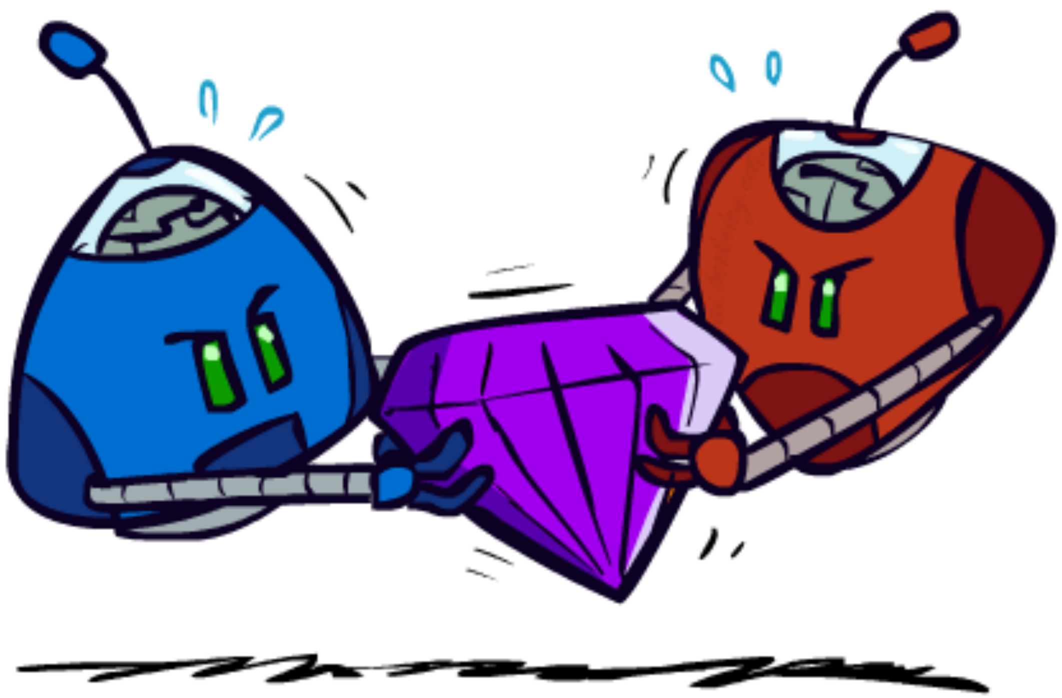
# Announcements

- HW1 due today at 11:59pm
- HW2 release today
- P0
  - Submission on OJ
  - Watch out if grade below 90
- P1 released
  - Start early! Maybe the longest of all projects
- Mid-term exam Oct 28 10-11:40am (D508)
- Recitation attendance

# Ve492: Introduction to Artificial Intelligence

## From Decision Theory to Game Theory




Paul Weng

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Slides adapted from http://ai.berkeley.edu, AIMA, UM, CMU

# Outline

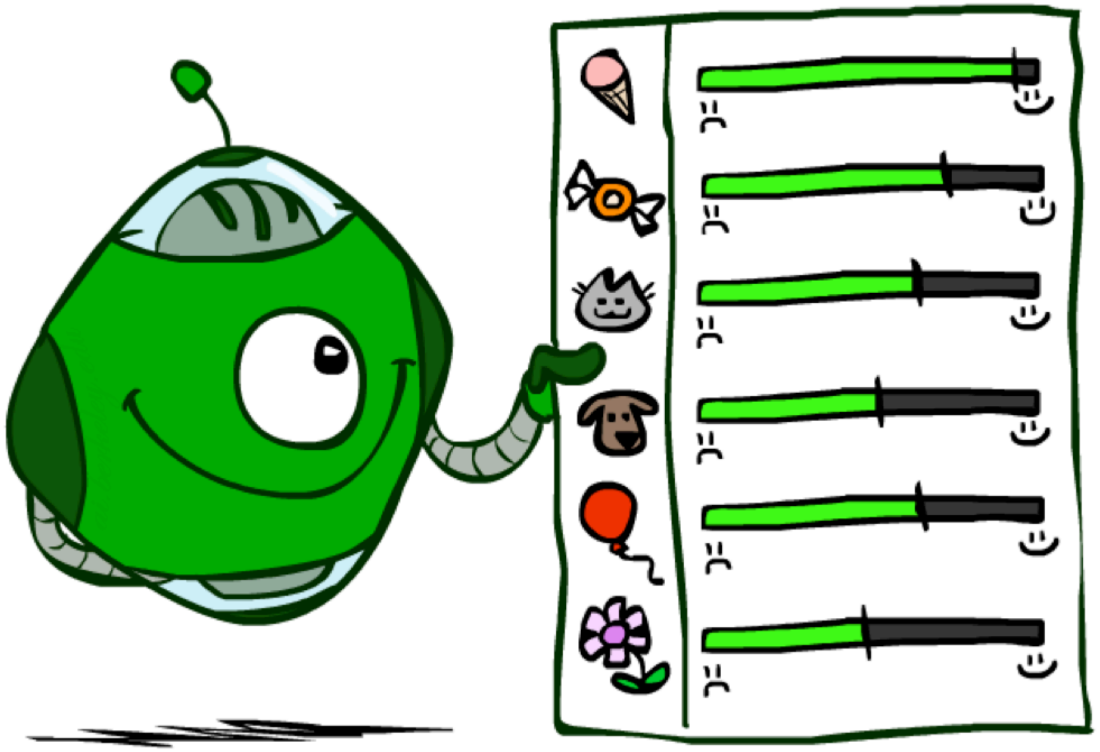
- Decision Theory
- Game Theory

# Decision Theory

- Decision problem:
  - Choose $a \in A$ assuming given preference relation $\succsim$ over $A$
- Often, choice has uncertain outcomes
  - Probability distribution over outcomes
- Here, we assume single-agent decision-making
- Which decision criterion should we choose?
  - Descriptive
  - Normative

*Decision Theory*

# Utilities

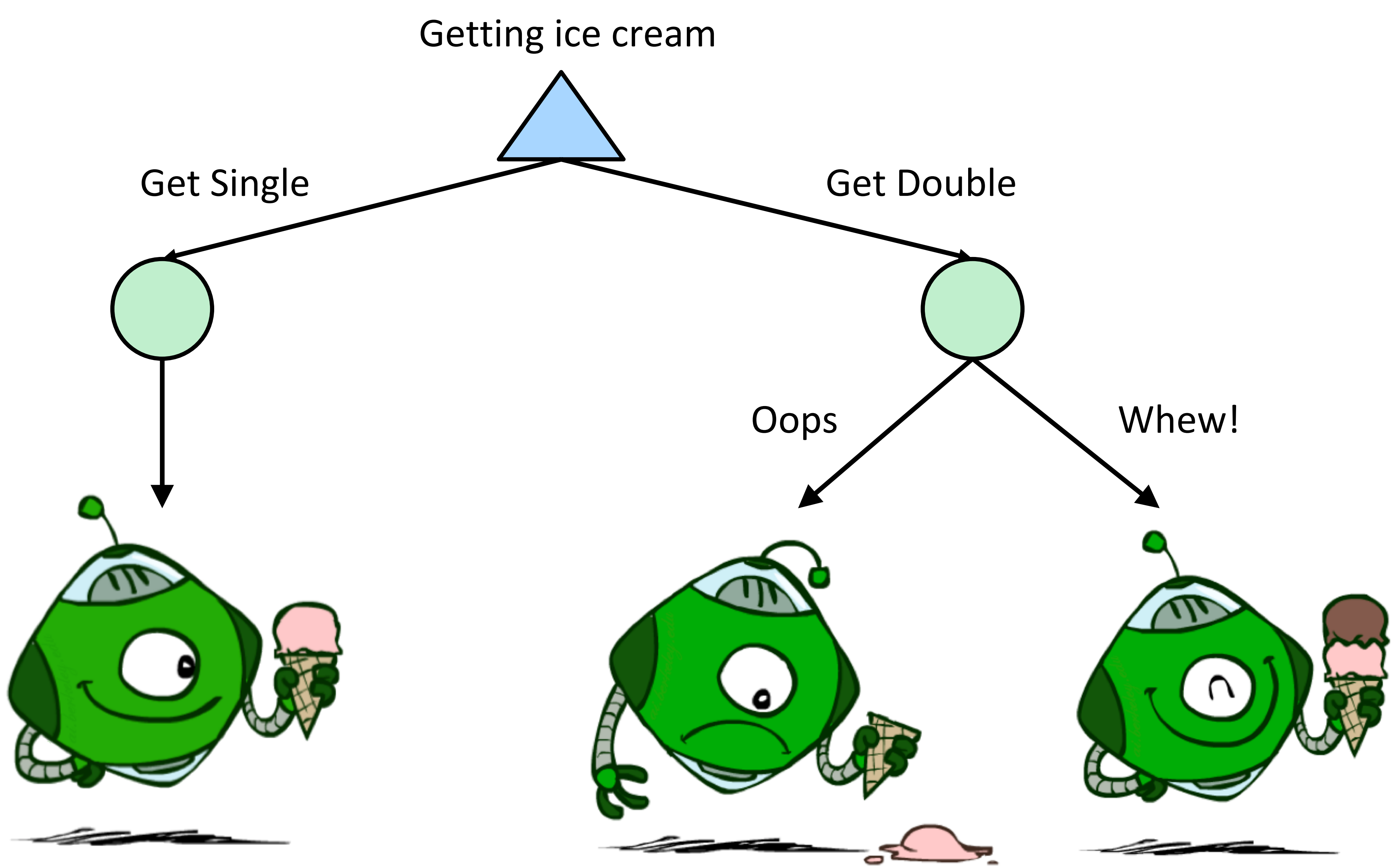


# Utilities

- Utilities are functions from outcomes (states of the world) to real numbers that describe an agent's preferences

- Where do utilities come from?
  - In a game, may be simple (+1/-1)
  - Utilities summarize the agent's goals
  - Theorem: any "rational" preferences can be summarized as a utility function
- We hard-wire utilities and let behaviors emerge
  - Why don't we let agents pick utilities?
  - Why don't we prescribe behaviors?

