

27

## VP390 Problem Set 7

Pan, Chongdan ID:516370910121

July 13, 2020

### 1 Problem 1

$$\begin{aligned}
 \hat{L}_y &= \hat{z}\hat{p}_x - \hat{x}\hat{p}_z, \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \\
 [\hat{L}_y, \hat{L}_z] &= \hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y = (\hat{z}\hat{p}_x - \hat{x}\hat{p}_z)(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x) - (\hat{x}\hat{p}_y - \hat{y}\hat{p}_x)(\hat{z}\hat{p}_x - \hat{x}\hat{p}_z) \\
 \hat{L}_y\hat{L}_z &= \hat{z}\hat{p}_x\hat{x}\hat{p}_y + \hat{x}\hat{p}_z\hat{y}\hat{p}_x - \hat{x}\hat{p}_z\hat{x}\hat{p}_y - \hat{z}\hat{p}_x\hat{y}\hat{p}_x = \hat{z}\hat{p}_y\hat{p}_x\hat{x} + \hat{p}_z\hat{y}\hat{x}\hat{p}_x - \hat{p}_z\hat{p}_y\hat{x}\hat{x} - \hat{y}\hat{z}\hat{p}_x\hat{p}_x \\
 -\hat{L}_z\hat{L}_y &= \hat{y}\hat{p}_x\hat{z}\hat{p}_x + \hat{x}\hat{p}_y\hat{x}\hat{p}_z - \hat{x}\hat{p}_y\hat{z}\hat{p}_x - \hat{y}\hat{p}_x\hat{x}\hat{p}_z = \hat{y}\hat{z}\hat{p}_x\hat{p}_x + \hat{p}_y\hat{p}_z\hat{x}\hat{x} - \hat{p}_y\hat{z}\hat{x}\hat{p}_x - \hat{y}\hat{p}_z\hat{p}_x\hat{x} \\
 [\hat{L}_y, \hat{L}_z] &= \hat{y}\hat{p}_z(\hat{x}\hat{p}_x - \hat{p}_x\hat{x}) - \hat{z}\hat{p}_y(\hat{x}\hat{p}_x - \hat{p}_x\hat{x}) = \hat{y}\hat{p}_z[\hat{x}, \hat{p}_x] - \hat{z}\hat{p}_y[\hat{x}, \hat{p}_x] \\
 [\hat{L}_y, \hat{L}_z] &= \hat{y}\hat{p}_z - \hat{z}\hat{p}_y[\hat{x}, \hat{p}_x] = i\hbar\hat{L}_x
 \end{aligned}$$

### 2 Problem 2

$$\cos \theta = \frac{L_z}{L} = \frac{m}{\sqrt{3(3+1)}} = \frac{\sqrt{3}}{2} \rightarrow \theta = \frac{\pi}{6}$$

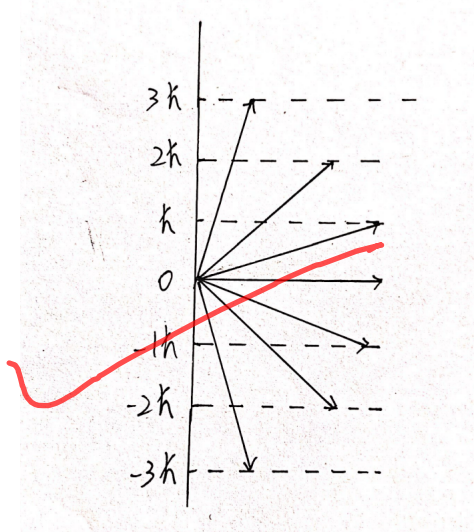


Figure 1: Projection of  $L$  on the  $z$ -axis

### 3 Problem 3

$$(a) \mu = \frac{m^2}{2m} = \frac{m}{2}$$

$$I = \mu(2a)^2 = 2ma^2$$

$$\hat{H}\psi = \frac{\hat{L}^2}{2I}\psi = E\psi \rightarrow \hat{L}^2\psi = 4Ema^2\psi$$

$$\text{Since } L = \hbar\sqrt{l(l+1)}$$

$$E_l = \frac{\hbar^2 l(l+1)}{4ma^2}$$

$$[-\frac{\hbar^2}{2\mu} \frac{\partial}{\partial r}(r^2 \frac{\partial}{\partial r}) + \frac{\hat{L}^2}{2\mu r^2} + V(r)]\psi = E\psi$$

$$[-\frac{\hbar^2}{2\mu} \frac{\partial}{\partial r}(r^2 \frac{\partial}{\partial r}) + \frac{\hbar^2 l(l+1)}{2\mu r^2} + V(r)]R(r) = ER(r)$$

$$\text{Then } E = -\frac{E_1}{n^2} \text{ where } E_1 = \frac{\mu e^4}{32\pi^2 \hbar^2 \epsilon_0} = \frac{me^4}{64\pi^2 \hbar^2 \epsilon_0} \text{ and } R_{nl} = Ae^{-\frac{r}{a_0 n}} r^l L_{nl}(\frac{r}{a_0})$$

