Chapter 1 – Special Theory of Relativity (I): Kinematics and Spacetime Interval

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Modern Physics (Summer 2020)
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Agenda

1. Introduction/Review
   - Newtonian Mechanics
   - Galilean Transformation
   - Is Newtonian Mechanics a Universal Theory?
2. The Special Theory of Relativity
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   - Lorentz Transformation
3. Relativistic Effects — Consequences of Lorentz Transformation
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4. Spacetime
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   - Four-Position and Spacetime Interval
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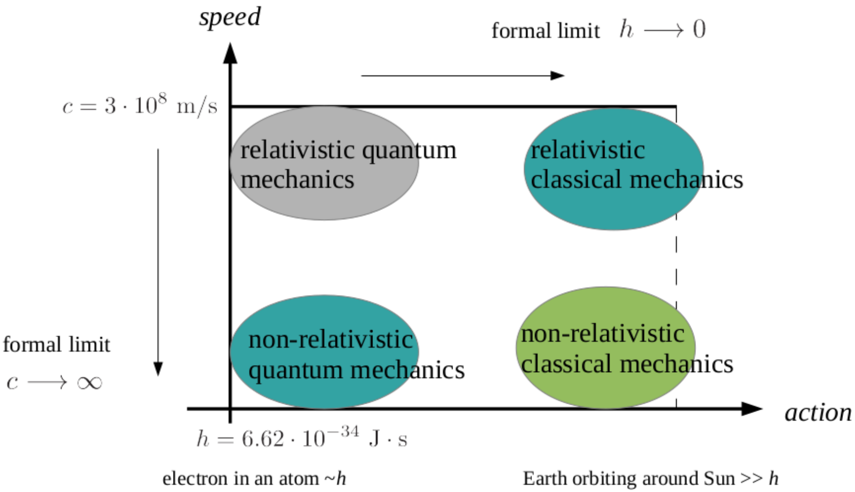
Introduction/Review

Newtonian Mechanics

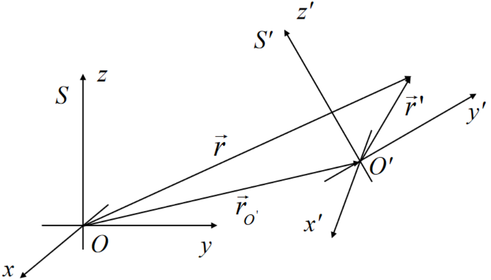
"Phase Diagram" of Mechanics



Newtonian Mechanics. First Law & Inertial FoR

Newton's First Law: *There exists an inertial FoR.*

Definition: An inertial FoR is a frame of reference in which $\bar{F} = 0$ implies $\bar{u} = \text{const}$ (i.e. $\bar{a} = 0$).



Two FoRs

S is an inertial FoR and S' moves with respect to S along a straight line with constant velocity $\bar{v}$.

Notation: $\dot{\bar{r}} = \bar{u}$, $\dot{\bar{r}}' = \bar{u}'$, $\ddot{\bar{r}} = \bar{a}$, $\ddot{\bar{r}}' = \bar{a}'$, $\dot{\bar{r}}_{O'} = \bar{v}$.

Addition of Velocities and Invariance of Newton's 2^{nd} law

For the position vectors in both FoRs

$$\bar{r} = \bar{r}_{O'} + \bar{r}' \qquad \text{or, equivalently,} \qquad \bar{r}' = \bar{r} - \bar{r}_{O'}$$

Differentiating the latter w.r.t. time yields

$$\boxed{\bar{u}' = \bar{u} - \bar{v}.}$$

Differentiating once again we get

$$\bar{a}' = \bar{a} - \dot{\bar{v}} = \bar{a}.$$

Multiply both sides by the mass of the particle

$$\boxed{m\bar{a}' = m\bar{a}.}$$

Conclusions

1. Velocities add according to $\bar{u}' = \bar{u} - \bar{v}$ (*velocity addition rule*).
2. Newton's 2^{nd} law has the same form in both S (here $\bar{F} = m\bar{a}$) and S' (here $\bar{F} = m\bar{a}'$).
3. Any FoR moving with constant velocity with respect to an inertial FoR is also inertial (consequence of the previous statement).

Comment. Non-Inertial FoR

In general, S' may move arbitrarily with respect to S, so that

$$\dot{\bar{r}}_{O'} = \bar{v} \neq \text{const.}$$

S' is then non-inertial. Recall that in non-inertial FoRs, Newton's 2^{nd} law needs to be modified into

$$m\bar{a}' = \bar{F} + \text{fictitious forces (a. forces of inertia).}$$

The *fictitious forces* are kinematic corrections (with the unit of the force, i.e. [N]) that must be introduced due to the fact that the FoR is non-inertial.

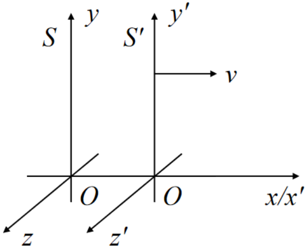
Typical examples

- FoR accelerating along a straight line (d'Alembert force),
- rotating FoR (Coriolis force, Euler force, centrifugal force),
- FoR moving along a curved trajectory (d'Alembert force).

Galilean Transformation

Galilean Transformation

Special case. $\bar{v} = (v, 0, 0) = \text{const}$. Moreover, assume that O coincides with O' at $t = 0$.



$$\boxed{\begin{aligned} x &= x' + vt \\ y &= y' \\ z &= z' \\ t &= t' \end{aligned}} \quad \text{or} \quad \boxed{\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}}$$

Consequently

$$\boxed{\begin{aligned} u'_x &= u_x - v \\ u'_y &= u_y \\ u'_z &= u_z \end{aligned}}$$

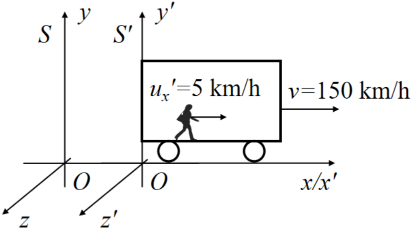
The transformation of the position vector given by the above equations is called the **Galilean transformation**.

