VP390 Problem Set 6

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1 Problem 1

For this problem, I choose the parameters of Gauss Wave Packet as:
$t0 = 1, \alpha = 10^{-34}, m = 9 \times 10^{-31}, p_0 = 10^{30}$
Then we get graphs when $t = 0, 1, 2, 3$ as:

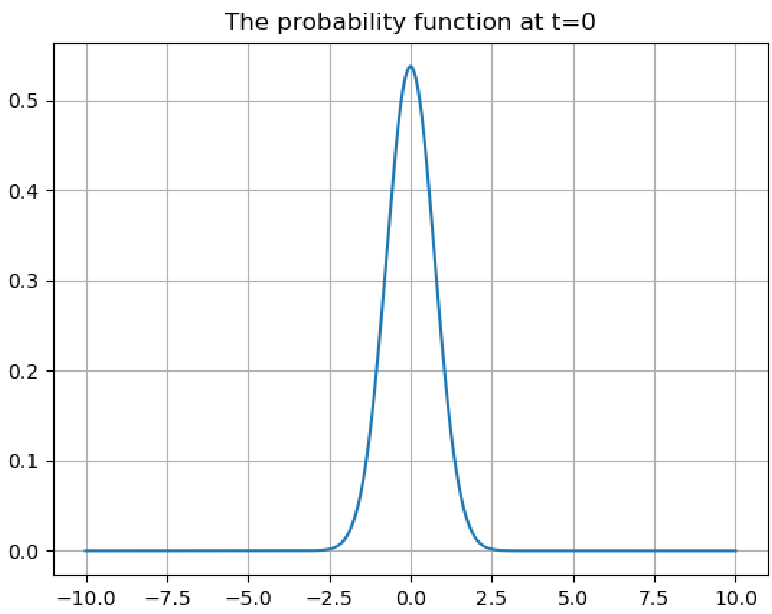

Figure 1: $t = 0$

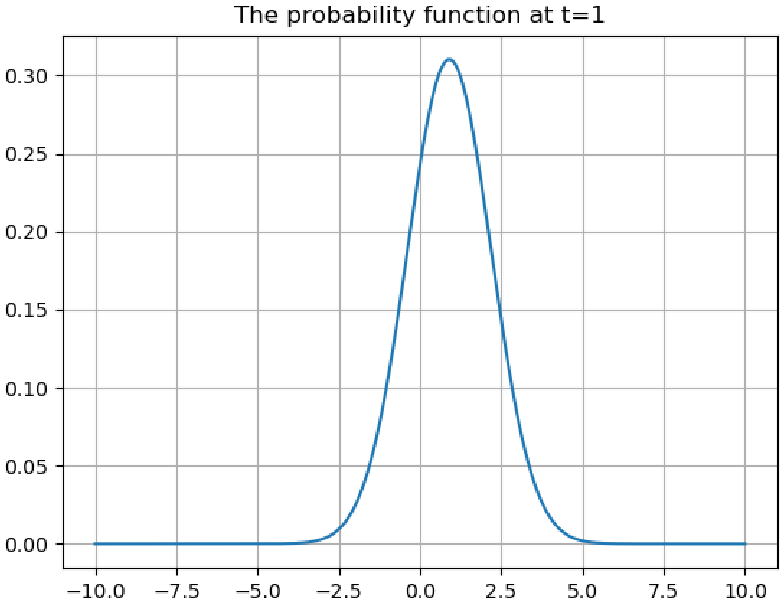

Figure 2: $t = 1$

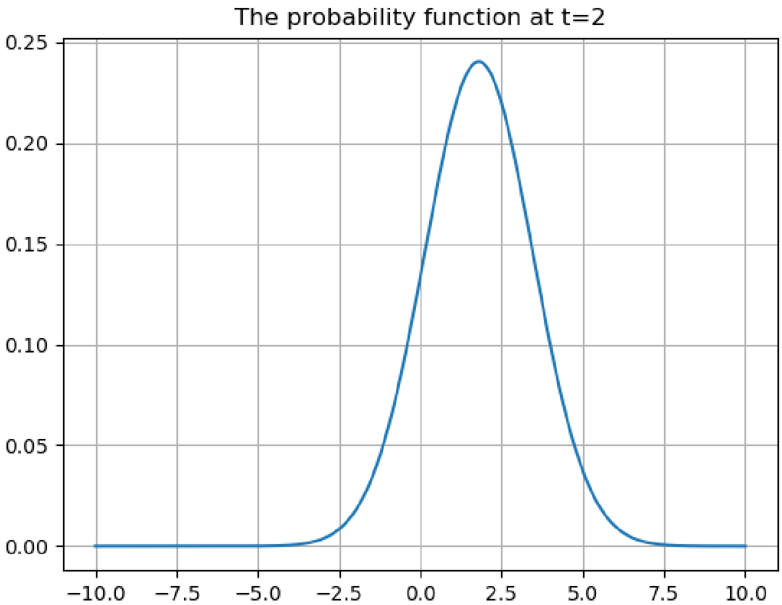

Figure 3: $t = 2$