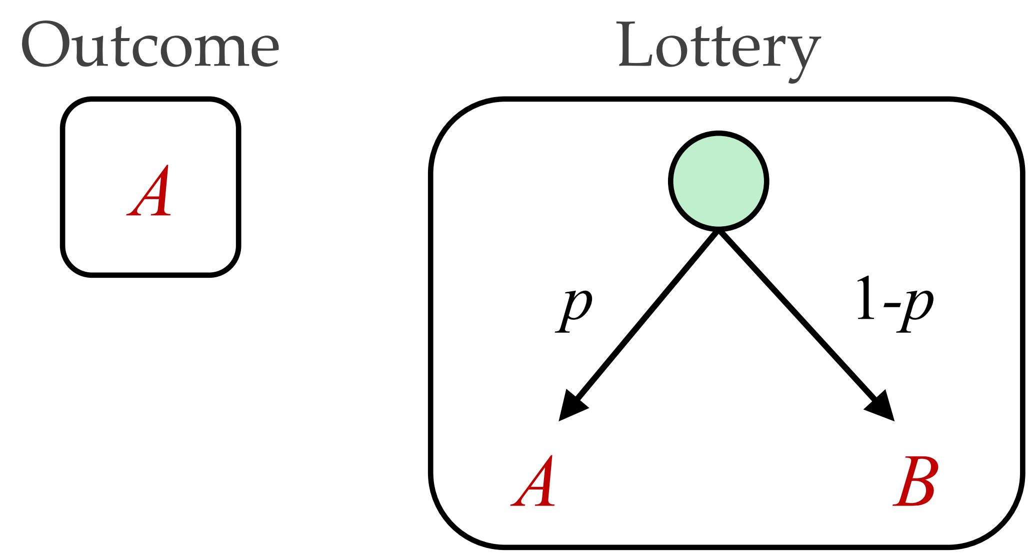
# Utilities: Uncertain Outcomes



# Preferences

- An agent must have preferences among:
  - Outcomes: A, B, etc.
  - Lotteries: situations with uncertain outcomes

$$L = [p, A; (1-p), B]$$




- Notations:
  - Preference: $A \succ B$
  - Indifference: $A \sim B$

# Maximum Expected Utility

- Principle of Maximum Expected Utility (MEU):
  - Rational agent should choose the action that maximizes its expected utility, given its knowledge
- Why should we average utilities? Why not minimax?
- Questions:
  - Where do utilities come from?
  - How do we know such utilities even exist?
  - How do we know that averaging even makes sense?
  - What if our behavior (preferences) can't be described by utilities?

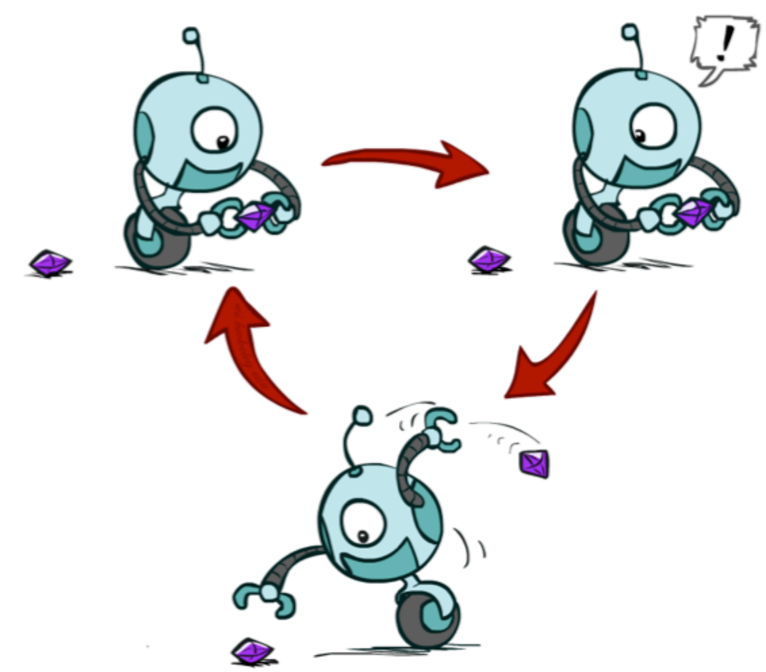
# Rational Preferences

- We want some conditions on preferences before we call them rational, such as:
  - Transitivity: $(A \succ B) \wedge (B \succ C) \Rightarrow (A \succ C)$
- Why is it important?
  - An agent with intransitive preferences can be induced to give away all of its money:

    If B > C, then an agent with C would pay (say) 1 cent to get B

    If A > B, then an agent with B would pay (say) 1 cent to get A

    If C > A, then an agent with A would pay (say) 1 cent to get C

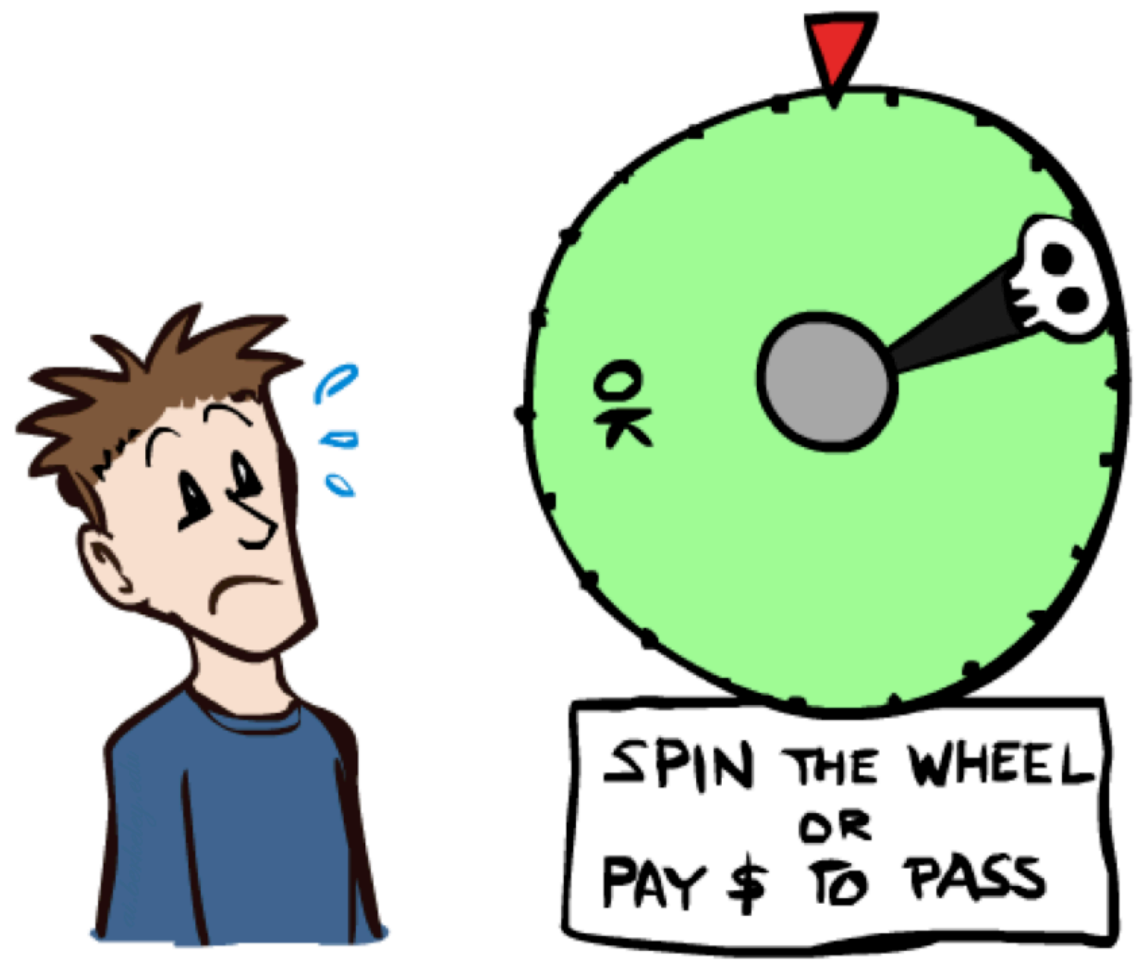


# Rational Preferences

- Axioms of Rationality [Machina, 1988]
  - Comparability: $(A \succ B) \wedge (B \succ A) \wedge (A \sim B)$
  - Transitivity: $(A \succ B) \wedge (B \succ C) \Rightarrow (A \succ C)$
  - Independence: $A \succsim B \Rightarrow [p, A; (1-p), C] \succsim [p, B; (1-p), C]$
  - Continuity: $A \succ B \succ C \Rightarrow \exists p, [p, A; (1-p), C] \sim B$
- Theorem [Ramsey, 1931; von Neumann & Morgenstern, 1944]
  - Axioms of rationality is equivalent to MEU principle

*Decision Theory*

# Human Utilities

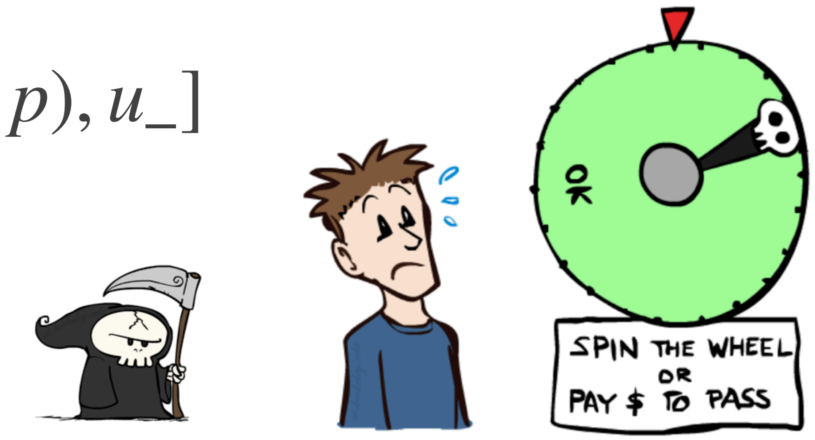


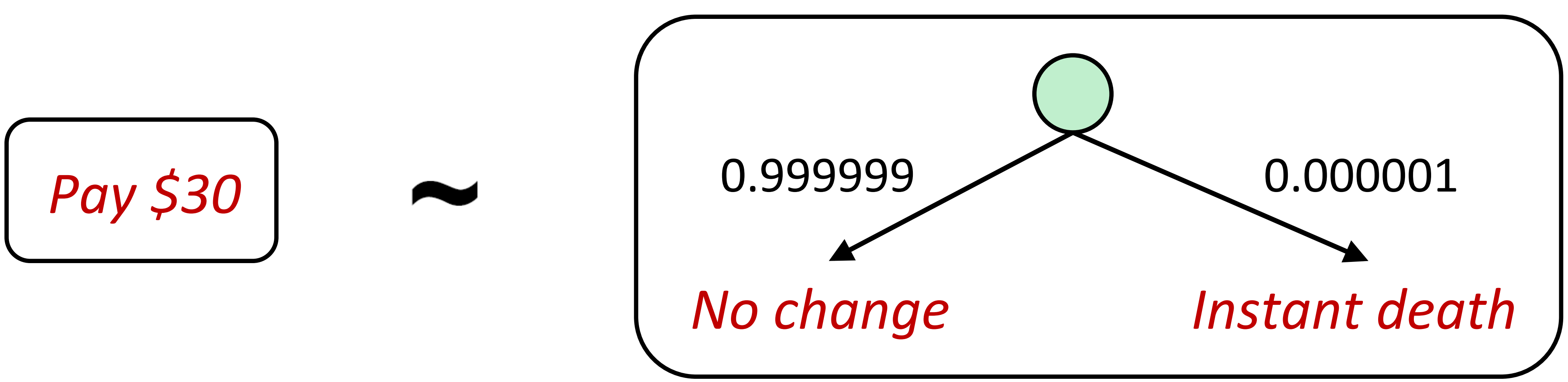
# Utility Elicitation

- Utility function maps outcome to real numbers. How to obtain them?

