Coordinate system, Scalar versus Vector fields Operators

- Position of the problem: Coordinate system
- Vector calculus: Basics
- Application to the scalar and vector **fields**

The ultimate goal of manipulating scalar and vector fields is to give a **mathematical view** of electricity and magnetism

Explain Coulomb – Poisson – Biot&Savart – Ampere – Gauss and Maxwell's laws

Position of the problem: Coordinate system

What do we need to evaluate scalar and vector fields at some point in space?

- Localize the source (charge or current distribution) in space
- Define a reference frame
- Localize the point where the effect of the source is to be evaluated
- Make use of **symmetry** if necessary to simplify the problem

Make a choice between

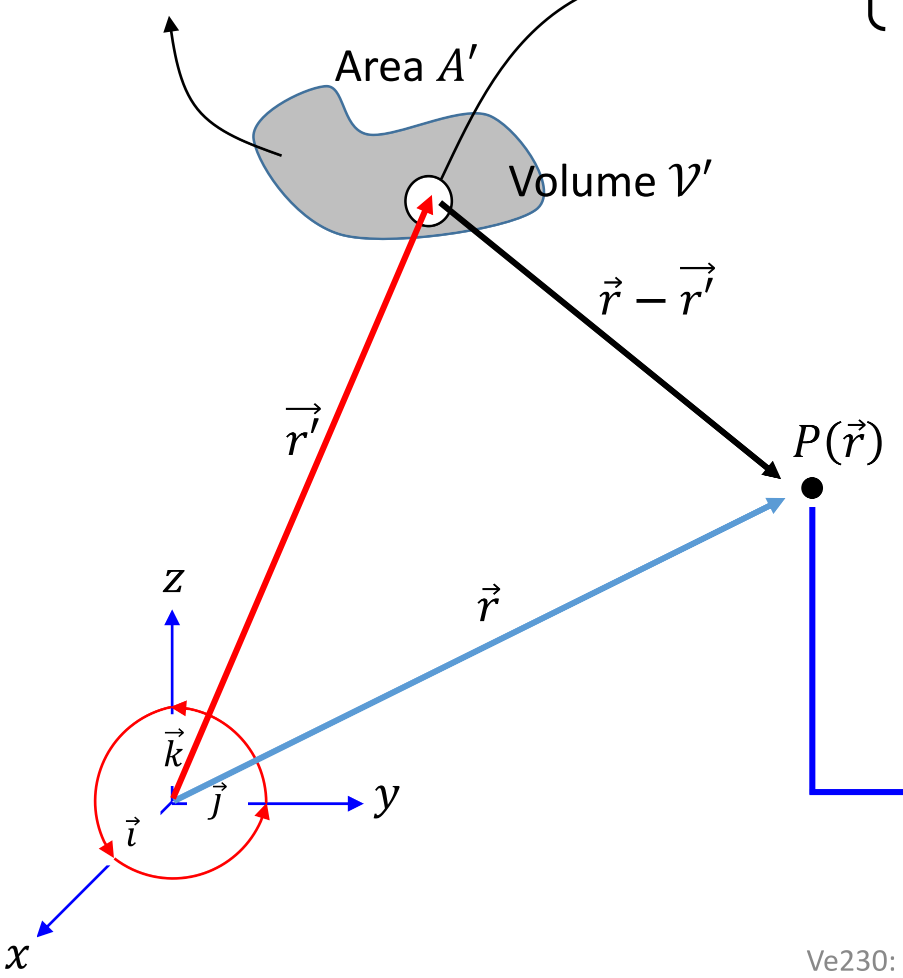
Cartesian **cylindrical** **spherical** representation

<u>Caution:</u> $\mathcal{V}'$ is a volume, V is a potential

Reference frame: localization in space

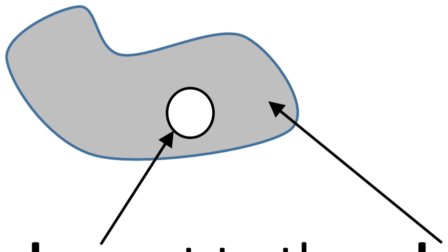
Source
- Charges immobile
- Charges in motion (current)

$$\begin{cases} d\mathcal{V}' \\ dA' \\ dl' \end{cases}$$



Effects

$$\begin{cases} V(\vec{r}) \\ \vec{E}(\vec{r}) \\ \vec{B}(\vec{r}) \\ \vec{A}(\vec{r}) \end{cases}$$



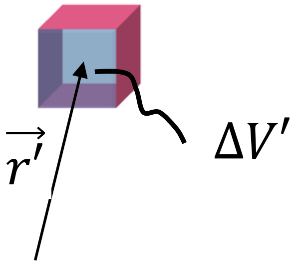
From the **element** to the **whole**
(Superposition principle)

$$\begin{cases} \int_{\mathcal{V}'} Source(\vec{r'}).\, d\mathcal{V}' \\ \int_{A'} Source(\vec{r'}).\, dA' \\ \int_{l'} Source(\vec{r'}).\, dl' \end{cases}$$

Operators: $\boxed{\text{Act on } \vec{r} \textbf{ NOT } \text{on } \vec{r'}}$

- Gradient: $\vec{\nabla} V(\vec{r})$ = Vector
- Divergence: $\vec{\nabla}.\, \vec{E}(\vec{r})$ = Scalar
- Curl: $\vec{\nabla} \times \vec{E}(\vec{r})$ = Vector

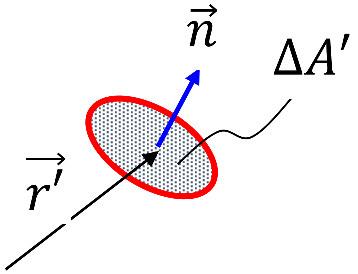
Charge and current distribution



$$\rho\left(\vec{r'}\right) = \lim_{\Delta\mathcal{V}' \to 0} \frac{Q_{\Delta\mathcal{V}'}}{\Delta\mathcal{V}'}$$

$\Delta\mathcal{V}'$ = **<u>volume</u>** element around $\vec{r'}$

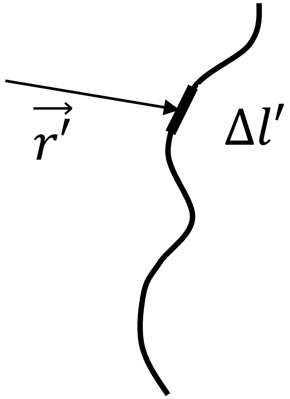
$Q_{\Delta\mathcal{V}'}$ = total charge enclosed in $\Delta V'$



$$\sigma\left(\vec{r'}\right) = \lim_{\Delta A' \to 0} \frac{Q_{\Delta A'}}{\Delta A'}$$

$\Delta A'$ = **<u>surface</u>** element around $\vec{r'}$

$Q_{\Delta A'}$ = total charge distributed on $\Delta A'$



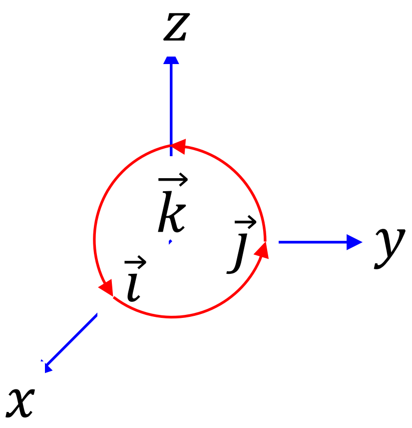
$$\lambda\left(\vec{r'}\right) = \lim_{\Delta l' \to 0} \frac{Q_{\Delta l'}}{\Delta l'}$$

$\Delta l'$ = **<u>line</u>** element around $\vec{r'}$

$Q_{\Delta A'}$ =total charge distributed along $\Delta l'$

Reference frame

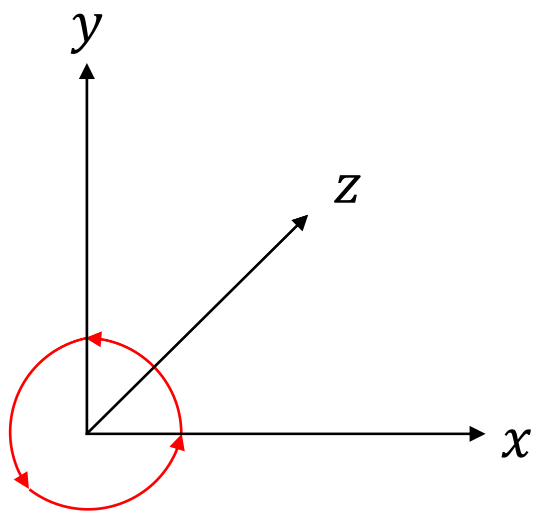
Rule



$\vec{\imath} \times \vec{\jmath} = \vec{k}$

$\vec{\jmath} \times \vec{k} = \vec{\imath}$

$\boxed{\vec{k} \times \vec{\imath} = \vec{\jmath}}$

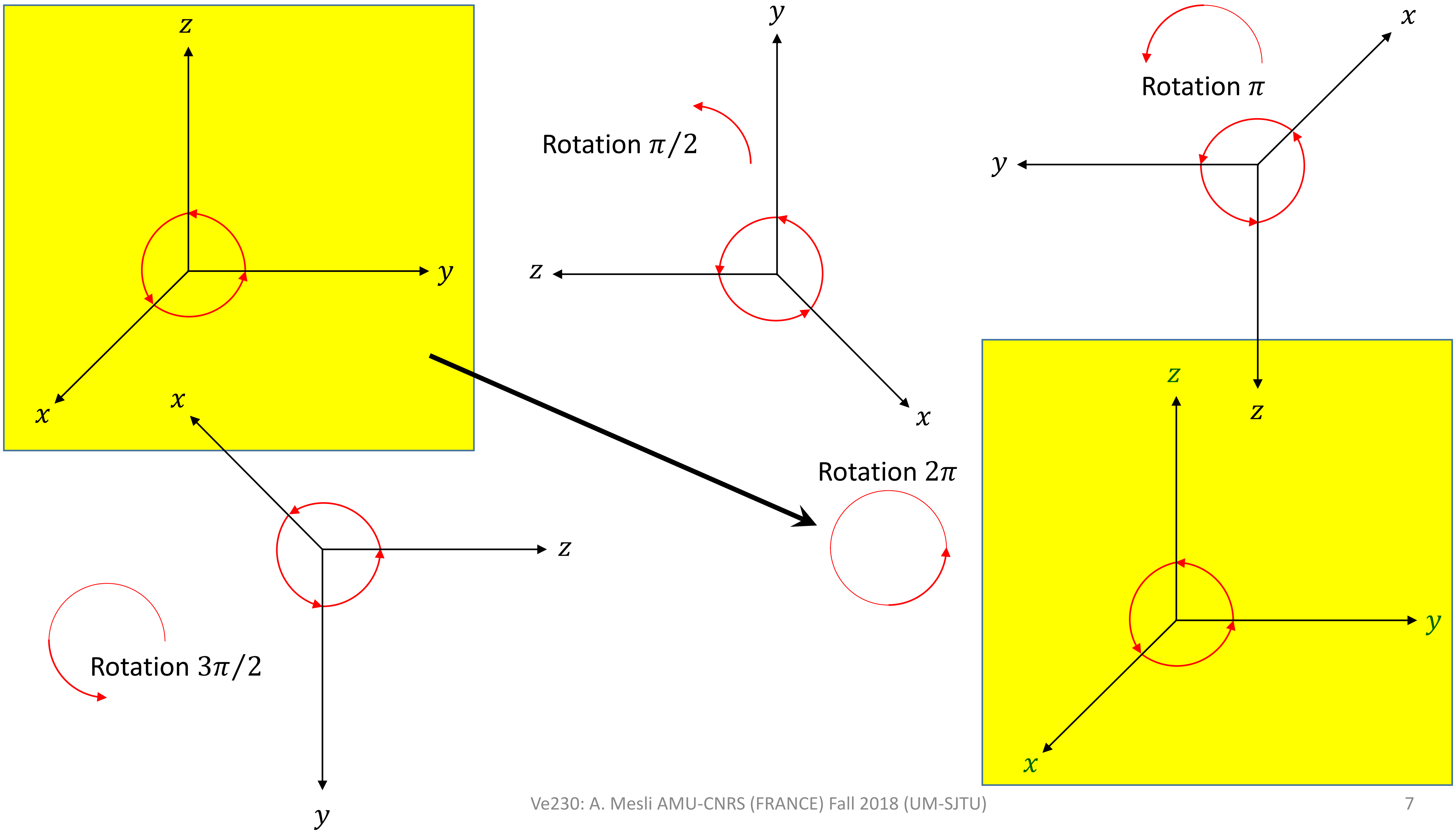


Forbidden

*** Question #1:** Why is this frame forbidden?

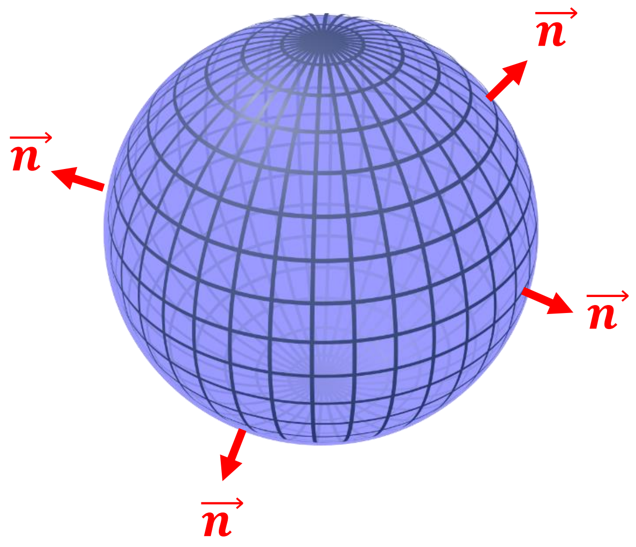
Answer to *Question 1 According to frame $\vec{\imath} \times \vec{k} = \vec{\jmath}$

