# Elements of Fluid Dynamics

# Outline

**Introduction**

**Fluids at rest**
    Density and pressure in a liquid
    Pascal's law
    Buoyancy and Archimedes principle

**Fluids in motion**
    Basic concepts; ideal liquid and continuity equation
    Bernoulli's equation
    Beyond the ideal liquid and laminar flow picture:
    viscosity and turbulence

# Introduction
# and
# Fluids at Rest

# Density of Mass

**Density of mass (bulk)**

$$\varrho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V}$$

In general, for *inhomogeneous* objects, the density of mass depends on position. If the density is constant throughout an object, then the object is said to be *homogeneous* (mass is distributed uniformly).

*Example*.　　The density of air at $20^o$ C is 1.2 kg/m³. What is the mass of the air in our classroom?
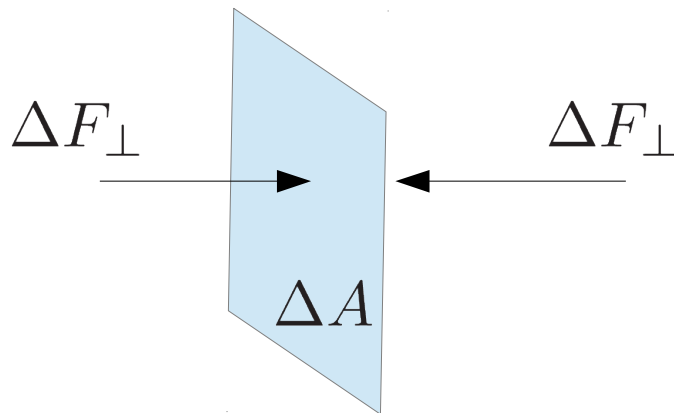
**Relative density ("specific gravity")**

$$r = \frac{\varrho}{\varrho_{\mathrm{H_2O},4^o\mathrm{C}}} = \frac{\varrho}{1000 \ \mathrm{kg/m^3}}$$

# Pressure in a Fluid
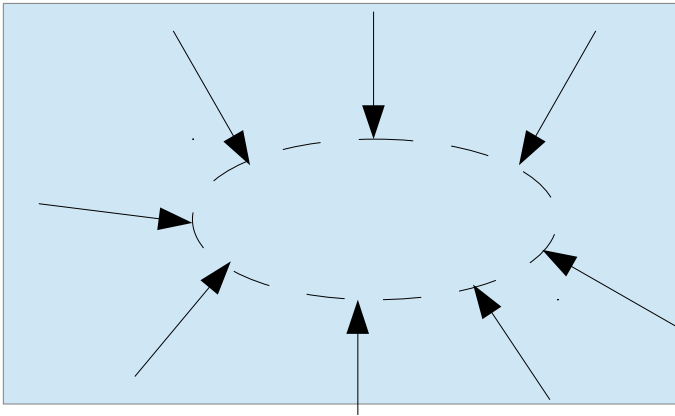
force exerted by molecules of the liquid; molecules in motion, constantly colliding with each other and the walls of the container

Assume that the fluid is at rest

$\Delta F_\perp$    $\Delta F_\perp$

$\Delta A$

both forces equal in magnitude; otherwise non-zero net force
→ motion

$$p \overset{\mathrm{def}}{=} \frac{\Delta F_\perp}{\Delta A} \quad [\mathrm{Pa} = \mathrm{N/m^2}]$$
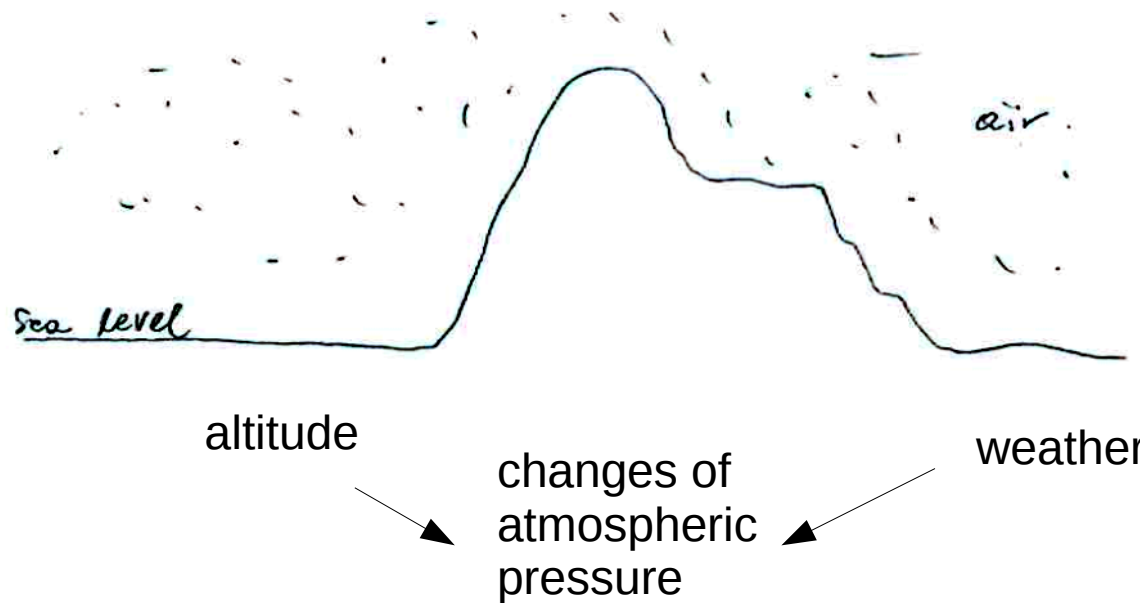
→ in general position-dependent $p = p(\vec{r})$

→ if constant at all points of surface *A*, then $p = \dfrac{F_\perp}{A}$
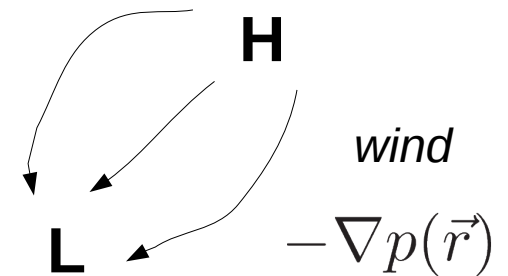
*Alternative units:* 1 bar = $10^5$ Pa
1 psi = 1 lb/in$^2$ = 6 895 Pa
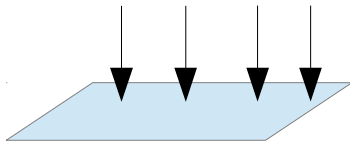
# Atmospheric Pressure



*Atmospheric pressure –* pressure due to the Earth's atmosphere

altitude

changes of atmospheric pressure

weather

*wind*

$-\nabla p(\vec{r})$

Normal atmospheric pressure a.s.l. (at the sea level):  $p_{atm}$ = 1.013 x $10^5$ Pa =

= 14.70 lb/$in^2$ = 1 atm

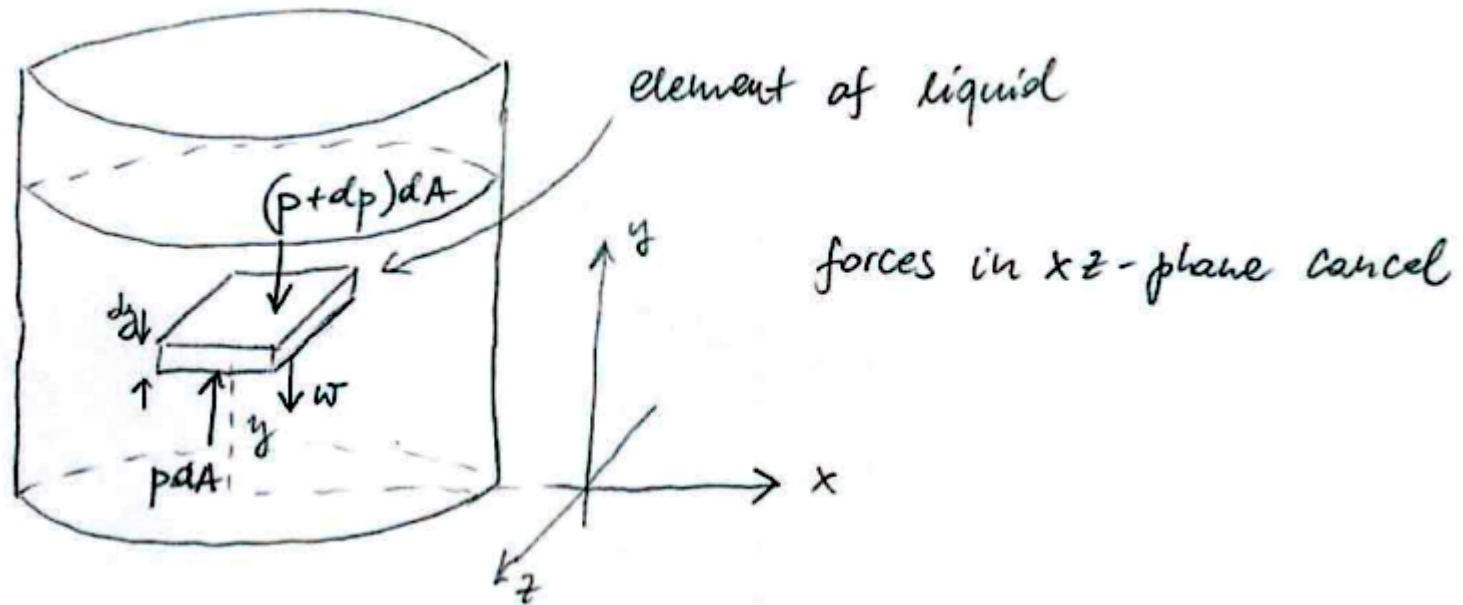*Example.*    Force on classroom's floor due to air at $p_{atm}$.

