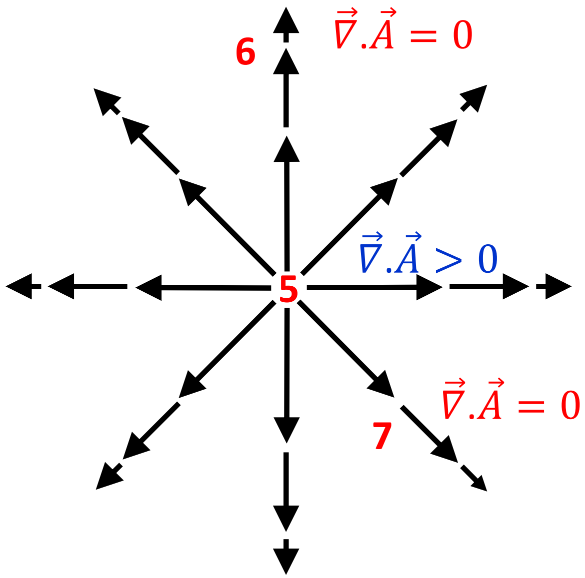
Gauss law

Outline

- Demonstrate that Gauss theorem does not depend on the shape of the Gauss surface
- Complete the first Maxwell's and Poisson equations
- Demonstrate that Coulomb's law is **exactly** inversely proportional to r^2
- Demonstrate that a static charge cannot be in equilibrium in the electric field of other charges
- Concept of work – Electric potential – Electric potential energy
- Apply Gauss law in various circumstances



- The field lines are clearly spreading from points **6** and **7**
- **BUT** they are getting shorter away from the center

Locally: Trade-off between <u>spreading</u> and <u>intensity of the field</u>

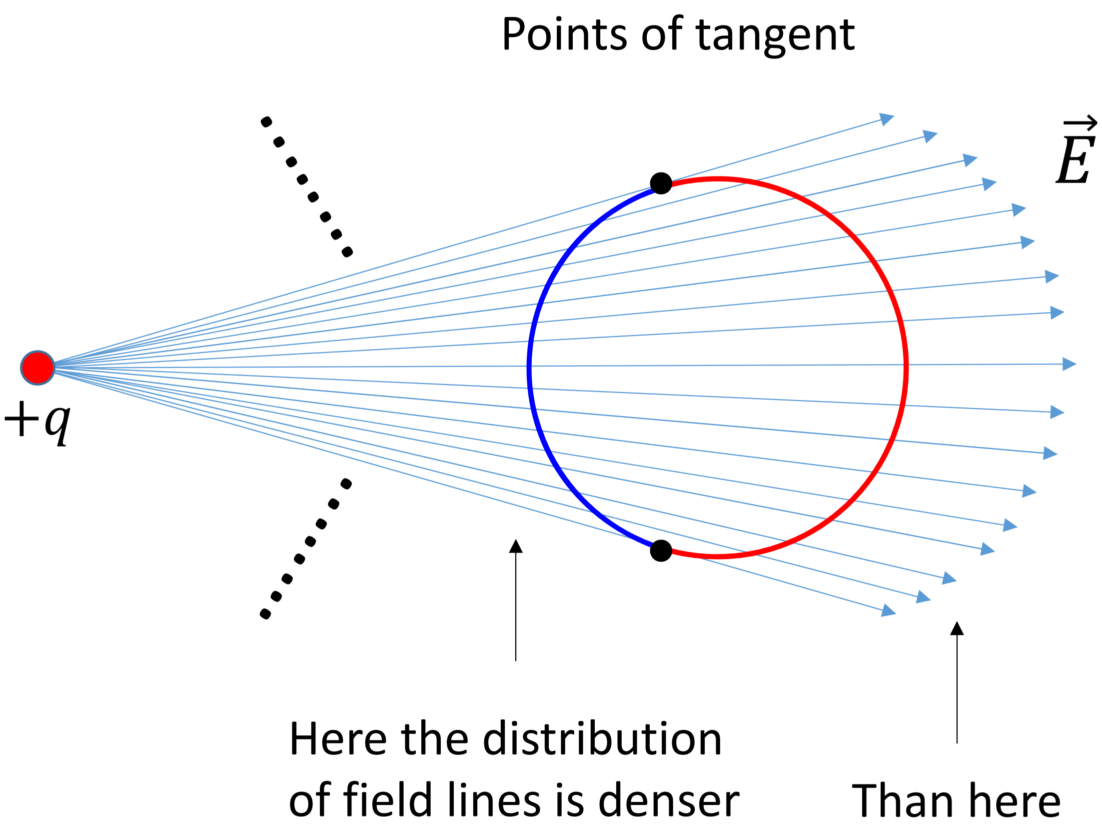
$\vec{A} \propto (1/r^2)\vec{u}_r$
Coulomb's law

Perfect compensation between **spreading** and **intensity** of the field at any point except around the source **(5)**

$\vec{\nabla}.\vec{A} = 0$ everywhere in space EXCEPT around the origin (source of the field)

$+q$ emits an electric field radially with an intensity given by Coulomb's law

$$E \propto \frac{1}{r^2}$$

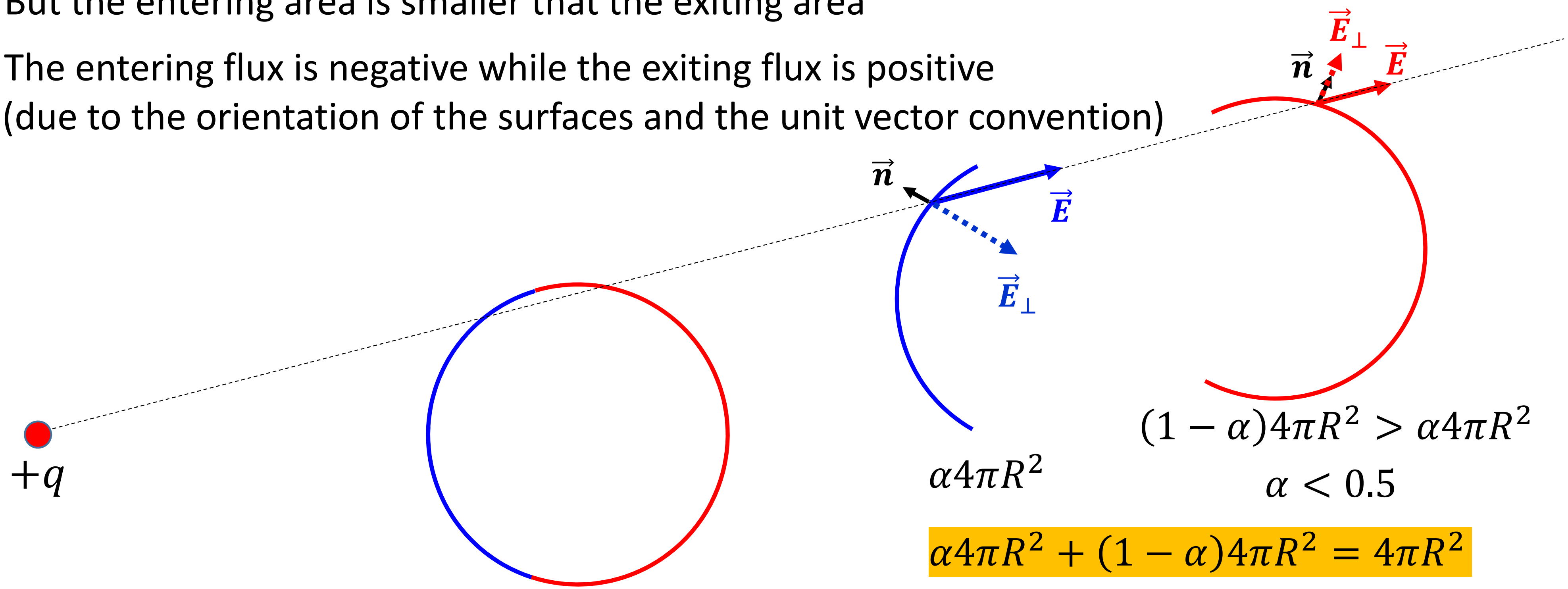


Coulomb's law ⇨

$\vec{E}$ entering the sphere from the left (blue portion) is stronger than $\vec{E}$ exiting from the right (red portion)

By splitting the entering and exiting parts (spherical caps) we see can several things

1. The entering field is larger in magnitude than the exiting field according to Coulomb's law
2. But the entering area is smaller that the exiting area
3. The entering flux is negative while the exiting flux is positive (due to the orientation of the surfaces and the unit vector convention)



$(1-\alpha)4\pi R^2 > \alpha 4\pi R^2$

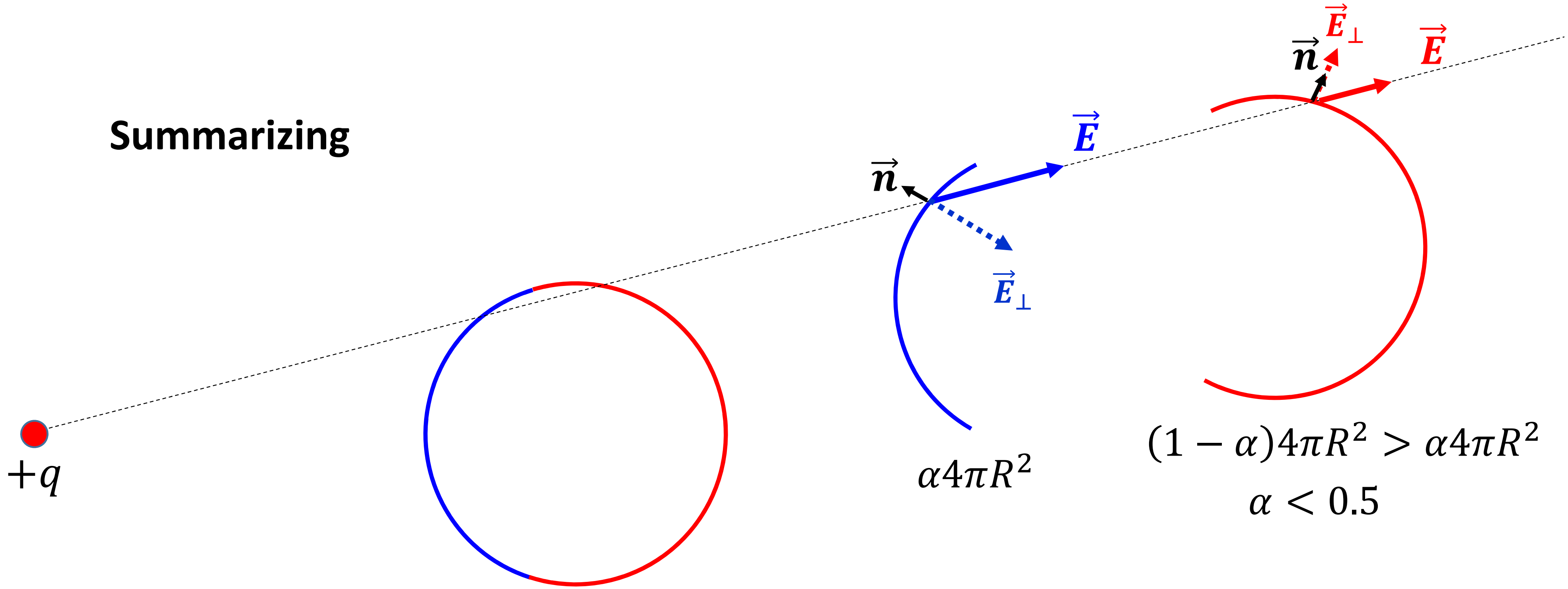
$\alpha < 0.5$

$\alpha 4\pi R^2 + (1-\alpha)4\pi R^2 = 4\pi R^2$

Therefore we may expect a compensation of the exiting flux by the entering flux

Remember that the flux is given by the **scalar product** $\vec{\boldsymbol{E}}.\vec{\boldsymbol{A}} = \boldsymbol{EAcos(\theta)} = \boldsymbol{E}_{\perp}.\boldsymbol{A}$

Summarizing



$$(1-\alpha)4\pi R^2 > \alpha 4\pi R^2$$
$$\alpha < 0.5$$

Qualitatively

$-\boldsymbol{E}_\perp . \boldsymbol{A}$	$+\boldsymbol{E}_\perp . \boldsymbol{A}$
$\boldsymbol{E}_\perp$ **large**	$\boldsymbol{E}_\perp$ **small**
$\boldsymbol{A}$ **small**	$\boldsymbol{A}$ **large**

Question:
Is this compensation total?
If the answer is yes, then the flux through the closed sphere is zero

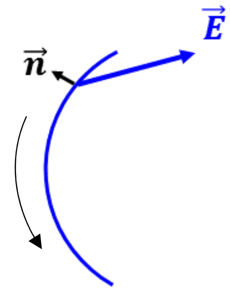
How to demonstrate that there is a total compensation?

Here the choice of the proper closed surface is crucial. Remember that Gauss's theorem does not specify any requirement regarding the shape of the surface.

We can easily see from previous slides that the choice of a spherical surface is not appropriate at all !

The reason is that to get the **TOTAL** flux from each spherical portion we need to **integrate** a **scalar product**

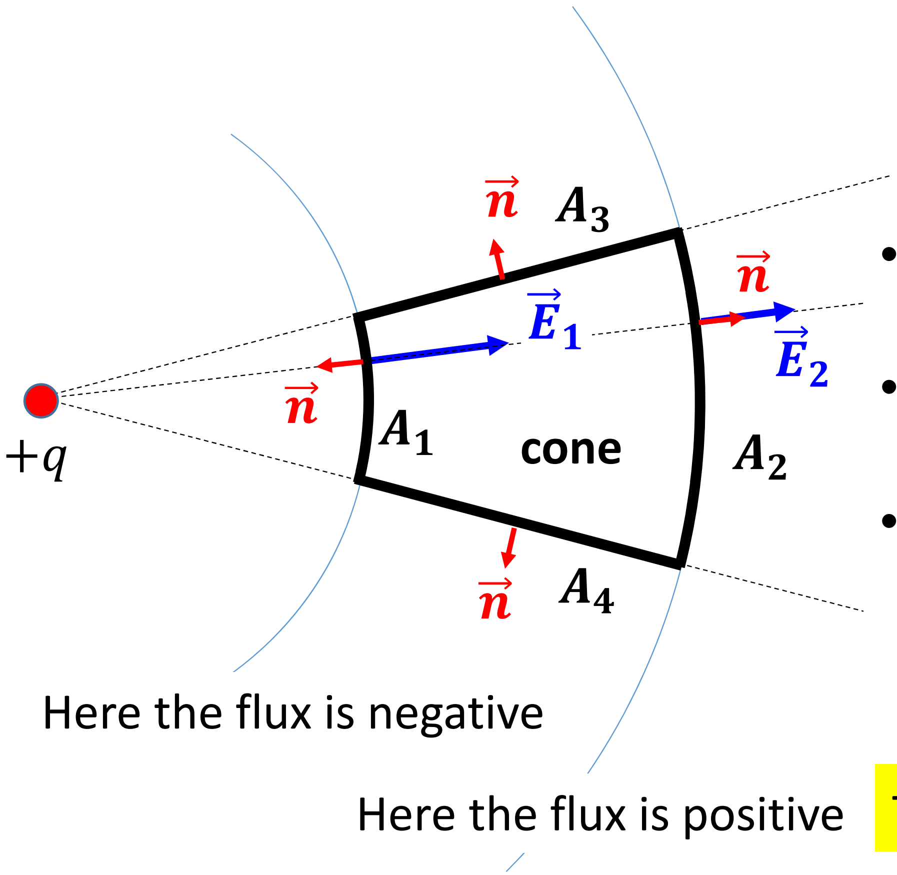
This operation requires to consider a changing angle between $\vec{E}$ and $\vec{n}$ while the vector field scans the whole area (blue or red)



Mathematically it is very tedious. Therefore another choice is demanded

As Gauss's theorem allows to take any **closed** surface let us consider two concentric spheres with the center located at the charge itself. These two surfaces cut a cone having its apex at the charge

The closed surface in bold is defined by a cone cut by two concentric spheres A_1 and A_2



Convention:
For a closed surface the unit vector is directed outwards

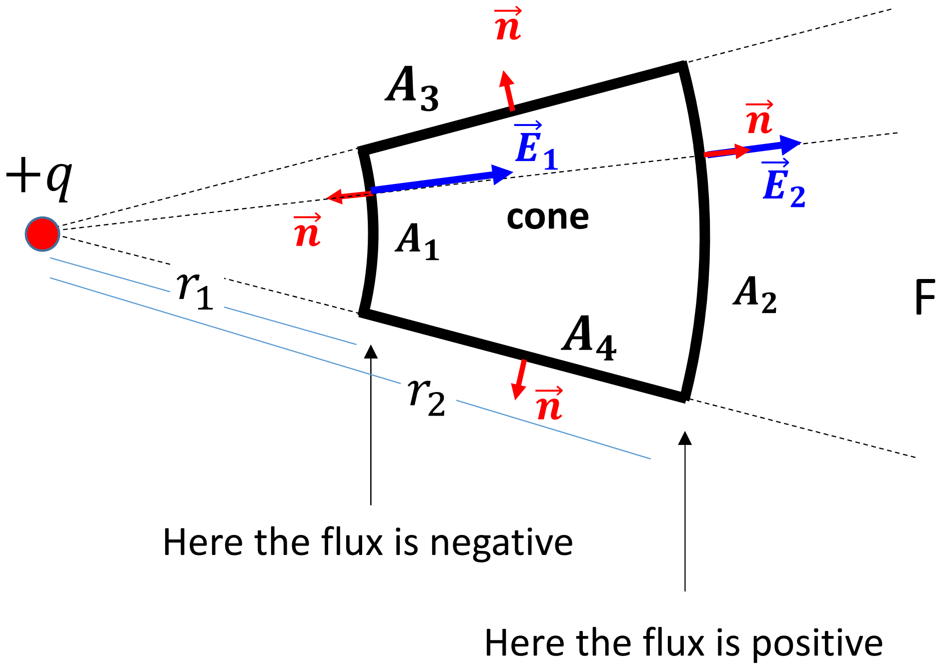
- At any point on A_1 and A_2, $\vec{E}$ is parallel to the unit vector
- At each point on A_1 and A_2 the magnitude $|\vec{E}|$ is constant
- On the two other sides $\boldsymbol{A_3}$ and $\boldsymbol{A_4}$ of the cone $\vec{E}$ is $\perp$ to the unit vector ⇨ **the flux through these two areas is zero**

This makes life much easier as the scalar product $\vec{E}d\vec{A} = EdA$

$$Flux = -\int_{A_1} E_1 dA + \int_{A_2} E_2 dA$$

As E_1 and $E_2$are constant along their respective areas A_1 and A_2

$$Flux = (-E_1A_1 + E_2A_2 + 0 + 0) \propto -\frac{A_1}{r_1^2} + \frac{A_2}{r_2^2}$$



What are A_1 and A_2?

