
Question1 (5 points)

(a) (1 point) Evaluate

$$\iint_{\mathcal{R}} \frac{x}{1+xy}\, dA,$$

where $\mathcal{R}$ is the rectangular region $\mathcal{R} = [0,1] \times [0,1]$.

Solution:

1M Applying Fubini's theorem, we have

$$\iint_{\mathcal{R}} \frac{x}{1+xy}\, dA = \int_0^1 \int_0^1 \frac{x}{1+xy}\, dy\, dx = \int_0^1 \ln(x+1)\, dx = 2\ln 2 - 1$$

(b) (1 point) Find the volume of the solid that lies under the paraboloid

$$z = x^2 + y^2$$

and above the region $\mathcal{D}$ in the xy-plane bounded by the line

$$y = 2x$$

and the parabola

$$y = x^2$$

Solution:

1M The region $\mathcal{D}$ is a type-I region,

$$y_1 = x^2 \qquad \text{and} \qquad y_2 = 2x$$

the upper and lower limits for x is given by

$$\left.\begin{aligned} y &= 2x \\ y &= x^2 \end{aligned}\right\} \implies x_1 = 0 \qquad \text{and} \qquad x_2 = 2$$

thus the volume is given by the following double integral by definition

$$\iint_{\mathcal{D}} \left(x^2 + y^2\right)\, dA$$

which can be converted into the following iterated integral, thus evaluated

$$\iint_{\mathcal{D}} \left(x^2 + y^2\right)\, dA = \int_0^2 \int_{x^2}^{2x} \left(x^2 + y^2\right)\, dy\, dx = \frac{216}{35}$$

(c) (1 point) Find the volume of the tetrahedron bounded by the planes

$$x + 2y + z = 2, \quad x = 2y, \quad x = 0, \quad \text{and } z = 0$$

Solution:

1M The volume is given by

$$\iint_{\mathcal{D}} (2 - x - 2y)\ dA$$

where the region $\mathcal{D}$ is bounded by

$$x = 0 \qquad \text{and} \qquad x = 2y$$

and

$$x + 2y + 0 = 2 \implies y = 1 - \frac{1}{2}x$$

Thus the region $\mathcal{D}$ is a type-I region,

$$y_1 = \frac{1}{2}x \qquad \text{and} \qquad y_2 = 1 - \frac{1}{2}x$$

and bounded on the left by $x = 0$ and on the right by $x = 1$ since

$$\left.\begin{aligned} y &= \frac{1}{2}x \\ y &= 1 - \frac{1}{2}x \end{aligned}\right\} \implies x = 1$$

Therefore the volume is equal to

$$\iint_{\mathcal{D}} (2 - x - 2y)\ dA = \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\ dy\, dx = \frac{1}{3}$$

(d) (1 point) Evaluate

$$\iint_{\mathcal{D}} \frac{\sin x}{x}\ dA,$$

where $\mathcal{D}$ is the triangular region

$$\{(x, y) \mid 0 \le y \le x,\ 0 \le x \le \pi\}$$

Solution:

1M This can be easily evaluated

$$\iint_{\mathcal{D}} \frac{\sin x}{x}\ dA = \int_0^{\pi} \int_0^{x} \frac{\sin x}{x}\, dy\, dx = \int_0^{\pi} \frac{\sin x}{x} \int_0^{x} dy\, dx = \int_0^{\pi} \sin x\, dx = 2$$

Notice this could be a problematic integral if we treat $\mathcal{D}$ as a type-II region.

(e) (1 point) If $f(x, y)$ is continuous on $[a, b] \times [c, d]$ and

$$g(x, y) = \int_a^x \int_c^y f(s, t)\ dt\ ds, \qquad \text{for } a < x < b,\ c < y < d.$$

Show that

$$g_{xy} = g_{yx} = f(x, y)$$

Solution:

1M Since f is continuous on this rectangular region, the function

$$g(x, y)$$

is a well defined function of x and y, and thus we can consider partial derivatives with respect to x and y, the fundamental theorem of calculus is applicable to partial integration as well, for fixing $y = y^*$, we just have a function of one variable in the plane $y = y^*$.

$$g_x = \frac{\partial}{\partial x} \int_a^x \int_c^y f(s, t)\, dt\, ds = \int_c^y f(x, t)\, dt$$

Similarly, we have

$$g_y = \frac{\partial}{\partial y} \int_a^x \int_c^y f(s, t)\, dt\, ds = \frac{\partial}{\partial y} \int_c^y \int_a^x f(s, t)\, ds\, dt = \int_a^x f(s, y)\, ds$$

Applying FTC, we have

$$g_{xy} = f(x, y) = g_{yx}$$

Question2 (5 points)

(a) (1 point) Let $f(x, y)$ be continuous. Find a single iterated integral that is equal to

$$\int_0^1 \int_0^{x^2} f(x, y)\, dy\, dx + \int_1^3 \int_0^{\frac{3-x}{2}} f(x, y)\, dy\, dx$$

by changing the order of integration.

