UM-SJTU JOINT INSTITUTE
PHYSICS LABORATORY
(VP141)

LABORATORY REPORT

EXERCISE 5
DAMPED AND DRIVEN OSCILLATIONS

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1 Objectives

In this exercise, I will study damped and driven oscillations in mechanical systems using the Pohl resonator. I'll also observe and quantify the mechanical resonance phenomenon.

2 Theoretical Background

If a periodically varying external force is applied to a damped harmonic oscillator, the resulting motion is called forced (or driven) oscillations, and the external force is called the driving force. Assuming that the driving force is of the form

$$F = F_0(sin\omega t + \delta)$$

with the amplitude F_0 and angular frequency ω, the resulting steady-state forced oscillations will be simple harmonic with the angular frequency equal to that of the driving force. The amplitude of these steady-state oscillations turns out to depend on the angular frequency of the the driving force, in particular on how far it is from the natural angular frequency, and the damping coefficient. The amplitude may become quite large, and this phenomenon is known as the mechanical resonance.

Another interesting property of driven steady-state oscillations is the fact that there is a phase lag between the driving force and the displacement from the equilibrium position of the oscillating particle. This phase lag reaches $\pi/2$ (a quarter of the cycle) when the system is driven at the natural angular frequency.

In this experiment, forced oscillation of a balance wheel will be studied. Damping in this system is provided by air drag and electromagnets inducing eddy currents in the wheel. It is a rotating system, hence the corresponding quantities (such as the force and the position) will be replaced by their angular counterparts.

When the balance wheel is acted upon a periodic driving torque $\tau_{dr} = \tau_0 cos\omega t$ and a damping torque $\tau_f = -b\frac{d\theta}{dt}$, in addition to the restoring torque $\tau = -k\theta$, its equation of motion is of the form

$$I\frac{d^2\theta}{dt^2} = k\theta - b\frac{d\theta}{dt} + \tau_0 cos\omega t \tag{1}$$

where I is the moment of inertia of the balance wheel, τ_0 is the amplitude of the driving torque, and ω is angular frequency of the driving torque. Introducing the symbols

$$\omega_0^2 = \frac{k}{I}, 2\beta = \frac{b}{I}, \mu = \frac{\tau_0}{I}$$

equation (1) can be rewritten as

$$\frac{d^2\theta}{dt^2} + 2\beta\frac{d\theta}{dt} + \omega_0^2\theta = \mu cos\omega t \tag{2}$$

which, in the absence of the driving torque ($\mu = 0$), is the equation of motion for a damped harmonic oscillator. If, additionally, there is no damping in the system ($\beta = 0$), equation

(2) describes a simple harmonic oscillator with the natural angular frequency ω_0. The solution to equation(2), in the general case of a damped and driven system, is of the form

$$\theta(t) = \theta_{tr}(t) = \theta_{st} cos(\omega t + \varphi) \tag{3}$$

where the former term θ_{dr} denotes the transient solution, that depends on the the initial conditions and vanishes exponentially as $t \to \infty$. The latter term describes steady-state oscillations, with the amplitude

$$\theta_{st} = \frac{\mu}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \tag{4}$$

The phase shift φ gives information about to what extent the displacement from the equilibrium position lags behind the the driving force. It can be found as

$$tan\varphi = \frac{2\beta\omega}{\omega^2 - \omega_0^2}$$

where $-\pi \leqslant \varphi < 0$. Note again that, the amplitude and the phase shift are determined by μ,ω,ω_0, and ω, and but not the initial conditions.

By finding the maximum of θ_{st}, as a function of ω, we can find the *resonance angular frequency* $\omega = \omega res = \sqrt{\omega_0^2 - 2\beta^2}$, and the corresponding amplitude

$$\theta_{res} = \theta_{st}(\omega_{res}) = \frac{\mu}{2\beta\sqrt{\omega_0^2 - \beta^2}}$$

For small values of the damping coefficient β, the resonance angular frequency is close to the the natural angular frequency, and the amplitude of steady-state oscillations becomes large. The dependence of both the amplitude and the phase shift on the driving angular frequency are shown in the left and right Figure 1, respectively, for different values of the damping coefficient. Please note that with increasing damping, (1) the resonance frequency moves away from the natural frequency towards smaller values, (2) the amplitude of the steady-state oscillations decreases.

