VP390 Problem Set 3

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1 Problem 1

(a) $|\Psi(x,t)|^2 = \Psi(x,t)\Psi^*(x,t) = Ce^{-\frac{i}{\hbar}Et}(x^2-a^2)Ce^{\frac{i}{\hbar}Et}(x^2-a^2) = C^2(x^2-a^2)^2$
$|\Psi(x,t)|^2 = \Phi(x)$ So Ψ represents a stationary state.

(b) $\int_{-a}^{a}|\Psi(x,t)|^2\mathrm{d}x = \int_{-a}^{a}C^2(x^2-a^2)^2\mathrm{d}x = C^2(\frac{2a^5}{5} - \frac{4a^5}{3} + 2a^5) = C^2\frac{16a^5}{15} = 1$
$C^2a^5 = \frac{15}{16}$

(c) $\int_{-a/2}^{a/2}C^2(x^2-a^2)^2\mathrm{d}x = C^2(\frac{a^5}{80} - \frac{a^5}{6} + a^5) = \frac{203}{256} \approx 0.793$

(d) Since $\Phi(x)$ is symmetric about x-axis, the probability is 0.5

(e) $\langle x\rangle_\Psi = \int_{-\infty}^{\infty}x|\Psi(x,t)|^2\mathrm{d}x = \int_{-a}^{a}C^2x(x^2-a^2)^2\mathrm{d}x = 0$ So it's time independent

(f) $\triangle_x = \sqrt{\langle x^2\rangle_\Psi - \langle x\rangle_\Psi^2} = C\sqrt{\int_{-a}^{a}x^2(x^-a^2)^2\mathrm{d}x} = C\sqrt{\frac{30a^7}{105} - \frac{84a^7}{105} + \frac{70a^7}{105}} = \sqrt{\frac{a^2}{7}}$

2 Problem 2

(a) $E_1 = 13.6\mathrm{eV}$, $a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2} \approx 5.29\times10^{-2}\mathrm{nm}$
$\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\int_0^{\infty}\Psi_{100}(r,\varphi,\theta,t)|^2\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi = C^2\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\int_0^{\infty}r^2\sin\varphi\exp(-\frac{2r}{a_0})\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi = 2\times 2\pi C^2\frac{a^3}{4}$
$C = \sqrt{\frac{1}{\alpha_0^3\pi}} = 1.467\times10^{15}$

(b) $C^2\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\int_0^{a_0}r^2\sin\varphi\exp(-\frac{2r}{a_0})\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi = \pi C^2(a^3 - 5a^3e^{-2}) = 1-5e^{-2}$

(c) The probability for the electron radius is r can be expressed :
$C^2\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}r^2\sin\varphi\exp(-\frac{2r}{a_0})\mathrm{d}\theta\mathrm{d}\varphi = 4\pi C^2r^2\exp(-\frac{2r}{a_0})$
$\frac{\mathrm{d}r^2\exp(-\frac{2r}{a_0})}{\mathrm{d}r} = 2r\exp(-\frac{2r}{a_0}) - \frac{2r^2}{a_0}\exp(-\frac{2r}{a_0}) = 0$
$r_1 = 0, r_2 = a_0, r_3 = \infty$, since r_1 and r_3 can lead to the minium value of $r^2\exp(-\frac{2r}{a_0})$, only $r = \alpha_0$ has the most probability value to find the electron

(d) $r_a = \int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\int_0^{\infty}r\Psi_{100}(r,\varphi,\theta,t)|^2\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi = C^2\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\int_0^{\infty}r^3\sin\varphi\exp(-\frac{2r}{a_0})\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi$
$= 4\pi C^2\frac{3a_0^4}{8} = \frac{3a_0}{2}$
$\triangle_r = \sqrt{\langle r^2\rangle_\Psi - r_a^2}$
$\langle r^2\rangle = 4\pi C^2\int_0^{\infty}r^4\exp(-\frac{2r}{a_0})\mathrm{d}r = 3\pi C^2a_0^5 = 3a_0^2$

$\triangle_r = \sqrt{3a_0^2 - \frac{9a_0^2}{4}} = \frac{\sqrt{3}a_0}{2}$
So the average value of r is $\frac{3a_0}{2}$, its standard deviation is $\frac{\sqrt{3}a_0}{2}$

3 Problem 3

(a) $\langle f, g\rangle = \int_{-\infty}^{\infty} f^*(u)g(u)\mathrm{d}u = \int_{-\infty}^{\infty} (g^*(u))^* f^*(u)\mathrm{d}u = (\int_{-\infty}^{\infty} g^*(u)f(u)\mathrm{d}u)^* = \langle g, f\rangle^*$

