Vv256 Lecture 15

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Definition

The inverse transform of $F(s)$, denoted by

$$\mathcal{L}^{-1}\left[F\right],$$

is the function $f(t)$ whose Laplace transform is $F(s)$.

- The formula for computing inverse Laplace transforms is

$$\mathcal{L}^{-1}\left[F\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{t(\alpha+\beta \mathrm{i})} F(\alpha+\beta \mathrm{i})\, d\beta$$

- Unfortunately, deriving and verifying this formula are beyond our current goal
- Alternatively, to avoid complex analysis, we can use the following formula,

$$\mathcal{L}^{-1}\left[F\right] = \lim_{k\to\infty} \frac{(-1)^k}{k!}\left(\frac{k}{t}\right)^{k+1} F^{(k)}\left(\frac{k}{t}\right)$$

for $t \geq 0$, where $F^{(k)}$ denotes the kth derivative of F with respect to s.

Exercise

Find the inverse Laplace transform of $F(s) = \dfrac{1}{s-a}$ for $s > a$.

Solution

- By inspection and induction, we can conclude the kth derivative is given by

$$F^{(k)} = k!(-1)^k (s-a)^{-(k+1)}$$

- Using the formula without complex integral with L'Hospital's rule, we have

$$\begin{aligned} f(t) &= \lim_{k\to\infty} \frac{(-1)^k}{k!} \left(\frac{k}{t}\right)^{k+1} F^{(k)}\left(\frac{k}{t}\right) \\ &= \lim_{k\to\infty} \left(\frac{k}{t}\right)^{k+1} \left(\frac{k}{t} - a\right)^{-(k+1)} \\ &= \lim_{k\to\infty} \left(1 - \frac{at}{k}\right)^{-(k+1)} = \exp\left(\lim_{k\to\infty} -\frac{\ln(1 - at/k)}{1/(k+1)}\right) = e^{at} \quad t \geq 0. \end{aligned}$$

- However, it can be shown with significantly less effort the following is true

Theorem

Suppose f and g are piecewise continuous on $[0, \infty)$, and are exponential order. If

$$\mathcal{L}\big[f(t)\big] = \mathcal{L}\big[g(t)\big]$$

for $s > a$, then $f(t) = g(t)$ for $t \geq 0$.

- This guarantees the inverse is unique, and by the definition of $\mathcal{L}\big[f(t)\big]$

$$f(t) = \mathcal{L}^{-1}\big[F\big] \iff \mathcal{L}\big[f(t)\big] = F(s)$$

- In other words, there is a one-to-one correspondence between

$$f(t) \qquad \text{and} \qquad F(s)$$

so expanding our table on the Laplace transform of various functions builds our ability to identify the inverse Laplace transform of various functions.

Exercise

Verify that e^{at} is the inverse transform of $\dfrac{1}{s-a}$ for $s > a$.

Solution

- $$\begin{aligned}\mathcal{L}\left[e^{at}\right] &= \int_0^\infty e^{-st}e^{at}\,dt \\ &= \lim_{b\to\infty}\int_0^b e^{-(s-a)t}\,dt \\ &= \lim_{b\to\infty}\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^b \\ &= \lim_{b\to\infty}\left(\frac{e^{-(s-a)b}}{-(s-a)} - \frac{e^0}{-(s-a)}\right) \\ &= \frac{1}{s-a} \qquad \text{for} \quad s > a\end{aligned}$$

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>> syms s t a
>> F = 1/(s-a);
>> ilaplace(F,s,t)

ans = exp(a*t)
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- Since the Laplace transform is linear, we expect the inverse is also linear.
- Consider

$$\mathcal{L}^{-1}[c_1 F_1 + c_2 F_2] \qquad \text{where } c_1 \text{ and } c_2 \text{ are two constants,}$$

  and F_1 and F_2 are inverse Laplace transforms of f_1 and f_2, respectively.
- Since

$$\mathcal{L}[c_1 f_1 + c_2 f_2] = c_1 F_1 + c_2 F_2$$

- Thus

$$\mathcal{L}^{-1}[c_1 F_1 + c_2 F_2] = c_1 f_1 + c_2 f_2 = c_1 \mathcal{L}^{-1}[F_1] + c_2 \mathcal{L}^{-1}[F_2]$$

- Therefore the inverse transform is linear.

