

# The concept of vector potential

Pushing the parallel between Electrostatic and Magnetostatic to the ultimate limit

The question that brings to the concept of vector potential

Giving the following equations

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

Can we solve them without requiring any special symmetry or intuitive guessing ?

In electrostatic we have:

Divergence

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho(\vec{r})}{\epsilon_0}$$

$\rho(\vec{r})$  source of the field

Curl

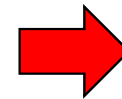
$$\vec{\nabla} \times \vec{E} = \vec{0}$$

Gradient

$$\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r}) \quad \text{Electric potential}$$

If we know the charge distribution  $\rho(\vec{r})$

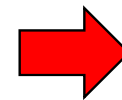
With symmetry



Gauss law

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho(\vec{r})}{\epsilon_0}$$

Without refereeing to symmetry



$$V(\vec{r}) = \frac{1}{4\mu\epsilon_0} \int \frac{\rho(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|} d\mathcal{V}'$$

$$\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r})$$

In magnetostatic:

By analogy

Divergence

$$\vec{\nabla} \cdot \vec{B} = 0$$

Curl

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}(\vec{r})$$

Gradient

???

“Magnetic potential”

$\vec{J}(\vec{r})$  is for magnetostatic what  $\rho(\vec{r})$  is for electrostatic: Source of the field

Is there any equivalent way to solving magnetostatic problems if we know  $\vec{J}(\vec{r})$ ?

The difference with electrostatic comes from these two sides

## Concept of Magnetic Potential

Stokes theorem

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

Differential form



$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$$

Integral form

**BUT**

Can we define a potential for magnetic field  
... as we have a potential for an electric field?

## Electrostatic

Electric field  $\vec{E}$

$$\vec{\nabla} \times \vec{E} = \vec{0}$$



**Scalar potential  $V$ ,  $\vec{E} = -\vec{\nabla}V$**

*B\_Lecture 4&7\_Coordinate system\_Scalar  
versus Vector fields\_Operators*

## Magnetostatic

Magnetic field  $\vec{B}$

$\vec{\nabla} \times \vec{B}$  is not always zero



Scalar potential  $\psi$ ,  ~~$\vec{B} = -\vec{\nabla}\psi$~~

**NOT ALWAYS POSSIBLE**

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{ALWAYS}$$



$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$\vec{A} = \underline{\text{Vector}}$  potential

$$\text{Mathematics} \Rightarrow \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

*Slide #62 B\_Lecture 4&7\_Coordinate system\_Scalar versus Vector fields\_Operators*

The parallel with electrostatic

In magnetostatic if we know the current density  $\vec{J}(\vec{r}) \Rightarrow \vec{A}(\vec{r}) \Rightarrow \vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$

$$B_x = (\vec{\nabla} \times \vec{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

$$B_y = (\vec{\nabla} \times \vec{A})_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}$$

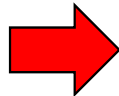
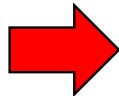
$$B_z = (\vec{\nabla} \times \vec{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

By analogy

## In electrostatic

Charge distribution  $\rho(\vec{r})$    $V(\vec{r})$    $\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$

## In magnetostatic

Current density  $\vec{J}(\vec{r})$    $\vec{A}(\vec{r})$    $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$



For electrostatic Poisson equation

$$\nabla^2 V(\vec{r}) = -\frac{\rho(\vec{r})}{\epsilon_0}$$

For Magnetostatic ...

$$\vec{A}(\vec{r}) \overset{?}{\longleftrightarrow} \vec{J}(\vec{r})$$

Neither of the quantities  $V(\vec{r})$  and  $\vec{A}(\vec{r})$  is unique !

## Vector potential $\vec{A}$ exists: Physical interpretation

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \longrightarrow \quad \text{Dimension Equation} \quad \longrightarrow \quad [A] = [B] \cdot [L]$$

Lorentz force  $F = qvB$   $\longrightarrow$   $[B] = \frac{[F]}{[qv]}$

$$[qA] = [\text{Momentum}]$$

*Although the vector potential result from mathematical considerations it does have a physical meaning*

## Non uniqueness of Vector potential $\vec{A}$ in magnetostatic

The vector potential  $\vec{A}$  is **NOT** unique



A lot of different  $\vec{A}$ 's give the same  $\vec{B}$



We can measure locally  $\vec{B}$  **BUT NOT**  $\vec{A}$

In electrostatic



$$V' = V + C$$

$C$  does not give rise to  $\vec{E}$

$$V(\infty) = 0$$

*$V$  is always defined within a constant  $C$*

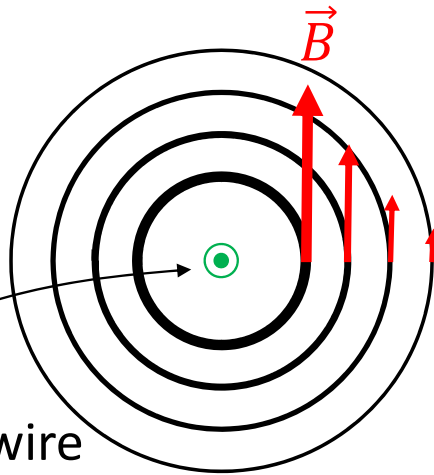
# Non uniqueness of Vector potential $\vec{A}$ in magnetostatic

$$\vec{B} = \vec{\nabla} \times \vec{A}$$



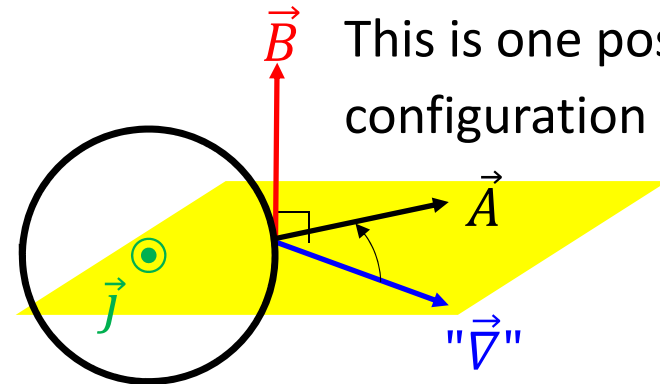
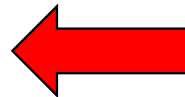
$$\vec{B} \perp \vec{A}$$

Straight current carrying wire



Infinite number of configurations  
in which  $\vec{B} \perp \vec{A}$

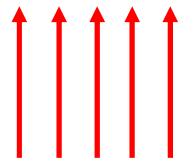
$$\vec{\nabla} \cdot \vec{B} = 0$$



This is one possible  
configuration for  $\vec{A}$ ...

... Among infinite possibilities

# Proof that the vector potential $\vec{A}$ is **NOT** unique

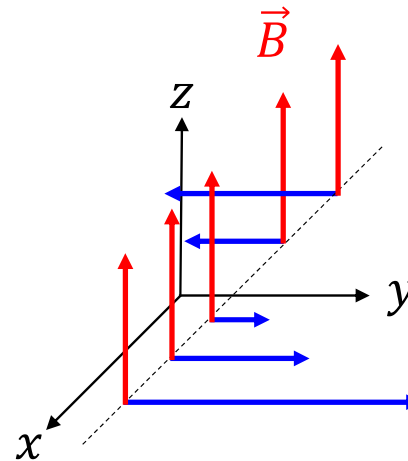


$$\begin{aligned}\vec{B} &= \vec{B}_z = B_z \vec{k} = \mathbf{b} \vec{k} \\ \vec{B}_y &= 0 \\ \vec{B}_x &= 0\end{aligned}$$

What is  $\vec{A}$  ?

$$B_z = (\vec{\nabla} \times \vec{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \mathbf{b}$$

**Possibility 1**



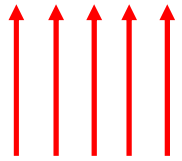
$$\vec{B} \perp \vec{A}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{A} = \vec{A}_y = A_y \vec{j} = \mathbf{b} x \vec{j}$$

Any of the **blue vectors**  $\vec{A}$  is okay: an infinity

# Proof that the vector potential $\vec{A}$ is **NOT** unique



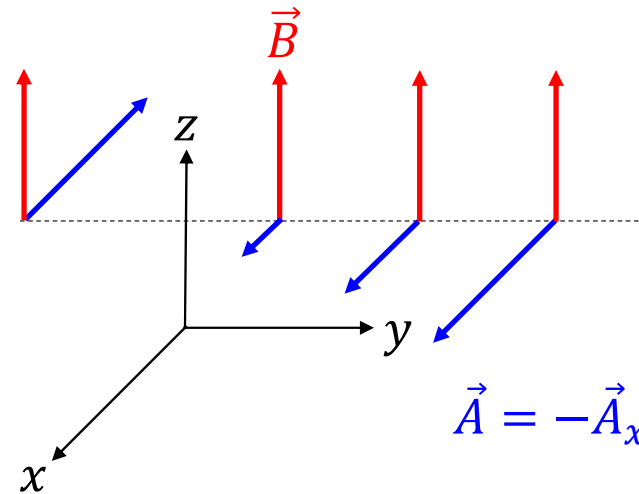
$$\begin{aligned}\vec{B} &= \vec{B}_z = B_z \vec{k} = \mathbf{b} \vec{k} \\ \vec{B}_y &= 0 \\ \vec{B}_x &= 0\end{aligned}$$

What is  $\vec{A}$  ?

$$B_z = (\vec{\nabla} \times \vec{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \mathbf{b}$$

Any of the black vectors  $\vec{A}$  is okay: an infinity

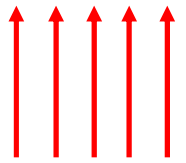
Possibility 2



$$\vec{B} \perp \vec{A}$$

$$\vec{A} = -\vec{A}_x = -A_x \vec{i} = -\mathbf{b}y\vec{i}$$

# Proof that the vector potential $\vec{A}$ is **NOT** unique



$$\begin{aligned}\vec{B} &= \vec{B}_z = B_z \vec{k} = \mathbf{b} \vec{k} \\ \vec{B}_y &= 0 \\ \vec{B}_x &= 0\end{aligned}$$

What is  $\vec{A}$  ?

