Vv256 Lecture 17

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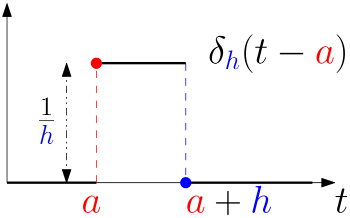
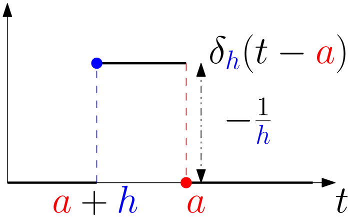
- Mechanical systems are often acted on by external forces of large magnitude that act only for a very short period of time. e.g. golf ball hit by a club.
- Such force can be modelled by step functions,

$$\delta_h(t-a) = \frac{u\Big(t-a\Big) - u\Big(t-(a+h)\Big)}{h}$$





Definition

This step function $\delta_h(t-a)$ is sometimes called a unit impulse function, since

$$\int_0^\infty \delta_h(t-a)\,dt = 1$$

Exercise

Find the Laplace transform of

$$\delta_h(t-a)$$

Solution

- By the definition of the unit impulse functiuon, we have

$$\begin{aligned}\mathcal{L}\left[\delta_h(t-a)\right] &= \mathcal{L}\left[\frac{u\Big(t-a\Big)-u\Big(t-(a+h)\Big)}{h}\right] \\ &= \frac{1}{h}\left(\mathcal{L}\left[1\cdot u\Big(t-a\Big)\right]-\mathcal{L}\left[1\cdot u\Big(t-(a+h)\Big)\right]\right)\end{aligned}$$

- By the alternative second translation theorem, we have

$$\mathcal{L}\left[\delta_h(t-a)\right] = \frac{e^{-as}-e^{-(a+h)s}}{hs} = e^{-as}\left(\frac{1-e^{-hs}}{hs}\right), \qquad s>0$$

Exercise

Solve the following initial-value problem

$$\ddot{y} + y = \delta_h(t - 2\pi), \qquad y(0) = 1, \qquad y'(0) = 0$$

Solution

- Taking the Laplace transform,

$$\begin{aligned}
\mathcal{L}[\ddot{y}] + \mathcal{L}[y] &= \mathcal{L}[\delta_h(t - 2\pi)] \\
s^2 Y(s) - s + Y(s) &= e^{-2\pi s}\left(\frac{1 - e^{-hs}}{hs}\right) \\
Y(s) &= \frac{s}{s^2 + 1} + \left[\frac{1}{s^2 + 1}\right] \cdot \left[e^{-2\pi s}\left(\frac{1 - e^{-hs}}{hs}\right)\right]
\end{aligned}$$

- Thus the solution according to the convolution theorem is given by

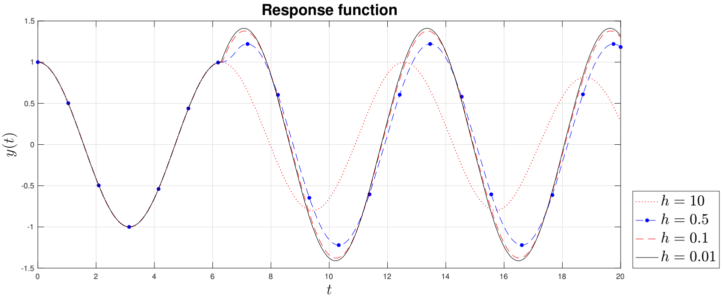
$$y = \cos t + \int_0^t \sin(\tau)\delta_h(t - \tau - 2\pi)\, d\tau$$

