Conductors and electrostatics

From dipoles to image force: How does induction work?

Main questions related to conductors

- How do static charges distribute in a conductor?
- How does the electric field look like <u>inside</u> and <u>outside</u> a conductor? Does the shape matter?
- What is the potential inside a conductor?
- What impact on an external field pattern when a conductor is placed in that field?

Conductor

- In a conductor the charges are free to move if subject to a force: <u>very high conductivity</u>

- **<u>When isolated</u>** ⇨ charge must be conserved: <u>local</u> as well as <u>overall</u> neutrality prevail

  - If charges are <u>added</u>, they redistribute minimizing the total energy but they remain in the conductor

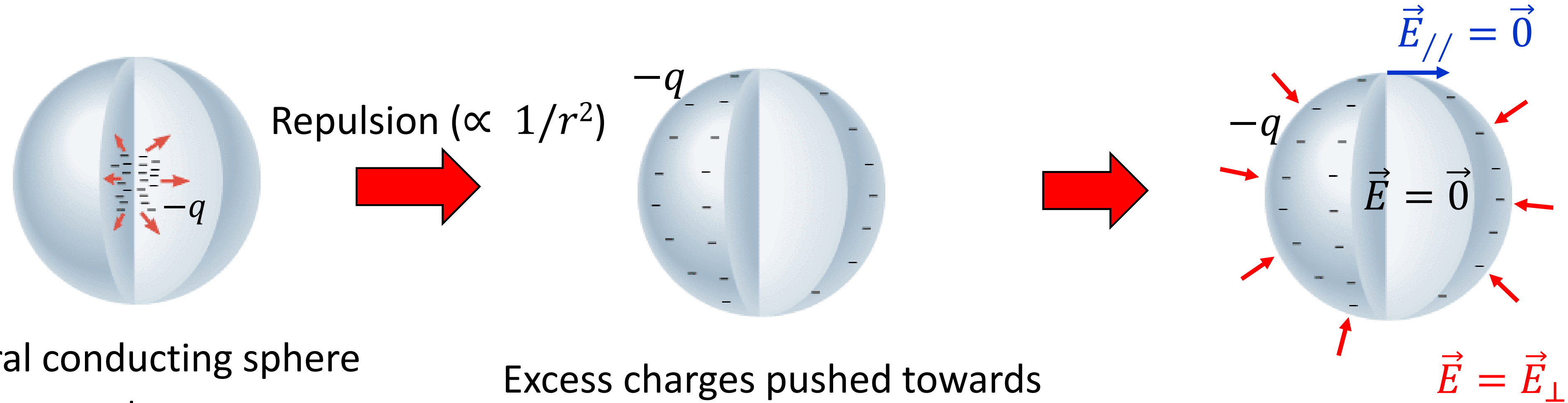
  - If charges are <u>induced</u>, they redistribute: **overall** neutrality prevails **BUT NOT locally** and once the field is removed the conductor retrieves its initial condition

Consequence of coulomb ($\propto\ 1/r^2$ law)

Charging a conductor

Isolated conductor: Charge conservation

Uniformly distributed **BECAUSE** of the spherical shape !

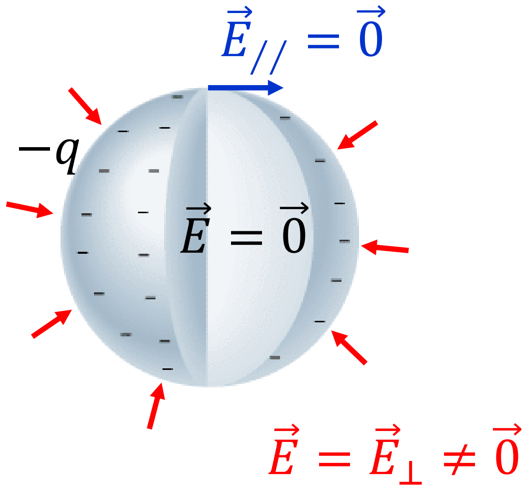


Neutral conducting sphere
+
Excess electrons added inside

Excess charges pushed towards surface (Coulomb repulsion) but cannot leave the conductor

→ **<u>Equilibrium</u>**

At <u>equilibrium</u> and <u>electrostatic conditions</u>, excess charges **MUST** reside on the surface of the conductor

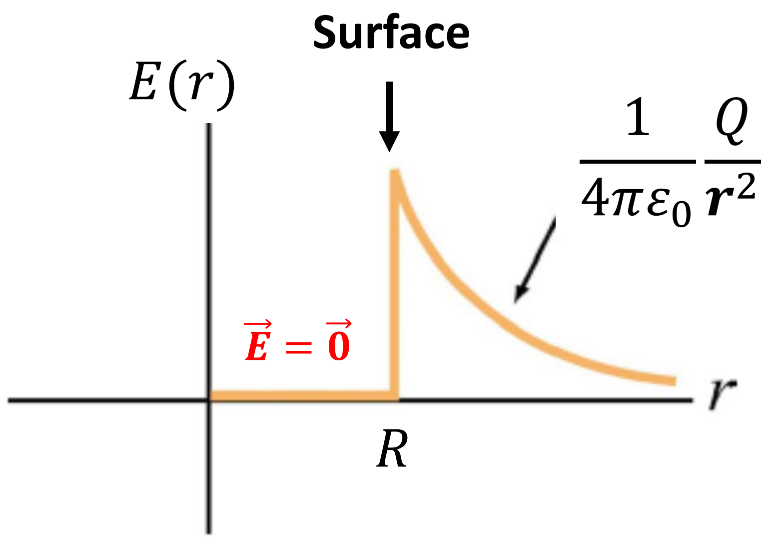


Why $\vec{E}_{//} = \vec{0}$?

Otherwise there would be a permanent current at the surface which is impossible

See the formal demonstration below

The hollow or solid conducting sphere is an example where:



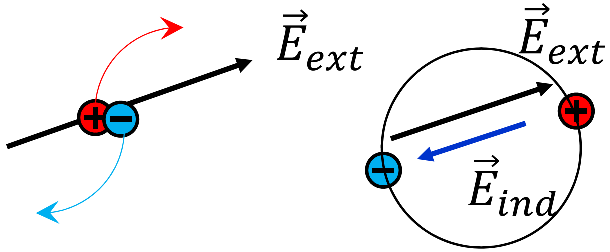
$$\oint \vec{E}.d\vec{A} = \int \vec{\nabla}.\vec{E}\, dV = 0$$

Closed surface inside the sphere

Volume enclosed by the surface

What does this mean?

Why is the electric field zero inside the conductor ?



$\vec{E}_{ext} + \vec{E}_{ind} = \vec{0}$ Charges accumulate at the surface

From Gradient

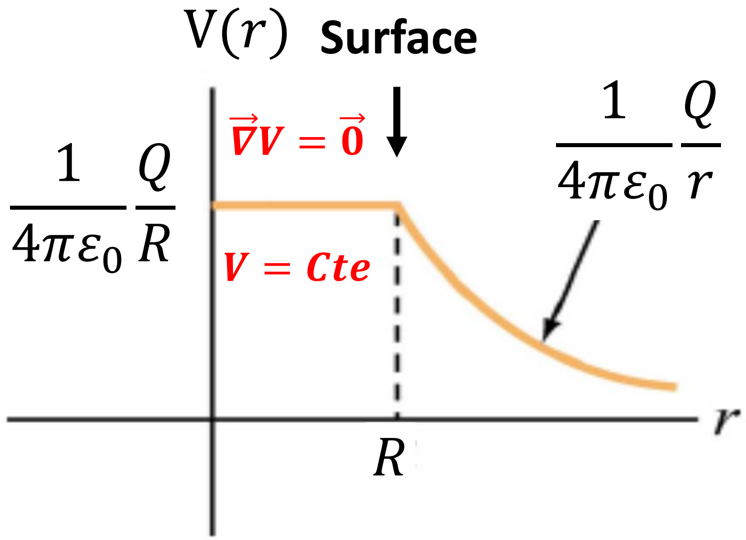
$\vec{E} = \vec{0}$ ⇨ $-\vec{\nabla}V = 0$ ⇨ $V = Cte$ ⇨ The whole conductor is an equipotential **Including the surface**

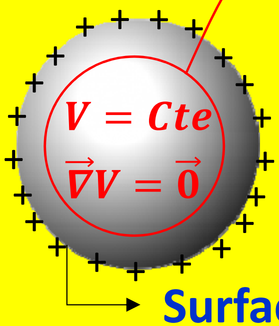
From Divergence

$\vec{E} = \vec{0}$ ⇨ $\vec{\nabla}.\vec{E} = 0$ ⇨ **No charge inside**

The hollow or solid conducting sphere is an example where:

$\Leftrightarrow$ **Volume** $\vec{E} = \vec{0}$, $\vec{\nabla}.\vec{E} = \vec{0}$





Surface $V = Cte$, $\vec{\nabla}V = \vec{0}$

$$dV = \vec{\nabla}V.d\vec{l} = 0$$

$$V = \int_a^b \vec{\nabla}V.d\vec{l} = -\int_a^b \vec{E}.d\vec{l} = 0$$

*Along any **open** or **closed** path on the surface*

What does this mean?

Why the electric field at the surface must be perpendicular?

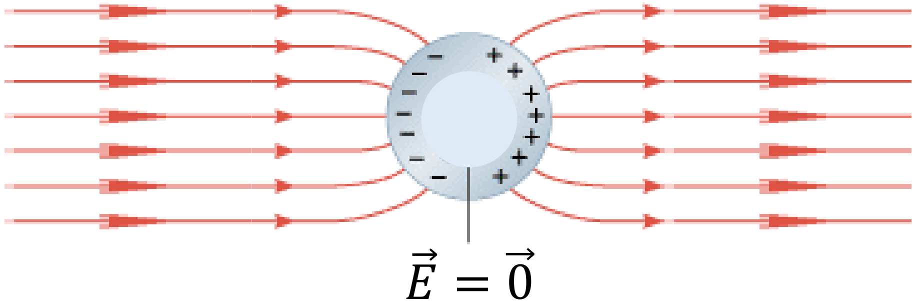
$$\int \vec{E}.d\vec{l} = \int -\vec{\nabla}V.d\vec{l} = 0$$

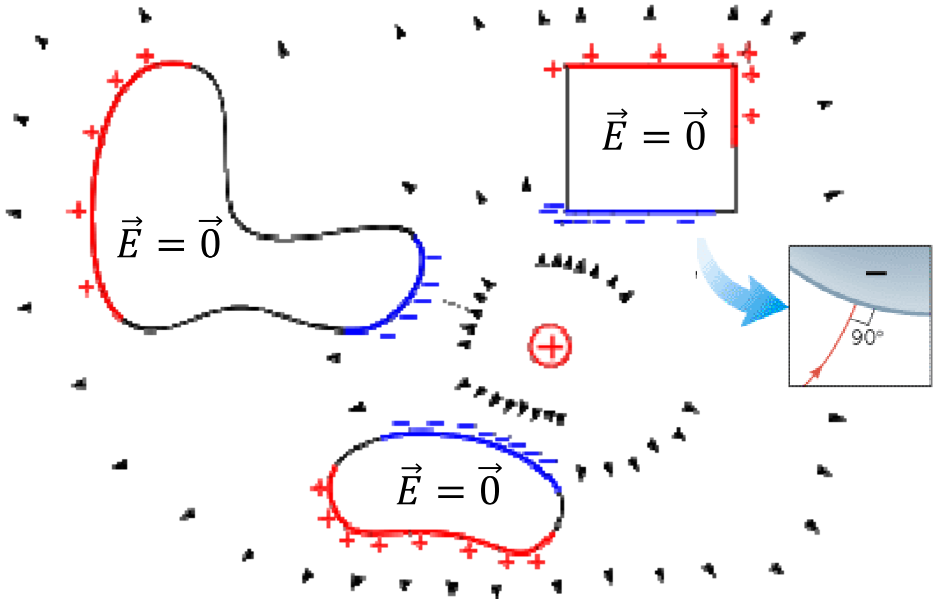
Path on the surface

$$\vec{E} \perp d\vec{l}$$

If not, there would be a current flowing on the surface, **and that surface would cease to be an equipotential**

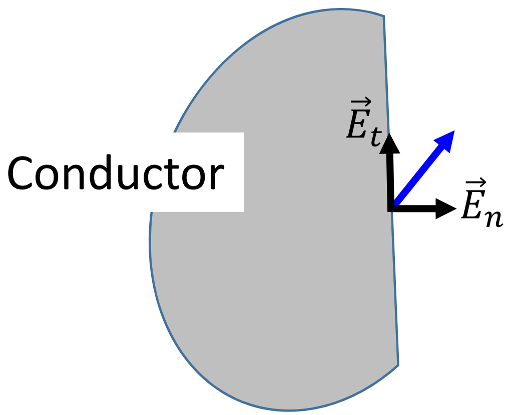
The external field lines MUST curve when approaching the conductor to hit it perpendicularly





Irrespective of the shape of the conductor

$\vec{E}_t$ cannot exist on the surface of a conductor: Interesting application

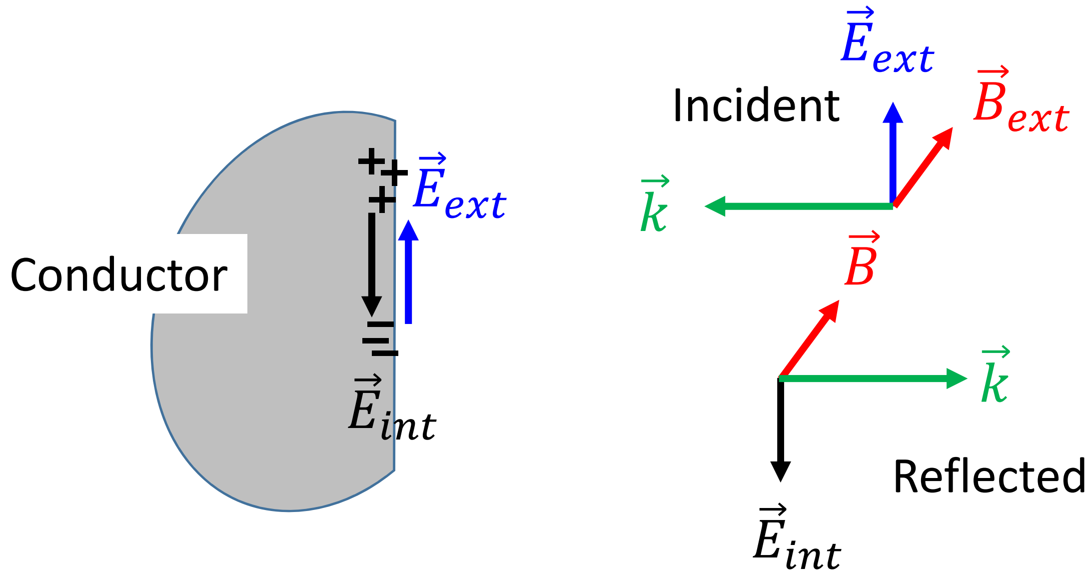


If $E_t \neq 0$ ⇨ charges will move along the surface
⇨ perpetual motion can be created with a charge outside !

***Question #1:

What does this property imply regarding the interaction of a conductor with Electromagnetic waves

Answer to Question #1: Conductors are very good reflectors

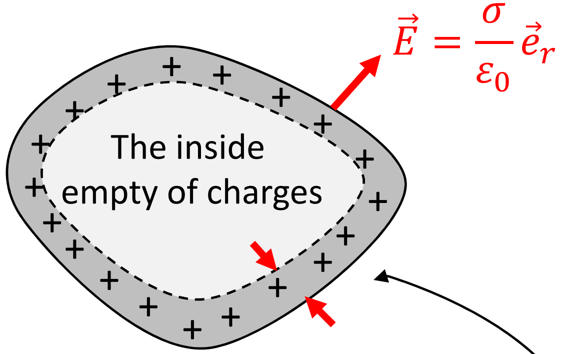


$$\vec{E}_{int} + \vec{E}_{ext} = \vec{0}$$

As the external field E_{ext} oscillates so does the internal field E_{int} and the conductor transforms into a source of Electromagnetic wave

Perfect reflector if the conductivity is infinite

Conductor and charge distribution



Excess charge in a conductor distribute on the surface with $\vec{E} = \vec{0}$ inside

How thick is the surface layer where all charges distribute?

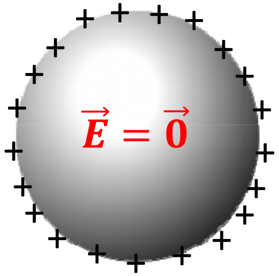
- $\sigma_C = 8.85 \times 10^{-12} \times 10^{6} = 10\mu C/m^2$ conductivity
- Charge of electron q $= 1.6 \times 10^{-19} C$
- Concentration of free electron in a metal N = $10^{28} m^3$
- $\sigma = \frac{nq}{A} = \frac{nqd}{Ad} = Nqd$ charge density
- $E = \frac{\sigma}{\varepsilon_0} = 10^5 V/m$

Thickness $\mathrm{d} = \frac{\sigma}{qN} = 10^{-14} m$ *size of a nucleus!*

Charges distribute really at the surface

That $\vec{E} = \vec{0}$ inside the conducting sphere is NOT trivial at all !

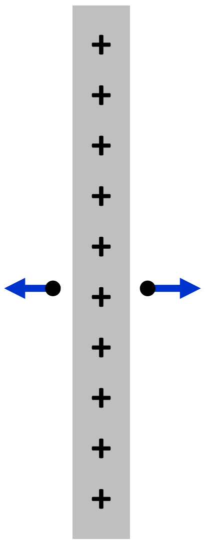
"<u>Conspiracy principle</u>"



Every charge on the sphere creates a field inside and outside the sphere **BUT** all individual effects when added cancel out inside.

Whereas

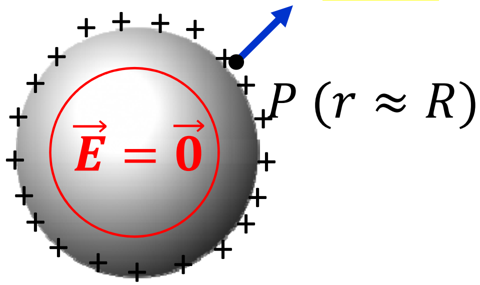
$$E = \frac{\sigma}{2\varepsilon_0} \qquad E = \frac{\sigma}{2\varepsilon_0}$$

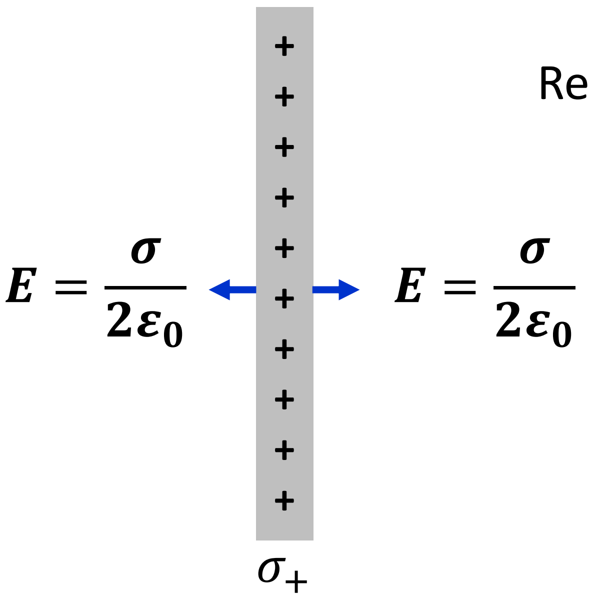


σ_+

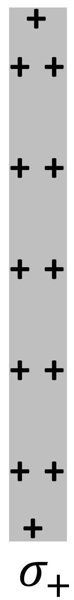
Consequence

$$\frac{1}{\varepsilon_0}\frac{Q}{4\pi r^2} \qquad E = \frac{\sigma}{\varepsilon_0}$$

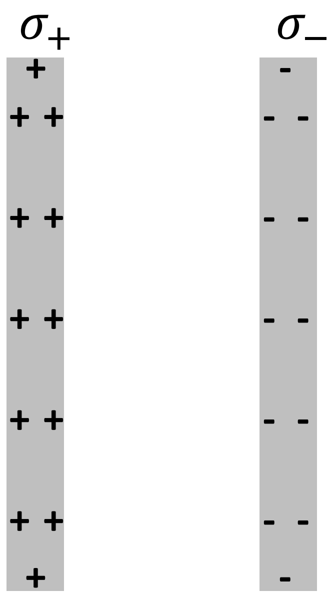




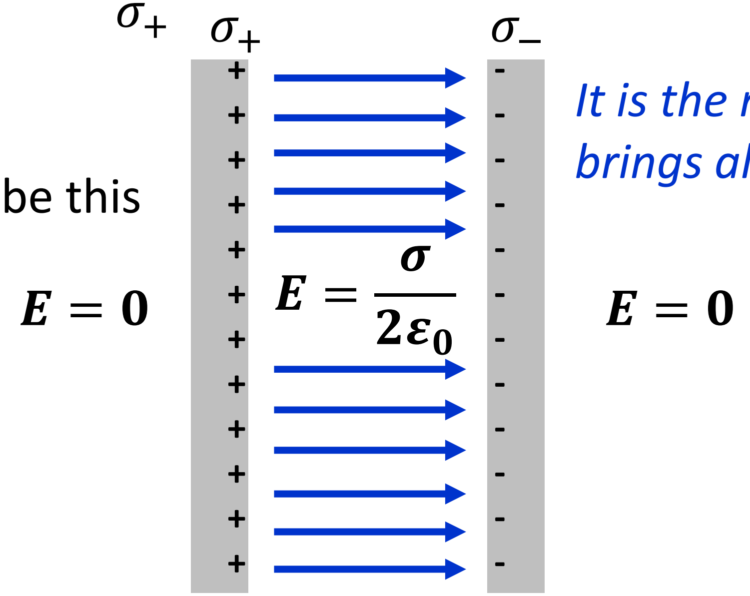
Real representation is this



Remember that a charge added to a conductor spreads all around the surface !