(b)

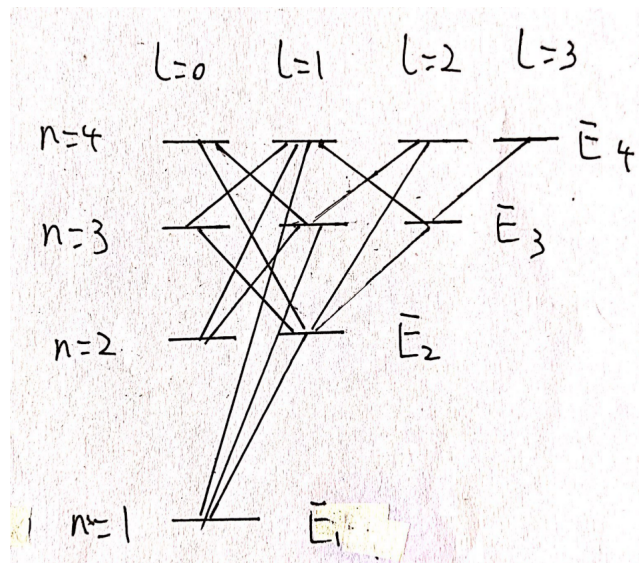


Figure 2: Energy level diagram

$$\text{For transition energy } E_1 = \frac{\hbar^2(l+1)}{2I} = \frac{\hbar^2(1+1)}{2I} = \frac{\hbar^2}{I}$$

$$\Delta E = E_{l+1} - E_l = \frac{\hbar^2(l+2)(l+1)}{2I} - \frac{\hbar^2 l(l+1)}{2I} = \frac{2\hbar^2(l+1)}{2I} = (l+1) \frac{\hbar^2}{I}$$

$$\text{Hence } E_n = nE_1 \text{ where } n = 1, 2, 3 \dots$$

$$(c) I = 2ma^2 = 2 \cdot 1.67 \cdot 10^{-27} \cdot (3.7 \cdot 10^{-11})^2 = 4.57 \cdot 10^{-48}$$

$$E_1 = \frac{\hbar^2}{2ma^2} = \frac{\hbar^2}{I} = 2.41 \times 10^{-21} \text{ J}$$

$$(d) \Delta E_{l \rightarrow l+1} = \frac{2\hbar^2(l+1)}{4ma^2}$$

$$\text{Hence when } l = 0 \text{ we have the lowest transition energy } \Delta E_{l \rightarrow l+1} = \frac{\hbar^2}{2ma^2} = 0.015 \text{ eV}$$

$$\Delta E = \frac{hc}{\lambda} \rightarrow \lambda = \frac{hc}{\Delta E} = \frac{6.63 \cdot 10^{-34} \cdot 3 \cdot 10^8}{2.41 \cdot 10^{-21}} = 8.25 \cdot 10^{-5} \text{ m}$$

It's in the visible light region close to Paschen region.

## 4 Problem 4

(a)  $L = r \times p$ , since  $r$  and  $p$  are perpendicular to each other in the  $xy$ -plane, then  $L$  is along the  $z$ -axis, hence it has two zero components

According to classic mechanic, for the total energy, since  $V(x) = 0$ ,  $E = K = \frac{1}{2}I\omega^2 = \frac{L^2}{2I}$

$$(b) \hat{H} = \frac{\hat{L}_z^2}{2I} + V = -\frac{\hbar^2}{2I} \frac{\partial^2}{\partial \varphi^2} + V$$

Since  $V = 0$ ,  $-\frac{\hbar^2}{2I} \frac{\partial^2 \psi}{\partial \varphi^2} = E\psi \rightarrow -\frac{\hbar^2}{2I} \frac{\partial^2 \phi(\varphi)}{\partial \varphi^2} = E\phi(\varphi) \rightarrow \phi = Ce^{\pm \sqrt{-\frac{2IE}{\hbar^2}}\varphi}$ ,  $E$  can be any real number

