Honors Calculus III

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- Recall for a functions of two variables $f(x,y)$ defined on a closed rectangle,

$$\begin{aligned} \mathcal{R} &= [a,b]\times[c,d] \\ &= \big\{(x,y) \mid a \le x \le b, c \le y \le d\big\} \\ \text{Subrectangles: } \mathcal{R}_{ij} &= [x_{i-1},x_i]\times[y_{j-1},y_j] \\ &= \big\{(x,y) \mid x_{i-1} \le x \le x_i,\ y_{j-1} \le y \le y_j\big\} \\ \text{Sample points: } & \quad \big\{(x_{11}^*,y_{11}^*),\dots,(x_{ij}^*,y_{ij}^*),\dots,(x_{mn}^*,y_{mn}^*)\big\} \end{aligned}$$

- The Riemann sum S_{mn} of $f(x,y)$ over $\mathcal{R}$

$$S_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*,y_{ij}^*)\Delta A_{ij}$$

$$S_\ell = \sum_{k=1}^{\ell} f(x_k^*,y_k^*)\Delta A_k$$

where $\Delta A_{ij} = \Delta x_i \Delta y_j$ and $\ell = m \times n$.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

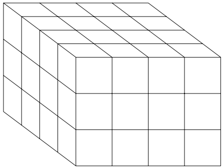
- A function of *3* variables $f(x,y,z)$ defined on a closed rectangular box,

$$\begin{aligned}\mathcal{B} &= [a,b]\times[c,d]\times[r,s]\\ &= \big\{(x,y,z)\mid a\le x\le b, c\le y\le d, r\le z\le s\big\}\\ \mathcal{B}_{ijk} &= [x_{i-1},x_i]\times[y_{j-1},y_j]\times[z_{k-1},z_k]\\ &= \big\{(x,y,z)\mid x_{i-1}\le x\le x_i, y_{j-1}\le y\le y_j, z_{k-1}\le z\le z_k\big\}\end{aligned}$$

Sample points: $\big\{(x^*_{111},y^*_{111},z^*_{111}),\ldots,(x^*_{ijk},y^*_{ijk},z^*_{ijk}),\ldots,(x^*_{\ell mn},y^*_{\ell mn},z^*_{\ell mn})\big\}$

- The Riemann sum $S_{\ell mn}$ of $f(x,y,z)$ over $\mathcal{B}$

$$\begin{aligned}S_{\ell mn} &= \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk},y^*_{ijk},z^*_{ijk})\Delta V_{ijk}\\ S_q &= \sum_{k=1}^{q} f(x^*_k,y^*_k,z^*_k)\Delta V_k\end{aligned}$$



where $\Delta V_{ijk}=\Delta x_i\Delta y_j\Delta z_k$ and $q=\ell\times m\times n$.

- If we consider the limit of the Riemann sum as usual, and it converges,

- Double integral

$$\iint\limits_{\mathcal{R}} f(x,y)\,dA = \lim_{m,n\to\infty} S_{mn} = \lim_{\ell\to\infty} S_{\ell}$$

- Triple integral

$$\iiint\limits_{\mathcal{B}} f(x,y,z)\,dV = \lim_{l,m,n\to\infty} S_{lmn} = \lim_{q\to\infty} S_{q}$$

- The limit processes above are the ones by which the number of subregions increases in such a way that the norm of the partition approaches 0.
- Similarly, a function $f(x,y,z)$ is called integrable if the limit actually exists and that its value does not depend on the choice of the tagged partition.
- The triple integral always exists if f is continuous over a simple region, and we will concern only those integrals.

Q: What does a triple integral represent geometrically?

- Just as for double integrals, the method for evaluating triple integrals over a rectangular region $\mathcal{B}$ is to express them as iterated integrals.

Fubini's theorem

If $f(x,y,z)$ is continuous in the rectangular region $\mathcal{B} = [a,b] \times [c,d] \times [r,s]$, then

$$\iiint\limits_{B} f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

- There are five other possible orders, all of which give the same value.

Exercise

Evaluate $\iiint\limits_{\mathcal{B}} 8xyz\,dV$, *where* $\mathcal{B} = [2,3] \times [1,2] \times [0,1]$.

Q: What is the essence of evaluating a double integral over Type-I/II regions?

