Question1 (1 points)

A curve has the property that at *each* of its points (x, y), its slope y' is equal to twice the sum of the coordinates of the point. Write down this property by means of a differential equation for such curves.

Solution:

1M From the description, it is clear that such curves can be found by solving

$$y' = 2(x + y)$$

0M Matlab can be used to solve such equations symbolically

```
>> syms y(x)
>> dsolve(diff(y,x) == 2*(x+y));
>> pretty(ans)

C1 exp(2 x)       1
----------- - x - -
     2            2
```

Question2 (2 points)

Find the direction field for the following differential equation

$$\dot{y} = e^{-t} + y$$

Based on the direction field, does the behaviour of y as $t \to \infty$ depend on the initial value

$$y(0) = y_0$$

Justify your answer without explicitly solving the differential equation.

Solution:

1M The direction field is

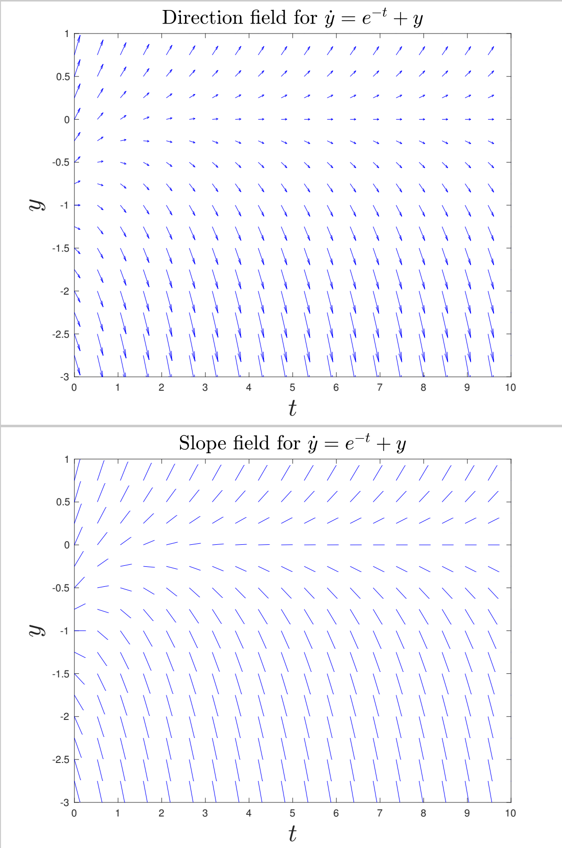
$$\mathbf{F} = \mathbf{e}_t + \left(e^{-t} + y\right)\mathbf{e}_y$$

1M Yes, it does depend on the initial value. If $y_0 \geq 0$, then $\dot{y} = e^{-t} + y$ is always positive, thus y is strictly increasing. However, if $y_0 < -1$, then $\dot{y}$ is always negative for $t > 0$, thus y is strictly decreasing. Considering the second derivative, we have

$$\ddot{y} = -e^{-t} + \dot{y} = y$$

which means y is concave up when $y_0 \geq 0$ and concave downward when $y_0 < -1$. So there is no upper bound or lower bound for y. Hence, without solving it, we know for sure that $y \to \infty$ as $t \to \infty$ when $y_0 \geq 0$, and $y \to -\infty$ as $t \to \infty$ when $y_0 < -1$. A similar conclusion can be obtain from a direction field or slope field is available. The explicit solution in terms of $y(0) = y_0$ is

$$y = (y_0 + 0.5)e^{t} - 0.5e^{-t}$$




```
>> clear all;
>> ft=@(t,y) 1 + t*0;
>> fy=@(t,y) exp(-t) + y;

>> t = linspace(0,10,20);
>> y = -3:0.25:1;
>> [tt, yy] = meshgrid(t,y);

>> ftt = ft(tt,yy);
>> fyy = fy(tt,yy);

>> quiver(tt, yy, ftt, fyy, 0.75, 'b');

>> xlim([0 10]);ylim([-3 1])

>> obj = xlabel('$t$', 'fontsize', 25);
>> set(obj,'Interpreter','Latex');
>> obj = ylabel('$y$', 'fontsize', 25,'rot',90);
>> set(obj,'Interpreter','Latex');

>> obj = title(...
'Direction field for $\dot{y} = e^{-t} + y$',...
'FontSize',20);
>> set(obj,'Interpreter','Latex');

%%%Normalisation

>> ftt = ftt./sqrt(ftt.^2+fyy.^2);
>> fyy = fyy./sqrt(ftt.^2+fyy.^2);
>> obj = quiver(tt, yy, ftt, fyy, 0.5, 'b');
>> set (obj, 'maxheadsize', 0);
>> xlim([0 10]);ylim([-3 1])

>> obj = xlabel('$t$', 'fontsize', 25);
>> set(obj,'Interpreter','Latex');
>> obj = ylabel('$y$', 'fontsize', 25,'rot',90);
>> set(obj,'Interpreter','Latex');

>> obj = title( ...
    'Slope field for $\dot{y} = e^{-t} + y$', ...
    'FontSize',20);
>> set(obj,'Interpreter','Latex');
```

