LEC011 Maximum Flow and Linear Program

VG441 SS2020

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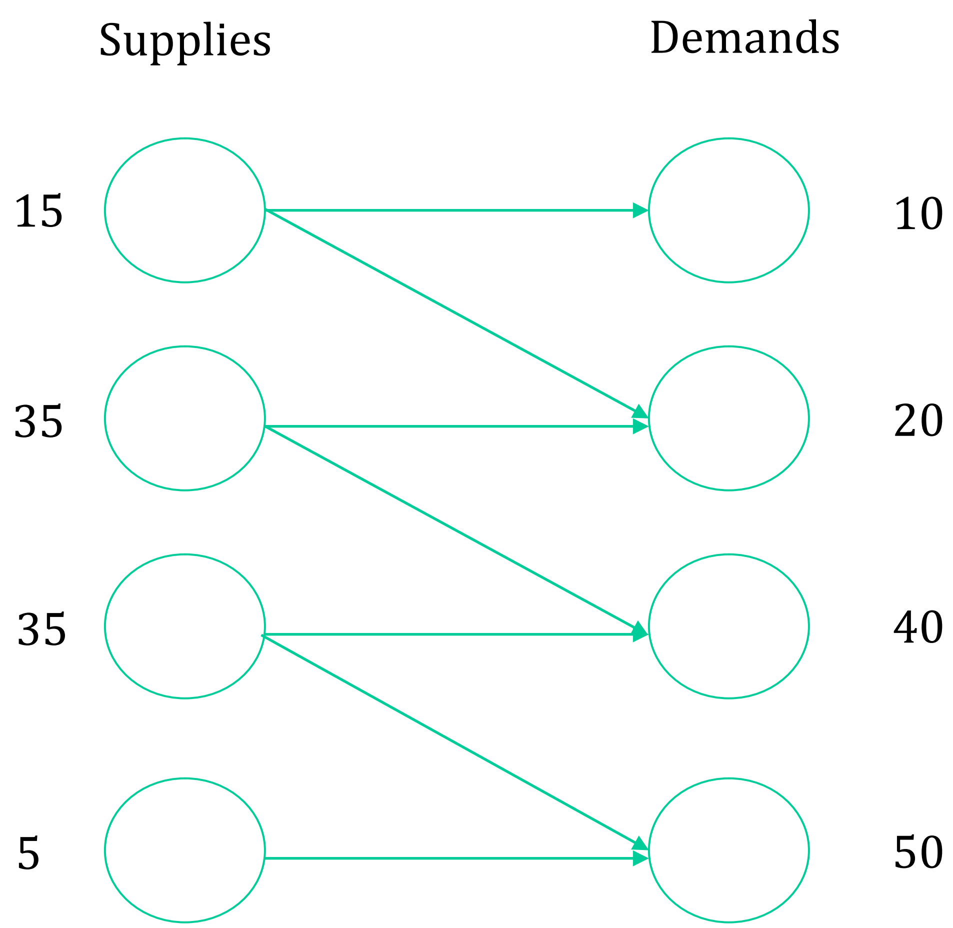
Max Flow Applications



Linear Programming

- Comparison to systems of linear equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

