Lecture 4: Optical waveguides

- Waveguide structures
- Waveguide modes
- Field equations
- Wave equations
- Guided modes in symmetric slab waveguides
- General formalisms for step-index planar waveguides

References: A significant portion of the materials follow "Photonic Devices," Jia-Ming Liu, Chapter 2

Waveguide structures

Waveguide structures

- The basic structure of a dielectric waveguide consists of a longitudinally extended high-index optical medium, called the *core*, which is transversely surrounded by low-index media, called the *cladding*. A guided optical wave propagates in the waveguide along its *longitudinal* direction.

- The characteristics of a waveguide are determined by the transverse profile of its dielectric constant $\varepsilon(x, y)/\varepsilon_o$, which is *independent of the longitudinal* (z) direction.

- For a waveguide made of optically isotropic media, we can characterize the waveguide with a single spatially dependent transverse profile of the index of refraction $n(x, y)$.

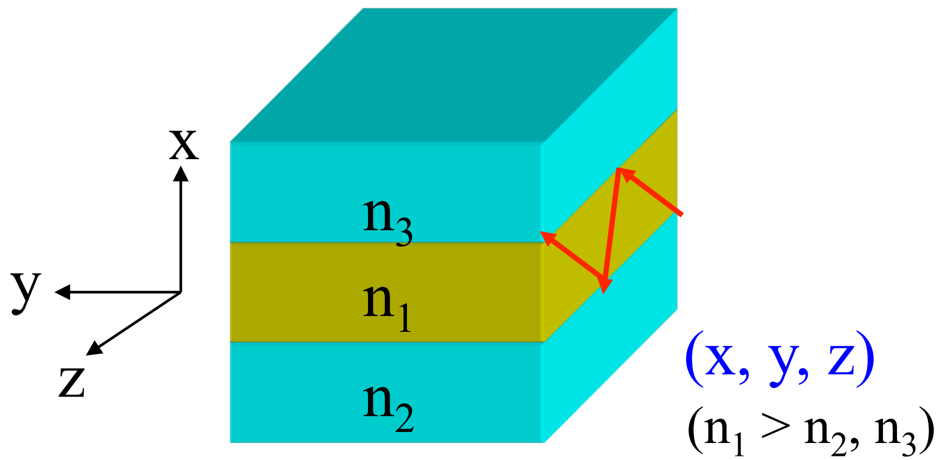
Nonplanar and planar waveguides

- There are two basic types of waveguides:
  - In a ***nonplanar waveguide*** of two-dimensional transverse optical confinement, the core is surrounded by cladding in *all transverse directions*, and n(x, y) is a function of both x and y coordinates.

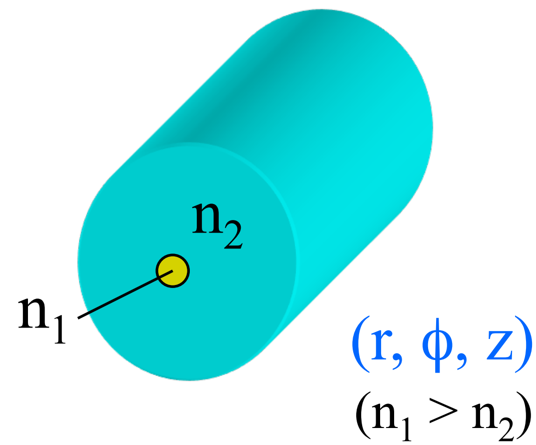
    E.g. the *channel waveguides* and the *optical fibers*

  - In a ***planar waveguide*** that has optical confinement in *only one transverse direction*, the core is sandwiched between cladding layers in only one direction, say the x direction, with an index profile n(x). The core of a planar waveguide is also called the *film*, while the upper and lower cladding layers are called the *cover* and the *substrate*.

Optical waveguides



Planar (slab) waveguides for integrated photonics (e.g. laser chips)

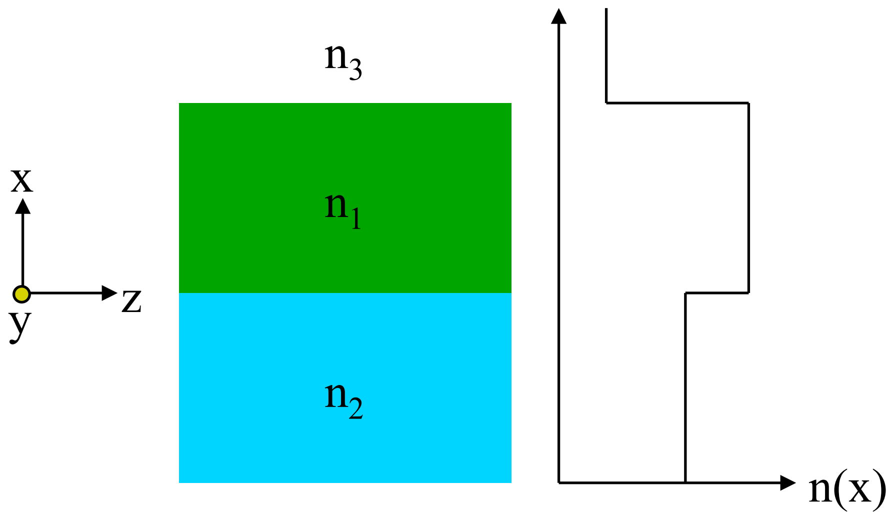


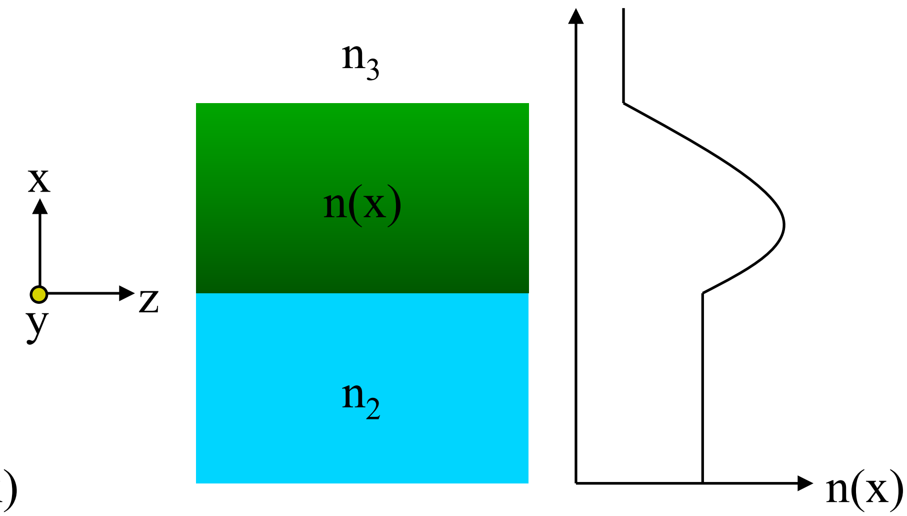
Cylindrical optical fibers

- Optical waveguides are the *basic elements for confinement and transmission of light* over various distances, ranging from tens or hundreds of μm in integrated photonics to hundreds or thousands of km in long-distance fiber-optic transmission. Optical waveguides *also form key structures in semiconductor lasers, and act as passive and active devices* such as waveguide couplers and modulators.

Index profiles

- A waveguide in which the index profile has abrupt changes between the core and the cladding is called a ***step-index waveguide***, while one in which the index profile varies gradually is called a ***graded-index waveguide***.
- We will focus on the step-index waveguide for our discussion.

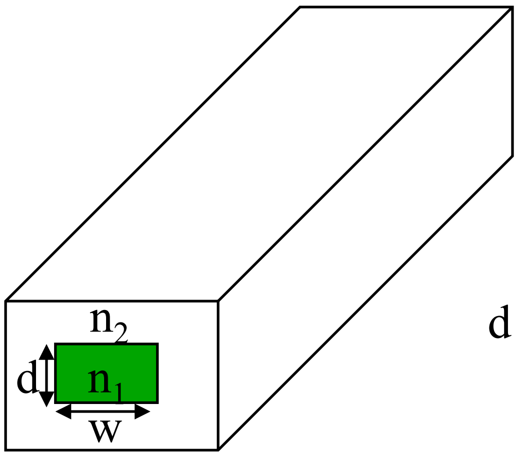




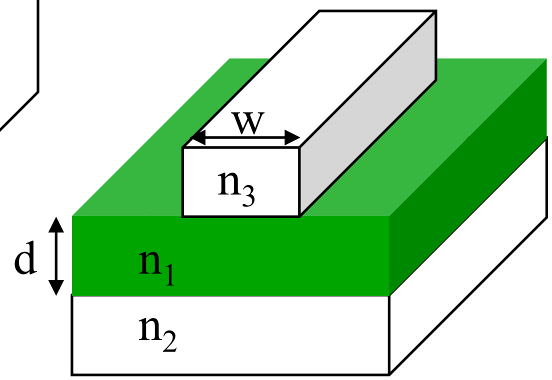
Channel waveguides

- Most waveguides used in device applications are *nonplanar* waveguides.
- For a nonplanar waveguide, the index profile n(x, y) is a function of both transverse coordinates x and y.
- There are many different types of nonplanar waveguides that are differentiated by the distinctive features of their index profiles.
- One very unique group is the *circular optical fibers* (to be discussed in Lecture 5).
- Another important group of nonplanar waveguides is the *channel waveguides*, which include
  - The buried channel waveguides
  - The strip-loaded waveguides
  - The ridge waveguides
  - The rib waveguides
  - The diffused waveguides.

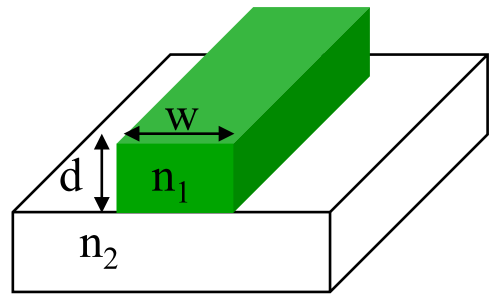
Representative channel waveguides



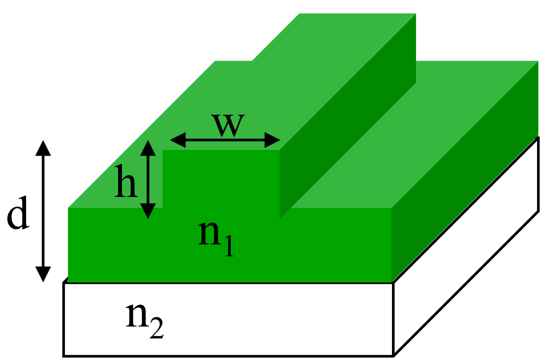
Buried channel waveguide



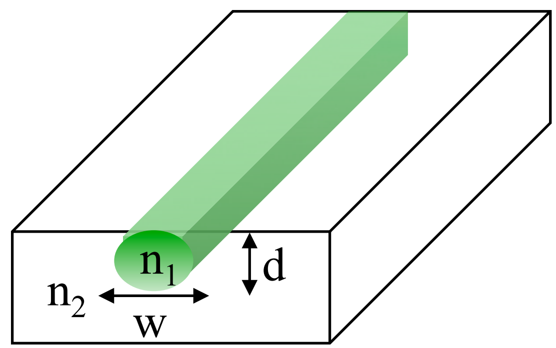
Strip-loaded waveguide



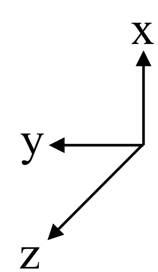
Ridge waveguide



rib waveguide



Diffused waveguide



Representative channel waveguides

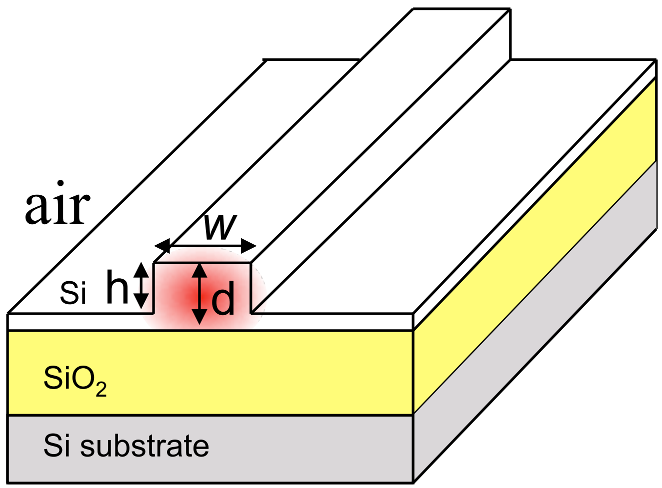
- A ***buried channel waveguide*** is formed with a high-index waveguiding core buried in a low-index surrounding medium. The waveguiding core can have any cross-sectional geometry though it is often a rectangular shape.
- A ***strip-loaded waveguide*** is formed by loading a planar waveguide, which already provides optical confinement in the x direction, with a dielectric strip of index $n_3<n_1$ or a metal strip to facilitate optical confinement in the y direction. The waveguiding core of a strip waveguide is the n_1 region under the loading strip, with its thickness d determined by the thickness of the n_1 layer and its width w defined by the width of the loading strip.
- A ***ridge waveguide*** has a structure that looks like a strip waveguide, but the strip, or the ridge, on top of its planar structure has a high index and is actually the waveguiding core. A ridge waveguide has strong optical confinement because it is surrounded on three sides by low-index air (or cladding material).