From the solid angle of the cone $\Omega = \frac{A_1}{r_1^2} = \frac{A_2}{r_2^2}$

$$A_2 = r_2^2 \frac{A_1}{r_1^2}$$

Plugging this in the flux given above, we get

$$Flux \propto -\frac{A_1}{r_1^2} + \frac{A_2}{r_2^2} = -\frac{A_1}{r_1^2} + \frac{A_1}{r_1^2} = 0$$

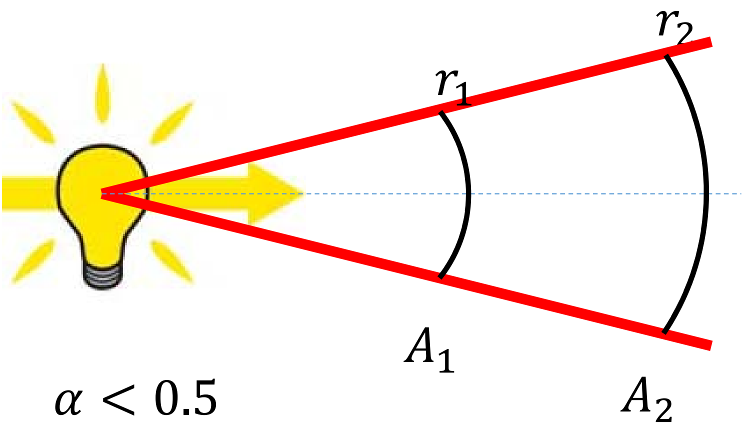
Gauss's theorem ⇒ $\vec{\nabla}.\vec{E} = 0$

Conclusion

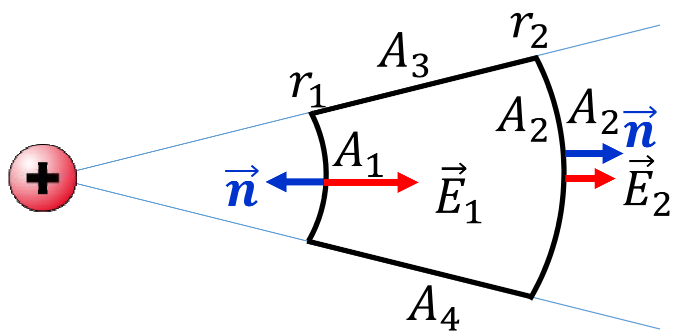
- Gauss's law would not be valid without Coulomb's law ($E \propto \frac{1}{r^2}$)
- The flux through a closed surface not containing any charge is zero no mater the shape of that surface
- Coulomb's law would not be valid without Gauss's law

Consequence

- These two laws are in perfect agreement with the **<u>law of energy conservation</u>**



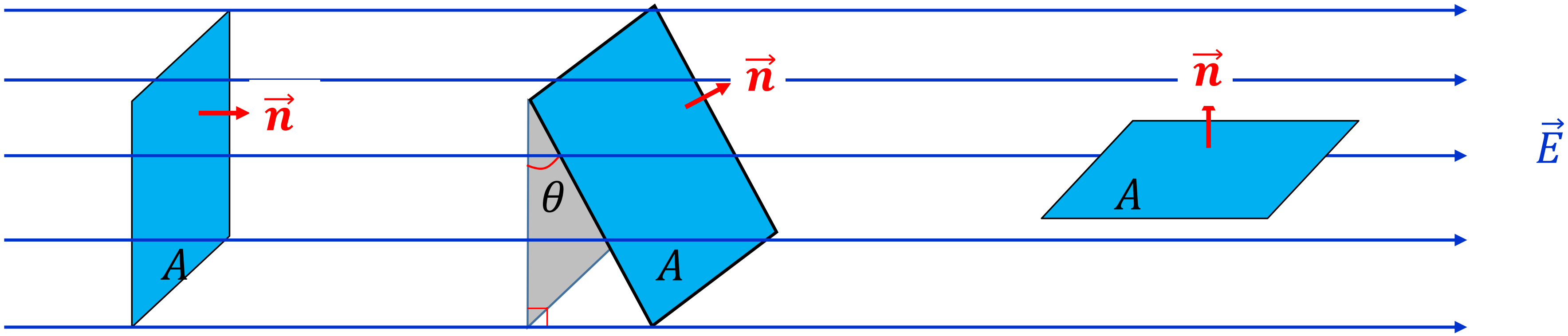
$$\frac{A_1}{{r_1}^2} = \frac{A_2}{{r_2}^2}$$



Energy passing through A_1 = Energy passing through A_2

Demonstrate that Gauss theorem does not depend on the shape of the Gauss surface

Tilting the surface with respect to the field changes the flux



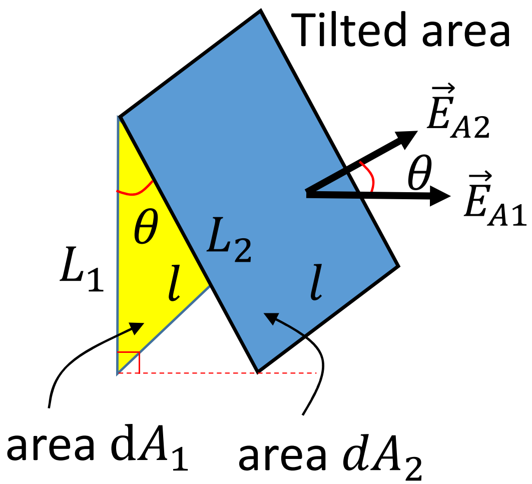
$A = A_\perp$ $\quad$ $A = A_\perp / cos\theta$ $\quad$ $A_\perp = 0$

$E_\perp = E$ $\quad$ $E_\perp = Ecos\theta$ $\quad$ $E_\perp = 0$

$$\Phi_E = \int_A \vec{E} d\vec{A}$$ Unchanged

Perfect compensation

Tilting the surface does not change anything



$$\left.\begin{aligned} \mathrm{d}A_1 &= lL_1 \\ dA_2 &= lL_2 \\ L_1 &= L_2 cos\theta \end{aligned}\right\} \quad \mathrm{d}A_1 = dA_2 cos\theta$$

$$\vec{E}_{A2} = \vec{E}_{A1} cos\theta$$

$$\vec{E}_{A1} = \frac{\vec{E}_{A2}}{cos\theta}$$

$$\vec{E}_{A1}.d\vec{A}_1 = \vec{E}_{A2}.d\vec{A}_2$$

$$\boxed{\int_{A_1} \vec{E}_{A1}.d\vec{A}_1 = \int_{A_2} \vec{E}_{A2}.d\vec{A}_2}$$

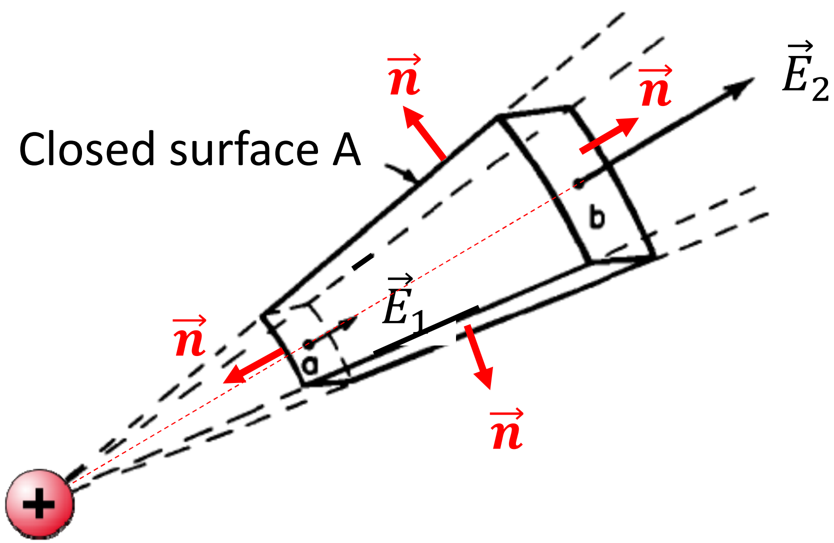
The flux through a closed surface does not depend on the shape of the surface

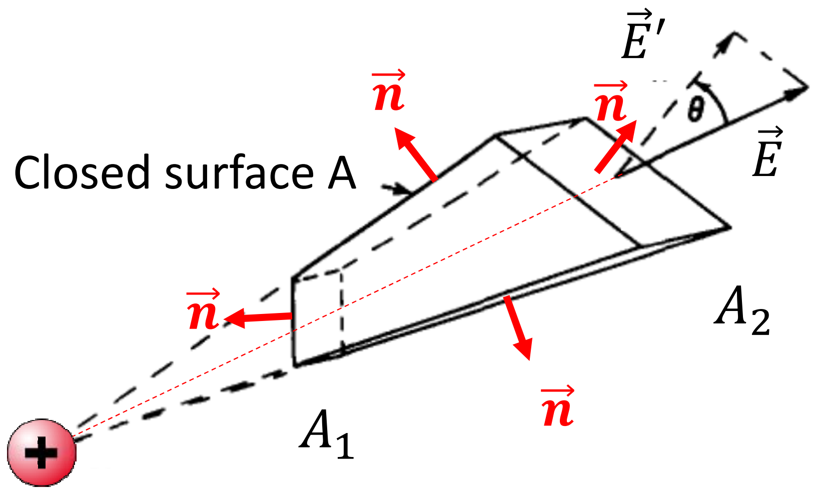
The flux of the electric field through closed surface

The miracle behind the $1/r^2$ dependence

$$\text{Flux } \Phi_E = \oiint \vec{E}\, d\vec{A}$$

Closed surface





Tilting the surfaces does it bring anything new ?

Demonstrate that the fluxes through these two surfaces are the same

First part of the first Maxwell's equation

Applying Gradient $\vec{E} = -\vec{\nabla}\mathrm{V}$

Applying Divergence $\vec{\boldsymbol{\nabla}}.\vec{E} = \vec{\boldsymbol{\nabla}}.(-\vec{\nabla}\mathrm{V})$

$\Rightarrow \quad \vec{\boldsymbol{\nabla}}.\vec{E} = -\nabla^2\mathrm{V}$

Not complete

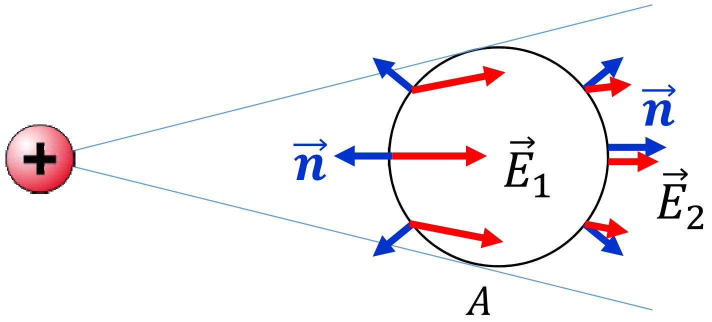
Towards Poisson and Laplace equations

Second Maxwell's equation

Applying Curl $\vec{\nabla} \times \vec{E} = \vec{0}$

Valid in Electrostatic only !

Charge outside

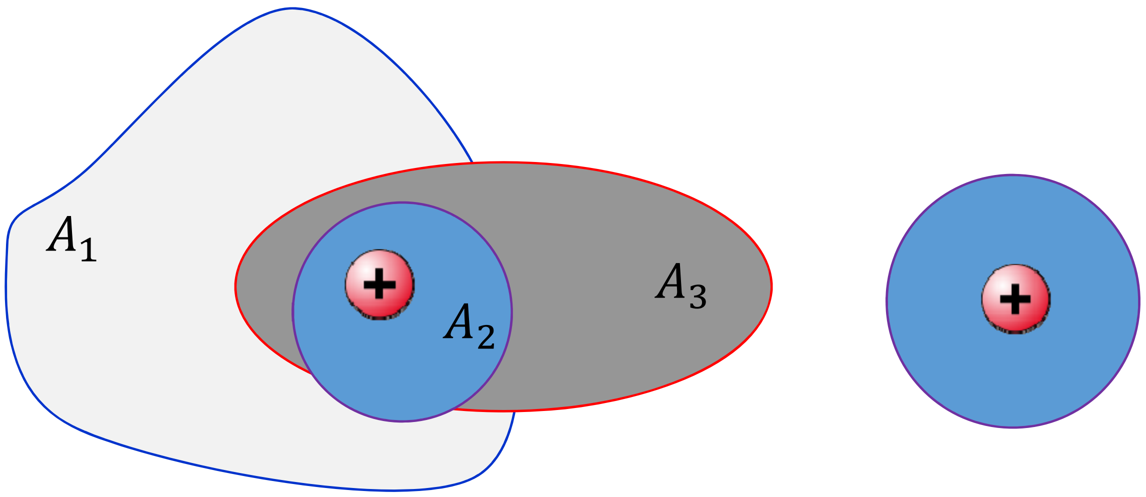


$$\text{Flux } \Phi = \oiint \vec{E} d\vec{A} = 0$$

Closed surface

Charge inside

The flux of the field produced by the charge $+q$ through the 3 surfaces is the same



$$\text{Flux } \Phi = \oiint \vec{E} d\vec{A} = \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right)(4\pi r^2) = \frac{q}{\varepsilon_0}$$

Closed surface

Gauss theorem

Enclosed in the volume

$$\text{Flux } \Phi = \oiint \vec{E}.d\vec{A} = \frac{\sum_i q_i}{\varepsilon_0}$$

$$\sum_i q_i = Q = \int_{\text{Volume}} \rho dV$$

$$\oiint_{\text{surface}} \vec{E} d\vec{A} = \int_{\text{Volume}} \frac{\rho}{\varepsilon_0} dV$$

Divergence theorem

$$\oiint_{\text{Surface}} \vec{E} d\vec{A} = \int_{\text{Volume}} \vec{\nabla}.\vec{E}\, dV$$

$$\boxed{\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}}$$

Gauss Law

Electrostatic field

If $\frac{\partial}{\partial t} = 0$ no time dependence

$$\boxed{\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}}$$

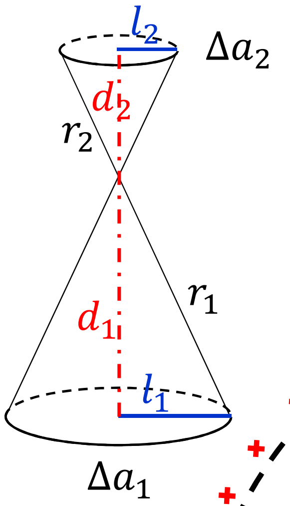
$$\vec{E} = -\vec{\nabla}\mathrm{V}$$ $\Leftrightarrow$ Electrostatic field is conservative

$$\vec{\nabla}.\vec{E} = -\vec{\nabla}.(\vec{\nabla}\mathrm{V}) = -\nabla^2\mathrm{V} = \frac{\rho}{\varepsilon_0} \Rightarrow \boxed{\nabla^2\mathrm{V} = -\frac{\boldsymbol{\rho}}{\boldsymbol{\varepsilon_0}}}$$

Poisson equation

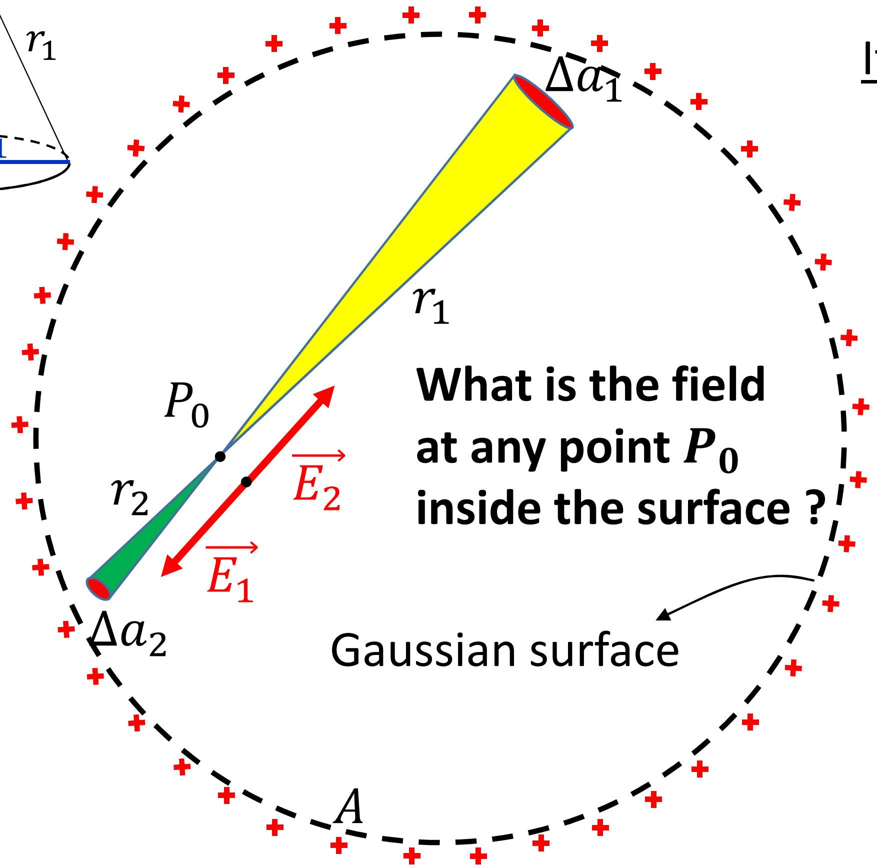
$$\boxed{\vec{\nabla} \times \vec{E} = 0} \Rightarrow \boxed{\mathrm{V} = -\int_a^b \vec{E}.d\vec{l}}$$

Is the field of a point charge exactly $1/r^2$?



Thales theorem

$$\frac{d_1}{d_2} = \frac{2l_1}{2l_2} = \frac{r_1}{r_2} \Rightarrow \frac{\pi l_1^2}{\pi l_2^2} = \frac{r_1^2}{r_2^2} = \Rightarrow \frac{\Delta a_1}{\Delta a_2} = \frac{{r_1}^2}{{r_2}^2}$$



Uniformly charged sphere

<u>If the charge on the sphere is uniform</u> $\frac{\Delta a_1}{\Delta a_2} = \frac{\Delta q_1}{\Delta q_2}$

$$\frac{\Delta q_1}{\Delta q_2} = \frac{{r_1}^2}{{r_2}^2} \Rightarrow \frac{\Delta q_1}{{r_1}^2} = \frac{\Delta q_2}{{r_2}^2} \Rightarrow \boxed{E_1 = E_2}$$

Electric Field inside the sphere = 0 $\quad \boldsymbol{E \propto 1/r^2}$

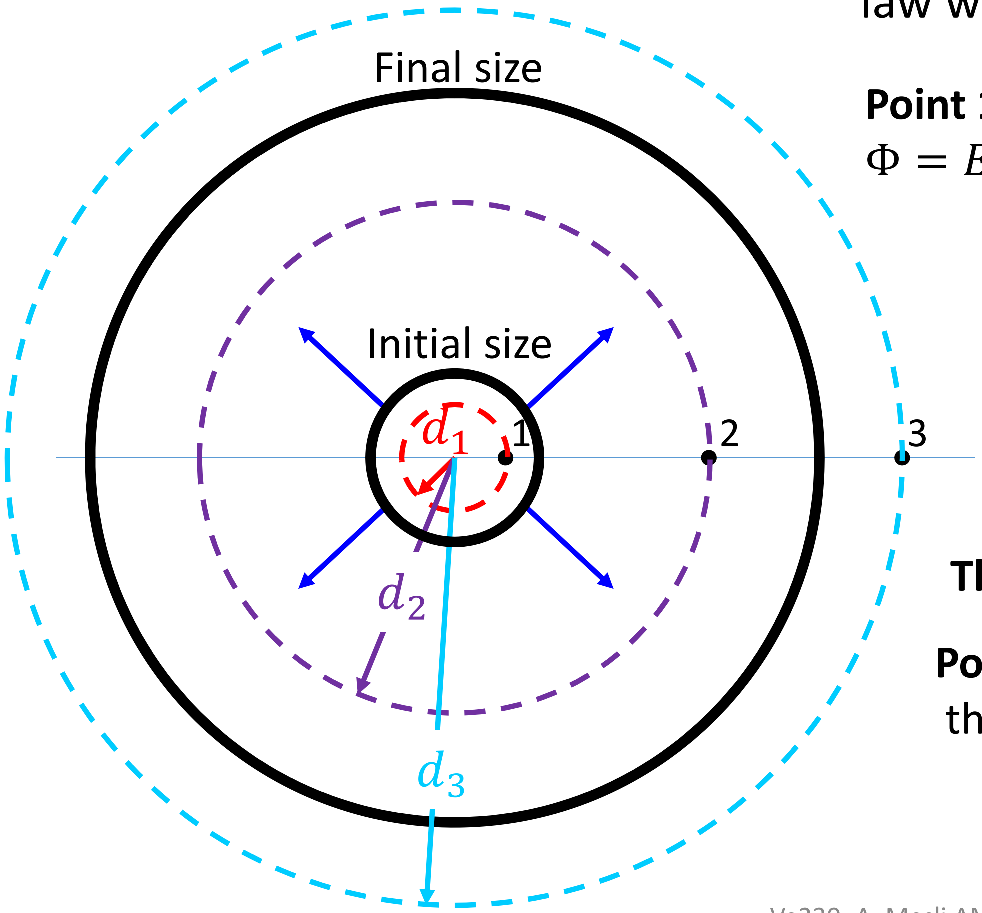
Gauss law = $\oint_A \vec{E}.d\vec{A} = 0 \Rightarrow \vec{E} = \vec{0}$

