Ve215 Electric Circuits

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Chapter 7

First-Order Circuits

7.1 Introduction

- In this chapter, we examine two types of simple circuits: a circuit comprising a resistor and a capacitor and a circuit comprising a resistor and an inductor. These are called *RC* and *RL* circuits, respectively.

- *RC* and *RL* circuits are characterized by first-order differential equations. Hence, the circuits are collectively called *first-order* circuits.
- There are two ways to excite the circuits. The first way is by energy initially stored in the capacitive or inductive element. The second way is by independent sources.

7.2 The Source-Free *RC* Circuit

- A source-free *RC* circuit occurs when its dc source is suddenly disconnected. The energy already stored in the capacitor is released to the resistor.

Consider the circuit in Fig. 7.1. The capacitor is initially charged, we assume that at time $t = 0$, the initial voltage is

$$v(0) = V_0$$

with the corresponding value of energy stored as

$$w_C(0) = \frac{1}{2} C V_0^2$$

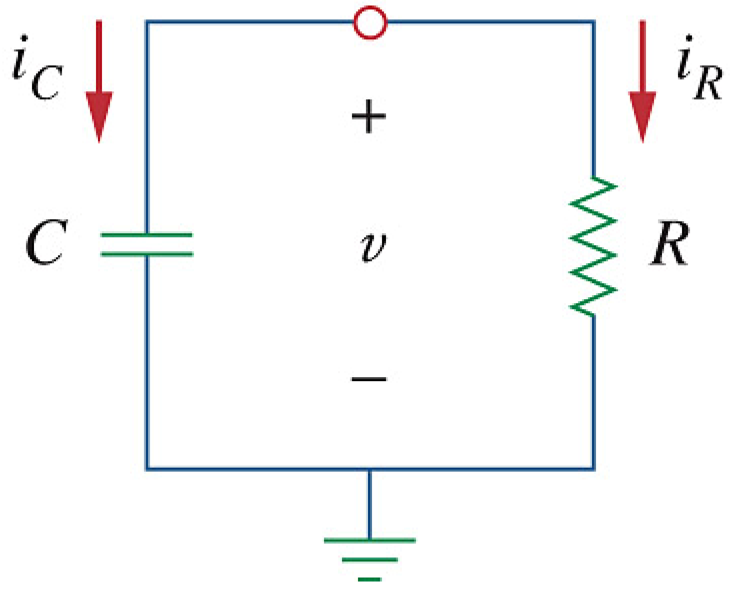

Figure 7.1 A source-free RC circuit.

Our objective is to determine the circuit response, say, the capacitor voltage *v*.

$$i_C + i_R = 0$$

$$i_C = C\frac{dv}{dt}, \, i_R = \frac{v}{R}$$

$$C\frac{dv}{dt} + \frac{v}{R} = 0$$

or

$$\frac{dv}{dt} + \frac{v}{RC} = 0$$

This is a first-order differential equation.

$$r + \frac{1}{RC} = 0$$

$$r = -\frac{1}{RC}$$

$$v = Ae^{rt} = Ae^{-\frac{t}{RC}}$$

$$v(0) = A = V_0$$

$$v = V_0 e^{-\frac{t}{RC}}$$

Since the response is due to V_0 (i.e., the *initial state* of the circuit) and not due to some external voltage or current source (i.e., the input to the circuit), it is called the *zero-input* response of the circuit.

The response of the source-free RC circuit is illustrated in Fig. 7.2. Note that at $t = 0$, $v = V_0$. As t increases, v decreases toward zero. The rapidity with which v decreases is expressed in terms of the *time constant*, denoted by τ.

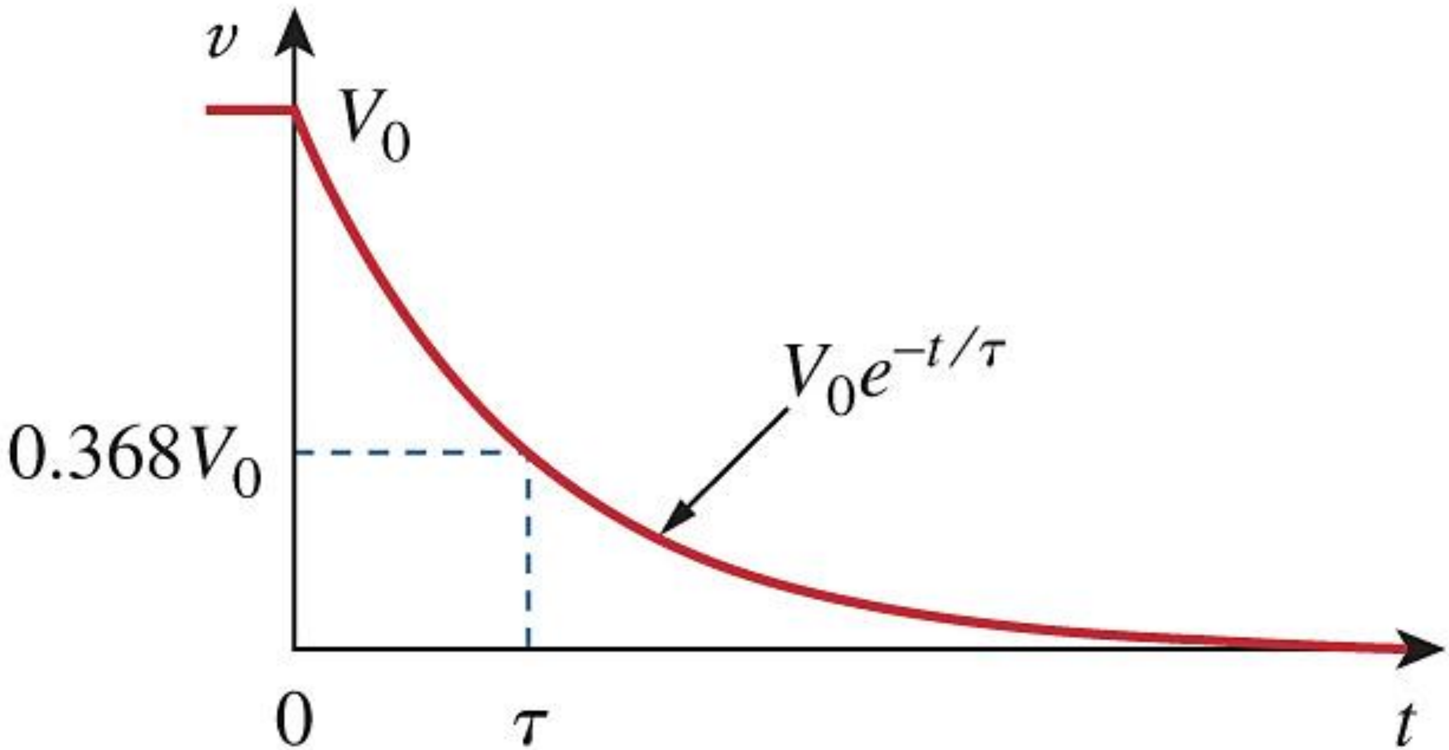

Figure 7.2 The volatge response of the source-free RC circuit.

The *time constant* of a circuit is the time required for the response to decay to a factor of $1/e$ or 38.6% of its initial value.

This implies that at $t = \tau$,

$$v = V_0 e^{-\frac{\tau}{RC}} = V_0 e^{-1}$$

or

$$\tau = RC$$

In terms of the time constant,

$$v = V_0 e^{-t/\tau}$$

The time constant may be viewed from another perspective. Evaluating the derivative of v/V_0 at $t = 0$, we obtain

$$\frac{d}{dt}\left(\frac{v}{V_0}\right)\bigg|_{t=0} = -\frac{1}{\tau}e^{-t/\tau}\bigg|_{t=0} = -\frac{1}{\tau}$$

Thus, the time constant is the initial rate of decay. To find τ from the response curve, draw the tangent to the curve at $t = 0$, as shown in Fig. 7.3. The tangent intercepts with the time axix at $t = \tau$.

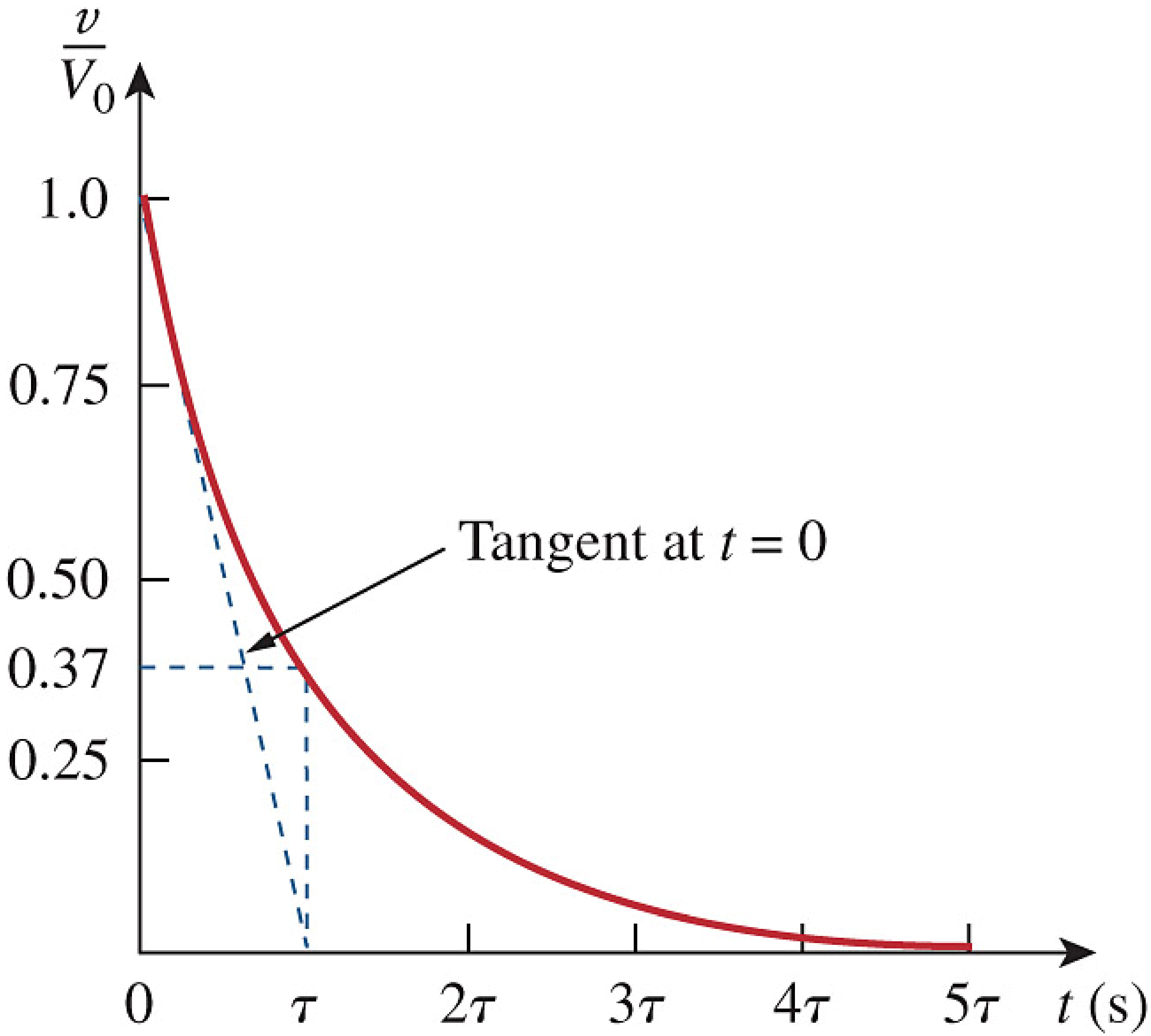

Figure 7.3 Graphical determination of the time constant from the response curve.

Five values of $v / V_0 = e^{-t/\tau}$ are shown in Table 7.1. It is evident from Table 7.1 that v is less than 1% of V_0 after 5τ. Thus, it is customary to assume that the capacitor is fully discharged after five time constants.

TABLE 7.1 Values of $v/V_0 = e^{-t/\tau}$

t	τ	2τ	3τ	4τ	5τ
v/V_0	0.36788	0.13534	0.04979	0.01832	0.00674

In other words, it takes 5τ for the circuit to reach its *final state* or *steady state* when no changes take place with time.

The smaller the time constant, the more rapidly the voltage decreases, that is, the faster the response. This is illustrated in Fig. 7.4. A circuit with a small time constant gives a fast response in that it reaches the steady state quickly due to quick dissipation of energy stored.

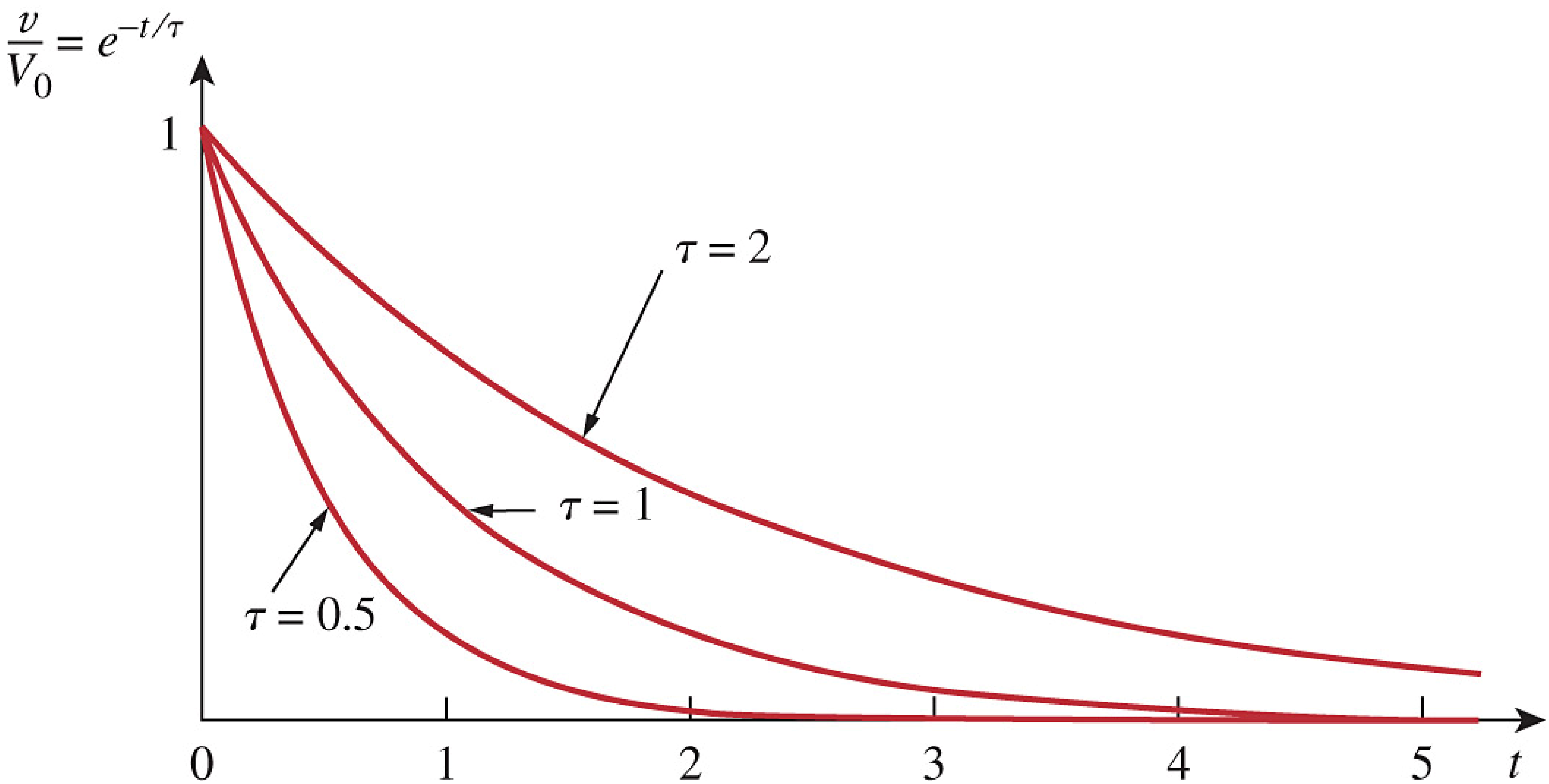

Figure 7.4 Plot of v/V_0 for various values of the time constant.

With the voltage $v = V_0 e^{-t/\tau}$, we can find the resistor current

$$i_R = \frac{v}{R} = \frac{V_0}{R} e^{-t/\tau}$$

The power dissipated in the resistor is

$$p = v i_R = \frac{V_0^2}{R} e^{-2t/\tau}$$

The energy absorbed by the resistor up to time t is

$$w_R = \int_0^t \left(\frac{V_0^2}{R} e^{-2t/\tau} \right) dt = \frac{V_0^2}{R} \frac{e^{-2t/\tau}}{-2/\tau} \Bigg|_0^t$$

$$= \frac{1}{2} C V_0^2 \left(1 - e^{-2t/\tau} \right)$$

Notice that as $t \to \infty$, $w_R \to \frac{1}{2} C V_0^2 = w_C(0)$, the energy initially stored in the capacitor.

