Question1 (10 points)

(a) Derive $\mathcal{L}[f(t)]$ from the definition and state the interval of convergence.

i. (1 point) $f(t) = te^{4t}$.

ii. (1 point) $f(t) = \begin{cases} \sin t & \text{if } 0 \le t < \pi \\ 0 & \text{if } t \ge \pi. \end{cases}$

iii. (1 point) $f(t) = \sinh t$.

Solution:

i. By Definition,

$$\begin{aligned}
\mathcal{L}\Big[f(t)\Big] = \int_0^\infty te^{4t}e^{-st}\,dt &= \int_0^\infty te^{(4-s)t}\,dt \\
&= \lim_{b\to\infty} \int_0^b te^{(4-s)t}\,dt \\
&= \lim_{b\to\infty} \left[\left[\frac{t}{4-s}e^{(4-s)t}\right]_0^b - \frac{1}{4-s}\int_0^b e^{(4-s)t}\,dt\right] \\
&= \lim_{b\to\infty} \left(\frac{be^{(4-s)b}}{4-s} - \frac{e^{(4-s)b}}{(4-s)^2} + \frac{1}{(4-s)^2}\right) \\
&= \begin{cases} \dfrac{1}{(4-s)^2} & s > 4 \\ \infty & \text{otherwise} \end{cases}
\end{aligned}$$

The interval of convergence is $s \in (4, \infty)$.

ii. By definition

$$\begin{aligned}
I = \mathcal{L}\Big[f(t)\Big] &= \int_0^\pi \sin(t)e^{-st}\,dt + \int_\pi^\infty 0 \cdot e^{-st}\,dt \\
&= \int_0^\pi \sin(t)e^{-st}\,dt \\
&= \left[\frac{1}{-s}\sin(t)e^{-st}\right]_0^\pi + \frac{1}{s}\int_0^\pi \cos(t)e^{-st}\,dt \\
&= 0 + \frac{-1}{s^2}\left[\cos te^{-st}\right]_0^\pi - \frac{1}{s^2}\int_0^\pi \sin(t)e^{-st}\,dt \\
&= \frac{e^{-\pi s}}{s^2} + \frac{1}{s^2} - \frac{1}{s^2}I \\
\Longrightarrow \mathcal{L}\Big[f(t)\Big] &= \frac{e^{-\pi s}+1}{s^2+1}
\end{aligned}$$

The interval of convergence is $(-\infty, \infty)$.

iii. By definition,

$$\begin{aligned}
\mathcal{L}\Big[f(t)\Big] &= \int_0^\infty \sinh(t)e^{-st}\,dt \\
&= \int_0^\infty \frac{e^t - e^{-t}}{2}e^{-st}\,dt \\
&= \int_0^\infty \frac{e^{(1-s)t} - e^{(-1-s)t}}{2}\,dt \\
&= \lim_{b\to\infty}\left[\frac{e^{(1-s)t}}{2(1-s)} - \frac{e^{(-1-s)t}}{2(-1-s)}\right]_0^b \\
&= \left[\frac{1}{2(-1-s)} - \frac{1}{2(1-s)}\right]\lim_{b\to\infty}\left[\frac{e^{(1-s)b}}{2(1-s)} - \frac{e^{(-1-s)b}}{2(-1-s)}\right] \\
&= \begin{cases} \dfrac{1}{s^2-1} & s > 1 \\ \infty & \text{otherwise} \end{cases}
\end{aligned}$$

The interval of convergence is $s \in (1, \infty)$.

(b) One definition of the gamma function is given by the improper integral

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\,dt, \qquad \alpha > 0$$

i. (1 point) Show $\Gamma(\alpha)$ converges for $0 < \alpha < 1$.

