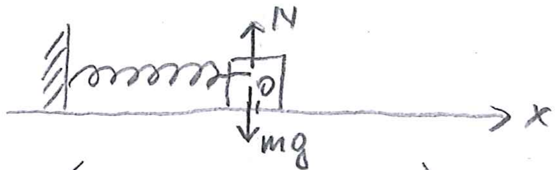
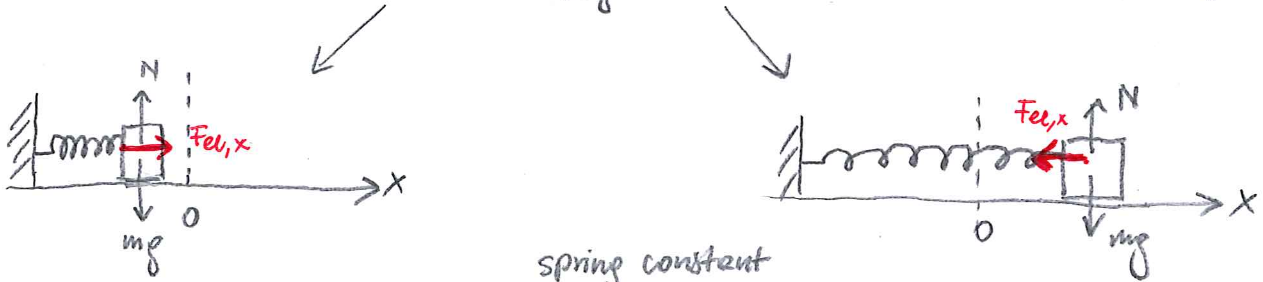
PERIODIC MOTION, HARMONIC OSCILLATOR AND MECHANICAL RESONANCE

Motivation: three systems

(A) horizontal spring-mass system (no friction; massless spring)



equilibrium ($x = 0$)
spring not streched/compressed



$x < 0 \Rightarrow F_{el,x} > 0$ $\qquad$ $x > 0 \Rightarrow F_{el,x} < 0$

spring constant

$$\boxed{F_{el,x} = -kx} \quad \text{(Hooke's law)}$$

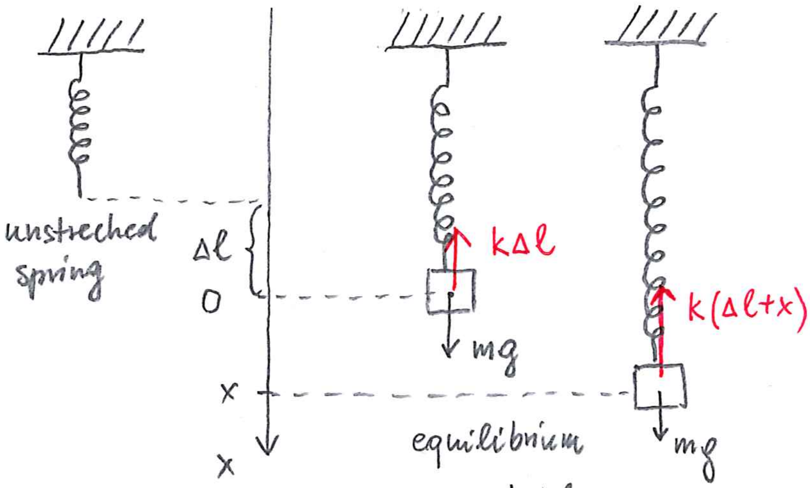
↳ restoring force (here – the elastic force)
tries to bring the mass back to equilibrium

Equation of motion

net force F_x

$$ma_x = F_{el,x} \iff \ddot{x} + \frac{k}{m}x = 0$$

(B) vertical spring-mass system (massless spring)



equilibrium
$mg - k\Delta l = 0$

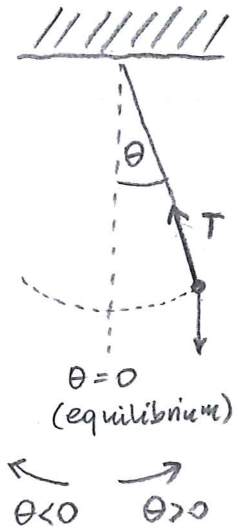
Net force

$$F_x = mg - k(x + \Delta l) =$$
$$= \underbrace{mg - k\Delta l}_{=0} - kx =$$
$$= -kx$$

Equation of motion

$$ma_x = -kx \iff \ddot{x} + \frac{k}{m}x = 0$$

(c) simple pendulum (or mathematical pendulum)