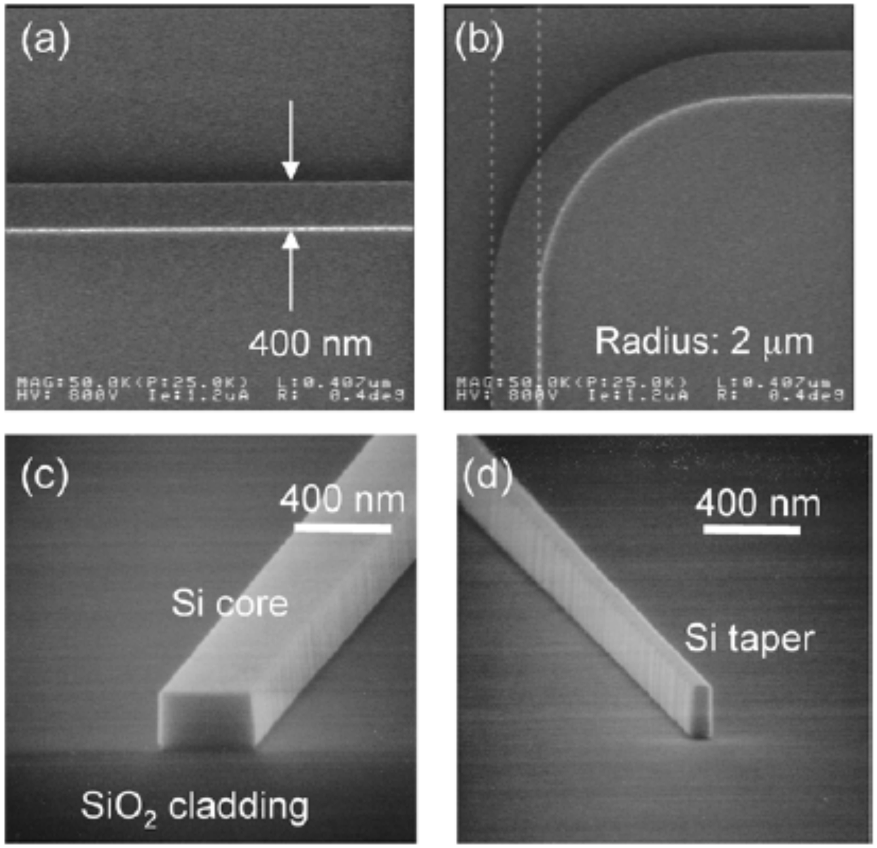
Representative channel waveguides

- A ***rib waveguide*** has a structure similar to that of a strip or ridge waveguide, but the strip has the same index as the high-index planar layer beneath it and is part of the waveguiding core.
- These four types of waveguides are usually termed *rectangular waveguides* with a thickness d in the x direction and a width w in the y direction, though their shapes are normally not exactly rectangular.
- A ***diffused waveguide*** is formed by creating a high-index region in a substrate through diffusion of dopants, such as $LiNbO_3$ waveguide with a core formed by Ti diffusion.
- Because of the diffusion process, the core boundaries in the substrate are not sharply defined.
- A diffused waveguide also has a thickness d defined by the diffusion depth of the dopant in the x direction and a width w defined by the distribution of the dopant in the y direction.

Silicon optical waveguides (nanophotonic wires)

A silicon rib waveguide





• Guide light by total internal reflection in a few 100 nm cross-section (propagation loss typically few - ~1 dB/cm)

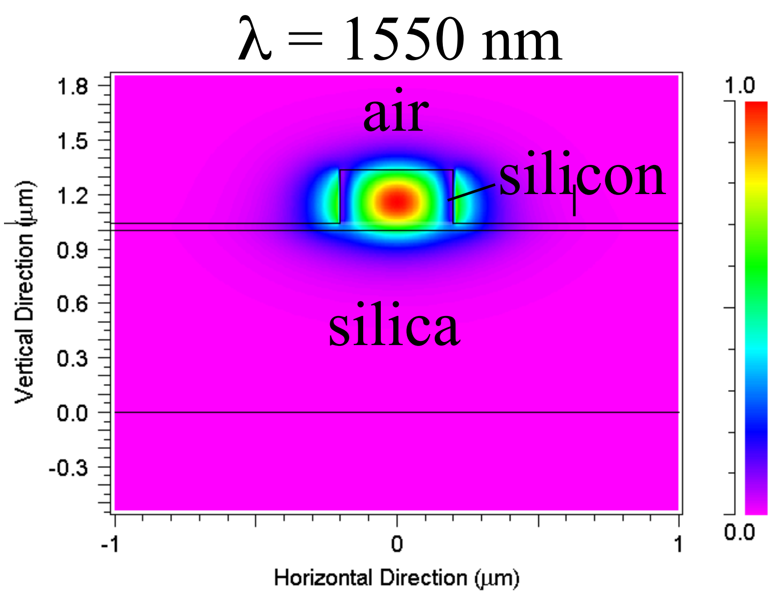
Tsuchizawa *et al.*, *IEEE J. Sel. Topic Quantum Electron.* **11**, 232-240 (2005).

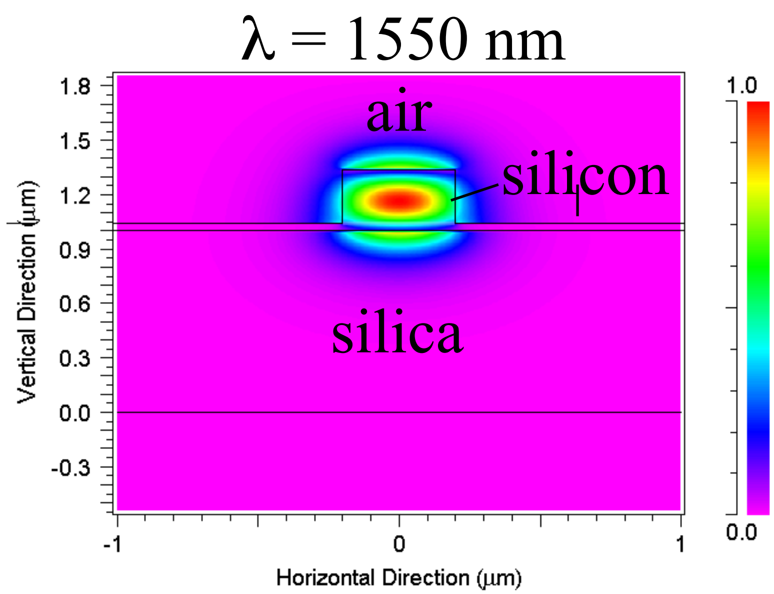
Remarks on nonplanar waveguides

- One distinctive property of *nonplanar* dielectric waveguides versus planar waveguides is that a ***nonplanar waveguide supports hybrid modes*** in addition to TE and TM modes, whereas a planar waveguide supports only TE and TM modes.
- Except for those few exhibiting special geometric structures, such as circular optical fibers, ***nonplanar dielectric waveguides generally do not have analytical solutions*** for their guided mode characteristics.
- Numerical methods, such as the ***beam propagation method***, exist for analyzing such waveguides.
- Here we discuss approximate solutions that give the mode characteristics. One of the methods is the ***effective index method***.

Solving waveguide modes by numerical methods

- Except for those few exhibiting special geometric structures, such as circular optical fibers, *non-planar* dielectric waveguides generally do *not* have analytical solutions for their guided mode characteristics.
- ***Numerical methods***, such as the *beam propagation method*, are typically used for analyzing such waveguides (e.g. silicon-on-insulator waveguides modes, TE and TM mode electric field distributions)





Waveguide modes

Waveguide modes

- ***Waveguide modes*** exist that are characteristic of a particular waveguide structure.
- A waveguide mode is a ***transverse field pattern*** whose *amplitude and polarization profiles* remain constant along the longitudinal z coordinate.
- Therefore, the electric and magnetic fields of a mode can be written as follows

$$E_\nu(r,t) = E_\nu(x,y)\exp i(\beta_\nu z - \omega t)$$

$$H_\nu(r,t) = H_\nu(x,y)\exp i(\beta_\nu z - \omega t)$$

where υ is the *mode index*, $E_\upsilon(x, y)$ and $H_\upsilon(x, y)$ are the *mode field profiles*, and β_υ is the *propagation constant* of the mode.

Waveguide modes

- For a *waveguide of two-dimensional transverse optical confinement*, there are *two* degrees of freedom in the transverse xy plane, and the mode index υ consists of two parameters for characterizing the variations of the mode fields in these two transverse dimensions. E.g. υ represents two mode numbers, υ = mn with integral m and n, for discrete guided modes.
- For the *planar waveguide*, the mode fields do not depend on the y coordinate.
- Thus, the electric and magnetic fields of a mode can be simplified to

$$E_\nu(r,t) = E_\nu(x)\exp i(\beta_\nu z - \omega t)$$

$$H_\nu(r,t) = H_\nu(x)\exp i(\beta_\nu z - \omega t)$$

- In this case, υ consists of *only* one parameter characterizing the field variation in the x dimension.

General idea of waveguide modes

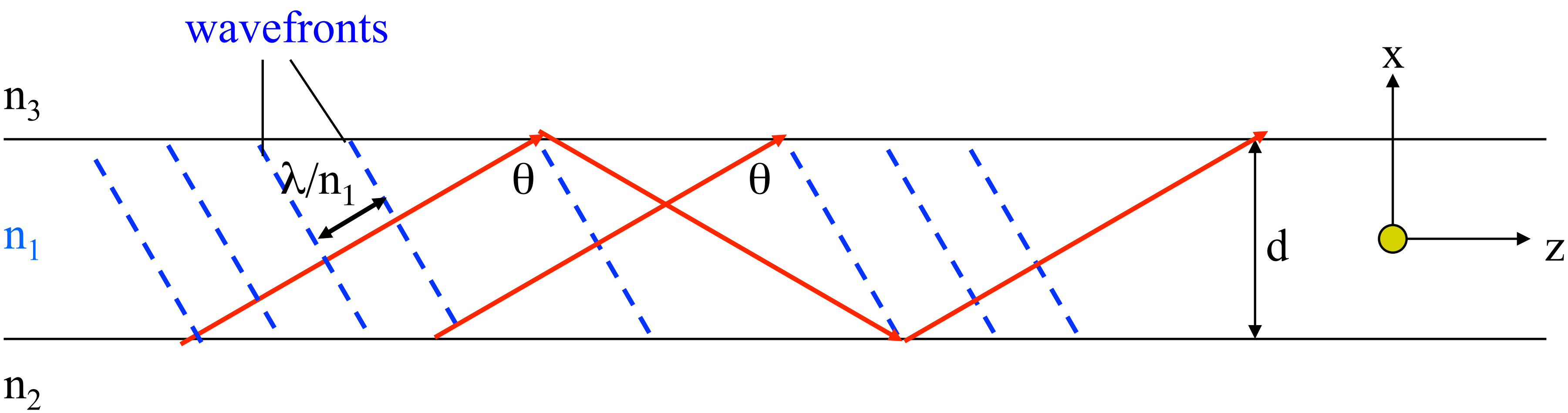
- Consider the qualitative behavior of an optical wave in an *asymmetric* planar step-index waveguide, where $n_1 > n_2 > n_3$.
- For an optical wave of angular frequency ω and free-space wavelength λ, the media in the three different regions of the waveguide define the following propagation constants:

$$k_1 = n_1\omega / c, k_2 = n_2\omega / c, k_3 = n_3\omega / c$$

where $k_1 > k_2 > k_3$

- We can obtain useful intuitive picture from considering the path of an *optical ray*, or a *plane optical wave*, in the waveguide.

Intuitive ray and wavefront picture



- There are two *critical angles* associated with the internal reflections at the lower and upper interfaces:

$$\theta_{c2} = \sin^{-1}\frac{n_2}{n_1} \qquad \theta_{c3} = \sin^{-1}\frac{n_3}{n_1}$$

  $\theta_{c2} > \theta_{c3}$ because $n_2 > n_3$

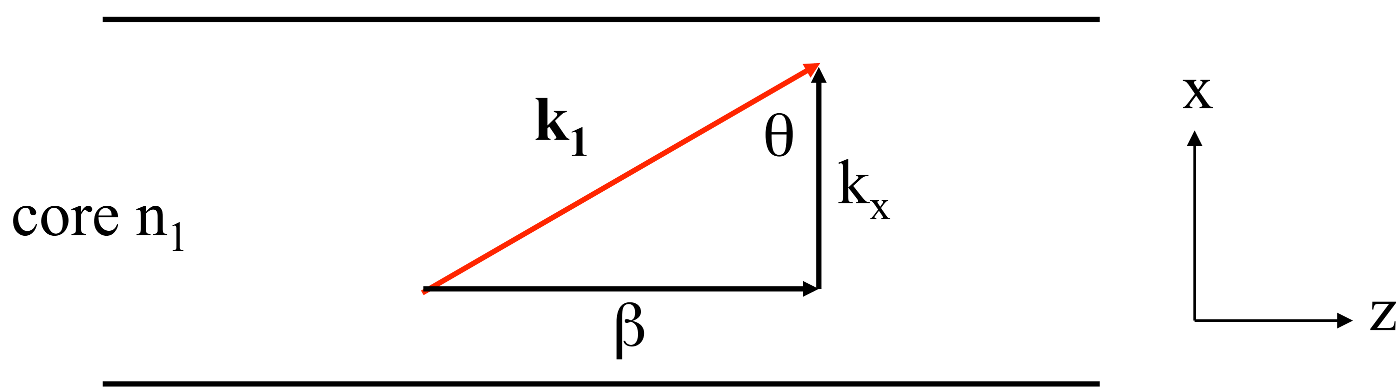
- If $\theta > \theta_{c2} > \theta_{c3}$, the wave inside the core is totally reflected at both interfaces and is trapped by the core, resulting in *guided modes*.

Guided modes

- *As the wave is reflected back and forth between the two interfaces, it interferes with itself.*
- ***A guided mode can exist only when a <u>transverse resonance condition</u> is satisfied s.t. the repeatedly reflected wave has constructive interference with itself.***
- In the core region, the x component of the wavevector is $k_x = k_1 \cos\theta$ for a ray with an angle of incidence θ, while the z component is $\beta = k_1 \sin\theta$.
- The phase shift in the optical field due to a round-trip transverse passage in the core of thickness d is $2k_1 d\cos\theta$.

