LEC008 Inventory Management III

VG441 SS2020

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Newsvendor Problem

- Let D be random demand with $\mu = E[D]$ and $\sigma^2 = V[D]$
- Let c be unit cost, $r > c$ selling price, $s < c$ salvage value
- Question: what is the optimal ordering quantity S?

variance of D

$V[D] = E[D^2] - (E[D])^2$

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$$\pi(S) = rE[\min(S, D)] + sE[(S - D)^+] - cS$$

$x^+ = \max(0, x)$

Transforming this objective into …

$$\pi(S) = (r - c)\mu - g(S)$$

where $$g(S) = \underbrace{(c - s)}_{h} E[(S - D)^+] + \underbrace{(r - c)}_{p} E[(D - S)^+]$$

Newsvendor Problem

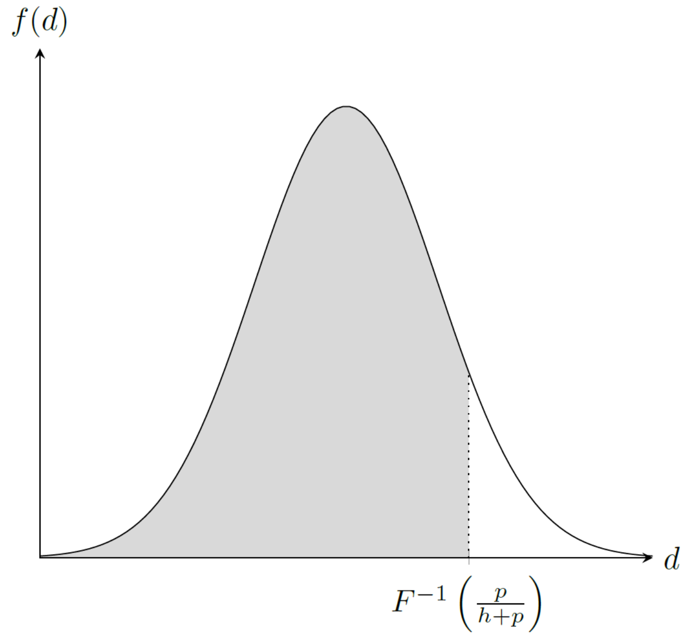
- Minimizing …

$$g(S) = hE[(S - D)^+] + pE[(D - S)^+]$$

- The optimal solution is called “newsvendor quantile”:

$$\frac{dg(S)}{dS} = hF(S) + p(F(S) - 1) = (h + p)F(S) - p = 0$$

$$S^* = F^{-1}\left(\frac{p}{h + p}\right)$$



Explicit Form for Normal Demand

- $D \sim N(50, 8^2), h = 0.18, p = 0.70$

$$S^* = F^{-1}\left(\frac{0.70}{0.18 + 0.70}\right) = F^{-1}(0.795) = 56.6$$

- Derivation…

$$F\left(S^*\right) = \frac{p}{h+p}$$

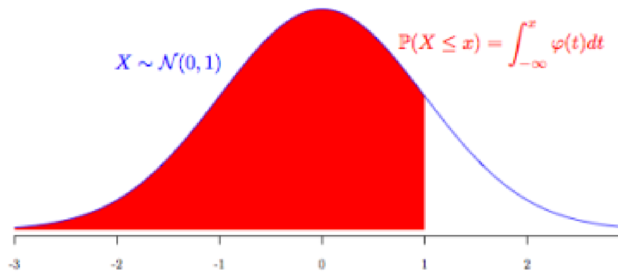
$$\Longleftrightarrow \quad \Phi\left(\frac{S^* - \mu}{\sigma}\right) = \frac{p}{h+p}$$

$$\Longleftrightarrow \quad S^* = \mu + \sigma\Phi^{-1}\left(\frac{p}{h+p}\right)$$

- Concept of cycle stock and safety stock

$$S^* = \mu + z_\alpha \sigma \quad \text{where } \alpha = p/(h+p) \text{ and } z_\alpha = \Phi^{-1}(\alpha)$$

Normal CDF Table



	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Forecast and STDEV

- Don't use the stdev of demand

$$\hat{\sigma}_t = \sqrt{\frac{1}{N-1} \sum_{i=t-N}^{t-1} (d_t - \hat{\mu}_t)^2} \text{ where } \hat{\mu}_t = \frac{1}{N} \sum_{i=t-N}^{t-1} d_t$$