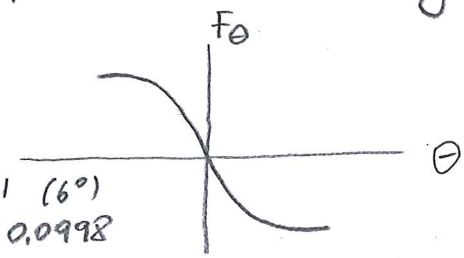
Tangential component of the net force (due to mg only)

$$F_\theta = -mg\sin\theta$$

For small angles:

$$\sin\theta \approx \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$$

keep the 1st term only (APPROXIMATION!)



$$F_\theta \approx -mg\theta$$

Eg. $\theta = 0{,}1$ ($6°$)
$\sin\theta = 0{,}0998$

Motion along the arc of the circle (equ. of motion for the tangential component)

$$m a_\theta = F_\theta \quad \overset{a_\theta = l\ddot{\theta}}{\Longleftrightarrow} \quad \cancel{m} l\ddot{\theta} \approx -\cancel{m} g\theta$$

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

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OBSERVATION

In all three cases the equation of motion is of the same form

$$(*) \qquad \ddot{x} + \omega_0^2 x = 0 \qquad \begin{cases} \omega_0^2 = \frac{k}{m} & \text{mass-spring} \\ \omega_0^2 = \frac{g}{l} & \text{simple pendulum} \end{cases}$$

Any system for which the equation of motion is of the form (*) is called a simple harmonic oscillator (SHO).

Mathematical problem to solve (case $\boxed{A}$ - only the restoring force acts)

$\boxed{A}$ $$\ddot{x} + \omega_0^2 x = 0 \qquad \text{find } x = x(t)$$

Note*: 2nd order ODE (linear, with constant coefficients)

How to solve? (guess & check)

Guess: $x(t) = \cos(\omega_0 t)$ & check: $\dot{x} = -\omega_0 \sin(\omega_0 t)$

$$\ddot{x} = -\omega_0^2 \cos(\omega_0 t) = -\omega_0^2 x$$
$$\Downarrow$$
$$\ddot{x} + \omega_0^2 x = 0 \quad \checkmark$$

Note that: $x(t) = A\cos(\omega_0 t) \Rightarrow \ddot{x} = -\omega_0^2 x$

and $x(t) = A\cos(\omega_0 t + \varphi) \Rightarrow \ddot{x} = -\omega_0^2 x$

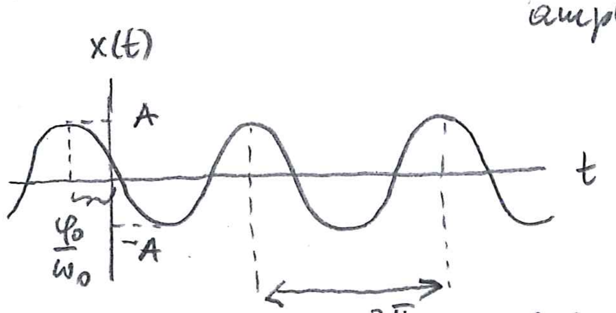
also satisfy the SHO equation of motion.

Hence, the most general solution

natural angular frequency (eg. mass-spring system $\omega_0 = \sqrt{k/m}$; pendulum $\omega_0 = \sqrt{g/\ell}$)

$$\boxed{x(t) = A\cos(\omega_0 t + \varphi_0)}$$

amplitude — phase shift



oscillatory (periodic) behavior

$T = \frac{2\pi}{\omega_0}$ period (e.g. mass spring system $T = 2\pi\sqrt{m/k}$
simple pendulum $T = 2\pi\sqrt{\ell/g}$)

Comment (see Problem Set 4): the most general solution can also be written as

$$\boxed{x(t) = B\cos(\omega_0 t) + C\sin(\omega_0 t)} \quad \text{(again two constants)}$$

The two constants A and φ (or B and C) are found by applying the initial conditions: $x(0) = x_0$ and $v_x(0) = v_{0x}$.

A general rule*: 2nd order ODEs have general solutions depending on 2 parameters (constants).

Question: Are there any other solutions? NO! (existence & uniqueness thms.)

