Ve401 Probabilistic Methods in Engineering

Summer 2018 — Assigment 6

Date Due: 12:10 PM, Wednesday, the 4th of July 2018

This assignment has a total of **(25 Marks)**.

Exercise 6.1

The percentage of titanium in an alloy used in aerospace castings is measured in 51 randomly selected parts. The sample standard deviation is $s = 0.37$. Construct a 95% two-sided confidence interval for σ.
(2 Marks)

Exercise 6.2 Non-Parametric Confidence Intervals

Let $X_1, \ldots, X_n$ be a random sample of size n from a continuous distribution with median M. Let

$$X_{\min} = \min_{1 \le i \le n} X_i, \qquad X_{\max} = \max_{1 \le i \le n} X_i.$$

i) Show that

$$P[X_{\min} \le M \le X_{\max}] = 1 - \left(\frac{1}{2}\right)^{n-1}.$$

(2 Marks)

ii) Suppose the sample data are arranged from smallest to largest, so that $X_1 \le X_2 \le \cdots \le X_n$. Calculate

$$P[X_{k+1} \le M \le X_{n-k}]$$

for $k = 0, \ldots, \lfloor n/2 \rfloor$.
(3 Marks)

Exercise 6.3 Fisher Test

Cloud seeding has been studied for many decades as a weather modification procedure.[1] It is claimed that mean rainfall from seeded clouds exceeds 25 acre-feet.

The rainfall in acre-feet from 20 clouds that were selected at random and seeded with silver nitrate follows:

18.0, 30.7, 19.8, 27.1, 22.3, 18.8, 31.8, 23.4, 21.2, 27.9,
31.9, 27.1, 25.0, 24.7, 26.9, 21.8, 29.2, 34.8, 26.7, 31.6.

Can you support the claim that mean rainfall from seeded clouds exceeds 25 acre-feet? What is the P-value of the test?
(3 Marks)

Exercise 6.4 Neyman-Pearson Decision Test

Medical researchers have developed a new artificial heart constructed primarily of titanium and plastic. The heart will last and operate almost indefinitely once it is implanted in the patients body, but the battery pack needs to be recharged about every four hours.

It is claimed that the mean battery life μ exceeds 4 hours and a researcher would like to test the hypotheses

$$H_0\colon \mu \le 4\,\text{hours} \qquad H_1\colon \mu \ge 4.5\,\text{hours}$$

A random sample of 50 battery packs is selected and subjected to a life test. Assume that battery life is normally distributed with standard deviation $\sigma = 0.2$ hours.

i) Using the sample mean life span $\overline{X}$ as a test statistic, what is the critical region if $\alpha = 5\%$ is desired?
(2 Marks)

ii) Find the power of the test, i.e., the probability of rejecting H_0 if H_1 is true.
(1 Mark)

iii) What sample size would be required to obtain a power of at least 0.97?
(1 Mark)

[1] For an interesting study of this subject, see the article in *Technometrics*, "A Bayesian Analysis of a Multiplicative Treatment Effect in Weather Modification", Vol. 17, pp. 161166.

iv) The sample mean life span turns out to be $\overline{x} = 4.05$ hours. Is H_0 rejected? Find a confidence interval for μ.
(2 Marks)

Exercise 6.5 NHST

Supercavitation is a propulsion technology for undersea vehicles that can greatly increase their speed. It occurs above approximately 50 meters per second, when pressure drops sufficiently to allow the water to dissociate into water vapor, forming a gas bubble behind the vehicle. When the gas bubble completely encloses the vehicle, supercavitation is said to occur.

Eight tests were conducted on a scale model of an undersea vehicle in a towing basin with the average observed speed $\bar{x} = 102.2$ meters per second. Assume that speed is normally distributed with known standard deviation $\sigma = 4$ meters per second.

i) Test the hypothesis $H_0 \colon \mu \leq 100$ versus $H_1 \colon \mu > 100$ using $\alpha = 0.05$.
(2 Marks)

ii) What is the P-value for the test in part (i)?
(1 Mark)

iii) Find the power of the test if the true mean speed is as low as 105 meters per second.
(1 Mark)

iv) What sample size would be required to detect a true mean speed as low as 105 meters per second if we wanted the power of the test to be at least 0.85?
(1 Mark)

v) Explain how the question in part (i) could be answered by constructing a one-sided confidence bound on the mean speed.
(1 Mark)

Exercise 6.6

Let $X_1, \ldots, X_n$ be i.i.d. exponential random variables with parameter β. Recall that $Y = X_1 + X_2 + \cdots + X_n$ follows a Gamma distribution with parameter $\alpha = n$ and β.

Deduce that

$$\frac{2}{\beta} \sum_{k=1}^{n} X_k$$

follows a chi-squared distribution with $2n$ degrees of freedom. Devise a test statistic and critical regions for testing $H_0 \colon \beta = \beta_0$ and $H_0 \colon \beta \leq \beta_0$ for given α.
(3 Marks)