Ve215 Electric Circuits

Ma Dianguang

Chapter 10

Sinusoidal Steady-State Analysis

10.1 Introduction

In this chapter, we want to see how nodal analysis, mesh analysis, Thevenin's theorem, Norton's theorem, superposition, and source transformations are applied in analyzing ac circuits.

Steps to Analyze AC Circuits

1. Transform the circuit to the phasor domain.
2. Find the circuit output using nodal analysis, mesh analysis, superposition, etc.
3. Transform the resulting phasor to the time domain.

Step 1 is not necessary if the problem is specified in the frequency domain. In step 2, the analysis is performed in the same manner as dc circuit analysis except that complex numbers are involved.

Example 10.1 Find i_x in the circuit of Fig. 10.1 using nodal analysis.

Solution :

$$20\cos 4t \Rightarrow 20\angle 0^\circ \text{ V}, \ \omega = 4 \text{ rad/s}$$

$$1 \text{ H} \Rightarrow j\omega L = j4 \ \Omega$$

$$0.5 \text{ H} \Rightarrow j\omega L = j2 \ \Omega$$

$$0.1 \text{ F} \Rightarrow \frac{1}{j\omega C} = -j2.5 \ \Omega$$

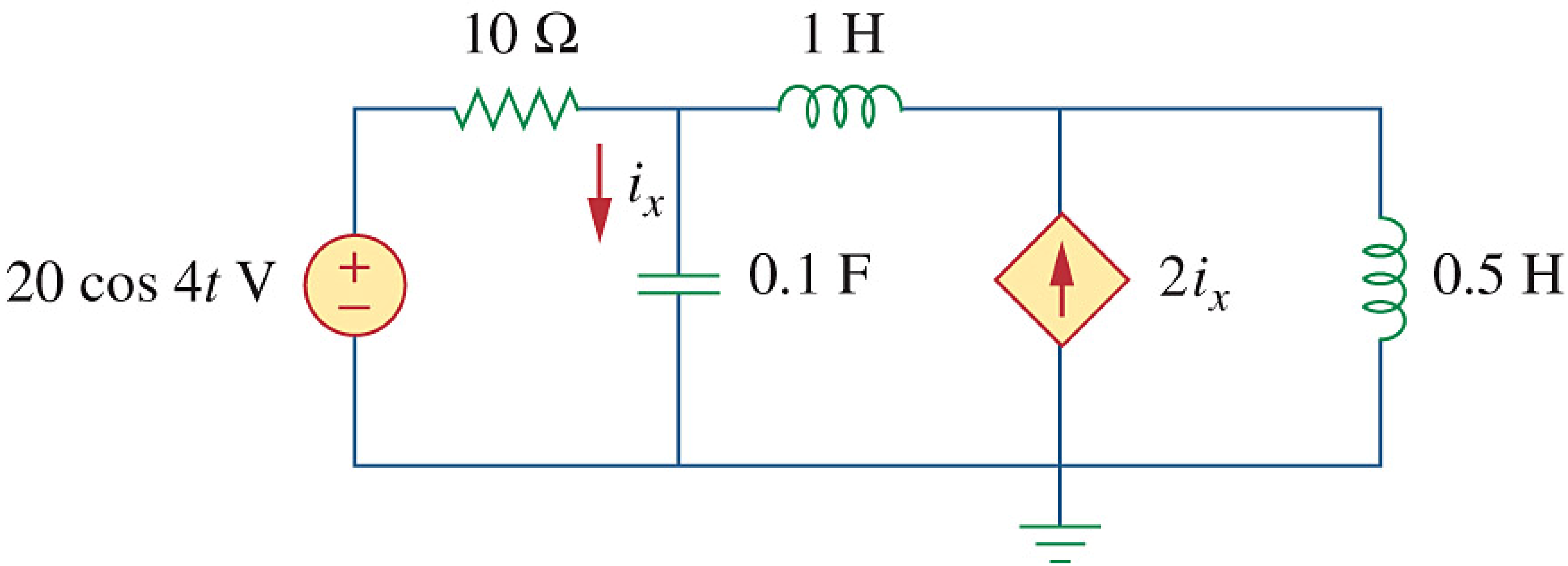

Figure 10.1

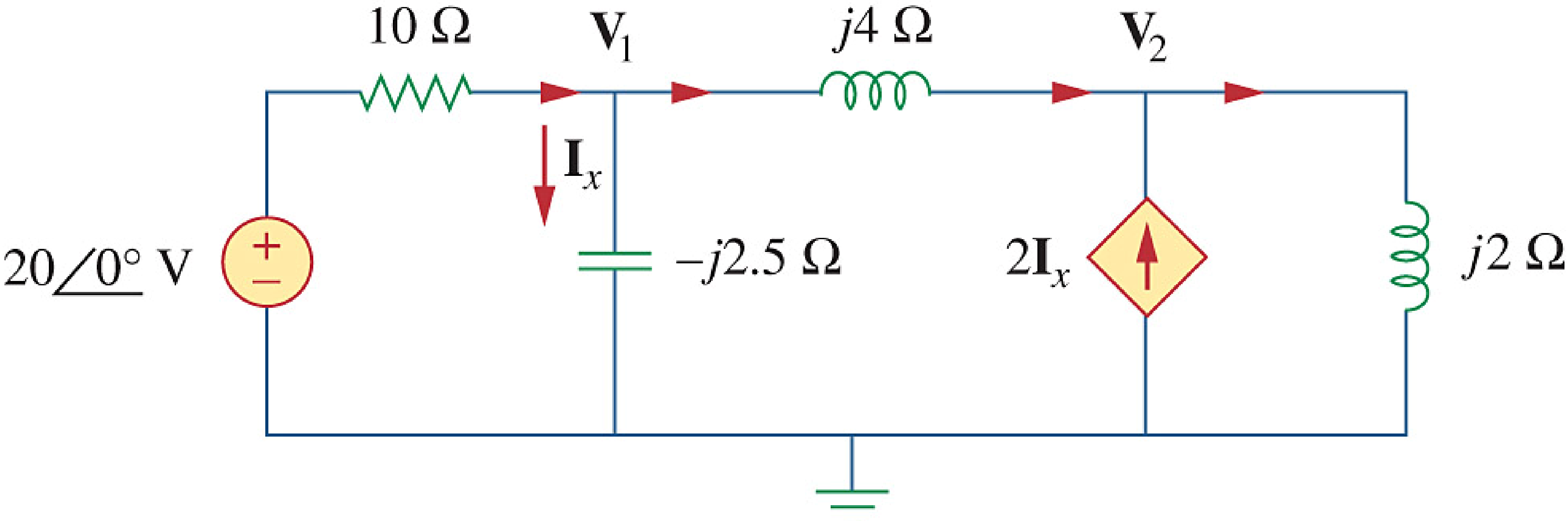

Figure 10.2 Frequency domain equivalent of the circuit in Fig. 10.1.

$$\begin{bmatrix} \frac{1}{10}+\frac{1}{j4}+\frac{1}{-j2.5} & -\frac{1}{j4} \\ -\frac{1}{j4} & \frac{1}{j4}+\frac{1}{j2} \end{bmatrix} \begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \end{bmatrix} = \begin{bmatrix} \frac{20\angle 0^\circ}{10} \\ 2\dot{I}_x \end{bmatrix}$$

$$\dot{I}_x = \frac{\dot{V}_1}{-j2.5}$$

$$\begin{bmatrix} \frac{1}{10}+\frac{1}{j4}+\frac{1}{-j2.5} & -\frac{1}{j4} \\ -\frac{1}{j4}-\frac{2}{-j2.5} & \frac{1}{j4}+\frac{1}{j2} \end{bmatrix} \begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \end{bmatrix} = \begin{bmatrix} \frac{20\angle 0^\circ}{10} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2+j3 & j5 \\ 11 & 15 \end{bmatrix} \begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \end{bmatrix} = \begin{bmatrix} 40 \\ 0 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 2+j3 & j5 \\ 11 & 15 \end{vmatrix} = 30 - j10$$

$$\Delta_1 = \begin{vmatrix} 40 & j5 \\ 0 & 15 \end{vmatrix} = 600$$

$$\dot{V}_1 = \frac{\Delta_1}{\Delta} = \frac{600}{30 - j10} = \frac{60}{3 - j}$$

$$\approx \frac{60}{3.1623\angle -18.43^\circ}$$

$$\approx 18.9735\angle 18.43^\circ \text{ (V)}$$

$$\dot{I}_x = \frac{\dot{V}_1}{-j2.5} = \frac{18.9735\angle 18.43^\circ}{2.5\angle -90^\circ}$$

$$\approx 7.59\angle 108.43^\circ \text{ (A)}$$

$$i_x = 7.59\cos(4t + 108.43^\circ) \text{ (A)}$$

10.3 Mesh Analysis

Example 10.3 Determine current $\dot{I}_o$ in the circuit of Fig. 10.7 using mesh analysis.

