LEC009 Inventory Management IV

VG441 SS2020

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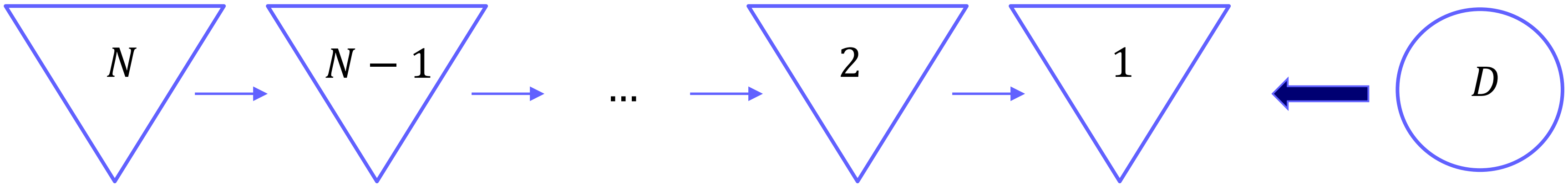
Multi-Echelon Problem

- N stages of a (serial) supply chain
- Demand rate is λ deterministically only at stage 1
- Stockouts are not allowed



$$
\begin{aligned}
k_j &= \text{fixed cost for orders of stage } j \\
h'_j &= \text{inventory holding-cost rate for stage } j \\
I'_j(t) &= \text{(local) inventory of stage } j \text{ at time } t \\
L'_j &= \text{leadtime for stage-}j\text{ orders } \;= 0 \text{ W.L.O.G.}
\end{aligned}
$$

Question: what is the optimal ordering strategy?

Echelons and Echelon Inventories

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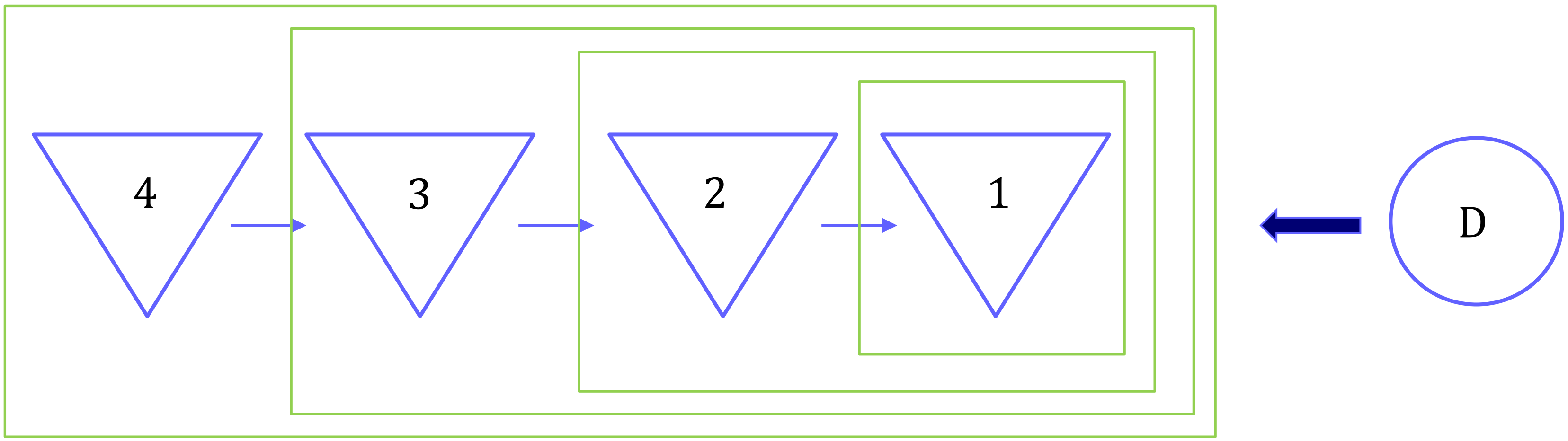
$$I'_j(t) = \text{ (local) inventory of stage } j \text{ at time } t$$



