Vv256
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Assignment 3
Due: Oct. 24, 2017

Question1 (5 points)

Consider the following initial-value problem

$$y'' - 3y' + 2y = 2xe^{3x} + 3\sin(x)$$

(a) (1 point) Find the complementary solution y_c of the given nonhomogeneous equation.

Solution:

1M The complementary solution is given by

$$r^2 - 3r + 2 = 0 \implies r_1 = 1 \qquad r_2 = 2 \implies y_c = C_1 e^x + C_2 e^{2x}$$

where C_1 and C_2 are two arbitrary constants.

(b) (1 point) State an annihilator for

$$2xe^{3x} + 3\sin(x)$$

Solution:

1M Since $2xe^{3x}$ is a solution to

$$\mathcal{L}_1 y = (D-3)^2 y = 0$$

and $\sin(x)$ is a solution to

$$\mathcal{L}_2 y = (D^2+1)y = 0$$

a possible annihilator is given by

$$\mathcal{L}_1\mathcal{L}_2 = (D-3)^2(D^2+1)$$

(c) (1 point) Find the general solution

$$y_h$$

to which a particular solution y_p of the given nonhomogeneous equation belongs.

Solution:

1M The general solution y_h is the general solution to

$$\underbrace{(D-3)^2(D^2+1)}_{\text{annihilator}}\underbrace{(D-1)(D-2)}_{\text{complementary}} y = 0$$

there is no overlap between the annihilator and the original differential operator,

$$y_h = e^{3x}(d_1 + d_2 x) + d_3 \sin x + d_4 \cos x + d_5 e^x + d_6 e^{2x}$$

(d) (1 point) Find a particular solution by determining all the coefficients of

$$y_p$$

Solution:

1M We can set $d_5 = d_6 = 0$ since those two terms are in y_c, we have

$$y_p = \underbrace{e^{3x}\left(d_1 + d_2 x\right)}_{\phi_1} + \underbrace{d_3 \sin x + d_4 \cos x}_{\phi_2}$$

It is clear

$$(D^2 - 3D + 2)\phi_1$$

will not have any trigonometric term, and

$$(D^2 - 3D + 2)\phi_2$$

will not have any exponential term, thus we can solve separately

$$(D^2 - 3D + 2)\phi_1 = 2xe^{3x} \qquad \text{and} \qquad (D^2 - 3D + 2)\phi_2 = 3\sin(x)$$

Differentiating ϕ_1 and ϕ_2, we have

$$\begin{aligned} \phi_1 &= e^{3x}\left(d_1 + d_2 x\right) \\ \phi_1' &= 3e^{3x}(d_1 + d_2 x) + d_2 e^{3x} \\ \phi_1'' &= 9e^{3x}(d_1 + d_2 x) + 6d_2 e^{3x} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \phi_2 &= d_3 \sin x + d_4 \cos x \\ \phi_2' &= d_3 \cos x - d_3 \sin x \\ \phi_2'' &= -d_3 \sin x - d_4 \cos x \end{aligned}$$

Substituting and simplifying, we have

$$\begin{aligned} (2d_1 + 3d_2)e^{3x} + 2d_2 x e^{3x} &= 2xe^{3x} \\ (d_3 + 3d_4)\sin(x) + (d_4 - 3d_3)\cos(x) &= 3\sin(x) \end{aligned}$$

Equating coefficients, we have

$$d_1 = -\frac{3}{2}; \quad d_2 = 1; \quad d_3 = \frac{3}{10}; \quad d_3 = \frac{9}{10}$$

Thus

$$y_p = -\frac{3}{2}e^{3x} + xe^{3x} + \frac{3}{10}\sin(x) + \frac{9}{10}\cos(x)$$

We could use exponential shift law to avoid the tedious substitution,

$$\begin{aligned} (D-1)(D-2)\phi_1 &= (D-1)(D-2)e^{3x}(d_1 + d_2 x) \\ &= e^{3x}(D+2)(D+1)(d_1 + d_2 x) \\ &= e^{3x}(D^2 + 3D + 2)(d_1 + d_2 x) \\ &= e^{3x}(3d_2 + 2d_1 + 2d_2 x) = 2xe^{3x} \\ \implies d_2 = 1 \qquad &\text{and} \quad d_1 = -\frac{3}{2} \end{aligned}$$