According to the rule $\vec{k} \times \vec{\imath} = \vec{\jmath} \Rightarrow \vec{\imath} \times \vec{k} = -\vec{\jmath}$

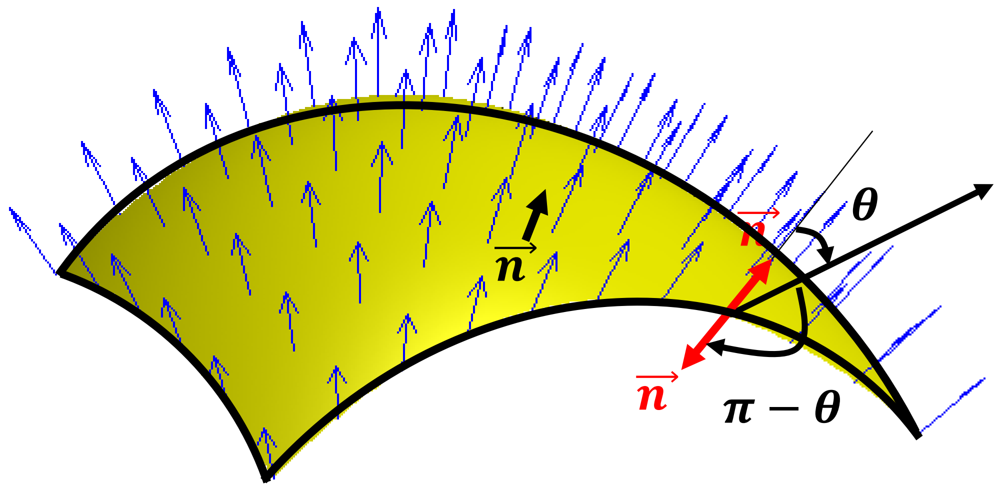

z
y
x
Rotation $\pi/2$
y
z
x
Rotation π
x
y
z
x
z
Rotation 2π
z
y
x
Rotation $3\pi/2$
y

Closed versus open surface

Closed surface



Open surface



$$cos\theta = -\cos(\pi - \theta)$$

Unit vector is always directed outwards

Both orientations of the unit vector are valid

Closed surface defines a volume

Open surface defines a closed path

Divergence and Gauss theorem

Curl and Stokes theorem

BUT

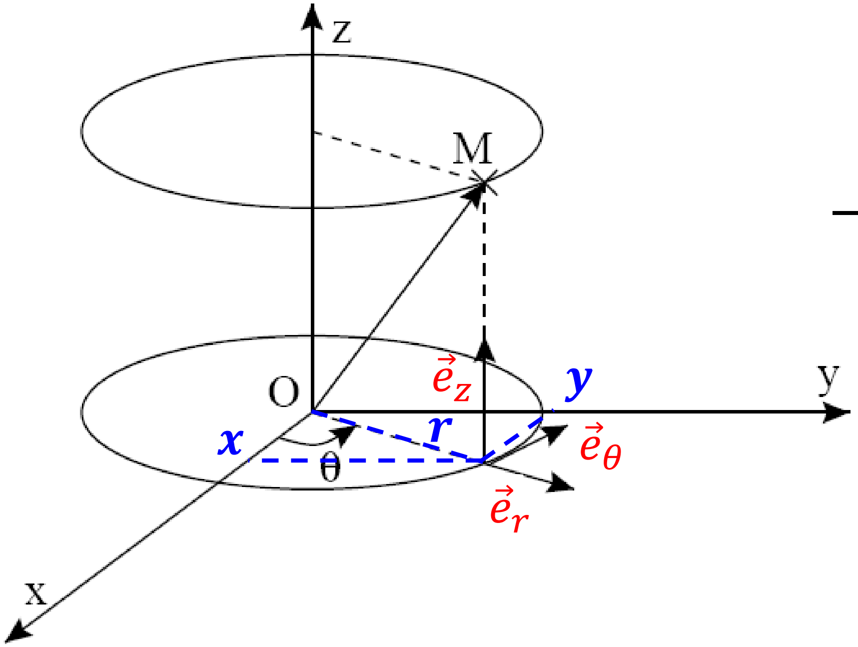
When the open surface closes

Divergence
Gauss theorem

↔

Curl
Stokes theorem

Surface and volume element in cylindrical coordinates

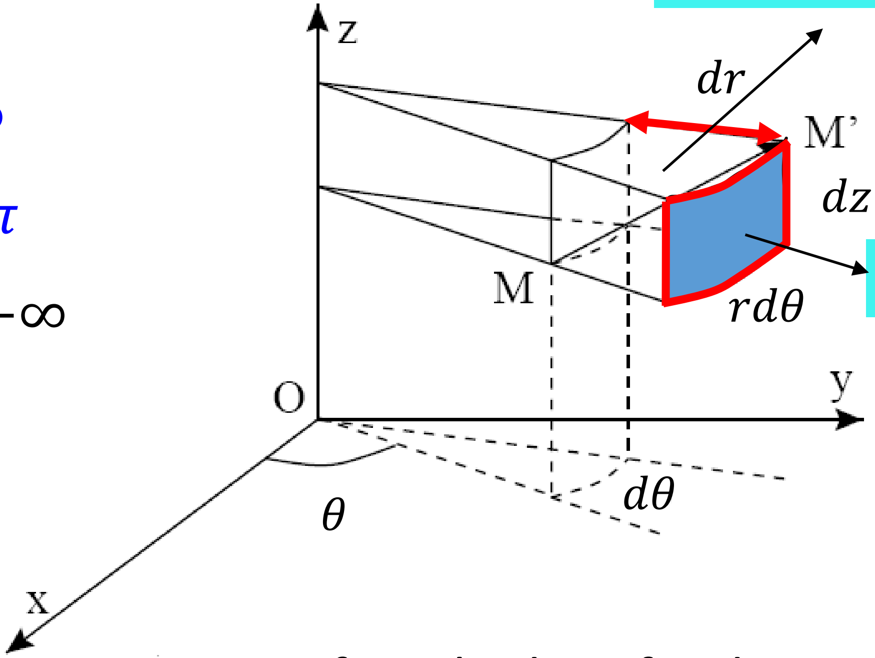


$0 < r < \infty$

$0 < \theta < 2\pi$

$-\infty < z < +\infty$

$dV = dr(rd\theta)dz$



$dA = (rd\theta)dz$

$x = r\cos\theta$

$y = r\sin\theta$

Polar coordinates

$z = z$

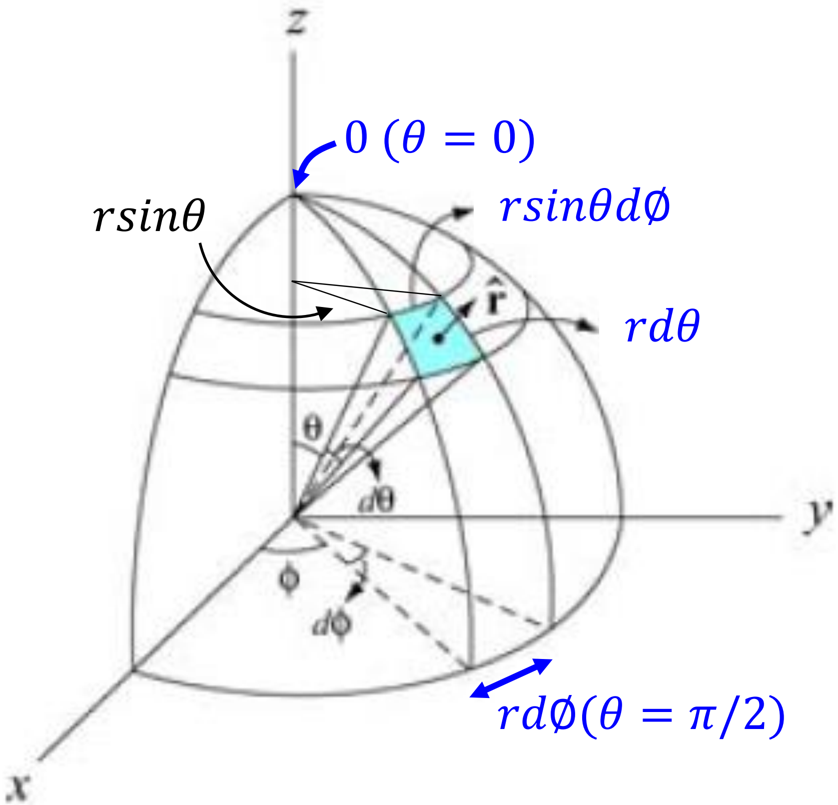
Area of a cylinder of radius R and height H

$$dA = (Rd\theta)dz \Rightarrow A = \int_0^{2\pi} Rd\theta \int_0^H dz = 2\pi RH$$

Volume of a cylinder of radius R and height H

$$dV = dr(rd\theta)dz \Rightarrow V = \int_0^R rdr \int_0^{2\pi} d\theta \int_0^H dz = \pi R^2 H$$

Surface and volume element in spherical coordinates



$0 < \theta < \pi$
$0 < \emptyset < 2\pi$

$x = rsin\theta cos\phi$
$y = rsin\theta sin\phi$
$z = rcos\theta$

$$dA = (rsin\theta d\phi)(rd\theta)$$

$$dA = r^2 sin\theta d\theta d\phi$$

$$dV = (dA)dr$$

$$dV = r^2 sin\theta dr d\theta d\phi$$

Area of a sphere of radius R

$$A = R^2 \int_0^{\pi} sin\theta d\theta \int_0^{2\pi} d\emptyset = 4\pi R^2$$

Volume of a sphere of radius R

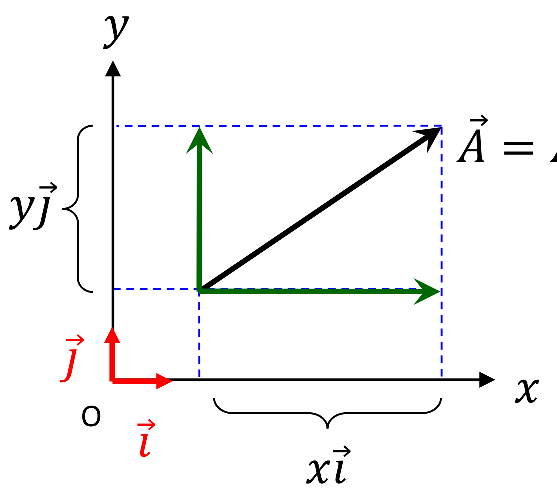
$$\mathcal{V} = \int_0^{R} r^2 dr \int_0^{\pi} sin\theta d\theta \int_0^{2\pi} d\emptyset$$

$$= \frac{4}{3}\pi R^3$$

Vector calculus: Basics

Vector and representation

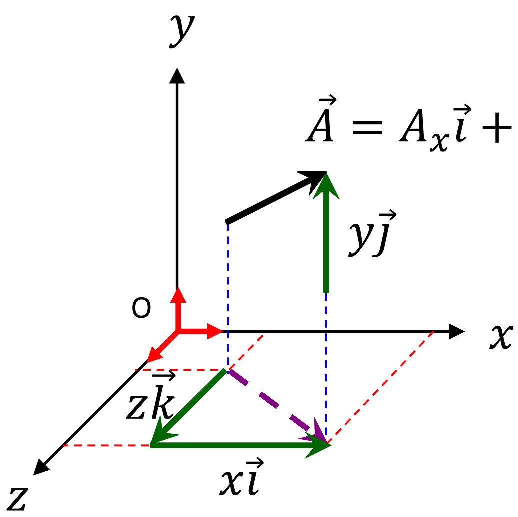
Representation in 2D **2 components**



$$\vec{A} = A_x\vec{i} + A_y\vec{j}$$

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2}$$

Representation in 3D **3 components**



$$\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{j}$$

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Unit vectors $\vec{i}$**,** $\vec{j}$ and $\vec{k}$ are unit vectors of magnitude 1 in Cartesian frame in direction along x, y, z axes

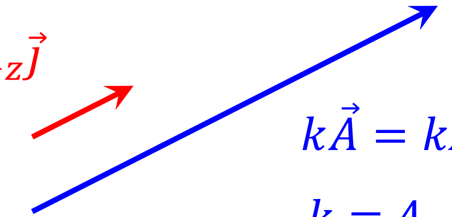
$\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$, $\vec{k} = (0,0,1)$

A_x, A_y, A_z are the components along the three axes

$|\vec{A}|$= **Magnitude** of the vector

Multiplying a vector by a scalar

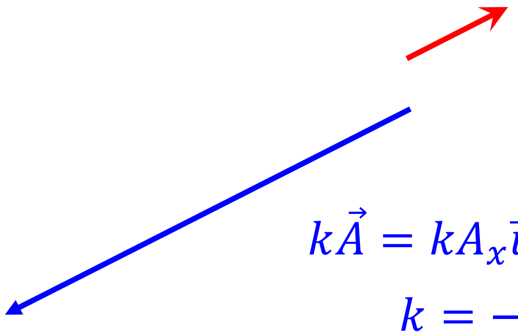
$\vec{A} = A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{\jmath}$