Coulomb law derives from Gauss law

Problem at midterm II fall 2017

A charged balloon expands as it is blown up, increasing in size from the initial to final diameter as shown. Do the electric fields at point 1, 2 and 3 increase, decrease, or stay the same? Explain

- Draw 3 Gaussian surfaces



- Symmetry argument is crucial here otherwise Gauss law would be of no help

Point 1: Remains always inside the charged balloon.
$\Phi = E_1.\boldsymbol{A_1} = 0$ as no charge inside Gaussian surface 1

$$\boxed{E_1 = 0}$$

Point 2: As long as the charged balloon **remains inside** Gaussian surface 2 and because of symmetry argument
$\Phi = E_2.\boldsymbol{A_2} = Q/\varepsilon_0 = Cte$

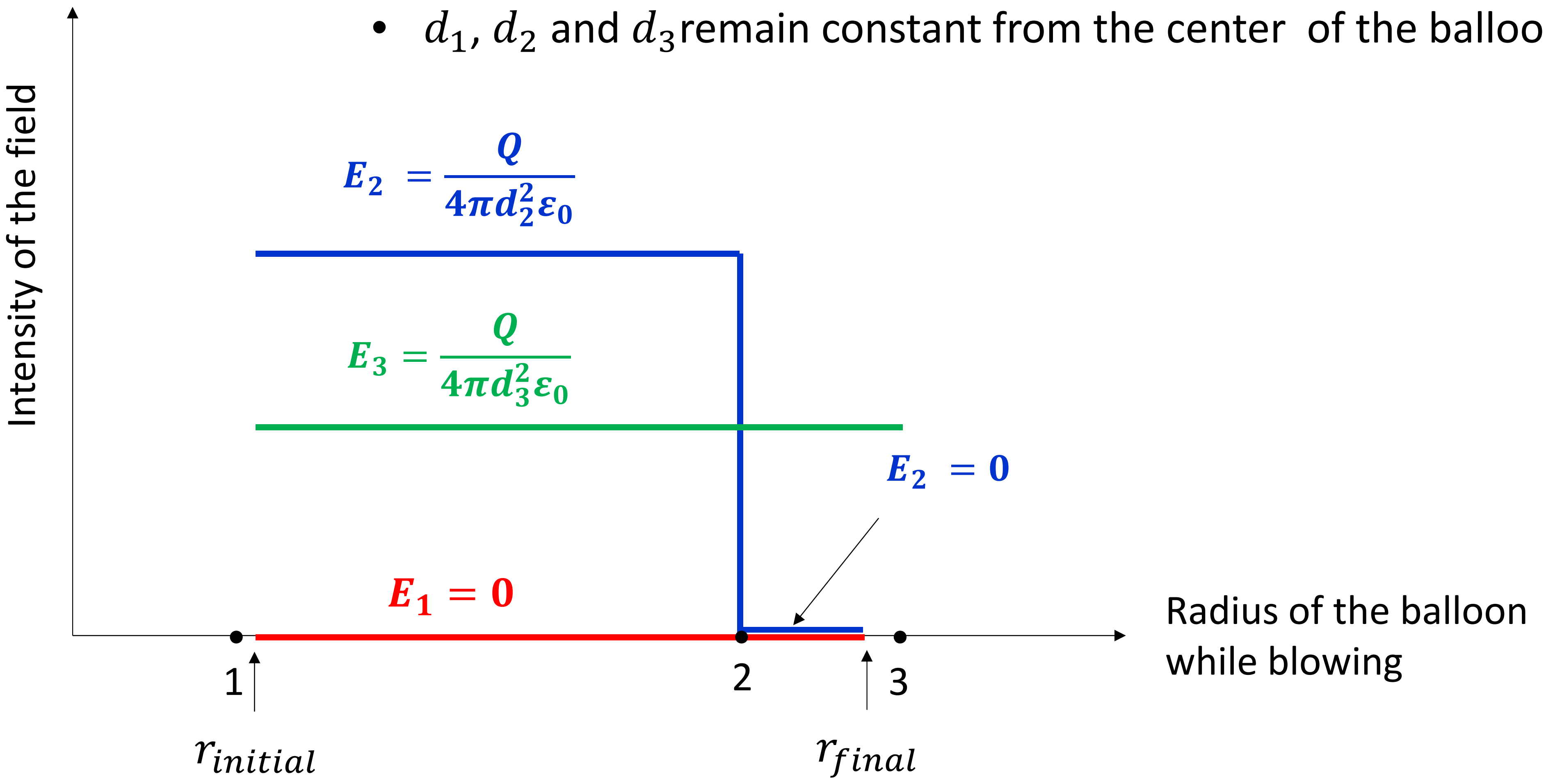
$$\boxed{E_2 = \frac{Q}{4\pi d_2^2 \varepsilon_0} = Cte}$$

Then it drops to 0 when the balloon reaches point 2 and beyond

Point 3: The charged balloon remains inside Gaussian surface 3 all the time. $\Phi = E_3.\boldsymbol{A_3} = Q/\varepsilon_0 = Cte$

$$\boxed{E_3 = \frac{Q}{4\pi d_3^2 \varepsilon_0} = Cte}$$

- The charge $\boldsymbol{Q}$ on the balloon does not change during the expansion
- d_1, d_2 and d_3 remain constant from the center of the balloon

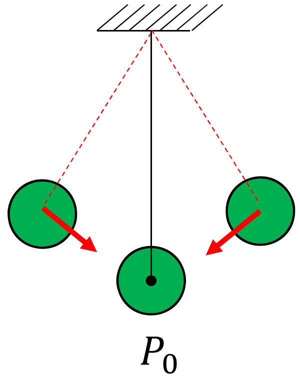


Consequence of Gauss's law

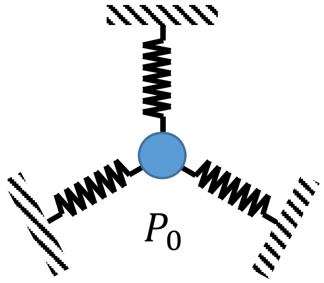
A static charge **CANNOT** be in equilibrium in the electric field of other **STATIC** charges?

Mechanical equilibrium

pendulum

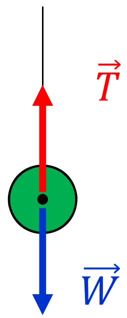


Stable equilibrium position

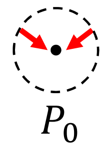


Mass – spring system

Two conditions must be fulfilled

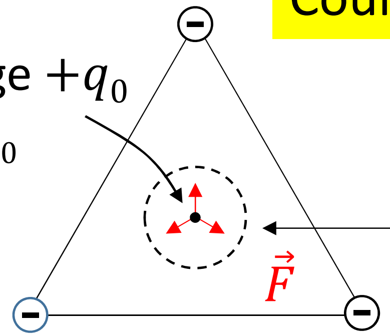


$$\sum \vec{F}_j = \vec{0}$$



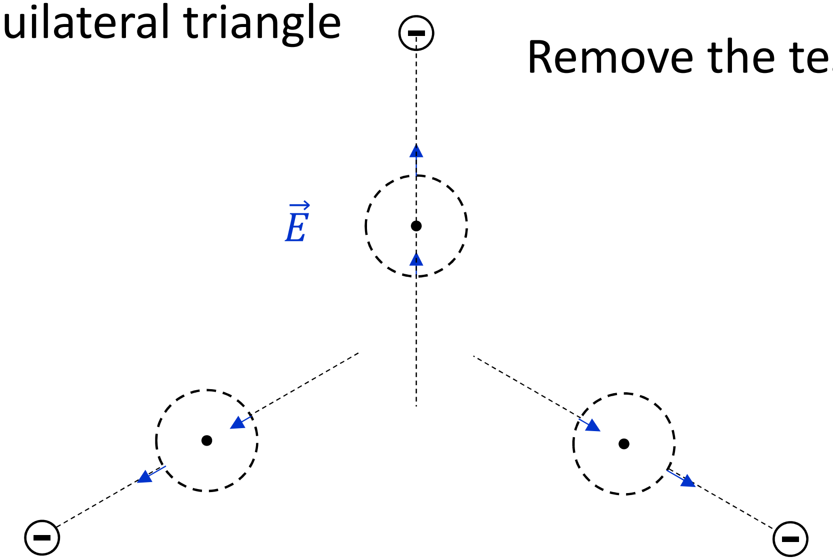
Restoring forces directed towards P_0

Could a positive charge test $+q_0$ placed at P_0 be in equilibrium ?

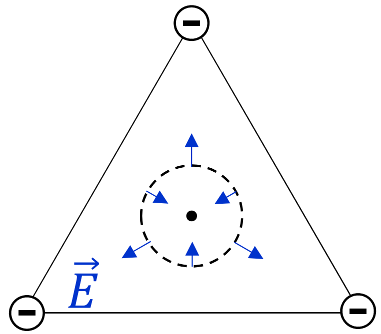


First condition fulfilled $\sum_i \vec{F}_i(P_0) = \vec{0}$

Equilateral triangle



Remove the test charge from point P_0



⇨ **Flux = 0 through area**

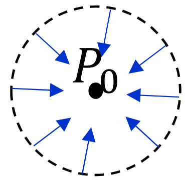
Principle of superposition

$$\vec{\nabla}.\vec{E} = 0$$

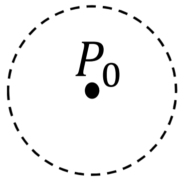
Could a positive charge test $+q_0$ placed at P_0 be in equilibrium ?

Second condition: *The restoring forces* should bring the charge back to P_0 if slightly displaced **NOT** fulfilled

The electric flux must be negative around P_0?



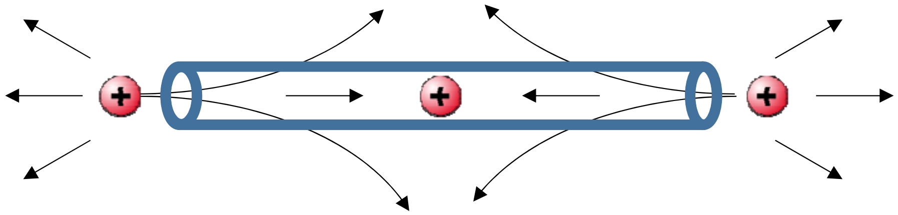
BUT this is <u>**impossible**</u> ! as there are no charges in the area



$$\vec{\nabla}.\vec{E} = 0$$

Static atom cannot be in stable equilibrium. It cannot exist

Devising a system in which stable equilibrium is possible although the flux of the field is zero?



This charge is in stable mechanical equilibrium although $\vec{\nabla}.\vec{E} = 0$

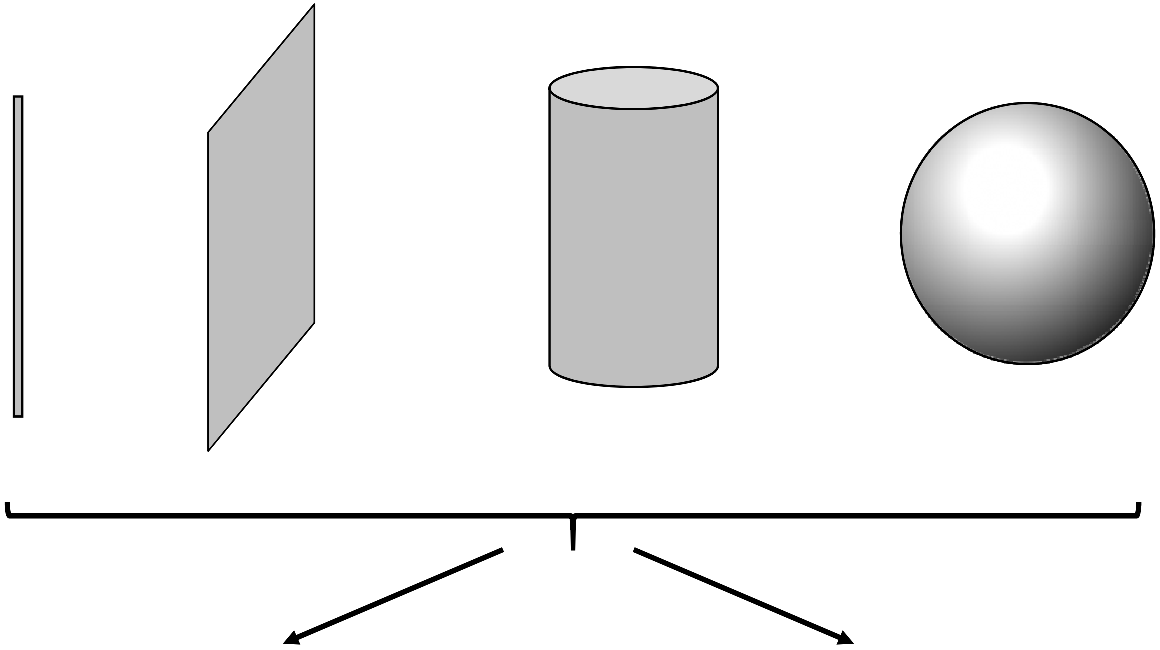
Why?

Because a mechanical constraint has been added

What is the Gauss's law made for?

Coulomb's law is the fundamental law for calculating electric field in any configuration…

In principle…



Complicated with Coulomb's law

Due to symmetry ⇨ straightforward with Gauss's law

But is sometimes useless

Electrostatics

- Coulomb law: superposition principle
  - o Is superposition principle an easy concept when dealing with vectors?

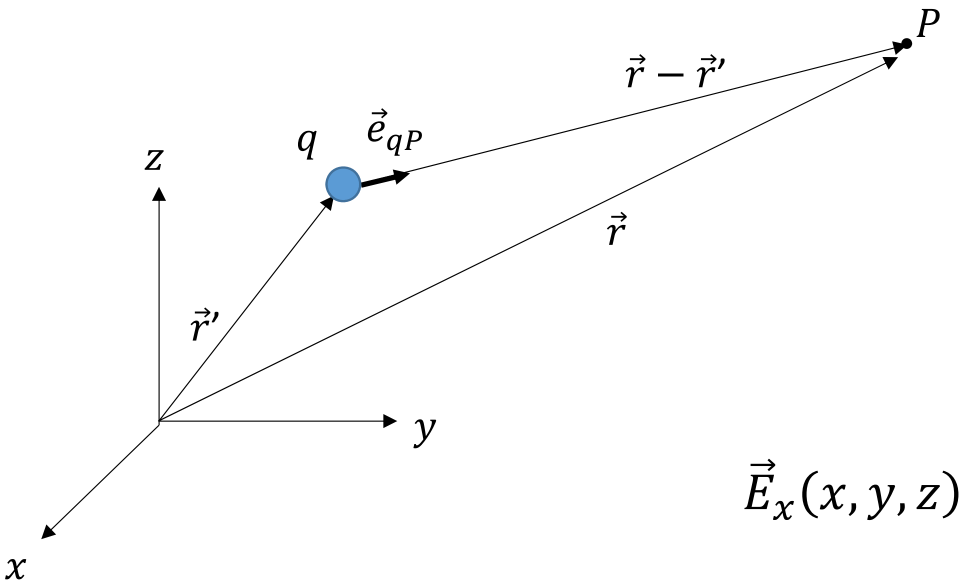
- Concept of work – Potential Energy

- Electrostatic potential and implication: $\vec{E} = -\vec{\nabla}\mathrm{V}$

- Electric field and potentials for various distributions of charges

Case of point charge

$$\vec{e}_{qP} = \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|}$$



$$\vec{r} - \overrightarrow{r'} = (x - x')\vec{\imath} + (y - y')\vec{\jmath} + (z - z')\vec{k}$$

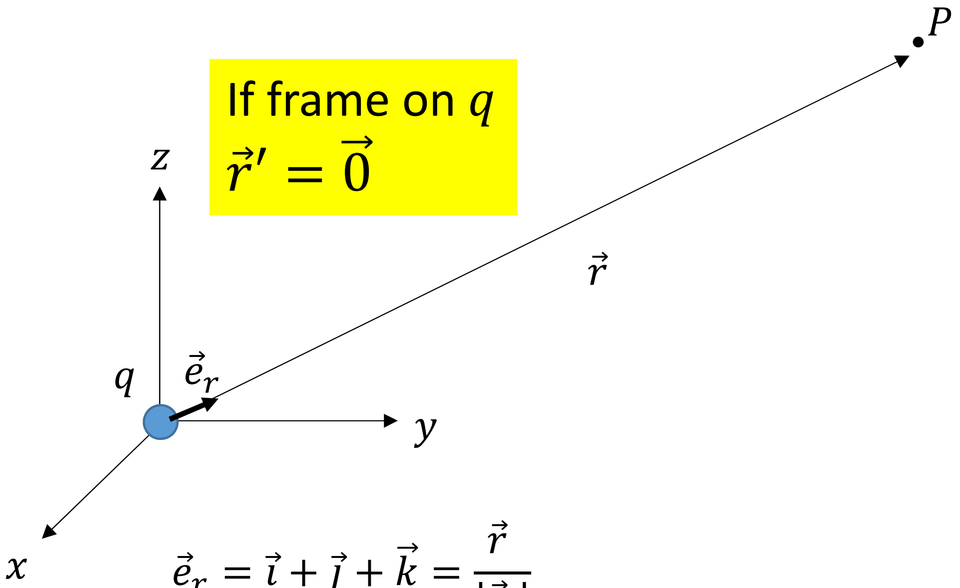
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r} - \vec{r}'|^2}\underbrace{\frac{\boldsymbol{\vec{r} - \vec{r}'}}{|\boldsymbol{\vec{r} - \vec{r}'}|}}_{\boldsymbol{\vec{e}_{qP}}} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

$$\vec{E}(\vec{r}) = \vec{E}_x(x, y, z) + \vec{E}_y(x, y, z) + \vec{E}_z(x, y, z)$$

$$\vec{E}_x(x, y, z) = \frac{q}{4\pi\varepsilon_0}\frac{x - x'}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}}\vec{\imath}$$

$\boldsymbol{\vec{E}_x(x, y, z)}$ = function of 3 variables

For a single point charge simplification is possible



If frame on q
$\vec{r}\,' = \vec{0}$

$$\vec{e}_r = \vec{\imath} + \vec{\jmath} + \vec{k} = \frac{\vec{r}}{|\vec{r}\,|}$$

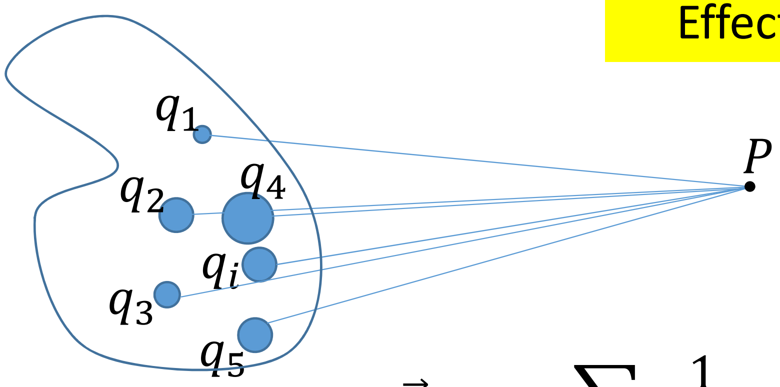
$$\vec{E}_x(x, y, z) = \frac{q}{4\pi\varepsilon_0}\frac{x}{[x^2 + y^2 + z^2]^{3/2}}\vec{\imath}$$

The same for the other components

$$\boxed{\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\vec{e}_r}$$

Principle of superposition

Effect of charge distribution



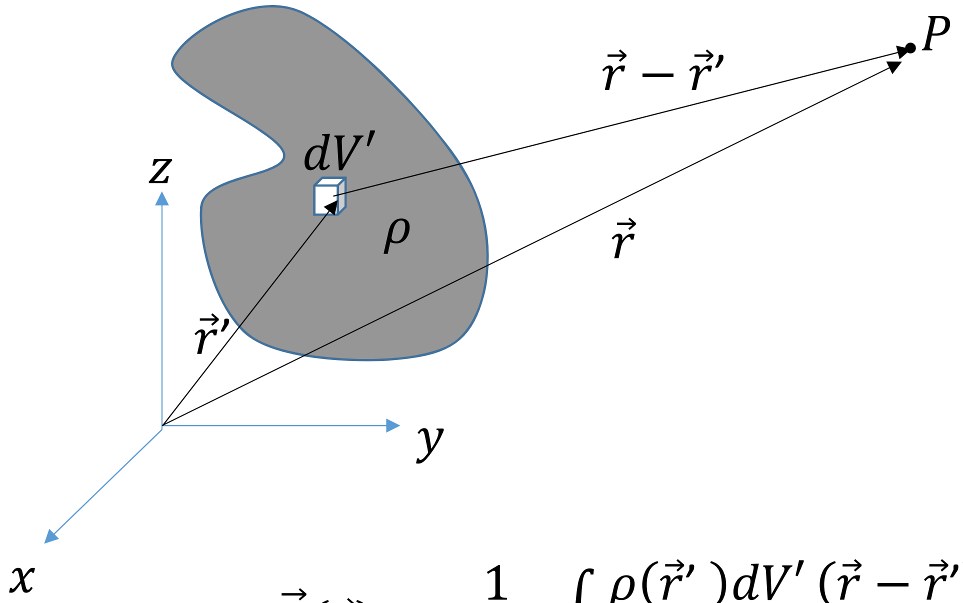
$$\vec{E}(P) = \sum_i \frac{1}{4\pi\varepsilon_0}\frac{q_i}{{r_{iP}}^2}\vec{e}_{iP}$$

Σ to $\int$

<u>Provided we look far away from the distribution</u>

<u>Remember the dipole:</u> A test charge close to it may feel one or the other charge, but far away it feels nothing

$$\sum q_i = 0$$



$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')dV'}{|\vec{r}-\vec{r}'|^2}\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|}$$

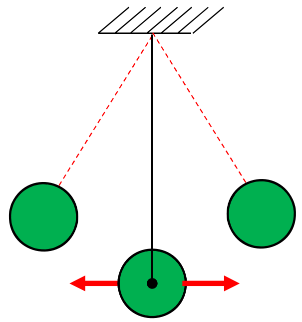
Difficult to carry integration on vectors

Concept of work – Electrostatic Potential Energy

What does a conservative force mean and imply?

