Chapter 2 – Theory of Relativity (II): Dynamics and Relativistic Energy

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Modern Physics (Summer 2020)
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Agenda

1. Dynamics
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   - Relativistic Momentum
   - Relativistic Equation of Motion
   - Example
2. Relativistic Energy
   - Relativistic Energy
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Dynamics

Motivation

Observation. The 2nd law of dynamics in the form $\bar{F} = m\bar{a}$ cannot be used if m is assumed to be invariant.

Arguments

- The Lorentz transformation for acceleration (see Problem Set)

$$
\begin{aligned}
a'_x &= \frac{a_x}{(1 - vu_x/c^2)^3}(1 - \frac{v^2}{c^2})^{\frac{3}{2}}, \\
a'_y &= \frac{a_y(1 - vu_x/c^2) + a_x v u_y/c^2}{(1 - vu_x/c^2)^3}\left(1 - \frac{v^2}{c^2}\right), \\
a'_z &= \frac{a_z(1 - vu_x/c^2) + a_x v u_z/c^2}{(1 - vu_x/c^2)^3}\left(1 - \frac{v^2}{c^2}\right).
\end{aligned}
$$

- $\bar{F} = m\bar{a}$ would imply $v \to \infty$ for motion with $\bar{F} = \text{const}$.

Remedy. Use $\bar{F} = \frac{\mathrm{d}\bar{p}}{\mathrm{d}t}$, but $\bar{p}$ (the momentum) has to be redefined.

Relativistic Momentum

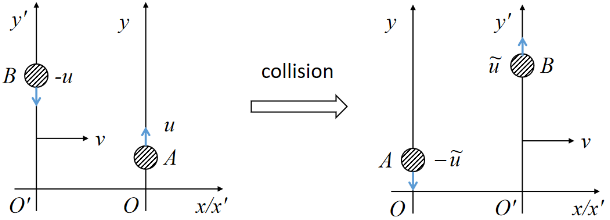
Relativistic Momentum

Find a new definition of the momentum by using the momentum conservation principle. In general, conservation principles are more fundamental than equations of motion. (They follow from symmetry of spacetime — Noether's theorem).

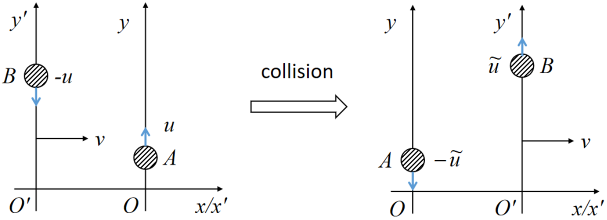
Goal. We need to find a new formula for the momentum, such that it

- satisfies the conservation of momentum law,
- and reduces to the non-relativistic case when $u/c \to 0$.

How? Consider a situation where the outcome is known due to symmetry. E.g., elastic collision of two identical balls.



Relativistic Momentum. Derivation



$u_x^A = 0 \quad u_{x'}^B = 0$ $\qquad$ $u_x^A = 0 \quad u_{x'}^B = 0$

$u_y^A = u \quad u_{y'}^B = -u$ $\qquad$ $u_y^A = -\tilde{u} \quad u_{y'}^B = \tilde{u}$

Rewrite all components of velocities in a single FoR, e.g. the unprimed, using the relativistic velocity addition formulas:

$u_x = \frac{u'_x + v}{1 + u'_x v/c^2}$ and $u_y = \frac{u'_y}{1 + u'_x v/c^2}\sqrt{1 - \frac{v^2}{c^2}}$.

$$u_x^A = 0, \quad u_x^B = v \qquad\qquad u_x^A = 0, \quad u_x^B = v$$

$$u_y^A = u, \quad u_y^B = -u\sqrt{1 - \frac{v^2}{c^2}} \qquad\qquad u_y^A = -\tilde{u}, \quad u_y^B = \tilde{u}\sqrt{1 - \frac{v^2}{c^2}}$$

Observation: If $\bar{p} = m\bar{u}$ ($m =$ const.), then $\bar{p}$ is not conserved!

