VE485, Optimization in Machine Learning (Summer 2020) **Homework: 1. Convex Set**

1. Convex Set

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Problem 1

Midpoint convexity. A set C is *midpoint convex* if whenever two points a, b are in C, the average or midpoint $(a+b)/2$ is in C. Obviously a convex set is midpoint convex. It can be proved that under mild conditions midpoint convexity implies convexity. As a simple case, prove that if C is closed and midpoint convex, then C is convex.

Answer.

Assume $\boldsymbol{C}$ is a closed midpoint convex.
If C only has one boundary point a, then C also has only one element $a = 2a/2$.

When C has two boundary points a, b, then $(a+b)/2, (3a+b)/4, (a+3b)/4, (7a+b)/8, (3a+5b)/8 \cdots \in \boldsymbol{C}$
Therefore, $\forall x \in \boldsymbol{C}, x = \sigma/2^n \cdot a + (1-\sigma/2^n)b \in \boldsymbol{C}$ where $\sigma \in N, 0 \le \sigma \le 2^n$

Let $z = \theta x_1 + (1-\theta)x_2$ where $x_1, x_2 \in \boldsymbol{C}, 0 \le \theta \le 1, \theta \in R$
Then $z = [\theta\sigma_1/2^{n_1} + (1-\theta)\sigma_2/2^{n_2}]a + [\theta(1-\sigma_1/2^{n_1}) + (1-\theta)(1-\sigma_2/2^{n_2})]b$
Therefore, $z = \gamma a + (1-\gamma)b$ where $\gamma = \theta\sigma_1/2^{n_1} + (1-\theta)\sigma_2/2^{n_2}$
Since $0 \le \gamma \le 1$, γ can be represented in the form of $\sigma_\gamma/2^{n_\gamma}$ where $n_\gamma \in N, \sigma_\gamma \in N, 0 \le \sigma_\gamma \le 2^{n_\gamma}$
$z = \sigma_\gamma/2^{n_\gamma} \cdot a + (1-\sigma_\gamma/2^{n_\gamma})b$

As a result, $z \in C$ and $\boldsymbol{C}$ is a convex. It still applies when C has more than 2 boundary points.

Problem 2

***Linear-fractional functions and convex sets.* Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be the linear-fractional function**

$$f(x) = (Ax+b)/(c^T x + d), \qquad \mathbf{dom} f = \{x \mid c^T x + d > 0\}.$$

In this problem we study the inverse image of a convex set C under f, i.e.,

$$f^{-1}(C) = \{x \in \mathbf{dom} f \mid f(x) \in C\}.$$

For each of the following sets $C \subseteq \mathbb{R}^n$, give a simple description of $f^{-1}(C)$.

1. **The halfspace $C = \{y \mid g^T y \le h\}$ (with $g \ne 0$).**
2. **The polyhedron $C = \{y \mid Gy \preceq h\}$.**

3. **The ellipsoid** $\{y \mid y^T P^{-1} y \le 1\}$ **(where** $P \in \mathbf{S}^n_{++}$**).**
4. **The solution set of a linear matrix inequality,** $C = \{y \mid y_1 A_1 + \cdots + y_n A_n \preceq B\}$**, where** $A_1, \ldots, A_n$**,** $B \in \mathbf{S}^p$**.**

Answer.

$f^{-1}(x) = P[RP^{-1}(x)]$ **where** $P(x)$ **is a perspective function and** $R : \mathbb{R}^{n+1} \to \mathbb{R}^{m+1}$

Let $Q = \begin{pmatrix} A & b \\ c^T & d \end{pmatrix} \in \mathbb{R}^{(m+1)\times(n+1)}$**, then if** $m > n$ **let** $R = (Q^T Q)^{-1} Q^T$

Then $f^{-1}(x)$ **is also a linear-fractional function, which conserves convexity.**

