Ve492: Introduction to Artificial Intelligence
Bayesian Networks: Independence

Paul Weng

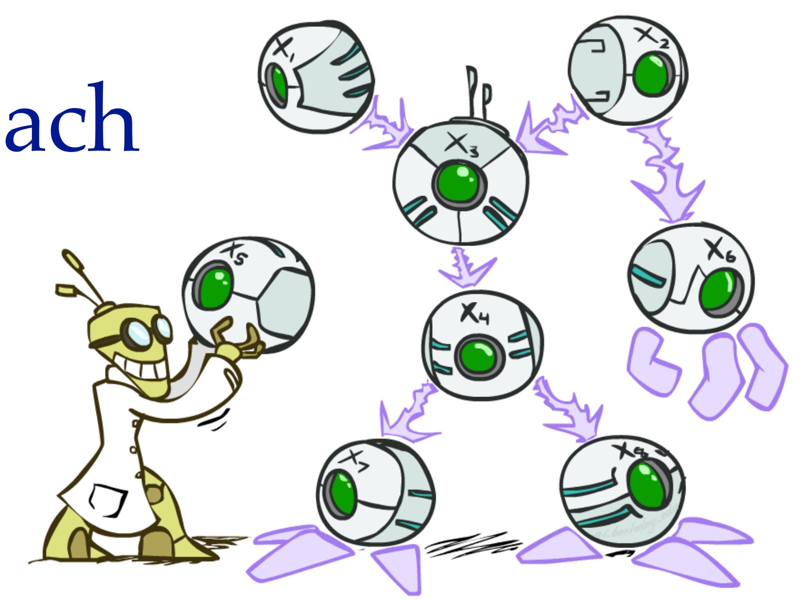
UM-SJTU Joint Institute

Slides adapted from http://ai.berkeley.edu, AIMA, UM, CMU

Bayes' Net

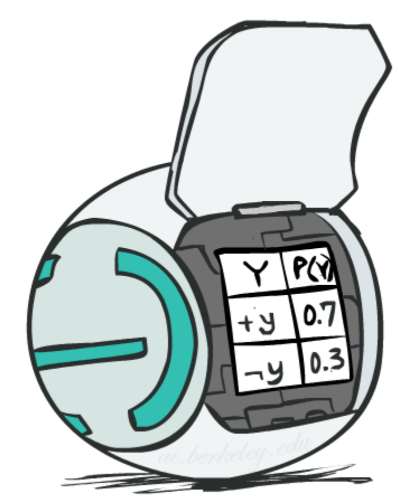
- A directed, acyclic graph, one node per random variable
- A conditional probability table (CPT) for each node
  - A collection of distributions over X, one for each combination of parents' values

$$P(X|a_1 \ldots a_n)$$



- Bayes' nets implicitly encode joint distributions
  - As a product of local conditional distributions
  - To see what probability a BN gives to a full assignment, multiply all the relevant conditionals together:

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i|\mathit{parents}(X_i))$$



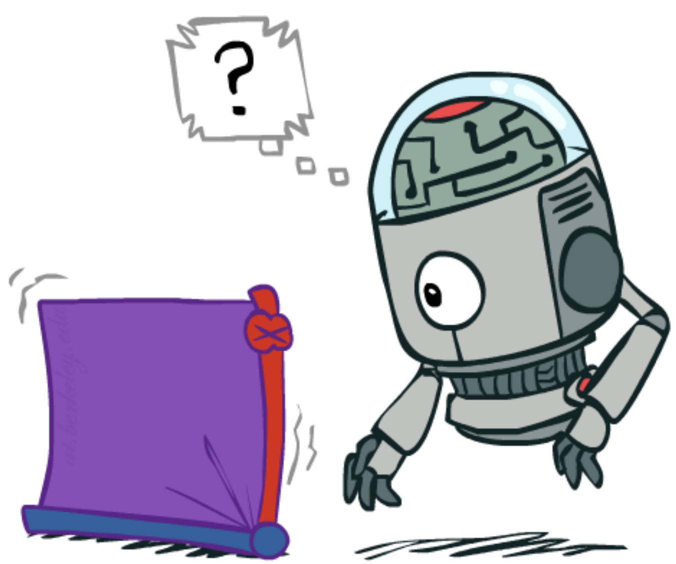
Size of a Bayes' Net

- How big is a joint distribution over N Boolean variables?

  2^N

- How big is an N-node net if nodes have up to k parents?

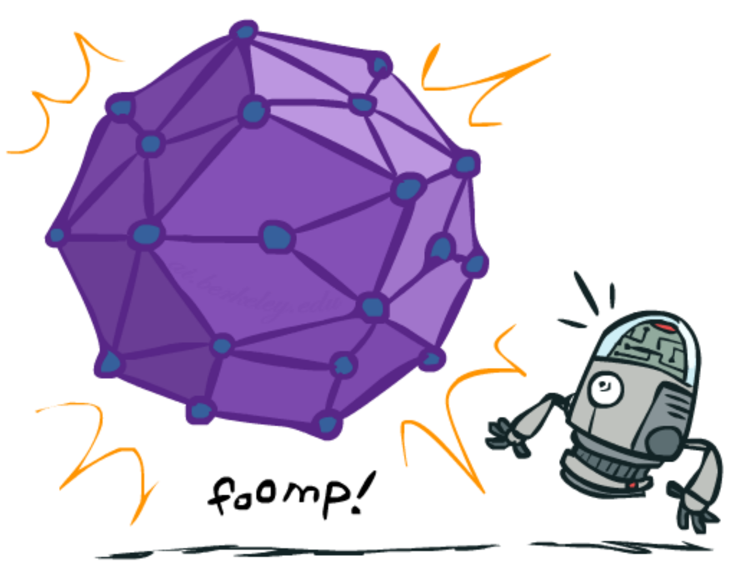
  $O(N * 2^{k+1})$



- Both give you the power to calculate

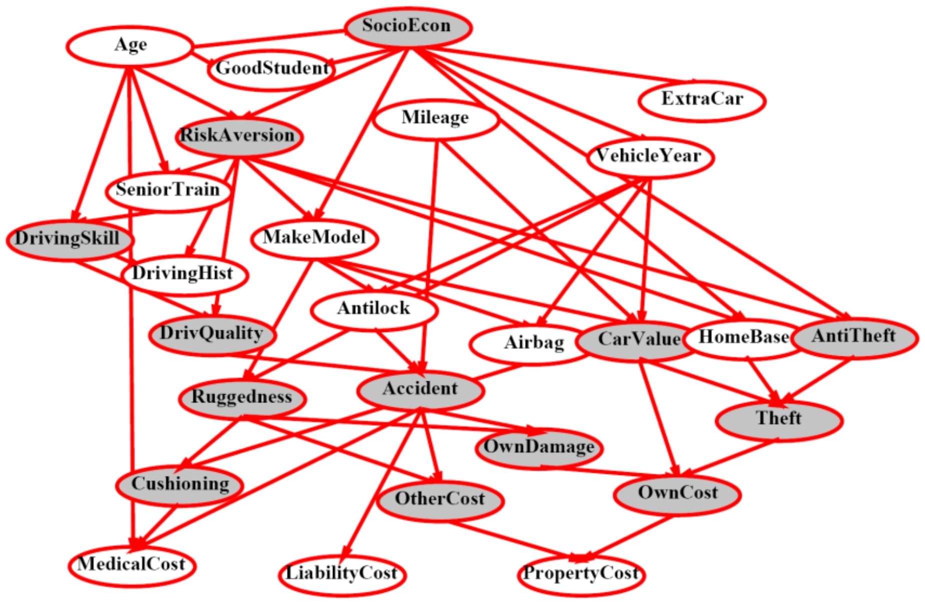
  $P(X_1, X_2, \ldots X_n)$

- BNs: Huge space savings!
- Also easier to elicit local CPTs
- Also faster to answer queries (coming)