For ϕ_2, we could use the formula in the bonus question part (b),

$$y(x) = a\frac{e^{rx}}{p(r)}$$

Since

$$e^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x \implies \sin(x) = \operatorname{Im}\left(e^{\mathrm{i}x}\right)$$

which means ϕ_2 will is the imaginary part of the particular solution to

$$(D^2 - 3D + 2)y = e^{\mathrm{i}x}$$

Applying the formula, we have

$$\begin{aligned}\phi_2 &= 3\operatorname{Im}\left(\frac{e^{\mathrm{i}x}}{\mathrm{i}^2 - 3\mathrm{i} + 2}\right)\\ &= 3\operatorname{Im}\left(\frac{\cos x + \mathrm{i}\sin x}{1 - 3\mathrm{i}}\right)\\ &= \frac{3}{10}\left(\sin x + 3\cos x\right)\end{aligned}$$

which lead us to the same y_p. The above two techniques are very useful when the number of coefficients are unbearably large.

(e) (1 point) State the general solution of the given nonhomogeneous equation

$$y = y_c + y_p$$

then find the particular solution that satisfies the following IVP

$$y'' - 3y' + 2y = 2xe^{3x} + 3\sin(x), \qquad y(0) = 1, \quad y'(0) = 1$$

Solution:

1M The general solution is

$$\begin{aligned}y &= y_c + y_p\\ &= C_1e^x + C_2e^{2x} - \frac{3}{2}e^{3x} + xe^{3x} + \frac{3}{10}\sin(x) + \frac{9}{10}\cos(x)\end{aligned}$$

Using the initial conditions, we have

$$C_1 = -1 \qquad C_2 = \frac{13}{5}$$

Question2 (1 points)

Suppose $\mathcal{L}(y) = 0$ is a homogeneous 4th-order linear equation with constant coefficient and

$$x^3e^{-x}$$

is a solution to the equation. Find the general solution y and the differential operator $\mathcal{L}$ of

$$\mathcal{L}(y) = 0$$

Solution:

1M Since we have a homogeneous equation, if x^3e^{-x} is a solution to the equation, then x^2e^{-x}, xe^{-x}, e^{-x} are also solutions of the equation. They form the general solution

$$y(x) = C_1x^3e^{-x} + C_2x^2e^{-x} + C_3xe^{-x} + C_4e^{-x}$$

and -1 is the root of the characteristic polynomial

$$p(\lambda) = (\lambda + 1)^4$$

Thus the corresponding differential operator is

$$\mathcal{L} = (D + 1)^4$$

Question3 (1 points)
Based on the method of variation of parameters for second-order equations, show how to use variation of parameters for solving the given third-order differential equation.

$$y''' + y' = \tan x$$

Solution:

1M Solving the complementary equation, we have

$$r^3 + r = 0 \implies \phi_1 = 1 \qquad \phi_2 = \cos x \qquad \phi_3 = \sin x$$

Suppose the following is a solution

$$y_p = u_1\phi_1 + u_2\phi_2 + u_3\phi_3$$

where u_1, u_2 and u_3 are arbitrary functions.

$$y_p' = u_1'\phi_1 + u_2'\phi_2 + u_3'\phi_3 + u_1\phi_1' + u_2\phi_2' + u_3\phi_3'$$

Let $u_1'\phi_1 + u_2'\phi_2 + u_3'\phi_3 = 0$, then

$$y_p'' = u_1'\phi_1' + u_2'\phi_2' + u_3'\phi_3' + u_1\phi_1'' + u_2\phi_2'' + u_3\phi_3''$$

Let $u_1'\phi_1' + u_2'\phi_2' + u_3'\phi_3' = 0$, then

$$y_p''' = u_1'\phi_1'' + u_2'\phi_2'' + u_3'\phi_3'' + u_1\phi_1''' + u_2\phi_2''' + u_3\phi_3'''$$

Substituting y', and y''' into the equation, we have

$$\begin{aligned} u_1'\phi_1' + u_2'\phi_2' + u_3'\phi_3' + u_1'\phi_1'' + u_2'\phi_2'' + u_3'\phi_3'' &= \tan x \\ u_1'\left(\phi_1' + \phi_1''\right) + u_2'\left(\phi_2' + \phi_2''\right) + u_3'\left(\phi_3' + \phi_3''\right) &= \tan x \end{aligned}$$