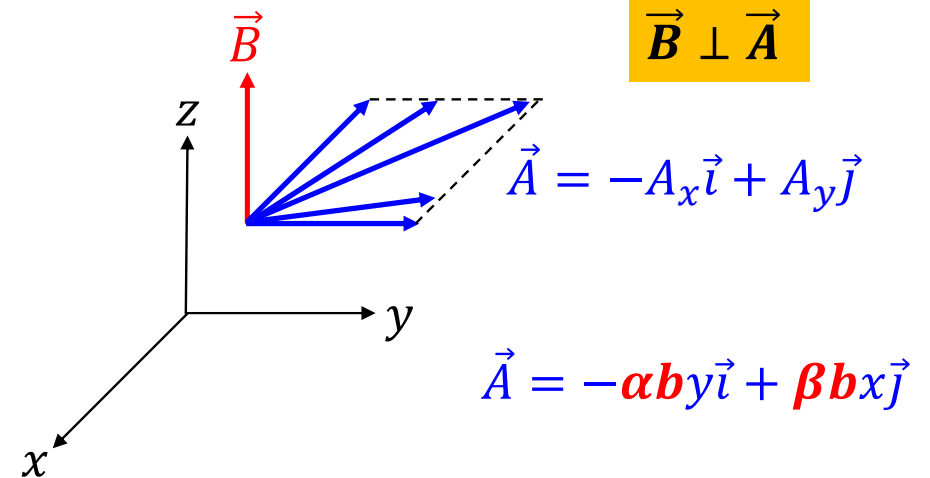
*Any combination of the previous ones*



$$\begin{aligned}\vec{B} &= \vec{B}_z = (\alpha + \beta) \mathbf{b} \vec{k} \\ (\alpha + \beta) &= 1\end{aligned}$$

Infinity of combinations

**Possibility 3**



$$B_z = (\vec{\nabla} \times \vec{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = (\alpha + \beta) \mathbf{b}$$

Any **BLUE** vectors  $\vec{A}$  is okay:  
an infinite number

## In electrostatic

Solution is not unique for  $V$  **BUT** must be unique for  $\vec{E}$

$$\vec{\nabla} \times \vec{E} = \vec{0} \quad \rightarrow \quad \vec{E} = -\vec{\nabla} V \quad V' = V + C$$

Unique  $\vec{E}$  for any arbitrary  $C$

Useful condition for  $V' = V + C$  that does not affect  $\vec{E}$


$$\vec{\nabla} \cdot C = 0$$

$$V(\infty) = 0$$



Likewise Solution is not unique for  $\vec{A}$  **BUT** must be unique for  $\vec{B}$

## Using Curl

$$\vec{B} = \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A}$$


Two possible solutions

$\vec{A}'$  and  $\vec{A}$  have the same curl

$$\vec{\nabla} \times \vec{A}' - \vec{\nabla} \times \vec{A} = \vec{\nabla} \times (\vec{A}' - \vec{A}) = \vec{0}$$

Looks like  
 $\vec{\nabla} \times \vec{E} = \vec{0}$

For which  
 $\vec{E} = -\vec{\nabla}V$

$$(\vec{A}' - \vec{A}) = \vec{\nabla}\psi$$

$$\vec{\nabla} \times (\vec{\nabla}\psi) = \vec{0}$$

$\vec{\nabla}\psi$  does not give rise to  $\vec{B}$

*Slide #74 B\_Lecture 4&7\_Coordinate system\_Scalar versus Vector fields\_Operators*

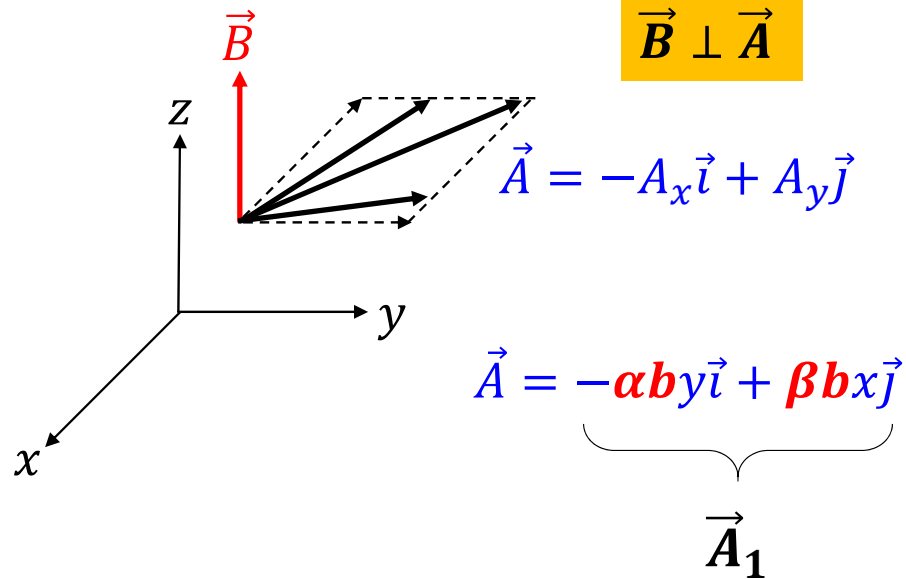
$$\vec{A}' = \vec{A} + \vec{\nabla}\psi$$

Unique  $\vec{B}$  for any arbitrary  $\psi$

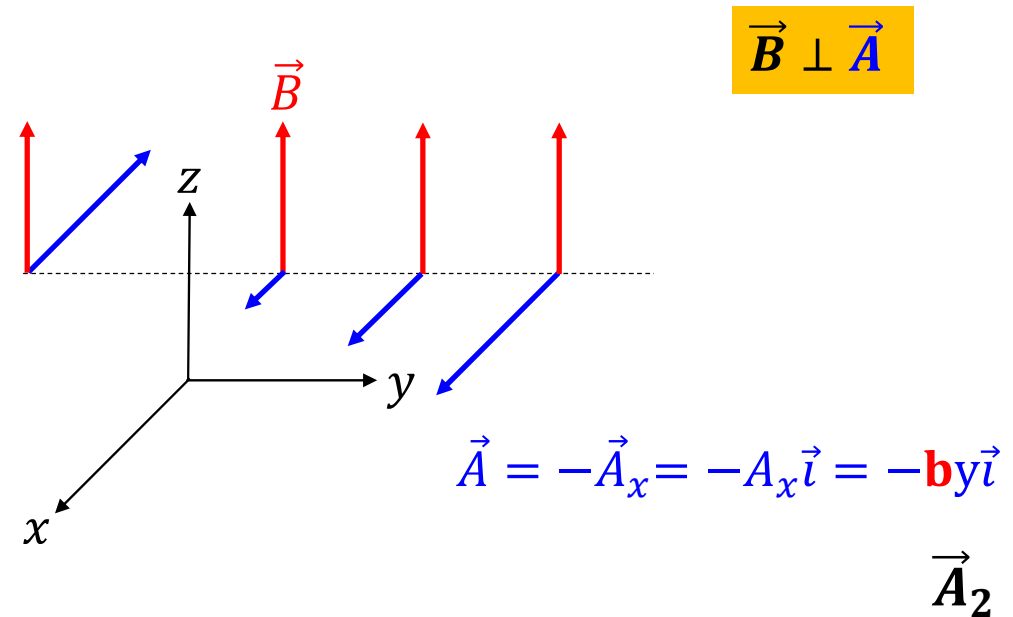
Electrostatic  $V' = V + C$

Example for  $\vec{A}' - \vec{A} = \vec{\nabla}\psi$

Possibility 3



Possibility 2



$$\vec{A}_1 - \vec{A}_2 = by(1 - \alpha)\vec{i} + \beta bx\vec{j} = \vec{\nabla} \left( \frac{bxy}{2} [1 - \alpha + \beta] \right) = \vec{\nabla} (bxy[1 - \alpha]) = \vec{\nabla} b\beta xy$$

$\psi$

$\vec{A}'$  and  $\vec{A}$  have the same Curl  $\Rightarrow \vec{B}$  is unique

**BUT**

**Using Div**

$$\vec{A}' = \vec{A} + \vec{\nabla}\psi \quad \longrightarrow \quad \vec{\nabla} \cdot \vec{A}' = \vec{\nabla} \cdot \vec{A} + \nabla^2 \psi$$

