
Question1 (5 points)

(a) (1 point) Evaluate

$$\iiint_E z\, dV$$

where E is the solid tetrahedron bounded by the four planes

$$x = 0, \quad y = 0, \quad z = 0, \quad \text{and} \quad x + y + z = 1$$

(b) (1 point) Evaluate the following integral

$$\iiint_{\mathcal{E}} (x + 2y - z)\, dV$$

where $\mathcal{E}$ is the solid region bounded between the graph of

$$z = x^2 + y^2$$

and the plane

$$3x + 5y + 2z = 12$$

(c) (1 point) Find the mass and center of mass of a solid, whose density is

$$\rho(x, y, z) = 1 + xyz$$

in the shape of the cube bounded $0 \le x \le 1$, $0 \le y \le 1$ and $0 \le z \le 1$.

(d) (1 point) Evaluate

$$\iiint_E |xyz|\, dV$$

over the solid ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$$

(e) (1 point) Suppose $f(x)$ is continuous, show that

$$\int_0^t \int_0^z \int_0^y f(x)\, dx\, dy\, dz = \frac{1}{2} \int_0^t (t - x)^2 f(x)\, dx$$

Question2 (5 points)

(a) (1 point) Find a double integral for the area of $\mathcal{D}$ in the first quadrant formed by

$$xy = 4; \qquad xy = 8; \qquad xy^3 = 5; \qquad xy^3 = 15$$

Then convert the double integral into an iterated integral with constant limits.

(b) (1 point) Evaluate

$$\iint_{\mathcal{S}} (x^4 - y^4) e^{xy}\, dA,$$

where $\mathcal{S}$ is the region in the first quadrant enclosed by the hyperbolas

$$xy = 1, \quad xy = 3, \quad x^2 - y^2 = 3, \quad x^2 - y^2 = 4.$$

(c) (1 point) Find the volume of the solid region lying below the surface

$$f(x,y) = \frac{xy}{1+x^2y^2}$$

and above the plane region bounded by $xy = 1$, $xy = 4$, $x = 1$, and $x = 4$.

(d) (1 point) Evaluate the following integral

$$\iiint_{\mathcal{E}} xyz\, dV$$

where $\mathcal{E}$ is a region formed by the following surfaces:

$$m = \frac{x^2+y^2}{z}; \quad n = \frac{x^2+y^2}{z}; \quad a^2 = xy; \quad b^2 = xy; \quad \alpha = \frac{y}{x}; \quad \beta = \frac{y}{x}$$

in the first octant, that is,

$$x > 0; \qquad y > 0; \qquad z > 0$$

and $0 < a < b$, $0 < \alpha < \beta$ and $0 < m < n$.

(e) (1 point) Suppose partial derivatives of $f(x, y, z)$ are continuous and satisfy

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 \le 1$$

Show that the following is true

$$f(x_0, y_0, z_0) - \frac{3}{4}R \le \bar{f} \le f(x_0, y_0, z_0) + \frac{3}{4}R$$

where $\bar{f}$ is the average value

$$\bar{f} = \frac{1}{V} \iiint_{\mathcal{S}} f(x, y, z)\, dV$$

over the spherical region $\mathcal{S}$ defined by

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \le R^2$$

and V is the volume of the region $\mathcal{S}$.