Vv417 Lecture 17

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- Suppose $\mathcal{V}$ is a finite-dimensional vector space, and

$$\text{both } \mathcal{B} \text{ and } \mathcal{B}' \text{ are bases for } \mathcal{V}.$$

and let $\mathbf{v}$ be a vector in $\mathcal{V}$, and

$$[\mathbf{v}]_{\mathcal{B}} \text{ and } [\mathbf{v}]_{\mathcal{B}'} \text{ are the coordinate vectors for } \mathbf{v}$$

relative to the bases $\mathcal{B}$ and $\mathcal{B}'$, respectively.

Q: How are the vectors $[\mathbf{v}]_{\mathcal{B}}$ and $[\mathbf{v}]_{\mathcal{B}'}$ related?

- For simplicity, we will consider the relation for two-dimensional spaces, but the same ideas applies to other finite-dimensional spaces.
- Suppose we have two bases,

$$\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2\} \qquad \text{and} \qquad \mathcal{B}' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$$

and let us call them the old and new bases, respectively.

- Let the coordinate vector of the old basis vectors relative to the new basis be

$$[\mathbf{u}_1]_{\mathcal{B}'} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad [\mathbf{u}_2]_{\mathcal{B}'} = \begin{bmatrix} c \\ d \end{bmatrix}$$

- That is, $\mathbf{u}_1$ and $\mathbf{u}_2$ are linear combinations of $\mathbf{u}_1'$ and $\mathbf{u}_2'$, so

$$\mathbf{u}_1 = a\mathbf{u}_1' + b\mathbf{u}_2'$$
$$\mathbf{u}_2 = c\mathbf{u}_1' + d\mathbf{u}_2'$$

- Suppose $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$ is the old coordinate vector, that is,

$$\mathbf{v} = k_1\mathbf{u}_1 + k_2\mathbf{u}_2$$

- To find the new coordinates of $\mathbf{v}$, we must find $\mathbf{v}$ in terms of the new basis.

$$\mathbf{v} = k_1(\underbrace{a\mathbf{u}_1' + b\mathbf{u}_2'}_{\mathbf{u}_1}) + k_2(\underbrace{c\mathbf{u}_1' + d\mathbf{u}_2'}_{\mathbf{u}_2}) = (k_1a + k_2c)\mathbf{u}_1' + (k_1b + k_2d)\mathbf{u}_2'$$

- Thus, the new coordinate vector for $\mathbf{v}$ is

$$[\mathbf{v}]_{\mathcal{B}'} = \begin{bmatrix} k_1 a + k_2 c \\ k_1 b + k_2 d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} [\mathbf{v}]_{\mathcal{B}}$$

- This states that the new coordinate vector $[\mathbf{v}]_{\mathcal{B}'}$ results when we multiply the old coordinate vector $[\mathbf{v}]_{\mathcal{B}}$ on the left by the matrix

$$\mathbf{P} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Q: Notice the columns of this matrix, what are they?

- The coordinate vectors for the old basis vectors relative to the new basis.

Q: Why this isn't surprising at all?

$$T\left([\mathbf{u}+\mathbf{v}]_{\mathcal{B}}\right) = [\mathbf{u}+\mathbf{v}]_{\mathcal{B}'} = [\mathbf{u}]_{\mathcal{B}'} + [\mathbf{v}]_{\mathcal{B}'} = T\left([\mathbf{u}]_{\mathcal{B}}\right) + T\left([\mathbf{v}]_{\mathcal{B}}\right)$$

$$T\left([\alpha\mathbf{v}]_{\mathcal{B}}\right) = [\alpha\mathbf{v}]_{\mathcal{B}'} = \alpha[\mathbf{v}]_{\mathcal{B}'} = \alpha T\left([\mathbf{v}]_{\mathcal{B}}\right)$$

where $\mathbf{u}$ is also in $\mathcal{V}$ and α in the field $\mathcal{F}$ of $\mathcal{V}$.