- Possible approach:
  - Compare outcome A to standard lottery $[p, u^+; (1-p), u\_]$

    where $u^+$ = best outcome, $u\_$ = worst outcome
  - For which $p^*$ do we have $A \sim [p, u^+; (1-p), u\_]$?
  - $p^* = u(A)$ if we assume that $u$ is normalized (in [0, 1])

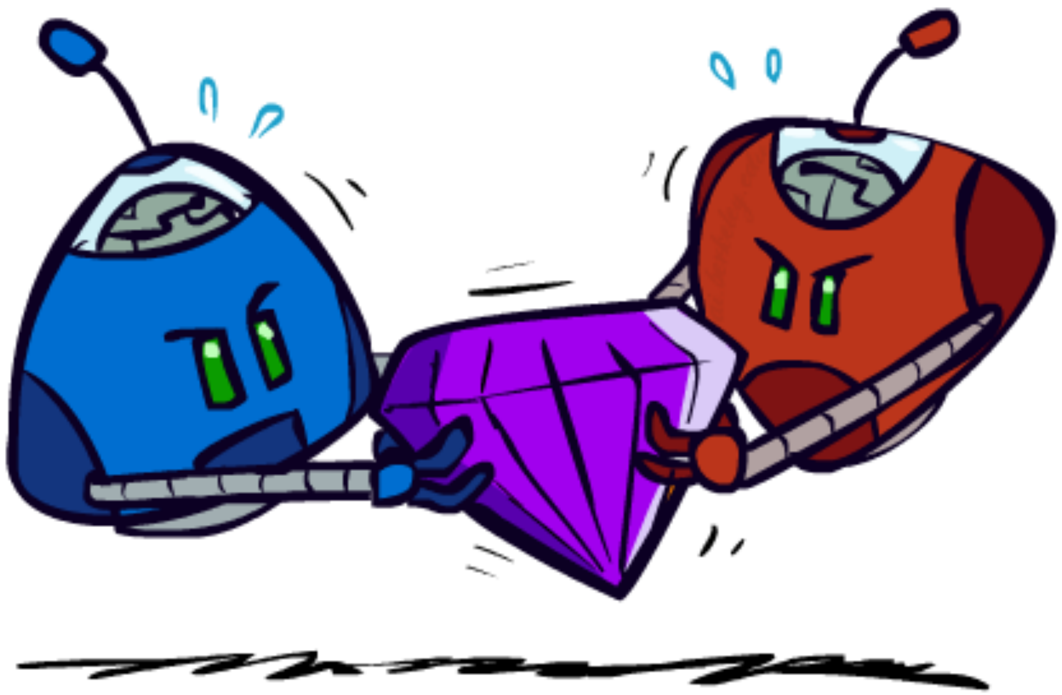


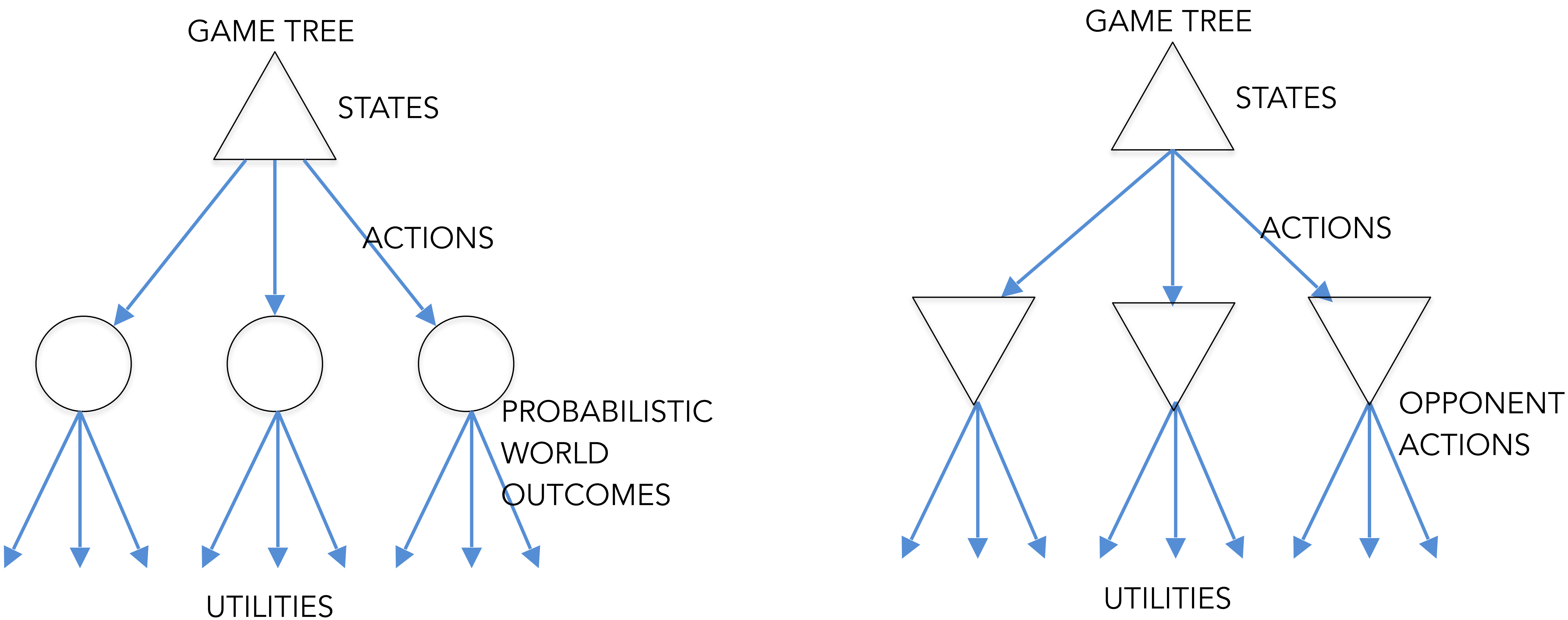
# Human decision-makers

- Often risk-averse: utility function is concave in EU
- However, does human follow MEU principle?
- Allais paradox (1953)
  - A: [0.8, \$4k;  0.2, \$0] $\succ$ B: [1.0, \$3k;  0.0, \$0] ?
  - C: [0.2, \$4k;  0.8, \$0] $\succ$ D: [0.25, \$3k;  0.75, \$0] ?
  - Most people prefer $B \succ A$, $C \succ D$
  - However, inconsistent with EU:
    - Assume u(\$0) = 0
    - $B \succ A \Rightarrow$ u(\$3k) > 0.8 u(\$4k)
    - $C \succ D \Rightarrow$ 0.8 u(\$4k) > u(\$3k)