Question3 (1 points)

The general solution of a differential equation is the family of functions that satisfy a given differential equation. However, A nonlinear ODE may sometimes have an additional solution that cannot be obtained from the general solution and is then called a singular solution. The following first-order ODE is of this kind.

$$(\dot{y})^2 - t\dot{y} + y = 0$$

Verify the given equation has the general solution of

$$y = ct - c^2$$

and the singular solution of

$$y = \frac{t^2}{4}$$

Solution:

0M The term "general solution" is well-understood for linear equations, it means a family of all possible solutions and typically includes arbitrary constants in the case of an ODE or arbitrary functions in the case of PDE. A solution without arbitrary constants/functions is called a particular solution. However, when comes to nonlinear equations, the usage of "general equation" is not universal. For a first-order equation, a " general solution" is defined to be an one-parameter family of solutions and a "singular solution" is a solution which is not contained in the "general solution".

1M For any constant c, we have

$$\text{LHS} = \left(\frac{d}{dt}\left(ct - c^2\right)\right)^2 - t\frac{d}{dt}\left(ct - c^2\right) + \left(ct - c^2\right) = (c)^2 - tc + ct - c^2 = 0 = \text{RHS}$$

which shows $y = ct - c^2$ is a general solution for any c. Substituting

$$y = \frac{t^2}{4}$$

into the equation, we have

$$\text{LHS} = \left(\frac{d}{dt}\left(\frac{t^2}{4}\right)\right)^2 - t\frac{d}{dt}\left(\frac{t^2}{4}\right) + \left(\frac{t^2}{4}\right) = \frac{t^2}{4} - \frac{t^2}{2} + \frac{t^2}{4} = \text{RHS}$$

It is clear that no c satisfies the following equation

$$ct - c^2 = \frac{t^2}{4}$$

So $y = \dfrac{t^2}{4}$ is a particular solution that is not included by $y = ct - c^2$, thus singular.

Question4 (1 points)

Given $y > 0$, solve the following initial-value problem

$$\dot{y} = -\frac{ty}{\ln y}, \qquad y(0) = e^2$$

Specify the maximum open interval on which your solution is valid.

Solution:

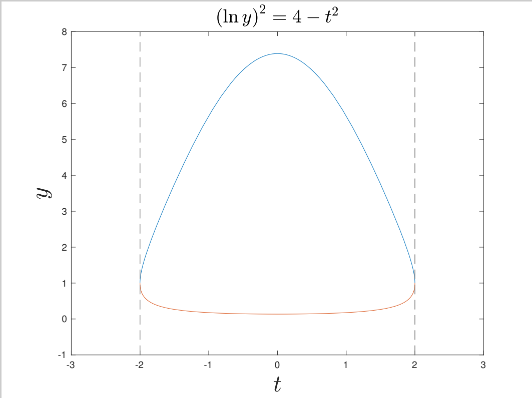
1M This is a separable equation, applying the formula, we have

$$\begin{aligned} \int \frac{\ln y}{y}\, dy &= \int -t\, dt \\ \implies \frac{(\ln y)^2}{2} &= -\frac{t^2}{2} + C_1 \\ \implies y &= \exp\left(\pm\sqrt{2C_1 - t^2}\right) \end{aligned}$$