Theorem

Let $\mathcal{B} = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ and $\mathcal{B}' = \{\mathbf{u}'_1, \cdots, \mathbf{u}'_n\}$ be bases for a vector space $\mathcal{V}$,

$$[\mathbf{v}]_{\mathcal{B}'} = \mathbf{P}[\mathbf{v}]_{\mathcal{B}}, \qquad \text{for all} \quad \mathbf{v} \in \mathcal{V},$$

where the columns of $\mathbf{P}$ are given by

$$[\mathbf{u}_1]_{\mathcal{B}'} \qquad [\mathbf{u}_2]_{\mathcal{B}'} \qquad \cdots \qquad [\mathbf{u}_n]_{\mathcal{B}'}$$

- The matrix $\mathbf{P}$ is the transformation matrix for $T : \mathbb{R}^n \to \mathbb{R}^n$, so the operation of changing the coordinate vector from old to new is clearly linear
- The matrix $\mathbf{P}$ is also called the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$. To stress,

$$\mathbf{P}_{\mathcal{B}\to\mathcal{B}'} = \begin{bmatrix} [\mathbf{u}_1]_{\mathcal{B}'} & [\mathbf{u}_2]_{\mathcal{B}'} & \cdots & [\mathbf{u}_n]_{\mathcal{B}'} \end{bmatrix}$$

- Similarly, the transition matrix from $\mathcal{B}'$ to $\mathcal{B}$ is

$$\mathbf{P}_{\mathcal{B}'\to\mathcal{B}} = \begin{bmatrix} [\mathbf{u}'_1]_{\mathcal{B}} & [\mathbf{u}'_2]_{\mathcal{B}} & \cdots & [\mathbf{u}'_n]_{\mathcal{B}} \end{bmatrix}$$

Exercise

Find $\mathbf{P}_{B'\to B}$ and $\mathbf{P}_{B\to B'}$ for the bases $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $\mathcal{B}' = \{\mathbf{u}_1', \mathbf{u}_2'\}$ on $\mathbb{R}^2$,

$$\text{where} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_1' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{u}_2' = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solution

- $\mathbf{P}_{\mathcal{B}'\to\mathcal{B}} = \begin{bmatrix} [\mathbf{u}_1']_{\mathcal{B}} & [\mathbf{u}_2']_{\mathcal{B}} \end{bmatrix}$
- $$\begin{aligned} \mathbf{u}_1' &= \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \end{aligned}$$
  $$\Longrightarrow \ [\mathbf{u}_1']_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- $\mathbf{P}_{\mathcal{B}\to\mathcal{B}'} = \begin{bmatrix} [\mathbf{u}_1]_{\mathcal{B}'} & [\mathbf{u}_2]_{\mathcal{B}'} \end{bmatrix}$
- $$\begin{aligned} \mathbf{u}_1 &= \beta_1 \mathbf{u}_1' + \beta_2 \mathbf{u}_2' \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \beta_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \end{aligned}$$
  $$\Longrightarrow \ [\mathbf{u}_1]_{\mathcal{B}'} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solution

- $$\mathbf{u}_2' = \alpha_3 \mathbf{u}_1 + \alpha_4 \mathbf{u}_2$$
$$\begin{bmatrix}2\\1\end{bmatrix} = \alpha_3 \begin{bmatrix}1\\0\end{bmatrix} + \alpha_4 \begin{bmatrix}0\\1\end{bmatrix}$$
$$= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix}$$
$$\Longrightarrow [\mathbf{u}_2']_{\mathcal{B}} = \begin{bmatrix}2\\1\end{bmatrix}$$
- So
$$\mathbf{P}_{\mathcal{B}'\to\mathcal{B}} = \begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}$$
- $$\mathbf{u}_2 = \beta_3 \mathbf{u}_1' + \beta_4 \mathbf{u}_2'$$
$$\begin{bmatrix}0\\1\end{bmatrix} = \beta_3 \begin{bmatrix}1\\1\end{bmatrix} + \beta_4 \begin{bmatrix}2\\1\end{bmatrix}$$
$$\Longrightarrow = \begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}\begin{bmatrix}\beta_3\\\beta_4\end{bmatrix}$$
$$\Longrightarrow [\mathbf{u}_2]_{\mathcal{B}'} = \begin{bmatrix}2\\-1\end{bmatrix}$$
- So
$$\mathbf{P}_{\mathcal{B}\to\mathcal{B}'} = \begin{bmatrix}-1 & 2\\1 & -1\end{bmatrix}$$
- In general, we need to solve a series of systems with the same coefficient matrix to obtain the transition matrix.
- Intuitively, you would expect