$$F_\perp = p \cdot A = 1.013 \times 10^5 \cdot 200 = 2.026 \times 10^7 \ \text{N}$$

corresponding to 2000 tonnes!

Why doesn't it collapse?    Upward force from the room downstairs (has the same magnitude if floor thickness is small.)

## Pressure at a Depth



If weight of the liquid could be neglected, then the pressure would be the same throughout the volume of the liquid; otherwise – pressure varies with depth.
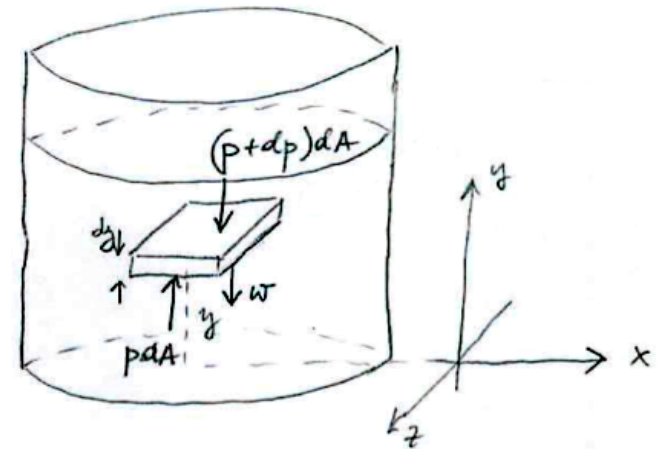
*Everyday experience:*    Force exerted by water on eardrums varies when we dive.

# Pressure at a Depth

In equilibrium, the net force on the element of the liquid is zero

$$F_x = 0, \;\; F_z = 0$$
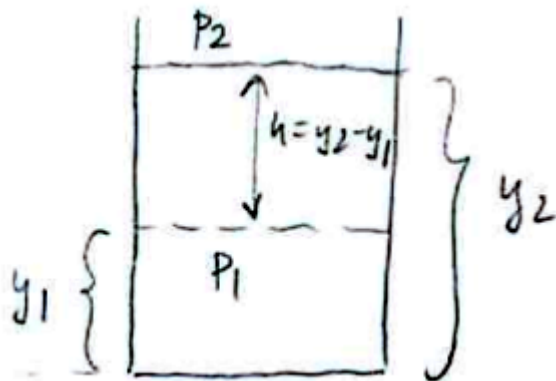
$$F_y = p \, \mathrm{d}A - (p + \mathrm{d}p)\, \mathrm{d}A - \underbrace{\varrho g \, \mathrm{d}A \, \mathrm{d}y}_{w} = 0$$

$$-\mathrm{d}p \, \mathrm{d}A = \varrho g \, \mathrm{d}A \, \mathrm{d}y \qquad \Longrightarrow \qquad \boxed{\mathrm{d}p = -\varrho g \, \mathrm{d}y}$$

pressure decreases as we move towards the liquid's surface

$$\int_{p_1}^{p_2} \mathrm{d}p = - \int_{y_1}^{y_2} \varrho g \mathrm{d}y \Longrightarrow \boxed{p_2 - p_1 = -\varrho g (y_2 - y_1)}$$

Or, introducing the symbols $\;\; p_1 = p, \;\; p_2 = p_0, \;\; y_2 - y_1 = h$

$$\boxed{p = p_0 + \varrho g h}$$

# Pressure at a Depth

Note.  Pressure does not depend on the shape of the container (cross-sectional area does not enter the formula).

$$P_A = P_B = P_C = P_0 + \rho g h$$
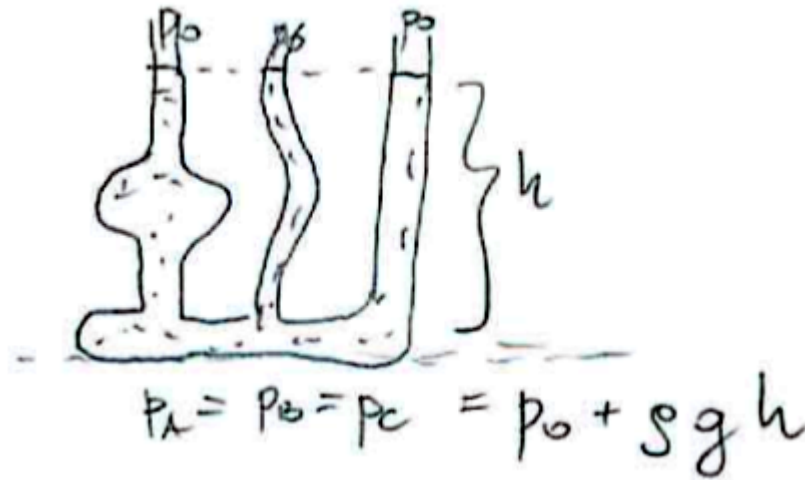
# Pressure at Depth. Example

Consider two non-mixing fluids (e.g. water and oil) in a U-shaped tube. If the boundary between the fluids is exactly at the bottom of the tube, what is the ratio of the heights of both liquids in the arms of the tube.



$$P_{atm} + \rho g h_{H_2O} = P_{atm} + \rho_{oil} g h_{oil}$$
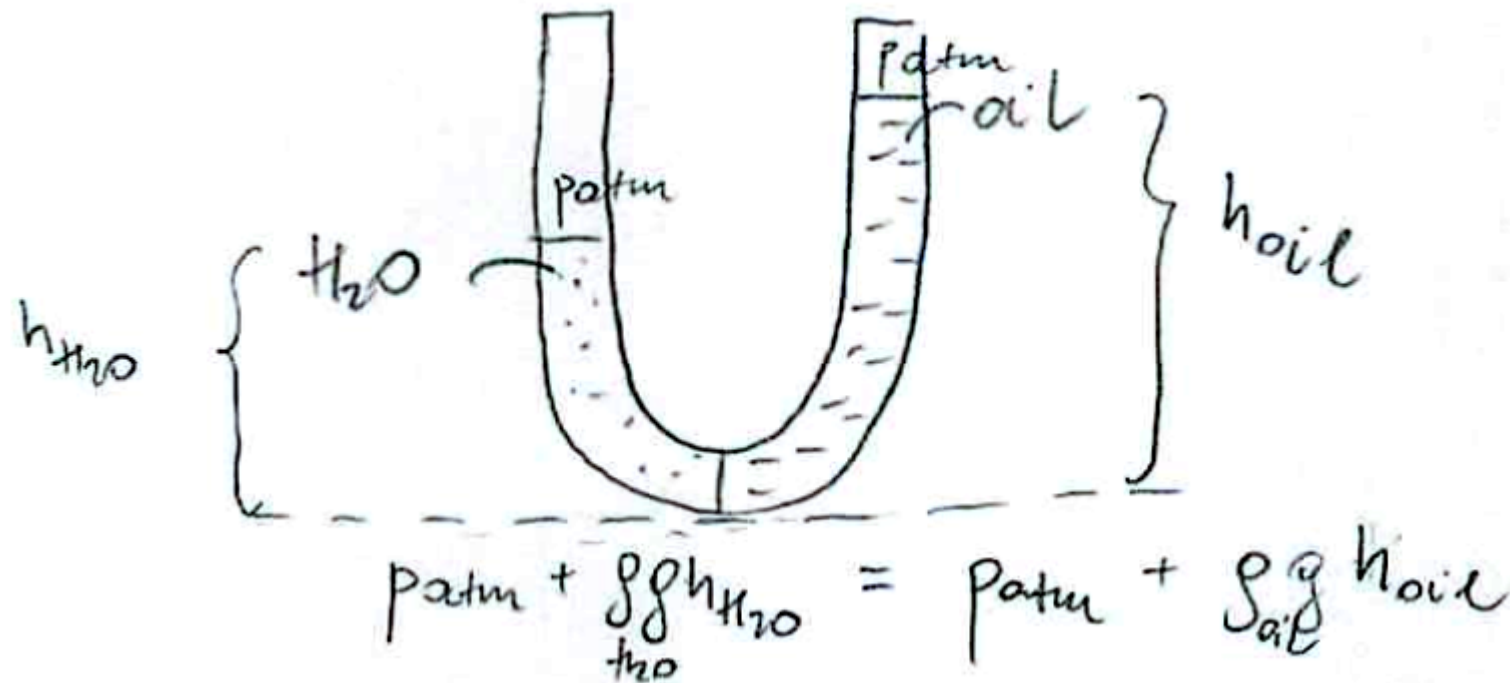
$$h_{oil} = \frac{\varrho_{H_2O}}{\varrho_{oil}} h_{H_2O} > h_{H_2O}$$