Exercise

Find $\mathcal{L}^{-1}\left[\dfrac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right]$, thus the solution to the last IVP.

Solution

- Using partial fraction decomposition,

$$Y(s) = \frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4}$$

$$= -\frac{16}{5}\frac{1}{s-1} + \frac{25}{6}\frac{1}{s-2} + \frac{1}{30}\frac{1}{s+4}$$

- Since we know the inverse transform is linear, and

$$\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a} \iff \mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at} \qquad \text{for} \quad s > a$$

- The inverse transform is

$$\mathcal{L}^{-1}\left[Y(s)\right] = -\frac{16}{5}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + \frac{25}{6}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + \frac{1}{30}\mathcal{L}^{-1}\left[\frac{1}{s+4}\right]$$

$$= -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t} \qquad \text{for} \quad t \geq 0$$

Transforms of some basic functions I

$f(t)$	$F(s)$	s
1	$\frac{1}{s}$	$s > 0$
t	$\frac{1}{s^2}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$t^n, \quad n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$	$s > 0$
$t^z, \quad z > -1$	$\frac{\Gamma(z+1)}{s^{z+1}}$	$s > 0$

Transforms of some basic functions II

$f(t)$	$F(s)$	s
$\sin at$	$\dfrac{a}{s^2+a^2}$	$s>0$
$\cos at$	$\dfrac{s}{s^2+a^2}$	$s>0$
$\dfrac{e^{at}-e^{-at}}{2}=\sinh at$	$\dfrac{a}{s^2-a^2}$	$s>\lvert a\rvert$
$\dfrac{e^{at}+e^{-at}}{2}=\cosh at$	$\dfrac{s}{s^2-a^2}$	$s>\lvert a\rvert$

Exercise

Solve $y^{(4)}-y=0, y'(0)=1, y(0)=y''(0)=y^{(3)}(0)=0$ by the Laplace method.

Solution

- Taking Laplace transforms:

$$y^{(4)} - y = 0 \implies \mathcal{L}\left[y^{(4)} - y\right] = \mathcal{L}\left[0\right] \implies \mathcal{L}\left[y^{(4)}\right] - \mathcal{L}\left[y\right] = \mathcal{L}\left[0\right]$$

$$\implies s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y^{(3)}(0) - Y(s) = 0$$

$$s^4Y(s) - s^2 - Y(s) = 0$$

$$\implies Y(s) = \frac{s^2}{s^4 - 1}$$

- Partial fraction decomposition leads to

$$Y(s) = \frac{As + B}{s^2 - 1} + \frac{Cs + D}{s^2 + 1} = \frac{1}{2}\frac{1}{s^2 - 1} + \frac{1}{2}\frac{1}{s^2 + 1}$$

- Look up in the table, we can conclude

$$y(t) = \frac{1}{2}\sinh(t) + \frac{1}{2}\sin(t) \qquad \text{for} \quad t \geq 0$$

First translation Theorem

Suppose $\mathcal{L}\big[f(t)\big] = F(s)$ and a is any real number, then

$$\mathcal{L}\big[e^{at} f(t)\big] = F(s-a) \qquad \text{for} \quad s > a$$

Proof

$$F(s) = \mathcal{L}\big[e^{at} f(t)\big] = \lim_{k\to\infty} \int_0^k e^{-(s-a)t} f(t)\, dt = F(s-a) \qquad \text{for} \quad s > a \quad \square$$

- In general, if we know the Laplace transform of a function f,

$$\mathcal{L}\big[f(t)\big] = F(s)$$

multiplying exponential multiple of f in the time domain t,

$$\mathcal{L}\big[e^{at} f(t)\big]$$

corresponds to translating $F(s)$ to $F(s-a)$ in the Laplace domain s.

Exercise

Find the inverse Laplace transform of $Y(s) = \dfrac{2s+5}{(s-3)^2}$.

Solution

- Partial fraction decomposition

$$\mathcal{L}^{-1}\left[F(s)\right] = \mathcal{L}^{-1}\left[\frac{A}{(s-3)^2} + \frac{B}{s-3}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{11}{(s-3)^2} + \frac{2}{s-3}\right] = 11\mathcal{L}^{-1}\left[\frac{1}{(s-3)^2}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{s-3}\right]$$

- The first term can be found by considering the Laplace transform of t,

$$\mathcal{L}\left[t\right] = \frac{1}{s^2} \implies \mathcal{L}\left[e^{3t}t\right] = \frac{1}{(s-3)^2}$$

with a translation of 3 units in the s-domain.

