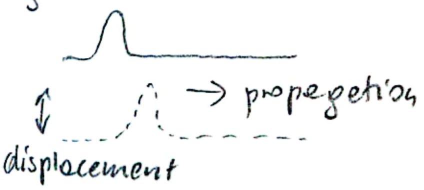
Mechanical waves

Wave – disturbance propagating through space
(mechanical waves need a <u>medium</u> to propagate)

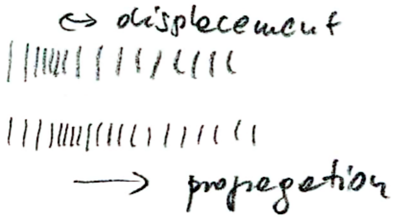
transverse wave
(direction of displacement of medium's particles perpendicular to the direction of wave propagation)

e.g. wave on a cord



longitudinal waves
(direction of displacement parallel to the direction of propagation)

e.g. sound



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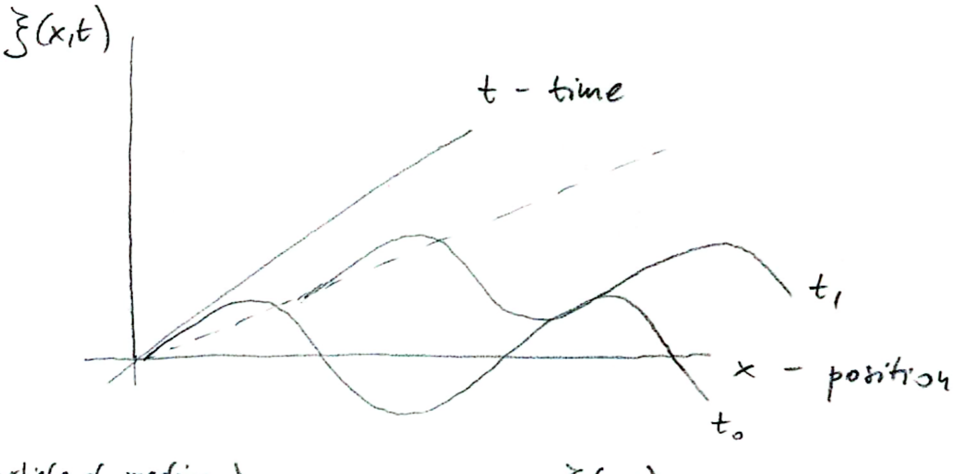
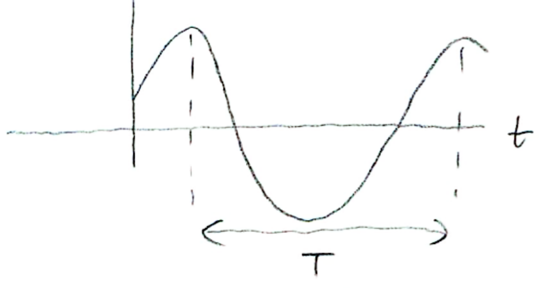
Sinusoidal (harmonic) waves (1D)



$\xi(\cdot, t)$ → fixed point (single particle of medium)

$$\xi(\cdot, t) = \cos\left(\frac{2\pi}{T} - \varphi_t\right)$$

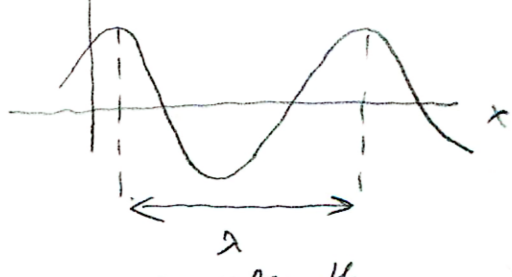


period

particle in harmonic motion about $\xi = 0$

$\xi(x, \cdot)$ fixed instant of time "snapshot" of the wave

$$\xi(x, \cdot) = \cos(kx - \varphi_x)$$



wavelength

$v_{ph} = \frac{\lambda}{T}$ – phase speed

Propagating (traveling) harmonic wave

$$\xi(x,t) = \xi_0 \cos\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right) =$$
$$= \xi_0 \cos(kx - \omega t)$$

where $k = \frac{2\pi}{\lambda}$ (wave number)

