# Vv417 Lecture 9

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Q: Why do we have to study Vector Spaces?

- It was realized that many mathematical objects of different sorts:

  vectors, matrices, polynomials, functions and operators

  were in fact quite similar.

- They are similar because they share some defining properties, hence they share the same "consequences" of those defining properties. e.g.

$$\alpha(\mathbf{u}+\mathbf{v}) = \alpha\mathbf{u}+\alpha\mathbf{v}; \qquad \beta(\mathbf{A}+\mathbf{B}) = \beta\mathbf{A}+\beta\mathbf{A}$$

- Rather than studying each objects separately, it is more efficient to study the common properties and their consequences instead the actual objects.

**Definition**

The set of vectors

$$\mathbb{R}^n$$

is also called the $n$-dimensional Euclidean space.

Q: Intuitively, what is the difference between

a Euclidean *Space* and a set *Set* of Euclidean vectors?

## Properties of addition and scalar mutliplication

- If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$ and $\alpha$ and $\beta$ are scalars in $\mathbb{R}$, then

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
3. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$
4. $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$
5. $(\alpha\beta)\mathbf{u} = \alpha(\beta\mathbf{u})$
6. $1\ \mathbf{u} = \mathbf{u}$
7. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
8. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
9. $\mathbf{u} + \mathbf{v}$ is in $\mathbb{R}^n$
10. $\alpha\mathbf{u}$ is in $\mathbb{R}^n$

- Those properties are the defining properties of Euclidean spaces.

- In general, a vector space consists four things,

  two sets $\mathcal{F}$ and $\mathcal{V}$, two operations called addition and scalar multiplication.

1. $\mathcal{F}$ is a scalar field.

- For us $\mathcal{F}$ is either the field $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers.

2. $\mathcal{V}$ is a non-empty set of mathematical objects called vectors.

- So in the general sense, matrices, polynomials, continuous and differentiable functions are all known as vectors as well.

3. Addition is an operation between elements of $\mathcal{V}$.

- By addition we mean a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{V}$ an object $\mathbf{u}+\mathbf{v}$, called the sum of $\mathbf{u}$ and $\mathbf{v}$;

4. Scalar multiplication is an operation between elements of $\mathcal{F}$ and $\mathcal{V}$.

- By scalar multiplication we mean a rule for associating with each object $\mathbf{u}$ in $\mathcal{V}$ and each scalar $\alpha$ in $\mathcal{F}$ an object $\alpha\mathbf{u}$, called the scalar multiple of $\mathbf{u}$ by $\alpha$.

Definition

A field $\mathcal{F}$ is a set on which two operations, addition and multiplication, are defined so that, for each pair of elements $\alpha$, $\beta$ in $\mathcal{F}$, there are unique elements $\alpha + \beta$ and $\alpha \cdot \beta$ in $\mathcal{F}$ for which the following axioms hold for all elements $\alpha$, $\beta$, and $\gamma$ in $\mathcal{F}$.

1. $\alpha + \beta = \beta + \alpha$
2. $\alpha \cdot \beta = \beta \cdot \alpha$
3. $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$
4. $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$
5. $0 + \alpha = \alpha$
6. $1 \cdot \alpha = \alpha$
7. $\alpha + (-\alpha) = 0$
8. $\alpha \cdot \alpha^{-1} = 1$ nonzero element
9. $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$

Q: Is the set of integers a field ?

Q: Is the set of real numbers of the following form a field?

$$a + b\sqrt{2} \quad \text{where } a \text{ and } b \text{ are rational numbers.}$$

- Unless stated otherwise, the usual operations of addition and multiplication are assumed. However, the usual ways are not the only way.

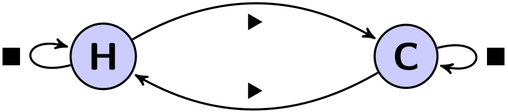
- Let $\mathbb{Z}_2$ be a set of two elements,

▶ and ■

- Consider the following two operations that associate any two elements in $\mathbb{Z}_2$
  1. Sequential operation **T**

     ■**T**■☞■ ■**T**▶☞▶ ▶**T**■☞▶ ▶**T**▶☞■
  2. Parallel operation **I**

     ■**I**■☞■ ■**I**▶☞■ ▶**I**■☞■ ▶**I**▶☞▶
- To understand the above two operations, imagine the following situation:




Q: With the above two operations, is the set of $\mathbb{Z}_2$ a field ?

Q: Can you identify which operation is addition and which is multiplication?

- Note that "■" is the additive identity, and "▶" is the multiplicative identity
  1. Sequential operation is the addition
     $$0+0=0 \qquad 0+1=1 \qquad 1+0=1 \qquad 1+1=0$$
  2. Parallel operation is the multiplication
     $$0\cdot 0=0 \qquad 0\cdot 1=0 \qquad 1\cdot 0=0 \qquad 1\cdot 1=1$$
- This field is known as a Galois field.
- Two properties of $\mathbb{Z}_2$ that are not possessed by a real field.
  1. Every element $\alpha$ of $\mathbb{Z}_2$ satisfies $\alpha+\alpha=0$, therefore
     $$-\alpha=\alpha$$
  2. Every element $\alpha$ of $\mathbb{Z}_2$ satisfies $\alpha\cdot\alpha=\alpha$, therefore
     $$\alpha^n=\alpha$$
- Evariste Galois (1811-1832) was a better mathematician than marksman.

