Problem Set 4

Due: 15 June 2020, 12.30 p.m.

Problem 1. In the space of square-integrable functions $\mathcal{L}^2(\mathbb{R})$ let us define two operators: $\hat{\pi}_x = \hbar\frac{\partial}{\partial x}$ (here we additionally assume that the functions are differentiable), and the scale operator $\hat{M}_c$, defined as $\hat{M}_c f(x) = f(cx)$, where f is a function and c — a number.

(a) Find $\hat{\pi}_x f$ and $\hat{M}_{\frac{1}{2}} f$, with $f(x) = e^{-x^2}$.

(b) Find the Hermitian conjugates of these operators. Are $\hat{\pi}_x$ and $\hat{M}_c$ Hermitian?

(1 + (2 × 2) points)

Problem 2. Show that for any operators $\hat{A}$, $\hat{B}$ defined on $\mathcal{L}^2(\mathbb{R})$ and any complex number α

(a) $(\alpha\hat{A})^\dagger = \alpha^*\hat{A}^\dagger$, (b) $\left(\hat{A}^\dagger\right)^\dagger = \hat{A}$, (c) $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$.

Hint. Use the definition of the Hermitian conjugate and properties of the inner product.

(3 × 3/2 points)

Problem 3. Problem 6-52 (Tipler).

(1 + (4 × 1/2) point)

Problem 4. At the initial instant of time $t = 0$, the wave function of a particle in an infinite potential well with walls at $x = 0$ and $x = a$ is $\Psi(x, 0) = \sqrt{\frac{1}{3}}\psi_3 + \sqrt{\frac{2}{3}}\psi_5$, where ψ_3 and ψ_5 are normalized solutions of the stationary Schrödinger equation, *i.e.* $\hat{H}\psi_n = E_n\psi_n$.

(a) Is Ψ normalized? Explain.

(b) Write down the explicit form of $\Psi(x, 0)$. What is the probability that the particle is in the region $0 < x < a/2$ (use a computer to calculate the integral).

(c) What are possible outcomes of a measurement of energy in the state described by $\Psi(x, 0)$? What are their probabilities?

(d) What is the average value of the particle's total energy in this state?

(e) What is the wave function of the particle for $t > 0$.

(1/2 + (4 × 1) point)

Problem 5. A particle moving in an infinite potential well with walls at $x = 0$ and $x = a$ is in its ground state. At $t = 0$, the right wall of the well is suddenly moved to $x = 2a$.

(a) Find the wave function of the particle $\Psi(x, t)$ for $t > 0$. Use a computer to calculate the integrals in the expansion coefficients and list the values of the first five coefficients $c_1, c_2, \ldots, c_5$. Plot $|\Psi(x, t)|^2$ for $t = 0$ and three other instants $t > 0$.

Hint. Is the ground-state wave function in the narrow well an eigenstate in the wide well?

(b) What is the probability that a measurement of particle's energy at $t = 0^+$ finds its value unchanged?

(5 + 2 points)

Problem 6. In class we have quoted the following two statements about solutions to the 1D stationary Schrödinger equation. Please justify one of them (choose whichever you want).

(a) The solutions of the stationary Schrödinger equation can always be taken as real valued functions (unlike the solutions to the time-dependent Schrödinger equation, which are necessarily complex).

Hint. Show that if a function ψ solves the stationary Schrödinger equation for a given energy, so too does it complex conjugate, and hence also the real linear combinations $(\psi + \psi^*)$ and $i(\psi - \psi^*)$.

Comment. This statement does not mean that every solution to the stationary Schrödinger equation is real. What is says is that if you have got one that is not, it can always be expressed as a linear combination of solutions (corresponding to the same energy) that are real. So you might as well stick to ψ's that are real.

(b) If the potential energy $V(x)$ is an even function of x, then the solutions of the stationary Schrödinger equation $\psi(x)$ can be taken to be either even or odd.

Hint. Show that if a function ψ solves the stationary Schrödinger equation for a given energy, so too does $\psi(-x)$, and hence also the even and odd linear combination $\psi(x) \pm \psi(-x)$.

(2 points)