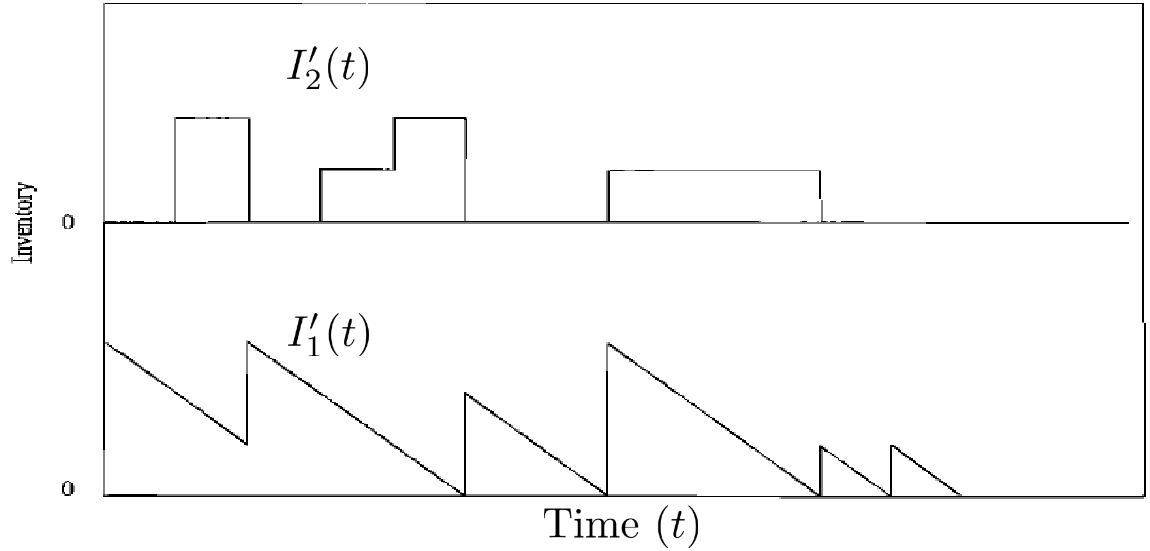
$$I_j(t) = \text{echelon imventory of stage } j \text{ at time } t = \Sigma_{i \leq j} I'_i(t)$$

$$h_j = \text{echelon-inventory holding-cost rate for stage } j = h'_j - h'_{j+1} \geq 0$$

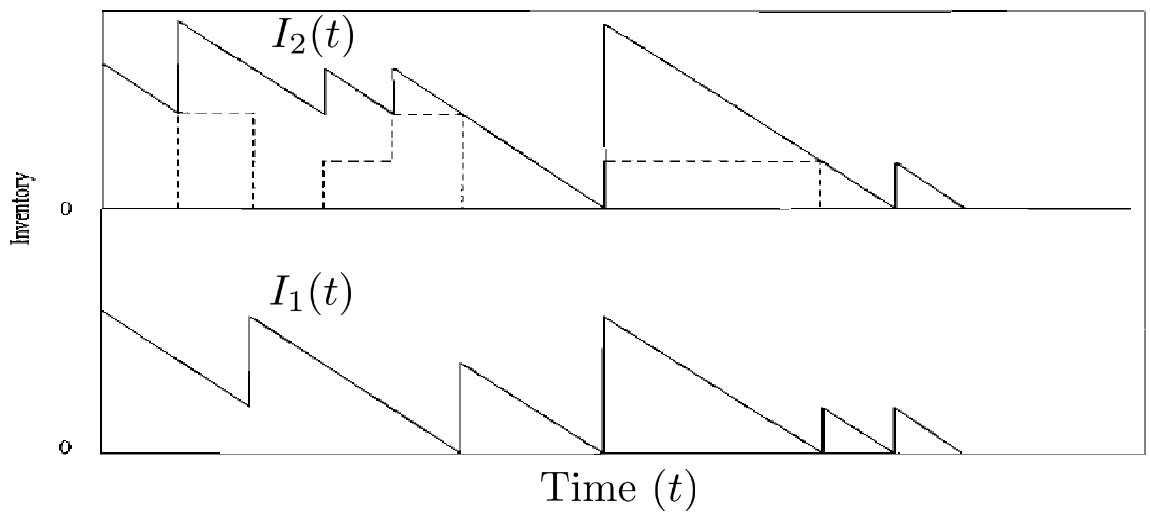
$$\Sigma_j h'_j I_j(t) = \Sigma_j h_j I_j(t) \quad \text{ for all } t$$