Note. In the above equations, time is treated as a *parameter*, exactly the same in both FoRs. In other words, it is implicitly assumed that $t = t'$.

Galilean Transformation. Illustration

Consider an inertial frame of reference S associated, e.g., with a train station platform. A train travels with constant velocity along the x-axis of FoR S, in the positive direction. Set the FoR of a train to be S'. There is a man in the train, walking in the train's direction of motion. The speed of the train in S is $v = 150$ km/h and the speed of the man in S' is $u'_x = 5$ km/h.

Qestion: *What is the speed of the man in S?*



The answer is just simply

$$u_x = u'_x + v = 155\text{km/h}$$

This simple illustration shows that the Galilean transformation works very well for objects moving at **small** speeds.

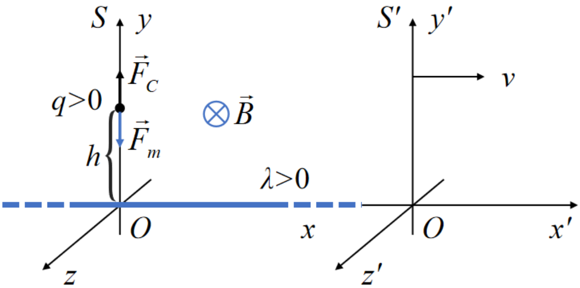
Is Newtonian Mechanics a Universal Theory?

Limitations of Newton's Theory (I)

Question: *Is Newtonian mechanics a universal theory?* $\mathbb{NO}$!

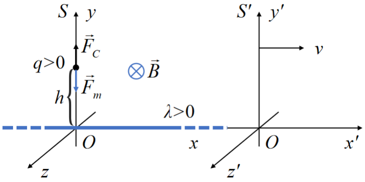
First hint: Maxwell's equations are not invariant under the Galilean transformation.

Illustration. A uniformly charged infinitely long rod and a point charge.



In FoR S, the net force exerted on the charge q is due to the Coulomb force due to the charged rod. Both objects are at rest. Hence

$$\boxed{\bar{F}} = \bar{F}_C = \boxed{\frac{1}{4\pi\varepsilon_0}\frac{2\lambda q}{h}\hat{n}_y.}$$



In FoR S', however, the rod moves in the $-x'$ direction with speed v, so an observer in S' will measure electric current through the rod

$$I = \frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\lambda \mathrm{d}x}{\mathrm{d}t} = \lambda v.$$

And the magnitude of magnetic field at the point charge q is

$$B = \frac{\mu_0 I}{2\pi h}.$$

So the magnetic force on q (which in S' is moving in the negative x' axis direction, with speed v) is

$$\bar{F}_m = q\bar{v}' \times \bar{B} = q(-\bar{v}) \times \bar{B} = -\frac{q\mu_0\lambda v^2}{2\pi h}\hat{n}_y.$$

Hence in S', the net (electromagnetic) force exerted on the charge is

$$\boxed{\bar{F}'} = \bar{F}_C + \bar{F}_m = \left(\frac{1}{4\pi\varepsilon_0} \frac{2\lambda q}{h} - \frac{q\mu_0 \lambda v^2}{2\pi h} \right) \hat{n}_y .$$

Therefore, now Newton's 2^{nd} law has a different form in S and S'

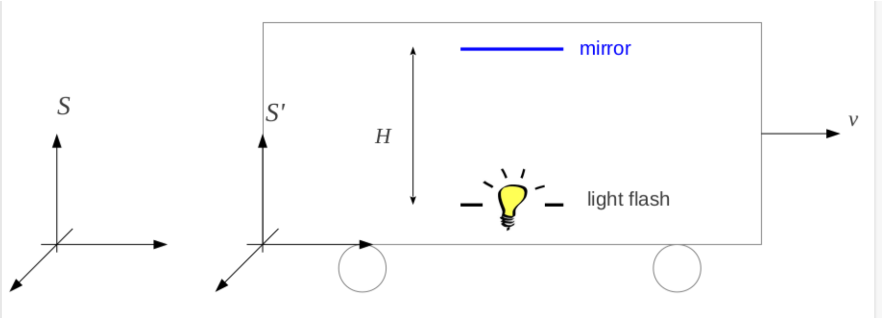
$$\boxed{\bar{F} \neq \bar{F}'.}$$

Which is a clear contradiction with the conclusion drawn from the Galilean transformation.

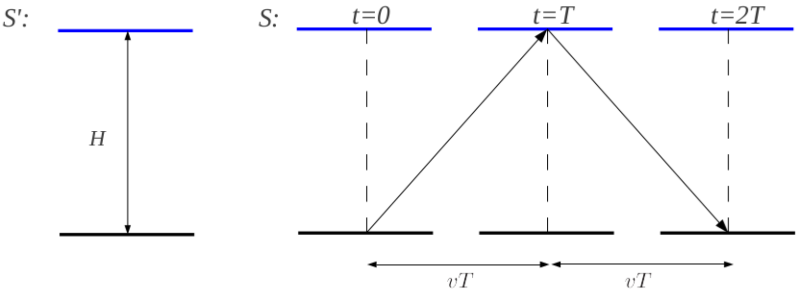
Limitations of Newton's Theory (II)

Second hint: Speed of light, quest for aether, and Michelson-Morley experiment.

Gedankenexperiment: Measurement of the speed of light.