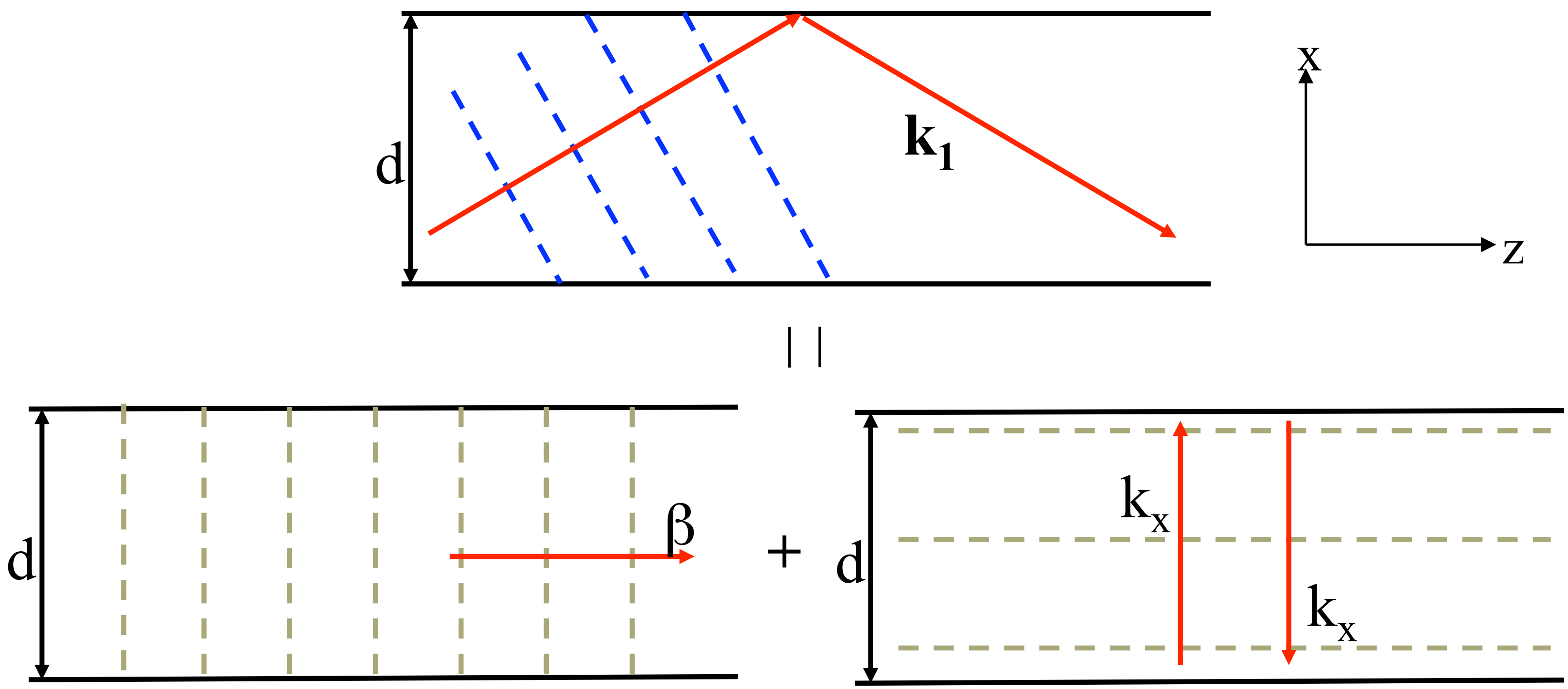
k-vector triangle

- The orthogonal components of the *propagation constant*, β and k_x, are related by the "**k**-vector triangle."



transverse component	$k_x = (n_1\omega/c) \cos\theta$
longitudinal component	$\beta = (n_1\omega/c) \sin\theta$
"**k**-vector triangle"	$\beta^2 + k_x^2 = (n_1\omega/c)^2$

k_x and β components



- We can consider the "zig-zag" wave in the waveguide as *two orthogonal components* traveling in the *longitudinal* (z) and *transverse* (x) directions.
- The *transverse* component of the plane wave is *reflected back and forth* in the x direction, *interfering with itself.*

Transverse resonance condition

- There are phase shifts φ_2 and φ_3 associated with the internal reflections in the lower and upper interfaces.
- Recall from Lecture 1 that these phase shifts can be obtained from the phase angle of r_s for a TE wave (s wave) and that of r_p for a TM wave (p wave) for a given $\theta > \theta_{c2}, \theta_{c3}$.
- Because φ_2 and φ_3 are functions of θ, the ***transverse resonance condition*** for constructive interference in a round-trip transverse passage is

$$2k_1 d\cos\theta + \varphi_2(\theta) + \varphi_3(\theta) = 2m\pi$$

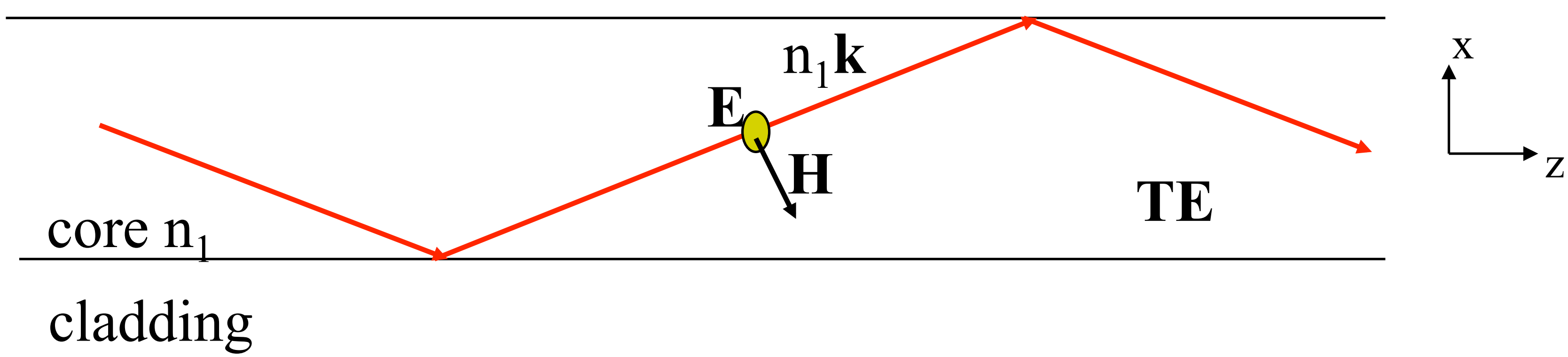
  where m is an integer = 0, 1, 2, …
- Because m can assume only integral values, only certain discrete values of θ can satisfy the transverse resonance condition.

Transverse electric polarization

- For slab waveguides, we define the x-z plane as the plane of incidence.

- An electric field pointing in the y direction corresponds to the perpendicular, or s, polarization.

- Waves with this polarization are labeled transverse electric (TE) fields because the electric field vector lies entirely in the x y plane (i.e. $E_z = 0$) that is *transverse* to the direction of net travel (the z direction).

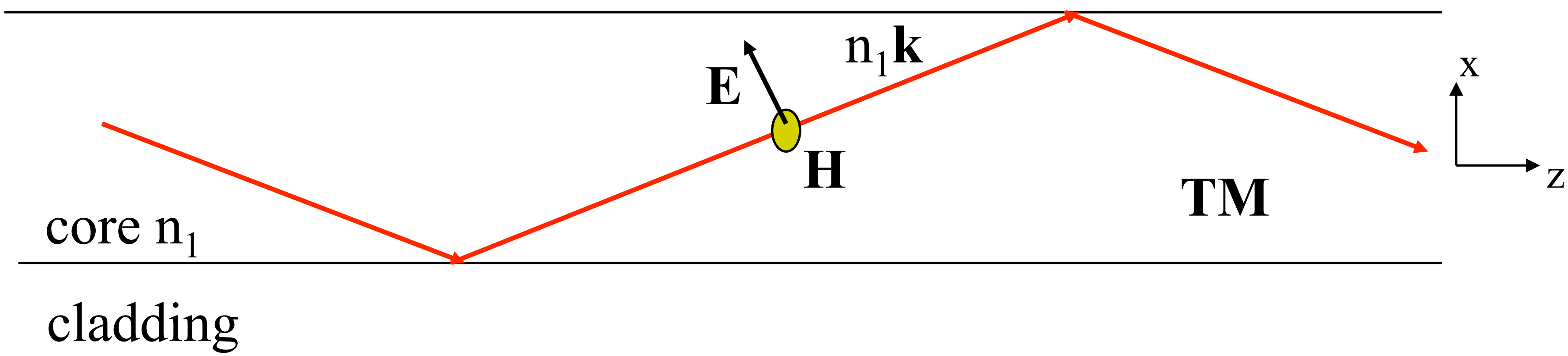


Transverse magnetic polarization

- For the parallel, or p, polarization, the electric field is no longer purely transverse. It has a component along the z direction.

- However, the magnetic field points in the y direction for this polarization is entirely transverse (i.e. $H_z = 0$).

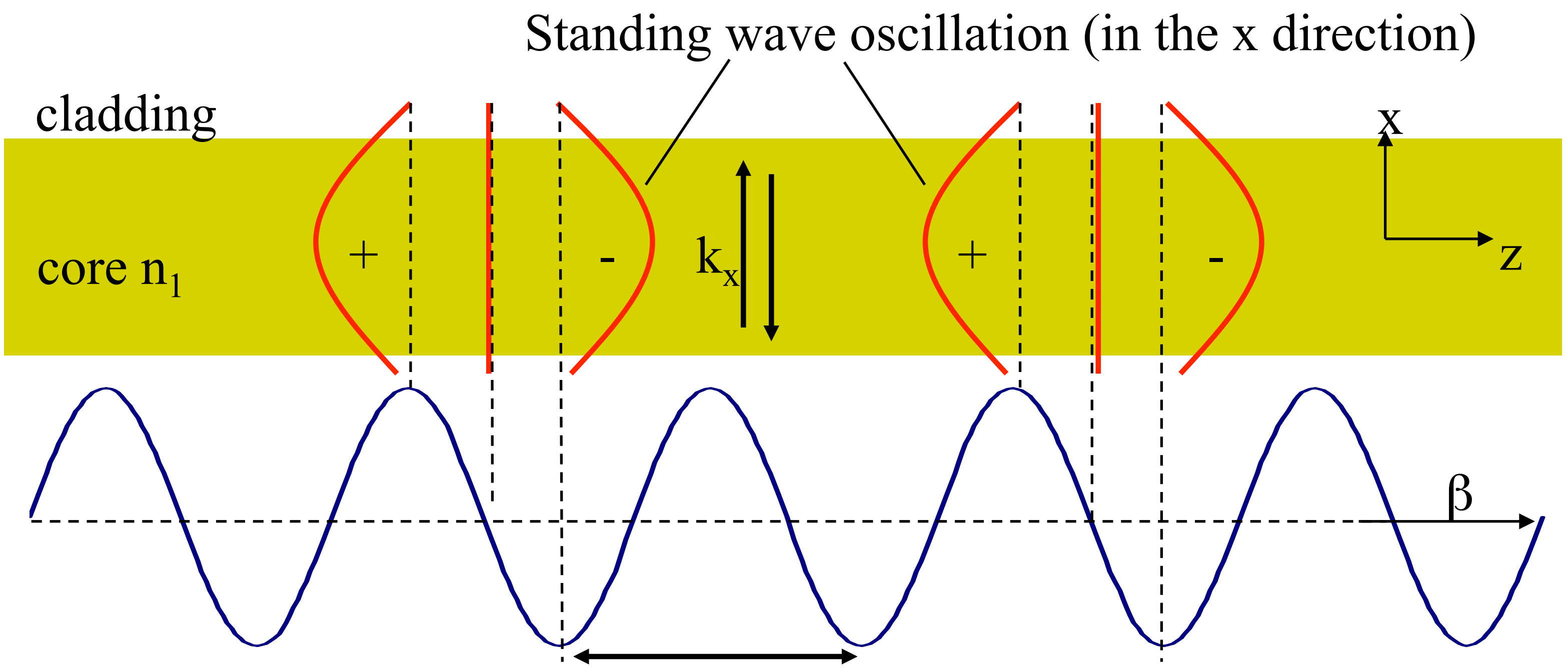
The p polarization is labeled transverse magnetic (TM) in the slab.



Discrete guided modes

- The *transverse resonance condition* results in *discrete* values of the propagation constant β_m for guided modes identified by the mode number m.
- Although the critical angles, θ_{c2} and θ_{c3}, do not depend on the polarization of the wave, the phase shifts, $\varphi_2(\theta)$ and $\varphi_3(\theta)$, caused by the internal reflection at a given angle θ depend on the *polarization*.
- Therefore, ***TE and TM waves have different solutions for the transverse resonance condition***, resulting in different β_m and different mode characteristics for a given mode number m.
- For a given polarization, solution of the transverse resonance condition yields a smaller value of θ and a correspondingly smaller value of β for a larger value of m. Therefore, $\beta_0 > \beta_1 > \beta_2 > \dots$
- The guided mode with m = 0 is called the ***fundamental mode*** and those with m ≠ 0 are ***higher-order modes***.

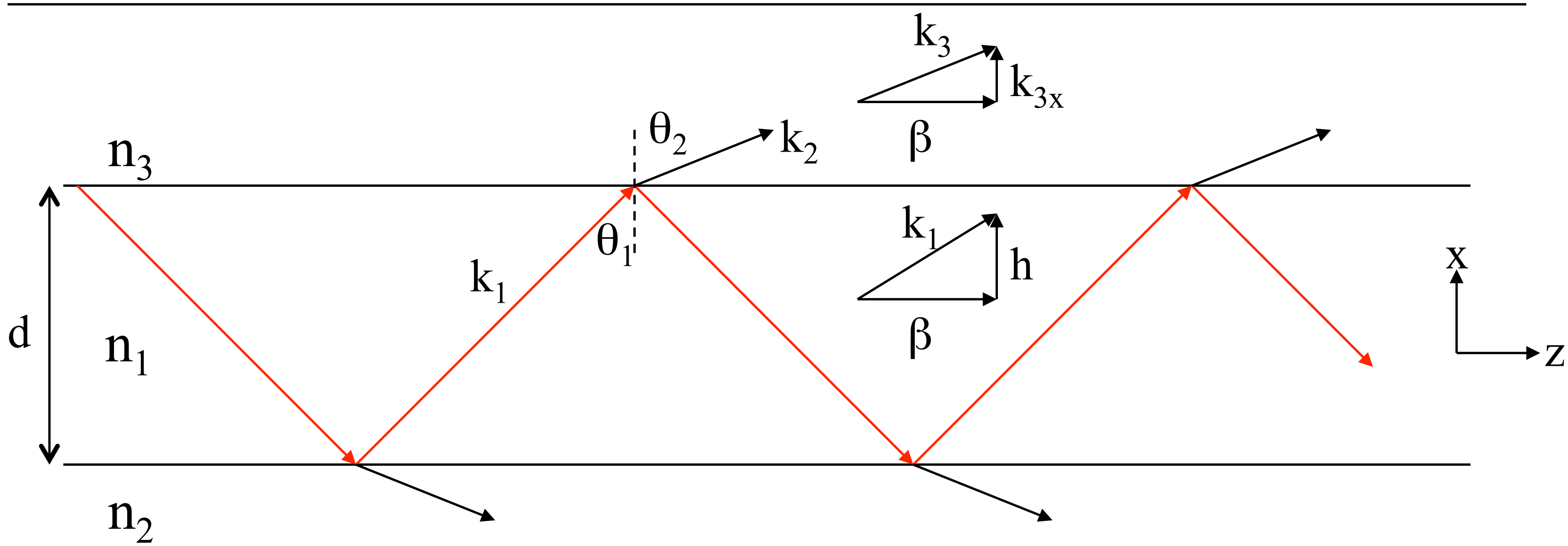
Qualitative picture of a waveguide mode



$\lambda_z = 2\pi/\beta_m = \lambda/(n_1 \sin\theta_m)$

- The stable field distribution in the transverse direction with only a periodic longitudinal dependence is known as a waveguide mode.