Solution

- The second term

$$\frac{1}{s-3}$$

is the Laplace transform of

$$e^{3t}$$

but it can also be treated as the Laplace transform of

$$1$$

with a translation of 3 units in the s-domain since

$$\mathcal{L}\left[1\right] = \frac{1}{s} \implies \mathcal{L}\left[e^{3t} \cdot 1\right] = \frac{1}{s-3}$$

- Either way

$$\mathcal{L}^{-1}\left[Y(s)\right] = 2e^{3t} + 11te^{3t} \qquad \text{for} \quad t \geq 0$$

Exercise

Find the inverse Laplace transform of

$$Y(s) = \frac{s/2 + 5/3}{s^2 + 4s + 6}$$

Solution

- This denominator is an irreducible quadratic factor. Completing the square,

$$s^2 + 4s + 6 = (s+2)^2 + 2$$

  reminds us the following Laplace transforms

$$\mathcal{L}\Big[\sin at\Big] = \frac{a}{s^2 + a^2} \qquad \text{and} \qquad \mathcal{L}\Big[\cos at\Big] = \frac{s}{s^2 + a^2}$$

- If we manipulate the numerators to create $s + 2$, we have

$$Y(S) = \frac{s/2 + 5/3}{(s+2)^2 + 2} = \frac{(s + 2 - 2)/2 + 5/3}{(s+2)^2 + (\sqrt{2})^2} = \frac{(s+2)/2 + 2/3}{(s+2)^2 + (\sqrt{2})^2}$$

Solution

- So if we consider -2 units of translation in the s-domain, and let $a = \sqrt{2}$,

$$\begin{aligned}
\mathcal{L}^{-1}[Y(s)] &= \mathcal{L}^{-1}\left[\frac{(s+2)/2 + 2/3}{(s+2)^2 + (\sqrt{2})^2}\right] \\
&= \mathcal{L}^{-1}\left[\frac{1}{2}\frac{(s+2)}{(s+2)^2 + (\sqrt{2})^2}\right] + \frac{2}{3}\mathcal{L}^{-1}\left[\frac{1}{(s+2)^2 + (\sqrt{2})^2}\right] \\
&= \frac{1}{2}\mathcal{L}^{-1}\left[\frac{(s-(-2))}{(s-(-2))^2 + (\sqrt{2})^2}\right] + \frac{\sqrt{2}}{3}\mathcal{L}^{-1}\left[\frac{\sqrt{2}}{(s-(-2))^2 + (\sqrt{2})^2}\right] \\
&= \frac{1}{2}e^{-2t}\cos(\sqrt{2}t) + \frac{\sqrt{2}}{3}e^{-2t}\sin(\sqrt{2}t) \qquad \text{for} \quad t \geq 0
\end{aligned}$$

Exercise

Solve the following initial-value problem using the Laplace's method

$$y'' + 4y' + 6y = 1 + e^{-t}, \qquad y(0) = 0, \quad y'(0) = 0$$

Solution

- Take the Laplace transform

$$\begin{aligned} \mathcal{L}[y''] + 4\mathcal{L}[y'] + 6\mathcal{L}[y] &= \mathcal{L}[1] + \mathcal{L}[e^{-t}] \\ s^2Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 6Y &= \frac{1}{s} + \frac{1}{s+1} \\ s^2Y(s) + 4sY(s) + 6Y(s) &= \frac{2s+1}{s(s+1)} \end{aligned}$$

- Make $Y(s)$ the subject, and compute the partial fraction decomposition.

$$Y(s) = \frac{2s+1}{s(s+1)(s^2+4s+6)} = \frac{1}{6}\frac{1}{s} + \frac{1}{3}\frac{1}{s+1} - \frac{s/2+5/3}{s^2+4s+6}$$

- Back transform

$$\begin{aligned} y(t) &= \frac{1}{6}\mathcal{L}^{-1}\left[\frac{1}{s}\right] + \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{s/2+5/3}{s^2+4s+6}\right] \\ &= \frac{1}{6} + \frac{1}{3}e^{-t} - \frac{1}{2}e^{-2t}\cos(\sqrt{2}t) - \frac{\sqrt{2}}{3}e^{-2t}\sin(\sqrt{2}t) \qquad \text{for} \quad t \geq 0 \end{aligned}$$