$\vec{A}'$  and  $\vec{A}$  do not need to have the same Divergence

$$\vec{\nabla} \cdot \vec{A}' = \nabla^2 \psi$$

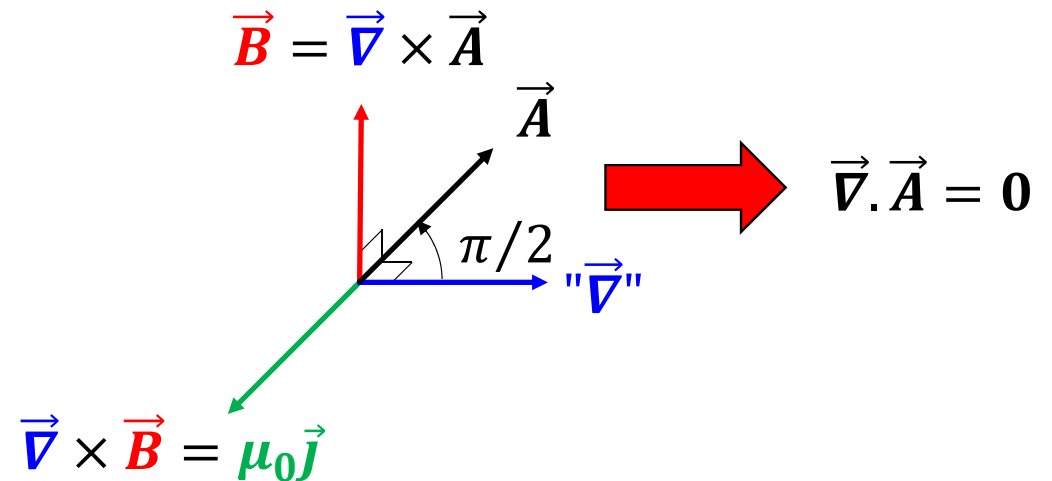
In magnetostatic, the greatest mathematical convenience requires

$$\begin{aligned} \vec{\nabla} \cdot \vec{A} &= 0 \\ \vec{\nabla} \times \vec{A} &= \vec{B} \end{aligned}$$

## Final step: From Mathematics to Physics

$$\vec{U} \times (\vec{V} \times \vec{W}) = \vec{V}(\vec{U} \cdot \vec{W}) - \vec{W}(\vec{U} \cdot \vec{V})$$

*Slide #29\_Lecture 4&7\_Coordinate system\_Scalar versus Vector fields\_Operators*



$$\boxed{\vec{V} \times (\vec{V} \times \vec{A})} = \boxed{\vec{V} \cdot (\vec{V} \cdot \vec{A})} - \nabla^2 \vec{A}$$

$\downarrow$   
 $\vec{V} \times \vec{B} = \mu_0 \vec{J}$

$\downarrow$   
 $\vec{0}$

$\rightarrow$

$\nabla^2 \vec{A} = -\mu_0 \vec{J}$

## Miracle of analogy

What relationship between  $\vec{A}$  and  $\mu_0 \vec{j}$ ?

$$\nabla^2 \vec{A} = -\mu_0 \vec{j}$$

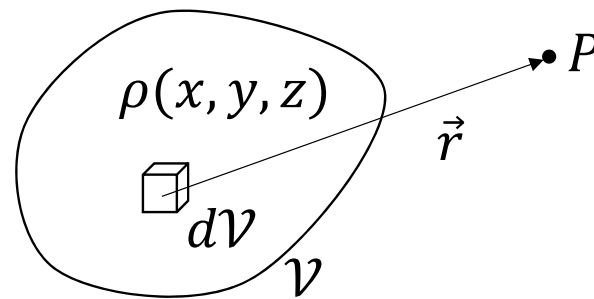
$$\left\{ \begin{array}{l} \nabla^2 A_x = -\mu_0 j_x \\ \nabla^2 A_y = -\mu_0 j_y \\ \nabla^2 A_z = -\mu_0 j_z \end{array} \right.$$

For electrostatic  
Poisson equation

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

$$(A_x, A_y, A_z) \Rightarrow (B_x, B_y, B_z)$$

Electrostatic  $\nabla^2 V = -\frac{\rho}{\epsilon_0}$



$P$  can be inside  $\mathcal{V}$

*$V$  and  $\mathcal{V}$  should not be confused*

Potential  $\uparrow$

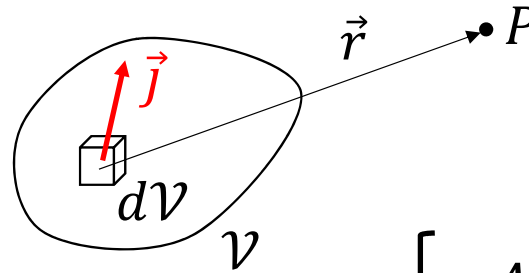
Volume  $\uparrow$

$$V(P) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho(x, y, z) d\mathcal{V}}{r}$$

Magnetostatic

$$\nabla^2 \vec{A} = -\mu_0 \vec{j}$$

Steady current  $\Rightarrow \vec{\nabla} \cdot \vec{j} = 0$



Imaginary “electrostatic” problem

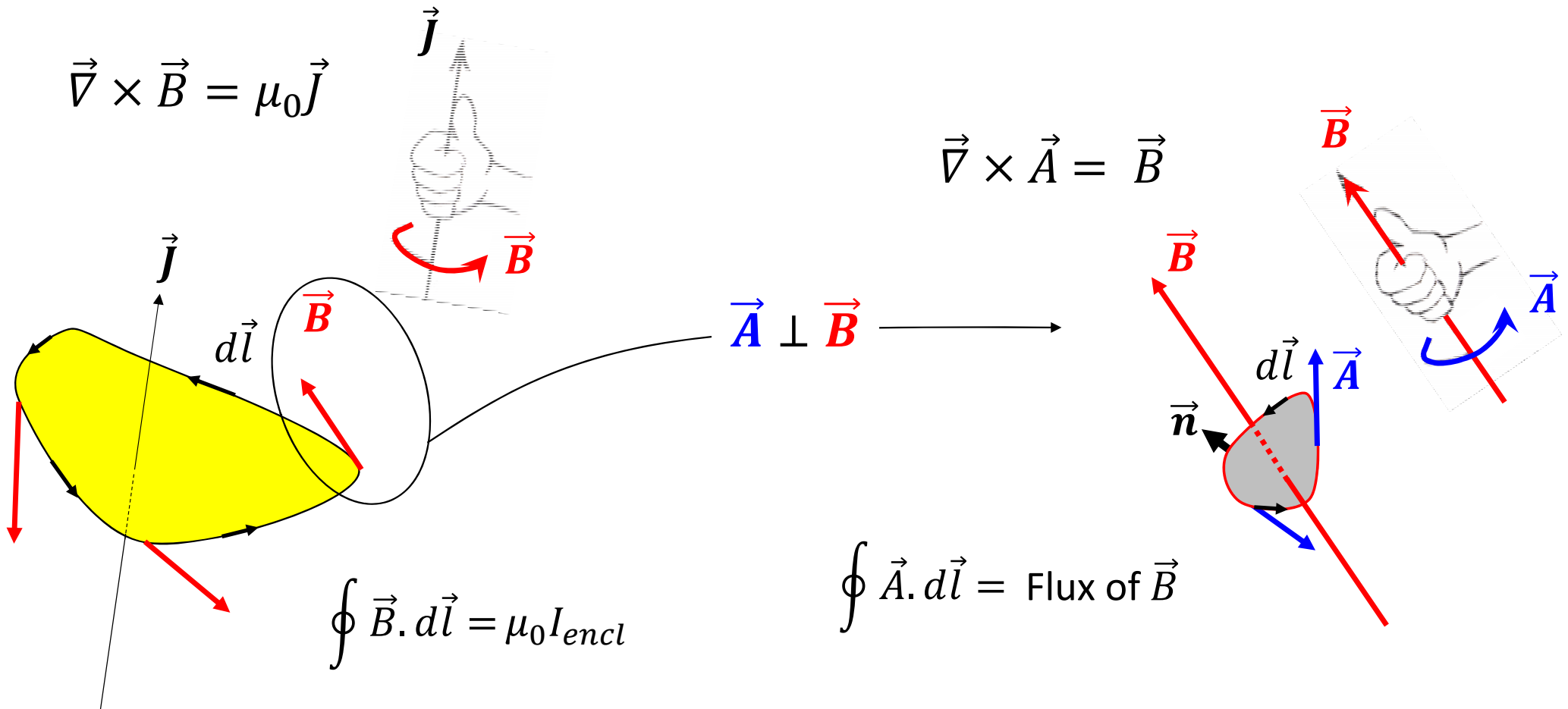
$$\left[ \begin{aligned} A_x(P) &= \frac{\mu_0}{4\pi} \int \frac{j_x(x, y, z) d\mathcal{V}}{r} \\ A_y(P) &= \frac{\mu_0}{4\pi} \int \frac{j_y(x, y, z) d\mathcal{V}}{r} \\ A_z(P) &= \frac{\mu_0}{4\pi} \int \frac{j_z(x, y, z) d\mathcal{V}}{r} \end{aligned} \right.$$

