Vv156 Lecture 4

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- Unlike for sequences, there are several types of limits for a function

$$y = f(x)$$

- We want to know the behaviour of f near a point $x = a$ or near infinity, e.g.

  We may be interested in knowing the behaviour of average speed near $t = 3$

$$\text{Average Speed} = \frac{\text{Distance travelled}}{\text{Time interval}} = \frac{s(t+\delta t) - s(t)}{\delta t}$$

- For an object that is dropped and falls straight down towards earth when the resistance of air is neglected, we have

$$s = \frac{1}{2}gt^2, \qquad \text{where} \quad g \approx 10\text{m/s}^2$$

δt	1.0000	0.5000	0.0100	0.0050	0.0001	0.00005
$s(3+\delta t) - s(3)$	35.0000	16.2500	0.3005	0.1501	0.0030	0.0015
Average Speed	35.0000	32.5000	30.0500	30.0250	30.0005	30.0002

Definition

The value L is the limit of $f(x)$ as x approaches a,

$$\lim_{x \to a} f(x) = L$$

if the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a, but not equal to a.

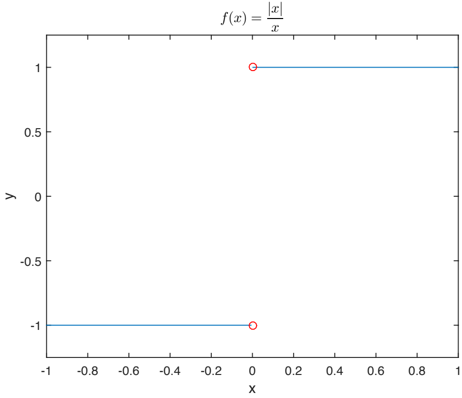
- There are two ways that x can approach a, from the left or from the right

$$\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

0.5000	0.0100	0.0050	0.0001	0.00005	0.00005	0.0001	0.0050	0.0100	0.5000
13.7500	0.2995	0.1499	0.0030	0.0015	0.0015	0.0030	0.1501	0.3005	16.2500
27.5000	29.9500	29.9750	29.9995	29.9997	30.0002	30.0005	30.0250	30.0500	32.5000

- The two-sided limit exists if and only if the one-sided limits exist and are equal.

- For example, consider $\lim_{x \to 0} \frac{|x|}{x}$



Matlab

```
>> syms x; ezplot('abs(x)/x',[-1,1]);
>> hold on; plot(0,1,'ro'); plot(0,-1,'ro'); hold off;
>> xlabel('x'); ylabel('y');
```

- The limit concerns the value of dependent variable y, $y = f(x)$, as the value of the independent variable x gets

closer and closer to a rather than the value of y at $x = a$.

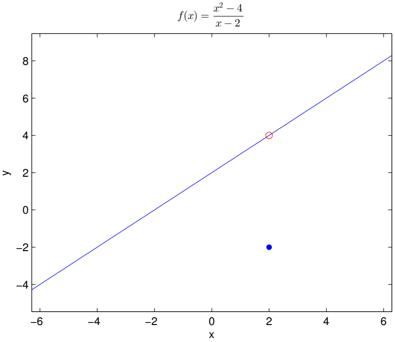
- It is clear that

$$f(x = 2) = -2$$

$$\lim_{x \to 2} f(x) = 4$$

Matlab

```
>> syms x;
>> ezplot('x+2');
>> hold on;
>> plot(2,4, 'ro');
>> plot(2,-2,'b.', 'MarkerSize', 15);
>> hold off;
>> xlabel('x'); ylabel('y');
```



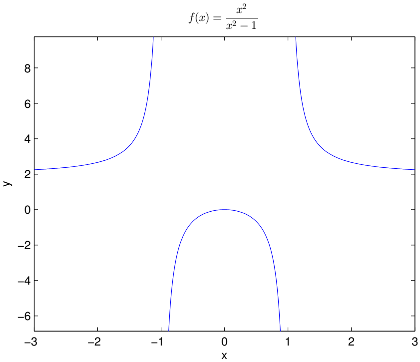
- Limits that are infinite and limits at infinity, for example, consider the following