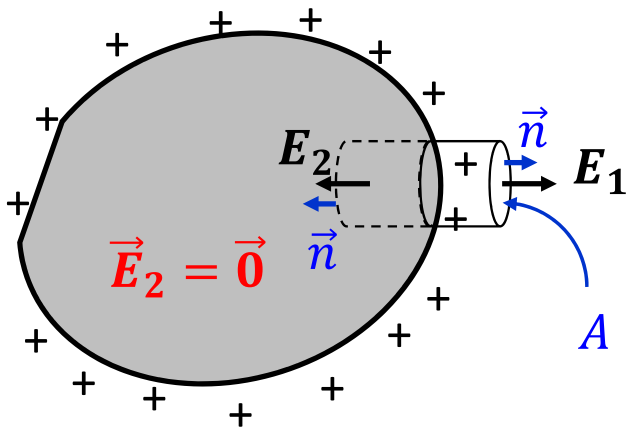
Turns out to be this



It is the mutual influence that brings all charges on one side!

$$\text{In both cases} \quad E_1 A + E_2 A = \frac{\sigma A}{\varepsilon_0}$$

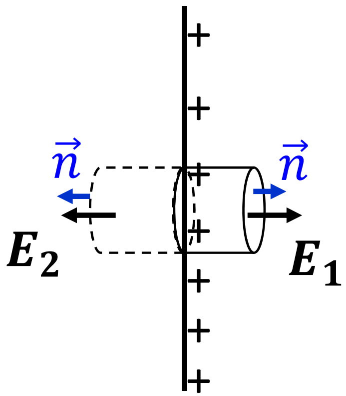
We evaluate the field in the immediate neighborhood of the conductor



All charges outside the pill box "conspire" to "kill" the field E_2

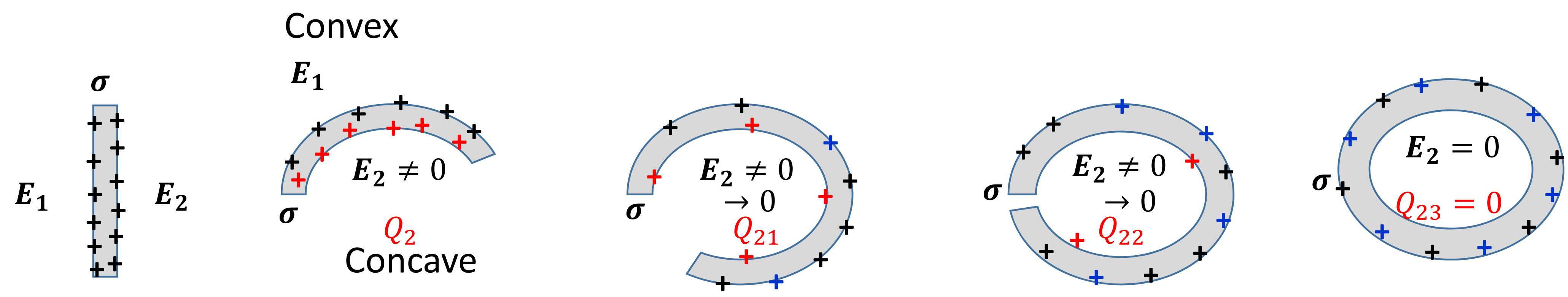
$$E_1 + E_2 = E_1 + 0 = \boxed{E = \frac{\sigma}{\varepsilon_0}}$$

"Conspiracy principle"



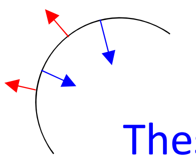
$$E_1 = E_2 = \boxed{E = \frac{\sigma}{2\varepsilon_0}}$$

“Conspiracy” principle

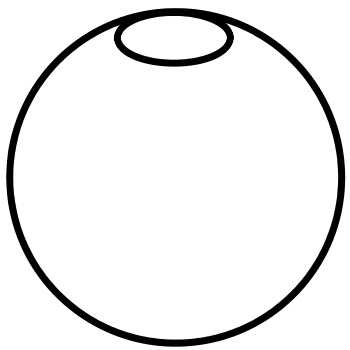


$E_1 = E_2 = \dfrac{\sigma}{2\varepsilon_0}$ $\quad$ $E_1 > E_2$ $\quad$ $E_1 > E_2$ $\quad$ $E_1 > E_2$ $\quad$ $E_1 = \dfrac{\sigma}{\varepsilon_0}$

These unit vectors diverge



These unit vectors converge

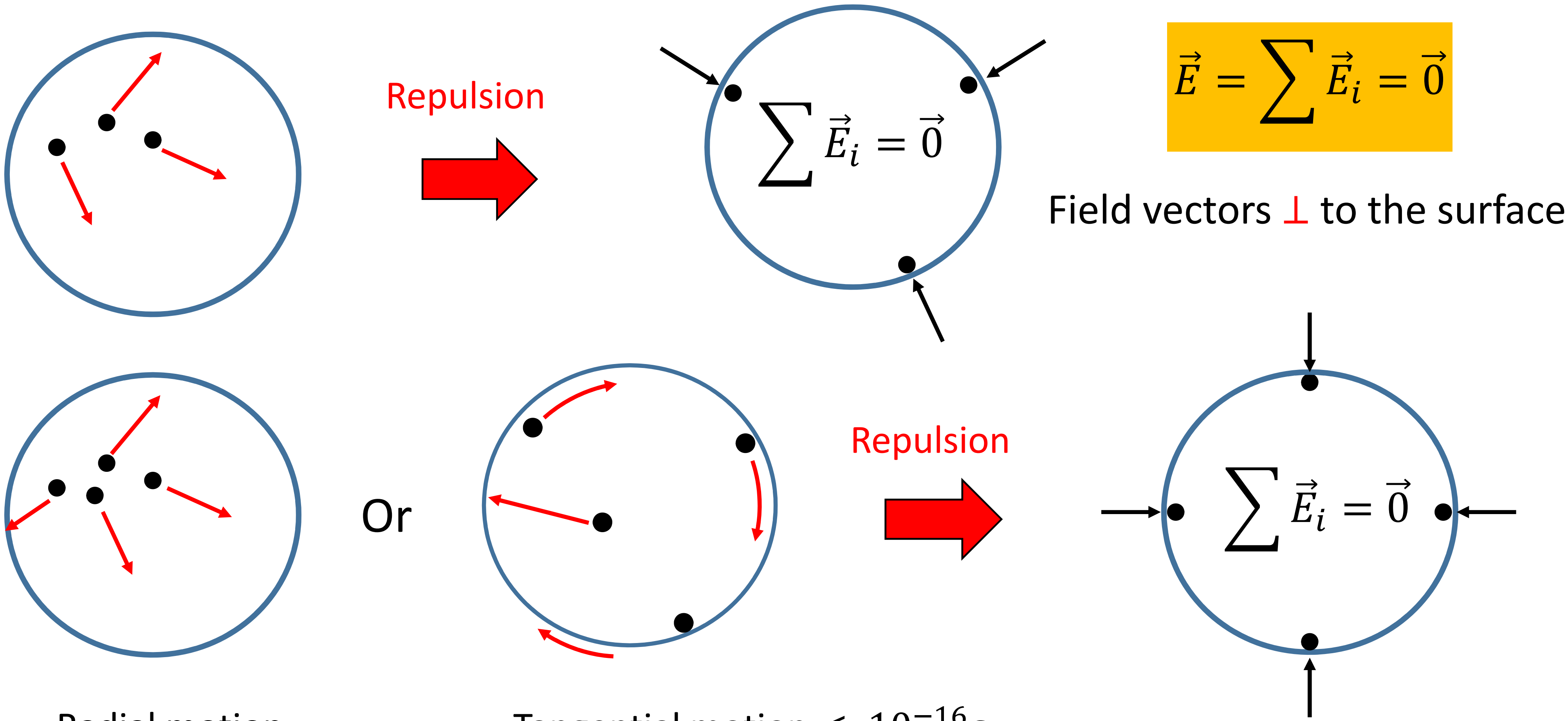


Remember the hollow sphere with an opening:
Field inside is not completely canceled

See E_Lectures 8&9_Electrostatics_Gauss law

Consider a 2D conducting disc

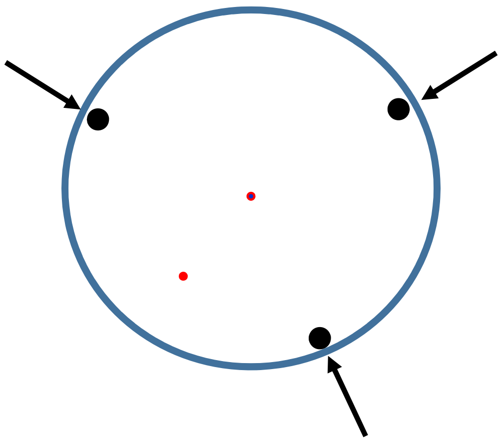
● Negative charge



Radial motion

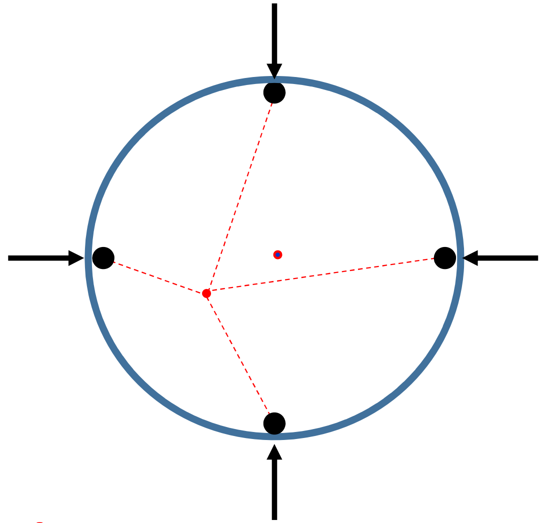
Tangential motion $< \ 10^{-16} s$

What about the potential in the conducting disc?



Inside the conducting disc

$$V = \sum V_i = \sum \frac{1}{4\pi\varepsilon_0}\frac{q}{r_i} =$$



See slide #20 ES_Lectures 8&9 Gauss law in Electrostatics

r_i is the distance from each charge to the point of consideration: In the case of the center

$$V = \frac{3q}{4\pi\varepsilon_0 R}$$

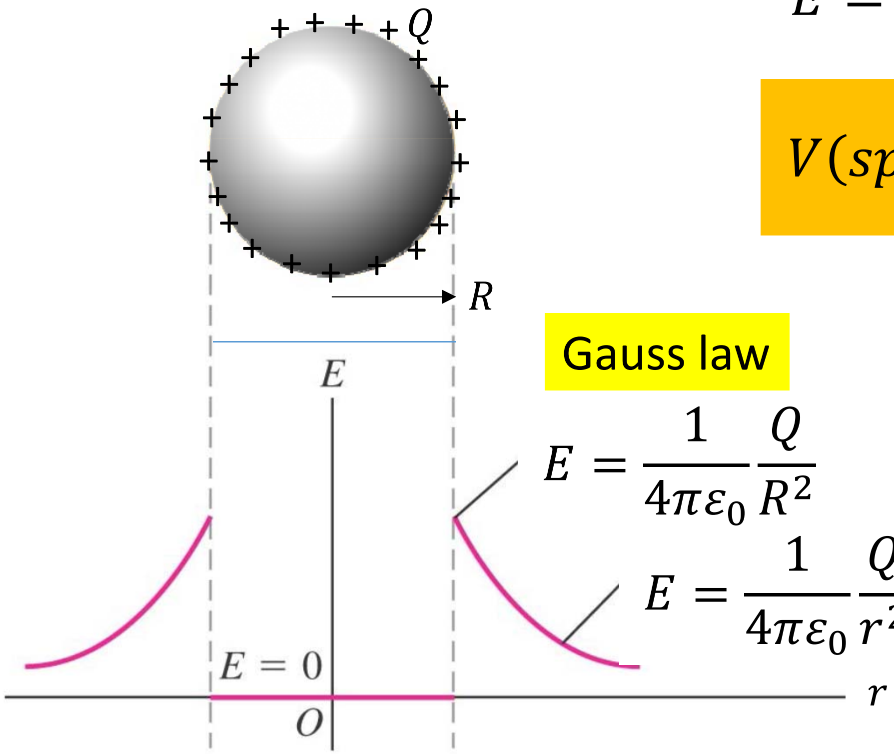
Everywhere inside the disc as $\sum \vec{E}_i = \vec{0}$ inside

$$V = \frac{4q}{4\pi\varepsilon_0 R}$$

The electric field and potential from the center of the sphere to infinity

$$\vec{E} = -\vec{\nabla}.V(r)$$

$$V(sphere) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R}$$

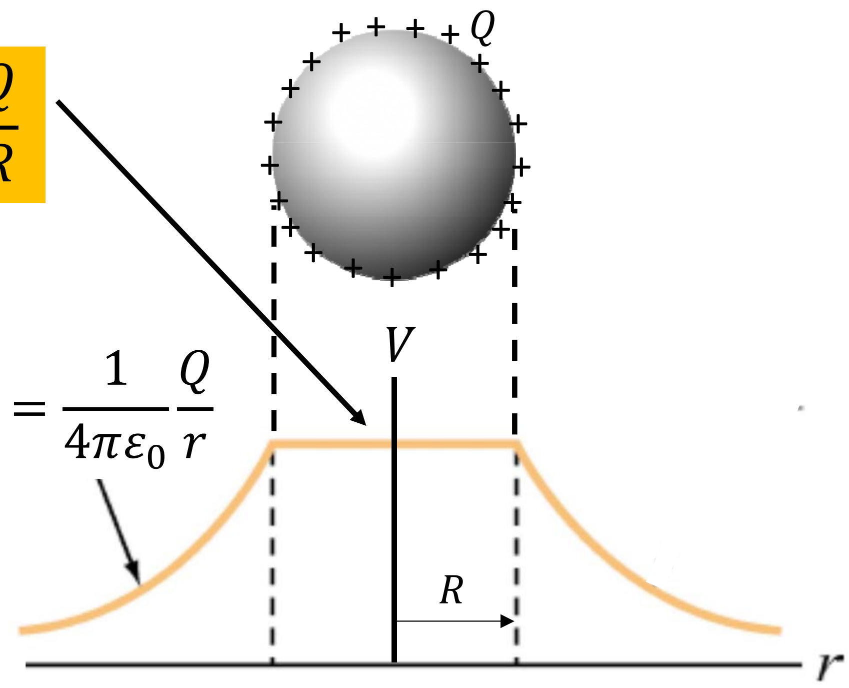


Gauss law

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2}$$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

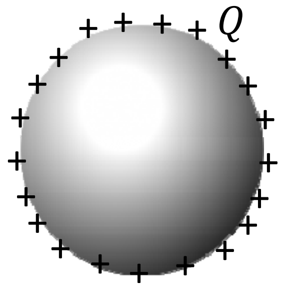
Discontinuity for E



$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

Continuity for V

The earth as an infinite sink for charges



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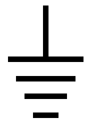
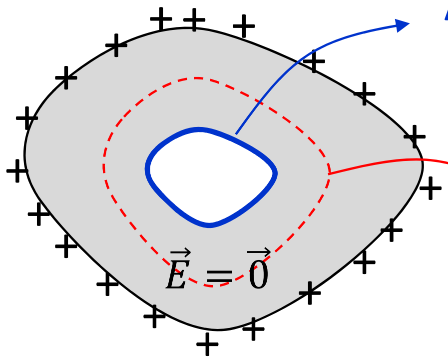


$$R = 6.4 \times 10^{3} m$$

$$V_G(earth) = 1.4 \times 10^{-12} Q \approx 0$$

$$Q = 10^{9} C \quad \rightarrow \quad \boldsymbol{V_G(earth) = 1.4 \times 10^{-3}\, V}$$

Conductor with a hollow and charges on the outer surface



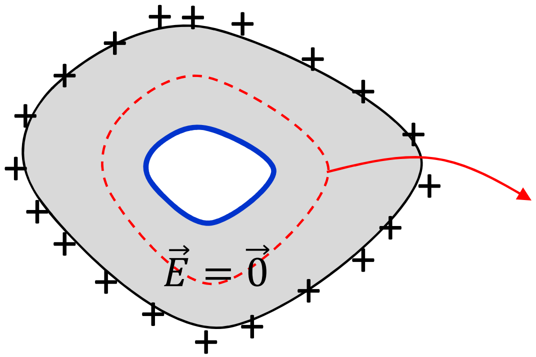
Are charges <u>induced</u> in the inner surface surrounding the hollow?

Gaussian surface

If **<u>no charge</u>** in the cavity AND because $\vec{E} = \vec{0}$ in the conductor

⇨ **no charge in the inner surface of the conductor**

How to prove it ?



Gaussian surface

$$\Phi = \int \vec{E}\, d\vec{A} = 0$$

Gaussian surface

Gauss law can only say that NET charge in the inner surface is zero

For a spherical shell

$$\Phi = EA = 0$$

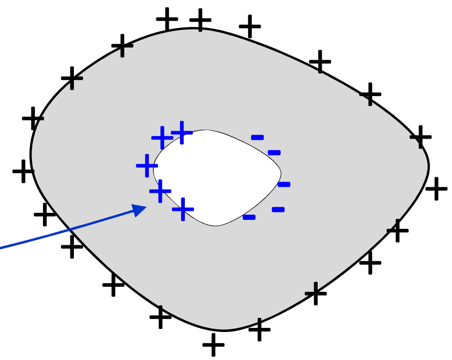
$$E = 0$$

No charge inside

Slide #20, E_Lectures 8&9
Gauss law in Electrostatics

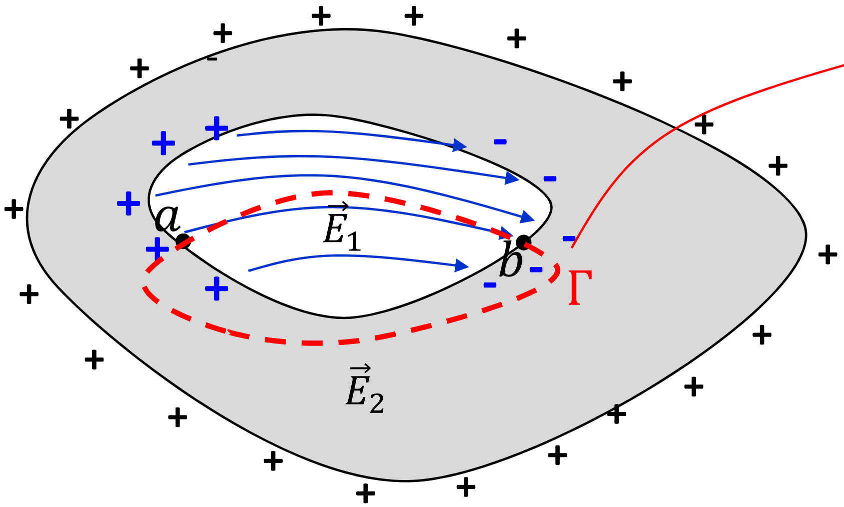
We cannot exclude this configuration
NET charge in the inner surface is zero

$$+Q - Q = 0$$



The most beautiful and elegant demonstration comes from Stoke's theorem

If charges are induced in the inner surface there must be field lines



Closed loop along a field line going from + to -

$$\boxed{\oint_\Gamma \vec{E}.d\vec{l} = ?}$$

Stoke's theorem

$$\oint_\Gamma \vec{E}.d\vec{l} = \iint_A (\vec{\nabla} \times \vec{E}).\vec{n}\, dA = \mathbf{0}$$

Because the field derives from a potential !

$\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \vec{\nabla} V = 0$

$$\oint \vec{E}.d\vec{l} = \int_a^b \vec{E}_1.d\vec{l} + \int_b^a \vec{E}_2.d\vec{l} = \mathbf{0}$$

$$\underset{?}{} \qquad \underset{\text{0}}{\shortparallel}$$

Cannot be 0 if there are field lines because there are charges!