Using the initial condition, we have

$$y(0) = e^2 \implies y = \exp\left(\sqrt{4 - t^2}\right)$$

The solution satisfy the differential equation at every (t, y), thus we know it is not differentiable at $y = 1$, which happens when $x = \pm 2$, so $(-2, 2)$ is the largest open interval on which the solution is valid. The implicit curve is given below




```
>> clear all
>> syms y(t) logy0
>> eqn = diff(y,t) == - t*y/log(y);
>> cond = y(0) == exp(logy0);
>> yc = dsolve(eqn, cond);

>> yp = simplify(subs(yc, logy0, 2));
>> fplot(yp(1))
>> hold on
>> fplot(yp(2))
>> hold off
>> xlim([-3 3]);ylim([-1 8])

>> obj = xlabel('$t$', 'fontsize', 25);
>> set(obj,'Interpreter','Latex');
>> obj = ylabel('$y$', 'fontsize', 25,'rot',90);
>> set(obj,'Interpreter','Latex');
>> obj = title( ...
>>     '$\left(\ln y\right)^2 = 4 - t^2 $', ...
>>     'FontSize',20);
>> set(obj,'Interpreter','Latex');

>> clear all
>> f = @(t,y) t.^2 - 4 + (log(y)).^2 ;
>> fimplicit(f, [-3 3 -1 8])
```

Question5 (1 points)

Solve the following initial-value problem (IVP).

$$\dot{y} - 2y = f(t), \qquad y(0) = 1, \qquad \text{where} \qquad f(t) = \begin{cases} 1 - t, & \text{if} \quad t < 1, \\ 0, & \text{if} \quad t \geq 1. \end{cases}$$

Solution:

1M With the following integrating factor,

$$\mu = A \exp\left(\int -2\, dt\right) = \exp(-2t)$$

the given linear equation can be turned into one that can integrated directly

$$y = \frac{\displaystyle\int \mu\gamma\, dt}{\mu\alpha} = \begin{cases} \dfrac{\displaystyle\int (1-t)\exp(-2t)\, dt}{\exp(-2t)}, & \text{if} \quad t < 1, \\ \dfrac{\displaystyle\int 0\, dt}{\exp(-2t)}, & \text{if} \quad t \geq 1. \end{cases}$$

$$= \begin{cases} C_1 e^{2t} + \dfrac{1}{2}t - \dfrac{1}{4}, & \text{if} \quad t < 1, \\ C_2 e^{2t}, & \text{if} \quad t \geq 1. \end{cases}$$

Using the initial condition, we have

$$C_1 = \frac{5}{4}$$

Notice $f(t)$ is continuous, and

$$\dot{y} = f(t) + 2y$$

so $\dot{y}$ is continuous if we make y continuous at $t = 1$, this is achieved if we set

$$C_2 = \frac{5}{4} + \frac{1}{4}e^{-2}$$

Therefore, the particular solution for all t is

$$y(t) = \begin{cases} \dfrac{5}{4}e^{2t} + \dfrac{1}{2}t - \dfrac{1}{4}, & \text{if} \quad t < 1, \\ \left(\dfrac{5}{4} + \dfrac{1}{4}e^{-2}\right) e^{2t}, & \text{if} \quad t \geq 1. \end{cases}$$

Note the input $f(t)$ is not differentiable at $t = 1$, but that does not prevent the output $y(t)$ to be differentiable.

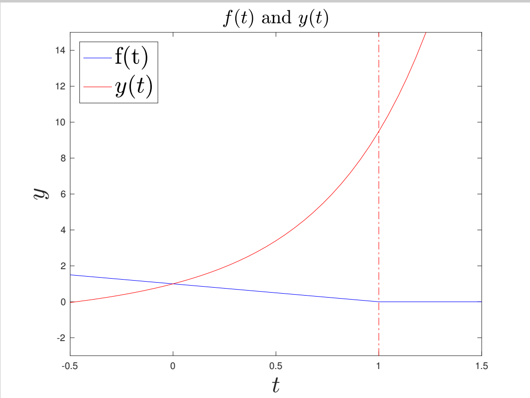
```
>> clear all
>> syms y(t)
>> eqn = diff(y,t) == 2*y + 1 - t;
>> cond = y(0) == 1;
>> y1 = dsolve(eqn, cond);

>> eqn = diff(y,t) == 2*y;
>> cond = y(1) == subs(y1,t,1);
>> y2 = dsolve(eqn, cond);

>> f(t) = piecewise(t<1, 1-t, t >=1, 0);
>> yp = piecewise(t<1, y1, t >=1, y2);
>> fplot(f,'b');
>> hold on
>> fplot(yp,'r');
>> hold off
>> xlim([-0.5 1.5]);ylim([-3 15])
>> obj = xlabel('$t$', 'fontsize', 25);
>> set(obj,'Interpreter','Latex');
>> obj = ylabel('$y$', 'fontsize', 25,'rot',90);
>> set(obj,'Interpreter','Latex');

>> obj = title( ...
>>    '$f(t)$ and $y(t)$', ...
>>    'FontSize',20);
>> set(obj,'Interpreter','Latex');

>> obj = legend({'f(t)','$y(t)$'}, ...
>>    'Location','northwest','FontSize',25);
>> set(obj,'Interpreter','Latex');
>> hold on
>> tt = linspace(1,1,15);
>> yy = linspace(-3,15,15);
>> obj = plot(tt,yy);
>> set(obj, 'color','red'); set(obj, 'LineStyle', '-.');
>> hold off
```