$$\mathbf{P}_{\mathcal{B}\to\mathcal{B}'}\mathbf{P}_{\mathcal{B}'\to\mathcal{B}} = \mathbf{I} = \mathbf{P}_{\mathcal{B}'\to\mathcal{B}}\mathbf{P}_{\mathcal{B}\to\mathcal{B}'}$$

Theorem

If $\mathbf{P}$ is the transition matrix from a basis $\mathcal{B}'$ to a basis $\mathcal{B}$ for a finite-dimensional vector space $\mathcal{V}$, then $\mathbf{P}$ is invertible and $\mathbf{P}^{-1}$ is the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$.

Proof

- For every vector $\mathbf{v} \in \mathcal{V}$ with bases $\mathcal{B} = \{\mathbf{u}_1 \cdots \mathbf{u}_n\}$ and $\mathcal{B}' = \{\mathbf{u}'_1 \cdots \mathbf{u}'_n\}$,

$$\mathbf{v} = \mathbf{B}'[\mathbf{v}]_{\mathcal{B}'} = \mathbf{B}[\mathbf{v}]_{\mathcal{B}} = \mathbf{v} \qquad \text{where } \mathbf{B}' \text{ and } \mathbf{B} \text{ are matrices}$$

  containing the vectors $\mathbf{u}_1 \cdots \mathbf{u}_n$ and $\mathbf{u}'_1 \cdots \mathbf{u}'_n$ as columns respectively.
- If $\mathbf{v}$ is the zero vector, then

$$\alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

- Since $\mathbf{u}_1, \cdots, \mathbf{u}_n$ are linearly independent,

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0 \implies [\mathbf{0}]_{\mathcal{B}} = \mathbf{0}$$

Proof

- Of course, it is also true the other way around,

$$[\mathbf{v}]_{\mathcal{B}} = \mathbf{0} \implies \mathbf{v} = \mathbf{0} \implies \mathbf{v} = \mathbf{0} \iff [\mathbf{v}]_{\mathcal{B}} = \mathbf{0}$$

- Similarly,

$$\mathbf{v} = \mathbf{0} \iff [\mathbf{v}]_{\mathcal{B}'} = \mathbf{0} \implies [\mathbf{v}]_{\mathcal{B}} = \mathbf{0} \iff [\mathbf{v}]_{\mathcal{B}'} = \mathbf{0}$$

- Now consider the transition matrix $\mathbf{P}_{\mathcal{B}' \to \mathcal{B}}$, if $\mathbf{v} = \mathbf{0}$,

$$\mathbf{P}_{\mathcal{B}' \to \mathcal{B}}[\mathbf{v}]_{\mathcal{B}'} = [\mathbf{v}]_{\mathcal{B}} \implies \mathbf{P}_{\mathcal{B}' \to \mathcal{B}}[\mathbf{v}]_{\mathcal{B}'} = \mathbf{0}$$

