Vv156 Lecture 6

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October 11, 2016

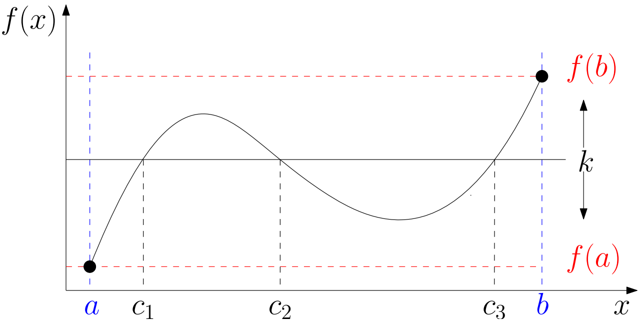
The Intermediate-Value theorem

If f is continuous on a closed interval $[a, b]$ and

$$k \text{ is any number strictly between } f(a) \text{ and } f(b), \text{ where } f(a) \neq f(b),$$

then there exits

$$\text{a number } c \text{ in the interval } (a, b) \text{ such that } f(c) = k.$$



Proof

- Let us assume

$$f(a) < k < f(b)$$

- Under this assumption, suppose $g(x) = f(x) - k$, then

$$g(a) < 0 \qquad \text{and} \qquad g(b) > 0$$

- To show there exists $a < c < b$ such that

$$f(c) = k,$$

  we need to show that there exist c such that

$$g(c) = 0 \qquad \text{for some} \quad a < c < b$$

- Let

$$\mathcal{S} = \Big\{ x \in [a, b] \mid g(x) < 0 \Big\}$$

Proof

- Note the set $\mathcal{S}$ is nonempty since $a \in \mathcal{S}$ and $\mathcal{S}$ is bounded from above by b.
- Since there is no gap in $\mathbb{R}$, then we take as given that $\sup(\mathcal{S})$ exist, and claim

$$c = \sup(\mathcal{S})$$

- To show this claim is correct, we need to show

$$g(x = c) = 0, \qquad \text{where} \quad c = \sup(\mathcal{S})$$

- Suppose $g(c) \neq 0$, since f and thus g is continuous at c, there exists $\delta > 0$

$$|g(x) - g(c)| < \frac{1}{2}|g(c)| \qquad \text{when} \quad |x - c| < \delta$$

- Now if $g(c) < 0$, then $c \neq b$ and for all x such that $|x - c| < \delta$.

$$g(c) + g(x) - g(c) < g(c) - \frac{1}{2}g(c) \iff g(x) < \frac{1}{2}g(c) < 0$$

Proof

- So the following is true, which means there are points $x \in \mathcal{S}$ bigger than c,

$$g(x) < 0 \qquad \text{for all } x \text{ such that} \quad |x - c| < \delta.$$

- That leads us to a contradiction of c being an upper bound of $\mathcal{S}$.
- Similarly, if $g(c) > 0$, then $c \neq a$ and for all x such that $|x - c| < \delta$

$$g(c) + g(x) - g(c) > g(c) - \frac{1}{2}g(c) \iff g(x) > \frac{1}{2}g(c) > 0$$

- Thus

$$g(x) > 0 \qquad \text{for all } x \text{ such that } |x - c| < \delta.$$

- It follows that there exists $\mu > 0$ such that $c - \mu \geq a$ and

$$g(x) > 0 \qquad \text{for} \quad c - \mu \leq x \leq c$$

- So $x = c - \mu$, which is less than c, is also an upper bound of $\mathcal{S}$, this leads to a contradiction of c being the least upper bound.

Proof

- The above forces us to conclude that

$$g(c) = 0$$

- Finally, $c \neq a$ and $c \neq b$ since g is nonzero at the endpoints

$$a < c < b$$

- If the original assumption is not true, that is, $f(b) < k < f(a)$ instead, then we apply the argument to the reflect of $y = f(x)$ about x-axis, and prove there is c

$$-f(c) = -k \quad \square$$

Q: Can you, in theory, slice a rain drop on a windscreen exactly in half no matter how irregular its shape with a single straight-line cut?

Q: In theory, can you always simultaneously slice two rain drops each exactly in half with a single straight-line cut, no matter the shapes of the rain drops nor their location on the windscreen.