Ve492: Introduction to Artificial Intelligence
Hidden Markov Models I

Paul Weng

UM-SJTU Joint Institute

Slides adapted from http://ai.berkeley.edu, AIMA, UM, CMU

Reasoning over Time or Space

Often, we want to reason about a sequence of observations

- Speech recognition
- Robot localization
- User attention
- Medical monitoring

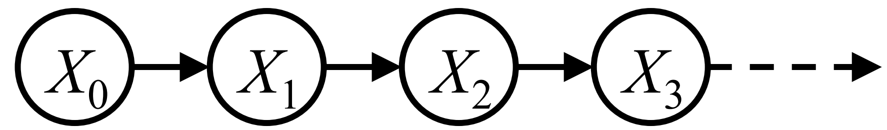
Need to introduce time (or space) into our models

Today

- Markov Models
  - Model
  - Stationary distribution
- Hidden Markov Models
  - Model
  - Forward algorithm for filtering

Markov Models

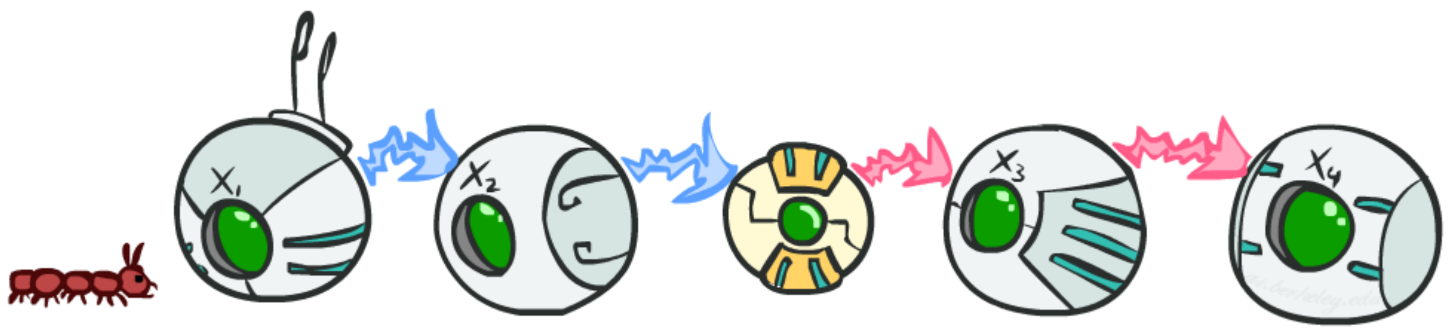
- Value of X at a given time is called the state



$P(X_0)$ $P(X_t \mid X_{t-1})$

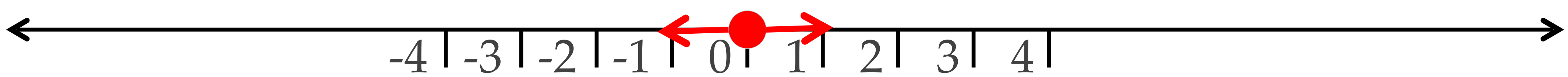
- **Parameters**:
  - Initial state probabilities
  - Transition probabilities (or dynamics) specify how state evolves over time
- **Stationarity assumption**: transition probabilities the same at all times $P(X' \mid X)$
- **Markov assumption**: "future is independent of the past given the present"
  - X_{t+1} is independent of $X_0, \ldots, X_{t-1}$ given X_t
  - First-order Markov model (k-th-order = dependencies on k earlier steps)
- Joint distribution $P(X_0, \ldots, X_T) = P(X_0) \prod_{t=1}^{T} P(X_t \mid X_{t-1})$

Relation to Previous Models



- Bayes' net
  - Markov chain is a (growable) BN
  - We can always use generic BN reasoning on it if we truncate the chain at a fixed length
- Markov decision process
  - Markov chain is an MDP with one action per state (or MDP with fixed policy)

Example: Random Walk in One Dimension



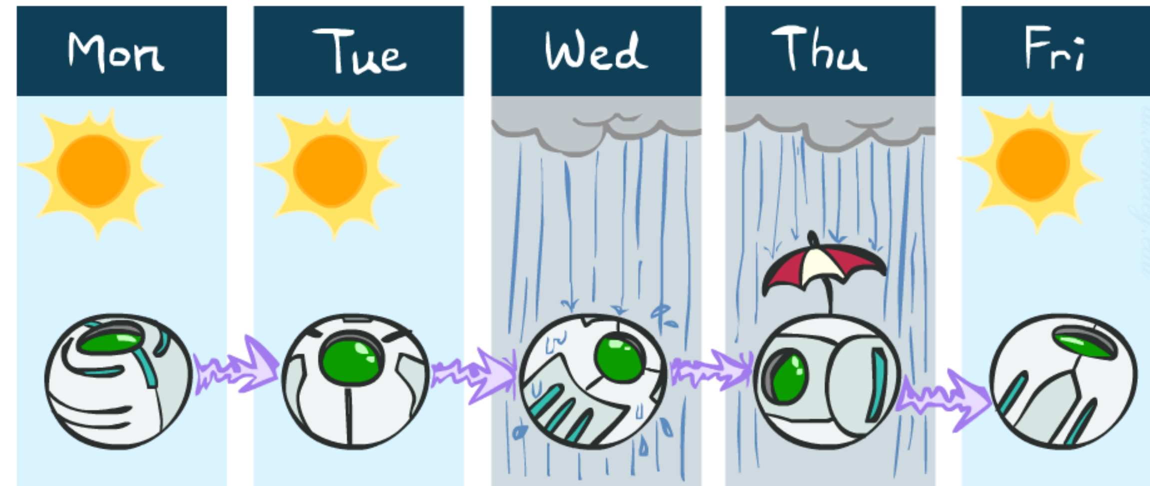
- **State**: location on the unbounded integer line
- **Initial probability**: starts at 0
- **Transition model**: $P(X_t = k\pm 1 \mid X_{t-1} = k) = 0.5$
- **Applications**: particle motion in crystals, stock prices, gambling, genetics, etc.
- **Questions**:
  - How far does it get as a function of t?
    - Expected distance is $O(\sqrt{t})$
  - Does it get back to 0 or can it go off for ever and not come back?
    - In 1D and 2D, returns w.p. 1; in 3D, returns w.p. 0.34053733

Example: n-gram Models

- **State**: word at position t in text (can also build letter n-grams)
- **Transition model** (probabilities come from empirical frequencies):
  - Unigram (zero-order): $P(Word_t = i)$
    - "logical are as are confusion a may right tries agent goal the was . . ."
  - Bigram (first-order): $P(Word_t = i \mid Word_{t\text{-}1} = j)$
    - "systems are very similar computational approach would be represented . . ."
  - Trigram (second-order): $P(Word_t = i \mid Word_{t\text{-}1} = j, Word_{t\text{-}2} = k)$
    - "planning and scheduling are integrated the success of naive Bayes model is . . ."
- **Applications**: text classification, spam detection, author identification, language classification, speech recognition

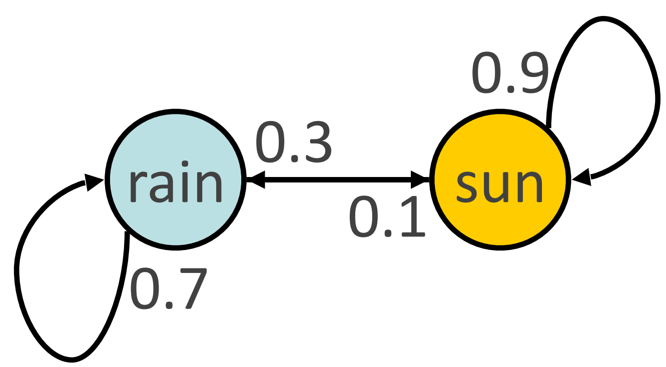
Example: Weather Prediction

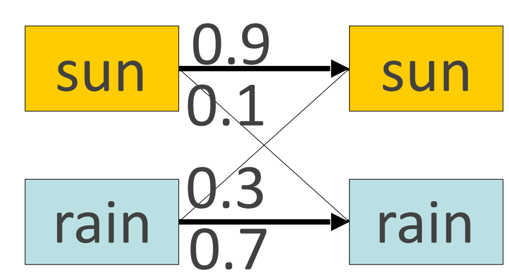
- **State**: sun or rain
- **Initial distribution**: <0.5,0.5>
- **Transition model**:



X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

Two new ways of representing the same CPT





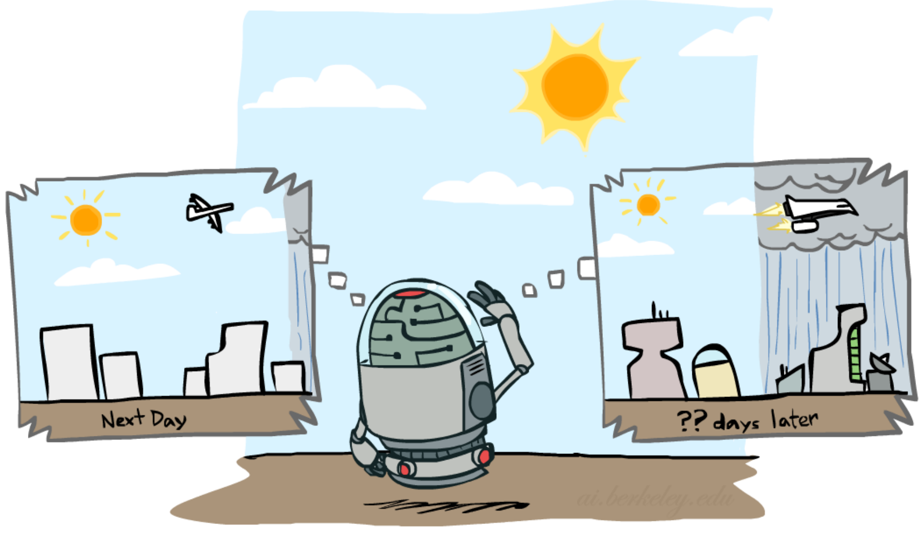
Weather Prediction ctd.