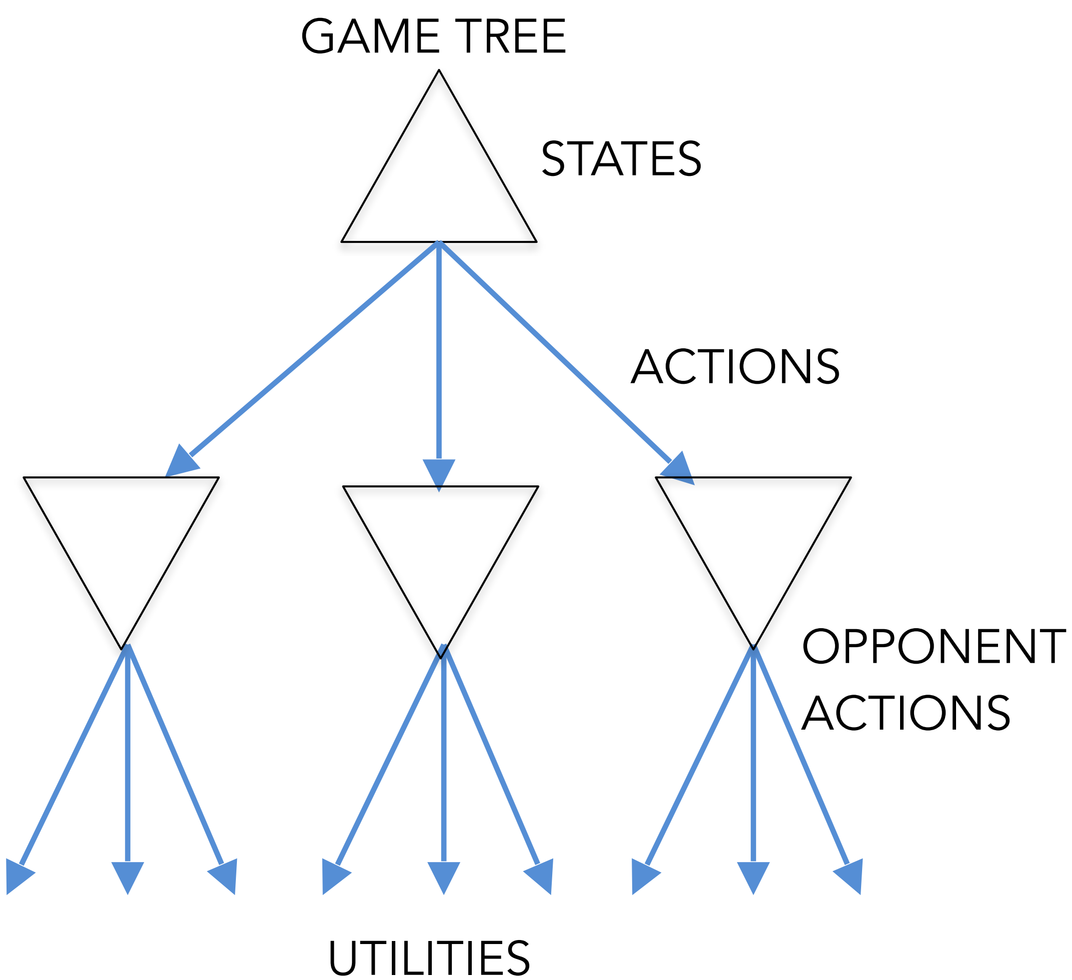
# Game Theory

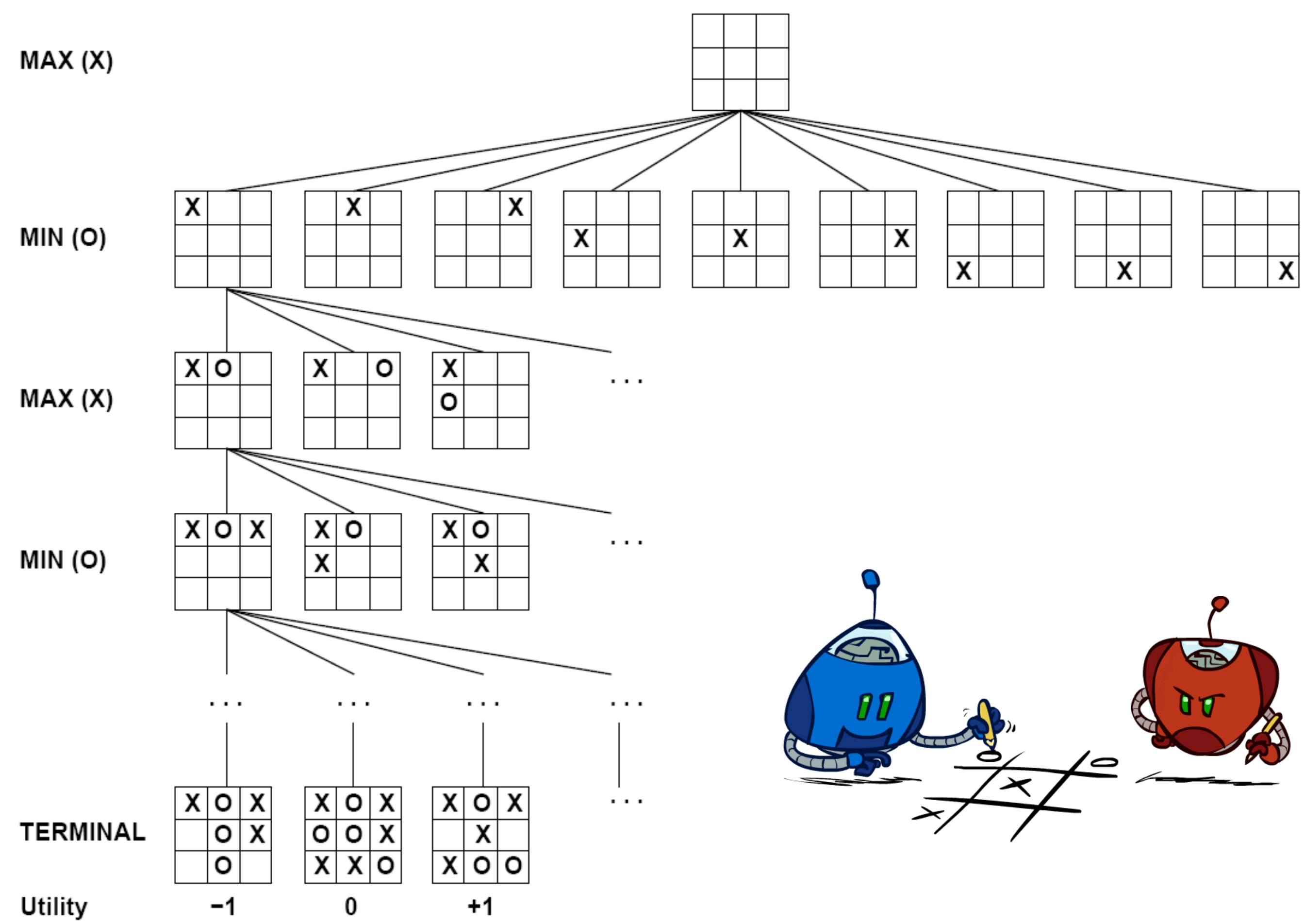


# Problems with Uncertainty vs Adversary

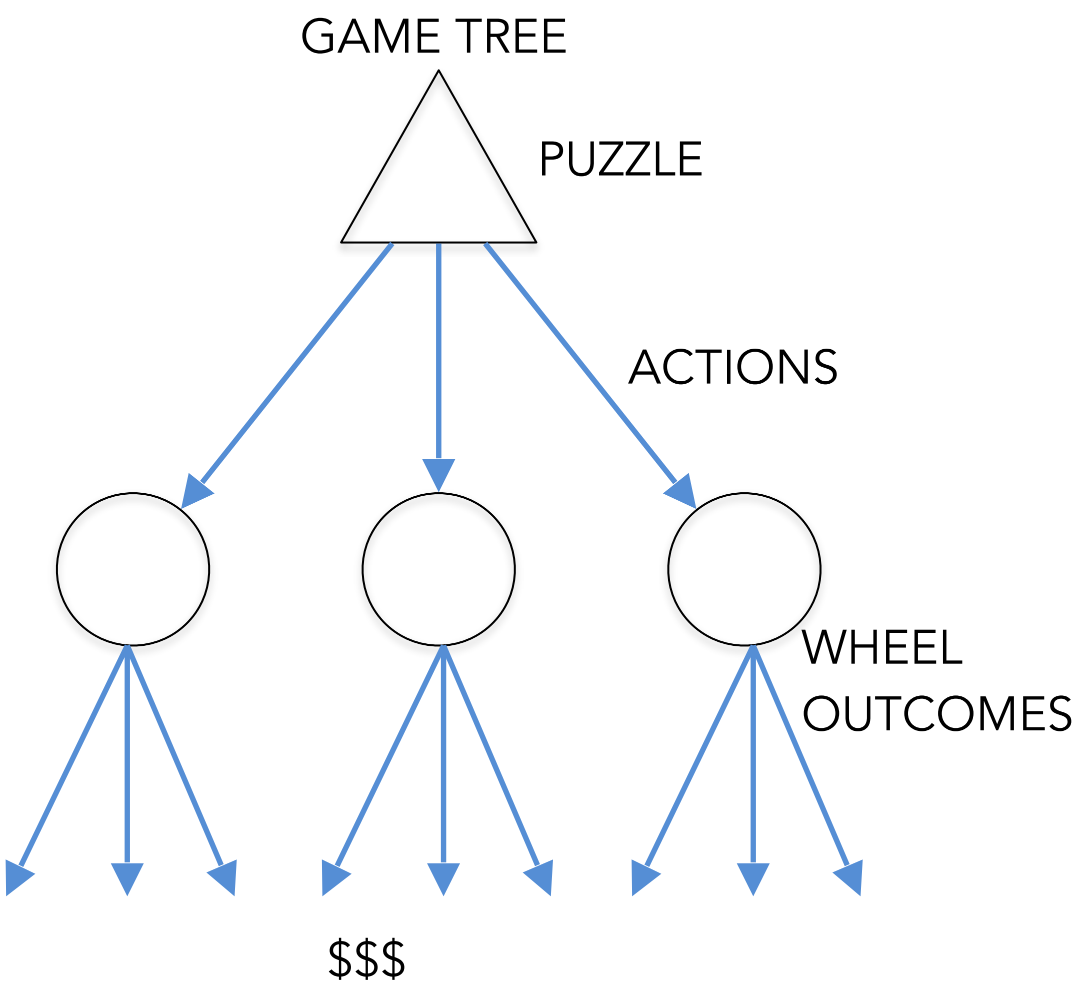


# Tic Tac Toe

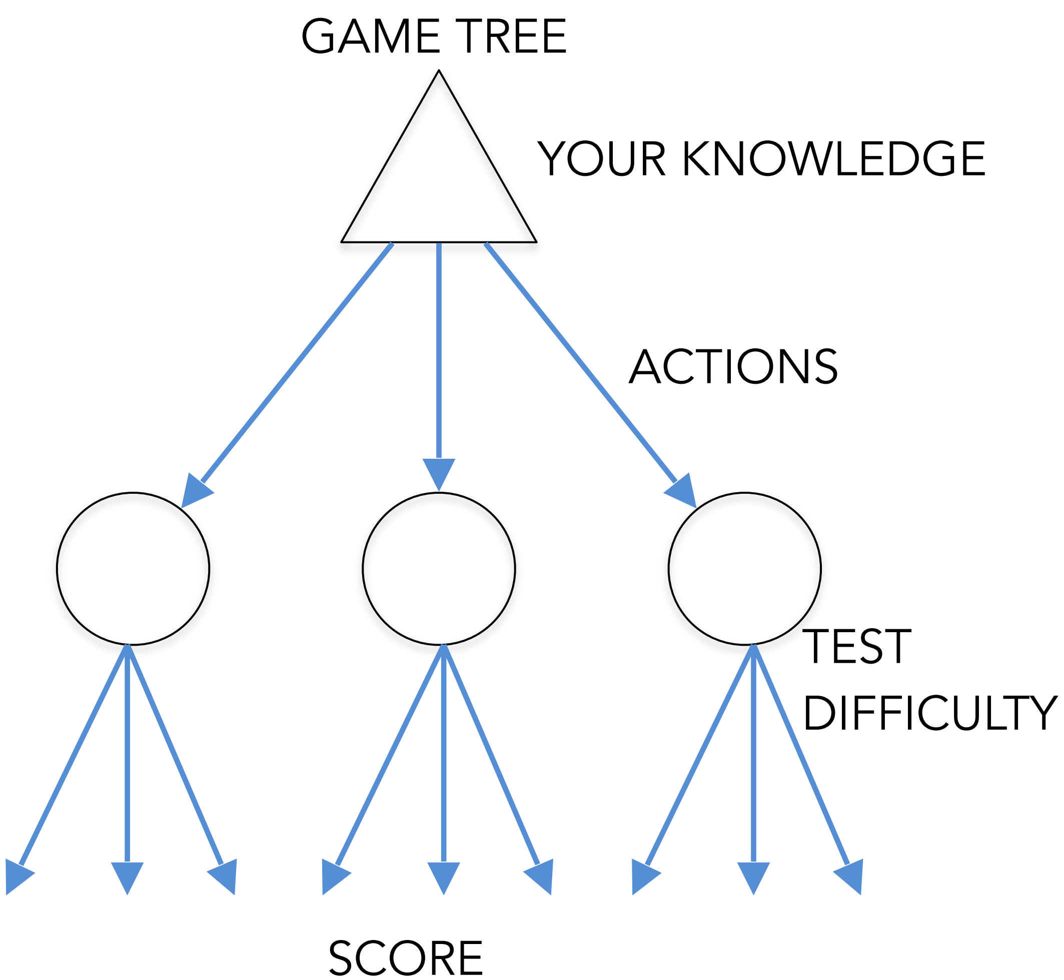

GAME TREE
STATES
ACTIONS
OPPONENT ACTIONS
UTILITIES


MAX (X)
MIN (O)
MAX (X)
MIN (O)
TERMINAL
Utility
−1
0
+1

# Wheel Of Fortune




# VE492 Exam



# Decision Theory vs Game Theory

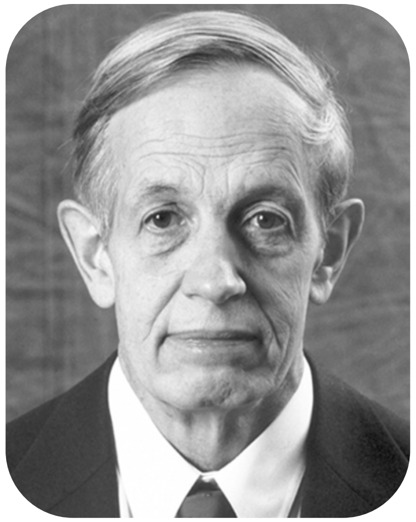
- Decision Theory: pick a strategy to maximize utility given world outcomes
- Game Theory: pick a strategy for player that maximizes his utility given the strategies of the other players
- Models are essentially the same
- Imagine the world is a player in the game!

# History of Game Theory

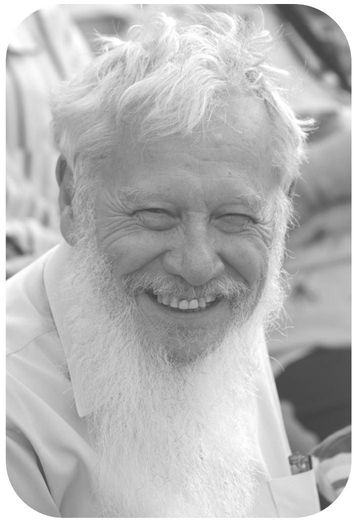