Solution

- Computing the integral, we have

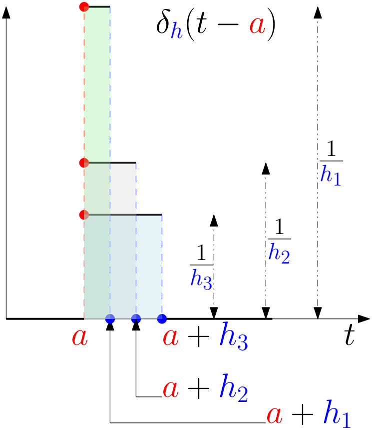
$$\begin{aligned} y &= \cos t + \int_0^t \sin(\tau)\delta_h(t-\tau-2\pi)\,d\tau \\ &= \cos t + \frac{1}{h}\Big[\Big(\cos(t-h)-1\Big)u\Big(t-(2\pi+h)\Big) - \Big(\cos(t)-1\Big)u\Big(t-2\pi\Big)\Big] \end{aligned}$$

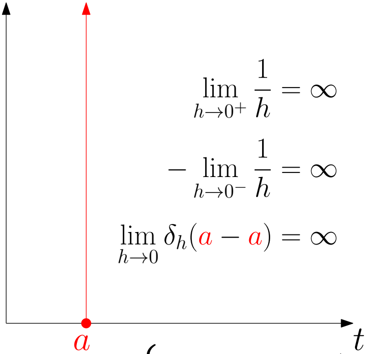
- The following shows how the solution changes as h approaches zero.



- In practice it is convenient to work with the limit of the unit impulse function

$$\delta(t-a) = \lim_{h\to 0} \delta_h(t-a)$$





$$\lim_{h\to 0^+} \frac{1}{h} = \infty$$

$$-\lim_{h\to 0^-} \frac{1}{h} = \infty$$

$$\lim_{h\to 0} \delta_h(a-a) = \infty$$

$$\delta(t-a) = \begin{cases} 0, & t \neq a \\ \text{undefined}, & t = a \end{cases}$$

Definition

This generalization $\delta(t-a) = \lim_{h\to 0} \delta_h(t-a)$ is called the Dirac delta function.

- The Dirac delta function $\delta(t-a)$ is not a function in the usual sense, and it is an idealization of what happens in practice. e.g. elastic collision.
- From the definition of $\delta_h(t-a)$, we can see that $\delta(t-a)$ is essentially,

$$\delta(t-a) = \lim_{h\to 0} \delta_h(t-a) = \begin{cases} \infty, & t = a \\ 0, & t \neq a \end{cases}$$

with the following property by construction,

$$\begin{aligned} \int_{-\infty}^{\infty} \delta(t-a)\,dt &= \int_{-\infty}^{\infty} \lim_{h\to 0} \delta_h(t-a)\,dt \\ &= \lim_{h\to 0} \int_{-\infty}^{\infty} \delta_h(t-a)\,dt = \lim_{h\to 0} 1 = 1 \end{aligned}$$

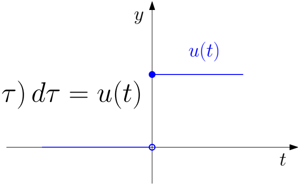
- Of course for an interval $[c,d]$ such that $c<a<d$, we have the same result

$$\int_c^d \delta(t-a)\,dt = \int_{-\infty}^{\infty} \delta(t-a)\,dt = 1$$

- In fact, it is also true for smaller and smaller interval around a, that is,

$$\begin{aligned} \int_{a^-}^{d} \delta(t-a)\,dt &= \lim_{c\to a^-} \int_c^d \delta(t-a)\,dt = 1 \\ \int_c^{a^+} \delta(t-a)\,dt &= 1 \\ \Longrightarrow \int_{a^-}^{a^+} \delta(t-a)\,dt &= 1 \end{aligned}$$

- In some sense $\dot{u}(t-a) = \delta(t-a)$ since $\displaystyle\int_{-\infty}^{t} \delta(\tau)\,d\tau = u(t)$



- With the definition we are using,

$$\delta(t-a) = \lim_{h\to 0} \delta_h(t-a)$$

- We have the following properties:

1. Because $\delta(t-a)$ is the limit of graphs of area 1, the area under its graph is 1

$$\int_c^d \delta(t-a)\,dt = \begin{cases} 1 & c < a < d, \\ 0 & \text{otherwise.} \end{cases}$$

2. For any continuous function $f(t)$, we have $f(t)\delta(t-a) = f(a)\delta(t-a)$ and

$$\int_c^d f(t)\delta(t-a)\,dt = \begin{cases} f(a) & c < a < d, \\ 0 & \text{otherwise.} \end{cases}$$

3. For a unit step function $u(t-a)$, the Dirac delta function

$$\delta(t-a) = u'(t-a)$$

is defined to be the generalized derivative.