3 Apparatus

The BG-2 Pohl resonator consists of two main parts: a vibrometer and a control box. The vibrometer is shown in Figure 2. A copper balance wheel is mounted on a supporting

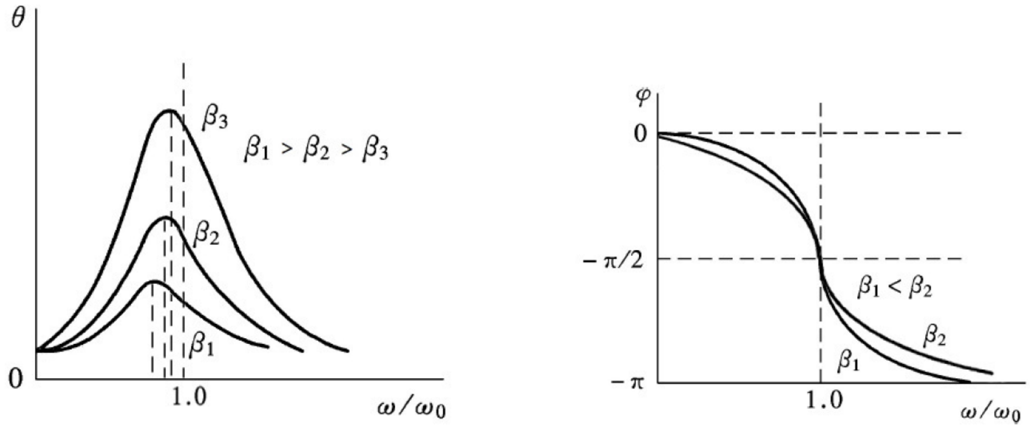


Figure 1: The dependence of the amplitude (left) and phase shift (right) of steady-state driven oscillations.

frame, and the axis of the balance wheel is attached to the supporting frame with a scroll spring. The spring provides an elastic restoring torque to the wheel, which makes the balance wheel rotating about an equilibrium position.

There are many notches on the edge of the balance wheel with one notch being much deeper than the others. A photoelectric detector is set above the deep notch. The detector is used to measure the amplitude and the period of oscillations, and it is connected to the electronic control box.

A pair of coils is placed at the bottom of the supporting frame, with the balance wheel fitting exactly into the gap between the two coils. Due to electromagnetic induction, the wheel will be acted upon an electromagnetic damping force when the coils are carrying current, and the magnitude of the damping force can be controlled by changing the current.

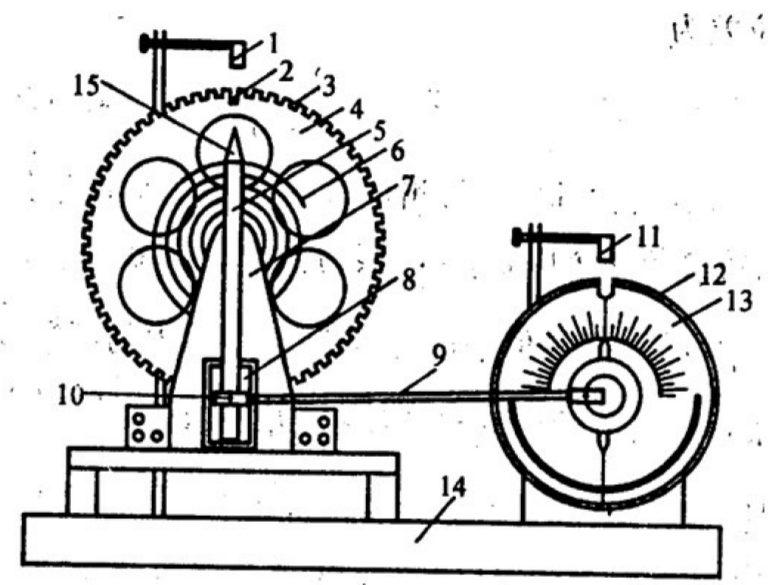
The device is equipped with a motor with an eccentric wheel and a rod used to drive the wheel.

