Vv256 Lecture 19

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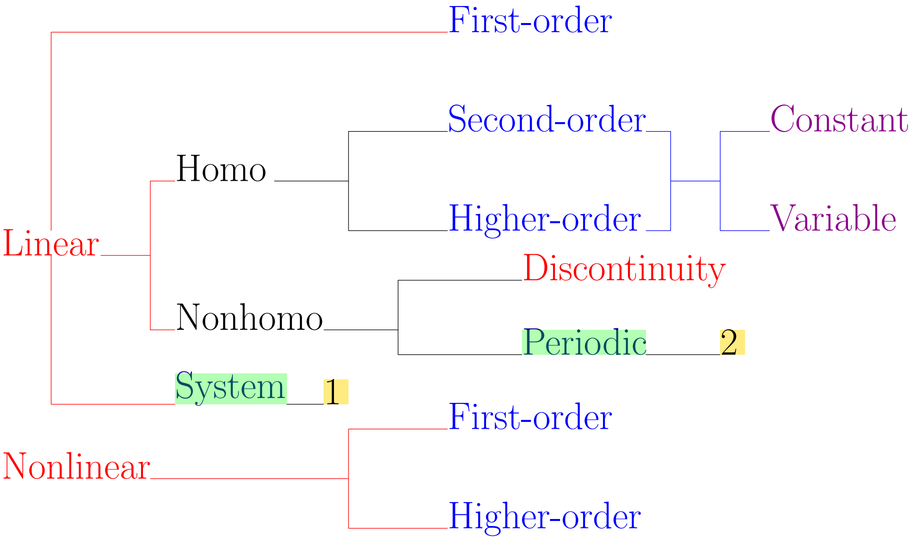
Overview


First-order
Second-order
Constant
Homo
Higher-order
Variable
Linear
Discontinuity
Nonhomo
Periodic
2
System
1
First-order
Nonlinear
Higher-order

Q: Why do we have to study linear algebra?

- It was realised that many mathematical objects of different sorts:

  vectors, matrices, polynomials, functions and operators

  were in fact quite similar.

- They are similar because they share some defining properties, hence they share the same "consequences" of those defining properties. e.g.

$$\alpha(\mathbf{u}+\mathbf{v}) = \alpha\mathbf{u}+\alpha\mathbf{v}; \qquad \beta(\mathbf{A}+\mathbf{B}) = \beta\mathbf{u}+\beta\mathbf{v}$$

- Rather than studying each objects separately, it is more efficient to study the common properties and their consequences instead the actual objects.

Definition

The set of vectors

$$\mathbb{R}^n$$

is known as the n-dimensional Euclidean space.

Q: Intuitively, what is the difference between a *Space* and a *Set* of vectors?

Properties of addition and scalar mutliplication

- If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$ and α and β are scalars in $\mathbb{R}$, then

1. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
2. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
3. $\alpha(\mathbf{u}+\mathbf{v})=\alpha\mathbf{u}+\alpha\mathbf{v}$
4. $(\alpha+\beta)\mathbf{u}=\alpha\mathbf{u}+\beta\mathbf{u}$
5. $(\alpha\beta)\mathbf{u}=\alpha(\beta\mathbf{u})$
6. $1\ \mathbf{u}=\mathbf{u}$
7. $\mathbf{u}+\mathbf{0}=\mathbf{u}$
8. $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$
9. $\mathbf{u}+\mathbf{v}$ is in $\mathbb{R}^n$
10. $\alpha\mathbf{u}$ is in $\mathbb{R}^n$

- Those properties are the defining properties of a basic Euclidean space.
- In general, structure can be introduced into a set by defining addition and scalar multiplication, different structure result different types of space.

- In general, a vector space consists four things,

  two sets $\mathcal{F}$ and $\mathcal{V}$, two operations called addition and scalar multiplication.

1. $\mathcal{F}$ is a scalar field.

- For us $\mathcal{F}$ is either the field $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers.

2. $\mathcal{V}$ is a non-empty set of mathematical objects called vectors.

- So in the general sense, matrices, polynomials, continuous and differentiable functions are all known as vectors as well.