Solution:

1M Suppose $0 < \alpha < 1$, for $0 \le t < \infty$, we have

$$\frac{t^0}{e^{t/2}} \le \frac{t^{\alpha-1}}{e^{t/2}} \le \frac{t^1}{e^{t/2}} \implies \frac{1}{e^{t/2}} \le \frac{t^{\alpha-1}}{e^{t/2}} \le \frac{t}{e^{t/2}}$$

Since

$$\lim_{t\to\infty}\frac{1}{e^{t/2}} = \lim_{t\to\infty}\frac{t}{e^{t/2}} = 0$$

By the squeeze theorem,

$$\lim_{t\to\infty}\frac{t^{\alpha-1}}{e^{t/2}} = 0$$

thus

$$\left|\frac{t^{\alpha-1}}{e^{t/2}} - 0\right| < 1 \implies t^{\alpha-1}e^{-t} < e^{-t/2} \qquad \text{for} \quad t > 1$$

By comparison, we know

$$\int_1^\infty t^{\alpha-1}e^{-t}\,dt < \int_1^\infty e^{-t/2}\,dt = 2e^{-1/2}$$

is convergent. Notice the following is actually improper as well

$$\int_0^1 t^{\alpha-1}e^{-t}\,dt$$

since for $0<\alpha<1$,

$$\lim_{t\to 0^+}\frac{e^{-t}}{t^{1-\alpha}}=\infty$$

Note, for $0\le t\le 1$,

$$0\le t^{\alpha-1}e^{-t}\le t^{\alpha-1}e^{-0}$$

thus, by comparison, we know

$$\int_0^1 t^{\alpha-1}e^{-t}\,dt\le\int_0^1 t^{\alpha-1}\,dt=\lim_{a\to 0^+}\left(\frac{1}{\alpha}-\frac{a^\alpha}{\alpha}\right)=\frac{1}{\alpha}$$

is convergent. Therefore

$$\int_0^\infty t^{\alpha-1}e^{-t}\,dt=\int_0^1 t^{\alpha-1}e^{-t}\,dt+\int_1^\infty t^{\alpha-1}e^{-t}\,dt$$

is convergent for $0<\alpha<1$, and together with what we have shown in lecture 15, we know Gamma is convergent for all

$$\alpha>0$$

Note Gamma actually is convergent for all complex numbers excepts non-positive integers. But we restrict ourselves to positive reals to stay in real.

ii. (1 point) Show that

$$\Gamma(\alpha+1)=\alpha\Gamma(\alpha).$$

Solution:

1M Using integration by parts, we have

$$\begin{aligned}\Gamma(\alpha+1)=\int_0^\infty t^\alpha e^{-t}\,dt&=\lim_{b\to\infty}\int_0^b t^\alpha e^{-t}\,dt\\&=\lim_{b\to\infty}\left[\left[-t^\alpha e^{-t}\right]_0^b+\int_0^b \alpha t^{\alpha-1}e^{-t}\,dt\right]\\&=0+\alpha\int_0^\infty t^{\alpha-1}e^{-t}\,dt\qquad\text{for}\quad\alpha>0\\&=\alpha\Gamma(\alpha)\end{aligned}$$

iii. (1 point) Show that

$$\mathcal{L}\left[t^\alpha\right]=\frac{\Gamma(\alpha+1)}{s^{\alpha+1}},\qquad\alpha>-1.$$

Solution:

1M By definition,

$$\mathcal{L}\left[t^\alpha\right] = \int_0^\infty t^\alpha e^{-st}\,dt$$

Let $u = st$, we have

$$\begin{aligned}\mathcal{L}\left[t^\alpha\right] = \int_0^\infty t^\alpha e^{-st}\,dt = \frac{1}{s^\alpha}\int_0^\infty (st)^\alpha e^{-st}\frac{s}{s}\,dt &= \frac{1}{s^{\alpha+1}}\int_0^\infty u^\alpha e^{-u}\,du \\ &= \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}\end{aligned}$$

using the convergence interval of Gamma function, we see $\alpha > -1$.

iv. (1 point) Show that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Solution:

1M By definition,

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2}e^{-t}\,dt$$

Let $t = u^2$,

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2}e^{-t}\,dt = 2\int_0^\infty e^{-u^2}\,du$$

Now consider

$$\begin{aligned}\left[\Gamma\left(\frac{1}{2}\right)\right]^2 &= 4\left(\int_0^\infty e^{-u^2}\,du\right)\left(\int_0^\infty e^{-v^2}\,dv\right) \\ &= 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\,du\,dv\end{aligned}$$

In polar, we have

$$\begin{aligned}\left[\Gamma\left(\frac{1}{2}\right)\right]^2 &= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} r\,dr\,d\theta = \pi \\ \implies \Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi}\end{aligned}$$

v. (1 point) Show that

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

Solution:

1M Notice this is just what we have had in the last part in reverse,

$$\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty t^{-1/2}e^{-t}\,dt = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

by using the substitution $x^2 = t$, thus

$$\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

(c) (1 point) Show that $f(t) = t\cos t$ is of exponential order.