Bayes' Nets

- A Bayes' net is an efficient encoding of a probabilistic model of a domain



- Questions we can ask:
  - Inference: given a fixed BN, what is P(X | e)?
  - Representation: given a BN graph, what kinds of distributions can it encode?
  - Modeling: what BN is most appropriate for a given domain?

Bayes' Nets

- ✓ Representation
- Conditional Independences
- Probabilistic Inference

Conditional Independence

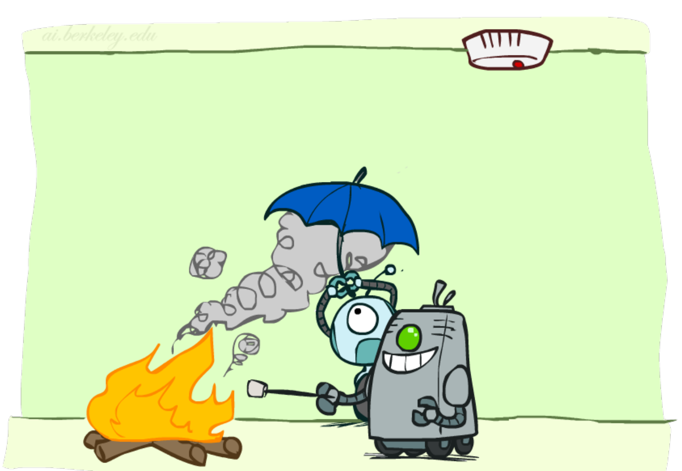
- X and Y are independent if

$$\forall x, y \;\; P(x,y) = P(x)P(y) \dashrightarrow X \perp\!\!\!\perp Y$$

- X and Y are conditionally independent given Z

$$\forall x, y, z \;\; P(x, y|z) = P(x|z)P(y|z) \dashrightarrow X \perp\!\!\!\perp Y|Z$$

- (Conditional) independence is a property of a joint distribution
- Example: $Alarm \perp\!\!\!\perp Fire | Smoke$



Bayes' Nets: Assumptions

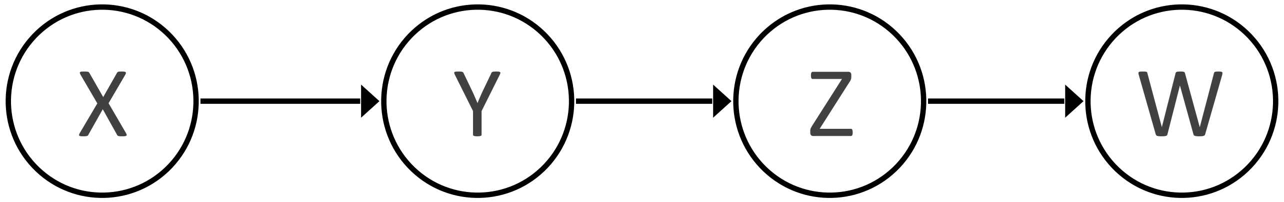
- Assumptions we are required to make to define the Bayes' net when given the graph:

$$P(x_i|x_1 \cdots x_{i-1}) = P(x_i|parents(X_i))$$

- Beyond the above conditional independence assumptions
  - Often additional conditional independences
  - They can be read off the graph
- Important for modeling: understand assumptions made when choosing a Bayes' net graph

Example

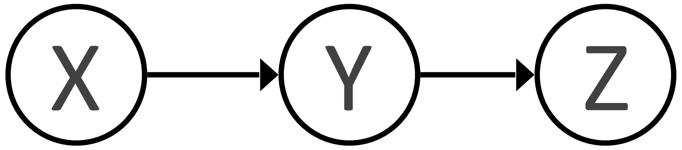
- Conditional independence assumptions directly from simplifications in chain rule:



- Additional implied conditional independence assumptions?