$k\vec{A} = kA_x\vec{\imath} + kA_y\vec{\jmath} + kA_z\vec{\jmath}$

$k = 4$

$\vec{A} = A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{\jmath}$



$k\vec{A} = kA_x\vec{\imath} + kA_y\vec{\jmath} + kA_z\vec{\jmath}$

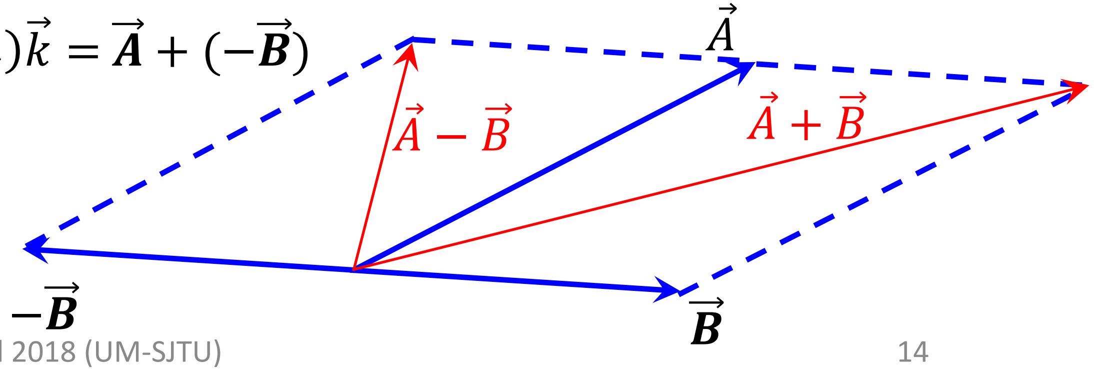
$k = -4$

Vectors $\vec{A}$ and $k\vec{A}$ are co-linear (parallel)

Addition and subtraction of vectors

$\vec{A} = A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{\jmath}$ $\quad$ $\vec{A} + \vec{B} = (A_x + B_x)\vec{\imath} + (A_y + B_y)\vec{\jmath} + (A_z + B_z)\vec{k}$

$\vec{B} = B_x\vec{\imath} + B_y\vec{\jmath} + B_z\vec{\jmath}$ $\quad$ $\vec{A} - \vec{B} = (A_x - B_x)\vec{\imath} + (A_y - B_y)\vec{\jmath} + (A_z - B_z)\vec{k} = \vec{\boldsymbol{A}} + (-\vec{\boldsymbol{B}})$



Scalar product

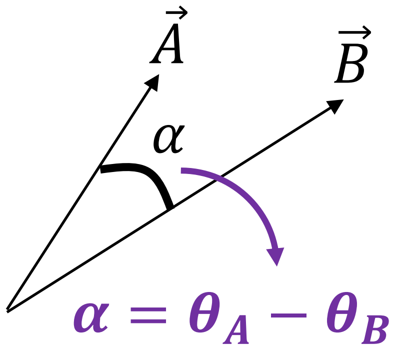
Cartesian vs polar coordinates

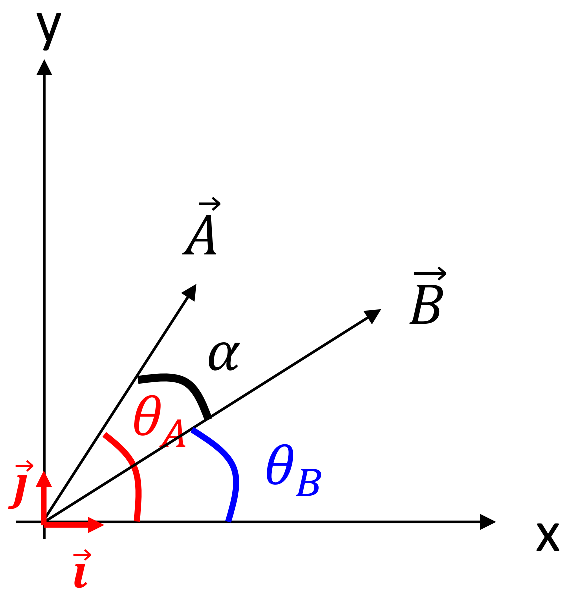
$\vec{A} = A_x\vec{i} + A_y\vec{j} \longrightarrow$ Cartesian

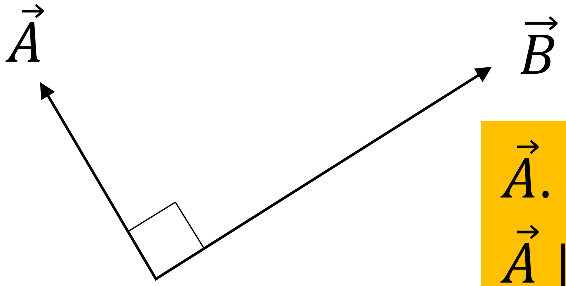
$\vec{A} = |\vec{A}|cos\theta_A\vec{i} + |\vec{A}|sin\theta_A\vec{j} \longrightarrow$ Polar

$$\vec{B} = B_x\vec{i} + B_y\vec{j}$$

$$\vec{B} = |\vec{B}|cos\theta_B\vec{i} + |\vec{B}|sin\theta_B\vec{j}$$







$$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|cos\theta_A cos\theta_B + |\vec{A}||\vec{B}|sin\theta_A sin\theta_B$$

$$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|[cos\theta_A cos\theta_B + sin\theta_A sin\theta_B]$$

$$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|[cos(\theta_A - \theta_B)] = |\vec{A}||\vec{B}|\cos\alpha$$

$$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|\cos\alpha$$

$$\vec{A} \parallel \vec{B} \Longrightarrow \vec{A}.\vec{B} = |\vec{A}||\vec{B}|$$

$$\vec{A} \perp \vec{B} \Longrightarrow \vec{A}.\vec{B} = 0$$

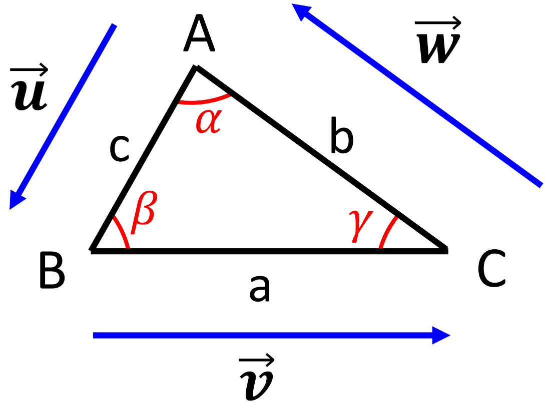
$$\boldsymbol{cos(\theta_A - \theta_B) = cos\theta_A cos\theta_B + sin\theta_A sin\theta_B}$$

Example: Al-Kashi's theorem

Given a triangle ABC, demonstrate on the basis of vector calculus Al-Kashi's theorem which states that:

$$a^2 = b^2 + c^2 - 2bccos(\alpha)$$

Where a is for BC, b for AC and c for AB and α is the angle $\widehat{BAC}$ etc…



$$\vec{u} + \vec{v} = -\vec{w}$$

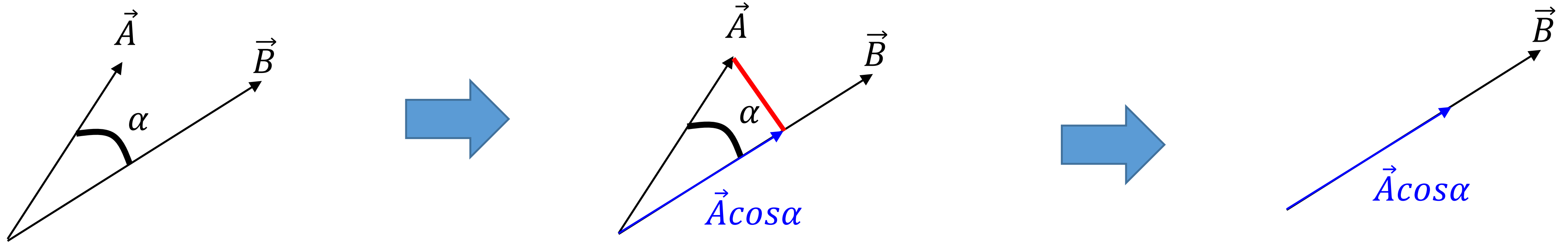
$$a^2 = \vec{v}.\vec{v} = (-\vec{w} - \vec{u})^2 = w^2 + v^2 + 2uwcos(\pi + \alpha)$$

$$a^2 = b^2 + c^2 - 2bccos(\alpha)$$

Vectors turning counterclockwise

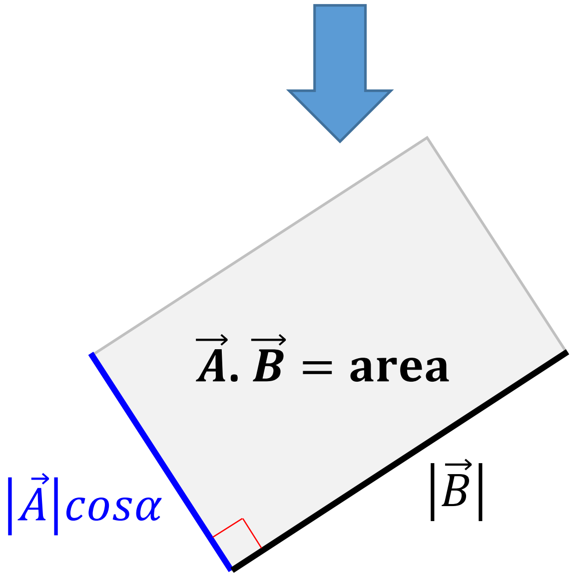
Pythagoras's theorem ⇨ $\alpha = \pi/2$ ⇨ $a^2 = b^2 + c^2$

Geometrical interpretation of Scalar product



The scalar product defines an area

$$\vec{A}.\vec{B} = \vec{B}.\vec{A}$$
$$\vec{A} \perp \vec{B} \Longrightarrow \vec{A}.\vec{B} = 0$$
$$\vec{A} \parallel \vec{B} \Longrightarrow \vec{A}.\vec{B} = |\vec{A}||\vec{B}|$$



The area may involve other dimensions than surface !

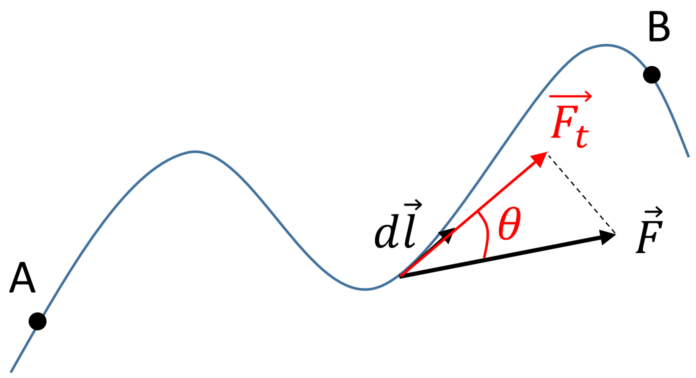
Example of scalar product in physics

***Question #2** What major physical quantity is expressed by scalar product?

Answer to *Question #2 Work done by a force on a particle moving along a path AB

$$dW = \vec{F}.d\vec{l} = F cos\theta dl$$

$$W = \int_A^B F cos\theta dl = \int_A^B F_t dl$$



Here the area has a new dimension: Energy

Line integral

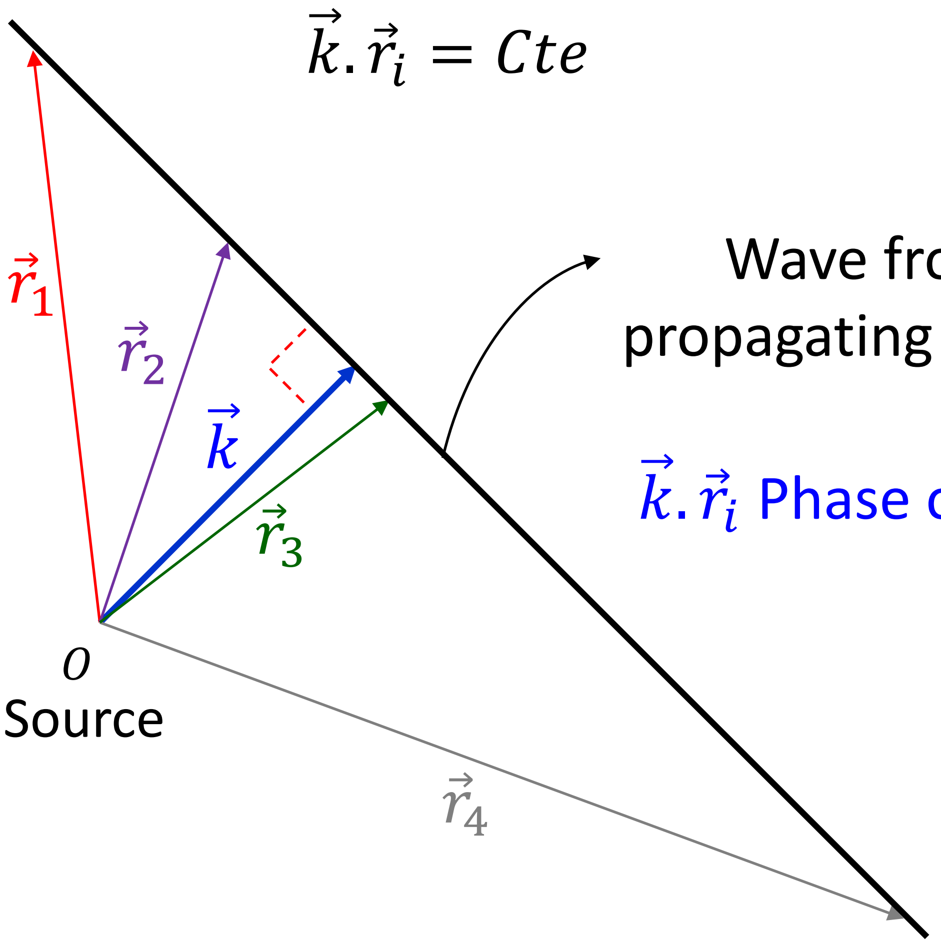
Close the path

Circulation

***Question #3** What is the value of the circulation in this case?

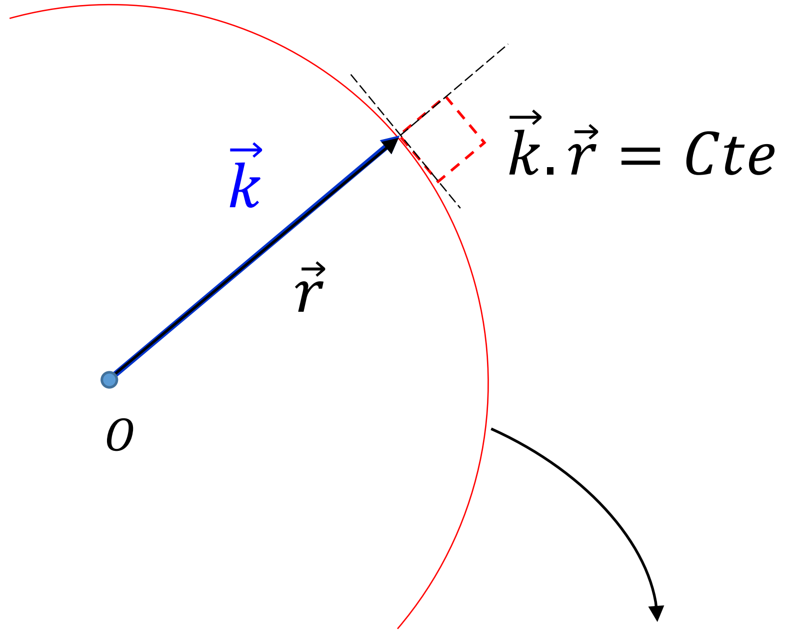
Answer to *Question #3 $= \mathbf{0}$: force is conservative ⇨ dW is exact differential