Why Echelon Inventories?

Local inventories over time



Echelon inventories over time



Policy Structures

- A policy is nested if for all j, whenever stage j orders, so does stages $j-1$.

- A policy is ZIO (zero-inventory-ordering) if order occurs only when its echelon inventory is zero.

Convince yourself that nested and ZIO are optimal.

Stationary-Interval Policies

u_j = order interval for item j
$\mathbf{u}$ = the vector $(u_j)_j$
$g_j = h_j \lambda$
$C(\mathbf{u})$ = average cost of the policy specified by $\mathbf{u}$

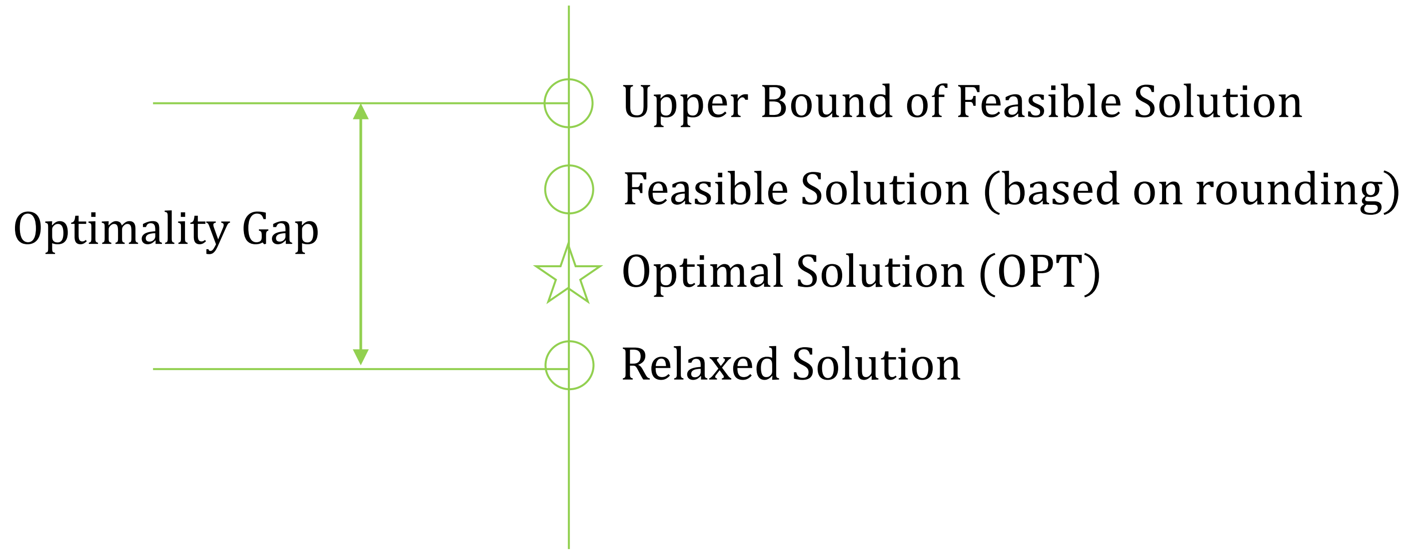
$$C(\mathbf{u}) = \sum_j \left[\frac{k_j}{u_j} + \frac{1}{2} g_j u_j \right]$$

$$\begin{array}{lll} \text{Minimize} & C(\mathbf{u}) & \\ \text{subject to} & u_j = \xi_j u_{j-1}, & \text{for all } j = 2, \ldots, J \\ & \xi_j \in \mathbb{Z}^+, & \text{for all } j = 2, \ldots, J \end{array}$$

Caveat: This MILP may be difficult to solve.

