VP390 Problem Set 4

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1 Problem 1

(a) $\hat{\pi_x} f(x) = \hbar \frac{\partial e^{-x^2}}{\partial x} = -2x\hbar e^{-x^2}$
$\hat{M}_{\frac{1}{2}} f = e^{-(\frac{1}{2}x)^2} = e^{-\frac{x^2}{4}}$

(b) Assume the conjugates exist and donate as $\hat{\pi_x}^\dagger, \hat{M_c}^\dagger$
$\langle \Psi, \hat{\pi_x}\Phi \rangle = \int_{-\infty}^{\infty} \Psi^* \hbar \frac{\partial \Phi}{\partial x} \mathrm{d}x$
$= [\hbar \Psi^* \Phi]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \hbar \frac{\partial \Psi^*}{\partial x} \Phi \mathrm{d}x = -\langle \hat{\pi_x}\Psi, \Phi \rangle$
So $\hat{\pi_x}^\dagger = -\hat{\pi_x}$, and it's not Hermitian

$\langle \Psi, \hat{M_c}\Phi \rangle = \int_{-\infty}^{\infty} \Psi^*(x)\Phi(cx)\mathrm{d}x$
Let $y = cx$, then $\langle \Psi, \hat{M_c}\Phi \rangle = \int_{-\infty}^{\infty} \frac{\Psi^*(\frac{y}{c})\Phi(y)}{c} \mathrm{d}y$
So $\hat{M_c}^\dagger = \frac{\hat{M_{1/c}}}{c}$, and it's not Hermitian

2 Problem 2

(a) $\langle \Psi, (\alpha \hat{A})\Phi \rangle = \alpha \langle \hat{A}^\dagger \Psi, \Phi \rangle = \langle (\alpha^* \hat{A}^\dagger)\Psi, \Phi \rangle$
So $(\alpha \hat{A})^\dagger = \alpha^* \hat{A}^\dagger$

(b) $\langle \Psi, \hat{A}\Phi \rangle = \langle \hat{A}^\dagger \Psi, \Phi \rangle = \langle \Phi, \hat{A}^\dagger \Psi \rangle = \langle (\hat{A}^\dagger)^\dagger \Phi, \Psi \rangle = \langle \Psi, (\hat{A}^\dagger)^\dagger \Phi \rangle$
So $(\hat{A}^\dagger)^\dagger = \hat{A}$

(c) $\langle (\hat{A}\hat{B})^\dagger \Psi, \Phi \rangle = \langle \Psi, (\hat{A}\hat{B})\Phi \rangle = \langle \Psi, \hat{A}(\hat{B}\Phi) \rangle = \langle \hat{A}^\dagger \Psi, \hat{B}\Phi \rangle = \langle \hat{B}^\dagger \hat{A}^\dagger \Psi, \Phi \rangle$
So $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger \hat{A}^\dagger$

3 Problem 3

(a) $E_1 = \frac{\hbar^2 \pi^2}{2mL^2} = \frac{(1.05\times 10^{-34})^2 \times 3.14^2}{2\times 1.67\times 10^{-27}(10^{-15})^2} \approx 3.25 \times 10^{-11} J = 203.4\text{MeV}$

(b) $E_1 = h\nu = h\frac{c}{\lambda}$
$\lambda_1 = h\frac{c}{E_1} = 6.63 \times 10^{-34} \frac{3\times 10^8}{3.3\times 10^{-11}} \approx 6.12 \times 10^{-15}\text{m}$

(c) $\triangle E = (2^2 - 1)E_1 = 3E_1$
$\lambda_2 = \frac{\lambda_1}{3} \approx 2.04 \times 10^{-15}\text{m}$

(d) $\triangle E = (3^2 - 2^2)E_1 = 5E_1$
$\lambda_3 = \frac{\lambda_1}{5} \approx 1.22 \times 10^{-15}\text{m}$

(e) $\triangle E = (3^2 - 1)E_1 = 8E_1$
$\lambda_4 = \frac{\lambda_1}{8} \approx 7.65 \times 10^{-16}\text{m}$

4 Problem 4

(a) $\langle \Psi, \Psi \rangle = \langle \sqrt{\frac{1}{3}}\psi_3, \sqrt{\frac{1}{3}}\psi_3 \rangle + \langle \sqrt{\frac{1}{3}}\psi_3, \sqrt{\frac{2}{3}}\psi_5 \rangle + \langle \sqrt{\frac{2}{3}}\psi_5, \sqrt{\frac{1}{3}}\psi_3 \rangle, \langle \sqrt{\frac{2}{3}}\psi_5 + \sqrt{\frac{2}{3}}\psi_5 \rangle$
$= \frac{1}{3}\langle \psi_3, \psi_3 \rangle + 0 + 0 + \frac{2}{3}\langle \psi_5, \psi_5 \rangle = \frac{1}{3} + \frac{2}{3} = 1$
So it's normalized