Measurement of the Speed of Light



Both observers want to find the speed of light impulse moving from the source (*t=0*) to the mirror (*t=T*) and back to the source (*t=2T*) in their frames of reference:

S': $c_1 = \frac{2H}{2T} = \frac{H}{T}$

S: $c_2 = \frac{2\sqrt{H^2 + (vT)^2}}{2T} = \sqrt{c_1^2 + v^2} > c_1$

Experiments negatively verify this result!

$$\boxed{c_1 = c_2 = c}$$

Quest for Aether. Michelson-Morley Experiment

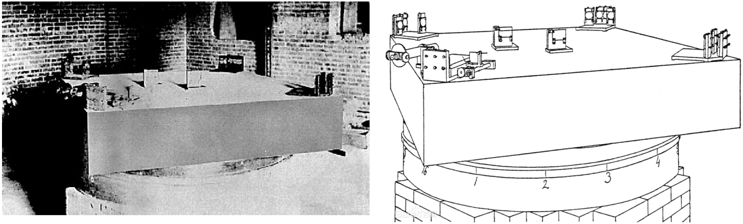
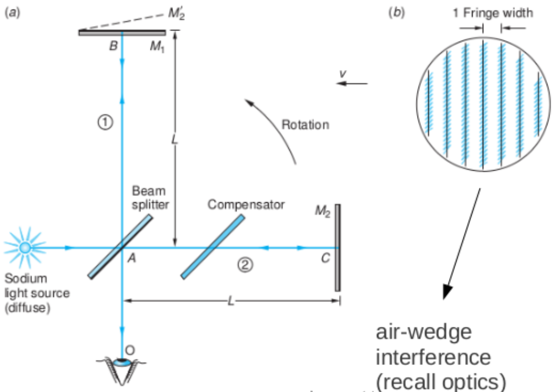


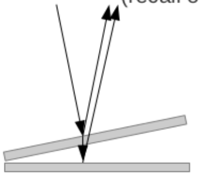
Figure: Michelson and Morley's interferometric setup.

Michelson-Morley experiment's failure shows that in a given medium, the speed of light coming from a source (no matter: moving or not) remains the same in every direction. Obviously it violates the addition rule of velocity derived from the Galilean transformation.

Quest for Aether. Michelson-Morley Experiment



air-wedge
interference
(recall optics)



Source of illustrations
(this and following pages): textbook

The Special Theory of Relativity

Special Theory of Relativity (STR)

Postulates of Special Theory of Relativity (Albert Einstein, 1905)

1. The laws of physics are the same in all inertial FoRs.
2. The light in vacuum propagates with the same speed c in all inertial FoRs.

Comments

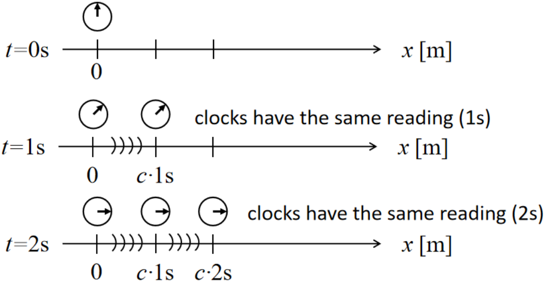
- The speed of light in vacuum $c = 1/\sqrt{\varepsilon_0 \mu_0} \approx 3 \times 10^8$ m/s (can be derived from classical electrodynamics).
- Extends the conclusion drawn (with respect to Newton's laws) from the Galilean transformation to include all laws of physics.
- The speed of light is invariant for all inertial observers in a given medium. It has the same value, regardless of the motion of the observer or the light source.
- Therefore, it is impossible to detect any "absolute motion".

Vocabulary of STR

$$\text{EVENT occurs at} \left\{ \begin{array}{l} \text{some point of space} \\ \text{some instant of time} \end{array} \right\} \text{in a given FoR}$$

An OBSERVER describes an event with respect to his FoR, using his "meter sticks" (at rest in his FoR) and his "clocks".

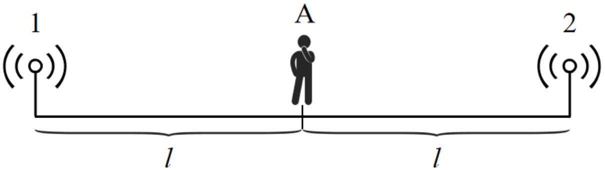
CLOCK SYNCHRONIZATION



A flash fired at $t = 0$ and $x = y = z = 0$ (at origin).

Vocabulary of STR

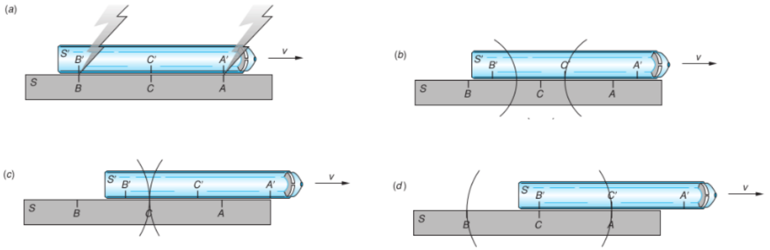
SIMULTANEITY



Firing flash 1 and 2 are simultaneous events for observer A, if the light signals from 1 and 2 arrive at the same instant of time (measured using A's own clocks).

Example. Einstein's example.

Einstein's Example



both flashes reach the observer *C* (standing mid-way) at the same instant, therefore ***C* concludes that the flashes WERE simultaneous**;

observer *C′* moved to the right while the flash was traveling; the front flash arrives at *C′* earlier than the rear one

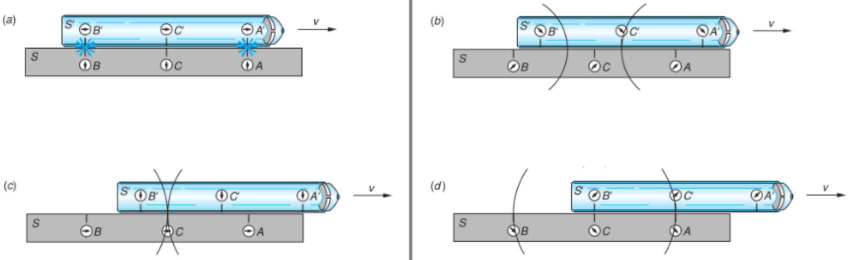
the speed of light is invariant, and *C′* sits in the middle of the car;

if the front flash arrives earlier than the rear one ***C′* has to conclude that the flashes WERE NOT simultaneous**; the front one happened earlier than the rear one

Simultaneity of events is relative!

Animation: `https://www.youtube.com/watch?v=wteiuxyqtoM`

Einstein's Example

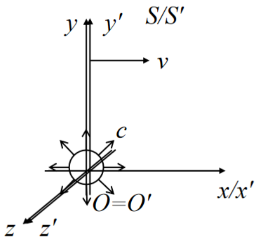


observer C: local clocks at A and B **are synchronized**

observer C': local clocks at A and B **are NOT synchronized**; clock at A runs ahead of that at B

Clock synchronization is relative!