- What is the weather like at time 1?

$$P(X_1) = \sum_{x_0} P(X_1, X_0 = x_0)$$

$$= \sum_{x_0} P(X_0 = x_0)\, P(X_1 \mid X_0 = x_0)$$

$$= 0.5\langle 0.9, 0.1\rangle + 0.5\langle 0.3, 0.7\rangle = \langle 0.6, 0.4\rangle$$



- What is the weather like at time 2?

$$P(X_2) = \sum_{x_1} P(X_2, X_1 = x_1)$$

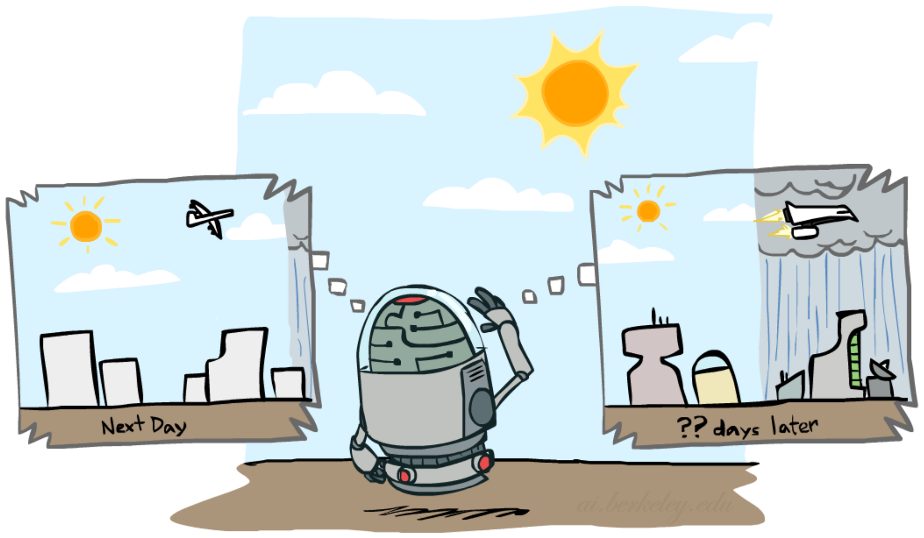
$$= \sum_{x_1} P(X_1 = x_1)\, P(X_2 \mid X_1 = x_1)$$

$$= 0.6\langle 0.9, 0.1\rangle + 0.4\langle 0.3, 0.7\rangle = \langle 0.66, 0.34\rangle$$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

Quiz: Weather Prediction

- Time 2: <0.66,0.34>

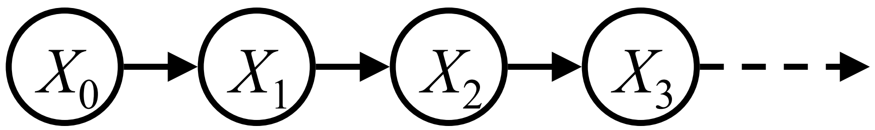


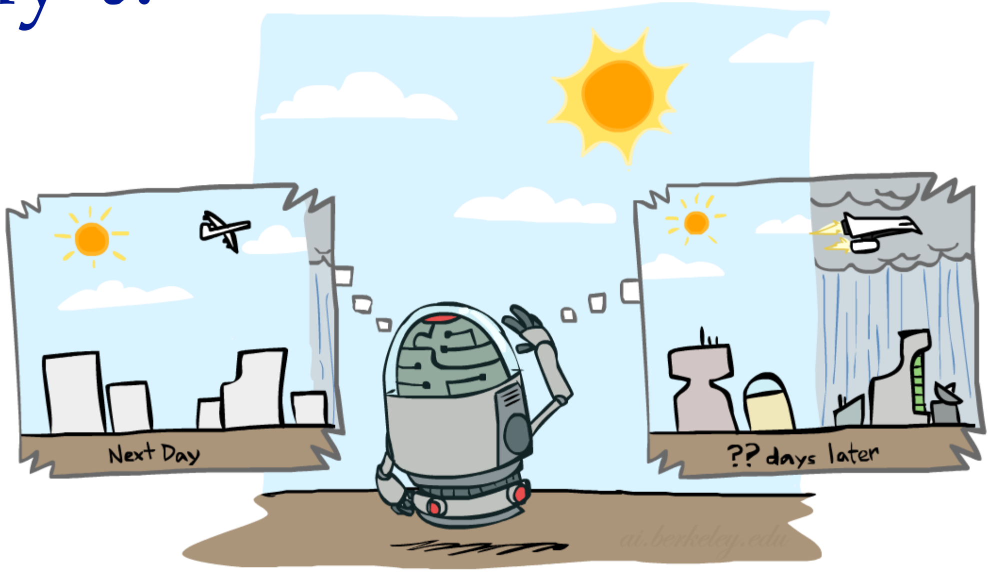
- What is the weather like at time 3?

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

Simple Forward Algorithm

- **Question:** What's P(X) on some day t?





$P(X_0)$ known

$P(X_t) = \sum_{x_{t-1}} P(X_t, X_{t-1}=x_{t-1})$

$= \sum_{x_{t-1}} P(X_{t-1}=x_{t-1})\ P(X_t \mid X_{t-1}=x_{t-1})$

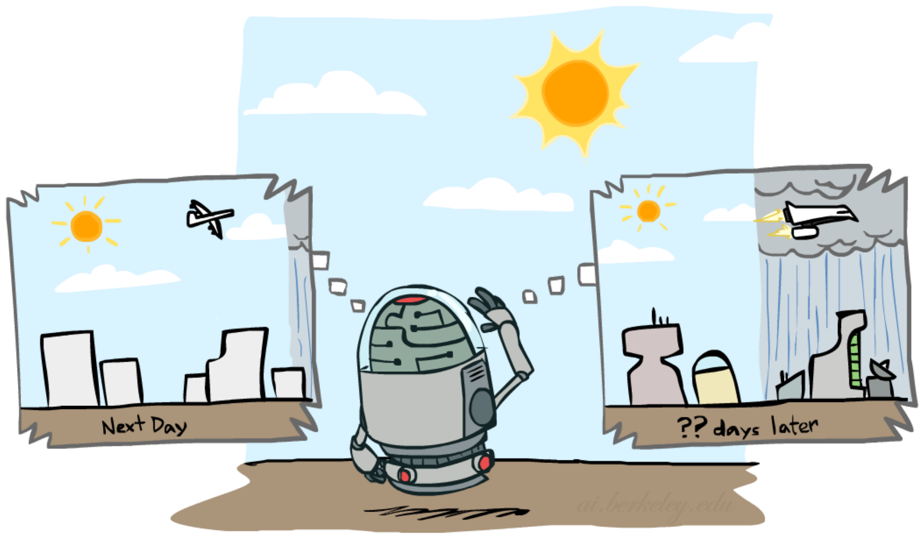
Probability from previous iteration

Transition model

Matrix Form

- What is the weather like at time 2?

  $P(X_2) = 0.6\langle 0.9,0.1\rangle + 0.4\langle 0.3,0.7\rangle = \langle 0.66,0.34\rangle$



- In matrix-vector form:

  $$P(X_2) = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.66 \\ 0.34 \end{pmatrix}$$

- More generally,

  $P(X_{t+1}) = T^T P(X_t)$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

Example Runs of Simple Forward Algorithm

- From initial observation of sun

$$\left\langle \begin{matrix} 1.0 \\ 0.0 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.9 \\ 0.1 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.84 \\ 0.16 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.804 \\ 0.196 \end{matrix} \right\rangle \Rightarrow \left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle$$

$$\mathrm{P}(X_0) \quad \mathrm{P}(X_1) \quad \mathrm{P}(X_2) \quad \mathrm{P}(X_3) \quad \mathrm{P}(X_\infty)$$

- From initial observation of rain

$$\left\langle \begin{matrix} 0.0 \\ 1.0 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.3 \\ 0.7 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.48 \\ 0.52 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.588 \\ 0.412 \end{matrix} \right\rangle \Rightarrow \left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle$$

$$\mathrm{P}(X_0) \quad \mathrm{P}(X_1) \quad \mathrm{P}(X_2) \quad \mathrm{P}(X_3) \quad \mathrm{P}(X_\infty)$$

- From yet another initial distribution P(X1):

$$\left\langle \begin{matrix} p \\ 1-p \end{matrix} \right\rangle \quad \ldots \Rightarrow \left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle$$

$$\mathrm{P}(X_0) \quad \mathrm{P}(X_\infty)$$

Stationary Distributions

- The limiting distribution (if it exists) is called the ***stationary distribution*** P_∞ of the chain
- It satisfies $P_\infty = P_{\infty+1} = T^{\mathrm{T}} P_\infty$
- Solving for P_∞ in the example:

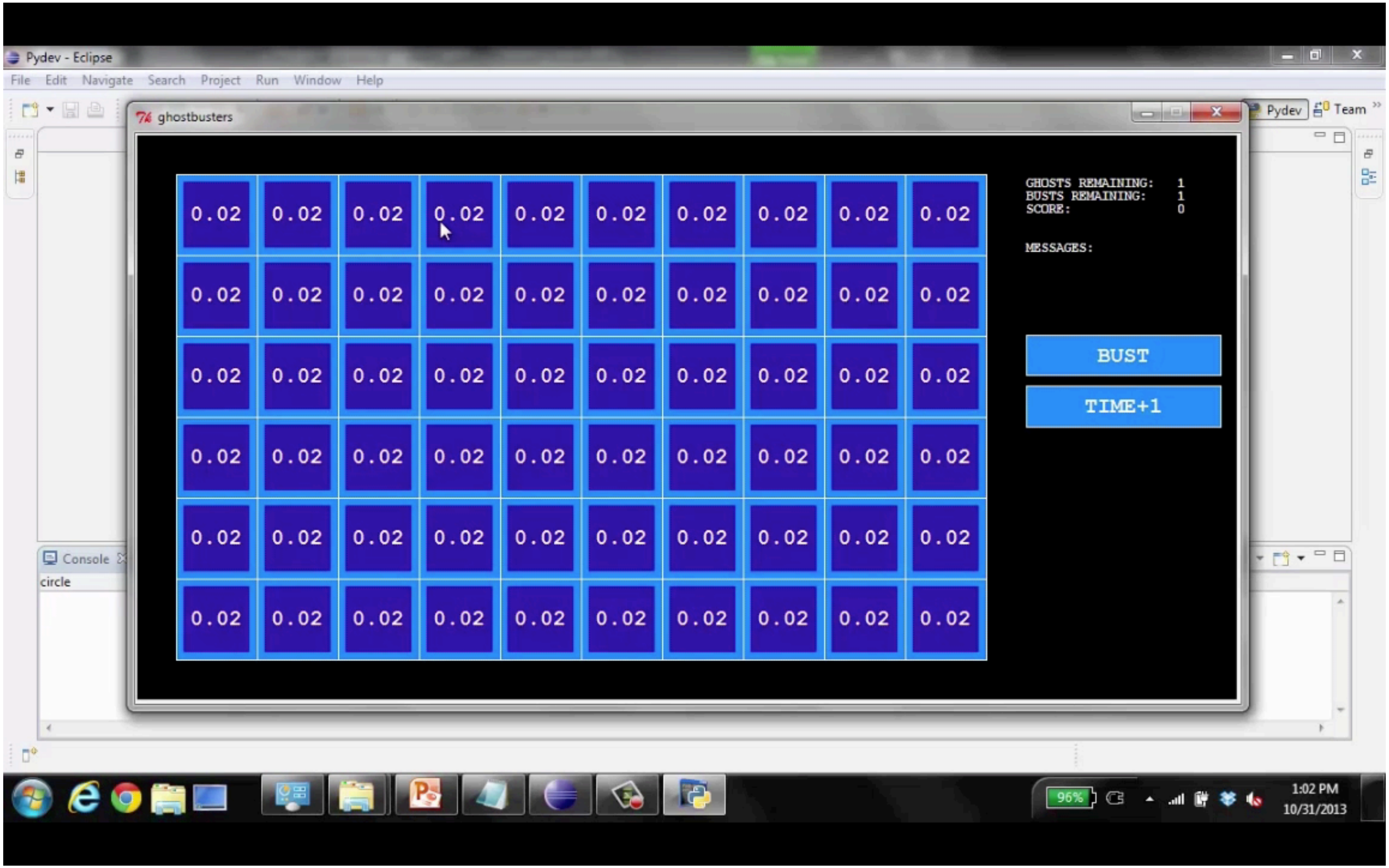
$$\begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} p \\ 1\text{-}p \end{pmatrix} = \begin{pmatrix} p \\ 1\text{-}p \end{pmatrix}$$

$0.9p + 0.3(1\text{-}p) = p$

$p = 0.75$

Stationary distribution is <0.75,0.25> ***regardless of starting distribution***

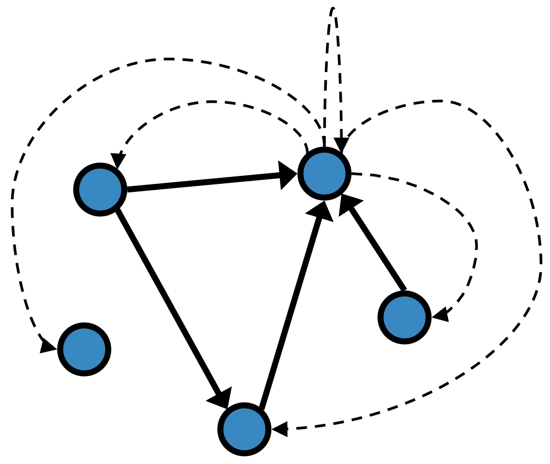
Ghostbusters - Circular Dynamics



Application of Stationary Distribution: Web Link Analysis

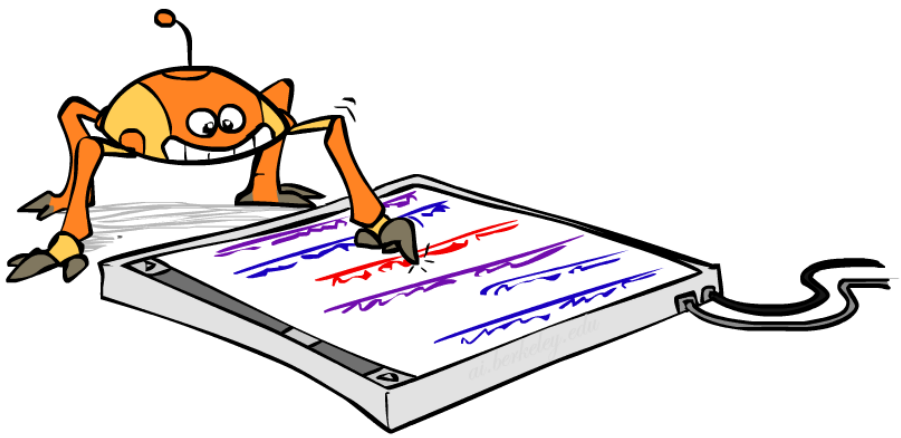
Web browsing

- State = webpage
- Initial distribution: uniform over pages
- Transitions:
  - With prob. c, uniform jump to a random page (dotted lines, not all shown)
  - With prob. 1-c, follow a random outlink (solid lines)



Stationary distribution

- Will spend more time on highly reachable pages
- Google 1.0 (Pagerank) returned the set of pages containing all your keywords in decreasing rank, now all search engines use link analysis along with many other factors (rank actually getting less important over time)

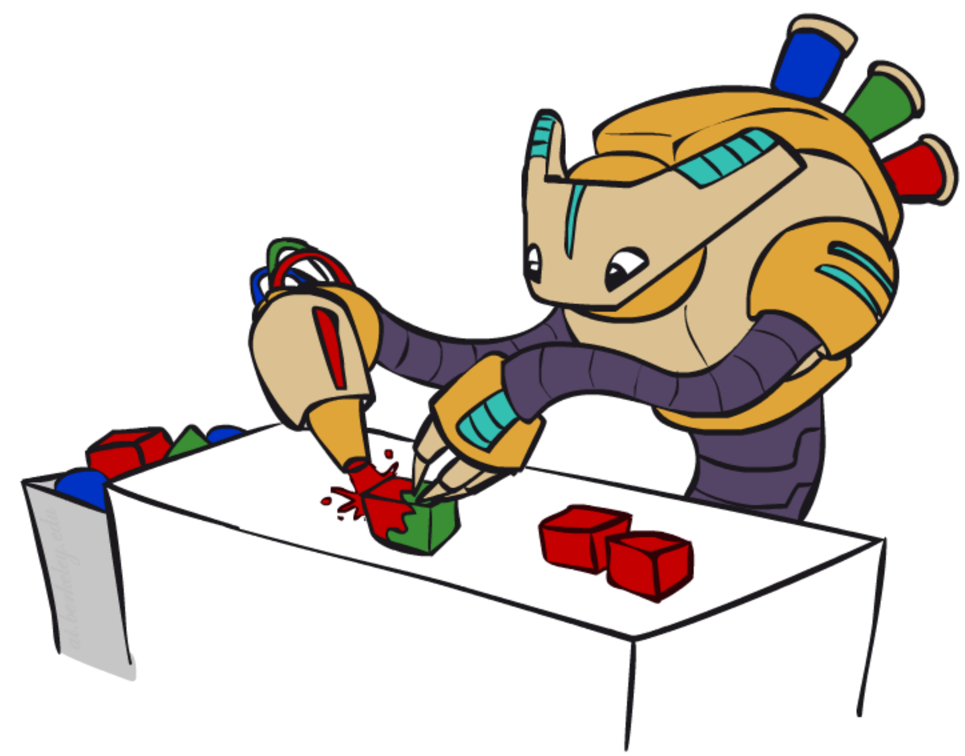


Application of Stationary Distributions: Gibbs Sampling*

Gibbs Sampling

- State = joint instantiation over all hidden and query variables; {X1, …, Xn} = H U Q
- Some initial distribution
- Transitions:
  - With probability 1/n resample variable X_j according to

$P(X_j \mid x_1, x_2, \ldots, x_{j-1,} x_{j+1}, \ldots, x_{n,} e_{1,} \ldots, e_m)$



Stationary distribution:

- Conditional distribution $P(X_1, X_2, \ldots, X_n \mid e_{1,} \ldots, e_m)$
- Means that when running Gibbs sampling long enough we get a sample from the desired distribution
- Requires some proof to show this is true!

