Vv256 Lecture 23

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Definition

A square matrix in which all the entries below the main diagonal are 0 is called an upper triangular matrix. In case of all the entries above the main diagonal are 0,

$$\mathbf{U} = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \dots & l_{nn} \end{bmatrix}$$

it is called a lower triangular matrix. A square matrix in which all the entries off the main diagonal are zero is called a diagonal matrix, which is also triangular.

$$\mathbf{D} = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix} = \operatorname{diag}(d_1, d_2, \dots, d_n)$$

Theorem

If $\mathbf{A}$ is triangular, then the eigenvalues are the entries on the main diagonal.

Definition

If $\mathbf{A}$ is a square matrix, then we define the nonnegative integer powers of $\mathbf{A}$ to be

$$\mathbf{A}^0 = \mathbf{I} \qquad \text{and} \qquad \mathbf{A}^n = \mathbf{A}\mathbf{A}\cdots\mathbf{A} \qquad [n \text{ factors}]$$

and if $\mathbf{A}$ is invertible, then we define the negative integer powers of $\mathbf{A}$ to be

$$\mathbf{A}^{-n} = \left(\mathbf{A}^{-1}\right)^n = \mathbf{A}^{-1}\mathbf{A}^{-1}\cdots\mathbf{A}^{-1} \qquad [n \text{ factors}]$$

- It is much easier to compute powers and inverses of diagonal matrices.

Exercise

Find the square of $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and the inverse of $\begin{bmatrix} -1 & 0 \\ 0 & 1/2 \end{bmatrix}$.

Solution

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^2 = \begin{bmatrix} 2^2 & 0 \\ 0 & 3^2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0.5 \end{bmatrix}^{-1} = \begin{bmatrix} 1/(-1) & 0 \\ 0 & 1/0.5 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Definition

A matrix $\mathbf{A}_{n\times n}$ is said to be diagonalizable if there is an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}, \qquad \text{where } \mathbf{P} \text{ is said to diagonalize } \mathbf{A}.$$

Theorem

An $n \times n$ matrix $\mathbf{A}$ is diagonalizable if and only if

$\mathbf{A}$ has n linearly independent eigenvectors.

Moreover, when $\mathbf{A}$ is diagonalizable, then

1. The columns of the diagonalizing matrix $\mathbf{P}$ are the n eigenvectors.
2. The diagonal elements of $\mathbf{D}$ are the corresponding eigenvalues.

Proof

- Suppose the matrix $\mathbf{A}$ has linearly independent eigenvector $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$.
- Let λ_i be the eigenvalue of $\mathbf{A}$ corresponding to $\mathbf{x}_i$ for each i, some of the λ_i's may be equal, and $\mathbf{P}$ be the matrix whose jth column is $\mathbf{x}_j$,

Proof

- If follows that $\mathbf{AP} = \begin{bmatrix}\mathbf{Ax}_1 & \cdots & \mathbf{Ax}_n\end{bmatrix}$ $= \begin{bmatrix}\lambda_n\mathbf{x}_1 & \cdots & \lambda_n\mathbf{x}_n\end{bmatrix}$

$$= \begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_n\end{bmatrix}\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$$

$$= \mathbf{PD}$$

- $\mathbf{P}$ has n linearly independent columns, so $\mathbf{P}$ is invertible and

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$$

- Let $\mathbf{A}$ be diagonalizable, then there exists a nonsingular matrix $\mathbf{P}$ such that

$$\mathbf{AP} = \mathbf{PD}$$

- If $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are the columns of $\mathbf{P}$, then

$$\mathbf{Ax}_i = \lambda_i\mathbf{x}_i, \qquad \lambda_i = d_{ii} \quad \text{for each } i$$

- Since $\mathbf{P}$ is invertible, thus $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent.

Exercise

Find a matrix $\mathbf{P}$ *that diagonalizes* $\mathbf{A} = \begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix} = \mathbf{PDP}^{-1}$ *where* $\mathbf{D}$ *is diagonal.*

Solution

- The eigenvalues of $\mathbf{A}$ are

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = -4$$

- Corresponding to λ_1 and λ_2, the eigenvectors are $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$,

$$\mathbf{P} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$$

You can easily verify that $\mathbf{A} = \mathbf{PDP}^{-1}$.