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Position, velocity, and acceleration in simple harmonic motion

[Fig. 14.11, 14.12]

Observations: (*) velocity shifted by 1/4 of a cycle ($\frac{\pi}{2}$-shift) w.r.t. position

(*) acceleration shifted by $\frac{1}{2}$ of a cycle (π-shift) w.r.t. position

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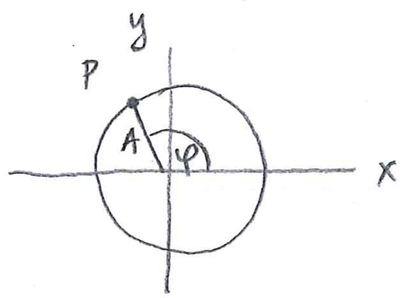
Simple harmonic motion and uniform circular motion



$$\frac{d\varphi}{dt} = \omega_0 = \frac{v}{A} = \text{const} \Rightarrow \varphi = \omega_0 t$$

(assume $\varphi(0)=0$ i.e. $\varphi_0 = 0$)

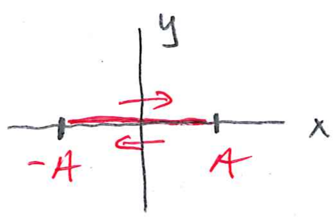
Hence

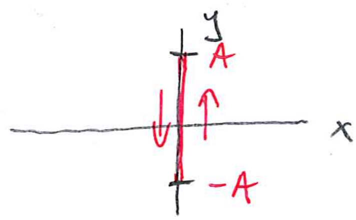
$$\begin{cases} x(t) = A\cos\varphi = A\cos(\omega_0 t) \\ y(t) = A\sin\varphi = A\sin(\omega_0 t) \end{cases}$$

Differentiate twice w.r.t. time

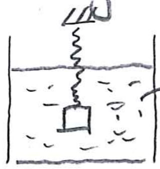
$$\begin{cases} a_x = -A\omega_0^2\cos(\omega_0 t) = -\omega_0^2 x \\ a_y = -A\omega_0^2\sin(\omega_0 t) = -\omega_0^2 y \end{cases}$$

Conclusion: The projection of point P onto the x axis (or y axis) moves as if it was in a simple harmonic motion





B More realistic model: restoring force + linear drag (damping)



Equ. of motion

$$m a_x = F_x = -kx - b v_x \qquad (b>0)$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

Recall that $\frac{k}{m} = \omega_0^2$, so

$$\boxed{\ddot{x} + \frac{b}{m}\dot{x} + \omega_0^2 x = 0}$$

(more complicated than before)
still linear eqn.!

How to solve? → guess & check ✗ (not easy to guess directly)

→ try $x(t) = e^{\lambda t}$ (needs to find λ) ✓
λ ↳ number

Then $\dot{x} = \lambda e^{\lambda t} = \lambda x$

$\ddot{x} = \lambda^2 e^{\lambda t} = \lambda^2 x$

Plug back into the eqn. of motion

$$\lambda^2 x + \frac{b}{m}\lambda x + \omega_0^2 x = 0$$

$$\Downarrow$$

$$\boxed{\lambda^2 + \frac{b}{m}\lambda + \omega_0^2 = 0} \quad (**)$$

Observation: A differential eqn. turned into an algebraic (quadratic) eqn — we know how to solve it!

Solution (roots) of (**) depends on the sign of

$$\Delta = \left(\frac{b}{m}\right)^2 - 4\omega_0^2$$

$\Delta < 0$ → 1° complex roots

$$\lambda_{1,2} = -\frac{b}{2m} \pm i\sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

$(i^2 = -1)$

$\Delta > 0$ → 2° real roots (different)

$$\lambda_{1,2} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \omega_0^2}$$

$\Delta = 0$ → 3° real root (repeated)

$$\lambda = -\frac{b}{2m}$$

Analyze these 3 cases

1° $\Delta < 0 \Rightarrow \left(\frac{b}{m}\right)^2 - 4\omega_0^2 < 0 \Rightarrow \boxed{\left(\frac{b}{m}\right)^2 < 4\omega_0^2}$ damping not too strong (underdamped regime)

General solution (linear combination of two solutions)

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = C_1 e^{-\frac{b}{2m}t} e^{i\sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}\,t} + C_2 e^{-\frac{b}{2m}t} e^{-i\sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}\,t}$$