Modification: $\bar{p} = m(u^2)\bar{u}$, where the square makes the mass insensitive to the direction.

Use conservation of momentum

$$\boxed{x} \qquad 0 + m[v^2 + u^2(1 - \frac{v^2}{c^2})]v \quad = \quad 0 + m[v^2 + \widetilde{u}^2(1 - \frac{v^2}{c^2})]v$$

$$\begin{aligned} \boxed{y} \qquad & m(u^2)u + m[v^2 + u^2(1 - \frac{v^2}{c^2})](-u)\sqrt{1 - \frac{v^2}{c^2}} \\ & = \quad m(\widetilde{u}^2)(-\widetilde{u}) + m[v^2 + \widetilde{u}^2(1 - \frac{v^2}{c^2})]\widetilde{u}\sqrt{1 - \frac{v^2}{c^2}} \end{aligned}$$

From $\boxed{x}$ we obtain that $u^2 = \widetilde{u}^2$ and hence $u = \widetilde{u}$ (compatibility with the non-relativistic limit).

Now, pluging $u = \widetilde{u}$ into $\boxed{y}$ yields

$$m(u^2)\cancel{u} = m[v^2 + u^2(1 - \frac{v^2}{c^2})]\cancel{u}\sqrt{1 - \frac{v^2}{c^2}} \quad \overset{u\to 0}{\Longrightarrow} \quad \boxed{m(u^2) = \frac{m(0)}{\sqrt{1 - u^2/c^2}}}$$

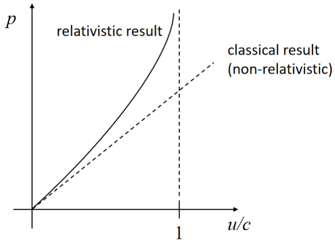
Relativistic Momentum. Discussion

Eventually, the momentum of a particle moving with velocity $\bar{u}$ is given by

$$\boxed{\bar{p} = \frac{m(0)\bar{u}}{\sqrt{1 - u^2/c^2}}}$$

Note. For $u \ll c$, that is $u/c \to 0$ we have $\bar{p} \to m(0)\bar{u}$ (non-relativistic formula recovered!). Here $m(0)$ is called the **rest mass** and will be denoted just as m. The quantity $m/\sqrt{1 - u^2/c^2}$ is called the **relativistic mass**.

Comparison between classical & relativistic momentum



1. The relativistic momentum can "fix" the 2nd Newton's law $\bar{F} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m\bar{u}}{\sqrt{1-u^2/c^2}}\right)$.
2. If $u = c$ and p is finite, then the rest mass must be zero ($m = 0$). Example: photons.

Dynamics

Equation of Motion

The relativistic equation of motion can be written down as

$$\bar{F} = \frac{\mathrm{d}\bar{p}}{\mathrm{d}t} \qquad \text{or} \qquad F_i = \frac{\mathrm{d}p_i}{\mathrm{d}t}, \quad i = x, y, z,$$

but the relativistic momentum must be used.

Let us rewerite the r.h.s.

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}[m(u^2)u_i] = \frac{\mathrm{d}[m(u^2)]}{\mathrm{d}t} u_i + m(u^2)\dot{u}_i$$

where $\dot{u}_i = a_i$. But

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}[m(u^2)] &= \frac{\mathrm{d}m(u^2)}{\mathrm{d}u^2} \cdot \frac{\mathrm{d}}{\mathrm{d}t}(u^2) = \frac{m}{(1-u^2/c^2)^{3/2}} \left(-\frac{1}{2}\right) \cdot \left(-\frac{1}{c^2}\right) 2u_j\dot{u}_j \\ &= \frac{1}{c^2} \frac{m}{(1-u^2/c^2)^{3/2}} u_j \dot{u}_j \end{aligned}$$