Lorentz Transformation



At $t = t' = 0$, both FoRs coincide ($S = S'$) and a light impulse is sent from the origin radially outwards. From *Postulate 2*, wave-front equations in both FoRs can be written as

$$S: \qquad x^2 + y^2 + z^2 = (ct)^2 \qquad (\star)$$

$$S': \quad (x')^2 + (y')^2 + (z')^2 = (ct')^2 \quad (\star\star)$$

Goal: Find a relationship between x, y, z, t and x', y', z', t'.

How? Use wave font equations and the fact that S' moves with respect to S with velocity $(v, 0, 0)$.

Observations

- $y = y'$ and $z = z'$ because motion is only in the x-direction.
- Look for a <u>linear</u> relation

$$\begin{cases} x = a_{11}x' + a_{12}t' \\ t = a_{21}x' + a_{22}t' \end{cases} \qquad (\star\star\star)$$

Why linear?

More specific goal: Find the four coefficients of the transformation.
Plug Eqs.($\star\star\star$) into Eq.($\star$)

$$(a_{11}x' + a_{12}t')^2 + (y')^2 + (z')^2 = c^2(a_{21}x' + a_{22}t')^2$$

$$(a_{11}^2 - c^2 a_{21}^2)(x')^2 + 2(a_{11}a_{12} - c^2 a_{21}a_{22})x't' + (y')^2 + (z')^2 = \left(a_{22}^2 - \frac{1}{c^2}a_{12}^2\right)(ct')^2$$

Compare the coefficients next to x', y', z', t' with those in Eq.($\star\star$) to find

$$a_{11}^2 - c^2 a_{21}^2 = 1 \tag{1}$$

$$a_{22}^2 - \frac{1}{c^2}a_{12}^2 = 1 \tag{2}$$

$$a_{11}a_{12} - c^2 a_{21}a_{22} = 0 \tag{3}$$

There are four unknowns in total, so we need another equation. The 4th equation can be found by looking at the motion of the origin O' (where $x'_{O'} = 0$):

$$\left.\begin{aligned} x_{O'} &= a_{12}t' \\ t &= a_{22}t' \end{aligned}\right\} \quad \Rightarrow \quad \frac{a_{12}}{a_{22}} = \frac{x_{O'}}{t} = v \tag{4}$$

Plug (4) into (2). Pay attention to the sign! S' moves in the positive x-axis direction

$$a_{22}^2\left(1-\frac{v^2}{c^2}\right)=1 \quad \Rightarrow \quad \boxed{\begin{aligned} a_{22} &= \tfrac{1}{\sqrt{1-v^2/c^2}} \\ a_{12} &= \tfrac{v}{\sqrt{1-v^2/c^2}} \end{aligned}}$$

Plug (4) into (3) to get

$$a_{11}va_{22}-c^2a_{21}a_{22}=0 \quad \Rightarrow \quad a_{21}=\frac{v}{c^2}a_{11}. \tag{$\square$}$$

Substitute ($\square$) into (1)

$$a_{11}^2-\frac{v^2}{c^2}a_{11}^2=1 \quad \Rightarrow \quad \boxed{a_{11}=\frac{1}{\sqrt{1-v^2/c^2}}.}$$

And, finally, using ($\square$) once again,

$$\boxed{a_{21}=\frac{v/c^2}{\sqrt{1-v^2/c^2}}.}$$

Lorentz Transformation

Now, as all the coefficients are found, we can write down the expression for the **Lorentz transformation**

$$\boxed{\begin{aligned} x &= \frac{x'+vt'}{\sqrt{1-v^2/c^2}} \\ y &= y' \\ z &= z' \\ t &= \frac{t'+vx'/c^2}{\sqrt{1-v^2/c^2}} \end{aligned}}$$

Since S moves with respect to S' with $-v$, so just replacing v in Lorentz transformation by $-v$ gives us the inverse Lorentz transformation

$$\boxed{\begin{aligned} x' &= \frac{x-vt}{\sqrt{1-v^2/c^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t-vx/c^2}{\sqrt{1-v^2/c^2}} \end{aligned}}$$

Lorentz Transformation. General Remarks

Spatial and temporal variables get “entangled”. Time is no longer a parameter, but a variable that varies in different FoRs.

TIME & SPACE $\Longrightarrow$ SPACETIME

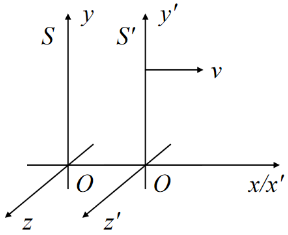
Observation. If $v \ll c$ ($v/c \to 0$), then

$$\boxed{\begin{aligned} t &= t' \\ x &= x' + vt \\ y &= y' \\ z &= z' \end{aligned}}$$

which is exactly the Galilean transformation. Therefore, the Galilean transformation is a special case of the Lorentz transformation, valid for small v (i.e. $v \ll c$).

Relativistic Effects — Consequences of Lorentz Transformation

(I) Temporal Effects. Relativity of Simultaneity



Consider two events A and B happening in FoR S'

$$\begin{array}{ll} A: & (t'_A, x'_A, y'_A, z'_A) \\ B: & (t'_B, x'_B, y'_B, z'_B) \end{array}$$

then $t'_B - t'_A$ is the time interval between A and B as measured in FoR S'.

An observer at rest in FoR S will measure the time interval between these two events as

$$\begin{aligned} t_B - t_A &= \frac{t'_B + vx'_B/c^2}{\sqrt{1 - v^2/c^2}} - \frac{t'_A + vx'_A/c^2}{\sqrt{1 - v^2/c^2}} \\ &= \frac{t'_B - t'_A}{\sqrt{1 - v^2/c^2}} + \frac{v(x'_B - x'_A)}{\sqrt{1 - v^2/c^2}} \end{aligned}$$

If there is a spatial separation ($x'_B \neq x'_A$), then two events A and B simultaneous in S' ($t'_A = t'_B$) cannot be simultaneous in S ($t_A = t_B$), and vice versa. **Simultaneity is relative.**

Consequently, clocks synchronized in S' are not synchronized in S, since $t_B - t_A$ depends on $x'_B - x'_A$.

