Problem Set 7

Due: 15 July 2020, 12.30 p.m.

Problem 1. Use the momentum–position commutation rule $[\hat{x}_\alpha, \hat{p}_\beta] = i\hbar\delta_{\alpha\beta}$, where $\alpha, \beta = x, y, z$ and to check that $[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x$.

(*3 points*)

Problem 2. For the case $l = 3$ sketch a vector model illustrating possible orientations of the angular momentum $\mathbf{L}$ in space and the possible values of L_z, i.e. its projection onto the z–axis.

What is the minimum angle between the angular momentum $\mathbf{L}$ and the z–axis?

(*2 points*)

Problem 3. Consider two particles with mass m each, connected by a massless rigid rod of length $2a$. The system is free to rotate about its center of mass which is fixed in space. The system is far away from any force fields, so it is purely kinetic. This model, mentioned in class, is known as the (linear) rigid rotor and it is used to discuss the rotational energy spectrum of diatomic molecules.

(a) Repeat the calculation we did in class to find the energy levels of this system.

(b) Make an energy-level diagram of these energies, and indicate the transitions that obey the *selection rule* $\Delta l = \pm 1$. Show that the allowed transition energies are $E_1, 2E_1, 3E_1, \ldots$, where $E_1 = \hbar^2/I$ and I is the moment of inertia with respect to the center of mass.

(c) The distance between the protons in the H_2 molecule is $2a \approx 0.074$ nm and proton's mass is $m \approx 1.67 \cdot 10^{-27}$ kg. Find the energy of the first excited state ($l = 1$) of the H_2 molecule, assuming that it is a rigid rotor.

(d) What is the lowest energy for which a transition between rotational levels in the H_2 molecule can be observed? Give the answer in eV. Calculate the corresponding length of electromagnetic radiation and identify the region of the electromagnetic spectrum this wavelength corresponds to.

(*1 + 2 + 2 + 1 points*)

Problem 4. A linear rigid rotor whose rotations are restricted to the xy plane (hence are about the z axis) is called the planar rigid rotor.

(a) Using classical mechanics write down a relation between the angular momentum and the total energy of this system, i.e. kinetic energy in rotational motion. How many non-zero components does the angular momentum of this planar rotor have?

(b) Write down the Hamiltonian of the corresponding quantum system. Find its eigenvalues and eigenfunctions (you may directly use all results we derived in class).

(c) At some instant of time the rotor is described by the wave function $\psi(\varphi) = C\cos^2\varphi$, where C is the normalizing constant (find its value). If measure the energy of the planar rotor in this state, what are the possible values and the corresponding probabilities?

(1 + 2 + 2 points)

Problem 5. A hydrogen atom is in the $6f$ state. (a) What are the values of n and l? (b) Compute the energy of the electron. (c) Compute the magnitude of $\mathbf{L}$. (d) Compute the possible values of L_z in this situation.

(4 × 1/2 points)

Problem 6. (a) Look up the corresponding tables and check that the radial probability density for the $n = 2$, $l = 1$, $m = 0$ state of the hydrogen atom is $P(r) \propto r^4 e^{-r/a_0}$, where a_0 is the radius of the first Bohr orbit. (b) Show that the electron in the hydrogen atom in this state is most likely to be found at $r = 4a_0$.

(3/2 + 3/2 points)

Problem 7. The radius of the proton is about $R_0 = 10^{-15}$ m. The probability that the electron is inside the volume occupied by the proton is $\int_0^{R_0} P(r)\,\mathrm{d}r$, where $P(r)$ is the radial probability density. Compute this probability for the ground state of the hydrogen atom. (*Hint.* Use the fact that $e^{-2r/a_0} \approx 1$ for $r \ll a_0$.)

(2 mark)

Problem 8. Assume that the electron is a classical particle, a sphere of radius 10^{-15} m and uniform mass density, that rotates about its axis of symmetry with angular momentum equal to the magnitude of the spin angular momentum $|\mathbf{S}| = \hbar\sqrt{s(s+1)} = \hbar\sqrt{\frac{3}{4}}$. Compute the linear speed of a point on „equator” of the electron.

How does it compare with the speed of light? Hence, is this classical understanding of spin as the angular momentum in rotational motion about the electron's axis of symmetry correct?

(*Hint.* The moment of inertia of a sphere about its axis of symmetry is $I = \frac{2}{5}mR^2$.)

(2 points)

Problem 9. (a) The angular momentum of the yttrium atom in the ground state is characterized by the quantum number $j = 3/2$. How many lines would you expect to see if you did a Stern-Gerlach experiment with yttrium atoms? (b) How many lines would you expect to see if the beam consisted of atoms with zero spin, but $l = 1$?

(3/2 + 3/2 points)

Problem 10. Suppose that the outer electron in a potassium atom is in a state with $l = 2$. Compute the magnitude of $\mathbf{L}$. What are the possible values of j and the possible magnitudes of $\mathbf{J}$?

(1 point)