Vv156 Lecture 10

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- So far we have been concerned with differentiating functions that are given by

equations of the form $y = f(x)$.

Definition

A function in which the dependent variable is written explicitly in terms of the independent variable is called an explicit function. We say

$$y \text{ is explicitly defined by } \quad y = f(x)$$

- Functions can be defined by equations in which y is not alone on one side, e.g

$$xy + y + 1 = x \tag{1}$$

is not of the form $y = f(x)$, but equation (1) still defines y as a function of x,

$$xy + y + 1 = x \implies y(x+1) = x - 1 \implies y = \frac{x-1}{x+1}$$

- Here we say y is implicitly defined as a function of x by equation (1).

Definition

An implicit equation is a relation between variables, which cannot, in general, be isolated on their own, or solved in terms of other variables. An implicit function is a function that is defined implicitly by an implicit equation.

- For example,

$$x^2 + y^2 = 1$$

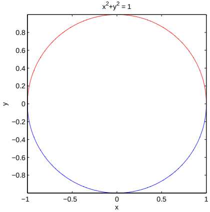
- An implicit equation can implicitly define more than one function of x.

Matlab

```
>> syms x y
>> obj = ezplot('x^2+y^2=1', [-1,1,0,1]);
>> set(obj, 'color','red'); clear obj
>> hold on
>> obj = ezplot('x^2+y^2=1', [-1,1,-1,0]);
>> set(obj, 'color','blue'); clear obj
>> hold off
>> axis([-1,1,-1,1])
>> axis equal tight
```

$$y = \sqrt{1 - x^2}$$

$$y = -\sqrt{1 - x^2}$$



- So here we have two functions implicitly defined by the equation

- In general, it is not necessary to solve an equation for y in terms of x in order to differentiate the functions defined implicitly. To illustrate this, consider

$$xy = 1$$

- One way to find dy/dx is to rewrite this equation as

$$xy = 1 \implies y = \frac{1}{x} \implies \frac{dy}{dx} = \frac{-1}{x^2}$$

- Another way is to differentiate both sides of the original equation

$$xy = 1 \implies \frac{d}{dx}(xy) = \frac{d}{dx}(1) \implies x\frac{dy}{dx} + y\frac{d}{dx}(x) = 0$$

$$\implies x\frac{dy}{dx} + y \cdot (1) = 0 \implies \frac{dy}{dx} = -\frac{y}{x}$$

- Then solve for y in terms of x, and make a substitution $\frac{dy}{dx} = -\frac{1/x}{x} = \frac{-1}{x^2}$

- This is known as the implicit differentiation.

Exercise

(a) *Use implicit differentiation to find y' for the implicit equation $5y^2 + \sin y = x^2$.*

(b) *Find an equation of the tangent line to the circle at the point $(3, 4)$.*

$$x^2 + y^2 = 25$$

(c) *Show that if a normal line to each point on an ellipse passes through the center of an ellipse, then the ellipse is circle.*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

- When differentiating implicitly, it is assumed that y represents a differentiable function of x. If this is not so, then the resulting calculations may be nonsense.

(d) Use implicit differentiation to find y' if

$$x^2 + y^2 + 1 = 0$$

Theorem

The natural logarithmic function $f(x) = \ln x$ is differentiable, and moreover

$$f'(x) = \frac{1}{x}, \qquad \text{for} \qquad x > 0.$$

Proof

- By definition,

$$\frac{d}{dx}(\ln x) = \lim_{h\to 0} \frac{\ln(x+h) - \ln(x)}{h} = \lim_{h\to 0} \frac{1}{h} \ln\left(\frac{x+h}{x}\right) = \lim_{h\to 0} \frac{1}{h} \ln\left(1 + \frac{h}{x}\right)$$

$$\text{Let } v = \frac{h}{x}, \text{ notice } v \to 0 \text{ as } h \to 0, \qquad = \lim_{v\to 0} \frac{1}{vx} \ln(1+v)$$