- Game theory is the study of strategic decision-making (of more than one player)
- Used in economics, political science etc.

John von Neumann

John Nash

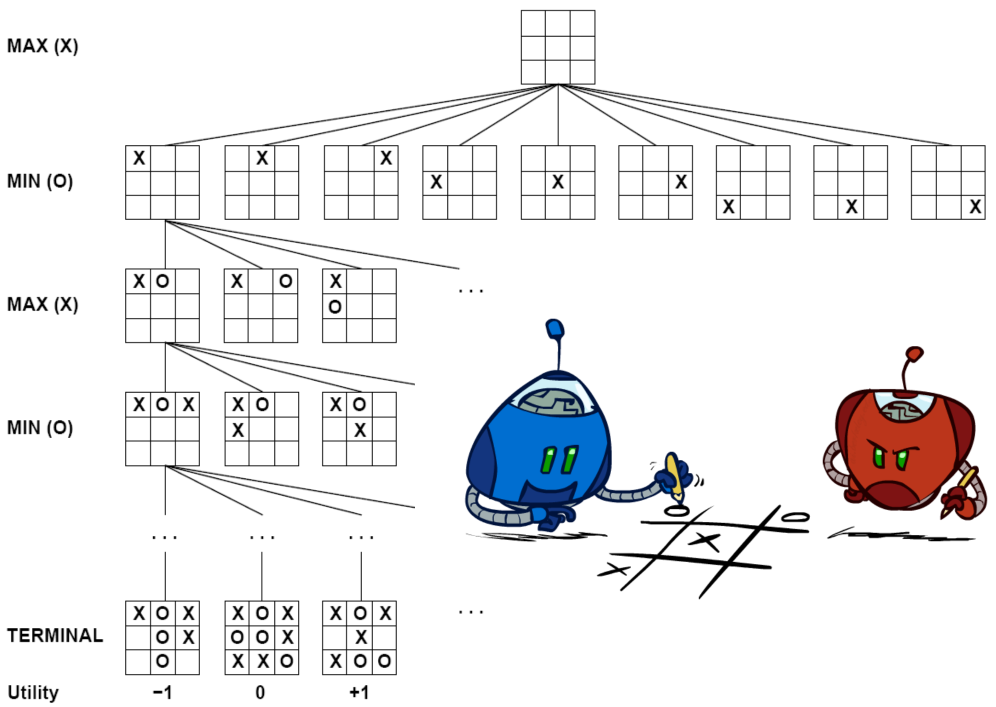


Robert Aumann



# Games: Extensive Form

- Representation:
    1. Set of all players of a game
    2. For every player, every opportunity they have to move
    3. What each player can do at each of their moves
    4. What each player knows/observes when making every move
    5. Payoffs received by everyone for all possible combo of moves



# Alternative Representation: Normal Form

- Represent games as single-shot decision-making problems
- Represent only strategies (e.g., actions or policies) and utilities
- Easier to determine particular properties of games