- If $\mathcal{E}$ lies between the graphs of two continuous functions of 2 variables, i.e.

$$\mathcal{E}_1 = \left\{(x,y,z) \mid (x,y) \in \mathcal{D},\, u_1(x,y) \le z \le u_2(x,y)\right\}$$

- If $\mathcal{D}$ lies between the graphs of two continuous functions of one variable

$$\mathcal{D} = \left\{(x,y) \mid x \in [a,b],\, g_1(x) \le y \le g_2(x)\right\}$$

Then we can evaluate the following integral

$$\iiint_{\mathcal{E}_1} f(x,y,z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz\, dy\, dx$$

- Of course, $\mathcal{E}$ can be bounded by two continuous function of y and z,

$$\mathcal{E}_2 = \left\{(x,y,z) \mid (y,z) \in D,\, u_1(y,z) \le x \le u_2(y,z)\right\}$$

- and $\mathcal{E}$ can be bounded by two continuous function of x and z

$$\mathcal{E}_3 = \left\{(x,y,z) \mid (x,z) \in D,\, u_1(x,z) \le y \le u_2(x,z)\right\}$$

Exercise

(a) *Evaluate* $\iiint\limits_{\mathcal{E}} z\,dV$, *where* $\mathcal{E}$ *be the region in the first octant that bounded by the cylinder* $y^2 + z^2 = 1$ *and the planes* $y = x$ *and* $x = 0$.

(b) *Find the volume of the region* $\mathcal{E}$ *enclosed by the surfaces*

$$z = x^2 + 3y^2 \qquad \text{and} \qquad z = 8 - x^2 - y^2.$$

(c) *Set up all six iterated integrals for*

$$\iiint\limits_{\mathcal{E}} e^x(y + 2z)\,dV$$

where $\mathcal{E}$ *is the region bounded by the planes*

$$z = x + y, \qquad z = 0, \qquad y = 0, \qquad y = x \qquad \text{and} \qquad x = 2$$

- Consider the following integral,

$$\int_2^3 \int_{1-z}^0 \int_{-y^2-z^2}^{y^2+z^2} f(x,y,z)\, dx\, dy\, dz$$

Q: Suppose $f(x,y,z)$ is a density function, what is the above integral represent?

$$\iiint_{\mathcal{W}} f(x,y,z)\, dV$$

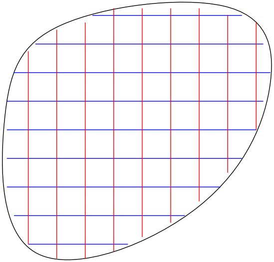
Q: Can you visualize the region $\mathcal{W}$?

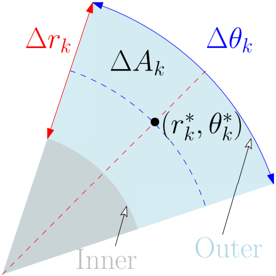
$$\begin{array}{rcccl} 2 & \leq & z & \leq & 3 \\ 1-z & \leq & y & \leq & 0 \\ -y^2-z^2 & \leq & x & \leq & y^2+z^2 \end{array}$$

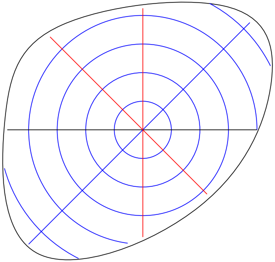
Q: Why is the following integral not making any sense?

$$\iiint_{\mathcal{W}} f(x,y,z)\, dV = \int_y^x \int_1^{3x} \int_0^1 f(x,y,z)\, dx\, dy\, dz$$

- When we encounter a cylinder, cone, or sphere, etc., we can often simplify our work by using cylindrical or spherical coordinates.
- The procedure for transforming to these two coordinates and evaluating the resulting triple integrals is similar to the procedure of using polar coordinates.
- Recall what we have done for the polar coordinates,







$$\Delta A_k = \Delta x_k \Delta y_k \qquad\qquad \Delta A_k' = r_k^* \Delta r_k \Delta\theta_k$$

- It allows us to convert double integrals into iterated integrals in polar form

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta x_k \Delta y_k = \iint_{\mathcal{D}} f(x,y)\ dA = \lim_{n\to\infty}\sum_{k=1}^{n} F(r_k^*, \theta_k^*) r_k^* \Delta r_k \Delta\theta_k$$

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)\ dx\, dy = \iint_{\mathcal{D}} f(x,y)\ dA = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} F(r,\theta) r\, dr\, d\theta$$