4. Suppose it is valid to interchange the order of limiting operations,

$$\mathcal{L}\Big[\delta(t-a)\Big] = \mathcal{L}\Big[\lim_{h\to 0}\delta_h(t-a)\Big] = \lim_{h\to 0}\mathcal{L}\Big[\delta_h(t-a)\Big]$$

then we can obtain the Laplace transform of the Dirac delta function

Theorem

$$\mathcal{L}\Big[\delta(t-a)\Big] = e^{-sa} \qquad \text{where} \qquad a > 0$$

Proof

- Assuming it is valid to interchange the order of limiting operations, we have

$$\begin{aligned}\mathcal{L}\Big[\delta(t-a)\Big] = \lim_{h\to 0}\mathcal{L}\left[\delta_h(t-a)\right] &= \lim_{h\to 0} e^{-as}\left(\frac{1-e^{-hs}}{hs}\right), \qquad s>0 \\ &= e^{-as}\cdot 1 \\ &= e^{-as}, \qquad s>0\end{aligned}$$

according the L'hospital's rule.

Exercise

Consider the following IVP again, this time with the Dirac delta function instead.

$$\ddot{y} + y = \delta(t - 2\pi), \qquad y(0) = 1, \qquad y'(0) = 0$$

Solution

- Taking the Laplace transform, we have

$$s^2 Y(s) - s + Y(s) = e^{-2\pi s} \implies Y(s) = \frac{s}{s^2+1} + e^{-2\pi s}\frac{1}{s^2+1}$$

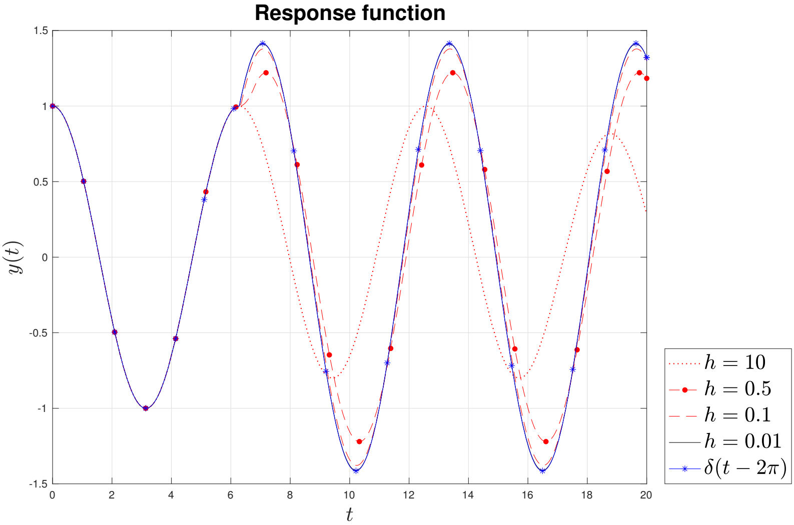
- Thus according to the convolution theorem, we have

$$y = \cos(t) + \int_0^t \delta(\tau - 2\pi)\sin(t-\tau)\,d\tau$$

- Applying property 4 of the Dirac delta function, we have

$$y(t) = \cos t + \sin(t-2\pi)u(t-2\pi) = \cos t + \sin(t)u(t-2\pi)$$

- Notice the solution to this IVP is fairly close to the cases with small h,



- So far, we have avoided having discontinuity at

$$t = 0$$

however, there is a need to distinguish the followings

$$0^-, \quad 0, \quad 0^+$$

when comes to the unilateral Laplace transform of a function in general.

- In general, we actually need the following

$$\mathcal{L}_-\Big[f(t)\Big] = \int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

when the singularity happens at $t = 0$ to avoid inconsistency, for example

$$\delta(t)$$