- Approaches ∞ or $-\infty$

$$\lim_{x\to\infty} \frac{x^2}{x^2-1} = 1$$
$$\lim_{x\to-\infty} \frac{x^2}{x^2-1} = 1$$

- Approahces to ∞ or $-\infty$

$$\lim_{x\to 1^-} \frac{x^2}{x^2-1} = -\infty$$
$$\lim_{x\to 1^+} \frac{x^2}{x^2-1} = \infty$$
$$\lim_{x\to -1^-} \frac{x^2}{x^2-1} = \infty$$
$$\lim_{x\to -1^+} \frac{x^2}{x^2-1} = -\infty$$



Matlab

```
>> syms x
>> num = 2*power(x,2); denom = power(x,2) -1;
>> f = num/denom;
>> ezplot(f,[-3 3])
>> xlabel('x'); ylabel('y');
```

Limit Laws

Assume that $\lim_{x \to a} f(x) = K$ and $\lim_{x \to a} g(x) = L$, and that c is constant,

1 The limit of a constant is the constant itself.

$$\lim_{x \to a} c = c$$

2 The limit of a sum/difference is the sum/difference of the limits.

$$\lim_{x \to a} \left[f(x) \pm g(x) \right] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = K \pm L$$

3 The limit of a product is the product of the limits.

$$\lim_{x \to a} \left[f(x) g(x) \right] = \left[\lim_{x \to a} f(x) \right] \left[\lim_{x \to a} g(x) \right] = KL$$

4 The limit of a quotient is the quotient of the limits.

$$\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{K}{L}, \qquad \text{provided } \lim_{x \to a} g(x) \neq 0$$

Limit Laws

Assume that $\lim_{x \to a} f(x) = K$ and $\lim_{x \to a} g(x) = L$, and that c is constant,

5 If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

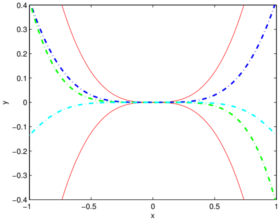
6 If $f(x) = g(x)$ for all x near a, possibly except at $x = a$, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x), \qquad \text{provided the limits exist}$$

The Squeeze Theorem

If $g(x) \leq f(x) \leq h(x)$ when x is near a, except possibly at a, and if

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L, \quad \text{then} \lim_{x \to a} f(x) = L$$



Exercise

(a) Evaluate the following limits

$$\lim_{x\to 1}\frac{x^2-1}{x-1} \qquad\qquad \lim_{x\to 1}\frac{2x^2+6x}{x^2-9}$$

(b) Use the squeeze theorem to show $\lim_{x\to 0} x\sin\frac{1}{x}=0$

- The following precise definition of limit removes any vagueness in the definition.