- Since $[\mathbf{v}]_{\mathcal{B}} = \mathbf{0}$ if and only if $[\mathbf{v}]_{\mathcal{B}'} = \mathbf{0}$, thus $\mathbf{P}_{\mathcal{B}' \to \mathcal{B}}[\mathbf{v}]_{\mathcal{B}'}$ is invertible.
- The existence of $\mathbf{P}^{-1}_{\mathcal{B}' \to \mathcal{B}}$ means

$$[\mathbf{v}]_{\mathcal{B}'} = \mathbf{P}^{-1}_{\mathcal{B}' \to \mathcal{B}}[\mathbf{v}]_{\mathcal{B}}$$

therefore $\mathbf{P}^{-1}_{\mathcal{B}' \to \mathcal{B}}$ is the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$. $\square$

Exercise

Suppose for $\mathcal{P}_2$ we want to change from $\mathcal{S} = \{1, x, x^2\}$ to $\mathcal{S}' = \{1, 2x, 4x^2 - 2\}$.

Solution

- Since $\mathcal{S}$ is the standard basis for $\mathcal{P}_2$, we have $\mathbf{P}_{\mathcal{S}' \to \mathcal{S}} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.
- The inverse of $\mathbf{P}_{\mathcal{S}' \to \mathcal{S}}$ will be the transition matrix from $\mathcal{S}$ to $\mathcal{S}'$.

$$\mathbf{P}_{E \to S} = \mathbf{P}_{S \to E}^{-1} = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$$

- For any $p(x) = a + bx + cx^2$, the coordinates of $p(x)$ with respect to $\mathcal{S}'$

$$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + \frac{1}{2}c \\ \frac{1}{2}b \\ \frac{1}{4}c \end{bmatrix} \implies p(x) = (a + \tfrac{1}{2}) + (\tfrac{1}{2}b)2x + \tfrac{1}{4}c(4x^2 - 2)$$

Theorem

Let $\mathcal{S}' = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be a basis for the vector space $\mathbb{R}^n$ and $\mathcal{S} = \{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Then the transition matrix is

$$\mathbf{P}_{\mathcal{S}' \to \mathcal{S}} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix}$$

- We have seen that each transition matrix is invertible. It follows from the above theorem any invertible matrix can be thought of as a transition matrix.
- If $\mathbf{A} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix}$ is any invertible $n \times n$ matrix, then $\mathbf{A}$ can be viewed as the transition matrix from the basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ for $\mathbb{R}^n$ to the standard basis $\mathcal{S} = \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ for $\mathbb{R}^n$. e.g, the invertible matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

is the transition matrix from the basis made of the columns to $\mathcal{S}$.

Exercise

Show that

$$\mathcal{B} = \{1, \cos t, \cos^2 t, \cdots, \cos^6 t\}$$

is a linearly independent set of functions defined on $\mathbb{R}$.

Solution

- Use MATLAB, we can compute the Wronskian easily,

```
>> syms t;
>> r = [ 1, cos(t), cos(t)^2, cos(t)^3, cos(t)^4, cos(t)^5, cos(t)^6];
>> A = [ r; diff(r,t,1); diff(r,t,2); diff(r,t,3); diff(r,t,4); diff(r,t,5); diff(r,t,6)];
>> W = det(A)

ans = (-24883200)*sin(t)^21
```

- The Wronskian is not identically zero, so $\mathcal{B}$ is linearly independent.

- Let $\mathcal{H}$ be the subspace of functions spanned by the functions in $\mathcal{B}$. Then

$\mathcal{B}$ is a basis for $\mathcal{H}$.

Exercise

Confirm the following trigonometric identities

$$\begin{aligned}
\cos 2t &= -1 + 2\cos^2 t \\
\cos 3t &= -3\cos t + 4\cos^3 t \\
\cos 4t &= 1 - 8\cos^2 t + 8\cos^4 t \\
\cos 5t &= 5\cos t - 20\cos^3 t + 16\cos^5 t \\
\cos 6t &= -1 + 18\cos^2 t - 48\cos^4 t + 32\cos^6 t
\end{aligned}$$

Solution

```
>> syms t;
>> combine( -1 + 2*cos(t)^2, 'sincos')

ans = cos(2*t)

>> combine( -3*cos(t) + 4*cos(t)^3, 'sincos')

ans = cos(3*t)
```

Exercise

Show that

$$\mathcal{C} = \{1, \cos t, \cos 2t, \cdots, \cos 6t\}$$

is a linearly independent set of functions defined on $\mathbb{R}$.