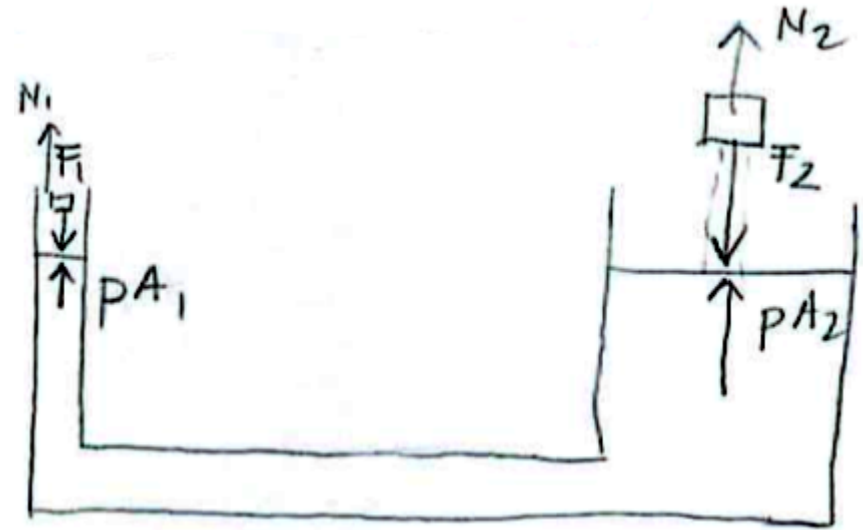because the density of water is greater than that of oil

# Pascal's Law

*Pressure applied to an enclosed incompressible fluid is transmitted undiminished to every portion of the liquid and the walls of the container.*

*Application*: the hydraulic lift

Statics: (magnitudes)
$$N_1 = F_1 = pA_1$$
$$N_2 = F_2 = pA_2$$

$$\frac{F_2}{F_1} = \frac{A_2}{A_1} \implies F_2 = F_1 \frac{A_2}{A_1}$$

If the ratio of cross-sectional areas $A_2/A_1$ is large, a small force **F**$_1$ applied to the surface 1 results in a large lifting force at surface 2.

Of course, the same work is done at both ends of the lift (surface $A_1$ needs to be moved over a longer distance).

Note. If the pistons are not at the same height, need to take into account the term $\rho gh$.

# Absolute Pressure vs. Gauge Pressure
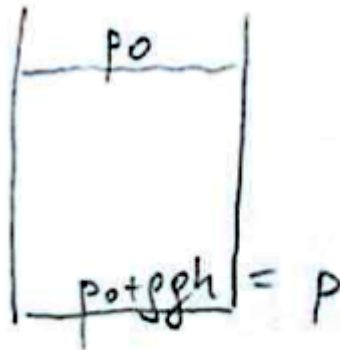
flat tire

$$p = p_{\text{atm}}$$

pumped tire

total (absolute) pressure

$$p = p_{\text{atm}} + p_{\text{gauge}}$$

gauge pressure

$$p_{\text{gauge}} = \underbrace{p - p_{\text{atm}}}_{>0}$$
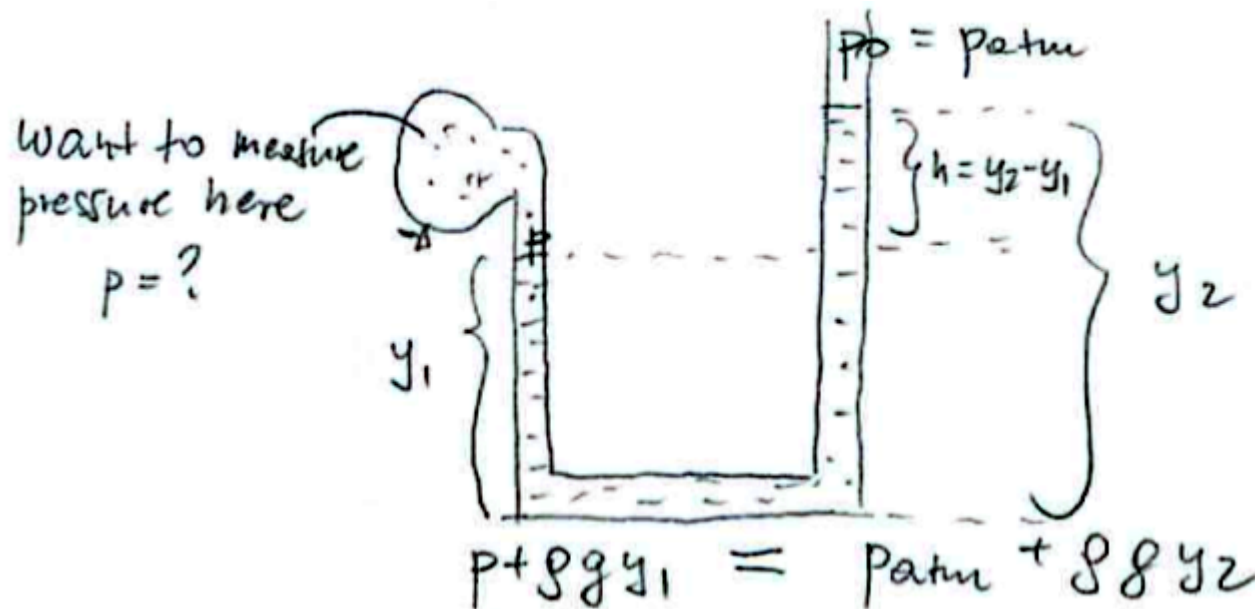
*Illustration.* Pressure at depth

$$p_0$$

$$p_0 + \rho g h = p$$

$$p - p_0 = \rho g h$$

gauge pressure
at the bottom of the container

# Pressure Measurement – Pressure Gauges

(1) Open-tube Manometer



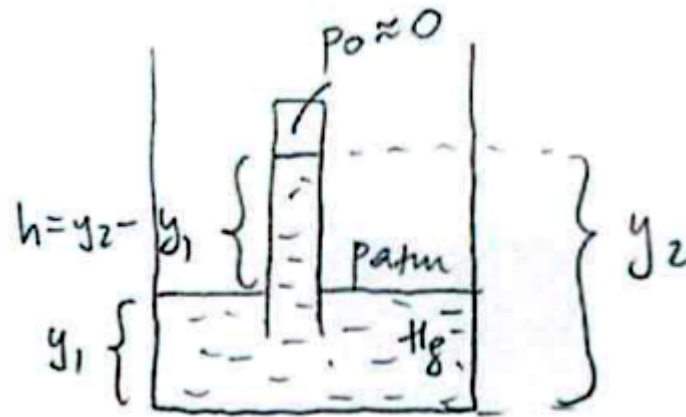Hand-written annotations:
Want to measure pressure here
$p = ?$

$p_0 = p_{atm}$

$h = y_2 - y_1$

$y_2$

$y_1$

$p + \rho g y_1 = p_{atm} + \rho g y_2$

$$\boxed{p_{\text{gauge}}} = p - p_{\text{atm}} = \varrho g(y_2 - y_1) = \boxed{\varrho g h}$$

excess height of the liquid in the right column

(2) Mercury Manometer



$$p_{\mathrm{atm}} = p_0 + \varrho g h = \varrho g h$$

*Alternative unit of pressure*: 1 mmHg = 1 torr = 133.3 Pa

*Disadvantage*: The density of mercury varies with temperature.

# Buoyancy. Archimedes Principle

Buoyancy results from the fact that because of the gravitational force, fluid pressure increases with depth and from the fact that the pressure is exerted in all directions (Pascal's law), so that there is a net upward force on the bottom of a submerged object.