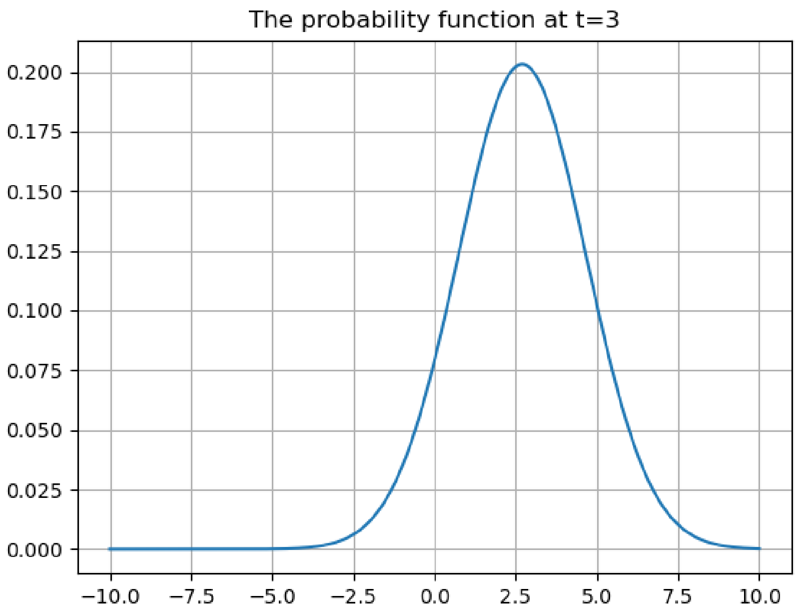

Figure 4: $t = 3$

2 Problem 2

(a) For the third maximum point, we get $E = V_0 + \frac{\hbar^2\pi^2}{8ma^2}3^2$
$100 \times 1.6 \times 10^{-19} = 9\frac{(1.054\times10^{-34})^2\pi^2}{8\times9.1\times10^{-31}\times(10^{-10})^2} + V_0$
$V_0 = 15.28\text{eV}$

(b) For $E \to V_0, T \to \frac{1}{1+\frac{2ma^2V_0}{\hbar^2}}$
Then $a = 1.0626 \times 10^{-9}$m, and the width is $2.13 \times 10^{-9}m$