3. Addition is an operation between elements of $\mathcal{V}$.

- By addition we mean a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{V}$ an object $\mathbf{u}+\mathbf{v}$, called the sum of $\mathbf{u}$ and $\mathbf{v}$;

4. Scalar multiplication is an operation between elements of $\mathcal{F}$ and $\mathcal{V}$.

- By scalar multiplication we mean a rule for associating with each object $\mathbf{u}$ in $\mathcal{V}$ and each scalar α in $\mathcal{F}$ an object $\alpha\mathbf{u}$, called the scalar multiple of $\mathbf{u}$ by α.

- The Euclidean space $\mathbb{R}^3$ over $\mathbb{R}$ is a vector space, where the scalars, vectors, addition and scalar multiplication all follow the usual definition.
- In some sense, a vector space is a generalisation of $\mathbb{R}^n$, where not everything is defined in the usual way, however, a vector space must possess the same structure as $\mathbb{R}^n$ in terms of addition and scalar multiplication.

Definition

If the addition and scalar multiplication operations satisfy the following properties by all objects $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathcal{V}$ and all scalars α and β in $\mathcal{F}$, then we call

$\mathcal{V}$ a vector space over $\mathcal{F}$.

1. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
2. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
3. $\alpha(\mathbf{u}+\mathbf{v})=\alpha\mathbf{u}+\alpha\mathbf{v}$
4. $(\alpha+\beta)\mathbf{u}=\alpha\mathbf{u}+\beta\mathbf{u}$
5. $(\alpha\beta)\mathbf{u}=\alpha(\beta\mathbf{u})$
6. $1\,\mathbf{u}=\mathbf{u}$
7. $\mathbf{u}+\mathbf{0}=\mathbf{u}$
8. $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$
9. $\mathbf{u}+\mathbf{v}$ is in $\mathcal{V}$
10. $\alpha\mathbf{u}$ is in $\mathcal{V}$

To Show That a Set with Two Operations Is a Vector Space

- Step 1. Identify the set $\mathcal{V}$ of objects that will become vectors.
- Step 2. Identify the addition and scalar multiplication operations on $\mathcal{V}$.
- Step 3. Verify Axiom 9 and Axiom 10; that is, checking
  - $\mathcal{V}$ is closed under addition
  - $\mathcal{V}$ is closed under scalar multiplication
- Step 4. Confirm that Axioms 6, 7, 8.
- Step 5. Check Axioms 1, 2, 3, 4, 5.

Exercise

Let $\mathcal{V}$ consist of a single object, which we denote by ❤, and define

$$❤ + ❤ = ❤ \quad \text{and} \quad \alpha ❤ = ❤ \qquad \text{where } \alpha \text{ is any scalar in } \mathcal{F}.$$

Is $\mathcal{V}$ a vector space over $\mathbb{R}$?

Solution

1. Here the set is simply that single object.
2. Addition and scalar multiplication are defined by the two equations

$$\heartsuit + \heartsuit = \heartsuit \quad \text{and} \quad \alpha\heartsuit = \heartsuit \qquad \text{for all real scalars } \alpha.$$

3. Since both the operations produce the object itself, $\mathcal{V}$ is closed under addition and closed under scalar multiplication.
4. In this case, the multiplicative identity can be any real numbers α. This is special, every other vector space has a unique multiplicative identity of 1.

   Here both the additive identity and the inverse are simply the object itself.

$$\mathbf{0} = \heartsuit \qquad \text{and} \qquad -\heartsuit = \heartsuit$$

5. Axioms 1–5 are clearly satisfied, since there is only one object in this set, the left-hand side and right-hand side are always equal.

- So $\mathcal{V}$ is a vector space, and it is known as the trivial or zero vector space.

Exercise

Let $\mathcal{V} = \mathbb{R}^2$ and the operations of addition and scalar multiplication are defined as

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix} \quad \text{and} \quad \alpha\mathbf{v} = \begin{bmatrix} \alpha v_1 \\ 0 \end{bmatrix}, \quad \text{where} \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

are in $\mathcal{V}$ and α is any scalar in $\mathcal{F}$. Is $\mathcal{V}$ a vector space over $\mathbb{R}$?