Solution:

1M We need to show there exist constant c, $M > 0$ and $T > 0$ such that

$$|t\cos t| \le Me^{ct} \qquad \text{for all} \quad t > T$$

Since $-1 \le \cos t \le 1$ for all t,

$$|t\cos t| \le |t|$$

Consider $g(t) = e^t - t$, we have

$$g(0) = 0 \qquad \text{and} \qquad g'(t) = e^t - 1 > 0 \quad \text{for} \quad t > 1$$

thus

$$\begin{aligned} e^t &\ge t && \text{for all} \quad t > 1 \\ |e^t| &\ge |t| && \text{for all} \quad t > 1 \\ |t\cos t| \le |t| &\le e^t && \text{for all} \quad t > 1 \end{aligned}$$

So $c = 1$, $M = 1$ and $T = 1$ works, and therefore f is of exponential order.

(d) (1 point) Solve by Laplace's method the initial value problem

$$\dot{y} = 5 - 2t, \qquad y(0) = 1$$

Solution:

1M Taking the Laplace transform

$$\begin{aligned} sY(s) - y(0) &= \frac{5}{s} - \frac{2}{s^2} \\ sY(s) - 1 &= \frac{5}{s} - \frac{2}{s^2} \\ Y(s) &= \frac{1}{s} + \frac{5}{s^2} - \frac{2}{s^3} \end{aligned}$$

Back transform $Y(s)$ using $\mathcal{L}\Big[t^n\Big] = \dfrac{n!}{s^{n+1}}$, we have

$$y(t) = 1 + 5t - t^2$$

is the solution.

Question2 (4 points)

(a) (1 point) Write $f(t) = \begin{cases} 2\sin t, & 0 \le t < 3 \\ 7\cos t, & 3 \le t < 5 \\ 3\sin t, & t \ge 5. \end{cases}$ in terms of unit step functions.

Solution:

1M Recall $u(t-a)$ "turns on" a function at a and $-u(t-b)$ "turns off" at b.

$$f(t) = 2\sin t\,(u(t) - u(t-3)) + 7\cos t\,(u(t-3) - u(t-5)) + 3\sin t u(t-5)$$

(b) (1 point) Find the convolution t^2 and $e^{2t}\sin 2t$.

Solution:

1M By definition of convolution and integration by parts,

$$t^2 * e^{2t}\sin 2t = \int_0^t e^{2\tau}\sin 2\tau(t-\tau)^2\,d\tau = \frac{1}{16}\left[(2t+1)^2 - e^{2t}\left(\sin 2t + \cos 2t\right)\right]$$

(c) (1 point) Show the operation of convolution is associative.

Solution:

1M This is to show

$$(f*g)*h = f*(g*h)$$

which can be done by using the definition of integral and substitution, however, let me invoke the convolution theorem,

$$\begin{aligned}(f*g)*h = \mathcal{L}^{-1}\Big[\mathcal{L}\left[f*g\right]H\Big] &= \mathcal{L}^{-1}\Big[FGH\Big] \\ &= \mathcal{L}^{-1}\Big[F\mathcal{L}\left[g*h\right]\Big] \\ &= f*(g*h)\end{aligned}$$

(d) (1 point) Find $\lim\limits_{t\to\infty} f(t)$, where $\mathcal{L}\left[f(t)\right] = \dfrac{1}{s(s+a)(s+b)}$.

Solution:

1M Applying final-value theorem, we have

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0}\frac{s}{s(s+a)(s+b)} = \lim_{s\to 0}\frac{1}{(s+a)(s+b)} = \frac{1}{ab}$$

Question3 (3 points)

Consider the equation

$$\phi(t) + \int_0^t h(t-\xi)\phi(\xi)\,d\xi = f(t),$$

in which f and h are known functions, and ϕ is to be determined. Since the unknown function ϕ appears under an integral sign, the given equation is called an integral equation; in particular, it belongs to a class of integral equations known as Volterra integral equations.