(b) $\langle f, \alpha g\rangle = \int_{-\infty}^{\infty} f^*(u)\alpha g(u)\mathrm{d}u = \alpha \int_{-\infty}^{\infty} f^*(u)g(u)\mathrm{d}u = \alpha \langle f, g\rangle$

(c) $\langle \alpha f, g\rangle = \langle g, \alpha f\rangle^* = \alpha^* \langle g, f\rangle^* = \alpha^* \langle f, g\rangle$

4 Problem 4

1. $\int_{-\infty}^{\infty} |\psi|^2 \mathrm{d}x = \int_0^L \frac{2}{L}\sin^2(\frac{n\pi x}{L})\mathrm{d}x = \frac{2}{L}(\frac{L}{2} - \frac{L\sin 2n\pi}{4n\pi}) = 1$
   So it is normalized
2. $\langle \psi_n, \psi_m\rangle = \int_0^L \frac{2}{L}\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\mathrm{d}x = \frac{1}{L}\int_0^L \cos\frac{(n-m)\pi x}{L} - \cos\frac{(n+m)\pi x}{L}\mathrm{d}x$
   $= \frac{1}{L}[\frac{L\sin(n-m)\pi}{(n-m)\pi} - \frac{L\sin(n+m)\pi}{(n+m)\pi}] = 0$
3. $\langle i\psi_1, -3\psi_1 + 2i\psi_2 - \psi_3\rangle = 3i\langle \psi_1, \psi_1\rangle - 2\langle \psi_1, \psi_2\rangle - i\langle \psi_1, \psi_3\rangle = 3i$
4. Assume $x = x_1\psi_1 + x_2\psi_2, y = y_1\psi_1 + y_2\psi_2$ where
   $x_1^2 + x_2^2 = 1, y_1^2 + y_2^2 = 1, x_1y_1 + x_2y_2 = 0$
   $x = \frac{\sqrt{2}}{2}\psi_1 - \frac{\sqrt{2}}{2}\psi_2, y = \frac{\sqrt{2}}{2}\psi_1 + \frac{\sqrt{2}}{2}\psi_2$

5 Problem 5

1. $\langle \psi, \psi\rangle = \frac{1}{2}[\langle e^{-\frac{i}{\hbar}E_1t}\psi_1(x), e^{-\frac{i}{\hbar}E_1t}\psi_1(x)\rangle + \langle e^{-\frac{i}{\hbar}E_2t}\psi_2(x), e^{-\frac{i}{\hbar}E_2t}\psi_2(x)\rangle]$
   $\frac{1}{2}(\langle \psi_1, \psi_1\rangle + \langle \psi_2, \psi_2\rangle) = 1$
   So it's normalized
2. $\Psi(x,t)^*\Psi(x,t) = \frac{1}{2}(e^{\frac{i}{\hbar}E_1t}\psi_1^*(x) + e^{\frac{i}{\hbar}E_2t}\psi_2^*(x))(e^{-\frac{i}{\hbar}E_1t}\psi_1(x) + e^{-\frac{i}{\hbar}E_2t}\psi_2(x))$
   $= \frac{1}{2}(\psi_1^*(x)\psi_1(x) + \psi_2^*(x)\psi_2(x)) + \frac{1}{2}(e^{\frac{i}{\hbar}(E_1-E_2)t}\psi_1^*(x)\psi_2(x) + e^{\frac{i}{\hbar}(E_2-E_1)t}\psi_1(x)\psi_2^*(x))$
   So the Ψ only describe a particle in a stationary state when $E_1 = E_2$
3. $\Psi(x,t) = \frac{1}{\sqrt{2}}(\cos\frac{E_1t}{\hbar}\psi_1(x) - i\sin\frac{E_1t}{\hbar}\psi_1(x) + \cos\frac{4E_1t}{\hbar}\psi_2(x) - i\sin\frac{4E_1t}{\hbar}\psi_2(x))$
   $T = \frac{2\pi}{\omega} = \frac{h}{E_1}$
   if $|\Psi(x,t+T)|^2 = |\Psi(x,t)|^2 =$ then
   $\frac{1}{2}(e^{-\frac{i}{\hbar}3t}\psi_1^*(x)\psi_2(x) + e^{\frac{i}{\hbar}3t}\psi_1(x)\psi_2^*(x) = \frac{1}{2}(e^{-\frac{i}{\hbar}3E_1(t+T)}\psi_1^*(x)\psi_2(x) + e^{\frac{i}{\hbar}3E_1(t+T)}\psi_1(x)\psi_2^*(x)$
   $T = \frac{2\pi}{\omega} = \frac{h}{3E_1}$