But $x(t)$ - physical quantity $\Rightarrow$ must be real, so

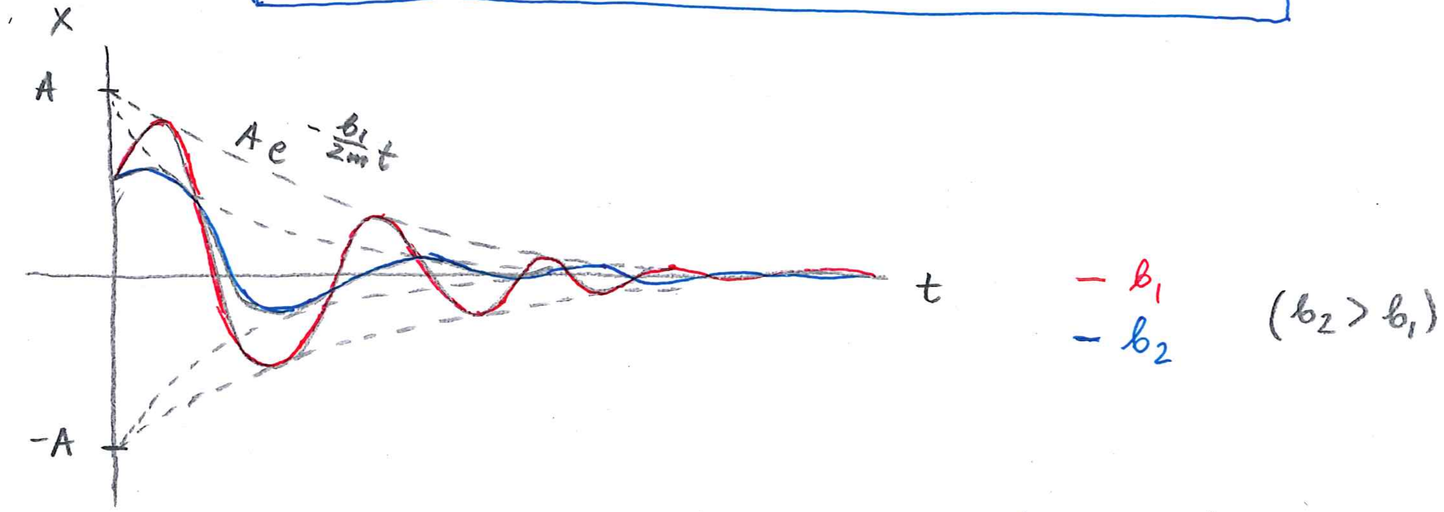
$$C_1 = \frac{1}{2}\underbrace{A}_{\text{real}} e^{i\varphi_0} = C_2^* \Rightarrow C_2 = \frac{1}{2} A\, e^{-i\varphi_0}$$

Then

$$x(t) = \frac{1}{2} A e^{-\frac{b}{2m}t} e^{i\left(\sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}\,t + \varphi_0\right)} + \frac{1}{2} A e^{-\frac{b}{2m}t} e^{-i\left(\sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} + \varphi_0\right)}$$

Hence (Recall: $e^{iu} = \cos u + i \sin u$)

$$\boxed{x(t) = A e^{-\frac{b}{2m}t} \cos\left(\sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}\,t + \varphi_0\right)}$$ (underdamped regime)



Effects of weak damping (underdamped regime)

* motion still periodic, but the amplitude of oscillations decreases exponentially with time

* the angular frequency of oscillations $\omega^2 = \omega_0^2 - \left(\frac{b}{2m}\right)^2 < \omega_0$, so it is smaller than ω_0 (for undamped oscillations). Consequently, the period increases ($T = 2\pi/\omega$)

2° $\Delta > 0 \Rightarrow \boxed{\left(\frac{b}{m}\right)^2 > 4\omega_0^2}$ strong damping (overdamped regime)

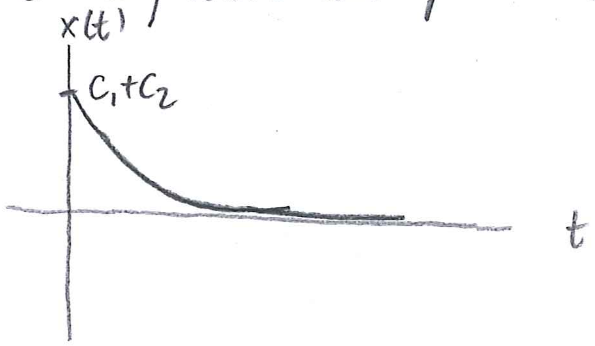
General solution:

$$\boxed{x(t) = C_1 e^{-\left(\frac{b}{2m} - \sqrt{\left(\frac{b}{2m}\right)^2 - \omega_0^2}\right)t} + C_2 e^{-\left(\frac{b}{2m} + \sqrt{\left(\frac{b}{2m}\right)^2 - \omega_0^2}\right)t}}$$

(overdamped regime)

Effects of strong damping (overdamped regime)

* motion is aperiodic - the system returns aperiodically to the equilibrium position