Knowing  $A_x(P)$   $A_y(P)$   $A_z(P)$

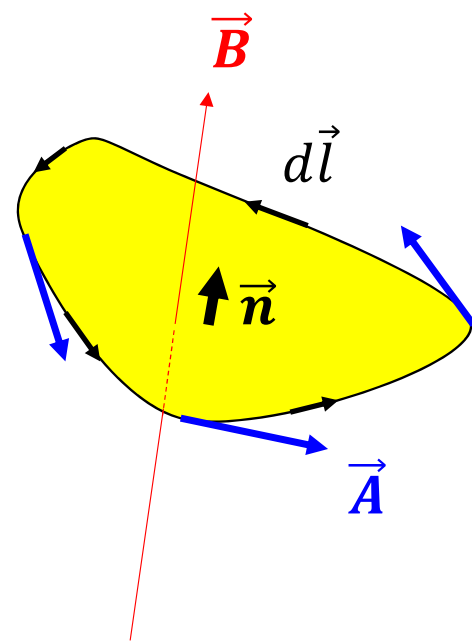
$$\vec{B}(P) = \vec{\nabla} \times \vec{A}(P) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \underbrace{\left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right)}_{B_x(P)} \vec{i} + \underbrace{\left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right)}_{B_y(P)} \vec{j} + \underbrace{\left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)}_{B_z(P)} \vec{k}$$



# Stoke's theorem: Power of analogy



Surface =  $S$  instead of  $A$  (to avoid giving the same label to the area and vector potential)



$$\oint \vec{A} \cdot d\vec{l} = \text{Flux of } \vec{B}$$

$$= \int (\vec{\nabla} \times \vec{A}) d\vec{S}$$

$$= \int \vec{B} d\vec{S}$$



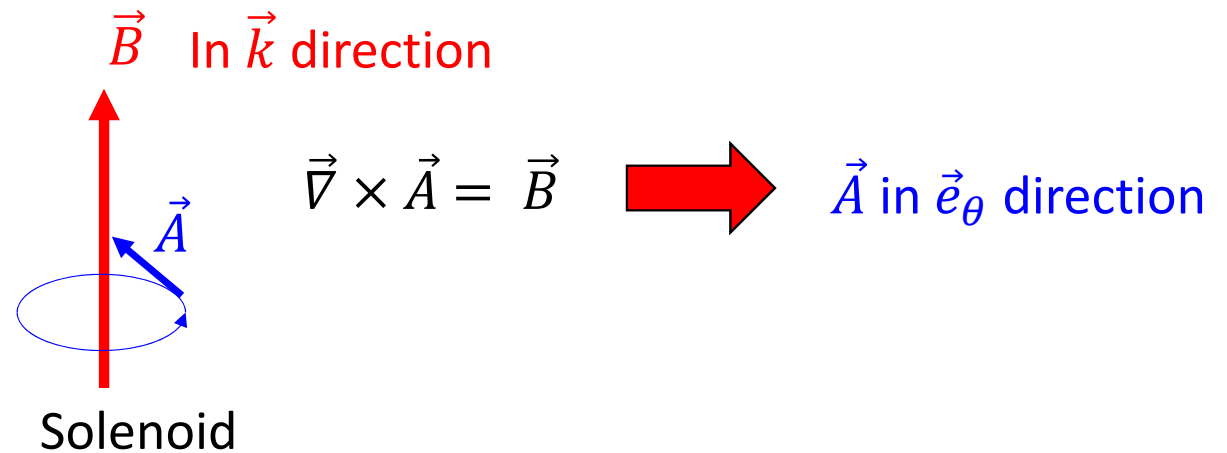
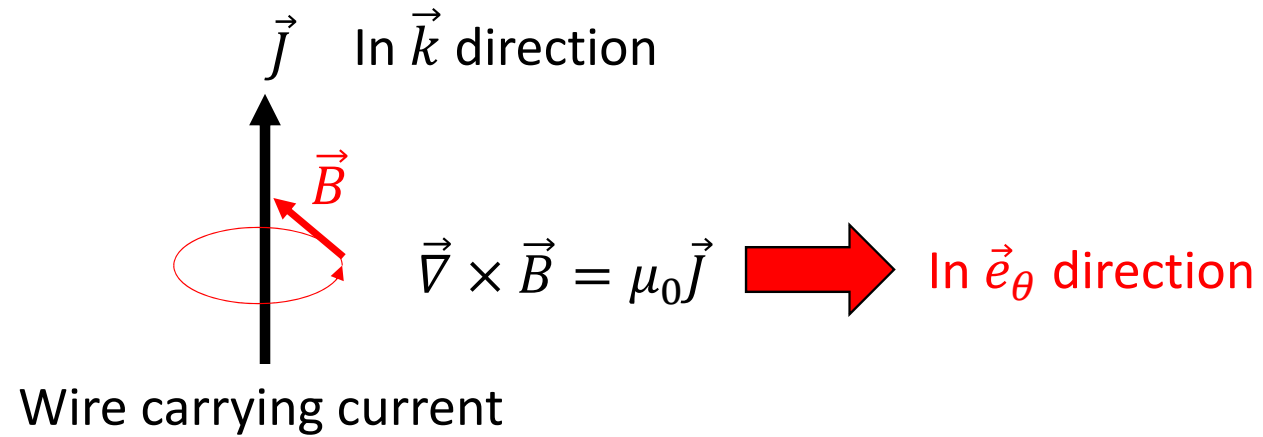
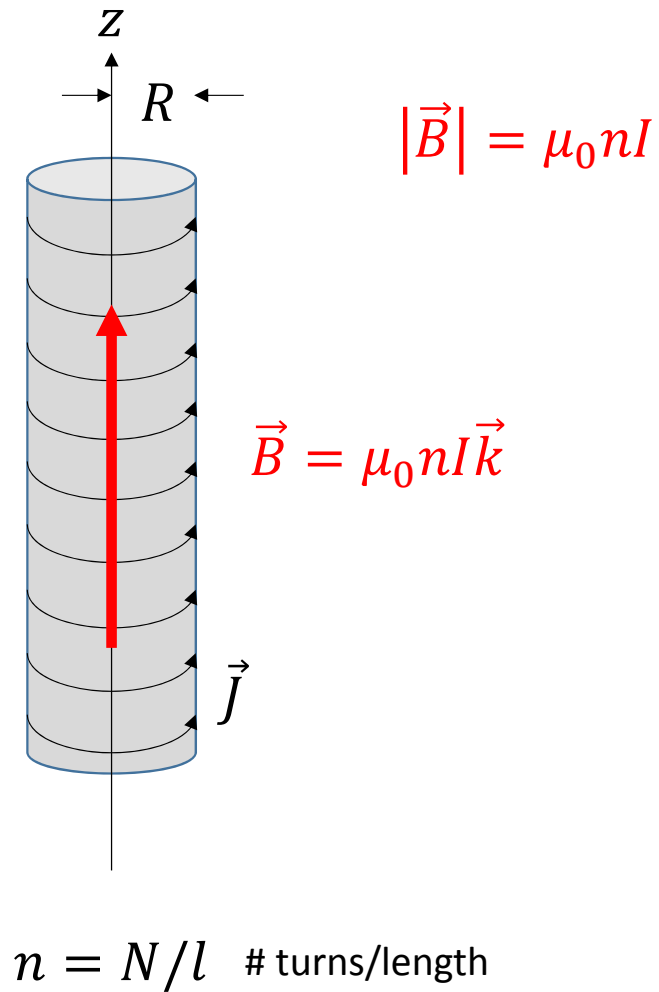
Thomb = direction of the flux of  $\vec{B}$

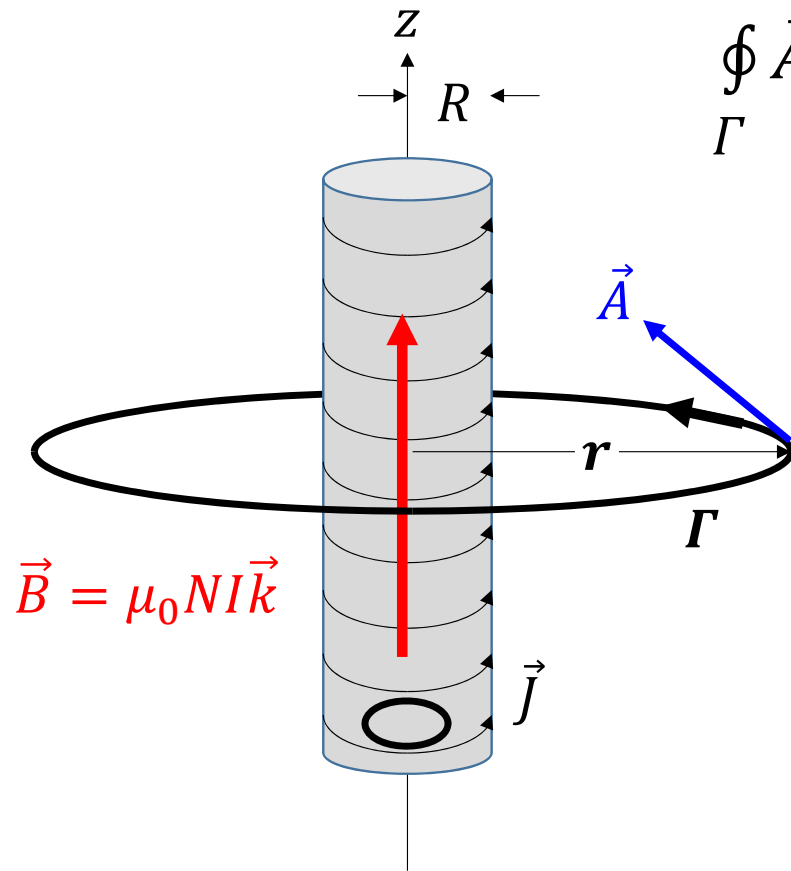
2 examples

Vector potential for a solenoid

Vector potential for a current carrying wire

## Analogy





$\oint_{\Gamma} \vec{A} \cdot d\vec{l} = \text{flux passing through the loop } \Gamma$