**Archimedes' Principle**

*When a body is immersed in a fluid, the fluid exerts an upward force on the body equal to the weight of the fluid displaced by the body.*
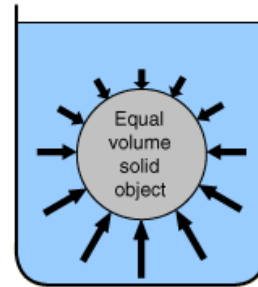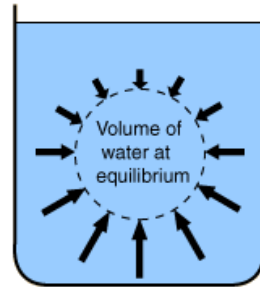
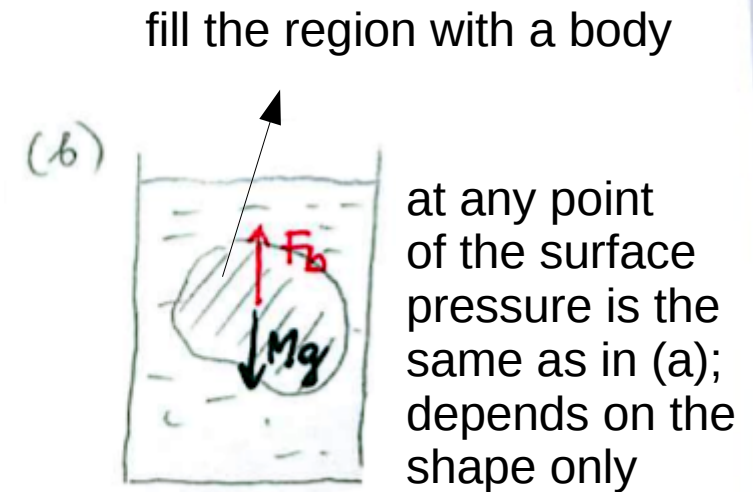$$F_{\mathrm{b}} = \underbrace{\varrho_{\mathrm{liquid}} V_{\mathrm{immersed}}}_{m_{\mathrm{liquid}}} g$$
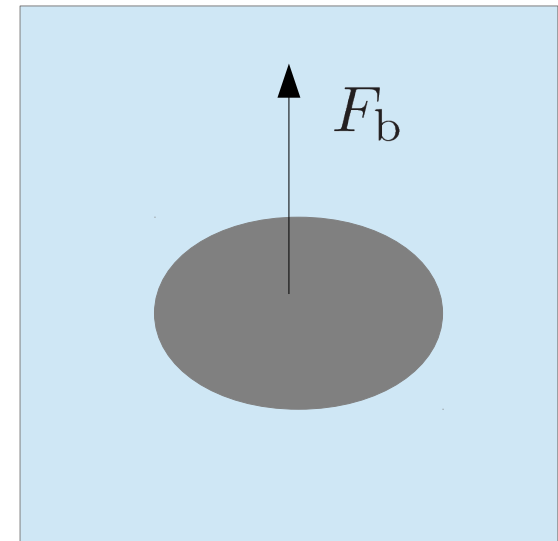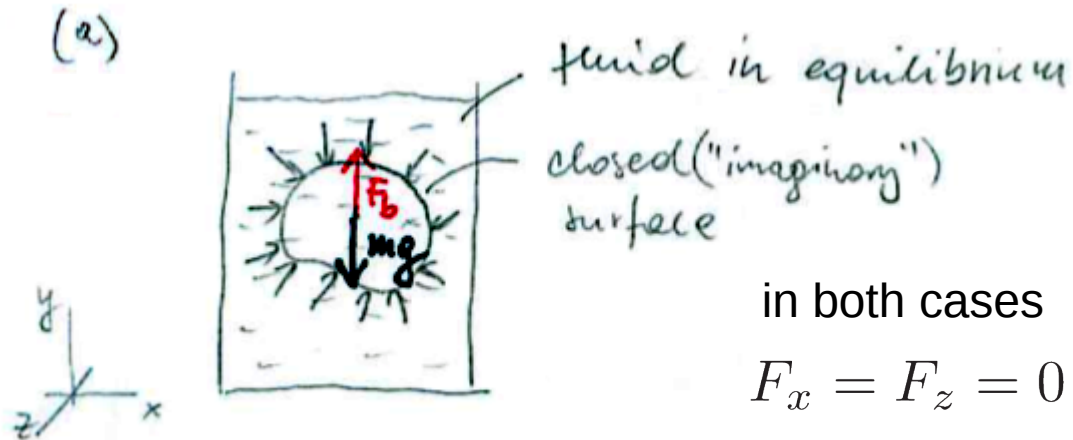
Note. All objects of *equal volume* experience *equal buoyant forces*.

# Archimedes Principle

*Justification* – idea

Volume of water at equilibrium

Equal volume solid object

All objects of *equal volume* experience *equal buoyant forces.*

fill the region with a body

(a)

fluid in equilibrium

closed ("imaginary") surface

(b)

at any point of the surface pressure is the same as in (a); depends on the shape only

in both cases

$$F_x = F_z = 0$$

mass of liquid

mass of the body

$$F_y = F_b - mg \implies$$

$$\implies F_b = mg = \varrho_{\text{liquid}} V g$$

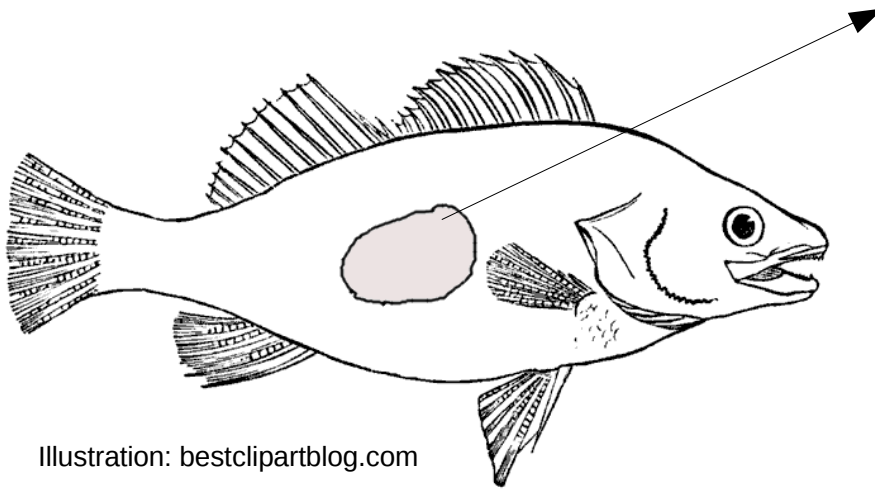$$F_y = F_b - Mg = (m - M)g =$$

$$= (\varrho_{\text{liquid}} - \varrho_{\text{body}})V g$$

# Archimedes Principle. Examples

(1) Floating fish



Illustration: bestclipartblog.com

Inflatable swim bladder controls the density of the body; can adjust the net force on the fish in the vertical direction
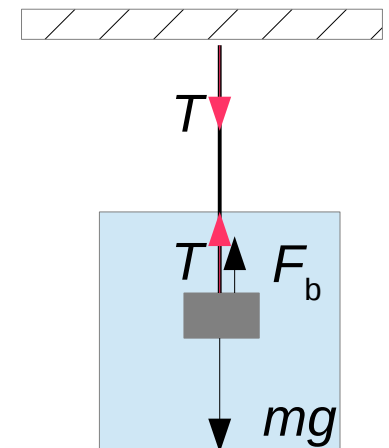
Although $\varrho_{\text{tissue}} > \varrho_{\text{H}_2\text{O}}$

filling the swim bladder with air can make $\varrho_{\text{fish}} \leq \varrho_{\text{H}_2\text{O}}$

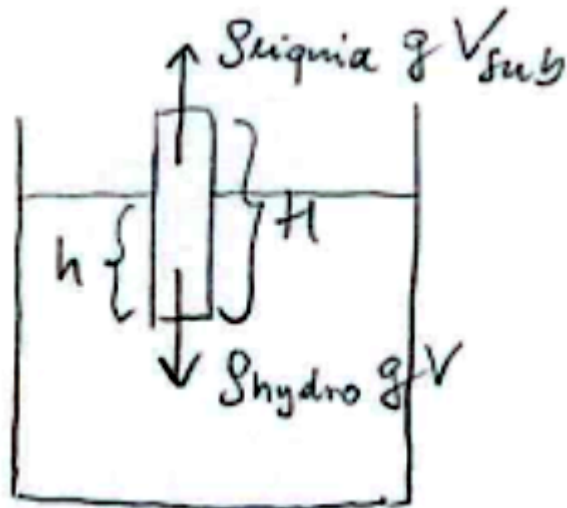and there will be a net upward force on the fish.