The inverse Laplace transform of $\mathcal{L}\{\phi(t)\}$ is the solution of the original integral equation.

Consider the following Volterra integral equation

$$\phi(t) + \int_0^t (t-\xi)\phi(\xi)\, d\xi = \sin 2t. \tag{1}$$

That is, $h(t-\xi) = t - \xi$ and $f(t) = \sin 2t$.

(a) (1 point) Solve Equation (1) by using the Laplace method.

Solution:

1M Taking the Laplace transform, and using the convolution theorem, we have

$$\begin{aligned}\mathcal{L}\Big[\phi\Big] + \mathcal{L}\Big[t\Big]\mathcal{L}\Big[\phi\Big] &= \frac{2}{s^2+4} \\ \mathcal{L}\Big[\phi\Big] &= \frac{2s^2}{(s^2+1)(s^2+4)} = -\frac{2}{3}\frac{1}{s^2+1} + \frac{4}{3}\frac{2}{s^2+4}\end{aligned}$$

Back transforming, we have

$$\phi(t) = \frac{1}{3}\left(4\sin 2t - 2\sin t\right)$$

(b) (1 point) Show that if u is a function such that $u''(t) = \phi(t)$, then

$$u''(t) + u(t) - tu'(0) - u(0) = \sin 2t.$$

Solution:

1M Since we have found ϕ, from which we can solve for u

$$\begin{aligned}u''(t) &= \frac{1}{3}\left(4\sin 2t - 2\sin t\right) \\ u'(t) &= \frac{2}{3}\left(-\cos 2t + \cos t\right) + C_1 \\ u(t) &= \frac{1}{3}\left(-\sin 2t + 2\sin t\right) + C_1 t + C_2\end{aligned}$$

substituting into the given equation, we can confirm u is the general solution.

(c) (1 point) Show that Equation (1) is equivalent to the initial value problem

$$u''(t) + u(t) = \sin 2t; \qquad u(0) = 0, u'(0) = 0.$$

Solution:

1M Simply apply the initial conditions, we see

$$\begin{aligned}& u''(t) + u(t) - tu'(0) - u(0) = \sin 2t. \\ \iff\ & u''(t) + u(t) = \sin 2t; \qquad u(0) = 0, u'(0) = 0. \\ \iff\ & \phi(t) + \int_0^t (t-\xi)\phi(\xi)\, d\xi = \sin 2t.\end{aligned}$$

since $u'' = \phi$ and

$$\int_0^t (t-\xi)\phi(\xi)\,d\xi = \frac{1}{3}\left(-\sin 2t + 2\sin t\right) = u$$

that satisfies $u(0) = 0$ and $u'(0) = 0$

Question4 (4 points)

Solve the following initial-value problem using the Laplace method.

(a) (1 point) $y' + 2y = u(t-\pi)\sin 2t, \quad y(0) = 3$

Solution:

1M Taking the Laplace transform, we have

$$\begin{aligned} sY(s) - y(0) + 2Y(s) &= e^{-\pi s}\frac{2}{s^2+4} \\ \Longrightarrow\ Y(s) &= \frac{3}{s+2} + e^{-\pi s}\frac{2}{(s+2)(s^2+4)} \\ &= \frac{3}{s+2} + e^{-\pi s}\left(\frac{1}{4}\frac{1}{s+2} - \frac{1}{4}\frac{s}{s^2+4} + \frac{1}{4}\frac{2}{s^2+4}\right) \end{aligned}$$

Back transforming, we have

$$y(t) = 3e^{-2t} + u(t-\pi)\left[\frac{1}{4}e^{-2(t-\pi)} - \frac{1}{4}\cos(2(t-\pi)) + \frac{1}{4}\sin(2(t-\pi))\right]$$

(b) (1 point) $y''+y = f(t), \quad y(0) = 0, \quad y'(0) = 0, \quad$ where $\quad f(t) = \begin{cases} 4, & 0 \le t < 2 \\ t+2, & 2 \le t \end{cases}$