$$r > R \quad A_{\theta} 2\pi r = B\pi R^2 = \mu_0 n I \pi R^2$$

$$\vec{A} = \frac{\mu_0 n I R^2}{2r} \vec{e}_{\theta}$$

$$r < R \quad A_{\theta} 2\pi r = B\pi r^2 = \mu_0 n I \pi r^2$$

$$\vec{A} = \frac{\mu_0 n I r}{2} \vec{e}_{\theta}$$

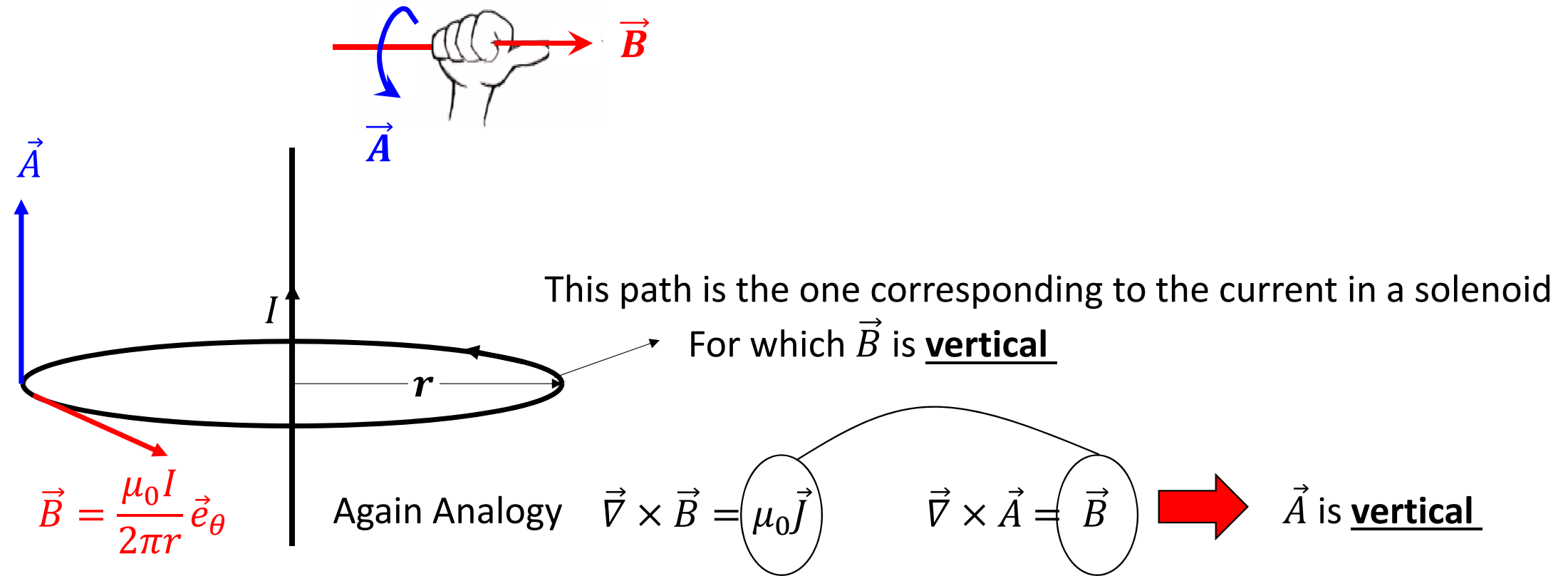
$r = R \Leftrightarrow \text{continuity}$

$$\text{Check: } \vec{\nabla} \times \vec{A} \Big|_z = \frac{1}{r} \left( \frac{\partial r A_\theta}{\partial r} - \frac{\partial A_r}{\partial \theta} \right) \begin{cases} r < R \\ r > R \end{cases} \quad \begin{matrix} \vec{B} = \mu_0 n I \vec{k} \\ \vec{B} = \vec{0} \end{matrix}$$

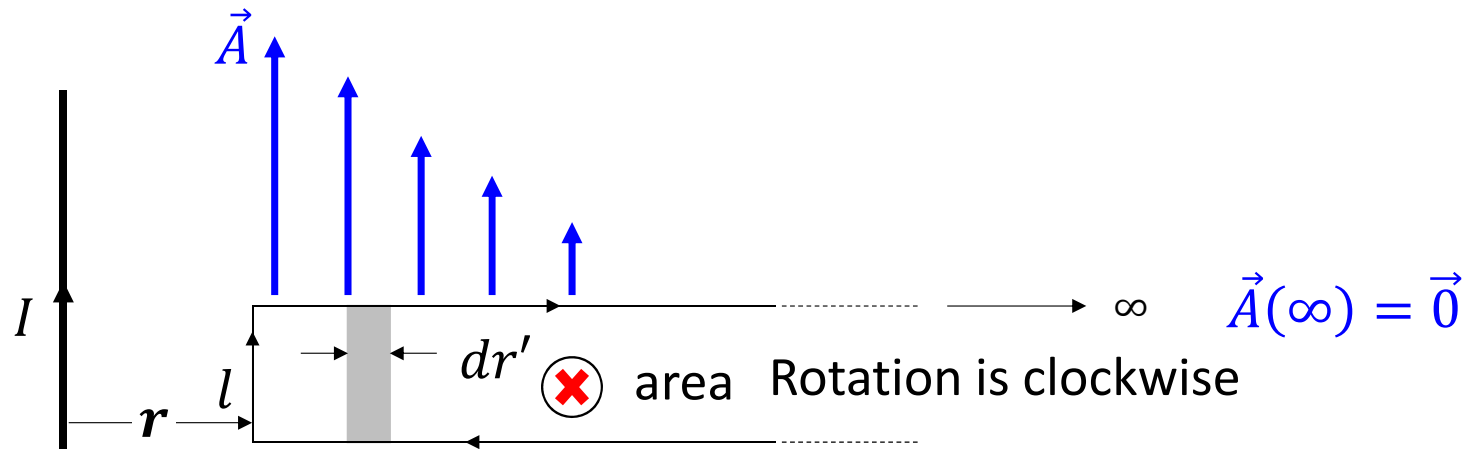
The vector potential  $\vec{A}$  exists in a region where there is no magnetic field !

$$\vec{\nabla} \times \vec{A} = \vec{B}$$

The existence of  $\vec{A}$  has been proven by quantum mechanics and a beautiful experiment:  
**Bohm and Aharanov 1956**



Direction of  $\vec{A}$  is known before any calculation



$\oint_{\Gamma} \vec{A} \cdot d\vec{l} = \text{flux of } \vec{B} \text{ through the area}$

$$\vec{B} = \frac{\mu_0 I}{2\pi r} \vec{e}_\theta$$

$$A(r)l = l \int_r^\infty \frac{\mu_0 I}{2\pi r'} dr' = \frac{l\mu_0 I}{2\pi} \ln r' \Big|_r^\infty = -\frac{l\mu_0 I}{2\pi} \ln r$$