Question. Given a time interval ($t_B - t_A$) and space interval ($x_B - x_A$) between A and B in S, is there an inertial FoR S'' such that both events are simultaneous in S''?

$$t''_B - t''_A = 0 \quad \Rightarrow \quad \frac{t_B - t_A}{\sqrt{1 - v^2/c^2}} - \frac{v(x_B - x_A)/c^2}{\sqrt{1 - v^2/c^2}} = 0$$

$$\frac{x_B - x_A}{t_B - t_A} = c \cdot \frac{c}{v}$$

Since $v < c$, then it is possible only if $x_B - x_A > c(t_B - t_A)$.

These types of questions will be discussed again using the concept of spacetime interval.

(I) Temporal Effects. Time Dilation

Assume that $x'_B = x'_A$, i.e. events A and B occur at the same location in S', e.g. at O'. From the Lorentz transformation, the time interval between A and B, measured by an observer in FoR S, moving with respect to S' is

$$t_B - t_A = \frac{t'_B - t'_A}{\sqrt{1 - v^2/c^2}},$$

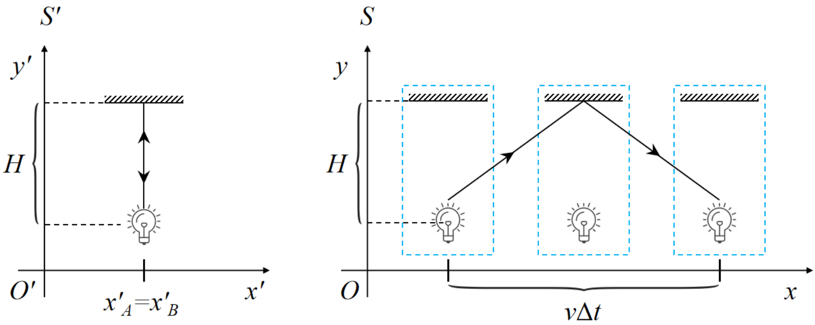
and, equivalently,

$$\boxed{t'_B - t'_A = \sqrt{1 - \frac{v^2}{c^2}}(t_B - t_A) < t_B - t_A}.$$

Hence, $t'_B - t'_A$ is the shortest possible time interval between the events (*proper time interval*) and an observer in S (moving w.r.t S') will measure a longer time interval. This effect is known as the **time dilation**.

Time Dilation. Illustration

Suppose there is a light bulb in FoR S' which moves with $\bar{v} = (v, 0, 0)$ with respect to S. Denote two events that the light is emitted from the bulb and the light returns to the bulb as A and B, respectively.



In FoR S': $\Delta t' = t_B - t_A = \frac{2H}{c}$

In FoR S: due to time dilation, $\Delta t = \frac{\Delta t'}{\sqrt{1 - v^2/c^2}}$

By geometry, the distance light travels in time Δt is

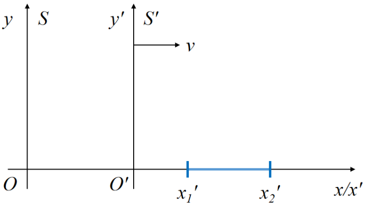
$$2\sqrt{H^2 + (v\frac{\Delta t}{2})^2}.$$

Hence its speed (as measured by observer in S) is

$$\begin{aligned}
c_S &= \frac{2\sqrt{H^2 + \frac{1}{4}(v\Delta t)^2}}{\Delta t} = \sqrt{\frac{4H^2}{(\Delta t)^2} + v^2} \\
&= \sqrt{\left(\frac{2H}{\frac{\Delta t'}{\sqrt{1-v^2/c^2}}}\right)^2 + v^2} = \sqrt{\underbrace{\left(\frac{2H}{\Delta t'}\right)}_{=c}^2 (1 - \frac{v^2}{c^2}) + v^2} \\
&= \sqrt{c^2 - v^2 + v^2} = c.
\end{aligned}$$

And we can see that the speed of light measured by the observer in S is $c_S = c$, exactly as it should be. (Recall the 2$^{\text{nd}}$ postulate of the STR.)

(II) Spatial Effects. Length Contraction



A rod of length L_0 rests in S'. The length $L_0 = x_2' - x_1'$, measured in S' (where the rod is at rest) is called the *proper length*.

Now we want to find the length $L = x_2 - x_1$ measured by an observer in S. Note that in this FoR the rod is moving. Therefore, such a measurement makes sense only if the positions of both ends are measured simultaneously (in S**!**), i.e. $t_2 = t_1$. Hence

$$
\begin{aligned}
L_0 = x_2' - x_1' \quad &= \quad \frac{x_2 - vt_2}{\sqrt{1 - v^2/c^2}} - \frac{x_1 - vt_1}{\sqrt{1 - v^2/c^2}} \\
&\underline{\underline{t_1 = t_2}} \quad \frac{x_2 - x_1}{\sqrt{1 - v^2/c^2}} = \frac{L}{\sqrt{1 - v^2/c^2}}
\end{aligned}
$$

Length Contraction. Example (a)

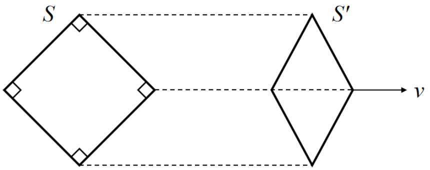
Therefore, the length L measured by the observer in S is.

$$\boxed{L = L_0\sqrt{1-\frac{v^2}{c^2}} < L_0}.$$

This apparent decrease in the length of the moving rod is a relativistic effect called the **length contraction** or **Lorentz-Fitzgerald contraction**. It indicates that the length of the rod is smaller when measured in a FoR with respect to which the rod is in motion.