Conservative force keeps **REVERSIBLE** the exchange between Kinetics and Potential energies

Pendulum



K Minimum, U maximum

U Minimum, K maximum

Conservative force conserves the total energy

$$K_1 + U_1 = K_2 + U_2 \quad \Rightarrow \quad \Delta K_{1\to 2} = -\Delta U_{1\to 2} \; or \; K_2 - K_1 = -(U_2 - U_1)$$

Work – energy theorem For **C**onservative **F**orce (CF) ⇨ $\boxed{\boldsymbol{W(CF)_{1\to 2} = \Delta K_{1\to 2}}}$

$$\boxed{\Rightarrow \quad \boldsymbol{W(CF)_{1\to 2} = -\Delta U_{1\to 2}}}$$

In the absence of friction, there are 3 fundamental conservative forces

- Gravitation
- Elastic
- Electric

If other external forces are involved ⇨ work – energy theorem says

⇨ $W(CF)_{1\to 2} + W(ext)_{1\to 2} = \Delta K_{1\to 2}$

When the work is done slowly to keep $\boldsymbol{\Delta K_{1\to 2} = 0}$ ⇨ $\boldsymbol{W(ext)_{1\to 2} = -W(CF)_{1\to 2}}$

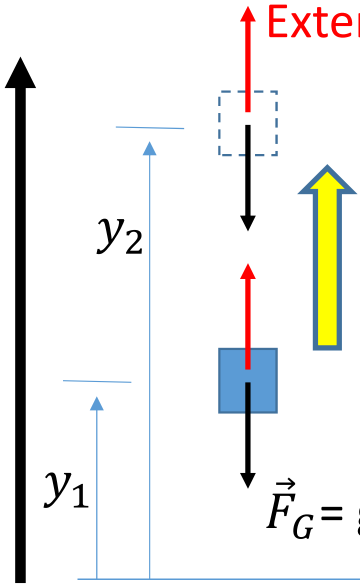
External forces do work **AGAINST** conservative forces

$$\boxed{W(ext)_{1\to 2} = -W(CF)_{1\to 2} = -(U_2 - U_1)}$$

Lifting slowly a body

Energy is injected into the system

External force: does positive work



Displacement **opposite** to gravitational force

$\vec{j}$

$$\vec{F}_G = -mg\vec{j}$$
$$d\vec{l} = dy\vec{j}$$

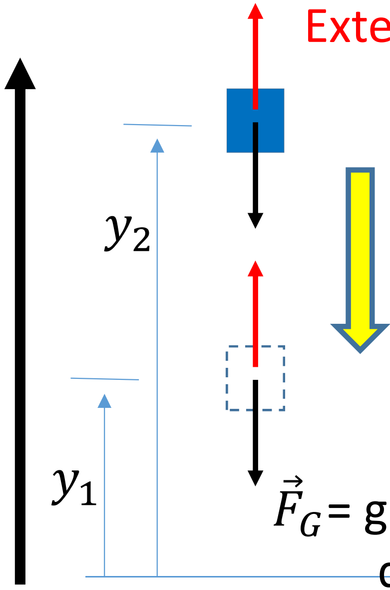
$\vec{F}_G$= gravitational force: does negative work

$$\boxed{W(CF)_{1\to 2} = \int_{y_1}^{y_2} \vec{F}_G d\vec{l} = -\mathrm{mg}(y_2 - y_1) < 0}$$

Lowering slowly a body

Energy is extracted from the system

External force: does negative work



Displacement and gravitational force In the **same direction**

$\vec{F}_G$= gravitational force does positive work

$$\boxed{W(CF)_{2\to 1} = \int_{y_2}^{y_1} \vec{F}_G d\vec{l} = -\mathrm{mg}(y_1 - y_2) > 0}$$

<u>Conservative mechanical force always acts</u> naturally to push the system towards **lower potential energy**

In case of one dimension $$W_{CF} = -\Delta U \Rightarrow F_{CF}.dx = -dU \Rightarrow \vec{F}_{CF} = -\frac{dU}{dx}\vec{i}$$

In case of three dimensions

$$\Rightarrow \boldsymbol{\vec{F}_{CF} = -\vec{\nabla}U}$$

Gradient directed towards higher potential energy

Nature acts to lower the potential energy

In the case of electrostatic $$\Rightarrow \boldsymbol{\vec{F}_{el} = -\vec{\nabla}U} \quad \Rightarrow \boldsymbol{q\vec{E}_{el} = -\vec{\nabla}q\mathrm{V}} \quad \Rightarrow \boldsymbol{\vec{E}_{el} = -\vec{\nabla}V}$$

Concept of work and Electric potential

In the presence of an electrostatic field $\vec{\boldsymbol{E}}$, a test charge q_0 feels a conservative force $\vec{\boldsymbol{F}}_{\boldsymbol{el}}$

work – energy theorem including external force ⇨ $W(\vec{\boldsymbol{F}}_{\boldsymbol{el}})_{1\to 2} + W(\vec{\boldsymbol{F}}_{\boldsymbol{ext}})_{1\to 2} = \Delta K_{1\to 2}$

⇨ $W(\vec{\boldsymbol{F}}_{\boldsymbol{el}})_{1\to 2} = -\Delta U_{1\to 2} = -(U_2 - U_1)$

To keep the test charge charge moving very slowly from 1 to 2

$$\Delta K_{1\to 2} = 0 \quad ⇨ \quad \boxed{W(\vec{F}_{ext})_{1\to 2} = -W(\vec{F}_{el})}$$

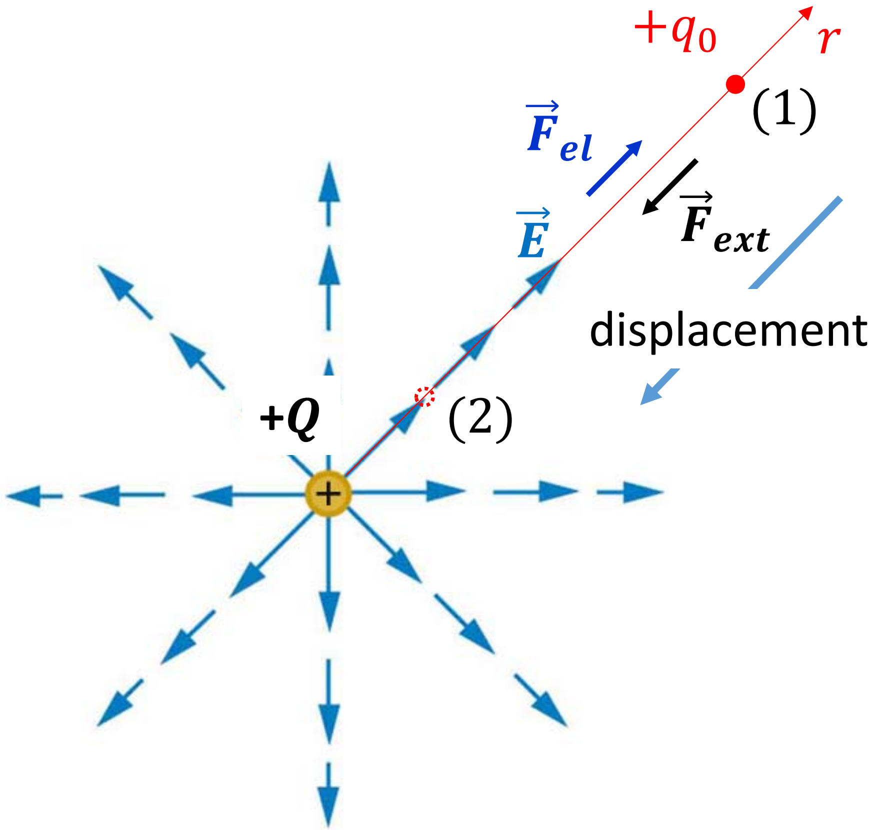
⇩

Work must be done against the electric force

$$\boxed{W(\vec{F}_{ext})_{1\to 2} = -\int_1^2 \vec{F}_{el}.\,d\vec{r}}$$

⇨ $W(\vec{F}_{el})_{1\to 2} = -\Delta U_{1\to 2} = -(U_2 - U_1)$ As external force acts against conservative force

⇨ $W(\vec{F}_{ext})_{1\to 2} = +\Delta U_{1\to 2} = +(U_2 - U_1)$

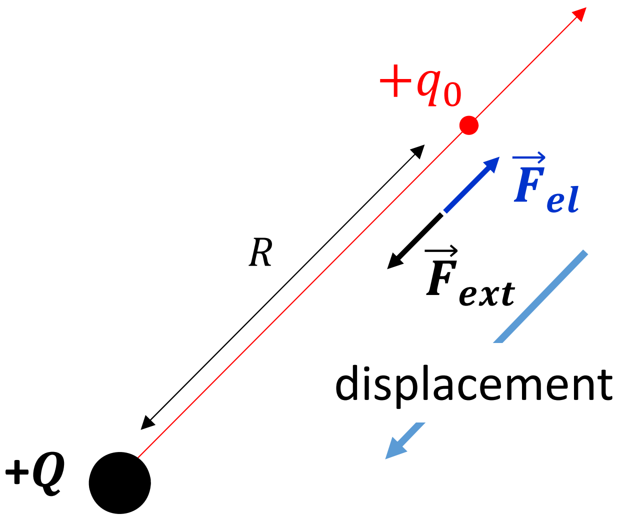


$+q_0$ is the charge test

Work done along a radial path

$\vec{F}_{ext}$ brings the test charge from: (1) to (2)

(1) Could be ∞



$W(\vec{F}_{ext}) > 0$ because energy is **<u>injected</u>** into the system: Why?

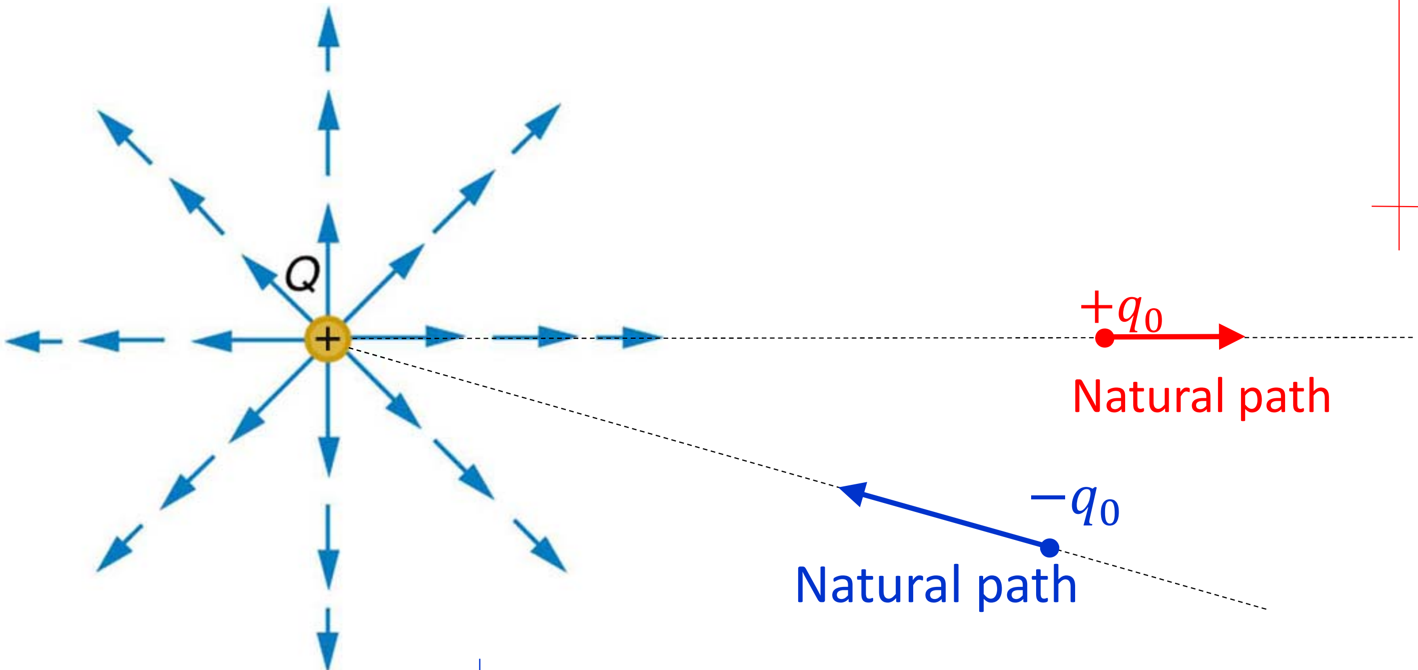
If charge q_0 is released, it will be repelled
(Energy is released by the system)

$$W(\vec{F}_{ext})_{\infty\to R} = \int_{\infty}^{R} \vec{F}_{ext}.d\vec{r} = -\int_{\infty}^{R} \vec{F}_{el}.d\vec{r} \qquad W(\vec{F}_{ext})_{\infty\to R} = \int_{R}^{\infty} \vec{F}_{el}.d\vec{r} = \frac{Qq_0}{4\pi\varepsilon_0}\int_{R}^{\infty}\frac{1}{r^2}dr$$

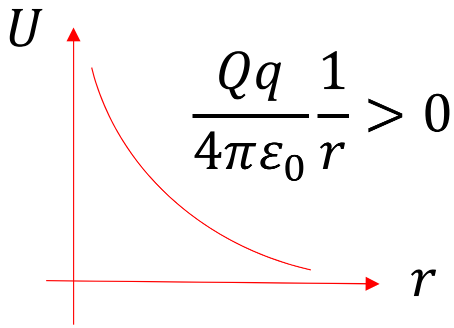
$$\boxed{W(\vec{F}_{ext})_{\infty\to R} = \frac{Qq_0}{4\pi\varepsilon_0}\frac{1}{R}} = U \text{ Potential energy}$$

- Q and $q_0 > 0$ Or Q and $q_0 < 0$ $\qquad U > 0$
- $Q > 0$ and $q_0 < 0$ Or vice versa $\qquad U < 0$

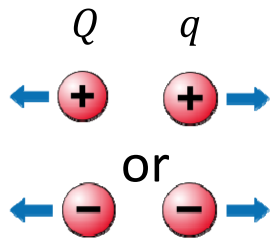
Potential energy

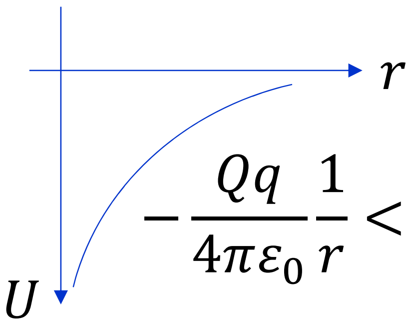


Towards decreasing potential energy

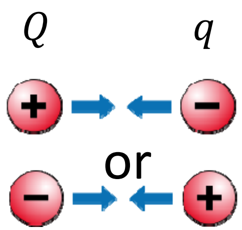


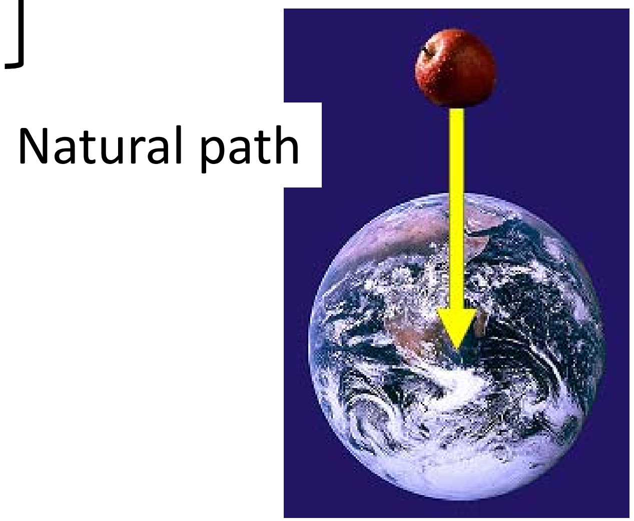
$$\frac{Qq}{4\pi\varepsilon_0}\frac{1}{r} > 0$$



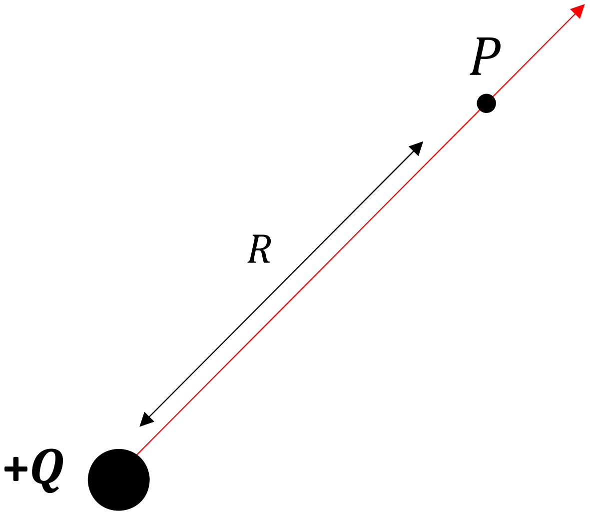


$$-\frac{Qq}{4\pi\varepsilon_0}\frac{1}{r} < 0$$





Concept of Electric potential



Concept of **Electric field** ⇨ what is the effect of $\boldsymbol{Q}$ on P when there is no test charge q_0 ? ⇨ There is $\vec{E}(P)$ ⇨ $\vec{F} = q_0\vec{E} = m\vec{a}$ ⇨ Energy

Likewise we define an **Electric Potential** of doing work on a test charge q_0 once placed at P