Disregarding infinity

$$A(r) = -\frac{\mu_0 I}{2\pi} \ln r$$

## Applications of the Vector potential

- Wire carrying current
- Magnetic dipole



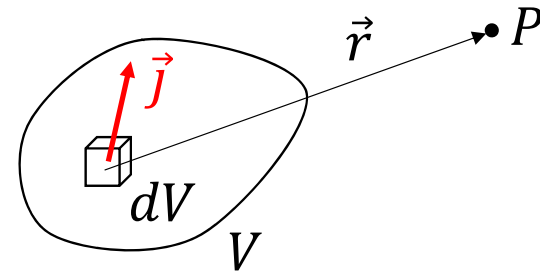
Magnetostatic

$$\nabla^2 \vec{A} = -\mu_0 \vec{j}$$

Ampere's law  $\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$

Steady current  $\Rightarrow \vec{\nabla} \cdot \vec{j} = 0$

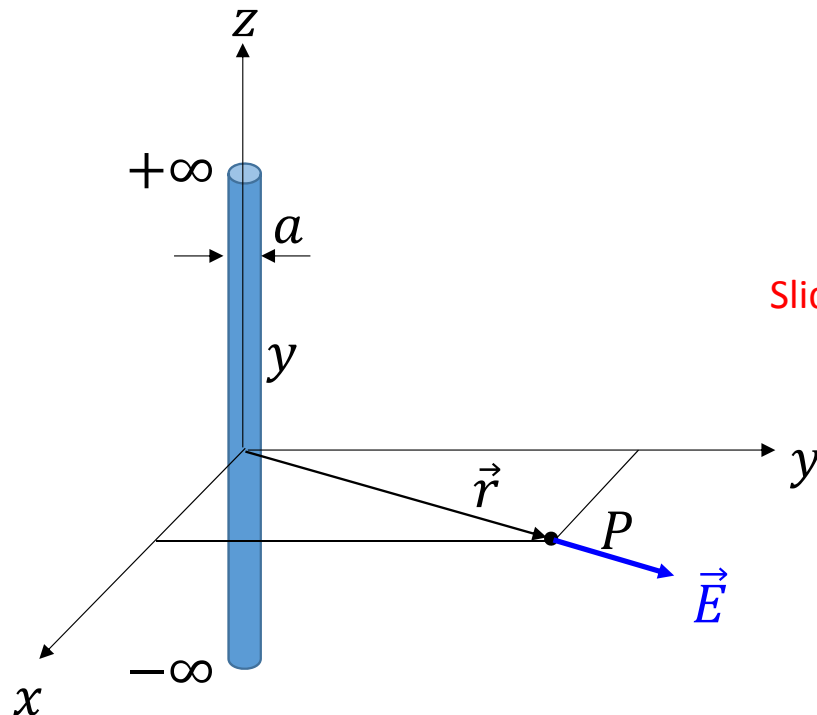
Imaginary electrostatic problem



$$\left[ \begin{aligned} A_x(P) &= \frac{\mu_0}{4\pi} \int \frac{j_x(x, y, z) dV}{r} \\ A_y(P) &= \frac{\mu_0}{4\pi} \int \frac{j_y(x, y, z) dV}{r} \\ A_z(P) &= \frac{\mu_0}{4\pi} \int \frac{j_z(x, y, z) dV}{r} \end{aligned} \right.$$

# Electrostatic versus magnetostatics: Analogy

A straight wire



Volume charge density

$$\rho \cdot \pi a^2 = \frac{dQ}{dl} = \lambda \text{ (linear charge density)}$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \quad \longrightarrow \quad V(r) = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{R}{r}$$

Slide #59 E\_Lectures 8&9 Gauss law in Electrostatics

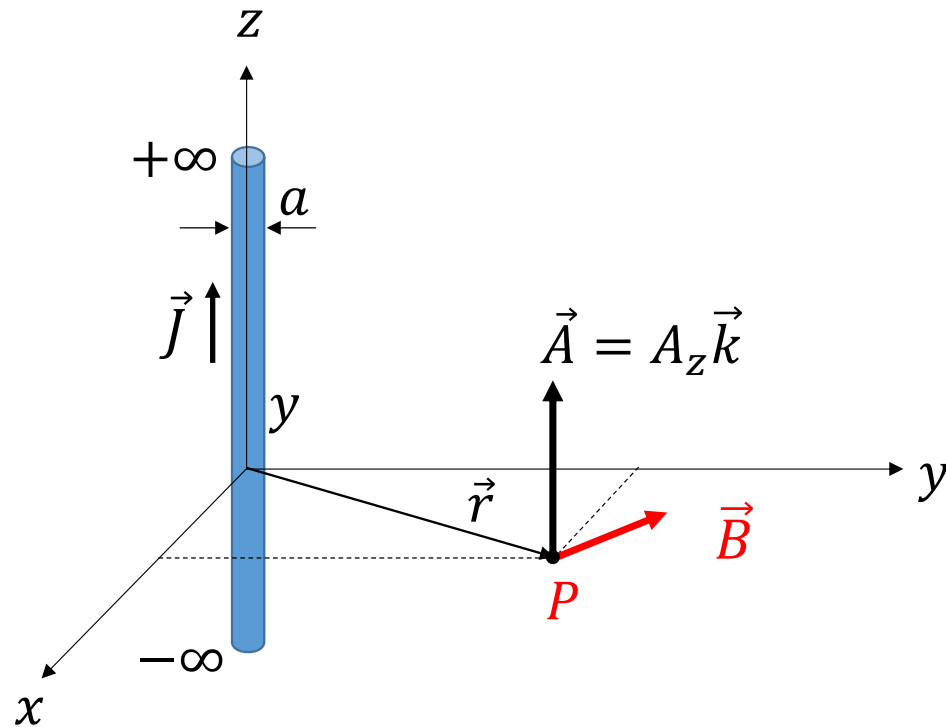
$$\vec{E} = -\vec{\nabla} V$$

$$\vec{E} = E_x \vec{i} + E_y \vec{j}$$

$$E_z = 0$$

# Electrostatic versus magnetostatics: Analogy

## A straight wire



## Magnetostatic

$$j_x = 0 \quad \vec{A}_x(P) = 0$$

$$j_y = 0 \quad \vec{A}_y(P) = 0$$

Electrostatic

$$V = -\frac{\lambda}{2\pi\epsilon_0} \ln r + \text{Constant}$$

Constant to be dropped

By analogy

$$j_z = \frac{I}{\pi a^2} \quad \vec{A}_z(P) = -\frac{\mu_0 j_z \pi a^2}{2\pi} \ln r$$

Knowing  $A_x(P)$   $A_y(P)$   $A_z(P)$

$$\vec{B}(P) = \vec{\nabla} \times \vec{A}(P) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \underbrace{\left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right)}_{B_x(P)} \vec{i} + \underbrace{\left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right)}_{B_y(P)} \vec{j} + \underbrace{\left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)}_{B_z(P)} \vec{k}$$

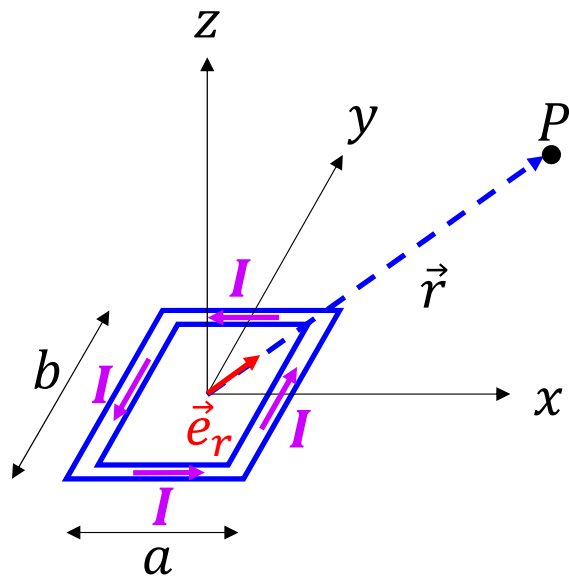
$$B_x = -\frac{\mu_0 I}{2\pi} \frac{y}{r^2} \quad B_y = +\frac{\mu_0 I}{2\pi} \frac{x}{r^2} \quad B_z = 0$$

$$B = \sqrt{(\vec{B}_x)^2 + (\vec{B}_y)^2 + (\vec{B}_z)^2} = \frac{\mu_0 I}{2\pi r}$$

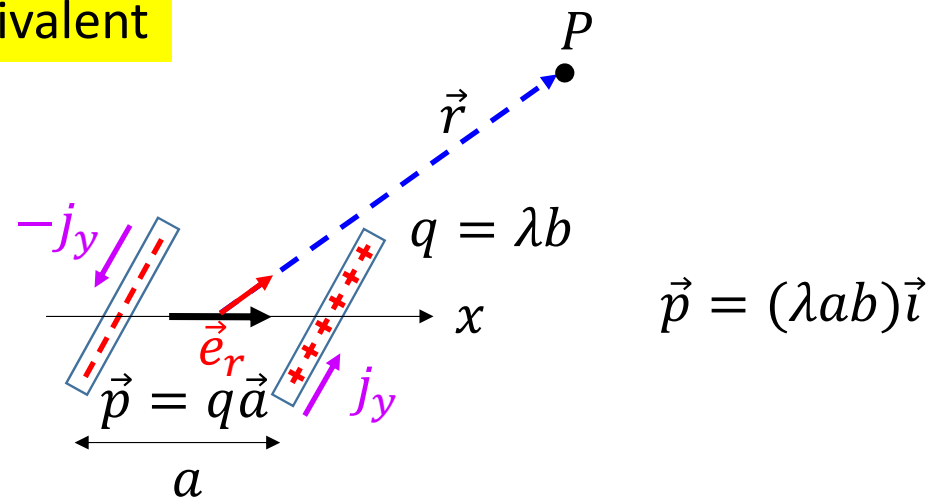
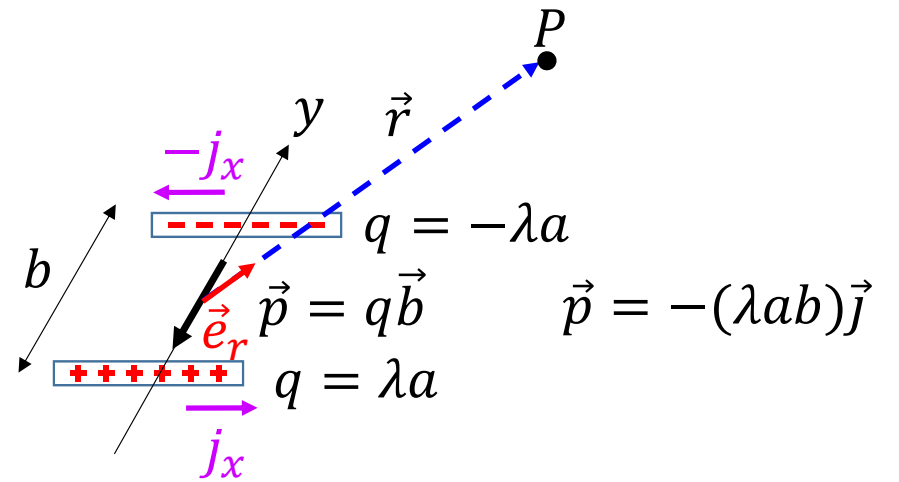
Of course more complicated than  
via simple Ampere's law !

