Vv255 Lecture 20

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- For $y = f(x)$, we often use the u-substitution for the integral

$$\int_a^b f(x)\, dx$$

- For example, consider the following integral,

$$\int_0^{\sqrt{\pi}} 2x \sin(x^2)\, dx$$

- Typically, in this case, we use the following substitution

$$u = g(x) = x^2$$

which is essentially a change of variables from x to u, "new in terms of old".

- Applying the substitution formula

$$\int_a^b f(x)\, dx = \int_a^b F(g(x))g'(x)\, dx = \int_{g(a)}^{g(b)} F(u)\, du$$

$$\implies \int_0^{\sqrt{\pi}} \sin(x^2) 2x\, dx = \int_0^{\pi} \sin(u)\, du = 2$$

- This can also be done using transformation as "old in terms of new", that is,

$$x = h(u) = \sqrt{u}$$

- Of course, the substitution formula still holds in this case,

$$\int_{g(a)}^{g(b)} F(u)\,du = \int_a^b F(g(x))g'(x)\,dx = \int_a^b f(x)\,dx = \int_{h^{-1}(a)}^{h^{-1}(b)} f(h(u))\,h'(u)\,du$$

- Apply this version of the formula, we surely obtain the same result

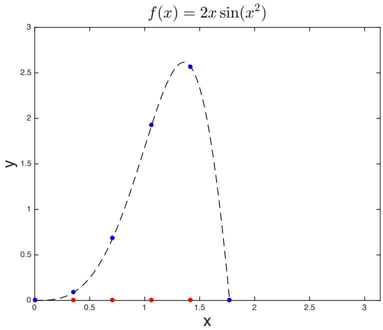
$$\int_0^{\sqrt{\pi}} 2x\sin(x^2)\,dx = \int_0^{\pi} 2\cdot\sqrt{u}\cdot\sin(u)\cdot\frac{1}{2\sqrt{u}}\,du = \int_0^{\pi}\sin(u)\,du = 2$$

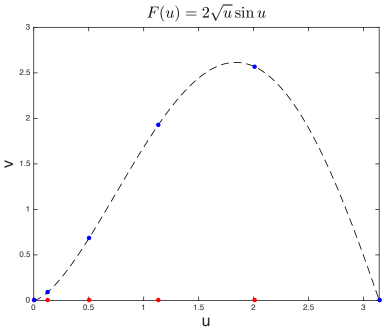
- This version offers insights into the change of variables

$$\int_a^b f(x)\,dx = \int_{h^{-1}(a)}^{h^{-1}(b)} f(\sqrt{u})\,\frac{1}{2\sqrt{u}}\,du$$

- Notice the effect of the change of variables

$$x = h(u) = \sqrt{u}$$





is to stretch the x-axis in a non-uniform way.

Q: What does this term $h'(u)$ actually do?

- Consider the definition of the definite integral with $x_i^* = x_{i-1}$,

$$\begin{aligned}
\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^n f(x_i^*)\Delta x_i &= \lim_{n\to\infty}\sum_{i=1}^n f(x_{i-1})(x_i - x_{i-1}) \\
&= \lim_{n\to\infty}\sum_{i=1}^n f(\sqrt{u_{i-1}})(\sqrt{u_i} - \sqrt{u_{i-1}}) \\
&= \lim_{n\to\infty}\sum_{i=1}^n f(\sqrt{u_{i-1}})\frac{u_i - u_{i-1}}{\sqrt{u_i} + \sqrt{u_{i-1}}} \\
&= \lim_{n\to\infty}\sum_{i=1}^n f\left(\sqrt{u_{i-1}^*}\right)\frac{u_i - u_{i-1}}{\boxed{2\sqrt{u_{i-1}^*}}} \\
&= \int_{h^{-1}(a)}^{h^{-1}(b)} f(h(u))\,\boxed{\frac{1}{2\sqrt{u}}}\,du
\end{aligned}$$

- The correction term, $\boxed{h'(u)}$, gives how much the operation of changing axes expanded or contracted the subinterval containing the sample points.

- Recall the correction term for changing into the polar coordinates is

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)\,dx\,dy = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} F(r,\theta)\, r \,dr\,d\theta$$

which was found by using the definition of double integral.

