{B}$, so there exists a invertible matrix $\mathbf{E}$ such that

$$\mathbf{A} = \mathbf{E}\mathbf{B}$$

- It is clear each row of $\mathbf{A}$ is a linear combination of rows of $\mathbf{B}$, and vice versa.
- Recall we have shown

$$\mathrm{span}\,(\mathcal{S}) = \mathrm{span}\,(\mathcal{S}^\star)$$

  if and only if each vector in $\mathcal{S}$ is a linear combination of those in $\mathcal{S}^\star$ and each vector in $\mathcal{S}^\star$ is a linear combination of those in $\mathcal{S}$.
- Hence we can conclude

$$\mathrm{row}(\mathbf{A}) = \mathrm{row}(\mathbf{B})$$

Proof

- Conversely, let $\mathbf{a}_i$ and $\mathbf{b}_i$ be the rows of $\mathbf{A}$ and $\mathbf{B}$, and $\mathrm{row}(\mathbf{A}) = \mathrm{row}(\mathbf{B})$,

$$\mathrm{span}(\mathbf{a}_1, \ldots, \mathbf{a}_m) = \mathrm{span}(\mathbf{b}_1, \ldots, \mathbf{b}_m),$$

- By the same theorem, each $\mathbf{a}_i$ is a linear combination of $\mathbf{b}_i$, and vice versa.
- Hence there is an invertible matrix $\mathbf{E}$ such that $\mathbf{A} = \mathbf{EB}$, and $\mathbf{A} \sim \mathbf{B}$. $\square$

Q: How to determine whether two matrices have the same column space?

Theorem

For matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size, $\mathrm{col}(\mathbf{A}) = \mathrm{col}(\mathbf{B})$ if and only if $\mathbf{A}^{\mathrm{T}} \sim \mathbf{B}^{\mathrm{T}}$.

Q: Rows of $\mathbf{A}$ span $\mathrm{row}(\mathbf{A})$ and columns of $\mathbf{A}$ span $\mathrm{col}(\mathbf{A})$, but is it possible to span these spaces with fewer vectors than the full set of rows and columns?

Q: Do row operations change the null space of a matrix?

Q: Do row operations change the row space of a matrix?

Q: Do row operations change the column space of a matrix?

- Consider

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} = \mathbf{B}$$

- Notice that

$$\operatorname{col}(\mathbf{A}) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} \qquad \text{while} \qquad \operatorname{col}(\mathbf{B}) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$

Exercise

Find the smallest spanning sets for the column space and the row space of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{B}$$

Solution

- It is clear that the first and the second row of $\mathbf{B}$ span the row space of $\mathbf{A}$ since row operations do not alter the row space.

Solution

- Now the columns space, one way at the moment is to use the fact that

$$\operatorname{col}(\mathbf{A}) = \operatorname{row}\left(\mathbf{A}^{\mathrm{T}}\right)$$

- By the gaussian elimination, $\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 1 & 1 \\ 3 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 5/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{C}$
- It is clear that the first and the second row of $\mathbf{C}$ span the row space of $\mathbf{A}^{\mathrm{T}}$, thus they also span the column space of $\mathbf{A}$.

Q: Row operations alter the column space, however, why row operations will not alter the linear dependency among the columns?

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{B}$$

Matlab

```
>> format rational

>> X = [ 1 2; 2 1; 3 1]
X =
1      2
2      1
3      1

>> Y = [ 1 0; 0 1; -1/3 5/3]
Y =
  1              0
  0              1
-1/3            5/3

>> mldivide(X, Y)
ans =
-1/3            2/3
 2/3           -1/3
```

Q: How to find a spanning set for $\mathrm{null}(\mathbf{A}^{\mathrm{T}})$?

- Of course, we could obtain it from the general solution of

$$\mathbf{A}^{\mathrm{T}}\mathbf{y} = \mathbf{0}$$

- However, $\mathbf{A}^{\mathrm{T}}\mathbf{y} = \mathbf{0}$ is connected to $\mathbf{A}\mathbf{x} = \mathbf{0}$, and we don't need to start from scratch and compute a new echelon form.

Theorem

Let $\mathbf{A}$ be a matrix of size $m \times n$, and $\mathrm{rank}(\mathbf{A}) = r$, and suppose

$$\mathbf{E}\mathbf{A} = \mathbf{U}, \qquad \text{where } \mathbf{E} \text{ is non-singular and } \mathbf{U} \text{ is in row echelon form,}$$

then the last $(m - r)$ rows of $\mathbf{E}$ span $\mathrm{null}(\mathbf{A}^{\mathrm{T}})$. In other words, if

$$\mathbf{E} = \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \end{bmatrix} \qquad \text{where } \mathbf{E}_2 \text{ is } (m-r) \times m,$$

then

$$\mathrm{null}(\mathbf{A}^{\mathrm{T}}) = \mathrm{row}(\mathbf{E}_2)$$

Proof

- If $\mathbf{U} = \begin{bmatrix} \mathbf{C} \\ \mathbf{0} \end{bmatrix}$, where $\mathbf{C}_{r\times n}$, then

$$\mathbf{EA} = \mathbf{U} \implies \mathbf{E}_2\mathbf{A} = \mathbf{0} \implies \mathbf{A}^{\mathrm{T}}\mathbf{E}_2^{\mathrm{T}} = \mathbf{0}^{\mathrm{T}} \implies \mathrm{row}(\mathbf{E}_2) \subset \mathrm{null}(\mathbf{A}^{\mathrm{T}})$$