Solution:

1M This is possible because the region is both type-I and type-II.

$$\int_0^1 \int_0^{x^2} f(x, y)\, dy\, dx + \int_1^3 \int_0^{\frac{3-x}{2}} f(x, y)\, dy\, dx = \int_0^1 \int_{\sqrt{y}}^{3-2y} f(x, y)\, dx\, dy$$

Such regions are known as simple regions.

(b) (1 point) Evaluate

$$\iint_{\mathcal{D}} \frac{1}{(1 + x^2 + y^2)^2}\, dA$$

where $\mathcal{D}$ is the region in the xy-plane outside the circle of radius 1 centred at the origin.

Solution:

1M In polar coordinates, we have

$$\iint_{\mathcal{D}} \frac{1}{(1 + x^2 + y^2)^2}\, dA = \int_0^{2\pi} \left(\lim_{\beta \to \infty} \int_1^{\beta} \frac{r}{(1 + r^2)^2}\, dr \right) d\theta = \int_0^{2\pi} \frac{1}{4}\, d\theta = \frac{\pi}{2}$$

(c) (1 point) Let $f(x)$ and $g(x)$ be continuous on $[a, b]$, and monotonically increasing.

$$\left[\int_a^b f(x)\,dx\right]\left[\int_a^b g(x)\,dx\right] \leq (b-a)\int_a^b f(x)g(x)\,dx$$

Show the above inequality is true.

Solution:

1M Let

$$\mathcal{D} = \{(x, y) \mid a \leq x \leq b, a \leq y \leq b\}$$

Consider the following two double integrals

$$\iint_{\mathcal{D}} f(x)g(y)\,dA = \left[\int_a^b f(x)\,dx\right]\left[\int_a^b g(y)\,dy\right] = \left[\int_a^b f(x)\,dx\right]\left[\int_a^b g(x)\,dx\right]$$
$$\iint_{\mathcal{D}} f(x)g(x)\,dA = (b-a)\int_a^b f(x)g(x)\,dx$$

Since both $f(x)$ and $g(x)$ is monotonically increasing on $[a, b]$, we have

$$I(x, y) = [f(x) - f(y)][g(x) - g(y)] \geq 0,$$

Integrate $I(x, y)$ over $\mathcal{D}$, and by the symmetry $\mathcal{D}$ in terms of x and y, we have

$$\iint_{\mathcal{D}} [f(x) - f(y)][g(x) - g(y)]\,dA \geq 0$$
$$\implies \iint_{\mathcal{D}} f(x)g(x)\,dA \geq \iint_{\mathcal{D}} f(x)g(y)\,dA$$

which implies the inequality that we need to show.

(d) (2 points) Find the value of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

at $s = 2$ by considering the values of

$$\int_0^{\pi/2}\int_0^1 \frac{\sin\theta}{\sqrt{1 - x^2\sin^2\theta}}\,dx\,d\theta \qquad \text{and} \qquad \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

Solution:

0M Let $u = g(x) = x\sin\theta$, and use u-substitution, then $g'(x) = \sin\theta$ and

$$\int_0^{\pi/2}\int_0^1 \frac{\sin\theta}{\sqrt{1 - x^2\sin^2\theta}}dx d\theta = \int_0^{\pi/2}\int_0^{\sin\theta} \frac{1}{\sqrt{1-u^2}}dx d\theta$$
$$= \int_0^{\pi/2} \sin^{-1}(\sin\theta)d\theta = \frac{\pi^2}{8}$$

1M If $y=\sqrt{1-x^2}$, then $y^2+x^2=1$ and

$$1-x^2\sin^2\theta=y^2+x^2\cos^2\theta$$

Now consider

$$h(x)=\int_0^{\pi/2}\frac{\sin\theta}{\sqrt{1-x^2\sin^2\theta}}d\theta$$

Let $t=x\cos\theta$.

$$h(x)=\frac{1}{x}\int_0^x\frac{1}{\sqrt{y^2+t^2}}dt=\frac{1}{xy}\int_0^x\frac{1}{\sqrt{1+\left(\frac{t}{y}\right)^2}}dt$$

Use u-substitution with $\tan u=\frac{t}{y}$.

$$h(x)=\frac{1}{x}\int_0^{\arctan(x/y)}\frac{\sec^2 u}{\sec u}du=\frac{1}{x}\ln\left(\frac{x}{y}+\sqrt{1+\left(\frac{x}{y}\right)^2}\right)$$

Notice that $1+\left(\frac{x}{y}\right)^2=\frac{1}{y^2}$.

$$h(x)=\frac{1}{x}\ln\left(\frac{1+x}{\sqrt{1-x^2}}\right)=\frac{1}{2x}\ln\left(\frac{1+x}{1-x}\right)=\frac{1}{2x}\left(\ln(1+x)-\ln(1-x)\right)$$

Use Taylor Series, we know that

$$\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$
$$\ln(1-x)=-\sum_{n=1}^{\infty}\frac{x^n}{n}$$