Solution

1. The set is all vectors in $\mathbb{R}^2$.
2. Addition and scalar multiplication are given above.
3. Addition and scalar multiplication are both clearly closed,
4. But, let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, where $u_2 \neq 0$, and the multiplicative identity $\alpha = 1$, we have $\alpha\mathbf{u} = \begin{bmatrix} u_1 \\ 0 \end{bmatrix} \neq \mathbf{u}$, so Axiom 6 is violated and $\mathcal{V}$ is not a vector space.

Exercise

Let $\mathcal{V}$ be the set of positive real numbers, and α be any scalar from $\mathcal{F}$. Suppose addition and scalar multiplication are defined to be

$$\mathbf{u} + \mathbf{v} = uv \quad \text{and} \quad \alpha\mathbf{v} = v^{\alpha}, \qquad \text{where} \quad \mathbf{u} = u \quad \text{and} \quad \mathbf{v} = v \quad \text{are in } \mathcal{V}.$$

Is $\mathcal{V}$ a vector space over $\mathbb{R}$?

Solution

1. Here the set is all positive real numbers.
2. Addition and scalar multiplication are given above.
3. Closure property for vector addition is satisfied since a usual product of positive real numbers is a positive real number.

   Closure property for scalar multiplication is satisfied since exponentiation of any positive real number is a positive real number.

Solution

4. Multiplicative identity is 1 since raising any number to the power of 1,

$$1\mathbf{u} = u^1 = u = \mathbf{u}$$

Additive identity is $\mathbf{0} = 1$ since multiplying any number to 1, we have

$$\mathbf{u} + \mathbf{0} = u \cdot 1 = u = \mathbf{u}$$

Negative $\mathbf{u}$ is $-\mathbf{u} = \dfrac{1}{u}$ since

$$\mathbf{u} + (-\mathbf{u}) = u \cdot \frac{1}{u} = 1 = \mathbf{0}$$

5. It is trivial to show axiom 1–5 are satisfied since they are basic properties of multiplication and exponentiation.

- Therefore $\mathcal{V}$ is a vector space over $\mathbb{R}$.

More examples of vector space

- $\mathbb{R}^{m\times n}$ of $m \times n$ matrices with the usual operations is a vector space over $\mathbb{R}$.
- $\mathbb{C}^{m\times n}$ of $m \times n$ matrices with the usual operations is a vector space over $\mathbb{C}$.
- In this course, you may assume $\mathcal{F}$ is $\mathbb{R}$, and usual definitions of addition and scalar multiplication unless otherwise specified, e.g. for functions, we have

$$\big(f+g\big)(x) = f(x) + g(x) \qquad \text{and} \qquad \big(\alpha f\big)(x) = \alpha f(x)$$

Q: Which of the following sets are vector spaces over $\mathbb{R}$?

1. The set of all functions mapping the interval $[0, 1]$ into $\mathbb{R}$.
2. The set of all real-valued functions that are continuous on $[0, 1]$.
3. The set of all real-valued functions that are differentiable on $(0, 1)$.
4. The set of all vector-valued functions define on $(-\infty, \infty)$.
5. The set of all polynomials.

Theorem

Let $\mathcal{V}$ be a vector space, $\mathbf{u}$ a vector in $\mathcal{V}$, and α a scalar, then

1. $0\mathbf{u} = \mathbf{0}$
2. $\alpha\mathbf{0} = \mathbf{0}$
3. $(-1)\mathbf{u} = -\mathbf{u}$
4. If $\alpha\mathbf{u} = \mathbf{0}$, then $\alpha = 0$ or $\mathbf{u} = \mathbf{0}$.