### Cancellation Laws

Suppose $\alpha$, $\beta$, and $\gamma$ are arbitrary elements in a field.

- If $\alpha + \beta = \gamma + \beta$, then
$$\alpha = \gamma$$
- If $\alpha \cdot \beta = \alpha \cdot \gamma$ and $\alpha \neq 0$, then
$$\beta = \gamma$$

### Theorem

The additive identity, $0$, the additive inverse $-\alpha$, the multiplicative identity, $1$, and the multiplicative inverse $\alpha^{-1}$, are unique.

### Proof

- Let there be a second $0^* \in \mathcal{F}$ such that $0^* + \alpha = \alpha$ for all $\alpha \in \mathcal{F}$,
$$0^* + \alpha = 0 + \alpha \implies 0^* = 0 \qquad \text{by the cancellation laws}$$
- So the additive identity must be unique. Similarly, the rest can be proved.

- Many of the properties of multiplication of real numbers are true in any field.

**Theorem**

Suppose $\alpha$ and $\beta$ are arbitrary elements of a field.

1. $\alpha \cdot 0 = 0$
2. $(-\alpha) \cdot \beta = \alpha \cdot (-\beta) = -(\alpha \cdot \beta)$
3. $(-\alpha) \cdot (-\beta) = \alpha \cdot \beta$

Q: How can we prove the theorem using the axioms?

**Proof**

- To prove 1., we apply axiom 5 twice

$$
\begin{aligned}
0 + \alpha \cdot 0 = \alpha \cdot 0 & \\
= \alpha\,(0+0) & \qquad \text{axiom 9} \\
= \alpha \cdot 0 + \alpha \cdot 0 & \qquad \text{CL 1}
\end{aligned}
$$

$$\Longrightarrow \ \alpha \cdot 0 = 0$$

## Definition

If the addition and scalar multiplication operations satisfy the following properties by all objects $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathcal{V}$ and all scalars $\alpha$ and $\beta$ in $\mathcal{F}$, then we call

$\mathcal{V}$ a vector space over $\mathcal{F}$.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
3. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$
4. $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$
5. $(\alpha\beta)\mathbf{u} = \alpha(\beta\mathbf{u})$
6. $1\ \mathbf{u} = \mathbf{u}$
7. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
8. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
9. $\mathbf{u} + \mathbf{v}$ is in $\mathcal{V}$
10. $\alpha\mathbf{u}$ is in $\mathcal{V}$

Q: Is a field a vector space?

- A field is a vector space over itself, with vector addition begin the field addition and scalar multiplication being the field multiplication.

Q: What is the difference between a field and a vector space?

### Exercise

Let $\mathcal{V} = \mathbb{R}^2$ and the operations of addition and scalar multiplication are defined as

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix} \quad \text{and} \quad \alpha\mathbf{v} = \begin{bmatrix} \alpha v_1 \\ 0 \end{bmatrix}, \quad \text{where} \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

are in $\mathcal{V}$ and $\alpha$ is any scalar in $\mathcal{F}$. Is $\mathcal{V}$ a vector space over $\mathbb{R}$?

### Solution

- Most of the axioms are satisfied, but consider $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ where $u_2 \neq 0$.
- It is clear that there is no the multiplicative identity $\alpha$ such that

$$\alpha\mathbf{u} = \mathbf{u}$$

according to the given scalar multiplication, so it is not a vector space.

Q: How can we prove any positive real number to the power of zero is 1?

### Theorem

Let $\mathcal{V}$ be a vector space, $\mathbf{u}$ a vector in $\mathcal{V}$, and $\alpha$ a scalar, then

1. $0\mathbf{u} = \mathbf{0}$
2. $\alpha\mathbf{0} = \mathbf{0}$
3. $(-1)\mathbf{u} = -\mathbf{u}$
4. If $\alpha\mathbf{u} = \mathbf{0}$, then $\alpha = 0$ or $\mathbf{u} = \mathbf{0}$.

### Proof

Let us prove 1., consider

$$\begin{aligned}
0\mathbf{u} + 0\mathbf{u} &= (0+0)\mathbf{u} && \text{Axiom 4} \qquad (\alpha+\beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u} \\
&= 0\mathbf{u} && \text{Axiom 5 of } \mathcal{F}. \\
(0\mathbf{u} + 0\mathbf{u}) + (-0\mathbf{u}) &= 0\mathbf{u} + (-0\mathbf{u}) && \text{Axiom 8, existence of a negative vector} \\
0\mathbf{u} + [0\mathbf{u} + (-0\mathbf{u})] &= 0\mathbf{u} + (-0\mathbf{u}) && \text{Axiom 2} \qquad \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w} \\
0\mathbf{u} + \mathbf{0} &= \mathbf{0} && \text{Axiom 8} \\
0\mathbf{u} &= \mathbf{0} && \text{Axiom 7, existence of a zero vector.}
\end{aligned}$$