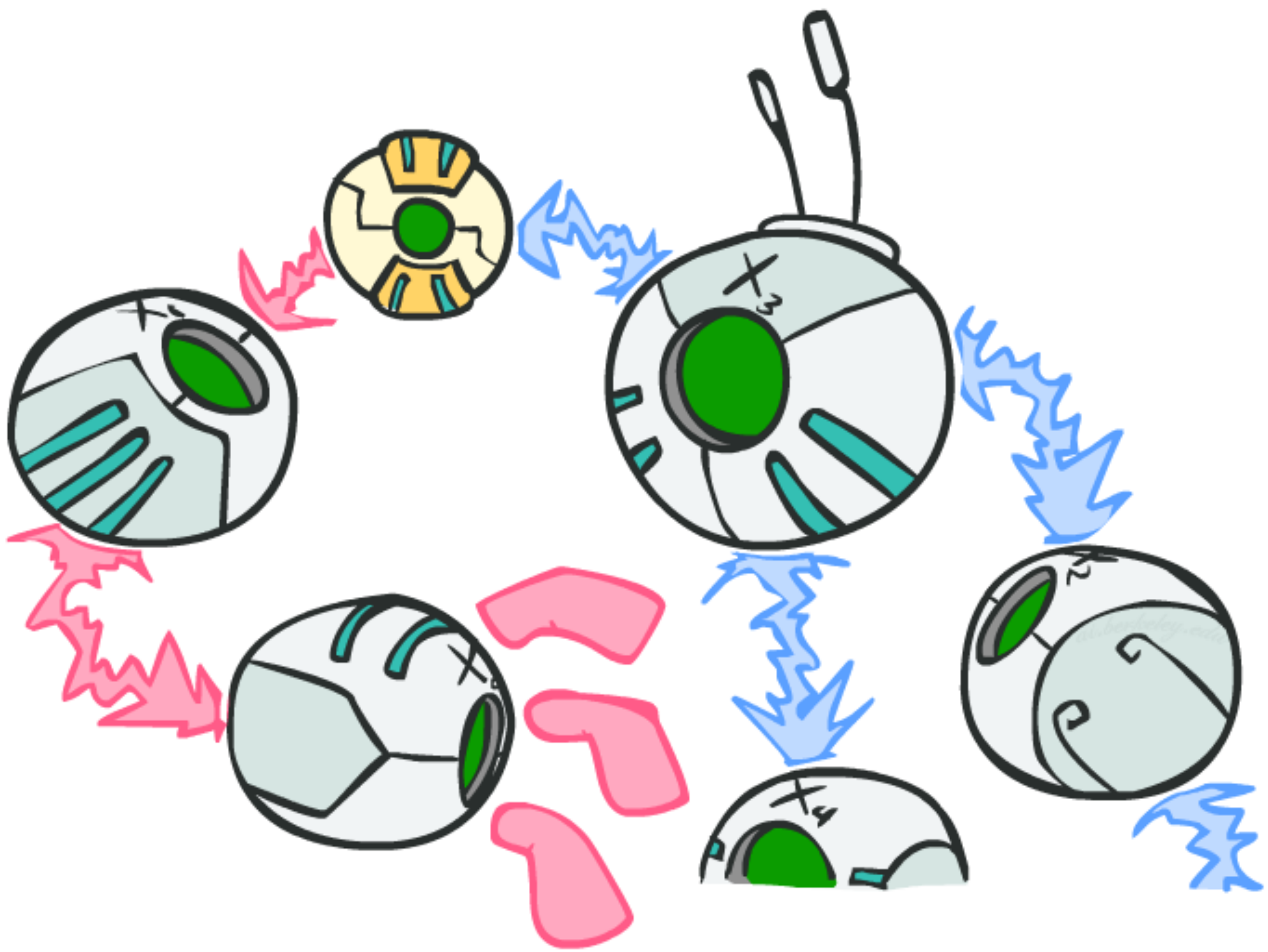
Independence in a BN

- Important question about a BN:
  - Are two nodes independent given certain evidence?
  - If yes, can prove using algebra (tedious in general)
  - If no, can prove with a counter example
  - Example:



  - Question: are X and Z necessarily independent?
    - Answer: No, e.g., low pressure causes rain, which causes traffic.
    - X can influence Z, Z can influence X (via Y)
    - Addendum: they *could* be independent: how?

D-separation: Outline



D-separation: Outline

- Study independence properties for triples
- Analyze complex cases in terms of member triples
- D-separation: a condition / algorithm for answering such queries

Serial Chains

- This configuration is a "serial chain"

X: Low pressure Y: Rain Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

- Is X always independent of Z ?
  - *No!*
  - One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed.
  - Counter-example:
    - Low pressure => rain => traffic, high pressure => no rain => no traffic
    - In numbers:

      P(+y | +x) = 1, P(-y | - x) = 1,

      P(+z | +y) = 1, P(-z | -y) = 1

Serial Chains

- This configuration is a "serial chain"

X: Low pressure Y: Rain Z: Traffic

$$P(x,y,z) = P(x)P(y|x)P(z|y)$$

- Is X always independent of Z given Y?

$$P(z|x,y) = \frac{P(x,y,z)}{P(x,y)}$$

$$= \frac{P(x)P(y|x)P(z|y)}{P(x)P(y|x)}$$

$$= P(z|y)$$

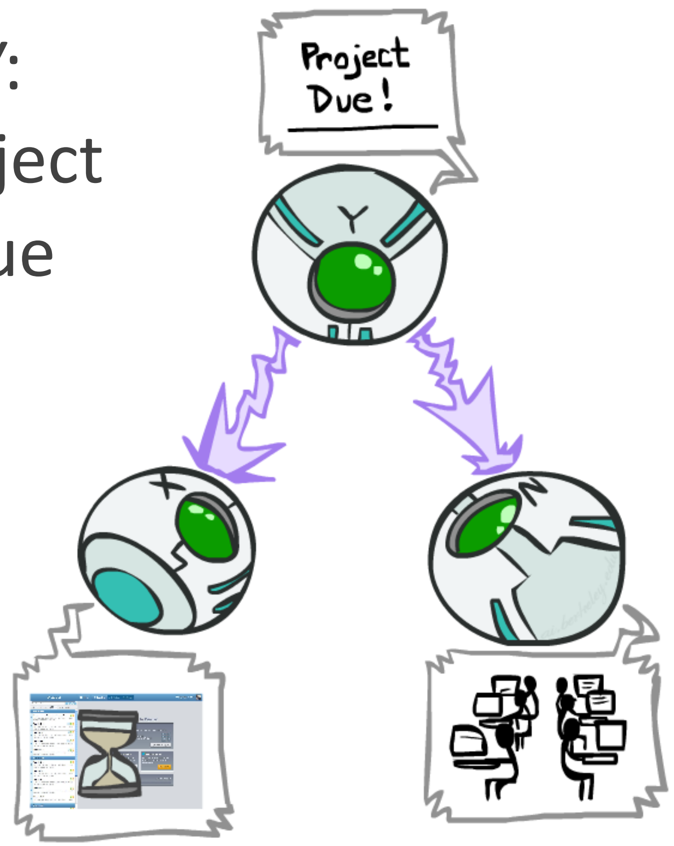
Yes!

- Evidence along the chain "blocks" the influence

Divergent Chain

- This configuration is a "divergent chain"

Y: Project due



X: Piazza busy

Z: OH busy

$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

- Is X always independent of Z ?
  - *No!*
  - One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed.
  - Counter-example:
    - Project due => Piazza busy and OH busy
    - In numbers:

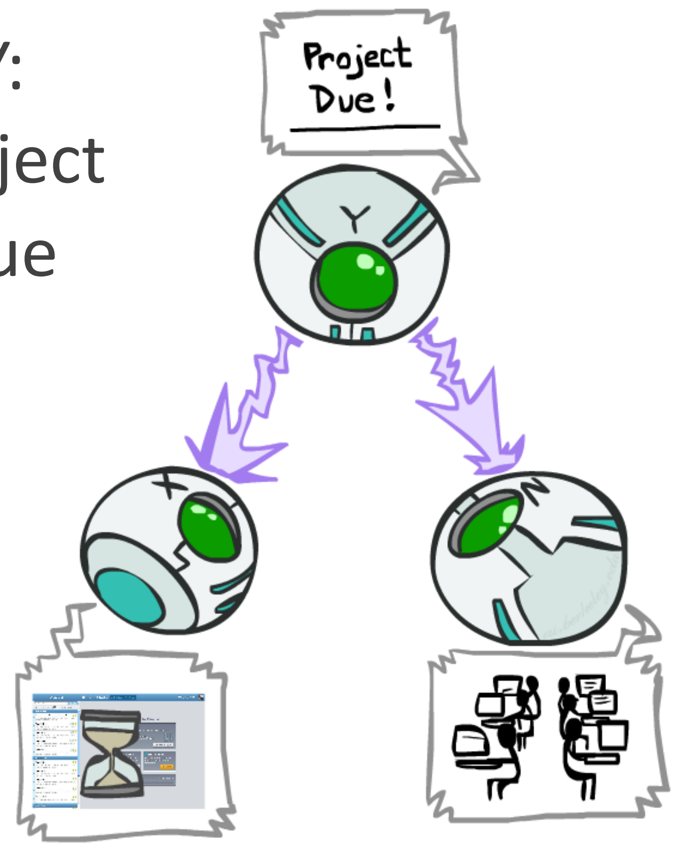
      P(+x | +y) = 1, P(-x | -y) = 1,

      P(+z | +y) = 1, P(-z | -y) = 1

Divergent Chain

- This configuration is a "divergent chain"