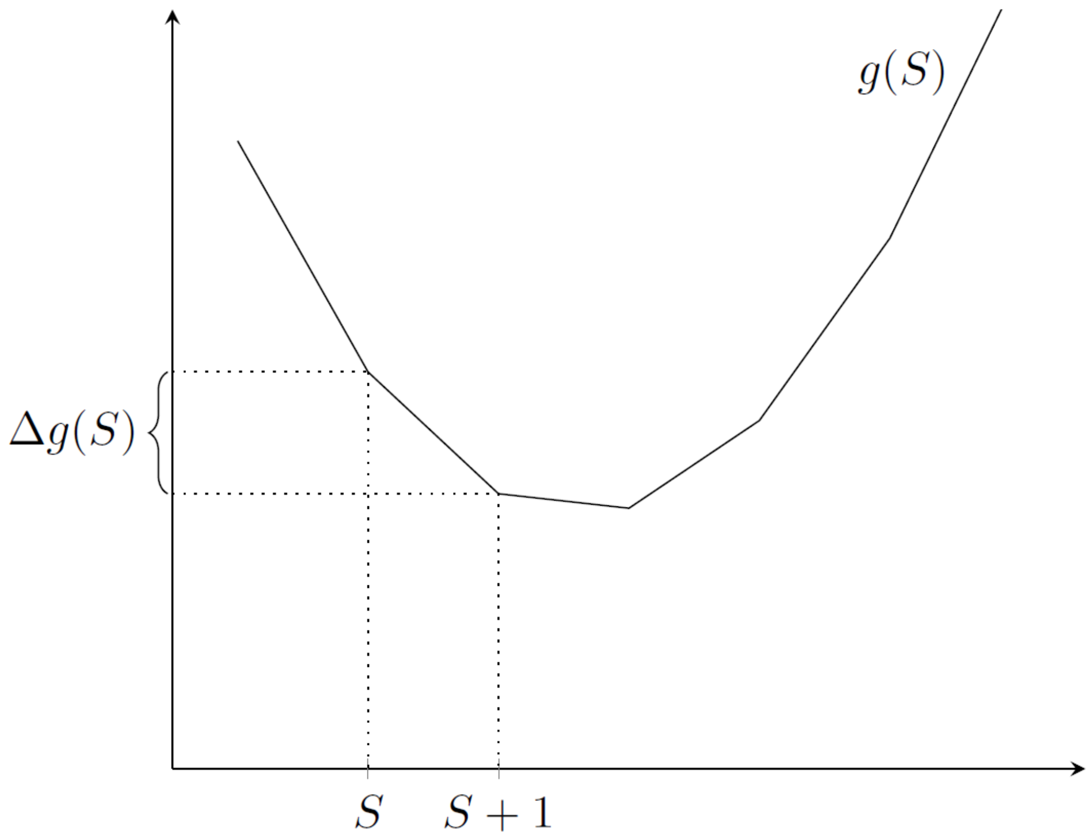
- Instead, use the stdev of forecast errors

$$e_t = \text{forecast}_t - d_t$$

$$\hat{\sigma}_{e,t} = \sqrt{\frac{1}{N-1} \sum_{i=t-N}^{t-1} (e_t - \hat{e}_t)^2} \text{ where } \hat{e}_t = \frac{1}{N} \sum_{i=t-N}^{t-1} e_t$$

What about Discrete Demand?

Find the smallest S such that $F(S) \geq \dfrac{p}{h+p}$



Multiple-Period Newsvendor

- $D_1, \dots, D_T$ i.i.d. demands over period $1, \dots, T$

- In each period $t = 1, \dots, T$
  - Raise the beginning inventory x to a new level y
  - Demand realizes and is satisfied to the max extent
  - Assess the cost (ordering, holding, backlogging)

- Objective: find the optimal ordering strategy

Multiple-Period Newsvendor

- Let $\theta_t(x)$ be the optimal expect cost for periods $[t, T]$ if we begin with inventory x in period t

$$\theta_t(x) = \min_{y \geq x} \{c(y-x) + g(y) + \gamma \mathbb{E}_D [\theta_{t+1}(y-D)]\}$$

- Boundary condition:

$$\theta_{T+1}(x) = -cx$$

- The optimal strategy is the called base-stock policy:

$$S^* = F^{-1}\left(\frac{p-(1-\gamma)c}{h+p}\right)$$

$g(y) = E_D[h(y-D)^+] + E_D[p(D-y)^+]$ newsvendor

Optimality of Base-Stock

$$\theta_{T+1}(x) = -cx$$

$$\theta_t(x) = \min_{y \geq x} \{H_t(y) - cx\}$$

where $H_t(y) = cy + g(y) + \gamma \mathbb{E}_D\left[\theta_{t+1}(y - D)\right]$

Claim:

If $\theta_{t+1}(x)$ is convex, then:
(a) $H_t(y)$ is convex.
(b) A base-stock policy is optimal in period t
and any minimizer of $H_t(y)$ is an optimal base-stock level.
(c) $\theta_t(x)$ is convex.

By assumption, $\theta_{T+1}(x)$ is convex. Therefore, by Claim, a base-stock policy is optimal in period T. Moreover, $\theta_T(x)$ is convex by Claim. This implies that a base-stock policy is optimal in period $T-1$ and that $\theta_{T-1}(x)$ is convex. Continuing this logic, a base-stock policy is optimal in every period.

Multiple-Period Newsvendor

$$\underbrace{\theta_t(x)}_{g(x)} = \min_{y \geq x} \left\{ \underbrace{H_t(y) - cx}_{f(x,y)} \right\}$$

Claim:

If $f(x, y)$ is jointly convex in x and y, and C is a convex set, then the function $g(x) = \min_{y \in C} f(x, y)$ is convext in x.