Solution:

1M Write f using step functions,

$$f(t) = 4u(t) + (t-2)u(t-2)$$

Taking the Laplace transform, we have

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + Y(s) &= \frac{4}{s} + e^{-2s}\frac{1}{s^2} \\ Y(s) &= -4\frac{s}{s^2+1} + \frac{4}{s} + e^{-2s}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right) \end{aligned}$$

Back transform,

$$y(t) = -4\cos t + 4 + u(t-2)\left[t - 2 - \sin(t-2)\right]$$

(c) (1 point) $y'' + 2y' + y = \delta(t-1), \quad y(0) = 0, \quad y'(0) = 0$

Solution:

1M Taking the Laplace transform, we have

$$s^2Y(s) - sy(0) - y'(0) + Y = e^{-s}$$
$$Y(s) = \frac{e^{-s}}{(s+1)^2}$$

Back transform

$$y(t) = u(t-1)e^{-(t-1)}(t-1)$$

(d) (1 point) $ty'' + 2(t-1)y' - 2y = 0, \quad y(0) = 0$

Solution:

1M Taking the Laplace transform, we have

$$-\frac{d}{ds}\left[s^2Y(s) - sy(0) - y'(0)\right] - 2\frac{d}{ds}\left[sY(s) - y(0)\right] - 2\left(sY(s) - y(0)\right) - 2Y(s)$$

Setting it to zero, we have

$$(-s^2 - 2s)Y' - (4s+4)Y = 0$$

Solving this first-order equation, we have

$$Y(s) = \frac{C}{s^2(s+2)^2}$$

Back transform, we have

$$y(t) = \frac{C}{4}\left(e^{-2t} + te^{-2t} + t - 1\right)$$

Question5 (1 points)

Suppose a tank holds 10 gallon of pure water. There are two sources of brine solution: the first source has concentration of 0.5g of salt per gallon while the second source has a concentration of 2.5g of salt per gallon. The first source pours into the tank at a rate of 1 gallon per minute for 5 minutes after which it is turned off. In the meanwhile, the second source is turned on at a rate of 1 gallon per minute. The well-mixed solution pours out of the tank at a rate of 1 gallon per minute. Find the amount of salt in the tank at time t.

Solution:

1M Letting $y(t)$ denote the amount of salt in the tank at time t, measured in g. The rate of change of $y(t)$ comes from the difference between the rate salt is being added and the rate salt is being removed, that is,

$$y'(t) = \text{rate in} - \text{rate out}$$

Recall that the input and output rates of salt are the product of the concentration of salt and the flow rates of the mixtures. The rate at which salt is being added depends on the interval of time. For the first five minutes, source one adds salt at a rate of 0.5 g/min, and after that, source two takes over and adds salt at a rate of 2.5 g/min. Since the flow rate in is 1 gal/min, the rate at which salt is being added is given by the function

$$f(t) = \begin{cases} 0.5 & 0 \leq t < 5 \\ 2.5 & 5 \leq t \end{cases}$$

The concentration of salt at time t is $\frac{y(t)}{10}$ g/gal, and since the flow rate out is 1 gal/min, it follows that the rate at which salt is being removed is $\frac{y(t)}{10}$ g/min. Since initially there is only pure water, it follows that $y(0) = 0$, and therefore, we have that $y(t)$ satiesfies the following initial value problem:

$$y' = f(t) - \frac{y(t)}{10}, \qquad y(0) = 0$$

Rewriting $f(t)$ using unit step functions

$$\begin{aligned} f(x) &= 0.5(u(t) - u(t-5)) + 2.5u(t-5) \\ &= 0.5 + 2u(t-5) \end{aligned}$$

Applying the Laplace transform to the differential equation and solving for

$$Y(s) = \frac{5}{s} - \frac{5}{s + \frac{1}{10}} + e^{-5s}\frac{20}{s} - e^{-5s}\frac{20}{s + \frac{1}{10}}$$

Back transform

$$y(t) = 5 - 5\exp\left(\frac{-t}{10}\right) + 20u(t-5) - 20u(t-5)\exp\left(-\frac{t-5}{10}\right)$$