Proof

Let us prove 1., consider

$$
\begin{aligned}
0\mathbf{u} + 0\mathbf{u} &= (0+0)\mathbf{u} && \text{Axiom 4} \qquad (\alpha+\beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u} \\
&= 0\mathbf{u} && \text{Properties of } \mathbb{R}. \\
(0\mathbf{u} + 0\mathbf{u}) + (-0\mathbf{u}) &= 0\mathbf{u} + (-0\mathbf{u}) && \text{Axiom 8, existence of a negative vector} \\
0\mathbf{u} + [0\mathbf{u} + (-0\mathbf{u})] &= 0\mathbf{u} + (-0\mathbf{u}) && \text{Axiom 2} \qquad \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w} \\
0\mathbf{u} + \mathbf{0} &= \mathbf{0} && \text{Axiom 8} \\
0\mathbf{u} &= \mathbf{0} && \text{Axiom 7, existence of a zero vector.}
\end{aligned}
$$

- You might be tempted to prove by working with components

$$0\mathbf{u} = \begin{bmatrix} 0u_1 \\ 0u_2 \\ \vdots \\ 0u_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}$$

- If we do, the resulting theorem would be applicable only to Euclidean spaces
- Using only axioms the theorem is proved to be valid for all vector spaces. This is the power of the axiomatic approach, one proof serves to establish theorems for many seemingly different mathematical objects.

Exercise

Do you realise that we have proved the following statement?

$$a^0 = 1, \qquad \text{where } a \text{ is a positive real number.}$$

Solution

- The theorem on page $\boxed{13}$ is applicable to any vector space.
- Applying the first statement of the theorem on page $\boxed{10}$ to $\mathcal{V}$ here.

$$0\mathbf{u} = \mathbf{0} \implies u^0 = 1$$

- Since 1 is the additive identity $\mathbf{0}$ for this vectors space.

- Often we are interested only in a subset of a known vector space.

Definition

A non-empty subset $\mathcal{H}$ of a vector space $\mathcal{V}$ over $\mathcal{F}$ is called a subspace of $\mathcal{V}$ if $\mathcal{H}$ is itself a vector space over $\mathcal{F}$ under the same addition and scalar multiplication.

- In general, to show that a set $\mathcal{H}$ with a scalar multiplication and an addition is a vector space, it is necessary to verify the 10 axioms.

- However, if $\mathcal{H}$ is a subset of $\mathcal{V}$ which is known to be a vector space, then not all axioms need to be verified. e.g. It is not necessary to verify that

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

holds in $\mathcal{H}$ because it holds for all vectors in $\mathcal{V}$ including those in $\mathcal{H}$.

Exercise

Which of those 10 axioms are NOT "inherited" ?

Solution

- Axiom 7 Existence of a zero vector in the subset
- Axiom 8 Existence of a negative in the subset for every vector in the subset
- Axiom 9 Closure of the subset under addition
- Axiom 10 Closure of the subset under scalar multiplication

- However, the next theorem shows only two of those need to be checked.

Theorem

If $\mathcal{H}$ is a non-empty subset a vector space $\mathcal{V}$ over $\mathcal{F}$, then $\mathcal{H}$ is a subspace of $\mathcal{V}$ over $\mathcal{F}$ if and only if the following conditions are satisfied.

- 1. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathcal{H}$, then $\mathbf{u} + \mathbf{v}$ is in $\mathcal{H}$.
- 2. If α is a scalar in $\mathcal{F}$ and $\mathbf{u}$ is a vector in $\mathcal{H}$, then $\alpha\mathbf{u}$ is in $\mathcal{H}$.

Proof

- Given $\mathcal{H}$ is a subspace, then those two conditions are satisfied by definition.
- It $\mathcal{H}$ is a subset of a vector space $\mathcal{V}$, then Axiom 1–6 are inherited.
- Since $\mathcal{V}$ is a vector space, the theorem on $\boxed{13}$ are true, specifically, we have

$$1. \quad 0\mathbf{u} = \mathbf{0} \qquad 2. \quad (-1)\mathbf{u} = -\mathbf{u}$$

- So 2. implies we have zero vector and a negative in $\mathcal{H}$, that is, Axiom 7–8.
- Hence we only need to make sure that 1. Axiom 9. is satisfied. □

Definition

If $\mathbf{u}$ is a vector in a vector space $\mathcal{V}$, then $\mathbf{u}$ is said to be a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_r$ in $\mathcal{V}$ if $\mathbf{u}$ can be expressed in the form

$$\mathbf{u} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_r \mathbf{v}_r$$

where α_1, α_2 , ..., α_r are scalars in $\mathcal{F}$. These scalars are called the coefficients.