## Current loop: Magnetic dipole

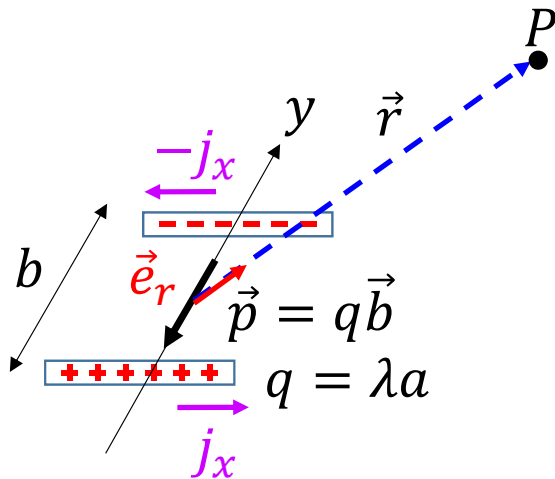
Current loop: Magnetic dipole



Electrostatic equivalent



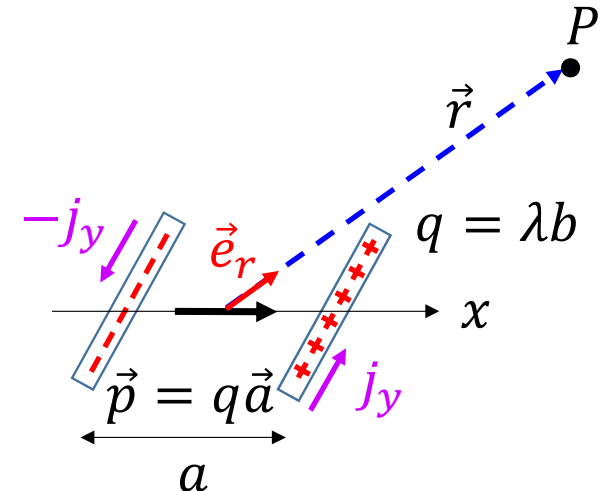
## Electrostatic equivalent



$$\vec{p} = -(\lambda ab)\vec{j}$$

$$V(P) = -\frac{1}{4\pi\epsilon_0} \frac{\lambda ab}{r^2} \frac{y}{r}$$

$$V(P) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \vec{e}_r}{r^2}$$

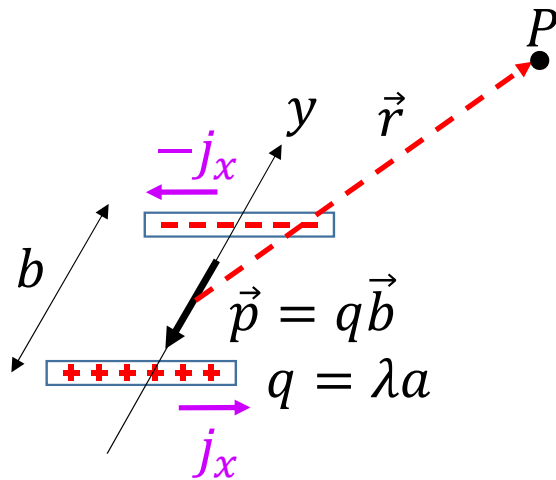


$$\vec{p} = (\lambda ab)\vec{i}$$

$$V(P) = \frac{1}{4\pi\epsilon_0} \frac{\lambda ab}{r^2} \frac{x}{r}$$

$$V(P) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \vec{e}_r}{r^2}$$

There is a  $\cos(\theta)$  here

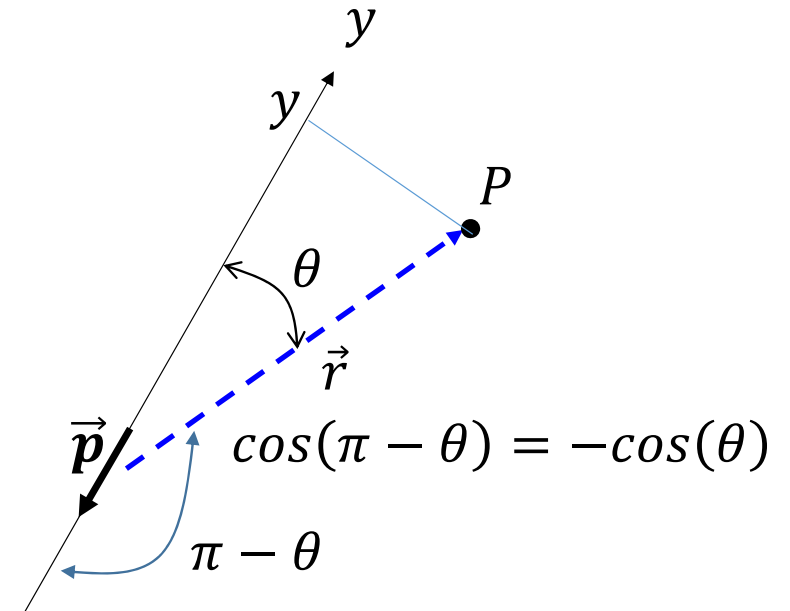


$$\vec{p} \cdot \vec{e}_r = \lambda ab \vec{j} \cdot \vec{e}_r$$



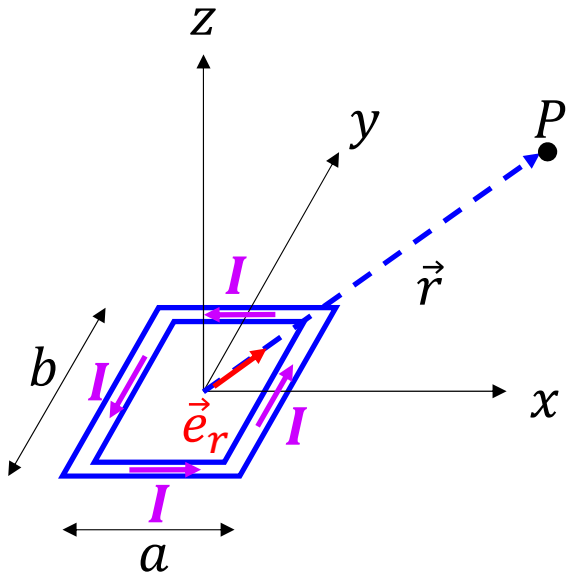
$$\vec{j} \cdot \vec{e}_r = \cos(\theta)$$

$$\cos(\theta) = \frac{y}{r}$$





## Current loop: From electrostatic to equivalent magnetostatic



$$V(P) = -\frac{\lambda ab}{4\pi\epsilon_0} \frac{y}{r^3}$$

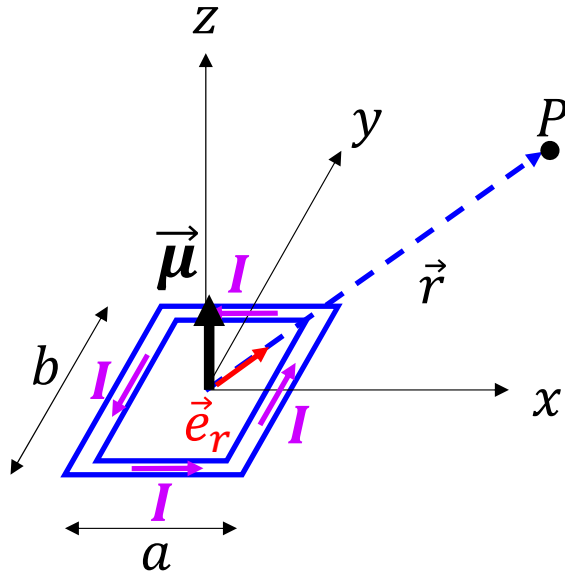
$$V(P) = \frac{\lambda ab}{4\pi\epsilon_0} \frac{x}{r^3}$$

$$A_x(P) = -\frac{\mu_0 I ab}{4\pi} \frac{y}{r^3}$$

$$A_y(P) = \frac{\mu_0 I ab}{4\pi} \frac{x}{r^3}$$

$$A_z(P) = 0 \text{ as } j_z(x, y, z) = 0$$

## Current loop: Magnetic dipole



$\mu = Iab \longrightarrow$  Vector  $\vec{\mu}$  along the  $z$ -axis

*The vector potential has a real physical meaning*

$$A_z(P) = 0 \text{ as } j_y(x, y, z) = 0$$

$$A_x(P) = -\frac{\mu_0 Iab}{4\pi} \frac{y}{r^3}$$

$$A_y(P) = \frac{\mu_0 Iab}{4\pi} \frac{x}{r^3}$$

$$\vec{\mu} = 0\vec{i} + 0\vec{j} + \mu\vec{k}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{A}(P) = \frac{\mu_0}{4\pi} \frac{\vec{\mu} \times \vec{r}}{r^3}$$