Solution:

$\dot{I}_3 = 5\angle 0° = 5\ (\mathrm{A})$

$20\angle 90° = j20$

$$\begin{bmatrix} 8+j10-j2 & j2 & -j10 \\ j2 & 4-j2-j2 & j2 \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ -j20 \end{bmatrix}$$

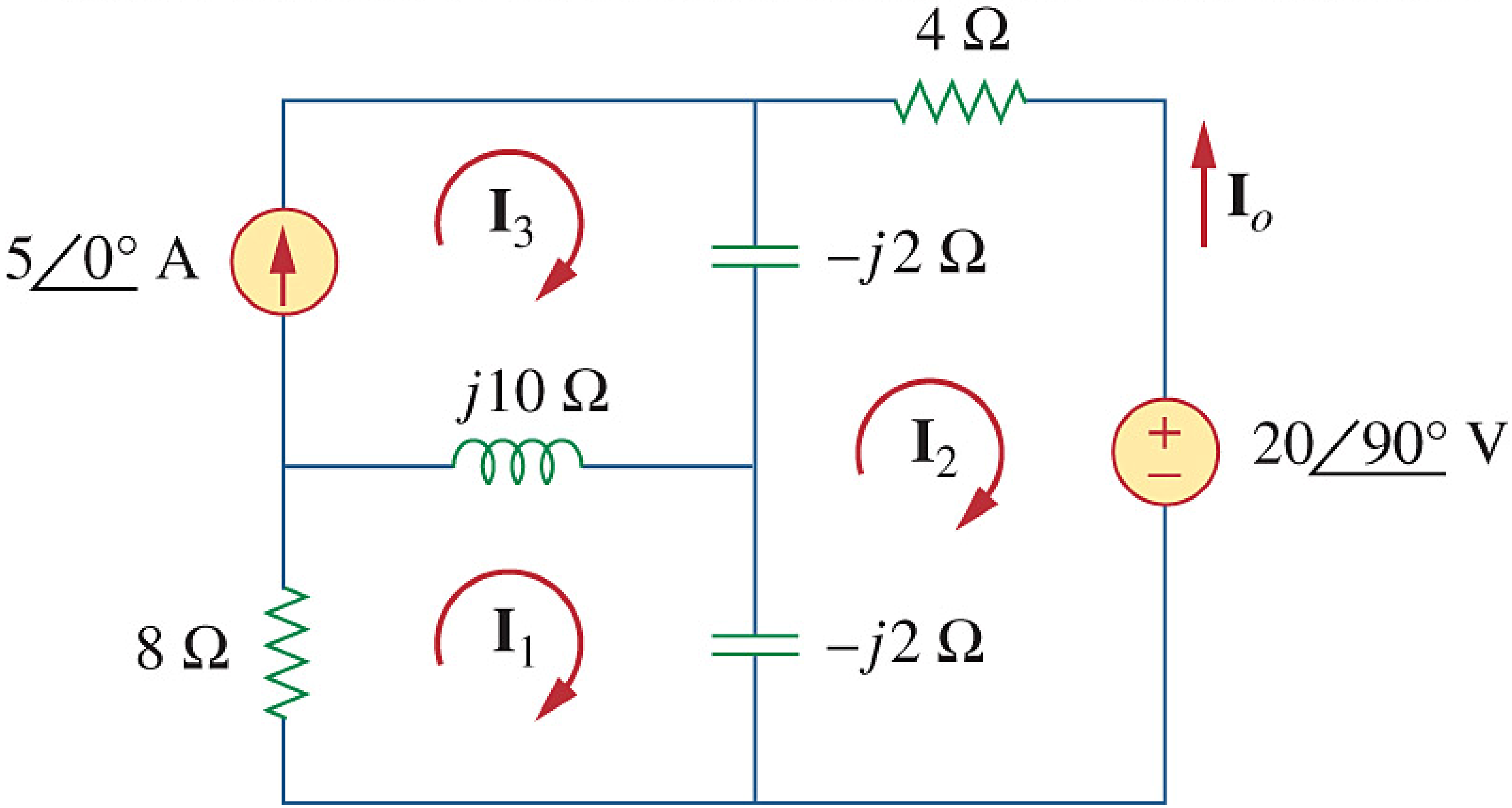
4 Ω
$\mathbf{I}_o$
$\mathbf{I}_3$
$5\angle 0°$ A
$-j2\ \Omega$
$j10\ \Omega$
$\mathbf{I}_2$
$20\angle 90°$ V
8 Ω
$\mathbf{I}_1$
$-j2\ \Omega$

Figure 10.7

$$\begin{bmatrix} 8+j8 & j2 & -j10 \\ j2 & 4-j4 & j2 \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ -j20 \end{bmatrix}$$

$$\begin{bmatrix} 8+j8 & j2 \\ j2 & 4-j4 \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \end{bmatrix} = \begin{bmatrix} j50 \\ -j30 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 8+j8 & j2 \\ j2 & 4-j4 \end{vmatrix} = 68$$

$$\Delta_2 = \begin{vmatrix} 8+j8 & j50 \\ j2 & -j30 \end{vmatrix} = 340 - j240$$

$$\approx 416.1730\angle -35.22^\circ$$

$$\dot{I}_2 = \frac{\Delta_2}{\Delta} = \frac{416.1730\angle -35.22^\circ}{68}$$

$$\approx 6.12\angle -35.22^\circ \text{ (A)}$$

$$\dot{I}_o = -\dot{I}_2 = 6.12\angle 144.78^\circ \text{ (A)}$$

10.4 Superposition Theorem

The superposition theorem becomes important if the circuit has sources operating at *different* frequencies. In this case, since the impedances depend on frequency, we must have a different frequency domain circuit for each frequency. The total response must be obtained by adding the individual responses in the *time* domain.

Example 10.6 Find v_o of the circuit of Fig. 10.13 using the superposition theorem.

Solution :

$$v_o = v_1 + v_2 + v_3$$

where v_1 is due to the 5-V dc source, v_2 is due to the $10\cos 2t$ V voltage source, and v_3 is due to the $2\sin 5t$ current source.