In summary, the key to working with a source-free *RC* circuit is to find

1. The initial voltage $v(0) = V_0$ across the capacitor.
2. The time constant $\tau = RC$, where R is often the equivalent resistance at the terminals of C.

With these two items, we obtain the response as the capacitor voltage $v_C = v = v(0)e^{-t/\tau}$.

Once v_C is first obtained, other variables (i_C, v_R, i_R) can be obtained.

Practice Problem 7.1 Refer to the circuit in Fig. 7.7. Let $v_C(0) = 45$ V. Determine v_C, v_x, and i_o for $t \geq 0$.

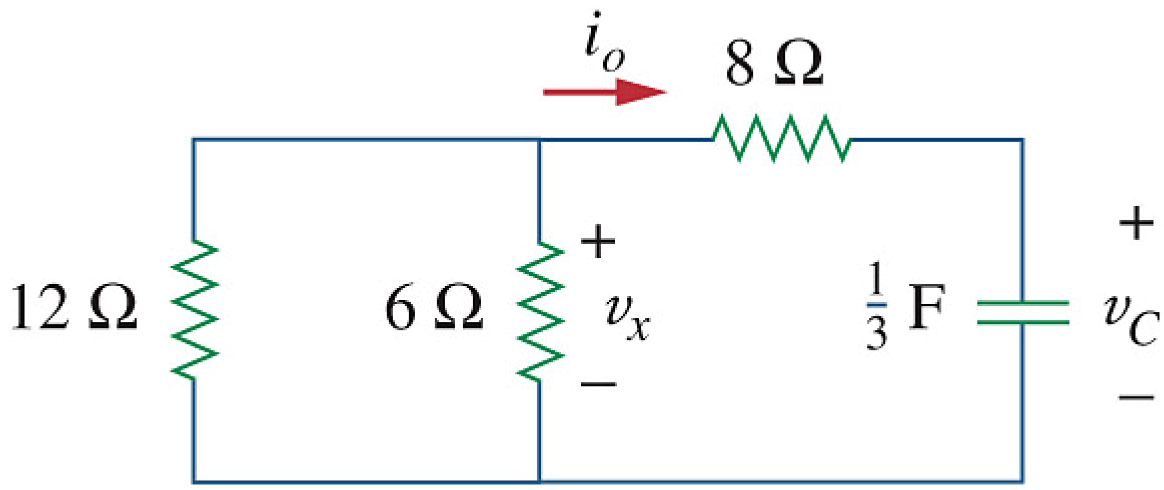

Figure 7.7 An *RC* circuit.

Solution :

The equivalent resistance seen at the terminals of the capacitor is

$$R_{eq} = 8 + 12 \,||\, 6 = 12\ (\Omega)$$

The time constant is

$$\tau = R_{eq}C = 12 \times \frac{1}{3} = 4\ (\text{s})$$

The capacitor voltage is

$$v_C = v_C(0)e^{-t/\tau} = 45e^{-t/4} = 45e^{-0.25t} \text{ (V)}$$

$$i_o = C\frac{dv_C}{dt} = \frac{1}{3} \times \frac{d}{dt}\left(45e^{-0.25t}\right)$$

$$= 15 \times \left(-0.25e^{-t/4}\right) = -3.75e^{-t/4} \text{ (A)}$$

$$v_x = 8i_o + v_C = 8 \times \left(-3.75e^{-0.25t}\right) + 45e^{-0.24t}$$

$$= 15e^{-0.25t} \text{ (V)}$$

Practice Problem 7.12 If the switch in Fig. 7.10 opens at $t = 0$, find $v(t)$ for $t \geq 0$ and $w_C(0)$.

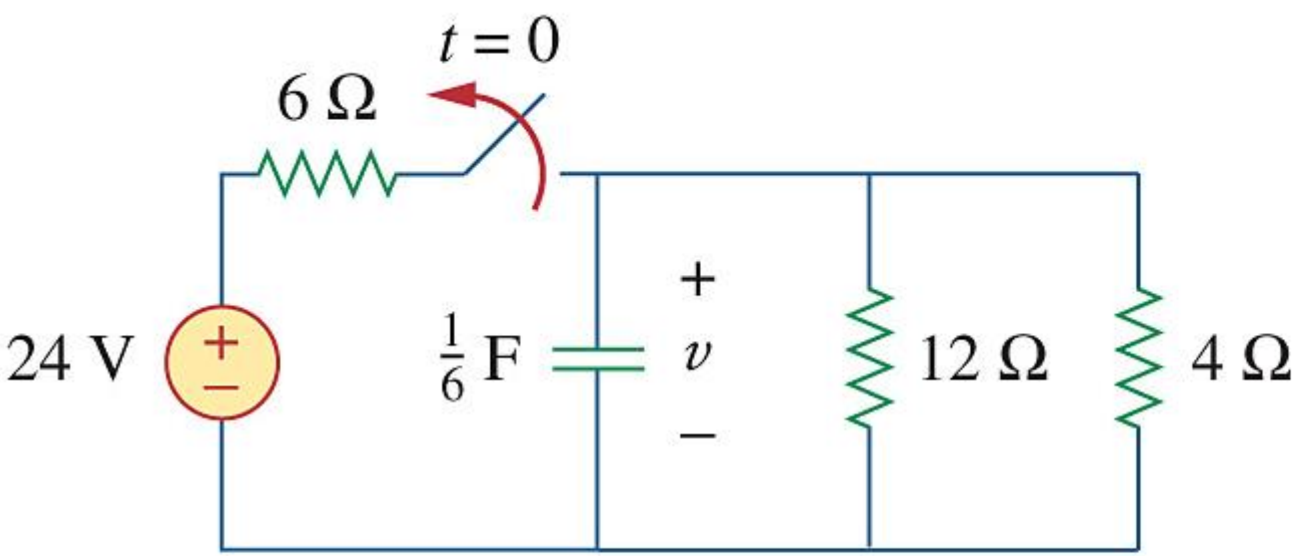

Figure 7.10

Solution :

When $t \le 0,$ the capacitor voltage

$$v(t) = 24 \times \frac{12 \,||\, 4}{6 + 12 \,||\, 4} = 8 \text{ (V)}$$

Hence, $v(0) = 8$ V.

When $t \ge 0$, the circuit becomes a source-free RC circuit with

$$\tau = R_{eq} C = \left(12 \,||\, 4\right) \times \frac{1}{6} = \frac{1}{2} \text{ (s)}$$

Therefore,

$$v(t) = v(0)e^{-t/\tau} = 8e^{-t/(1/2)} = 8e^{-2t} \text{ (V)}$$

$$w_C(0) = \frac{1}{2}C\left(v(0)\right)^2 = \frac{1}{2} \times \frac{1}{6} \times 8^2$$

$$= \frac{16}{3} \approx 5.33 \text{ (J)}$$

7.3 The Source-Free *RL* Circuit

Consider the circuit in Fig. 7.11. Our goal is to determine the circuit response, which we assume to be the current i through the inductor. The inductor has an initial current

$$i(0) = I_0$$

with the corresponding energy

$$w_L(0) = \frac{1}{2} L I_0^2$$

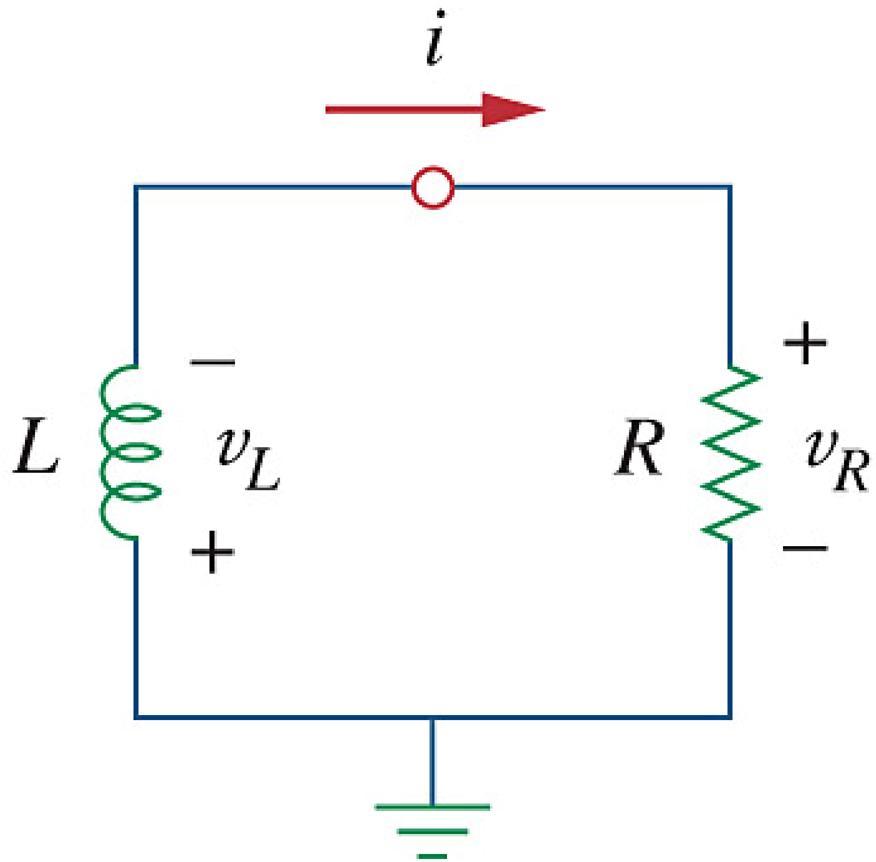

Figure 7.11 A source-free *RL* circuit.

$$v_L + v_R = 0$$

$$v_L = L\frac{di}{dt},\, v_R = iR$$

$$L\frac{di}{dt} + iR = 0$$

or

$$\frac{di}{dt} + \frac{R}{L}i = 0$$

$$r + \frac{R}{L} = 0$$

$$r = -\frac{R}{L}$$

$$i = Be^{rt} = Be^{-\frac{R}{L}t} = Be^{-\frac{t}{(L/R)}}$$

$$i(0) = B = I_0$$

$$i = I_0 e^{-\frac{t}{(L/R)}}$$

Let $\tau = L/R$, we have

$$i = I_0 e^{-t/\tau}$$

The current response is shown in Fig. 7.12.

Since the response is due only to I_0 (i.e., the initial state of the circuit), it is called the zero-input response of the circuit.

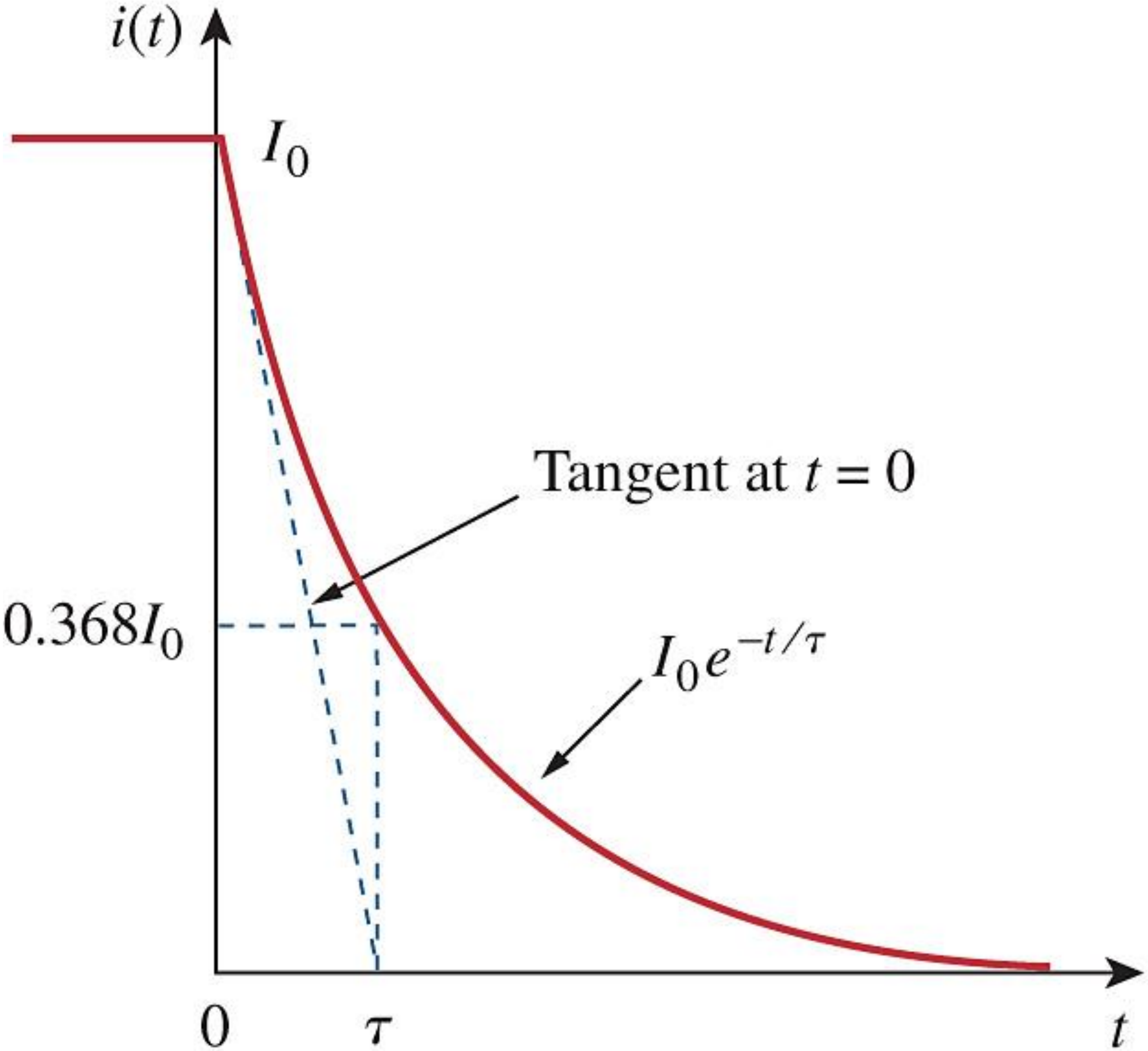

Figure 7.12 The current response of the source-free RL circuit.

With the current $i = I_0 e^{-t/\tau}$, we can find the resistor voltage

$$v_R = iR = I_0 R e^{-t/\tau}$$

The power dissipated in the resistor is

$$p = v_R i = I_0^2 R e^{-2t/\tau}$$

The energy absorbed by the resistor up to time t is

$$w_R = \int_0^t \left(I_0^2 R e^{-2t/\tau} \right) dt = I_0^2 R \left. \frac{e^{-2t/\tau}}{-2/\tau} \right|_0^t$$

$$= \frac{1}{2} L I_0^2 \left(1 - e^{-2t/\tau} \right)$$

Notice that as $t \to \infty$, $w_R \to \tfrac{1}{2} L I_0^2 = w_L(0)$, the energy initially stored in the inductor.

The key to working with a source-free *RL* circuit is to find

1. The initial current $i(0) = I_0$ through the inductor.
2. The time constant $\tau = L / R$, where R is often the equivalent resistance at the terminals of L.

With the two items, we obtain the response as the inductor current $i_L = i = i(0)e^{-t/\tau}$.

Once i_L is first obtained, other variables (v_L, v_R, i_R) can be obtained.

Practice Problem 7.4 For the circuit in Fig. 7.18, find $i(t)$ for $t > 0$.