$\omega = \frac{2\pi}{T}$ (angular frequency)

$$\boxed{\xi(x,t) = \xi_0 \cos(kx - \omega t)}$$

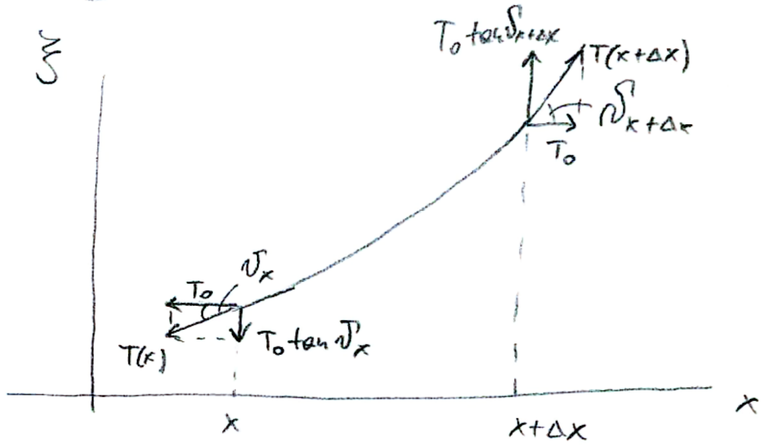
(sinusoidal) harmonic wave traveling to the right with phase speed $v_{ph} = \frac{\omega}{k}$

$kx - \omega t = \text{const} \Rightarrow k\dot{x} - \omega = 0$

$\dot{x} = \frac{\omega}{k}$

The wave equation

Wave on a string of linear density ρ: element of string with length Δx



(sketch exaggerated)

Longitudinal components of the tension equal (T_0), because no motion in that direction

Equ. of motion for vertical direction ($ma = F$)

$$\rho \Delta x \underbrace{\frac{\partial^2 \xi}{\partial t^2}}_{\substack{\text{acceleration} \\ \text{in vertical direction}}} = T_0 \tan\vartheta_{x+\Delta x} - T_0 \tan\vartheta_x$$

But $\tan\vartheta = \frac{\partial \xi}{\partial x}$, and

$$\rho \Delta x \frac{\partial^2 \xi}{\partial t^2} = T_0 \overbrace{\left[\frac{\partial \xi}{\partial x}\Big|_{x+\Delta x} - \frac{\partial \xi}{\partial x}\Big|_{x} \right]}^{\approx \frac{\partial^2 \xi}{\partial x^2}\Delta x} =$$

$$= T_0 \frac{\partial^2 \xi}{\partial x^2} \Delta x$$

Eventually

$$\frac{\partial^2 \xi(x,t)}{\partial t^2} = \frac{T_0}{\rho} \frac{\partial^2 \xi}{\partial x^2}$$

or

$$\boxed{\frac{\partial^2 \xi(x,t)}{\partial x^2} - \frac{\rho}{T_0} \frac{\partial^2 \xi(x,t)}{\partial t^2} = 0}$$

where $\frac{\rho}{T_0} = \frac{1}{v_{ph}^2}$. (or $v_{ph} = \sqrt{\frac{T_0}{\rho}}$) — T_0: measure of the restoring force; ρ: measure of inertia

$\rho \uparrow \quad v_{ph} \downarrow$

$T_0 \uparrow \quad v_{ph} \uparrow$

Comments: (1) the harmonic wave satisfies the wave equation

$$\xi(x,t) = \xi_0 \cos(kx-\omega t)$$

$$\frac{\partial \xi}{\partial x} = -k\,\xi_0 \sin(kx-\omega t)\,; \qquad \frac{\partial^2 \xi}{\partial x^2} = -k^2 \xi_0 \cos(kx-\omega t) = -k^2 \xi(x,t)$$

$$\frac{\partial \xi}{\partial t} = \omega\,\xi_0 \sin(kx-\omega t)\,; \qquad \frac{\partial^2 \xi}{\partial t^2} = -\omega^2 \xi(x,t)$$

Check

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{1}{v_{ph}^2}\frac{\partial^2 \xi}{\partial t^2} = -k^2 \xi(x,t) - \frac{1}{\left(\frac{\omega}{k}\right)^2}(-\omega^2)\,\xi(x,t) \equiv 0 \;\checkmark$$

(2) wave equ. is linear $\Rightarrow$ superposition principle

If ξ_1 and ξ_2 are solutions of the same wave equ.