A very important property of scalar product in electromagnetism



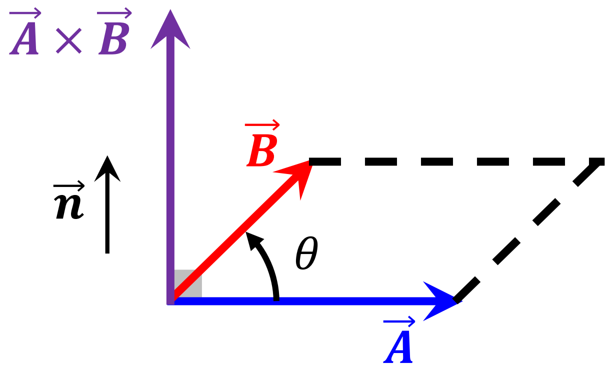
Wave front of a propagating <u>plane wave</u>

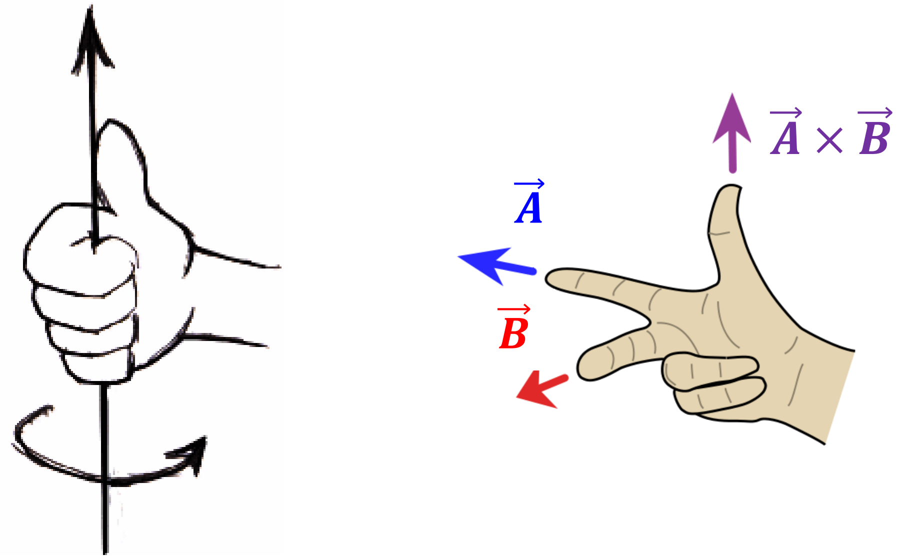
$\vec{k}.\vec{r}_i$ Phase of the wave



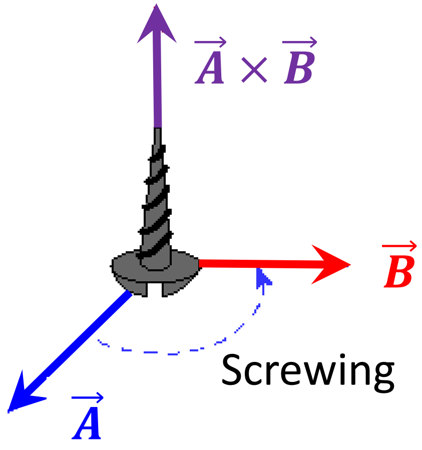
Wave front of a propagating <u>spherical wave</u>

Cross product





Right hand rule



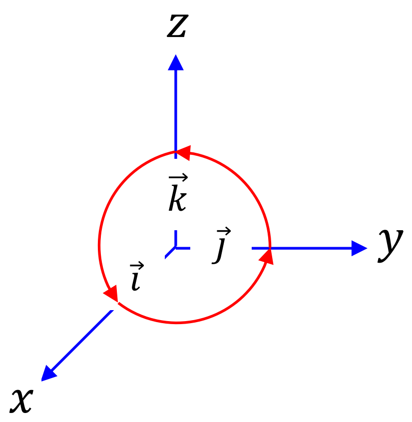
Screwing and unscrewing rule

- Screwing ⇨ $\vec{A} \times \vec{B}$
- Unscrewing ⇨ $\vec{B} \times \vec{A} = -\vec{A} \times \vec{B}$

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| sin\theta . \vec{n}$$

The vector “cross product” is **always** perpendicular to the plan containing the vectors $\vec{A}$ and $\vec{B}$

Rules



Scalar product

$$\vec{i}.\vec{i} = 1$$

$$\vec{j}.\vec{j} = 1$$

$$\vec{k}.\vec{k} = 1$$

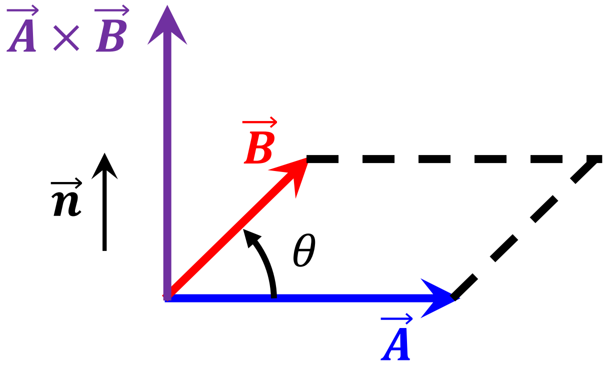
$$\vec{i}.\vec{j} = \vec{j}.\vec{k} = \vec{k}.\vec{i} = 0$$

Cross product

$$\boxed{\begin{aligned} \vec{i} \times \vec{j} &= \vec{k} \\ \vec{j} \times \vec{k} &= \vec{i} \\ \vec{k} \times \vec{i} &= \vec{j} \end{aligned}}$$

$$\boxed{\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}}$$

Calculating Cross product in Cartesian coordinates

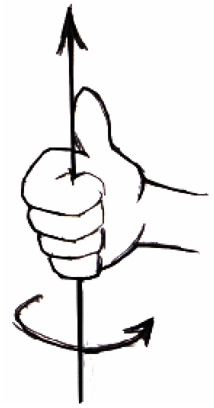


Cartesian notation

$\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k}$

$\vec{B} = B_x\vec{i} + B_y\vec{j} + B_z\vec{k}$

+



$$\vec{A} \times \vec{B} = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{pmatrix} = \begin{pmatrix} A_y & A_z \\ B_y & B_z \end{pmatrix}\vec{i} - \begin{pmatrix} A_x & A_z \\ B_x & B_z \end{pmatrix}\vec{j} + \begin{pmatrix} A_x & A_y \\ B_x & B_y \end{pmatrix}\vec{k}$$

$$\vec{A} \times \vec{B} = (A_yB_z - A_zB_y)\vec{i} - (A_xB_z - A_zB_x)\vec{j} + (A_xB_y - A_yB_x)\vec{k}$$

Scalar versus Vector product

Scalar product

$\vec{A}.\vec{B} = \vec{B}.\vec{A}$

$\vec{A} \perp \vec{B} \Longrightarrow \vec{A}.\vec{B} = 0$

$\vec{A} \parallel \vec{B} \Longrightarrow \vec{A}.\vec{B} = |\vec{A}||\vec{B}|$

$\boldsymbol{Cos}$ is involved

Cross product

$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

$\vec{A} \perp \vec{B} \Longrightarrow |\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|$

$\vec{A} \parallel \vec{B} \Longrightarrow \vec{A} \times \vec{B} = \vec{0}$

$\boldsymbol{Sin}$ is involved

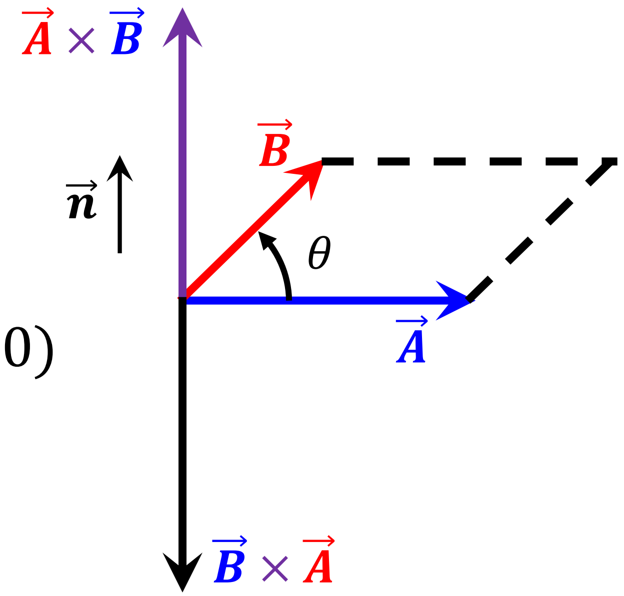
$$\vec{A} \times \vec{A} = \vec{0} \qquad (\theta = 0 \text{ then } \sin(\theta) = 0)$$

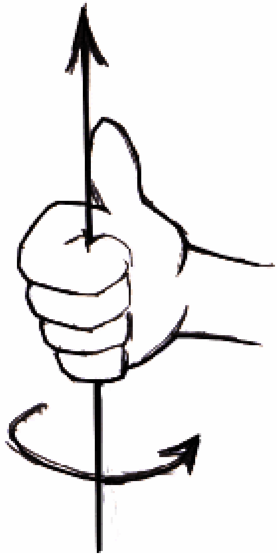
$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

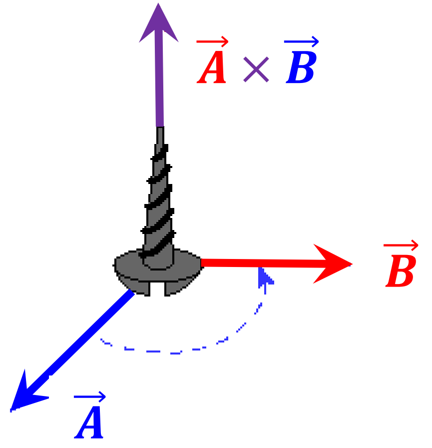
$$\boxed{\vec{A}.\left(\vec{A} \times \vec{B}\right) \ = \mathbf{0}}$$

Scalar product $\quad (\theta = \pi/2 \text{ then } \cos(\theta) = 0)$

$$\boxed{\vec{B}.\left(\vec{A} \times \vec{B}\right) \ = \mathbf{0}}$$







A property that did not escape to Maxwell's attention !

