Vv255 Lecture 26

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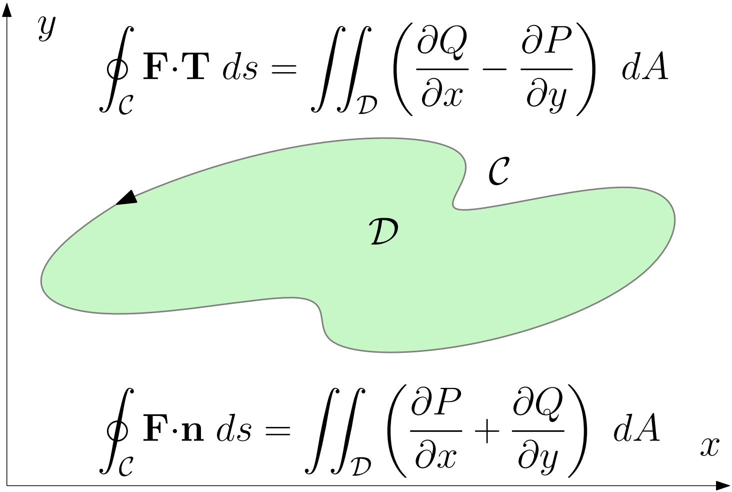
- We have learned Green's theorem serves as a bridge between a line integral over a closed plane curve $\mathcal{C}$ and a double integral over a region $\mathcal{D}$, it relates

$$\oint_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\, ds = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\, dA$$



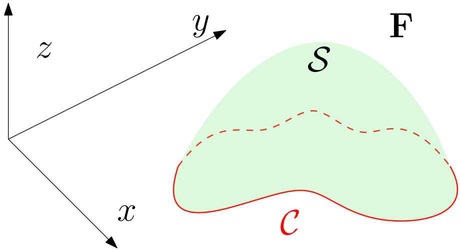
$$\oint_{\mathcal{C}} \mathbf{F}\cdot\mathbf{n}\, ds = \iint_{\mathcal{D}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\, dA$$

1. A line integral of $\mathbf{F}$ in $\mathbb{R}^2$ over a closed plane curve $\mathcal{C}$ to an integral of a scalar component of $\operatorname{curl}\mathbf{F}$ over the region that is enclosed by $\mathcal{C}$.

$$\oint_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\,ds = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA = \iint_{\mathcal{D}} (\operatorname{curl}\mathbf{F})\cdot\mathbf{e}_z\,dA$$

- Our next goal is to generalize this result to $\mathbb{R}^3$, that is, to relate

1. A line integral of $\mathbf{F}$ in $\mathbb{R}^3$ over a closed space curve $\mathcal{C}$ to an integral of a scalar component of $\operatorname{curl}\mathbf{F}$ over the surface $\mathcal{S}$ enclosed by $\mathcal{C}$.



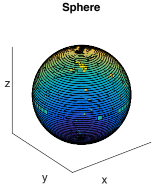
- The surface $\mathcal{S}$ is enclosed by $\mathcal{C}$ in the sense that $\mathcal{C}$ is the only boundary of $\mathcal{S}$.

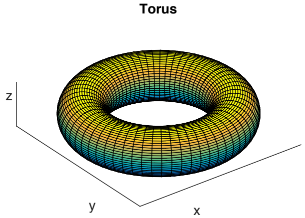
2. A line integral of the normal component of $\mathbf{F}$ in $\mathbb{R}^2$ over a closed curve $\mathcal{C}$ to a double integral of $\operatorname{div}\mathbf{F}$ over the region that is enclosed by $\mathcal{C}$.

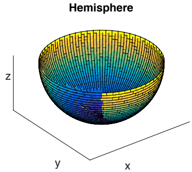
$$\oint_{\mathcal{C}} \mathbf{F}\cdot\mathbf{n}\,ds = \iint_{\mathcal{D}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\,dA = \iint_{\mathcal{D}} \operatorname{div}\mathbf{F}\,dA$$

- We wish to generalize this result as well to $\mathbb{R}^3$, that is, to relate

2. An integral of the normal component of $\mathbf{F}$ in $\mathbb{R}^3$ *over a closed surface* $\mathcal{S}$ to a triple integral of $\operatorname{div}\mathbf{F}$ over the solid $\mathcal{E}$ that is enclosed by $\mathcal{S}$.