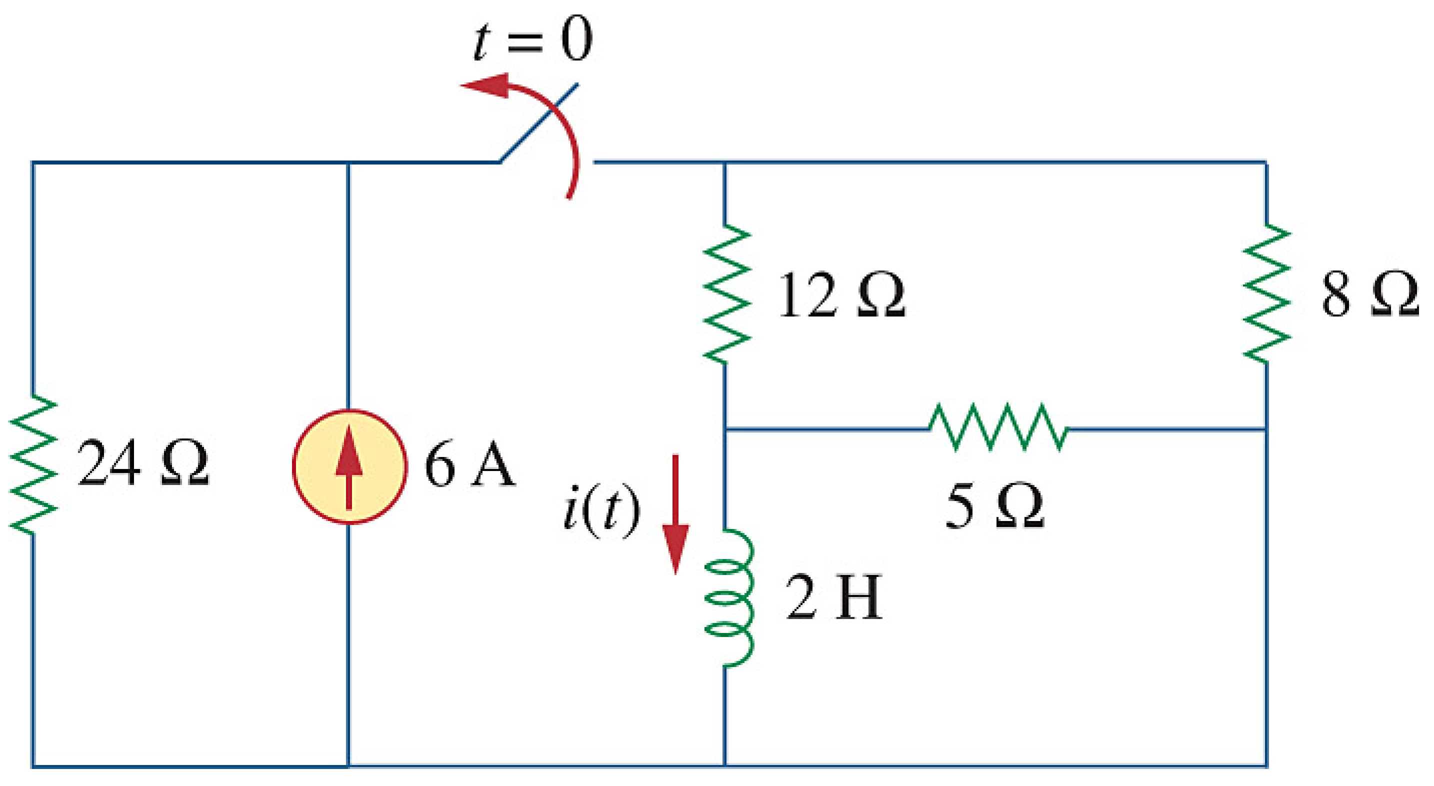

Figure 7.18

Solution :

When $t < 0,$ the current through the 5-Ω resistor is zero.

$$i(t) = 6\times\frac{24\,||\,8}{24\,||\,8+12} = 2\ (\mathrm{A})$$

When $t > 0,$

$$R_{eq} = 5\,||\,(12+8) = 4\ (\Omega)$$

$$\tau = L\,/\,R_{eq} = 2\,/\,4 = 0.5\ (\mathrm{s})$$

$$i(t) = i(0)e^{-t/\tau} = 2e^{-t/0.5} = 2e^{-2t}\ (\mathrm{A})$$

7.4 Singularity Functions

- Singularity functions (also called *switching functions*) are functions that either are discontinuous or have discontinuous derivatives. They serve as good approximations to switching operations.
- The three most widely used singularity functions in circuit analysis are the *unit step*, the *unit impulse*, and the *unit ramp* functions.

The unit step function $u(t)$ is 0 for negative values of t and 1 for positive values of t. In mathematical terms,

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

The unit step function is not defined at $t = 0$, where it changes abruptly from 0 to 1.

Figure 7.23 depicts the unit step function.

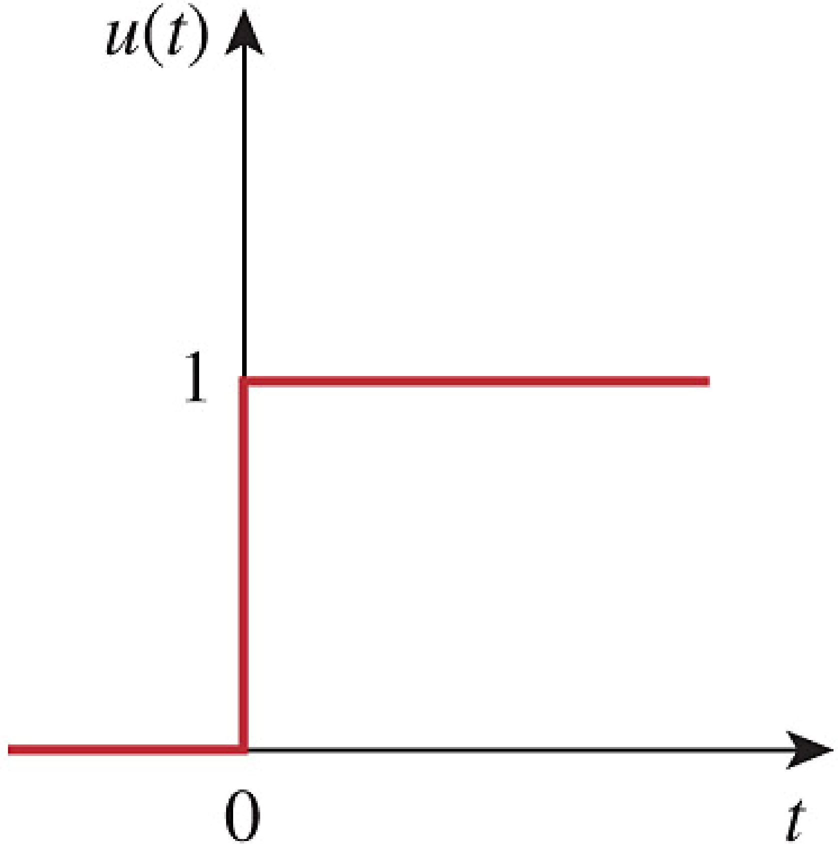

Figure 7.23 The unit step function.

If the abrupt change occurs at $t = t_0$ (where $t_0 > 0$) instead of $t = 0$, the mathematical representation becomes

$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases}$$

meaning that $u(t)$ is delayed by t_0 seconds.

If the abrupt change occurs at $t = -t_0$ (where $t_0 > 0$) instead of $t = 0$, the mathematical representation becomes

$$u(t+t_0) = \begin{cases} 0, & t < -t_0 \\ 1, & t > -t_0 \end{cases}$$

meaning that $u(t)$ is advanced by t_0 seconds.

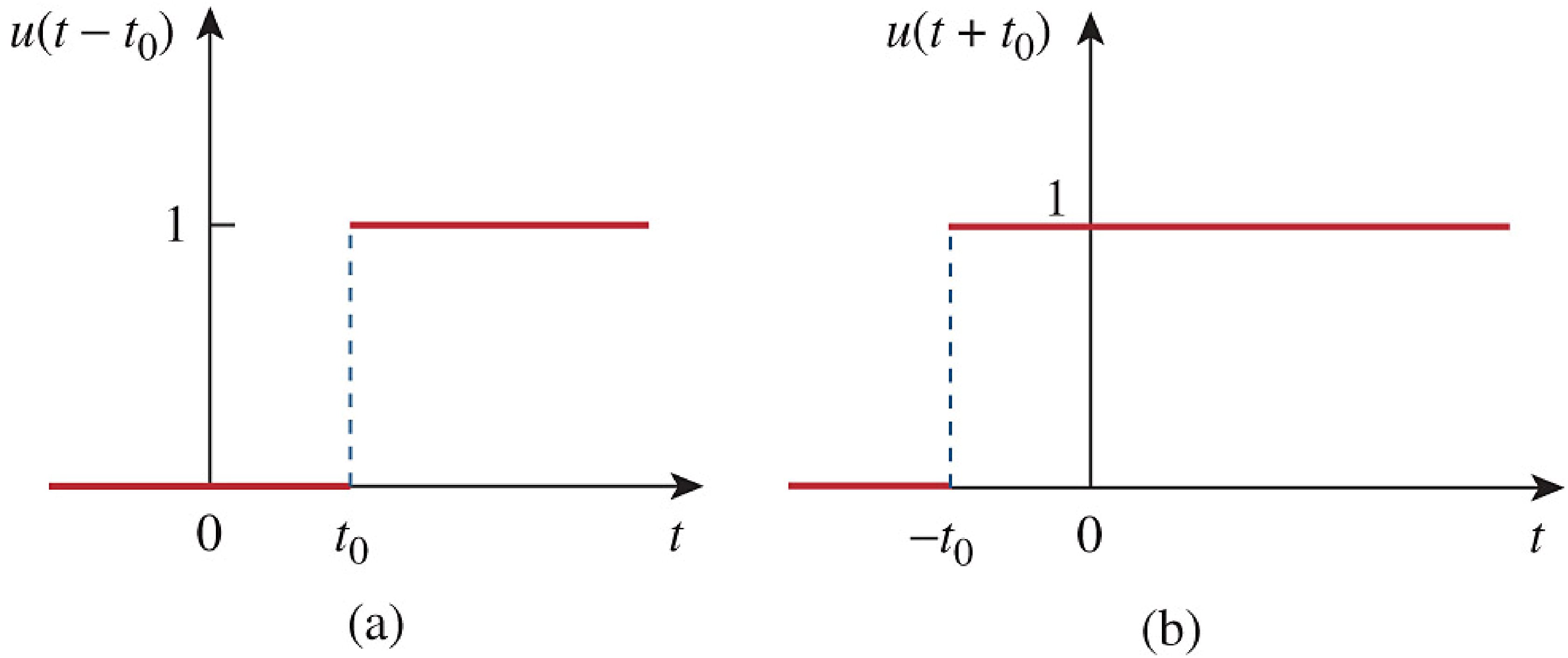

Figure 7.24 (a) The delayed version of the unit step, (b) the advanced version of the unit step.

In general, the step function is $Ku(t)$, where K is a constant.

Switching operations create abrupt changes in voltages and currents. An abrupt change can be represented by the step function. Two examples are given in Figures 7.25 and 7.26.

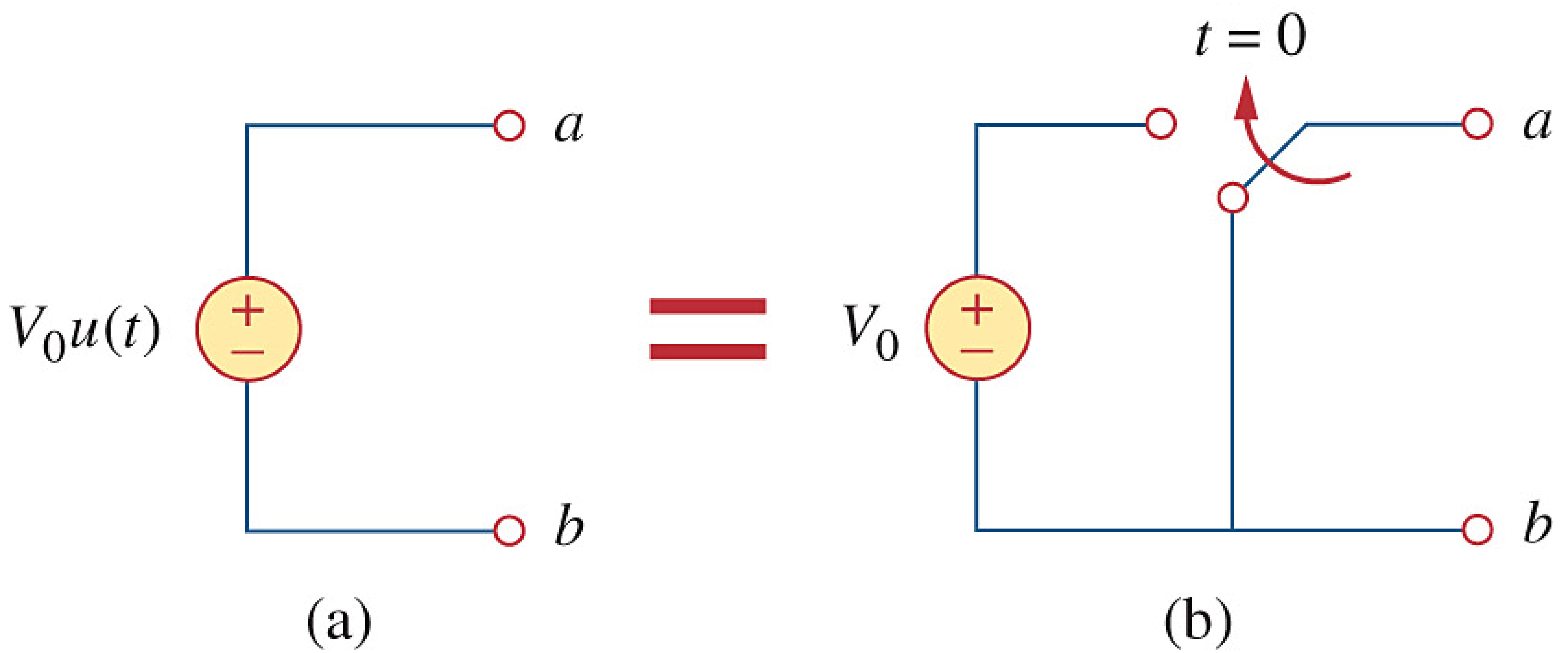

Figure 7.25 (a) Voltage source of $V_0 u(t)$, (b) its equivalent circuit.

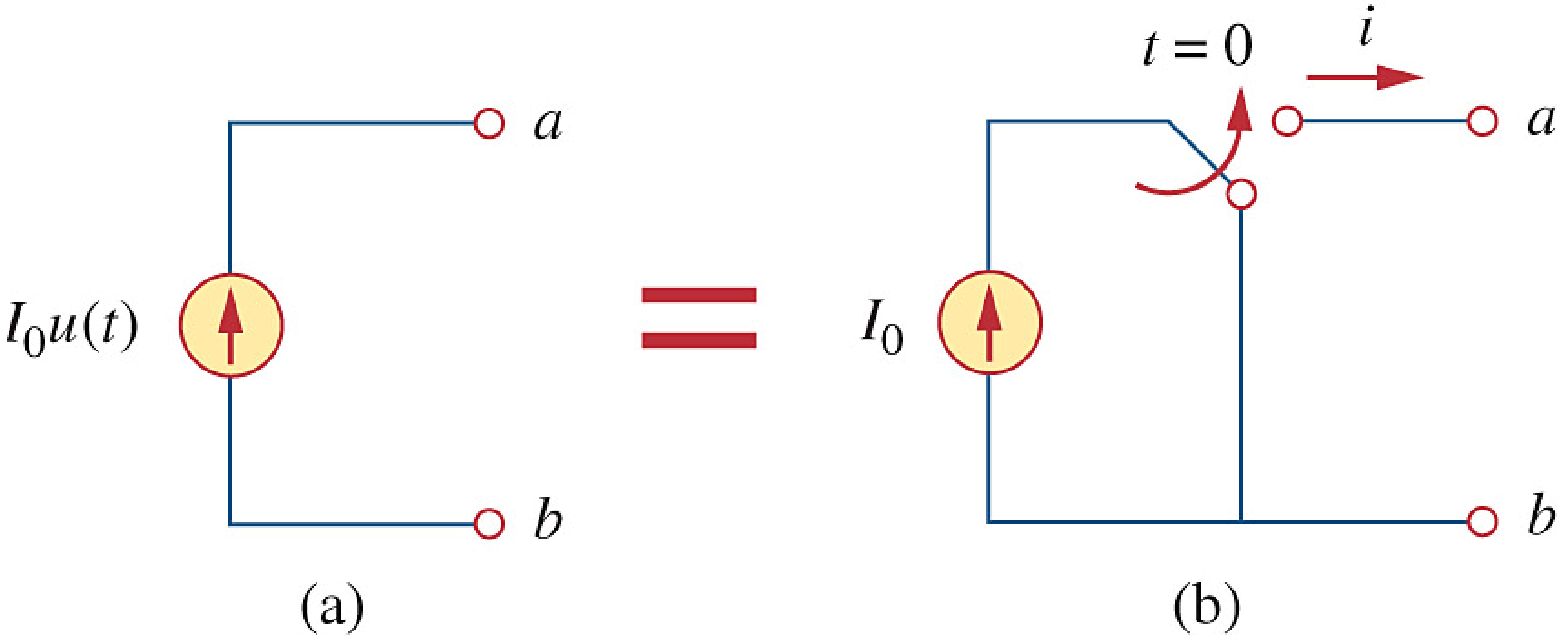

Figure 7.26 (a) Current source of $I_0u(t)$, (b) its equivalent circuit.