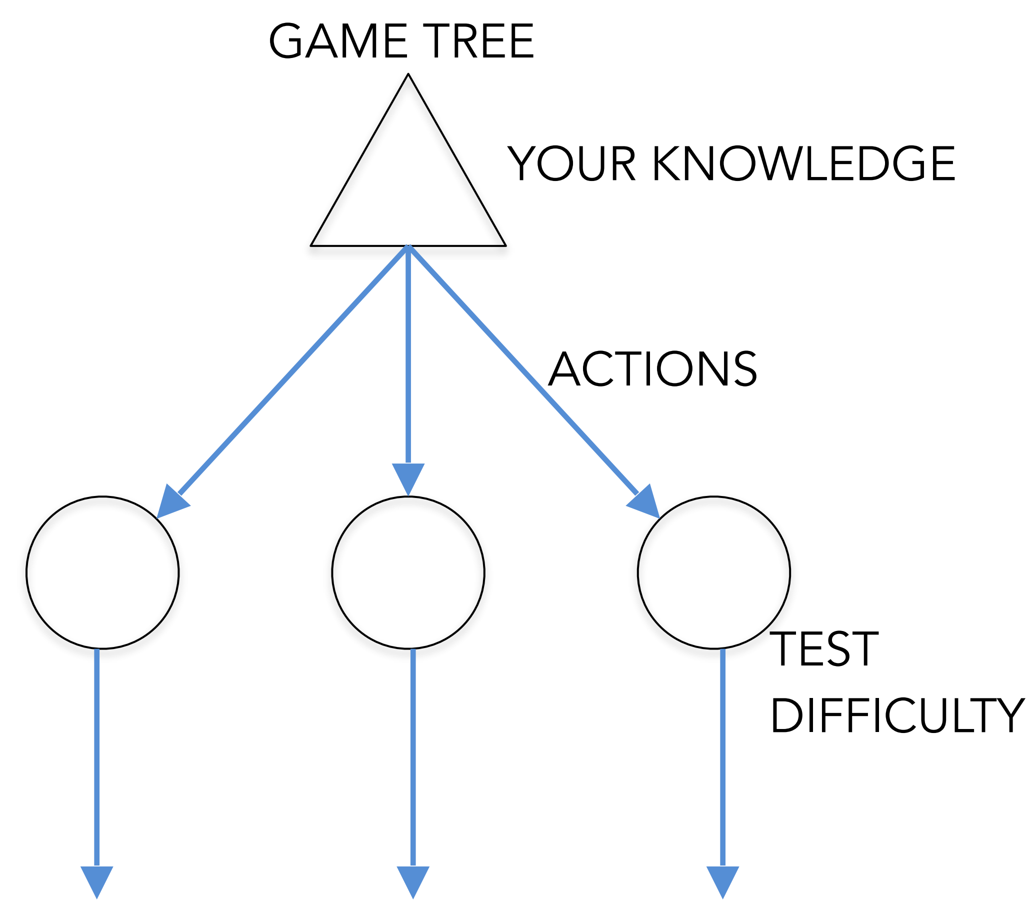
# Studying – Normal Form Game

- Represent games as single shot
- Represent only strategies and utilities

STRATEGIES

| | CRAM | DO HW | PLAY GAME |
|---|---|---|---|
| GRADE | 98 | 100 | 85 |

UTILITIES



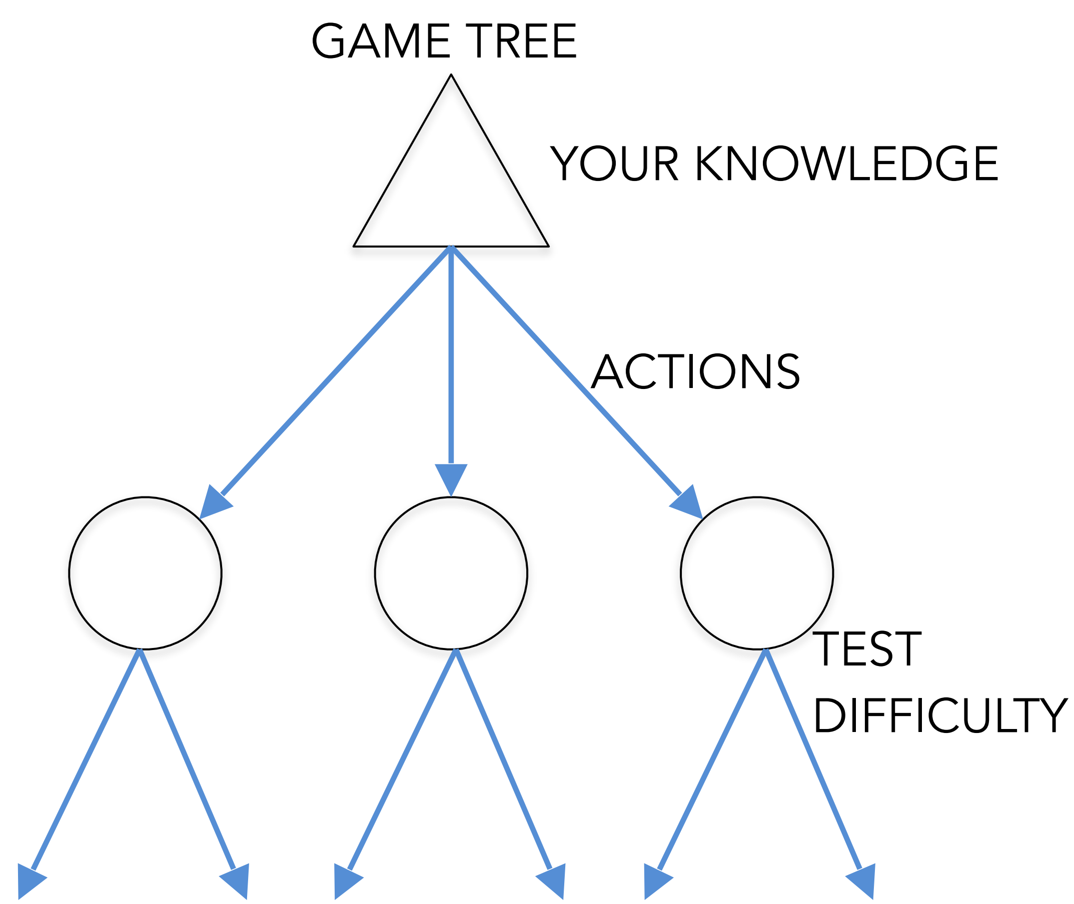
# Studying – Normal Form Game

- Represent games as single shot
- Represent only strategies and utilities

STRATEGIES

| | CRAM | DO HW | PLAY GAME |
|---|---|---|---|
| EASY | 98 | 100 | 85 |
| HARD | 97 | 90 | 65 |

UTILITIES



# Studying - Strategies and Utilities

- The world acts at the same time as you choose a strategy

World outcomes

STRATEGIES

| | CRAM | DO HW | PLAY GAME |
|---|---|---|---|
| EASY | 98 | 100 | 85 |
| HARD | 97 | 90 | 65 |

UTILITIES

# Strategy/Utility Notations

- Strategy $k$ for player $= \pi_k \in \Pi$ where $\Pi$ is finite
- Utility $u(\pi_k, s)$ where $s$ is a state of the world

STRATEGIES

| | CRAM | DO HW | PLAY GAME |
|---|---|---|---|
| EASY | 98 | 100 | 85 |
| HARD | 97 | 90 | 65 |

UTILITIES

# Questions you may ask…

- Which strategy should I adopt?
  - Maximize the expected utility based on state probabilities
- Is it beneficial to choose a strategy in a random way?

STRATEGIES

| | CRAM | DO HW | PLAY GAME |
|---|---|---|---|
| EASY | 98 | 100 | 85 |
| HARD | 97 | 90 | 65 |

UTILITIES

# Mixed Strategies

- Pure strategies $\Pi$
- Mixed strategies $\Delta(\Pi)$ = set of probability distributions over $\Pi$
- Goal: Pick strategy that maximizes expected utility given exam probability

Pure Strategy: deterministic (p=1) selection of actions

STRATEGIES

| | CRAM | DO HW | PLAY GAME |
|---|---|---|---|
| EASY | 98 | 100 | 85 |
| HARD | 97 | 90 | 65 |

UTILITIES

Strategy 1: Always cram
Strategy 2: Always do HW
Strategy 3: Always play games
Strategy 4: ½ cram, ½ do HW
…

Mixed Strategy: randomized selection

# Calculating Utilities of Pure Strategies

- What is the utility of pure strategy: CRAM?

$$u(\text{CRAM}) = \text{P}(\text{Easy}) \cdot u(\text{CRAM}, \text{Easy}) + \text{P}(\text{Hard}) \cdot u(\text{CRAM}, \text{Hard})$$

- General formula:

$$u(\pi) = \sum_{s} \text{P}(s) \cdot u(\pi, s)$$

| CRAM | DO HW | PLAY GAME |
|---|---|---|
| 98 | 100 | 85 |
| 97 | 90 | 65 |

$\text{P}(\text{Easy}) = .2$

$\text{P}(\text{Hard}) = .8$

# Calculating Utilities of Pure Strategies

- What is the utility of pure strategy: DO HW?
- What is the utility of pure strategy: PLAY GAME?

| CRAM | DO HW | PLAY GAME |
|---|---|---|
| 98 | 100 | 85 |
| 97 | 90 | 65 |

P(Easy) = .2
P(Hard) = .8

# Calculating Utilities of Mixed Strategies

- What is the utility of mixed strategy: $\sigma$ = (½ CRAM, ½ DO HW)?

$$u(\sigma) = \mathrm{P}_\sigma(\mathrm{CRAM})\Big(\sum_s \mathrm{P}(s)u(\mathrm{CRAM}, s)\Big) + \mathrm{P}_\sigma(\mathrm{DO\ HW})\Big(\sum_s \mathrm{P}(s)u(\mathrm{DO\ HW}, s)\Big)$$

- General formula:

$$u(\sigma) = \sum_k \sum_s \mathrm{P}(s)\mathrm{P}_\sigma(\pi_k)u(\pi_k, s) = \sum_k \mathrm{P}_\sigma(\pi_k) \sum_s \mathrm{P}(s)u(\pi_k, s)$$

| CRAM | DO HW | PLAY GAME |
|---|---|---|
| 98 | 100 | 85 |
| 97 | 90 | 65 |

P(Easy) = .2
P(Hard) = .8

# Quiz: Grocery Shopping Transportation Decision

Suppose you want to decide how to get groceries from the store

| | BIKE | WALK | BUS | DRIVE |
|---|---|---|---|---|
| SUN | 1 | 2 | 1 | 1 |
| RAIN | -2 | -4 | -1 | 0 |

1. How many pure strategies to do you have?

   A) 1 B) 2 C) 3 D) 4 E) Infinite

2. How many mixed strategies do you have?

   A) 4 B) 8 C) 16 D) 64 E) Infinite

3. What is your best pure strategy?

   A) Bike B) Walk C) Bus D) Drive E) It depends

# Quiz: Grocery Shopping Transportation Decision

Suppose you want to decide how to get groceries from the store

| | BIKE | WALK | BUS | DRIVE | |
|---|---|---|---|---|---|
| SUN | 1 | 2 | 1 | 1 | P=.5 |
| RAIN | -2 | -4 | -1 | 0 | P=.5 |

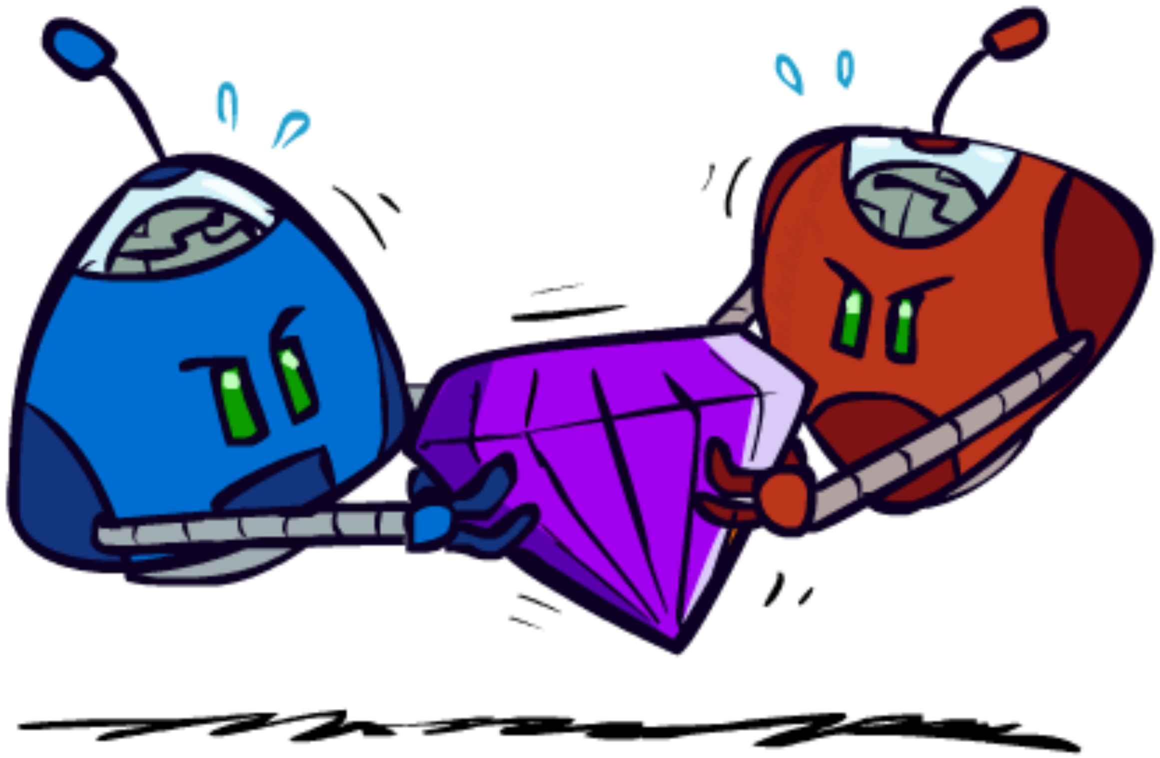
4. What is your best pure strategy?