Hidden Markov Models

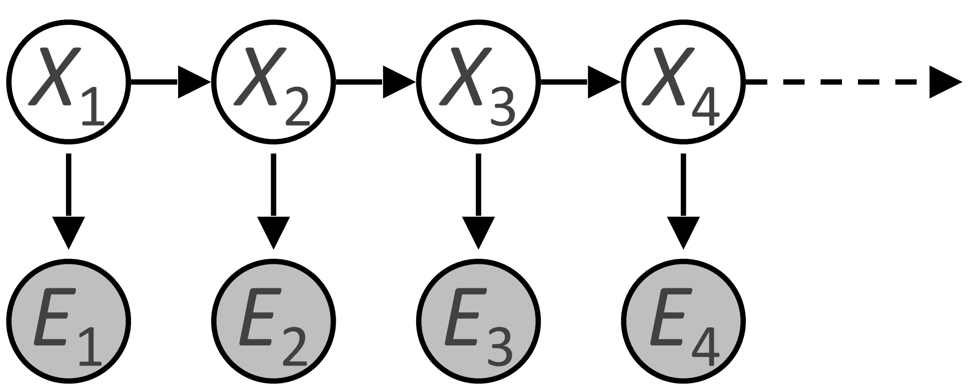
Hidden Markov Models

Markov chains not so useful for most agents

- Need observations to update your beliefs

Hidden Markov models (HMMs)

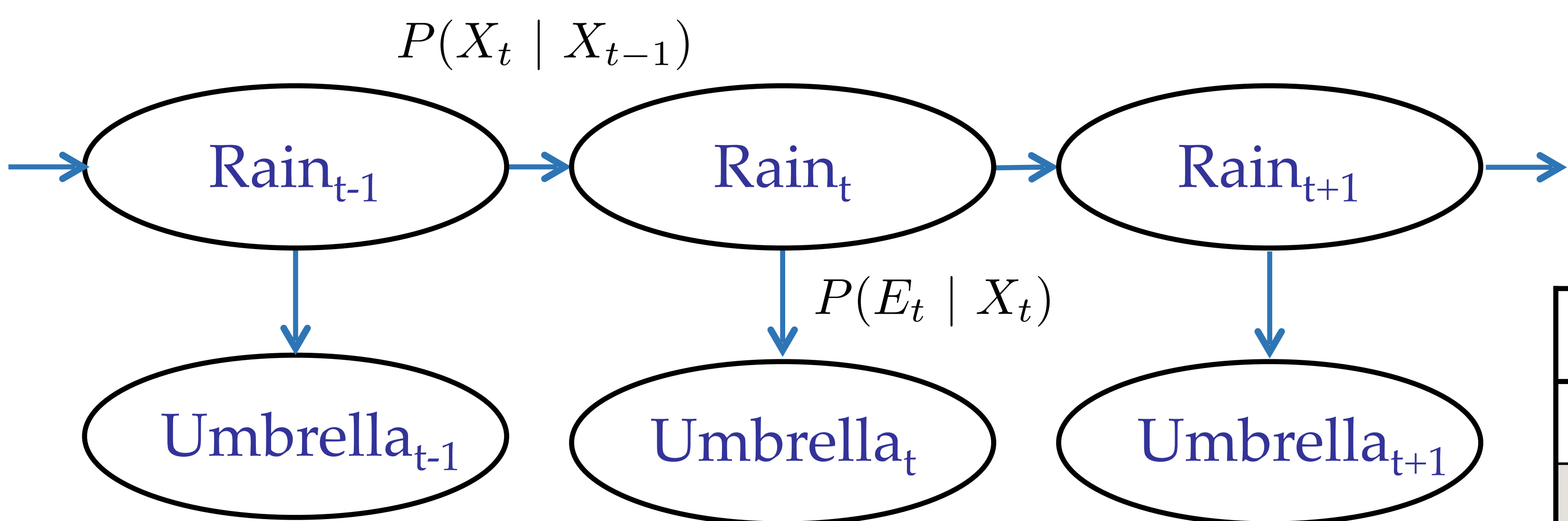
- Underlying Markov chain over states X
- You observe outputs (effects) at each time step



HMM Definition and Weather Example

An HMM is defined by:

- Initial distribution: $P(X_1)$
- Transitions: $P(X_t \mid X_{t-1})$
- Emissions: $P(E_t \mid X_t)$



R_{t-1}	R_t	$P(R_t \mid R_{t-1})$
+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7

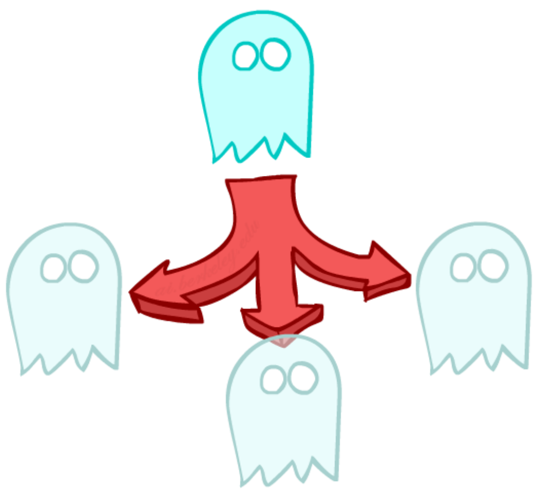
R_t	U_t	$P(U_t \mid R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

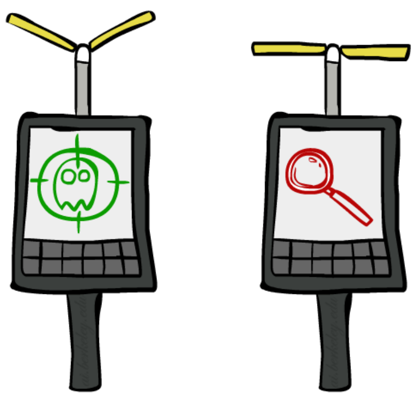
Example: Ghostbusters HMM

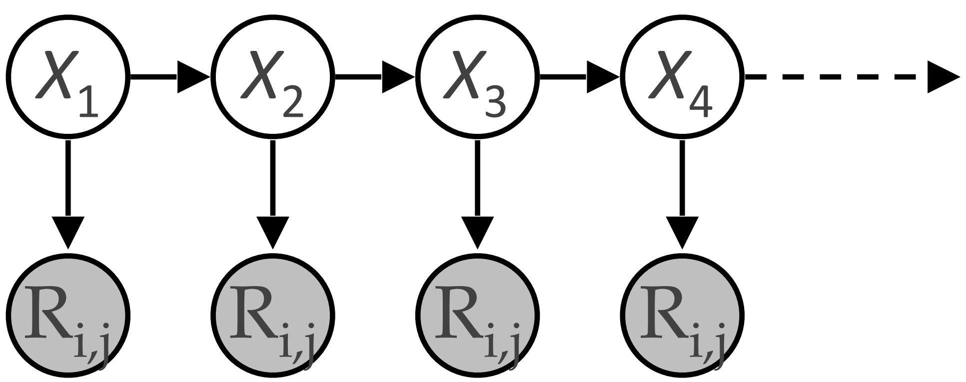
$P(X_0)$ = uniform

$P(X' \mid X)$ = usually move clockwise, but sometimes move in a random direction or stay in place

$P(R_{ij} \mid X)$ = same sensor model as before: red means close, green means far away.







1/9	1/9	1/9
1/9	1/9	1/9
1/9	1/9	1/9

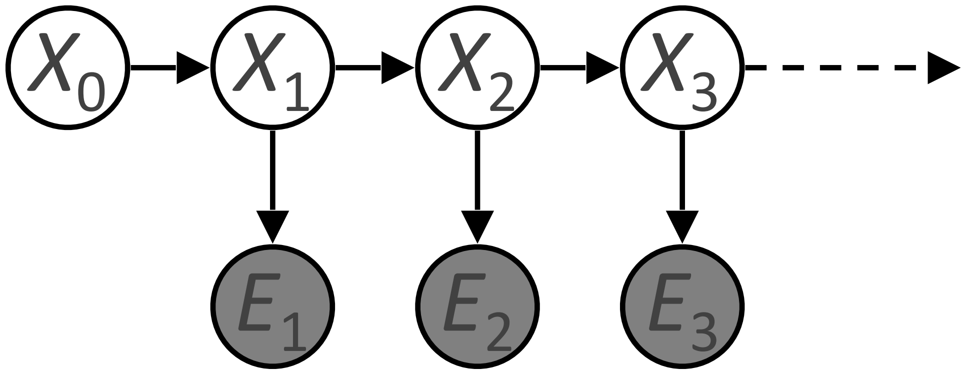
$P(X_0)$

1/6	1/6	1/2
0	1/6	0
0	0	0

$P(X' \mid X=\langle 1,2 \rangle)$

HMM as Probability Model

- Joint distribution for Markov model: $P(X_0,\ldots,X_T) = P(X_0) \prod_{t=1:T} P(X_t \mid X_{t-1})$
- Joint distribution for hidden Markov model:
  $P(X_0,X_1,\ldots,X_T,E_1,\ldots,E_T) = P(X_0) \prod_{t=1:T} P(X_t \mid X_{t-1})\, P(E_t \mid X_t)$
- Markov assumption
  - Future states are independent of the past given the present
  - Current evidence is independent of everything else given current state
- Are evidence variables independent of each other?



Useful notation:

$X_{a:b} = X_a, X_{a+1}, \ldots, X_b$

Ghostbusters – Circular Dynamics - HMM

Real HMM Examples

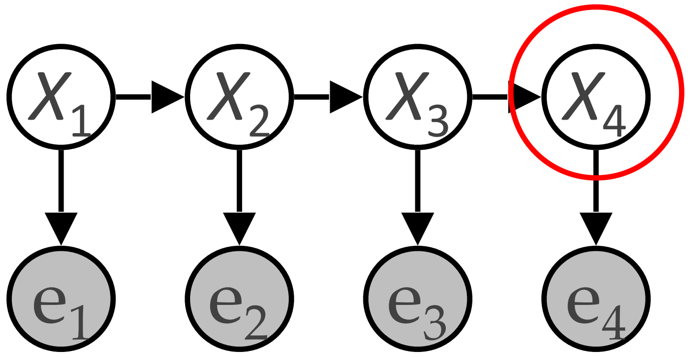
- Speech recognition HMMs:
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)
- Machine translation HMMs:
  - Observations are words (tens of thousands)
  - States are translation options
- Robot tracking:
  - Observations are range readings (continuous)
  - States are positions on a map (continuous)
- Molecular biology:
  - Observations are nucleotides ACGT
  - States are coding/non-coding/start/stop/splice-site etc.