Y: Project due



X: Forums busy

Z: Lab full

$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

- Guaranteed X and Z independent given Y?

$$P(z|x, y) = \frac{P(x, y, z)}{P(x, y)}$$

$$= \frac{P(y)P(x|y)P(z|y)}{P(y)P(x|y)}$$

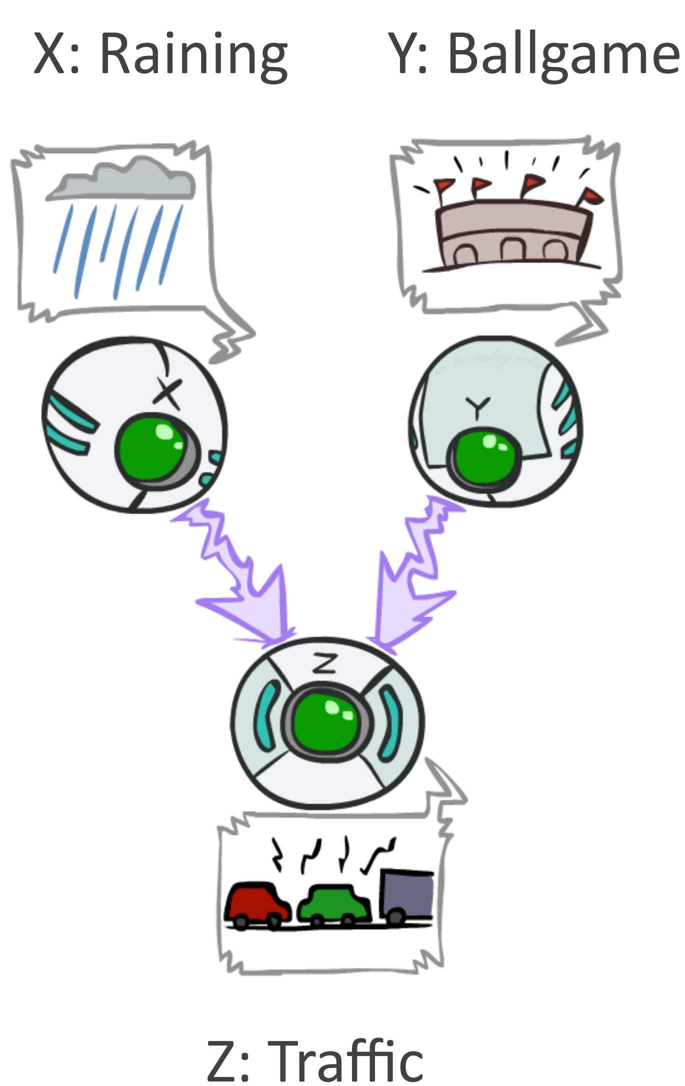
$$= P(z|y)$$

Yes!

- Observing the cause blocks influence between effects.

Convergent Chain

- Last configuration: "convergent chain" (v-structures)

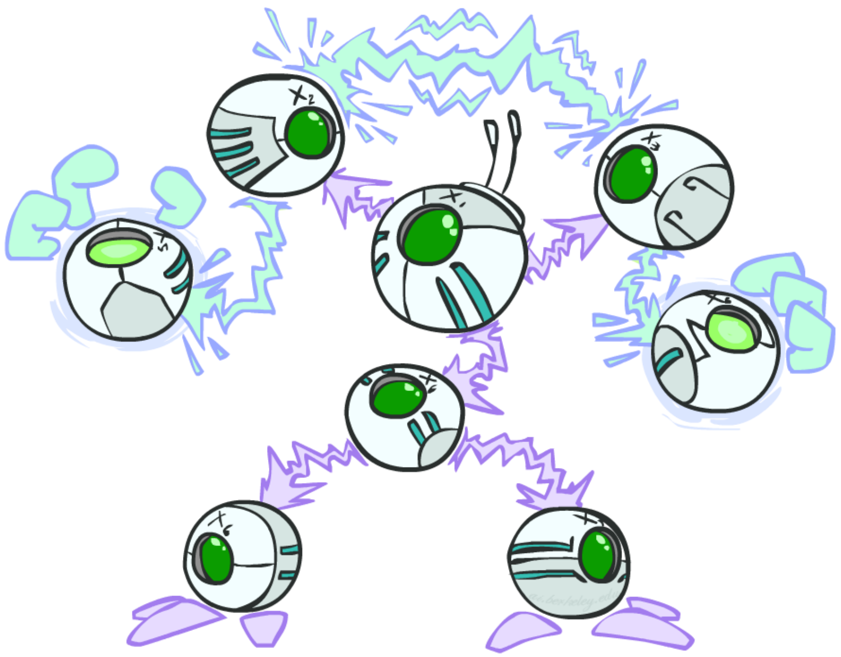


- Are X and Y independent?
  - *Yes*: the ballgame and the rain cause traffic, but they are not correlated
  - Still need to prove they must be (try it!)
- Are X and Y independent given Z?
  - *No*: seeing traffic puts the rain and the ballgame in competition as explanation.
- This is backwards from the other cases
  - Observing an effect activates influence between possible causes.

The General Case

The General Case

- **General question**: in a given BN, are two variables independent (given evidence)?



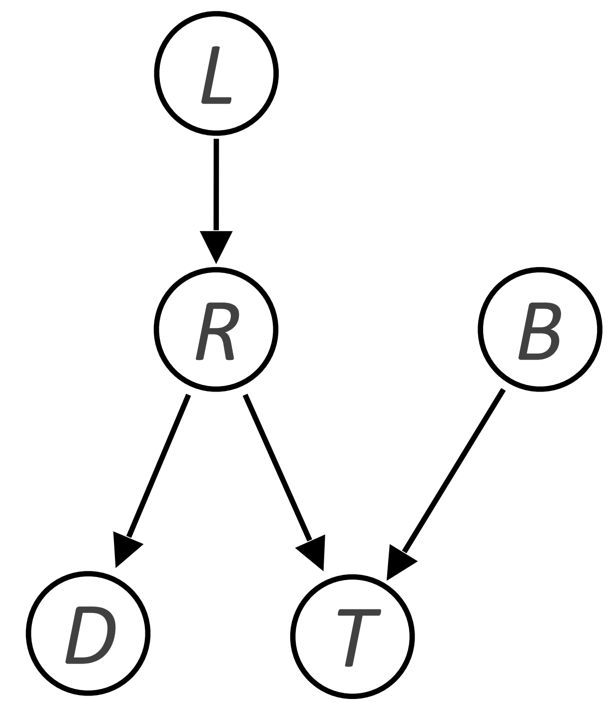
- **Solution**: analyze the graph
- Any complex example can be broken into repetitions of the three canonical cases