Linear Programming

- Ingredients of a Linear Program

1. Decision variables $x_1, \ldots, x_n \in \mathbb{R}$

2. Linear constraints, each of the form

$$\sum_{j=1}^{n} a_{ij} x_j \quad (*) \quad b_i$$

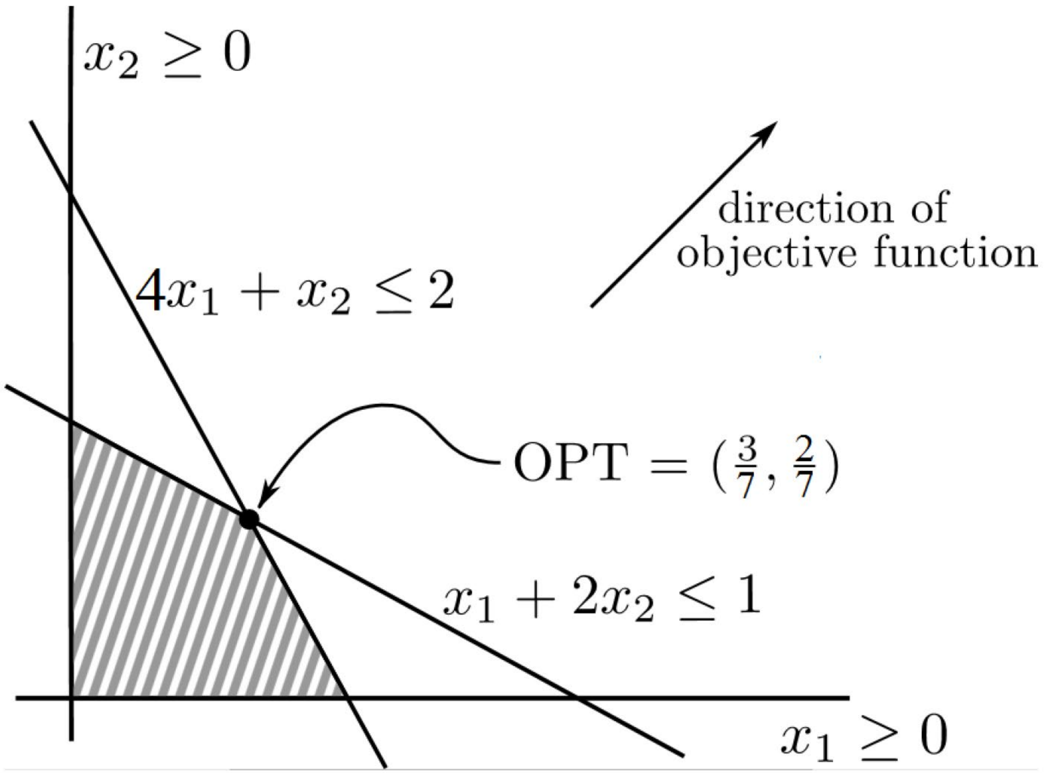
where $(^*)$ could be $\leq$, $\geq$, or $=$

3. A linear objective function, of the form

$$\max \sum_{j=1}^{n} c_j x_j \qquad \text{or} \qquad \min \sum_{j=1}^{n} c_j x_j$$

A Simple Example

$$
\begin{aligned}
\max \quad & x_1 + x_2 \\
\text{s.t.} \quad & 4x_1 + x_2 \leq 2 \\
& x_1 + 2x_2 \leq 1 \\
& x_1, x_2 \geq 0
\end{aligned}
$$



Python + Gurobi Time!

MaxFlow is a LP

- Decision variables

$$\{f_e\}_{e\in E}$$

- Constraints $(2m + n - 2)$

$$\underbrace{\sum_{e\in\delta^-(v)} f_e}_{\text{flow in}} - \underbrace{\sum_{e\in\delta^-(v)} f_e}_{\text{flow out}} = 0$$