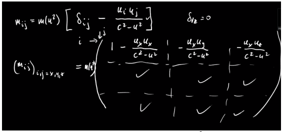
Hence

$$F_i = \frac{\mathrm{d}p_i}{\mathrm{d}t} = m(u^2)\left[\dot{u}_i + \frac{1}{c^2 - u^2} u_i u_j \dot{u}_j\right]$$
$$= m(u^2)\left[\delta_{ij} + \frac{1}{c^2 - u^2} u_i u_j\right]\dot{u}_j = m_{ij}\dot{u}_j.$$

Eventually

$$\boxed{F_i = m_{ij}\dot{u}_j} \qquad \text{with} \qquad \boxed{m_{ij} = m(u^2)\left[\delta_{ij} + \frac{1}{c^2 - u^2} u_i u_j\right]}$$

where $\delta_{\alpha\beta}$ is the Krönecker delta: $\delta_{\alpha\beta} = \begin{cases} 1, & \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases}$ The matrix m_{ij} is called the **relativistic mass tensor**.

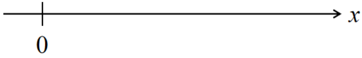


Comments

- In general $\bar{F} \nparallel \bar{a}$.
- If $u/c \to 0$ then $m_{ij} = m\delta_{ij}$, i.e. the usual scalar mass (equal to the rest mass) can be used. Now $\bar{F}_i = m\dot{u}_i$.

Example. Motion in 1D under Constant Force

$F_x = \text{const}$



Initial conditions:
$u_x(0) = x(0) = 0.$

Rewriting the equation of motion $\frac{\mathrm{d}p_x}{\mathrm{d}t} = F_x = \text{const.}$ as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{mu_x}{\sqrt{1-u_x^2/c^2}}\right) = F_x \quad \Longrightarrow \quad \int_0^{\frac{mu_x(t)}{\sqrt{1-u_x^2(t)/c^2}}} \mathrm{d}\left(\frac{mu_x}{\sqrt{1-u_x^2/c^2}}\right) = \int_0^t F_x\,\mathrm{d}t$$

we obtain

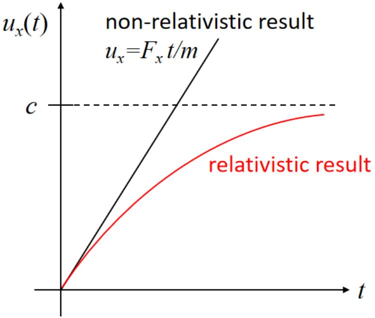
$$\frac{mu_x(t)}{\sqrt{1-u_x^2(t)/c^2}} = F_x t.$$

Solve for $u_x(t)$ to find

$$[mu_x(t)]^2 = (F_x t)^2\left(1 - \frac{u_x^2(t)}{c^2}\right)$$

$$\boxed{u_x(t) = \frac{F_x t/m}{\sqrt{1+(F_x t/mc)^2}}}$$

Discussion



$$\boxed{u_x(t) = \frac{F_x t/m}{\sqrt{1 + (F_x t/mc)^2}}}$$

- In the initial phase of motion, t is small so $F_x t/mc \ll 1$, then $u_x(t) \sim F_x t/m$ as in the non-relativistic case.
- However, as t increases, $F_x t/mc \gg 1$ and $u_x \to c$.

Relativistic Energy

Relativistic Energy. Derivation

From the work-kinetic energy theorem:

$$\delta W = \bar{F} \cdot \mathrm{d}\bar{r} = \mathrm{d}K,$$

where δW is the elementary work done by the net force. Rate of work done

$$\frac{\delta W}{\mathrm{d}t} = \bar{F} \cdot \frac{\mathrm{d}\bar{r}}{\mathrm{d}t} = F_i u_i = m_{ij} \dot{u}_j u_i = \frac{\mathrm{d}K}{\mathrm{d}t}.$$