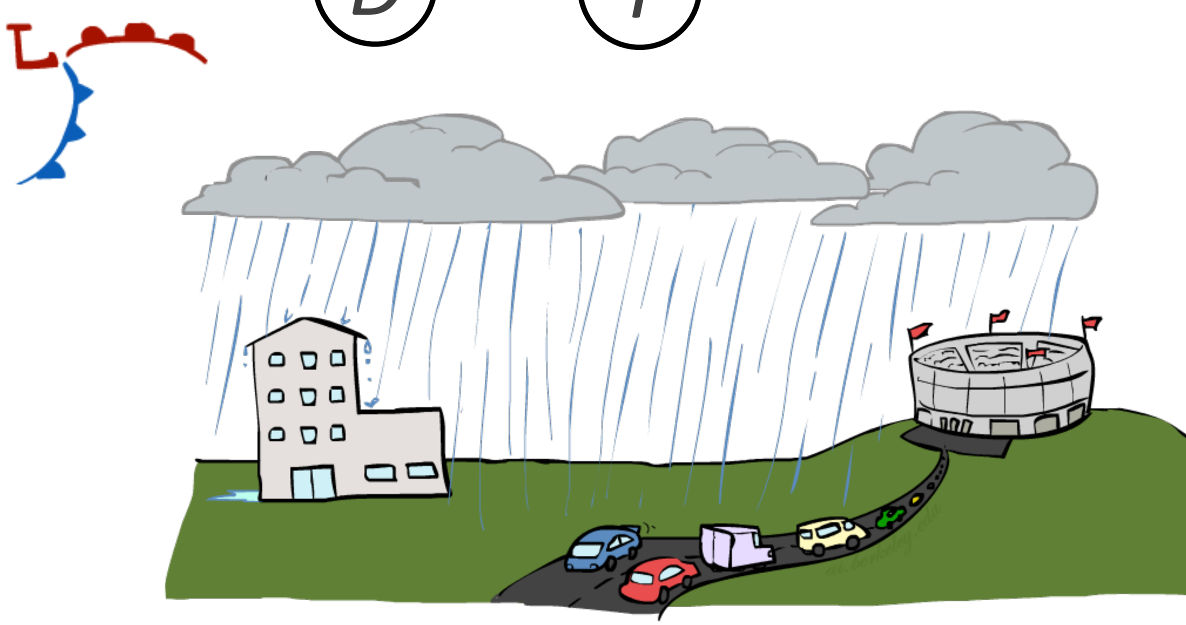
Reachability

- **Recipe**: shade evidence nodes, look for paths in the resulting graph

- **Attempt 1**: if two nodes are connected by an undirected path not blocked by a shaded node, they are conditionally independent

- Almost works, but not quite
  - Where does it break?
  - Answer: the v-structure at T doesn't count as a link in a path unless "active"





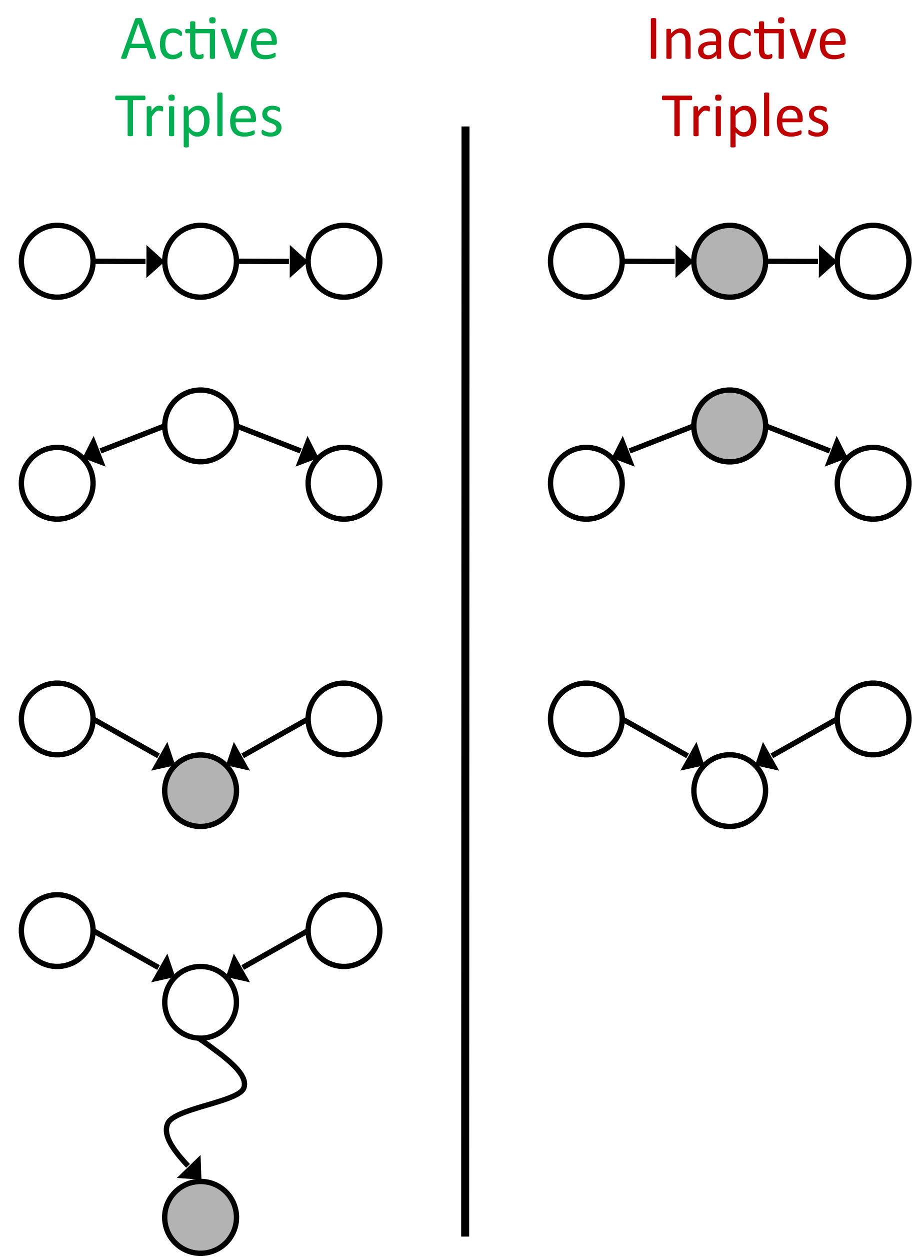
Active / Inactive Paths

- Question: Are X and Y conditionally independent given evidence variables {Z}?
  - Yes, if X and Y "d-separated" by Z
  - Consider all (undirected) paths from X to Y
  - No active paths = independence!

