VE485, Optimization in Machine Learning (Summer 2020) **Homework: 1. Convex Set**

1. Convex Set

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Problem 1

Midpoint convexity. A set C is *midpoint convex* if whenever two points a, b are in C, the average or midpoint $(a+b)/2$ is in C. Obviously a convex set is midpoint convex. It can be proved that under mild conditions midpoint convexity implies convexity. As a simple case, prove that if C is closed and midpoint convex, then C is convex.

Answer.

Assume $\boldsymbol{C}$ is a closed midpoint convex.
If C only has one boundary point a, then C also has only one element $a = 2a/2$.

When C has two boundary points a, b, then $(a+b)/2, (3a+b)/4, (a+3b)/4, (7a+b)/8, (3a+5b)/8 \cdots \in \boldsymbol{C}$
Therefore, $\forall x \in \boldsymbol{C}, x = \sigma/2^n \cdot a + (1-\sigma/2^n)b \in \boldsymbol{C}$ where $\sigma \in N, 0 \leq \sigma \leq 2^n$

Let $z = \theta x_1 + (1-\theta)x_2$ where $x_1, x_2 \in \boldsymbol{C}, 0 \leq \theta \leq 1, \theta \in R$
Then $z = [\theta\sigma_1/2^{n_1} + (1-\theta)\sigma_2/2^{n_2}]a + [\theta(1-\sigma_1/2^{n_1}) + (1-\theta)(1-\sigma_2/2^{n_2})]b$
Therefore, $z = \gamma a + (1-\gamma)b$ where $\gamma = \theta\sigma_1/2^{n_1} + (1-\theta)\sigma_2/2^{n_2}$
Since $0 \leq \gamma \leq 1$, γ can be represented in the form of $\sigma_\gamma/2^{n_\gamma}$ where $n_\gamma \in N, \sigma_\gamma \in N, 0 \leq \sigma_\gamma \leq 2^{n_\gamma}$
$z = \sigma_\gamma/2^{n_\gamma} \cdot a + (1-\sigma_\gamma/2^{n_\gamma})b$

As a result, $z \in C$ and $\boldsymbol{C}$ is a convex. It still applies when C has more than 2 boundary points.

Problem 2

Linear-fractional functions and convex sets. **Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be the linear-fractional function**

$$f(x) = (Ax+b)/(c^Tx+d), \qquad \mathbf{dom} f = \{x \mid c^Tx + d > 0\}.$$

In this problem we study the inverse image of a convex set C under f, i.e.,

$$f^{-1}(C) = \{x \in \mathbf{dom} f \mid f(x) \in C\}.$$

For each of the following sets $C \subseteq \mathbb{R}^n$, give a simple description of $f^{-1}(C)$.

1. **The halfspace $C = \{y \mid g^Ty \leqslant h\}$ (with $g \neq 0$).**
2. **The polyhedron $C = \{y \mid Gy \preceq h\}$.**

3. **The ellipsoid $\{y \mid y^T P^{-1} y \leqslant 1\}$ (where $P \in \mathbf{S}^n_{++}$).**

4. **The solution set of a linear matrix inequality, $C = \{y \mid y_1 A_1 + \cdots + y_n A_n \preceq B\}$, where A_1, ..., A_n, $B \in \mathbf{S}^p$.**

Answer.

The domain of x is a halfspace created by a hyperplane. So $f^{-1}(C)$ is C's inverse image projected on the hyperplane. Let $z \in C$ and $f(x) = z$, which means $Ax + b = z(c^T x + d)$
Then $(A - zc^T)x = (zd - b)$

1. **a**
2. **b**
3. **c**
4. **d**

Problem 3

Give an example of two closed convex sets that are disjoint but cannot be strictly separated.

Answer.