- For example, the vector $\mathbf{u}$ is a linear combination of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ below

$$\underbrace{\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}}_{\mathbf{u}} = 3 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_1} + 2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}_2} \quad \text{or} \quad \underbrace{3x^2 + 2x + 3}_{\mathbf{u}} = 3 \underbrace{(x^2 + 1)}_{\mathbf{v}_1} + 2 \underbrace{(x)}_{\mathbf{v}_2}$$

Theorem

If $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is a non-empty set of vectors in a vector space $\mathcal{V}$, then the set $\mathcal{H}$ of all possible linear combinations of the vectors in $\mathcal{S}$ is a subspace of $\mathcal{V}$.

Proof

- Let $\mathcal{H}$ be the set of all possible linear combinations of the vectors in $\mathcal{S}$.

$$\mathbf{u} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_r\mathbf{v}_r$$

- The set $\mathcal{S}$ only contains vectors in $\mathcal{V}$, and $\mathcal{V}$ is a vector space, which is closed under addition and scalar multiplication, so $\mathcal{H}$ is a subset of $\mathcal{V}$.
- So we only need to show $\mathcal{H}$ is closed under addition and scalar multiplication
- Let $\mathbf{u} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_r\mathbf{v}_r$ and $\mathbf{w} = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \cdots + \beta_r\mathbf{v}_r$, then

$$\begin{aligned}\mathbf{u} + \mathbf{w} &= (\alpha_1 + \beta_1)\mathbf{v}_1 + (\alpha_2 + \beta_2)\mathbf{v}_2 + \cdots + (\alpha_r + \beta_r)\mathbf{v}_r \\ \gamma\mathbf{u} &= (\gamma\alpha_1)\mathbf{v}_1 + (\gamma\alpha_2)\mathbf{v}_2 + \cdots + (\gamma\alpha_r)\mathbf{v}_r\end{aligned}$$

  where α_i, β_i and γ are any scalars in $\mathcal{F}$.
- So $\mathbf{u} + \mathbf{w}$ and $\gamma\mathbf{u}$ are linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_r$, and they are in $\mathcal{H}$
- That shows $\mathcal{H}$ is closed under addition and scalar multiplication. $\square$

Defintion

The subspace $\mathcal{H}$ of all possible linear combinations of vectors in $\mathcal{S} \subset \mathcal{V}$ is called the subspace of $\mathcal{V}$ generated by $\mathcal{S}$, and we say the set $\mathcal{S}$ spans $\mathcal{H}$, or $\mathcal{H}$ is the subspace spanned by $\mathcal{S}$. We denote this subspace $\mathcal{H}$ as

$$\mathcal{H} = \operatorname{span}(\mathcal{S})$$

Alternatively, we denote it by

$$\mathcal{H} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} \qquad \text{where} \qquad \mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$$

The set $\mathcal{S}$ is known as the spanning set for $\mathcal{H}$.

- Let us denote that the standard unit vectors in $\mathbb{R}^n$ as

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

- The subspace

$$\mathcal{H} = \operatorname{span}\{\mathbf{e}_1\}$$

is the subspace consisting of all scalar multiples of $\mathbf{e}_1$.

Q: What is the geometric interpretation of $\mathcal{H} = \operatorname{span}\{\mathbf{e}_1\}$?

- The subspace

$$\mathcal{H} = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2\}$$

is the subspace containing all linear combinations $\mathbf{e}_1$ and $\mathbf{e}_2$,

$$\alpha\mathbf{e}_1 + \beta\mathbf{e}_2, \qquad \text{where } \alpha \text{ and } \beta \text{ are in } \mathcal{F}.$$

Q: What is the geometric interpretation of $\mathcal{H} = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2\}$?

- Of course, we can go on adding more $\mathbf{e}_i$ into the set $\mathcal{S}$, we will just end up with subspaces that are hyperplanes determined by those vectors in $\mathcal{S}$.
- If we put all n of those standard unit vectors in the $\mathcal{S}$, then clearly

$$\operatorname{span}(\mathcal{S}) = \mathbb{R}^n$$

- So the n standard unit vectors span $\mathbb{R}^n$ since every vector $\mathbf{u}$ in $\mathbb{R}^n$ is a linear combination of those $\mathbf{e}_i$s, and the set $\{\mathbf{e}_1 \cdots \mathbf{e}_n\}$ is a spanning set for $\mathbb{R}^n$.