Integral in rectangular= Double Integral =Integral in polar

- The essential idea is the same, starting from some other ways of partitioning $\mathbb{R}^3$ into smaller regions instead of simple boxes.
- Moreover, we want figure how the volume of those small regions

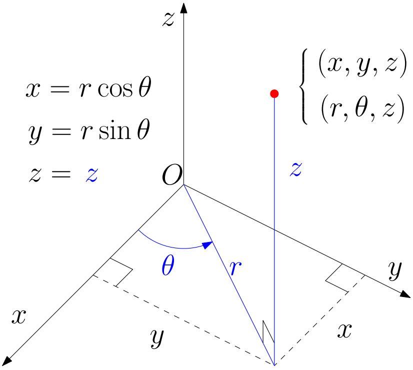
$$\Delta V_k$$

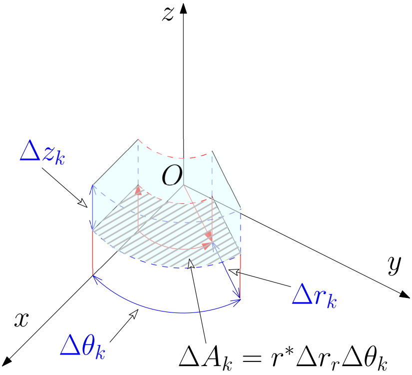
changes with respect to small changes of the new coordinates.

Definition

Cylindrical coordinates represent a point P in $\mathbb{R}^3$ by (r, θ, z) in which

1. r and θ are the polar coordinates for the projection P onto the xy-plane.
2. z is the Cartesian vertical coordinate.





- When computing triple integrals in cylindrical coordinates, we partition the region into n small cylindrical blocks, rather than into rectangular boxes.
- The volume of such a cylindrical block ΔV_k is obtained by taking the area ΔA_k of its base in terms of Δr_k and $\Delta \theta_k$ and multiplying by the height Δz_k

$$\Delta V_k = \Delta A_k \Delta z_k = r_k^* \Delta r_k \Delta \theta_k \Delta z_k$$

- Triple integral can be defined as a limit of Riemann sums using these blocks.

$$\iiint_{\mathcal{E}} f(x,y,z)\, dV = \begin{cases} \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta x_k \Delta y_k \Delta z_k \\ \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(r_k^* \cos\theta_k^*, r_k^* \sin\theta_k^*, z_k^*) r_k^* \Delta r_k \Delta \theta_k \Delta z_k \end{cases}$$

- Therefore in terms of iterated integrals, we have

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)\, dx\, dy\, dz = \int_{z_1}^{z_2}\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} F(r,\theta,z) r\, dr\, d\theta\, dz$$

Exercise

Find the iterated integral in cylindrical coordinates for integrating

$$f(r, \theta, z)$$

over the region $\mathcal{E}$ bounded above by the paraboloid

$$z = x^2 + y^2$$

and below by the plane $z = 0$, laterally by the circular cylinder

$$x^2 + (y-1)^2 = 1$$

Matlab

```
>> f_func = @(x,y,z) ...
0.* x + 0.*y + 0.* z + 1; xmin = 0; xmax = 2;
>> ymin = @(x) - sqrt(1-(x-1).^2);
>> ymax = @(x)  sqrt(1-(x-1).^2);
>> zmin = 0; % can be functions of two variables
>> zmax = @(x,y) x.^2 + y.^2 ;
>> q = integral3( f_func, xmin, xmax, ...
ymin, ymax, zmin, zmax)

q = 4.7124
```

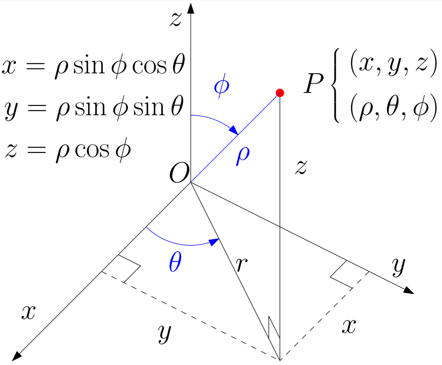
```
% In polar coordinates
>> f_polar_func = @(theta, r,z) ...
f_func( r.*cos(theta), r.*sin(theta), z).* r;
>> thetamin = 0; thetamax = pi;
>> rmin = 0; rmax = @(theta) 2.*sin(theta);
>> zmin = 0; zmax = @(theta, r) r.^2;
>> q = integral3( f_polar_func, thetamin,...
thetamax, rmin, rmax, zmin, zmax)

q = 4.7124
```

