3. Convex Optimization

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Problem 1

Consider the following *Square LP*,

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b, \end{array}$$

with A square and nonsingular. Show that the optimal value is given by

$$p^\star = \begin{cases} c^T A^{-1} b & A^{-T} c \preceq 0 \\ -\infty & \end{cases}$$

Answer:
$p^\star = c^T A^{-1}(Ax) = (A^{-T}c)^T(Ax)$
When $A^{-T}c \succ 0$, $p^\star = -\infty$ when $Ax = -\infty$
When $A^{-T}c \preceq 0$, $p^\star = c^T A^{-1} b$ when $Ax = b$

Problem 2

Relaxation of Boolean LP. In a *Boolean linear program*, the variable x is constrained to have components equal to zero or one:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & x_i \in \{0, 1\}, \quad i = 1, \ldots, n. \end{array} \tag{1}$$

In general, such problems are very difficult to solve, even though the feasible set is finite (containing at most 2^n points).

In a general method called *relaxation*, the constraint that x_i be zero or one is replaced with the linear inequalities $0 \leqslant x_i \leqslant 1$:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & 0 \leqslant x_i \leqslant 1, \quad i = 1, \ldots, n. \end{array} \tag{2}$$

We refer to this problem as the *LP relaxation* of the Boolean LP (1). The LP relaxation is far easier to solve than the original Boolean LP.

1. Show that the optimal value of the LP relaxation (2) is a lower bound on the optimal value of the Boolean LP (1). What can you say about the Boolean LP if the LP relaxation is infeasible?

2. It sometimes happens that the LP relaxation has a solution with $x_i \in \{0, 1\}$. What can you say in this case?

Answer:

1. **Assume $c^T x_1$ is the optimal value of Boolean LP, then $0 \leq x_{1i} \leq 1$ since $x_{1i} \in \{0, 1\}$**
Therefore, x_1 satisfies LP relaxation's constraints.
Assume $c^T x_2$ is the optimal value of LP relaxation, then $c^T x_2 \leq c^T x_1$
Hence $c^T x_2$ is a lower bound on the optimal value of the Boolean LP

   If LP relaxation is infeasible, then there is no x satisfying its constraints or dom.
The Boolean LP's feasible solution set is contained in the LP relaxation's, which is empty.
Hence Boolean LP's feasible solution set is also empty, and the problem is infeasible

2. **The LP relaxation's optimal value is also Boolean LP's, and the solution happens to be both problems' optimal solution.**

Problem 3

Consider the QCQP

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & x^T x \leqslant 1, \end{array}$$

with $P \in \mathbf{S}^n_{++}$. Show that $x^\star = -(P + \lambda I)^{-1} q$ where $\lambda = \max\{0, \bar{\lambda}\}$ and $\bar{\lambda}$ is the largest solution of the nonlinear equation

$$q^T (P + \lambda I)^{-2} q = 1.$$

Answer:
$(1/2)x^T P x + q^T x + r = \frac{1}{2}(x + P^{-1}q)^T P (x + P^{-1}q) + r - \frac{1}{2} q^T P^{-1} q$
So we just need to minimize $(x + P^{-1}q)^T P (x + P^{-1}q) = (Px + q)^T P^{-1} (Px + q)$
Since $P \in \mathbf{S}^n_{++}, (Px + q)^T P^{-1}(Px + q) \geq 0$**, and it's minimal when** $||Px + q||_2$ **is minimal**
Since x **is located in a unit circle,** $||Px + q||_2$ **is minimal when** $Px + q = \lambda x, x = -(P + \lambda I)^{-1} q$

When $\bar{\lambda} < 0, q$ **is the interior point of ellipsoid** $q^T P^{-2} q \leq 1$
$x^T x = q^T P^{-2} q \leq 1$**, satisfying the constraint**
$\lambda = 0, x = -P^{-1}q, (Px + q)^T P^{-1} (Px + q) = 0$
The optimal value is $r - \frac{1}{2} q^T P^{-1} q$

When $\bar{\lambda} \geq 0, q$ **is a boundary point of a larger ellipsoid** $q^T (P + \bar{\lambda} I)^{-2} q \leq 1$
$\forall \lambda < \bar{\lambda}, q^T (P + \lambda I)^{-2} q = x^T x > 1$ **because the axis is shorter**
$\forall \lambda > \bar{\lambda}, ||\lambda x||_2 > ||\bar{\lambda} x||_2$**, so** $||Px + q||_2$ **is minimal when** $\lambda = \bar{\lambda}$
Therefore when $\lambda = \max\{0, \bar{\lambda}\}$**, we can get the optimal value.**