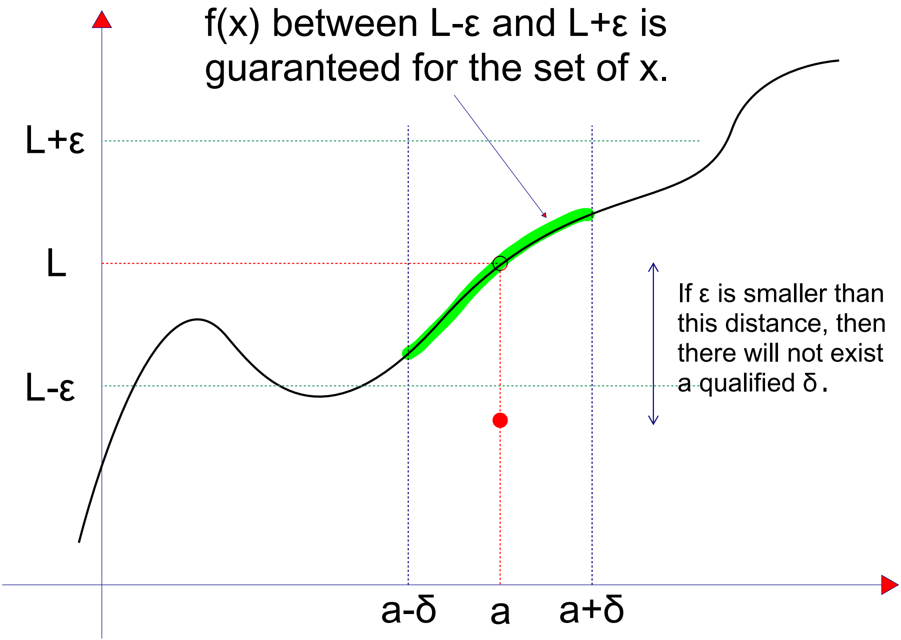
Epsilon-Delta definition of limit

Let f be a function defined on some open interval that contains the number a, except possibly at a itself. The value of L is the limit of $f(x)$ as x approaches a,

$$\lim_{x\to a} f(x)=L$$

if for every number $\epsilon>0$ there is a number $\delta>0$ such that

$$|f(x)-L|<\epsilon \qquad \text{if} \qquad 0<|x-a|<\delta$$


f(x) between L-ε and L+ε is guaranteed for the set of x.
L+ε
L
L-ε
If ε is smaller than this distance, then there will not exist a qualified δ.
a-δ
a
a+δ

- This precise definition of limit removes any vagueness, and thus can be used to prove or establish results or theorems regarding limits.
- For example, consider

$$\lim_{x \to -1} (x^2 + 3) = 4$$

- For every $\epsilon > 0$, we need to find $\delta > 0$ (which depends on ϵ) such that

$$|f(x) - 4| < \epsilon \qquad \text{if} \qquad 0 < |x - (-1)| = |x + 1| < \delta$$

- Since δ is the upper bound of $|x + 1|$, we need to know how $|x + 1|$ behaves.
- Specifically, we need to find the upper bound in terms of ϵ.
- This can be done by investigating what leads to $|f(x) - 4| < \epsilon$

$$\begin{aligned} |f(x) - 4| < \epsilon \qquad & \text{if and only if} & |x^2 + 3 - 4| < \epsilon \\ & \text{if and only if} & |x^2 - 1| < \epsilon \\ & \text{if and only if} & |(x - 1)(x + 1)| < \epsilon \\ & \text{if and only if} & |x - 1||x + 1| < \epsilon \end{aligned}$$

- So we have

$$|f(x) - 4| < \epsilon \quad \text{if and only if} \quad |x-1||x+1| < \epsilon$$

- We will now "replace" the term $|x-1|$ with an appropriate constant and keep the term $|x+1|$ since $|x+1|$ is what we are after.
- To do so, we will arbitrarily assume that $\delta \leq 1$. This is a valid assumption to make since, in general, once we find a δ that works, all smaller values of δ also work.

  Based on this assumption, then

$$\begin{aligned} |x+1| < \delta \leq 1 &\implies |x+1| < 1 \\ &\implies -1 < x+1 < 1 \\ &\implies -2 < x < 0 \\ &\implies 1 < |x-1| < 3 \end{aligned}$$