Work done on a unit charge q_0, $U = q_0 V$

$$W(\vec{\boldsymbol{F}}_{\boldsymbol{ext}})_{1\to2}\Big|_{q_0} = \frac{W}{q_0}(\vec{\boldsymbol{F}}_{\boldsymbol{ext}})_{1\to2} = (V_2 - V_1)$$

$$W(\vec{\boldsymbol{F}}_{\boldsymbol{el}})_{1\to2}\Big|_{q_0} = \frac{W}{q_0}(\vec{\boldsymbol{F}}_{\boldsymbol{el}})_{1\to2} = -(V_2 - V_1)$$

$$\boxed{V(R) = \frac{W}{q_0} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{R}}$$

$$\boxed{V(\infty) = 0}$$

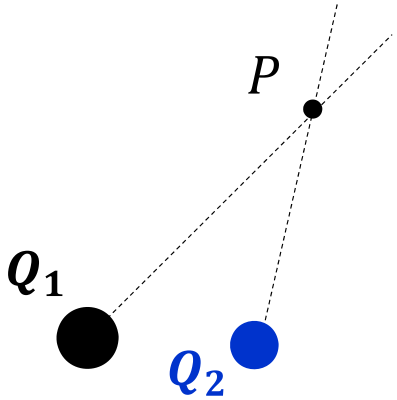
Superposition principle

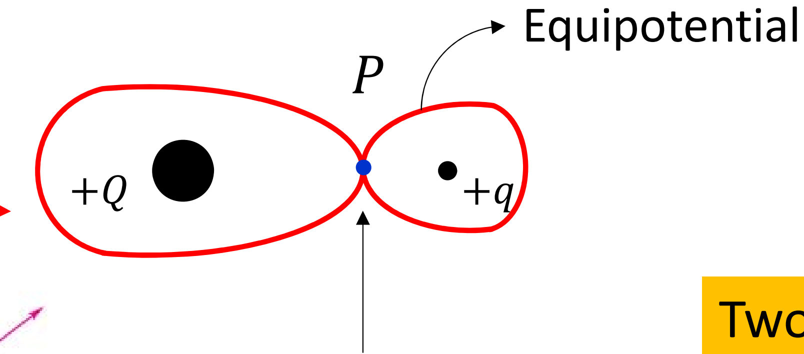
$$V(P) = V_{Q1} + V_{Q2}$$

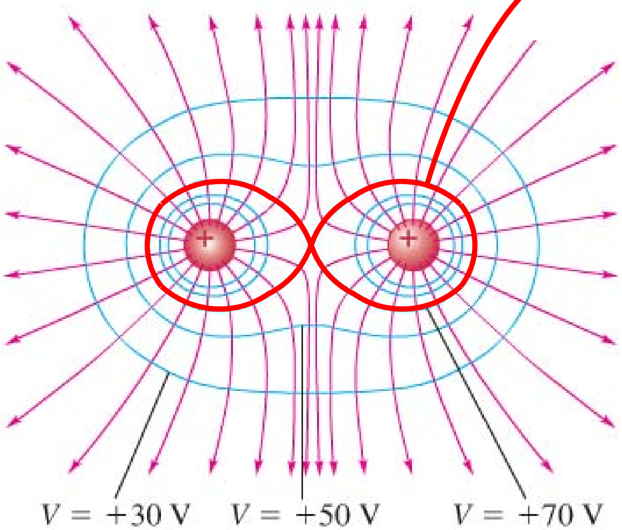
If charge at P is closer to $\boldsymbol{Q_1}$ it will feel V_{Q1}

If charge at P is closer to $\boldsymbol{Q_2}$ it will feel V_{Q2}

Far away it will feel $V(\boldsymbol{Q_1} + \boldsymbol{Q_2})$





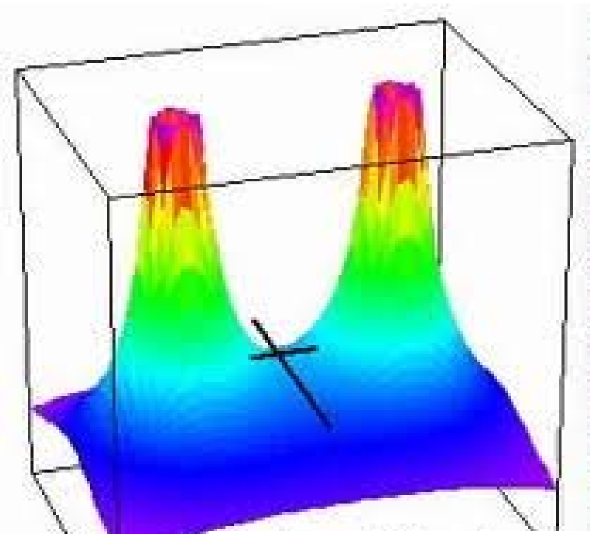


$\vec{E}(P) = \vec{0}$

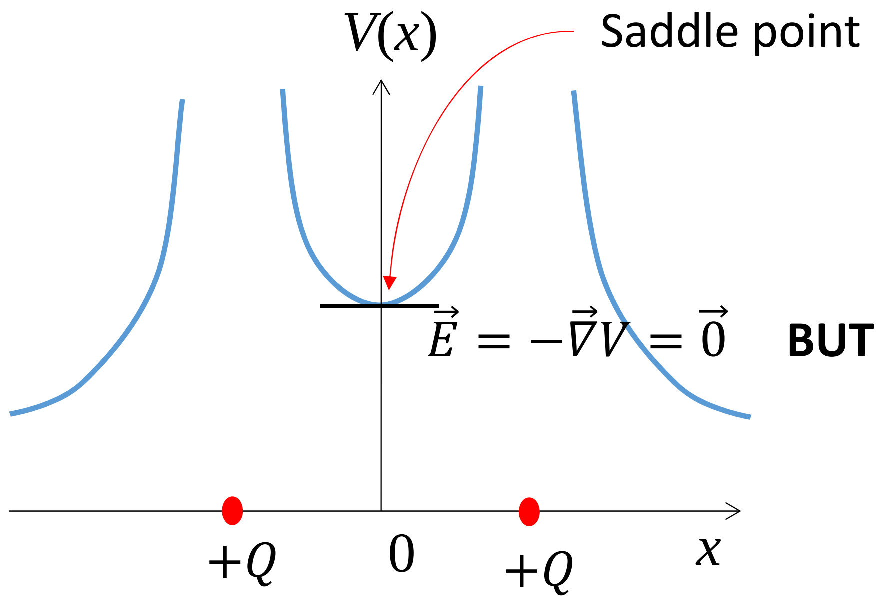
$\boldsymbol{V \neq 0}$

$|Q| \neq |q|$

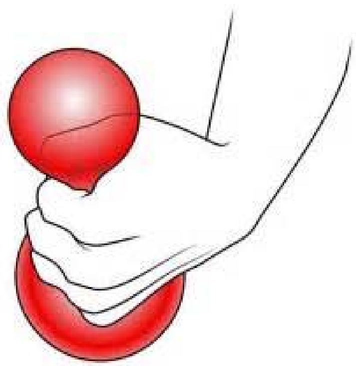
Two **different** equipotentials **NEVER** cross like two electric field lines

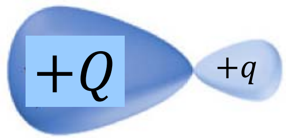


At the saddle point the gradient (field) is zero



$\vec{E} = -\vec{\nabla} V = \vec{0}$ **BUT** $V \neq 0$ **!**





Summarizing

The key word is the work done by the **field**

The potential energy $U = qV$ is zero when the charges are infinitely separated

So U is the work done on the test charge $\boldsymbol{q_0}$ by the **field** of $\boldsymbol{Q}$ $(\sum_{=1}^{n} q_i)$ when moving $\boldsymbol{q_0}$ from r to ∞

$W(\vec{F}_{el})_{r\to\infty} > 0$ if $\boldsymbol{Q}$ and $\boldsymbol{q_0}$ are of the same sign and $U(r) > 0$

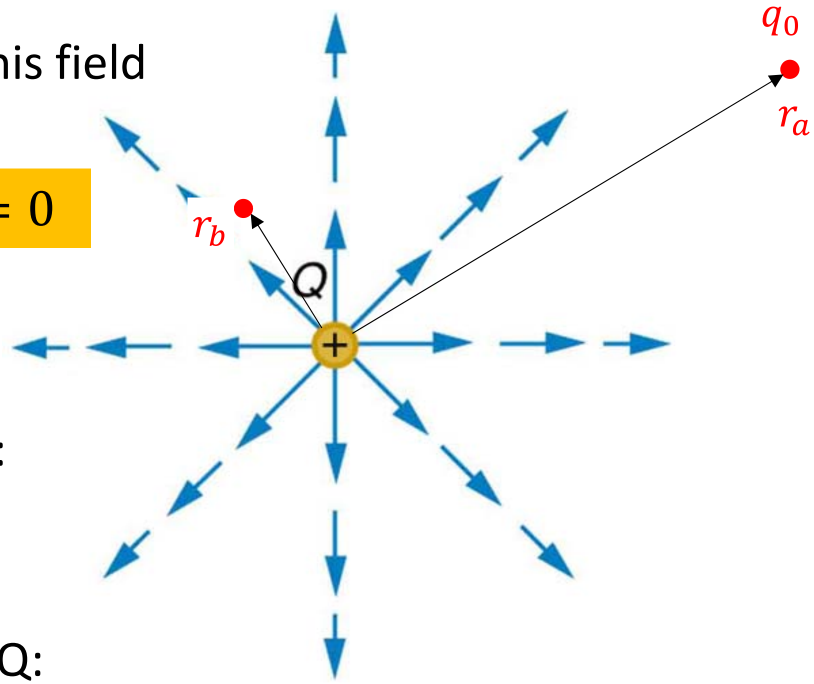
$W(\vec{F}_{el})_{r\to\infty} < 0$ if $\boldsymbol{Q}$ and $\boldsymbol{q_0}$are of opposite sign and $U(r) < 0$

Electrostatic potential and implication: $\vec{E} = -\vec{\nabla}\mathrm{V}$

Work done along an arbitrary path

The positive charge $\boldsymbol{Q}$ creates a field all around

A test charge $\boldsymbol{q_0}$ will undergo a force (attractive or repulsive) in this field



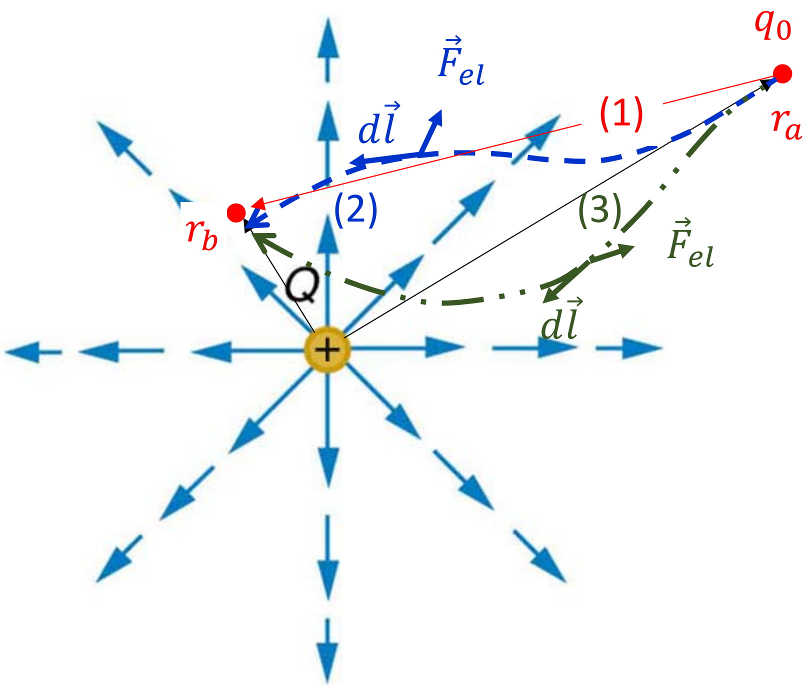
To keep moving very slowly the charge $\boldsymbol{q_0}$ from $\boldsymbol{r_a}$ to $\boldsymbol{r_b}$

$\Delta K = 0$

If $\boldsymbol{q_0 > 0}$:
a repulsion takes place with increasing intensity as we approach Q:
We need to push harder and harder

If $\boldsymbol{q_0 < 0}$:
an attraction takes place with increasing intensity as we approach Q:
We need to retain harder and harder

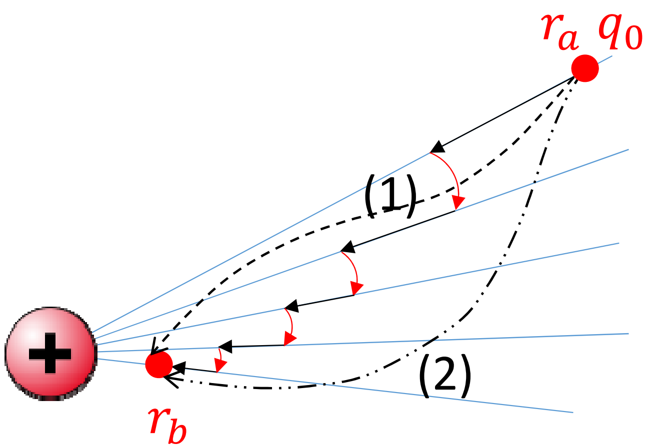
- The opposite happens if we need to go away from $\boldsymbol{Q}$
- In both cases the work is done against the electric force



$$W = -\int_{r_a}^{r_b} \vec{F}_{el} . d\vec{l}$$

Does the work depend on the path from r_a to r_b ?

$$W = -\int_{r_a}^{r_b} \vec{F}_{el}.d\vec{l} \quad \Rightarrow \quad \frac{W}{q_0} = W(unit)_{a\to b} = -\int_{r_a}^{r_b} \vec{E}.d\vec{l}$$



Along the radial lines $\vec{E} // d\vec{l}$ Work = maximum

Along the curved lines $\vec{E} \perp d\vec{l}$ Work = 0

Work does not depend on the path

$$W(unit)_{a\to b} = -\int_{r_a}^{r_b} \vec{E}.d\vec{l} = \int_{r_a}^{r_b} dV$$

$$\boxed{W(unit)_{a\to b} = V(r_b) - V(r_a)}$$

Another way of demonstrating the relation $\vec{E} = -\vec{\nabla}V$

Application of vector calculus !

$$\int_{r_a}^{r_b} dV = \int_{r_a}^{r_b} \left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \right) = \int_{r_a}^{r_b} \left(\frac{\partial V}{\partial x} \vec{\imath} + \frac{\partial V}{\partial y} \vec{\jmath} + \frac{\partial V}{\partial z} \vec{k} \right) \left(dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k} \right)$$

$$\int_{r_a}^{r_b} \vec{\nabla} V . d\vec{l}$$

$$\int_{r_a}^{r_b} dV = \int_{r_a}^{r_b} -\vec{E} . d\vec{l}$$

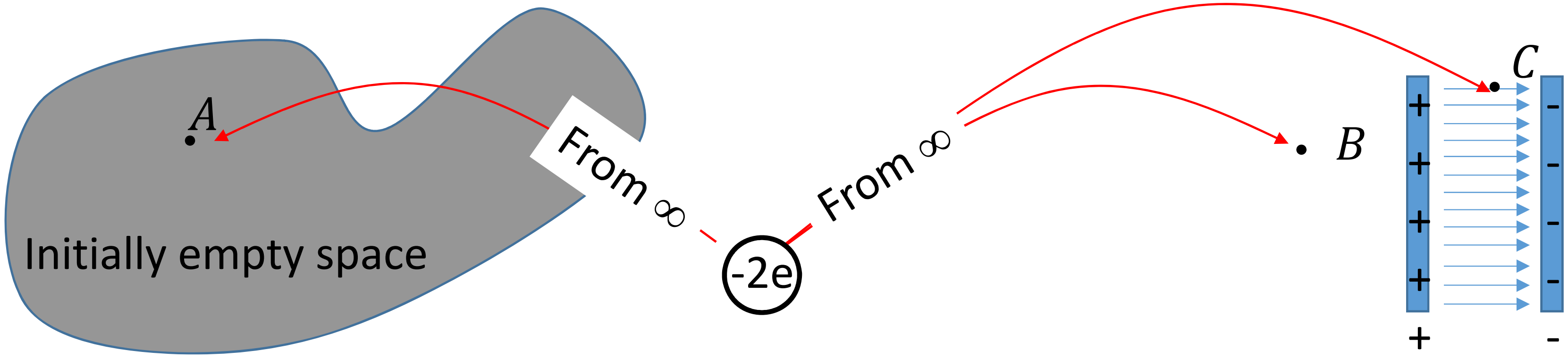
$$\int_{r_a}^{r_b} dV = \int_{r_a}^{r_b} \vec{\nabla} V . d\vec{l}$$

$$\Rightarrow \boxed{\vec{E} = -\vec{\nabla}V}$$

Static Electric field is conservative

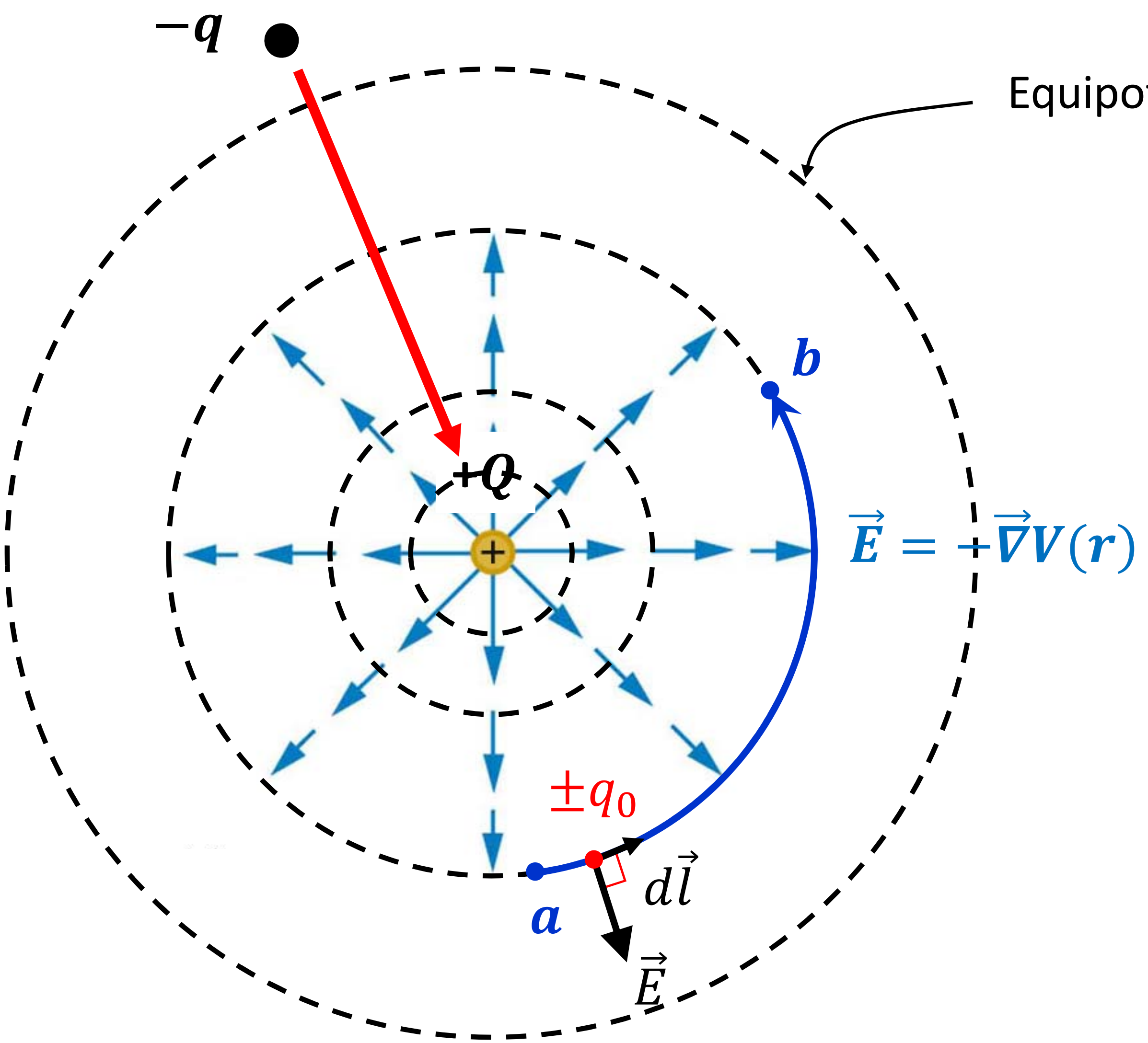
Questions

Does it require any work to bring the charge $-q$ from ∞ to point A, B and C?