3 Problem 3

For the barrier problem we get three equations:

$$\begin{cases} \psi_1(x) = Ae^{ik_1x} + Be^{-ik_1x} & x \le -a \\ \psi_2(x) = Ce^{ik_2x} + De^{-ik_2x} & |x| < a \\ \psi_3(x) = Fe^{ik_1x} & x \ge a \end{cases}$$

where $k_1 = \frac{\sqrt{2mE}}{\hbar}$ and $k_2 = \frac{\sqrt{2m(E-V_0)}}{\hbar}$
Since $\psi(x)$ and $\psi'(x)$ are continuous at $x = \pm a$, we get:

$$\begin{cases} Ae^{-ik_1a} + Be^{ik_1a} = Ce^{-ik_2a} + De^{ik_2a} & (1) \\ Ce^{ik_2a} + De^{-ik_2a} = Fe^{ik_1a} & (2) \\ k_1(Ae^{-ik_1a} - Be^{ik_1a}) = k_2(Ce^{-ik_2a} - De^{ik_2a}) & (3) \\ k_2(Ce^{ik_2a} - De^{-ik_2a}) = k_1Fe^{ik_1a} & (4) \end{cases}$$

$k_2 * (2) - (4)$ we get $D = \frac{k_2 - k_1}{2k_2} F e^{i(k_1+k_2)a}$
$k_2 * (2) + (4)$ we get $C = \frac{k_1 + k_2}{2k_2} F e^{i(k_1-k_2)a}$
$k_1 * (1) + (3)$ we get $2k_1 A = (k_1 + k_2) C e^{i(k_1-k_2)a} + (k_1 - k_2) D e^{i(k_1+k_2)a}$
Then $\frac{A}{F} = \frac{1}{4k_1k_2}[(k_1+k_2)^2 e^{i(k_1-2k_2)a} + (k_1-k_2)^2 e^{i(k_1+2k_2)a}]$
$\frac{F}{A} = \frac{4k_1k_2}{(k_1+k_2)^2 e^{i(k_1-2k_2)a} + (k_1-k_2)^2 e^{i(k_1+2k_2)a}} = \frac{4k_1k_2 e^{i(-k_1+2k_2)a}}{(k_1+k_2)^2 + (k_1-k_2)^2 e^{4ik_2a}}$

$k_1 * (1) - (3)$ we get $2k_1 B = (k_1 - k_2) C e^{-i(k_1+k_2)a} + (k_1 + k_2) D e^{i(k_2-k_1)a}$
$\frac{B}{F} = \frac{1}{4k_1k_2}[(k_1^2 - k_2^2) e^{-i2k_2a} + (k_2^2 - k_1^2) e^{i2k_2a}]$
$\frac{B}{A} = \frac{(k_1^2-k_2^2)e^{-i2k_2a} + (k_2^2-k_1^2)e^{i2k_2a}}{(k_1+k_2)^2 e^{i(k_1-2k_2)a} + (k_1-k_2)^2 e^{i(k_1+2k_2)a}} = \frac{(k_1^2-k_2^2)e^{-ik_1a} + (k_2^2-k_1^2)e^{i(-k_1+4k_2a)}}{(k_1+k_2)^2 + (k_1-k_2)^2 e^{4ik_2a}}$
Hence $T + R = (\frac{F}{A})^2 + (\frac{B}{A})^2 = 1$

4 Problem 4

For the barrier problem we get three equations:

$$\begin{cases} \psi_1(x) = Ae^{ik_1x} + Be^{-ik_1x} & x \leq -a \\ \psi_2(x) = Ce^{ik_2x} + De^{-ik_2x} & |x| < a \\ \psi_3(x) = Fe^{ik_1x} & x \geq a \end{cases}$$

where $k_1 = \frac{\sqrt{2mE}}{\hbar}$ and $k_2 = \frac{\sqrt{2m(E+V_0)}}{\hbar}$

We can plug them into the solution of rectangular barrier since the equations are in same forms by replacing V_0 with $-V_0$
Then we get $\frac{1}{T} = 1 + \frac{1}{4}\frac{V_0^2}{E(E+V_0)}\sin^2(2\sqrt{\frac{2ma^2}{\hbar^2}(E+V_0)})$
Hence we get the result $\frac{1}{T} = 1 + \frac{1}{4}\frac{V_0^2}{E(E+V_0)}\sin^2(2k_2a)$