- In order to realize the two generalizations, we need to define integration over

  piecewise smooth surfaces,

  just as we now know how to integrate over

  piecewise smooth curves.

- Recall a space curve $\mathcal{C}$ is conveniently described parametrically by

$$\mathbf{r}(t) = x(t)\mathbf{e}_x + y(t)\mathbf{e}_y + z(t)\mathbf{e}_z, \qquad \text{where} \quad a \le t \le b,$$

Definition

The vector function together with the domain $\mathcal{D}$,

$$\mathbf{r}(u, v) = x(u, v)\mathbf{e}_x + y(u, v)\mathbf{e}_y + z(u, v)\mathbf{e}_z$$

is called a parametric surface.

- Recall if $\mathbf{r}(t)$ is continuously differentiable and $\mathbf{r}'(t) \neq 0$, then we say

  a smooth curve

Q: Can you think of a sensible definition of smoothness for parametric surfaces?

Definition

A parametric surface is smooth provided the following two conditions are satisfied:

$$\mathbf{r}(u, v) = x(u, v)\mathbf{e}_x + y(u, v)\mathbf{e}_y + z(u, v)\mathbf{e}_z$$

1. The partial derivatives are continuous

$$\frac{\partial \mathbf{r}}{\partial u} \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial v}$$

2. The cross product between partial derivatives is non-zero

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \neq \mathbf{0}$$

in the interior of the domain of $\mathbf{r}(u, v)$.

Q: What does the second condition mean? Why we need it?

- We used the following formula for finding the area of a parametric surface $\mathcal{S}$

$$S = \iint_{\mathcal{D}} \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \, dA$$

  defined by the vector-valued function of u and v,

$$\mathbf{r}(u, v) = x(u, v)\mathbf{e}_x + y(u, v)\mathbf{e}_y + z(u, v)\mathbf{e}_z$$

  over a region of $\mathcal{D}$ in the uv-plane.

- Recall we partition the domain of the surface defined by

$$z = f(x, y) \qquad \text{over a region } \mathcal{D} \text{ in the } xy\text{-plane.}$$

  we approximate each small area on the surface by the area of a parallelogram

$$\Delta S_i \approx \Delta T_i = |\mathbf{a} \times \mathbf{b}|$$

Q: Do you remember what $\mathbf{a}$ and $\mathbf{b}$ are in the above cross product?

- The area is defined to be the limit of the Riemann sum

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta T_i = \lim_{n\to\infty} \sum_{i=1}^{n} \left| \left(-\frac{\partial f}{\partial x}\mathbf{e}_x - \frac{\partial f}{\partial y}\mathbf{e}_y + \mathbf{e}_z \right) \right| \Delta x_i \Delta y_i$$
$$= \iint_{\mathcal{D}} \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dA$$

- Now suppose we have a parametrization for the surface

$$\mathbf{r}(u,v) = x(u,v)\mathbf{e}_x + y(u,v)\mathbf{e}_y + z(u,v)\mathbf{e}_z$$

over a region of $\mathcal{D}^*$ in the uv-plane instead of

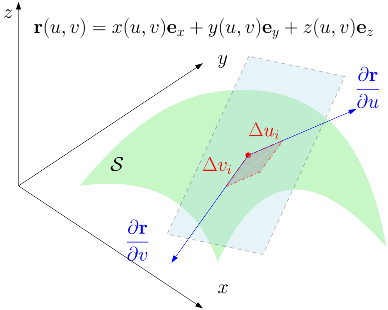
$z = f(x,y)$ over a region $\mathcal{D}$ in the xy-plane.

Q: How to find the small change in area ΔT_i on the tangent plane induced by

Δu_i and Δv_i

in the uv-plane?