Singularity functions can be used to write mathematical expressions for signals. For example, the *rectangular pulse* in Fig. 7(i) can be expressed as

$$x_\Delta(t) = \frac{1}{\Delta}\left(u(t+\Delta/2) - u(t-\Delta/2)\right)$$

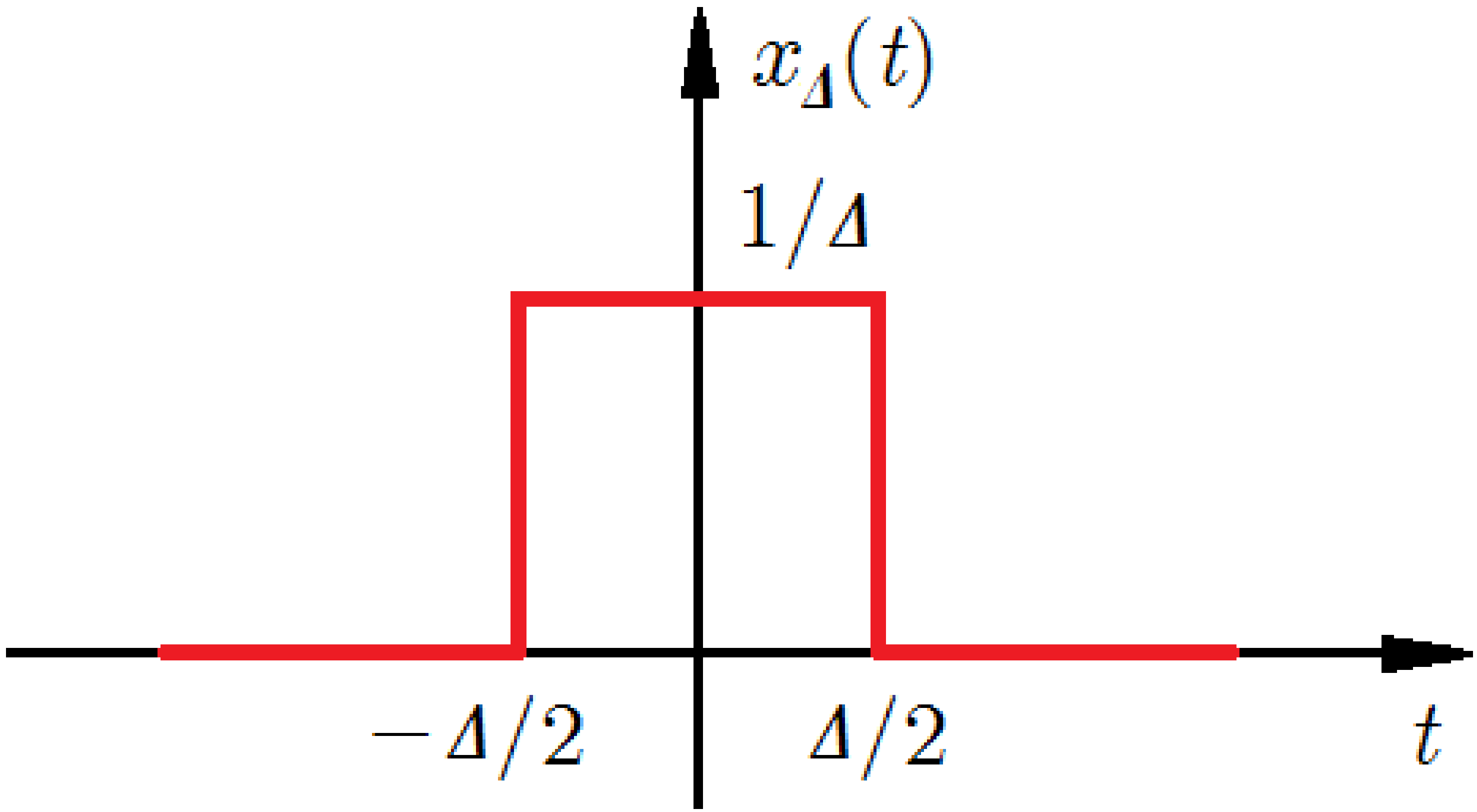

Figure 7(i) A rectangular pulse.

We view the *unit impulse* function $\delta(t)$ as the limiting form of any pulse, say the rectangular pulse $x_\Delta(t)$ in Fig. 7(i), that is an even function of time t with unit area:

$$\delta(t) = \lim_{\Delta \to 0} x_\Delta(t)$$

The derivative of the unit step function $u(t)$ is the unit impulse function $\delta(t)$, which we write as

$$\delta(t) = \frac{d}{dt} u(t)$$

That implies

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

Proof :

Without loss of generality, we take the rectangular pulse as an example.

$$\delta(t) = \lim_{t \to 0} x_\Delta(t)$$

$$= \lim_{t \to 0} \frac{1}{\Delta}\big(u(t+\Delta/2) - u(t-\Delta/2)\big)$$

$$= \lim_{t \to 0} \frac{u(t+\Delta/2) - u(t-\Delta/2)}{(t+\Delta/2) - (t-\Delta/2)} = \frac{d}{dt}u(t)$$

An impulse function is a signal of infinite amplitude and zero duration. Such signals don't exist in nature, but some circuit signals come very close to approximating this definition, so we find a mathematical model of an impulse useful.

The unit impulse function is also called the *delta* function. We use an arrow to symbolize the unit impulse function, as in Fig. 7.27. The area under $\delta(t)$ is unity:

$$\int_{-\infty}^{\infty} \delta(t)dt = \int_{0^-}^{0^+} \delta(t)dt = 1$$

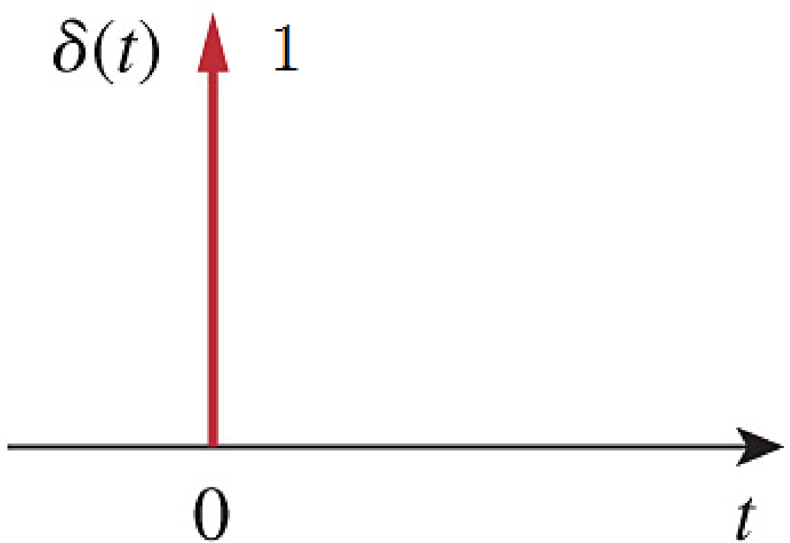

Figure 7.27 The unit impulse function.

where $t = 0^-$ denotes the time just before $t = 0$ and $t = 0^+$ denotes the time just after $t = 0$. For this reason, it is customary to write "1" beside the arrow. The area is known as the *strength* of the function. Figure 7.28 shows the impulse functions $5\delta(t+2)$, $10\delta(t)$, and $-4\delta(t-3)$.

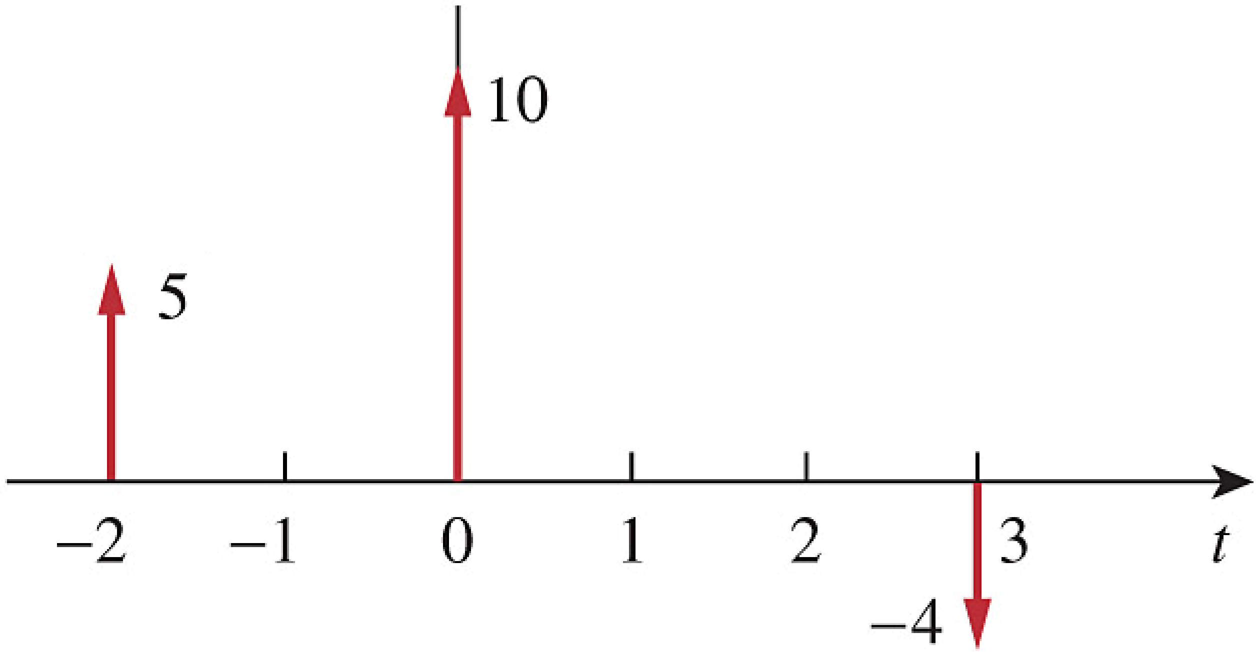

Figure 7.28

The unit impulse function has a property:

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$$

where $f(t)$ is a function that is continuous at $t = t_0$.

Proof :

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = \int_{-\infty}^{\infty} f(t_0)\delta(t-t_0)dt$$

$$= f(t_0)\int_{-\infty}^{\infty} \delta(t-t_0)dt = f(t_0)$$

This shows that when a function is integrated with the impulse function, we obtain the value of the function at the point where the impulse occurs. This property is known as the *sifting* property.

When $t = t_0$, the equation becomes

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

The *unit ramp function* is zero for negative values of t and has a unit slope for positive values of t.

$$r(t) = \begin{cases} 0, & t \le 0 \\ t, & t \ge 0 \end{cases}$$

Figure 7.29 shows the unit ramp function.

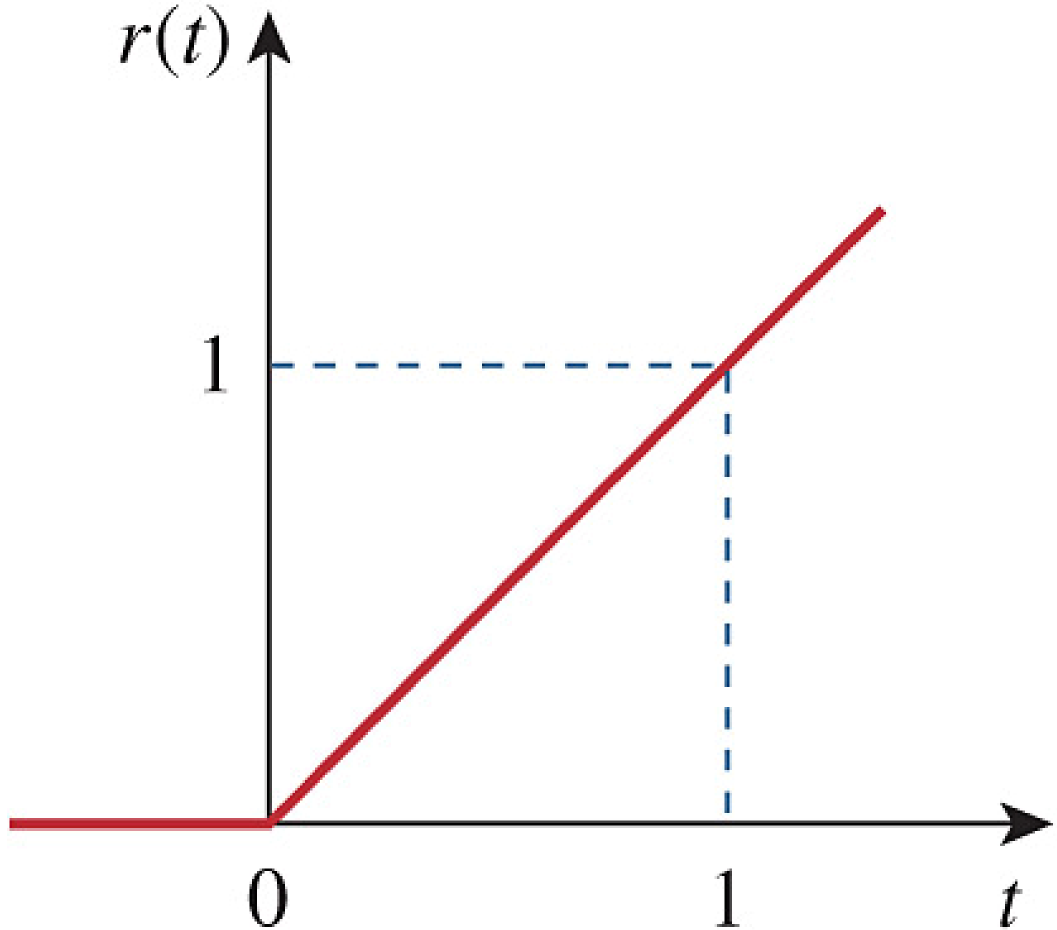

Figure 7.29 The unit ramp function.

Integrating the unit step function $u(t)$ results in the unit ramp function $r(t)$; we write

$$r(t) = \int_{-\infty}^{t} u(\tau)d\tau = tu(t)$$

The unit ramp function may be delayed or advanced as shown in Fig. 7.30.

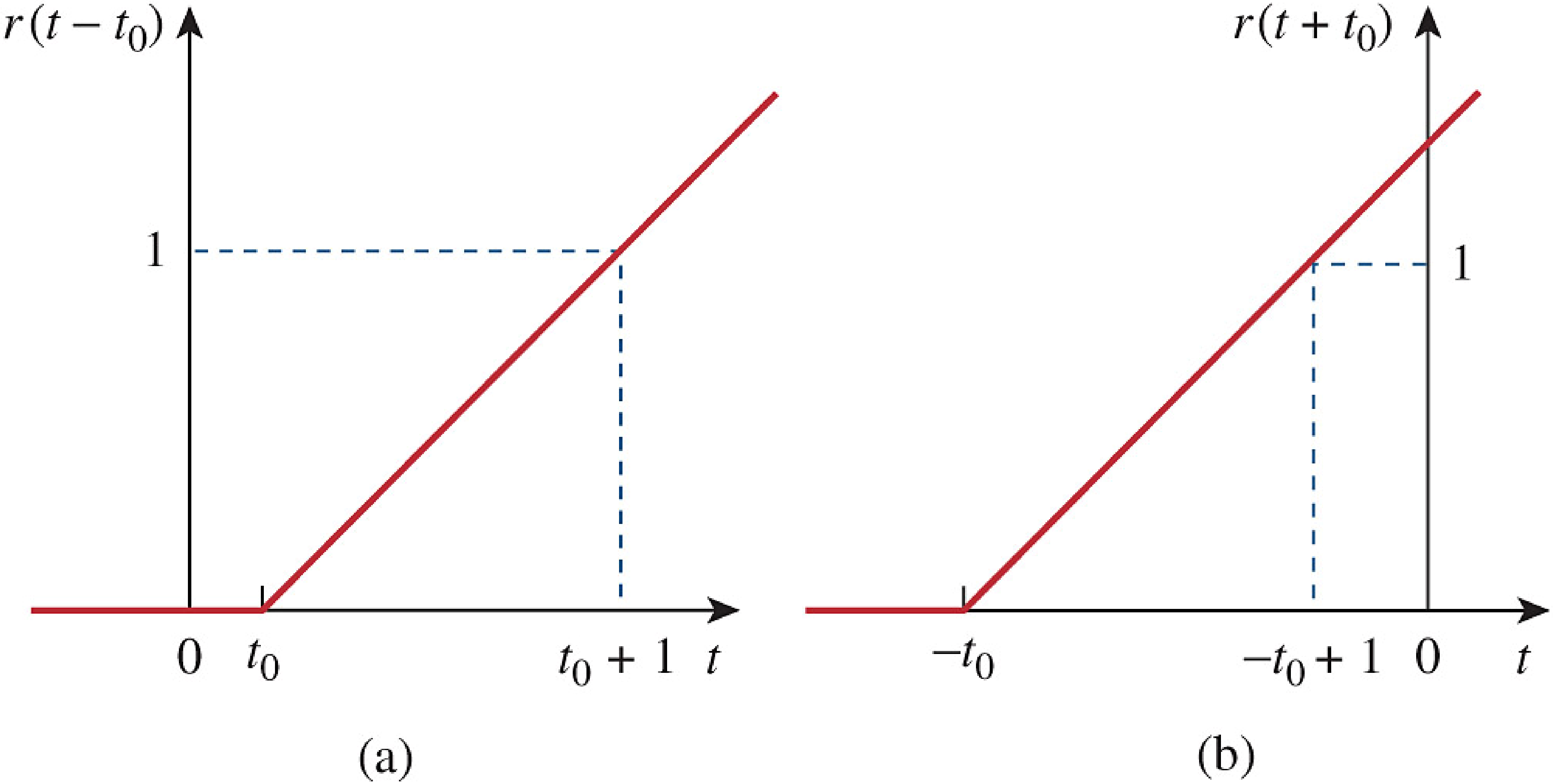

Figure 7.30 The unit ramp function (a) delayed by t_0, (b) advanced by t_0.

Example 7.6 Express the voltage pulse in Fig. 7.31 in terms of the unit step. Calculate its derivative and sketch it.