Using the expression for the relativistic mass tensor, we get

$$\begin{aligned}
m_{ij} \dot{u}_j u_i &= \frac{m}{\sqrt{1 - u^2/c^2}} [\delta_{ij} + \frac{1}{c^2 - u^2} u_i u_j] u_i \dot{u}_j \\
&= \frac{m}{\sqrt{1 - u^2/c^2}} [u_j \dot{u}_j + \frac{1}{c^2 - u^2} u^2 u_j \dot{u}_j] \\
&= \frac{m}{\sqrt{1 - u^2/c^2}} [1 + \frac{u^2}{c^2 - u^2}] u_j \dot{u}_j = \frac{m}{(1 - u^2/c^2)^{3/2}} u_j \dot{u}_j \\
&= \frac{\mathrm{d}}{\mathrm{d}t} [m(u^2) c^2].
\end{aligned}$$

Where in the last step the result $\frac{\mathrm{d}}{\mathrm{d}t}[m(u^2)] = \frac{1}{c^2} \frac{m}{(1-u^2/c^2)^{3/2}} u_j \dot{u}_j$ was used.

Hence

$$\frac{\mathrm{d}K}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}[m(u^2)c^2] \qquad \Longrightarrow \qquad K = \frac{mc^2}{\sqrt{1-u^2/c^2}} + \text{const.}$$

Finally the constant is found from the requirement relativistic formulas become non-relativistic ones if $u \ll c$, so that in this limit, $K \to mu^2/2 = K_{\text{non-rel}}$. Therefore,

$$K = mc^2(1 - (-\frac{1}{2})\frac{u^2}{c^2} + ...) + \text{const} \xrightarrow{u/c \to 0} mc^2 + K_{\text{non-rel}} + \text{const.}$$

Hence we find const $= -mc^2$ and, eventually,

$$\boxed{K = \frac{mc^2}{\sqrt{1-u^2/c^2}} - mc^2.}$$

Relativistic Energy. Discussion

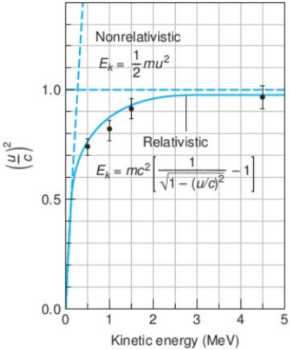
The formula for the relativistic kinetic energy can also be rewritten as

$$K + mc^2 = \frac{mc^2}{\sqrt{1 - u^2/c^2}} = E,$$

where E is the total energy and is conserved for isolated systems.

In particular, if $\bar{u} = 0$ (the particle is at rest), then $\boxed{E = mc^2}$.

FIGURE 2-3 Experimental confirmation of the relativistic relation for kinetic energy. Electrons were accelerated to energies up to several MeV in large electric fields and their velocities were determined by measuring their time of flight over 8.4 m. Note that when the velocity $u \ll c$, the relativistic and nonrelativistic (i.e., classical) relations are indistinguishable. [*W. Bertozzi, American Journal of Physics*, **32**, *551 (1964)*.]

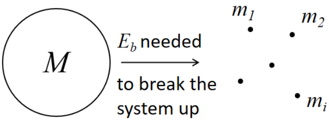


Mass-Energy Equivalence & Binding Energy

Whenever an additional energy ΔE in any form is stored in an object, the rest mass of the object is increased by $\Delta m = \Delta E / c^2$.

Implication. The mass of a bound system (e.g. nucleus) is less than that of separated particles (e.g. protons & neutrons) by E_b / c^2, where E_b is the **binding energy**.

Example. $\alpha-$particle (He^{2+})