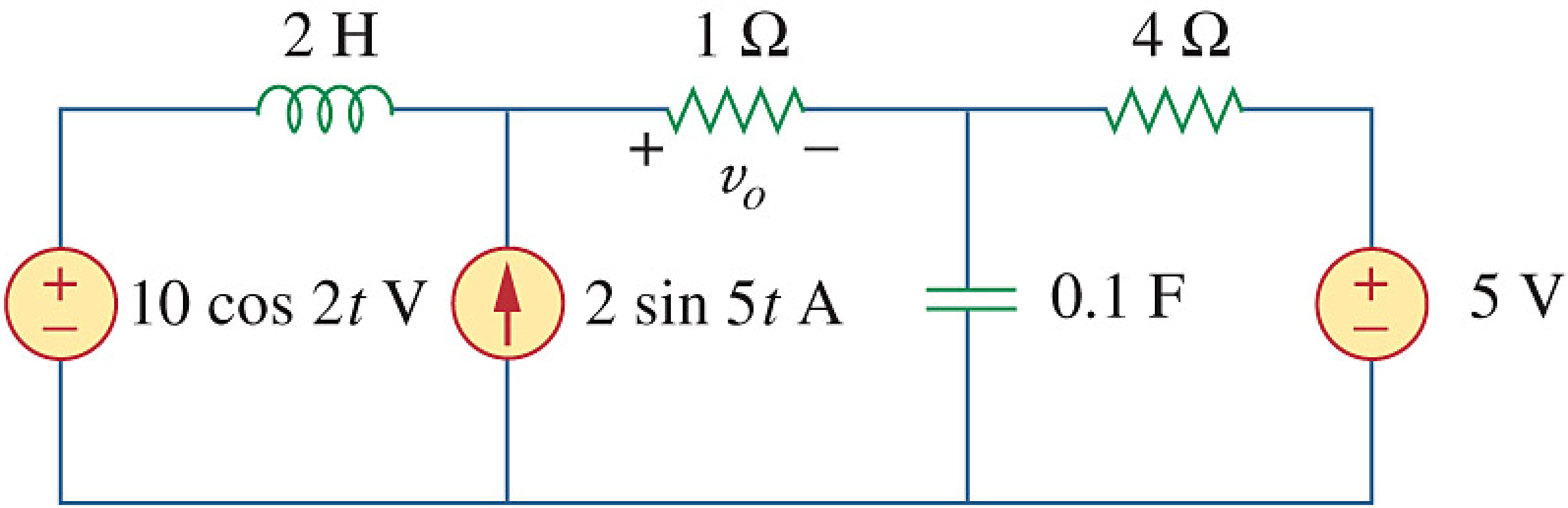

Figure 10.13

$$(1)\ v_1 = -5 \times \frac{1}{1+4} = -1\ (\text{V})$$

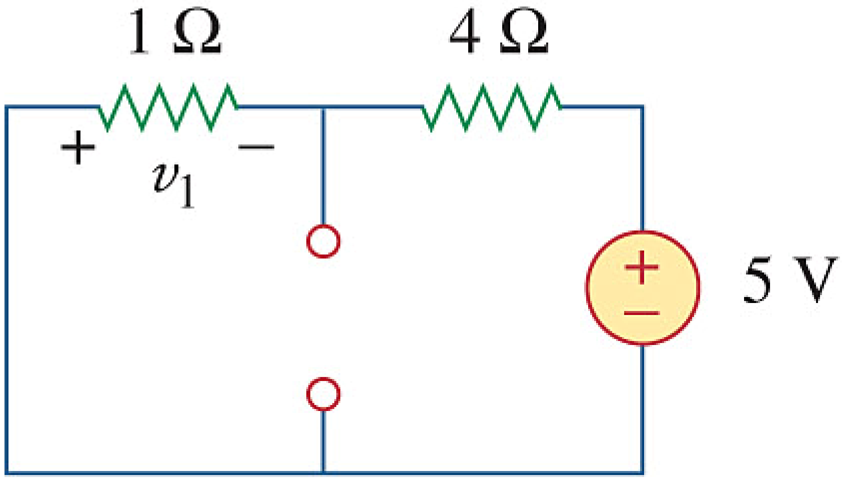

Figure 10.14(a)

(2) v_2

$$10\cos 2t\ \text{V} \Rightarrow 10\angle 0^\circ\ \text{V},\ \omega = 2\ \text{rad/s}$$

$$2\ \text{H} \Rightarrow j\omega L = j4\ (\Omega)$$

$$0.1\ \text{F} \Rightarrow 1/(j\omega C) = -j5\ (\Omega)$$

$$\dot{V}_2 = 10\angle 0^\circ \frac{1}{j4 + 1 + (-j5) \,||\, 4}$$

$$(-j5) \,||\, 4 = \frac{(-j5)\times 4}{(-j5)+4} \approx \frac{20\angle -90^\circ}{6.4031\angle -51.34^\circ}$$

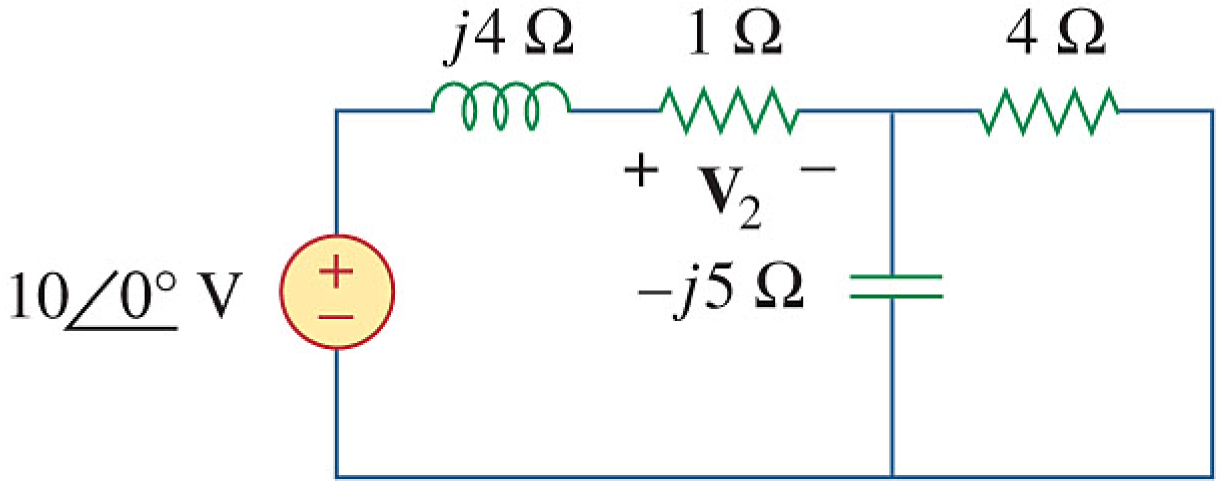

Figure 10.14(b)

$$\approx 3.1235\angle -38.66° \approx 2.4390 - j1.9512$$

$$\dot{V}_2 = 10\angle 0° \frac{1}{j4 + 1 + (2.4390 - j1.9512)}$$

$$= \frac{10\angle 0°}{3.4390 + j2.0488} = \frac{10\angle 0°}{4.0030\angle 30.78°}$$

$$\approx 2.50\angle -30.78° \text{ (V)}$$

$$v_2 = 2.50\cos(2t - 30.78°) \text{ (V)}$$

(3) v_3

$$2\sin 5t \text{ A} \Rightarrow 2\angle -90^\circ \text{ A},\ \omega = 5 \text{ rad/s}$$