Plane wave representation for a planar waveguide



Upward field: $E_u = E_0 \exp(i\mathbf{k_u} \bullet \mathbf{r}) = E_0 \exp(ihx) \exp(i\beta z)$

Downward field: $E_d = E_0 \exp(i\mathbf{k_d} \bullet \mathbf{r}) = E_0 \exp(-ihx) \exp(i\beta z)$

where $\mathbf{r} = x\mathbf{a_x} + z\mathbf{a_z}$, $\mathbf{k_u} = h\mathbf{a_x} + \beta\mathbf{a_z}$, $\mathbf{k_d} = -h\mathbf{a_x} + \beta\mathbf{a_z}$

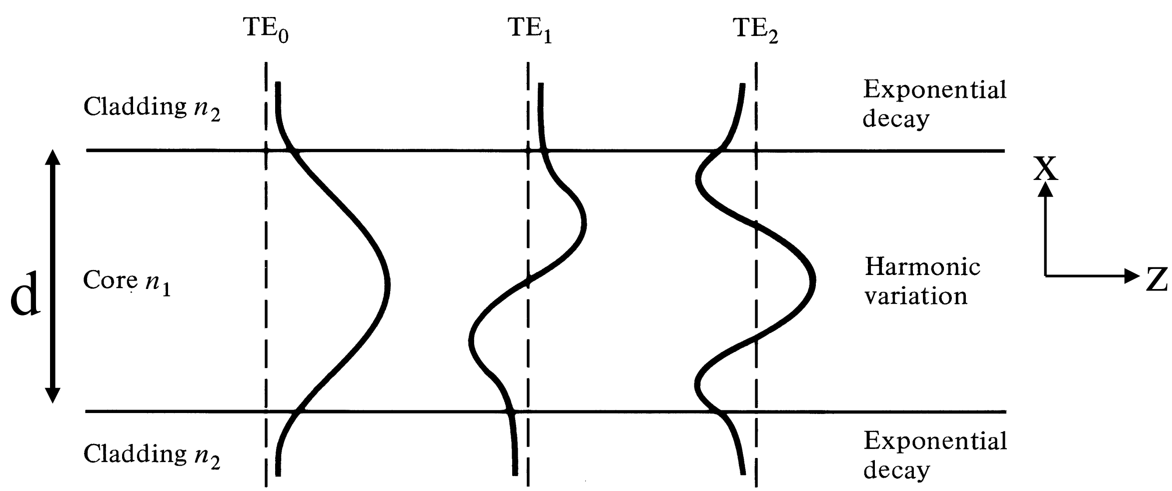
Plane wave representation

- The mode field is found through the superposition of the plane wave components. Assuming that these are in phase,

$$E_t = E_{upward} + E_{downward} = 2E_0 \cos(hx) \exp(i\beta z)$$

- In *real instantaneous* form,

$$E_t(\mathbf{r}, t) = 2E_0 \cos(hx) \cos(\beta z - \omega t)$$

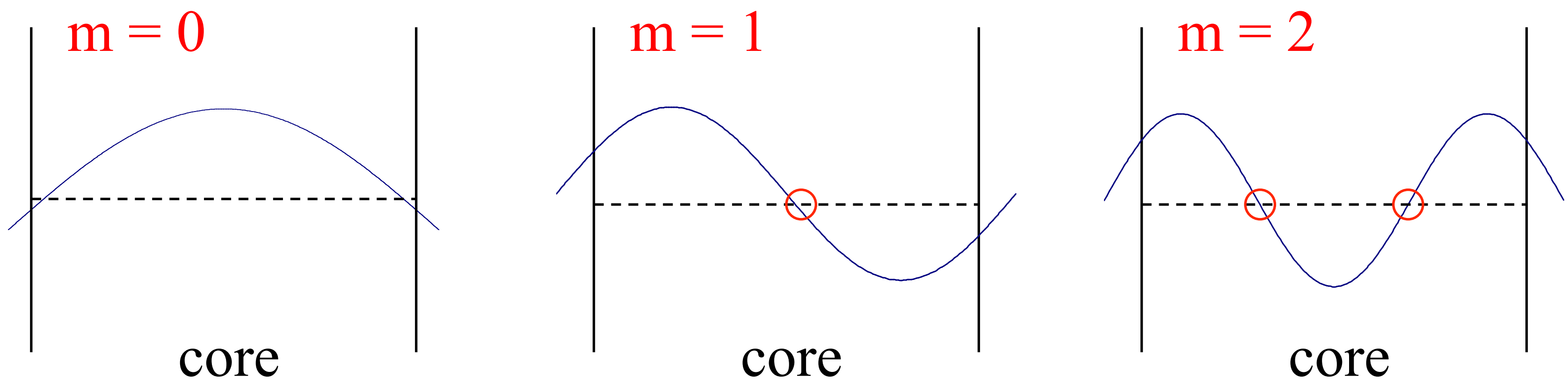


Discrete waveguide modes

- Because m can assume only integral values, only certain *discrete* values of $\theta = \theta_m$ can satisfy the resonance condition.

- This results in *discrete values of the propagation constant* β_m for guided modes identified by the mode number m.

- The guided mode with m = 0 is called the *fundamental mode* and those with m = 1, 2, … are *higher-order modes.*



Surface waves and the reflective phase shift

- An electric field component in the upper cladding region assumes the phasor form:

$$E_3 = E_{30} \exp(i\mathbf{k_3}\cdot\mathbf{r}) = E_{30} \exp(ik_{3x}x) \exp(i\beta z)$$

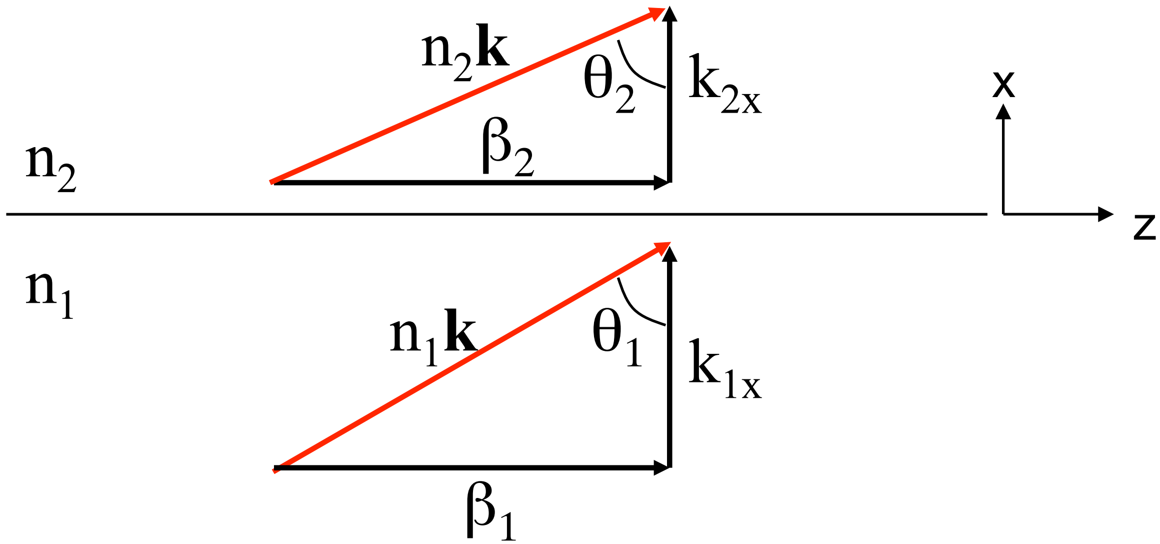
- When $\theta_1 > \theta_{c3}$, k_{3x} becomes *imaginary* and can be expressed in terms of a *real attenuation coefficient* κ as

$$k_{3x} = i\kappa \qquad (\theta_1 > \theta_{c3})$$

$$E_3 = E_{30} \exp(-\kappa x) \exp(i\beta z)$$

- This is the phasor expression for a *surface* or *evanescent* wave – propagates only in the z direction, decays in the direction normal to the interface.

Phase-matching at an interface



- As the *spatial rate of change of phase at the boundary* (or *the projection of the wavefront propagation*) on the n_1 side must match with that on the n_2 side, we have $\beta_1 = \beta_2 = \beta$. This condition is known as "phase-matching." Phase-matching allows *coupling of oscillating field between two media*.

$$\beta_1 = \beta_2 \;\Rightarrow\; n_1 k \sin\theta_1 = n_2 k \sin\theta_2 \quad \text{(Snell's Law)}$$

Evanescent field in total internal reflection

- Phase-matching at TIR $\theta_1 > \theta_c$ (i.e. $\sin\theta_1 > n_2/n_1$)

$$\beta_1 = n_1 k \sin\theta_1 = \beta_2 > n_2 k$$

- k-vector triangle in n_2

$$k_{2x} = [(n_2 k)^2 - (n_1 k \sin\theta_1)^2]^{1/2}$$

$$k_{2x} = i[(n_1 k \sin\theta_1)^2 - (n_2 k)^2]^{1/2}$$

$$= ik[(n_1 \sin\theta_1)^2 - n_2^2]^{1/2} = i\kappa$$

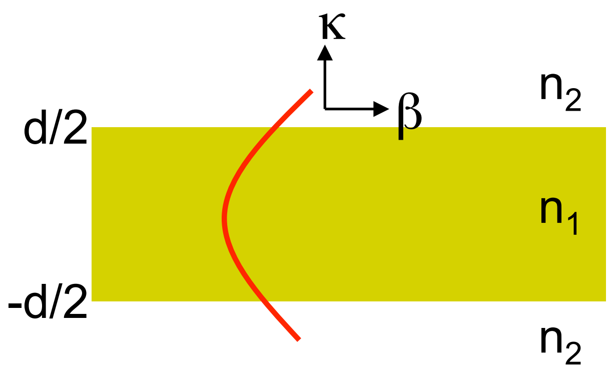
- evanescent field in the transverse direction

$$E \propto \exp -\kappa x \exp i(\beta z - \omega t)$$

Evanescent fields in the waveguide cladding

- Evanescent fields outside the waveguide core decay exponentially with an *attenuation factor* given by

$$\kappa = k\,(n_1^2 \sin^2 \theta_1 - n_2^2)^{1/2}$$



- In the *upper* layer ($x \geq d/2$)

$$E = E_2 \exp -\kappa(x - d/2) \exp i(\beta z - \omega t)$$

- In the *lower* layer ($x \leq -d/2$)

$$E = E_2 \exp \kappa(x + d/2) \exp i(\beta z - \omega t)$$

where E_2 is the peak value of the electric field at the lower ($x = -d/2$) and upper ($x = d/2$) boundaries.

Waveguide effective index

- We can define the waveguide phase velocity v_p as

$$v_p = \omega / \beta$$

- We now define an effective refractive index n_{eff} as the free-space velocity divided by the waveguide phase velocity.

$$n_{eff} = c / v_p$$

$$\text{Or} \quad n_{eff} = c\beta / \omega = \beta / k$$

$$\Rightarrow n_{eff} = n_1 \sin \theta$$

- The effective refractive index is a key parameter in *guided propagation*, just as the refractive index is in unguided wave travel.

- For waveguiding at n_1-n_2 interface, we see that $n_2 \le n_{eff} \le n_1$

At $\theta = 90^o$, $n_{eff} = n_1$ => A ray traveling parallel to the slab (core) has an effective index that depends on the guiding medium alone.

At $\theta = \theta_c$, $n_{eff} = n_2$ => The effective index for critical-angle rays depends only on the outer material n_2.

- The <u>effective refractive index changes with the wavelength</u> (i.e. dispersion) in a way related to that the bulk refractive index does.

- The wavelength as measured in the waveguide is

$$\lambda_z = \lambda/n_{eff}$$

Field equations

Field equations

- For a linear, isotropic dielectric waveguide characterized by a spatial permittivity distribution ε(x, y), Maxwell's equations can be written as

$$\nabla \times E = -\mu_0 \frac{\partial H}{\partial t}$$

$$\nabla \times H = \varepsilon \frac{\partial E}{\partial t}$$

- Because the optical fields in the waveguide have the form of

$$E_v(r,t) = E_v(x,y) \exp i(\beta_v z - \omega t)$$

$$H_v(r,t) = H_v(x,y) \exp i(\beta_v z - \omega t)$$

Field equations

- These two Maxwell's equations can be written as

$$\frac{\partial E_z}{\partial y} - i\beta E_y = i\omega\mu_0 H_x \qquad \frac{\partial H_z}{\partial y} - i\beta H_y = -i\omega\varepsilon E_x$$

$$i\beta E_x - \frac{\partial E_z}{\partial x} = i\omega\mu_0 H_y \qquad i\beta H_x - \frac{\partial H_z}{\partial x} = -i\omega\varepsilon E_y$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega\mu_0 H_z \qquad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = -i\omega\varepsilon E_z$$

- The *transverse* components of the electric and magnetic fields can be expressed in terms of the longitudinal components

$$(k^2 - \beta^2)E_x = i\beta\frac{\partial E_z}{\partial x} + i\omega\mu_0\frac{\partial H_z}{\partial y} \qquad (k^2 - \beta^2)H_x = i\beta\frac{\partial H_z}{\partial x} - i\omega\varepsilon\frac{\partial E_z}{\partial y}$$

$$(k^2 - \beta^2)E_y = i\beta\frac{\partial E_z}{\partial y} - i\omega\mu_0\frac{\partial H_z}{\partial x} \qquad (k^2 - \beta^2)H_y = i\beta\frac{\partial H_z}{\partial y} + i\omega\varepsilon\frac{\partial E_z}{\partial x}$$

Field equations

- $k^2 = \omega^2 \mu_0 \varepsilon(x, y)$ is a function of x and y to account for the transverse spatial inhomogeneity of the waveguide structure.

- In a waveguide, ***once the longitudinal field components, E_z and H_z, are known, all field components can be obtained***.

- The relations on the transverse components of the **E** and **H** fields are generally true for a *longitudinally homogeneous waveguide* of any transverse geometry and any transverse index profile where $\varepsilon(x, y)$ is not a function of z. They are equally true for step-index and graded-index waveguides.
- In waveguides that have circular cross sections, such as optical fibers, the x and y coordinates of the rectangular system can be transformed to the r and ϕ coordinates of the cylindrical system for similar relations.