$\Downarrow$

Any linear combination $\alpha\xi_1 + \beta\xi_2$ is also a solution

$$\frac{\partial^2(\alpha\xi_1+\beta\xi_2)}{\partial x^2} - \frac{1}{v_{ph}^2}\frac{\partial^2(\alpha\xi_1+\beta\xi_2)}{\partial t^2}$$

linearity of partial derivatives

$$= \alpha\underbrace{\left(\frac{\partial^2 \xi_1}{\partial x^2} - \frac{1}{v_{ph}^2}\frac{\partial^2 \xi_1}{\partial t^2}\right)}_{\equiv 0} + \beta\underbrace{\left(\frac{\partial^2 \xi_2}{\partial x^2} - \frac{1}{v_{ph}^2}\frac{\partial^2 \xi}{\partial t^2}\right)}_{\equiv 0} \equiv 0 \;\checkmark$$

(3) the shape of the wave impulse needs not to be sinusoidal

$$\xi(x,t) = f(x-vt)$$

↳ any twice differentiable function $f = f(u)$

$$\frac{\partial \xi}{\partial x} = f'\,; \quad \frac{\partial^2 \xi}{\partial x^2} = f''$$

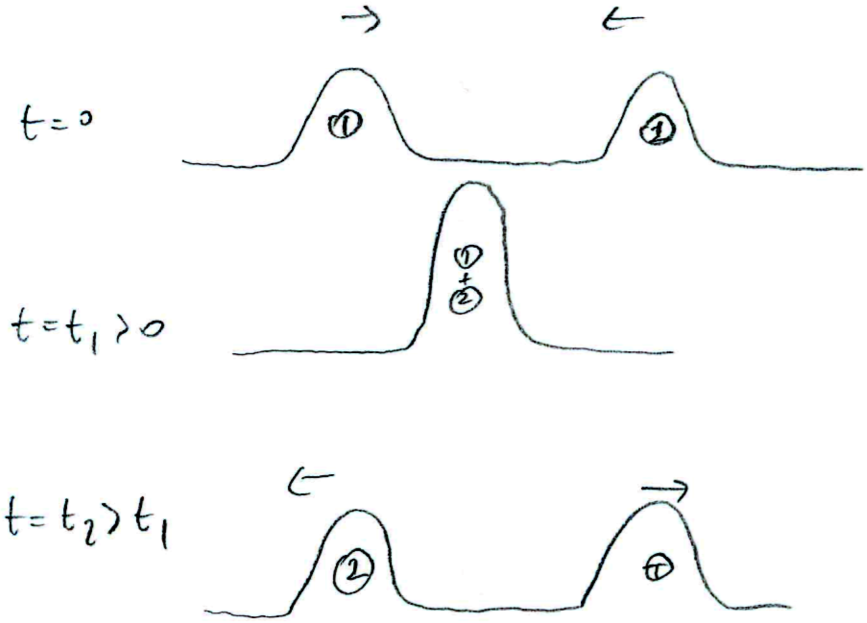
$$\frac{\partial \xi}{\partial t} = -v f'\,; \quad \frac{\partial^2 \xi}{\partial t^2} = v^2 f''$$

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \xi}{\partial t^2} = 0 \;\checkmark$$

Interference of waves

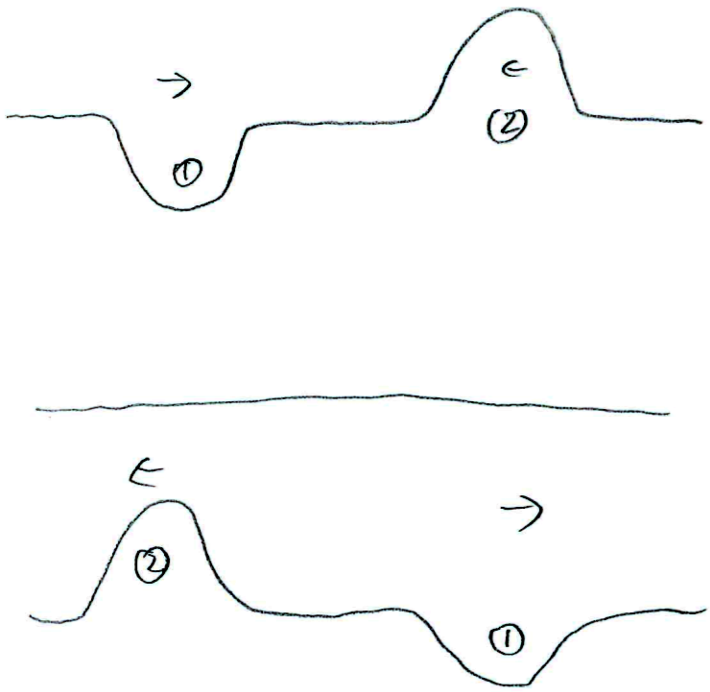
Idea: two impulses $\xi(x,t) = \xi_1(x,t) + \xi_2(x,t)$