Solution

- Again compute Wronskian, we have

```
>> syms t;
>> r = [ 1, cos(t), cos(2*t), cos(3*t), cos(4*t), cos(5*t), cos(6*t)];
>> A = [ r; diff(r,t,1); diff(r,t,2); diff(r,t,3); diff(r,t,4); diff(r,t,5); diff(r,t,6)];
>> W = simplify(det(A))

ans = (-815372697600)*sin(t)^21
```

- The Wronskian is not identically zero, so $\mathcal{C}$ is linearly independent.

Q: Why $\mathcal{C}$ is a basis for $\mathcal{H}$ as well?

- Since $\mathcal{B}$ is a basis for $\mathcal{H}$, then every vector $\mathbf{v}$ in $\mathcal{H}$ has a coordinate vector,

$$[\mathbf{v}]_B = \begin{bmatrix} b_1 & b_2 & \cdots & b_7 \end{bmatrix}^{\mathrm{T}}$$

- Since $\mathcal{C}$ is linearly independent, then every vector $\mathbf{u}$ in the span of $\mathcal{C}$ has

$$[\mathbf{u}]_{\mathcal{C}} = \begin{bmatrix} c_1 & c_2 & \cdots & c_7 \end{bmatrix}^{\mathrm{T}}$$

- The trigonometric identities mean that every $\mathbf{u}$ in the span of $\mathcal{C}$

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -3 & 0 & 5 & 0 \\ 0 & 0 & 2 & 0 & -8 & 0 & 18 \\ 0 & 0 & 0 & 4 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & -48 \\ 0 & 0 & 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 32 \end{bmatrix}$$

- The matrix $\mathbf{P}$ is invertible, why?

$$[\mathbf{u}]_{\mathcal{C}} = \mathbf{P}^{-1}[\mathbf{v}]_B$$

Exercise

Find the indefinite integral $\int \left(5\cos^3 t - 6\cos^4 t + 5\cos^5 t - 12\cos^6 t\right) dt.$

Solution

- This is tedious to compute using integration by parts or trig formulae.

```
>> format rational
>> P = diag([ 1 1 2 4 8 16 32])
>> P(1,3) = -1; P(1,5) = 1; P(1,7) = -1; P(2, 4) = -3;
>> P(2,6) = 5; P(3,5) = -8; P(3,7) = 18; P(4, 6) = -20; P(5, 7) = -48
>> c = inv(P) * [ 0; 0; 0; 5; -6; 5; -12]
c =       -6       55/8      -69/8       45/16       -3        5/16       -3/8
```

- Thus the integral may be written as

$$\int \left(-6 + \frac{55}{8}\cos t - \frac{69}{8}\cos 2t + \frac{45}{16}\cos 3t - 3\cos 4t + \frac{5}{16}\cos 5t - \frac{3}{8}\cos 6t\right) dt$$

```
>> syms t;
>> r = [ 1, cos(t), cos(2*t), cos(3*t), cos(4*t), cos(5*t), cos(6*t)];
>> int( r*c, t) - int( 5*cos(t)^3 - 6*cos(t)^4 + 5*cos(t)^5 - 12*cos(t)^6, t)
ans = 0
```

Theorem

For each pair of vector spaces $\mathcal{U}$ and $\mathcal{V}$ over $\mathcal{F}$, the set

$$\mathcal{L}\left(\mathcal{U}, \mathcal{V}\right)$$

of all linear transformations from $\mathcal{U}$ and $\mathcal{V}$ is a vector space over $\mathcal{F}$.