$$2 \text{ H} \Rightarrow j\omega L = j10\ (\Omega)$$

$$0.1 \text{ F} \Rightarrow 1/(j\omega C) = -j2\ (\Omega)$$

$$\dot{V}_3 = (2\angle -90^\circ)\times(j10)\times\frac{1}{j10+1+(-j2)\,\|\,4}$$

$$(-j2)\,\|\,4 = \frac{(-j2)\times 4}{(-j2)+4} = 0.8 - j1.6$$

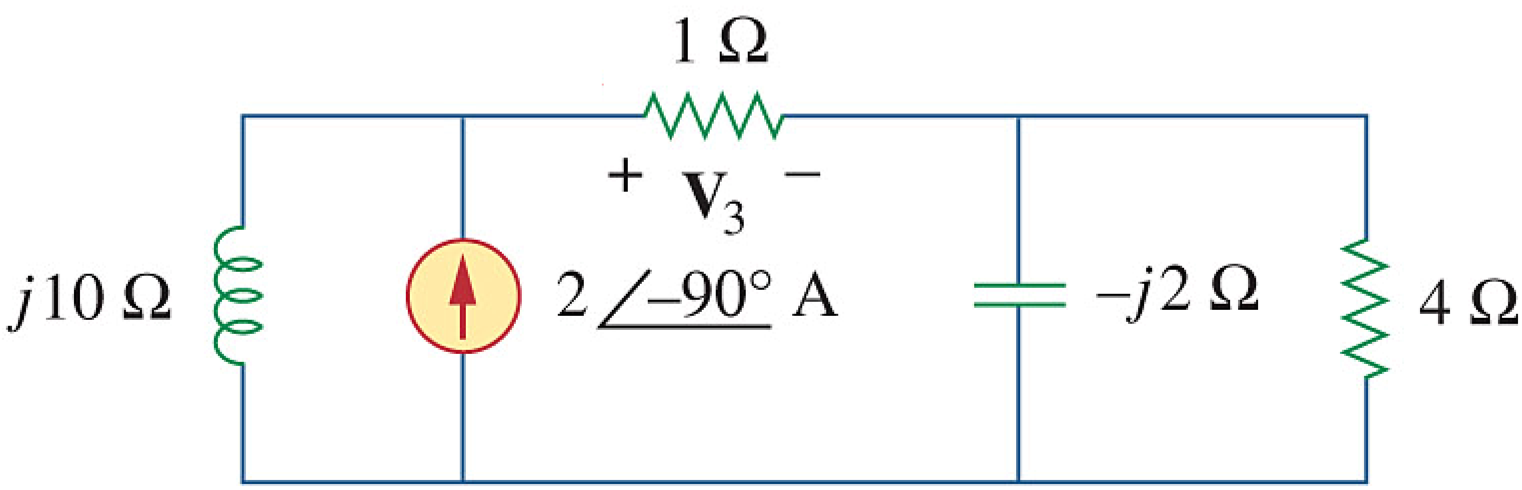

Figure 10.14(c)

$$\dot{V}_3 = (2\angle -90°) \times (j10) \times \frac{1}{j10 + 1 + (0.8 - j1.6)}$$

$$= \frac{20}{1.8 + j8.4} \approx \frac{20}{8.5907\angle 77.91°}$$

$$\approx 2.33\angle -77.91° \text{ (V)}$$

$$v_3 = 2.33\cos(5t - 77.91°) \text{ V}$$

$$v = -1 + 2.50\cos(2t - 30.78°)$$

$$+2.33\cos(5t - 77.91°) \text{ (V)}$$

10.5 Source Transformation

As Fig. 10.16 shows, source transformation in the frequency domain involves transforming a voltage source in series with an impedance to a current source in parallel with an impedance, or vice versa.

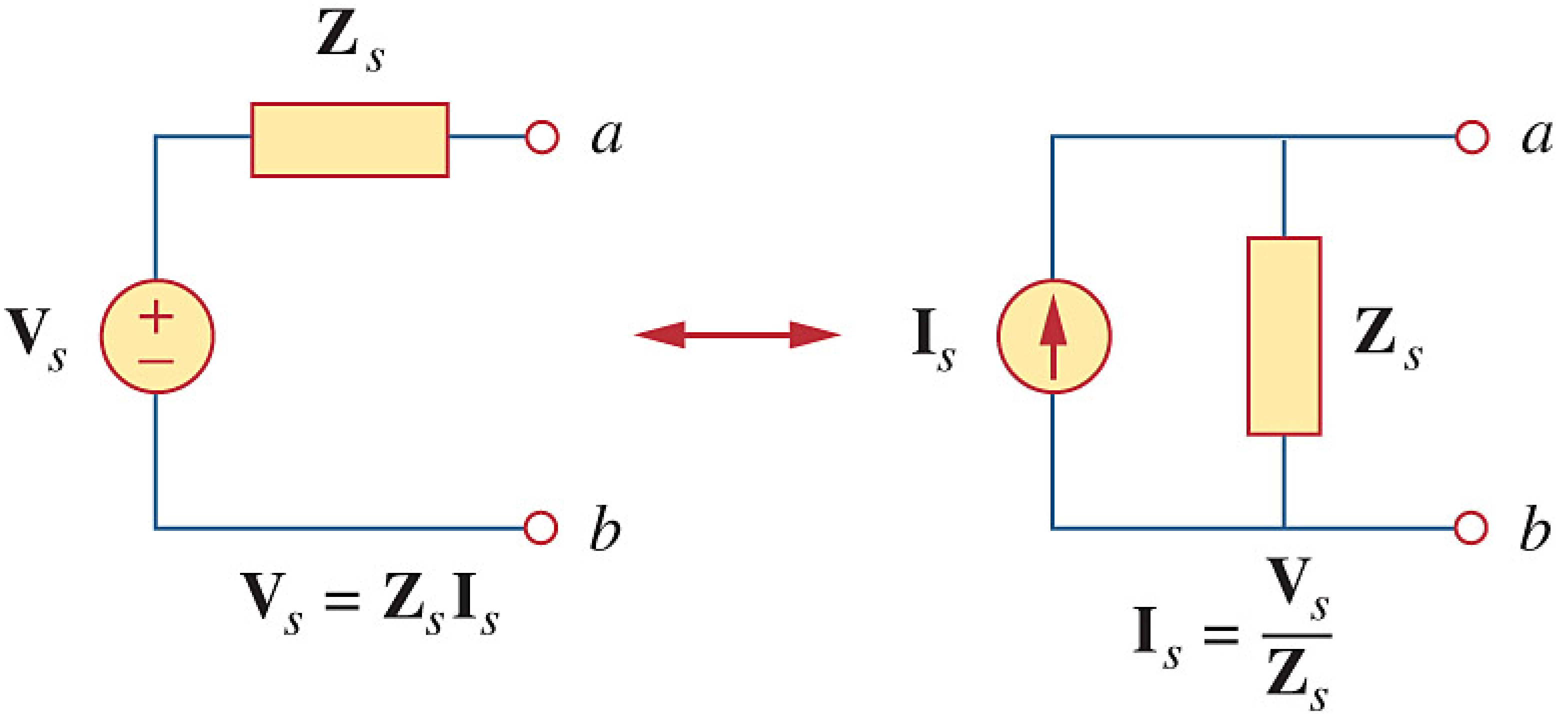

Figure 10.16 Source transformation.

Practice Problem 10.7 Find $\dot{I}_o$ in the circuit of Fig.10.19 using the concept of source transformation.

Solution :

$$\begin{cases} \dot{I}_1 = 8\angle 90^\circ \text{ A} \\ Z_1 = 4 - j3\ \Omega \end{cases} \Rightarrow \begin{cases} \dot{V}_1 = 24 + j32\ (\text{V}) \\ Z_1 = 4 - j3\ \Omega \end{cases}$$

$$\dot{V}_1 = \dot{I}_1 Z_1 = 8\angle 90^\circ \times (4 - j3) = j8 \times (4 - j3)$$