(a)



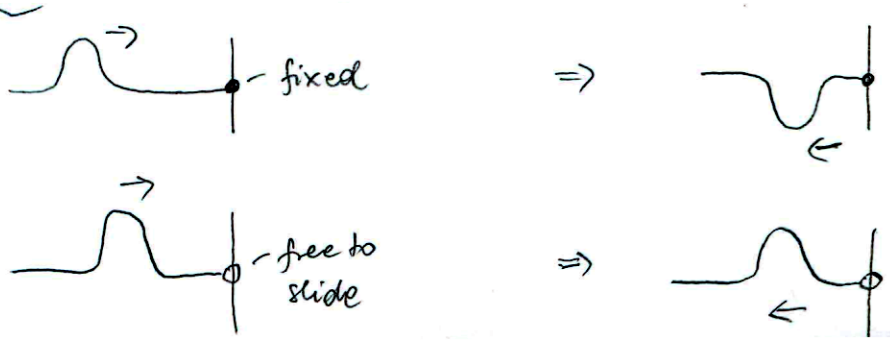
"constructive"

(b)



"destructive"

Digression: reflection of waves – wave pulse on a rope

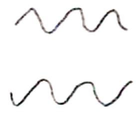


different boundary conditions!

Interference of sinusoidal waves – special case: standing waves

Suppose: two sinusoidal waves with the same wavelength propagating in opposite directions

$$\xi_1(x,t) = -\xi_0 \cos(kx + \omega t) \qquad \leftarrow$$
$$\xi_2(x,t) = \xi_0 \cos(kx - \omega t) \qquad \rightarrow$$

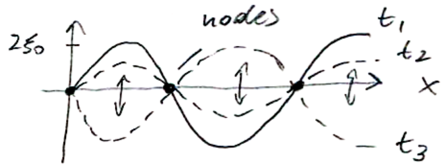


Superposition

$$\xi(x,t) = \xi_1(x,t) + \xi_2(x,t) = \xi_0\left[-\cos(kx+\omega t) + \cos(kx - \omega t)\right] =$$

$$= -2\xi_0 \sin\frac{kx - \cancel{\omega t} + kx + \cancel{\omega t}}{2} \sin\frac{\cancel{kx} - \omega t - \cancel{kx} - \omega t}{2} =$$

$$= 2\xi_0 \underbrace{\sin kx}_{\text{space-dependence}} \underbrace{\sin \omega t}_{\text{time-dependence}}$$

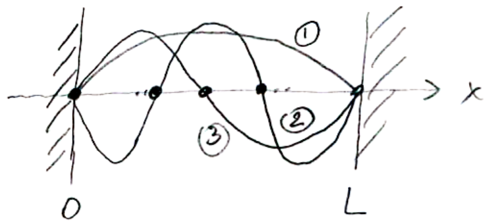


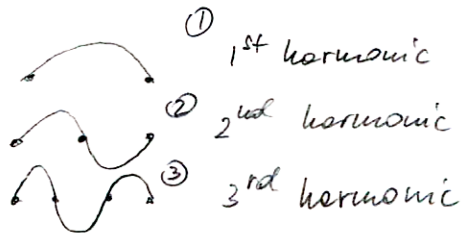
nodes are fixed: $x_{node} = \frac{n\pi}{k} = \lambda \frac{n}{2}$, $n = 0, 1, 2, \ldots$

Standing wave

Example Standing wave on a string of length L clamped at both ends.
What are the possible wavelengths?

Boundary conditions: $\xi(0,t) = \xi(L,t) = 0$ for all t





Possible wavelengths:

$L = n \cdot \frac{\lambda}{2}$ (length of the string accomodates multiples of $\frac{\lambda}{2}$)

$\lambda = \frac{2L}{n} = \lambda_n$

1st harmonic: $\lambda_1 = 2L$
2nd harmonic: $\lambda_2 = L$
3rd harmonic: $\lambda_3 = \frac{2}{3}L$ ...