Figure 2: The dependence of the amplitude (left) and phase shift (right) of steady-state driven oscillations.

There is a **Period Selection** switch and a **Period of Driving Force** knob on the electric control box, which allow to control the speed of the motor precisely. Another photoelectric detector is set above the turntable and connected to the control box to measure the period of driving force.

The phase shift can be measured using the glass turntable with an angle scale and a strobe light. The strobe is controlled by the photoelectric detector above the wheel. When the deep notch passes the equilibrium position, the detector sends a signal and the strobe ashes. In a steady state, a line on the angle scale will be highlighted by the ash of the strobe and the phase difference can be read from the angle scale directly.

The amplitude of oscillations is measured by counting the notches on the wheel, and this measurement is performed by a photoelectric detector with the result displayed on the electronic control box.



1. Photoelectric detector (光电门)	2. Deep notch (深凹槽)	3. Shallow notch (浅凹槽)
4. Copper balance wheel (铜质摆轮)	5. Swinging rod (摇杆)	6. Scroll spring (蜗卷弹簧)
7. Supporting frame (支撑架)	8. Damping coil (阻尼线圈)	9. Connecting rod (连杆)
10. Adjustment screw (调节螺钉)	11. Photoelectric detector	12. Angle scale (角度盘)
13. Glass turntable (玻璃转盘)	14. Pedestal (底座)	15. Locking screw (夹持螺钉)

Figure 3: The vibrometer.

The front panel and the rear panel of the control box are shown in Figures 3 and 4, respectively.

The function **Amplitude Display** shows the oscillation amplitude of the balance wheel and Period Display shows the oscillation period in two modes. When the **Period Selection** switch is at position "1", a single oscillation period will be displayed; when the **Period**

Selection switch is at "10", the time of 10 oscillation periods will be displayed. The reset button works only when the **Period Selection** button is at "10".

The period of the driving force can be changed precisely by using the **Period of the driving force** knob, but please pay attention that the scale on the knob is not very accurate.

The **Damping Selection** knob changes the damping force by adjusting the electric current through the coils at the bottom of the wheel. There are six options, ranging from "0" (no current) to "5" (current of about 0.6 A). You will use "2", "3" or "4" in this exercise.

The strobe generates a ash that allows you to read the phase difference from the angle scale directly. To protect the strobe, you should turn on the **Strobe** switch only when measuring the phase difference.

The **Motor Switch** is used to control the motor. You should turn the motor off when measuring the damping coefficient and the natural angular frequency.