1. $$f^{-1}(C) = \{x \in \mathbf{dom} f | g^T f(x) \le h\}$$
$$f^{-1}(C) = \{x \in \mathbf{dom} f | g^T (Ax + b)/(c^T x + d) \le h\}$$
$$f^{-1}(C) = \{x \in \mathbf{dom} f | (g^T A - hc^T)x \le hd - g^T b\}$$
$$f^{-1}(C) = \{x \in \mathbf{dom} f | (A^T g - ch^T)^T x \le hd - g^T b\}$$
Let $g' = A^T g - ch^T, h' = hd - g^T$
The inverse image is the intersection of a halfspace and domain: $\{x | g'^T f(x) \le h'\} \cap \mathbf{dom} f$**.**

2. $$f^{-1}(C) = \{x \in \mathbf{dom} f | G f(x) \preceq h\}$$
$$f^{-1}(C) = \{x \in \mathbf{dom} f | G(Ax + b)/(c^T x + d) \preceq h\}$$
$$f^{-1}(C) = \{x \in \mathbf{dom} f | (GA - hc^T)x \preceq hd - Gb\}$$
Let $G' = GA - hc^T, h' = hd - Gb$
The inverse image is the intersection of a polyhedron and domain: $\{x | G'x \preceq h'\} \cap \mathbf{dom} f$**.**

3. $$f^{-1}(C) = \{x \in \mathbf{dom} f | f(x)^T P^{-1} f(x) \le 1\}$$
$$f^{-1}(C) = \{x \in \mathbf{dom} f | (\frac{(Ax + b)^T}{c^T x + d}) P^{-1} \frac{Ax + b}{c^T x + d} \le 1\}$$
$$f^{-1}(C) = \{x \in \mathbf{dom} f | (Ax + b)^T P^{-1} (Ax + b) \le (c^T x + d)^2\}$$
$$f^{-1}(C) = \{x \in \mathbf{dom} f | [x^T (A^T P^{-1} A)x + b^T P^{-1} Ax + x^T A^T P^{-1} b + b^T P^{-1} b] \le (x^T cc^T x + dc^T x + x^T dc + d^2\}$$
$$f^{-1}(C) = \{x \in \mathbf{dom} f | [x^T (A^T P^{-1} A - cc^T)x + (b^T P^{-1} A - dc^T)x + x^T (A^T P^{-1} b - dc)] \le d^2 - b^T P^{-1} b\}$$
$$f^{-1}(C) = \{x \in \mathbf{dom} f | (x + x_c)^T P'^{-1} (x + x_c) \le d^2 - b^T P^{-1} b + x_c^2\}$$
where $P'^{-1} = A^T P^{-1} A - cc^T, x_c = P'(A^T P^{-1} b - dc)$
Then $f^{-1}(C) = \{x \in \mathbf{dom} f | (x + x_c)^T M^{-1} (x + x_c) \le 1\}$
where $M^{-1} = \frac{P'^{-1}}{d^2 - b^T P^{-1} b + x_c^2}$

The inverse image is the intersection of a ellipsoid and domain: $\{x | (x+x_c)^T M^{-1} (x+x_c) \le 1\} \cap \mathbf{dom} f$ **(where** $M \in \mathbf{S}^n_{++}$**).**

4.

$$f^{-1}(C) = \{x_1 \cdots x_n \in \textbf{dom} f | (a_1 x + b_1)A_1 + \cdots + (a_n x + b_n)A_n \preceq B(c^T x + d)\}$$

where a_i **represents for** A **in** $(Ax+b)/(c^T x + d)$

$$f^{-1}(C) = \{x_1 \cdots x_m \in \textbf{dom} f | x_1 A'_1 + \cdots + x_m A'_m \preceq B'\}$$

The inverse image is the solution set of a linear matrix inequality and domain: $\{x | x_1 A'_1 + \cdots + x_m A'_m \preceq B'\} \cap \textbf{dom} f$ **(where** $A'_i \in \mathbf{S}^p$**,** $B' = Bd - b_1 A_1 - \cdots - b_n A_n$)

Problem 3

Give an example of two closed convex sets that are disjoint but cannot be strictly separated.

$\{(x,y) | y \geq 2^x\}$ **and** $\{(x,y) | y \leq 0\}$