For perfect transmission, we need $2k_2a = 2\sqrt{\frac{2ma^2}{\hbar^2}(E+V_0)} = n\pi$
Then $E = \frac{\pi^2\hbar^2}{8ma^2}n^2 - V_0$

5 Problem 5

$E = V_0 + \frac{\hbar^2\pi^2}{8ma^2}n^2$
$a = 0.5 \times 10^{-10}, n = 1, E = 0.9 \times 1.6 \times 10^{-19}$
$V_0 = -5.885 \times 10^{-18}$J

6 Problem 6

(a) $-[\hat{B}, \hat{A}] = -\hat{B}\hat{A} + \hat{A}\hat{B} = \hat{A}\hat{B} - \hat{B}\hat{A} = [\hat{A}, \hat{B}]$

(b) $\hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B} = \hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} + \hat{A}\hat{C}\hat{B} - \hat{C}\hat{A}\hat{B} = (\hat{A}\hat{B})\hat{C} - \hat{C}(\hat{A}\hat{B}) = [\hat{A}\hat{B}, \hat{C}]$

(c) $[\hat{A}, \hat{C}] + [\hat{B}, \hat{C}] = \hat{A}\hat{C} - \hat{C}\hat{A} + \hat{B}\hat{C} - \hat{C}\hat{B} = (\hat{A} + \hat{B})\hat{C} - \hat{C}(\hat{A} + \hat{B}) = [\hat{A} + \hat{B}, \hat{C}]$

(d) $[\hat{A}, \gamma] = \hat{A}\gamma - \gamma\hat{A} = \gamma\hat{A} - \gamma\hat{A} = 0$

(e) $[\hat{A}-\alpha, \hat{B}-\beta] = (\hat{A}-\alpha)(\hat{B}-\beta)-(\hat{B}-\beta)(\hat{A}-\alpha) = \hat{A}\hat{B}-\hat{B}\hat{A}-\alpha(\hat{B}-\hat{B})-\beta(\hat{A}-\hat{A}) = [\hat{A}, \hat{B}]$

(f) $[\hat{A}, f(\hat{A})] = \hat{A}f(\hat{A}) - f(\hat{A})\hat{A}$
Since $f(\hat{A})$ is an analytic function, it can be expressed in the form $f(\hat{A}) = \sum_{i=0}^{\infty} k_i \hat{A}^i$
Then $\hat{A}f(\hat{A}) - f(\hat{A})\hat{A} = \sum_{i=0}^{\infty} k_i \hat{A}^{i+1} - \sum_{i=0}^{\infty} k_i \hat{A}^{i+1} = 0$
Hence $[\hat{A}, f(\hat{A})] = 0$

7 Problem 7

(a) $[\hat{H}, \hat{p}]\psi(x) = (\hat{H}\hat{p} - \hat{p}\hat{H})\psi(x)$
Since $\hat{H} = \hat{K} + \hat{V}, \hat{V} = 0, \hat{K} = \frac{\hat{p}^2}{2m}$
Then $[\hat{H}, \hat{p}]\psi(x) = (\frac{\hat{p}^3}{2m} - \frac{\hat{p}^3}{2m})\psi(x) = 0$

Since $\hat{V} = 0$, we can consider $\hat{H} = f(\hat{p})$ and f is analytic function, then according to $[\hat{A}, f(\hat{A})] = 0$, we know the result is 0.

(b) For the momentum operator, the eigenfunction is $\phi_{p_x}(x) = Ce^{\frac{i}{\hbar}p_x x}$
For Hamiltonian, since it's a free particle, $\hat{H} = \hat{K} = \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$
$\hat{H}\phi_{p_x}(x) = -\frac{\hbar^2}{2m}\frac{\partial^2 \phi_{p_x}(x)}{\partial x^2} = \frac{C}{2m}p_x^2\phi_{p_x}(x)$

Hence any eigenfunction of the 1D momentum operator is also an eigenfunction of the Hamiltonian of a free particle.