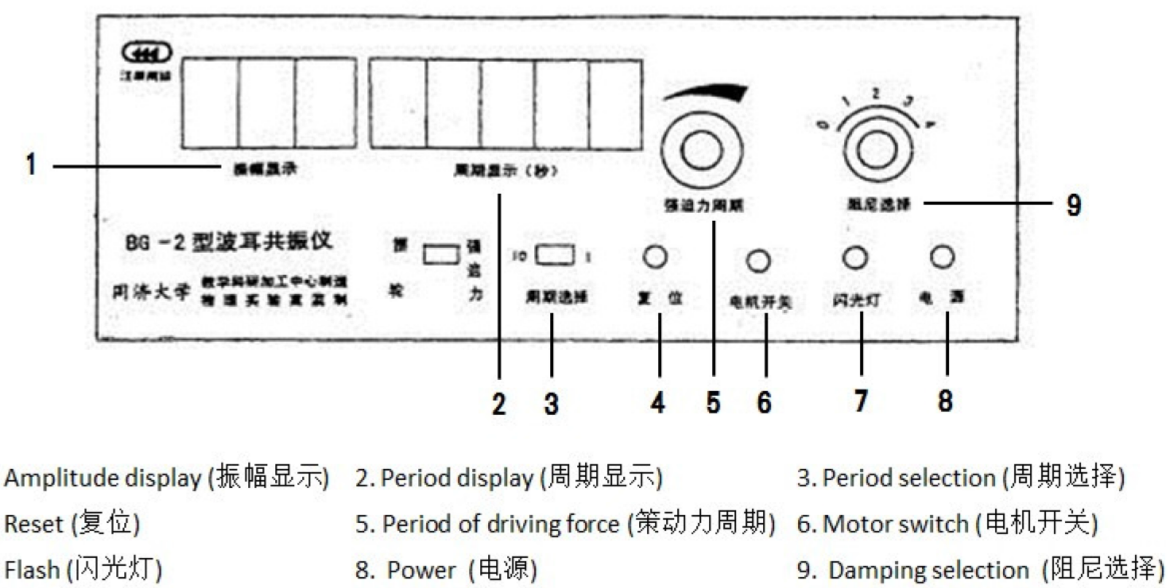


Figure 4: The front panel of the control box.

4 Measurement Procedure

4.1 Natural Angular Frequency

1. Turn the **Damping Selection** knob to "0".

2. Carefully rotate the balance wheel to the initial angular position $\theta_0 \approx 150^o$ and release it. Record the time of 10 periods.

3. Repeat for four times and calculate the natural angular frequency ω_0.

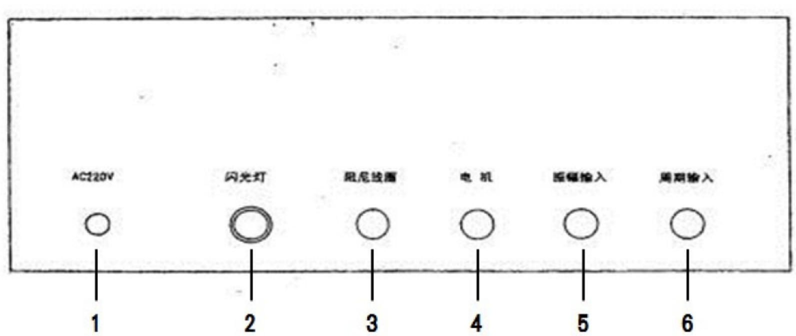
4.2 Damping Coefficient

1. Turn the **Damping Selection** knob to "2", and the selection should not be changed during this part.

2. Carefully rotate the balance wheel to the initial amplitude of approximately 150^o and release it. Record the amplitude of each period (start from the second amplitude after you release the wheel) and the time of 10 periods.

3. The solution to the homogeneous equation of motion (2), with the corresponding initial conditions, is $\theta(t) = \theta_0 e^{-\beta t} cos(\omega_f t + \alpha)$. Hence $\theta_1 = \theta_0 e^{-\beta T}, \theta_2 = \theta_0 e^{-\beta(2T)}, \cdots \theta_n = \theta_0 e^{-\beta(nT)}$. The damping coefficient β can then be calculated as

$$ln\frac{\theta_i}{\theta_j} = ln\frac{\theta_0 e^{-\beta(iT)}}{\theta_0 e^{-\beta(jT)}} = (j-i)\beta T$$



1. 220V AC power (AC220V) 2. Flash (闪光灯) 3. Damping coil (阻尼线圈)
4. Motor (电机) 5. Amplitude input(振幅输入) 6. Period input (周期输入)

Figure 5: The rear panel of the control box.

4. The value of T should be the average period, and $ln\frac{\theta_i}{\theta_{i+5}}$should be obtained by the successive difference method as

$$\beta = \frac{1}{5T} ln\frac{\theta_i}{\theta_{i+5}}$$

4.3 θ_{st} vs. ω and φ vs. ω Characteristics of Forced Oscillations

1. Keep the **Damping Selection** at "2", and set the speed of the motor (record the position of the motor knob in case you need to repeat the measurement). Record the amplitude θ_{st}, the period T, and the phase shift φ when the oscillation reaches a steady state.

2. Repeat the steps above by changing the speed of the motor. It will result in a change of the phase shift φ. To make your plots more accurate, you should collect more data when φ and θ_{st} change rapidly. At least 15 data should be collected for plotting.

