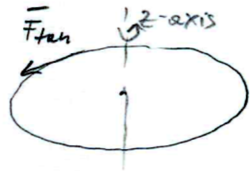
# Work and power in rotational motion

Idea:





Work done by $\bar{F}_{tan}$ over distance $ds = R\,d\theta$ ($\bar{F}_{tan} \parallel d\bar{r}$, and $ds = |d\bar{r}|$)

$$\delta W = F\,ds = \underbrace{F_{tan} R}_{\tau_z}\,d\theta = \tau_z\,d\theta$$

Total work ($\theta_1 \to \theta_2$)

$$W_{\theta_1\to\theta_2} = \int_{\theta_1}^{\theta_2} \tau_z\,d\theta \qquad \text{if } \tau_z = \text{const} \Rightarrow W_{\theta_1\to\theta_2} = \tau_z(\theta_2-\theta_1)$$

Units $[N\cdot m]$

Note. Axial (parallel to axis of rotation) and radial components do no work.

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## Rotational analogue of work-k.e. theorem

Similarly, use the $2^{nd}$ law of dynamics for rotational motion

$$\delta W = \underset{\substack{\downarrow\\ \text{net torque}}}{\tau_z}\,d\theta = (I\varepsilon_z)\,d\theta = I\frac{d\omega_z}{dt}\,d\theta = I\underbrace{\frac{d\theta}{dt}}_{\omega_z}\,d\omega_z = I\omega_z\,d\omega_z = d\left(\tfrac{1}{2}I\omega_z^2\right)$$

$$\boxed{\delta W = d\left(\tfrac{1}{2}I\omega_z^2\right)} = dK_{rot}$$

↳ kinetic energy of rotational motion

Hence the total work $W = K_2 - K_1$

↳ final ↳ initial

Power

$$\delta W = \tau_z\,d\theta \Rightarrow \frac{\delta W}{dt} = \tau_z\frac{d\theta}{dt} = \tau_z\omega_z$$

$$\boxed{P = \tau_z\omega_z}$$

# Angular momentum

Start with $2^{nd}$ law of dynamics for a particle

$$\underbrace{\bar{F}}_{\text{net force}} = \frac{d\bar{p}}{dt} \quad / \; \bar{r} \times$$

$$\underbrace{\bar{r} \times \bar{F}}_{\text{net torque}} = \bar{r} \times \frac{d\bar{p}}{dt}$$

Note : $\frac{d}{dt}(\bar{r} \times \bar{p}) = \underbrace{\frac{d\bar{r}}{dt}}_{=\bar{v}} \times \underbrace{\bar{p}}_{m\bar{v}} + \bar{r} \times \frac{d\bar{p}}{dt} = \bar{r} \times \frac{d\bar{p}}{dt}$ , hence

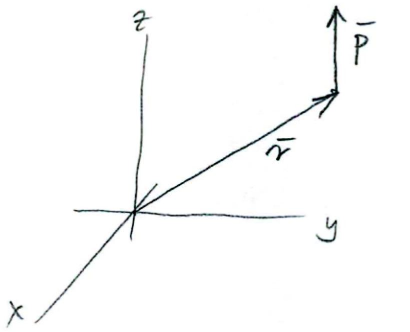
$\frac{d\bar{r}}{dt} \parallel \bar{p}$

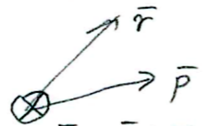
$$\bar{r} \times \bar{F} = \frac{d}{dt}(\bar{r} \times \bar{p}) \quad (*)$$

Define $\boxed{\bar{L} = \bar{r} \times \bar{p}}$ - angular momentum (units: $kg \frac{m^2}{s}$)
with respect to the origin

Rewrite $(*)$

$$\bar{r} \times \bar{F} = \frac{d\bar{L}}{dt} \quad \text{or} \quad \boxed{\bar{\tau} = \frac{d\bar{L}}{dt}} \quad \text{(single particle)}$$




$\bar{L} = \bar{r} \times \bar{p}$ ; $\bar{L} \perp \bar{r}$, $\bar{L} \perp \bar{p}$

Conclusion: the rate of change of angular momentum of a particle equals the torque of the net force acting on it

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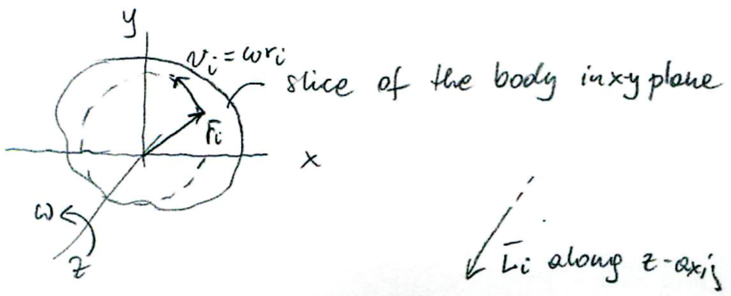
## Angular momentum of a rigid body

Use $\bar{L} = \bar{r} \times \bar{p}$ to find the total angular momentum of a rigid body rotating about z axis with angular velocity $\bar{\omega}$




for all particles in the slice

$\bar{r}_i \perp \bar{v}_i \Rightarrow \bar{r}_i \perp \bar{p}_i = m_i \bar{v}_i$