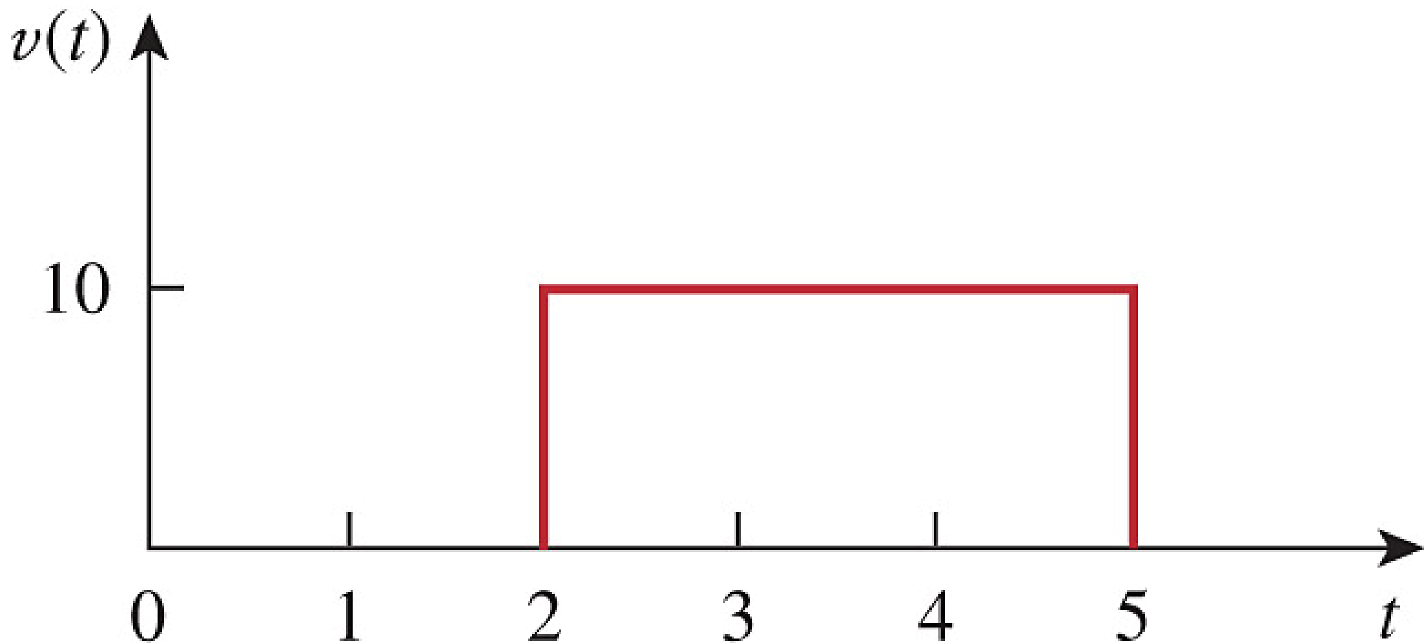

Figure 7.31

Solution :

$$v(t) = 10\left[u(t-2) - u(t-5)\right]$$

$$\frac{dv}{dt} = 10\left[\delta(t-2) - \delta(t-5)\right]$$, see Fig. 7.32(b).

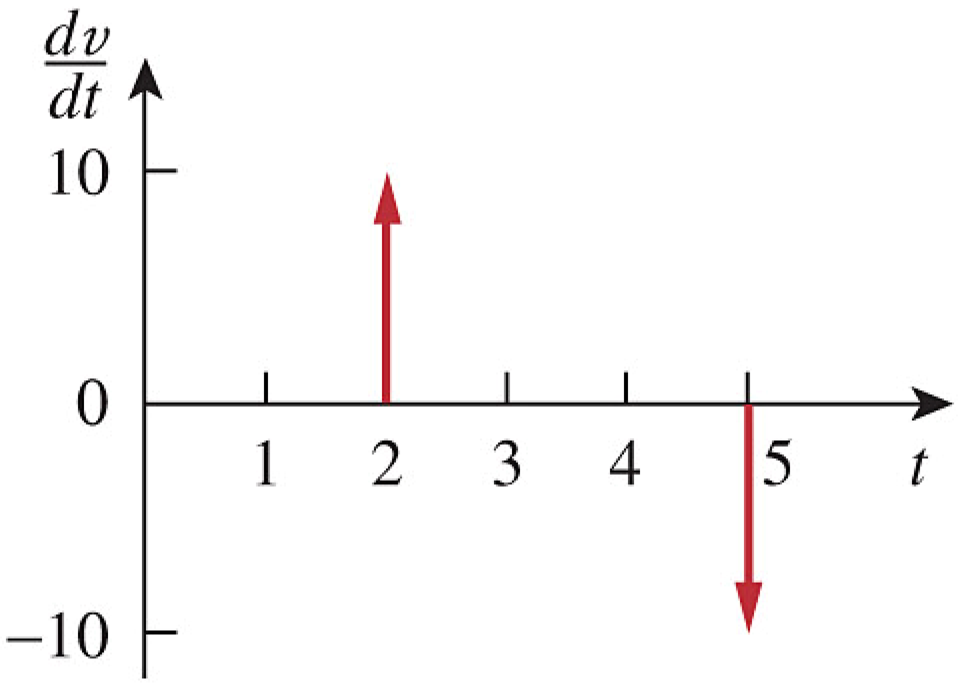

Figure 7.32(b)

Example 7.7 Express the *sawtooth* function shown in Fig. 7.35 in terms of singularity functions.

Solution :

$$v(t) = 5r(t) - 5r(t-2) - 10u(t-2).$$

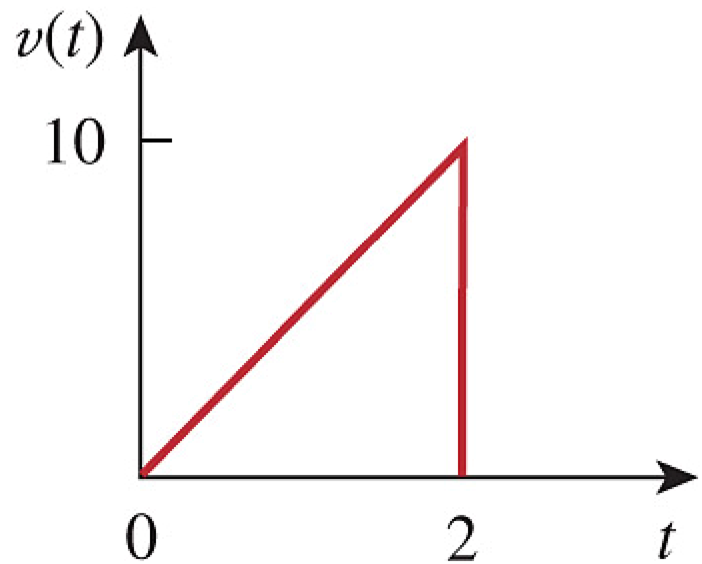

Figure 7.35

7.5 An *RC* Circuit with Step Input

Consider the *RC* circuit in Fig. 7.40(a) which can be replaced by the circuit in Fig. 7.40(b), where V_s is a constant dc voltage source. Again, we select the capacitor voltage v as the circuit response to be determined. We assume an initial voltage V_0 on the capacitor.

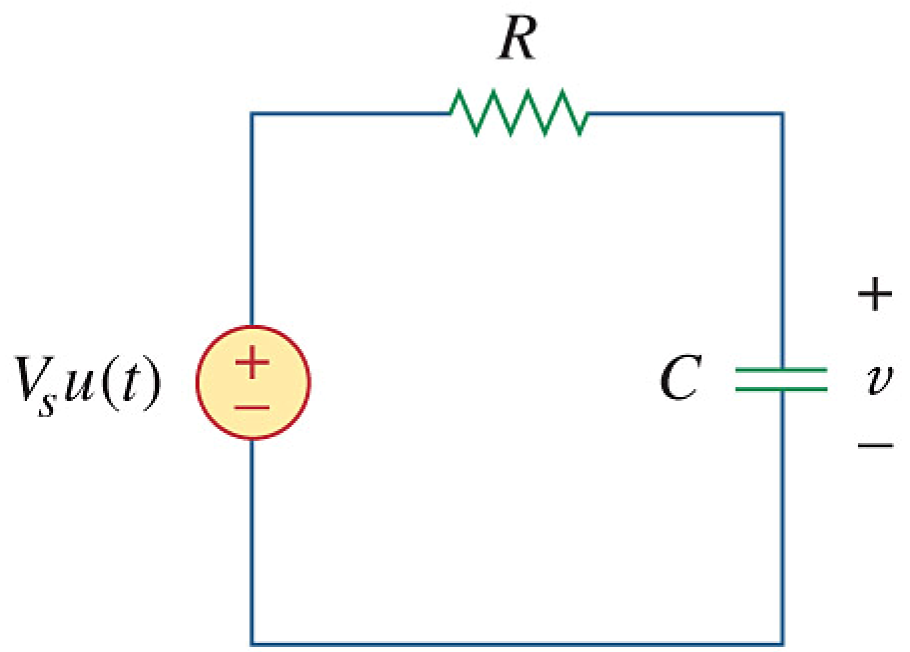

Figure 7.40 An RC circuit with voltage step input.

Since the voltage across the capacitor cannot change instantaneously,

$$v(0^+) = v(0^-) = V_0$$

For $t > 0$,

$$\left(C\frac{dv}{dt} \right) R + v = V_s$$

or

$$\frac{dv}{dt} + \frac{1}{\tau} v = \frac{V_s}{\tau}, \text{ where } \tau = RC$$

$$r + \frac{1}{\tau} = 0 \Rightarrow r = -\frac{1}{\tau}$$

The homogeneous solution or *natural response*

$$v_n(t) = Ae^{-t/\tau}$$

Suppose the particular solution or *forced response*

$$v_f(t) = B$$

$$\frac{dB}{dt} + \frac{1}{\tau} B = \frac{V_s}{\tau} \Rightarrow B = V_s$$

The complete solution or *complete response* or *total response*

$$v(t) = v_n(t) + v_f(t) = Ae^{-t/\tau} + V_s$$

When $t = 0^+$,

$$v(0^+) = A + V_s = V_0 \Rightarrow A = V_0 - V_s$$

Hence,

$$v(t) = (V_0 - V_s)e^{-t/\tau} + V_s$$

This is the response of the *RC* circuit to a sudden application of a constant dc source, assuming the capacitor is initially charged. Assuming that $V_s > V_0 > 0$, a plot of $v(t)$ is shown in Fig. 7.41.

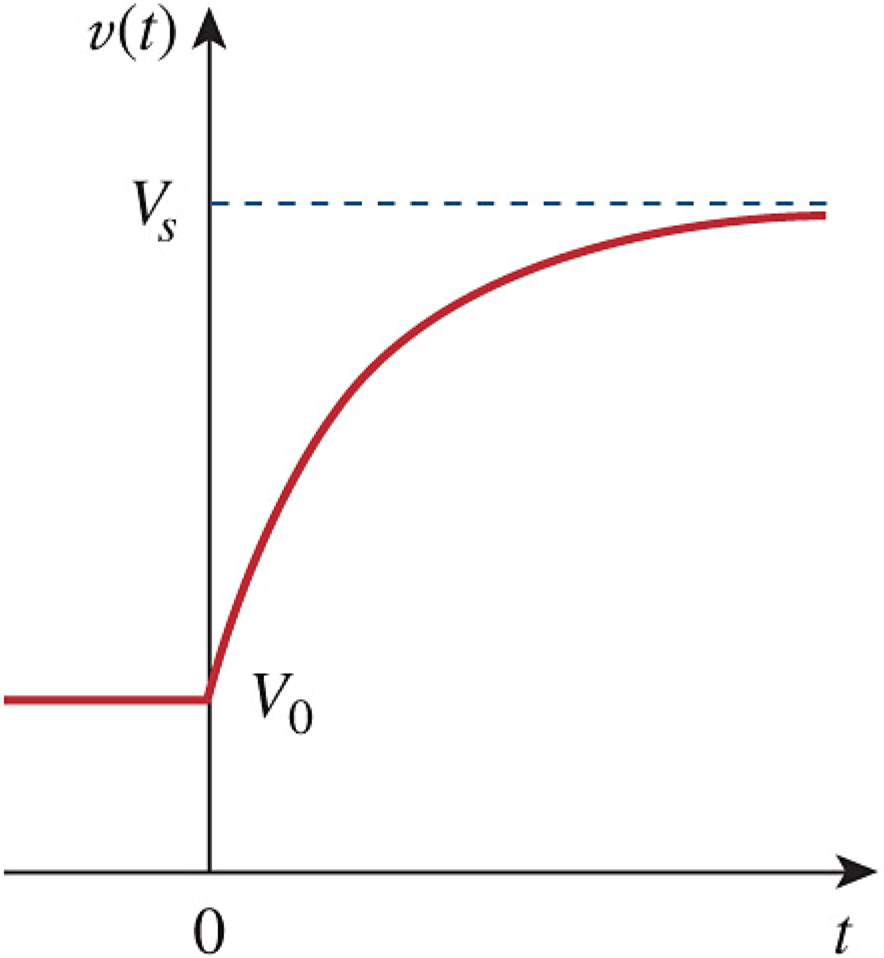

Figure 7.41 Response of an RC circuit.

If the capacitor is uncharged initially, i.e., $V_0 = 0$, then

$$v(t) = V_s(1 - e^{t/\tau}), \quad t > 0$$

or

$$v(t) = V_s(1 - e^{t/\tau})u(t)$$

This is the *zero-state response*. The zero-state response corresponding to a unit-step input is called the *unit-step response*.

The complete response consists of two components:

(a) $v(t) = v_n(t) + v_f(t)$

where $v_n(t) = (V_0 - V_s)e^{-t/\tau}$ is the natural response (homogeneous solution), and $v_f(t) = V_s$ is the forced response (particular solution). Basically, $v_f(t)$ has the same form as the input (external "force").

(b) $v(t) = v_{zp}(t) + v_{zs}(t)$

where $v_{zp}(t) = V_0 e^{-t/\tau}$ is the $zero\text{-}input$ *response* (due to the initial state), and $v_{zs}(t) = V_s(1 - e^{-t/\tau})$ is the zero-state response (due to the input).

(c) $v(t) = v_t(t) + v_{ss}(t)$

where $v_t(t) = (V_0 - V_s)e^{-t/\tau}$ is the temporary response (will die out with time) and $v_{ss}(t)$ $= V_s$ is the response as $t \to \infty$.

Whichever way we look at it, the complete response may be writen as

$$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-t/\tau}$$

where $v(0^+)$ is the *initial value* (i.e., the value at $t = 0^+$) and $v(\infty)$ is the *final value* (or steady-state value).

Thus, to find the response of a first-order *RC* circuit requires three things:

1. The initial capacitor voltage $v(0^+)$.
2. The final capacitor voltage $v(\infty)$.
3. The time constant $\tau = RC$.

This is know as the *three - factor method*.

Example 7.10 The switch in Fig.7.43 has been in position A for a long time. At $t = 0$, the switch moves to B. Determine $v(t)$ for $t > 0$ and calculate its value at $t = 1$ s and 4s.

Solution :

For $t < 0$,

$$v(t) = 24 \times \frac{5}{3+5} = 15 \text{ (V)}$$

$$v(0^-) = 15 \text{ V}$$

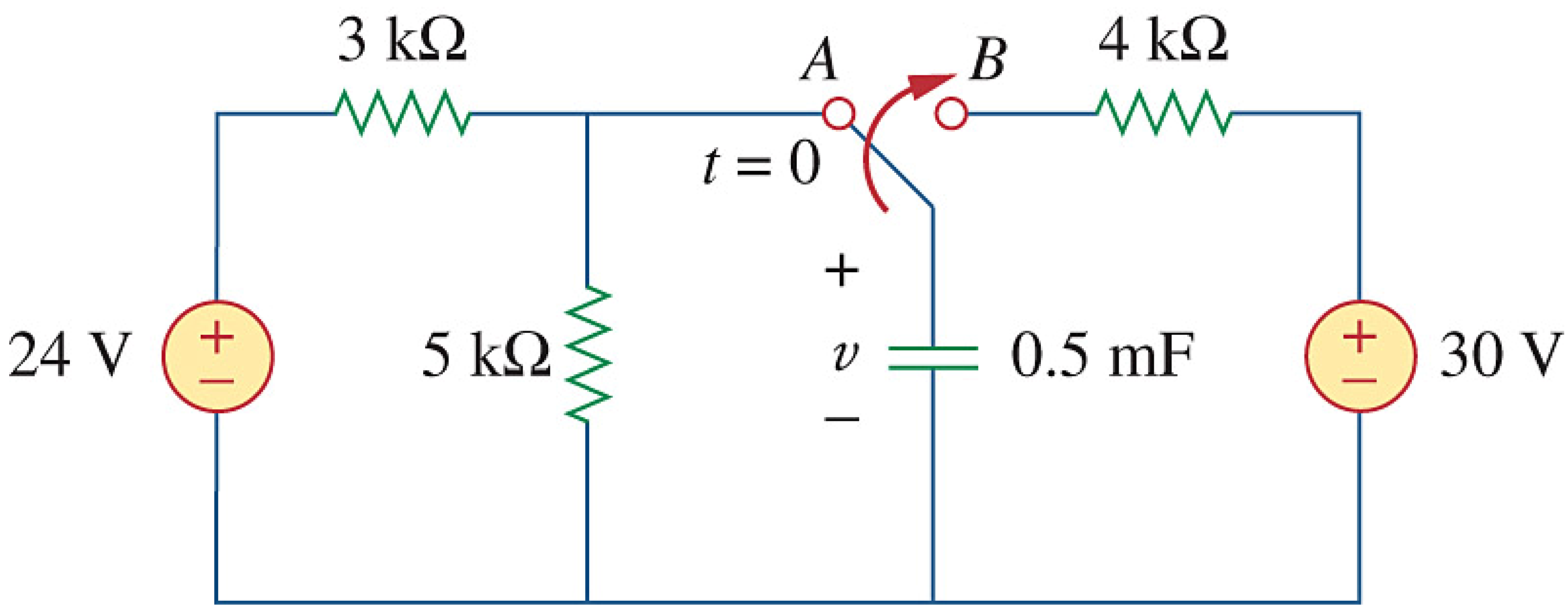

Figure 7.43

For $t > 0$,

$$v(0^+) = v(0^-) = 15 \text{ V}$$

$$v(\infty) = 30 \text{ V}$$

$$\tau = 4 \times 10^3 \times 0.5 \times 10^{-3} = 2 \text{ (s)}$$

$$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-t/\tau}$$

$$= 30 + (15 - 30)e^{-t/2} = 30 - 15e^{-0.5t} \text{ (V)}$$

$$v(1) = 30 - 15e^{-0.5\times1} \approx 20.90 \text{ (V)}$$

$$v(4) = 30 - 15e^{-0.5\times4} \approx 27.97 \text{ (V)}$$

Practice Problem 7.11 The switch in Fig. 7.43 is closed at $t = 0$. Find $i(t)$ and $v(t)$ for all time.