$$= 24 + j32\ (\text{V})$$

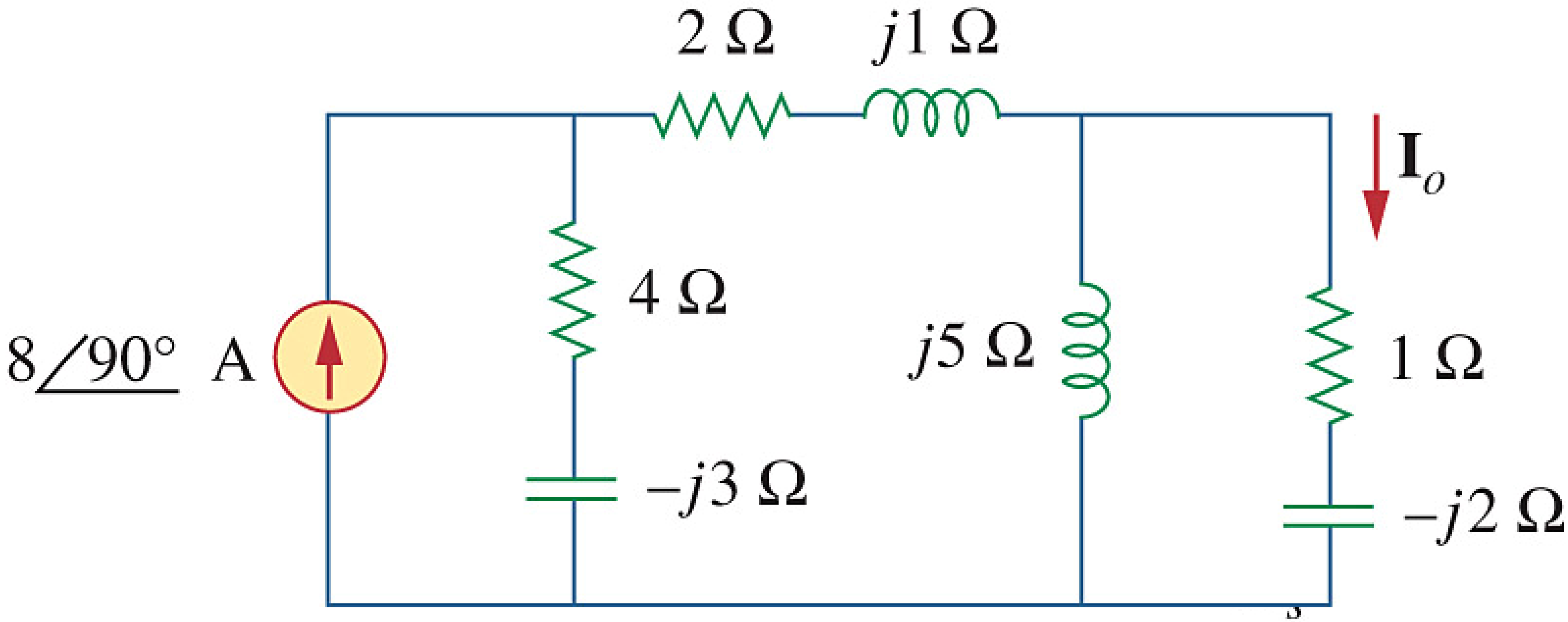

Figure 10.19

$$Z_2 = Z_1 + (2 + j1) = (4 - j3) + (2 + j1)$$
$$= 6 - j2\ (\Omega)$$
$$\begin{cases} \dot{V}_1 = 24 + j32\ \text{V} \\ Z_2 = 6 - j2\ \Omega \end{cases} \Rightarrow \begin{cases} \dot{I}_2 = 6.3245\angle 71.56^\circ\ \text{A} \\ Z_2 = 6 - j2\ \Omega \end{cases}$$
$$\dot{I}_2 = \frac{\dot{V}_1}{Z_2} = \frac{24 + j32}{6 - j2} = \frac{40\angle 53.13^\circ}{6.3246\angle -18.43^\circ}$$
$$\approx 6.3245\angle 71.56^\circ\ (\text{A})$$

$$\dot{I}_o = \dot{I}_2 \frac{1/(1-j2)}{1/(1-j2)+1/(j5)+1/(6-j2)}$$

$$\frac{1/(1-j2)}{1/(1-j2)+1/(j5)+1/(6-j2)}$$

$$= \frac{1}{1+(1-j2)/(j5)+(1-j2)/(6-j2)}$$

$$= \frac{1}{1+(-2-j)/5+(1-j)/4}$$

$$
\begin{aligned}
&= \frac{20}{17 - j9} \\
&\approx 1.0398\angle 27.90^\circ \\
\dot{I}_o &= \dot{I}_2 \times 1.0398\angle 27.90^\circ \\
&= 6.3245\angle 71.56^\circ \times 1.0398\angle 27.90^\circ \\
&\approx 6.58\angle 99.46^\circ \text{ (A)}
\end{aligned}
$$

10.6 Thevenin and Norton Equivalent Circuits

The frequency domain version of a Thevenin equivalent circuit is depicted in Fig. 10.20. The Norton equivalent circuit is illustrated in Fig. 10.21.

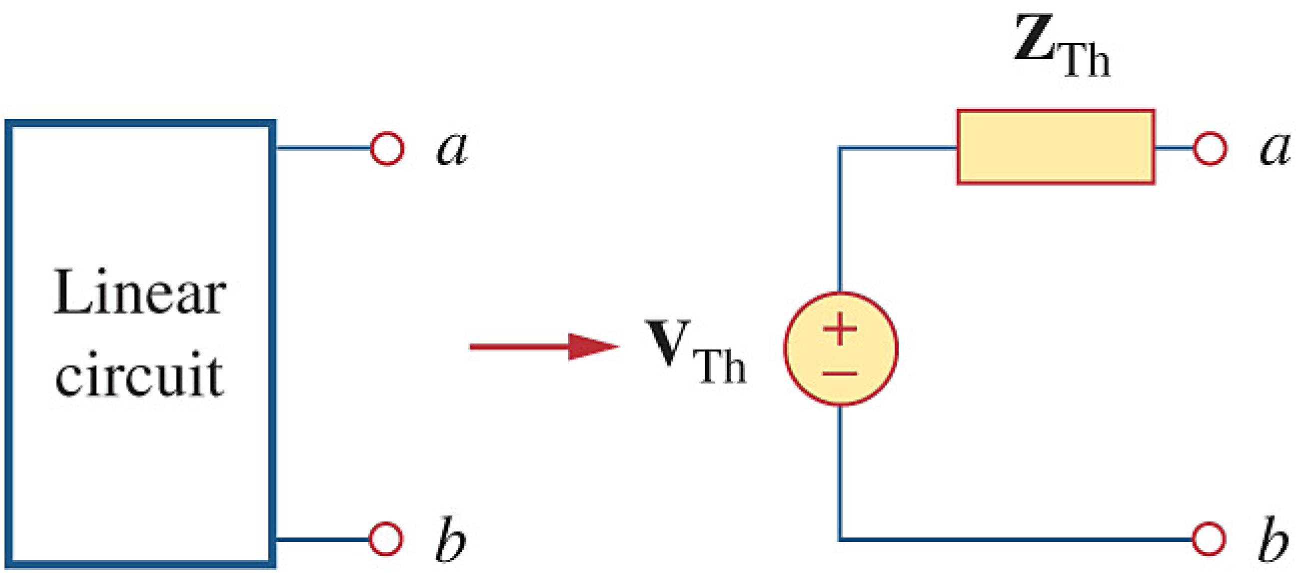

Figure 10.20 Thevenin equivalent.