Conductor

To make $\int_a^b \vec{E}_1 . d\vec{l} = 0$, we must clear up the field lines thus the charges

We actually do not need a formal demonstration. The reason is that any equal and opposite charges on the inner surface would slide around to meet each other, cancelling out completely

In an **EMPTY** cavity of a conductor there can be

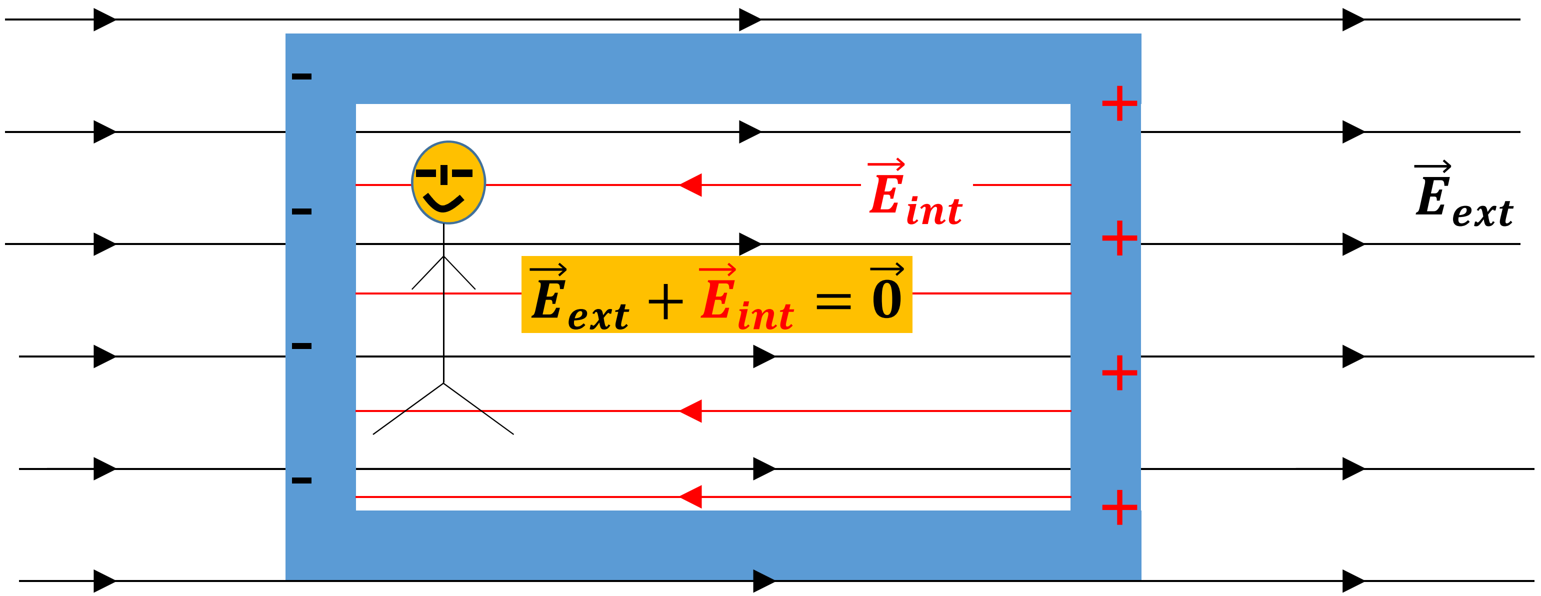
- **NO** charges in the inner surface thus **NO** field
- **NO** matter what is outside or on the outer surface

***Question 2: What can technology make out with this physical property?

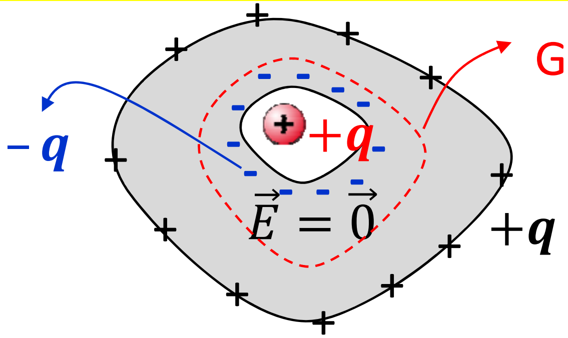
Answer to ***Question 2: Faraday cage (shielding) which prevents you from electric shocks

Shielding Electric Fields

A box or room made of metal liner can shield its interior from external electric fields. Mobile electrons in the metal will respond to the field and reorient themselves until the field inside the box no longer exists. The external field (black) points right. This causes a charge separation in the box (e^-'s migrating left), which produces its own field (red), negating the external field. Thus, the net field inside is zero. **Outside, the field persists.**



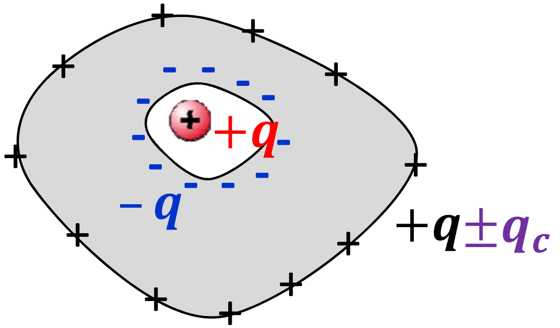
What about the opposite: Hollow conductor containing a charge in the cavity and **NO** charge or field outside



1) Gauss theorem AND because $\vec{E} = \vec{0}$ in the conductor

⇨ A negative charge MUST $-q$ builds up in the inner surface of the conductor

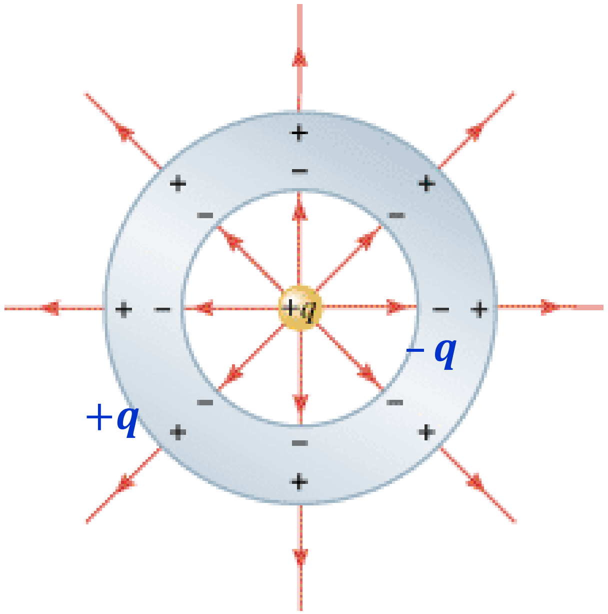
2) And because of the neutrality of the conductor positive charge $+q$ must build up at the outer surface



3) If a charge $\pm q_c$ is added on the outer surface, **NOTHING will change inside the hollow conductor containing charge** $+q$

Hollow conductor containing a charge in the cavity and **NO** charge or field outside

Case 1: charge at the center of the cavity

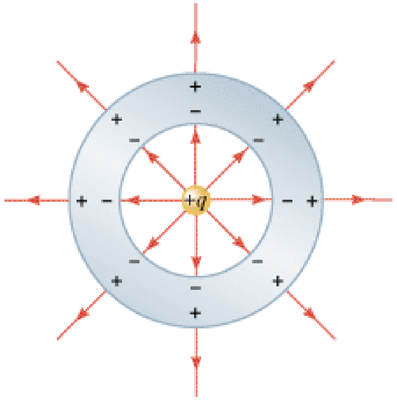


- The field lines emanate from positive charge $+\boldsymbol{q}$ and stop at the inner surface of the conductor (In the conductor $E = 0$).
- As they must stop at negative charges, a charge $-\boldsymbol{q}$ is induced in the inner surface of the cavity.
- Because the conductor is initially neutral, it must remain so. A positive charge $+q$ is then induced in the outer surface

It looks like the field outside is just and extension of the field inside, in spite of the discontinuity brought by the conductor

But nature is much subtle than that !

Spherical shell with a charge in the center of the cavity



- The field lines hit the inner surface of the conductor <u>perpendicularly</u>
- Outside they emerge <u>perpendicularly</u> from the outer to the surface

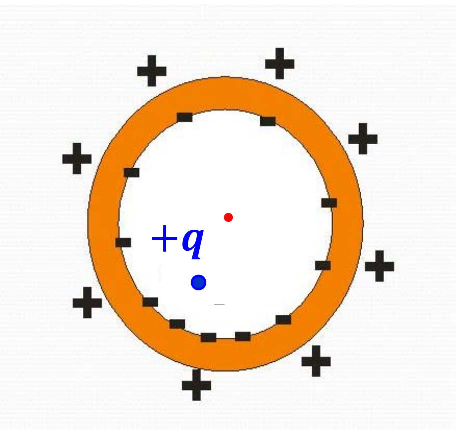
- The conducting shell introduces **discontinuity** in the field lines **BUT** does **NOT** <u>seem</u> to perturb the field pattern created by the charge inside the cavity

$+q$ in the center = $+q$ on the outer surface $\Rightarrow$ # fields lines in the cavity = # lines outside the sphere

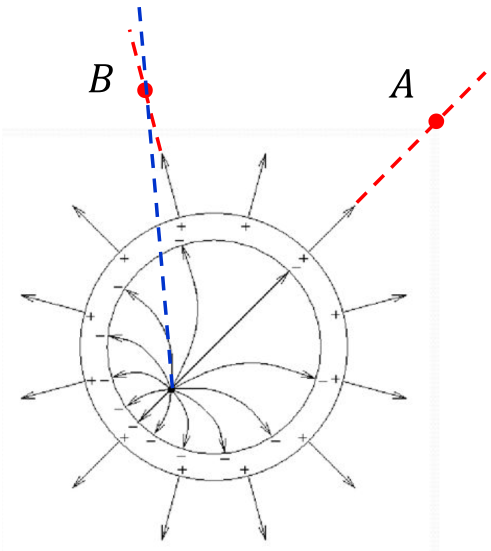
Do the fields inside the cavity and outside the sphere have something to do with each other?

Are they dependent or independent from each other?

What happens if the charge is moved off the center of the cavity?



- The field lines **MUST** hit the inner surface of the conductor perpendicularly
- The induced charges on the outer surface **MUST** distribute uniformly to minimize the energy
- And **MUST** emerge outside perpendicularly from the outer the surface

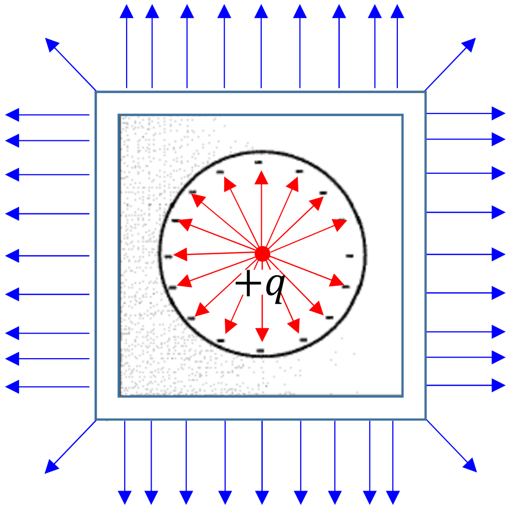


The outer field acting at points A and B is totally different from the inner field due to the charge inside the cavity

These two fields are completely independent from each other

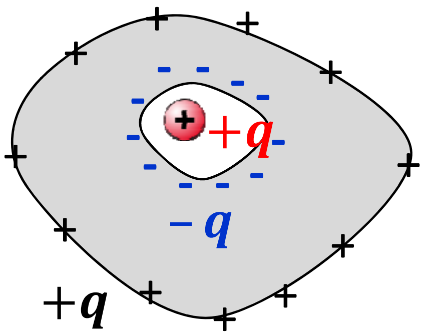
Inside the cavity the charge can be moved in any direction, outside the field stays unperturbed

What happens if we change the shape of the conductor: cubic shape?

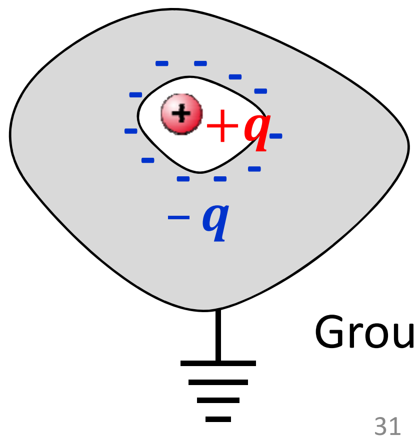


Here the field lines show a spherical symmetry when emanating from the charge at the **center** of the cavity

Outside the cube shaped conductor the field lines are parallel on each side, except at the corners where they show a different orientation and have greater density near the corners due to stronger bending of the surface



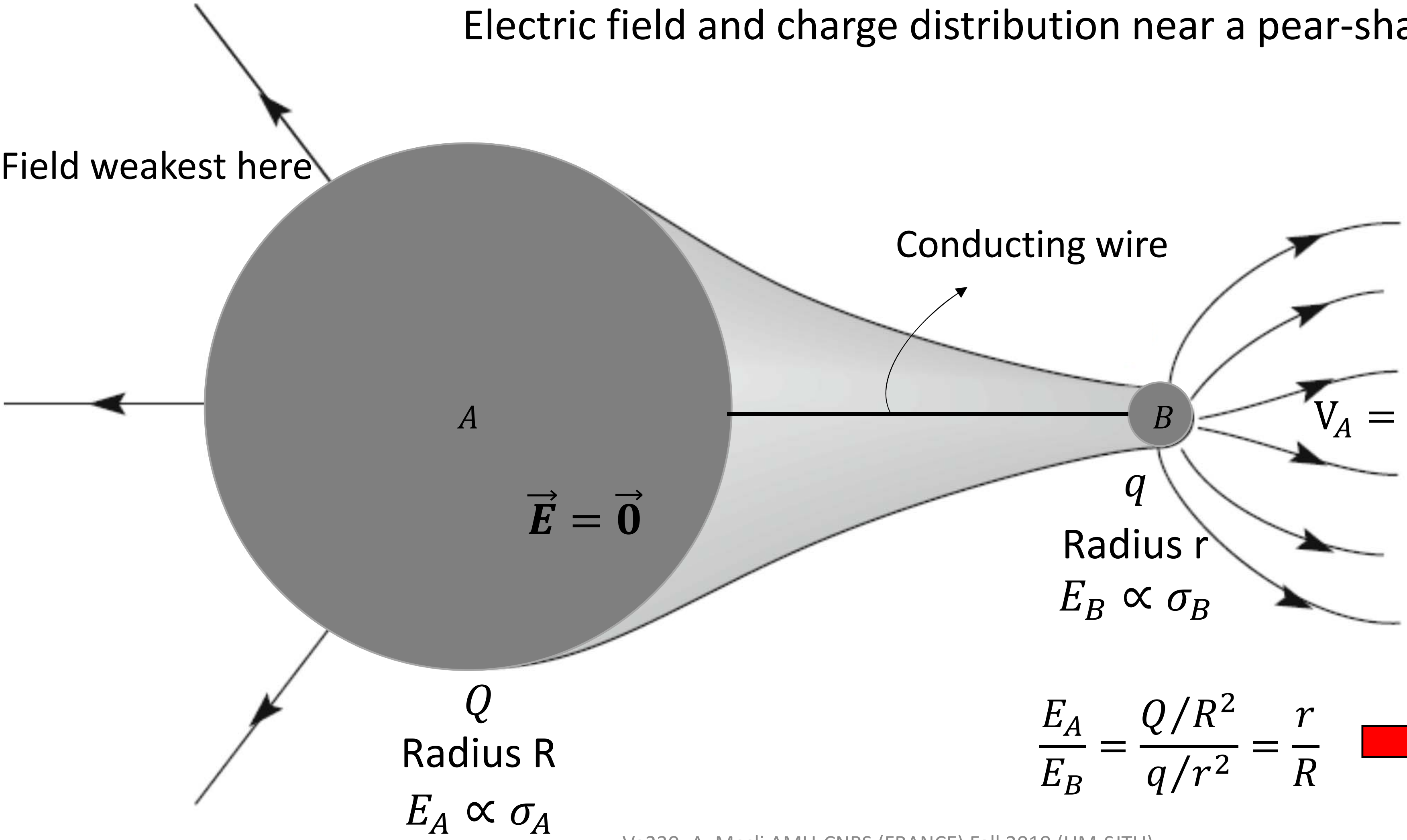
$+q$ is **responsible** for the field outside **BUT does not shape it***. How can we avoid that field while the charge is still in the cavity?



** The fields inside and outside are independent !*

Conductor shape dependent electric field

Electric field and charge distribution near a pear-shaped conductor



$$V_A = V_B \Rightarrow \frac{Q}{R} = \frac{q}{r}$$

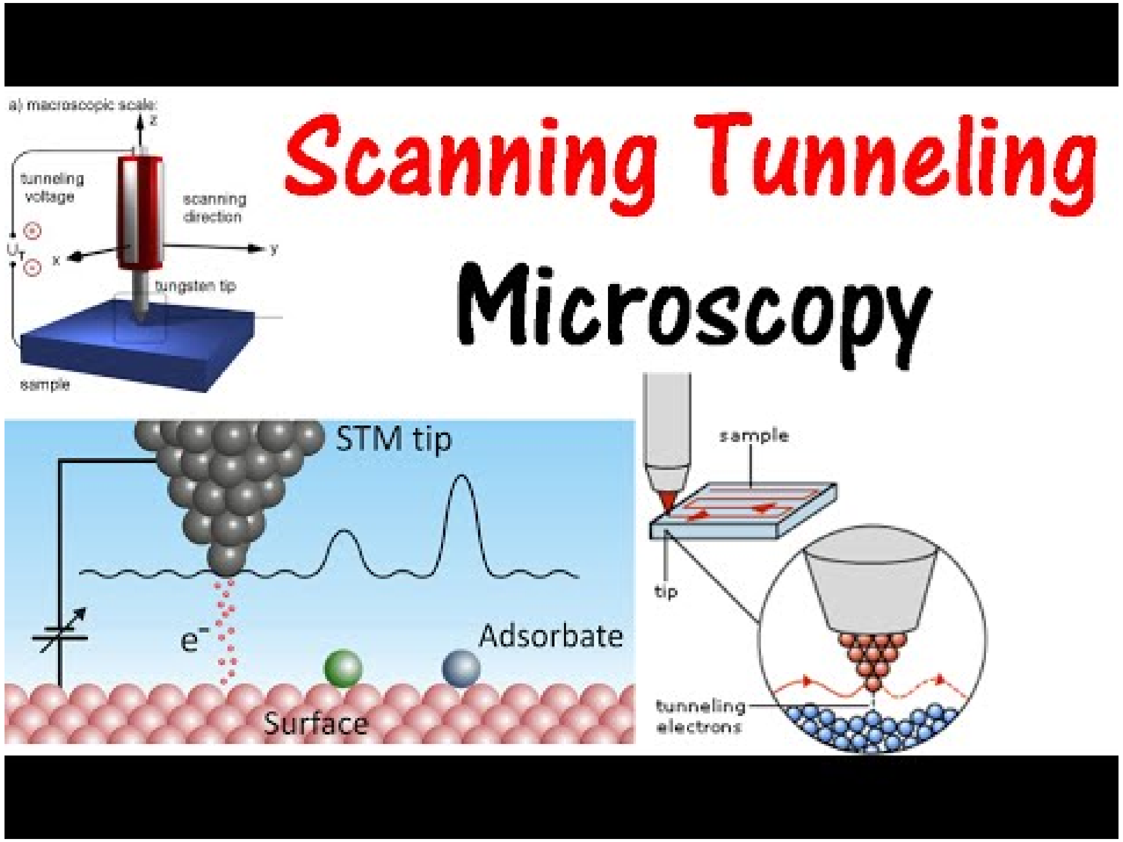
$$\frac{E_A}{E_B} = \frac{Q/R^2}{q/r^2} = \frac{r}{R} \Rightarrow \boxed{E_B = E_A \frac{R}{r}}$$

***Question 3: Are the charges distributed uniformly on each sphere?

Answer to ***Question 3:

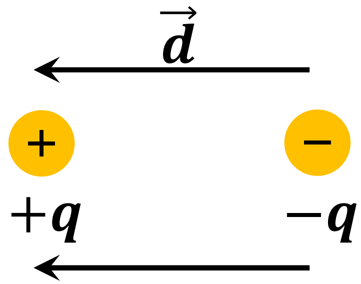
We have just seen that two charge distributions disturb each other

One major application of a tip-shaped conductor: observing atoms



<u>Nobel price in 1986:</u> Gerd Binning and Heinrich Roher (IBM Zürich, Switzerland)

Electric dipole

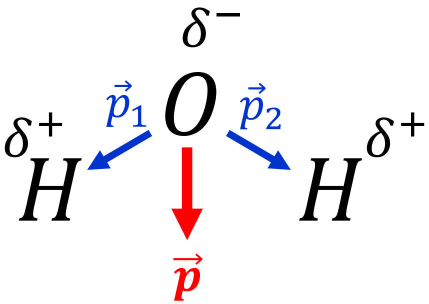


$\vec{p} = q\vec{d}$ Dipole moment

- Useful to explain dielectrics
- Useful to explain charge induction in conductors

Why is the dipole so important?

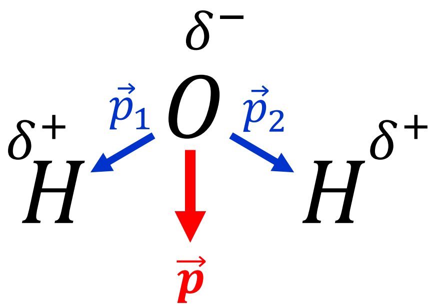
Water molecules



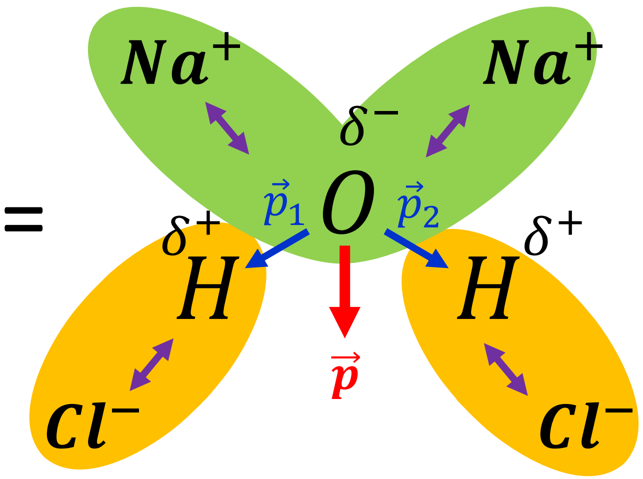
$\vec{p} = \vec{p}_1 + \vec{p}_2$

Net dipole moment

Why is water a good solvent for Na^+Cl^- ?