Solution :

For $t < 0$,

$i(t) = 0 \text{ A}$

$v(t) = 20 \text{ V}$

$v(0^{-}) = 20 \text{ V}$

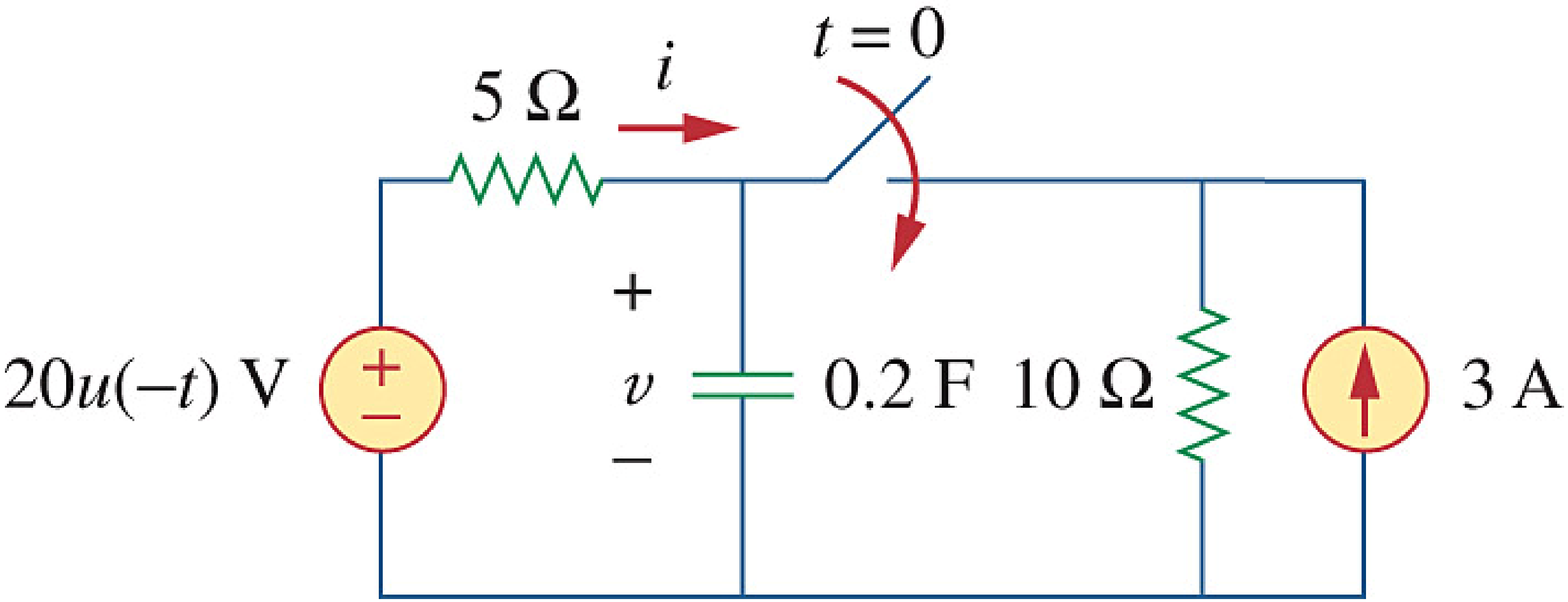

Figure 7.47

For $t > 0$,

$$v(0^+) = v(0^-) = 20 \text{ V}$$

$$v(\infty) = 3 \times (5 \| 10) = 10 \text{ (V)}$$

$$\tau = (5 \| 10) \times 0.2 = \frac{2}{3} \text{ (s)}$$

$$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-t/\tau}$$

$$= 10 + (20 - 10)e^{-t/(2/3)} = 10(1 + e^{-1.5t}) \text{ (V)}$$

$$i(t) = -\frac{v(t)}{5} = -2(1 + e^{-1.5t}) \text{ (A)}$$

7.5 An *RL* Circuit with Step Input

Consider the *RL* circuit in Fig. 7.48(a) which can be replaced by the circuit in Fig. 7.48(b), where V_s is a constant dc voltage source. We select the inductor current i as the circuit response to be determined. We assume an initial current I_0 in the inductor.

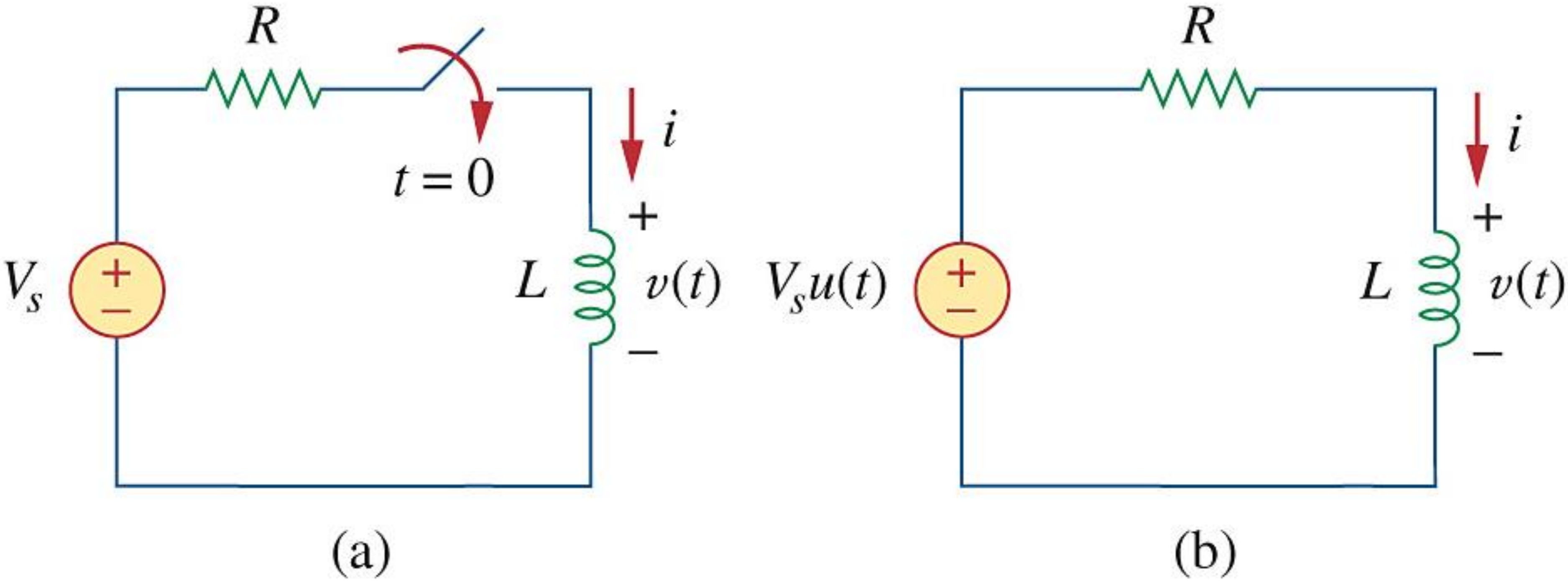

Figure 7.48 An *RL* circuit with a step input voltage.

Since the current through the inductor cannot change instantaneously,

$$i(0^+) = i(0^-) = I_0$$

For $t > 0$,

$$iR + L\frac{di}{dt} = V_s$$

or

$$\frac{di}{dt} + \frac{1}{\tau}i = \frac{V_s / R}{\tau}, \text{ where } \tau = \frac{L}{R}$$

Since this differential equation has the same form as that describing the *RC* circuit,

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R} \right) e^{-t/\tau}$$

It is evident from Fig. 7.48 that,

$$i(\infty) = \frac{V_s}{R}$$

The complete response can be written as

$$i(t) = i(\infty) + \left[i(0^+) - i(\infty) \right] e^{-t/\tau}$$

Thus, to find the response of a first-order *RL* circuit requires three things:

1. The initial inductor current $i(0^+)$.
2. The final inductor current $i(\infty)$.
3. The time constant $\tau = L / R$.

The response of the *RL* circuit to a sudden application of a constant dc source is shown in Fig, 4.49, assuming that

$$I_0 > \frac{V_s}{R} > 0$$

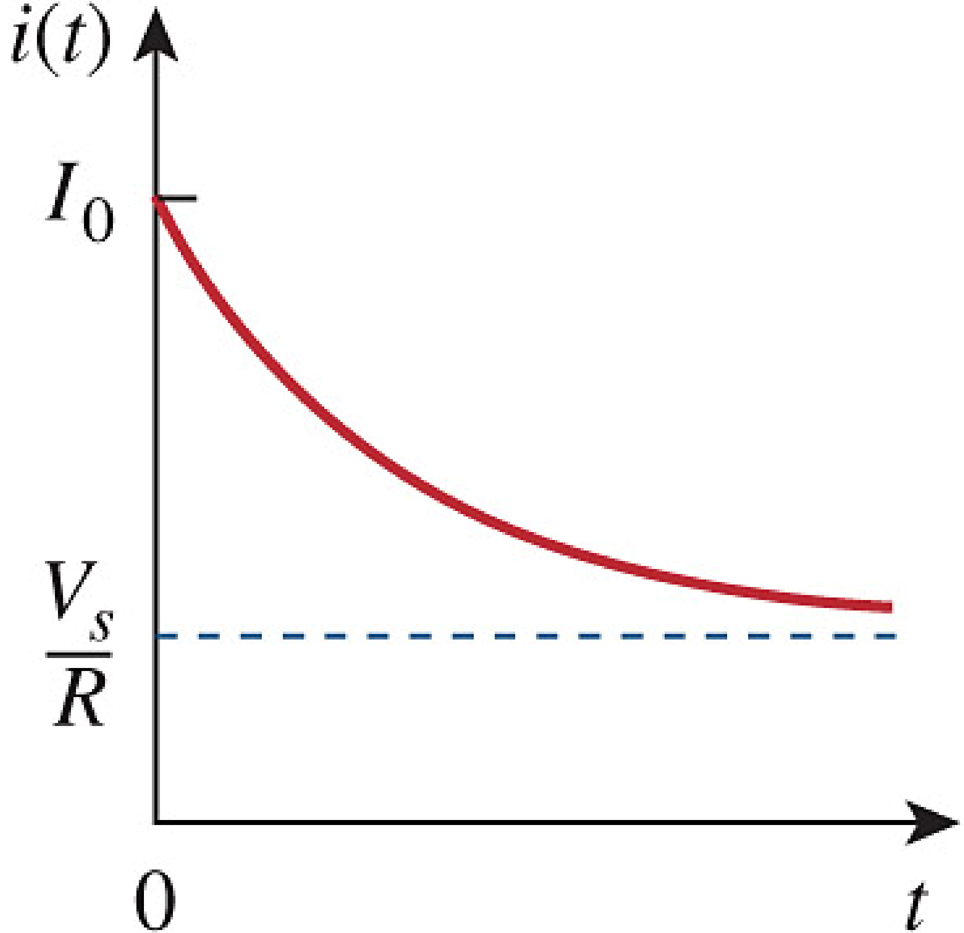

Figure 7.49 Total response of the *RL* circuit.

Example 7.13 At $t = 0$, switch 1 in Fig. 7.53 is closed, and switch 2 is closed 4 s later. Find $i(t)$ for $t > 0$. Calculate i for $t = 2$ s and $t = 5$ s.

Solution :

For $t < 0$,

$$i(t) = 0$$

$$i(0^-) = 0$$

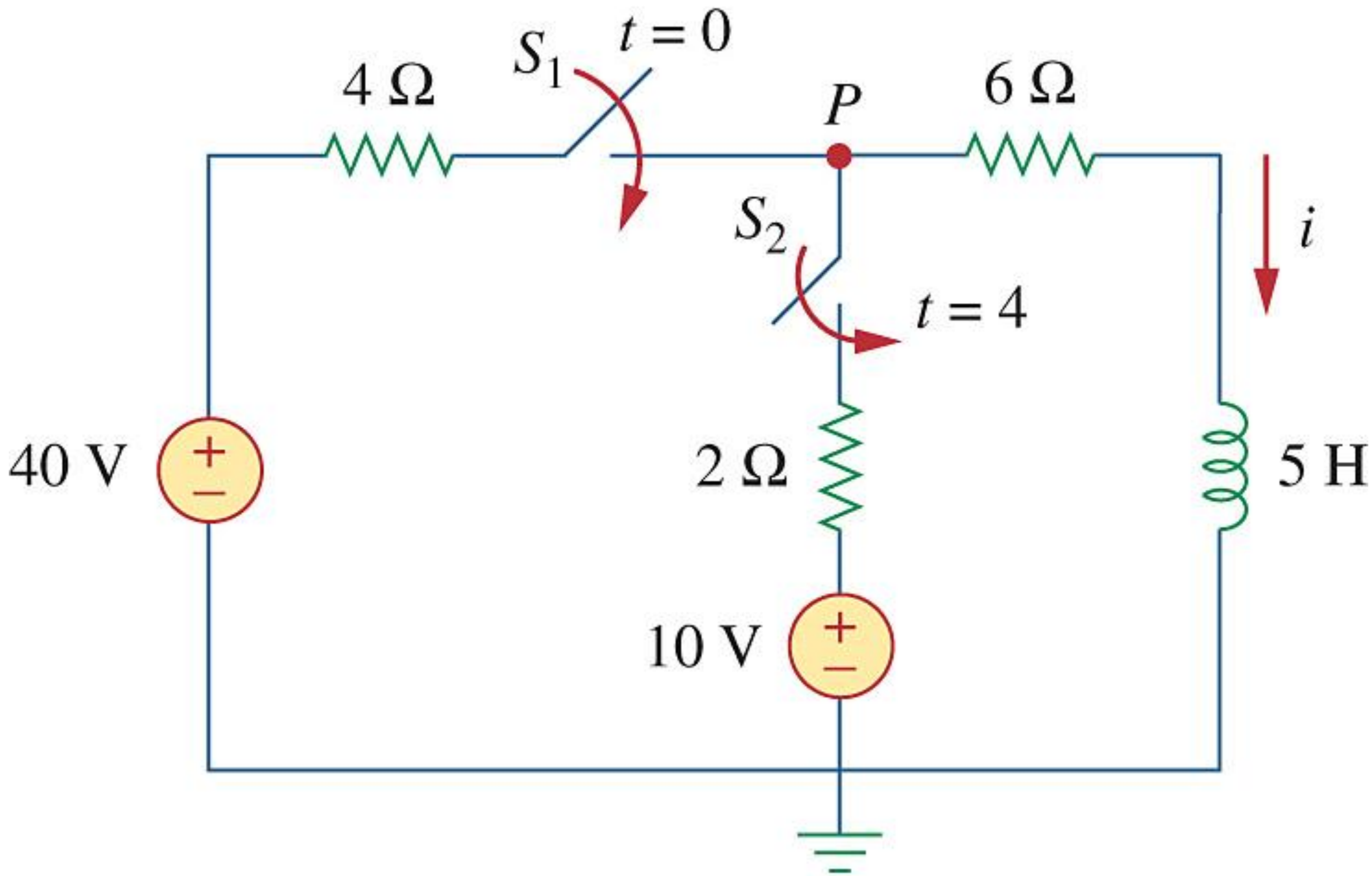

Figure 7.53

For $t > 0$,

$$i(0^+) = i(0^-) = 0$$

For $0 < t \leq 4$,

$$i(\infty) = \frac{40}{4+6} = 4 \text{ (A)}$$

$$\tau = \frac{5}{4+6} = 0.5 \text{ (s)}$$

$$i(t) = 4 + (0-4)e^{-t/0.5} = 4(1-e^{-2t}) \text{ (A)}$$

$$i(2) = 4(1-e^{-2\times 2}) \approx 3.93 \text{ (A)}$$

For $t > 4$,

$$i(4^+) = i(4^-) = 4(1 - e^{-2\times 4}) = 4(1 - e^{-8})\ \text{(A)}$$

$$v_P(\infty) = 40 \times \frac{2 \,||\, 6}{4 + 2 \,||\, 6} + 10 \times \frac{4 \,||\, 6}{2 + 4 \,||\, 6}$$

$$= \frac{180}{11}\ \text{(V)}$$

$$i(\infty) = \frac{v_P(\infty)}{6} = \frac{30}{11}\ \text{(A)}$$

$$\tau = \frac{5}{6+4\,||\,2} = \frac{15}{22}\ \text{(s)}$$

$$i(t) = \frac{30}{11} + \left(4(1-e^{-8}) - \frac{30}{11}\right)e^{-(t-4)/(15/22)}$$

$$\approx 2.7273 + 1.2714e^{-1.4667(t-4)}\ \text{(A)}$$

$$i(5) = 2.7273 + 1.2714e^{-1.4667(5-4)} \approx 3.02\ \text{(A)}$$

7.7 First-Order Op Amp Circuits

Example 7.14 For the op amp circuit in Fig. 7.55(a), find v_o for $t > 0$, given that $v(0) = 3$ V. Let $R_f = 80$ kΩ, $R_1 = 20$ kΩ, and $C = 5\ \mu$F.