- To show equality, we demonstrate containment in the opposite direction

$$\mathrm{null}(\mathbf{A}^{\mathrm{T}}) \subset \mathrm{row}(\mathbf{E}_2)$$

- Suppose $\mathbf{y} \in \mathrm{null}(\mathbf{A}^{\mathrm{T}})$, and let $\mathbf{E}^{-1} = \begin{bmatrix} \mathbf{F}_1 & \mathbf{F}_2 \end{bmatrix}$, then

$$\begin{aligned} \mathbf{0}^{\mathrm{T}} = \mathbf{y}^{\mathrm{T}}\mathbf{A} = \mathbf{y}^{\mathrm{T}}\mathbf{E}^{-1}\mathbf{U} = \mathbf{y}^{\mathrm{T}} \begin{bmatrix} \mathbf{F}_1 & \mathbf{F}_2 \end{bmatrix} \begin{bmatrix} \mathbf{C} \\ \mathbf{0} \end{bmatrix} = \mathbf{y}^{\mathrm{T}}\mathbf{F}_1\mathbf{C} &\implies \mathbf{C}^{\mathrm{T}}\mathbf{F}_1^{\mathrm{T}}\mathbf{y} = \mathbf{0} \\ &\implies \mathbf{0} = \mathbf{F}_1^{\mathrm{T}}\mathbf{y} \end{aligned}$$

- The last step used the fact $\mathrm{rank}(\mathbf{A}) = \mathrm{rank}(\mathbf{A}^{\mathrm{T}})$ and $\mathbf{C}$ is full rank, so

$$\mathrm{null}(\mathbf{C}^{\mathrm{T}}) = \{\mathbf{0}\}$$

Proof

- In order to show $\mathbf{y} \in \text{row}\left(\mathbf{E}_2\right)$ from $\mathbf{0} = \mathbf{F}_1^{\mathrm{T}}\mathbf{y}$, we need the following result.
- Notice $\mathbf{E}\mathbf{E}^{-1} = \mathbf{I} = \mathbf{E}^{-1}\mathbf{E}$, where

$$\begin{bmatrix}\mathbf{F}_1 & \mathbf{F}_2\end{bmatrix}\begin{bmatrix}\mathbf{E}_1 \\ \mathbf{E}_2\end{bmatrix} = \mathbf{I} \implies \mathbf{F}_1\mathbf{E}_1 + \mathbf{F}_2\mathbf{E}_2 = \mathbf{I} \implies \mathbf{F}_1\mathbf{E}_1 = \mathbf{I} - \mathbf{F}_2\mathbf{E}_2$$

- Now back to $\mathbf{0} = \mathbf{F}_1^{\mathrm{T}}\mathbf{y}$ from which, we have

$$\begin{aligned}\mathbf{0}^{\mathrm{T}} = \mathbf{y}^{\mathrm{T}}\mathbf{F}_1 &\implies \mathbf{0}^{\mathrm{T}} = \mathbf{y}^{\mathrm{T}}\mathbf{F}_1\mathbf{E}_1 = \mathbf{y}^{\mathrm{T}}(\mathbf{I}_m - \mathbf{F}_2\mathbf{E}_2) \\ &\implies \mathbf{y}^{\mathrm{T}} = \mathbf{y}^{\mathrm{T}}\mathbf{F}_2\mathbf{E}_2 = (\mathbf{y}^{\mathrm{T}}\mathbf{F}_2)\mathbf{E}_2 \\ &\implies \mathbf{y}^{\mathrm{T}} \in \text{row}(\mathbf{E}_2) \quad \text{and thus} \quad \mathbf{y} \in \text{row}(\mathbf{E}_2)\end{aligned}$$

- Since $\mathbf{y}$ is any vector $\in \text{null}(\mathbf{A}^{\mathrm{T}})$,

$$\text{null}(\mathbf{A}^{\mathrm{T}}) \subset \text{row}(\mathbf{E}_2) \quad \square$$

Exercise

Find a spanning set for $\text{null}(\mathbf{A}^{\mathrm{T}})$, where $\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{bmatrix}$.

Solution

- We want to find the matrix $\mathbf{E}$ such that

$$\mathbf{EA} = \mathbf{U} \qquad \text{where } \mathbf{U} \text{ is a row echelon form of } \mathbf{A}.$$

- Thus let us consider the following matrix, and apply row operations on it

$$\left[\begin{array}{cccc|ccc} 1 & 2 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 1 & 3 & 0 & 1 & 0 \\ 3 & 6 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|ccc} 1 & 2 & 0 & 1 & -1/3 & 2/3 & 0 \\ 0 & 0 & 1 & 1 & 2/3 & -1/3 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & -5/3 & 1 \end{array}\right] \leftarrow \mathbf{x} \quad = \left[\begin{array}{c|c} \mathbf{C} & \mathbf{E}_1 \\ \mathbf{0} & \mathbf{E}_2 \end{array}\right]$$

- Therefore

$$\text{null}\left(\mathbf{A}^{\mathrm{T}}\right) = \text{row}\left(\mathbf{E}_2\right) = \text{span}\left(\mathbf{x}\right)$$