3. Choose **Damping Selection** "1" or "3". Repeat the above steps.

4. Plot the $\theta_{st}(\omega)$ characteristics, with ω/ω_0 on the horizontal axis and θ_{st} on the vertical axis. Two sets of data should be plotted on the same graph. Plot the $\varphi(\omega)$ characteristics, with ω/ω_0 on the horizontal axis and φ on the vertical axis. Two sets of data should be plotted on the same graph.

5 Calculation and Results

5.1 Measurement Data

5.1.1 Measurement of the Natural Frequency

We first record the time of ten periods for four times.

	10T [s] $\pm$ 0.001 [s]
1	15.733
2	15.734
3	15.733
4	15.733
Average	15.733

Table 1: Measurement of the natural frequency

5.1.2 Measurement of the damping coefficient

When we measure the damping coefficient, the damping selection is 2.

Amplitude [^{o}] $\pm$ 1 [^{o}]		Amplitude [^{o}] $\pm$ 1 [^{o}]		$\ln(\theta_i/\theta_{i+5})$
θ_0	162	θ_5	102	0.4626
θ_1	148	θ_6	93	0.4646
θ_2	135	θ_7	85	0.4626
θ_3	123	θ_8	77	0.4684
θ_4	112	θ_9	71	0.4558
The average value of $\ln(\theta_i/\theta_{i+5})$				0.4628

Table 2: Measurement of the damping coefficient

$10T$=15.814 [s] $\pm$ 0.001 [s]

5.1.3 Damping Selection 2 Data for θ vs. ω and φ vs. ω characteristics

	10T [s] $\pm$ 0.001 [s]	ω/ω_0	$\varphi[^0] \pm 1[^o]$	$\theta[^0] \pm 1[^o]$
1	15.162	1.039	164	41
2	15.375	1.024	155	58
3	15.519	1.015	145	81
4	15.593	1.010	135	100
5	15.644	1.007	126	118
6	15.680	1.004	116	130
7	15.710	1.002	106	139
8	15.732	1.001	100	142
9	15.748	1.000	95	144
10	15.773	0.998	89	145
11	15.792	0.997	84	144
12	15.812	0.996	80	142
13	15.860	0.993	71	136
14	15.920	0.989	61	127
15	15.981	0.985	54	116
16	16.076	0.980	44	100
17	16.232	0.970	32	77
18	16.378	0.962	25	62
19	16.582	0.950	18	48

Table 3: θ vs. ω and φ vs. ω characteristics

5.1.4 Damping Selection 3 Data for θ vs. ω and φ vs. ω characteristics

	10T [s] $\pm$ 0.001 [s]	ω/ω_0	$\varphi[^0] \pm 1[^o]$	$\theta[^0] \pm 1[^o]$
1	15.105	1.043	165	41
2	15.367	1.025	156	618
3	15.532	1.014	145	87
4	15.608	1.009	134	110
5	15.655	1.006	121	132
6	15.687	1.004	111	143
7	15.706	1.003	105	148
8	15.721	1.002	100	152
9	15.734	1.001	95	154
10	15.760	0.999	90	155
11	15.778	0.998	85	154
12	15.797	0.997	80	152
13	15.825	0.995	74	148
14	15.842	0.994	70	146
15	15.893	0.991	62	136
16	16.967	0.986	52	123
17	16.071	0.980	43	103
18	16.233	0.970	30	78
19	16.439	0.958	21	59

Table 4: θ vs. ω and φ vs. ω characteristics

5.2 Calculations

5.2.1 Natural Angular Frequency

$$\omega_o = \frac{10 \times 2\pi}{T_{average}} = \frac{20\pi}{15.733} = 3.99[rad/s]$$

5.2.2 Damping Coefficient

$$\beta = \frac{2}{T_{10}} ln \frac{\theta_i}{\theta_{i+5}} = \frac{2 \times 0.4628}{15.733} = 0.059[s^{-1}]$$