- Linear transformations possess coordinates in the same way vectors do.

Theorem

Suppose $\mathcal{B}_{\mathcal{U}} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ and $\mathcal{B}_{\mathcal{V}} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ are bases for $\mathcal{U}$ and $\mathcal{V}$, respectively, and let B_{ji} be the linear transformation from $\mathcal{U}$ into $\mathcal{V}$ defined by

$$B_{ji}(\mathbf{u}) = \gamma_j \mathbf{v}_i \quad \text{where} \quad \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix} = [\mathbf{u}]_{\mathcal{B}_{\mathcal{U}}}, \quad \text{then} \quad \mathcal{B}_{\mathcal{L}} = \left\{ B_{ji} \right\}_{j=1\ldots n}^{i=1\ldots m}$$

is a basis for $\mathcal{L}(\mathcal{U}, \mathcal{V})$.

Proof

- To prove $\mathcal{B}_{\mathcal{L}}$ is a basis, we need to show it is a linearly independent spanning set for $\mathcal{L}(\mathcal{U},\mathcal{V})$, first consider,

$$\sum_{j,i} \alpha_{ji} B_{ji} = \mathbf{0}, \qquad \text{for scalars } \alpha_{ji} \text{ in } \mathcal{F}.$$

- For each $\mathbf{u}_k \in \mathcal{B}_{\mathcal{U}}$,

$$B_{ji}(\mathbf{u}_k) = \begin{cases} \mathbf{v}_i & \text{if } j = k \\ \mathbf{0} & \text{if } j \neq k \end{cases} \implies \mathbf{0} = \left(\sum_{j,i} \alpha_{ji} B_{ji}\right)(\mathbf{u}_k) = \sum_{j,i} \alpha_{ji} B_{ji}(\mathbf{u}_k)$$

$$= \sum_{i}^{m} \alpha_{ki} \mathbf{v}_i$$

- The independence of $\mathcal{B}_{\mathcal{V}}$ implies that $\alpha_{ki} = 0$ for each i and k, since every other vector in $\mathcal{U}$ is a linear combination of $\mathbf{u}_k$'s, $\mathcal{B}_{\mathcal{L}}$ is linearly independent.

Proof

- To see that $\mathcal{B}_{\mathcal{L}}$ spans $\mathcal{L}(\mathcal{U}, \mathcal{V})$, let $T \in \mathcal{L}(\mathcal{U}, \mathcal{V})$,

$$T(\mathbf{u}) = T\left(\sum_j^n \gamma_j \mathbf{u}_j\right) = \sum_j^n \gamma_j T(\mathbf{u}_j) = \sum_j^n \gamma_j \sum_i^m \beta_{ij} \mathbf{v}_i = \sum_{i,j} \beta_{ij} \gamma_j \mathbf{v}_i$$
$$= \sum_{i,j} \beta_{ij} B_{ji}(\mathbf{u})$$

- This holds for all $\mathbf{u} \in \mathcal{U}$, so $\mathcal{B}_{\mathcal{L}}$ spans $\mathcal{L}(\mathcal{U}, \mathcal{V})$
- Therefore $\mathcal{B}_{\mathcal{L}}$ is a basis for $\mathcal{L}(\mathcal{U}, \mathcal{V})$. $\square$

Q: What is the dimension of $\mathcal{L}(\mathcal{U}, \mathcal{V})$?

- It now makes sense to talk about the coordinates of $T \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ w.r.t. $\mathcal{B}_{\mathcal{L}}$.

Q: What are the coordinates going to be?