- Now if we combine $|x-1|<3$ with the result

$$|f(x)-4|<\epsilon \quad \text{if and only if} \quad |x-1||x+1|<\epsilon,$$

then we know

$$|f(x)-4|<\epsilon \quad \text{if and only if} \quad (3)|x+1|<\epsilon$$

- This means an upper bound of $\frac{\epsilon}{3}$ for $|x+1|$ will guarantee

$$|f(x)-4|<\epsilon, \qquad \text{provided that } \delta \leq 1.$$

- Therefore choosing $\delta=\min\{1,\frac{\epsilon}{3}\}$ will guarantee both assumptions made about δ in the course of this proof are simultaneously taken into account, and will guarantee that

$$|f(x)-4|<\epsilon \qquad \text{if} \qquad 0<|x+1|<\delta$$

for all $\epsilon>0$. $\square$

- More importantly, the Epsilon-Delta definition is used to establish limit laws.
- For example, for the following limit laws:

1. $\lim\limits_{x \to a} c = c$
2. $\lim\limits_{x \to a} cf(x) = c \lim\limits_{x \to a} f(x) = cK$, where $K = \lim\limits_{x \to a} f(x)$.

Proof

- For the first part, let $f(x)$ be the constant function, that is $f(x) = c$. We need to show that, for every number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$|f(x) - c| < \epsilon \qquad \text{if} \qquad 0 < |x - a| < \delta$$

- The left inequality is always satisfied for any x since $f(x) = c$.

Thus for any $\epsilon > 0$, not only there is a number $\delta > 0$ such that

$$|f(x) - c| < \epsilon \qquad \text{if} \qquad 0 < |x - a| < \delta$$

actually every $\delta > 0$ is fine.

Proof

- For the second part, if $c = 0$ then $cf(x) = 0$, and $\lim_{x \to a} [0f(x)] = \lim_{x \to a} 0$.

  It reduces to a special case of limit law 1., with $c = 0$. Therefore we know 2. is true for $c = 0$ and so we can assume that $c \neq 0$ for the remainder of this proof.

- Suppose $\epsilon > 0$, then $\frac{\epsilon}{|c|} > 0$. Because $\lim_{x \to a} f(x) = K$ by the precise definition of the limit there exists a $\delta_1 > 0$ such that,

$$|f(x) - K| < \frac{\epsilon}{|c|} \qquad \text{if} \qquad 0 < |x - a| < \delta_1$$

  Now suppose $\delta = \delta_1$ is a valid choice, to finish we need to show that

$$|cf(x) - cK| < \epsilon \qquad \text{if} \qquad 0 < |x - a| < \delta$$

  Assume that $0 < |x - a| < \delta$, then $0 < |x - a| < \delta_1$, which means

$$|f(x) - K| < \frac{\epsilon}{|c|} \implies |c||f(x) - K| < \epsilon \implies |cf(x) - cK| < \epsilon \quad \square$$

The Squeeze Theorem

If $g(x) \leq f(x) \leq h(x)$ when x is near a, except possibly at a, and if

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L, \quad \text{then} \quad \lim_{x \to a} f(x) = L$$

Proof

- Suppose that $g(x) \leq f(x) \leq h(x)$ for all $x \neq a$ in near a and also that

$$\lim_{x \to a} g(x) = L = \lim_{x \to a} h(x) = L$$

- Let $\epsilon > 0$. Since

$$\lim_{x \to a} g(x) = L$$

- there is $\delta_g > 0$ so that

$$|g(x) - L| < \epsilon \qquad \text{if} \qquad 0 < |x - a| < \delta_g$$

- Notice that this implies that

$$-\epsilon < g(x) - L < \epsilon \implies g(x) > L - \epsilon$$

Proof

- Similarly, since

$$\lim_{x \to a} h(x) = L$$

- there is $\delta_h > 0$ so that

$$|h(x) - L| < \epsilon \qquad \text{if} \qquad 0 < |x - a| < \delta_h$$

  and this implies that $h(x) < L + \epsilon$.