- Note the diagonalizing matrix $\mathbf{P}$ is not unique.

Exercise

Consider $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & 0 \\ -3 & 6 & 2 \end{bmatrix}$, are they diagonalizable?

Solution

- You can easily verify that both $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues,

$$\lambda_1 = 4 \qquad \text{and} \qquad \lambda_2 = \lambda_3 = 2$$

- The eigenspace of $\mathbf{A}$ corresponding to $\lambda_1 = 4$ is spanned by $\mathbf{e}_2$, however, the other eigenspace of $\mathbf{A}$ is spanned by $\mathbf{e}_3$. So there are only two linearly independent eigenvectors for $\mathbf{A}$ instead of three.
- On the other hand, $\mathbf{B}$ has three linearly independent eigenvectors

$$\mathbf{x}_1 = \mathbf{e}_2 + 3\mathbf{e}_3 \qquad \mathbf{x}_2 = 2\mathbf{e}_1 + \mathbf{e}_2 \qquad \mathbf{x}_3 = \mathbf{e}_3$$

Therefore $\mathbf{B}$ is diagonalizable while $\mathbf{A}$ is not diagonalizable.

Q: Why diagonalization is useful for solving a linear homogeneous system?

Exercise

Solve the system of differential equations by diagonalization:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \qquad \text{where} \qquad \mathbf{A} = \begin{bmatrix} 2 & 3 \\ 4 & -2 \end{bmatrix}$$

Solution

- Find the eigenvalues and eigenvectors for $\mathbf{A}$, we find that $\mathbf{A}$ is diagonalizable,

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1/4 & 1/8 \\ -1/4 & 3/8 \end{bmatrix} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$$

- So the system of differential equations can be written as

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \mathbf{P} \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix} \mathbf{P}^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution

- If we define a new vector $\mathbf{y}$,

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{P}^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Thus, the system can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \mathbf{P} \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix} \mathbf{y} \implies \mathbf{P}^{-1}\dot{\mathbf{x}} = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix} \mathbf{y}$$

- Then we realise $\dot{\mathbf{y}}$ is simply $\mathbf{P}^{-1}\dot{\mathbf{x}}$

$$\frac{d}{dt}\left[\mathbf{P}^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right] = \frac{d}{dt}\left[x_1 \begin{bmatrix} 1/4 \\ -1/4 \end{bmatrix} + x_2 \begin{bmatrix} 1/8 \\ 3/8 \end{bmatrix}\right] = \dot{x}_1 \begin{bmatrix} 1/4 \\ -1/4 \end{bmatrix} + \dot{x}_2 \begin{bmatrix} 1/8 \\ 3/8 \end{bmatrix} = \mathbf{P}^{-1}\dot{\mathbf{x}}$$

- Therefore, our new vector $\mathbf{y}$ satisfies,

$$\dot{\mathbf{y}} = \mathbf{D}\mathbf{y}$$

Solution

- This system is not coupled and can be solve directly

$$\mathbf{y} = \begin{bmatrix} c_1 e^{4} \\ c_2 e^{-4t} \end{bmatrix}$$

- Now back transform to $\mathbf{x}$,

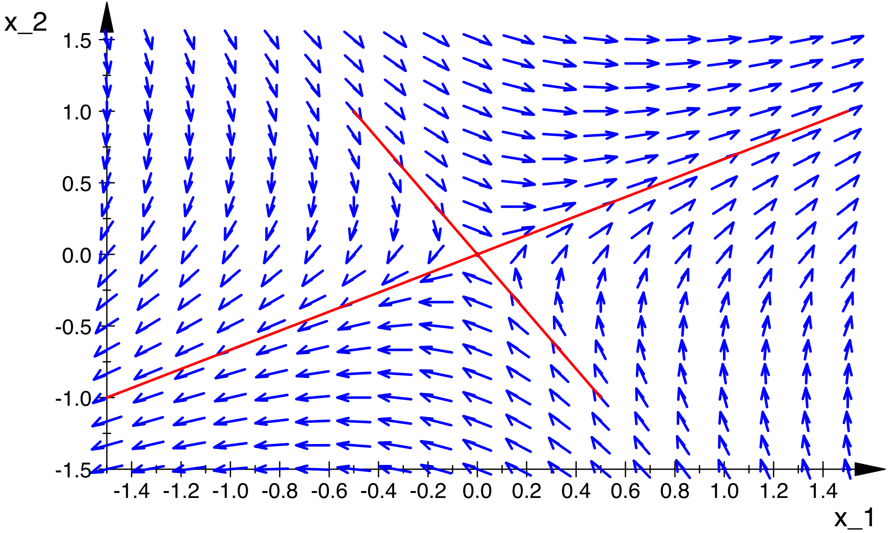
$$\mathbf{y} = \mathbf{P}^{-1}\mathbf{x} \implies \mathbf{x} = \mathbf{P}\mathbf{y} = \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} c_1 e^{4} \\ c_2 e^{-4t} \end{bmatrix}$$