Q: Can we find the the correction term using differentiation like

$$\int_{x_1}^{x_2} f(x)\,dx = \int_{u_1}^{u_2} f(h(u))\, h'(u)\, du = \int_{u_1}^{u_2} F(u)\, h'(u)\, du$$

- First let us consider the correction term for a general transformation in $\mathbb{R}^2$.

$$x = x(u,v), \qquad y = y(u,v)$$

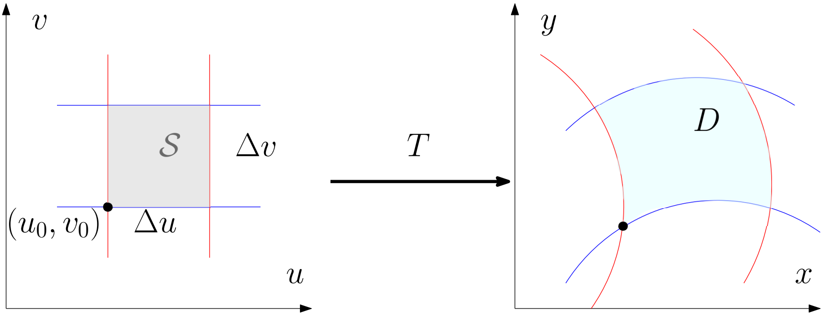
- Using vector notation,

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x(u,v) \\ y(u,v) \end{bmatrix} = x(u,v)\mathbf{e}_x + y(u,v)\mathbf{e}_y$$

- To find the correction term between $\Delta x \Delta y$ and $\Delta u \Delta v$ for

$$\mathbf{r} = x(u,v)\mathbf{e}_x + y(u,v)\mathbf{e}_y$$

we will need to find how regions in one plane become distorted when they are transformed into another plane, for example,

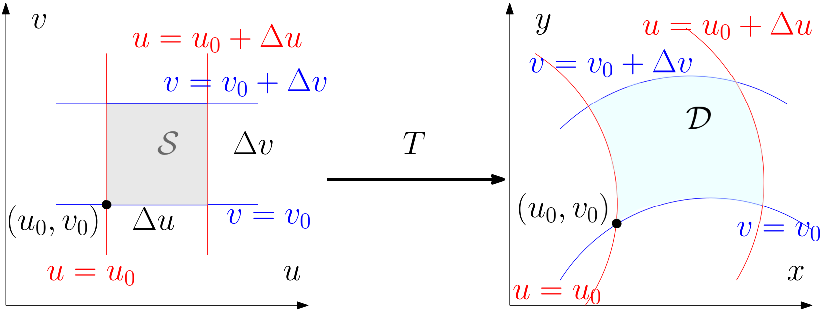


Q: How can we find ΔA for $\mathcal{S}$ in terms of ΔA^* for $\mathcal{D}$?

- If x and y are differentiable functions of u and v,

$$\mathbf{r} = x(u,v)\mathbf{e}_x + y(u,v)\mathbf{e}_y$$

then we expect no sudden and drastic change going from $\mathcal{S}$ to $\mathcal{D}$.

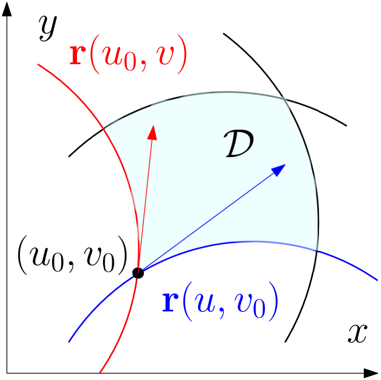


Q: What are the graph of

$$\mathbf{r}(u, v_0) \qquad \text{and} \qquad \mathbf{r}(u_0, v)$$

- For small enough Δu and Δv, we expect the following curves to be fairly flat

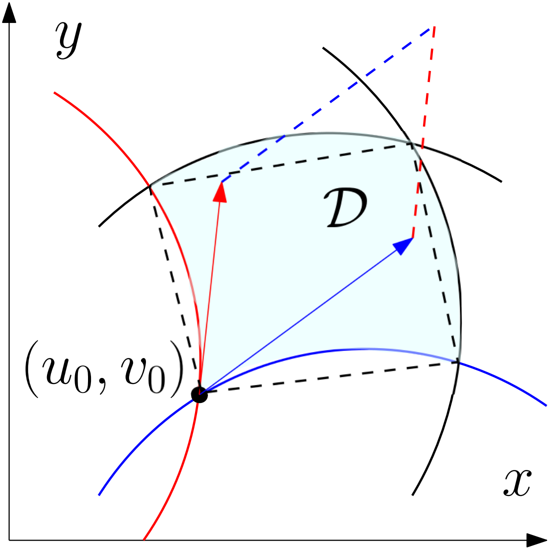
$$\mathbf{r}(u, v_0) \qquad \text{and} \qquad \mathbf{r}(u_0, v)$$