- Since $\dfrac{\partial \mathbf{r}}{\partial u}$ and $\dfrac{\partial \mathbf{r}}{\partial v}$ are the tangent vectors



- Thus the area can be approximated by

$$\Delta S_i \approx \Delta T_i = \left| \Delta u_i \frac{\partial \mathbf{r}}{\partial u} \times \Delta v_i \frac{\partial \mathbf{r}}{\partial v} \right| = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \Delta u_i \Delta v_i$$

- Now the limit of the Riemann sum is

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta T_i = \lim_{n\to\infty} \sum_{i=1}^{n} \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \Delta u_i \Delta v_i = \iint_{\mathcal{D}} \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \, dA$$

Definition

If a smooth parametric surface $\mathcal{S}$ is defined by

$$\mathbf{r}(u,v), \qquad \text{where} \quad (u,v) \in \mathcal{D}$$

and $\mathcal{D}$ is a region in the uv-plane, then the surface area of $\mathcal{S}$ is

$$S = \iint_{\mathcal{D}} \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \, dA$$

Exercise

Find the area of the surface $\mathcal{S}$ that is the part of the plane

$$2x + 5y + z = 10$$

that lies inside the cylindrical

$$x^2 + y^2 = 9$$

Solution

- It is clear that we can use the following parametrization

$$\mathbf{r}(u,v) = (u\cos v)\mathbf{e}_x + (u\sin v)\mathbf{e}_y + (10 - u(2\cos v + 5\sin v))\,\mathbf{e}_z$$

  where $0 \le u \le 3$ and $0 \le v \le 2\pi$.

- Compute the partial derivatives,

$$\frac{\partial \mathbf{r}}{\partial u} = (\cos v)\mathbf{e}_x + (\sin v)\mathbf{e}_y - (2\cos v + 5\sin v)\mathbf{e}_z$$

$$\frac{\partial \mathbf{r}}{\partial v} = (-u\sin v)\mathbf{e}_x + (u\cos v)\mathbf{e}_y + u(2\sin v - 5\cos v)\mathbf{e}_z$$

- Use the formula,

$$A(s) = \int_0^3 \int_0^{2\pi} \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \, du\, dv = \int_0^3 \int_0^{2\pi} u\sqrt{30}\, du dv = 9\pi\sqrt{30}$$

- Suppose $\mathcal{C}$ is a smooth curve in $\mathbb{R}^3$ that is parametrized by a function

$$\mathbf{r}(t) = x(t)\mathbf{e}_x + y(t)\mathbf{e}_y + z(t)\mathbf{e}_z, \qquad a \le t \le b,$$

then the arc length of $\mathcal{C}$ is given by the integral

$$\int_a^b |\mathbf{r}'(t)|\, dt$$

- The integral of a scalar-valued function $f(x, y, z)$ along $\mathcal{C}$ is given by

$$\int_{\mathcal{C}} f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))\; |\mathbf{r}'(t)|\, dt$$

Q: Is there a similar link between a surface integral and the surface area ?

$$\iint_{\mathcal{D}} \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \, dA$$

Q: What is the meaning of integrating over a smooth surface $\mathcal{S}$?

- Suppose a smooth surface $\mathcal{S}$ is defined by

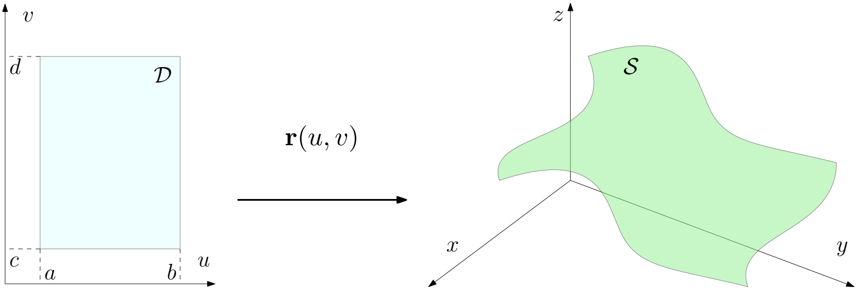
$$\mathbf{r}(u,v) = x(u,v)\mathbf{e}_x + y(u,v)\mathbf{e}_y + z(u,v)\mathbf{e}_z \quad \text{for} \quad (u,v) \in \mathcal{D}$$

and the density per unit area of the surface is given by the continuous

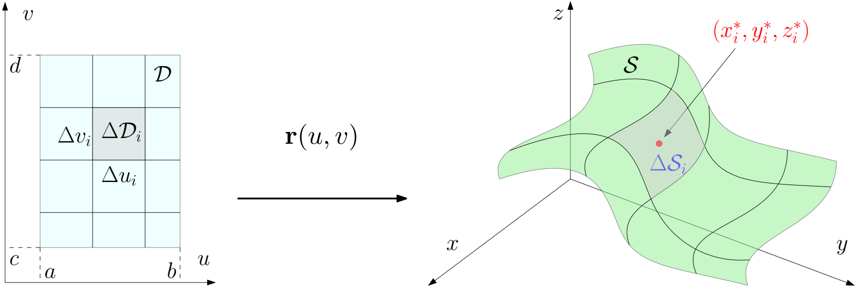
$$f(x,y,z)$$

Q: What is the mass of the surface?