Definition

The unit step function or Heaviside function $u(t-a)$ or $u_a(t)$ is defined to be

$$u(t-a) = \begin{cases} 0 & \text{if} \quad t < a, \\ 1 & \text{if} \quad t \geq a. \end{cases}$$

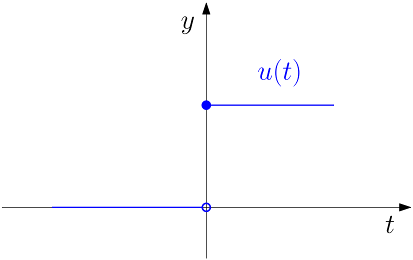
- There are various convention when comes to defining $u(t-a)$,

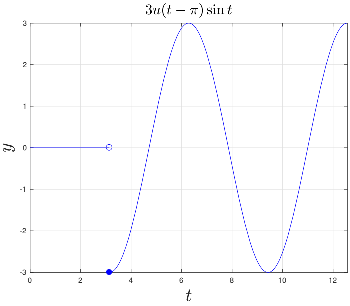
$$u(t-a) = \begin{cases} 0 & \text{if} \quad t < a, \\ 1 & \text{if} \quad t > a. \end{cases} \qquad \text{or} \qquad u(t-a) = \begin{cases} 0 & \text{if} \quad t < a, \\ 0.5 & \text{if} \quad t = a, \\ 1 & \text{if} \quad t > a. \end{cases}$$

- It doesn't make a great deal of difference to us, and we'll use the first with

$$a \geq 0$$

being our assumption, and consider cases where $t \geq 0$.





- The step function is useful in engineering because it can be used to describe

on and off effect

Q: Consider the graph of

$$f(t) = 3\sin t$$

what does the graph of $u(t-\pi)f(t)$ look like?

Q: How to turn off a function after $t = a$ using $u(t-a)$?

- The step function can also be used to write piecewise-defined functions in a compact form. Consider a general piecewise-defined function of the type

$$f(t) = \begin{cases} g(t) & \text{if} \quad 0 \leq t < a, \\ h(t) & \text{if} \quad t \geq a. \end{cases}$$

- The above function is the same as the following function

$$f(t) = g(t) - g(t)u(t-a) + h(t)u(t-a)$$

Exercise

Express $f(t) = \begin{cases} 0 & \text{if } 0 \leq t < a \\ g(t) & \text{if } a \leq t < b \\ 0 & \text{if } t \geq b. \end{cases}$ in terms of unit step functions and $g(t)$.

Solution

$$f(t) = g(t)u(t-a) - g(t)u(t-b)$$

- Recall the first translation theorem:

$$\mathcal{L}\left[e^{at}f(t)\right] = F(s-a)$$

Q: What happens if an exponential function is multiplied to $F(s)$?

Second Translation Theorem

If $F(s) = \mathcal{L}\big[f(t)\big]$ and $a > 0$, then

$$\mathcal{L}\big[f(t-a)u(t-a)\big] = e^{-as}F(s)$$

Proof

- $$\begin{aligned}\mathcal{L}[f(t-a)u(t-a)] &= \lim_{b\to\infty}\int_0^b e^{-st}f(t-a)u(t-a)\,dt \\ &= \int_0^a e^{-st}f(t-a)\cdot 0\,dt + \lim_{b\to\infty}\int_a^b e^{-st}f(t-a)\cdot 1\,dt \\ &= \lim_{b\to\infty}\int_a^b e^{-st}f(t-a)\,dt\end{aligned}$$

Proof

- Now let $v = t - a$, then $\dot{v} = 1$ and

$$v(a) = 0 \qquad \text{and} \qquad v(b) = b - a$$

$$\begin{aligned}
\mathcal{L}[f(t-a)u(t-a)] &= \lim_{b\to\infty} \int_a^b e^{-st} f(t-a)\, dt \\
&= \lim_{b\to\infty} \int_0^{b-a} e^{-s(v+a)} f(v)\, dv \\
&= e^{-as} \lim_{b\to\infty} \int_0^{b-a} e^{-sv} f(v)\, dv \\
&= e^{-as} \int_0^{\infty} e^{-sv} f(v)\, dv = e^{-as} \mathcal{L}\big[f(t)\big] = e^{-as} F(s) \qquad \square
\end{aligned}$$

Exercise

Find the Laplace transform of $f(t) = t^2 u(t-2)$.