(b) $\Psi(x, 0) = \sqrt{\frac{2}{3a}} \sin \frac{3\pi}{a} x + \sqrt{\frac{4}{3a}} \sin \frac{5\pi}{a} x$
$\Psi(x, 0)\Psi(x, 0)^* = (\sqrt{\frac{2}{3a}} \sin \frac{3\pi}{a} x + \sqrt{\frac{4}{3a}} \sin \frac{5\pi}{a} x)^2$
$= \frac{2}{3a} \sin^2 \frac{3\pi}{a} x + \frac{2\sqrt{2}}{3a} \sin \frac{3\pi}{a} x \sin \frac{5\pi}{a} x + \frac{4}{3a} \sin^2 \frac{5\pi}{a} x$
$\int_0^{a/2} \Psi(x, 0)^2 \mathrm{d}x = \frac{4}{15}$
So the probability is $\frac{1}{2}$

(c) The possible outcome of energy is $E_3 = \frac{9\hbar^2\pi^2}{2ma^2}, E_5 = \frac{25\hbar^2\pi^2}{2ma^2}$
The probability to get E_3 is $\frac{1}{3}$, for E_5 is $\frac{2}{3}$

(d) $E = \frac{1}{3}E_3 + \frac{2}{3}E_5 = \frac{59\hbar^2\pi^2}{6ma^2}$

(e) $\Psi(x, t) = \sqrt{\frac{1}{3}} e^{\frac{i}{\hbar}E_3 t} \psi_3(x) + \sqrt{\frac{2}{3}} e^{\frac{i}{\hbar}E_5 t} \psi_5(x)$

5 Problem 5

(a) $\Psi(x, 0) = \Phi(x) = \sum_n c_n \psi_n(x)$, where
$$\Phi(x) = \begin{cases} \sqrt{\frac{2}{a}} \sin \frac{\pi}{a} x & 0 \le x \le a \\ 0 & \text{otherwise} \end{cases}$$
$c_m = \langle \psi_m, \Phi \rangle$, where
$$\psi_m(x) = \begin{cases} \sqrt{\frac{2}{2a}} \sin \frac{m\pi}{2a} x & 0 \le x \le 2a \\ 0 & \text{otherwise} \end{cases}$$
$c_m = \int_{-\infty}^{\infty} \psi_m^* \Phi \mathrm{d}x = \frac{\sqrt{2}}{a} \int_0^a \sin \frac{\pi}{a} x \sin \frac{m\pi}{2a} x \mathrm{d}x$
Therefore $c_1 = \frac{4\sqrt{2}}{3\pi}, c_2 = \frac{\sqrt{2}}{2}, c_3 = \frac{4\sqrt{2}}{5\pi}, c_4 = 0, c_5 = \frac{-4\sqrt{2}}{21\pi}$
When $0 = t_1 < t_2 < t_3 < t_4$, we can the plot:

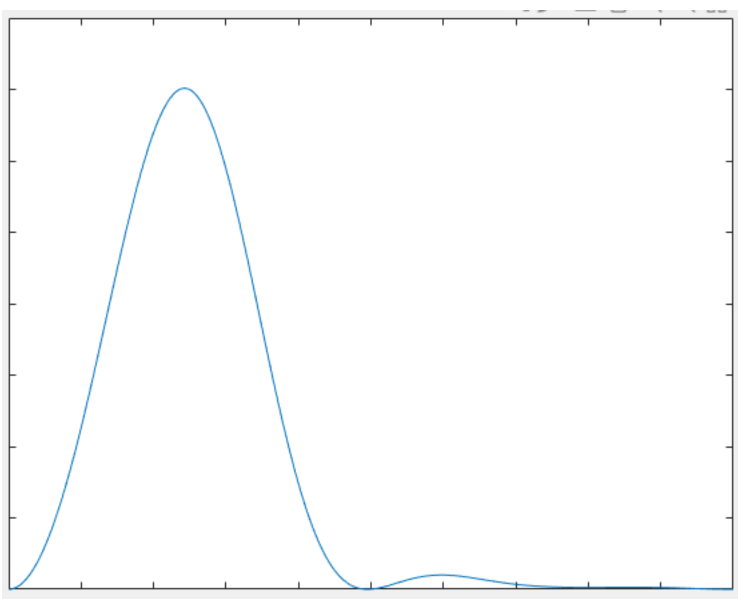

Figure 1: Plot for $t = t_1 = 0$

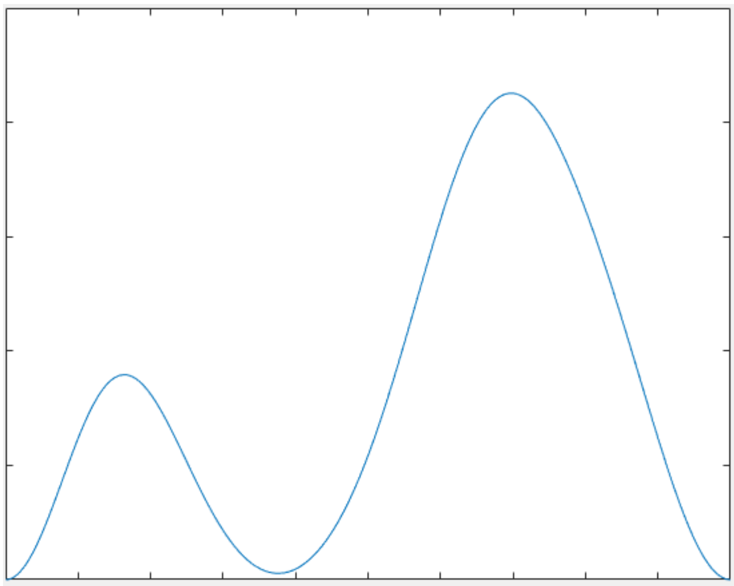

Figure 2: Plot for $t = t_2$

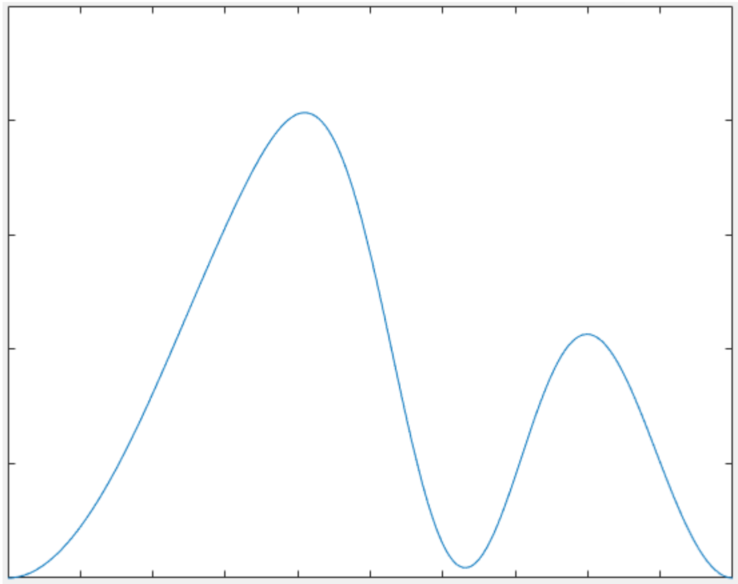

Figure 3: Plot for $t = t_3$

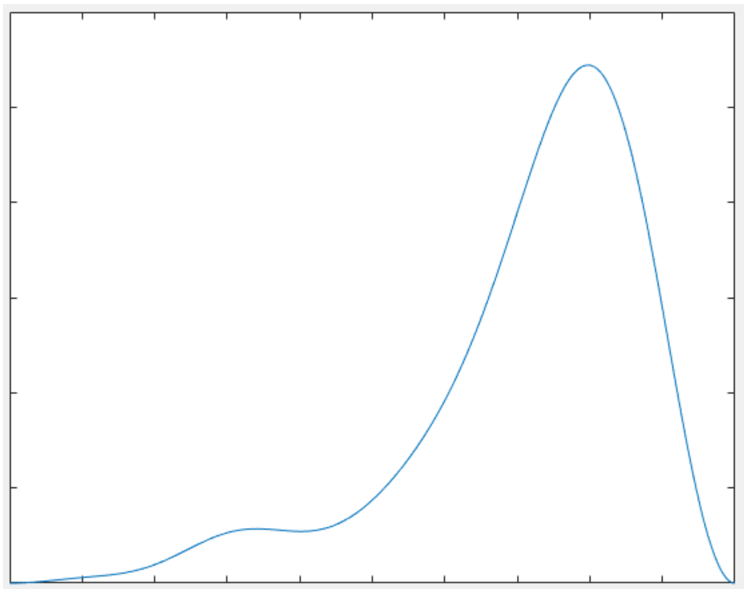

Figure 4: Plot for $t = t_4$

(b) When $t < 0, E = \frac{\hbar^2\pi^2}{2ma^2}$
When $t \geq 0, E_2 = \frac{\hbar^2\pi^2}{2m(2a)^2}n^2 = E$
So the probability is $c_2^2 = 0.5$

6 Problem 6

(b) Assume ψ is the solution for $-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + V(x)\psi(x) = E\psi(x)$
Then $-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(-x)}{\mathrm{d}x^2} + V(-x)\psi(-x) = E\psi(-x)$
Since $V(x) = V(-x)$
$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(-x)}{\mathrm{d}x^2} + V(x)\psi(-x) = E\psi(-x)$
$\psi(-x)$ also satisfies this stationery equation.
Therefore $\psi(x) \pm \psi(-x)$ satisfies this equation as well.