$$\text{Let } u = \frac{1}{v}, \text{ notice } u \to \infty \text{ as } v \to 0, \qquad = \frac{1}{x} \lim_{v\to 0} \ln(1+v)^{\frac{1}{v}}$$

$$\text{Recall } \lim_{x\to\infty} \left(1 + \tfrac{1}{x}\right)^x = e, \qquad = \frac{1}{x} \lim_{u\to\infty} \ln\left(1 + \frac{1}{u}\right)^u$$

$$\Longrightarrow \frac{d}{dx}(\ln x) = \frac{1}{x} \ln\left(\lim_{u\to\infty}\left(1 + \frac{1}{u}\right)^u\right) = \frac{1}{x} \ln e = \frac{1}{x} \qquad \text{for} \qquad x > 0.$$

- Implicit differentiation and the derivative of the logarithmic function can be used to prove the general power rule.

The General Power Rule

If r is any real number, then

$$\frac{d}{dx}\left(x^r\right) = rx^{r-1}$$

Proof

- Let $y = x^r$, so $\ln y = r \ln x$, then we apply implicit differentiation,

$$\frac{1}{y}\frac{dy}{dx} = \frac{r}{x}$$
$$\implies \frac{dy}{dx} = r\frac{y}{x} = rx^{r-1}$$

$\square$

Theorem

For the general logarithmic function

$$\frac{d}{dx}\Big(\log_b x\Big) = \frac{1}{x\ln b}, \qquad \text{for } x > 0, \text{ and } b > 0.$$

Proof

- Starting from the left-hand side,

$$\begin{aligned}
\frac{d}{dx}\Big(\log_b x\Big) &= \frac{d}{dx}\left(\frac{\ln x}{\ln b}\right) && \text{Property of logarithmic function} \\
&= \frac{1}{\ln b}\frac{d}{dx}(\ln x) \\
&= \frac{1}{x\ln b}
\end{aligned}$$

Exercise

(a) *Find the derivative function for*

$$y = \frac{x^2 \sqrt[3]{7x - 14}}{(1 + x^2)^4}$$

(b) *Find the values of h, k, and a that make the circle*

$$(x - h)^2 + (y - k)^2 = a^2$$

tangent to the parabola $y = x^2 + 1$ at the point $(1, 2)$, that is, they share the same tangent line, and that also make the second derivatives y'' have the same value on both curves there. Such circles are called osculating circles (from the Latin osculari, meaning "to kiss").

(c) *Suppose that f is an one-to-one differentiable function such that*

$$f(2) = 1 \qquad \textit{and} \qquad f'(2) = 3/4$$

Evaluate $\left(f^{-1}\right)'(1)$.

Theorem

In general, if f is a differentiable and one-to-one function, then

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)} \qquad \text{provided} \quad f'\left(f^{-1}(x)\right) \neq 0$$

Proof

- Notice that $y = f^{-1}(x)$ is equivalent to $x = f(y)$, if we differentiate implicitly,

$$1 = \frac{d}{dy} f(y) \frac{dy}{dx} \implies \frac{dy}{dx} = \frac{1}{\frac{d}{dy}\left(f(y)\right)} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

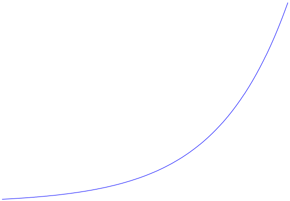
- Using $x = f(y)$, we have $\dfrac{dx}{dy} = f'(y)$, so essentially this theorem states,

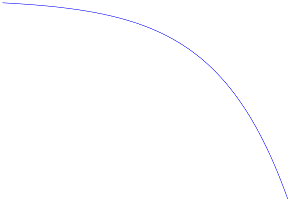
$$\frac{dy}{dx} = \frac{1}{dx/dy}$$

Theorem

Suppose that the domain of a function $f(x)$ is an open interval

1. on which $f'(x) > 0$
2. on which $f'(x) < 0$.