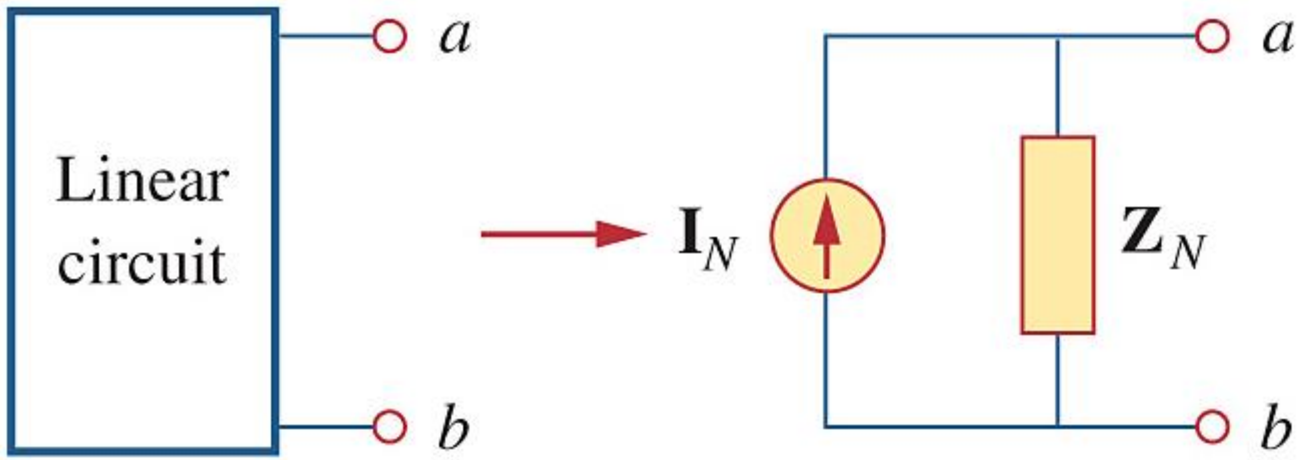

Figure 10.21 Norton equivalent.

Practice Problem 10.10 Determine the Norton equivalent of the circuit in Fig. 10.30 as seen from terminals *a*-*b*. Use the equivalent to find $\dot{I}_o$.

Solution:

$$Z_N = (4+j2) \parallel (8+1-j3)$$

$$= \frac{(4+j2)(9-j3)}{(4+j2)+(9-j3)} = \frac{6(7+j)}{13-j} = \frac{6(9+j2)}{17}$$

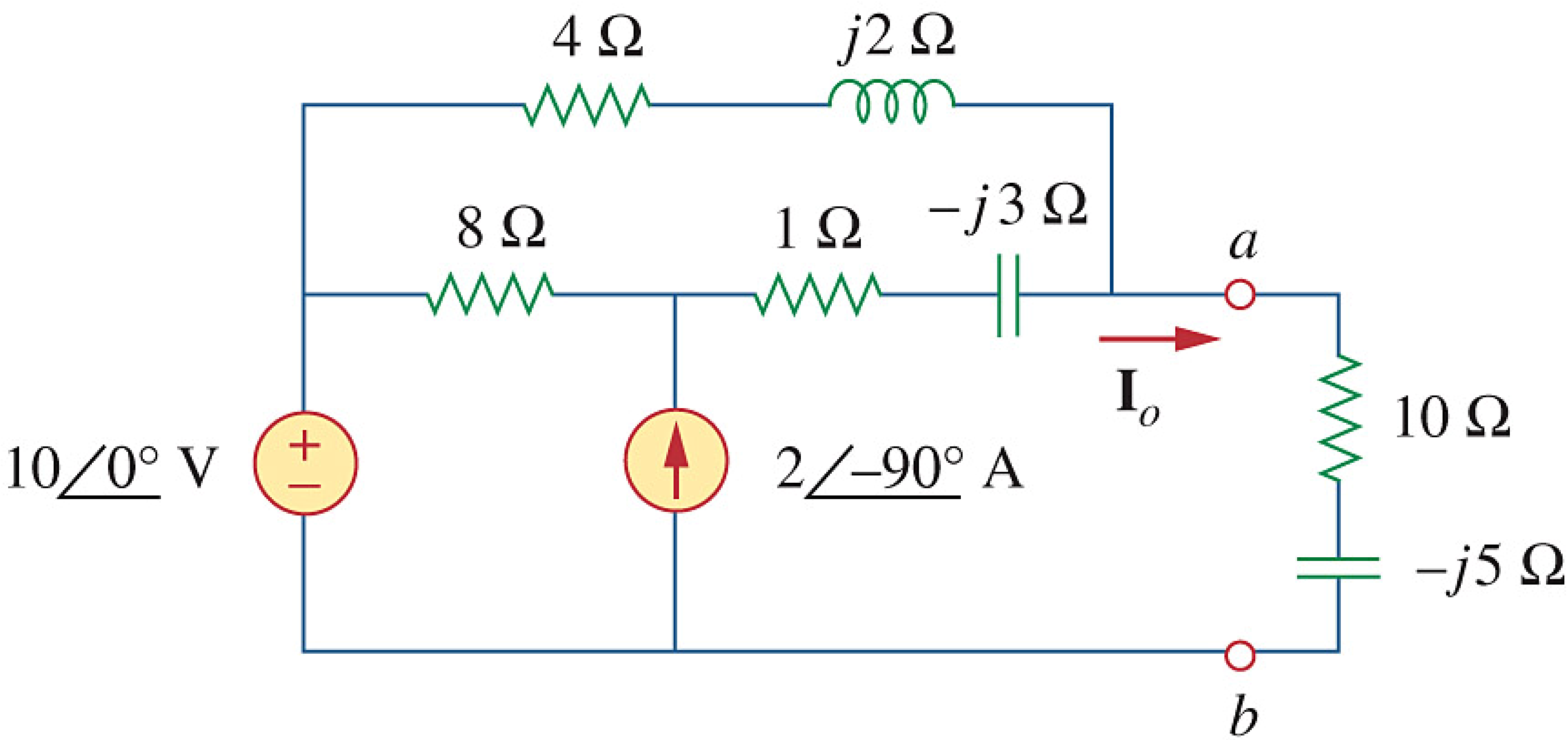

Figure 10.30

$$\approx 3.1765 + j0.7059\ (\Omega)$$

Use superposition to find $\dot{I}_N$:

$$\dot{I}_N = \dot{I}'_N + \dot{I}''_N$$

$$\dot{I}'_N = \frac{10\angle 0^\circ}{(4 + j2) \,||\, (8 + 1 - j3)}$$

$$= \frac{10}{6(7 + j)/(13 - j)} = \frac{9 - j2}{3}$$

$$\approx 3 - j0.6667\ (\text{A})$$

$$\dot{I}_N'' = 2\angle -90^\circ \times \frac{8}{8+1-j3}$$

$$\approx 0.5333 - j1.6 \text{ (A)}$$

$$\dot{I}_N = 3.5333 - j2.2667$$

$$\approx 4.1979\angle -32.68^\circ \text{ (A)}$$

$$\dot{I}_o = \dot{I}_N \frac{Z_N}{Z_N + (10 - j5)} = 4.1979\angle -32.68^\circ \times$$

$$\frac{3.1765 + j0.7059}{(3.1765 + j0.7059) + (10 - j5)}$$

$$
\begin{aligned}
&= 4.1979\angle -32.68^\circ \times \frac{3.1765 + j0.7059}{13.1765 - j4.2941} \\
&\approx 4.1979\angle -32.68^\circ \times \frac{3.1765 + j0.7059}{13.1765 - j4.2941} \\
&\approx 4.1979\angle -32.68^\circ \times \frac{3.2540\angle 12.53^\circ}{13.8586\angle -18.05^\circ} \\
&\approx 0.99\angle -2.10^\circ \text{ (A)}
\end{aligned}
$$

10.7 Op Amp AC Circuits

The three steps stated in Section 10.1 also apply to op amp circuits, as long as the op amp is in the linear region.

Practice Problem 10.11 Find v_o and i_o in the op amp circuit of Fig. 10.32. Let $v_s = 4\cos 5000t$ V.