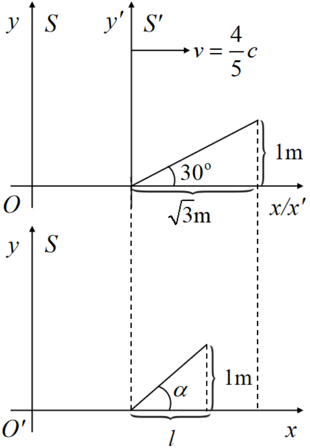
Examples

(a) square $\rightarrow$ parallelogram



Length Contraction. Example (b)

(b) inclined moving rod



In FoR S'

The inclined rod is at rest, which means that it moves at $\bar{v} = (4c/5, 0, 0)$ with respect to S.

In FoR S

At some instant of time, an observer in S measured the inclined rod:

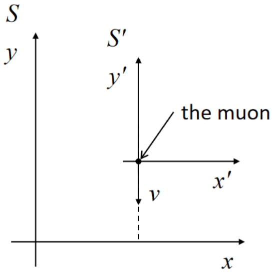
$$L = \sqrt{3}\ [m] \cdot \sqrt{1 - \frac{v^2}{c^2}} = \sqrt{3} \cdot \frac{3}{5}\ [m].$$

Therefore in S, the rod is observed to be inclined at the angle

$$\alpha = \arctan \frac{5}{3\sqrt{3}}.$$

Length Contraction. Example (c): Muons

The mean lifetime of muons measured in the FoR where they rest is $\tau \approx 2\ \mu\mathrm{s}$. Muons are particles created in the Earth's atmosphere at the altitude of 9 km above the sea level and they move at $v = 0.998c$. If we perform a non-relativistic calculation they can travel just $s = v\tau \approx 600$ [m]. They are, however, detected at the sea level. The question is, *how do muons make it to the sea level*?



(*Method I*) Denote the FoR where the muons rest by S' and the FoR of the Earth by S. In FoR S, due to time dilation, the mean life time of the muons is

$$\tau_S = \frac{\tau}{\sqrt{1 - v^2/c^2}} \approx 32\ [\mu\mathrm{s}].$$

Thus the distance travelled by the muons measured in S is

$$s_S = v\tau_s \approx 9.6\ [\mathrm{km}].$$

(*Method II*)

There is also another way to answer the question, involving the concept of length contraction. In FoR S', the Earth moves towards the muon with speed v.Therefore, the distance to the moving Earth appears shorter by the factor of $\sqrt{1 - v^2/c^2}$, and for the muon it needs to travel only

$$s_{S'} = s_S \sqrt{1 - \frac{v^2}{c^2}} = v\tau \approx 600 \text{ m}$$

to reach the sea level. In the FoR of the Earth this distance is

$$s_S = \frac{600 \text{ m}}{\sqrt{1 - v^2/c^2}} \approx 9.6 \text{ km}.$$

Short Break. Relativity of Time in Literature :-)

The European and the African have an entirely different concept of time. In the European worldview, time exists outside man, exists objectively, and has measurable and linear characteristics.

According to Newton, time is absolute: Absolute, true, mathematical time of itself and from its own nature, it flows equably and without relation to anything external. The European feels himself to be time's slave, dependent on it, subject to it. To exist and function, he must observe its ironclad, inviolate laws, its inflexible principles and rules. (...) An unresolvable conflict exists between man and time, one that always ends with man's defeat — time annihilates him.

Africans apprehend time differently. For them, it is a much looser concept, more open, elastic, subjective. It is man who influences time, its shape, course, and rhythm (...). Time is even something that man can create outright, for time is made manifest through events, and whether an event takes place or not depends, after all, on man alone.

Ryszard Kapuscinski, *The Shadow of the Sun* (chapter: *The road to Kumasi*)

Relativistic Velocity Addition Rule

Recall the formulas of the inverse Lorentz transformation and find the total differentials of the variables

$$\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathrm{d}x' &= \frac{\mathrm{d}x - v\,\mathrm{d}t}{\sqrt{1 - v^2/c^2}} \\ \mathrm{d}y' &= \mathrm{d}y \\ \mathrm{d}z' &= \mathrm{d}z \\ \mathrm{d}t' &= \frac{\mathrm{d}t - v\,\mathrm{d}x/c^2}{\sqrt{1 - v^2/c^2}} \end{aligned}$$

Note (chain rule). $x' = x'(x, t)$ and $\mathrm{d}x' = \frac{\partial x'}{\partial x}\mathrm{d}x + \frac{\partial x'}{\partial t}\mathrm{d}t$ is the total differential.

Hence we have

$$u'_x = \frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\mathrm{d}x - v\,\mathrm{d}t}{\mathrm{d}t - v\,\mathrm{d}x/c^2} = \frac{\mathrm{d}x/\mathrm{d}t - v}{1 - v\,\mathrm{d}x/c^2\mathrm{d}t} = \frac{u_x - v}{1 - vu_x/c^2}$$

$$\Rightarrow \boxed{u'_x = \frac{u_x - v}{1 - vu_x/c^2}.}$$

For the y component of the velocity

$$u'_y \quad = \quad \frac{\mathrm{d}y'}{\mathrm{d}t'} = \frac{\mathrm{d}y}{\mathrm{d}t - v\,\mathrm{d}x/c^2}\sqrt{1-\frac{v^2}{c^2}} = \frac{u_y}{1 - vu_x/c^2}\sqrt{1-\frac{v^2}{c^2}}$$

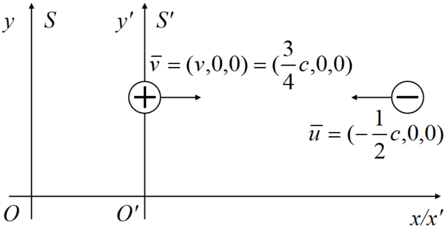
$$\Rightarrow \quad \boxed{u'_y = \frac{u_y}{1 - vu_x/c^2}\sqrt{1-\frac{v^2}{c^2}}.}$$

And, analogously, for the z component

$$\boxed{u'_z = \frac{u_z}{1 - vu_x/c^2}\sqrt{1-\frac{v^2}{c^2}}}.$$

Note. Unlike for the y and z components of the position vector, even though S' moves only in the x–direction, the y and z components of velocity transform as well.