   A) Bike B) Walk C) Bus D) Drive E) It depends

5. What is the utility of a ¼ walk, ¼ bike, and ½ drive strategy?

   A) -1/8 B) -1/4 C) -1/2 D) 1/8 E) 1/2

# Game Theory



# Game: Rock, Paper, Scissors

- Each player simultaneously picks rock, paper, or scissors
- Rock beats scissors, scissors beats paper, paper beats rock

P1's Strategies
$\Pi_1 = \{\text{rock, paper, scissors}\}$

P2's Strategies
$\Pi_2 = \{\text{rock, paper, scissors}\}$

# Joint Utilities

- When both players choose their actions, they receive a utility based on both of their choices

P2's ACTIONS

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK | 0,0 | -1,1 | 1,-1 |
| | PAPER | 1,-1 | 0,0 | -1,1 |
| | SCISSORS | -1,1 | 1,-1 | 0,0 |

JOINT UTILITIES

# Normal Form Notation

- Players: $\{1, \ldots, N\}$
- Pure strategies for each player $i$
  - $\pi_{i,1}, \ldots, \pi_{i,n_i}$
- Utility functions that maps a strategy per player to a reward for player $i$
  - $u_i(\pi_1, \ldots, \pi_N) = u_i(\vec{\pi})$
- Strategy profile:
  - $\vec{\pi} = (\pi_1, \pi_2, \ldots, \pi_N) \quad \vec{\pi}_{-i} = (\pi_1, \ldots, \pi_{i-1}, \pi_{i+1}, \ldots, \pi_N)$

P2's ACTIONS

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK | 0,0 | -1,1 | 1,-1 |
| | PAPER | 1,-1 | 0,0 | -1,1 |
| | SCISSORS | -1,1 | 1,-1 | 0,0 |

JOINT UTILITIES

# Solution Concepts

- Solution concept
  - Subset of outcomes of the games that are possibly interesting
  - Generally assumes that players are rational
- Nash equilibrium (NE)
  - With pure strategies vs mixed strategies
  - Weak vs strict NE
- Pareto-optimal solutions
- Correlated equilibrium

# Strategies for Games

- Best response against $\overrightarrow{\pi}_{-i}$
  - Strategy for player *i* that maximizes her utility given the strategy of the other players

Pure Strategies:

P2 always picks rock
P1 should ___________

P2 always picks paper
P1 should ___________

P2's ACTIONS

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK | 0,0 | -1,1 | 1,-1 |
| | PAPER | 1,-1 | 0,0 | -1,1 |
| | SCISSORS | -1,1 | 1,-1 | 0,0 |

JOINT UTILITIES

# Strategies for Games

- Best response against $\overrightarrow{\pi}_{-i}$
  - Strategy for player *i* that maximizes her utility given the strategy of the other players of the other players

Mixed Strategies:

P2 randomly chooses between 50% rock and 50% paper
P1 should ___________

P2's ACTIONS

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK | 0,0 | -1,1 | 1,-1 |
| | PAPER | 1,-1 | 0,0 | -1,1 |
| | SCISSORS | -1,1 | 1,-1 | 0,0 |

JOINT UTILITIES

# Zero-Sum Games

- If each cell in the table sums to 0, the game is zero-sum:

$$\forall \vec{\pi}, \sum_i u_i(\vec{\pi}) = 0$$

- Is Rock, Paper, Scissors zero-sum?
- Is Tic Tac Toe zero-sum?

P2's ACTIONS

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK | 0,0 | -1,1 | 1,-1 |
| | PAPER | 1,-1 | 0,0 | -1,1 |
| | SCISSORS | -1,1 | 1,-1 | 0,0 |

JOINT UTILITIES

# Dominant Strategies

- A strategy $\pi_{i,k}$ for player $i$ is strictly dominant if it is better than all other strategies for player $i$ no matter any opponent's strategy:
  - $\forall k' \neq k, u_i\left(\pi_{i,k}, \overrightarrow{\pi}_{-i}\right) > u_i\left(\pi_{i,k'}, \overrightarrow{\pi}_{-i}\right)$

| | A | B | C | D | E |
|---|---|---|---|---|---|
| i | 2,10 | 4,7 | 4,6 | 5,2 | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | 1,3 | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | 4,1 | 3,0 |
| iv | 6,7 | 9,5 | 7,5 | 8,5 | 5,5 |

# Dominant Strategies

- A strategy $\pi_{i,k}$ for player $i$ is weakly dominant if it is better than all other strategies for player $i$ no matter any opponent's strategy
  - $\forall k' \neq k, u_i\left(\pi_{i,k}, \overrightarrow{\pi}_{-i}\right) \geq u_i\left(\pi_{i,k'}, \overrightarrow{\pi}_{-i}\right)$

| | A | B | C | D | E |
|---|---|---|---|---|---|
| i | 2,10 | 4,7 | 4,6 | 5,2 | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | 1,3 | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | 4,1 | 3,0 |
| iv | 6,7 | 9,5 | 7,5 | 8,5 | 5,5 |

# Dominant Strategies

- For player Column, strategy A's utilities are the highest compared to B,C,D,E for all of player Row's strategies
  - $\forall \pi_r \in \{i, ii, iii, iv\}, \forall \pi_c \neq A, u_c(\pi_r, A) > u_c(\pi_r, \pi_c)$
- Column should always play A!

| | A | B | C | D | E |
|---|---|---|---|---|---|
| i | 2,10 | 4,7 | 4,6 | 5,2 | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | 1,3 | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | 4,1 | 3,0 |
| iv | 6,7 | 9,5 | 7,5 | 8,5 | 5,5 |

# Dominant Strategies

- For player Row, strategy iv's utilities are the highest compared to i,ii,iii for all of player Column's strategies
  - $\forall \pi_c \in \{A, B, C, D, E\}, \forall \pi_r \neq iv, u_r(iv, \pi_c) > u_r(\pi_r, \pi_c)$
- Row should always play iv!

| | A | B | C | D | E |
|---|---|---|---|---|---|
| i | 2,10 | 4,7 | 4,6 | 5,2 | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | 1,3 | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | 4,1 | 3,0 |
| iv | 6,7 | 9,5 | 7,5 | 8,5 | 5,5 |

# Is there always a dominant strategy?

- No! There is no dominant strategy in Tic Tac Toe, for example.

P2's ACTIONS

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAY ER 1 | ROCK | 0,0 | -1,1 | 1,-1 |
| | PAPER | 1,-1 | 0,0 | -1,1 |
| | SCISSORS | -1,1 | 1,-1 | 0,0 |

JOINT UTILITIES

# Prisoner's Dilemma

- 2 Players {1,2}
- Each as 2 strategies {Cooperate,Defect}
- Utilities in table:

| | | PRISONER 2 | |
|---|---|---|---|
| | | Cooperate | Defect |
| PRISONER 1 | Cooperate | -1,-1 | -6,0 |
| | Defect | 0,-6 | -3,-3 |

- Is there a dominant strategy?
  - Yes!
- What is the best joint strategy for both prisoners?
  - Best joint strategy: prisoners cooperate

# Measure of Social Welfare

- The sum of the utilities of the players is the social welfare

- SW(C,C) = -2
- SW(C,D) = -6
- SW(D,D) = -6

<table>
<tr><td colspan="2"></td><td colspan="2">PRISONER 2</td></tr>
<tr><td colspan="2"></td><td>Cooperate</td><td>Defect</td></tr>
<tr><td rowspan="2">PRISONER 1</td><td>Cooperate</td><td>-1,-1</td><td>-6,0</td></tr>
<tr><td>Defect</td><td>0,-6</td><td>-3,-3</td></tr>
</table>

# Prisoner’s Dilemma

- Compute best responses

| | | PRISONER 2 | |
|---|---|---|---|
| | | Cooperate | Defect |
| PRISONER 1 | Cooperate | -1,-1 | -6,0 |
| | Defect | 0,-6 | -3,-3 |

# Prisoner's Dilemma

- Strategy profile (C, C) is not stable
- Each prisoner would profit by switching to defection assuming that the other prisoner continues to cooperate

| | | PRISONER 2 | |
|---|---|---|---|
| | | Cooperate | Defect |
| PRISONER 1 | Cooperate | -1,-1 | -6,0 |
| | Defect | 0,-6 | -3,-3 |

# Prisoner's Dilemma

- If they both trust that the other prisoner will cooperate, each should defect. But both defecting results in lower scores!

<table>
<tr><td></td><td></td><td colspan="2">PRISONER 2</td></tr>
<tr><td></td><td></td><td>Cooperate</td><td>Defect</td></tr>
<tr><td rowspan="2">PRISONER 1</td><td>Cooperate</td><td>-1,-1</td><td>-6,0</td></tr>
<tr><td>Defect</td><td>0,-6</td><td>-3,-3</td></tr>
</table>

# Nash Equilibrium

- Nash Equilibria: strategy profiles $\overrightarrow{\pi}$ where none of the participants benefit from unilaterally changing their decisions:

$$\forall i, u_i(\overrightarrow{\pi}) \geq u_i(\pi_i', \overrightarrow{\pi}_{-i})$$

| | | PRISONER 2 | |
|---|---|---|---|
| | | Cooperate | Defect |
| PRISONER 1 | Cooperate | -1,-1 | -6,0 |
| | Defect | 0,-6 | -3,-3 |

# Nash Equilibrium

- NOT A NASH EQUILIBRIUM - participants benefit from unilaterally changing their decision

| | | PRISONER 2 | |
|---|---|---|---|
| | | Cooperate | Defect |
| PRISONER 1 | Cooperate | -1,-1 | -6,0 |
| | Defect | 0,-6 | -3,-3 |

# Nash Equilibrium

- Strict Nash Equilibria are Nash Equilibria where the "neighbor" strategy profiles have strictly less utility.
  - $\forall i, u_i(\overrightarrow{\pi}) > u_i(\pi_i', \overrightarrow{\pi}_{-i})$

| | | PRISONER 2 | |
|---|---|---|---|
| | | Cooperate | Defect |
| PRISONER 1 | Cooperate | -1,-1 | -6,0 |
| | Defect | 0,-6 | -3,-3 |

# Quiz: Professor's Dilemma!

- What is / are the Nash equilibrium / equilibria?
- Which are strict Nash equilibria?

| | | Student | |
|---|---|---|---|
| | | Study | Games |
| Professor | Effort | 1000,1000 | 0,-10 |
| | Slack | -10,0 | 0,0 |

# Finding a Pure Nash Equilibrium

## Pure Nash Equilibria are composed of pure strategies

- Option 1: Examine each state and determine if it fits the criteria
- Option 2: Find a dominating strategy and eliminate all other row or columns and recurse
- Option 3: Remove a strictly dominated strategy and recurse