If $\vec{E} \propto \frac{1}{r^n}\vec{e}_r$ with $n \neq 2$ would the work done be path independent?

For any radial and spherically symmetric force the work done does not depend on the path and there exist a potential. There could be any r dependence of $\vec{E} \propto \frac{1}{r^n}\vec{e}_r$. The existence of the potential and the fact that $\vec{\nabla} \times \vec{E} = \vec{0}$ is due to that only



Equipotential surface $\mathrm{V}(r)$

$$\int_a^b \vec{E}.d\vec{l} = 0$$

$$\oint \vec{E}.d\vec{l} = 0$$

Γ = equipotential path

From Stoke's theorem

$$\vec{\nabla} \times \vec{E} = \mathbf{0}$$

Curl-free

Applications of Coulomb and Gauss law

Electric field and potentials for various distributions of charges

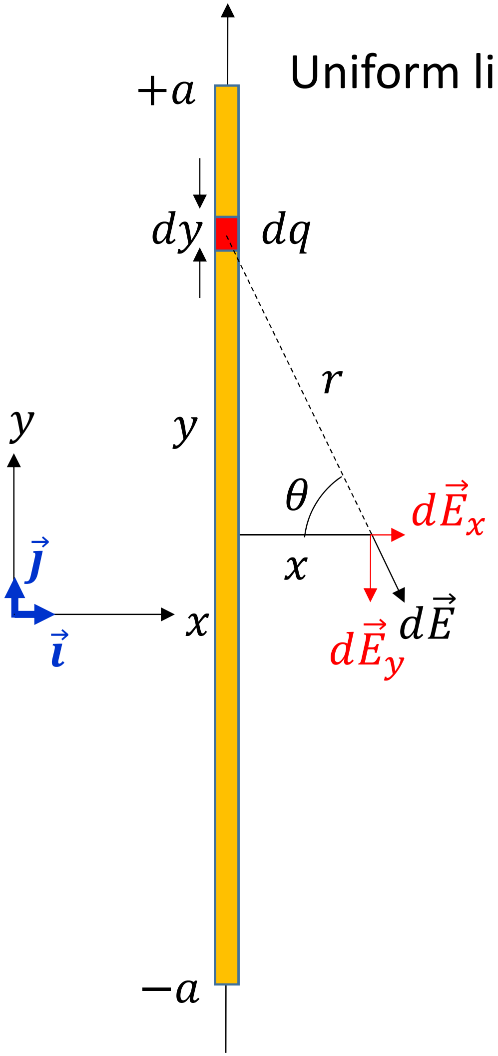
Electric field and potentials for various distributions of charges

Make use of symmetry whenever possible

- Point charge
- Line of charges
- Ring of charges
- Disk of charges
- Sheet of charges
- Plane (infinite disk) of charges
- Cylindrical distribution of charges
- Spherical distribution of charges
- Rectangular distribution of charges

The field of a line charge

By Coulomb' law



Uniform linear charge density $\lambda = \frac{dq}{dy} = \frac{Q}{2a}$

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2a}\frac{dy}{x^2+y^2}$$

$$dE_x = dE\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2a}\frac{xdy}{(x^2+y^2)^{3/2}}$$

$$dE_y = -dE\sin\theta = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2a}\frac{ydy}{(x^2+y^2)^{3/2}}$$

$$\boxed{E_x = \int_{-a}^{+a} dE_r = \frac{1}{4\pi\varepsilon_0}\frac{Q}{x}\frac{1}{\sqrt{(x^2+y^2)}}}$$

$$\boxed{E_y = \int_{-a}^{+a} dE_y = 0}$$

Why is this result trivial ?

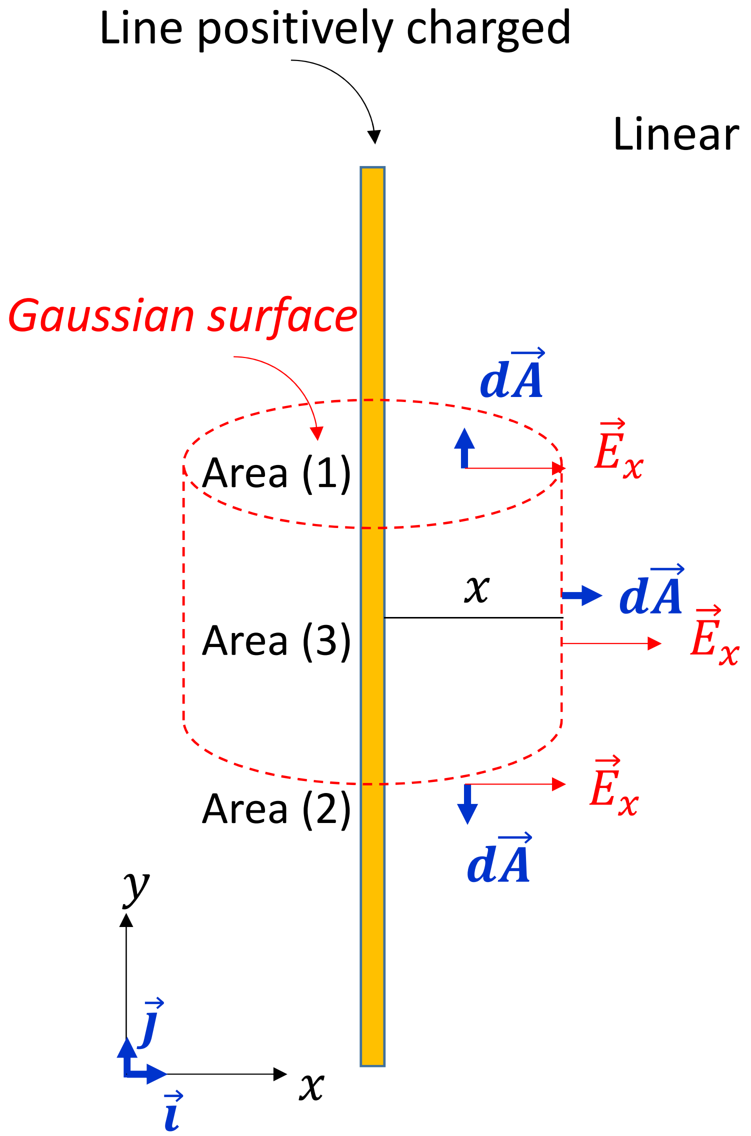
$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{x}\frac{1}{\sqrt{(x^2+a^2)}}\vec{\imath}}$$

If $a << x$ $\quad \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{x^2}\vec{\imath}$ Point charge

If $a = \infty$ $\quad \vec{E} = \frac{1}{2\pi\varepsilon_0}\left(\frac{Q}{2a}\right)\frac{1}{x}\vec{u} = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{x}\vec{\imath}$

The field of a line charge

By Gauss law



Linear charge density $\lambda = \frac{Q}{l}$ Charge enclosed in Gaussian surface $Q = \lambda l$

Through areas (1) and (2) $\vec{E}_x \perp d\vec{A}$

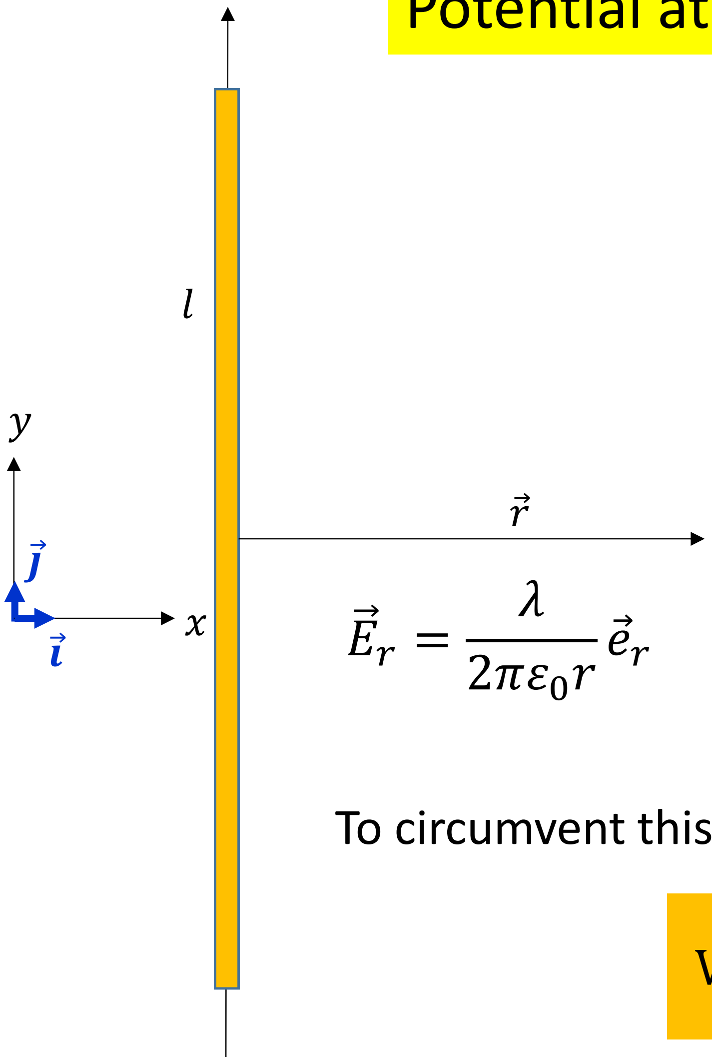
$$\Phi_E = \underbrace{\int \vec{E}_x d\vec{A}}_{\substack{\text{Area (1)}\\ 0}} - \underbrace{\int \vec{E}_x d\vec{A}}_{\substack{\text{Area (2)}\\ 0}} = 0$$

Through area (3) $\vec{E}_x \parallel d\vec{A}$

$$\Phi_E = \int \vec{E}_x d\vec{A} = E_x . 2\pi x l = \frac{Q}{\varepsilon_0} = \frac{\lambda l}{\varepsilon_0}$$

$$\boxed{E_x = \frac{\lambda}{2\pi\varepsilon_0 x}} \qquad \boxed{\vec{E}_x = \frac{\lambda}{2\pi\varepsilon_0 x}\vec{i}}$$

Potential at distance r from a very long line of charge



$$\vec{E}_r = \frac{\lambda}{2\pi\varepsilon_0 r}\vec{e}_r$$

$$\mathrm{V}_b - \mathrm{V}_a = -\int_{r_a}^{r_b} \vec{E}.d\vec{l} = -\int_{r_a}^{r_b} \frac{\lambda}{2\pi\varepsilon_0 r} dr = \frac{\lambda}{2\pi\varepsilon_0} Ln\frac{r_a}{r_b}$$

If we set $r_a = \infty$ and $\mathrm{V}_a = 0$ $\quad \mathrm{V}_b = \frac{\lambda}{2\pi\varepsilon_0} Ln\frac{\infty}{r_b} = \infty$ **!**

The problem comes for the fact that $\vec{E}_r = \frac{\lambda}{2\pi\varepsilon_0 r}\vec{e}_r$

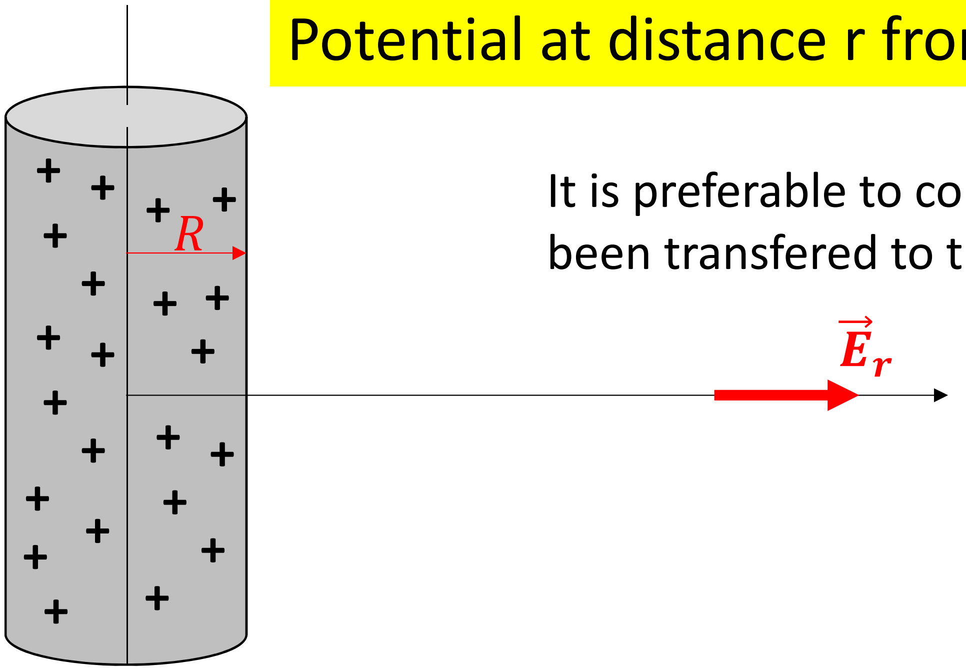
Assumes that the charge distribution extents to infinity

To circumvent this difficulty we consider that $\mathrm{V}_a = 0$ at an arbitrary radial distance r_0

$$\boxed{\mathrm{V}_r = \frac{\lambda}{2\pi\varepsilon_0} Ln\frac{r_0}{r}}$$

Where V_r decreases when r increases

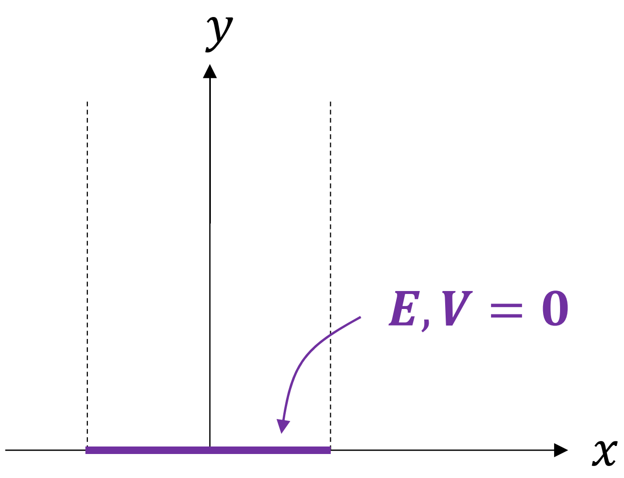
Potential at distance r from a very long charged <u>conducting</u> cylinder



It is preferable to consider that the <u>linear charge density</u> has been transfered to the cylinder <u>⇨ we can still use λ</u>

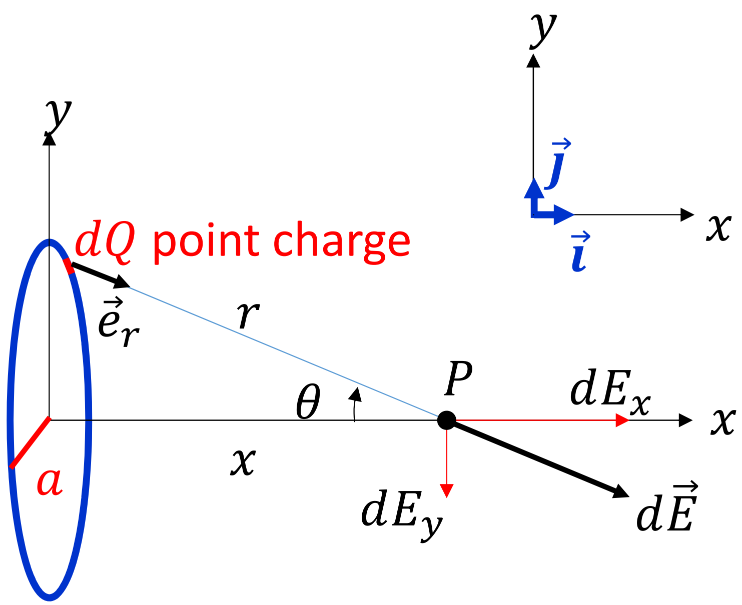
Gauss law ⇨ $\boxed{V_r = \dfrac{\lambda}{2\pi\varepsilon_0} Ln \dfrac{R}{r}}$

$$\vec{E}_r = -\vec{\nabla} V_r \qquad \boxed{\vec{E}_r = \frac{\lambda}{2\pi\varepsilon_0 r}\vec{e}_r} \qquad \boldsymbol{r > R}$$



$$\boldsymbol{r < R} \qquad V_r = 0 \quad ⇨ \quad \vec{E}_r = 0$$

The field of a charged ring



$$r = \sqrt{x^2 + a^2}$$

$$cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2 + a^2}}$$

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{x^2 + a^2}\vec{e}_r = dEcos\theta\vec{\imath} + \underbrace{dEsin\theta\vec{\jmath}}_{=0}$$

symmetry

$$\vec{E} = E_x\vec{\imath} = \frac{1}{4\pi\varepsilon_0}\int\frac{xdQ}{(x^2 + a^2)^{3/2}}\vec{\imath}$$

Running around the ring keeps x unchanged

$$\vec{E} = E_x\vec{\imath} = \frac{1}{4\pi\varepsilon_0}\frac{xQ}{(x^2 + a^2)^{3/2}}\vec{\imath}$$

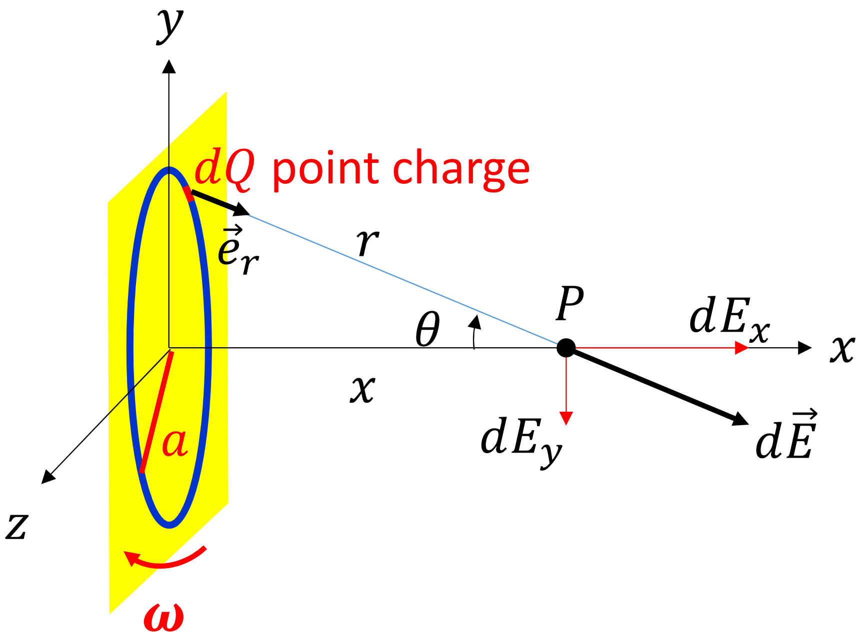
If $x >> a$ $\quad \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{x^2}\vec{\imath}$

Far away the ring appears like a point charge

At the center of the ring $x = 0$ ⇨ $\vec{E} = \vec{0}$

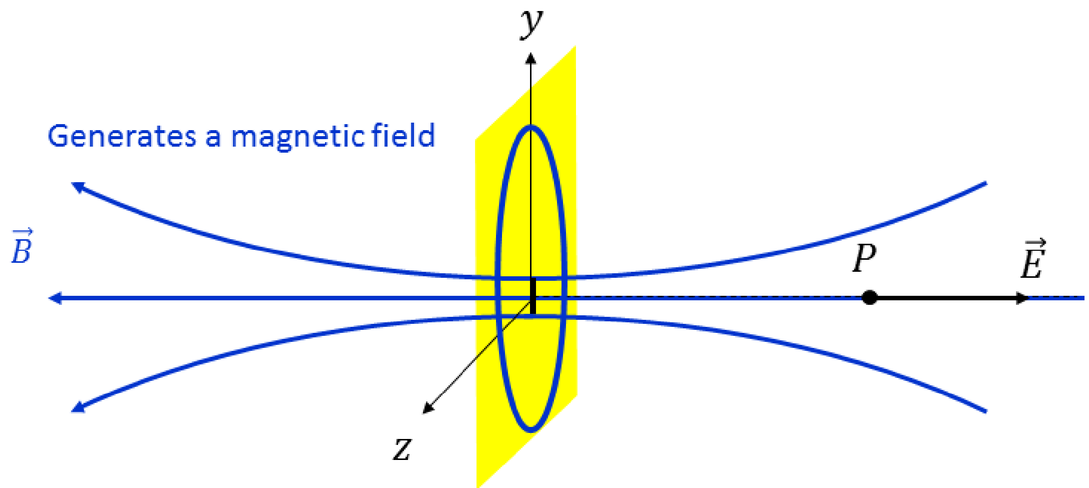
Why this is trivial ?