Here is the Plan

- First, we solve a simpler "relaxed" problem.
- Solution of the relaxed problem is a lower bound.
- We round off this relaxed solution to obtain a feasible solution.
- We get an upper bound on this feasible solution.
- Show that the two bounds are close together.



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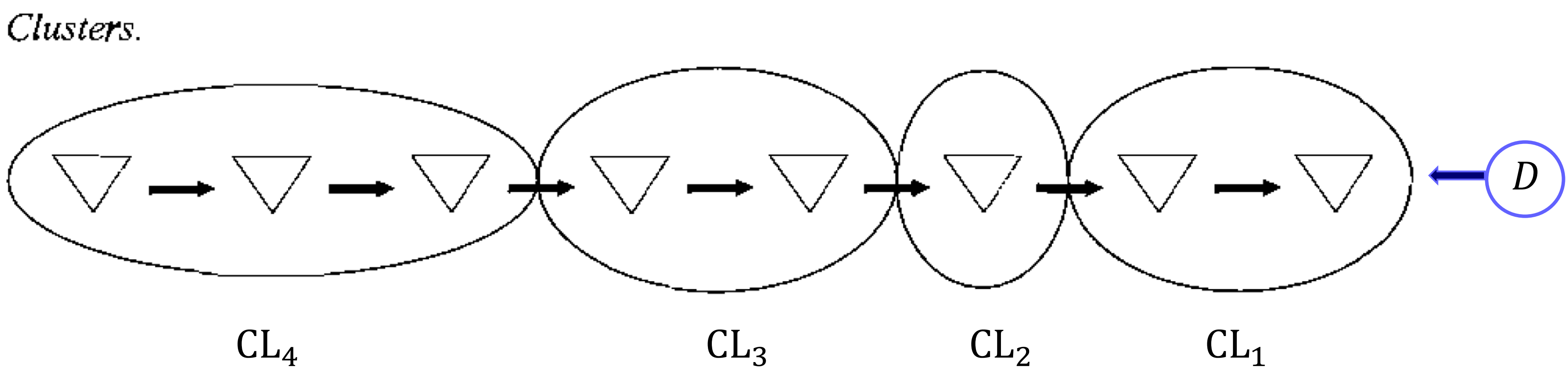
Relax the constraints …

$$\begin{array}{lll} \text{Minimize} & C(\mathbf{u}) & \\ \text{subject to} & u_j \geq u_{j-1}, & \text{for all } j = 2, \ldots, J \end{array}$$

Relaxed and Feasible Solutions

$$\begin{array}{lll} \text{Minimize} & C(\mathbf{u}) & \\ \text{subject to} & u_j \geq u_{j-1}, & \text{for all } j = 2, \ldots, J \end{array}$$



$u_8 = u_7 = u_6$ $u_5 = u_4$ u_3 $u_2 = u_1$

Feasible solution: round to the nearest power-of-2

Gap between this feasible solution and the relaxed solution: ~1.06

Deterministic Demand (Nonstationary)

Input: T period demands $d_1, \dots, d_T$
Decisions: $q_j(t), I_j(t)$

$$
\begin{array}{llr}
\min & \sum_j \sum_t I_j(t) h_j + \sum_j \sum_t k_j \mathbb{1}\left(q_j(t) > 0\right) & \\
\text{s.t.} & I_j(t) = I_j(t-1) + q_j(t) - d_t & \forall t,\ \forall j \\
& I_j(t-1) + q_j(t) \geq I_{j-1}(t-1) + q_{j-1}(t) & \forall t,\ \forall j \\
& I_j(t) \geq 0 & \forall t,\ \forall j \\
& I_j(0) = 0 & \forall j
\end{array}
$$