Solution

- From the last theorem, we know

$$\mathcal{L}\big[f(t-2)u(t-2)\big] = e^{-2s}F(s)$$

- So we have to do some manipulation so that we have a function of $t-2$.

$$t^2 = t^2 - 4t + 4 + 4t - 4 = (t-2)^2 + 4(t-2) + 4$$

- Thus the Laplace transform of the given function is

$$\begin{aligned}
\mathcal{L}\big[t^2u(t-2)\big] &= \mathcal{L}\big[(t-2)^2u(t-2)\big] + 4\mathcal{L}\big[(t-2)u(t-2)\big] + 4\mathcal{L}\big[u(t-2)\big] \\
&= e^{-2s}\frac{2!}{s^3} + 4e^{-2s}\frac{1}{s^2} + 4e^{-2s}\frac{1}{s} \\
&= 4e^{-2s}\left(\frac{1}{2s^3} + \frac{1}{s^2} + \frac{1}{s}\right) \qquad \text{for} \quad s > 0
\end{aligned}$$

- The following gives an alternative version of the second translation theorem.

Alternative Second Translation Theorem

If $a > 0$, then

$$\mathcal{L}\big[g(t)u(t-a)\big] = e^{-as}\mathcal{L}\big[g(t+a)\big]$$

Proof

- $$\mathcal{L}[g(t)u(t-a)] = \lim_{b\to\infty}\int_0^b e^{-st}g(t)u(t-a)\,dt = \lim_{b\to\infty}\int_a^b e^{-st}g(t)\,dt$$
- Let $v = t - a$, then $\dot{v} = 1$, and $v(a) = 0$ and $v(b) = b - a$

$$\mathcal{L}[g(t)u(t-a)] = \lim_{b\to\infty}\int_0^{b-a} e^{-s(v+a)}g(v+a)\,dv = e^{-as}\int_0^{\infty} e^{-vs}g(v+a)\,dv = e^{-as}\mathcal{L}\big[g(t+a)\big] \quad \square$$

Exercise

Solve the following initial-value problem.

$$\dot{y} + y = f(t), \quad y(0) = 5, \quad \text{where} \quad f(t) = \begin{cases} 0 & \text{if } 0 \le t < \pi \\ 3\cos t & \text{if } t \ge \pi. \end{cases}$$

Solution

- Rewrite the forcing function using the unit step function

$$f(t) = 3u(t-\pi)\cos t$$

- Take the Laplace transform by using the alternative 2nd translation theorem

$$\mathcal{L}\Big[f(t)\Big] = \mathcal{L}\Big[3u(t-\pi)\cos t\Big] = 3e^{-\pi s}\mathcal{L}\Big[\cos(t+\pi)\Big] = -3e^{-\pi s}\mathcal{L}\Big[\cos t\Big]$$

$$= -3e^{-\pi s}\frac{s}{s^2+1}$$

Solution

- Now consider the Laplace transform of the left-hand side of the equation

$$\mathcal{L}\Big[\dot{y} + y\Big] = sY(s) - y(0) + Y(s)$$

- Equate the two expressions in the s-domain, and make $Y(s)$ the subject

$$sY(s) - y(0) + Y(s) = -3e^{-\pi s}\frac{s}{s^2+1} \qquad \text{where} \quad y(0) = 5$$

$$\begin{aligned}\implies Y(s) &= 5 \cdot \frac{1}{s+1} - 3 \cdot e^{-\pi s}\frac{s}{(s^2+1)(s+1)} \\ &= 5 \cdot \frac{1}{s+1} - 3 \cdot e^{-\pi s} \cdot \frac{1}{2}\left(\frac{-1}{s+1} + \frac{1}{s^2+1} + \frac{s}{s^2+1}\right) \\ &= 5 \cdot \frac{1}{s+1} + \frac{3}{2} \cdot e^{-\pi s}\frac{1}{s+1} \\ &\qquad - \frac{3}{2} \cdot e^{-\pi s}\frac{1}{s^2+1} - \frac{3}{2} \cdot e^{-\pi s}\frac{s}{s^2+1}\end{aligned}$$

