Question1 (1 points)
Find the LU decomposition of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Solution:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}, \mathbf{U} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question2 (2 points)
Let $\mathbf{A}$ be the $n \times n$ matrix whose entry in the i_{th} row and j_{th} column is

$$\frac{1}{min(i,j)}$$

for $1 \leq i, j \leq n$. Compute $det(\mathbf{A})$

Solution:

Let r_i denote the i_{th} row of $\mathbf{A}$. The determinant will not be changed if we replace r_i by $r_i - r_{i-1}$ for $i \geq 2$

Because r_i and r_{i-1} agree in their first $i-1$ entries, the result of this row operation will be an upper triangular matrix.

So what we need to do is to calculate the entries on the main diagonal, which is $a'_{ii} = a_{ii} - a_{(i-1)i} = \frac{1}{i} - \frac{1}{i-1} = -\frac{1}{i(i-1)}$ for $i \geq 2$ and 1 for $i = 1$, so the determinant is

$$det\mathbf{A} = \prod_{i=2}^{n} \left(-\frac{1}{i(i-1)}\right) = \frac{(-1)^{n-1}}{n!(n-1)!}$$

Question3 (1 points)
Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & -1 & 2 & 4 & 2 \\ 0 & 0 & 4 & 0 & 0 \\ -3 & -6 & -9 & -12 & 4 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Solution:

$det(\mathbf{A}) = 28$

Question4 (1 points)

(a) Suppose $\mathbf{A}$ is a 2×2 invertible matrix whose entries are all integers. Show that

$$det(\mathbf{A}) = 1 \text{ or } -1$$

given that the entries of $\mathbf{A}^{-1}$ are all integers too.

Solution:

From the lecture, we know that the inverse of matrix $\mathbf{A}$ can be represented as

$$\mathbf{A}^{-1} = \frac{1}{det(\mathbf{A})} adj(\mathbf{A}) = \begin{bmatrix} \frac{a_{22}}{det(\mathbf{A})} & -\frac{a_{12}}{det(\mathbf{A})} \\ -\frac{a_{21}}{det(\mathbf{A})} & \frac{a_{11}}{det(\mathbf{A})} \end{bmatrix}$$

Since it is given that $\mathbf{A}^{-1}$'s entries are all integers, it can be deduced that all entries of $\mathbf{A}$ are multiples of $det(\mathbf{A})$

Now consider $det(\mathbf{A}) = a_{11}a_{22} - a_{12}a_{21}$, because all four entries are multiples of $det(\mathbf{A})$ and we know that $det(\mathbf{A})$ is not zero since $\mathbf{A}$ is invertible, we can see that $det(\mathbf{A})$ is a multiple of $det(\mathbf{A})^2$, then it is obvious that

$$det(\mathbf{A}) = 1 \text{ or } -1$$

(b) (bonus) Suppose $\mathbf{A}, \mathbf{B}$ are 2×2 matrices whose entries are all integers, and

$$\mathbf{A}, \mathbf{A} + \mathbf{B}, \mathbf{A} + \mathbf{2B}, \mathbf{A} + \mathbf{3B} \text{ and } \mathbf{A} + \mathbf{4B}$$

are all invertible and the entries of their inverse are all integers. Show that $\mathbf{A} + \mathbf{5B}$ is also invertible and the entries of its inverse are all integers.