- Therefore the solution $\mathbf{x}$ is

$$\mathbf{x} = c_1 e^{4t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

- This is the same as what we have got earlier. This also explains the connection between the long-run behaviour and eigenvalues/eigenvectors.

Phase plane with Eigenvectors



- Recall the coefficient matrix for the sensitive couple model,

$$\mathbf{A} = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{a/b} & -\sqrt{a/b} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{ab} & 0 \\ 0 & -\sqrt{ab} \end{bmatrix} \begin{bmatrix} \sqrt{b/4a} & 1/2 \\ -\sqrt{b/4a} & 1/2 \end{bmatrix} = \mathbf{PDP}^{-1}$$

- So the system of diffenetial equations can be written as

$$\begin{bmatrix} \dot{R} \\ \dot{J} \end{bmatrix} = \mathbf{P} \begin{bmatrix} \sqrt{ab} & 0 \\ 0 & -\sqrt{ab} \end{bmatrix} \mathbf{P}^{-1} \begin{bmatrix} R \\ J \end{bmatrix}$$

- If we define a new vector $\mathbf{x}$,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{P}^{-1} \begin{bmatrix} R \\ J \end{bmatrix}, \qquad \text{where } x_1 \text{ and } x_2 \text{ are two new functions of } t.$$

- Thus, the system can be written as

$$\begin{bmatrix} \dot{R} \\ \dot{J} \end{bmatrix} = \mathbf{P} \begin{bmatrix} \sqrt{ab} & 0 \\ 0 & -\sqrt{ab} \end{bmatrix} \mathbf{x} \Longrightarrow \mathbf{P}^{-1} \begin{bmatrix} \dot{R} \\ \dot{J} \end{bmatrix} = \begin{bmatrix} \sqrt{ab} & 0 \\ 0 & -\sqrt{ab} \end{bmatrix} \mathbf{x}$$

- If we consider the derivative of the vector

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \implies \dot{\mathbf{x}} = \dot{x_1}\mathbf{e}_1 + \dot{x_2}\mathbf{e}_2$$

we have

$$\begin{aligned}\dot{\mathbf{x}} = \frac{d}{dt}\mathbf{x} = \frac{d}{dt}\left(\mathbf{P}^{-1}\begin{bmatrix} R \\ J \end{bmatrix}\right) &= \frac{d}{dt}\left(R\begin{bmatrix} \sqrt{b/4a} \\ -\sqrt{b/4a} \end{bmatrix} + J\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}\right) \\ &= \frac{dR}{dt}\begin{bmatrix} \sqrt{b/4a} \\ -\sqrt{b/4a} \end{bmatrix} + \frac{dJ}{dt}\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \\ &= \mathbf{P}^{-1}\begin{bmatrix} \dot{R} \\ \dot{J} \end{bmatrix}\end{aligned}$$

- Therefore, we can rewrite the system again,

$$\dot{\mathbf{x}} = \begin{bmatrix} \sqrt{ab} & 0 \\ 0 & -\sqrt{ab} \end{bmatrix}\mathbf{x}$$

- The new system of differential equations, which is not coupled, can be solve,

$$\dot{\mathbf{x}} = \begin{bmatrix} \sqrt{ab} & 0 \\ 0 & -\sqrt{ab} \end{bmatrix} \mathbf{x} \implies \begin{matrix} \dot{x_1} = \sqrt{ab}\; x_1 \\ \dot{x_2} = -\sqrt{ab}\; x_2 \end{matrix} \implies \begin{matrix} x_1 = c_1 e^{\sqrt{ab}t} \\ x_2 = c_2 e^{-\sqrt{ab}t} \end{matrix}$$