What should we expect if the this ring starts rotating with an angular velocity $\boldsymbol{\omega}$?



Charges are set into motion

Current

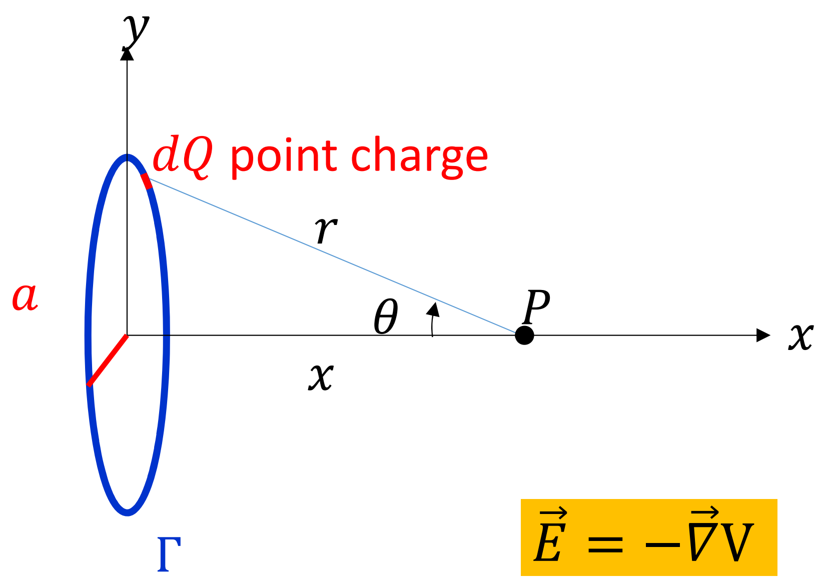


Wait for Magnetostatic

The potential of a charged ring

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{r}$$

$$V = \frac{1}{4\pi\varepsilon_0}\oint_{\Gamma}\frac{dQ}{r} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\oint_{\Gamma} dQ = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{x^2+a^2}}$$



If $x >> a$ $\quad V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{x}$

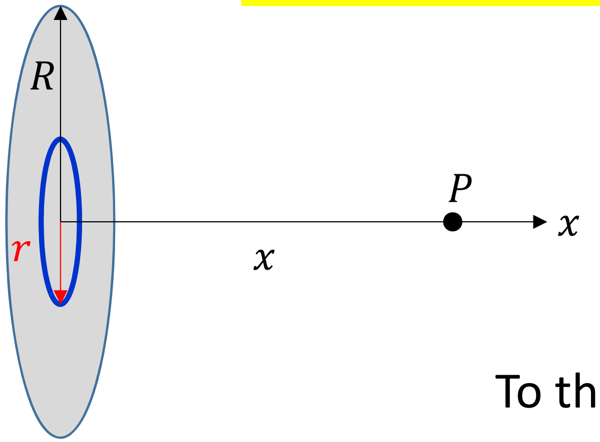
Far away the ring appears like a point charge

At the center of the ring $x = 0$ ⇨ $V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{a}$

$\vec{E} = -\vec{\nabla}V$

Works both ways: We can find $\vec{E}$ from V and vice versa

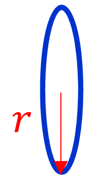
The field of an infinite sheet of charge: from uniformly charged disk



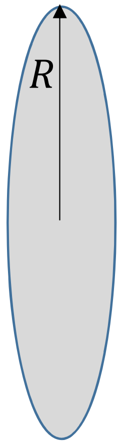
$\sigma = \frac{Q}{A}$

Charge element of the ring

$dQ = \sigma 2\pi r dr$

From the ring 

$$d\vec{E}_x = \frac{1}{4\pi\varepsilon_0} \frac{x dQ}{(x^2 + r^2)^{3/2}} \vec{\imath}$$

To the disk 

$$\vec{E}_x = \frac{1}{4\pi\varepsilon_0} \int_0^R \frac{x\sigma 2\pi r dr}{(x^2 + r^2)^{3/2}} \vec{\imath} = \frac{\sigma x}{2\varepsilon_0} \int_0^R \frac{r dr}{(x^2 + r^2)^{3/2}} \vec{\imath}$$

$$\boxed{\vec{E}_x = \frac{1}{4\pi\varepsilon_0} \int_0^R \frac{x\sigma 2\pi r dr}{(x^2 + r^2)^{3/2}} \vec{\imath} = \frac{\sigma}{2\varepsilon_0} \left[1 - \frac{1}{\sqrt{R^2/x^2 + 1}} \right] \vec{\imath}}$$

The field lines are not straight for a finite disk

If $R \gg x$

$$\vec{E} = \frac{\sigma}{2\varepsilon_0} \vec{\imath}$$

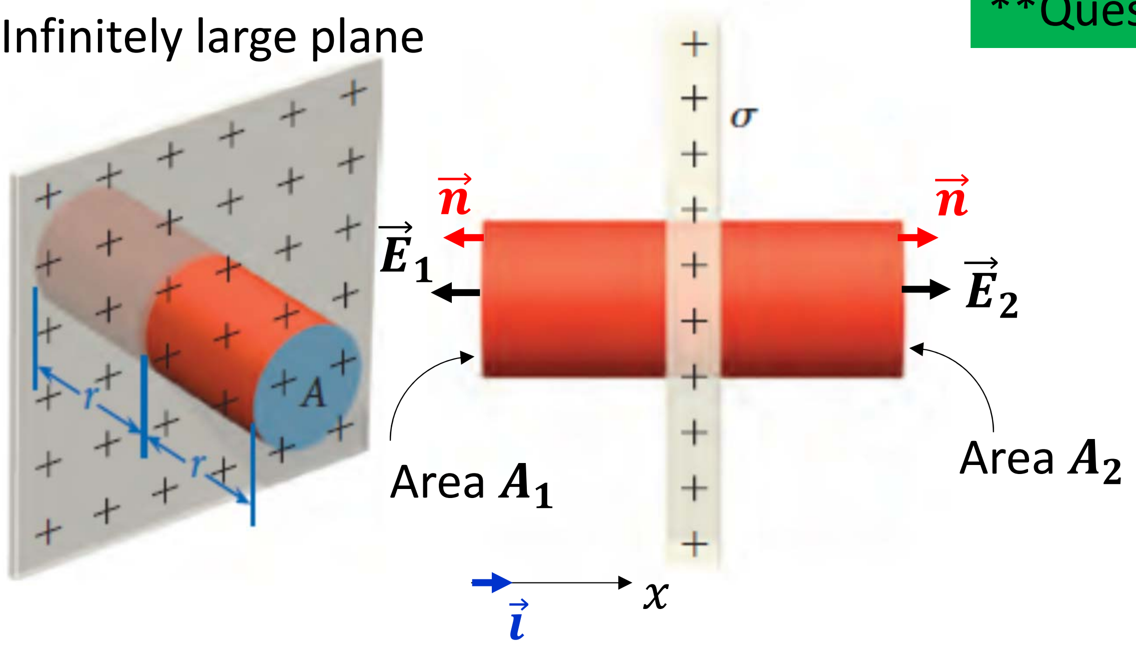
The field of an infinite sheet of uniformly distributed charges: Using Gauss law

Main tool = symmetry argument ⇨ Gauss's surface = cylinder

$$\Phi_E = \oint \vec{E} d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$$

Total area

Infinitely large plane



**Question #1: Why is the field horizontal thus flux = 0 through the cylinder wall?

Answer to Question #1:

Because we consider **infinitely large plane** and look at distance close to the plane

$\vec{E}$ **uniform** $\quad Q_{enc} = \sigma A$

**Question #2:

How does the field look like very far away from the plate,
Knowing that it has a finite size?

Answer to Question #2:

Decaying as ${}^{\sigma A}/_{r^2}$ as the plate looks as a point charge

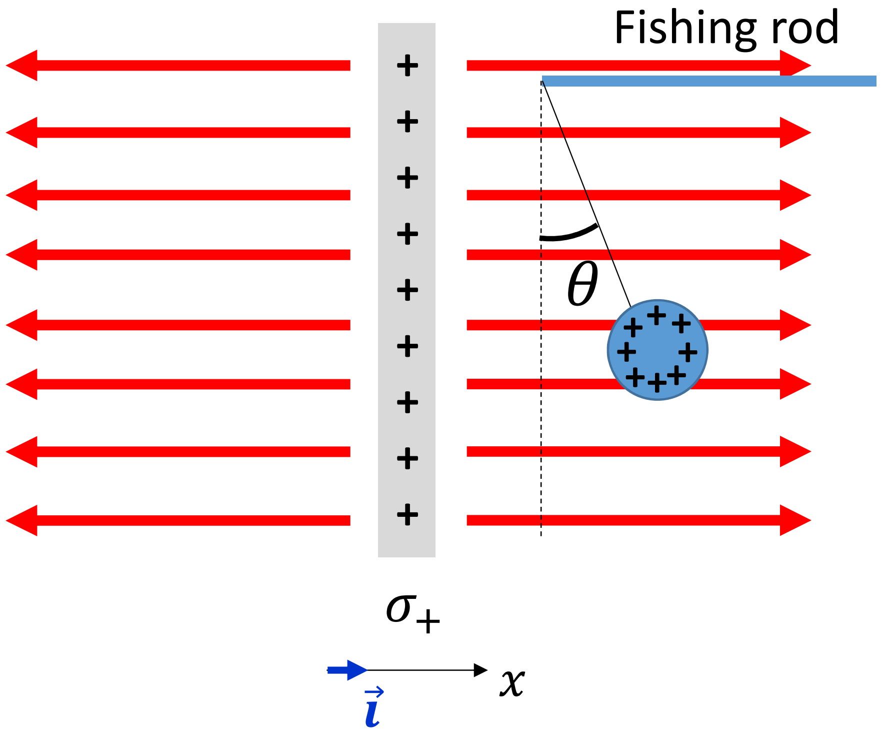
***Question #3:

What kind of simple experiment can we do to prove that the field is uniform at reasonable distance from the charged plate?

Answer to Question #3:

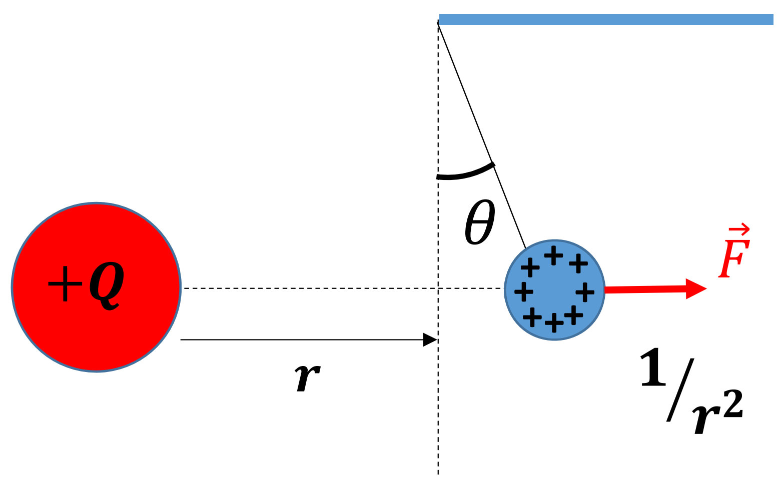
Use charge induction concept

$$\vec{E} = -\frac{\sigma}{\varepsilon_0}\vec{\imath} \qquad \vec{E} = \frac{\sigma}{\varepsilon_0}\vec{\imath}$$



The angle θ remains constant for reasonable distance of the pendulum from the plate

A spherical metal initially charged by induction



θ drops quickly when r increases

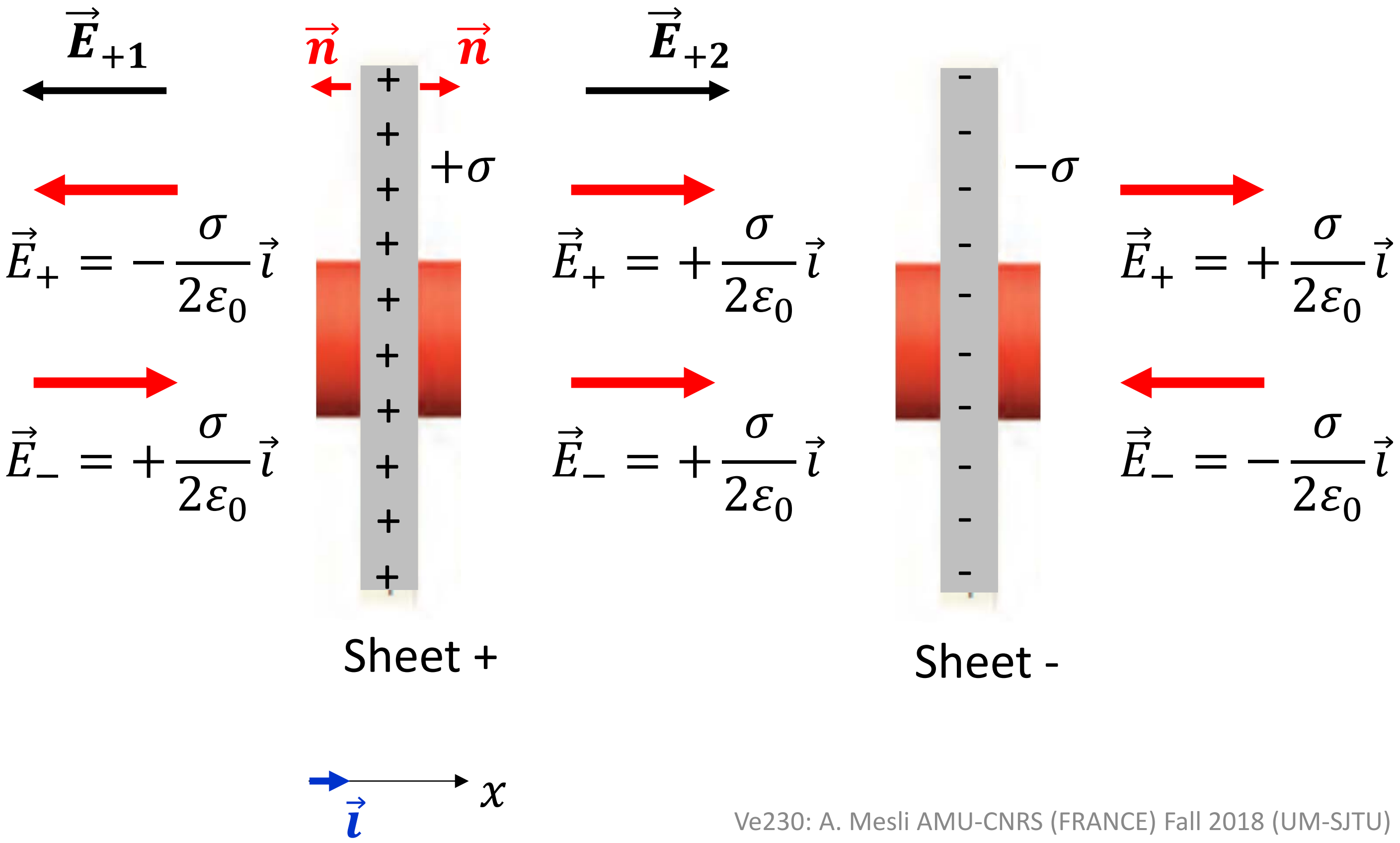
**Question #4: How do we proceed to charge a conducting sphere by induction?

Answer to Question #4: **See slide 73**

The field due to two parallel sheets of opposite charges: Superposition principle

$$\vec{E}_{+1}.\vec{A}_1 + \vec{E}_{+2}.\vec{A}_2 = E_{+1}.A + E_{+2}.A = \frac{\sigma A}{\varepsilon_0}$$

$$\boxed{\vec{E}_+ = \frac{\sigma}{2\varepsilon_0}\vec{\imath}}$$



From positive sheet (plate)

From negative sheet (plate)

$$\boxed{\vec{E} = \frac{\sigma}{\varepsilon_0}\vec{\imath}}$$

<u>One single type</u> of charge distributed <u>uniformly</u> on a <u>conducting</u> sphere

*Question #5: How can we obtain such distribution?

Answer to *Question #5: There are two possibilities

<u>By friction or contact</u>

All deposited charges will spread uniformly around the sphere as total energy must be minimized (repulsion will do the job).

Stable equilibrium*

<u>By induction via an external field</u>

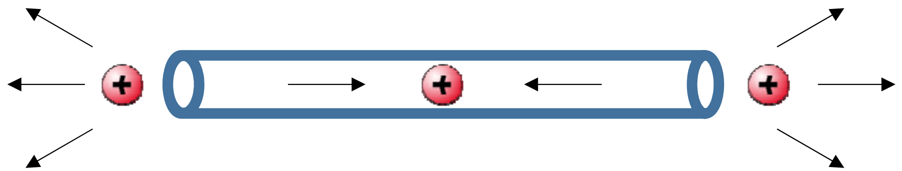
BUT induction induces charges by **<u>pair</u>** to keep neutrality (conservation of charge)

Unstable equilibrium*

Stable equilibrium* ??

We have claimed several times that there is **NO** equilibrium for electrostatic charges

BUT remember



Slide #29

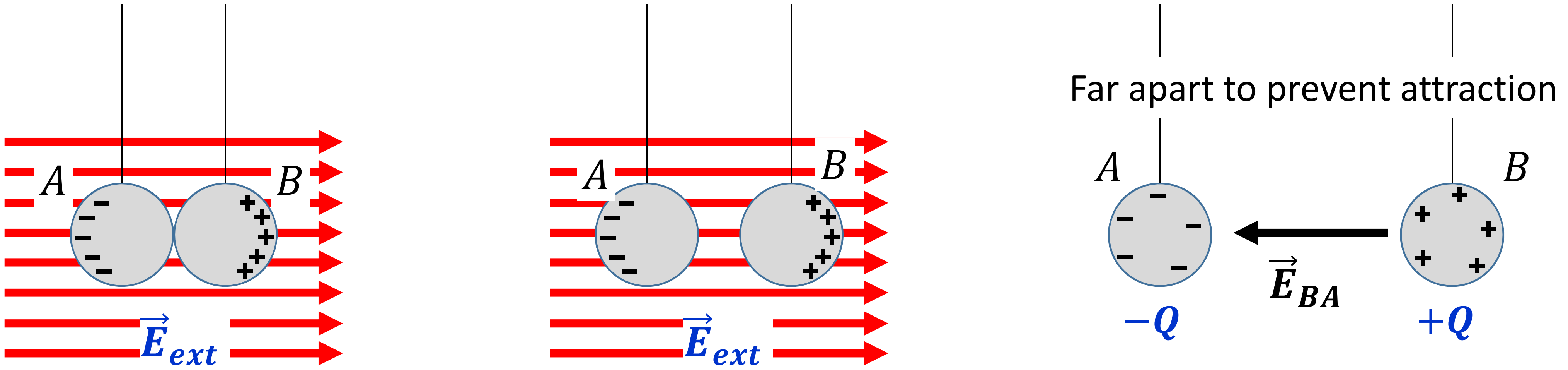
Other forces are acting to maintain equilibrium

- In the hollow tube above mechanical force
- In the case of the conducting sphere, forces at the surface prevent the charges from leaving the conductor

Why induction by an external field leads to **Unstable equilibrium*** ??