Solution

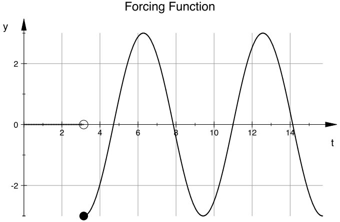
- By using the second translation theorem, we have

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\Big[Y(s)\Big] \\
&= \mathcal{L}^{-1}\left[5\left[\frac{1}{s+1}\right] + \frac{3}{2}\left[e^{-\pi s}\left[\frac{1}{s+1}\right] - e^{-\pi s}\left[\frac{1}{s^2+1}\right] - e^{-\pi s}\left[\frac{s}{s^2+1}\right]\right]\right] \\
&= 5e^{-t} + \frac{3}{2}\Big(e^{-(t-\pi)}u(t-\pi) - \sin(t-\pi)u(t-\pi) - \cos(t-\pi)u(t-\pi)\Big) \\
&= 5e^{-t} + \frac{3}{2}\Big(e^{-(t-\pi)} - \sin(t-\pi) - \cos(t-\pi)\Big)\, u(t-\pi) \\
&= 5e^{-t} + \frac{3}{2}\Big(e^{-(t-\pi)} + \sin(t) + \cos(t)\Big)\, u(t-\pi) \\
&= \begin{cases} 5e^{-t} & 0 \le t < \pi \\ 5e^{-t} + \dfrac{3}{2}\Big(e^{-(t-\pi)} + \sin t + \cos t\Big) & t \ge \pi \end{cases}
\end{aligned}
$$

Solution

- Note that even though the forcing function is discontinuous,

$$f(t) = \begin{cases} 0 & \text{if } 0 \le t < \pi \\ 3\cos t & \text{if } t \ge \pi. \end{cases}$$



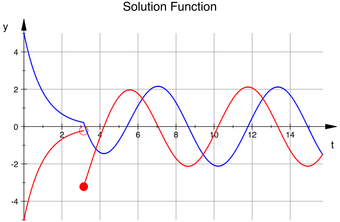
$$0 = \lim_{t \to \pi^-} f(t) \neq \lim_{t \to \pi^+} f(t) = -3$$

the solution $y(t)$ found by using Laplace, is a continuous, while $\dot{y}(t)$ is not

$$5e^{-\pi} = \lim_{t \to \pi^-} y = \lim_{t \to \pi^+} y = 5e^{-\pi}$$

$$\lim_{t \to \pi^-} \dot{y} \neq \lim_{t \to \pi^+} \dot{y}$$

$$-5e^{-\pi} \neq -5e^{-\pi} - 3$$



- So the method of Laplace transform may give solutions, e.g.

$$y(t) = \begin{cases} 5e^{-t} & 0 \le t < \pi \\ 5e^{-t} + \frac{3}{2}\left(e^{-(t-\pi)} + \sin t + \cos t\right) & t \ge \pi \end{cases}$$

  that do not fulfil the differential equation in the usual sense for all $t > 0$, we shall consider the above as a generalised solution, extending beyond $t = \pi$.

$$y(t) = 5e^{-t} \qquad \text{for} \quad 0 < t < \pi$$

- This generalisation makes sense since there is one and only one solution that makes $y(t)$ continuous at $t = \pi$, and this is given by the Laplace transform.
- Alternatively, in this case, we can also get this generalised solution from

$$y(t) = C\mu^{-1} + \mu^{-1} \int_{t^{**}}^{t} \mu(\xi) Q(\xi)\, d\xi, \qquad \mu = \exp\left(\int_{t^{*}}^{t} P(\xi)\, d\xi\right)$$

- The definition of solution to a 2nd-order linear equation is extended similarly

$$\ddot{y} + P(t)\dot{y} + Q(t)y = f(t)$$

where P and Q are continuous, but f is only piecewise continuous. Instead of demanding a solution to be twice differentiable in an open interval, we say

$$y = \phi(t)$$

is a solution if ϕ is twice differentiable and satisfies the equation on each open interval where f is continuous, while ϕ and $\dot{\phi}$ are both continuous at the points of discontinuity of f.

- Notice ϕ is usually not twice differentiable at points of discontinuity of f.
- Similar things can be said to higher order linear equations with piecewise continuous forcing functions. The highest order derivative that is present in the equation may have discontinuities at the same places as the forcing function, while the solution and lower order derivative are continuous.

Q: What types of discontinuities the highest order derivative may have?