# Finding a Pure Nash Equilibrium

- Option 1: Examine each state and determine if it fits the criteria
- Option 2: Find a dominating strategy and eliminate all other row or columns and recurse
- Option 3: Remove a strictly dominated strategy and recurse

| | L | C | R |
|---|---|---|---|
| U | 10,3 | 1,5 | 5,4 |
| M | 3,1 | 2,4 | 5,2 |
| D | 0,10 | 1,8 | 7,0 |

→

| | L | C |
|---|---|---|
| U | 10,3 | 1,5 |
| M | 3,1 | 2,4 |
| D | 0,10 | 1,8 |

→

| | L | C |
|---|---|---|
| U | 10,3 | 1,5 |
| M | 3,1 | 2,4 |

→

| | C |
|---|---|
| U | 1,5 |
| M | 2,4 |

→

| | C |
|---|---|
| M | 2,4 |

# Finding Nash Equilibrium Example 1

| | A | B | C | D | E |
|---|---|---|---|---|---|
| i | 2,10 | 4,7 | 4,6 | 5,2 | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | 1,3 | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | 4,1 | 3,0 |
| iv | 6,7 | 9,5 | 7,5 | 8,5 | 4,5 |

# Finding Nash Equilibrium Example 1

| | A | B | C | D | E |
|---|---|---|---|---|---|
| i | 2,10 | 4,7 | 4,6 | 5,2 | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | 1,3 | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | 4,1 | 3,0 |
| iv | 6,7 | 9,5 | 7,5 | 8,5 | 4,5 |

# Finding Nash Equilibrium Example 2

| | A | B | C | D | E |
|---|---|---|---|---|---|
| i | 2,**4** | 4,7 | 4,6 | 5,2 | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | 1,3 | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | 9,1 | 3,0 |
| iv | 6,7 | 9,5 | **5**,5 | 8,5 | 4,5 |

No longer strict dominant strategies!

# Finding Nash Equilibrium Example 2

| | A | B | C | D | E |
|---|---|---|---|---|---|
| i | 2,4 | 4,7 | 4,6 | 5,2 | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | 1,3 | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | 9,1 | 3,0 |
| iv | 6,7 | 9,5 | 5,5 | 8,5 | 4,5 |

>

D is strictly dominated by A

# Finding Nash Equilibrium Example 2

| | A | B | C | D | E |
|---|---|---|---|---|---|
| i | 2,4 | 4,7 | 4,6 | 5,2 | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | 1,3 | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | 9,1 | 3,0 |
| iv | 6,7 | 9,5 | 5,5 | 8,5 | 4,5 |

D is weakly dominated by B

# Finding Nash Equilibrium Example 2

| | A | B | C | E |
|---|---|---|---|---|
| i | 2,4 | 4,7 | 4,6 | 3,8 |
| ii | 3,8 | 6,4 | 5,2 | 2,6 |
| iii | 5,3 | 3,1 | 2,2 | 3,0 |
| iv | 6,7 | 9,5 | 5,5 | 4,5 |

# Finding Nash Equilibrium Example 2

| | A | B | C | | E | |
|---|---|---|---|---|---|---|
| i | 2,4 | 4,7 | 4,6 | | 3,8 | |
| ii | 3,8 | 6,4 | 5,2 | | 2,6 | |
| iii | 5,3 | 3,1 | 2,2 | | 3,0 | < |
| iv | 6,7 | 9,5 | 5,5 | | 4,5 | |

iii is strictly dominated by iv

# Finding Nash Equilibrium Example 2

| | A | B | C | E | |
|---|---|---|---|---|---|
| i | 2,4 | 4,7 | 4,6 | 3,8 | |
| ii | 3,8 | 6,4 | 5,2 | 2,6 | |
| iii | 5,3 | 3,1 | 2,2 | 3,0 | < |
| iv | 6,7 | 9,5 | 5,5 | 4,5 | |

i is strictly dominated by iv

# Finding Nash Equilibrium Example 2

| | A | B | C | E |
|---|---|---|---|---|
| | | | | |
| ii | 3,8 | 6,4 | 5,2 | 2,6 |
| | | | | |
| iv | 6,7 | 9,5 | 4,5 | 4,5 |

# Finding Nash Equilibrium Example 2

| | A | B | C | | E |
|---|---|---|---|---|---|
| | | | | | |
| ii | 3,8 | 6,4 | 5,2 | | 2,6 |
| | | | | | |
| iv | 6,7 | 9,5 | 4,5 | > | 4,5 |

E is strictly dominated by A

# Finding Nash Equilibrium Example 2

| | A | B | C | | E |
|---|---|---|---|---|---|
| | | | | | |
| ii | 3,8 | 6,4 | 5,2 | | 2,6 |
| | | | | | |
| iv | 6,7 | > 9,5 | 4,5 | | 4,5 |

C is strictly dominated by A

# Finding Nash Equilibrium Example 2

| | A | B | C | | E |
|---|---|---|---|---|---|
| | | | | | |
| ii | 3,8 | 6,4 | 5,2 | | 2,6 |
| | | | | | |
| iv | 6,7 | 9,5 | 4,5 | | 4,5 |

>

B is strictly dominated by A

# Finding Nash Equilibrium Example 2

| | A | | |
|---|---|---|---|
| | | | |
| ii | 3,8 | | |
| | | | |
| iv | 6,7 | | |

ii is strictly dominated by iv

# Finding Nash Equilibrium Example 2

| | A | | |
|---|---|---|---|
| | | | |
| | | | |
| | | | |
| iv | 6,7 | | |

# Finding Nash Equilibrium Example 3 (Battle of Sexes)

| | Opera | Football |
|---|---|---|
| Opera | (3, 2) | (0, 0) |
| Football | (0, 0) | (2, 3) |

# Finding Nash Equilibrium: Rock, Paper, Scissors

- Nash Equilibrium?
    - Not with pure strategies!

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK | 0,0 | -1,1 | 1,-1 |
| | PAPER | 1,-1 | 0,0 | -1,1 |
| | SCISSORS | -1,1 | 1,-1 | 0,0 |

# Nash Equilibria always exist in finite games

- Theorem (Nash, 1950)
  - If there are a finite number of players and each player has a finite number of actions, there always exists a Nash Equilibrium.
- The NE may be pure or it may be a mixed strategy.

# Calculating Utilities of Mixed Strategies

- Decision Theory Version:

$$u(\sigma) = \sum_k \sum_s \mathrm{P}(s)\mathrm{P}_\sigma(\pi_k)u(\pi_k, s)$$

- Game Theory Version:

$$u(\overrightarrow{\sigma}) = \sum_{\pi_1,\ldots,\pi_N} \prod_i \mathrm{P}_{\sigma_i}(\pi_i)u(\pi_1, \ldots, \pi_N)$$

# Example: Calculating Utilities

- What is $u_1$ for $\sigma_1 = (1/2,1/2,0)$ and $\sigma_2 = (0,1/2,1/2)$?
- Is $[\sigma_1 = (1/2,1/2,0),\ \sigma_2 = (0,1/2,1/2)]$ a mixed strategy equilibrium?

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK | 0,0 | -1,1 | 1,-1 |
| | PAPER | 1,-1 | 0,0 | -1,1 |
| | SCISSORS | -1,1 | 1,-1 | 0,0 |

# Quiz: Calculating Utilities

- What is $u_2$ for $\sigma_1 = (1/3, 1/3, 1/3)$ and $\sigma_2 = (1/3, 1/3, 1/3)$?

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK | 0,0 | -1,1 | 1,-1 |
| | PAPER | 1,-1 | 0,0 | -1,1 |
| | SCISSORS | -1,1 | 1,-1 | 0,0 |

# Finding the Mixed Strategy Nash Equilibrium

- What features of a mixed strategy qualify it as a NE?
- There is no reason for either player to deviate from their strategy, which occurs when the utilities of the weighted actions are equal!

# Finding the Mixed Strategy Nash Equilibrium

- P1
- P2

| | | PLAYER 2 | | |
|---|---|---|---|---|
| | | ROCK | PAPER | SCISSORS |
| PLAYER 1 | ROCK | 0,0 | -1,1 | 1,-1 |
| | PAPER | 1,-1 | 0,0 | -1,1 |
| | SCISSORS | -1,1 | 1,-1 | 0,0 |

# Another Mixed Strategy NE

- P1

- P2

| | | Receiver | |
|---|---|---|---|
| TENNIS | | F | B |
| Server | F | 90,10 | 20,80 |
| | B | 30,70 | 60,40 |

# Other Properties of Strategies

- Correlated Equilibrium

- Pareto Optimal / Dominated

# Correlated Equilibrium

- Suppose a mediator computes the best combined strategy $\sigma \in \Delta(\Pi_1 \times \Pi_2)$ for P1 and P2, and shares $\sigma_1$ with P1 and $\sigma_2$ with P2
- The strategy is a CE if $\forall \pi_1' \in \Pi_1$
- $$\sum_{\pi_2} \mathrm{P}_\sigma(\pi_1, \pi_2) u_1(\pi_1, \pi_2) \geq \sum_{\pi_2} \mathrm{P}_\sigma(\pi_1, \pi_2) u_1(\pi_1', \pi_2)$$
- And the same for s2.

# Game of Chicken

| | Dare | Chicken out |
|---|---|---|
| Dare | (0, 0) | (7, 2) |
| Chicken out | (2, 7) | (6, 6) |

# Pareto Optimal and Pareto Dominated

- An outcome $u(\overrightarrow{\sigma}) = \left(u_1(\overrightarrow{\sigma}), \ldots, u_1(\overrightarrow{\sigma})\right)$ is Pareto optimal if there is no other outcome that all players would prefer, i.e., each player gets higher utility
  - At least one player would be disappointed in changing strategy
- An outcome $u(\overrightarrow{\sigma}) = \left(u_1(\overrightarrow{\sigma}), \ldots, u_1(\overrightarrow{\sigma})\right)$ is Pareto dominated by another outcome if all the players would prefer the other outcome

# Summary

- Vocabulary
- Pure/Mixed Strategies (and calculating them)
- Zero-Sum Games
- Dominant vs Dominated Strategies
- Strict/Weak Nash Equilibrium
- Prisoner's dilemma
- Correlated Equilibrium
- Pareto Optimal/Dominated
- Social Welfare

# To Go Further

- Repeated games
- Stackelberg games
- Cooperative games
- Allocation, fairness
- Auctions
- Mechanism design