Polarization modes

- The fields in a waveguide can have various vectorial characteristics.
- They can be classified based on the characteristics of the longitudinal field components
  - A ***transverse electric and magnetic mode***, or **TEM** mode, has $E_z = 0$ and $H_z = 0$. Dielectric waveguides do *not* support TEM modes.
  - A ***transverse electric mode,*** or **TE** mode, has $E_z=0$ and $H_z\neq0$
  - A ***transverse magnetic mode,*** or **TM** mode, has $H_z=0$ and $E_z\neq0$
  - A ***hybrid mode*** has both $E_z \neq 0$ and $H_z \neq 0$. Hybrid modes do *not* appear in planar waveguides but exist in nonplanar waveguides of two-dimensional transverse optical confinement. E.g. We will see that the HE and EH modes of optical fibers are hybrid modes.

Wave equations

Wave equations

- The common approach to finding E_z and H_z is to solve the *wave equations* together with *boundary conditions*.
- For the case of a *linear*, *isotropic* waveguide with a spatially dependent ε(x, y), the two Gauss' laws for **E** and **H** can be written as

$$\nabla \bullet (\varepsilon E) = 0 \Rightarrow \nabla \bullet E = -\frac{\nabla \varepsilon}{\varepsilon} \bullet E$$

$$\nabla \bullet H = 0$$

- Note that ∇•E does *not* vanish in general because ε(x, y) is spatially dependent.
- Using the four Maxwell's equations, we have

$$\nabla^2 E + k^2 E = -\nabla\left(\frac{\nabla \varepsilon}{\varepsilon} \bullet E\right) \qquad \nabla^2 H + k^2 H = -\frac{\nabla \varepsilon}{\varepsilon} \times \nabla \times H$$

Wave equations

- The three components E_x, E_y and E_z for the electric field are generally coupled together because $\nabla\varepsilon \neq 0$ in a waveguide. For the same reason H_x, H_y and H_z are also coupled.
- This fact indicates that the vectorial characteristics of a mode field in a waveguide are strongly dependent on the geometry and index profile of the waveguide.

- In the case of a TE mode, $\nabla\varepsilon \perp E$ s.t. $\nabla\varepsilon \bullet E = 0$
- Thus, each component of the electric field of a TE mode satisfies a *homogeneous scalar differential equation*. The magnetic field components of a TE mode are still coupled.

Wave equations for step-index waveguides

- The index profile of a step-index waveguide is *piecewise constant*.
- We can write a *homogeneous* wave equation separately for each region of constant ε because ∇ε = 0 within each region.
- Assuming **E** and **H** of the harmonic guided wave form, and with ∇ε = 0 for each region of constant ε, we obtain for the longitudinal components

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + (k_i^2 - \beta^2) E_z = 0$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + (k_i^2 - \beta^2) H_z = 0$$

where $k_i^2 = \omega^2 \mu_0 \varepsilon_i = n_i^2 \omega^2 / c^2$ is a constant for region i, which has a constant index of refraction n_i.

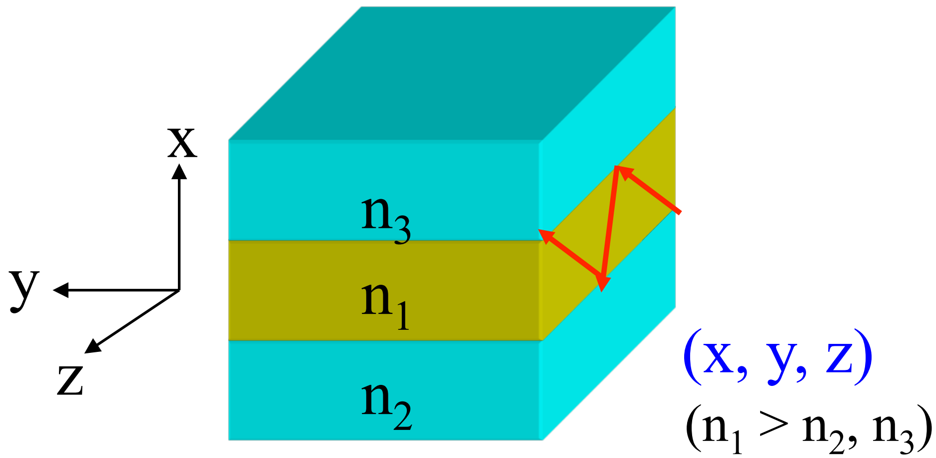
Wave equations for step-index waveguides

- A homogeneous equation in the same form can be written for each of the other four field components, E_x, E_y, H_x and H_y.
- However, it is not necessary to solve the wave equations for all field components because the transverse field components can be found from E_z and H_z.
- ***Therefore, the mode field pattern can be obtained*** by solving the two equations above for each region of *constant index* and by requiring the fields to satisfy the boundary conditions at the interfaces between neighboring regions.

- *However, note that this approach does not work for graded-index waveguides because* $\nabla\varepsilon \neq 0$ *for such waveguides*.

Wave equations for planar waveguides

- *Homogeneous wave equations* exist for *planar waveguides* of *any* index profile n(x).
- For a planar waveguide, the modes are either TE or TM.
- Furthermore, we consider $\partial/\partial y = 0$ because the index profile is independent of the y coordinate. The wave equations are substantially simplified.



Planar (slab) waveguides for integrated photonics (e.g. laser chips)

Wave equations for planar waveguides: TE modes

- For any TE mode of a planar waveguide, $E_z = 0$.
- Such that from relations of the transverse components to the longitudinal components, we see that $E_x = H_y = 0$ because $\partial H_z/\partial y = 0$.
- The *only nonvanishing* field components are H_x, E_y and H_z.
- Because there is only one nonvanishing electric field component E_y, the wave equation for E_y is naturally decoupled from the other field components. Therefore, we have

$$\frac{\partial^2 E_y}{\partial x^2} + (k^2 - \beta^2)E_y = 0$$

where $k^2 = \omega^2\mu_0\varepsilon(x) = (\omega^2/c^2)n^2(x)$.

- H_x and H_z can be obtained from E_y using the field equations:

$$H_x = -\frac{\beta}{\omega\mu_0}E_y \qquad\qquad H_z = \frac{1}{i\omega\mu_0}\frac{\partial E_y}{\partial x}$$

Wave equations for planar waveguides: TM modes

- For any TM mode of a planar waveguide, $H_z = 0$.
- Then, $H_x = E_y = 0$ because $\partial E_z/\partial y = 0$.
- The only nonvanishing field components are E_x, H_y and E_z.
- The wave equation for H_y

$$\frac{\partial^2 H_y}{\partial x^2} + (k^2 - \beta^2) H_y = \frac{1}{\varepsilon}\frac{d\varepsilon}{dx}\frac{\partial H_y}{\partial x}$$

where $k^2 = k^2(x) = (\omega^2/c^2)n^2(x)$.

- E_x and E_z can be obtained from H_y:

$$E_x = \frac{\beta}{\omega\varepsilon} H_y \qquad\qquad E_z = -\frac{1}{i\omega\varepsilon}\frac{\partial H_y}{\partial x}$$

Guided modes in symmetric slab waveguides

The solutions

- The solutions to the wave equation for the *three* regions have the form (***assuming TE polarization***)

$$E = \hat{y} E_y(x) \exp i(\beta z - \omega t)$$

- Substituting into the wave equation (*assuming no y dependence*)

$$\frac{\partial^2 E_y}{\partial x^2} + (k_i^2 - \beta^2) E_y = 0$$

- We can solve the second-order differential equation (*with constant coefficients*) for E_y

$$E_y \propto \exp\left(\pm ix\sqrt{k_i^2 - \beta^2}\right)$$

Symmetric slab waveguide solutions

- The solutions are *sinusoidal* or *exponential* according to

  Sinusoidal $\quad k_i^2 > \beta^2$

  Exponential $\quad k_i^2 < \beta^2$

- **Guided modes**: cladding region has *exponential* solutions $\beta > k_2 > k_3$ while core region has *sinusoidal* solutions $\beta < k_1$
- For x → ±∞, we require the solutions to remain finite.
- The solutions have the general form (*assuming* $n_2 = n_3$):

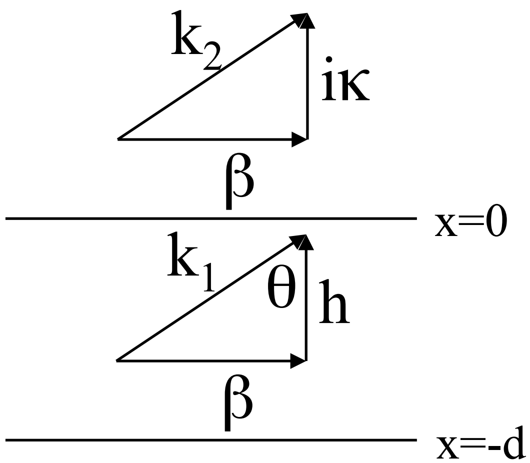
Cladding (x ≥ 0) $\quad E_y = A\exp(-\kappa x)$

Core (-d ≤ x ≤ 0) $\quad E_y = B\cos(hx) + C\sin(hx)$

Cladding (x ≤ -d) $\quad E_y = D\exp(\kappa(x+d))$

Symmetric slab waveguide solutions

- The *transverse* propagation constants



$$\kappa = (\beta^2 - k_2^2)^{1/2}$$

$$h = (k_1^2 - \beta^2)^{1/2} = k_1 \cos\theta$$

- In order to determine the *allowed* β and the unspecified constants A, B, C and D, we need to match the solution in cladding with the solution in core.
- Therefore, ***boundary conditions*** must be specified at the core-cladding interfaces.
- We expect at least *one* arbitrary constant in the final solution given by the overall field strength.

Boundary conditions

- E_y *continuous* at $x = 0 \rightarrow A = B$
- $\partial E_y/\partial x$ (H_z) *continuous* at $x = 0 \rightarrow C = (-\kappa/h)A$
- E_y *continuous* at $x=-d \rightarrow D = A\cos(hd) + (\kappa A/h)\sin(hd)$
- $\partial E_y/\partial x$ (H_z) *continuous* at $x=-d \rightarrow \tan(hd) = 2\kappa h / (h^2 - \kappa^2)$

The first *three* results can be used to solve the ***electric field distributions*** within the waveguide core and cladding.

The *fourth* result gives a *transcendental* equation that allows us to solve for the ***allowed β*** *graphically*.

TE mode field distributions in a symmetric slab waveguide

x ≥ 0 $$E = \hat{y}A\exp(-\kappa x)\exp i(\beta z - \omega t)$$

−d ≤ x ≤ 0 $$E = \hat{y}A[\cos(hx) - \frac{\kappa}{h}\sin(hx)]\exp i(\beta z - \omega t)$$

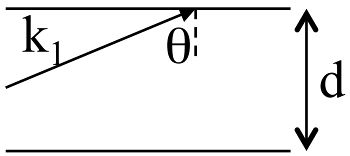
x ≤ −d $$E = \hat{y}A\exp(\kappa(x+d))[\cos(hd) + \frac{\kappa}{h}\sin(hd)]\exp i(\beta z - \omega t)$$

- Note that *all* the electric field distributions propagate along the z-direction because of the factor exp i(βz - ωt) even though the cladding field decays exponentially in the transverse direction.
- Next we solve for the allowed β, κ and h.

Eigenvalue equations for symmetric slab waveguides

- The ***transverse resonance condition*** for a *symmetric* waveguide

$$2k_1 d \cos\theta + 2\varphi(\theta) = 2m\pi$$



- The *reflection phase angle* for the TE polarization (s wave) is given as (see lecture 1)

$$\tan(\varphi_{TE}/2) = (n_1^2 \sin^2\theta - n_2^2)^{1/2} / (n_1 \cos\theta)$$

- The *reflection phase angle* for the TM polarization (p wave) is given as

$$\tan(\varphi_{TM}/2) = (n_1^2/n_2^2)\,(n_1^2 \sin^2\theta - n_2^2)^{1/2} / (n_1 \cos\theta)$$

Eigenvalue equations for symmetric slab waveguides

Recall the reflection coefficient $r_{TE} = \exp -i\varphi_{TE} \Rightarrow \quad \varphi(\theta) = -\varphi_{TE}(\theta)$

$r_{TM} = -\exp -i\varphi_{TM} \Rightarrow \quad \varphi(\theta) = -\varphi_{TM}(\theta)$

$\Rightarrow \quad k_1 d \cos\theta - m\pi = -\varphi(\theta) = \varphi_{TE,TM}(\theta)$

$\Rightarrow \quad (k_1 d \cos\theta)/2 - m\pi/2 = \varphi_{TE,TM}/2$

$\Rightarrow \quad \tan[(k_1 d \cos\theta)/2 - m\pi/2] = \tan(\varphi_{TE,TM}/2)$

TE:

$$\tan[(k_1 d \cos\theta)/2 - m\pi/2] = (n_1^2 \sin^2\theta - n_2^2)^{1/2} / (n_1 \cos\theta)$$

TM:

$$\tan[(k_1 d \cos\theta)/2 - m\pi/2] = (n_1^2/n_2^2)\,(n_1^2 \sin^2\theta - n_2^2)^{1/2} / (n_1 \cos\theta)$$

$m = 0, 1, 2, \ldots,$

Normalized waveguide parameters

- The mode properties of a waveguide are commonly characterized in terms of *dimensionless* normalized waveguide parameters.
- The ***normalized frequency***, also known as the ***V number***, of a step-index planar waveguide is defined as

$$V = (2\pi/\lambda)\, d\, (n_1^2 - n_2^2)^{1/2} = (\omega/c)\, d\, (n_1^2 - n_2^2)^{1/2}$$

where d is the core thickness (*or diameter in the case of optical fibers*).

- The propagation constant β can be represented by the ***normalized guide index***:

$$b = (\beta^2 - k_2^2)/(k_1^2 - k_2^2) = (n_{eff}^2 - n_2^2)/(n_1^2 - n_2^2)$$

recall $n_{eff} = c\beta/\omega = \beta\lambda/2\pi$ is the *effective refractive index* of the waveguide mode that has a propagation constant β.

