VP390 Problem Set 7

Pan, Chongdan ID:516370910121

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1 Problem 1

$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z, \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$
$[\hat{L}_y, \hat{L}_z] = \hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y = (\hat{z}\hat{p}_x - \hat{x}\hat{p}_z)(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x) - (\hat{x}\hat{p}_y - \hat{y}\hat{p}_x)(\hat{z}\hat{p}_x - \hat{x}\hat{p}_z)$
$\hat{L}_y\hat{L}_z = \hat{z}\hat{p}_x\hat{x}\hat{p}_y + \hat{x}\hat{p}_z\hat{y}\hat{p}_x - \hat{x}\hat{p}_z\hat{x}\hat{p}_y - \hat{z}\hat{p}_x\hat{y}\hat{p}_x = \hat{z}\hat{p}_y\hat{p}_x\hat{x} + \hat{p}_z\hat{y}\hat{x}\hat{p}_x - \hat{p}_z\hat{p}_y\hat{x}\hat{x} - \hat{y}\hat{z}\hat{p}_x\hat{p}_x$
$-\hat{L}_z\hat{L}_y = \hat{y}\hat{p}_x\hat{z}\hat{p}_x + \hat{x}\hat{p}_y\hat{x}\hat{p}_z - \hat{x}\hat{p}_y\hat{z}\hat{p}_x - \hat{y}\hat{p}_x\hat{x}\hat{p}_z = \hat{y}\hat{z}\hat{p}_x\hat{p}_x + \hat{p}_y\hat{p}_z\hat{x}\hat{x} - \hat{p}_y\hat{z}\hat{x}\hat{p}_x - \hat{y}\hat{p}_z\hat{p}_x\hat{x}$
$[\hat{L}_y, \hat{L}_z] = \hat{y}\hat{p}_z(\hat{x}\hat{p}_x - \hat{p}_x\hat{x}) - \hat{z}\hat{p}_y(\hat{x}\hat{p}_x - \hat{p}_x\hat{x}) = \hat{y}\hat{p}_z[\hat{x}, \hat{p}_x] - \hat{z}\hat{p}_y[\hat{x}, \hat{p}_x]$
$[\hat{L}_y, \hat{L}_z] = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y[\hat{x}, \hat{p}_x] = i\hbar\hat{L}_z$

2 Problem 2

$\cos\theta = \frac{L_z}{L} = \frac{m}{\sqrt{3(3+1)}} = \frac{\sqrt{3}}{2} \to \theta = \frac{\pi}{6}$

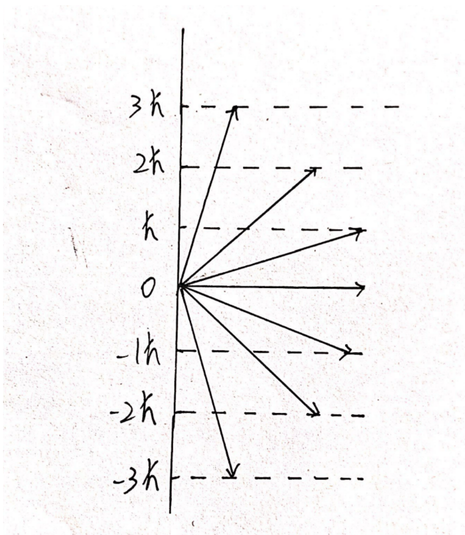

Figure 1: Projection of L on the z-axis

3 Problem 3

(a)$\mu = \frac{m^2}{2m} = \frac{m}{2}$
$I = \mu(2a)^2 = 2ma^2$
$\hat{H}\psi = \frac{\hat{L}^2}{2I}\psi = E\psi \to \hat{L}^2\psi = 4Ema^2\psi$
Since $L = \hbar\sqrt{l(l+1)}$
$E_l = \frac{\hbar^2 l(l+1)}{4ma^2}$
$[-\frac{\hbar^2}{2\mu}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{\hat{L}}{2\mu r^2} + V(r)]\psi = E\psi$
$[-\frac{\hbar^2}{2\mu}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{\hbar^2 l(l+1)}{2\mu r^2} + V(r)]R(r) = ER(r)$
Then $E = -\frac{E_1}{n^2}$ where $E_1 = \frac{\mu e^4}{32\pi^2\hbar^2\epsilon_0} = \frac{me^4}{64\pi^2\hbar^2\epsilon_0}$ and $R_{nl} = Ae^{-\frac{r}{a_0 n}}r^l L_{nl}(\frac{r}{a_0})$
(b)