(2) Tension in a cord supporting a submerged object with density greater than that of the liquid

$$T + F_{\text{b}} - mg = 0 \qquad \Longrightarrow \qquad T = mg - F_{\text{b}}$$

# Archimedes Principle. Examples

(3) Practical application: the hydrometer

The hydrometer is in equilibrium when

$$\varrho_{\text{liquid}} V_{\text{sub}} g = \varrho_{\text{hydro}} V g$$

Hence

$$\varrho_{\text{liquid}} = \varrho_{\text{hydro}} \frac{V}{V_{\text{sub}}}$$

If the hydrometer has a cylindrical shape with cross sectional area $A$, then

$$V = A \cdot H, \qquad V_{\text{sub}} = A \cdot h$$

$$\varrho_{\text{liquid}} = \varrho_{\text{hydro}} \frac{H}{h}$$

Marking the side of the cylinder allows us to directly read out the density of the liquid.
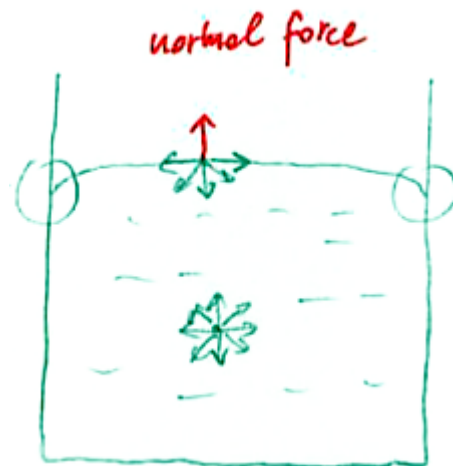
# Surface Tension

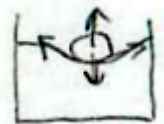Demonstration: A paper clip floating on the surface of the liquid.



edtech.boisestate.edu

*Origin*: attractive interaction between molecules of the liquid.



normal force

interaction between neighboring molecules (cohesion force)

Note. In the weightless state, the liquid will form a ball – the solid that, with a given volume, has minimum surface area, hence minimizes the energy due to surface tension.
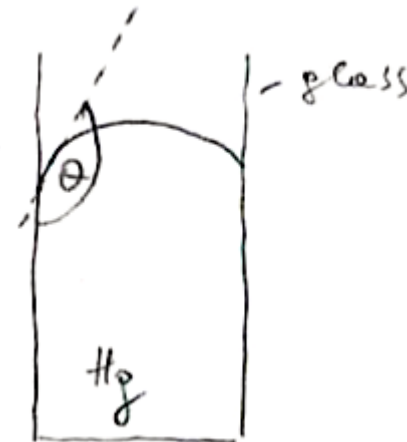
# Meniscus and capillarity

As a result of competition between cohesion forces (forces between the molecules of the liquid) and adhesion forces (forces between the liquid and the container), the surface of the liquid may take up various shapes.



wetting liquid $0 < \theta < \dfrac{\pi}{2}$

*adhesion forces dominate*

non-wetting liquid $\dfrac{\pi}{2} < \theta < \pi$

*cohesion forces dominate*

**Capillarity** (capilar action) –   ability of a liquid to flow in narrow tubes in opposition external forces (e.g. gravity).

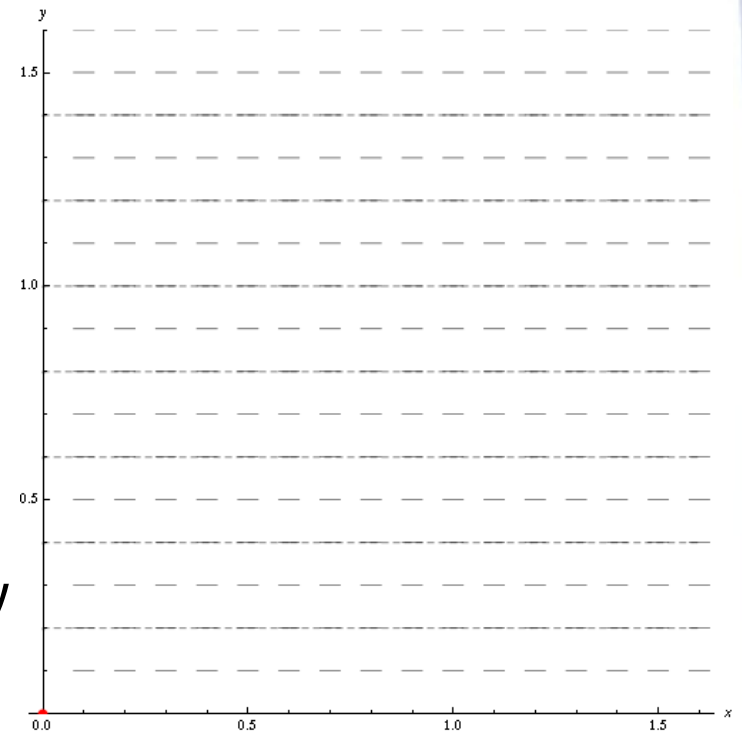# Fluids in Motion

# Vocabulary

## Ideal Fluid

fluid density does not change (the fluid is incompressible); experiences no internal friction (no viscosity)

## Flow Lines (Path Lines)

trajectories of individual particles in a fluid

## Stream Lines

family of curves such that are instantaneously tangential to the velocity vector field



Creative Commons

*Note.* When the flow pattern changes, flow lines do not coincide with stream lines. We will discuss only steady-flow situation, when the two do coincide.
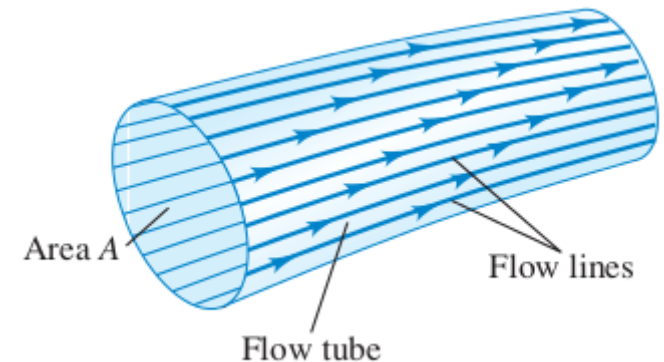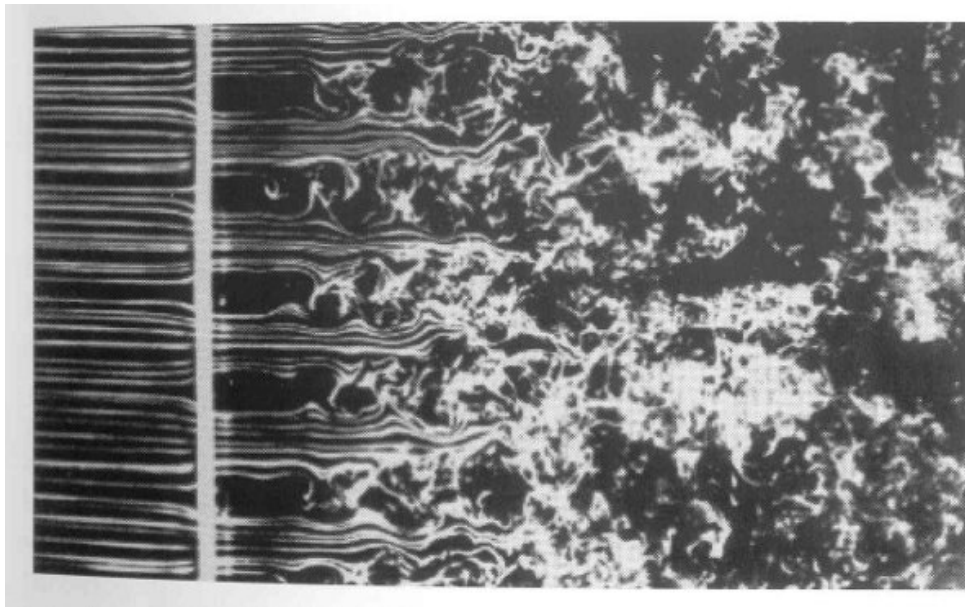
# Vocabulary

## Flow Tube

a tube formed by flow lines passing through the edge of an imaginary element of area. Note that in steady (laminar) flow no fluid can cross the side walls of a flow tube; fluids in different flow tubes cannot mix.