$$M - \sum m_i = -\frac{E_b}{c^2}$$

$$m_{\mathrm{He}^{2+}} = 3727.39 \mathrm{\ MeV}/c^2$$

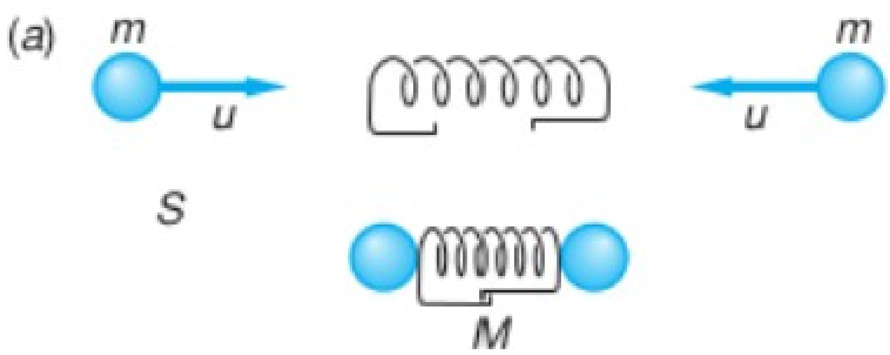
$$\begin{aligned} 2m_p + 2m_n &= 2 \times (938.272 + 939.565) \\ &= 3755.67 \mathrm{\ MeV}/c^2 \end{aligned}$$

$$m_{\mathrm{He}^{2+}} - 2(m_p + m_n) = -28.28 \mathrm{\ MeV}/c^2$$

Mass-Energy Equivalence: Example – Inelastic Collision

Whenever additional energy ΔE in any form is stored in an object, the rest mass of this object increases.

Consider a collision of two identical massive balls with a massless elastic spring that locks (perfectly inelastic collision).



S: laboratory FoR

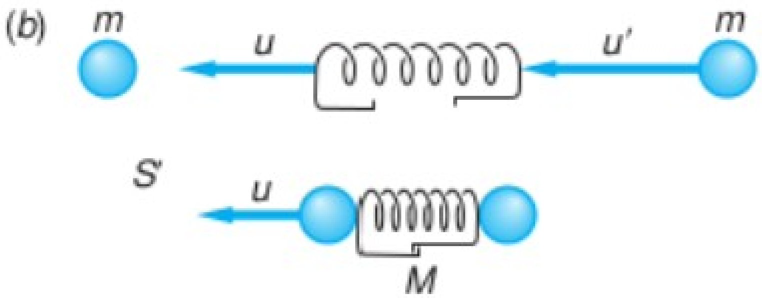
non-relativistic description: *Kinetic energy of both particles transforms into the (elastic) potential energy of the system.*

relativistic description: *Kinetic energy of the particles is stored in the excess rest mass of the system.*

$$M = 2m + \Delta m, \qquad \Delta m = \frac{K}{c^2}$$

Mass-Energy Equivalence: Example – Inelastic Collision

Goal: Find the mass change



S': FoR associated with the left mass; (moving to the right with v = u in the lab FoR)

$$u' = \frac{-u - v}{1 - \frac{-uv}{c^2}} \overset{v=u}{=} \frac{-2u}{1 + \frac{u^2}{c^2}}$$

Momentum (in S') before and after the collision

$$p'_i = \frac{mu'}{\sqrt{1 - \frac{u'^2}{c^2}}} = \frac{2mu}{1 - \frac{u^2}{c^2}} \qquad p'_f = \frac{Mu}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Conservation of momentum $p'_i = p'_f$

$$\frac{2mu}{1 - \frac{u^2}{c^2}} = \frac{Mu}{\sqrt{1 - \frac{u^2}{c^2}}} \Longrightarrow M = \frac{2m}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$\Delta m = M - 2m$$

$$\Delta m = 2m\left(\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} - 1\right) = \frac{K}{c^2}$$

Four-Momentum. Energy–Momentum Relationship

The (contravariant) four–momentum is defined as

$$p^{\mu} = \left(\frac{E}{c}, \bar{p}\right)^{T}.$$

For massive particles $(m \neq 0)$

$$\bar{p} = \frac{m\bar{u}}{\sqrt{1 - u^2/c^2}}, \qquad E = \frac{mc^2}{\sqrt{1 - u^2/c^2}}, \qquad p^{\mu} = mu^{\mu}$$

where u^{μ} is the four-velocity.