Conversely, for the eigenfunction of $\hat{H}$ is $\psi(x) = A\cos kx + B\sin kx$ where $k = \sqrt{\frac{2mE}{\hbar}}$
$\hat{p}\psi(x) = -i\hbar\frac{\partial(A\cos kx + B\sin kx)}{\partial x} = ik\hbar(A\sin kx - B\cos kx)$

Hence the eigenfunction of the Hamiltonian of a free particle is not always the eigenfunction of the 1D momentum operator

(c) In this case, $\hat{H} = \hat{K} + \hat{V} = f(\hat{p}) + \hat{V}$
$[\hat{H}, \hat{p}]\psi(x) = [f(\hat{p}) + \hat{V}, \hat{p}]\psi(x) = ([f(\hat{p}), \hat{p}] + [\hat{V}, \hat{p}])\psi(x) = [\hat{V}, \hat{p}]\psi(x)$
$[\hat{H}, \hat{p}]\psi(x) = V(x) - i\hbar\frac{\partial\psi(x)}{\partial x} + i\hbar\frac{\partial V(x)\psi(x)}{\partial x} = -i\hbar V(x)\psi(x)' + [i\hbar V(x)\psi(x)' + i\hbar V(x)'\psi(x)] = i\hbar V(x)'\psi(x)$
For two operators $\hat{A}, \hat{B}$ where one is a function while another takes one order derivative, their commutator will take the derivative of the operator's function instead of the input function, because it's cancelled out.

8 Problem 8

According to the general uncertainty principle $\triangle_E\triangle_x \geq \frac{1}{2}|\langle\Psi, i[\hat{H}, \hat{x}]\Psi\rangle|$
$i[\hat{H}, \hat{x}] = i(\hat{H}\hat{x} - \hat{x}\hat{H}) = i\{[-i\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)]x - xV(x) + xi\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\} = \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}x - x\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$
$i[\hat{H}, \hat{x}] = \frac{\hbar^2}{2m}(\frac{2\partial}{\partial x} + x\frac{\partial^2}{\partial x^2} - x\frac{\partial^2}{\partial x^2} = \frac{\hbar^2}{m}\frac{\partial}{\partial x} = i\frac{\hbar}{m}(-i\hbar\frac{\partial}{\partial x}) = i\frac{\hbar}{m}\hat{p}$
Hence $\triangle_E\triangle_x \geq \frac{1}{2}|\langle\Psi, i\frac{\hbar}{m}\hat{p}\Psi\rangle| = \frac{\hbar}{2m}\langle\hat{p}\rangle$

9 Problem 9

1. No, since the polarization direction of A and B are perpendicular to each other, the vertically polarized light can't penetrate through B

2. (i) According to Malus's Law, $I = I_0 \cos^2 \frac{\pi}{2} \cos^2 \frac{\pi}{2}$, behind the B we can get light with intensity $I = \frac{1}{4} I_0$

   (ii) With the increase of θ, the light intensity behind C is $I = I_0 \cos^2 \theta \sin^2 \theta$, which is max when $\theta = \frac{\pi}{2}$ and minimum when $\theta = 0$

   According to our intuition, we should never get light behind B because the polarization direction of A and B are perpendicular to each other, however, it fails when we put another polarizer C between then when $\theta \neq 0$

   In this guess, I guess the polarizer C plays a role like measurement or observation for the polarization of the light when $\theta \neq 0$, and it disturb the light's original polarization. Hence, Malus's law is an example showing how observation will cause the collapse of wave function

Python Code for Problem 1

```
import numpy as np
import matplotlib.pyplot as plt
import math
t0=1
a=1e34
h=1.05e-34
m=9e-31
p0=1e30
t=30
x=np.linspace(-100,100,10000)
b=a*h*(1+t*2/(t0**2))**0.5
psi=math.e**(-(x-p0*m*t)**2/(b**2))/(b*math.pi**0.5)
title='The probability function at t=' + str(t)
plt.title(title)
plt.grid(1)
plt.plot(x,psi)
plt.show()
```