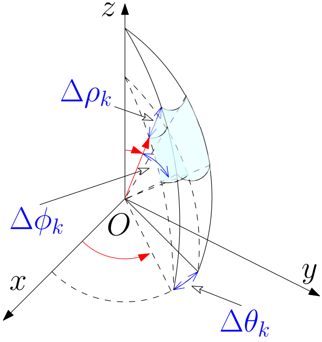
Definition

Spherical coordinates represent a point P in $\mathbb{R}^3$ by (ρ, θ, ϕ) in which

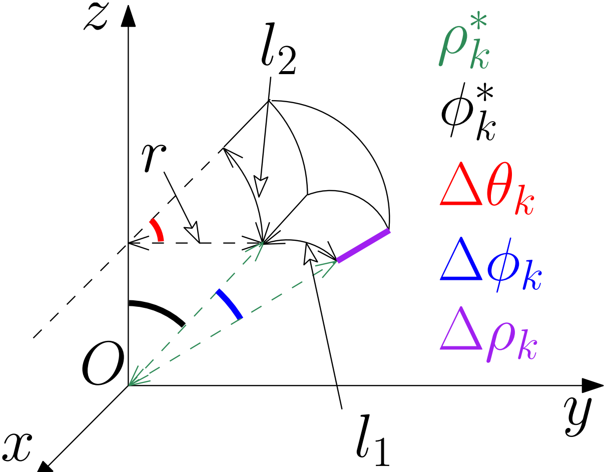
1. ρ is the distance between the point P to the origin O.
2. θ is the angular coordinate for the projection of P on the xy-plane.
3. ϕ is the angle $\overrightarrow{OP}$ makes with the positive z-axis.

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$





It is not difficult to work out the volume of the the spherical blocks in terms of spherical coordinates, and their increments.



Q: What is ℓ_1?

$$\ell_1 = \rho_k^* \Delta\phi_k$$

Q: What is r?

$$r = \rho_k^* \sin\phi_k^*$$

Q: What is ℓ_2?

$$\ell_2 = \rho_k^* \sin\phi_k^* \Delta\theta_k$$

Q: What is ΔV_k?

$$\Delta V_k \approx \ell_1 \times \ell_2 \times \Delta\rho_k = (\rho_k^* \Delta\phi_k)(\rho_k^* \sin\phi_k^* \Delta\theta_k)\Delta\rho_k = {\rho_k^*}^2 \sin\phi_k^* \Delta\phi_k \Delta\theta_k \Delta\rho_k$$

- Therefore the volume of such a spherical block ΔV_k can be approximated by

$$\Delta V_k \approx {\rho_k^*}^2 \sin\phi_k^* \Delta\phi_k \Delta\theta_k \Delta\rho_k$$

- Triple integral $\iiint\limits_{\mathcal{E}} f(x, y, z)\, dV$ can be defined as a limit of Riemann sums

$$\begin{cases} \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta x_k \Delta y_k \Delta z_k \\ \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(r_k^* \cos\theta_k^*, r_k^* \sin\theta_k^*, z_k^*) r_k^* \Delta r_k \Delta\theta_k \Delta z_k \\ \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(\rho_k^* \sin\phi_k^* \cos\theta_k^*, \rho_k^* \sin\phi_k^* \sin\theta_k^*, \rho_k^* \cos\phi_k^*) {\rho_k^*}^2 \sin\phi_k^* \Delta\phi_k \Delta\theta_k \Delta\rho_k \end{cases}$$

- Therefore in terms of iterated integrals, we have

$$\iiint\limits_{\mathcal{E}} f(x, y, z)\, dV = \int_{\rho_1}^{\rho_2} \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} F(\rho, \theta, \phi) \rho^2 \sin\phi \, d\phi\, d\theta\, d\rho$$

Coordinate Conversion Formulas

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\,dy\,dz \\ &= r\,dz\,dr\,d\theta \\ &= \rho^2\sin\phi\,d\rho\,d\theta\,d\phi \end{aligned}$$

Exercise

Find the volume of the "ice cream cone" $\mathcal{E}$ cut

from the solid sphere $\rho \leq 1$ by the cone $\phi = \pi/3$.