$+ \quad Na^+Cl^- \quad =$



If water were not polar it would have been a poor solvent

.... **BUT not only !** The microwave would have been useless

Electrostatic

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$$

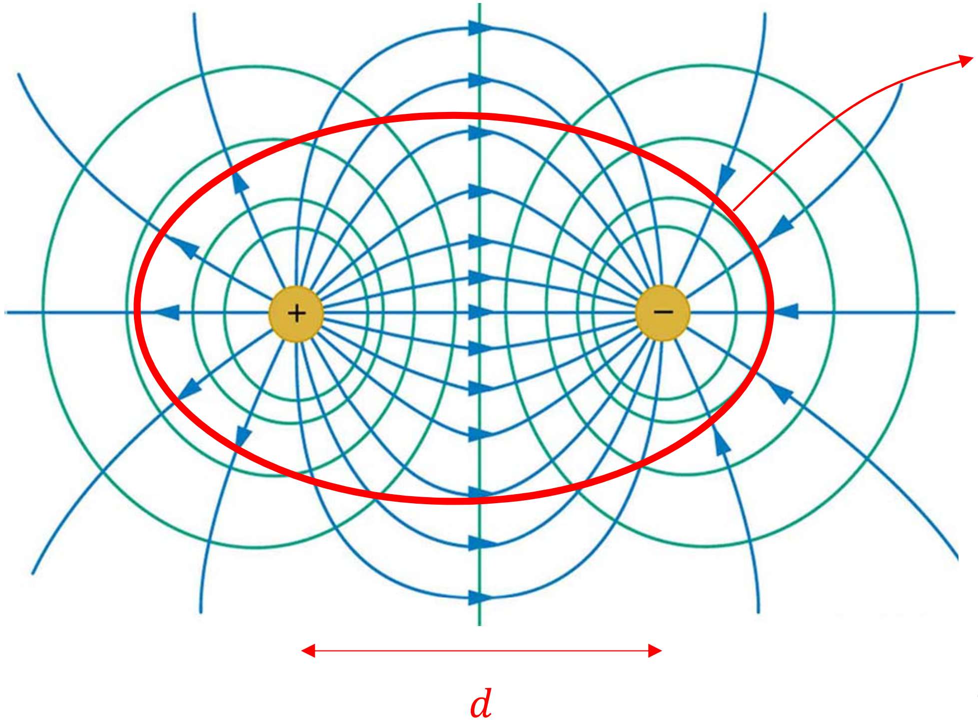
$$\vec{\nabla} \times \vec{E} = 0 \quad \Rightarrow \quad \vec{E} = -\vec{\nabla}\mathrm{V} \quad \Rightarrow \quad \vec{\nabla}.\vec{E} = -\vec{\nabla}.\left(\vec{\nabla}\mathrm{V}\right) = -\nabla^2\mathrm{V} = \frac{\rho}{\varepsilon_0}$$

Poisson equation $\nabla^2\mathrm{V}(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_0}$

If we know $\rho(\vec{r})$ ⇨ we extract $\mathrm{V}(\vec{r})$ and ⇨ we get $\vec{E}(\vec{r})$

<u>Superposition principle</u> $\mathrm{V}(\vec{r}) = \int \frac{\rho\left(\overrightarrow{r'}\right)dV'}{4\pi\varepsilon_0\left|\vec{r}-\overrightarrow{r'}\right|} \; \frac{\vec{r}-\overrightarrow{r'}}{\left|\vec{r}-\overrightarrow{r'}\right|}$

Gauss' law is of no help for a dipole



Gauss surface $\Phi_E = \int \vec{E}.d\vec{A} = \dfrac{\rho}{\varepsilon_0} = 0$

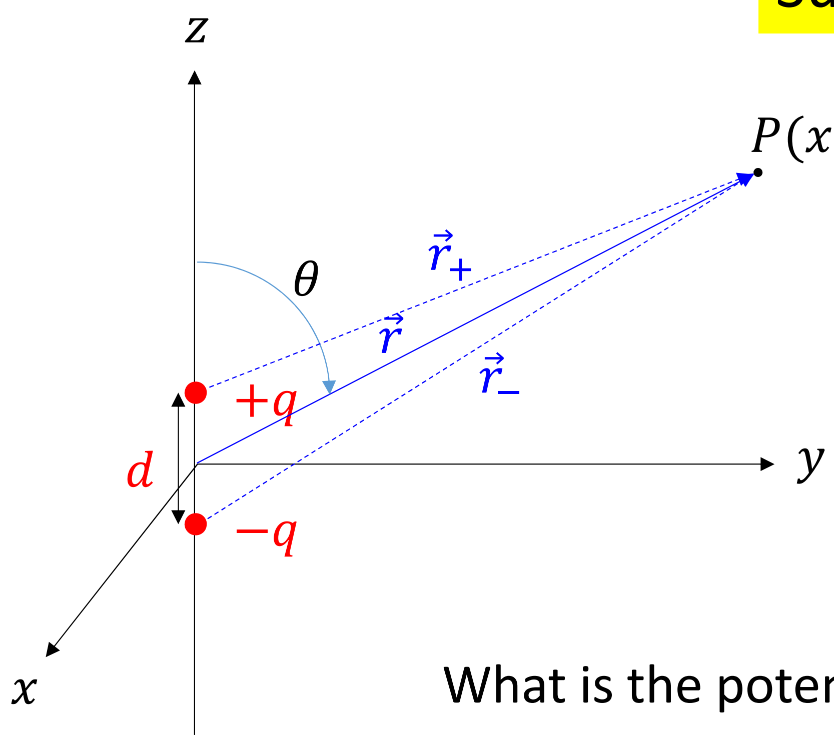
It does not mean that $\vec{E} = \vec{0}$

To have $\vec{E} = \vec{0}$ everywhere

$d = 0$ charges $+q$ and $-q$ on top of each other

But this is no longer a dipole !

Superposition principle



$$V(P) = \sum_i V_i(P) \qquad V(P) = V_+ + V_-$$

$$V(P) = \frac{1}{4\pi\varepsilon_0}\left[\frac{\boldsymbol{q}}{\sqrt{\underbrace{\left(z-\frac{d}{2}\right)^2 + x^2 + y^2}_{\boldsymbol{r}_+}}} - \frac{\boldsymbol{q}}{\sqrt{\underbrace{\left(z+\frac{d}{2}\right)^2 + x^2 + y^2}_{\boldsymbol{r}_-}}}\right]$$

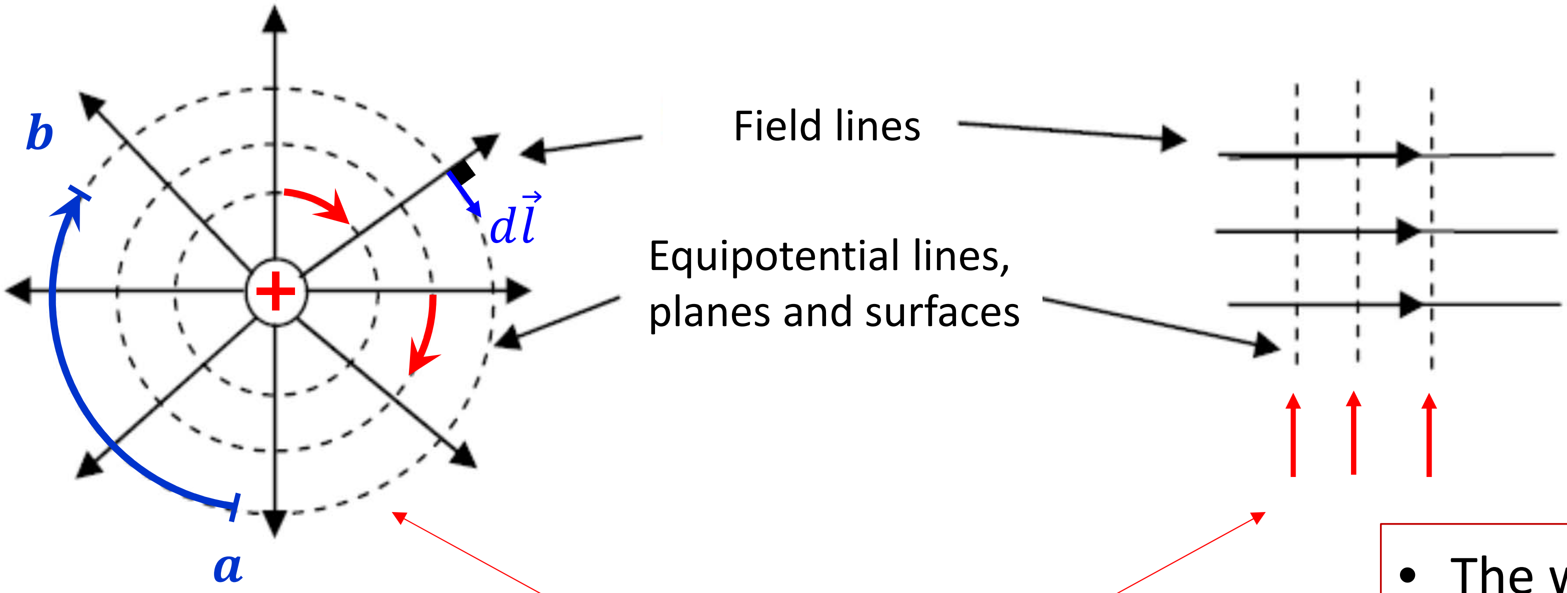
What is the potential in the $x - y$ plane ?

$z = 0$ ➡ Along the $x - y$ plane $V = 0$

BUT only because the two charges are equal in magnitude

<u>Definition of a dipole</u>

Field and equipotential lines, planes and surfaces



$\mathrm{V}(\boldsymbol{b}) = \mathrm{V}(\boldsymbol{a})$

Moving a charge along these paths does not require work

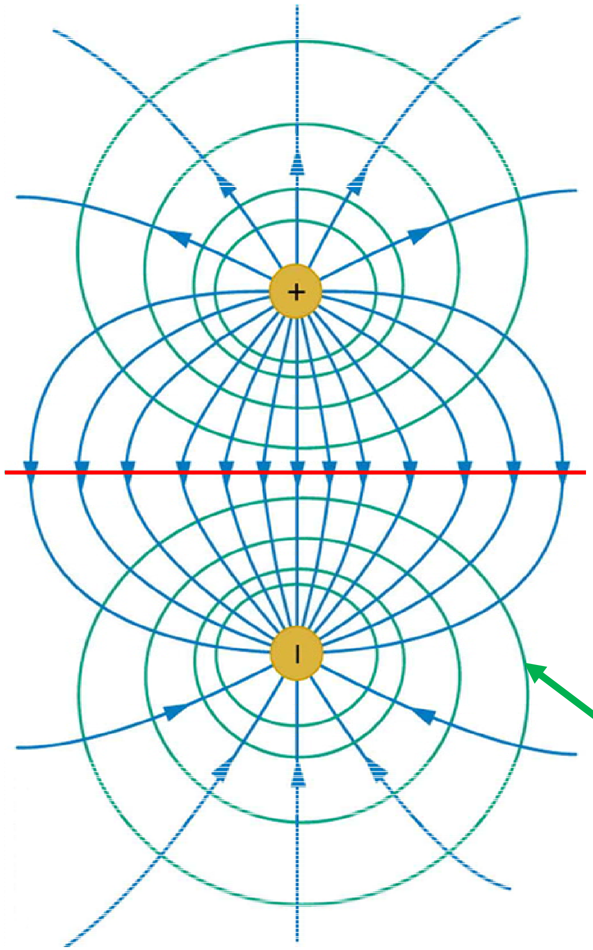
- The work is done against $\vec{E}$,

  $W = -\int \vec{E}.d\vec{l}$
- $\vec{E} \perp d\vec{l}$
- $W_{a \to b} = \mathrm{V}(b) - \mathrm{V}(a) = 0$

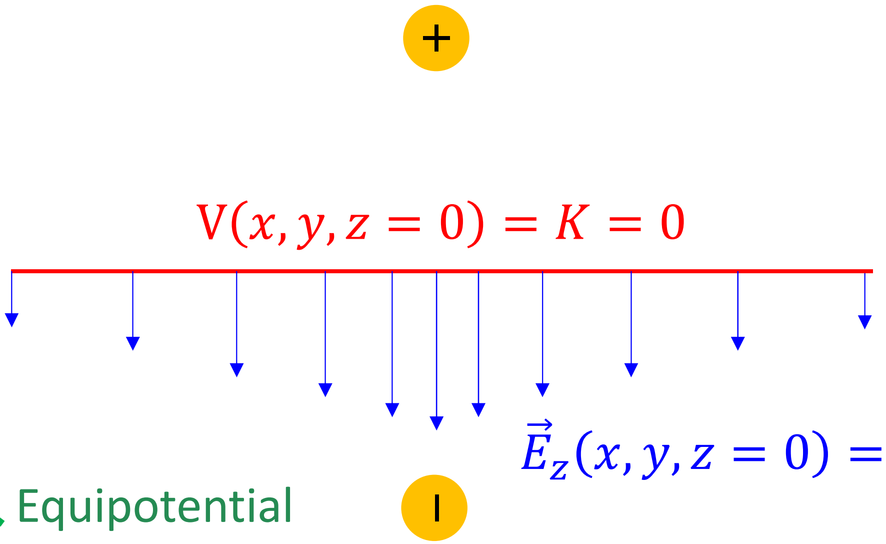
  $\mathrm{V}(b) = \mathrm{V}(a)$

$$\vec{E} = -\left[\underbrace{\vec{\nabla}_x V(x,y,z)\vec{\imath}}_{=0} + \underbrace{\vec{\nabla}_y V(x,y,z)\vec{\jmath}}_{=0} + \vec{\nabla}_z V(x,y,z)\vec{k}\right]$$

$$\vec{E}_z(x,y,z=0) = -\vec{\nabla}_z\, V(x,y,z=0) = -\boldsymbol{E_z}\vec{\boldsymbol{k}}$$

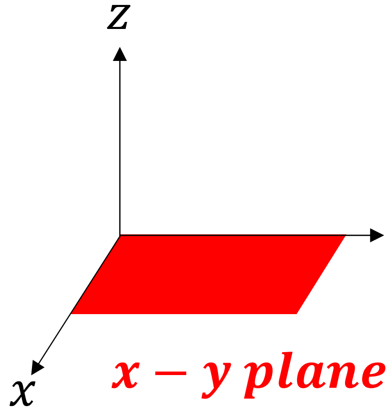


Equipotential Surface



$$V(x,y,z=0) = K = 0$$

$$\vec{E}_z(x,y,z=0) = -\vec{\nabla}_z V$$



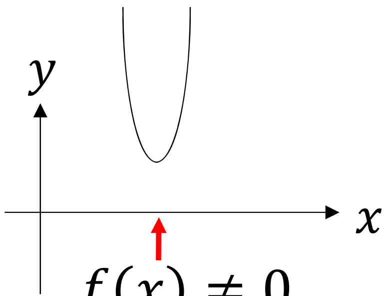
$$V(x,y,z) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{\left(z-{}^{d}/_{2}\right)^2 + x^2 + y^2}} - \frac{1}{\sqrt{\left(z+{}^{d}/_{2}\right)^2 + x^2 + y^2}}\right] = K$$

If at $\vec{r} = \vec{r}_0$, $\mathrm{V}(\vec{\boldsymbol{r}}_{\mathbf{0}}) = \mathbf{0}$, what happens to $\vec{\boldsymbol{\nabla}}\mathrm{V}(\vec{\boldsymbol{r}}_{\mathbf{0}})$?

V is a local property

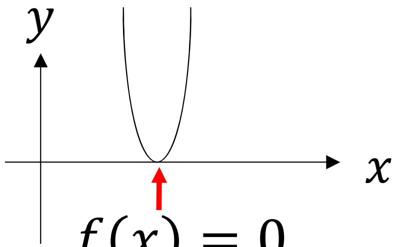
In 1D

If $f(x)$ is a function that cancels at $x = x_0$, it does not mean that $\left.\frac{df}{dx}\right|_{x=x_0} = 0$



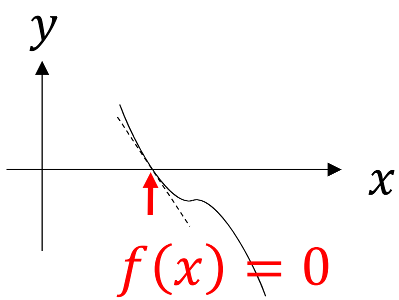
$f(x) \neq 0$

$$\frac{df}{dx} = 0$$



$f(x) = 0$

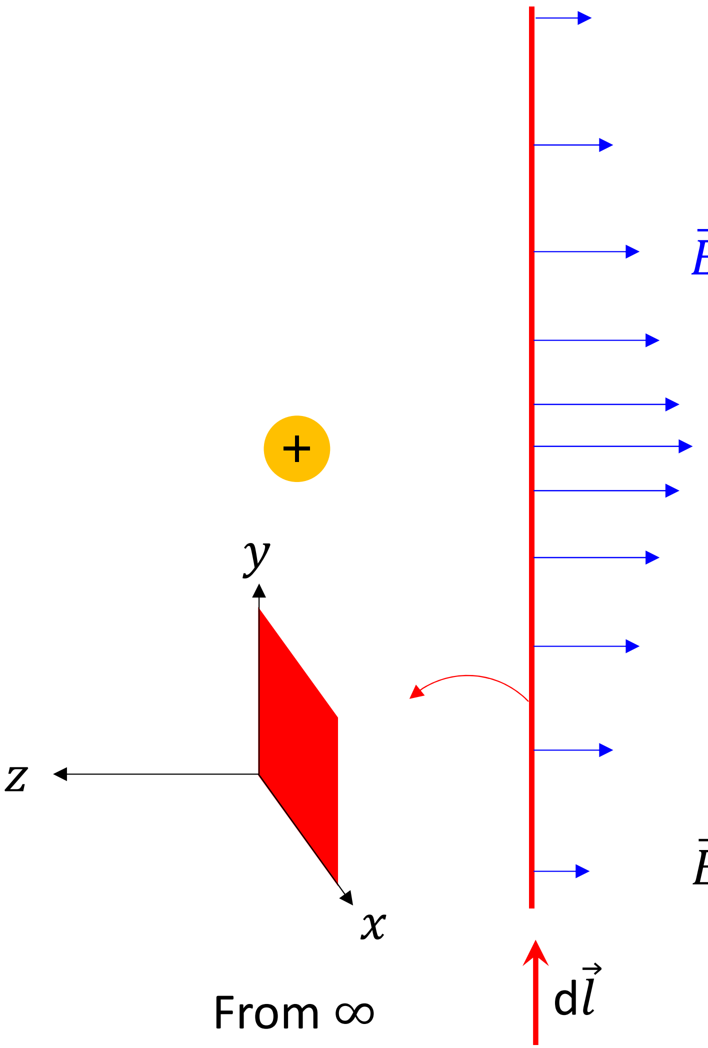
$$\frac{df}{dx} = 0$$



$f(x) = 0$

$$\frac{df}{dx} \neq 0$$

Work done from infinity to the middle along the line or the plane of equidistance is **zero**



$$\vec{E}_z = -\vec{\nabla}_z \mathrm{V}$$

$$\vec{E} = \vec{E}_z \perp \mathrm{d}\vec{l}$$

The work done between two points in space depends on the potentials at that points only **NOT** on the path

$$\vec{\nabla} \times \vec{E} = \vec{0}$$

$$\mathrm{V}(b) - \mathrm{V}(a) = -\int_a^b \vec{E}\, d\vec{l}$$

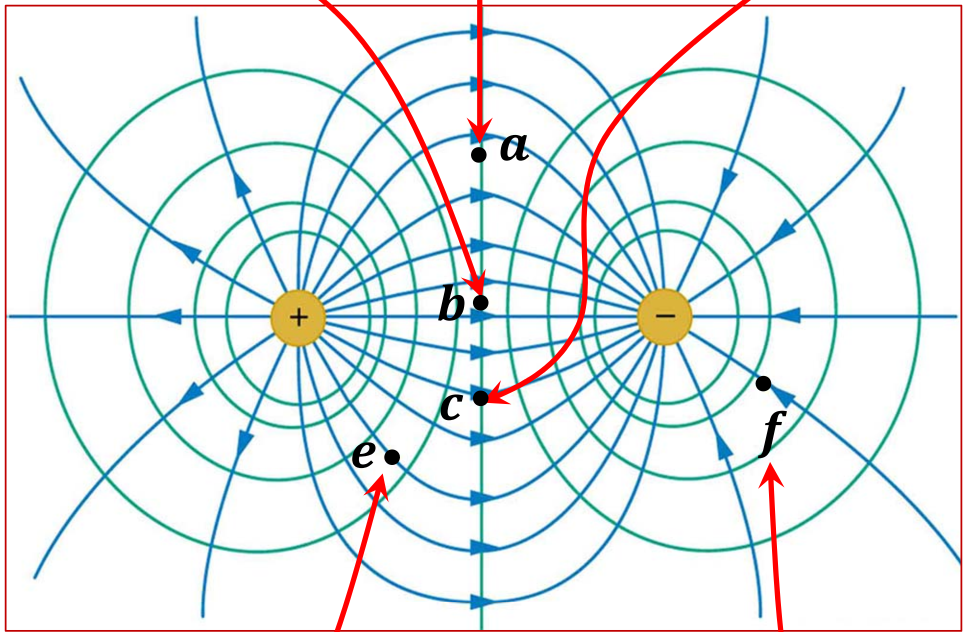
Any path

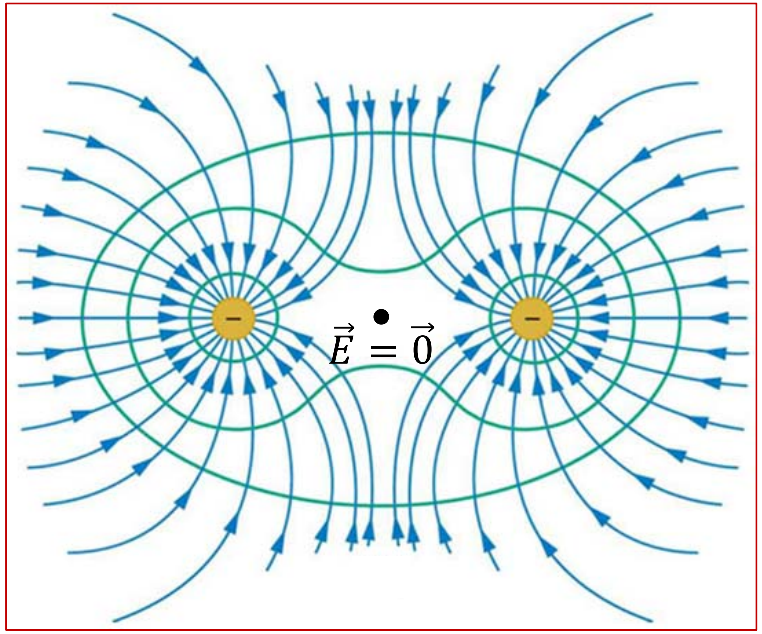
Because electrostatic field is conservative