Solution :

$$4\cos 5000t \text{ V} \Rightarrow 4\angle 0° \text{ V}, \ \omega = 5000 \text{ rad/s}$$

$$10 \text{ nF} \Rightarrow \frac{1}{j\omega C} = \frac{1}{j5000\times 10\times 10^{-9}}$$

$$= -j2\times 10^{4} \ (\Omega) = -j20 \ (\text{k}\Omega)$$

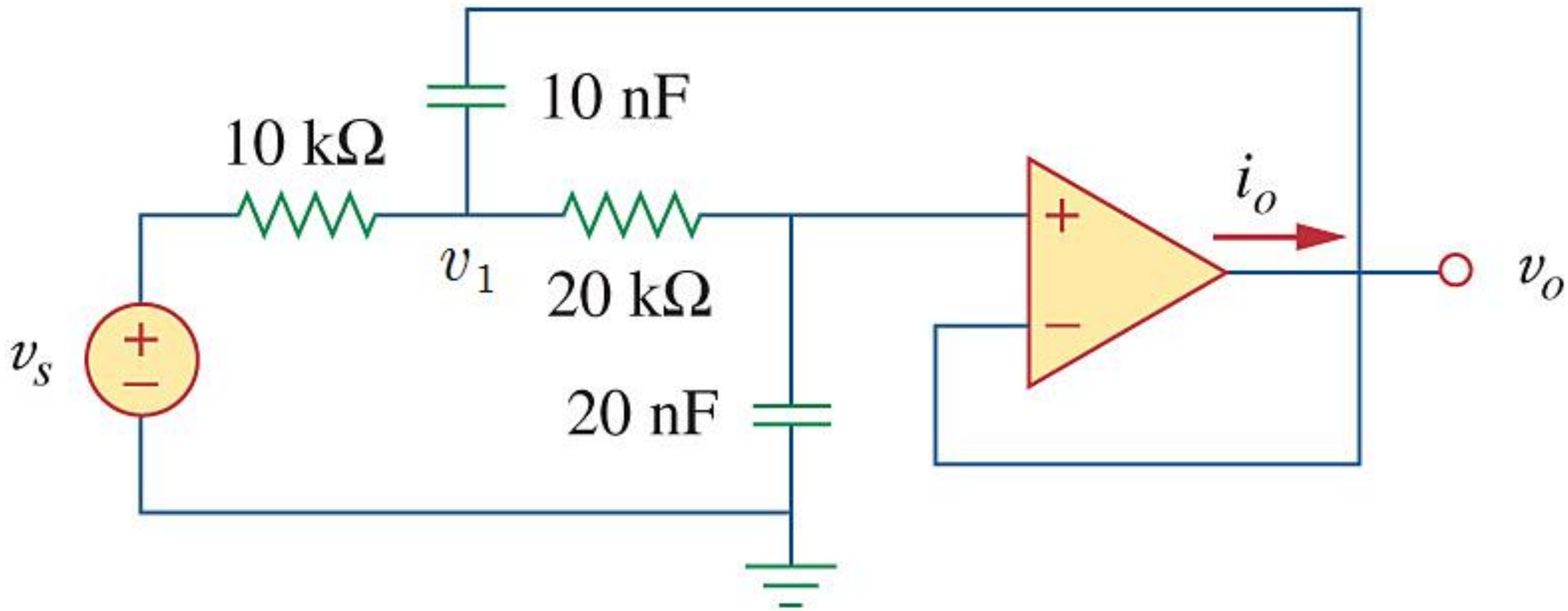

Figure 10.32

$$20\text{ nF} \Rightarrow \frac{1}{j\omega C} = \frac{1}{j5000 \times 20 \times 10^{-9}}$$

$$= -j1 \times 10^{4}\ (\Omega) = -j10\ (\text{k}\Omega)$$

$$\begin{cases} \dfrac{\dot{V}_s - \dot{V}_1}{10} = \dfrac{\dot{V}_1 - \dot{V}_o}{-j20} + \dfrac{\dot{V}_o}{-j10} \\ \dot{V}_1 = \dot{V}_o \dfrac{20 - j10}{-j10} = \dot{V}_o(1 + j2) \end{cases}$$

$$-j2(\dot{V}_s-\dot{V})=(\dot{V}_1-\dot{V}_o)+2\dot{V}_o$$

$$-j2\dot{V}_s=\dot{V}_1(1-j2)+\dot{V}_o$$

$$-j2\dot{V}_s=\dot{V}_o(1+j2)(1-j2)+\dot{V}_o=6\dot{V}_o$$

$$\dot{V}_o=\frac{-j2\dot{V}_s}{6}=\frac{-j2\times 4\angle 0^\circ}{6}$$

$$\approx 1.3333\angle -90^\circ\ (\mathrm{V})$$

$$v_o=1.33\cos(5000t-90^\circ)$$

$$=1.33\sin 5000t\ (\mathrm{V})$$

$$\dot{I}_o = \frac{\dot{V}_o - \dot{V}_1}{-j20} = \frac{\dot{V}_o - \dot{V}_o(1+j2)}{-j20} = 0.1\dot{V}_o$$

$$= 0.1 \times 1.3333\angle -90^\circ \approx 0.13\angle -90^\circ \ (\text{mA})$$

$$i_o = 0.13\cos(5000t - 90^\circ)$$

$$= 0.13\sin 5000t \ (\text{mA})$$

10.9 Applications

We study two practical ac circuits: the capacitance multiplier and the sine wave oscillator.

Capacitance Multiplier The circuit in Fig. 10.41 is used in integrated-circuit technology to produce a large capacitance.

