Problem Set 3

Due: 8 June 2020, 12.30 p.m.

Problem 1. One dimensional motion of a quantum particle is restricted to the region $|x| \leq a$. The particle is in a state represented by the wave function

$$\Psi(x,t) = Ce^{-\frac{i}{\hbar}Et}(x^2 - a^2),$$

for $|x| \leq a$ (and zero otherwise). E and C are real constants.

(a) Does Ψ represent a stationary state? Why or why not?
(b) Normalize Ψ.
(c) Find the probability that the particle is in the region $|x| \leq a/2$
(d) Find the probability that the particle is in the region $0 \leq x \leq a$.
(e) Calculate $\langle x \rangle_\Psi$. Does it depend on time?
(f) Calculate Δ_x.

(*1/2 + 1 + 1 + 3/2 + 2 points*)

Problem 2. As we will show later in class, the ground-state wave function of the electron in the hydrogen atom is

$$\Psi_{100}(r,\varphi,\theta,t) = Ce^{-\frac{i}{\hbar}E_1 t}\exp\left(-\frac{r}{a_0}\right),$$

where $a_0 = 4\pi\varepsilon_0\hbar^2/me^2$ is the Bohr radius (what is the value of a_0 in nanometers?), E_1, C are constants (what is the value of E_1 in eV?).

Calculate the

(a) normalization constant C,
(b) probability of finding the electron inside the sphere with radius a_0 around the nucleus,
(c) most probable value of the radial coordinate of the electron r (i.e. the most probable distance of the electron from the nucleus),
(d) average value of the radial coordinate r and its standard deviation.

(*1 + 2 + 2 + 3 points*)

Problem 3. For square–integrable functions f and g, we have defined $\langle f, g\rangle = \int\limits_{-\infty}^{\infty} f^*(u)g(u)\,\mathrm{d}u$.

(a) Check that $\langle f, g\rangle = \langle g, f\rangle^*$.
(b) Let α be any complex number. Check that $\langle f, \alpha g\rangle = \alpha\langle f, g\rangle$.
(c) Use (a) and (b) to show that $\langle \alpha f, g\rangle = \alpha^*\langle f, g\rangle$.

(*1 + 1 + 1 points*)

Problem 4. Two functions f and g are said to be *orthogonal* if the inner product $\langle f, g \rangle = 0$.
Let $\psi_n(x) = \sqrt{2/L} \sin(n\pi x/L)$ for $0 \le x \le L$ and zero otherwise.

(a) Check that ψ_n is normalized.

(b) Show that ψ_n and ψ_m are orthogonal for $n \neq m$.

(c) Find $\langle i\psi_1, -3\psi_1 + 2i\psi_2 - \psi_3 \rangle$, where i denotes the imaginary unit.

(d) Construct two linear combinations of ψ_1 and ψ_2 that are normalized and orthogonal.

(2 + 3 + 1 + 2 points)

Note. (a) and (b) can be written in a compact form as $\langle \psi_n, \psi_m \rangle = \delta_{nm}$, where δ_{nm} is the Krönecker symbol.

Problem 5. Consider the wave function $\Psi(x,t) = \left(e^{-\frac{i}{\hbar}E_1 t}\psi_1(x) + e^{-\frac{i}{\hbar}E_2 t}\psi_2(x)\right)/\sqrt{2}$, where E_1, E_2 are constants, and ψ_1, ψ_2 are normalized and orthogonal.

(a) Check that Ψ is normalized.

(b) Does Ψ describe a particle in a stationary state? Explain.

(c) If $E_2 = 4E_1$ what is the minimum time T such that $\Psi(x, t+T) = \Psi(x,t)$.

(3/2 + 3/2 + 2 points)