- A path is active if each triple is active:
  - Serial chain A → B → C where B is unobserved (either direction)
  - Divergent chain A ← B → C where B is unobserved
  - Convergent chain (aka v-structure)

  A → B ← C where B *or one of its descendants* is observed

- All it takes to block a path is a single inactive segment



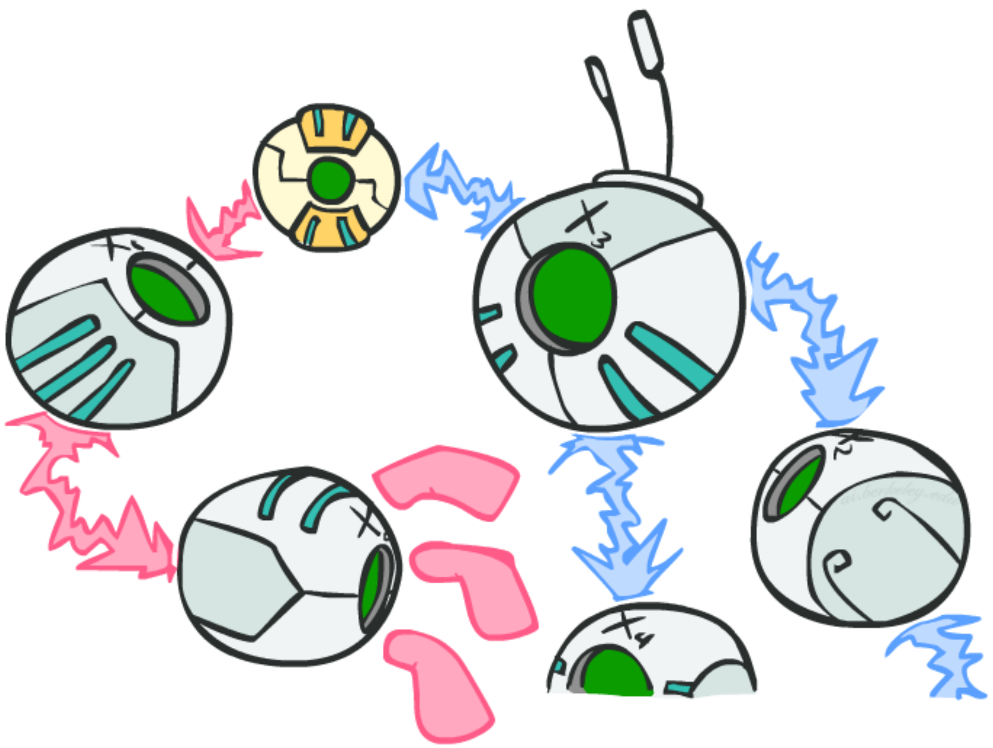
D-Separation

- Query: $X_i \perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$?
- Check all (undirected!) paths between X_i and X_j
  - If one or more active, then independence not guaranteed

$$X_i \not\perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$$

  - Otherwise (i.e. if all paths are inactive), then independence is guaranteed

$$X_i \perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$$

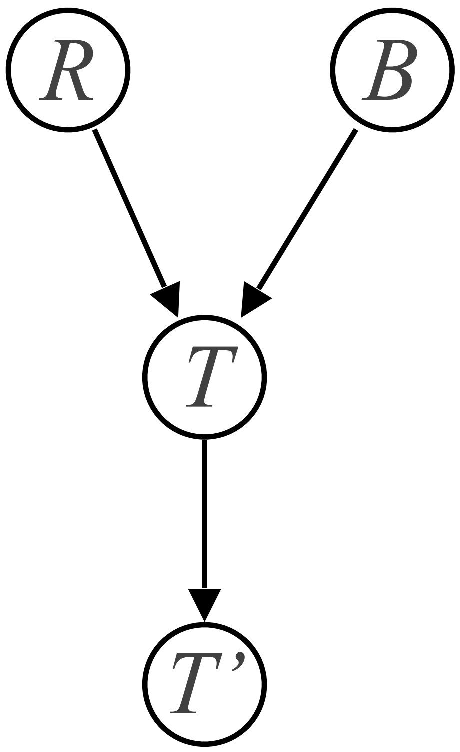


Example

$R \perp\!\!\!\perp B$ *Yes*

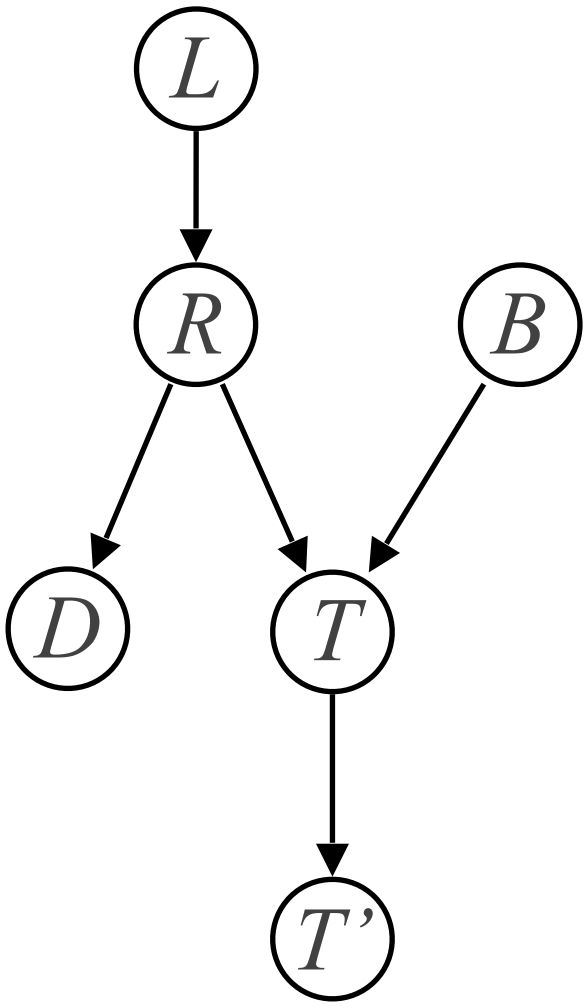
$R \perp\!\!\!\perp B | T$ *No*

$R \perp\!\!\!\perp B | T'$ *No*



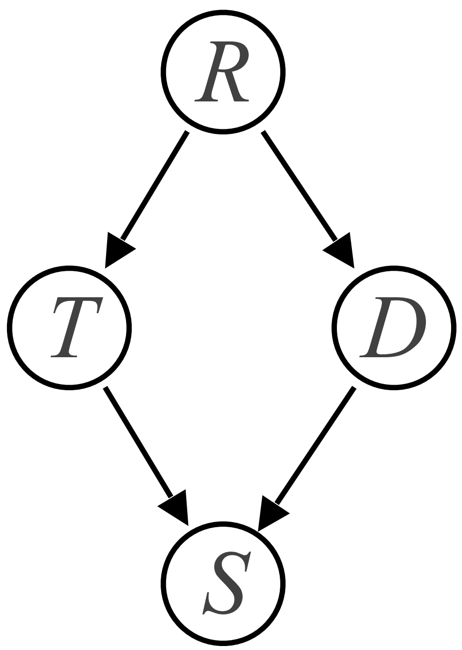
Example

$L \perp\!\!\!\perp T' \mid T$	*Yes*
$L \perp\!\!\!\perp B$	*Yes*
$L \perp\!\!\!\perp B \mid T$	*No*
$L \perp\!\!\!\perp B \mid T'$	*No*
$L \perp\!\!\!\perp B \mid T, R$	*Yes*