Inference tasks

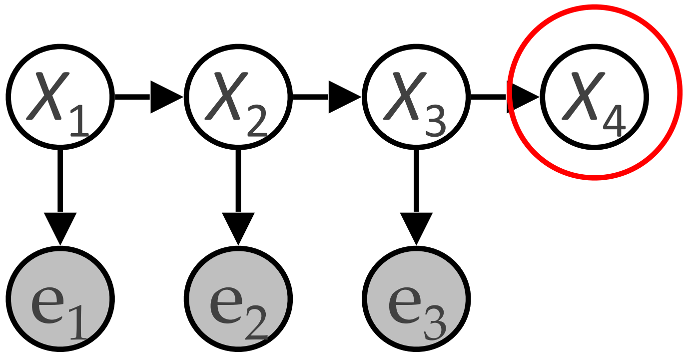
- Filtering: $P(X_t \mid e_{1:t})$
  - *Belief state*—input to the decision process of a rational agent
- Prediction: $P(X_{t+k} \mid e_{1:t})$ for $k > 0$
  - Evaluation of possible action sequences; like filtering without the evidence
- Smoothing: $P(X_k \mid e_{1:t})$ for $0 \leq k < t$
  - Better estimate of past states, essential for learning
- Most likely explanation: $\arg\max_{x_{1:t}} P(x_{1:t} \mid e_{1:t})$
  - Speech recognition, decoding with a noisy channel

HMM Queries

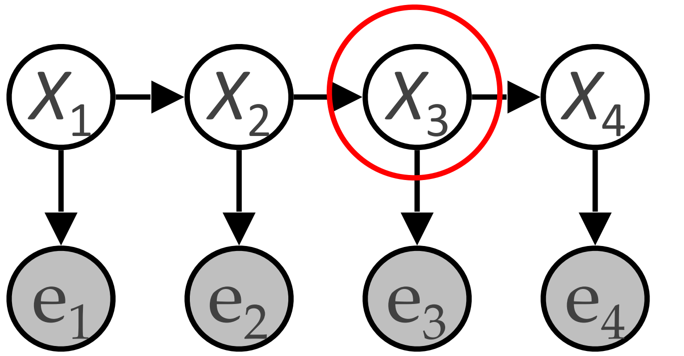
Filtering: $P(X_t \mid e_{1:t})$



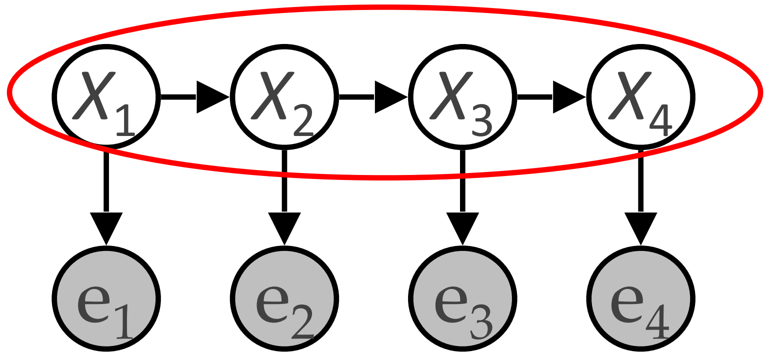
Prediction: $P(X_{t+h} \mid e_{1:t-1})$



Smoothing: $P(X_t \mid e_{1:N})$, $t<N$



Explanation: $P(X_{1:N} \mid e_{1:N})$



Filtering

- Filtering (or monitoring or state estimation) is the task of maintaining the distribution $\boldsymbol{f}_{1:t} = P(X_t \mid e_{1:t})$ over time

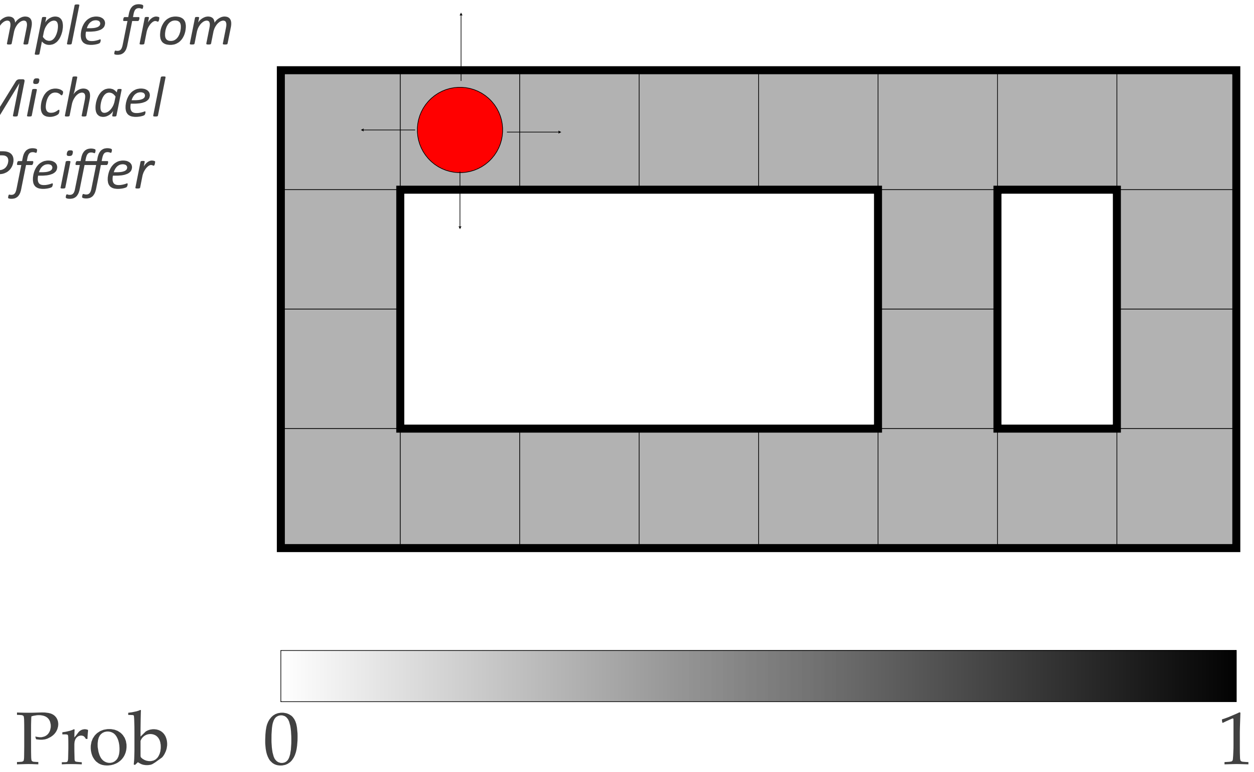
- We start with $\boldsymbol{f}_0$ in an initial setting, usually uniform

- Filtering is a fundamental task in engineering and science

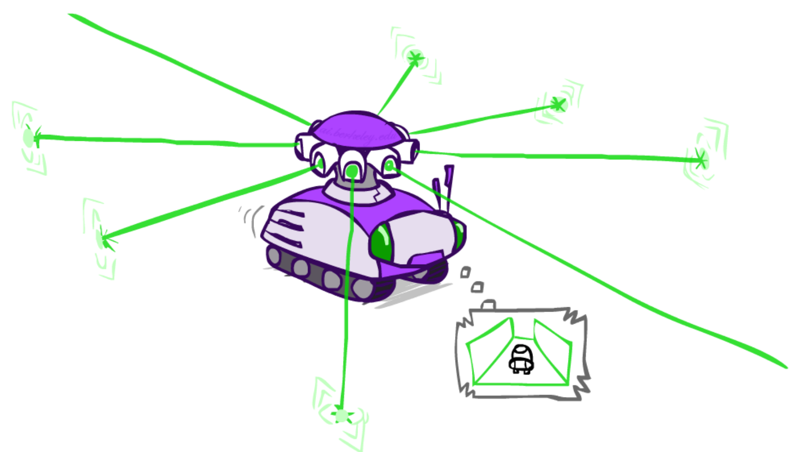
- The Kalman filter (continuous variables, linear dynamics, Gaussian noise) was invented in 1960 and used for trajectory estimation in the Apollo program; core ideas used by Gauss for planetary observations