## Laminar vs. Turbulent Flow



Area A
Flow lines
Flow tube

http://num3sis.inria.fr

# Continuity Equation

*Follows from the fact that mass is conserved (mass of a fluid element does not change as it flows)*

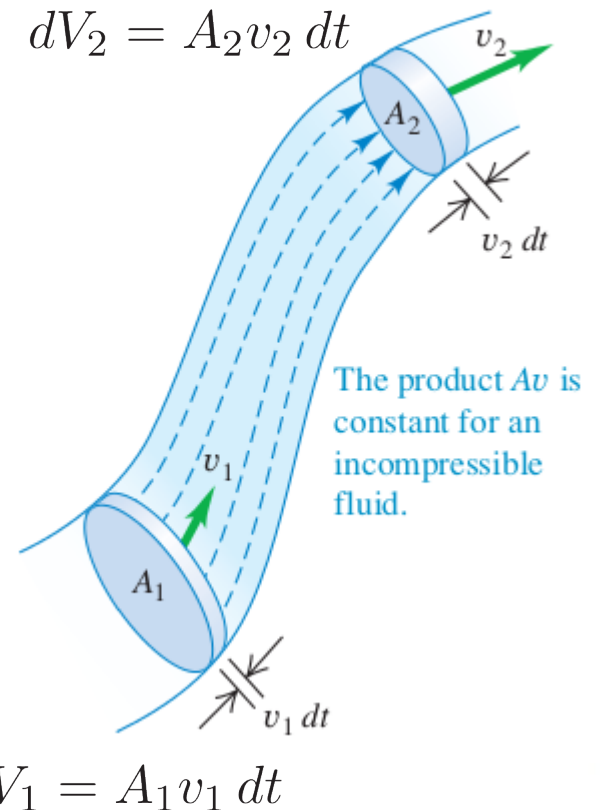$$\varrho A_1 v_1 \, dt = \varrho A_2 v_2 \, dt$$

For a homogeneous, incompressible fluid (steady flow)
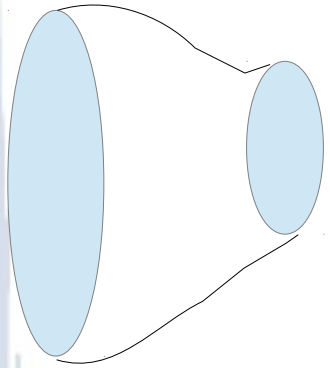
$$\varrho = \text{const}$$

$$A_1 v_1 = A_2 v_2$$

$$\frac{dV}{dt} \longrightarrow \text{volume flow rate}$$

$dV_2 = A_2 v_2 \, dt$

$v_2$

$A_2$

$v_2 \, dt$

The product $Av$ is constant for an incompressible fluid.

$v_1$

$A_1$

$v_1 \, dt$

$dV_1 = A_1 v_1 \, dt$

*Note.* In case of a fluid that is not incompressible   $\varrho_1 A_1 v_1 = \varrho_2 A_2 v_2$
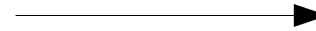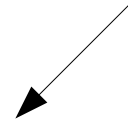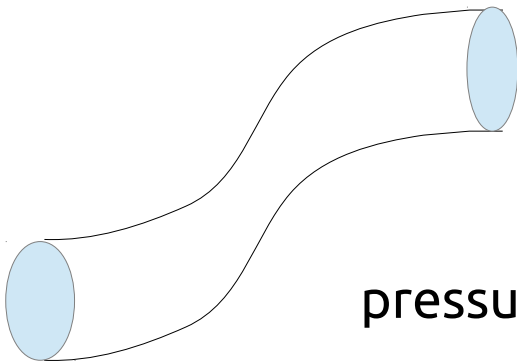
# Bernoulli's Equation. Motivation

when an incompressible fluid flows through a horizontal tube of variable cross-section, its speed must change (continuity equation)

speed change means acceleration (force due to surrounding liquid)

pressure is different in regions of different cross-section

pressure difference due to difference in height

*Note*
In the following discussion we assume: incompressible fluid, steady flow, no viscosity.

# Bernoulli's Equation.

**Idea:** Find the change in the (gravitational) potential and the kinetic energy of an element of liquid's volume. It is equal to the work of the force due to pressure in the liquid.

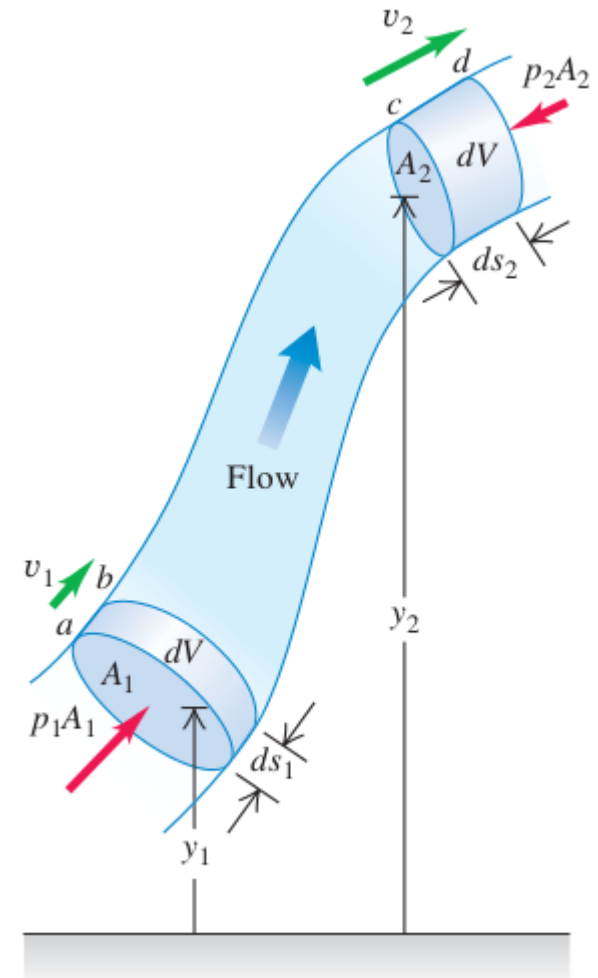element of volume that is displaced from a-c to b-d

$$dV = A_1\,ds_1 = A_2\,ds_2$$

work done on this element by the surrounding liquid

$$\delta W = p_1 \underbrace{A_1\,ds_1}_{dV} - p_2 \underbrace{A_2\,ds_2}_{dV}$$

on the other hand

$$\delta W = dE = d(K + U_{\mathrm{grav}}) = dK + dU_{\mathrm{grav}}$$

# Bernoulli's Equation. Derivation

*change in the kinetic energy*

[contribution due to the volume between *b* and *c* cancels out]

<u>initial</u>
(bottom slice in motion)

mass

<u>final</u>
(top slice in motion)

$$\varrho A_1 \, ds_1$$

$$\varrho A_2 \, ds_2$$

$$\frac{1}{2} \varrho \left( A_1 \, ds_1 \right) v_1^2$$

kinetic
energy

$$\frac{1}{2} \varrho \left( A_2 \, ds_2 \right) v_2^2$$

$$dK = \frac{1}{2} \varrho \left( v_2^2 - v_1^2 \right) \, dV$$
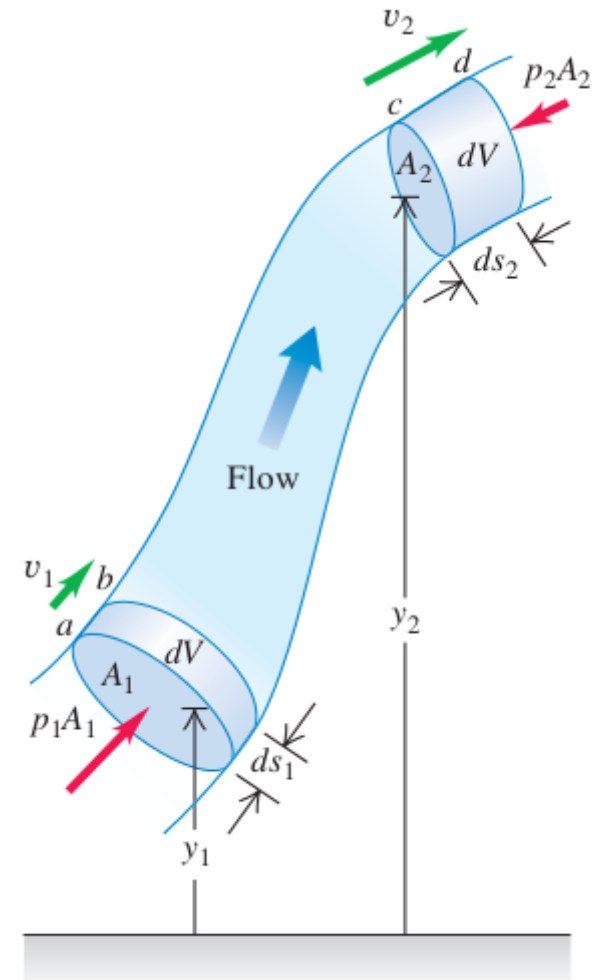
# Bernoulli's Equation. Derivation