- Let $\delta = \min\{\delta_g, \delta_h\}$ and $0 < |x - a| < \delta$. Then

$$L - \epsilon < g(x) \leq f(x) \leq h(x) < L + \epsilon$$

$$L - \epsilon < f(x) < L + \epsilon$$

- Hence, $|f(x) - L| < \epsilon$. Since $\epsilon > 0$ is arbitrary, $\lim_{x \to a} f(x) = L$. □

- We can definite limit at infinity and infinite limit precisely. For example,

Definition

Let f be function defined on some open interval that contains the number a, except possibly at a itself. Then the limit of $f(x)$ approaches infinity, written as,

$$\lim_{x\to a} f(x) = \infty \qquad \text{or} \qquad f(x) \to \infty \quad \text{as} \quad x \to a$$

if for every number $M > 0$ there exists a number $\delta > 0$ such that

$$f(x) > M \qquad \text{if} \qquad 0 < |x - a| < \delta$$

Definition

Let f be function defined on some open interval (a, ∞). Then the limit of $f(x)$

$$\lim_{x\to \infty} f(x) = L \qquad \text{or} \qquad f(x) \to L \quad \text{as} \quad x \to \infty$$

if for every number $\epsilon > 0$ there exists a number $M \in (a, \infty)$ such that

$$|f(x) - L| < \epsilon \qquad \text{if} \qquad x > M$$

- Many of our limits laws can be modified to accommodate infinity. For example,

Theorem

Suppose $f(x)$ and $g(x)$ are two functions such that

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a} g(x) = L$$

1. The limit of the sum/difference is infinity

$$\lim_{x\to a} [f(x) \pm g(x)] = \infty$$

2. The limit of the product is infinity if $L > 0$ and negative infinity if $L < 0$

$$\lim_{x\to a} [f(x)g(x)] = \pm\infty$$

3. The limit of the quotient is infinity if $L > 0$ and negative infinity if $L < 0$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \pm\infty \qquad \text{and} \qquad \lim_{x\to a} \frac{g(x)}{f(x)} = 0$$

Proof

- For $M > 0$, we know there exists δ_f such that if $0 < |x - a| < \delta_f$, we have

$$f(x) > \frac{2M}{L}$$

- We also know that there exists δ_g such that if $0 < |x - a| < \delta_g$, we have

$$0 < |g(x) - L| < \frac{L}{2} \implies \frac{L}{2} < g(x) < \frac{3L}{2}$$

- Now let $\delta = \min\{\delta_f, \delta_g\}$, so if $0 < |x - a| < \delta$ we know from the above,

$$f(x) > \frac{2M}{L} \qquad \text{and} \qquad g(x) > \frac{L}{2}$$

- This gives us

$$f(x)g(x) > \left(\frac{2M}{L}\right)\left(\frac{L}{2}\right) = M \quad \square$$

- All the limit laws hold when the limits are taken as $x \to \infty$ instead of $x \to a$.

Theorem

If r is a positive rational number, then $\lim\limits_{x\to\infty} \dfrac{1}{x^r} = 0$

Proof

- For every $\epsilon > 0$, we need to show that there exists a number M such that

$$\left|\frac{1}{x^r} - 0\right| < \epsilon \qquad \text{when} \quad x > M$$

- We know the root $\sqrt[r]{\frac{1}{\epsilon}}$ will exist since ϵ is positive, if we let $x > M = \sqrt[r]{\frac{1}{\epsilon}}$, then

$$x > \sqrt[r]{\frac{1}{\epsilon}} \implies x^r > \frac{1}{\epsilon} \implies \frac{1}{x^r} < \epsilon \implies \left|\frac{1}{x^r} - 0\right| < \epsilon \quad \square$$

Theorem

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n, then

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_n x^n$$

Exercise

(a) Find $\displaystyle\lim_{x\to\infty} \frac{x^2+5x+1}{2x^2-10}$

Theorem

The limit of a rational function as $x \to \infty$ is the limit of the quotient of the terms of highest degree in the numerator and the denominator as $x \to \infty$.

Exercise

(b) Find $\displaystyle\lim_{x\to\infty} \frac{3x^2+1}{4x^3+2x+1}$