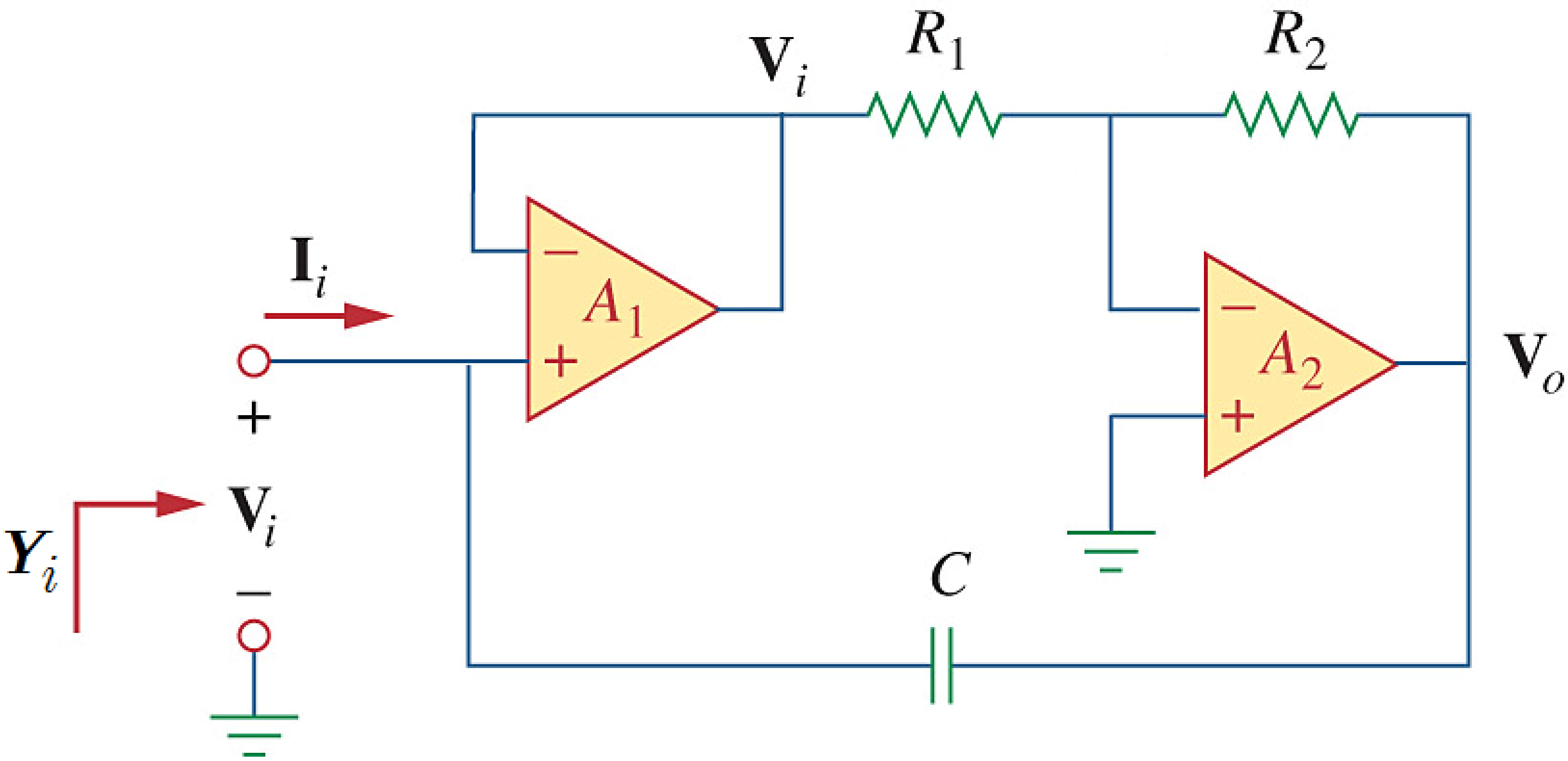

Figure 10.41 Capacitance multiplier.

$$\begin{cases} Y_i = \dfrac{\dot{I}_i}{\dot{V}_i} = \dfrac{j\omega C(\dot{V}_i - \dot{V}_o)}{\dot{V}_i} = \dfrac{j\omega C(\dot{V}_i - \dot{V}_o)}{\dot{V}_i} \\ \dfrac{\dot{V}_o}{\dot{V}_i} = -\dfrac{R_2}{R_1} \end{cases}$$

$$Y_i = j\omega C\left(1 + \frac{R_2}{R_1}\right)$$

$$C_{eq} = C\left(1 + \frac{R_2}{R_1}\right)$$

An *oscillator* is a circuit that produces an ac waveform as output when powered by a dc supply.

In order for sine wave oscillators to sustain oscillations, they must meet the *Barkhausen criteria* :

1. The overall gain of the oscillator must be unity or greater.

2. The overall phase shift must be zero.

The *Wien-bridge oscillator* is widely used for generating sinusoids in the frequency range below 1 MHz. As shown in Fig. 10.42, the oscillator consists of an amplifier and a frequency-selective network.

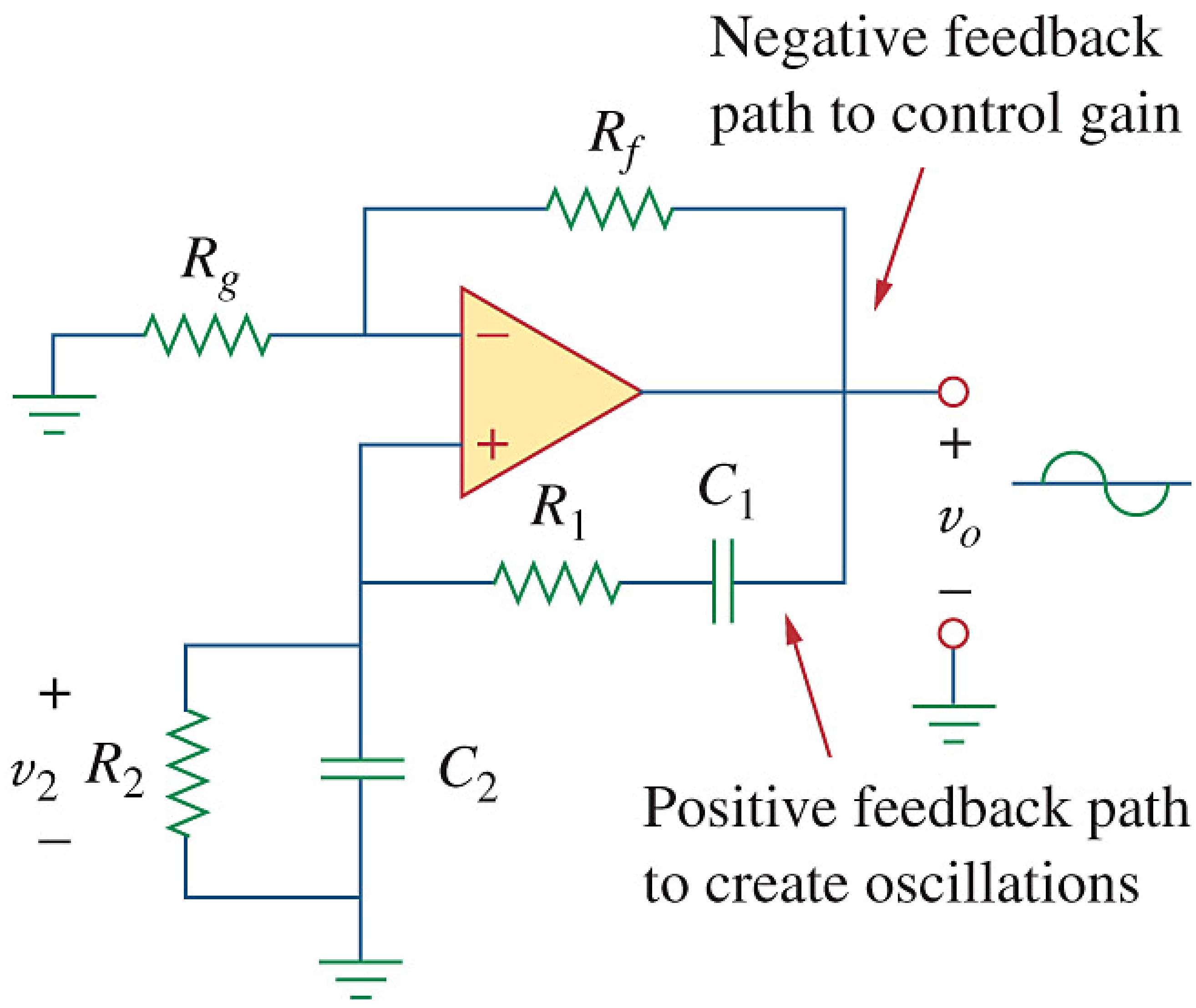

Figure 10.42 Wien-bridge oscillator.

$$\frac{\dot{V}_2}{\dot{V}_o} = \frac{R_2 \parallel \left(\dfrac{1}{j\omega C_2} \right)}{R_2 \parallel \left(\dfrac{1}{j\omega C_2} \right) + \left(R_1 + \dfrac{1}{j\omega C_1} \right)}$$

$$= \frac{R_2 / (1 + j\omega R_2 C_2)}{R_2 / (1 + j\omega R_2 C_2) + (1 + j\omega R_1 C_1) / (j\omega C_1)}$$

$$= \frac{j\omega R_2 C_1}{j\omega R_2 C_1 + (1 + j\omega R_1 C_1)(1 + j\omega R_2 C_2)}$$

$$= \frac{\omega R_2 C_1}{\omega(R_2 C_1 + R_1 C_1 + R_2 C_2) + j(\omega^2 R_1 R_2 C_1 C_2 - 1)}$$

To satisfy the second Barkhausen criterion, $\dot{V}_2 / \dot{V}_o$ must be real. Setting the imaginary part equal to zero gives the oscillation frequency as

$$\omega_0 = 2\pi f_0 = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}}$$

In most practical applications, $R_1 = R_2 = R$ and $C_1 = C_2 = C$, so that

$$\omega_0 = \frac{1}{RC} \text{ and } \frac{\dot{V}_2}{\dot{V}_o} = \frac{1}{3}$$

Thus, in order to satisfy the first Barkhausen criterion, the amplifier must provide a gain of 3 or greater, i.e.,

$$\frac{\dot{V}_o}{\dot{V}_2} = 1 + \frac{R_f}{R_g} = 3 \Rightarrow R_f = 2R_g$$