*change of the (gravitational) potential energy*
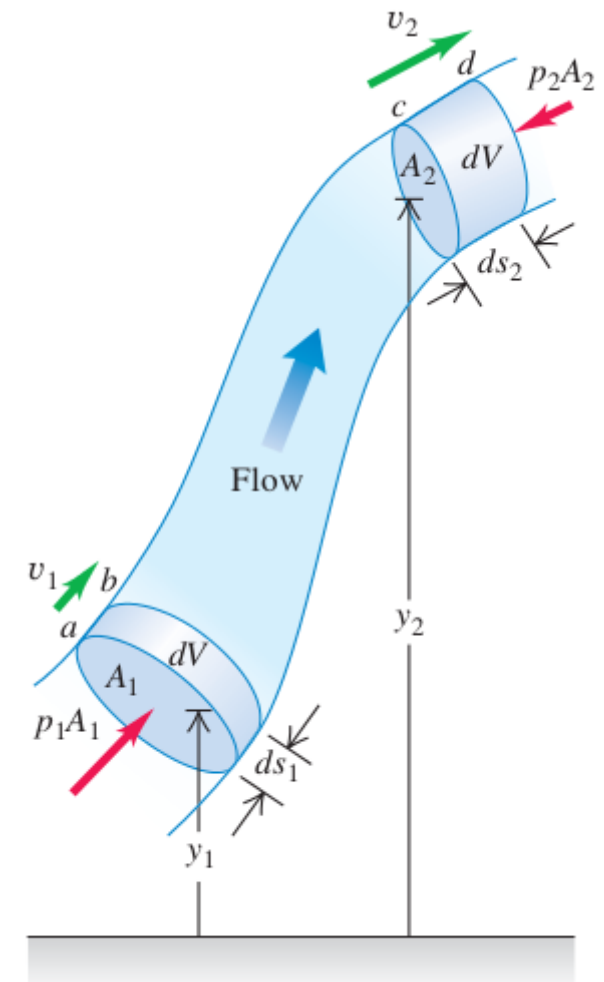[again, contribution due to the volume between *b* and *c* cancels out]

<u>initial</u>
(bottom slice)

<u>final</u>
(top slice)

$$gy_1 \, dm = \varrho g y_1 \, dV$$

$$gy_2 \, dm = \varrho g y_2 \, dV$$

$$dU_{\text{grav}} = \varrho g \left( y_2 - y_1 \right) \, dV$$

# Bernoulli's Equation

combining both

$$\delta W = dK + dU_{\text{grav}}$$

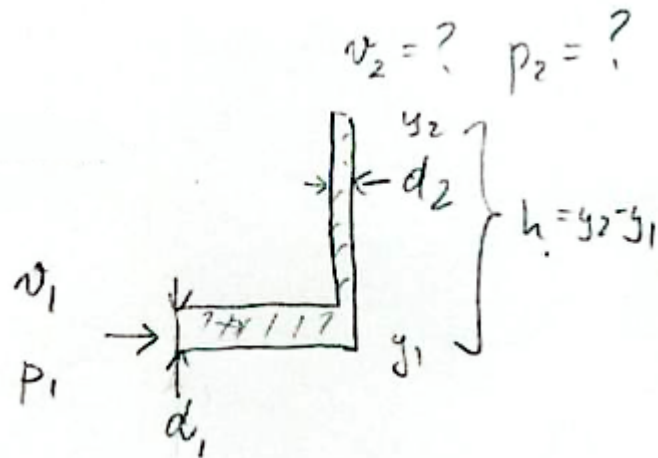$$(p_1 - p_2) \, dV = \frac{1}{2} \varrho \left( v_2^2 - v_1^2 \right) \, dV + \varrho g \left( y_2 - y_1 \right) \, dV$$

$$(p_1 - p_2) = \frac{1}{2} \varrho \left( v_2^2 - v_1^2 \right) + \varrho g \left( y_2 - y_1 \right)$$

$$p_1 + \frac{1}{2} \varrho v_1^2 + \varrho g y_1 = p_2 + \frac{1}{2} \varrho v_2^2 + \varrho g y_2$$

$$p + \frac{1}{2} \varrho v^2 + \varrho g y = \text{const}$$
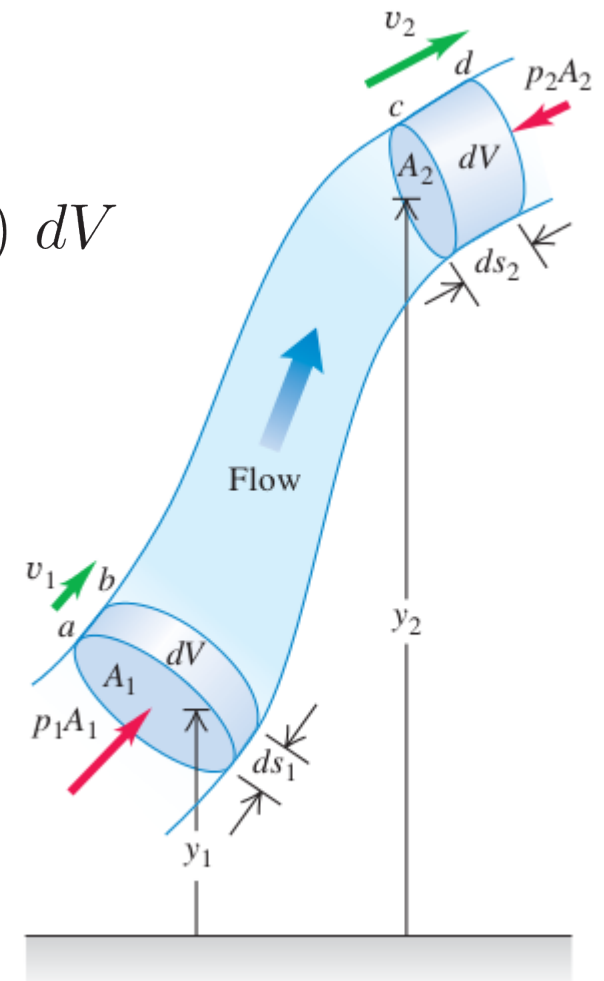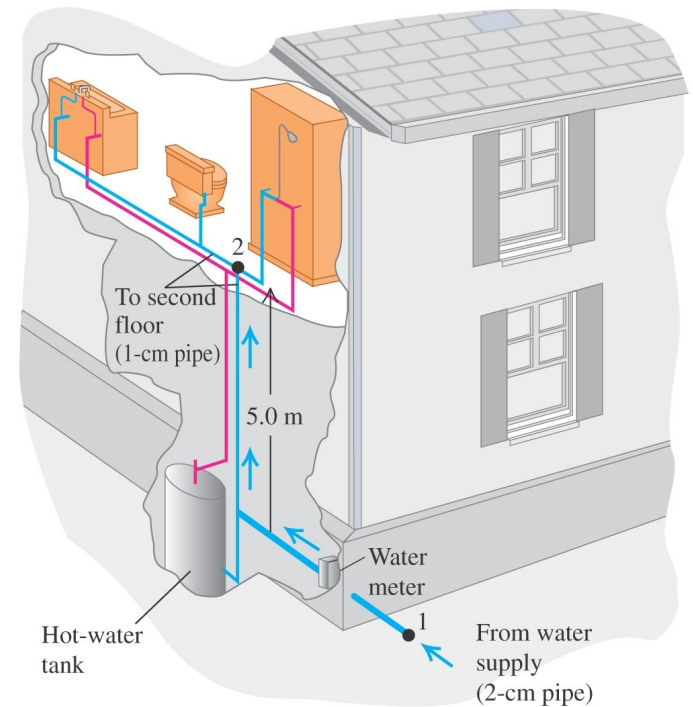
# Example. Water Pressure in a Building



*continuity equation*  $A_1 v_1 = A_2 v_2$

$$\pi \frac{d_1^2}{4} v_1 = \pi \frac{d_2^2}{4} v_2 \qquad \implies \qquad v_2 = \left(\frac{d_1}{d_2}\right)^2 v_1$$