$$f_e \le u_e$$

- Objectives

$$f_e \ge 0$$

$$\max \textstyle\sum_{e\in\delta^+(s)} f_e$$

Generalization of MaxFlow

- Min-Cost MaxFlow

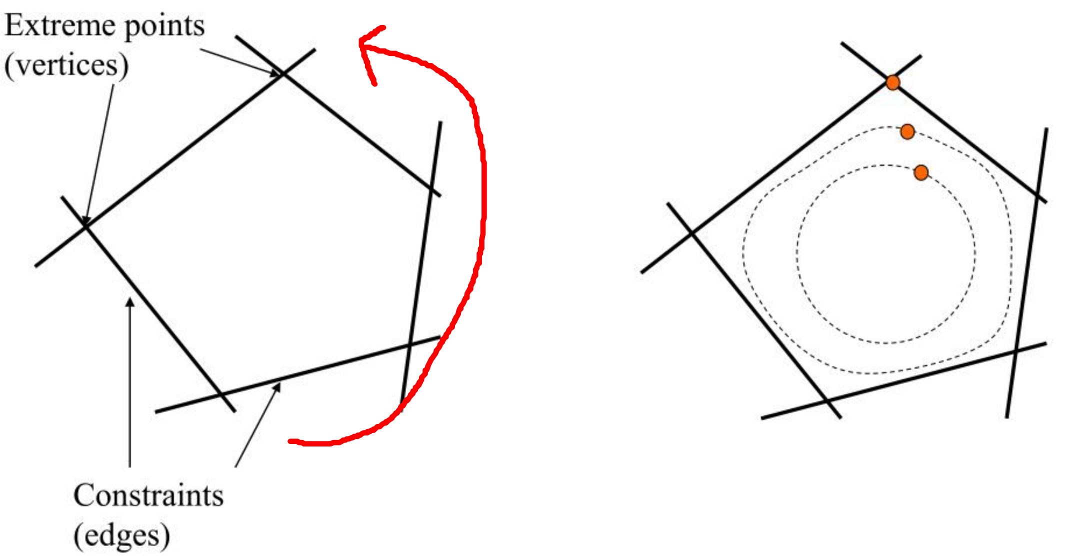
$$\min \sum_{e \in E} c_e f_e$$

- Easy to change the LP formulation!

Solution Approaches

- Simplex methods
- Interior point methods

Interior point methods



Simplex: search from vertex to vertex along the edges

Interior-point methods: go through the inside of the feasible space

LP Duality

- Question: given a feasible solution, how can we know whether it is optimal or close to optimal?

$$\boxed{\begin{array}{lll} \max & x_1 + x_2 & \\ \text{s.t.} & 4x_1 + x_2 \le 2 & \\ & x_1 + 2x_2 \le 1 & \\ & x_1, x_2 \ge 0 & \end{array}}$$

$$\underbrace{x_1 + x_2}_{\text{objective}} \le 4x_1 + x_2 \le \underbrace{2}_{\text{upper bound}}$$

$$\underbrace{x_1 + x_2}_{\text{objective}} \le x_1 + 2x_2 \le \underbrace{1}_{\text{upper bound}}$$

Can we get an even better upper bound?

$$x_1 + x_2 \le \frac{1}{7}\underbrace{(4x_1 + x_2)}_{\le 2 \text{ by } (2)} + \frac{3}{7}\underbrace{(x_1 + 2x_2)}_{\le 1 \text{ by } (3)} \le \frac{1}{7}\cdot 2 + \frac{3}{7}\cdot 1 = \frac{5}{7}$$

y

Deriving the Dual LP

Compact form

$$\begin{array}{ll} \max & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge 0 \end{array}$$

$$\begin{array}{lll} \max & \sum_{j=1}^{n} c_j x_j & \\ \text{s.t.} & \sum_{j=1}^{n} a_{1j} x_j \le b_1 & \longrightarrow \quad y_1 \ge 0 \\ & \sum_{j=1}^{n} a_{2j} x_j \le b_2 & \longrightarrow \quad y_2 \ge 0 \\ & \vdots \le \vdots & \\ & \sum_{j=1}^{n} a_{mj} x_j \le b_m & \longrightarrow \quad y_m \ge 0 \\ & x_1, \ldots, x_n \ge 0 & \end{array}$$

"Multipliers"

The idea is to "dominates the objective coefficients": $\sum_{i=1}^{m} y_i a_{ij} \ge c_j$

More compactly, Find $\mathbf{y} \ge 0$ such that $\mathbf{A}^T\mathbf{y} \ge \mathbf{c}$

Deriving the Dual LP

$$\begin{array}{ll} \max & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge 0 \end{array}$$

Find $\mathbf{y} \ge 0$ such that $\mathbf{A}^T\mathbf{y} \ge \mathbf{c}$

$$\begin{aligned} \underbrace{\sum_{j=1}^{n} c_j x_j}_{\text{x's obj fn}} &\le \sum_{j=1}^{n} \left(\sum_{i=1}^{m} y_i a_{ij} \right) x_j \\ &= \sum_{i=1}^{m} y_i \cdot \left(\sum_{j=1}^{n} a_{ij} x_j \right) \\ &\le \underbrace{\sum_{i=1}^{m} y_i b_i}_{\text{upper bound}} \end{aligned}$$

More compactly, $\mathbf{c}^T\mathbf{x} \le \left(\mathbf{A}^T\mathbf{y}\right)^T \mathbf{x} = \mathbf{y}^T(\mathbf{A}\mathbf{x}) \le \mathbf{y}^t\mathbf{b}$