5.2.3 Graph for θ_{st} vs. ω Characteristics of Forced Oscillations

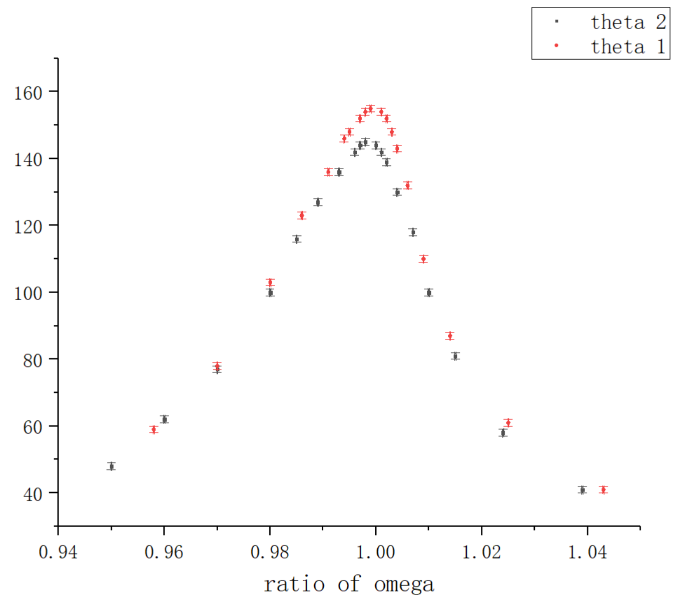


Figure 6: θ_{st} vs. ω Characteristics of Forced Oscillations

5.2.4 Graph for φ vs. ω Characteristics of Forced Oscillations

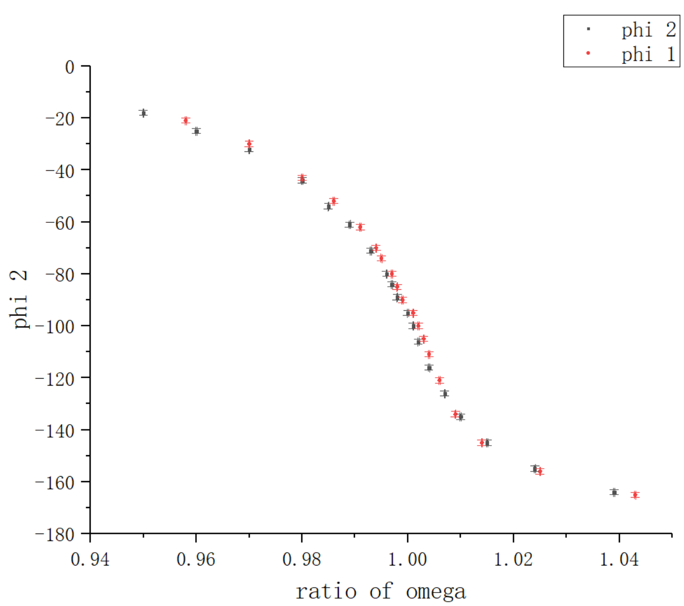


Figure 7: φ vs. ω Characteristics of Forced Oscillations

6 Measurement Uncertainty Analysis

6.1 Uncertainty of Natural Angular Frequency

$$s_X = 5 \times 10^{-4}$$
$$\triangle_A = \frac{S_X t_{0.95}}{\sqrt{n}} = 5 \times 10^{-4} \times 1.59 = 7.95 \times 10^{-4}[s]$$
$$\triangle_B = 0.001[s]$$
$$u_{10T} = \sqrt{\triangle_A^2 + \triangle_B^2} = \sqrt{(7.95 \times 10^{-4})^2 + (0.001)^2} = 1.28 \times 10^{-3}[s]$$
$$\omega_0 = \frac{20\pi}{T_{10}}$$
$$u_\omega = \frac{\partial \omega_0}{\partial T_{10}} \cdot u_T = \frac{20\pi}{15.733^2} \times 1.28 \times 10^{-3} = 3.25 \times 10^{-4}[rad/s]$$
$$u_r = \frac{u_\omega}{\omega_0} \times 100\% = \frac{3.25 \times 10^{-4}}{4.00} \times 100\% = 0.01\%$$

