Question1 (1 points)

Find two different row echelon forms of $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$.

Question2 (1 points)

Construct a linear system of 4 unknowns with 4 equations for which the corresponding coefficient matrix has 16 different elements, but row 3 and 4 of the coefficient matrix become zero in elimination. How many solutions do you have for your system? How many solutions do you have if your system is turned into a homogeneous system?

Question3 (1 points)

Suppose a system of linear equations has a 3×5 augmented matrix whose fifth column is not a pivot column. Is the system consistent? Justify your answer.

Question4 (3 points)

Consider the system of linear equations $\begin{aligned} x + \quad\; y &= 0 \\ x + \frac{401}{400} y &= 20 \end{aligned}$.

(a) (1 point) Use elimination to solve it, round each calculation to four significant digits.

(b) (1 point) Do part (a) again, but his time use rational arithmetic.

(c) (1 point) What can you conclude from part (a) and (b) about solving linear systems?

Question5 (1 points)

Use Gauss-Jordan elimination to solve the linear system, state all the elementary matrices

$$\begin{aligned} 4x_1 + 3x_2 + 2x_3 - \;x_4 &= \;\;4 \\ 5x_1 + 4x_2 + 3x_3 - \;x_4 &= \;\;4 \\ -2x_1 - 2x_2 - \;x_3 + 2x_4 &= -3 \\ 11x_1 + 6x_2 + 4x_3 + \;x_4 &= \;\;11 \end{aligned}$$

using the notations $\mathbf{E}_{i,j}$, $\mathbf{E}_{(\alpha)i}$ and $\mathbf{E}_{(\alpha)i,j}$.

Question6 (1 points)

Find a reduced row echelon matrix which is row equivalent to

$$\mathbf{A} = \begin{bmatrix} 1 & -\mathrm{i} \\ 2 & 2 \\ \mathrm{i} & 1+\mathrm{i} \end{bmatrix} \qquad \text{where } \mathrm{i} = \sqrt{-1}.$$

What are the solutions of $\mathbf{Ax} = \mathbf{0}$?

Question7 (1 points)

Give an example of matrices such that $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{B})$ while $\operatorname{rank}(\mathbf{AA}) \neq \operatorname{rank}(\mathbf{BB})$.

Question8 (2 points)

Suppose $\mathbf{BA} = \mathbf{I}$, where $\mathbf{I}$ is the $n \times n$ identity matrix.

(a) (1 point) Show that the equation

$$\mathbf{Ax} = \mathbf{0}$$

has only the trivial solution.

(b) (1 point) Explain why $\mathbf{A}$ cannot have more columns than rows.

Question9 (1 points)

Solve the matrix $\mathbf{CA} + \mathbf{B} = \mathbf{C}$, where $\mathbf{A} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 2 \\ 3 & 0 \end{bmatrix}$.

Question10 (1 points)

Find the third column of the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$ by elimination if it exists.

Question11 (1 points)

Suppose $\mathbf{A}_{n\times n}$ is singular, is it possible that $\mathbf{A}^3$ is an identity matrix? Justify your answer.

Question12 (1 points)

Suppose $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are matrices where the following matrix multiplications are defined, and $\mathbf{A}$ is invertible. Show

$$\mathbf{C}(\mathbf{A} - \mathbf{B}) = \mathbf{A}^{-1}\mathbf{B}$$

given

$$(\mathbf{A} - \mathbf{B})\mathbf{C} = \mathbf{B}\mathbf{A}^{-1}$$

Question13 (0 points)

(a) (1 point (bonus)) Suppose

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with a, b, c, $d \in \mathbb{R}$. Prove $\mathbf{A}$ has reduced echelon form of

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

if and only if $ad - bc \neq 0$.

(b) (1 point (bonus)) Consider the naive gaussian elimination with back substitution when floating-point arithmetic is used. Find a matrix such that the algorithm gives a wildly inaccurate solution. Use a sensible way to report the error. You may use the Matlab files on Canvas, and may find the following concept useful

ill-conditioned matrix

(c) (1 point (bonus)) Divide the interval $[0, 1]$ into $n = 5$ equal subintervals. Derive a linear system such that the solution of a similar system for large n numerically approximates the solution of the following boundary value problem reasonably well.

$$\ddot{y} - \dot{y} = 125t, \qquad y(0) = y(1) = 0$$

(d) (1 point (bonus)) Show $\mathbf{I} + \mathbf{BA}$ and $\mathbf{I} + \mathbf{AB}$ are both invertible or both singular.