Solution:

$$\left(C \frac{dv}{dt} \right) R_1 + v = 0, \quad t > 0$$

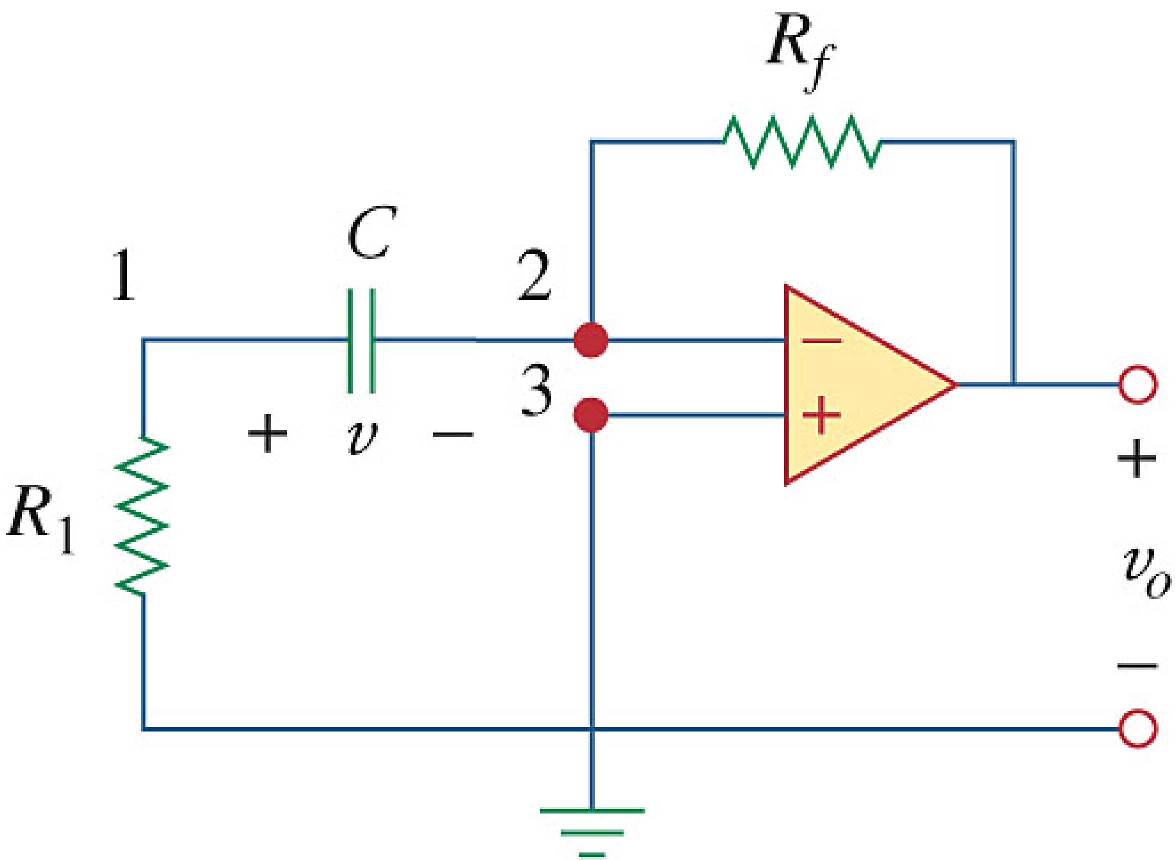

Figure 7.55(a)

or

$$\frac{dv}{dt} + \frac{1}{\tau} v = 0$$

where,

$$\tau = R_1 C = 20 \times 10^3 \times 5 \times 10^{-6} = 0.1\ (\mathrm{s})$$

$$v = v(0) e^{-t/\tau} = 3e^{-t/0.1} = 3e^{-10t}\ (\mathrm{V})$$

$$
\begin{aligned}
v_o &= -\left(C\frac{dv}{dt} \right) R_f \\
&= -\left(5\times 10^{-6}\frac{d}{dt}(3e^{-10t}) \right)\times 80\times 10^{3} \\
&= -0.4\times\left(3\times\left(-10e^{-10t}\right)\right) \\
&= 12e^{-10t}\ \ (\mathrm{V})
\end{aligned}
$$

Example 7.15 Determine $v(t)$ and $v_o(t)$ in the circuit of Fig. 7.57.

Solution :

For $t < 0$,

$$v_1(t) = 0, v_o(t) = 0, v(t) = 0$$

$$v(0^-) = 0$$

For $t > 0$,

$$v_1(t) = 3 \times \frac{20}{10+20} = 2 \text{ (V)}$$

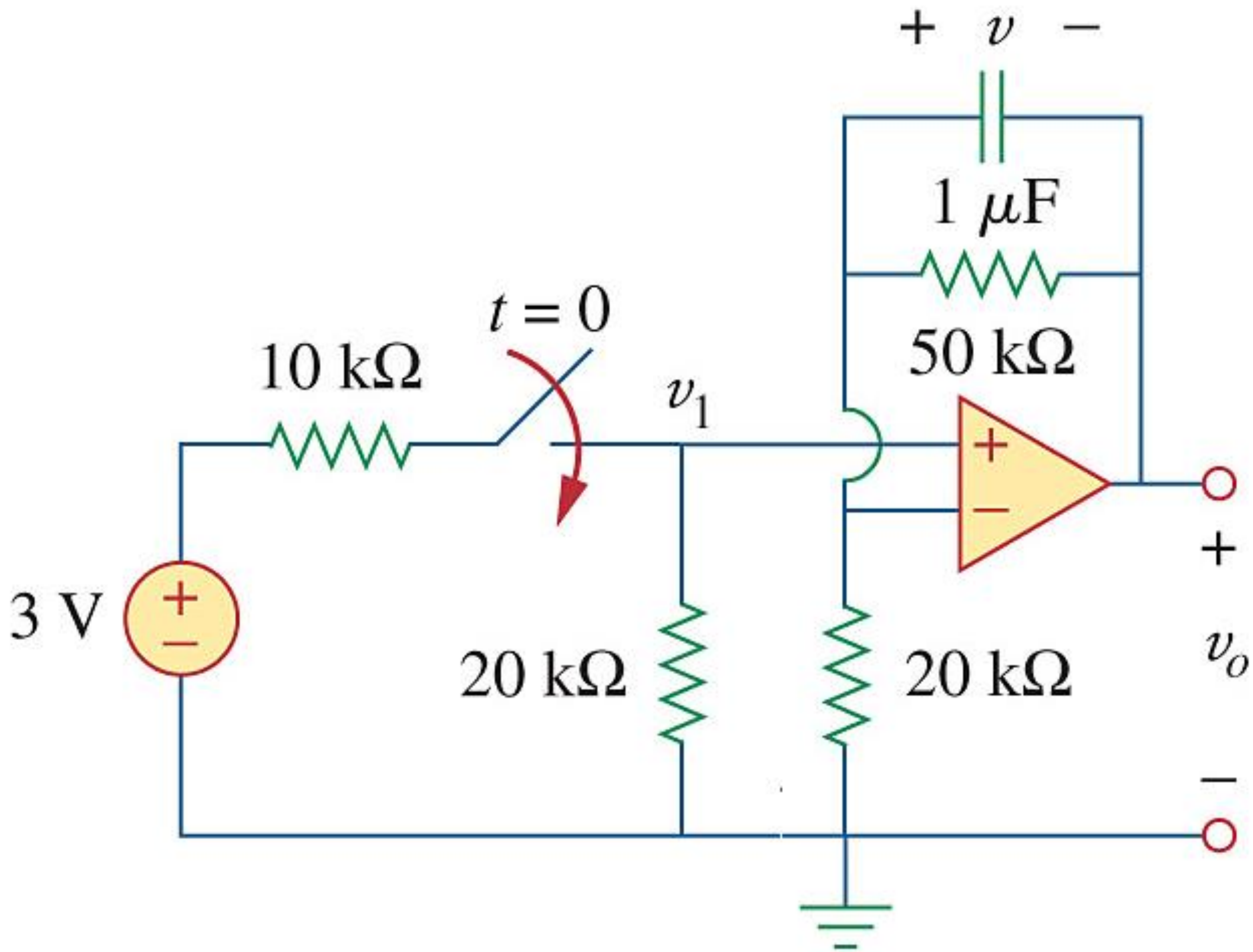

Figure 7.57

$$(1\times10^{-6})\frac{dv}{dt}+\frac{v}{50\times10^{3}}=-\frac{2}{20\times10^{3}}$$

$$\frac{dv}{dt}+20v=-100$$

$$v(t)=Ae^{-t/20}+B=5e^{-t/20}-5\ (\mathrm{V})$$

$$v_o(t)=-v(t)+2=7-5e^{-t/20}\ (\mathrm{V})$$

7.9 Applications

7.9.1. Delay Circuits An RC circuit can be used to provide various delays. Figure 7.73 shows a *glow - lamp relaxation oscillator* circuit.

In operation, C charges through $R_1 + R_2$ until the breakdown voltage (70 V) of the neon lamp is reached.

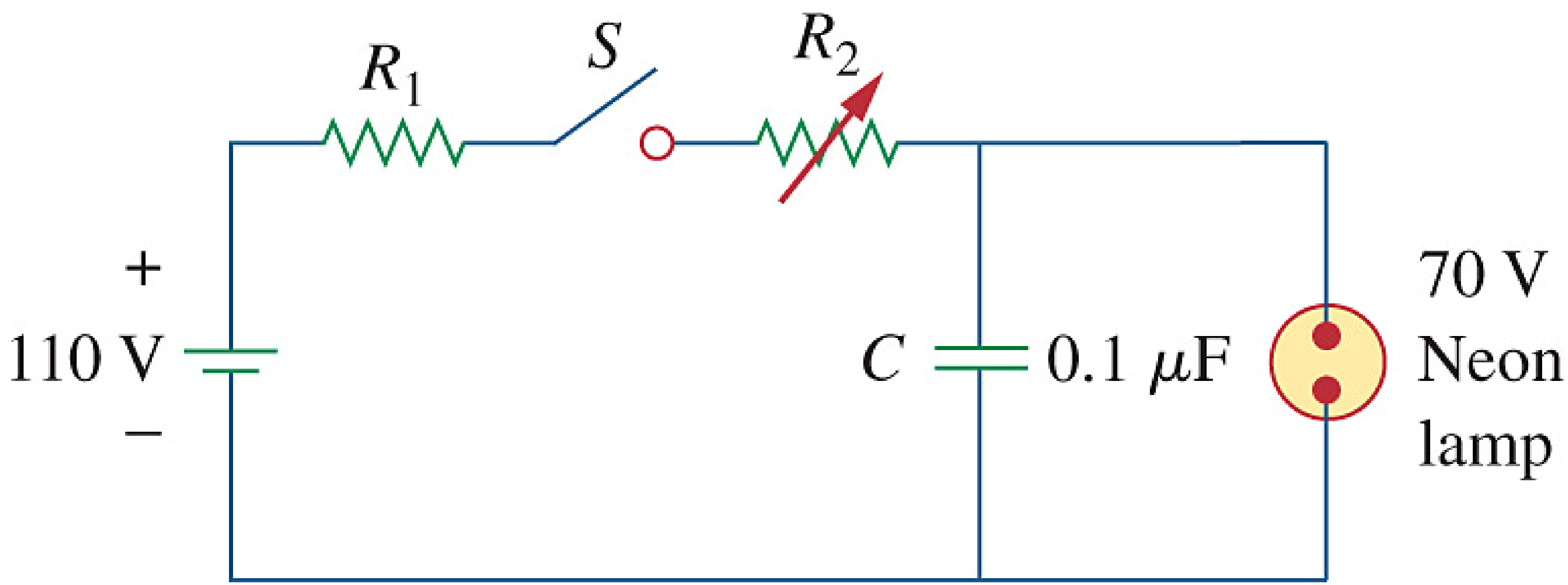

Figure 7.23 An RC dealy circuit. The lamp acts as an open circuit until the voltage across it exceeds the berakdown voltage. When the voltage level is reached, the lamp becomes highly conductive and glows with a characteristic color.

At that point, C discharges through the lamp and produces an orange flash. When the voltage across C drops below the voltage necessary to keep the lamp conducting, say 50 V, the lamp goes dark. Then C begins to charge and the cycle repeats.

To see the neon-lamp flash, $R_1 + R_2$ should be greater than 1 MΩ. Otherwise the flash rate will be faster than the 18 pulses per second discernible by the human eye and the lamp will appear continuously on.

Example 7.19 Consider the circuit in Fig. 7.73, and assume that $R_1 = 1.5\ \text{M}\Omega$, $0 < R_2 < 2.5\ \text{M}\Omega$. (a) Calculate the extreme limits of the time constant of the circuit. (b) How long does it take for the lamp to glow for the first time after the switch is closed? Let R_2 assume its largest value.

Solution :

(a) $\tau = (R_1 + R_2)C$

$$\tau_{\min} = (1.5\times10^6 + 0)\times 0.1\times10^{-6}$$

$$= 0.15\ \text{(s)}$$

$$\tau_{\max} = (1.5\times10^6 + 2.5\times10^6)\times 0.1\times10^{-6}$$

$$= 0.4\ \text{(s)}$$

(b) $v_C(0^+) = 0,\ v_C(\infty) = 110\ \text{V},\ \tau = 0.4\ \text{s}$

$$v_C(t) = v_C(\infty) + \left[v_C(0^+) - v_C(\infty)\right]e^{-t/\tau}$$

$$t = \tau \ln\left(\frac{v_C(0^+) - v_C(\infty)}{v_C(t) - v_C(\infty)}\right)$$

$$= 0.4 \ln\left(\frac{0 - 110}{70 - 110}\right) \approx 0.4046\ (\text{s})$$

The lamp glows after the switch is closed for 0.4046 secomds.

7.9.2 Photoflash Unit Figure 7.75 shows a simplified circuit. It consists of a high-voltage dc supply, a current-limiting large resistor R_1, and a capacitor C in parallel with the flashlamp of low resistance R_2.

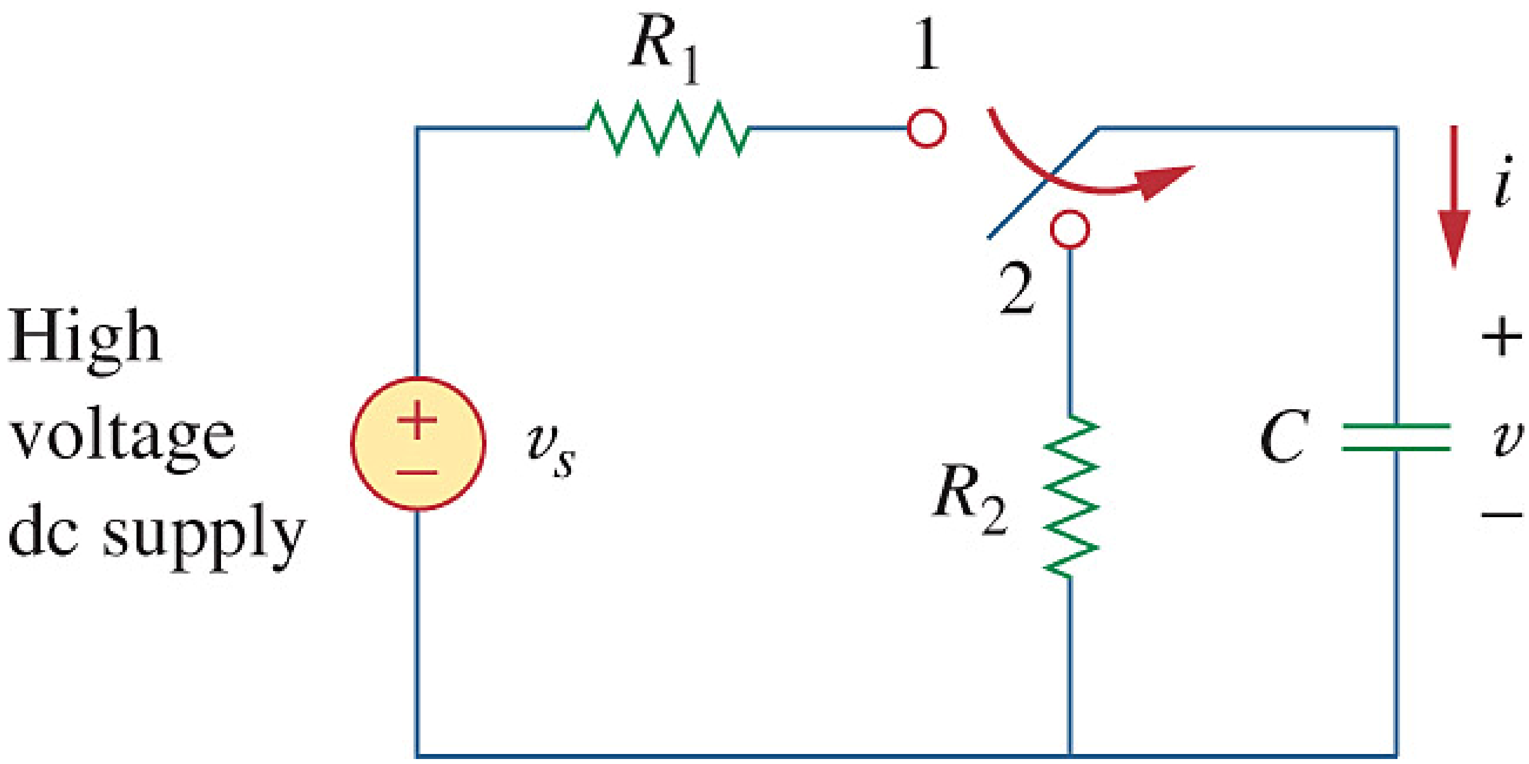

Figure 7.75 Circuit for a flash unit providing slow charge in position 1 and fast discharge in position 2.