Example



What is the velocity of the negative charge with respect to the positive one?

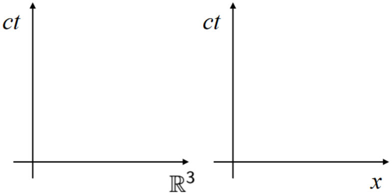
$$\bar{u}' = \left(\frac{-\frac{1}{2}c - \frac{3}{4}c}{1 - \frac{3}{4}c(-\frac{1}{2}c)/c^2}, 0, 0 \right) = \left(\frac{-\frac{5}{4}c}{1 + \frac{3}{8}}, 0, 0 \right) = \left(\frac{10}{11}c, 0, 0 \right)$$

Comment. The result derived using the non-relativistic velocity addition rule (i.e. based on the Galilean transformation) is $(-5c/4, 0, 0)$, hence the relative speed of the charge exceeds the speed of light. This result is therefore absolutely incorrect.

Spacetime

Spacetime (or Minkowski) Diagrams

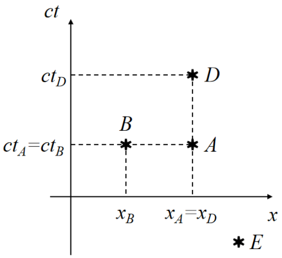
Idea: represent events in a certain inertial FoR by points in a 4-dimensional space (ct, x, y, z).



Remarks

1. The units of ct are the units of length.
2. A diagram with ct– and x–axes, is actually a cross-section of the 4-dimensional space. However, it is enough for us to study scenarios with one FoR moving with respect to another along a single axis (e.g. the x–axis).

Events on a Spacetime Diagram



For the events (in a certain FoR) shown on the spacetime diagram on the left

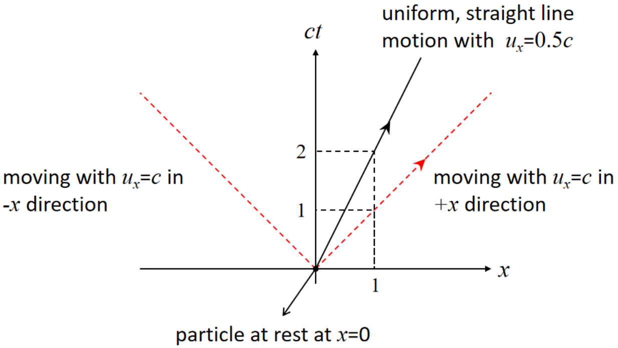
- A,B are simultaneous, but happen at different locations.
- A,D happen at the same location, and D follows A in time (or, equivalently, A precedes D)
- If $t = 0$ represents the present, then A, B and D are in the future and E is in the past.

Comments

- If two events are simultaneous in any FoR, then there is no possible causal relation between them (one cannot be caused by the other).
- On contrary, if one event follows another, then there *may be* a causal relation between them.

Trajectories of Particles on Space Diagrams. Worldlines

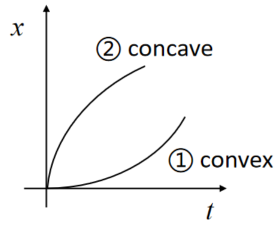
A trajectory of a particle moving in the 4-dimensional spacetime is called a **worldline**.



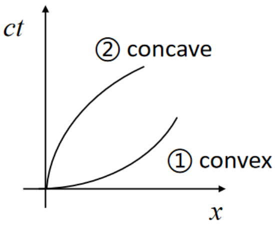
Note that

$$u_x = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{c\,\mathrm{d}x}{c\,\mathrm{d}t} \Rightarrow \frac{\mathrm{d}(ct)}{\mathrm{d}x} = \frac{c}{u_x}.$$

Since $\mathrm{d}(ct)/\mathrm{d}x$ is the slope of the worldline, the *speed* $\propto 1/$*slope*. And, since, $-c \leqslant u_x \leqslant c$, the slope $\leqslant$ -1 or $\geqslant$ 1.

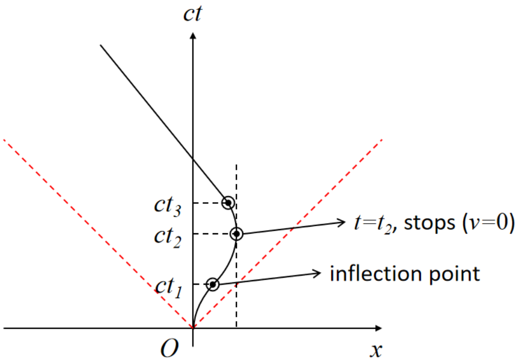


For a $x = x(t)$ diagram of a particle moving in the positive x–axis direction, we know that if the curve is convex, then $\ddot{x} > 0$ and hence the particle is speeding up (accelerates). On the other hand, if the curve is concave, then $\ddot{x} < 0$ and the the particle is slowing down.



For the worldlines, the coordinates of x and t have been switched. So now if the curve is convex, then $\ddot{x} < 0$ and the curve shows the worldline of a particle that is slowing down. On contrary, if the curve is concave, then $\ddot{x} > 0$ and that corresponds to a particle that is speeding up.

Spacetime Diagram. Example



1. $0 \leqslant t < t_1$, accelerates in $+x$ direction;
2. $t_1 \leqslant t < t_2$, slows down;
3. $t_2 \leqslant t < t_3$, accelerates in $-x$ direction;
4. $t \geqslant t_3$, moves with constant speed.

Four-Position and Spacetime Interval

Four-Position

The object

$$X^\mu \stackrel{def}{=\!=} (ct, x, y, z)^T,$$

where $\mu = 0, 1, 2, 3$ and $X^0 = ct, X^1 = x, X^2 = y, X^3 = z$, is called the (*contravariant*) **position four–vector**, or simply, the **contravariant four–position**. Its components transform according to the Lorentz transformation between FoRs, i.e.