Definition

Suppose

$$\mathcal{B}_{\mathcal{U}} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\} \qquad \text{and} \qquad \mathcal{B}_{\mathcal{V}} = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$$

are bases for $\mathcal{U}$ and $\mathcal{V}$, respectively. The coordinate matrix of $T \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ with respect to the pair $(\mathcal{B}_{\mathcal{U}}, \mathcal{B}_{\mathcal{V}})$ is defined to be the $m \times n$ matrix

$$[T]_{\mathcal{B}_{\mathcal{U}}\mathcal{B}_{\mathcal{V}}} = \begin{bmatrix} [T(\mathbf{u}_1)]_{\mathcal{B}_{\mathcal{V}}} & [T(\mathbf{u}_2)]_{\mathcal{B}_{\mathcal{V}}} & \cdots & [T(\mathbf{u}_n)]_{\mathcal{B}_{\mathcal{V}}} \end{bmatrix}$$

- In other words, if $T(\mathbf{u}_j) = \sum_{i,j} \beta_{ij} B_{ji}(\mathbf{u}_j) = \sum_{i,j} \beta_{ij}\gamma_j \mathbf{v}_i = \sum_{i}^{m} \beta_{ij}\mathbf{v}_i$, then

$$[T(\mathbf{u}_j)]_{\mathcal{B}_{\mathcal{V}}} = \begin{bmatrix} \beta_{1j} \\ \beta_{2j} \\ \vdots \\ \beta_{mj} \end{bmatrix} \quad \text{and} \quad [T]_{\mathcal{B}_{\mathcal{U}}\mathcal{B}_{\mathcal{V}}} = \begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m1} & \beta_{m2} & \cdots & \beta_{mn} \end{bmatrix}$$

- When T is a linear operator on $\mathcal{U}$, and when there is only one basis involved, $[T]_{\mathcal{B}}$ is used in place of $[T]_{\mathcal{B}\mathcal{B}}$ to denote the coordinate matrix of T w.r.t. $\mathcal{B}$.

Theorem

Let $T \in \mathcal{L}(\mathcal{U}, \mathcal{V})$, and let $\mathcal{B}_{\mathcal{U}}$ and $\mathcal{B}_{\mathcal{V}}$ be bases for $\mathcal{U}$ and $\mathcal{V}$, respectively. For each $\mathbf{u} \in \mathcal{U}$, the action of T on $\mathbf{u}$ is given by matrix multiplication between their coordinates in the sense that

$$[T(\mathbf{u})]_{\mathcal{B}_{\mathcal{V}}} = [T]_{\mathcal{B}_{\mathcal{U}}\mathcal{B}_{\mathcal{V}}}[\mathbf{u}]_{\mathcal{B}_{\mathcal{U}}}$$

$$\begin{tikzcd}
\mathbf{u} \in \mathcal{U} \arrow[rr, "\text{Direct}", "\text{Computation}"'] \arrow[d] & & T(\mathbf{u}) \in \mathcal{V} \\
{[\mathbf{u}]_{\mathcal{B}_{\mathcal{U}}} \in \mathbb{R}^n} \arrow[rr, "\text{Indirect}", "\text{Matrix transformation}"'] & & {[T(\mathbf{u})]_{\mathcal{B}_{\mathcal{V}}} \in \mathbb{R}^m} \arrow[u]
\end{tikzcd}$$

Proof

- Let $\mathcal{B}_{\mathcal{U}} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{B}_{\mathcal{V}} = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ be two bases for $\mathcal{U}$ and $\mathcal{V}$.
- Let $\mathbf{u} = \sum_{j}^{n} \gamma_j \mathbf{u}_j$ and $T(\mathbf{u}_j) = \sum_{i}^{m} \beta_{ij} \mathbf{v}_i$, then the coordinate vector and matrix are $[\mathbf{u}]_{\mathcal{B}_{\mathcal{U}}} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix}$ and $[T]_{\mathcal{B}_{\mathcal{U}}\mathcal{B}_{\mathcal{V}}} = \begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m1} & \beta_{m2} & \cdots & \beta_{mn} \end{bmatrix}$
- We have derived earlier that

$$T(\mathbf{u}) = \sum_{i,j} \beta_{ij} B_{ji}(\mathbf{u}) = \sum_{i,j} \beta_{ij} \gamma_j \mathbf{v}_i = \sum_{i}^{m} \left(\sum_{j}^{n} \beta_{ij} \gamma_j \right) \mathbf{v}_i$$