Quiz: Conditional Independence

- Variables:
  - R: Raining
  - T: Traffic
  - D: Roof drips
  - S: I'm sad



- Questions:

$T \perp\!\!\!\perp D$ No

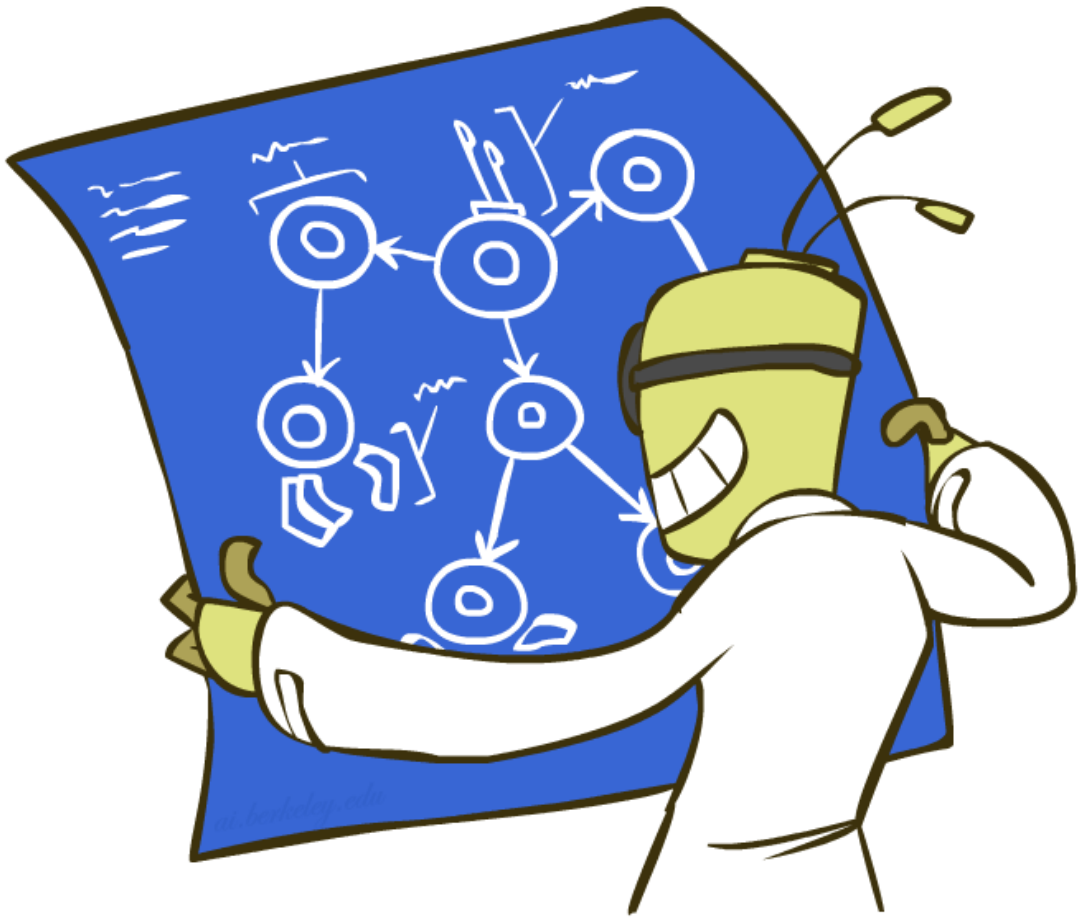
$T \perp\!\!\!\perp D | R$ Yes

$T \perp\!\!\!\perp D | R, S$ No

Structure Implications

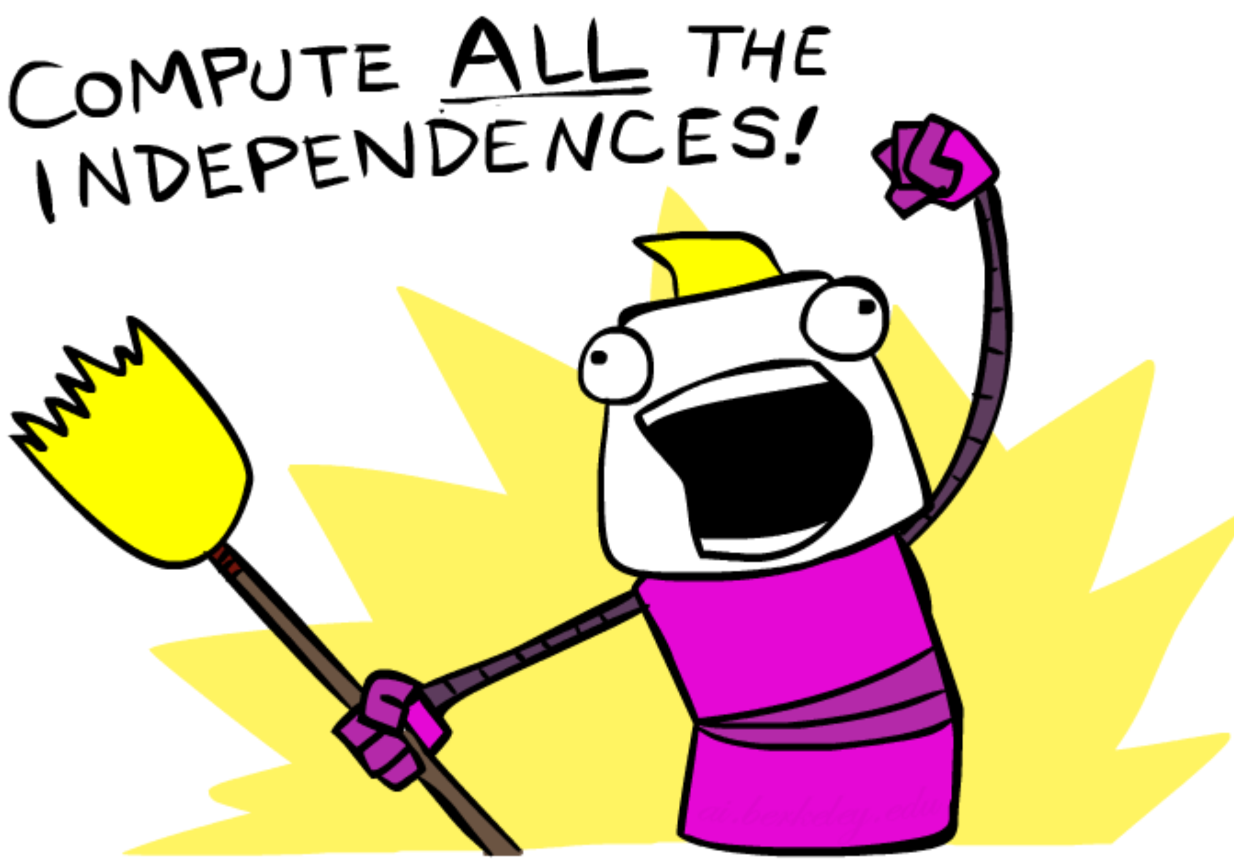
- Given a Bayes net structure, can run d-separation algorithm to build a complete list of conditional independences that are necessarily true of the form