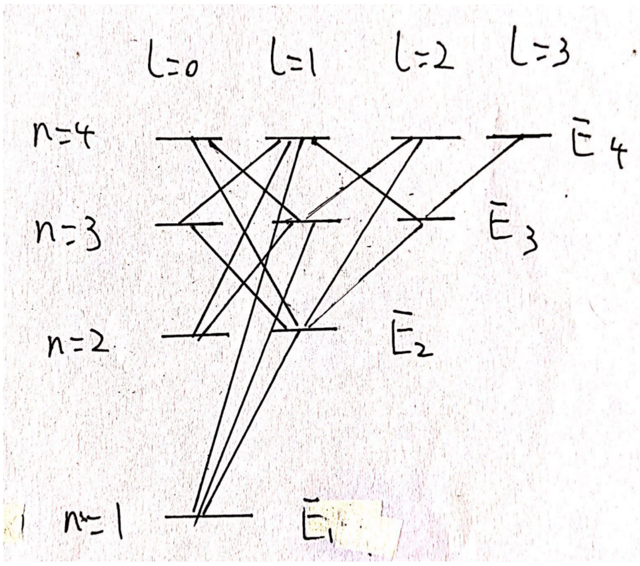

Figure 2: Energy level diagram

For transition energy $E_1 = \frac{\hbar^2(l+1)}{2I} = \frac{\hbar^2(1+1)}{2I} = \frac{\hbar^2}{I}$
$\triangle E = E_{l+1} - E_l = \frac{\hbar^2(l+2)(l+1)}{2I} - \frac{\hbar^2 l(l+1)}{2I} = \frac{2\hbar^2(l+1)}{2I} = (l+1)\frac{\hbar^2}{I}$
Hence $E_n = nE_1$ where $n = 1, 2, 3\cdots$
(c) $I = 2ma^2 = 2\cdot 1.67\cdot 10^{-27}\cdot(3.7\cdot 10^{-11})^2 = 4.57\cdot 10^{-48}$
$E_1 = \frac{\hbar^2}{2ma^2} = \frac{\hbar^2}{I} = 2.41\times 10^{-21}$J
(d) $\triangle E_{l\to l+1} = \frac{2\hbar^2(l+1)}{4ma^2}$
Hence when $l = 0$ we have the lowest transition energy $\triangle E_{l\to l+1} = \frac{\hbar^2}{2ma^2} = 0.015$eV
$\triangle E = \frac{hc}{\lambda} \to \lambda = \frac{hc}{\triangle E} = \frac{6.63\cdot 10^{-34} 3\cdot 10^8}{2.41\times 10^{-21}} = 8.25\cdot 10^{-5}m$
It's in the visible light region close to Paschen region.

4 Problem 4

(a)$L = r \times p$, since r and p are perpendicular to each other in the xy-plane, then L is along the z-axis, hence it has two zero components

According to classic mechanic, for the total energy, since $V(x) = 0, E = K = \frac{1}{2}I\omega^2 = \frac{L^2}{2I}$

(b) $\hat{H} = \frac{\hat{L_z}^2}{2I} + V = -\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\varphi^2} + V$

Since $V = 0, -\frac{\hbar^2}{2I}\frac{\partial^2\psi}{\partial\varphi^2} = E\psi \to -\frac{\hbar^2}{2I}\frac{\partial^2\phi(\varphi)}{\partial\varphi^2} = E\phi(\varphi) \to \phi = Ce^{\pm\sqrt{-\frac{2IE}{\hbar^2}}\varphi}, E$ can be any real number