Dynamic Programming

- Define $F(i, s, t)$ as the optimal cost of subproblem defined for stages $i, \dots, 1$ and periods $[s, t)$
- ZIO: $I_i(s-1) = I_i(t-1) = 0$
- Nested: $I_j(s-1) = I_j(t-1) = 0$, for all $j < i$
- Goal: $F(N, 1, T+1)$
- Boundary Conditions:

$$F(0, \cdot, \cdot) = 0$$

$$F(\cdot, T+1, \cdot) = 0$$

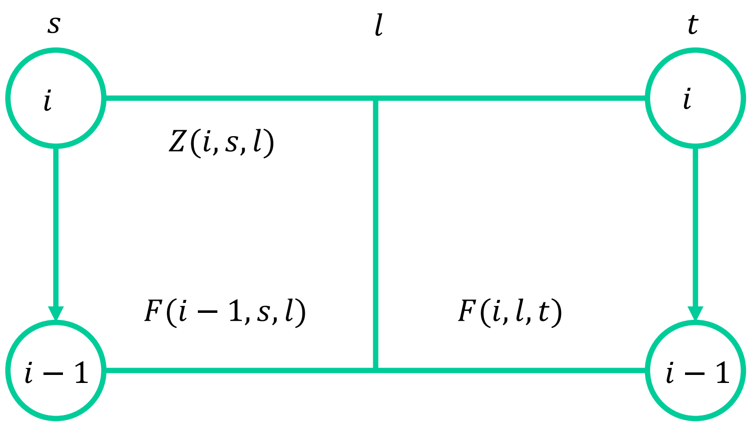
$$F(i, s, s+1) = \sum_{j=1}^{i} K_j \text{ for all } i$$

DP (Graphical Model)

- For each time $s < t$ and stage j, the cost of covering $d_s, \dots, d_{t-1}$

$$Z_{s,t}^{i} = K_i + \sum_{a=s}^{t-2} h_a \sum_{b=a-1}^{t-1} d_b$$

- Dynamic Programming (on graphical models)



$$F(i,s,t) = \min_{s \le l \le t-1} \left\{ Z_{s,l}^{i} + F(i-1,s,l) + F(i,l,t) \right\}$$

Stochastic Model

- Stage 1 faces stochastic demand D per period and penalty p
- Lead times $L_1, \dots, L_N$ and lead time demand $D_1, \dots, D_N$
- Echelon base-stock policy is optimal.

Theorem: Let $\underline{g}_0(x) = (p + h_1')\, x^-$. For $j = 1, \dots, N$, let

$$\begin{aligned}
\hat{g}_j(x) &= h_j x + \underline{g}_{j-1}(x) \\
g_j(y) &= \mathbb{E}\left[\hat{g}_j\left(y - D_j\right)\right] \\
S_j^* &= \operatorname{argmin}\left\{g_j(y)\right\} \\
\underline{g}_j(x) &= g_j\left(\min\left\{S_j^*, x\right\}\right)
\end{aligned}$$

Then $\mathbf{S}^* = \left(S_j^*\right)_{j=1}^N$ is the optimal base-stock vector and $g_N\left(S_N^*\right)$ is the corresponding optimal cost

Stochastic Model

- Use the following heuristics:

Theorem 6.4 (Shang and Song (2003): For any j and y

(a) $g_j^l(y) \le g_j(y) \le g_j^u(y)$

(b) $S_j^l \le S_j^* \le S_j^u$

$$S_j^l = \tilde{F}_j^{-1}\left(\frac{p+\sum_{i=j+1}^N h_i}{p+\sum_{i=1}^N h_i}\right) \quad \le \quad S_j^* \quad \le \quad S_j^u = \tilde{F}_j^{-1}\left(\frac{p+\sum_{i=j+1}^N h_i}{p+\sum_{i=j}^N h_i}\right)$$