Final exam Fall 2017

« Chance favors prepared minds » (Louis Pasteur)

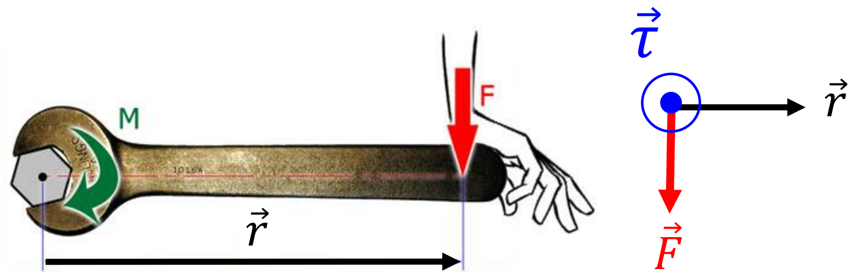
****Question #3

Give four examples in physics involving cross product:
Two from classical mechanics and two from Vp260

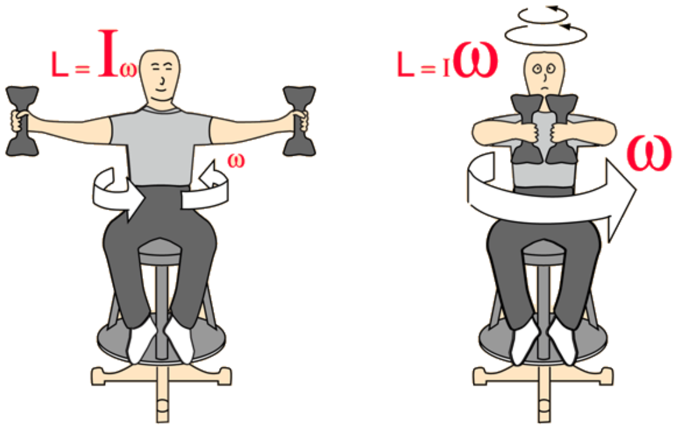
Answer to ****Question #3

From classical mechanics

Torque : $\vec{\tau} = \vec{r} \times \vec{F}$



Angular Momentum: $\vec{L} = \vec{r} \times m\vec{v}$

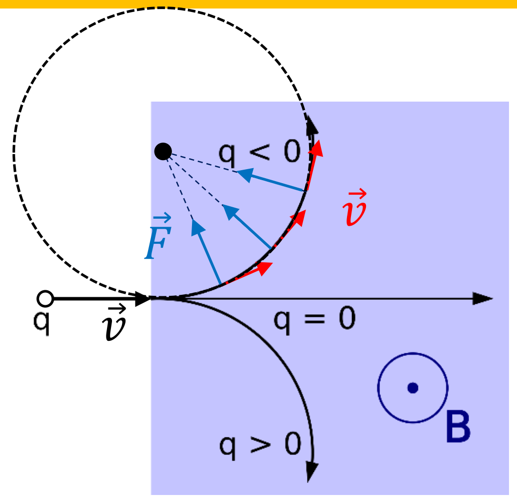


Example of cross product

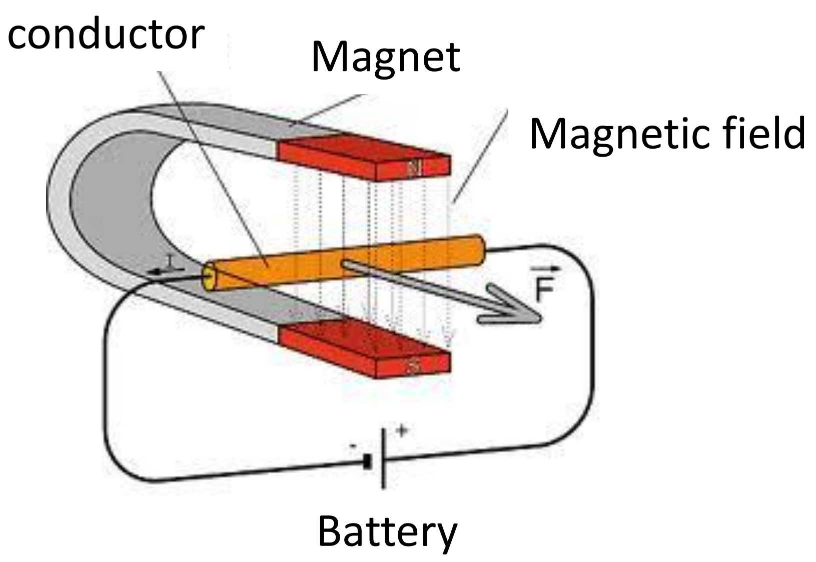
Answer to ****Question #3

From magnetostatic

Lorentz force: $\vec{F} = q\vec{v} \times \vec{B}$



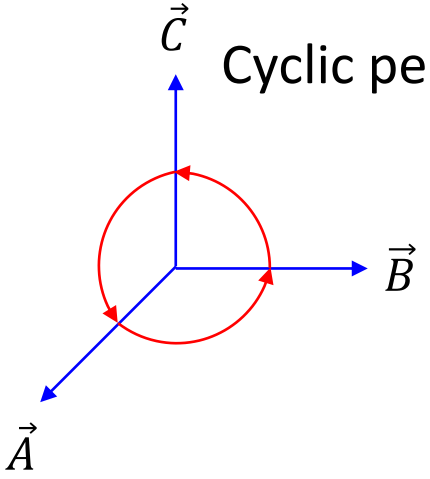
Laplace force: $\vec{F} = I\vec{l} \times \vec{B}$



Some geometrical interpretation of Cross product

Scalar triple product

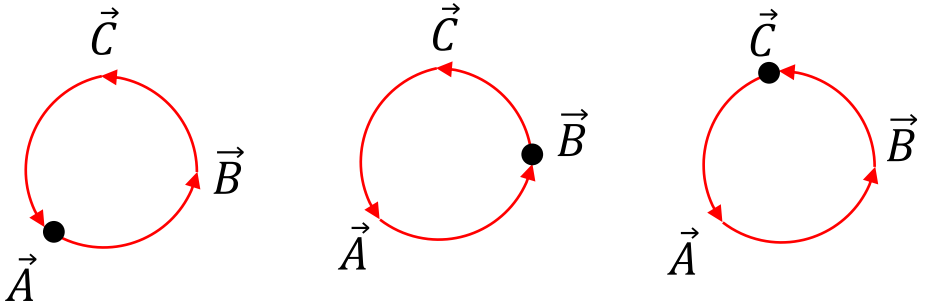
$$\vec{\boldsymbol{A}}.(\vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}})$$



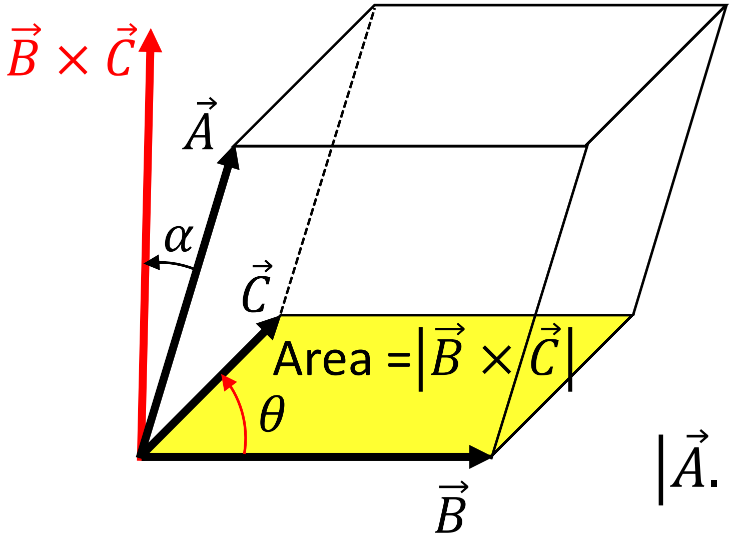
Cyclic permutation

$$\vec{A}.(\vec{B} \times \vec{C}) = \vec{B}.(\vec{C} \times \vec{A}) = \vec{C}.(\vec{A} \times \vec{B})$$

Commutativity of dot product



Counterclockwise

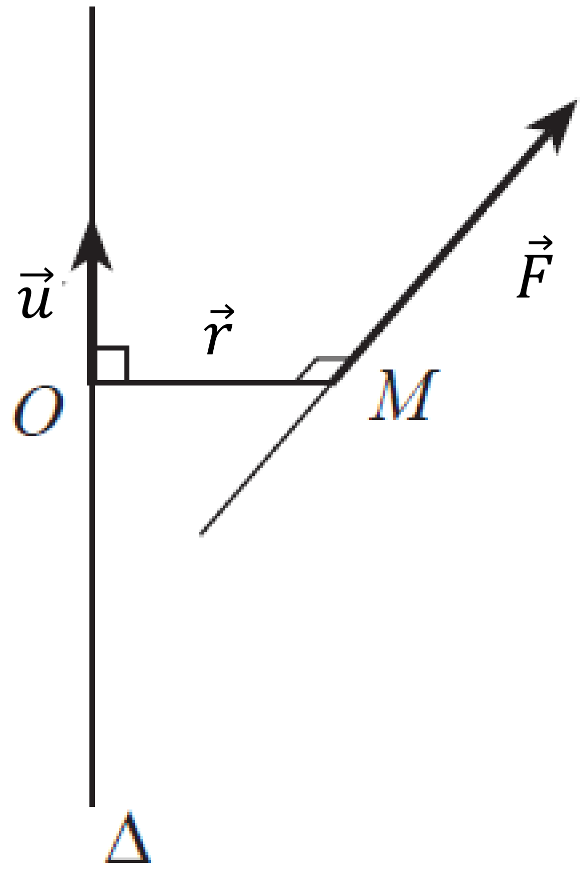


$$\vec{A}.(\vec{B} \times \vec{C}) = \begin{pmatrix} A_X & A_Y & A_Z \\ B_X & B_Y & B_Z \\ C_X & C_Y & C_Z \end{pmatrix} = V$$

If $\alpha = 0$ and $\theta = \pi/2$ $\rightarrow$ Cube

$|\vec{A}.(\vec{B} \times \vec{C})| = |\vec{A}||\vec{B}||\vec{C}|(\cos\alpha)(\sin\theta)$ = volume of the parallelepiped

Example of a Scalar triple product



Moment of a force about an axis

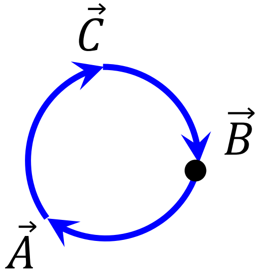
$$\Gamma_{\Delta} = \vec{u}.\left(\vec{r} \times \vec{F}\right)$$

Cross triple product

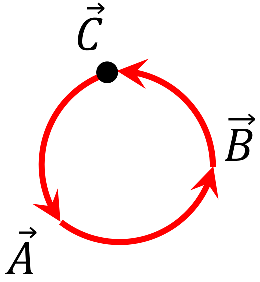
These are NOT scalar product !

Scalar times Vector = Vector

$$\vec{A} \times (\vec{B} \times \vec{C}) \quad = \quad \vec{B}(\vec{A}.\vec{C}) \quad - \quad \vec{C}(\vec{A}.\vec{B})$$



clockwise



counterclockwise

Caution ! *When dealing with vector fields and operators,* $\vec{C}(\vec{A}.\vec{B})$ *may lose physical meaning then we use* *<u>commutativity</u>* $\underbrace{(\vec{A}.\vec{B})}_{a}\vec{C}$