then $f(x)$ is one-to-one, and moreover

$$f^{-1} \text{ is differentiable at all values of } x \text{ in the range of } f.$$

- Our next objective is to show the following theorem

Theorem

The general exponential function

$$b^x, \qquad \text{where } b > 0,$$

is differentiable everywhere, and moreover

$$\frac{d}{dx}\left(b^x\right) = b^x \ln b$$

Proof

- b^x is the inverse of $\log_b x$, and we know the derivative of this inverse,

$$\frac{d}{dx}\left(\log_b x\right) = \begin{cases} \dfrac{1}{x \ln b} > 0 & \text{for all } x > 0 \text{ and } b > 1, \\ \dfrac{1}{x \ln b} < 0 & \text{for all } x > 0 \text{ and } 0 < b < 1, \end{cases}$$

Proof

- So given a certain value of b, the derivative of $\log_b x$ is either always positive or always negative, thus the theorem P11 guarantees that b^x is differentiable.
- To obtain the formula for the derivative, we implicitly differentiate w.r.t x

$$
\begin{aligned}
x &= \log_b y \\
\implies 1 &= \frac{1}{y \ln b} \frac{dy}{dx} \\
\implies \frac{dy}{dx} &= y \ln b = b^x \ln b
\end{aligned}
$$

- In the special case when $b = e$, we have

$$\ln e = 1$$

- Thus

$$\frac{d}{dx} e^x = e^x \cdot 1 = e^x$$

- If u is differentiable function of x and $b > 0$, then

$$\frac{d}{dx}\left(b^u\right) = \frac{d}{du}\left(b^u\right) \cdot \frac{du}{dx} = b^u \ln b \frac{du}{dx}$$

- You might be tempted to use this result to find

$$\frac{d}{dx}\left[\left(x^2+1\right)^{\sin x}\right] = (x^2+1)^{\sin x} \ln(x^2+1) \frac{d}{dx} \sin x$$

- This is not correct! Because the base b is not a constant.
- The correct way, we let $y = (x^2+1)^{\sin x}$, then

$$\ln y = \sin x \ln(x^2+1)$$

- Differentiate implicitly with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = \cos x \ln(x^2+1) + \frac{2x \sin x}{x^2+1}$$

$$\Longrightarrow \frac{dy}{dx} = y\left[\cos x \ln(x^2+1) + \frac{2x \sin x}{x^2+1}\right]$$

- The theorem P10 is useful to find the derivative of inverse trig function, e.g.

$$\frac{d}{dx}\left(\sin^{-1} x\right)$$

- Since $\sin x$ is dfferentiable for all x, thus $f^{-1}(x) = \sin^{-1}(x)$ is differentiable at any point of x such that

$$\cos\left(\sin^{-1}(x)\right) \neq 0$$

that is

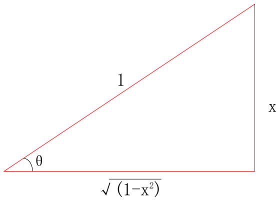
$$\sin^{-1}(x) \neq -\frac{\pi}{2} \qquad \text{and} \qquad \sin^{-1}(x) \neq \frac{\pi}{2}$$

so $\sin^{-1}(x)$ is differentiable on $(-1, 1)$

- By the theorem P10

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\cos\left(\sin^{-1}(x)\right)} \qquad \text{for} \qquad -1 < x < 1$$

- Consider the triangle below,



- Notice

$$\sin\theta = \frac{x}{1} = x \implies \sin^{-1} x = \theta$$

- Also

$$\cos\theta = \sqrt{1-x^2} = \cos\left(\sin^{-1} x\right)$$

- Thus

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \text{for} \qquad -1 < x < 1$$

Basic Differentiation Formulas

$$\frac{d}{dx}c = 0 \qquad \frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}a^x = a^x \ln a$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x} \qquad \frac{d}{dx}\log_a x = \frac{1}{x \ln a}$$

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\csc x = -\csc x \cot x \qquad \frac{d}{dx}\sec x = \sec x \tan x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2}$$