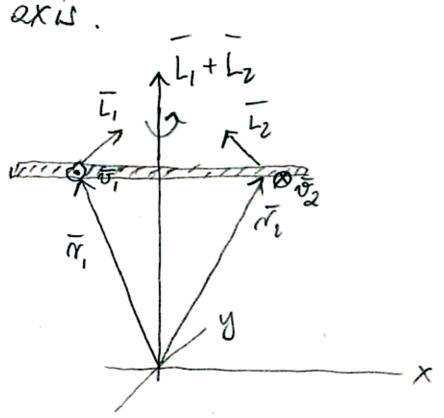
hence the contribution to the total angular momentum

$$L_i = r_i p_i = r_i m_i v_i = r_i m_i \omega r_i = m_i r_i^2 \omega$$

The total angular momentum of the slice in the xy plane ($\bar{L} \parallel z$-axis)

$$\underbrace{L}_{z\text{-component}} = \sum_{i=1}^{N_{slice}} L_i = \Big(\sum_{i=1}^{N_{slice}} m_i \underbrace{r_i^2}_{= r_{i\perp}^2}\Big)\omega = I_{slice}\,\omega$$

We can repeat this procedure for all slices, but in general if we add contributions of all slices, there will be a component of $\bar{L}$ perpendicular to $z$-axis.




$\bar{r}_1, \bar{r}_2$ - symmetric points (w.r.t. $z$-axis)

However, if z-axis is the axis of symmetry (mass distribution) of the body, then the perpendicular components of angular momentum cancel, and the total angular momentum is still along z-axis

Hence

$$\bar{L} = I\bar{\omega} \qquad \text{(rotation about a symmetry axis)}$$

(that is $\bar{L} \parallel \bar{\omega}$)

$$\bar{L} = (0, 0, L)$$
$$\bar{\omega} = (0, 0, \omega)$$

Note. In general (rotation about an arbitrary axis) $\bar{L} \not\parallel \bar{\omega}$



car tire wobbles $\Rightarrow$ balancing a car tire $\equiv$ adjusting the axis of symmetry to the axle

# Conservation of angular momentum

In general (valid for any axis of rotation)

$$\frac{d\bar{L}}{dt} = \frac{d}{dt}\left(\sum_{i=1}^{N} \bar{L}_i\right) = \frac{d}{dt}\sum_{i=1}^{N} \bar{r}_i \times \bar{p}_i = \overbrace{\sum_{i=1}^{N} \underbrace{\frac{d\bar{r}_i}{dt}}_{=\bar{v}_i} \times \bar{p}_i}^{=0} + \sum_{i=1}^{N} \bar{r}_i \times \underbrace{\frac{d\bar{p}_i}{dt}}_{}\ \underset{\text{law}}{\overset{2^{nd}}{=}}$$

$$= \sum_{i=1}^{N} \bar{r}_i \times \bar{F}_i = \sum_{i=1}^{N} \bar{\tau}_i = \bar{\tau}^{ext}$$

↳ we have shown before that internal forces do not contribute to the total torque, $\bar{\tau}^{ext} = \sum_{i=1}^{N} \bar{r}_i \times \bar{F}_i^{ext}$

## Conclusion (conservation of angular momentum)

When the net external torque on a system is zero, then the total angular momentum of the system is conserved.

For rotation about an axis of symmetry (choose z-axis)
(or planar objects about an axis ⊥ to the plane)

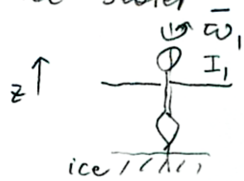
$$L_z = I_z \omega_z \qquad (z\text{-component})$$

$$\left.\frac{dL_z}{dt} = \tau_z^{ext}\right\} \overset{\tau_z^{ext}=0}{\Longrightarrow} \boxed{L_z = I_z\omega_z = const}$$

(rotation about axis of symmetry)

## Examples

(a) ice-skater



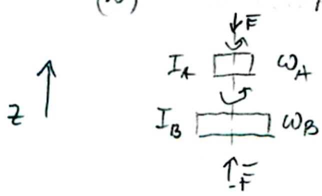

$I\omega = const$

$I_1\omega_1 = I_2\omega_2$

$$\boxed{\omega_2 = \frac{I_1}{I_2}\omega_1 > \omega_1}$$


Note that the kinetic energy $\frac{1}{2}I\omega^2$ increases here! (skater does work when pulling his arms inward) increase by a factor of $\frac{I_1}{I_2}$

(b) "rotational perfectly inelastic collision"

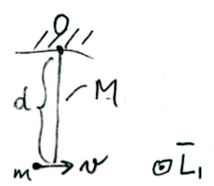



↳ no torque due to $\bar{F}$, $-\bar{F}$ ⇒ $\bar{L}$ conserved


$$I_A\omega_A + I_B\omega_B = (I_A + I_B)\,\omega$$

$$\boxed{\omega = \frac{I_A\omega_A + I_B\omega_B}{I_A + I_B}}$$

(c)




$L_1 = mvl$


$L_2 = \left(\frac{1}{3}Md^2 + md^2\right)\omega$

$L_1 = L_2$

$$\boxed{\omega = \frac{mlv}{\frac{1}{3}Md^2 + md^2}}$$