$$\mathrm{V}(b) = \mathrm{V}(a) = 0$$

$W_{\infty\to b} = 0$ $W_{\infty\to a} = 0$ $W_{\infty\to c} = 0$

Work done to from ∞ to any point





$V > 0$

$W_{\infty\to e} > 0$

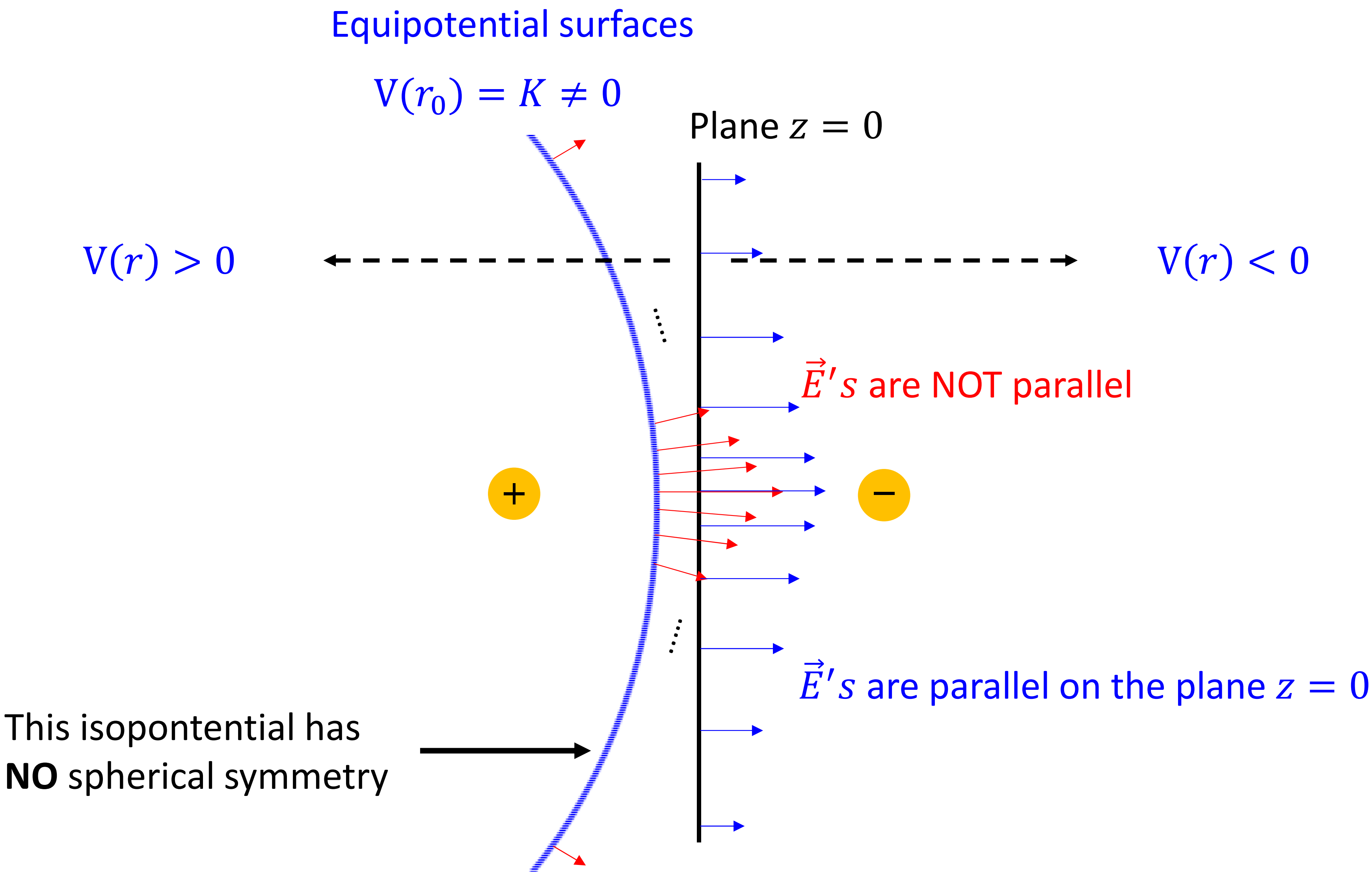
$V = 0$

$V < 0$

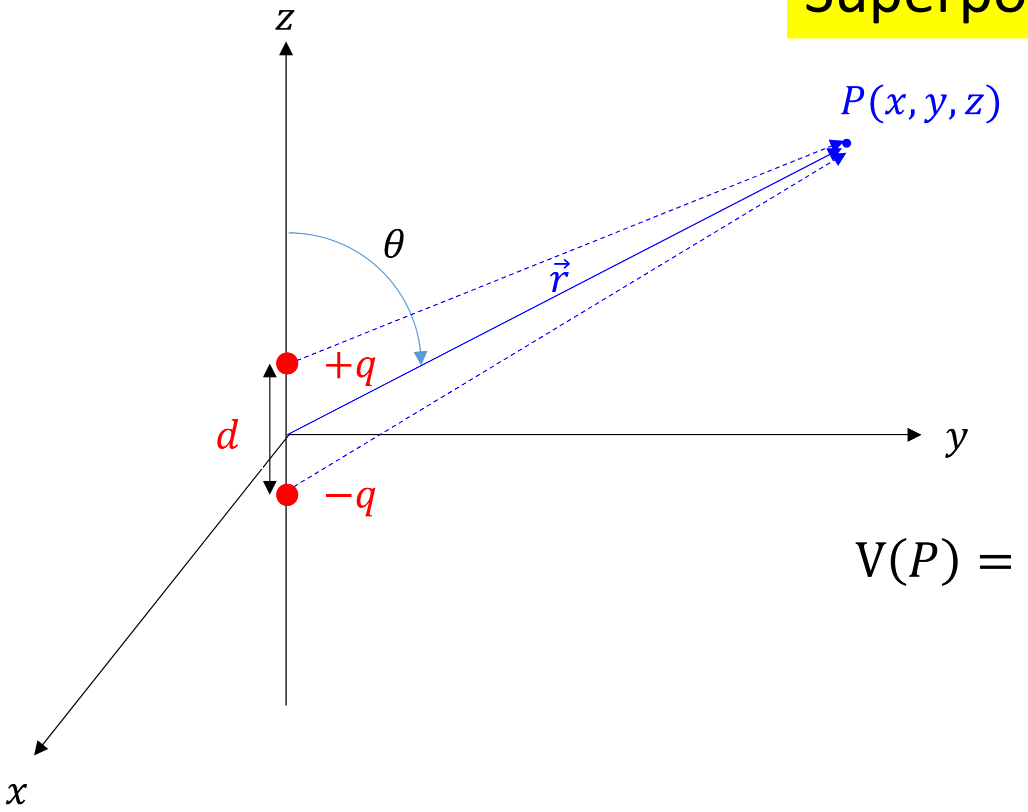
$W_{\infty\to f} < 0$

Work done from ∞ to any point is never 0

$V < 0$


Equipotential surfaces
$V(r_0) = K \neq 0$
Plane $z = 0$
$V(r) > 0$
$V(r) < 0$
$\vec{E}'s$ are NOT parallel
+
−
$\vec{E}'s$ are parallel on the plane $z = 0$
This isopontential has
NO spherical symmetry

Superposition principle



$$V(P) = \sum_i V_i(P)$$

$$V(P) = V_+ + V_-$$

$$V(P) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{\left(z-\frac{d}{2}\right)^2 + x^2 + y^2}} - \frac{q}{\sqrt{\left(z+\frac{d}{2}\right)^2 + x^2 + y^2}}\right]$$

Close to the dipole, $P(x, y, z)$ "sees" $+q$ and $-q$

What does $P(x, y, z)$ see at distances $r >> d$?

Looking far from the dipole

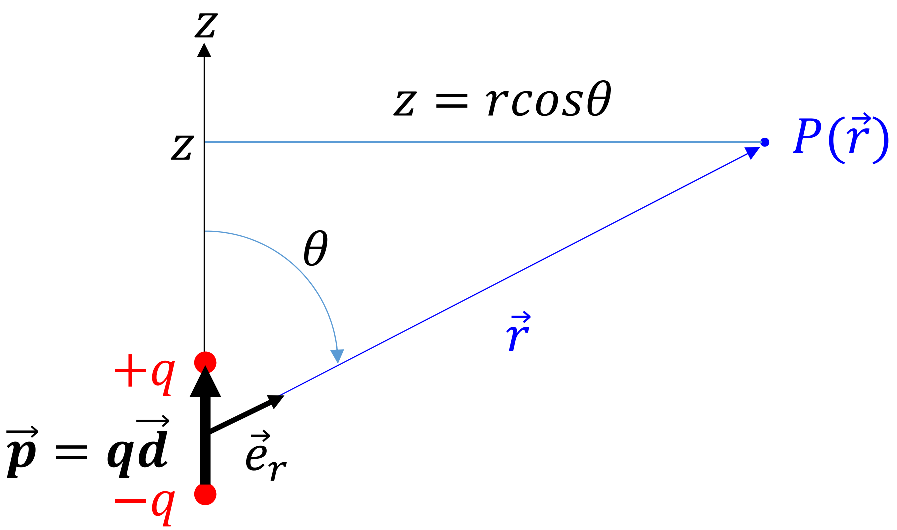
d **small but not 0**

$$\left(z-\frac{d}{2}\right)^2 \approx z^2 - zd \quad \Rightarrow \quad \sqrt{\left(z-\frac{d}{2}\right)^2 + x^2 + y^2} = \sqrt{r^2 - zd} = r\left(1 - \frac{zd}{r^2}\right)^{1/2} = r\left(1 - \frac{zd}{2r^2}\right)$$

$$\left(z+\frac{d}{2}\right)^2 \approx z^2 + zd \quad \Rightarrow \quad \sqrt{\left(z+\frac{d}{2}\right)^2 + x^2 + y^2} = \sqrt{r^2 + zd} = r\left(1 + \frac{zd}{r^2}\right)^{1/2} = r\left(1 + \frac{zd}{2r^2}\right)$$

$$\left.\begin{aligned} V_+(P) &= \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\left(1 + \frac{zd}{2r^2}\right) \\ V_-(P) &= -\frac{q}{4\pi\varepsilon_0}\frac{1}{r}\left(1 - \frac{zd}{2r^2}\right) \end{aligned}\right\} \quad \boxed{V(P) = \frac{q}{4\pi\varepsilon_0}\frac{zd}{r^3}} \quad \boxed{r >> d}$$

$$V(r) = \frac{q}{4\pi\varepsilon_0}\frac{zd}{r^3} \quad + \quad \begin{matrix} z = r\cos\theta \\ p = qd \end{matrix} \quad \Rightarrow \quad V(r) = \frac{\boldsymbol{p}\cos\theta}{4\pi\varepsilon_0}\frac{1}{r^2}$$



In the $x - y$ plane, $\theta = \frac{\pi}{2}, V(r)|_{z=0} = 0$

$$\boldsymbol{p}\cos\theta \; 1 = \vec{p}.\vec{e}_r$$

Dipole $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}.\vec{e}_r}{r^2}$$ Spherical coordinate

An interesting thing emerges !

Dipole $\mathrm{V}(r) \propto \frac{1}{r^2}$

Point charge $\mathrm{V}(r) \propto \frac{1}{r}$

Point charge $\boldsymbol{E}(\boldsymbol{r}) \propto \frac{\mathbf{1}}{\boldsymbol{r}^{\mathbf{2}}}$ Coulomb's law

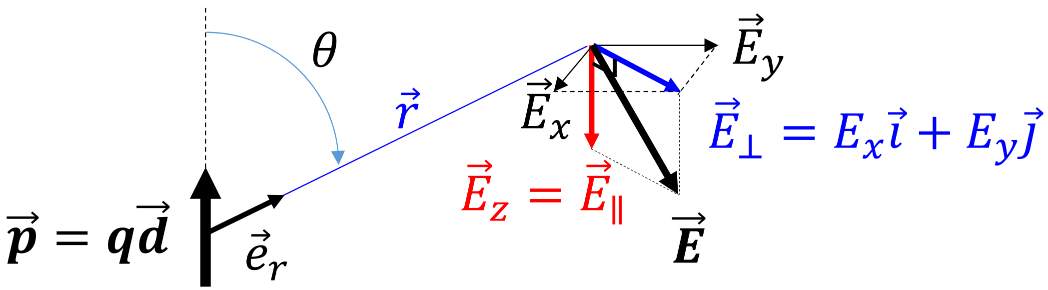
Is it just a simple coincidence?
See next

What are the three Cartesian components of the field of a dipole?

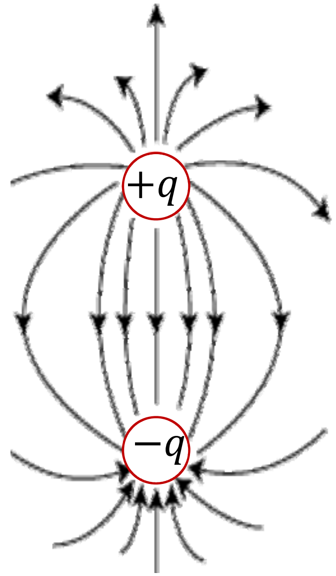
$$\vec{E}(x,y,z) = -\vec{\nabla}\mathrm{V}(x,y,z) = -\left(+\underbrace{\frac{\partial \mathrm{V}}{\partial x}\vec{\imath}}_{E_x} + \underbrace{\frac{\partial \mathrm{V}}{\partial y}\vec{\jmath}}_{E_y} + \underbrace{\frac{\partial \mathrm{V}}{\partial z}\vec{k}}_{E_z}\right)$$

$$\boxed{\mathrm{V}(r) = \frac{\boldsymbol{qd}}{4\pi\varepsilon_0}\frac{z}{r^3} = \frac{\boldsymbol{p}}{4\pi\varepsilon_0}\frac{z}{r^3} \qquad r^2 = x^2 + y^2 + z^2}$$

$$\boxed{E_z = \frac{p}{4\pi\varepsilon_0}\frac{3\cos^2\theta - 1}{r^3}}$$



$$\left.\begin{array}{l} \boxed{E_x = \dfrac{p}{4\pi\varepsilon_0}\dfrac{3zx}{r^5}} \\[2ex] \boxed{E_y = \dfrac{p}{4\pi\varepsilon_0}\dfrac{3zy}{r^5}} \end{array}\right\} = 0 \; if \; z = 0 \; (x - y \; plane, \theta = \pi/2)$$



Dipole in spherical coordinates

$\vec{\nabla}$ in spherical coordinates

$$\vec{\nabla} = \left(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \vec{e}_\varphi \frac{1}{r sin\theta}\frac{\partial}{\partial \emptyset}\right)$$

$$\mathrm{V}(r) = \frac{p cos\theta}{4\pi\varepsilon_0}\frac{1}{r^2}$$

$$E_r(r) = -\frac{\partial \mathrm{V}(r)}{\partial r} = \frac{2p cos\theta}{4\pi\varepsilon_0}\frac{1}{r^3}$$

$$E_\theta(r) = -\frac{1}{r}\frac{\partial \mathrm{V}(r)}{\partial \theta} = \frac{p sin\theta}{4\pi\varepsilon_0}\frac{1}{r^3}$$

$$E_\emptyset(r) = 0$$

Gauss Law = Flux through a closed surface

$$\Phi = \int \vec{E}.\vec{n} dA$$

$$\vec{E} = E_r\vec{e}_r + E_\theta\vec{e}_\theta \qquad dA = r^2 sin\theta d\theta d\emptyset$$

$$0 < \theta < \pi$$
$$0 < \emptyset < 2\pi$$

$\Phi = 0$ As expected

Gauss law can also be checked through divergence

$$\vec{\nabla}.\vec{E} = \underbrace{\frac{1}{r^2}\frac{\partial r^2 E_r}{\partial r}}_{-\frac{2pcos\theta}{4\pi\varepsilon_0}\frac{1}{r^3}} + \underbrace{\frac{1}{rsin\theta}\frac{\partial sin\theta E_\theta}{\partial\theta}}_{+\frac{2pcos\theta}{4\pi\varepsilon_0}\frac{1}{r^3}} + \underbrace{\frac{1}{rsin\theta}\frac{\partial E_\emptyset}{\partial\emptyset}}_{=0}$$

$$\underbrace{-\frac{2pcos\theta}{4\pi\varepsilon_0}\frac{1}{r^3} + \frac{2pcos\theta}{4\pi\varepsilon_0}\frac{1}{r^3}}_{=0}$$

$\vec{\nabla}.\vec{E} = 0$

NO net charge as expected

A beautiful illustration of the principle of superposition

Next 7 slides only for those who love physics and mathematics !

Dipole potential as a gradient: An interesting aspect

$$V(\vec{r}) = \frac{\vec{p}.}{4\pi\varepsilon_0}\boxed{\frac{\vec{e}_r}{r^2}} = \frac{1}{4\pi\varepsilon_0}\vec{p}.\boxed{\vec{\nabla}\left(-\frac{1}{r}\right)}$$

The gradient is taken along $\vec{e}_r$

$$\vec{\nabla}\left(-\frac{1}{r}\right) = \frac{\vec{e}_r}{r^2}$$

$$V(\vec{r}) = -\vec{p}.\vec{\nabla}\boxed{\left(\frac{1}{4\pi\varepsilon_0 r}\right)}$$

Potential of a **unit** charge

$$V(\vec{r}) = \vec{p}.\boxed{\vec{E}_{unit\ charge}}$$

$$-\vec{\nabla}\left(V_{unit\ charge}\right)$$

Potential of a dipole = scalar product of

- The dipole moment and the gradient of the potential of a **unit** charge

Or

- The dipole moment and the electric field created by a unit charge

For a dipole: $\mathrm{V}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}.\vec{e}_r}{r^2}$ $\Rightarrow$ $\mathrm{V}(\vec{r}) = -\vec{p}.\vec{\nabla}\left(\frac{1}{4\pi\varepsilon_0 r}\right)$ Scalar

Gradient of the potential of a **unit** charge

Formal Similarity

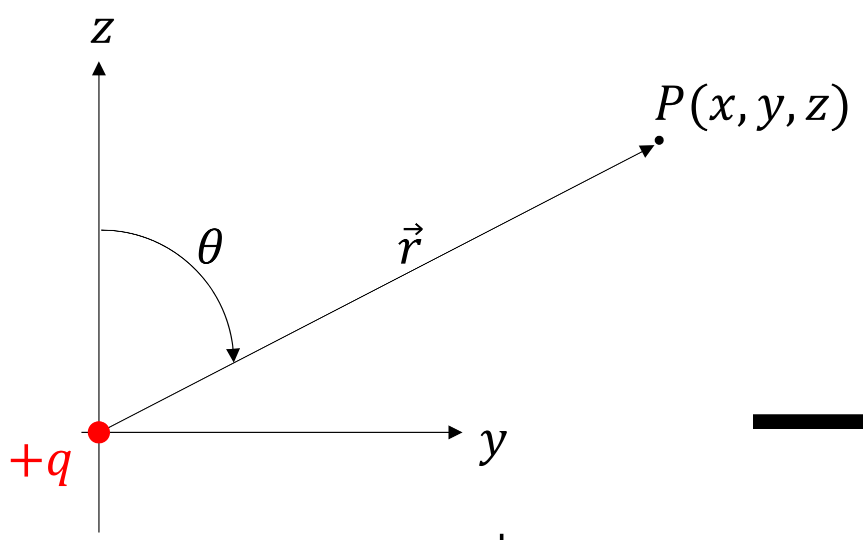
For a point charge: $\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q\vec{e}_r}{r^2}$ $\Rightarrow$ $\vec{E}(\vec{r}) = -q\vec{\nabla}\left(\frac{1}{4\pi\varepsilon_0 r}\right)$ Vector

$$\vec{E}(\vec{r}) = -\vec{\nabla}\mathrm{V}(\vec{r})$$

Is there any physical reason behind such a formal similarity
Or is it just a coincidence?

Let's us consider a 2D case

In the calculation below we drop $\frac{1}{4\pi\varepsilon_0}$



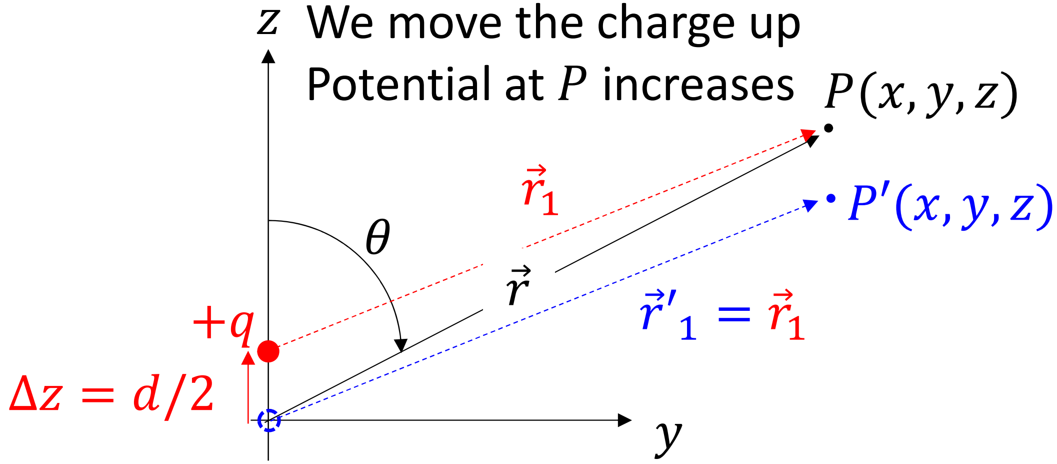
$$V(P) \propto \frac{+q}{r}$$

charge at origine

We move the charge up

Potential at P increases

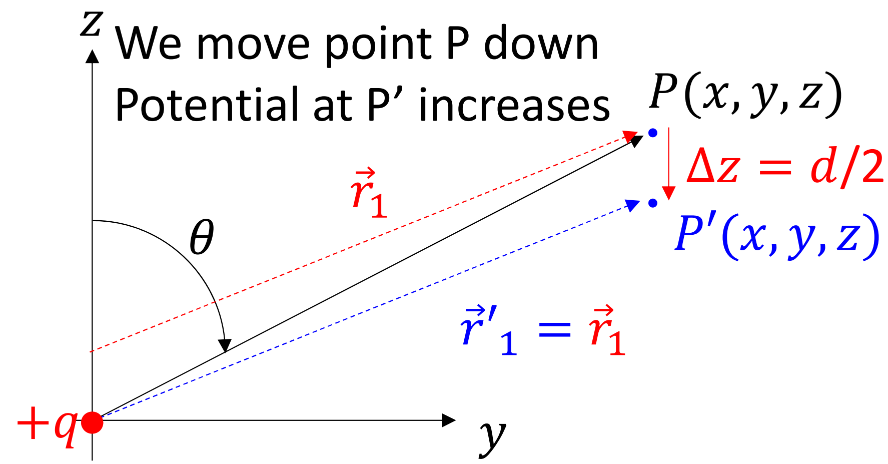
$|\vec{r}_1| < |\vec{r}|$

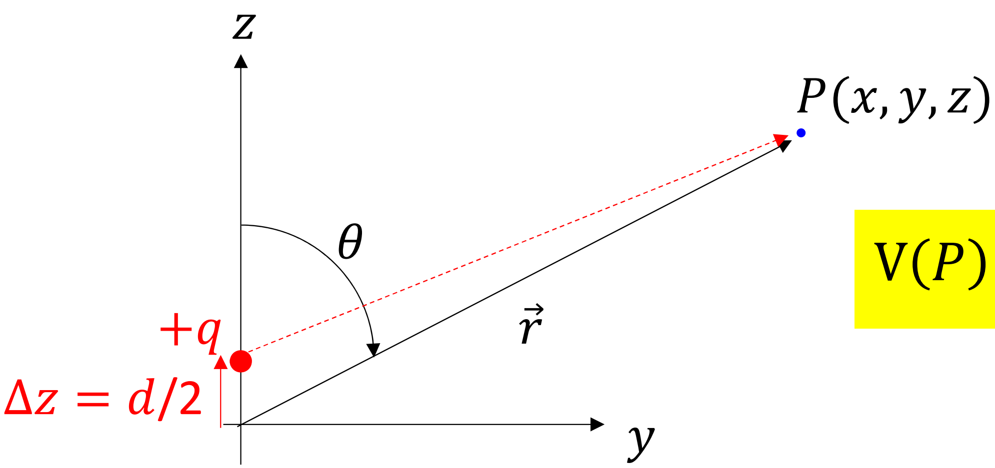


$$V(\boldsymbol{P})\Big|_{\boldsymbol{charge\ at+d/2}} = V(\boldsymbol{P'})\Big|_{\boldsymbol{charge\ at\ 0}}$$