Example: Robot Localization

Example from Michael Pfeiffer



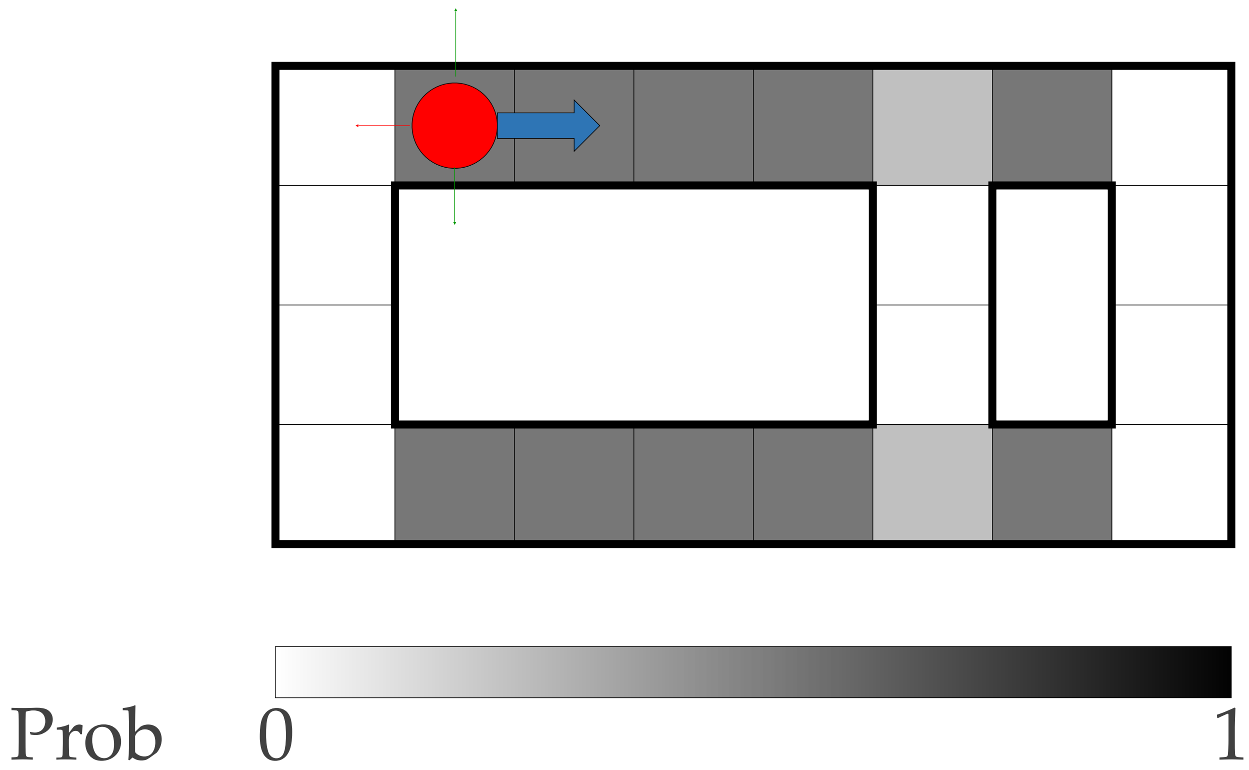
t=0



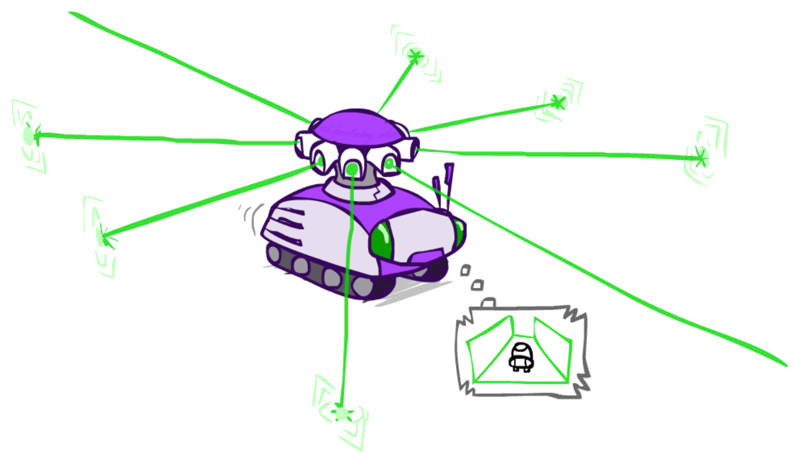
Sensor model: can read in which directions there is a wall, never more than 1 mistake

Motion model: may not execute action with small prob.

Example: Robot Localization

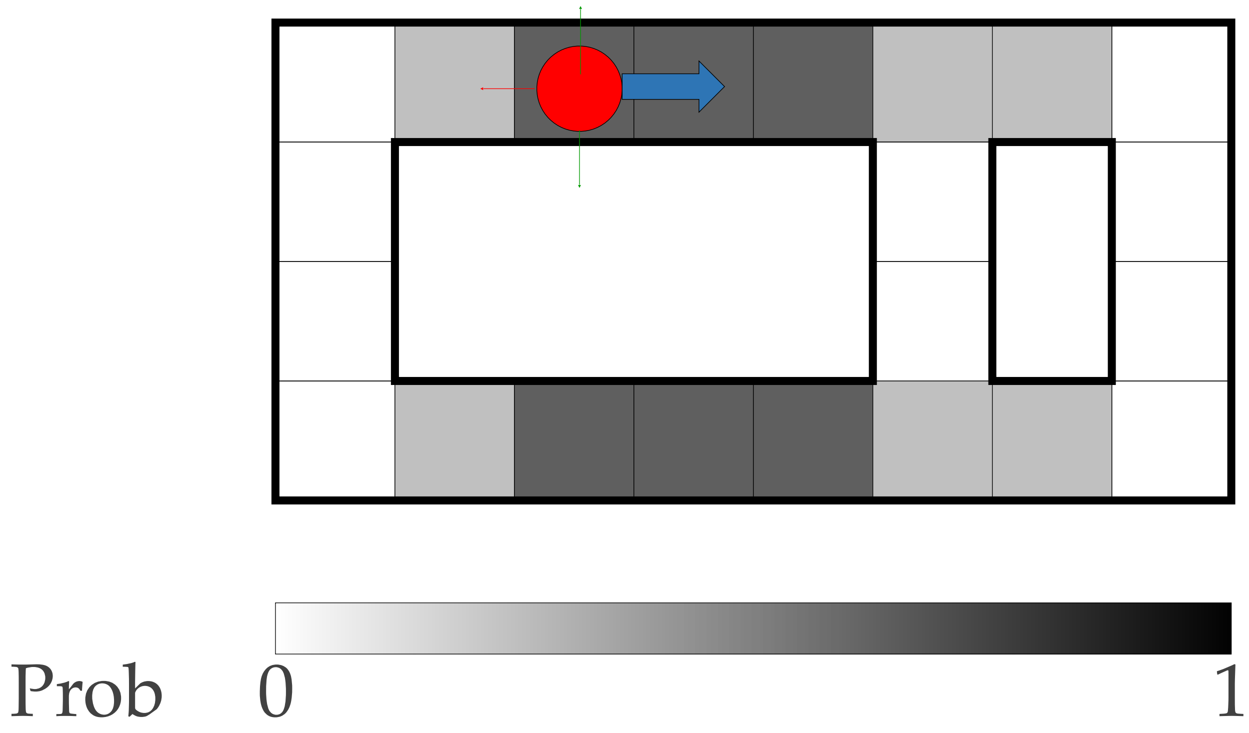


t=1

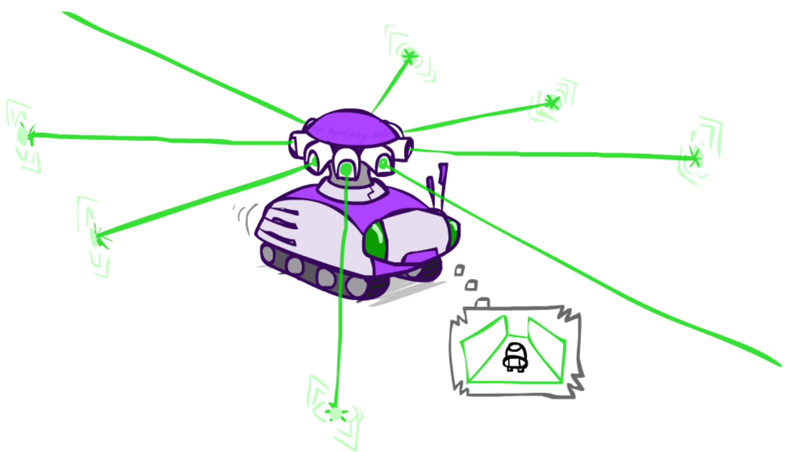


Lighter grey: was **possible** to get the reading, but **less likely** b/c required 1 mistake