Eigenvalue equations in terms of normalized frequency

TE: $\tan(hd/2 - m\pi/2) = (V^2 - h^2d^2)^{1/2}/hd$

TM: $\tan(hd/2 - m\pi/2) = (n_1^2/n_2^2)(V^2 - h^2d^2)^{1/2}/hd$

$m = 0, 1, 2, \ldots,$

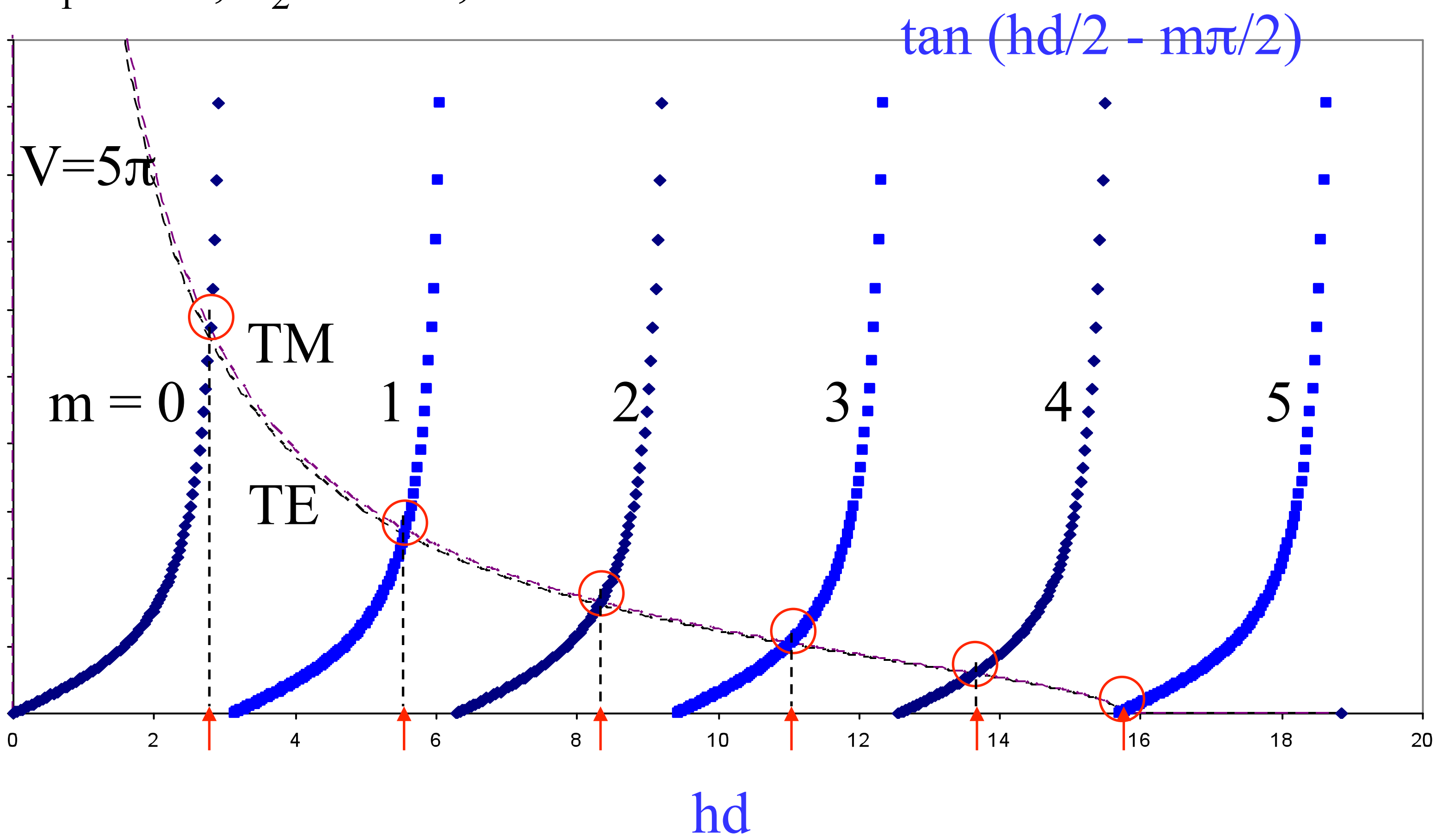
- The eigenvalue equations are in the form of *transcendental* equations, which are usually solved graphically by plotting their left- and right-hand sides as a function of hd.
- The solutions yield the *allowed values of hd* for a given value of the waveguide parameter V for TE/TM modes.

Example #1: Symmetric weakly guiding slab waveguides

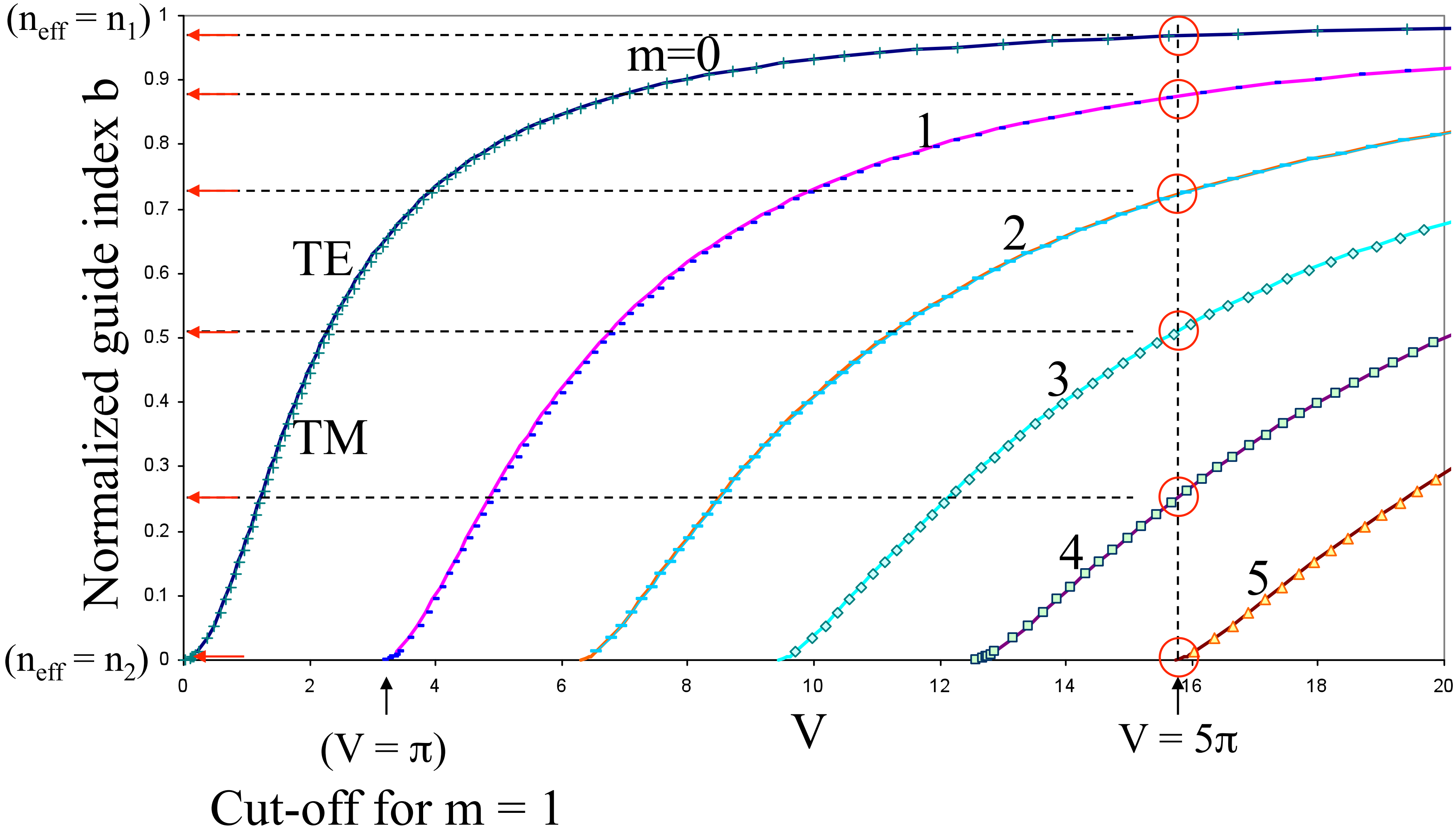
- Consider a ***weakly guiding*** waveguide $n_1 - n_2 << n_1$
- Here we choose $n_1 = 3.6$ and $n_2 = 3.55$. These values are characteristic of an AlGaAs double heterojunction light-emitting diode or laser diode.
- The critical angle for this structure is $\theta_c = \sin^{-1}(n_2/n_1) \sim 80^o$
- The range of angles for trapped rays is then $80^o \leq \theta \leq 90^o$.
- The range of *waveguide effective refractive index* is $3.55 \leq n_{eff} \leq 3.6$

Graphic solutions for the eigenvalues of guided TE and TM modes of a weakly guiding symmetric slab waveguide

$n_1 = 3.6$, $n_2 = 3.55$, $V = 5\pi$



Mode chart for the first six TE and TM modes ($m = 0 – 5$) of symmetric slab waveguides in AlGaAs ($n_1 = 3.6$, $n_2 = 3.55$)



Example #2: Symmetric strongly guiding slab waveguides

- Consider a ***strongly guiding*** waveguide $n_1 - n_2 >> 0$

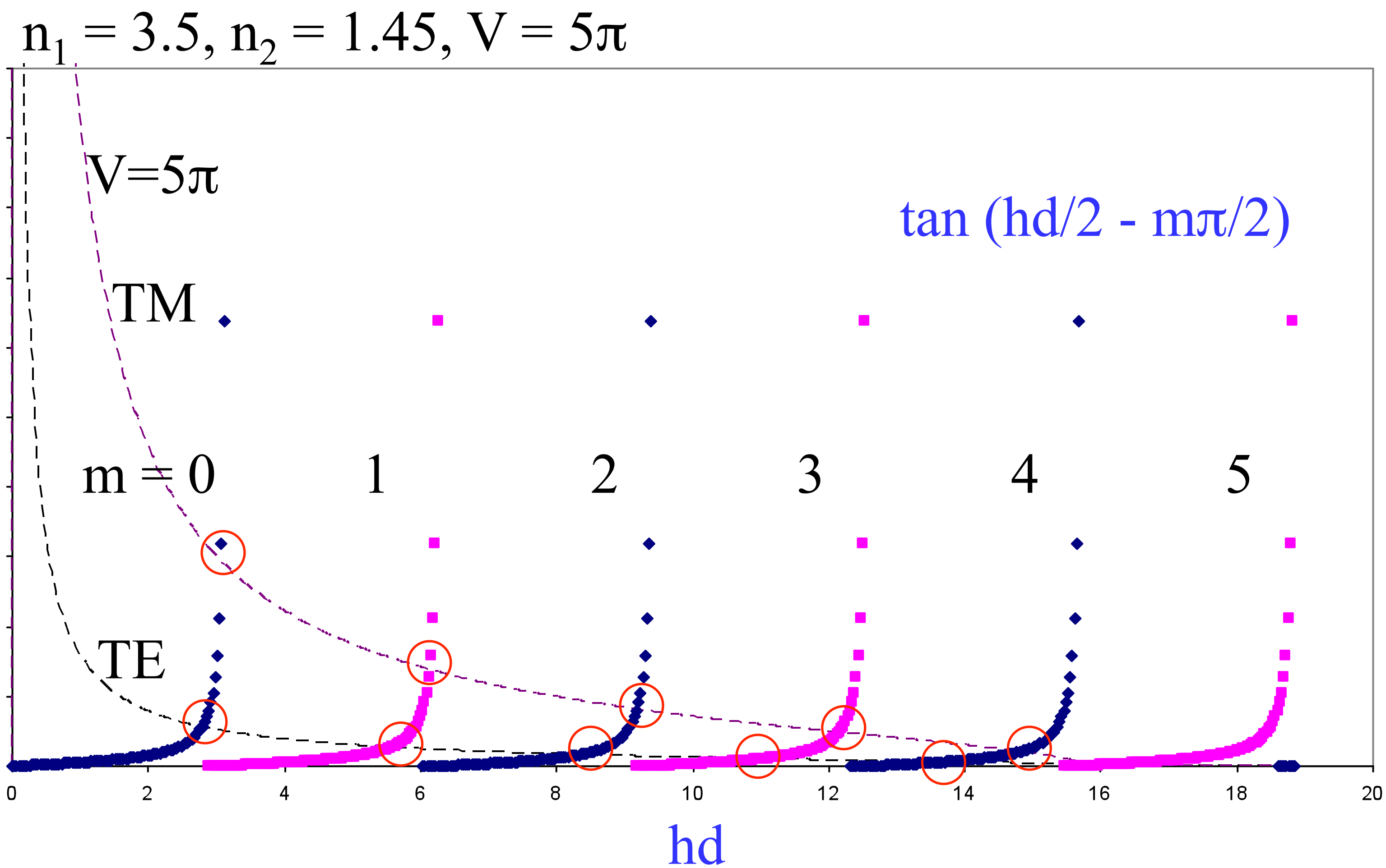
- Here we choose $n_1 = 3.5$ and $n_2 = 1.45$. These values are characteristic of an silicon-on-insulator (SOI) waveguide.

- The critical angle for this structure is $\theta_c = \sin^{-1}(n_2/n_1) \sim 24.5^o$

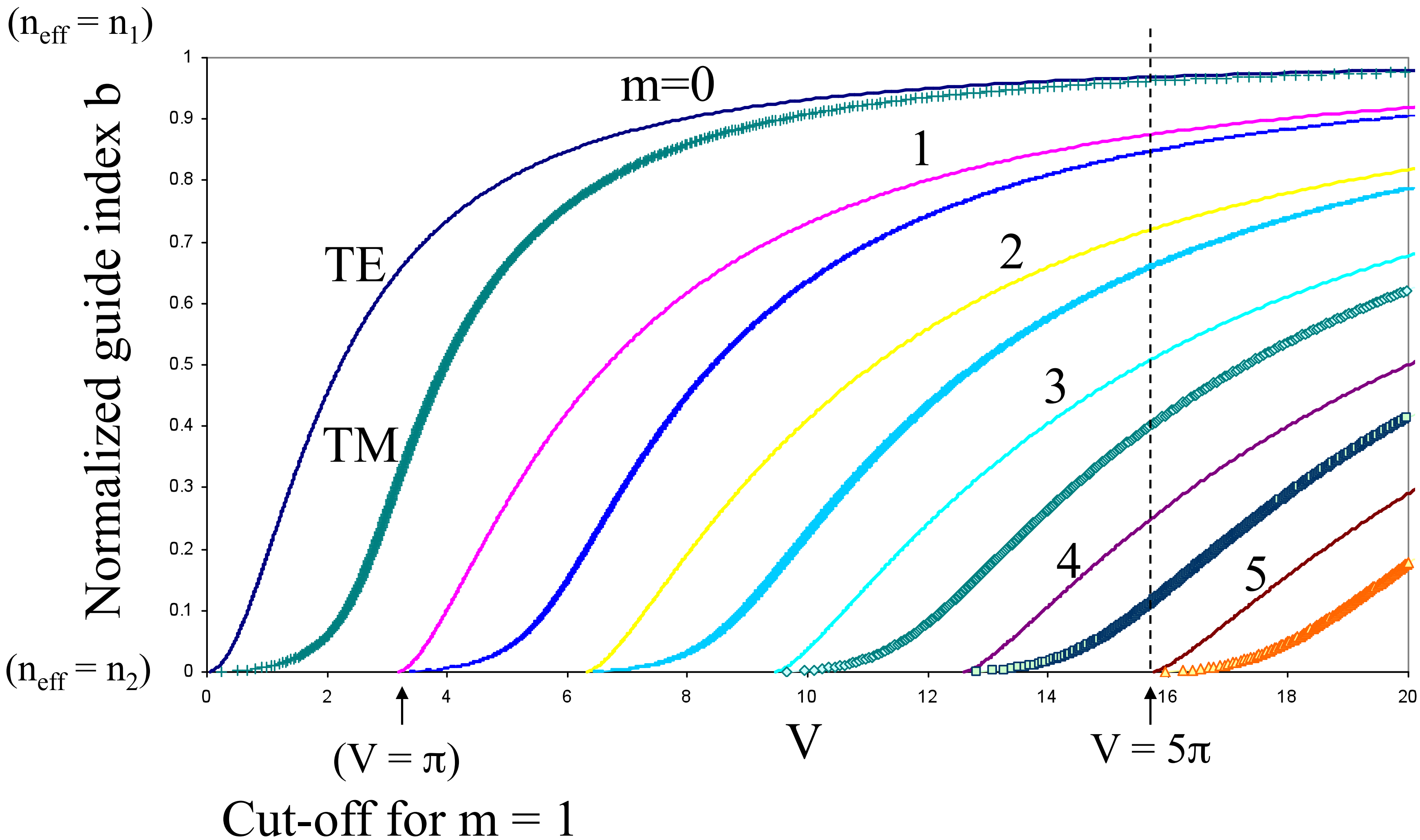
- The range of angles for trapped rays is then $24.5^o \leq \theta \leq 90^o$.