So we need to solve the following system,

$$\begin{aligned} u_1'\phi_1 + u_2'\phi_2 + u_3'\phi_3 &= 0 \\ u_1'\phi_1' + u_2'\phi_2' + u_3'\phi_3' &= 0 \\ u_1'\left(\phi_1' + \phi_1''\right) + u_2'\left(\phi_2' + \phi_2''\right) + u_3'\left(\phi_3' + \phi_3''\right) &= \tan x \end{aligned}$$

Since $\phi_1 = 1$, $\phi_2 = \cos x$, and $\phi_3 = \sin x$, we have

$$\begin{aligned} u_1' + u_2' \cos x + u_3' \sin x &= 0 \\ -u_2' \sin x + u_3' \cos x &= 0 \\ -u_2' (\cos x + \sin x) + u_3' (\cos x - \sin x) &= \tan x \end{aligned}$$

Solving the system, we have

$$u_1' = \tan x; \qquad u_2' = -\sin x; \qquad u_3' = \cos x - \sec x$$

Integrating each, we have

$$u_1 = \ln|\sec x|; \qquad u_2 = \cos x; \qquad u_3 = \sin x - \ln|\sec x + \tan x|$$

Thus

$$y = C_1 + C_2 \cos x + C_3 \sin x + \ln|\sec x| - \sin x \ln|\sec x + \tan x|$$

Since $\cos^2 x + \sin^2 x = 1$.

Question4 (3 points)

Find the radius of convergence for each of the following series

(a) (1 point) $\displaystyle\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n}$

Solution:

1M Applying the ratio test, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x+1)^{2(n+1)}}{9^{n+1}} \frac{9^n}{(x+1)^{2n}}\right| = \left|\frac{(x+1)^2}{9}\right| \to \frac{(x+1)^2}{9} \qquad \text{as } n \to \infty$$

thus the series is convergent for $-4 < x < 2$, and the radius of convergence is 3.

(b) (1 point) $\displaystyle\sum_{n=0}^{\infty} (\ln x)^n$

Solution:

1M Applying the root test, we have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|(\ln x)^n|} = \lim_{n\to\infty} \sqrt[n]{|\ln x|^n} = \lim_{n\to\infty} |\ln x| = |\ln x|$$

thus the series is convergent for $1/e < x < e$, and the radius of convergence is

$$\frac{1}{2}(e - \frac{1}{e})$$

(c) (1 point) $\displaystyle\sum_{n=0}^{\infty} c_n x^n$, where c_n takes the value 1 if n is odd, and 2 if n is even.

Solution:

1M In order to apply the root test, we need to consider

$$\lim_{n\to\infty} \sqrt[n]{|a_n|}$$

Break the sequence $\{\sqrt[n]{|a_n|}\}$ into two subsequences, when n is odd,

$$b_k = \sqrt[n]{|a_n|} \qquad \text{where } n = 2k+1 \text{ for } k \in \mathbb{N}_0,$$

and when n is even,

$$c_k = \sqrt[n]{|a_n|} \qquad \text{where } n = 2k \text{ for } k \in \mathbb{N}_0,$$

If $\{b_k\}$ and $\{c_k\}$ converge to the same value L, then $\sqrt[n]{|a_n|}$ converges to L as well.

$$\lim_{k\to\infty} b_k = \lim_{k\to\infty} \sqrt[2k+1]{|x^{2k+1}|} = \lim_{k\to\infty} \sqrt[2k+1]{|x^{2k+1}|} = |x|$$

For $\{c_k\}$, we have

$$\lim_{k\to\infty} c_k = \lim_{k\to\infty} \sqrt[2k]{|2x^{2k}|} = \lim_{k\to\infty} \sqrt[2k]{2}\,|x| = |x|$$

Since

$$\lim_{k\to\infty} \sqrt[2k]{2} = \lim_{k\to\infty} \exp\left(\frac{1}{2k}\ln 2\right) = \exp\left(\ln 2 \lim_{k\to\infty} \frac{1}{2k}\right) = \exp(0) = 1$$

So

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = |x|$$

which means the original series is convergent for

$$-1 < x < 1$$

by the root test, and the radius of convergent is 1.