Deriving the Dual LP

$$\begin{array}{ll} \max & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array}$$

Find $\mathbf{y} \geq 0$ such that $\mathbf{A}^T\mathbf{y} \geq \mathbf{c}$

$$\mathbf{c}^T\mathbf{x} \leq \left(\mathbf{A}^T\mathbf{y}\right)^T\mathbf{x} = \mathbf{y}^T(\mathbf{A}\mathbf{x}) \leq \mathbf{y}^t\mathbf{b}$$

$$\begin{array}{ll} \min & \mathbf{b}^T\mathbf{y} \\ \text{s.t.} & \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq 0 \end{array}$$

A Simple Example

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & \mathbf{b}^T\mathbf{y} \\ \text{s.t.} \quad & \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & 4x_1 + x_2 \leq 2 \\ & x_1 + 2x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & 2y_1 + y_2 \\ \text{s.t.} \quad & 4y_1 + y_2 \cancel{\leq} \geq 1 \\ & y_1 + 2y_2 \cancel{\leq} \geq 1 \\ & y_1, y_2 \geq 0 \end{aligned}$$

$x_1^* = 3/7, x_2^* = 2/7$

$y_1^* = 1/7, y_2^* = 3/7$

Optimal objectives are the same!

Weak and Strong Duality

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & \mathbf{b}^T\mathbf{y} \\ \text{s.t.} \quad & \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq 0 \end{aligned}$$

At optimality, we must have

- Weak Duality $\quad \mathbf{c}^T\mathbf{x}^* \leq \mathbf{b}^T\mathbf{y}^*$
- Strong Dualiy $\quad \mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$

Proof = Separating hyperplance theorem + Farkas's Lemma

More General Form

Primal	Dual
variables $x_1, \ldots, x_n$	n constraints
m constraints	variables $y_1, \ldots, y_m$
objective function $\mathbf{c}$	right-hand side $\mathbf{c}$
right-hand side $\mathbf{b}$	objective function $\mathbf{b}$
$\max \mathbf{c}^T \mathbf{x}$	$\min \mathbf{b}^T \mathbf{y}$
constraint matrix $\mathbf{A}$	constraint matrix $\mathbf{A}^T$
i-th constraint is "$\leq$"	$y_i \geq 0$
i-th constraint is "$\geq$"	$y_i \leq 0$
i-th constraint is "$=$"	$y_i \in \mathbb{R}$
$x_j \geq 0$	j-th constraint is "$\geq$"
$x_j \leq 0$	
$x_j \in \mathbb{R}$	

$$
\begin{aligned}
\textbf{(P)} \quad \min \quad & \sum_{i \in F} f_i y_i + \sum_{i \in F, j \in D} d_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{i \in F} x_{ij} \geq 1 \quad \forall j \in D \\
& x_{ij} \leq y_i \quad \forall i \in F, j \in D \\
& x_{ij}, y_i \geq 0
\end{aligned}
$$

Here x_{ij} is the fraction of demand j that is served by facility i, and y_i is the (continuous) decision of whether facility i should be opened. Note that y_i should be binary $\{0, 1\}$ but we relax it to be $y_i \geq 0$.

Task 1: Assign dual variables α_j for every demand $j \in D$, and dual variables β_{ij} for every (facility, demand) pair. Derive the dual linear program **(D)** in terms of these α's and β's.

Task 2: Think of α_j as the amount of money demand j is willing to contribute to the solution, and β_{ij} as the amount of money demand j's contributes towards opening facility i. Try to interpret the dual linear program **(D)**.

Back to MaxFlow

- Let the decision variable be flow on an s-t path

$$
\begin{aligned}
\max \quad & \sum_{P \in \mathcal{P}} f_P \\
\text{s.t.} \quad & \underbrace{\sum_{P \in \mathcal{P} : e \in P} f_P \le u_e}_{\text{total flow on } e} \text{ for all } e \in E \\
& f_P \ge \text{ for all } P \in \mathcal{P}
\end{aligned}
$$

Question: how to write its dual and what is its interpretation?