*Bernoulli's equation*

$$p_1 + \frac{1}{2}\varrho v_1^2 = p_2 + \varrho g h + \frac{1}{2}\varrho v_2^2$$

$$p_2 = p_1 - \frac{1}{2}\varrho v_1^2 \left[\left(\frac{d_1}{d_2}\right)^4 - 1\right] - \varrho g h$$
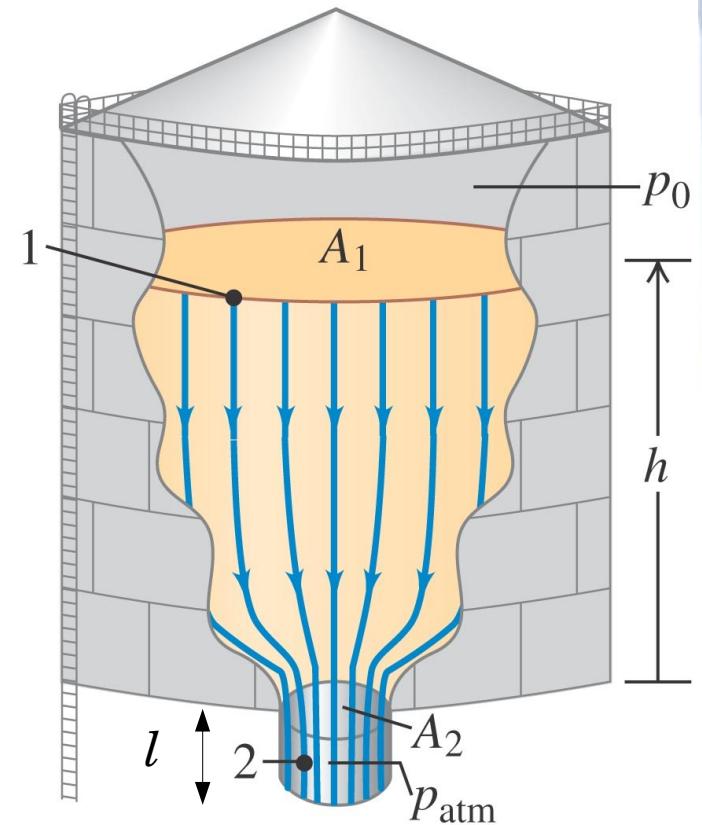
# Example. Speed of Efflux

*Bernoulli's equation*

$$p_0 + \frac{1}{2}\varrho v_1^2 + \varrho g(h + l) = p_{\text{atm}} + \frac{1}{2}\varrho v_2^2$$

taking into account

$$l \ll h \qquad \text{and} \qquad A_1 \gg A_2 \implies v_1 = \frac{A_2}{A_1}v_2 \approx 0$$

$$p_0 + \varrho g h = p_{\text{atm}} + \frac{1}{2}\varrho v_2^2$$

$$v_2 = \sqrt{2\left(\frac{p_0 - p_{\text{atm}}}{\varrho} + gh\right)}$$

*Note.* If the lid is open $p_0 = p_{\text{atm}}$ and $v_2 = \sqrt{2gh}$

# Example: Venturi Meter

*How to measure flow speed in a tube?*

*continuity equation*

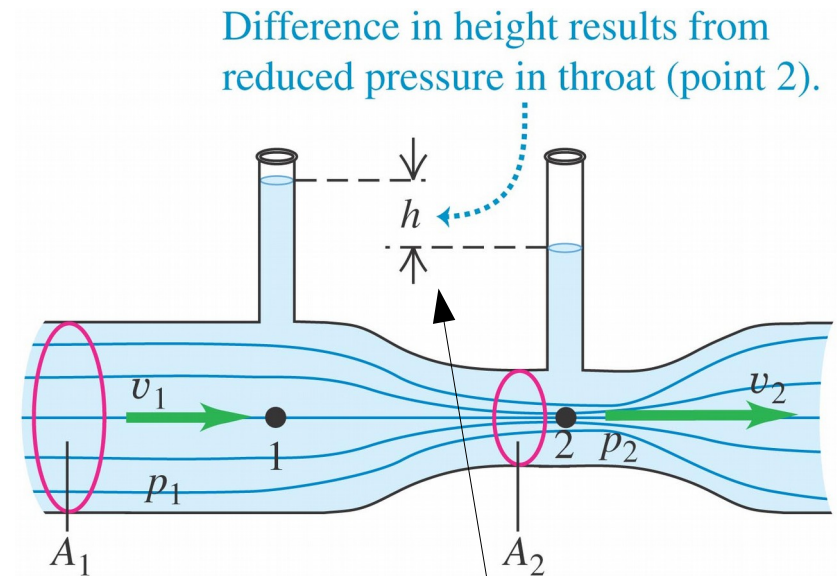$$v_1 A_1 = v_2 A_2 \implies v_2 = \frac{A_1}{A_2} v_1$$

*Bernoulli's equation*

$$p_1 + \frac{1}{2}\varrho v_1^2 = p_2 + \frac{1}{2}\varrho v_2^2$$

$$p_1 - p_2 = \frac{1}{2}\varrho v_1^2 \left[ \left(\frac{A_1}{A_2}\right)^2 - 1 \right]$$

*on the other hand*

$$p_1 - p_2 = \varrho g h$$

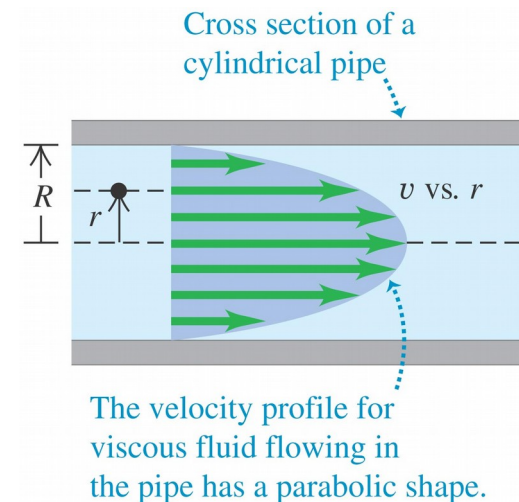Difference in height results from reduced pressure in throat (point 2).



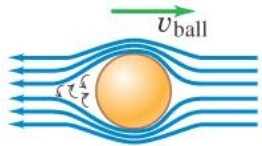$$v_1 = \sqrt{\frac{2gh}{\left(\frac{A_1}{A_2}\right)^2 - 1}}$$

# Viscosity and Turbulence

- *Viscosity* is internal friction in a fluid.

- *Turbulence* is irregular chaotic flow that is no longer laminar.

Cross section of a cylindrical pipe

$R$

$r$

$v$ vs. $r$

The velocity profile for viscous fluid flowing in the pipe has a parabolic shape.

# Example: A Curve Ball

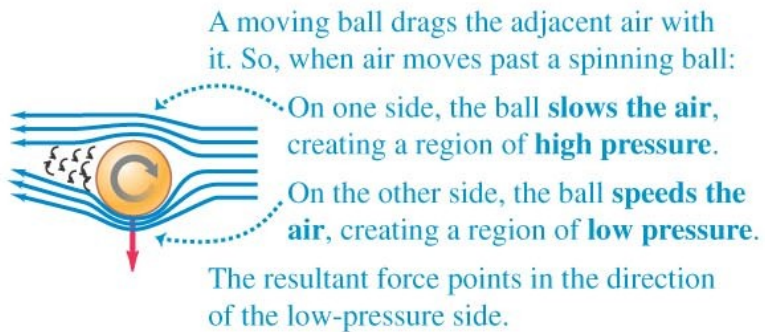**(a)** Motion of air relative to a nonspinning ball

$v_{ball}$

**(b)** Motion of a spinning ball
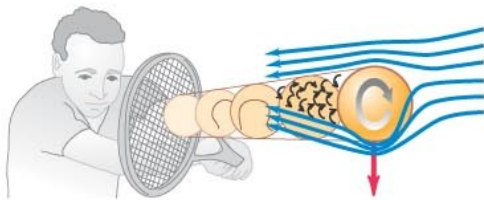
This side of the ball moves opposite to the airflow.
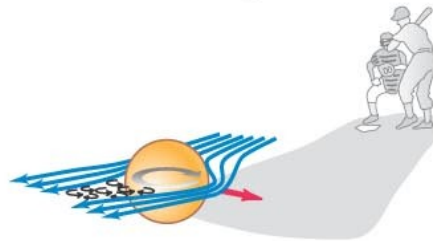
This side moves in the direction of the airflow.

**(c)** Force generated when a spinning ball moves through air

A moving ball drags the adjacent air with it. So, when air moves past a spinning ball:

On one side, the ball **slows the air**, creating a region of **high pressure**.

On the other side, the ball **speeds the air**, creating a region of **low pressure**.

The resultant force points in the direction of the low-pressure side.

**(d)** Spin pushing a tennis ball downward

**(e)** Spin causing a curve ball to be deflected sideways

**(f)** Backspin of a golf ball