$\Rightarrow a\vec{C}$

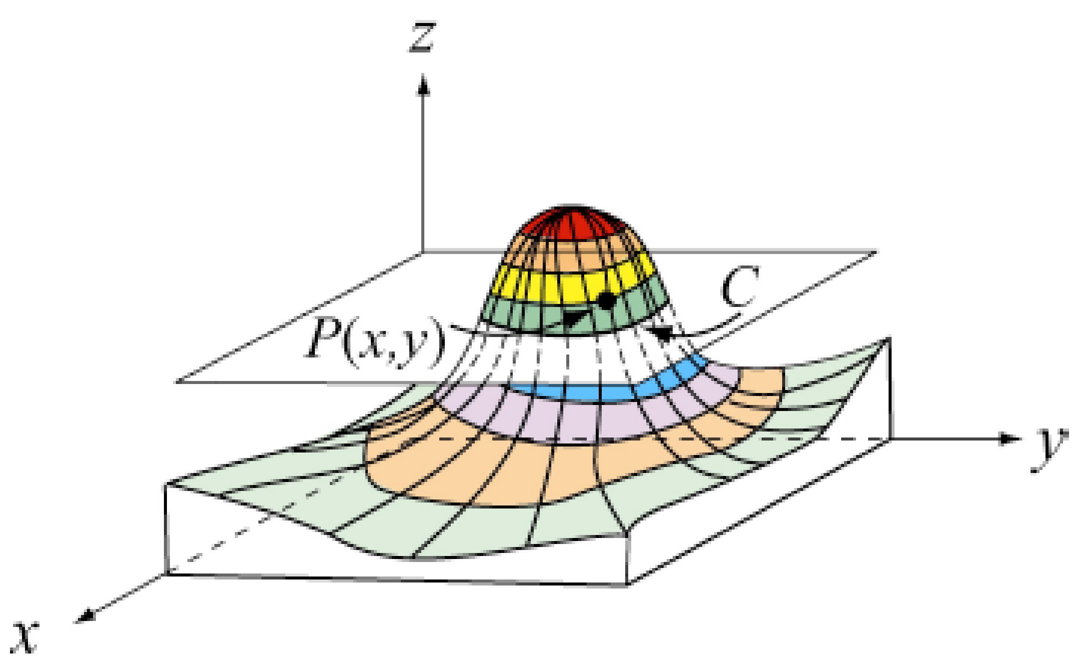
Manipulating scalar and vector fields

The central master piece in Electromagnetism

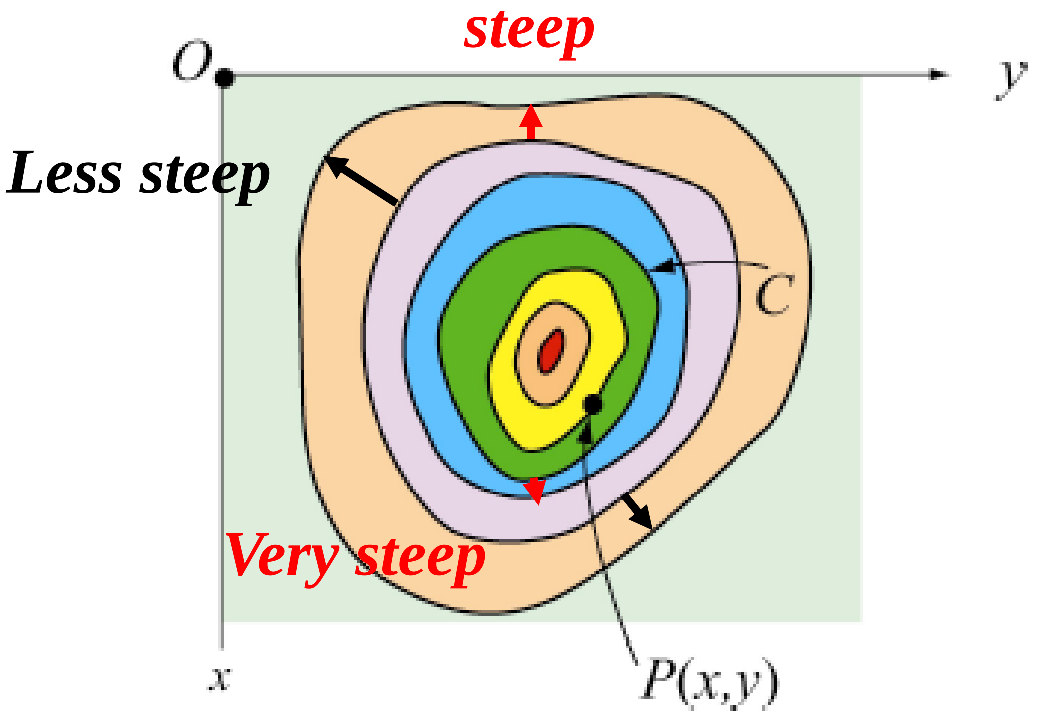
$$\vec{\nabla}$$

The Nabla or Del operator

Topographic map of a mountain: Hiking in the mountain



side view of a mountain

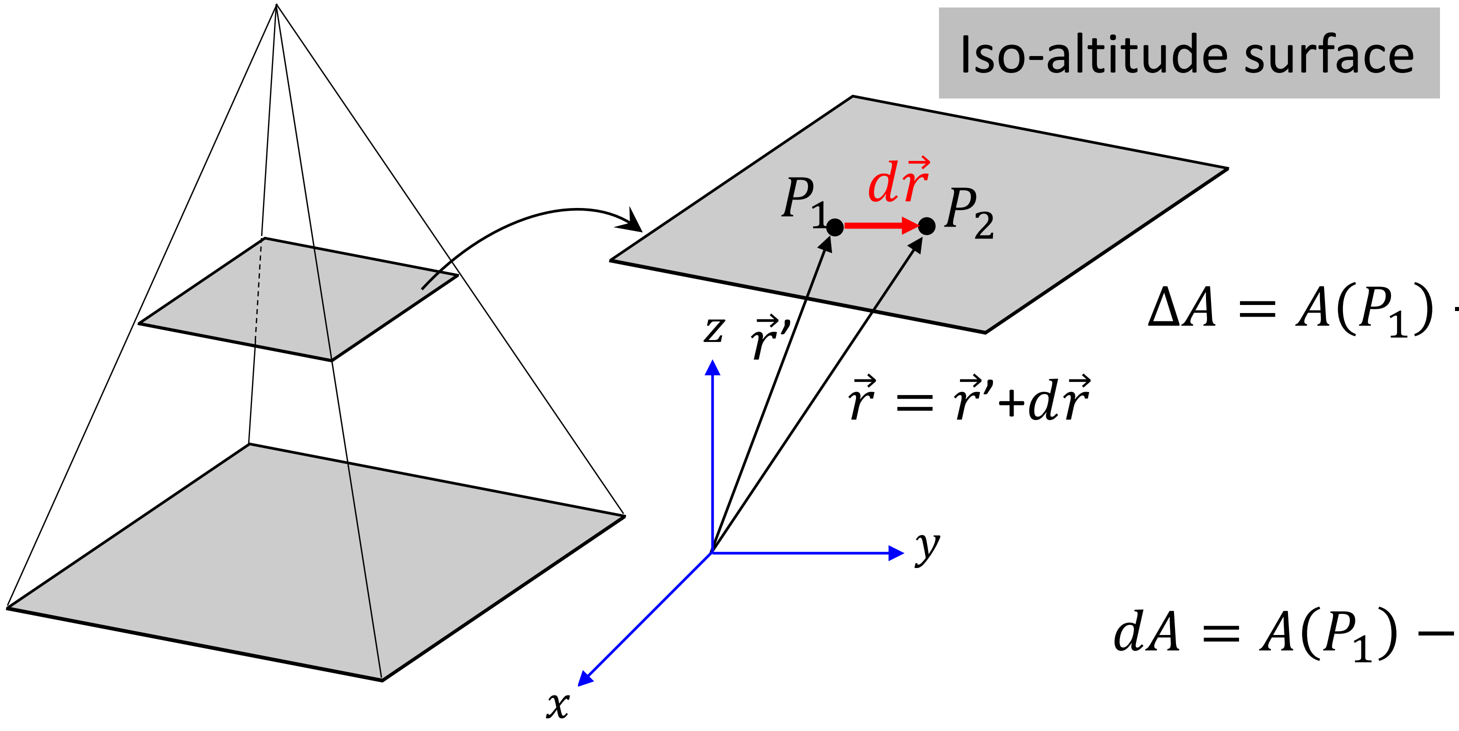


top view of iso-altitude surfaces

Where the arrow is small, the path is steep

How can we link the steepness to the iso-altitude surfaces?

Another way of looking at the gradient



$$P_1 \qquad\qquad \rightarrow \qquad\qquad P_2$$

$$A(x, y, z) \rightarrow A(x + \Delta x, y + \Delta y, z + \Delta z)$$

$A =$ Altitude

$$\Delta A = A(P_1) - A(P_2) = \frac{\partial A}{\partial x}\Delta x + \frac{\partial A}{\partial y}\Delta y + \frac{\partial A}{\partial z}\Delta z = \mathbf{0}$$

$$\Delta x, \Delta y, \Delta z \rightarrow 0$$

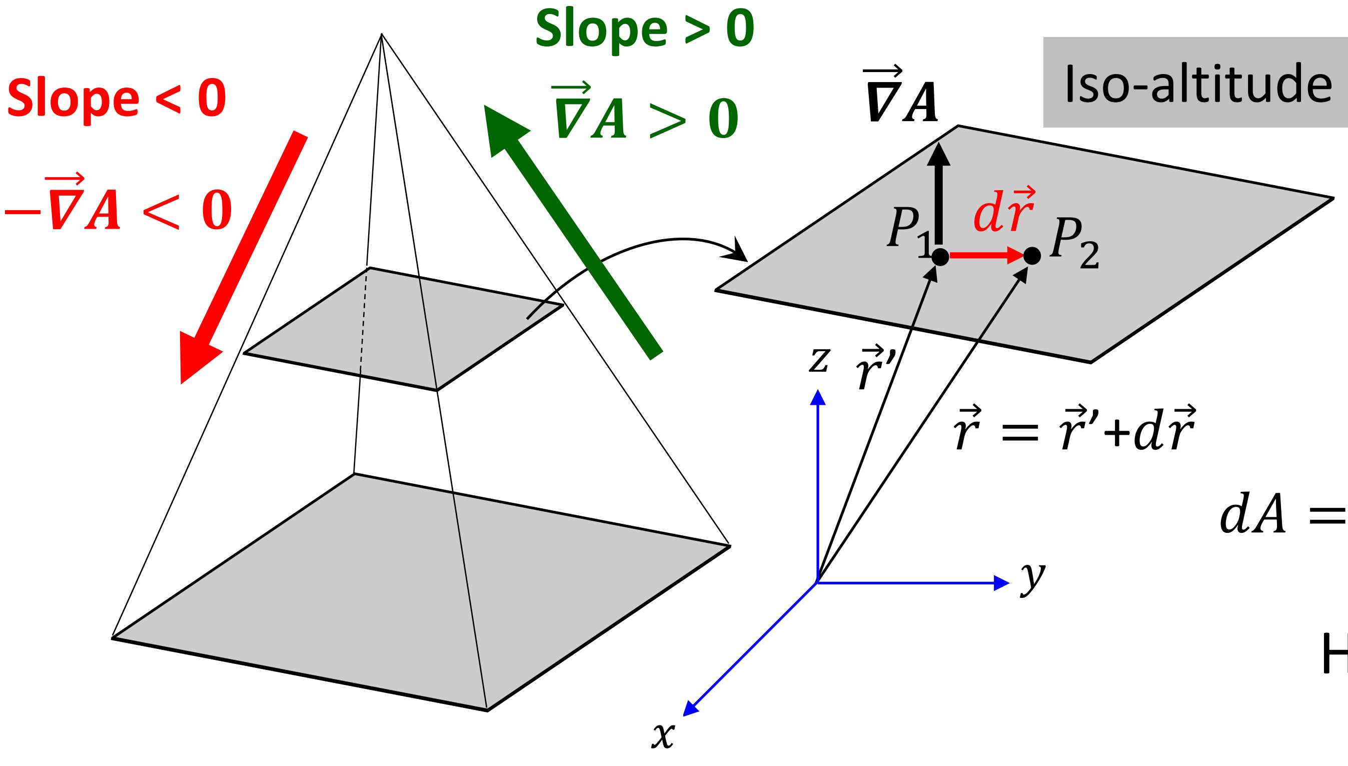
$$dA = A(P_1) - A(P_2) = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy + \frac{\partial A}{\partial z}dz = \mathbf{0}$$

What do dx, dy and dz represent?

The components of the vector displacement $d\vec{r}$

$$\boldsymbol{d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}}$$

Another way of looking at the gradient



$$P_1 \rightarrow P_2$$

$$A(x, y, z) \rightarrow A(x + dx, y + dy, z + dz)$$

$$dA = A(P_1) - A(P_2) = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy + \frac{\partial A}{\partial z}dz = \mathbf{0}$$

How can we transform this sum of products?

$$dA = \left(\frac{\partial A}{\partial x}\vec{\imath} + \frac{\partial A}{\partial y}\vec{\jmath} + \frac{\partial A}{\partial z}\vec{k}\right).\left(dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}\right) = \mathbf{0}$$

$$dA = \vec{\nabla}A.d\vec{r} = \mathbf{0} \quad \Rightarrow \quad \vec{\nabla}\boldsymbol{A} \perp \boldsymbol{d\vec{r}}$$

In mathematics two sets of three numbers each, (a, b, c) and (a', b', c'), arranged as:

$aa' + bb' + cc'$ = scalar product of two vectors

$$\vec{U} = a\vec{\imath} + b\vec{\jmath} + c\vec{k} \qquad \vec{V} = a'\vec{\imath} + b'\vec{\jmath} + c'\vec{k}$$

$$\boxed{\vec{U}.\vec{V} = aa' + bb' + cc'}$$

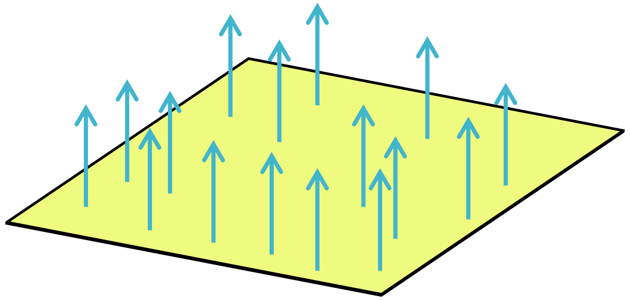
The same concerns two sets of on or two numbers:

(a) and (a') $\longrightarrow$ $\vec{U} = a\vec{\imath}$ $\quad \vec{V} = a'\vec{\imath}$ $\quad \boxed{\vec{U}.\vec{V} = aa'}$

(a, b) and (a', b') $\longrightarrow$ $\vec{U} = a\vec{\imath} + b\vec{\jmath}$ $\quad \vec{V} = a'\vec{\imath} + b'\vec{\jmath}$ $\quad \boxed{\vec{U}.\vec{V} = aa' + bb'}$

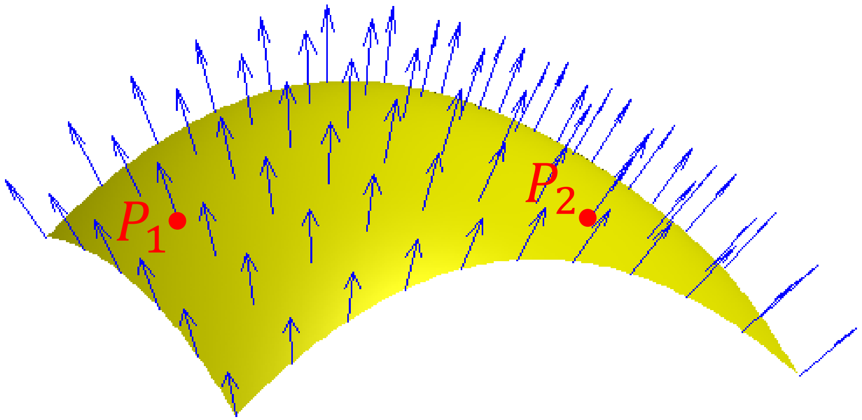
Property of the gradient of a scalar field $f(x,y,z)$ at a given point $P(x,y,z)$, $\vec{\boldsymbol{\nabla}} \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$

- If the iso-surface is flat, the vector is perpendicular everywhere to this surface



Gradient vectors parallel everywhere on the iso-surface

- If the iso-surface is curved, the vector $\vec{\nabla} f(x,y,z)$ is perpendicular to surface at the specified point $P(x,y,z)$



Gradient vectors at P_1 and P_2 are not parallel on the same iso-surface

The gradient is a <u>local property</u>

Water flows naturally from top to bottom $\longrightarrow$ $\vec{W}(x,y,z,t) = -K_w \vec{\nabla} A(x,y,z,t)$

$A(x,y,z,t)$ = Iso-altitude surface

Heat flows naturally from hot to cold $\longrightarrow$ $\vec{h}(x,y,z,t) = -K_h \vec{\nabla} T(x,y,z,t)$

$T(x,y,z,t)$ = Iso-thermal surface

Electric field also "flows" from $+q$ to $-q$ $\longrightarrow$ $\vec{E}(x,y,z,t) = -\vec{\nabla} V(x,y,z,t)$

$V(x,y,z,t)$ = Iso-potential surface

The same equation governs electromagnetism, water flow, heat flow etc...

See A_Lecture 1_General

*Question #4 What is for a point charge the shape of the iso-potential surface ?