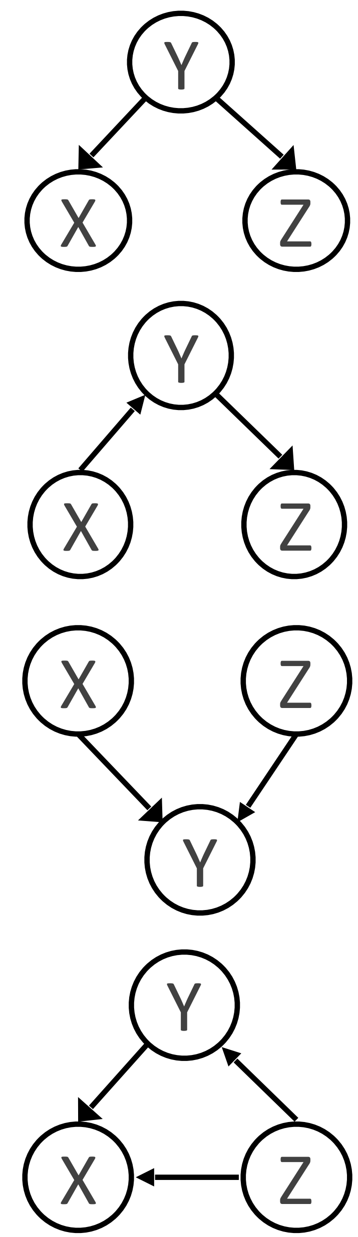
$$X_i \perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$$



- This list determines the set of probability distributions that can be represented

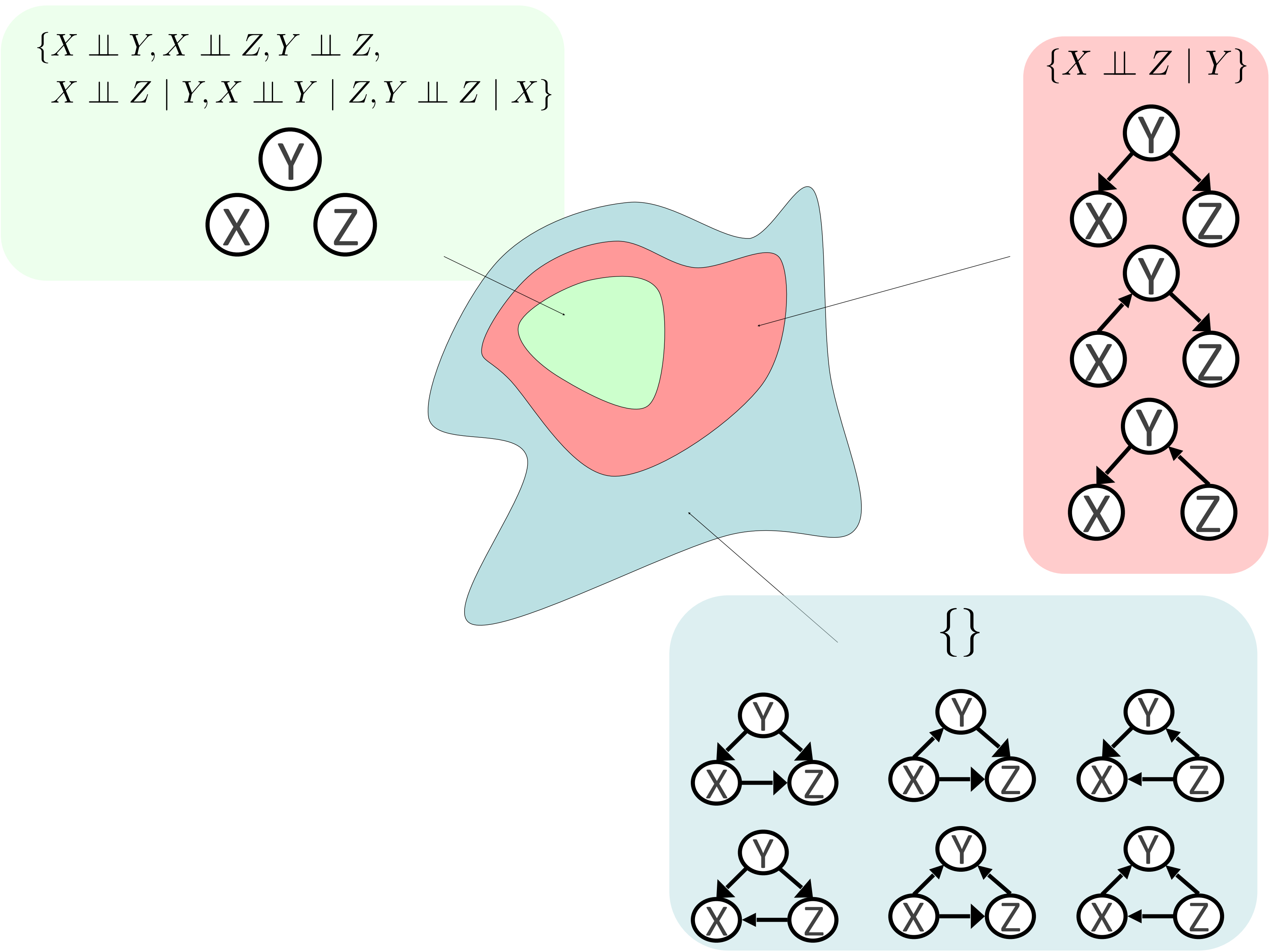
Computing All Independences





Topology Limits Distributions

- Given some graph topology G, only certain joint distributions can be encoded
- The graph structure guarantees certain (conditional) independences
- (There might be more independence)
- Adding arcs increases the set of distributions, but has several costs
- Full conditioning can encode any distribution



Bayes Nets Representation Summary

- Bayes nets compactly encode joint distributions
- Guaranteed independencies of distributions can be deduced from BN graph structure
- D-separation gives precise conditional independence guarantees from graph alone
- A Bayes' net's joint distribution may have further (conditional) independence that is not detectable until you inspect its specific distribution

Bayes' Nets

- ✓ Representation
- ✓ Conditional Independences
- Probabilistic Inference
  - Enumeration (exact, exponential complexity)
  - Variable elimination (exact, worst-case exponential complexity, often better)
  - Probabilistic inference is NP-complete
  - Approximate inference (sampling)