When the switch is in position 1, C charges. As shown in Fig. 7.76, the capacitor voltage rises gradually from zero to V_s, while its current decreases gradually from $I_1 = V_s / R_1$ to zero. The charging time is approximately

$$t_{\text{charge}} = 5R_1C$$

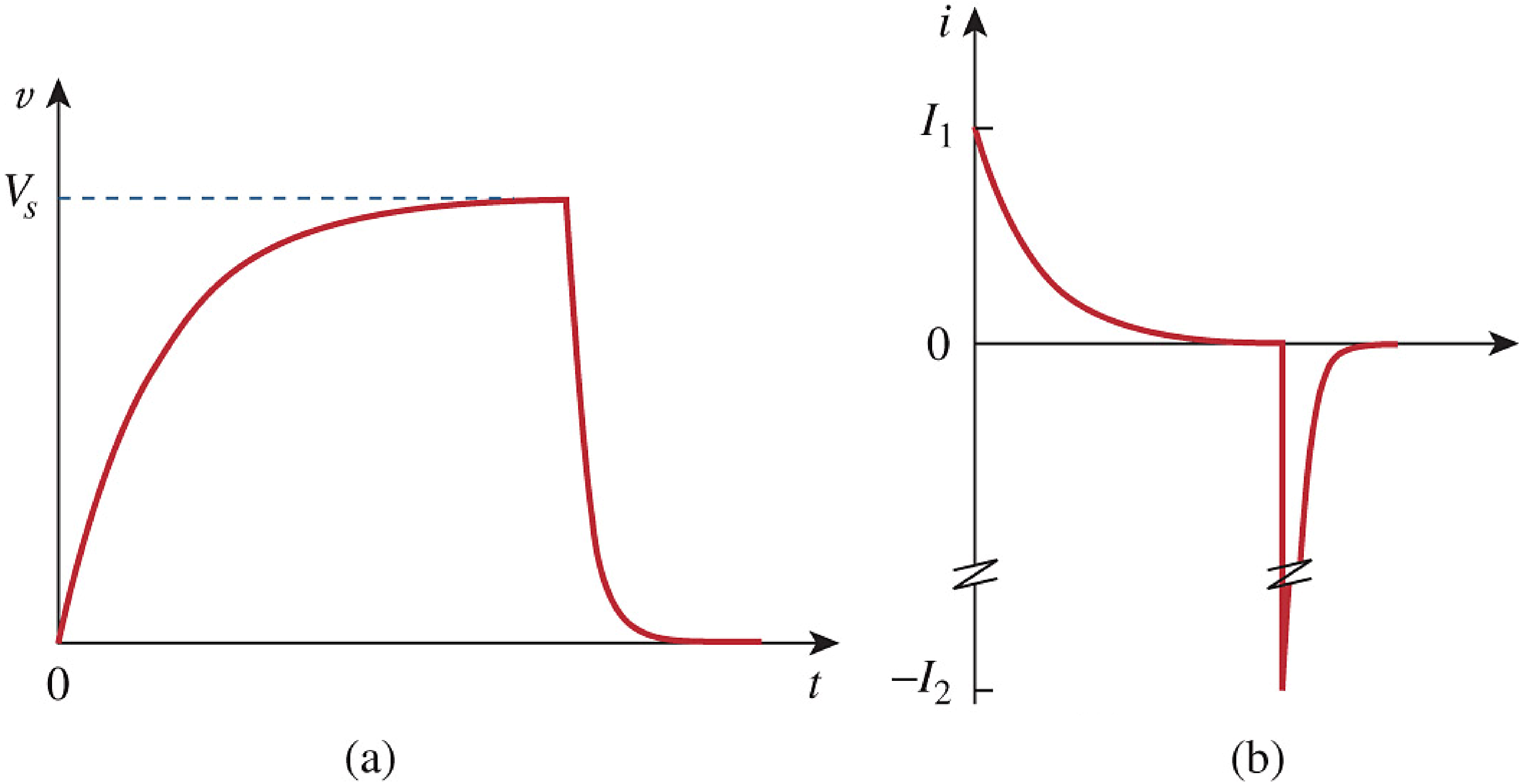

Figure 7.76 (a) Capacitor voltage showing slow charge and fast discharge, (b) capacitor current showing low charging current I_1 and high discharge current I_2.

With the switch in position 2, C is discharged. The low resistance R_2 of the photolamp permits a high discharge current with peak $I_2 = V_s / R_2$ in a short duration. Discharging takes place in approximately $t_{\text{discharge}} = 5R_2C$

Example 7.20 An electronic flashgun has a current-limiting 6-kΩ resistor and 2000-μF electrolytic capacitor charged to 240 V. If the lamp resistance is 12 Ω, find (a) the peak charging current, (b) the time required for the capacitor to fully charge, (c) the peak discharging current, (d) the total energy stored in the capacitor, and (e) the average power dissipated by the lamp.

Solution :

(a) $I_1 = V_s / R_1 = 240/6 = 40 \text{ (mA)}$

(b) $t_{\text{charge}} = 5R_1C = 5\times 6\times 10^3 \times 2000\times 10^{-6} = 60 \text{ (s)}$

(c) $I_2 = V_s / R_2 = 240/12 = 20 \text{ (A)}$

(d) $w = \dfrac{1}{2}CV_s^2 = \dfrac{1}{2}\times 2000\times 10^{-6}\times 240^2 = 57.6 \text{ (J)}$

(e) $t_{\text{discharge}} = 5R_2C = 5\times 12\times 2000\times 10^{-6} = 0.12 \text{ (s)}$

$$p = \frac{w}{t_{\text{discharge}}} = \frac{57.6}{0.12} = 480 \text{ (W)}$$

7.9.3 Relay Circuits A magnetically controlled switch is called a *relay*. A relay is essentially an electromagnetic device used to open or close a switch that controls another circuit. Figure 7.77(a) shows a typical relay circuit.

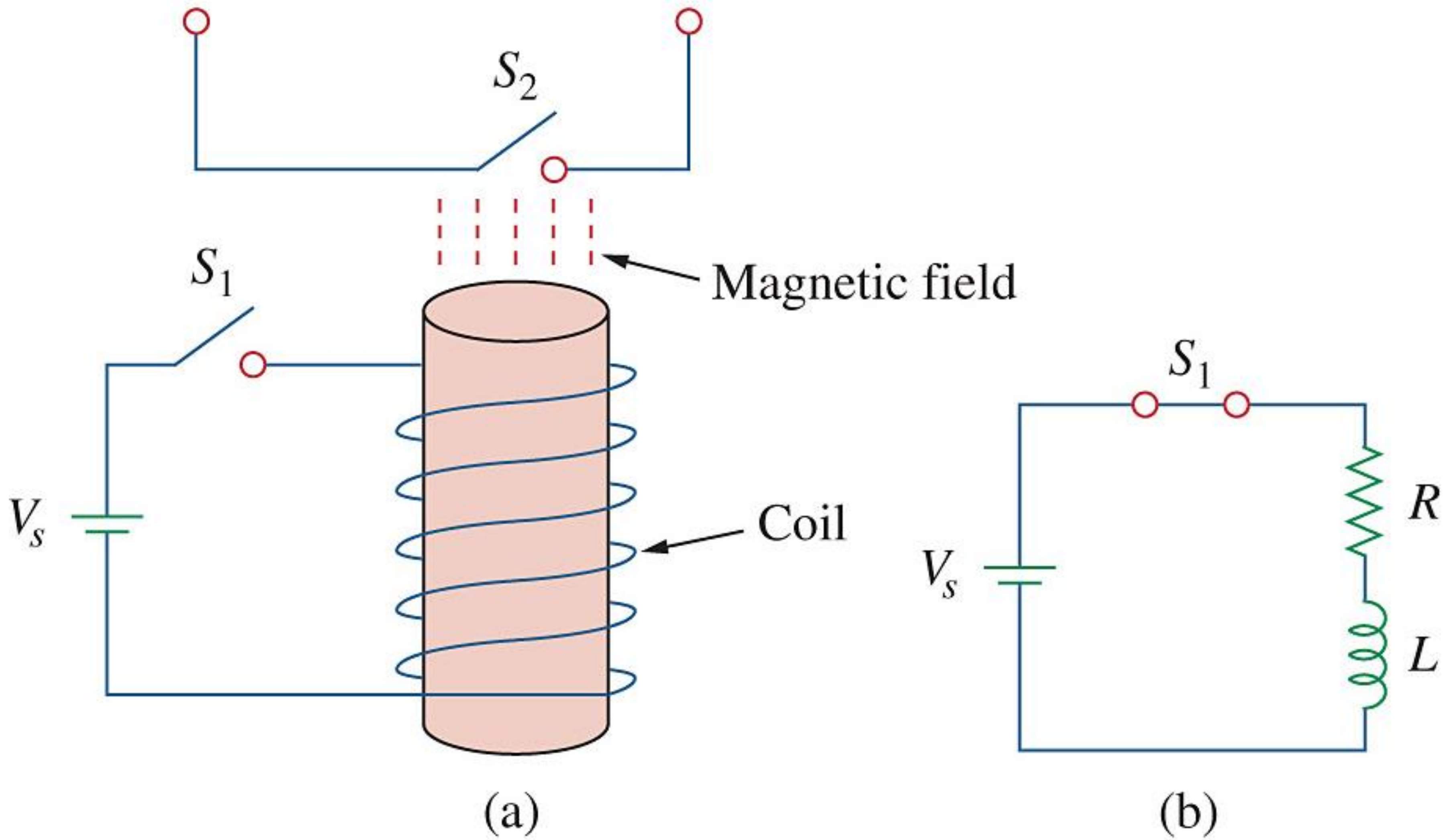

Figure 7.77 A relay circuit. The relay consists of a coil of wire wrapped around a soft iron core, an iron yoke, a movable iron armature, and one set of contacts.

When switch S_1 is closed, the coil is energized. The coil current gradually increases and produces a magnetic field. Eventually the magnetic field is sufficiently strong to pull the armature (which is mechanically linked to the movable contanct) and close switch S_2.

Example 7.21 The coil of a certain relay is operated by a 12-V battery. If the coil has a resistance of 150 Ω and an inductance of 30 mH and the current needed to pull in is 50 mA, calculate the the relay delay time.

Solution :

$$i(0^{+}) = 0$$

$$i(\infty) = 12/150 = 0.08 \text{ (A)}$$

$$i(t) = i(\infty) + [i(0^+) - i(\infty)]e^{-t/\tau}$$

$$t = \tau \ln\left(\frac{i(0^+) - i(\infty)}{i(t) - i(\infty)}\right)$$

$$= 2 \times 10^{-4} \ln\left(\frac{0 - 0.08}{50 \times 10^{-3} - 0.08}\right)$$

$$\approx 1.9617 \times 10^{-4}\ \text{(s)}$$

$$\approx 0.1962\ \text{(ms)}$$

7.9.4 Automobile Ignition Circuit The gasoline engine of an automobile requires that the fuel-air mixture in each cylinder be ignited at proper times. Figure 7.78 shows a circuit. When the ignition switch is closed, the inductor (spark coil) is energized. After $t_{\text{charge}} = 5L/R$ seconds, $i = V_s / R$, where $V_s = 12$ V, and $v = 0$ V.

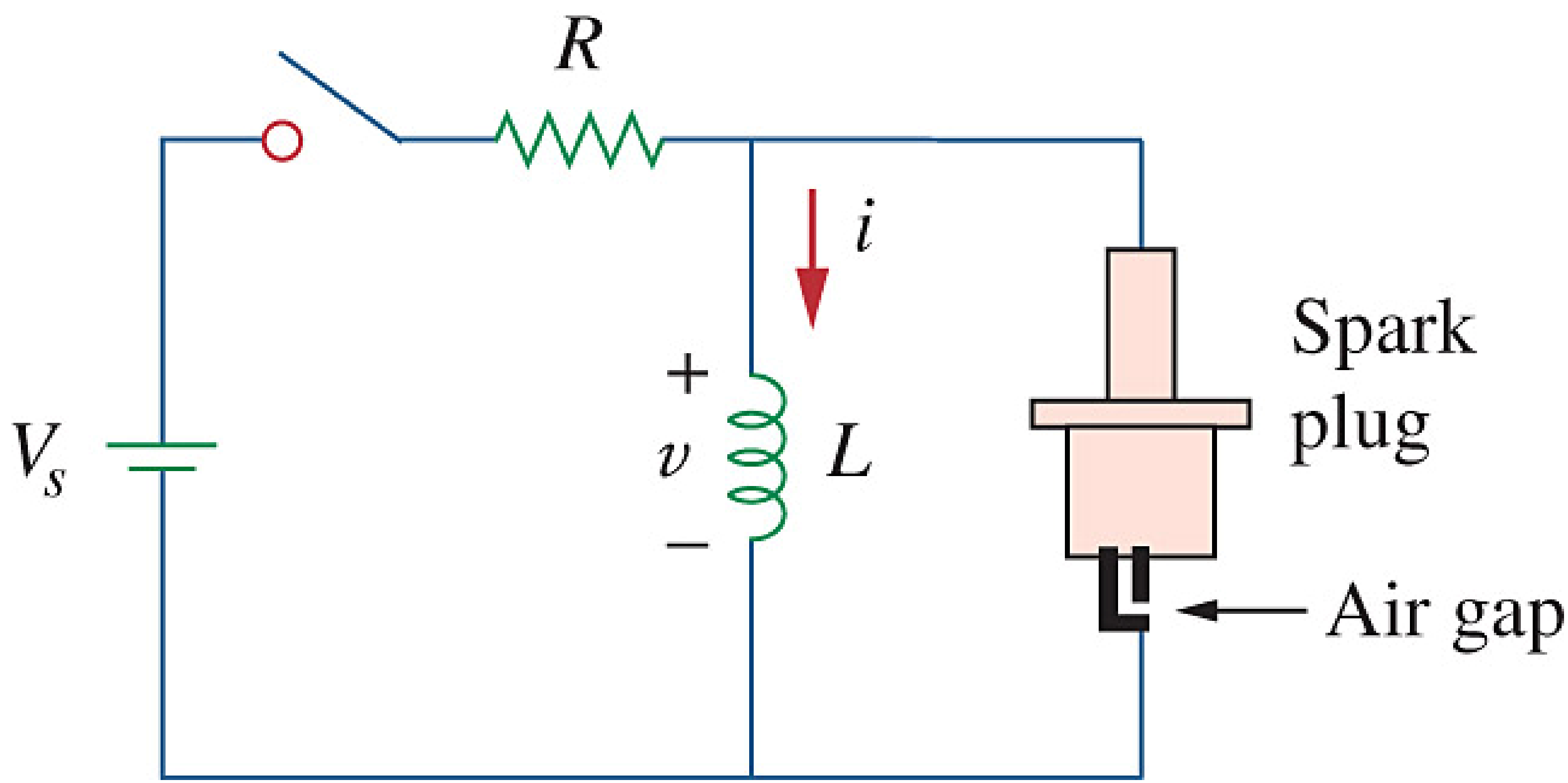

Figure 7.78 Circuit for an automobile ignition system.

When the switch suddenly opens, a large voltage is developed across the inductor causing a spark in the air gap of the spark plug. The spark continues until the energy stored in the inductor is dissipated in the spark discharge.

Example 7.22 A solenoid with resistance of 4 Ω and inductance 6 mH is used in an automobile ignition circuit. If the battery supples 12 V, determine: the final current through the solenoid when the switch is closed, the energy stored in the coil, and the voltage across the air gap, assuming that the switch takes 1 μs to open.

Solution :

$$i = V_s / R = 12 / 4 = 3 \text{ (A)}$$

$$w = \frac{1}{2} L i^2 = \frac{1}{2} \times 6 \times 10^{-3} \times 3^2 = 0.027 \text{ (J)}$$

$$v = L \frac{di}{dt} \approx L \frac{\Delta i}{\Delta t} = 6 \times 10^{-3} \times \frac{3}{1 \times 10^{-6}}$$

$$= 1.8 \times 10^4 \text{ (V)} = 18 \text{ kV}$$