- Use linear approximations,

$$\begin{aligned}
\Delta \mathbf{r_u} &= \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \\
&\approx \left.\frac{\partial \mathbf{r}}{\partial u}\right|_{u_0, v_0} \Delta u \\
&= \left(\frac{\partial x}{\partial u}\mathbf{e}_x + \frac{\partial y}{\partial u}\mathbf{e}_y\right)\Bigg|_{u_0, v_0} \Delta u \\
\Delta \mathbf{r_v} &= \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \\
&\approx \left.\frac{\partial \mathbf{r}}{\partial v}\right|_{u_0, v_0} \Delta v \\
&= \left(\frac{\partial x}{\partial v}\mathbf{e}_x + \frac{\partial y}{\partial v}\mathbf{e}_y\right)\Bigg|_{u_0, v_0} \Delta v
\end{aligned}$$

- It follows that the area of the region $\mathcal{D}$, denote by ΔA^*, is roughly given by



- If $\mathbf{r}$ is thought of as vector in $\mathbb{R}^3$ with zero components of $\mathbf{e}_z$

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \implies \frac{\partial \mathbf{r}}{\partial u} = \begin{bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ 0 \end{bmatrix} \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial v} = \begin{bmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ 0 \end{bmatrix}$$

and the derivatives are evaluated at (u_0, v_0), then the area expressed as

$$\Delta A^* \approx \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \Delta u \Delta v$$

- Computing the cross product, we obtain

$$\left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| = \left| \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right| = \left| \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \right|$$

Q: Have you seen this before?

Definition

The Jacobian of the coordinate transformation $x = x(u, v)$, $y = x(u, v)$ is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

It gives how much the transformation is expanding or contracting an infinitesimal area at a point in uv-plane as the point is transformed into xy-plane.

Theorem

If $f(x, y)$, and $x(u, v)$ and $y(u, v)$ have continuous partial derivatives and $J(u, v)$ is zero only at isolated points, if at all, then

$$\iint_R f(x,y)\,dA = \iint_S f(g(u,v),h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

- The ABSOLUTE VALUE of the Jacobian severs to correct the distortion.

Theorem

If $f(x, y)$, and $x(u, v)$ and $y(u, v)$ have continuous partial derivatives and $J(u, v)$ is zero only at isolated points, if at all, then

$$\iint_R f(x,y)\, dA = \iint_S f(g(u,v), h(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

Q: Why the correct term for $y = f(x)$ can be both positive and negative?

Exercise

(a) *Find the Jacobian for the polar coordinate transformation*

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

and write the Cartesian integral $\displaystyle\iint_D f(x,y)\, dA$ *as a polar integral.*

(b) *Evaluate* $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\, dy\, dx.$

- Similar procedures can be applied to substitutions in triple integrals.

Definition

For an one-to-one transformation that maps a region in $\mathbb{R}^3$ onto a region in $\mathbb{R}^3$,

$$x = g(u, v, w) \quad y = h(u, v, w) \quad z = k(u, v, w)$$

the Jacobian is

$$J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

This determinant measures how much the volume near a point is being expanded or contracted by the transformation from (u, y, w) to (x, y, z) coordinates.

- For cylindrical coordinates r, θ, and z,

$$\iiint\limits_E F(x,y,z)\,dV = \iiint\limits_G H(r,\theta,z)\,|r|\,dr\,d\theta\,dz$$

- We can drop the absolute value signs whenever $r \geq 0$.

Matlab

```
>> syms r theta z
>> J_r_theta_z = jacobian( ...
[r*cos(theta), r*sin(theta) , z], [r, theta, z])
J_r_theta_z =
[ cos(theta), -r*sin(theta), 0]
[ sin(theta),  r*cos(theta), 0]
[          0,             0, 1]

>> simplify(det(J_r_theta_z))
ans = r
```

- For spherical coordinates, ρ, θ, and ϕ,

$$\iiint_E F(x, y, z)\, dV = \iiint_G H(\rho, \theta, \phi)\, |\rho^2 \sin\phi|\, d\rho\, d\theta\, d\phi$$

- We can drop the absolute value signs because $\sin\phi$ is never negative.

Matlab

```
>> syms r t p
>> J_rho_theta_phi = jacobian(...
[rho*sin(p)*cos(t), rho*sin(p)*sin(t), rho*cos(p)], [r, t, p])
J_rho_theta_phi =

[ cos(t)*sin(p), -r*sin(p)*sin(t), r*cos(p)*cos(t)]
[ sin(p)*sin(t),  r*cos(t)*sin(p), r*cos(p)*sin(t)]
[        cos(p),                0,       -r*sin(p)]

>> simplify(det(J_rho_theta_phi))

ans =  - r^2*sin(p)
```

Exercise

Evaluate

$$\int_0^3 \int_0^4 \int_{y/2}^{y/2+1} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dx\, dy\, dz$$

by applying the transformation

$$u = (2x-y)/2, \qquad v = y/2, \qquad w = z/3$$

and integrating over an appropriate region in uyw*-space.*