Observations

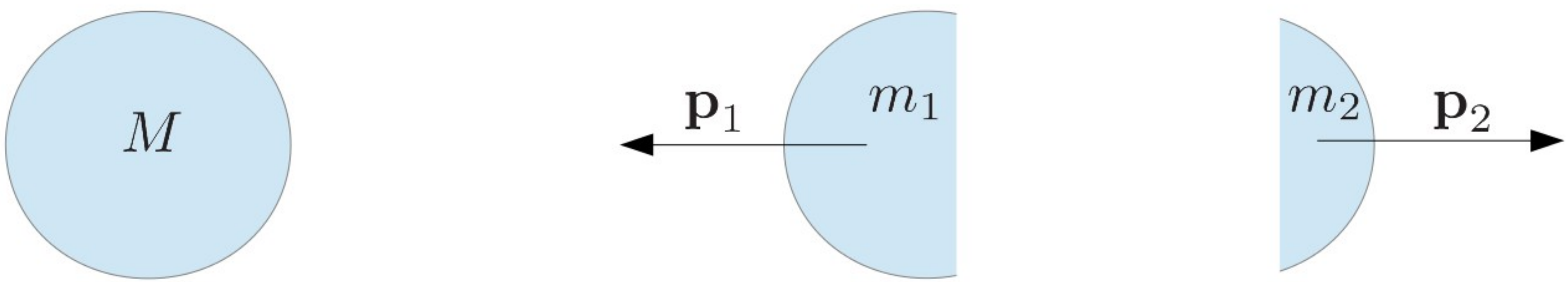
- p^{μ} transforms according to Lorentz transformation, i.e. it is a four-vector.
- Note that $p^{\mu} p_{\mu} = \left(\frac{E}{c}\right)^2 - p^2 = \frac{(mc)^2}{1-u^2/c^2} - \frac{m^2 u^2}{1-u^2/c^2} = m^2 c^2$.

  Therefore $\boxed{E^2 = p^2 c^2 + m^2 c^4}$ or $\boxed{E = \sqrt{p^2 c^2 + m^2 c^4}}$.

  - ⋆ For massless relativistic particles $(m = 0$ and $u = c) \Rightarrow E = pc$.
  - ⋆ For ultra-relativistic particles $(u \approx c) \Rightarrow E \approx pc$.

Relativistic Energy and Momentum: Example – Particle Decay

Problem: A resting particle with mass M decays into two parts with rest masses m_1 and m_2, respectively.
Find the kinetic energies of the products of this decay.



Solution: Isolated system – use conservation of total energy and momentum; write both in the FoR, where the particle before the decay rests (in the lab FoR)

$$Mc^2 = E_1 + E_2$$

E_1, E_2 – total energies

$$0 = \mathbf{p}_1 + \mathbf{p}_2$$

$\mathbf{p}_1$, $\mathbf{p}_2$ – and momenta of decay products

Relativistic Energy and Momentum: Example – Particle Decay

$$\boxed{Mc^2 = E_1 + E_2}$$

$$0 = \mathbf{p}_1 + \mathbf{p}_2$$

The latter equation yields $p_1^2 = p_2^2$

Hence, from the relation $E^2 = (pc)^2 + (mc^2)^2$

$$E_1^2 - (m_1c^2)^2 = E_2^2 - (m_2c^2)^2 \implies E_1^2 - E_2^2 = c^4(m_1^2 - m_2^2)$$

Combining with conservation of energy

$$\boxed{E_1 - E_2 = \frac{c^2(m_1^2 - m_2^2)}{M}}$$

Relativistic Energy and Momentum: Example – Particle Decay

System of two linear equations

$$Mc^2 = E_1 + E_2$$

$$E_1 - E_2 = \frac{c^2(m_1^2 - m_2^2)}{M}$$

Solution of the system

$$E_1 = \frac{M^2c^2 + c^2(m_1^2 - m_2^2)}{2M} \implies K_1 = E_1 - m_1c^2 = \frac{c^2}{2M}\left[(M - m_1)^2 - m_2^2\right]$$

$$E_2 = \frac{M^2c^2 - c^2(m_1^2 - m_2^2)}{2M} \implies K_2 = E_2 - m_2c^2 = \frac{c^2}{2M}\left[(M - m_2)^2 - m_1^2\right]$$