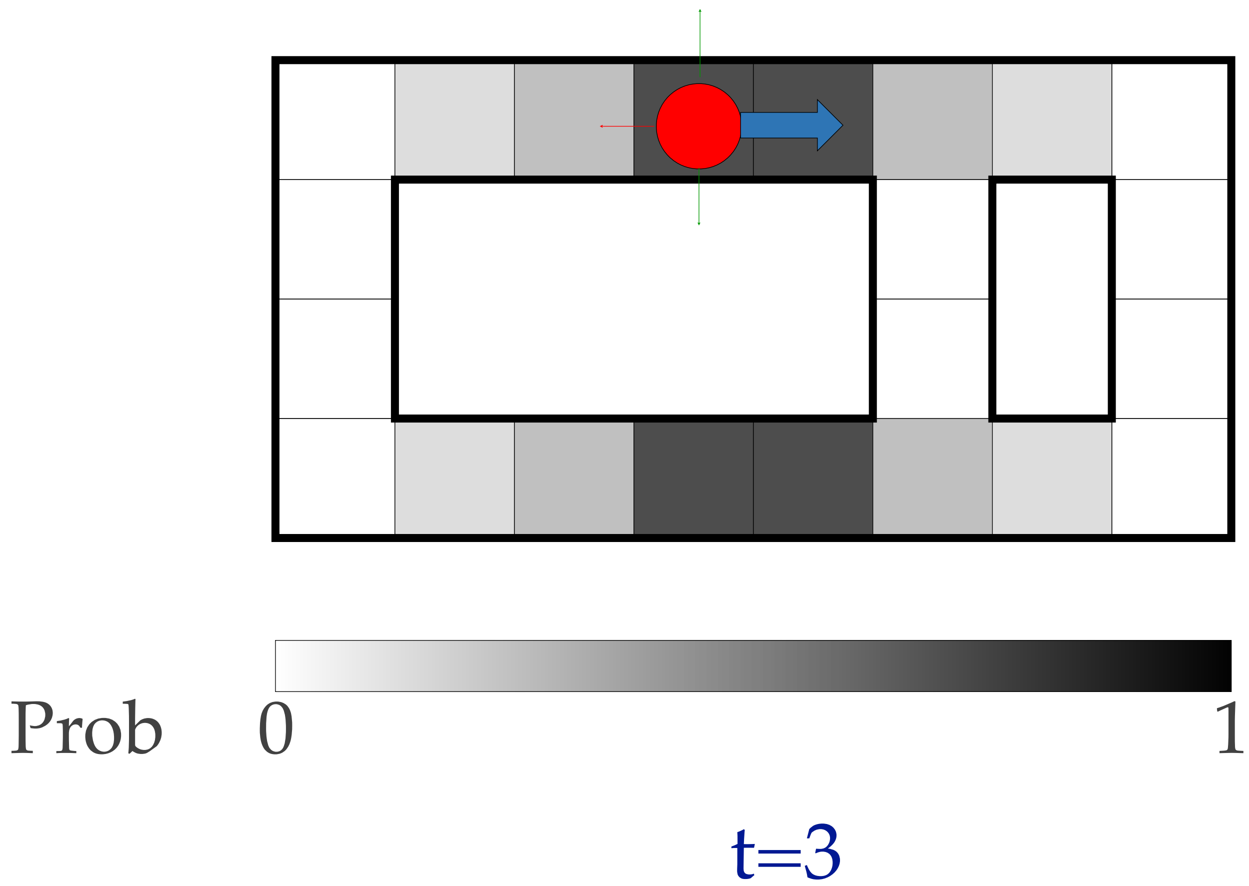
Example: Robot Localization

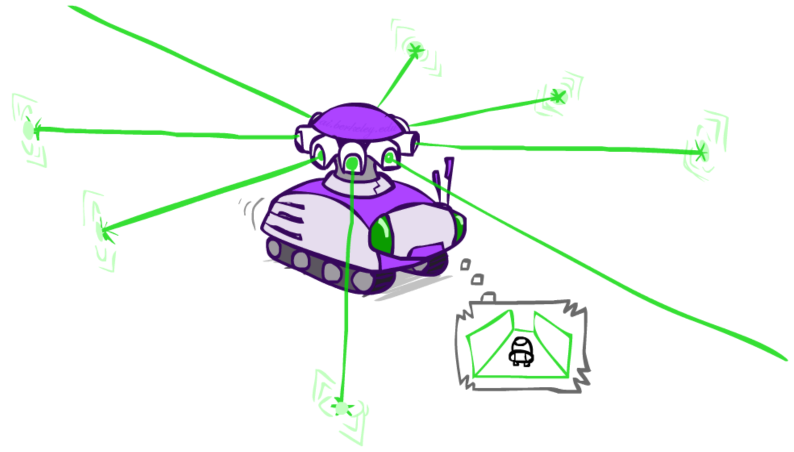


t=2

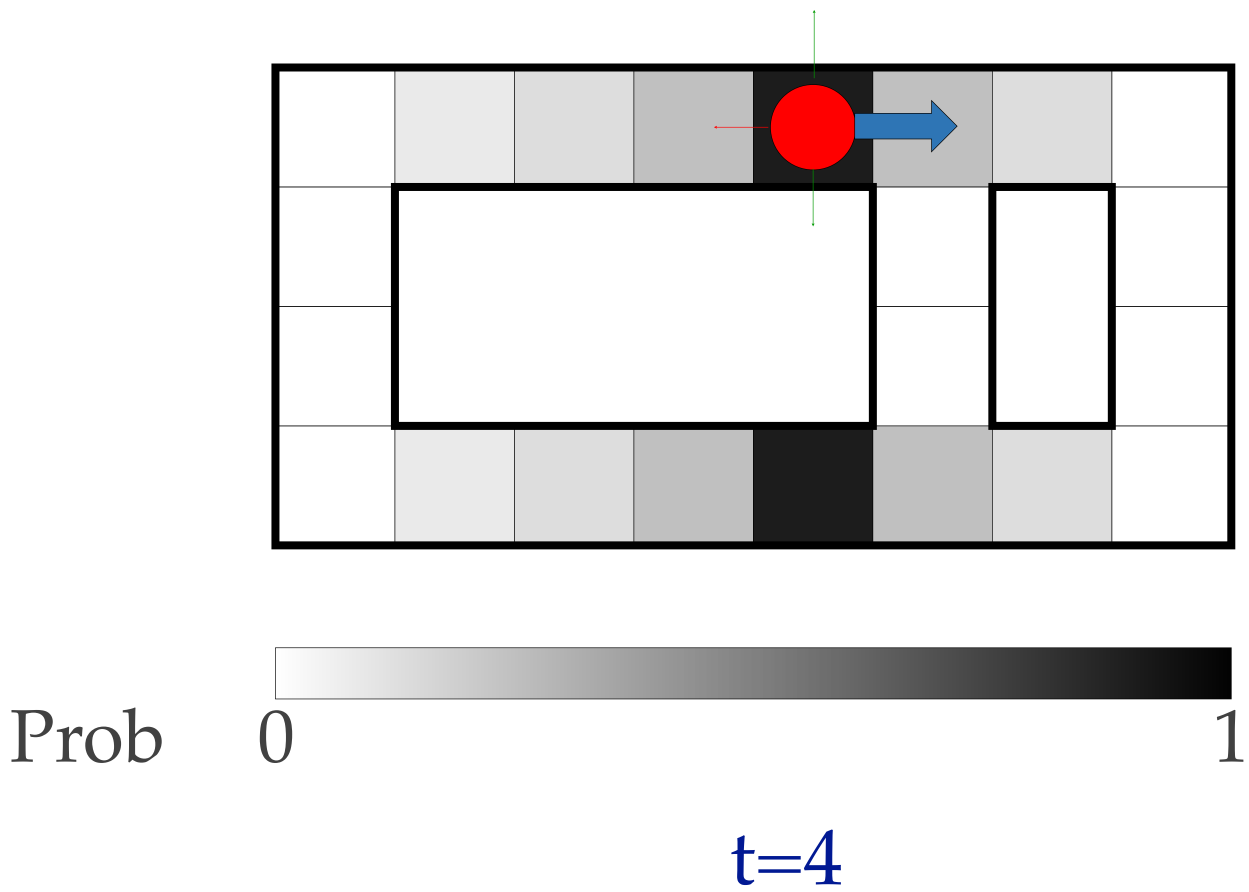


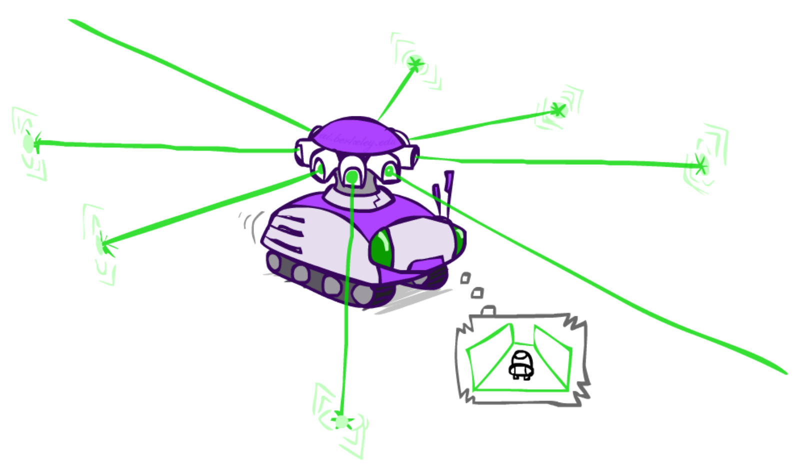
Example: Robot Localization



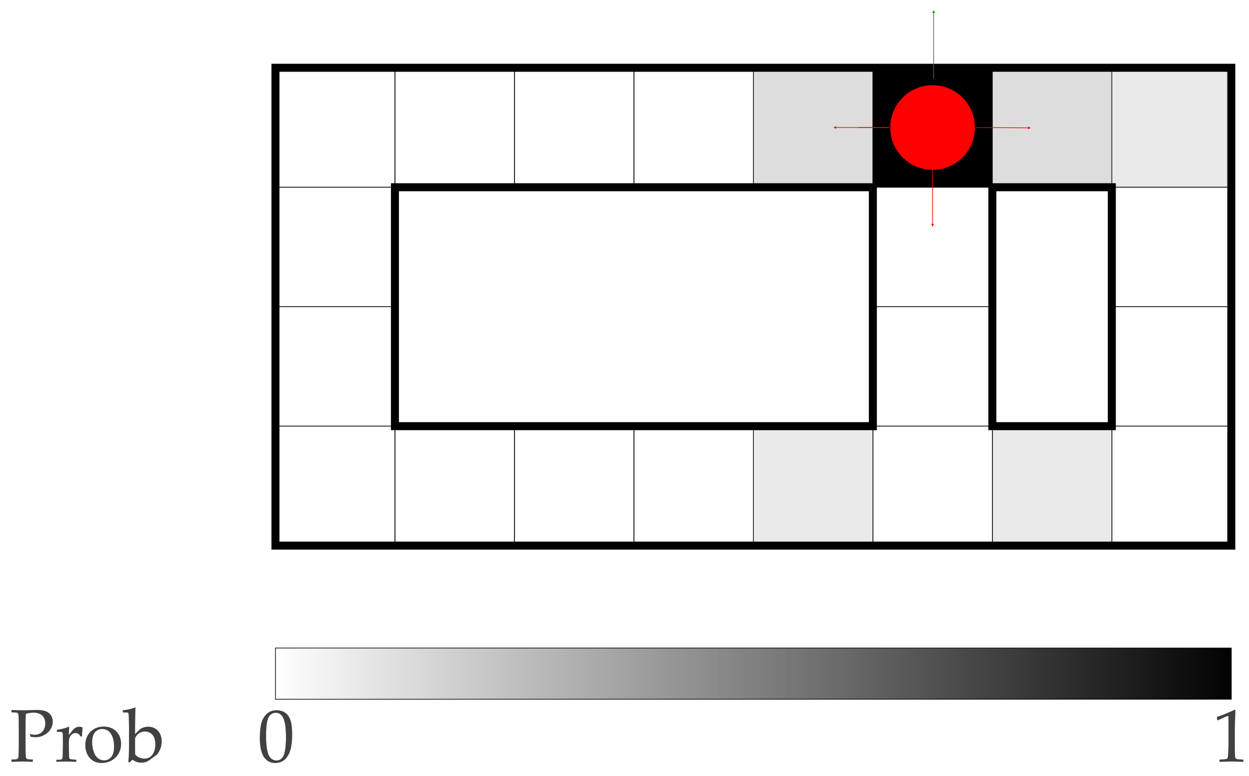


Example: Robot Localization

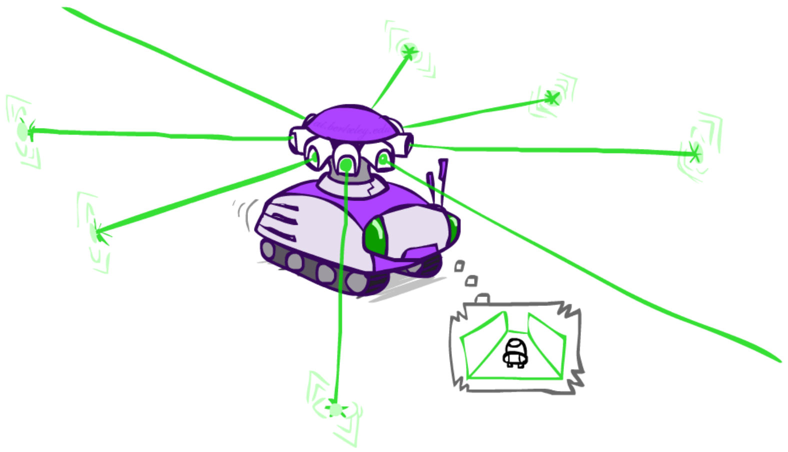




Example: Robot Localization



t=5



Filtering Algorithm

- Goal: design a ***recursive filtering*** algorithm of the form
  - $P(X_{t+1} \mid e_{1:t+1}) = g(e_{t+1}, P(X_t \mid e_{1:t}))$

- $P(X_{t+1} \mid e_{1:t+1}) = P(X_{t+1} \mid e_{1:t}, e_{t+1})$

$$= \alpha\, P(e_{t+1} \mid X_{t+1}, e_{1:t})\, P(X_{t+1} \mid e_{1:t})$$

Bayes rule
α normalization

$$= \alpha\, P(e_{t+1} \mid X_{t+1})\, P(X_{t+1} \mid e_{1:t})$$

Conditional ind.

$$= \alpha\, P(e_{t+1} \mid X_{t+1}) \sum_{x_t} P(X_t \mid e_{1:t})\, P(X_{t+1} \mid x_t, e_{1:t})$$

Law of total prob.

$$= \alpha\, P(e_{t+1} \mid X_{t+1}) \sum_{x_t} P(x_t \mid e_{1:t})\, P(X_{t+1} \mid x_t)$$

Conditional ind.

Forward Algorithm

- $P(X_{t+1} | e_{1:t+1}) = \underbrace{\alpha}_{\text{Normalize}} \underbrace{P(e_{t+1} | X_{t+1})}_{\text{Update}} \underbrace{\sum_{x_t} P(x_t \mid e_{1:t}) P(X_{t+1} | x_t)}_{\text{Predict}}$

- $f_{1:t+1} = \text{FORWARD}(f_{1:t}, e_{t+1})$
- Cost per time step: $O(|X|^2)$ where $|X|$ is the number of states
- Time and space costs are ***constant***, independent of t
- $O(|X|^2)$ is infeasible for models with many state variables
- We get to invent really cool approximate filtering algorithms

Matrix Form

- Transition matrix T, observation matrix O_t
  - Observation matrix has state likelihoods for E_t along diagonal