We move point P down

Potential at P' increases





$$\mathrm{V}(P)\Big|_{charge\ at\ d/2} = \mathrm{V}(P)\Big|_{charge\ at\ 0} + \Delta\mathrm{V}$$

$$\frac{+q}{r}$$

$$\Delta\mathrm{V} = dV = \frac{\partial \mathrm{V}}{\partial r}dr$$

From r to z: other coordinates remain unchanged

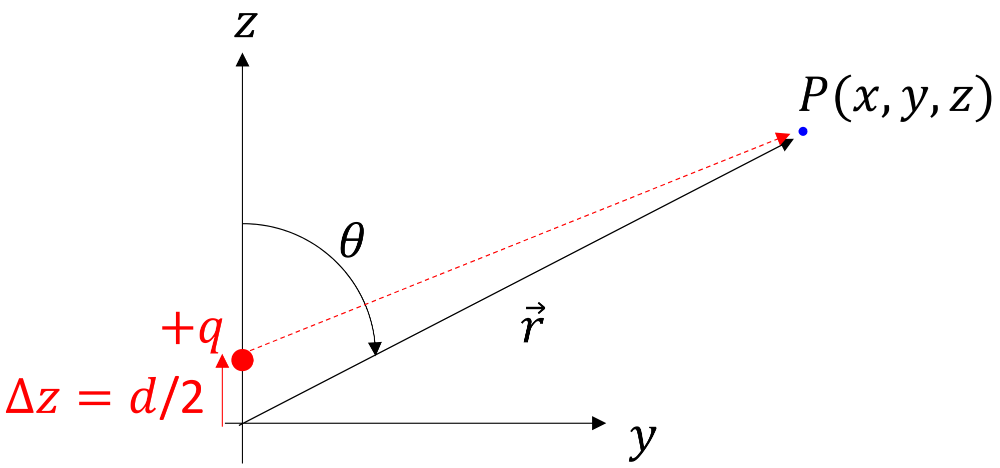
$$= {}^{d}\!/_{2}$$

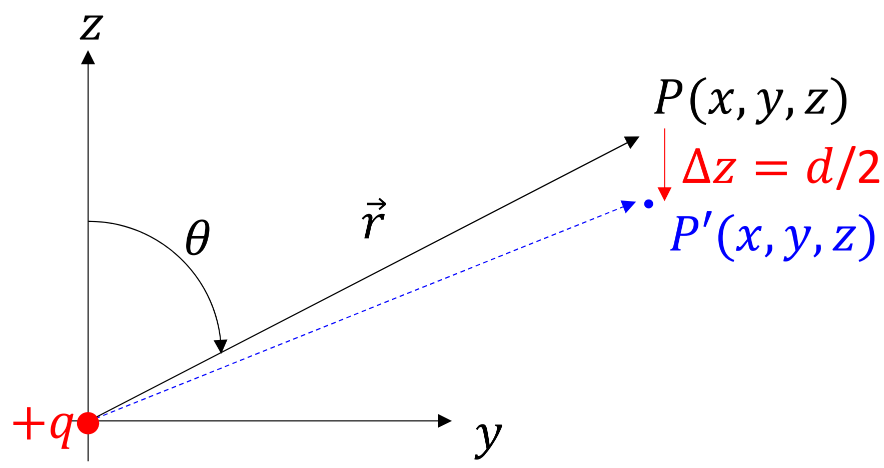
$$-\frac{\partial}{\partial z}\left(\frac{+q}{r}\right)\Delta z$$

We want the potential to increase

$$\mathrm{V}(P) \propto +q\left[\frac{1}{r} - \left[\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right]\frac{d}{2}\right]$$

charge at + d/2





$$\mathrm{V}(P) \propto +q\left[\frac{1}{r} - \left[\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right]\frac{d}{2}\right]$$

Charge remains at origin $+d/2$

$P(x,y,z)$ remains fixed

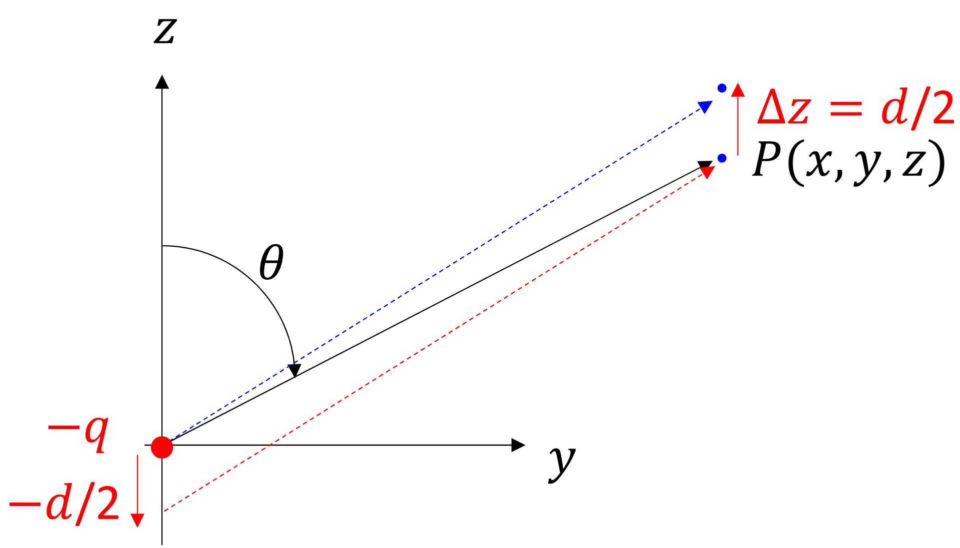
$$\mathrm{V}(P') \propto +q\left[\frac{1}{r} - \left[\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right]\frac{d}{2}\right]$$

Charge remains at origin

$P(x,y,z)$ moves to $P'(x,y,z)$

$$\mathrm{V}(\boldsymbol{P})\Big|_{\boldsymbol{charge\ at+d/2}} = \mathrm{V}(\boldsymbol{P'})\Big|_{\boldsymbol{charge\ at\ 0}}$$

$$\boxed{\mathrm{V}_{+}(r) \propto +q\left[\frac{1}{r} - \left[\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right]\frac{d}{2}\right]}$$



Decrease of the potential

$$\boxed{V_-(r) \propto \frac{-q}{r} + \frac{\partial}{\partial z}\left(\frac{-q}{r}\right)\frac{d}{2}}$$

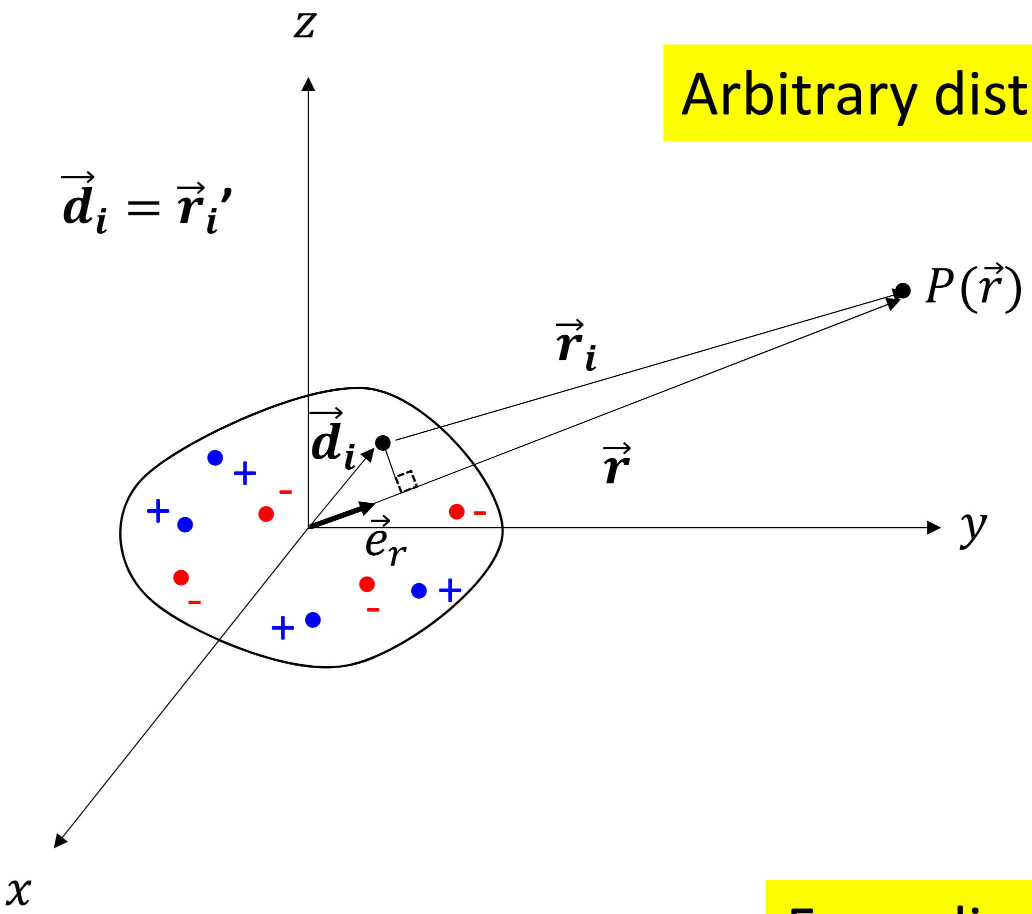
principle of superposition $\quad V(r) = V_+(r) + V_-(r) \quad \Rightarrow \quad V(r) = -\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\boldsymbol{qd}$

$\boldsymbol{p}$

Potential of a dipole

Extension to 3D

$$\boxed{V(\boldsymbol{r}) = -\vec{\boldsymbol{p}}.\vec{\boldsymbol{\nabla}}\left(\frac{\mathbf{1}}{\boldsymbol{r}}\right)}$$



Arbitrary distribution of dipoles

$$\mathrm{V}(r) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{|\vec{r} - \vec{d}_i|} = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r_i}$$

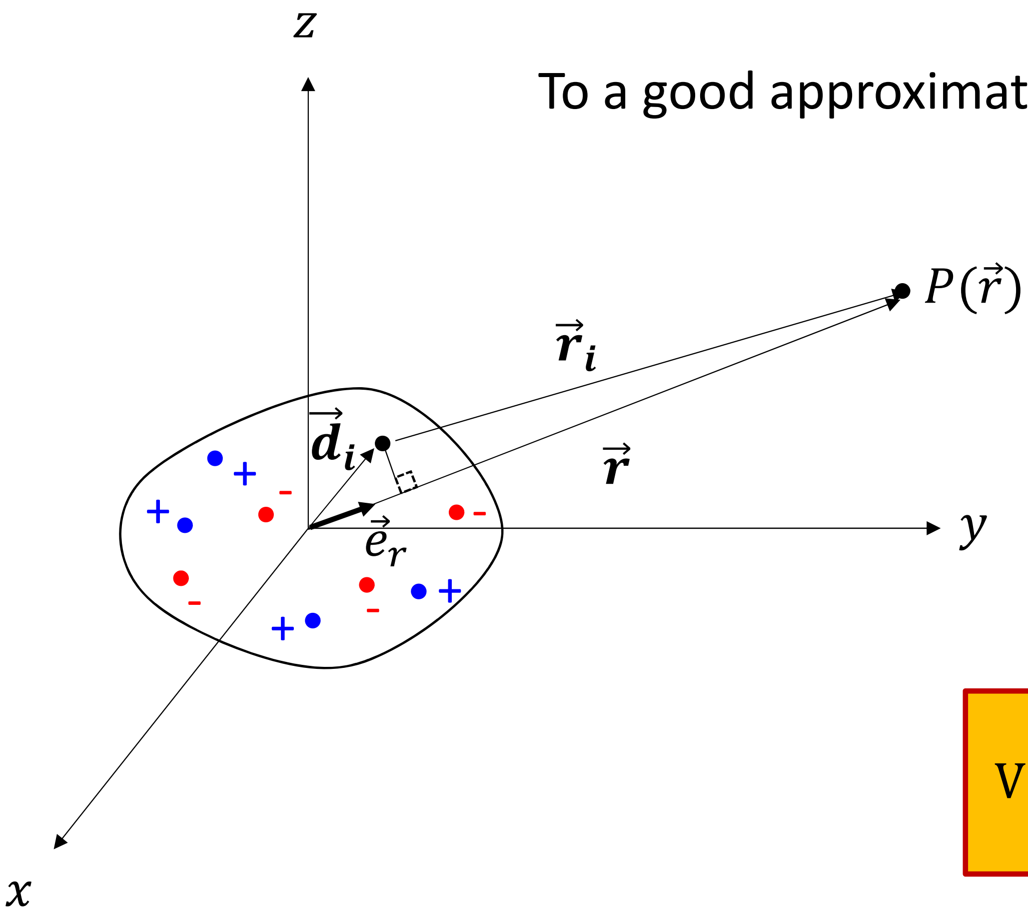
For $r \gg d_i$

$$\mathrm{V}(r) = \frac{Q}{4\pi\varepsilon_0 r} \qquad Q = \int \rho(d')d\mathcal{V}'$$

If $Q = 0 \Rightarrow \mathrm{V} = 0$!

For a dipole, $r \gg d/2$

$Q = 0$ but $\mathrm{V} \neq 0$ **!**



To a good approximation $r - d_i \approx r - \vec{d}_i.\vec{e}_r$

Projection of $\vec{d}_i$ on $\vec{r}$

$$\frac{1}{|\vec{r} - \vec{r}_i|} \approx \frac{1}{\left|r - \vec{d}_i.\vec{e}_r\right|} \approx \frac{1}{r}\left(1 + \frac{\vec{d}_i.\vec{e}_r}{r}\right)$$

$$\mathrm{V}(r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \sum_i q_i \frac{\vec{d}_i.\vec{e}_r}{r^2} \ldots\right)$$

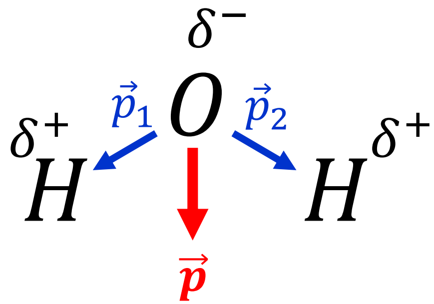
For a distribution of dipoles $Q = 0$

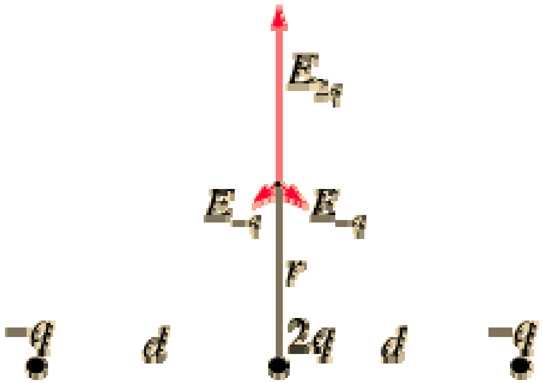
$$\boxed{\mathrm{V}(r) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i\vec{d}_i.\vec{e}_r}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{(\sum_i \vec{p}_i).\vec{e}_r}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{P}.\vec{e}_r}{r^2}}$$

Dipole moment of the distribution $\boxed{\vec{P} = \left(\sum_i \vec{p}_i\right)}$ → Towards Dielectric

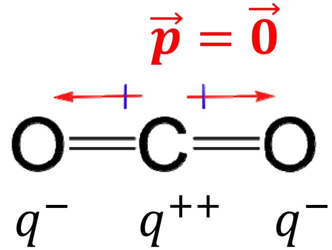
$$\boxed{\vec{P} = \left(\sum_i \vec{p}_i\right)}$$

Example water: Net dipole moment $\vec{\boldsymbol{p}}$





Linear quadrupole



NO net dipole moment

$$E_{2q}\ at\ r = \frac{2q}{4\pi\varepsilon_0 r^2}$$

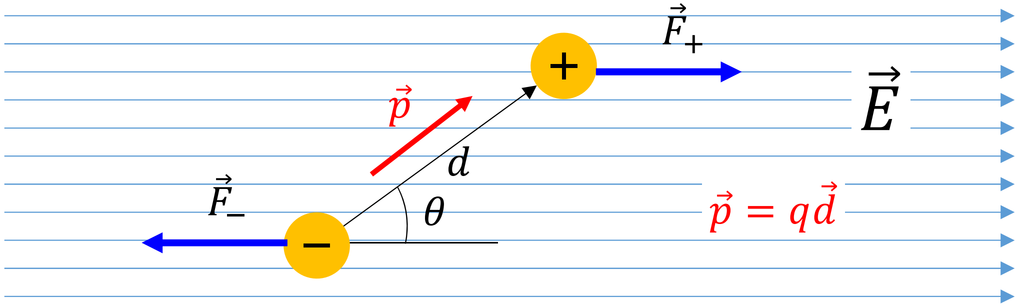
$$E_{-q}\ at\ r = \frac{-q}{4\pi\varepsilon_0 (r^2 + d^2)^{3/2}}$$

$$E_r = \frac{2q}{4\pi\varepsilon_0}\left[\frac{1}{r^2} - \frac{r}{(r^2 + d^2)^{3/2}}\right]$$

For $\frac{d^2}{r^2} \ll 1$ $\quad \left(1 + \frac{d^2}{r^2}\right)^{-3/2} = 1 - \frac{3}{2}\frac{d^2}{r^2}$

$$E_r = \frac{3qd^2}{4\pi\varepsilon_0 r^4} \qquad \boxed{\mathrm{V}(r) = \frac{qd^2}{4\pi\varepsilon_0 r^3}}$$

Force, Torque and work done on dipole: **Uniform field**



Net force $\sum_i \vec{F}_i = \vec{0}$ ➡ No translation

BUT the forces do not act on the same line ➡ **Torque is exerted** ➡ Rotation

$$\vec{\tau}_+ = \frac{\vec{d}}{2} \times \vec{F}_+$$

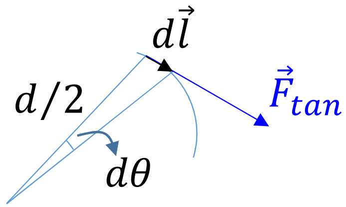
$$\vec{\tau}_- = \frac{\vec{d}}{2} \times \vec{F}_-$$

$$\vec{\tau} = \sum_i \vec{\tau}_i = \vec{d} \times \vec{F}_+ = qdE_+ sin\theta$$ ➡ $\boxed{\vec{\tau} = \vec{p} \times \vec{E}}$

No torque if $\vec{p} \parallel \vec{E}$

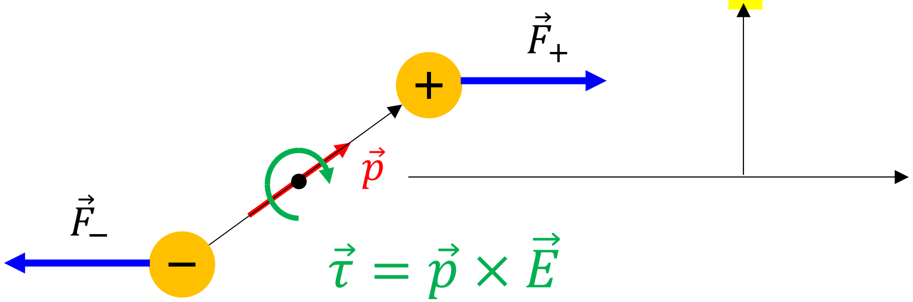
Force, Torque and work done on dipole: **Uniform field**

Work done by torque $dW = F_{tan}dl = F_{tan}\left(\frac{d}{2}\right)d\theta = \tau d\theta$



$$dW = \tau d\theta = -pE\sin\theta d\theta$$

+ Convention for angle orientation



$$\vec{\tau} = \vec{p} \times \vec{E}$$

Because $\vec{\tau}$ tends to decrease θ

⇨ lowering the energy of the system

$$W = \tau d\theta = \int_{\theta_1}^{\theta_2} -pE\sin\theta d\theta = pE\cos\theta_2 - pE\cos\theta_1$$

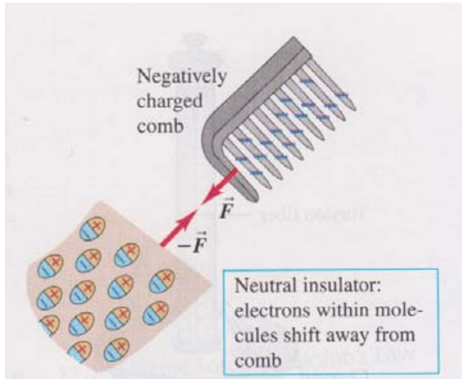
Work – energy theorem $\quad W = -\Delta U \quad \rightarrow \quad \Delta U = -pE\cos\theta_2 - pE\cos\theta_1$ $\quad \boxed{U = -pE\cos\theta}$

Dipole in a uniform field

Microwave oven

- Water molecules in an external electric field ⇨ Experience a torque ⇨ Begin to rotate
- Direction of the external electric field changes very rapidly ⇨ water molecules perform rotational oscillations
- Water molecules gain energy which they dissipate in the form of heat
- Frequency of 2.45 GHz for the oscillating electric field

Torque done on dipole: **Nonuniform field**

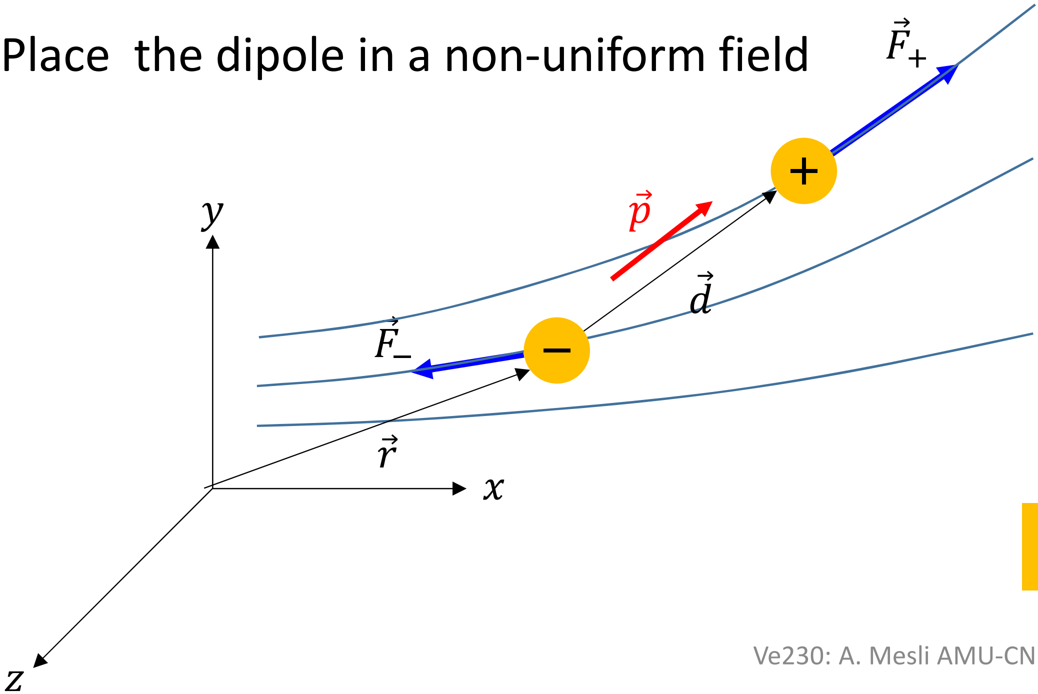


Question

How can we do to attract a neutral (polarizable) object?