Definition

Given three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^3$, if there exist three unique scalars

$$\alpha, \quad \beta, \quad \text{and} \quad \gamma$$

such that

$$\mathbf{v} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} \qquad \text{for any arbitrary vector } \mathbf{v} \text{ in } \mathbb{R}^3.$$

then we say that $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ form a basis for the space $\mathbb{R}^3$. The scalars

$$\alpha, \quad \beta, \quad \text{and} \quad \gamma$$

are called the components/coordinates of $\mathbf{v}$ with respect to the basis $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

- The notions of

  "basis vectors" and "coordinate systems"

  can be extended naturally to a general vector space $\mathcal{V}$.

Definition

The vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are said to be linearly independent if the only way to have

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n = \mathbf{0}$$

is for all the α's to be zero,

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

Defintion

The dimension of a vector space $\mathcal{V}$ is defined to be the largest number of linearly independent vectors in $\mathcal{V}$, often denoted by

$$\dim \mathcal{V}$$

Definition

In general, a basis for a vector space $\mathcal{V}$ is a linearly independent spanning set of $\mathcal{V}$

$$\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

- The idea of Wronskian can be easily extended to more than two functions.

Definition

If $\mathbf{f}_1 = f_1(x)$, $\mathbf{f}_2 = f_2(x)$, $\cdots$, $\mathbf{f}_n = f_n(x)$ are functions that are $n-1$ times continuously differentiable on the interval $(-\infty, \infty)$, then the determinant

$$W(x) = \det \left(\begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix} \right)$$

is called the Wronskian of f_1, f_2, ..., f_n.

Theorem

If the functions $\mathbf{f}_1, \mathbf{f}_2, \cdots, \mathbf{f}_n$ have $n-1$ continuous derivatives on $(-\infty, \infty)$, and if the Wronskian of these functions is not identically zero on $(-\infty, \infty)$, then these functions form a linearly independent set of vectors in $C^{n-1}(-\infty, \infty)$.

Exercise

Find a basis and the dimension of the solution space of

$$\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)y=0$$

Solution

- The solution space is the set of all solutions of the equation, and it is easy to show that it is a subspace of $F(-\infty,\infty)$ by considering the general solution.
- We know the general solution of the equation is

$$y=c_1e^{0t}+c_2e^{1t}+c_3e^{2t}$$

- Notice that the set of all solutions is simply,

$$\mathcal{H}=\operatorname{span}(\mathcal{S}), \qquad \text{where} \quad \mathcal{S}=\{1,e^t,e^{2t}\}$$

thus must be a vector space, and hence the subspace of $F(-\infty,\infty)$.

Solution

- To find a basis for $\mathcal{H}$, we need to find a linearly independent spanning set of $\mathcal{H}$, the set $\mathcal{S}$ is a spanning set of $\mathcal{H}$ by definition, so if $\mathcal{S}$ is linearly independent, then $\mathcal{S}$ is a basis for $\mathcal{H}$ and the dimension of $\mathcal{H}$ is 3.
- Consider the Wronskian $W(t)$,

$$W = \det\left(\begin{bmatrix} 1 & e^t & e^{2t} \\ 0 & e^t & 2e^{2t} \\ 0 & e^t & 4e^{2t} \end{bmatrix}\right) = 1\det\left(\begin{bmatrix} e^t & 2e^{2t} \\ e^t & 4e^{2t} \end{bmatrix}\right)$$
$$= 4e^{3t} - 2e^{3t} = 2e^{3t} \neq 0, \qquad \text{for all } t.$$

- By theorem $\boxed{24}$, the set $\mathcal{S}$ is linearly independent as well as being the span.
- Essentially we have been looking for a basis for the solution space when we are solving a differential equation,
- Note $\mathcal{S}$ is the fundamental set of solutions. In general, the fundamental set of solutions is a basis for the solution space by definition.