Hence $\psi = Ce^{\pm\sqrt{-\frac{2IE}{\hbar^2}}\varphi}$

(c) $\psi(\varphi + 2\pi) = \psi(\varphi) \to \int_0^{2\pi} C^2 \cos^4(\varphi)d\varphi = 1 \to C^2\frac{3\pi}{4} = 1 \to C = \pm\frac{2}{\sqrt{3\pi}}$

5 Problem 5

(a) $n = 6, l = 3$

(b) $E_6 = -\frac{E}{n^2} = -\frac{13.6}{36} = 0.38\text{eV}$

(c) $|L| = \sqrt{L^2} = \sqrt{3(3+1)\hbar^2} = 2.3 \times 10^{-33}$

(d) $L_z = m\hbar = \pm1.989 \times 10^{-33}, \pm1.326 \times 10^{-33}, \pm6.63 \times 10^{-34}, 0$

6 Problem 6

(a)$\psi_{210} = \frac{1}{4\sqrt{2\pi}}(\frac{Z}{a_0})^{3/2}\frac{Zr}{a_0}e^{-Zr/2a_0}\cos\theta$

Since it's hydrogen atom, $Z = 1, \psi_{210} = \frac{1}{4\sqrt{2\pi}}(\frac{1}{a_0})^{3/2}\frac{r}{a_0}e^{-r/2a_0}\cos\theta$

$P(r) = \psi_{210}^*\psi_{210}4\pi r^2 = \frac{r^4}{8a_0^5}e^{-r/a_0}\cos^2\theta$, hence $P(r) \propto r^4e^{-r/a_0}$

(b) $\frac{\mathrm{d}P(r)}{\mathrm{r}} = \frac{r^3}{2a_0^5}e^{-r/a_0}\cos^2\theta - \frac{1}{a0}\frac{r^4}{8a_0^5}e^{-r/a_0}\cos^2\theta = 0$

$r = 4a_0$ or $r = 0$

7 Problem 7

$\psi_{000} = \frac{1}{\sqrt{\pi}}(\frac{Z}{a_0})^{3/2}e^{-Zr/a_0}$

Since $Z = 1$, then $P(r) = 4r^2\frac{1}{a_0^3}e^{-2r/a_0}$

Since $a_0 = 5.29 \cdot 10^{-11} >> R_0 = 10^{-15}, e^{-2r/a_0} \approx 0$

$\int_0^{R_0} P(r)\mathrm{d}r = \int_0^{10^{-15}} 4r^2\frac{1}{a_0^3}\mathrm{d}r = 9 \cdot 10^{-15}$

8 Problem 8

$I = \frac{2}{5}mR^2 = \frac{2}{5}9.1 \cdot 10^{-31}(10^{-15})^2 \approx 3.64 \cdot 10^{-60}$

$\omega = \frac{|L|}{I} = \frac{|S|}{I} = \frac{\hbar\sqrt{\frac{3}{4}}}{3.64 \cdot 10^{-60}} \approx 2.5 \cdot 10^{25}\text{rad/s}$

$v = \omega R = 2.5 \cdot 10^{25}10^{-15} = 2.5 \cdot 10^{10}\text{m/s}$

The speed is larger than the speed of light, hence the result from classic mechanic is incorrect.

9 Problem 9

(a) We should expect to see 4 lines with $j = \pm\frac{3}{2}$ because $l = 0, s = \frac{3}{2}$
Hence the lines corresponds $m_s = \pm\frac{3}{2}, \pm\frac{1}{2}$
(b) We should expect to see 3 lines with $m = \pm 1, 0$

10 Problem 10

$\mathbf{L} = \hbar\sqrt{l(l+1)} = 1.05 \cdot 10^{-34}\sqrt{2 \cdot 3} \approx 2.58 \cdot 10^{-34}\mathrm{N} \cdot \mathrm{m} \cdot \mathrm{s}$
$j = l \pm \frac{1}{2} = \frac{5}{2}$ or $\frac{3}{2}$
The corresponding $\mathbf{J} \approx 3.11 \cdot 10^{-34}$ or $2.03 \cdot 10^{-34}\mathrm{N} \cdot \mathrm{m} \cdot \mathrm{s}$