⇨ how do we do to make the dipole **translate**?

Place the dipole in a non-uniform field



$$\vec{F} = \sum_i \vec{F}_i = q\vec{E}(\vec{r}+\vec{d}) - q\vec{E}(\vec{r})$$

$$\vec{E}(\vec{r}+\vec{d}) = \vec{E}(\vec{r}) + \vec{d}.\nabla\vec{E}(\vec{r})$$

Scalar matrix

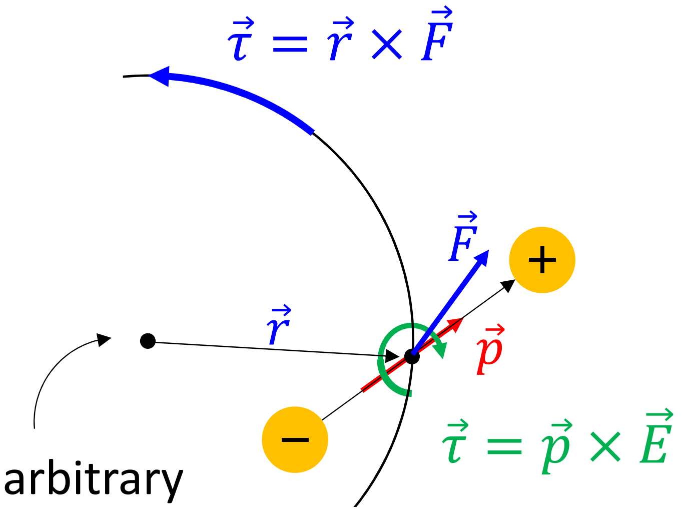
$$\vec{F} = q\vec{d}.\vec{E}(\vec{r}) = \vec{p}.\nabla\vec{E}(\vec{r})$$

Uniform field ⇔ $\nabla\vec{E}(\vec{r}) = 0$ ⇔ $\vec{F} = \vec{0}$

To observe an effect of nonuniform field on a dipole, the scale of nonuniformity of $\vec{E}$ MUST be large compared to $\boldsymbol{d}$

Torque and work done on dipole: **nonuniform field**

The principle of the boomerang



This point is arbitrary

Induction: How does it work?

Conductor in the field of a charge

- How do the induced charges distribute at the surface of a conductor (isopotential)?

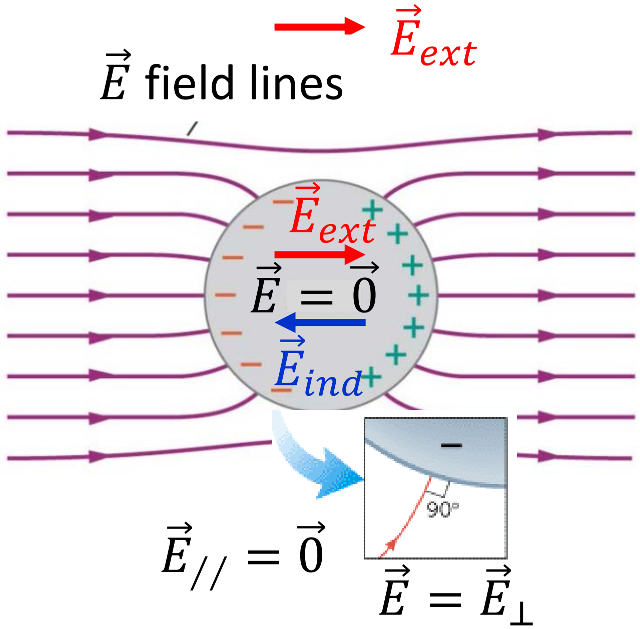
- What is the net force between the charge (field) and the conductor?

These questions are far from being trivial !!

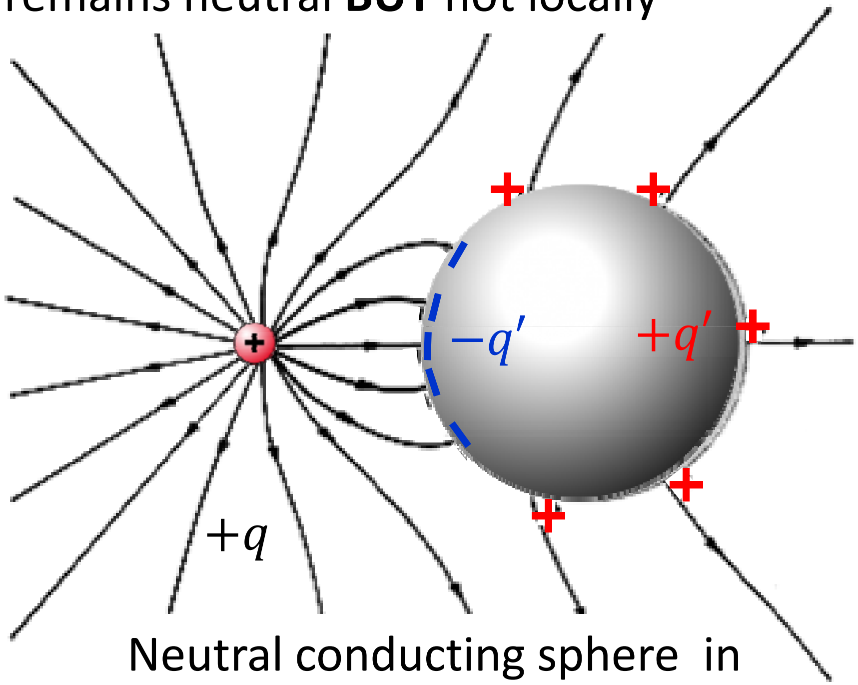
The concept of image charge

Inducing charges in a conductor

Isolated conductor: Charge conservation ⇨ overall conductor remains neutral **BUT** not locally



Uniform field outside



Neutral conducting sphere in
The field of a point charge

Non uniform field outside

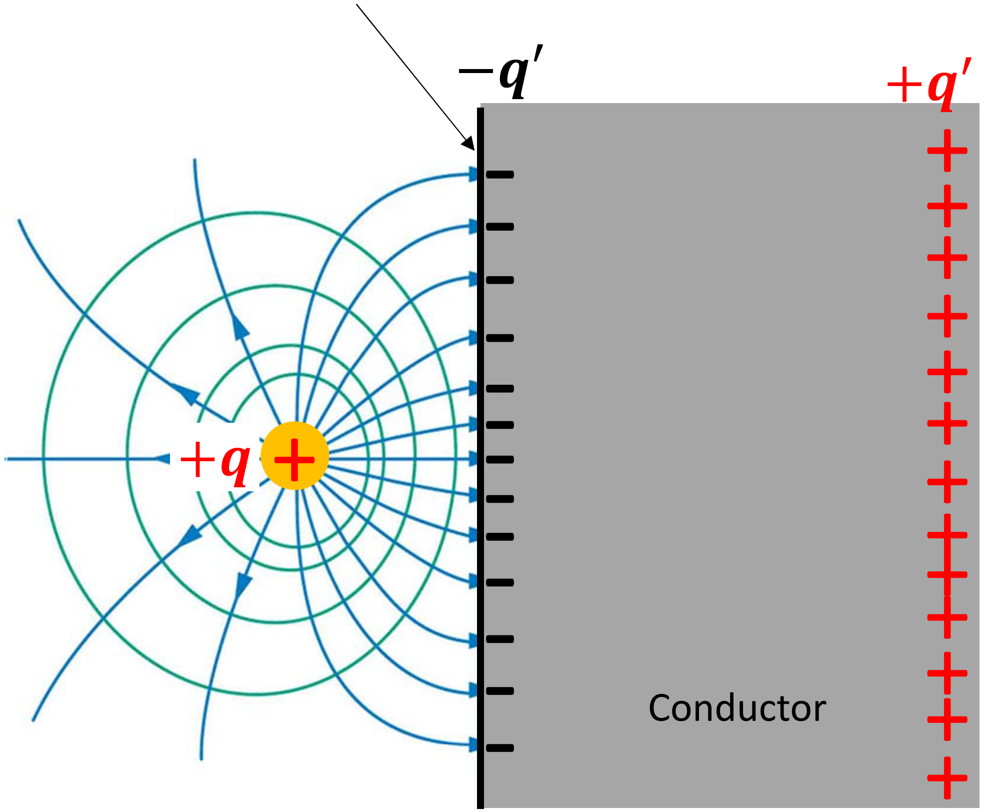
**Question 4: Does the sphere as a whole feel a force from these two external fields?

Answer to **Question 4:

YES in the case of non uniform field. Attraction because $-q'$ is closer to $+q$ than $+q'$

A point charge near a conducting plane: Application of the method of image

Halfway between the two charges the potential is zero



Remove charge $-q$ and put
a conducting plane as shown

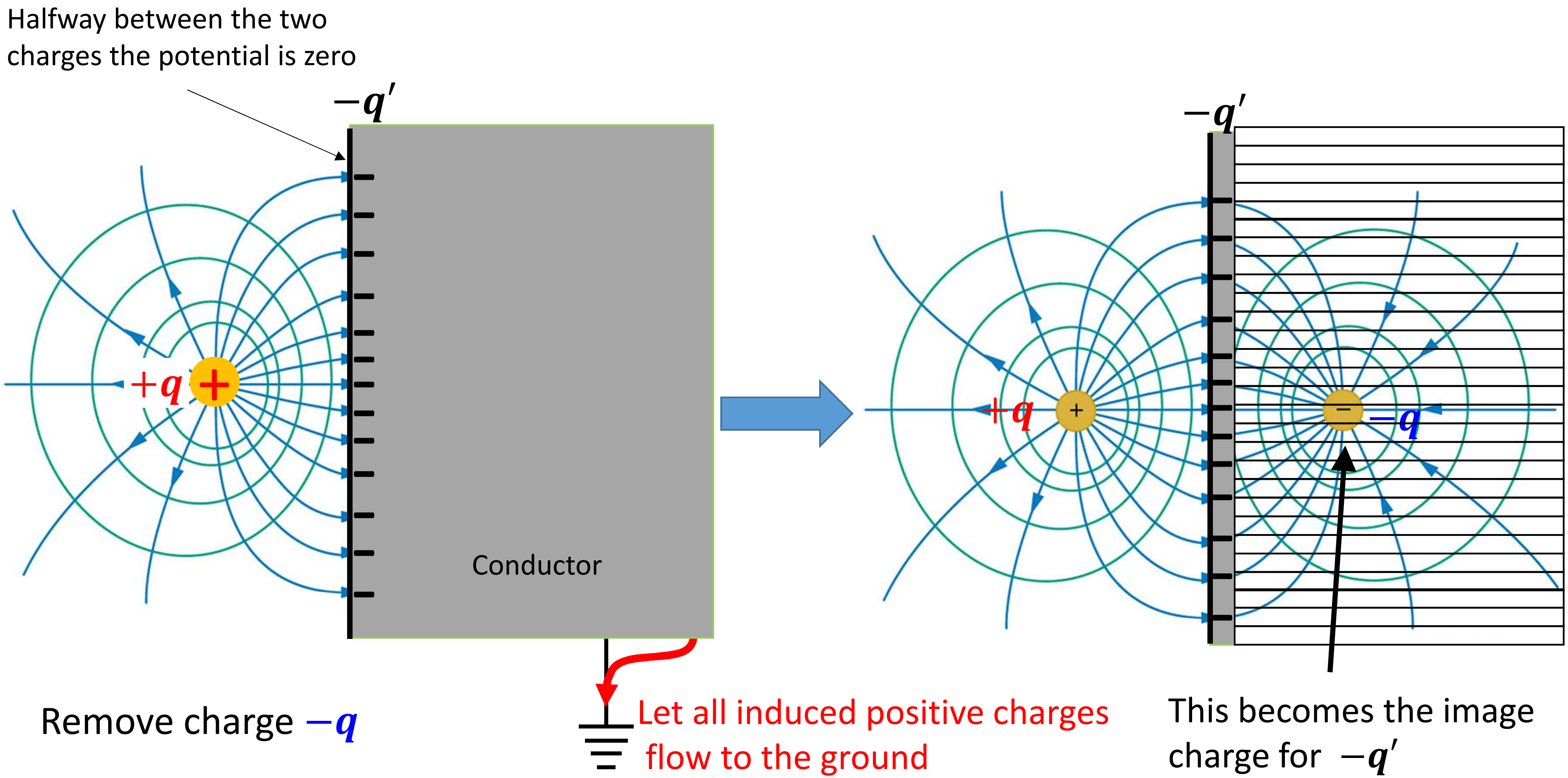
Equal and opposite charges are induced
on the surface of the conductor

The charges induced on the **left** do not distribute
uniformly along the surface of the conductor

The distribution follows the field lines

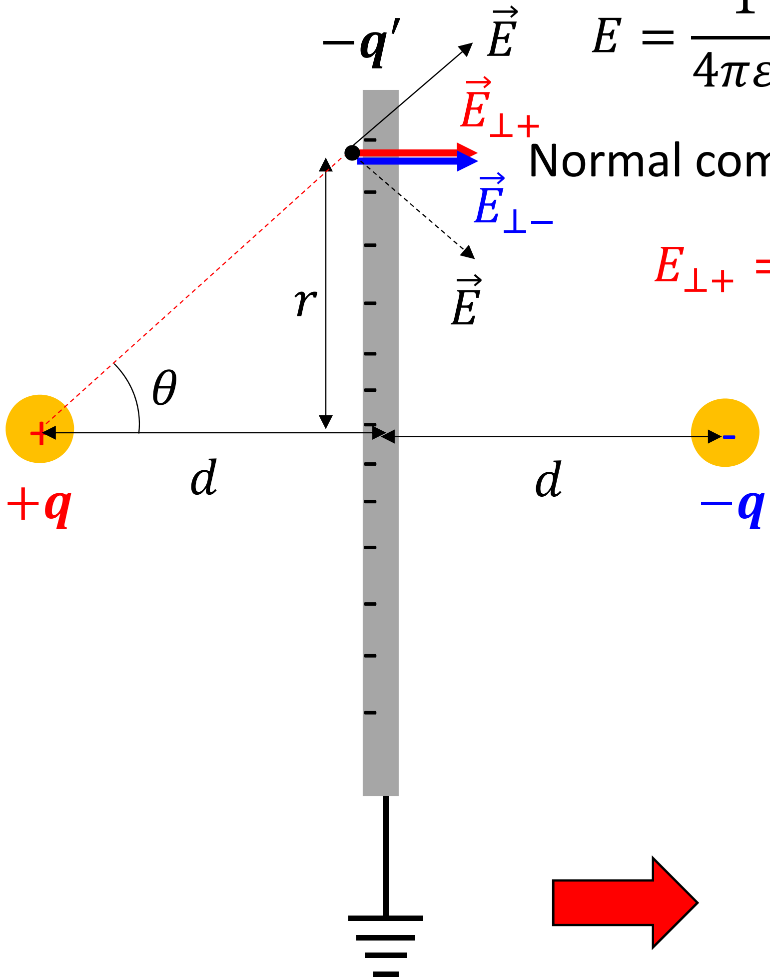
What about the opposite charges on the right?

They are free to distribute uniformly: No constraint



Halfway between the two
charges the potential is zero
$-q'$
$+q$
Conductor
$-q'$
$+q$
$-q$
Remove charge $-q$
Let all induced positive charges
flow to the ground
This becomes the image
charge for $-q'$

The conducting plane has an infinite size



$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{(d^2+r^2)^2}$$ Coulomb's law

Normal component of the field du to charge $+q$ (perpendicular to the plane)

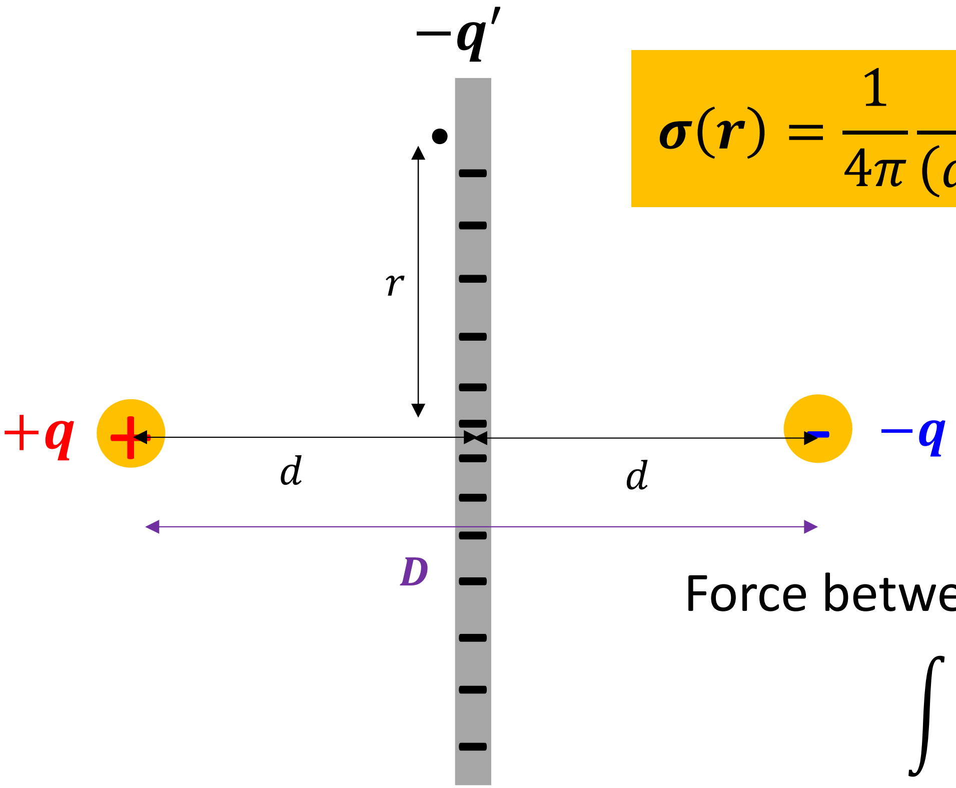
$$E_{\perp+} = E\cos(\theta) \Rightarrow E_{\perp+} = \frac{1}{4\pi\varepsilon_0}\frac{qd}{(d^2+r^2)^{3/2}}$$

To this field we must add the contribution from charge – q' due to the charged conducting plane, whose image charge is $-q$ on the other side of the plane

$$E_{\perp-} = \frac{1}{4\pi\varepsilon_0}\frac{qd}{(d^2+r^2)^{3/2}}$$

$$\Rightarrow \boxed{E(r) = E_{\perp+} + E_{\perp-} = \frac{1}{4\pi\varepsilon_0}\frac{2qd}{(d^2+r^2)^{3/2}}} = \frac{\boldsymbol{\sigma(r)}}{\boldsymbol{\varepsilon_0}}$$

From which we get the **NONUNIFORM** charge distribution



$$\boldsymbol{\sigma}(\boldsymbol{r}) = \frac{1}{4\pi}\frac{2qd}{(d^2+r^2)^{3/2}}$$

$$\int \boldsymbol{\sigma}(\boldsymbol{r})\boldsymbol{dA} = q = q'$$

Surface of the plane

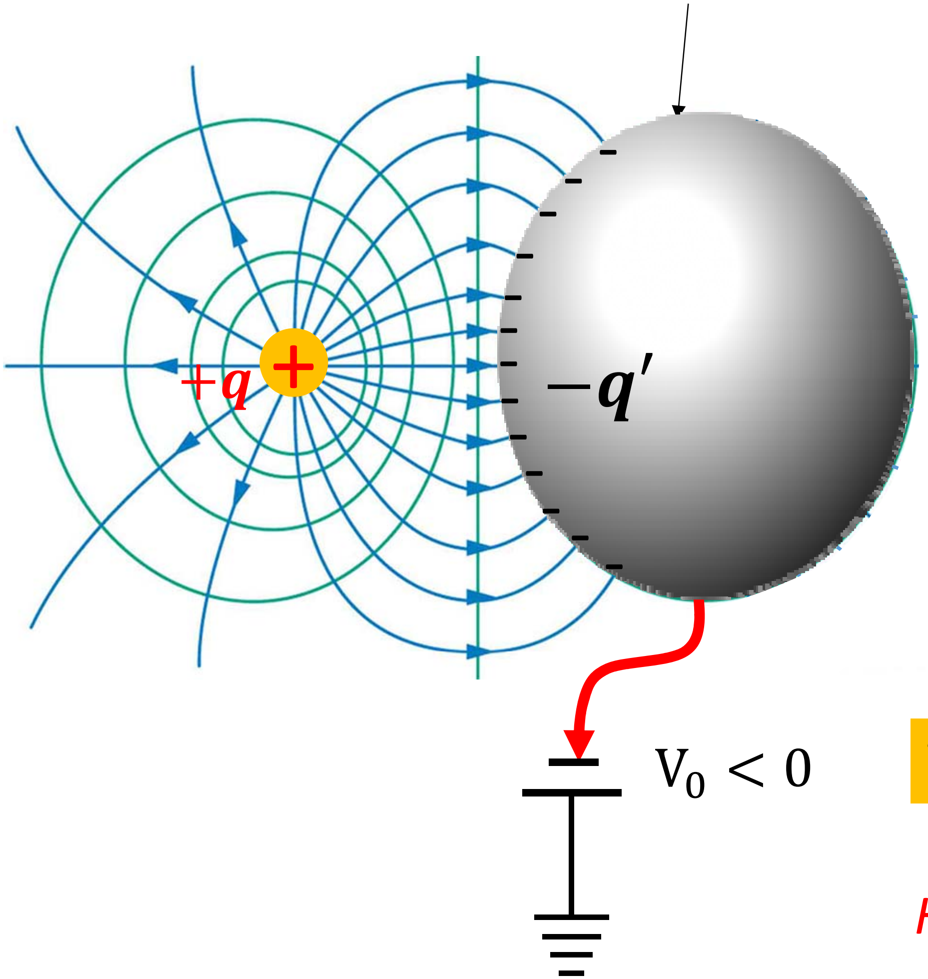
$$dq(r) = \boldsymbol{\sigma}(\boldsymbol{r})2\pi r dr$$

Force between charge $(+\boldsymbol{q})$ and the conductor $(-\boldsymbol{q}')$

$$\int dq(r)E(r) = -\frac{1}{4\pi\varepsilon_0}\frac{qq'}{(2d)^2} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{D^2}$$

Coulomb's law between two opposite charge $+q$ an $-q$
as if the conducting plane was not there !