- Thus

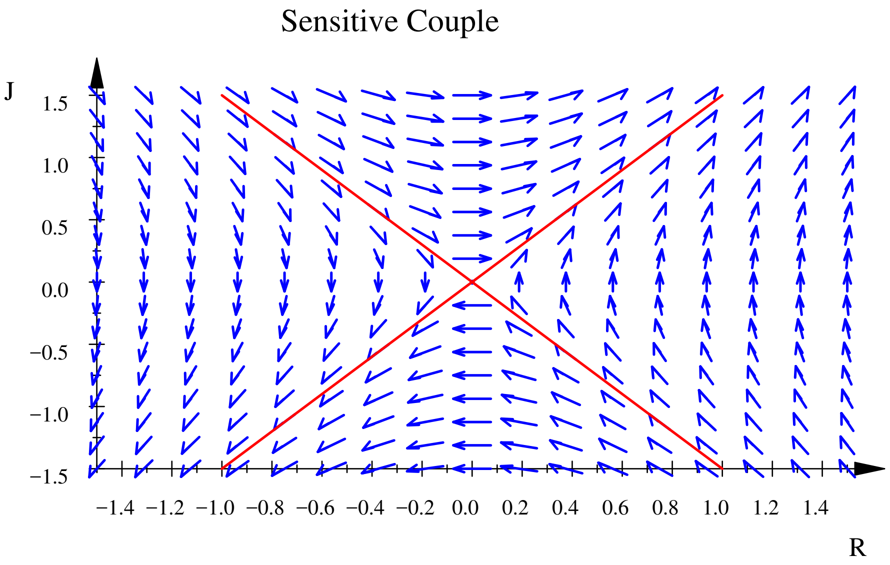
$$\mathbf{x} = \begin{bmatrix} c_1 e^{\sqrt{ab}t} \\ c_2 e^{-\sqrt{ab}t} \end{bmatrix}$$

- Now back transform to R and J,

$$\mathbf{x} = \mathbf{P}^{-1} \begin{bmatrix} R \\ J \end{bmatrix} \implies \begin{bmatrix} R \\ J \end{bmatrix} = \mathbf{P}\mathbf{x} = \begin{bmatrix} \sqrt{a/b} & -\sqrt{a/b} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{\sqrt{ab}t} \\ c_2 e^{-\sqrt{ab}t} \end{bmatrix}$$

- Hence the exact solution R and J are,

$$\begin{bmatrix} R \\ J \end{bmatrix} = \begin{bmatrix} \sqrt{a/b}\left(c_1 e^{\sqrt{ab}t} - c_2 e^{-\sqrt{ab}t}\right) \\ c_1 e^{\sqrt{ab}t} + c_2 e^{-\sqrt{ab}t} \end{bmatrix} = c_1 e^{\sqrt{ab}t} \begin{bmatrix} \sqrt{a/b} \\ 1 \end{bmatrix} + c_2 e^{-\sqrt{ab}t} \begin{bmatrix} -\sqrt{a/b} \\ 1 \end{bmatrix}$$



Sensitive Couple
J
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−1.4 −1.2 −1.0 −0.8 −0.6 −0.4 −0.2 0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4
R

Eigenvalues and Eigenvectors Method

- In general, if the coefficient matrix $\mathbf{A}$ is diagonalizable, and has eigenvalues

$$\lambda_1, \lambda_2, \cdots, \lambda_n, \quad \text{and corresponding eigenvectors} \quad \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$$

  then the system can always be solved by decoupling

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}\mathbf{x} \implies \mathbf{P}^{-1}\dot{\mathbf{x}} = \mathbf{D}\mathbf{P}^{-1}\mathbf{x}$$

- Since $\dot{\mathbf{y}} = \dfrac{d}{dt}(\mathbf{y}) = \dfrac{d}{dt}\left(\mathbf{P}^{-1}\mathbf{x}\right) = \mathbf{P}^{-1}\dot{\mathbf{x}}$, then

$$\dot{\mathbf{y}} = \mathbf{D}\mathbf{y}, \quad \text{where } \mathbf{y} = \mathbf{P}^{-1}\mathbf{x}.$$

- Thus $\mathbf{y}$ in terms of t and therefore $\mathbf{x}$ in terms of t are given by

$$\mathbf{y} = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix} \implies \mathbf{x} = \mathbf{P} \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n$$

