Problem Set 5

Due: 19 June 2020, 12.30 p.m.

Problem 1. In class we have quoted the following two statements about solutions to the 1D stationary Schrödinger equation. Please justify one of them (choose whichever you want).

(a) The solutions of the stationary Schrödinger equation can always be taken as real valued functions (unlike the solutions to the time-dependent Schrödinger equation, which are necessarily complex).

Hint. Show that if a function ψ solves the stationary Schrödinger equation for a given energy, so too does it complex conjugate, and hence also the real linear combinations $(\psi + \psi^*)$ and $i(\psi - \psi^*)$.

Comment. This statement does not mean that every solution to the stationary Schrödinger equation is real. What is says is that if you have got one that is not, it can always be expressed as a linear combination of solutions (corresponding to the same energy) that are real. So you might as well stick to ψ's that are real.

(b) If the potential energy $V(x)$ is an even function of x, then the solutions of the stationary Schrödinger equation $\psi(x)$ can be taken to be either even or odd.

Hint. Show that if a function ψ solves the stationary Schrödinger equation for a given energy, so too does $\psi(-x)$, and hence also the even and odd linear combination $\psi(x) \pm \psi(-x)$.

(*2 points*)

Problem 2. (a) Find bound states of a particle moving in the potential field with

$$V(x) = \begin{cases} \infty & \text{for} \quad x < 0 \\ -V_0 & \text{for} \quad 0 \le x \le a \\ 0 & \text{for} \quad x > a \end{cases}$$

where a, V_0 are positive constants. For the energy levels it is enough if you write down a system of transcendental equations obtained from continuity conditions. Normalize the wave functions (use a computer to evaluate integrals).

(b) Is the solution to this problem in any way related to the solution for a finite well of depth V_0 and walls placed at $x = \pm a$? In other words, if you were told the solution for the finite well, could you use it to tell what is the solution to the present problem? Justify your answer.

(c) Discuss the solution of the system of transcendental equations from part (a) graphically. Is it possible to set the width and the depth of the well so that there are no bound states? One bound state?

(d) We want to design a semiconductor device based on this model, using a pair of semiconducting materials with energy gap mismatch of 3.6 eV (that in our model corresponds to the depth of the well). What should the thickness of the in-well layer (that is a) be, if we want to design a structure with exactly one bound state (energy level)? Assume that the particle confined in this structure is an electron (with mass m equal to the mass of a free electron).

(*6 + 2 + 3/2 + 3/2 points*)

Problem 3. Show that in the singular potential field with $V(x) = -V_0\delta(x)$, where $V_0 > 0$ there is only one bound state. Find its energy and the corresponding normalized wave function.

Hint. Since this potential is singular, the requirement for continuity of ψ' needs to be relaxed. Find the discontinuity of ψ' at $x = 0$ by integrating both sides of the stationary Schrödinger equation over an infinitesimal interval $[-\varepsilon, \varepsilon]$, where $\varepsilon > 0$.

(*6 points*)

Problem 4. Solve Problem 4 from the previous assignment for a particle moving in a harmonic potential well $V(x) = m\omega^2 x^2/2$. (In part (b) skip the question about probability).

(*1/2 + 1/2 + 1/2 + 1/2 + 1 points*)

Problem 5. For the n-th eigenstate of the harmonic oscillator

(a) check that the wave function (see lecture notes) is normalized,

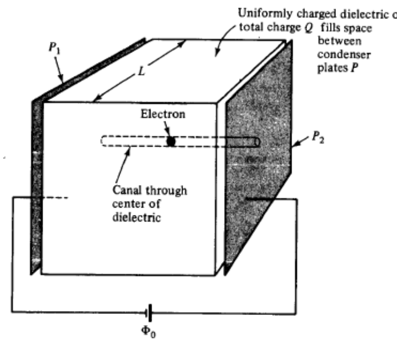
(b) find $\Delta_x\Delta_p$ and observe that for the ground state ($n = 0$) it is equal to the minimum possible value allowed by the Heisenberg uncertainty principle.

(c) calculate the average kinetic energy $\langle K\rangle$ and the average potential energy $\langle V\rangle$ in the n-th eigenstate. Comment on the result.

Hint. You may use a computer algebra system to evaluate the intergrals. If you do, please attach the code and to your solution.

(*1 + 3 + 2 points*)

Problem 6. A large dielectric cube with edge length L is uniformly charged throughout its volume so that it carries a total charge Q. It fills the space between capacitor plates which have a potential difference Φ_0. An electron is free to move in a small canal drilled in the dielectric, normal to the plates (see figure below). The Hamiltonian of the electron is $\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{Ax^2}{2} + \frac{e\Phi_0}{L}x$.



(a) How can we estimate the value of the spring constant A in terms of the total charge Q?

(b) What are the eigenenergies and eigenfunctions of $\hat{H}$?

Hints:
In part (a) you may find Gauss's law useful.
In part (b) do not solve the Schrödinger equation. Instead, just rewrite the Hamiltonian a little bit to see that it in fact describes a harmonic oscillator, and use what you already know about the solution of the quantum harmonic oscillator problem.

(*1 + 3 points*)