- The range of *waveguide effective refractive index* is $1.45 \leq n_{eff} \leq 3.5$

Graphic solutions for the eigenvalues of guided TE and TM modes of a strongly guiding symmetric slab waveguide



Mode chart for the first six TE and TM modes ($m = 0 – 5$) of symmetric slab waveguides in SOI ($n_1 = 3.5$, $n_2 = 1.45$)



Normalized guide index vs. V number

- When the V number is *very small* (e.g. $d/\lambda << 1$), the guided ray travels *close to the critical angle* ($b << 1$) . The effective index is close to that of the cladding layer n_2.

=>The wave penetrates deeply into the cladding layers, because the rays are near the critical angle. The evanescent decay is slow.

- As the V number increases, the ray travels more nearly parallel to the waveguide axis, and the effective refractive index lies between n_1 and n_2.

- For a *very large* V number (e.g. $d/\lambda >> 1$) the effective index is near that of the core index n_1. The wave in the cladding layer decays very rapidly for evanescent waves traveling at angles far above the critical angle.

Cutoff conditions

- For example, consider V = 15 on the mode chart, the TE_5/TM_5 modes could not propagate because V was not large enough to intersect with the b vs. V curves.

=> The TE_5/TM_5 modes, and *all higher-ordered modes*, are *cut off*.

- *Cutoff* occurs when the propagation angle for a given mode just equals the *critical angle* θ_c --- a guided mode transits to an *unguided* radiation mode.

- This corresponds to the condition that $\beta = k_2$ ($b = 0$) and $\kappa = 0$.

- The fields *extend to infinity* for $\kappa = 0$ (i.e. the fields become unguided!). This defines the *cutoff condition* for guided modes.

Cutoff conditions

- The cutoff value $V = V_c$ for a particular guided mode is the value of V at the point where b = 0 (i.e. the b vs. V relation intersects with the axis b = 0).
- For $\kappa = 0$,

$$V^2 \equiv k_1^2 d^2 - k_2^2 d^2 = k_1^2 d^2 - \beta^2 d^2 = h^2 d^2$$

$$\Rightarrow \quad V_c = hd$$

Substitute this relation to the TE/TM modes eigenvalue equations:

$$\tan\,(hd/2 - m\pi/2) = 0$$

$$\Rightarrow V_c = m\pi,\ m = 0, 1, 2, \ldots$$

Number of modes

- TE and TM modes of a *symmetric* waveguide have the *same* cutoff condition:

$$V_c = m\pi \qquad \text{for the } m_{th} \text{ TE and TM modes}$$

- Because cutoff of the fundamental mode (TE_0/TM_0) occurs at *zero* thickness ($V = 0$), *neither fundamental TE nor fundamental TM mode in a symmetric waveguide has cutoff*.

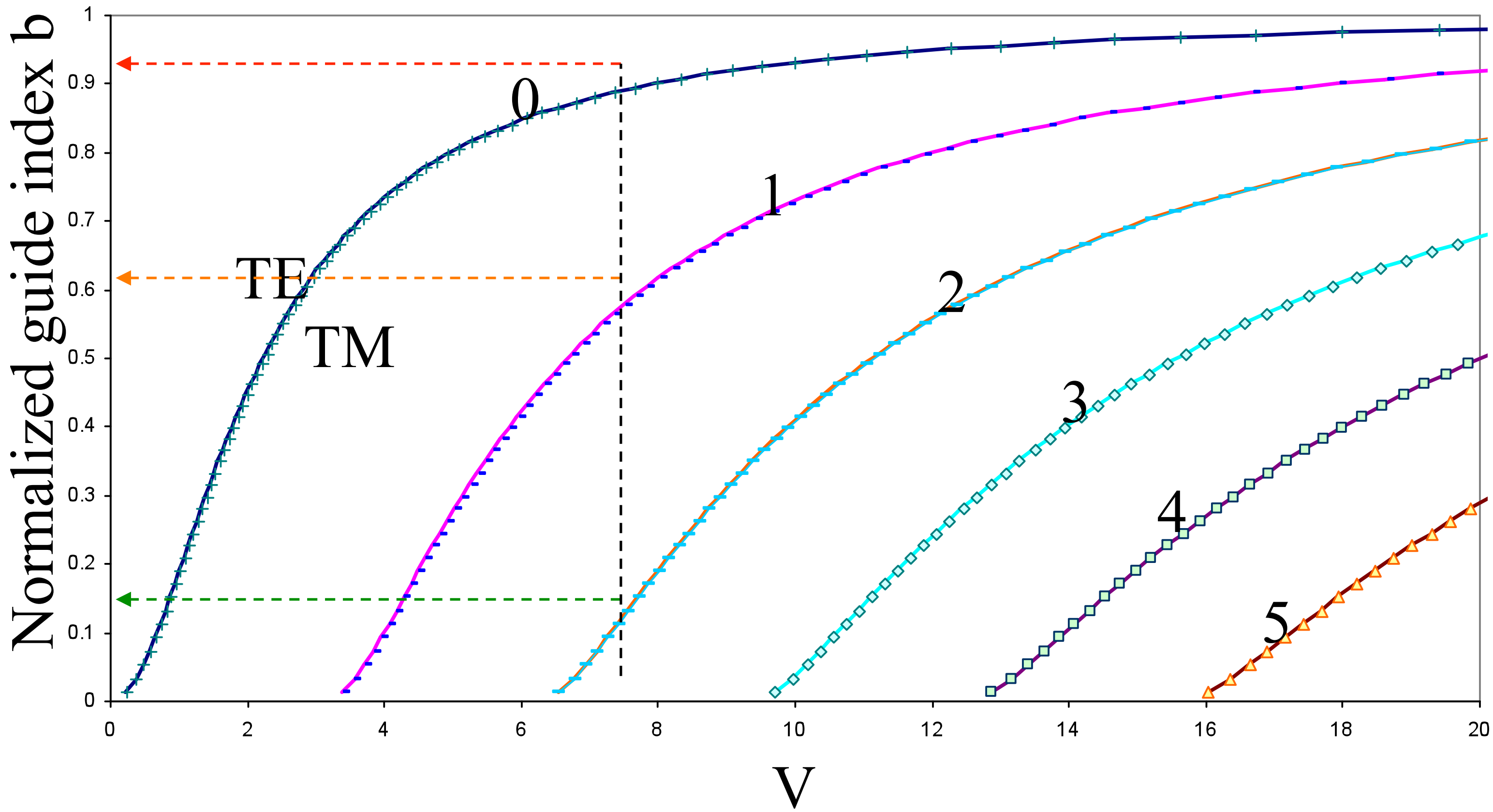
  => Any symmetric dielectric waveguide supports <u>at least one TE</u> and <u>one TM mode</u>.
- The number of TE modes supported by a given symmetric waveguide is the same as that of the TM modes and is

$$M_{TE} = M_{TM} = [V/\pi]_{integer}$$

Nearest integer larger than the bracket value

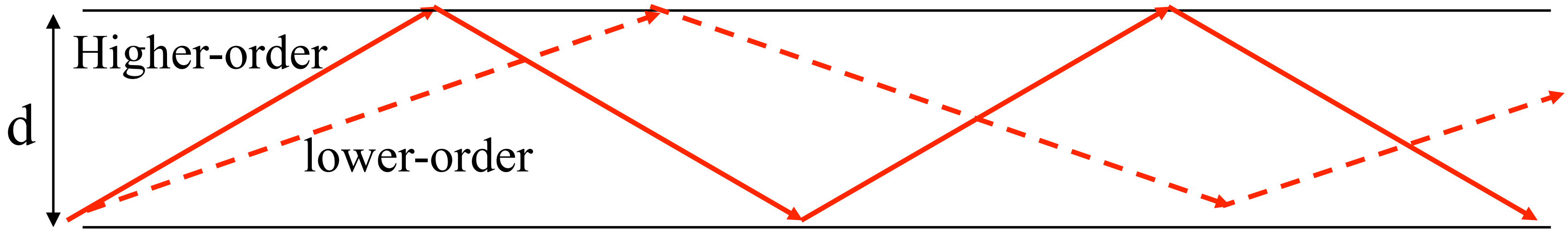
Number of modes

e.g. Find the number of TE/TM modes in an AlGaAs waveguide if d = 1.64 μm. The free-space wavelength is $\lambda = 0.82$ μm. $V = (2\pi d/\lambda)(n_1^2 - n_2^2)^{1/2} \approx 7.5$. $M_{TE} = M_{TM} = [7.5/\pi]_{int} = 3$. For this value, the mode chart yields three solutions – TE_0/TM_0, TE_1/TM_1, and TE_2/TM_2.



Singlemode vs. multimode waveguides

- A <u>multimode waveguide</u> is one that supports more than one propagating mode.
- For a multimode waveguide, at a fixed thickness the *higher*-ordered modes propagate with *smaller* β values than the lower-ordered modes.



*If we wish to propagate <u>only the TE_0/TM_0 mode</u>, then we must have

$$V < \pi$$ single-mode condition

=> This *cuts off* the m = 1 mode and *all* higher-order modes.

Further remarks on TE/TM mode charts

- For the *weakly guiding* AlGaAs waveguide, TE and TM modes have about the same *effective index* and *propagation angle*, but their electric-field vectors point in orthogonal directions.

- Two *modes having the same propagation constant* β are said to be *degenerate*. In the example of AlGaAs waveguide, TE and TM modes of the same order are *nearly* degenerate.

- Even when n_1 is *not* close to n_2 (i.e. high-index contrast), the cutoff values for the TE_m and TM_m mode are still the same ($V_c = m\pi$ *still* applies for the TE/TM modes in high-index contrast waveguides.)

$\Rightarrow$The number of propagating TM modes equals that of the TE modes. (i.e. the *total* number of allowed modes is *twice* the number of modes found from the equation $[V/\pi]_{int}$)

Remarks on modes in asymmetric slab waveguides

- Cutoff of the fundamental mode (TE_0/TM_0) does *not* occur at zero thickness ($V = 0$), as it does for the symmetric case.
- Because the core n_1 and the cladding n_2 and n_3 are all different, the TE and TM modes are *not* degenerate. A *truly singlemode waveguide exists* if the TM_0 mode (but not the TE_0 mode) is cut off.

- ***Integrated optic circuits*** often adopt *asymmetric waveguide structures* and normally *singlemode waveguiding is desirable*
(the *tradeoff* is that the waveguides become polarization dependent and only one polarization mode propagates!)

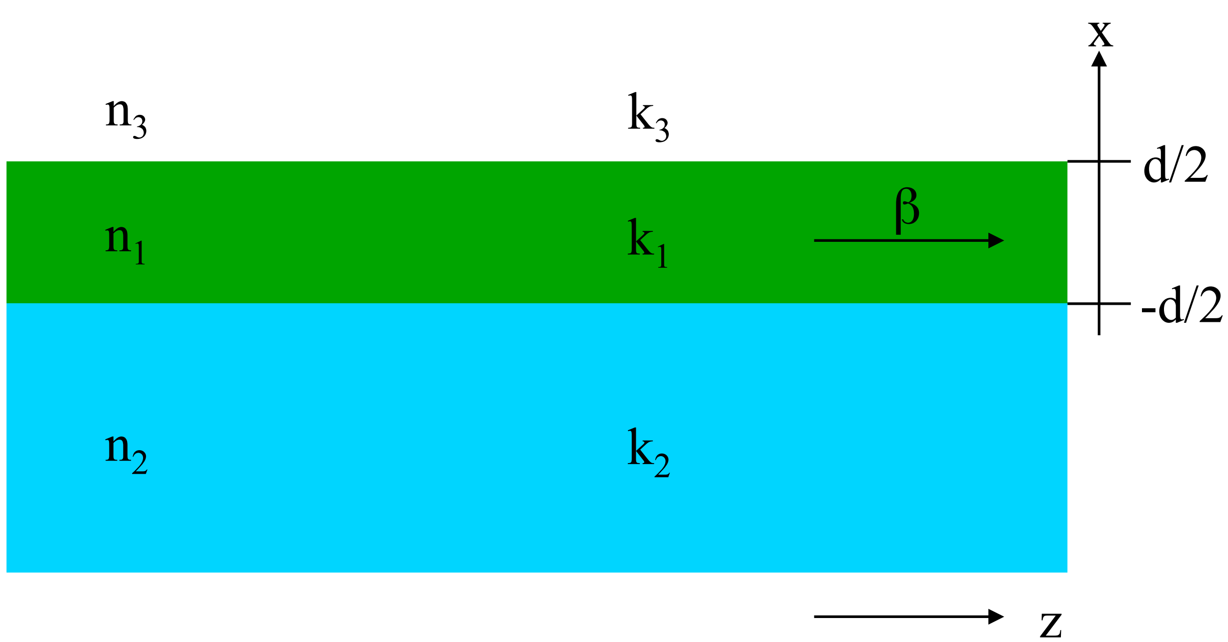
- Mode patterns for the asymmetric waveguide are similar to those of the symmetric waveguide, except that the asymmetry causes the fields to have *unequal* amplitudes at the two boundaries and to decay at different rates in the two cladding layers.