Question6 (1 points)

Test whether the following equation is exact.

$$\left(\arctan xy + \frac{xy - 2xy^2}{1 + x^2y^2}\right) + \frac{x^2 - 2x^2y}{1 + x^2y^2}\frac{dy}{dx} = 0$$

If so, find the general solution; if not, state the reason.

Solution:

1M Let $M = \arctan xy + \dfrac{xy - 2xy^2}{1 + x^2y^2}$ and $N = \dfrac{x^2 - 2x^2y}{1 + x^2y^2}$, it is exact if

$$M_y = N_x$$

by the conservative field test.

Computing both of them, we have

$$\begin{aligned}
\frac{\partial}{\partial y}\left(\arctan xy + \frac{xy-2xy^2}{1+x^2y^2}\right) &= \frac{x}{1+x^2y^2} + \frac{(x-4xy)(1+x^2y^2)-2x^2y(xy-2xy^2)}{(1+x^2y^2)^2} \\
&= \frac{2x(1-y)}{(1+x^2y^2)^2} \\
\frac{\partial}{\partial x}\left(\frac{x^2-2x^2y}{1+x^2y^2}\right) &= \frac{(2x-4xy)(1+x^2y^2)-2xy^2(x^2-2x^2y)}{(1+x^2y^2)^2} \\
&= \frac{2x(1-y)}{(1+x^2y^2)^2}
\end{aligned}$$

which shows that is is exact. Thus the following vector field is conservative,

$$\mathbf{F} = M\mathbf{e}_x + N\mathbf{e}_y$$

So the line integral

$$\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{r}, \qquad \text{where } \mathcal{C} \text{ is a curve from } (0,0) \text{ to } (x,y)$$

can be computed by using any path between $(0,0)$ and (x,y), and its value is equal to the difference between the potential function at (x,y) and $(0,0)$.

$$\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{r} = \Phi(x,y) - \Phi(0,0)$$

If we set the potential function at $(0,0)$ to zero, we have

$$\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{r} = \Phi(x,y)$$

Notice M along $y=0$ is zero, so by using the path consisting of x-axis and the vertical line $x=x$, we have

$$\begin{aligned}
\Phi(x,y) = \int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{r} &= \int_0^x M(u, y=0)\,du + \int_0^y N(x=x, v)\,dv \\
&= \int_0^x 0\,du + \int_0^y x^2\frac{1-2v}{1+x^2v^2}\,dv \\
&= x^2\left(\int_0^y \frac{1}{1+x^2v^2}\,dv - 2\int_0^y \frac{v}{1+x^2v^2}\,dv\right) \\
&= x^2\left(\frac{1}{x}\arctan(xy) - \frac{1}{x^2}\ln(1+x^2y^2)\right) \\
&= x\arctan(xy) - \ln(1+x^2y^2)
\end{aligned}$$

Therefore the general solution is given by

$$x\arctan(xy) - \ln(1+x^2y^2) = C$$

Question7 (1 points)

Solve the following equation using substitution.

$$y' = 2\left(\frac{y+2}{x+y-1}\right)^2$$

Solution:

1M Let $v = y+2$, then

$$\frac{dv}{dx} = 2\left(\frac{v}{x+v-3}\right)^2$$

if we let $u = x-3$, then

$$\frac{dv}{du} = 2\left(\frac{v}{u+v}\right)^2$$

which is a homogeneous equation, so by considering $z = \frac{v}{u}$, we have

$$z + u\frac{dz}{du} = 2\left(\frac{z}{1+z}\right)^2$$

which is separable, applying the formula, we have

$$\frac{(1+z)^2}{z(1+z^2)} \cdot \frac{dz}{du} = -\frac{1}{u}$$

$$\int \frac{(1+z)^2}{z(1+z^2)}\,dz = -\ln|u|$$

$$\ln|z| - 2\arctan\left(\frac{z+1}{z-1}\right) + C = -\ln|u|$$

$$\ln|y+2| = -2\arctan\frac{y+x-1}{y-x+5} + C$$

Question8 (1 points)
Consider a large vat that contains 100 gal of sugar water. The amount flowing in is the same as the amount flowing out, so there are always 100 gallons in the vat. Sugar water with concentration of 5g/gal enters the vat through pipe A at a rate of 2 gal/min. Sugar water with concentration of 10g/gal enters the vat through pipe B at a rate of 1 gal/min. Sugar water leaves the vat through pipe C at a rate of 3 gal/min. The vat is kept well mixed, so the sugar concentration is uniform throughout the vat. Find the concentration $C(t)$ of sugar in the vat at time t measured in g/gal. Assume the concentration of sugar in the vat is 0 g/gal initially.

Solution:

1M Let $S(t)$ denote the total amount of sugar in the vat at time t, and $C(t) = \frac{1}{100}S(t)$ be the concentration. The rate of change of S with respect to t is the difference between the amount of sugar being added and being removed. The amount of sugar being added is given by

$$\begin{aligned}\frac{dS}{dt} = \frac{d}{dt}(S_A + S_B) + \frac{dS_C}{dt} &= \frac{dS_A}{dW_A}\frac{dW_A}{dt} + \frac{dS_B}{dW_B}\frac{dW_B}{dt} + \frac{dS_B}{dW_B}\frac{dW_B}{dt} \\ &= 5 \cdot 2 + 10 \cdot 1 - \frac{S(t)}{100} \cdot 3 \\ &= 20 - \frac{3}{100}S(t) \\ \implies \dot{S} + \frac{3}{100}S = 20 \end{aligned}$$

where S_A, S_B and S_C are the amount of sugar due to pipe A, B and C, and W_A, W_B and W_C are the amount of water due pipe A, B and C. Solving this linear equation,

$$S(t) = Ce^{-0.03t} + \frac{2000}{3} \implies C(t) = \frac{A}{100}e^{-0.03t} + \frac{20}{3}$$

Applying the initial condition $C(0) = 0$, we get the constant $A = -\frac{20}{3}$

$$C(t) = -\frac{20}{3}e^{-0.03t} + \frac{20}{3}$$

Question9 (1 points)

There is a small boat moving from one side of the river to the other as the figure described. We know that the speed of the small boat v^* is bigger than the speed of the water v_0. And the boat always moving towards the destination O. The speed of the boat in stationary reference frame is the simple superposition of ship speed and water speed. Find the path of the boat using polar coordinates.

Solution:

1M Decomposing v_0 into two orthogonal components, one along $v_0 \cos\varphi$ and the other orthogonal $v_0 \sin\varphi$ to the direction that the boat is traveling, we have

$$\begin{aligned}\dot{r} &= -v^* + v_0 \cos\varphi \\ r\dot{\varphi} &= -v_0 \sin\varphi\end{aligned}$$

Using the chain rule, we have

$$\frac{dr}{d\varphi} = \frac{\dot{r}}{\dot{\varphi}} = r \cdot \frac{-v^* + v_0 \cos\varphi}{-v_0 \sin\varphi}$$

which is a separable equation,

$$\int \frac{1}{r}\, dr = \frac{v^*}{v_0}\int \frac{1}{\sin\varphi}\, d\varphi - \int \frac{1}{\tan\varphi}\, d\varphi$$

Integrating and using the initial conditions r_0 and φ_0, we have

$$r = r_0 \left(\frac{\tan\frac{\varphi}{2}}{\tan\frac{\varphi_0}{2}}\right)^{\frac{v^*}{v_0}} \cdot \frac{\sin\varphi_0}{\sin\varphi}$$