- So the ith element of $[T(\mathbf{u})]_{\mathcal{B}_{\mathcal{V}}}$ is $\sum_{j}^{n} \beta_{ij} \gamma_j$, which is the ith element of

$$[T]_{\mathcal{B}_{\mathcal{U}}\mathcal{B}_{\mathcal{V}}} [\mathbf{u}]_{\mathcal{B}_{\mathcal{U}}} \quad \square$$

Exercise

Let $T\colon \mathcal{P}_2 \to \mathcal{P}_2$ be the linear operator defined by

$$T(p(x)) = p(3x-5)$$

that is,

$$T(c_0 + c_1 x + c_2 x^2) = c_0 + c_1(3x-5) + c_2(3x-5)^2$$

(a) Find $[T]_{\mathcal{B}}$ relative to the basis

$$\mathcal{B} = \{1, x, x^2\}$$

(b) Use the indirect procedure to compute

$$T(1 + 2x + 3x^2)$$

(c) Check the result by computing $T(1 + 2x + 3x^2)$ directly.

Solution

- Find the image of the vectors in the basis $\mathcal{B}$ under T,

$$T(1) = 1, \quad T(x) = 3x - 5 \quad \text{and} \quad T(x^2) = (3x-5)^2 = 9x^2 - 30x + 25$$

- Find the coordinate vector of the images with respect to $\mathcal{B}$

$$[T(1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad [T(x)]_{\mathcal{B}} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} \quad \text{and} \quad [T(x^2)]_{\mathcal{B}} = \begin{bmatrix} 25 \\ -30 \\ 9 \end{bmatrix}$$

- Thus the coordinate matrix with respect to $\mathcal{B}$,

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix}$$

- The coordinate vector of $\mathbf{p} = 1 + 2x + 3x^2$ with respect to $\mathcal{B}$ is $[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Solution

- Perform the matrix transformation,

$$[T(\mathbf{p})]_{\mathcal{B}} = [T]_{\mathcal{B}}[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 66 \\ -84 \\ 27 \end{bmatrix}$$

- Reconstructing $T(\mathbf{p}) = T(1 + 2x + 3x^2)$ from $[T(\mathbf{p})]_{\mathcal{B}}$

$$T(1 + 2x + 3x^2) = 66 - 84x + 27x^2$$

- By direction computation,

$$\begin{aligned} T(1 + 2x + 3x^2) &= 1 + 2(3x - 5) + 3(3x - 5)^2 \\ &= 1 + 6x - 10 + 27x^2 - 90x + 75 = 66 - 84x + 27x^2 \end{aligned}$$

Exercise

Show how the action of the simple differential operation $D(\mathbf{p}) = p'(x)$ on the polynomial space of degree three or less is given by the matrix multiplication.

Solution

- The coordinate matrix of D w.r.t. the standard basis $\mathcal{S} = \{1, t, t^2, t^3\}$ is

$$[T]_{\mathcal{B_U}\mathcal{B_V}} = \begin{bmatrix} [T(\mathbf{u}_1)]_{\mathcal{B_V}} & \cdots & [T(\mathbf{u}_n)]_{\mathcal{B_V}} \end{bmatrix}$$

$$\Longrightarrow [D]_{\mathcal{S}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- If $\mathbf{p} = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, then $D(\mathbf{p}) = a_1 + 2a_2 x + 3a_3 x^2$, so that

$$[D(\mathbf{p})]_{\mathcal{S}} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = [D]_{\mathcal{S}} [\mathbf{p}]_{\mathcal{S}}$$

- So differentiation on finite-dimensional $\mathcal{P}_n$ is simply a matrix multiplication.