General formalisms for step-index planar waveguides

for asymmetric and symmetric slab waveguides

Step-index planar waveguides

- A step-index planar (slab) waveguide



(assume $n_1 > n_2 > n_3$)

Normalized waveguide parameters

- The measure of the asymmetry of the waveguide is represented by an ***asymmetry factor*** a, which depends on the polarization of the mode. For TE modes, we have

$$a_E = \frac{n_2^{\,2} - n_3^{\,2}}{n_1^{\,2} - n_2^{\,2}}$$

- For TM modes, we have

$$a_M = \frac{n_1^{\,4}}{n_3^{\,4}} \frac{n_2^{\,2} - n_3^{\,2}}{n_1^{\,2} - n_2^{\,2}}$$

- Note that for a given asymmetric structure, $a_M > a_E$.
- For *symmetric* waveguides, $n_3 = n_2$ and $a_E = a_M = 0$.
- V and b have the same definitions as in symmetric waveguides (using refractive indices n_1 and n_2).

Mode parameters

- For a *guided mode*, $k_1 > \beta > k_2 > k_3$. Therefore, positive real parameters h_1, κ_2 and κ_3 exist s.t.

$$k_1^{\,2} - \beta^2 = h_1^{\,2}$$
$$\beta^2 - k_2^{\,2} = \kappa_2^{\,2}$$
$$\beta^2 - k_3^{\,2} = \kappa_3^{\,2}$$

- We can see that $h_1 = k_1 \cos\theta$, which has the meaning of the *transverse* component of the wavevector in the core region of a refractive index n_1.
- For a *guided mode*, the transverse components of the wavevectors in the substrate and upper-cladding regions given by $h_2 = (k_2^2-\beta^2)^{1/2}$ and $h_3 = (k_3^2-\beta^2)^{1/2}$ are purely *imaginary*.

  => the guided field decays exponentially outside the core region

Mode parameters

- For a *substrate radiation mode*, $k_1>k_2>\beta>k_3$. Then h_2 can be chosen to be *real* and *positive*.

$$k_2^{\ 2} - \beta^2 = h_2^{\ 2}$$

- For a *substrate-upper-cladding radiation mode*, $k_1>k_2>k_3>\beta$. Then both h_2 and h_3 are *real* and *positive*.

$$k_3^{\ 2} - \beta^2 = h_3^{\ 2}$$

- The transverse field pattern of a mode is characterized by the transverse parameters h_1, κ_2 (or h_2) and κ_3 (or h_3).

Longitudinal propagation constant β

- Because k_1, k_2 and k_3 are well-defined parameters of a given waveguide, the only parameter that has to be determined for a particular waveguide mode is the ***longitudinal propagation constant*** β.

- Once the value of β is found, the parameters associated with the transverse field pattern are completely determined.

- Therefore, ***a waveguide mode is completely specified by its*** $\boldsymbol{\beta}$.

- Alternatively, if h_1 is determined, the value of its β is also determined by the *k-vector triangle* $k_1^2 - \beta^2 = h_1^2$. It turns out that this is often the approach.

Guided TE modes

- The fields of a TE mode are obtained by solving the *wave equation* for E_y, and then for H_x and H_z.
- The *boundary conditions* require that E_y, H_x and H_z be continuous at the interfaces at $x = \pm d/2$ between layers of different refractive indices. This is equivalent to requiring E_y and $\partial E_y/\partial x$ be continuous at these interfaces.
- For a guided mode, we have to use h_1, κ_2 and κ_3.
- The solutions of the wave equation and the boundary conditions requirement yield the *mode field distribution*

$$E_y = C_{TE}\cos(h_1 d/2-\psi)\exp[\kappa_3(d/2-x)] \qquad x > d/2$$

$$E_y = C_{TE}\cos(h_1 x-\psi) \qquad -d/2 < x < d/2$$

$$E_y = C_{TE}\cos(h_1 d/2+\psi)\exp[\kappa_2(d/2+x)] \qquad x < -d/2$$

Guided TE modes

- And the following *eigenvalue equations*:

$$\tan(h_1 d) = \frac{h_1(\kappa_2 + \kappa_3)}{h_1^2 - \kappa_2\kappa_3}$$

$$\tan(2\psi) = \frac{h_1(\kappa_2 - \kappa_3)}{h_1^2 + \kappa_2\kappa_3}$$

($\tan(2\psi) = 0$ for symmetric waveguides)

- In order to normalize the mode field, we apply the normalization relation to the field.

$$C_{TE} = \sqrt{\frac{\omega\mu_0}{\beta d_E}}$$

where $d_E = d + 1/\kappa_2 + 1/\kappa_3$ is the *effective waveguide thickness* for a guided TE mode.

Guided TM modes

- For a guided TM mode, the solutions of the wave equation and the requirement of the boundary conditions yield the following *mode field distribution*:

$$H_y = C_{TM} \cos(h_1 d/2 - \psi) \exp[\kappa_3 (d/2 - x)] \qquad x > d/2$$

$$H_y = C_{TM} \cos(h_1 x - \psi) \qquad -d/2 < x < d/2$$

$$H_y = C_{TM} \cos(h_1 d/2 + \psi) \exp[\kappa_2 (d/2 + x)] \qquad x < -d/2$$

- and the *eigenvalue equations*:

$$\tan(h_1 d) = \frac{(h_1/n_1^2)(\kappa_2/n_2^2 + \kappa_3/n_3^2)}{(h_1/n_1^2)^2 - \kappa_2\kappa_3/n_2^2 n_3^2}$$

$$\tan(2\psi) = \frac{(h_1/n_1^2)(\kappa_2/n_2^2 - \kappa_3/n_3^2)}{(h_1/n_1^2)^2 + \kappa_2\kappa_3/n_2^2 n_3^2}$$

($\tan(2\psi) = 0$ for symmetric waveguides)

Guided TM modes

- To normalize the mode field,

$$C_{TM} = \sqrt{\frac{\omega \varepsilon_0 n_1^2}{\beta d_M}}$$

where the *effective waveguide thickness* is

$$d_M = d + \frac{1}{\kappa_2 q_2} + \frac{1}{\kappa_3 q_3}$$

and

$$q_2 = \frac{\beta^2}{k_1^2} + \frac{\beta^2}{k_2^2} - 1$$

$$q_3 = \frac{\beta^2}{k_1^2} + \frac{\beta^2}{k_3^2} - 1$$

Modal dispersion

- Guided modes have discrete allowed values of β.
- They are determined by the allowed values of h_1 because β and h_1 are directly related to each other through $k_1^2 - \beta^2 = h_1^2$.
- Because κ_2 and κ_3 are uniquely determined by β, they are also uniquely determined by h_1.
- In terms of the normalized waveguide parameters, we have

$$\kappa_2^2 d^2 = \beta^2 d^2 - k_2^2 d^2 = V^2 - h_1^2 d^2$$

$$\kappa_3^2 d^2 = \beta^2 d^2 - k_3^2 d^2 = (1 + a_E) V^2 - h_1^2 d^2$$

- Therefore, there is *only* one independent variable h_1 in the eigenvalue equations for TE modes and for TM modes.

Modal dispersion

- The solutions of the eigenvalue equation for TE modes yield the allowed parameters for guided TE modes, while those of the eigenvalue equation for TM modes yield the allowed parameters for guided TM modes.
- A ***transcendental equation*** is usually solved graphically by plotting its left- and right-hand sides as a function of h_1d.
- The solutions yield the allowed values of β, or the normalized guide index b, as a function of the parameters *a* and V.
- For a given waveguide, a guided TE mode has a larger propagation constant than the corresponding TM mode of the same order:

$$\beta_m^{TE} > \beta_m^{TM}$$

- For ordinary dielectric waveguides where n_1-n_2 << n_1, the difference is very small.

Modal dispersion

- For a given waveguide, the values of a_E and a_M, and those of d and n_1^2-n_2^2 are fixed.
- Then, β is a function of optical frequency ω because V depends on ω.
- Thus, the propagation constant of a waveguide mode has a frequency dependence contributed by the *structure* of the waveguide in addition to that due to *material* dispersion.
- This extra contribution also causes different modes to have different dispersion properties, resulting in the phenomenon of ***modal dispersion***.
- ***Polarization dispersion*** also exists because TE and TM modes generally have different propagation constants. Polarization dispersion is very small in *weakly guiding waveguides* where n_1-$n_2 << n_1$.

Example

- An *asymmetric* slab waveguide is made of a polymer layer of thickness d = 1 μm deposited on a silica substrate. At 1μm optical wavelength, $n_1 = 1.77$ for the polymer guiding layer, $n_2 = 1.45$ for the silica substrate, and $n_3 = 1$ for the air upper cladding. Find the propagation constants of the guided TE and TM modes of this waveguide. Plot the mode field distributions.

- We find that $V = 6.378$, $a_E = 1.07$, and $a_M = 10.5$

- Also $k_1 = 2\pi n_1/\lambda = 11.12\ \mu m^{-1}$, $k_2 = 2\pi n_2/\lambda = 9.11\ \mu m^{-1}$ and $k_3 = 2\pi n_3/\lambda = 6.28\ \mu m^{-1}$

- To find the propagation constant, the parameter h_1 has to be found by solving the eigenvalue equations.

- To solve the eigenvalue equations, we express κ_2d and κ_3d in terms of h_1d:

$$\kappa_2 d = (V^2 - h_1^{\,2} d^2)^{1/2}$$

$$\kappa_3 d = [(1 + a_E)V^2 - h_1^{\,2} d^2]^{1/2}$$

- Then the eigenvalue equation for the TE mode:

$$\tan(h_1 d) = h_1 d \frac{(V^2 - h_1^{\,2} d^2)^{1/2} + [(1 + a_E)V^2 - h_1^{\,2} d^2]^{1/2}}{h_1^{\,2} d^2 - (V^2 - h_1^{\,2} d^2)^{1/2}[(1 + a_E)V^2 - h_1^{\,2} d^2]^{1/2}}$$

- The eigenvalue equation for the TM mode:

$$\tan(h_1 d) = h_1 d \frac{n_1^{\,2} n_3^{\,2} (V^2 - h_1^{\,2} d^2)^{1/2} + n_1^{\,2} n_2^{\,2} [(1 + a_E)V^2 - h_1^{\,2} d^2]^{1/2}}{n_2^{\,2} n_3^{\,2} h_1^{\,2} d^2 - n_1^{\,4} (V^2 - h_1^{\,2} d^2)^{1/2}[(1 + a_E)V^2 - h_1^{\,2} d^2]^{1/2}}$$

- These equations yield only discrete eigenvalues for given values of waveguide parameters n_1, n_2, n_3, V and a_E.
- They are transcendental equations that have to be solved graphically or numerically.

- With given waveguide parameters, numerical solution yields two eigenvalues for each of the two equations, indicating two guided TE modes (TE_0, TE_1) and two guided TM modes (TM_0, TM_1).
- Once the eigenvalues for h_1d are found, h_1, κ_2 and κ_3 are found. Then, ψ and β can be found.

Cutoff conditions

- κ_2 and κ_3 are real and positive for a guided mode, s.t. the fields of the mode decay exponentially in the transverse direction outside the core region and remain bound to the core.
- This is equivalent to the condition that $\theta > \theta_{c2} > \theta_{c3}$ in the ray optics picture s.t. the ray in the core is totally reflected by both interfaces.
- Because $\theta_{c2} > \theta_{c3}$, the transition from a guided mode to an unguided radiation mode occurs when $\theta = \theta_{c2}$.
- This corresponds to the condition that $\beta = k_2$ and $\kappa_2 = 0$.
- The fields extend to infinity on the substrate side for $\kappa_2 = 0$. This defines the ***cutoff condition*** for guided modes.

Cutoff conditions

- At cutoff, $V = V_c$. The cutoff value V_c for a particular guided mode is the value of V at the point where the curve of its b vs. V dispersion relation intersects with the horizontal axis b = 0.
- By setting $\kappa_2 = 0$, we find from the normalized waveguide parameters relations that

$$h_1 d = V_c \qquad \text{and} \qquad \kappa_3 d = \sqrt{a_E} V_c$$

- For a guided TE mode (*from the eigenvalue equation*)

$$\tan V_c = \sqrt{a_E}$$

- The cutoff condition for the mth guided TE mode is

$$V_m^c = \tan^{-1}\sqrt{a_E} + m\pi \qquad m = 0, 1, 2, \ldots$$

Cutoff conditions

- A similar mathematical procedure yields the cutoff condition for the mth guided TM mode:

$$V_m^{\ c} = \tan^{-1}\sqrt{a_M} + m\pi \qquad \mathrm{m} = 0, 1, 2, \ldots$$

- Because $a_M > a_E$ for a given asymmetric waveguide, the value of V_c for a TM mode is larger than that for a TE mode of the same order.
- We can write

$$V_m^{\ c} = \frac{2\pi}{\lambda_m^{\ c}} d\sqrt{n_1^{\ 2} - n_2^{\ 2}} = \frac{\omega_m^{\ c}}{c} d\sqrt{n_1^{\ 2} - n_2^{\ 2}}$$

where λ_m^c is the *cutoff wavelength*, and ω_m^c is the *cutoff frequency* of the mth mode.

Cutoff conditions

- The mth mode is not guided at a wavelength longer than λ_m^c, or a frequency lower than ω_m^c.
- For given optical wavelength, we can use the cutoff conditions to determine the waveguide parameters that allow the existence of a particular guided mode.
- For given waveguide parameters and optical wavelength, we can use the cutoff conditions to determine the number of guided modes for the waveguide.
- For a given optical wavelength and a waveguide with a given V number, the total number of guided TE/TM modes is

$$M_{TE} = \left[\frac{V}{\pi} - \frac{1}{\pi}\tan^{-1}\sqrt{a_E}\right]_{\text{int}}$$

$$M_{TM} = \left[\frac{V}{\pi} - \frac{1}{\pi}\tan^{-1}\sqrt{a_M}\right]_{\text{int}}$$

The nearest integer larger than the value in the bracket

Singlemode condition

- For *asymmetric* slab waveguides, it is possible to propagate a *single* fundamental TE mode in the case that V satisfies

$$\tan^{-1}\sqrt{a_E} \le V < \tan^{-1}\sqrt{a_M}$$ **Singlemode condition**

- For asymmetric slab waveguides, it also possible to propagate *no* modes at all (i.e. *all modes are cutoff*!) in the case that V satisfies

$$V < \tan^{-1}\sqrt{a_E}$$ **No guided modes**