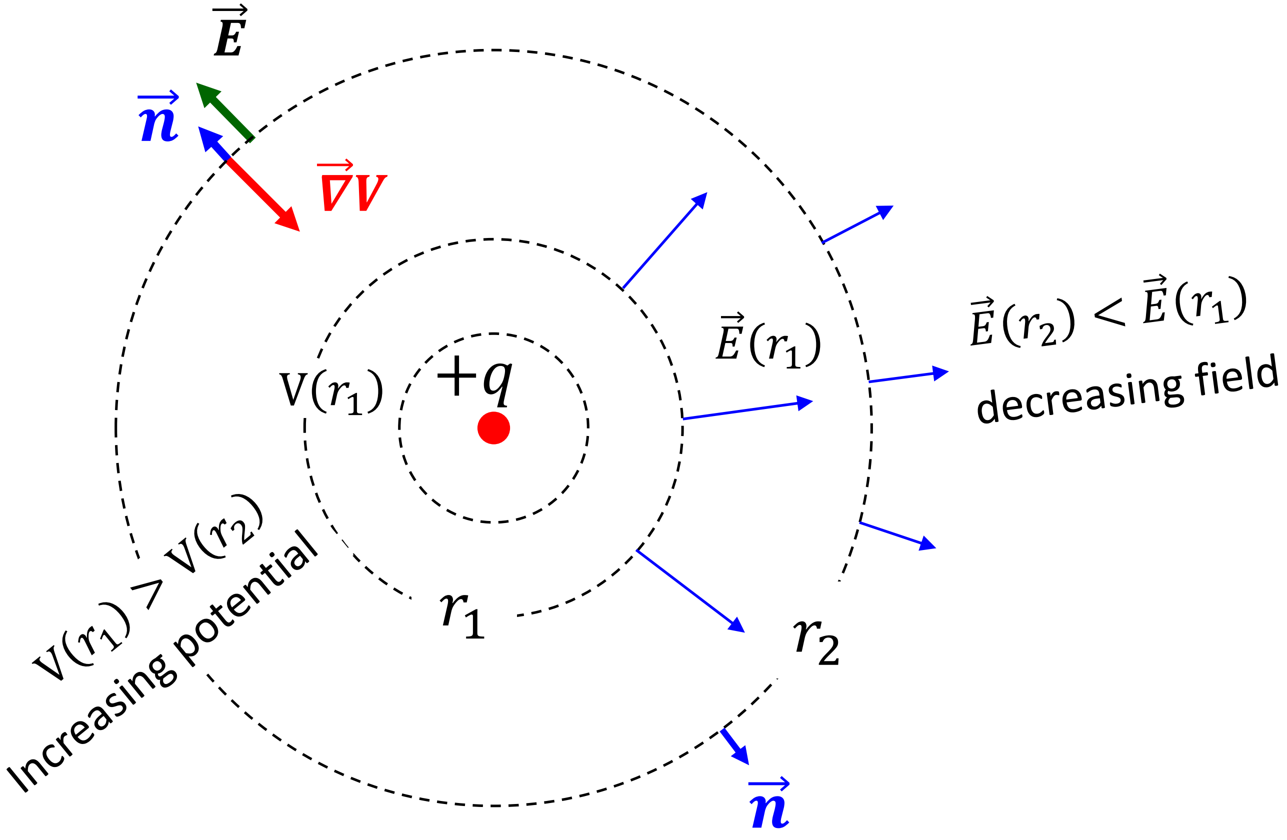
Answer to *Question #4 Spheres centered on the charge

$\vec{E} \parallel \vec{n} \perp$ iso-potential surface

$$\vec{E}(x,y,z,t) = -\vec{\nabla}\mathrm{V}(x,y,z,t)$$

$$\mathrm{V}(x,y,z,t) = \text{cte}$$

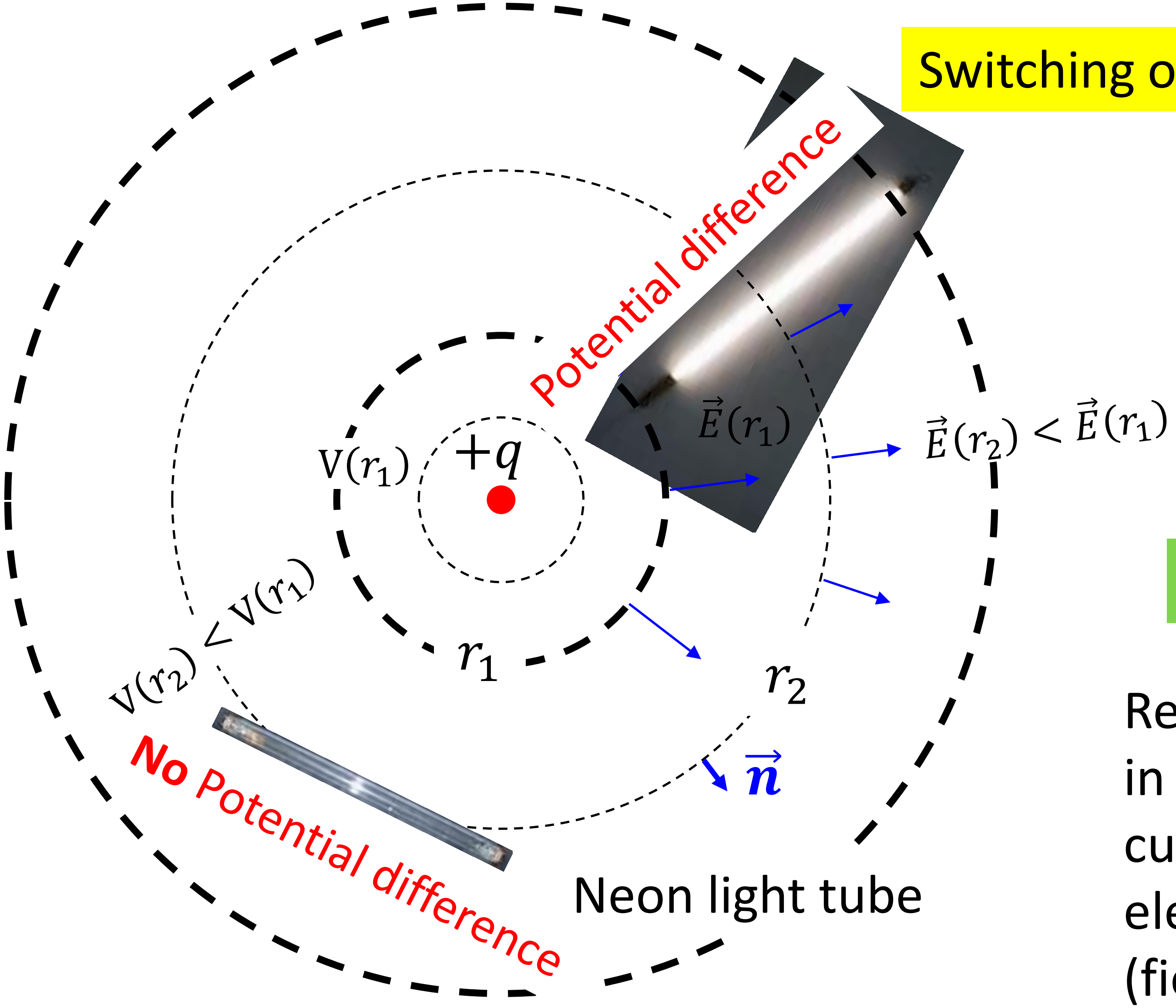
Iso-potential surface



First part of the first Maxwell's equation

$$\vec{E} = -\vec{\nabla}V$$

Valid in Electrostatic only



Switching on the light without any closed circuit !

***Question #5

Does it flash or remains ON ?

Answer to ***Question #5

Remains ON as long as discharge in gas is acting. In a pure metal the current would vanish quickly as electrons stop moving (field inside = 0)

The Nabla or Del operator

$$\vec{\boldsymbol{\nabla}} = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}$$

This operator alone means nothing, just as…

$$\sqrt{}$$

means nothing

Both operators must have something to act on

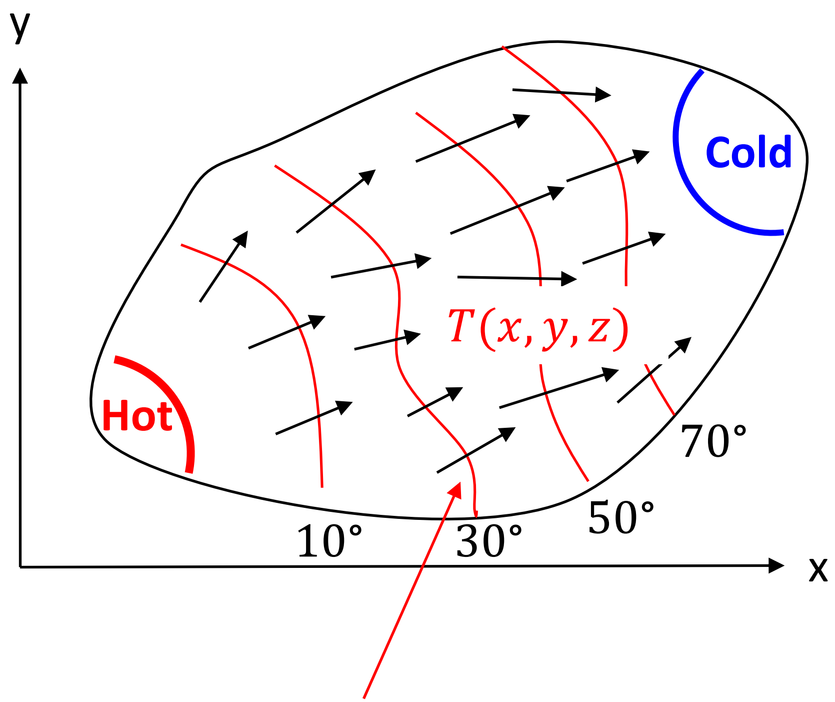
The best way to visualize the scalar and vector fields is via heat propagation

How do we get the flux?

Scalar field: Temperature in 3D space

*Question #6

Does temperature move or change from one position in space to another?



Isotherm = isothermal line (2D) or surface (3D)

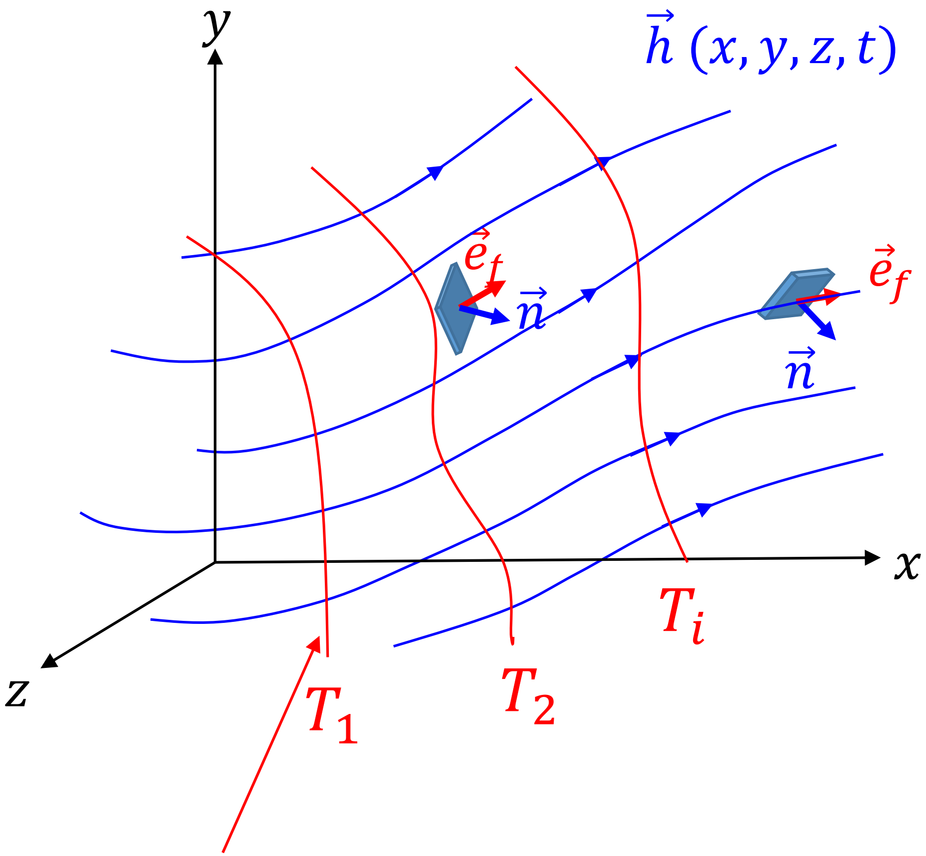
*Answer to question #6

$T(x,y,z)$ does not move: **It changes**

- Temperature $T(x,y,z)$ is a scalar field
- If it depends on time ⇨ $T(x,y,z,t)$

Defining the concept of Flux

Vector field: Heat flow in 3D space



Heat $\vec{\boldsymbol{h}}\,(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t})$ is a vector field. It flows in 3D space

It gives rise to a scalar field: temperature in 3D space $\boldsymbol{T}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t})$

$\left|\vec{\boldsymbol{h}}\,(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t})\right|$ = How much heat is <u>flowing</u>

= Thermal energy/time/area

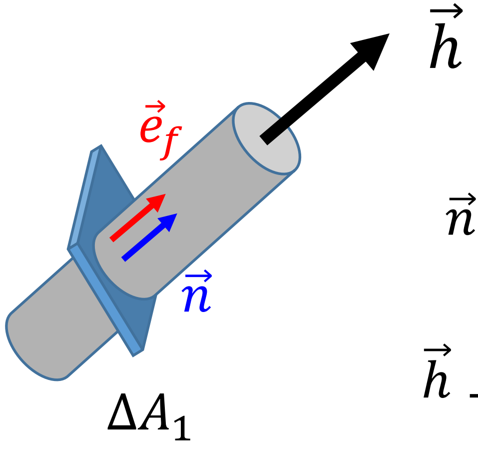
$\vec{\boldsymbol{h}}\,(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t}) = (\text{scalar quantity})\vec{e}_f$ **Heat <u>flow</u>**