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & \frac{v/c}{\sqrt{1-v^2/c^2}} & 0 & 0 \\ \frac{v/c}{\sqrt{1-v^2/c^2}} & \frac{1}{\sqrt{1-v^2/c^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

The relationship can also be written in a compact form, using Einstein summation convention, by which sum runs over any repeated Greek index appearing as both subscript and superscript

$$X^\mu = \Lambda^\mu_\nu (X^\nu)'.$$

Comment*. Contravariant vs. Covariant Four-position

The contravariant four-position is represented by a column vector, whereas the covariant four-position is a row vector. In general, when transforming between contra– and covariant four–vectors, we lower/raise the Greek index and change the sign of spatial variables.

$$X^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \qquad\qquad X_\mu = (ct, -x, -y, -z)$$

"contravariant" (upper index) "covariant" (lower index)

Examples

The matrix

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

is called the *metric tensor* with the signature (+ - - -).

It is easy to see that the following hold

1. $g_{\mu\nu} X^\nu \xlongequal{E.conv.} \sum_{\nu=0}^{3} g_{\mu\nu} X^\nu = (ct, -x, -y, -z) = X_\mu$ which is exactly the covariant 4-position.
2. $g_{\mu\nu} X^\nu X^\mu \xlongequal{E.conv.} \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu} X^\nu X^\mu = \sum_{\mu=0}^{3} X_\mu X^\mu = X_\mu X^\mu$ which is a kind of the "scalar product" of the 4-position with itself.

Examples (contd.)

Explicitly

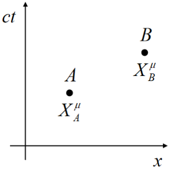
$$X_\mu X^\mu = (ct, -x, -y, -z) \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$
$$= (ct)^2 - x^2 - y^2 - z^2$$

Comments

- Be aware of the squares in the formula.
- Unlike the usual scalar (dot) product of a vector with itself, the number $X_\mu X^\mu$ is not always non-negative.

Spacetime Interval

Consider two events: event A described by $X_A^\mu = (ct_A, x_A, y_A, z_A)$ and event B with $X_B^\mu = (ct_B, x_B, y_B, z_B)$.



Introduce the 4-vector

$$\Delta X^\mu \stackrel{\text{def}}{=\!=} X_B^\mu - X_A^\mu = \begin{pmatrix} c(t_B - t_A) \\ x_B - x_A \\ y_B - y_A \\ z_B - z_A \end{pmatrix} .$$

The **spacetime interval** (between events A and B) is defined as

$$\boxed{\Delta s^2 \stackrel{def}{=\!=} \Delta X_\mu \Delta X^\mu}.$$

Explicitly

$$\begin{aligned} \Delta s^2 &= c^2(t_B - t_A)^2 - [(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2] \\ &= c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2. \end{aligned}$$

Invariance of the Spacetime Interval

Fact (important)

The spacetime interval is an **invariant** of the Lorentz transformation, $\Delta s^2 = \text{inv.}$ That is, it has the same value in all inertial FoRs.

Check

$$
\begin{aligned}
\Delta s^2 &= c^2\Delta t^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \\
&\underset{trans.}{\overset{Lor.}{=\!=}} c^2\left(\frac{\Delta t' + v\Delta x'/c^2}{\sqrt{1-v^2/c^2}}\right)^2 - \left(\frac{\Delta x' + v\Delta t'}{\sqrt{1-v^2/c^2}}\right)^2 - (\Delta y')^2 - (\Delta z')^2 \\
&= c^2(\Delta t')^2\left(\frac{1}{1-v^2/c^2} - \frac{v^2/c^2}{1-v^2/c^2}\right) \\
&\quad -(\Delta x')^2\left(-\frac{v^2/c^2}{1-v^2/c^2} + \frac{1}{1-v^2/c^2}\right) - (\Delta y')^2 - (\Delta z')^2 \\
&= c^2(\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2 = (\Delta s^2)' = \text{inv.}
\end{aligned}
$$

Three Types of Spacetime Interval

The value of the spacetime interval determines the relationship between events. There are three different possible cases for Δs^2.

1. $\Delta s^2 > 0$
   The spacetime interval is said to be **time-like**, which means that the events are *separated temporally*.
2. $\Delta s^2 < 0$
   In this case the spacetime interval is called **space-like**, which means that the events are *separated spatially*.
3. $\Delta s^2 = 0$
   The spacetime interval is **light-like**.

We will now comment on the three cases, in particular looking at whether there is a possible causal relation between the two events.

Spacetime Interval and Relationships between Events

Case 1

It is not possible to find a FoR such that these events happen at the same instant of time. (Consequently, there may be a possible causal relation between them.)

To prove this statement, assume that such a FoR (call it S') exists, i.e. $\Delta t' = 0$ in S'. Then, in S',

$$(\Delta s^2)' = c^2 \Delta t'^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2 \leqslant 0.$$

However, for a time-like spacetime interval $\Delta s^2 > 0$, and from space-time interval invariance $(\Delta s^2)' = \Delta s^2 > 0$. Contradiction! Hence it is not possible to find such a FoR.

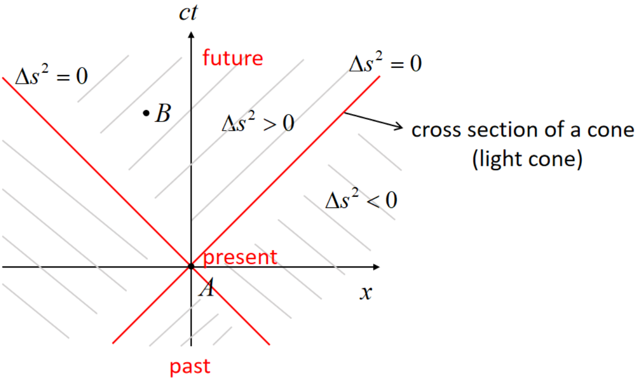
Case 2

Analogously, now it is not possible to find a FoR where these events happen at the same place, but they may be made to happen at the same time. Therefore, a causal relation is not possible.

Case 3

Events lie on a worldline of a light signal.

Illustration. "Light Cone"



$$\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2 \begin{cases} > 0 & \text{interior} \\ < 0 & \text{exterior} \\ = 0 & \text{lateral surface} \end{cases}$$