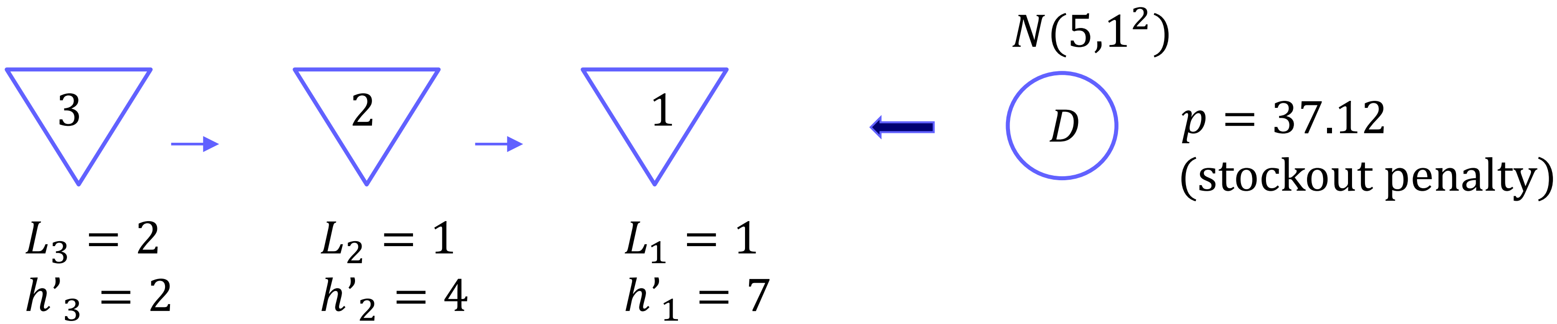
$$\tilde{D}_j = \sum_{i=1}^j D_i$$

$$\downarrow$$

Lead-time demand with lead time $\sum_{i=1}^j L_i$

Computational optimality gap ≤ 1%

A Simple Example



$$
\begin{aligned}
\tilde{D}_1 &\sim N\left(5 \cdot 1, 1^2 \cdot 1\right) = N(5,1) \\
\tilde{D}_2 &\sim N\left(5 \cdot 2, 1^2 \cdot 2\right) = N(10,2) \\
\tilde{D}_3 &\sim N\left(5 \cdot 4, 1^2 \cdot 4\right) = N(20,4)
\end{aligned}
$$

We have $(h_1, h_2, h_3) = (3, 2, 2)$. Therefore:

$$
\begin{aligned}
S_1^u &= \tilde{F}_1^{-1}\left(\frac{37.12+4}{37.12+7}\right) = 6.49 & S_1^l &= \tilde{F}_1^{-1}\left(\frac{37.12+4}{37.12+7}\right) = 6.49 \\
S_2^u &= \tilde{F}_2^{-1}\left(\frac{37.12+2}{37.12+4}\right) = 12.35 & S_2^l &= \tilde{F}_2^{-1}\left(\frac{37.12+2}{37.12+7}\right) = 11.71 \\
S_3^u &= \tilde{F}_3^{-1}\left(\frac{37.12+0}{37.12+2}\right) = 23.27 & S_3^l &= \tilde{F}_3^{-1}\left(\frac{37.12+0}{37.12+7}\right) = 22.00
\end{aligned}
$$

Taking the mean, we have

$$
\begin{aligned}
\tilde{S}_1 &= \tfrac{1}{2}(6.49 + 6.49) = 6.49 \\
\tilde{S}_2 &= \tfrac{1}{2}(12.35 + 11.71) = 12.03 \\
\tilde{S}_3 &= \tfrac{1}{2}(23.27 + 22.00) = 22.63
\end{aligned}
$$

These values are very close to $\mathrm{S}^* = (6.49, 12.02, 22.71)$ and indeed their costs are very similar: $g(\tilde{\mathbf{S}}) = 47.66$, compared to $g\left(\mathbf{S}^*\right) = 47.65$