Thermal energy/time/area

Isotherm

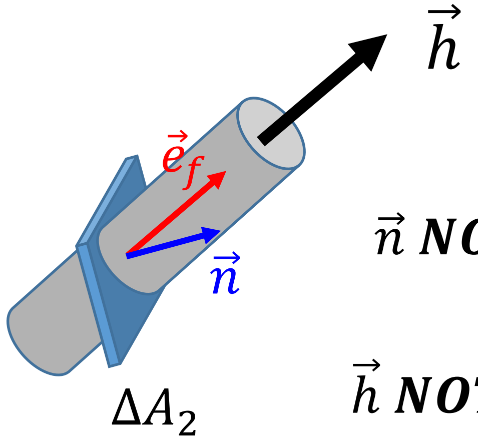
- Need to define a unit vector along the flow $\vec{e}_f$ at a given location in space
- Need to define a unit vector $\vec{n}$ for the area under consideration at a given location in space

$$\vec{h} \parallel \vec{e}_f$$



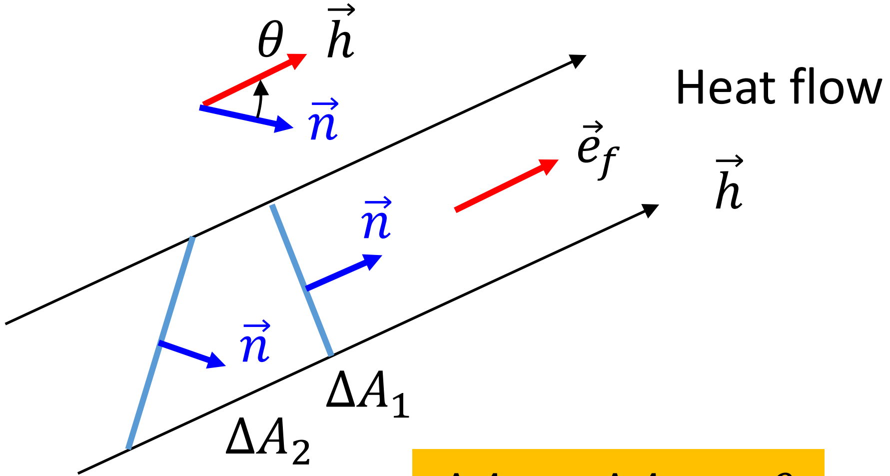
$\vec{n} \parallel \vec{e}_f$

$\vec{h} \perp \Delta A_1$



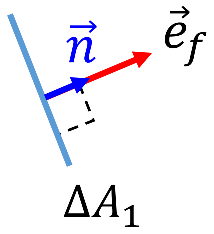
$\vec{n}$ ***NOT*** $\parallel \vec{e}_f$

$\vec{h}$ ***NOT*** $\perp \Delta A_1$



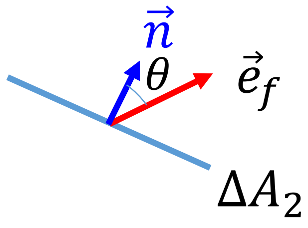
$$\Delta A_1 = \Delta A_2 cos\theta$$

Definition of heat flow $\vec{h}$



$$\vec{\boldsymbol{h}}\,(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{t}) = \frac{P}{\Delta A_1}\vec{e}_f = |\vec{h}\,|\vec{e}_f$$

P = Thermal energy/time



What is the flow per unit area through ΔA_2 ?

?

Scalar product !

$$\frac{P}{\Delta A_2} = \frac{P}{\Delta A_1}cos\theta = \vec{h}.\vec{n} = |\vec{h}_\perp|$$

$$\Delta A_1 = \Delta A_2 cos\theta \quad |\vec{h}\,| = |\vec{h}_\perp| \text{ if } \theta = 0 \Leftrightarrow \vec{e}_f = \vec{n}$$

$$\boxed{Flux = \int_A \vec{h}.\vec{n}dA}$$

or

$$\boxed{Flux = \int_A |\vec{h}_\perp|\, dA}$$

Applying Del operator $\vec{\boldsymbol{\nabla}}$ to heat flow

Gradient of a <u>scalar field</u> $T(x,y,z)$

<u>Differentiate</u> a stationary scalar field $T(x,y,z)$

$T(x,y,z) = \text{Cte} \Leftrightarrow$ Isotherm or isothermal surface

$$\underbrace{\Delta T(x,y,z)}_{\text{scalar}} = \underbrace{\frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial y}\Delta y + \frac{\partial T}{\partial z}\Delta z}_{\text{dot product of two vectors}}$$

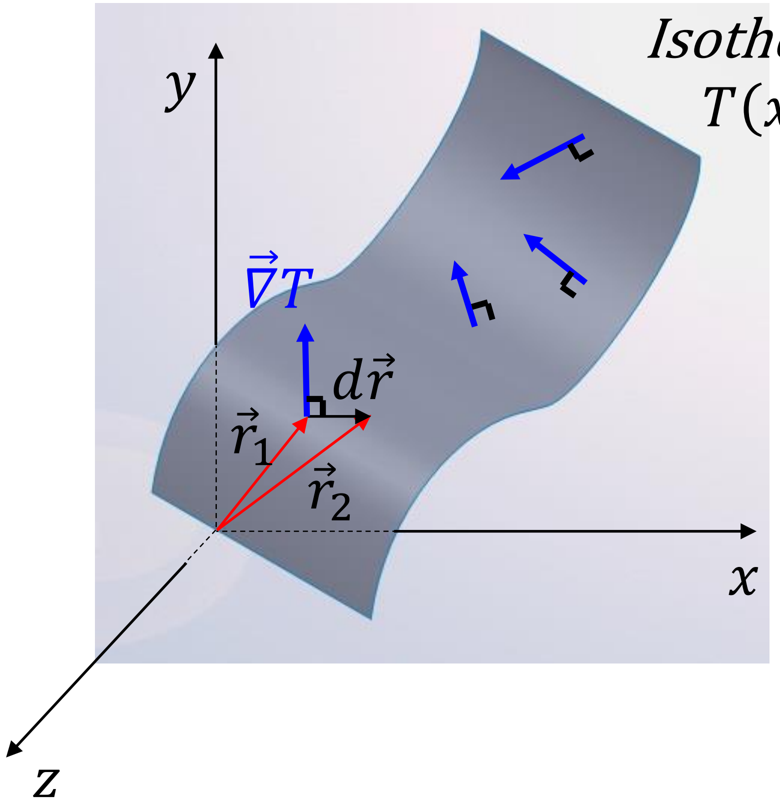
$$\underbrace{\left(\frac{\partial T}{\partial x}\vec{i} + \frac{\partial T}{\partial y}\vec{j} + \frac{\partial T}{\partial z}\vec{k}\right)}_{\text{Vector}} . \underbrace{\left(\Delta x\vec{i} + \Delta y\vec{j} + \Delta z\vec{k}\right)}_{\text{Vector displacement}}$$

$$\underbrace{\left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)}_{\text{Vector Operator } \vec{\nabla}} T(x,y,z)$$

Reminder

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)$$

Is **NOT** a vector as long as it does not operate on a scalar !



Isothermal surface

$T(x,y,z) = T_0 \quad \Rightarrow \quad \Delta T(x,y,z) = T(x,y,z) - T_0 = 0$

Temperature is everywhere the same on this surface

(Surface is not necessarily flat !)

$$\Delta T(x,y,z) = \left(\frac{\partial T}{\partial x}\vec{\imath} + \frac{\partial T}{\partial y}\vec{\jmath} + \frac{\partial T}{\partial z}\vec{k}\right).\left(\Delta x\vec{\imath} + \Delta y\vec{\jmath} + \Delta z\vec{k}\right) = 0$$

$$\Delta T(x,y,z) = \vec{\nabla}T.d\vec{r} = \left|\vec{\nabla}T\right|.\left|d\vec{r}\right|.cos(\theta) = 0$$

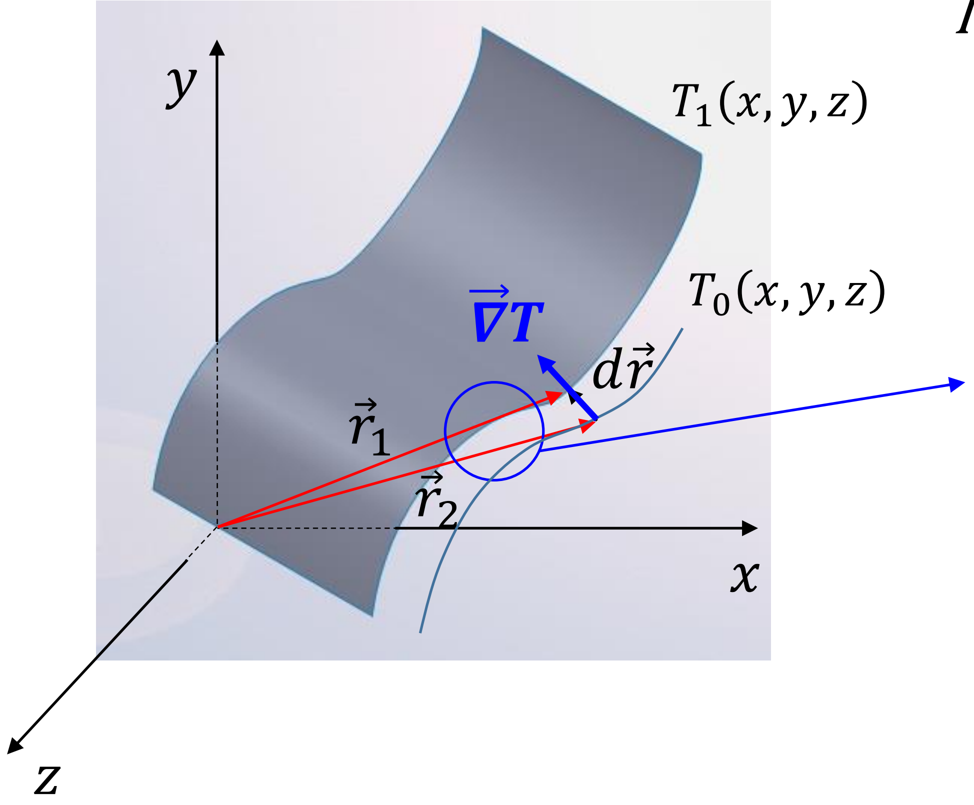
!

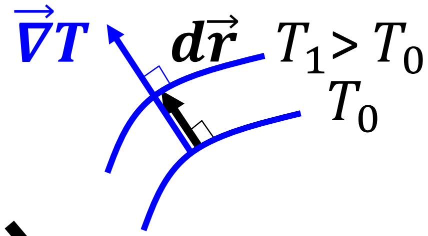
$\vec{r}_1$ and $\vec{r}_2$ indicate two close points **<u>on the isothermal surface</u>**

$\theta = \pi/2$ $\vec{\nabla}T$ and $d\vec{r}$ are $\perp$ everywhere on the isothermal surface

- $d\vec{r}$ is tangent to the isothermal surface at (x,y,z)
- $\vec{\nabla}T$ is $\perp$ to the isothermal surface **at $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$**

Isothermal surfaces T_1 *and* T_0





$$\Delta T(x,y,z) = \vec{\nabla}T . d\vec{r} = \nabla T . dr . cos(\theta)$$
$$= \nabla T . dr = Max > 0$$
$$\theta = 0$$

Natural flow of heat $\vec{h} \propto -\vec{\nabla}T$

$$\Delta T > 0, dr > 0, \Rightarrow \vec{\nabla}T \parallel d\vec{r}, \theta = 0$$

$\vec{r}_1$ and $\vec{r}_2$ indicate two close points
<u>on two close isothermal surfaces</u>

The gradient is oriented towards the increasing field

$\vec{\nabla}$ is a vector operator: What can we do with it?

$\boxed{\vec{E} = -\vec{\nabla}\mathrm{V}}$

$f(x, y, z)$ is a scalar $\quad \vec{\nabla}.f$

Gradient
(Vector)

$\boxed{\vec{\boldsymbol{\nabla}}.\vec{E} = \vec{\boldsymbol{\nabla}}.(-\vec{\nabla}\mathrm{V})}$

$\boxed{\vec{\boldsymbol{\nabla}} \times \vec{E} = \vec{\boldsymbol{\nabla}} \times (-\vec{\nabla}\mathrm{V})}$

$\vec{A}(x, y, z)$ is a vector

$\vec{\nabla}.\vec{A}$

Divergence
(Scalar)

$\vec{\nabla} \times \vec{A}$

Curl
(Vector)

<u>All these things are local properties</u>: They are defined at a specific point $P(x, y, z)$
Gradient, Divergence and Curl might be $> \mathbf{0}, < \mathbf{0}$ ***or*** $= \mathbf{0}$

Del operator $\vec{\nabla}$ transforms a scalar field into a vector field

Gradient of a scalar field $T(x, y, z), \mathrm{V}(x, y, z)$ etc… ⇔ $\vec{\nabla} T, \vec{\nabla} \mathrm{V}$, etc…

($\vec{\nabla}$ = vector **operating** on a **scalar** field)

Del operator $\vec{\nabla}$ transforms a vector field into a scalar field: New concept, **Divergence**

Divergence of a vector field $\vec{A}(x, y, z)$ ⇔ $\vec{\nabla} . \vec{A}$ (= **"scalar product"**)

($\vec{\nabla}$ = vector **operating** on a **vector** field)

- What is the divergence of a vector and what is its physical meaning ?