Hence  $\psi = Ce^{\pm \sqrt{-\frac{2IE}{\hbar^2}}\varphi}$

$$(c) \psi(\varphi + 2\pi) = \psi(\varphi) \rightarrow \int_0^{2\pi} C^2 \cos^4(\varphi) d\varphi = 1 \rightarrow C^2 \frac{3\pi}{4} = 1 \rightarrow C = \pm \frac{2}{\sqrt{3\pi}}$$

## 5 Problem 5

$$(a) n = 6, l = 3$$

$$(b) E_6 = -\frac{E}{n^2} = -\frac{13.6}{36} = 0.38 \text{ eV}$$

$$(c) |L| = \sqrt{L^2} = \sqrt{3(3+1)\hbar^2} = 2.3 \times 10^{-33}$$

$$(d) L_z = m\hbar = \pm 1.989 \times 10^{-33}, \pm 1.326 \times 10^{-33}, \pm 6.63 \times 10^{-34}, 0$$

## 6 Problem 6

$$(a) \psi_{210} = \frac{1}{4\sqrt{2\pi}} \left(\frac{Z}{a_0}\right)^{3/2} \frac{Zr}{a_0} e^{-Zr/2a_0} \cos \theta$$

Since it's hydrogen atom,  $Z = 1$ ,  $\psi_{210} = \frac{1}{4\sqrt{2\pi}} \left(\frac{1}{a_0}\right)^{3/2} \frac{r}{a_0} e^{-r/2a_0} \cos \theta$

$$P(r) = \psi_{210}^* \psi_{210} 4\pi r^2 = \frac{r^4}{8a_0^5} e^{-r/a_0} \cos^2 \theta, \text{ hence } P(r) \propto r^4 e^{-r/a_0}$$

$$(b) \frac{dP(r)}{dr} = \frac{r^3}{2a_0^5} e^{-r/a_0} \cos^2 \theta - \frac{r^4}{a_0^5} e^{-r/a_0} \cos^2 \theta = 0$$

$$r = 4a_0 \text{ or } r = 0$$

## 7 Problem 7

$$\psi_{000} = \frac{1}{\sqrt{\pi}} \left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}$$

Since  $Z = 1$ , then  $P(r) = 4r^2 \frac{1}{a_0^3} e^{-2r/a_0}$

Since  $a_0 = 5.29 \cdot 10^{-11} > R_0 = 10^{-15}$ ,  $e^{-2r/a_0} \approx 0$

$$\int_0^{R_0} P(r) dr = \int_0^{10^{-15}} 4r^2 \frac{1}{a_0^3} dr = 9 \cdot 10^{-15}$$

## 8 Problem 8

$$I = \frac{2}{5}mR^2 = \frac{2}{5}9.1 \cdot 10^{-31} (10^{-15})^2 \approx 3.64 \cdot 10^{-60}$$

$$\omega = \frac{|L|}{I} = \frac{|S|}{I} = \frac{\hbar \sqrt{\frac{3}{4}}}{3.64 \cdot 10^{-60}} \approx 2.5 \cdot 10^{25} \text{ rad/s}$$

$$v = \omega R = 2.5 \cdot 10^{25} 10^{-15} = 2.5 \cdot 10^{10} \text{ m/s}$$

The speed is larger than the speed of light, hence the result from classic mechanic is incorrect.



0.5

## 9 Problem 9

- (a) We should expect to see 4 lines with  $j = \pm \frac{3}{2}$  because  $l = 0, s = \frac{3}{2}$   
Hence the lines corresponds  $m_s = \pm \frac{3}{2}, \pm \frac{1}{2}$   
(b) We should expect to see 3 lines with  $m = \pm 1, 0$

## 10 Problem 10

$$\mathbf{L} = \hbar \sqrt{l(l+1)} = 1.05 \cdot 10^{-34} \sqrt{2 \cdot 3} \approx 2.58 \cdot 10^{-34} \text{N} \cdot \text{m} \cdot \text{s}$$

$$j = l \pm \frac{1}{2} = \frac{5}{2} \text{ or } \frac{3}{2}$$

$$\text{The corresponding } \mathbf{J} \approx 3.11 \cdot 10^{-34} \text{ or } 2.03 \cdot 10^{-34} \text{N} \cdot \text{m} \cdot \text{s}$$