## Magnetic dipole

$$B_x = (\vec{\nabla} \times \vec{A})_x$$

$$B_x = \frac{\mu_0 \mu}{4\pi} \frac{3xz}{r^5}$$

$$B_y = (\vec{\nabla} \times \vec{A})_y$$

$$B_y = \frac{\mu_0 \mu}{4\pi} \frac{3yz}{r^5}$$

$$B_z = (\vec{\nabla} \times \vec{A})_z$$

$$B_z = -\frac{\mu_0 \mu}{4\pi} \left( \frac{1}{r^3} - \frac{3z^2}{r^5} \right)$$

$$\vec{\mu} = I\vec{A} \quad \text{magnetic dipole moment}$$

## Electric dipole

$$E_x = \frac{p}{4\pi\epsilon_0} \frac{3xz}{r^5}$$

$$E_y = \frac{p}{4\pi\epsilon_0} \frac{3yz}{r^5}$$

$$E_z = -\frac{p}{4\pi\epsilon_0} \left( \frac{1}{r^3} - \frac{3z^2}{r^5} \right)$$

$$\vec{p} = q\vec{d} \quad \text{electric dipole moment}$$

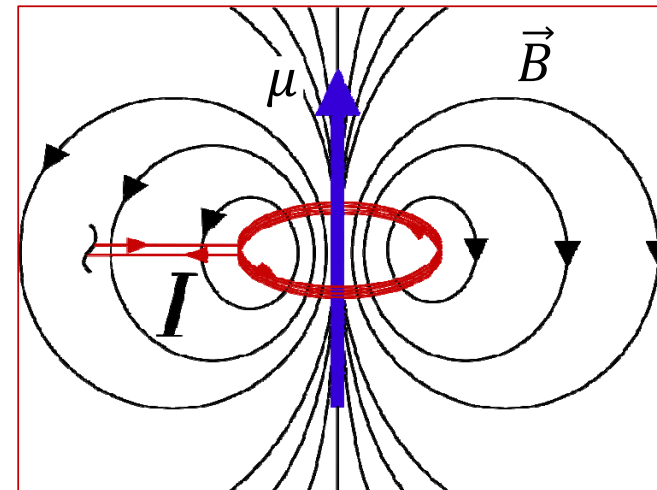
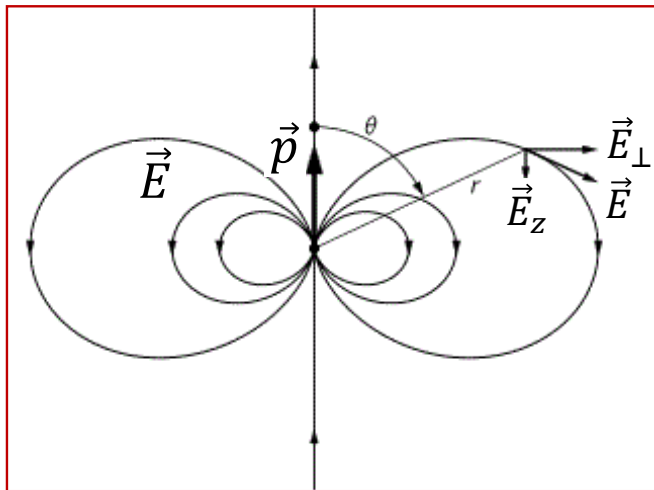
## Electrostatic

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

## Magnetostatic

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

Starting with completely different laws  
and ending up with the same kind of field



## Some important remarks

In the case of dipoles

Electric  $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

Magnetic  $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

Completely different laws



Lead to the same kind of fields

Why ?

Because the dipole fields appear only when we are far from the charges and currents

 Through most of the relevant space there are no charges nor currents

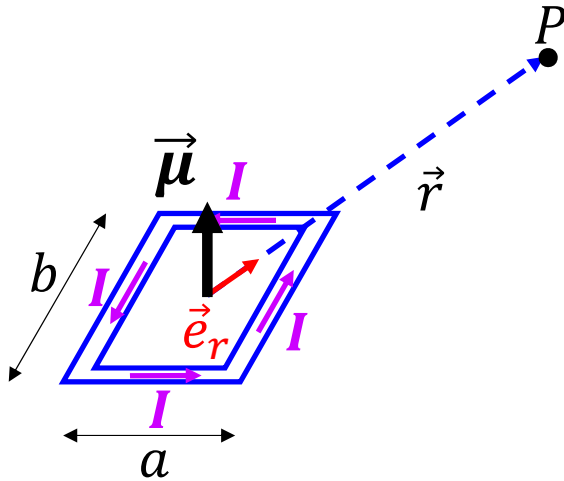
$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= 0 \\ \vec{\nabla} \times \vec{E} &= 0\end{aligned}$$

Both give the  
same solutions

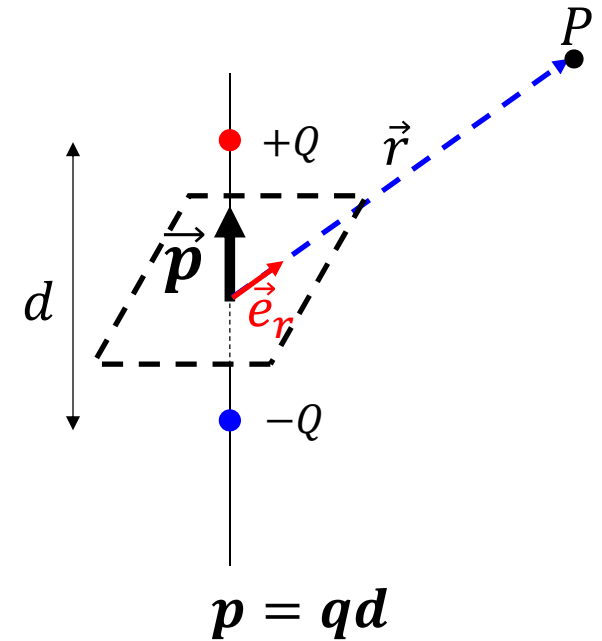
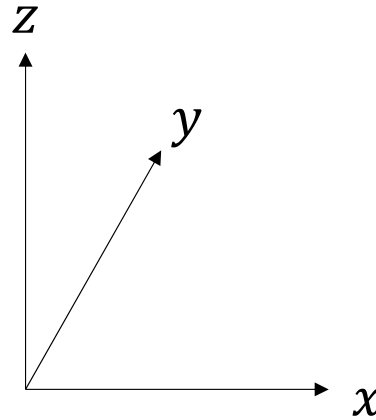
$$\begin{aligned}\vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{B} &= 0\end{aligned}$$

$\vec{E}$  and  $\vec{B}$  identical

What do we observe far away?



$$\mu = Iab$$

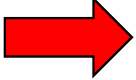


$$p = qd$$

Far away we see only dipole moments  $\vec{\mu}$  or  $\vec{p}$

What is the advantage of the vector potential?

$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$   Integrals of  $\vec{B}$  are very complicated

$\vec{B} = \vec{\nabla} \times \vec{A}$   Derivatives of  $\vec{A}$  much simpler  
Integrals of  $\vec{A}$  are those of electrostatics (**known**)

## Static Maxwell's equations in vacuum

### electrostatic

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\ \vec{\nabla} \times \vec{E} &= \vec{0}\end{aligned}$$

### Magnetostatic

$$\begin{aligned}\vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{B} &= \mu_0 \vec{J}\end{aligned}$$



What is true only in a static world and what is true always?

Electric and Magnetic phenomena are no longer two distinct subjects in a **DYNAMIC** world: Maxwell

## What is true and what is wrong in what we have learned so far?

Wrong in general (true only for static)

Coulomb's law  $\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \vec{e}_r$

$$\vec{\nabla} \times \vec{E} = \vec{0}$$

$$\vec{E} = -\vec{\nabla} V$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}') \vec{e}_r}{r^2} dV'$$

For conductor,  $E = 0, V = \text{constant}, Q = CV$

True always

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \quad \text{Lorentz force}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \text{Gauss' law}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

In conductor,  $E$  makes currents

## What is true and what is wrong in what we have learned so far?

Wrong in general (true only for static)

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad \text{Ampere's law}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}') \times \vec{e}_r}{r^2} dV' \quad \text{Biot \& Savart's law}$$

True always

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \quad \text{Maxwell's law}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{No magnetic charges}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

## What is true and what is wrong in what we have learned so far?

Wrong in general (true only for static)

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \quad \text{Poisson's law}$$

$$\nabla^2 \vec{A} = -\mu_0 \vec{J} \quad \text{With}$$

$$\vec{\nabla} \cdot \vec{A} = 0$$

True always

$$\nabla^2 V - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0} \quad \text{And}$$

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} \quad \text{With}$$

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = 0$$

# End of static world