6.2 Uncertainty of Damping Coefficient

6.2.1 Uncertainty of ln(θ_i/θ_{i+5})

$$\mathsf{ln}(\theta_i/\theta_{i+5}) = 0.4628$$
$$s_X = 4.57 \times 10^{-3}$$
$$\triangle_A = 4.57 \times 10^{-3} \times 1.204 = 5.51 \times 10^{-3}$$
$$\triangle_{B_i} = \sqrt{(\frac{\partial ln(\theta_i/\theta_{i+5})}{\partial \theta_i} \cdot u_{theta_i})^2 + (\frac{\partial ln(\theta_i/\theta_{i+5})}{\partial \theta_{i+5}} \cdot u_{theta_{i+5}})^2} = \sqrt{(\frac{u_{\theta_i}}{\theta_i})^2 + (\frac{u_{\theta_{i+5}}}{\theta_{i+5}})^2}$$

When i=0:

$$\triangle_{B_0} = \sqrt{(\frac{1}{162})^2 + (1/102)^2} = 0.012$$

i	$\triangle_B$
0	0.012
1	0.013
2	0.014
3	0.015
4	0.017
average	0.014

Table 5: $\triangle_{B_i}$

$$u = \sqrt{\triangle_A^2 + \triangle_B^2} = \sqrt{(5.51 \times 10^{-3})^2 + 0.014^2} = 0.015$$

6.2.2 Uncertainty of Damping Coefficient

$$\beta = \frac{2}{T_{10}} ln \frac{\theta_i}{\theta_{i+5}}$$

$$u_\beta = 2\sqrt{(\frac{0.4628}{15.814^2} \times 0.001)^2 + (\frac{0.015}{15.814})^2} = 1.9 \times 10^{-3}[s^{-1}]$$

$$u_r = \frac{u_\beta}{\beta} \times 100\% = \frac{1.9 \times 10^{-3}}{0.059} \times 100\% = 3.22\%$$

6.3 Uncertainty of ω/ω_0

$$u_1 = \sqrt{(\frac{u_\omega}{\omega_0})^2 + (\frac{u_{\omega_0} \cdot \omega^2}{\omega_0})^2} = \sqrt{(\frac{3 \times 10^{-4}}{4})^2 + (\frac{3 \times 10^{-4} \times 4.144^2}{4})^2} = 1.1 \times 10^{-4}$$

$$u_r = \frac{1.1 \times 10^{-4}}{1.039} \times 100\% = 0.01\%$$

6.4 Uncertainty of Data for θ_{st} vs. ω in Section 2

The uncertainty can be calculated similarly like natural angular frequency.

	$\omega[rad/s]$	$u_\omega[rad/s]$	u_r	ω/ω_0	u	u_r
1	4.144	3×10^{-4}	0.01%	1.039	1.1×10^{-4}	0.01%
2	4.087	3×10^{-4}	0.01%	1.024	1.1×10^{-4}	0.01%
3	4.049	3×10^{-4}	0.01%	1.015	1×10^{-4}	0.01%
4	4.029	3×10^{-4}	0.01%	1.010	1×10^{-4}	0.01%
5	4.016	3×10^{-4}	0.01%	1.007	1×10^{-4}	0.01%
6	4.007	3×10^{-4}	0.01%	1.004	1×10^{-4}	0.01%
7	3.999	3×10^{-4}	0.01%	1.002	1×10^{-4}	0.01%
8	3.994	3×10^{-4}	0.01%	1.001	1×10^{-4}	0.01%
9	3.990	3×10^{-4}	0.01%	1.000	1×10^{-4}	0.01%
10	3.984	3×10^{-4}	0.01%	0.998	1×10^{-4}	0.01%
11	3.792	3×10^{-4}	0.01%	0.997	1×10^{-4}	0.01%
12	3.974	3×10^{-4}	0.01%	0.996	1×10^{-4}	0.01%
13	3.962	3×10^{-4}	0.01%	0.993	1×10^{-4}	0.01%
14	3.947	2×10^{-4}	0.01%	0.989	1×10^{-4}	0.01%
15	3.931	2×10^{-4}	0.01%	0.985	1×10^{-4}	0.01%
16	3.908	2×10^{-4}	0.01%	0.980	1×10^{-4}	0.01%
17	3.871	2×10^{-4}	0.01%	0.970	1×10^{-4}	0.01%
18	3.836	2×10^{-4}	0.01%	0.962	1×10^{-4}	0.01%
19	3.789	2×10^{-4}	0.01%	0.950	1×10^{-4}	0.01%

Table 6: θ vs. ω and φ vs. ω characteristics

6.5 Uncertainty of Data for θ_{st} vs. ω in Section 1

The uncertainty can be calculated similarly like natural angular frequency.