This is a negative equipotential surface V_0

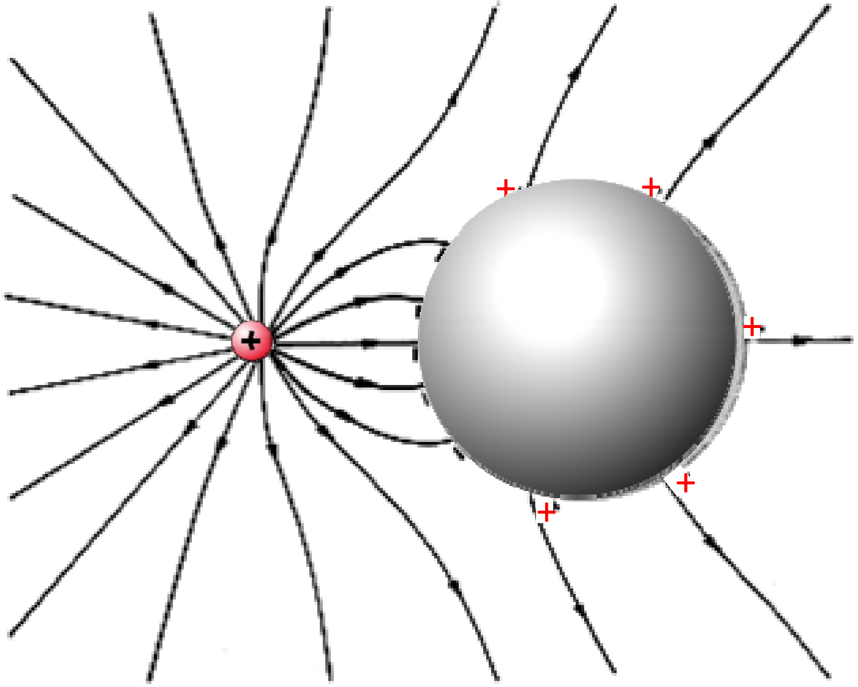


- We replace the charge $-q$ by a conductor fitting with this surface potential
- Charges $-\boldsymbol{q}'$ and $+\boldsymbol{q}'$ are induced on the surface of the conductor due to the field of charge $+\boldsymbol{q}$
- The conductor is placed at the same potential V_0

This configuration reproduces exactly the dipole

However, the equipotential is not exactly a sphere

Interaction of a point charge with a spherical conductor

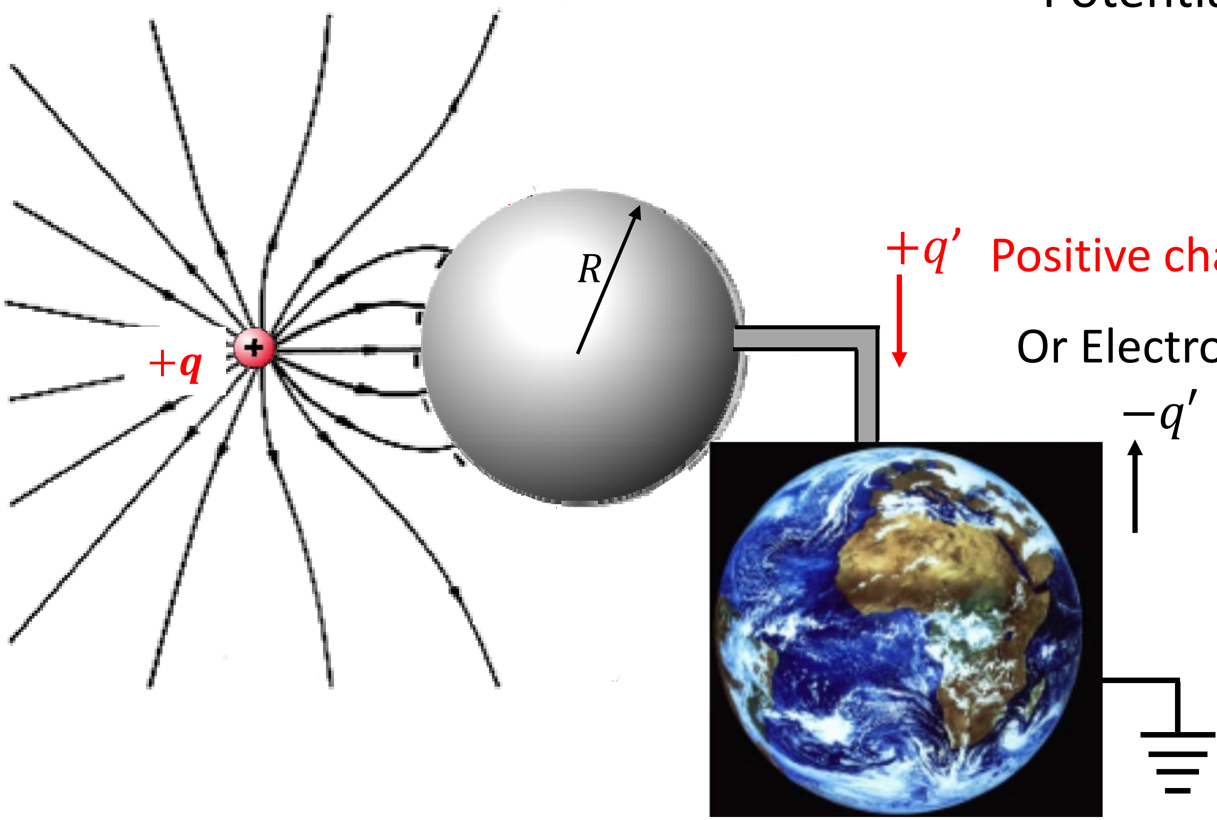


The external charge (field) induces both types of charges on the sphere
⇔ Charge conservation

What happens if we ground the sphere?

Inducing charges on a **grounded** conductor

Potential of any sphere $V = \frac{Q}{4\pi\varepsilon_0 R}$



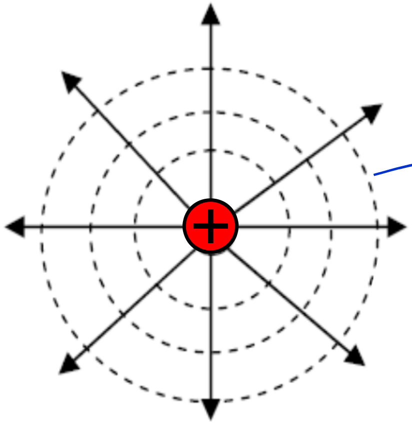
Positive charges flow to the earth

Or Electrons are provided to the sphere

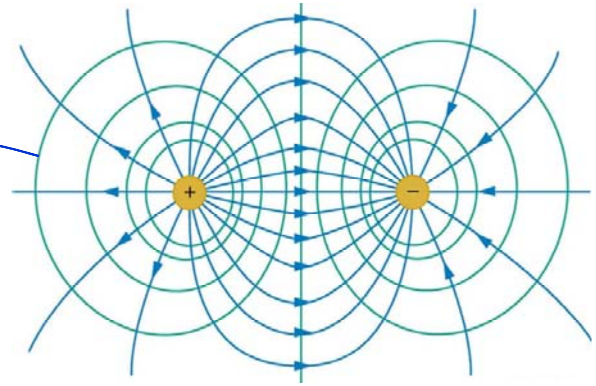
The process is inverted if the External charge is negative

For the earth $R \to \infty \Rightarrow V = 0$ for any charge q

The concept of image force

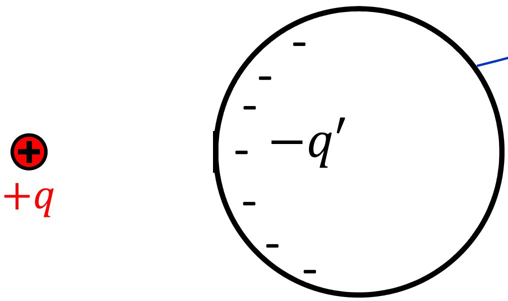


Equipotential

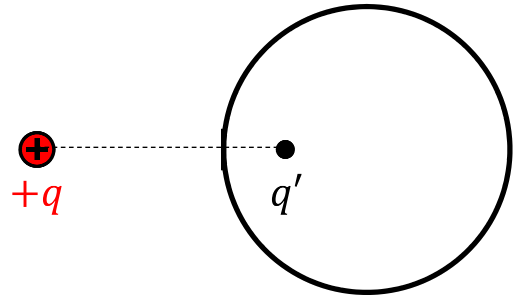


A single charge generates spherical equipotential shapes

Two equal and opposite charges do not generate spherical equipotential shapes



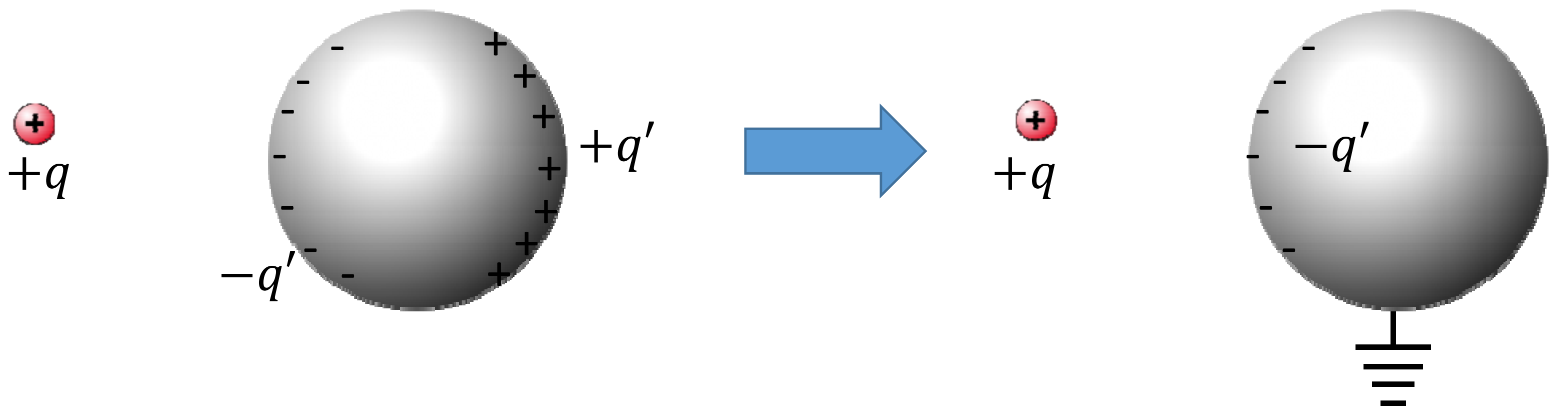
Two ***unequal*** and opposite charges may generate spherical equipotential



q' = Image charge representing all negative charges induced at the surface of the conductor

How do the induced charges distribute at the surface of the conductor so that the spherical potential of the surface is constant ?

Image method

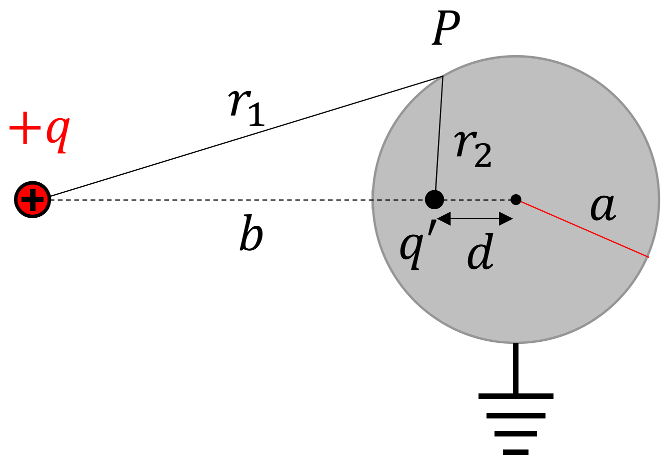


$\mathrm{V} = 0$

Everywhere in the sphere

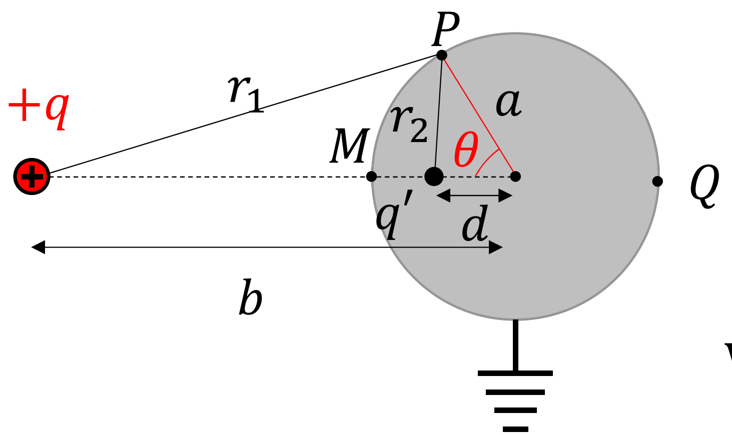
The grounding removes the positive charges

- The charge $-\boldsymbol{q}'$ induced on the sphere is equivalent to an image charge q' placed at d from the center



$$\mathrm{V}(P) = 0 \quad \Rightarrow \quad \mathrm{V}(P) = \frac{q}{r_1} + \frac{q'}{r_2} = 0 \qquad \left(\text{we drop } \frac{1}{4\pi\varepsilon_0}\right)$$

The negative charge induced on the surface is equivalent to q' placed at position $b - d$



V = 0 everywhere in the sphere (equipotential)

$\Rightarrow \quad V(P) = V(M) = V(Q) = 0$

Image charge $q' < 0$

$$V(M) \quad \Rightarrow \quad \frac{+q}{b-a} + \frac{q'}{a-d} = 0 \quad \Rightarrow \quad q' = -\frac{a-d}{b-a}q$$

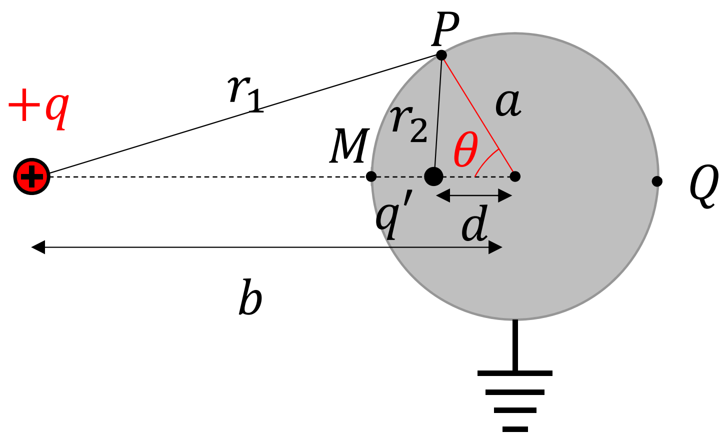
Superposition principle

On any point on the surface ⇨ use **Al-Kashi's theorem**

We know a and b but **NOT** d

Slide #16 D_Lectures 4-7 Coordinate system Scalar versus Vector fields Operators

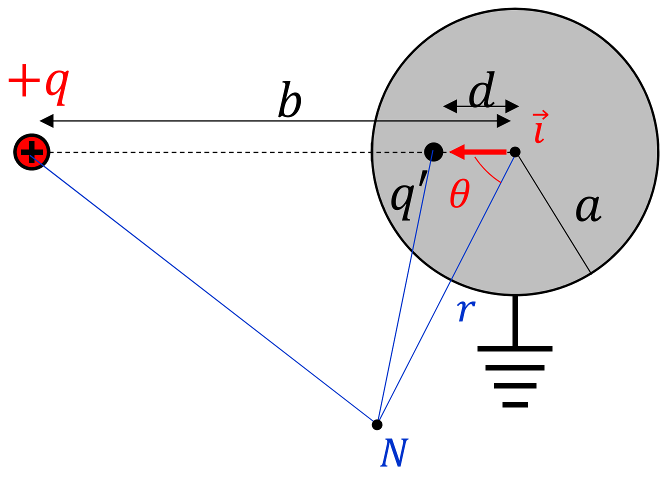
$$r_2 = \sqrt{a^2 + d^2 - 2ad\cos\theta} \qquad r_1 = \sqrt{a^2 + b^2 - 2ab\cos\theta}$$



$$\mathrm{V}(M) = \mathrm{V}(Q) = 0 \Rightarrow \begin{cases} d = \dfrac{a^2}{b} \\ q' = -\dfrac{a}{b}q \end{cases}$$

q' increases with the radius of the sphere (more lines hit the sphere) and decreases with distance b

The field of two unequal charges has an equipotential surface that is a **sphere !**



On a point N Outside the sphere at any point (r, θ, ϕ) from the center of the sphere

$$\mathrm{V}(N) = q\left[\frac{1}{\sqrt{r^2 + b^2 - 2brcos\theta}} - \frac{1}{\sqrt{\left(\frac{rb}{a}\right)^2 + a^2 - 2brcos\theta}}\right]$$

If N is on the surface

$$\mathrm{V}(N) = 0 \quad \Rightarrow \quad r = a$$

What is the force between $+q$ and the image charge q' ?

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{(b-d)^2}\vec{\imath} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2ab}{(b^2-a^2)^2}\vec{\imath}$$

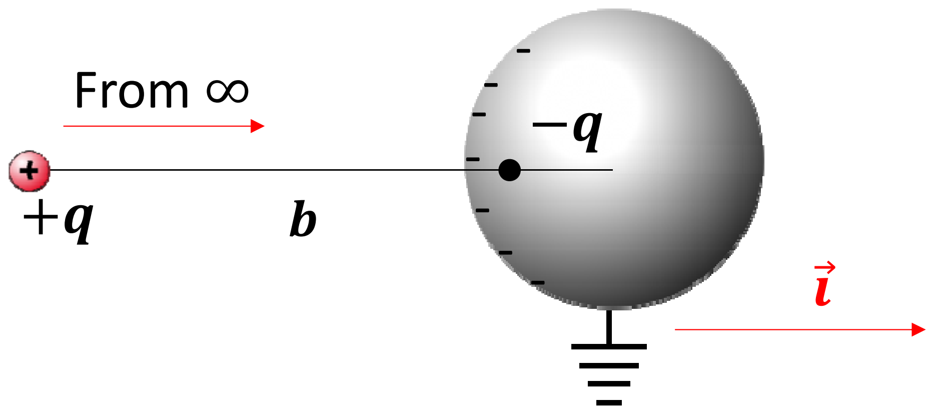
The work done to bring the charge $+q$ from ∞ to a distance $\boldsymbol{b}$ to the center of the grounded conductor: $\boldsymbol{b}$ **is the variable**

The origin is taken at the center of the sphere

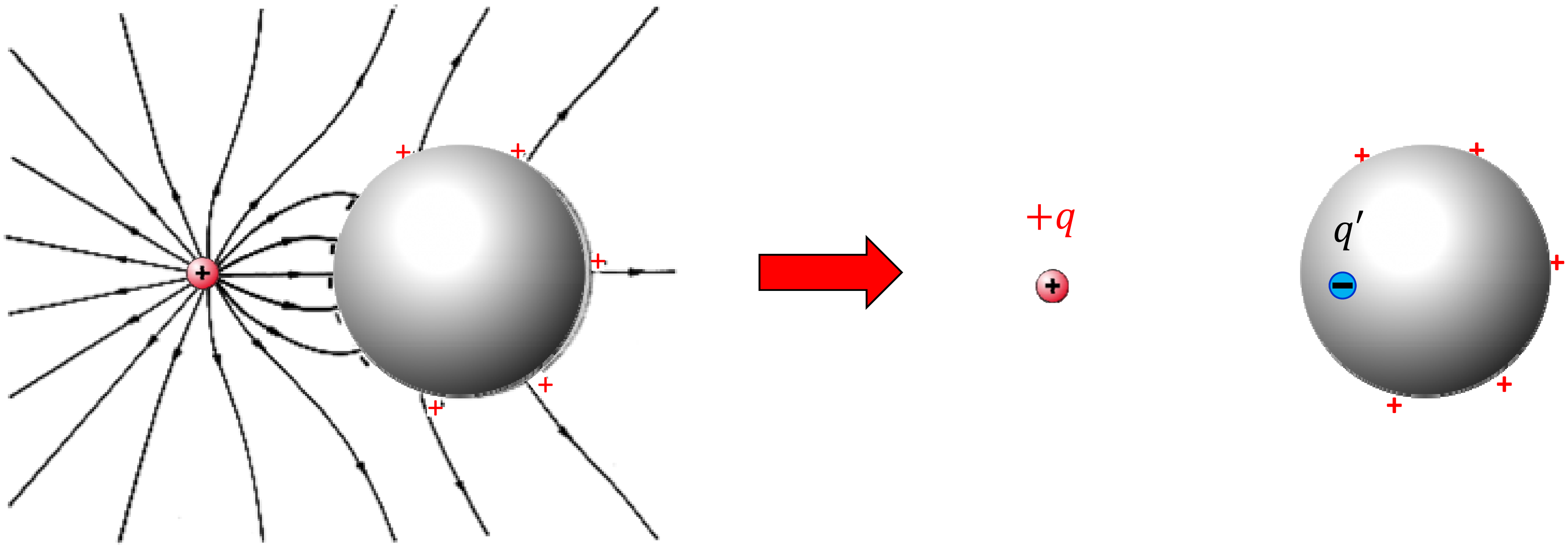
$$W = -\int_{\infty}^{b}\vec{F}.d\vec{l} = \frac{1}{4\pi\varepsilon_0}\int_{\infty}^{b}\frac{q^2ax}{(x^2-a^2)^2}dx$$

$$\boxed{W = -\frac{1}{8\pi\varepsilon_0}\frac{aq^2}{b^2-a^2} < 0}$$

Energy is extracted from the system



Question: What would happen if we do not ground the conducting sphere



- How should we account for the induced positive charges?
- Should we use another image charge q''?
- And if so where should it be placed?