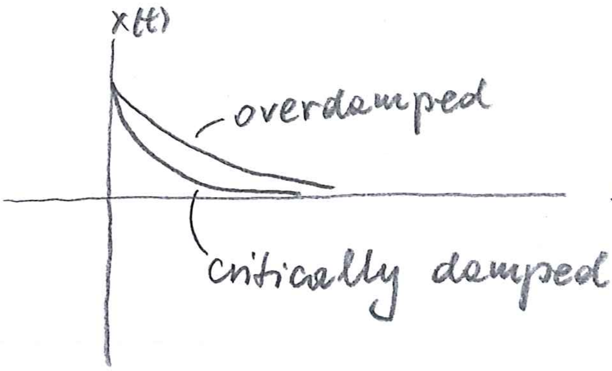
3° $\Delta = 0 \Rightarrow \boxed{\left(\frac{b}{m}\right)^2 = 4\omega_0^2}$ critical damping

$$\boxed{x(t) = D_1 e^{-\frac{b}{2m}t} + D_2 t e^{-\frac{b}{2m}t}}$$

provides another (linearly) independent solution $te^{-\lambda t}$

Effects of critical damping

* aperiodic motion
* the system may pass through the equilibrium position at most once (see Problem Set).



C forced (or driven) oscillations & mechanical resonance

Now, one more element in the model:

restoring force + linear drag + driving force F_{dr}

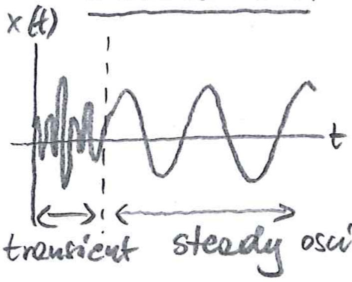
Simplest case: sinusoidally-varying force $F_{dr} = F_0 \cos(\omega_{dr} t)$

ω_{dr} ↳ driving frequency

Equation of motion:

$$m a_x = F_x = -kx - b v_x + F_0 \cos(\omega_{dr} t)$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \underbrace{\frac{k}{m}}_{=\omega_0^2} x = \frac{F_0}{m}\cos(\omega_{dr} t) \qquad (\square)$$



Observation: After some time, oscillations stabilize, and the particle (system) oscillates with the angular frequency of the driving force (there may be a phase-shift)

Solution to $(\square)$

$$x(t) = \underbrace{\text{solution to equ. of motion from } \boxed{B}}_{\text{vanishes as } t \to \infty} + \underbrace{\text{periodic, steady-state oscillations with angular frequency } \omega_{dr}}_{x_s(t)}$$

The steady-state solution

$$\boxed{x_s(t) = A \cos(\omega_{dr} t + \varphi)}$$

ω_{dr}: driving frequency

φ ↳ phase shift (assume $\varphi < 0$, so that it represents phase-lag)

Detailed calculations (skipped here) show that

$$\boxed{\begin{aligned} A &= \frac{F_0}{m\sqrt{(\omega_0^2 - \omega_{dr}^2)^2 + \left(\frac{b\,\omega_{dr}}{m}\right)^2}} = A(\omega_{dr}) \\[2ex] \tan\varphi &= \frac{b\,\omega_{dr}}{m(\omega_{dr}^2 - \omega_0^2)} \end{aligned}}$$

$A(\omega_{dr})$ ↳ depends on ω_{dr}

↳ in general $\varphi \neq 0 \Rightarrow F_{dr}$ & x_s are NOT in phase

Discussion of the results

$$A = \frac{F_0}{m\sqrt{(\omega_0^2-\omega_{dr}^2)^2 + \left(\frac{b\omega_{dr}}{m}\right)^2}}$$

resonance curves

[FIG. 14, 28 /p. 460]

Features:

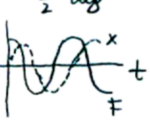
A

* peak in the curve $A = A(\omega_{dr})$ at $\omega_{dr} = \omega_{res} = \sqrt{\omega_0^2 - \frac{b^2}{2m^2}}$
  resonance frequency

  sharp increase in the amplitude of oscillations when $\omega_{dr} \approx \omega_{res}$ is called the (mechanical) resonance

* if $\omega_{dr} \to 0$, then $A \to \frac{F_0}{m\omega_0^2} = \frac{F_0}{k}$
  (i.e. $T_{dr} \to \infty$)
  constant force

$\frac{\pi}{2}$-lag



φ

* if $\omega_{dr} \to \omega_0$ then

$$\tan\varphi = \frac{b\omega_{dr}}{m(\omega_{dr}^2-\omega_0^2)} \Rightarrow \varphi \to -\pi/2$$

the response ($x(t)$) lags the drive ($F(t)$) by 1/4 of the cycle

* if $\omega_{dr} \to \infty$ (high frequencies)

π-lag

$$\varphi \to -\pi$$

the response lags the drive by 1/2 of the cycle (displacement and drive are in antiphase)