	$\omega[rad/s]$	$u_\omega[rad/s]$	u_r	ω/ω_0	u	u_r
1	4.160	3×10^{-4}	0.01%	1.043	1.1×10^{-4}	0.01%
2	4.089	3×10^{-4}	0.01%	1.025	1.1×10^{-4}	0.01%
3	4.045	3×10^{-4}	0.01%	1.014	1×10^{-4}	0.01%
4	4.026	3×10^{-4}	0.01%	1.009	1×10^{-4}	0.01%
5	4.014	3×10^{-4}	0.01%	1.006	1×10^{-4}	0.01%
6	4.005	3×10^{-4}	0.01%	1.004	1×10^{-4}	0.01%
7	4.000	3×10^{-4}	0.01%	1.003	1×10^{-4}	0.01%
8	3.997	3×10^{-4}	0.01%	1.002	1×10^{-4}	0.01%
9	3.993	3×10^{-4}	0.01%	1.001	1×10^{-4}	0.01%
10	3.987	3×10^{-4}	0.01%	0.999	1×10^{-4}	0.01%
11	3.982	3×10^{-4}	0.01%	0.998	1×10^{-4}	0.01%
12	3.977	3×10^{-4}	0.01%	0.997	1×10^{-4}	0.01%
13	3.970	3×10^{-4}	0.01%	0.995	1×10^{-4}	0.01%
14	3.966	3×10^{-4}	0.01%	0.994	1×10^{-4}	0.01%
15	3.953	2×10^{-4}	0.01%	0.991	1×10^{-4}	0.01%
16	3.935	2×10^{-4}	0.01%	0.986	1×10^{-4}	0.01%
17	3.910	2×10^{-4}	0.01%	0.980	1×10^{-4}	0.01%
18	3.871	2×10^{-4}	0.01%	0.970	1×10^{-4}	0.01%
19	3.822	2×10^{-4}	0.01%	0.958	1×10^{-4}	0.01%

Table 7: θ vs. ω and φ vs. ω characteristics

7 Conclusion and Discussion

7.1 Conclusion

In this exercise, I have a rough idea about the damped and driven oscillations. I also learn the usage of resonator. Also, I calculate the natural angular frequency and damping coefficient. $\omega_0 = 4.00\pm3.25\times10^{-4}[s^{-1}]$ with relative error 0.01%. $\beta = 0.059\pm1.9\times10^{-3}[s^{-1}]$ with relative error 3.22%

I also measured the data of $10T$, φ, and θ to draw graphs for θ vs. ω and φ vs. ω characteristics. The figure is similar to the first picture in figure 1. θ reaches its maximum value when $\omega/\omega_0 \approx 1$, and β_1 is large than β_2.

7.2 Discussion

7.2.1 Suggestions and Improvement

At first, we did the exercise quite slow because it's hard for us to find the relation between driven force and φ. I think I should be more familiar with the resonator in advance.

Since the most important part of the graph is when ω is close to ω_0, which means φ is close to 90, so we measured more data at that time.

7.2.2 Error Analysis

β's relative error is 3.22%, which is quite large and can't be neglected. I think it's air damping that lead to such large error.

The ω_0's error is not so large, I think it comes from the resonator itself.

8 Data Sheet

Data sheet is attach to the report

9 Reference

- Young, H.D., Freedman R.A. University Physics. Chapter 21,23,24.

- Qin Tian, Zeng Ming, Zhao Xijian, Krzyzosiak,M. Lab Manual of Exercise 5.

- Qin Tian, Zeng Ming, Zhao Xijian, Krzyzosiak,M. Handbook-Uncertainty Analysis.