- For simplicity, suppose $\mathcal{D} = [a,b] \times [c,d]$ is a rectangle in the uv-plane.



- If we divide $\mathcal{D}$ into sub-rectangles, then $\mathcal{S}$ will be divided into small patches



- The vector-valued function

$$\mathbf{r}(u, v)$$

maps ith sub-rectangle into ith surface patch that has area ΔS_i.

- Thus the mass of each surface patch can be approximated by

$$f(x_i^*, y_i^*, z_i^*)\Delta S_i$$

where (x_i^*, y_i^*, z_i^*) is any sample point on the ith surface patch.

- If we use the approximation for surface area we derive earlier,

$$\Delta S_i \approx \Delta T_i = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \Delta u_i \Delta v_i$$

then the Riemann sum over all possible i gives an approximation for the mass

$$\sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta S_i = \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*) \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \Delta u_i \Delta v_i$$

we expect the sum to converge as $n \to \infty$ since $\mathbf{r}$ and f are well-behaved.

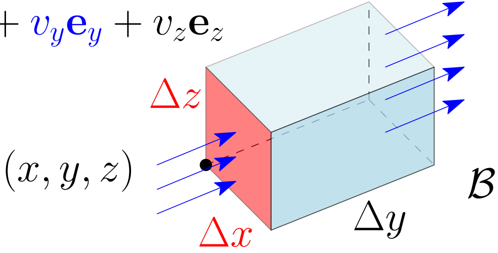
Definition

Suppose $\mathcal{S}$ is a smooth parametric surface defined by $\mathbf{r}(u,v)$ over $\mathcal{D}$ and $f(x,y,z)$ is a continuous function, then the surface integral of f over $\mathcal{S}$ is defined to be

$$\iint_{\mathcal{S}} f(x,y,z)\, dS = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta S_i = \iint_{\mathcal{D}} f(x,y,z) \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| dA$$

- Recall to find the total amount of fluid across a plane,

$$\mathbf{V} = v_x\mathbf{e}_x + v_y\mathbf{e}_y + v_z\mathbf{e}_z$$



we simply compute the approximation at (x, y, z)

$$\underbrace{\rho v_y \Delta x \Delta z}_{\text{flux}} \Delta t$$

where $\mathbf{V}$ is the velocity field of the flow and $\rho(x, y, z)$ is the density function.

Q: What would be the complication of computing the flux if we have a smooth parametric surface $\mathcal{S}$ instead of a plane that is parallel to a coordinate plane?

- Recall the flux in $\mathbb{R}^2$ across a positively oriented closed curve $\mathcal{C}$ is defined by

$$\oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{n}\, ds \qquad \text{where} \quad \mathbf{n} = \frac{dy}{ds}\mathbf{e}_x - \frac{dx}{ds}\mathbf{e}_y$$

- Roughly speaking, it sums only the normal component of $\mathbf{F}$ along $\mathcal{C}$.
- Of course, we do the same to flux across a smooth surface in $\mathbb{R}^3$,

$$\iint_{\mathcal{S}} (\rho \mathbf{V}) \cdot \mathbf{n}\, dS$$

where $\mathbf{n}$ is a unit normal vector to the smooth surface $\mathbf{r}(u, v)$.

Definition

Suppose $\mathbf{F}$ is a continuous vector field and $\mathcal{S}$ is an oriented smooth surface, then

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} \qquad \text{where } \mathbf{n} \text{ is unit normal to } \mathcal{S}.$$

is known as the surface integral of $\mathbf{F}$ over $\mathcal{S}$, a.k.a the flux integral of $\mathbf{F}$ across $\mathcal{S}$.