Exercise

Solve the system of differential equations,

$$\dot{x} = x + 2y - 3z, \qquad \dot{y} = -5x + y - 4z, \qquad \dot{z} = -2y + 4z$$

Solution

- The corresponding coefficient matrix is $\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ -5 & 1 & -4 \\ 0 & -2 & 4 \end{bmatrix}$.
- Solve the eigenvalue problem to determine whether it is diagonalizable,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \implies \lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3$$

the corresponding eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} -8 \\ 15 \\ 10 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$

- Since we have 3 distinct eigenvalues, $\mathbf{A}$ must be diagonalizable.

$$x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z = c_1 e^t \mathbf{v}_1 + c_2 e^{2t} \mathbf{v}_2 + c_3 e^{3t} \mathbf{v}_3$$

Exercise

Solve the system of differential equations, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

Solution

- Solve the eigenvalue problem to determine whether it is diagonalizable,

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0 \implies \lambda_1 = \lambda_2 = 1, \quad \lambda_3 = 4$$

  the corresponding eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

- We have 3 linearly independent eigenvectors, so $\mathbf{A}$ must be diagonalizable,

$$\mathbf{x} = c_1 e^{t} \mathbf{v}_1 + c_2 e^{t} \mathbf{v}_2 + c_3 e^{4t} \mathbf{v}_3$$

- Notice having repeated eigenvalues does not change the form of our solution.

Exercise

Solve the system of differential equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = \begin{bmatrix} -1 & -4 & 2 \\ 3 & 1 & -2 \\ 1 & -4 & 1 \end{bmatrix}$.

Solution

- Solve the eigenvalue problem to determine whether it is diagonalizable,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^3 - \lambda^2 + \lambda - 1 = 0 \implies \lambda_1 = 1, \qquad \lambda_{2,3} = \pm\mathrm{i},$$

  the corresponding eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3+\mathrm{i} \\ 2-\mathrm{i} \\ 5 \end{bmatrix}$, $\overline{\mathbf{v}}_2 = \begin{bmatrix} 3-\mathrm{i} \\ 2+\mathrm{i} \\ 5 \end{bmatrix}$

- We have 1 real and 2 complex eigenvalues, so $\mathbf{A}$ must be diagonalizable, and

$$\mathbf{x} = c_1 e^{t} \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} + d_2 e^{\mathrm{i}t} \begin{bmatrix} 3+\mathrm{i} \\ 2-\mathrm{i} \\ 5 \end{bmatrix} + d_3 e^{-\mathrm{i}t} \begin{bmatrix} 3-\mathrm{i} \\ 2+\mathrm{i} \\ 5 \end{bmatrix}$$

Solution

- Express complex coefficients and complex exponential functions in real form,

$$
\begin{aligned}
\mathbf{x} &= c_1 e^t \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} + d_2 e^{\mathrm{i}t} \begin{bmatrix} 3+\mathrm{i} \\ 2-\mathrm{i} \\ 5 \end{bmatrix} + d_3 e^{-\mathrm{i}t} \begin{bmatrix} 3-\mathrm{i} \\ 2+\mathrm{i} \\ 5 \end{bmatrix} \\
&= c_1 e^t \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} + d_2(\cos t + \mathrm{i}\sin t) \begin{bmatrix} 3+\mathrm{i} \\ 2-\mathrm{i} \\ 5 \end{bmatrix} + d_3(\cos -t + \mathrm{i}\sin -t) \begin{bmatrix} 3-\mathrm{i} \\ 2+\mathrm{i} \\ 5 \end{bmatrix} \\
&= c_1 e^t \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} + d_2 \begin{bmatrix} 3\cos t - \sin t \\ 2\cos t + \sin t \\ 5\cos t \end{bmatrix} + \mathrm{i}d_2 \begin{bmatrix} \cos t + 3\sin t \\ -\cos t + 2\sin t \\ 5\sin t \end{bmatrix} \\
&= c_1 e^t \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} + c_2 \begin{bmatrix} 3\cos t - \sin t \\ 2\cos t + \sin t \\ 5\cos t \end{bmatrix} + c_3 \begin{bmatrix} \cos t + 3\sin t \\ -\cos t + 2\sin t \\ 5\sin t \end{bmatrix}
\end{aligned}
$$