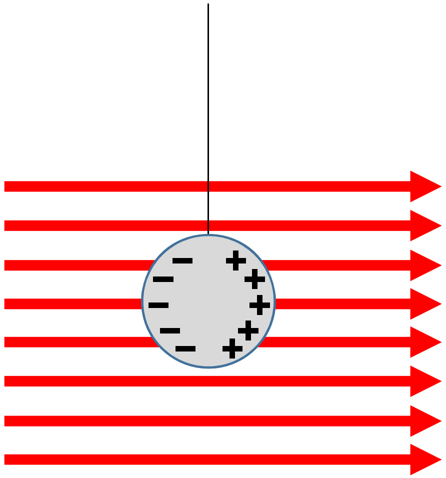
Because if we switch off the external field, the charges will redistribute themselves and annihilation takes place bringing the conducting sphere to neutrality

But we want to induce **permanent** charge on the conducing sphere by an external field !



Step 1: Contact and induction **Step 2:** Separation of the spheres **Step 3:** Switch off the external field

***Question #6: Something is **FUNDAMENTALLY WRONG** with this representation?



Answer to ***Question #6:

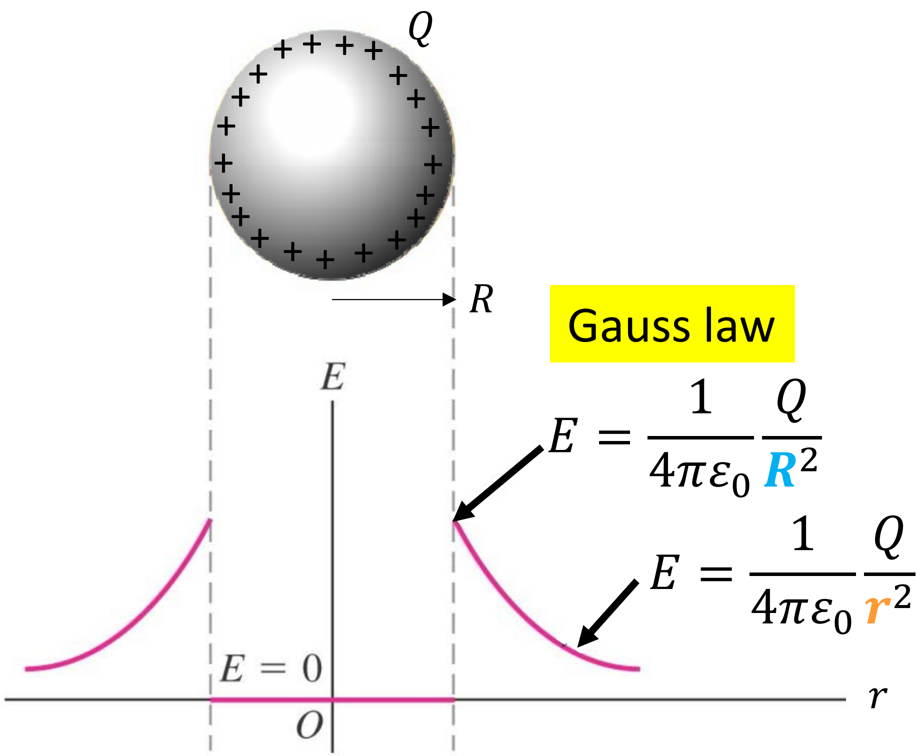
The field lines must bend close to the conducting spheres . Why?

See next

The field of a <u>surface</u> charged <u>conducting</u> sphere: Using Gauss law

<u>Electric</u> field and <u>potential</u> from the center of the sphere to infinity

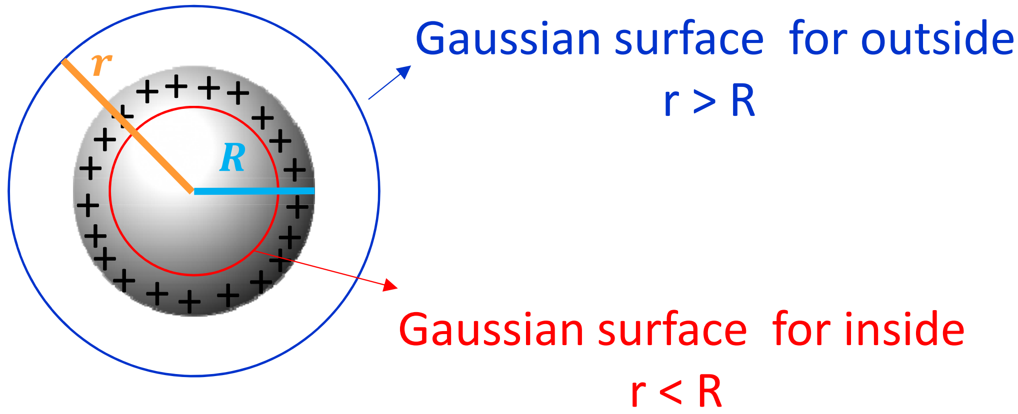
Could be a hollow metallic sphere



Gauss law

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2}$$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$



$$\vec{E} = -\vec{\nabla}.\mathrm{V}(r) \Rightarrow \mathrm{V}(sphere) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R}$$

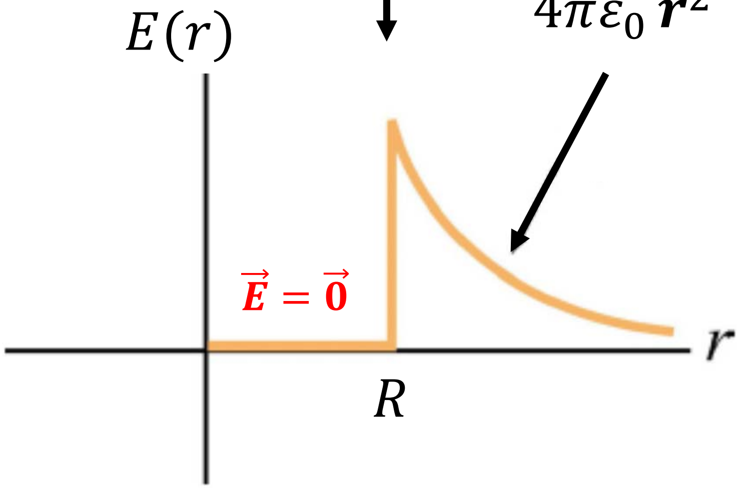
The conducting sphere is an equipotential body

$$\mathrm{V} = cte$$

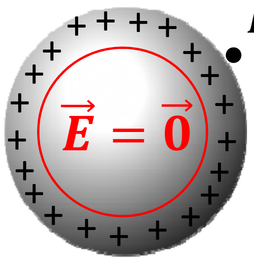
The hollow or solid conducting sphere is an example where

Surface

$$\frac{1}{4\pi\varepsilon_0}\frac{Q}{\boldsymbol{r}^2} = \frac{1}{\varepsilon_0}\frac{Q}{4\pi\boldsymbol{r}^2}$$

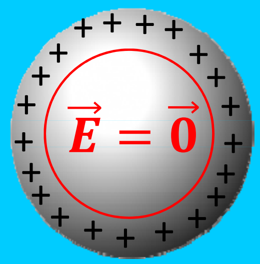


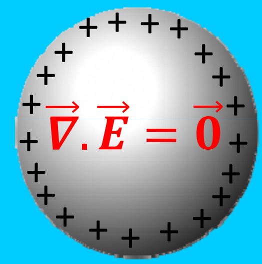
$$\boldsymbol{\sigma} = \frac{\boldsymbol{Q}}{\boldsymbol{A}}$$



$P\ (r \approx R)$

$$\boldsymbol{E} = \frac{\boldsymbol{\sigma}}{\boldsymbol{\varepsilon_0}}$$





$$\oint \vec{E}.d\vec{A} = \int \vec{\nabla}.\vec{E}\ dV = 0$$

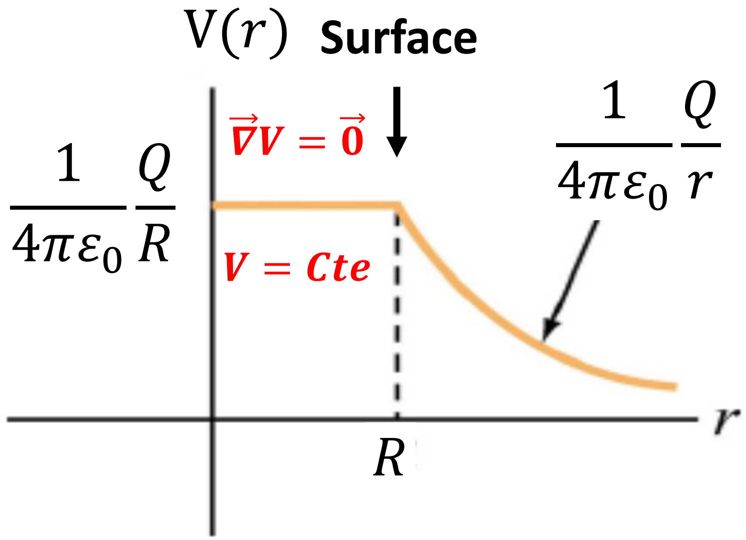
Closed surface inside the sphere

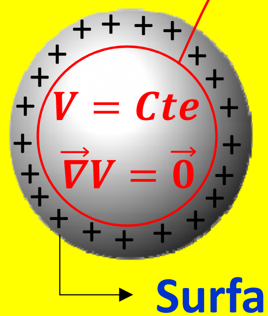
Volume enclosed by the surface

What does this mean?

The hollow or solid conducting sphere is an example where

$\Leftrightarrow$ **Volume** $\vec{E} = \vec{0}$, $\vec{\nabla}.\vec{E} = \vec{0}$





Surface $V = Cte$, $\vec{\nabla}V = \vec{0}$

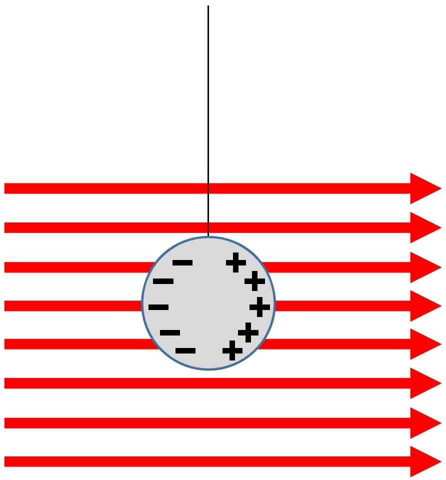
$$dV = \vec{\nabla}V.d\vec{l} = 0$$

$$V = \int_a^b \vec{\nabla}V.d\vec{l} = -\int_a^b \vec{E}.d\vec{l} = 0$$

Along any **open** or **closed** path on the surface

What does this mean?

***Question #7: Still no clue as to why this representation is **FUNDAMENTALLY WRONG**?

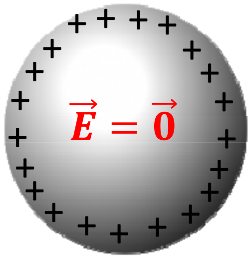


Answer to ***Question #7:

See chapter on conductors

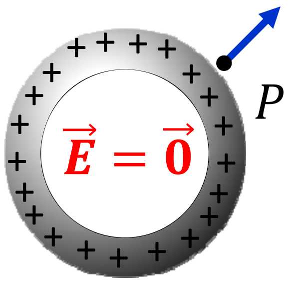
That $\vec{E} = \vec{0}$ *inside the conducting sphere is NOT trivial at all !*

“Conspiracy” principle



Every charge on the sphere creates a field inside and outside the sphere **BUT** all individual effects when added **cancel out inside**

Consequence For a hollow sphere

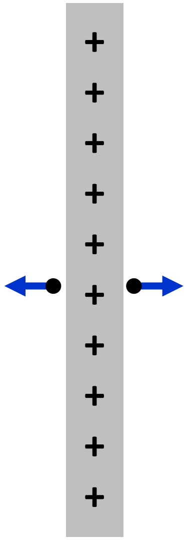


$$E = \frac{1}{\varepsilon_0}\frac{Q}{4\pi r^2} = \frac{\sigma}{\varepsilon_0}$$

$P\ (r \approx R)$

Whereas

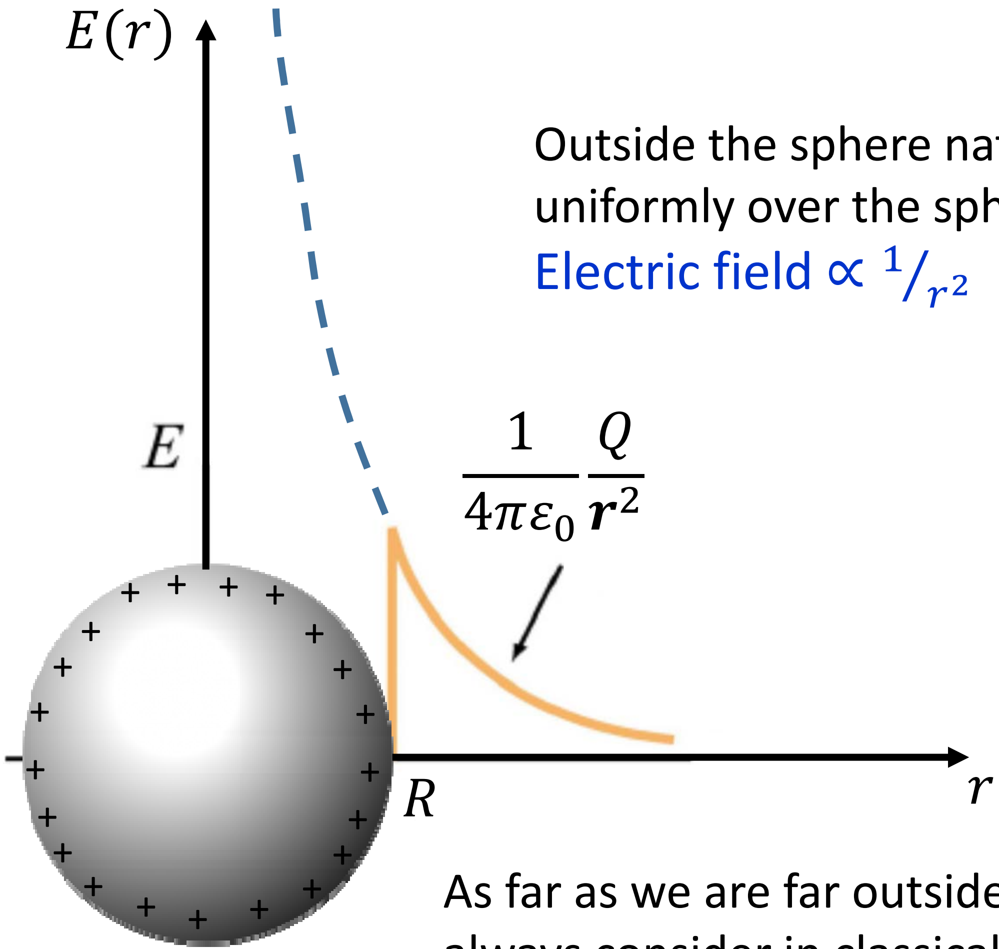
$$E = \frac{\sigma}{2\varepsilon_0}$$



$$E = \frac{\sigma}{2\varepsilon_0}$$

σ_+

See lectures on conductors



An interesting point

Outside the sphere nature cannot decide whether the charge is distributed uniformly over the sphere or concentrated on a point at the center because Electric field $\propto {}^{1}/_{r^2}$

Gravitational field $\propto {}^{1}/_{r^2}$

⬇

Planet in a form of a hollow sphere

⬇

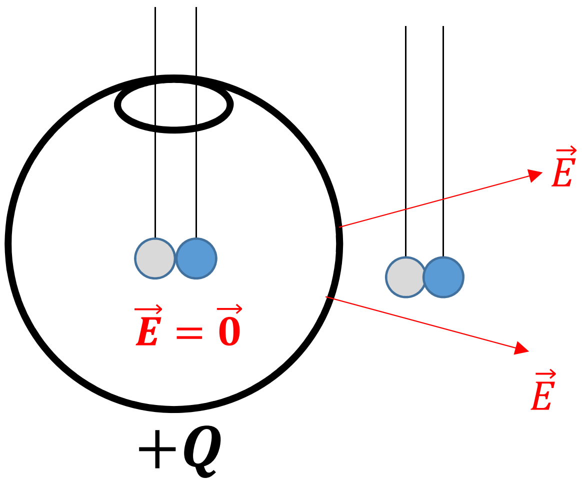
No gravitation inside

As far as we are far outside the planet with a mass uniformly distributed, we always consider in classical mechanics that the point mass approximation is valid.

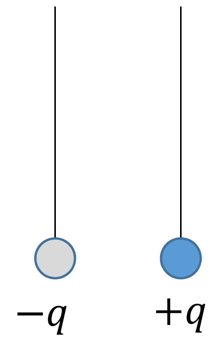
- It took 20 years to Newton to prove this statement
- 100 years later Gauss's law proved it in 3 seconds

Question #8: How can we check experimentally that the field inside a hollow sphere is really zero? <u>Theoretically it will be done in the next lecture</u>

Answer to Question #8: Using the double pendulum with two little metallic spheres and a small opening in the hollow sphere



1) Bring the double pendulum close to the hollow sphere
   ⇨ charges are induced on the double pendulum

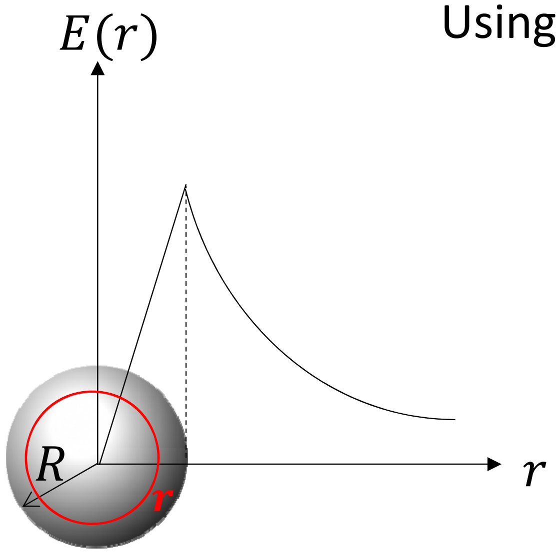


2) Inserted into the hollow sphere no charges are induced
   ⇨ field inside is zero

Rigorously speaking the charge distribution on the hollow sphere is no longer uniform because of the opening. There is field inside but very weak

The field of a bulk charged **non-**conducting sphere (dielectric): Using Gauss law

Uniformly charged



Using a Gaussian surface inside and outside the charged sphere

$$\boldsymbol{r < R} \quad \Rightarrow \quad Q = \rho\left(\frac{4\pi}{3}\boldsymbol{r}^3\right), \; E(r)4\pi r^2 = \frac{Q}{\varepsilon_0},$$

$$\Rightarrow \quad E(r) = \frac{\rho}{3\varepsilon_0} r \quad \Rightarrow \quad \boxed{E(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^3} r}$$

$$\boldsymbol{r > R} \quad \Rightarrow \quad Q = \rho\left(\frac{4\pi}{3}\boldsymbol{R}^3\right), \qquad E(r)4\pi r^2 = \frac{Q}{\varepsilon_0}$$

$$\Rightarrow \quad \boxed{E(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}}$$

Charge density $\boldsymbol{\rho} = \mathbf{Q}/(\frac{4\pi}{3}\boldsymbol{R}^3)$