VE485, Optimization in Machine Learning (Summer 2020) Homework: 2. Convex Function

2. Convex Function

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Problem 1

Definition of convexity. Suppose $f : \mathbb{R} \to \mathbb{R}$ is convex, and $a,\ b \in \mathbf{dom} f$ with $a < b$.

1. Show that
$$f(x) \leqslant \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$$
for all $x \in [a, b]$.

2. Show that
$$\frac{f(x)-f(a)}{x-a} \leqslant \frac{f(b)-f(a)}{b-a} \leqslant \frac{f(b)-f(x)}{b-x}$$
for all $x \in (a, b)$. Draw a sketch that illustrates this inequality.

3. Suppose f is differentiable. Use the result in 2 to show that
$$f'(a) \leqslant \frac{f(b)-f(a)}{b-a} \leqslant f'(b).$$
Note that these inequalities also follow from (3.2) in the textbook :
$$f(b) \geqslant f(a) + f'(a)(b-a), \qquad f(a) \geqslant f(b) + f'(b)(a-b).$$

4. Suppose f is twice differentiable. Use the result in 3 to show that $f''(a) \geqslant 0$ and $f''(b) \geqslant 0$.

Answer.

Problem 2

Composition with an affine function. Show that the following functions $f : \mathbb{R}^n \to \mathbb{R}$ are convex.

1. $f(x) = \|Ax - b\|$, where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $\|\cdot\|$ is a norm on $\mathbb{R}^m$.
2. $f(x) = -\left(\det(A_0 + x_1 A_1 + \cdots + x_n A_n)\right)^{1/m}$, on $\{x \mid A_0 + x_1 A_1 + \cdots + x_n A_n \succ 0\}$, where $A_i \in \mathbf{S}^m$.
3. $f(X) = \mathbf{tr}\left(A_0 + x_1 A_1 + \cdots + x_n A_n\right)^{-1}$, on $\{x \mid A_0 + x_1 A_1 + \cdots + x_n A_n \succ 0\}$, where $A_i \in \mathbf{S}^m$. (Use the fact that $\mathbf{tr}(X^{-1})$ is convex on $\mathbf{S}^m_{++}$; see exercise 3.18 in the text book.)

Answer.

Problem 3

Young's inequality. Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function, with $f(0) = 0$, and let g be its inverse. Define F and G as

$$F(x) = \int_0^x f(a)\, da, \qquad G(y) = \int_0^y g(a)\, da.$$

Show that F and G are conjugates, which then leads to the Young's inequality,

$$xy \leqslant F(x) + G(y).$$

Answer.