  - e.g., for U_1 = true, $O_1 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.9 \end{pmatrix}$
- Filtering algorithm becomes
  - $\boldsymbol{f}_{1:t+1} = \alpha\, O_{t+1} T^{\top} \boldsymbol{f}_{1:t}$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Example: Prediction Step

As time passes, uncertainty "accumulates"

Transition model: ghosts usually go clockwise

<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	1.00	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01

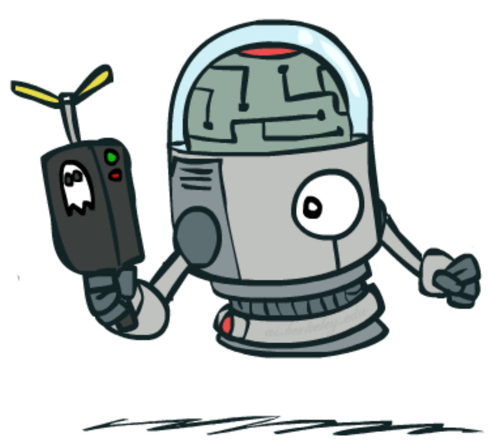
T = 1

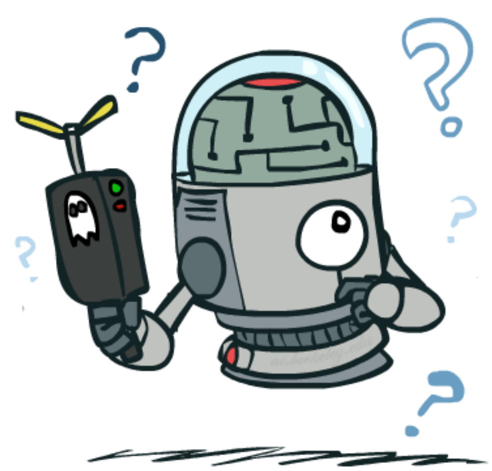
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	0.06	<0.01	<0.01	<0.01
<0.01	0.76	0.06	0.06	<0.01	<0.01
<0.01	<0.01	0.06	<0.01	<0.01	<0.01

T = 2

0.05	0.01	0.05	<0.01	<0.01	<0.01
0.02	0.14	0.11	0.35	<0.01	<0.01
0.07	0.03	0.05	<0.01	0.03	<0.01
0.03	0.03	<0.01	<0.01	<0.01	<0.01

T = 5





Example: Update Step

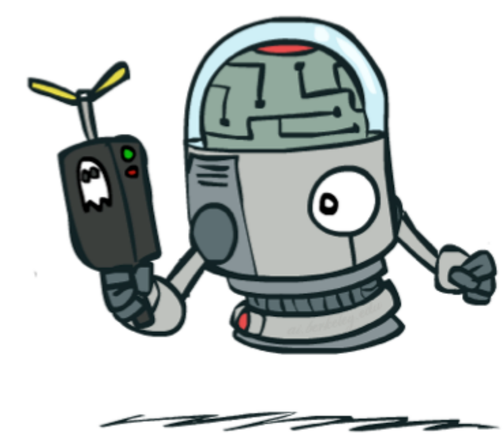
As we get observations, beliefs get reweighted, uncertainty "decreases"

0.05	0.01	0.05	<0.01	<0.01	<0.01
0.02	0.14	0.11	0.35	<0.01	<0.01
0.07	0.03	0.05	<0.01	0.03	<0.01
0.03	0.03	<0.01	<0.01	<0.01	<0.01

Before observation

<0.01	<0.01	<0.01	<0.01	0.02	<0.01
<0.01	<0.01	<0.01	0.83	0.02	<0.01
<0.01	<0.01	0.11	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01

After observation



Example: Weather HMM

$f(+r) = 0.5$
$f(-r) = 0.5$

predict

$f'(+r) = 0.5$
$f'(-r) = 0.5$

update

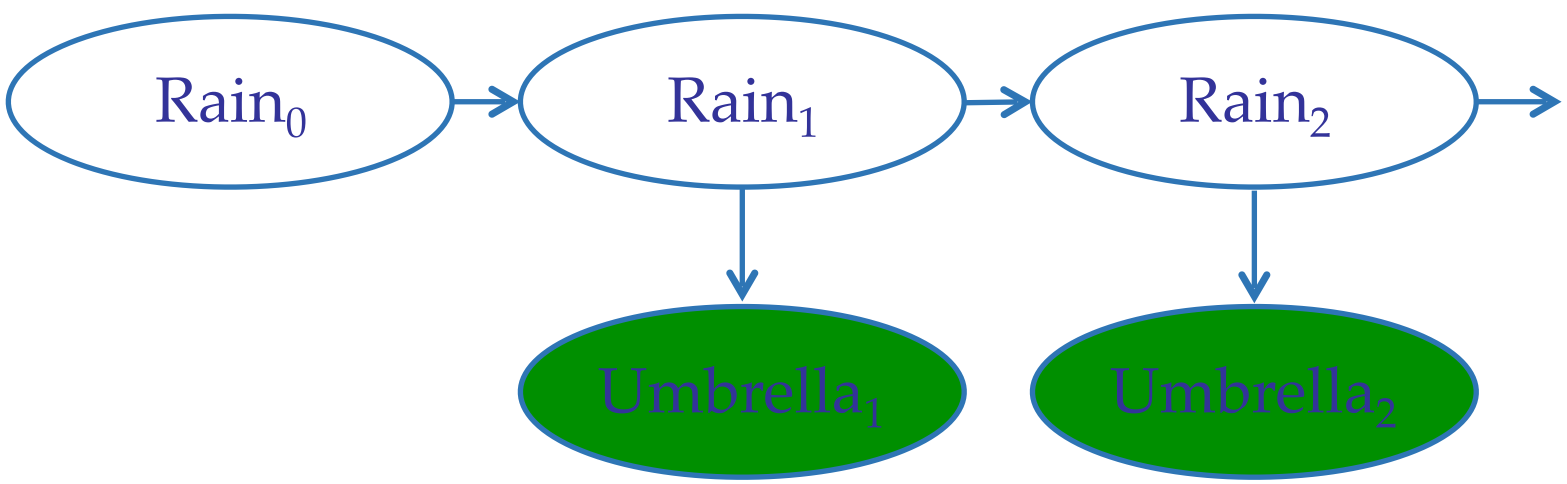
$f(+r) = 0.818$
$f(-r) = 0.182$

predict

$f'(+r) = 0.627$
$f'(-r) = 0.373$

update

$f(+r) = 0.883$
$f(-r) = 0.117$



R_t	R_{t+1}	$P(R_{t+1} \| R_t)$
+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7

R_t	U_t	$P(U_t \| R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

Pacman – Sonar