- Just as the orientation of a curve was relevant to the line integral

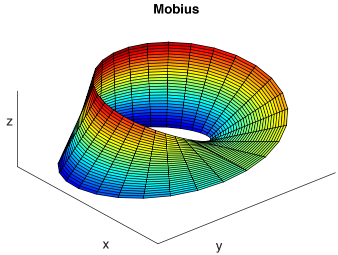
$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = - \int_{-\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

  the orientation of a surface is relevant to the surface integral.

Definition

A smooth surface $\mathcal{S}$ is said to be orientable if $\mathcal{S}$ is two-sided, and non-orientable if $\mathcal{S}$ is a one-sided surface.

- It is to exclude surfaces such as Moebius, for which flux is not defined



Matlab

```
>> u  = linspace(0, 2*pi, 30);
>> v  = linspace(-1, 1, 30);

>> [U, V] = meshgrid(u, v);

>> x = cos(U) + (V./2).*cos(U./2).*cos(U);
>> y = sin(U) + (V./2).*cos(U./2).*sin(U);
>> z = (V./2).*sin(U./2);

>> surf(x, y, z); axis equal;
```

- An orientable surface $\mathbf{r}(u,v)$ has two sides and thus two orientations.

Q: How can we find the unit vector normal to $\mathcal{S}$?

$$\mathbf{n}_1 = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \qquad \text{and} \qquad \mathbf{n}_2 = -\mathbf{n}_1 = \frac{\mathbf{r}_v \times \mathbf{r}_u}{|\mathbf{r}_v \times \mathbf{r}_u|}$$

the first is known as the positive and the second as the negative orientation.

- If we compute the flux in the positive orientation, that is, $\mathbf{n} = \mathbf{n}_1$, then

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{\mathcal{D}} \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

$$= \iint_{\mathcal{D}} \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

- Of course the flux in the reverse orientation is

$$\iint_{-\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = -\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = -\iint_{\mathcal{D}} \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

Q: How should we define the orientation of non-parametric surfaces?

- For a smooth surface $\mathcal{S}$ that is explicitly defined,

$$z = f(x, y), \qquad \text{or} \qquad y = g(x, z), \qquad \text{or} \qquad x = h(y, z)$$

we can use the followings vector-valued functions to represent, respectively,

$$\mathbf{r} = u\mathbf{e}_x + v\mathbf{e}_y + f\mathbf{e}_z \qquad \mathbf{r} = v\mathbf{e}_x + g\mathbf{e}_y + u\mathbf{e}_z \qquad \mathbf{r} = h\mathbf{e}_x + u\mathbf{e}_y + v\mathbf{e}_z$$

then the positive orientation is defined to be

$$\mathbf{n}_1 = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

- We can also write these surfaces as the followings,

$$\begin{matrix} \Phi(x, y, z) = z - f(x, y) = 0 \\ \Phi(x, y, z) = y - g(x, z) = 0 \\ \Phi(x, y, z) = x - h(y, z) = 0 \end{matrix} \implies \mathbf{n}_1 = \frac{\nabla\Phi}{|\nabla\Phi|} \implies \nabla\Phi = \mathbf{r}_u \times \mathbf{r}_v$$

- When no explicit orientation is given, we assume the positive orientation of the surface to be used, that is, using $\mathbf{n}_1$ defined above.

Exercise

(a) *Given the vector field* $\mathbf{F} = x\mathbf{e}_x + y\mathbf{e}_y + z^4\mathbf{e}_z$, *find the following integral*

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$$

where $\mathcal{S}$ *is the part of the cone* $z = \sqrt{x^2 + y^2}$ *that lies below the plane* $z = 1$ *with downward orientation.*

Definiton

For a closed surface $\mathcal{S}$, which is the boundary of a solid region $\mathcal{E}$, the positive orientation of $\mathcal{S}$ is defined to be the choice of $\mathbf{n}$ that consistently point outward from $\mathcal{E}$, while the inward-pointing normals define the negative orientation.

Exercise

(b) *Given the vector field* $\mathbf{F}(x, y, z) = y\mathbf{e}_x + (z - y)\mathbf{e}_y + x\mathbf{e}_z$, *find* $\displaystyle\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ *where* $\mathcal{S}$ *is the tetrahedron with vertices* $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.