Thus, we know that

$$h(x)=\sum_{n=0}^{\infty}\frac{x^{2n}}{2n+1}$$

Then we evaluate the outer integral term by term which means

$$\int_0^1 h(x)dx=\int_0^1\left(\sum_{n=0}^{\infty}\frac{x^{2n}}{2n+1}\right)dx=\sum_{n=0}^{\infty}\left(\int_0^1\frac{x^{2n}}{2n+1}dx\right)$$
$$=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

1M Combine the result from the last question, we know that

$$\frac{\pi^2}{8}=\int_0^{\pi/2}\int_0^1\frac{\sin\theta}{\sqrt{1-x^2\sin^2\theta}}dx\,d\theta=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=1}^{\infty}\frac{1}{(2n)^2}$$

In this way, we can calculate the

$$\zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{4}{3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{6}$$

Question3 (5 points)

(a) (1 point) Find the surface area of the part of the surface

$$z = x^2 + 2y$$

that lies above the triangle in the xy-plane with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

Solution:

1M By definition, the surface area is given by

$$\begin{aligned} S &= \iint_{\mathcal{D}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \\ &= \int_0^1 \int_0^x \sqrt{1 + 4x^2 + 4}\, dy\, dx \\ &= \int_0^1 x\sqrt{1 + 4x^2 + 4}\, dx = \frac{9}{4} - \frac{5\sqrt{5}}{12} \end{aligned}$$

(b) (1 point) Find the area of the part of the sphere

$$x^2 + y^2 + z^2 = 4z$$

that lies inside the paraboloid

$$z = x^2 + y^2$$

Solution:

1M The intersection is given by

$$z(z-3) = 0 \implies z_1 = 0 \qquad \text{and} \qquad z_2 = 3$$

thus the region over which we shall integrate is

$$\mathcal{D} = \left\{(x,y) \mid x^2 + y^2 \le 3\right\}$$

The sphere is centred at $(0,0,2)$ and has a radius of 2

$$x^2 + y^2 + (z-2)^2 = 2^2$$

The partial derivatives are given by

$$\begin{aligned} \frac{\partial z}{\partial x} &= -\frac{F_x}{F_z} = \frac{x}{2-z} \\ \frac{\partial z}{\partial y} &= -\frac{F_y}{F_z} = \frac{y}{2-z} \end{aligned}$$

The surface area is given by

$$\begin{aligned} S &= \iint_{\mathcal{D}} \sqrt{1 + \frac{x^2}{(2-z)^2} + \frac{y^2}{(2-z)^2}}\, dA \\ &= \iint_{\mathcal{D}} \sqrt{1 + \frac{x^2+y^2}{4-(x^2+y^2)}}\, dA \\ &= \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{1 + \frac{r^2}{4-r^2}}\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{\frac{4}{4-r^2}}\, r\, dr\, d\theta = 4\pi \end{aligned}$$

(c) (1 point) Find the area of the portion of the paraboloid

$$\mathbf{r}(u,v) = \begin{bmatrix} u\cos v \\ u\sin v \\ u^2 \end{bmatrix},$$

for which $1 \le u \le 2$ and $0 \le v \le 2\pi$.

Solution:

1M When the surface is defined parametrically

$$\mathbf{r}(u,v)$$

there is an easy way to find the surface area. Since the partial derivatives

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial u} &= \cos(v)\mathbf{e}_x + \sin(v)\mathbf{e}_y + 2u\mathbf{e}_z \\ \frac{\partial \mathbf{r}}{\partial v} &= -u\sin(v)\mathbf{e}_x + u\cos(v)\mathbf{e}_y \end{aligned}$$

gives the tangent vectors in direction of increasing u and v,

$$\begin{aligned} S &= \int_1^2 \int_0^{2\pi} \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| dv\, du \\ &= \int_1^2 \int_0^{2\pi} u\sqrt{1+4u^2}\, dv\, du = \frac{\left(17\sqrt{17} - 5\sqrt{5}\right)\pi}{6} \end{aligned}$$

(d) (2 points) Let $\mathcal{D}$ be the half-annulus

$$9 \le x^2 + y^2 \le 16 \qquad \text{where} \quad y \ge 0.$$

Suppose we have a lamina whose shape is $\mathcal{D}$ and has uniform density. Find the centroid.

Solution:

1M Symmetry suggests that

$$\bar{x} = 0$$

Area $= (16\pi - 9\pi)/2 = 7\pi/2$. The moment about the x-axis and $\bar{y}$ are

$$\iint_D x\, dA \qquad \bar{y} = \frac{1}{\text{Area}} \iint_D x\, dA$$

Convert the integral into polar,

$$\int_0^{\pi} \int_3^4 r^2 \sin\theta\, dr\, d\theta$$

1M Evaluate the integral to find

$$\bar{y} = \frac{74}{3} \cdot \frac{2}{7\pi} = \frac{148}{21\pi}$$