Ve460/Vm461 Automatic Control Systems
Chapter 4 Modeling of Physical Systems

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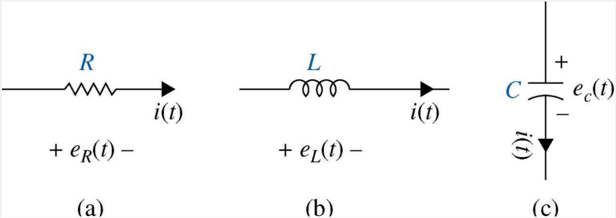
Two common methods:

TF	$\rightarrow$	LTI
State Space	$\rightarrow$	Linear and Nonlinear

4-2 Electrical Networks

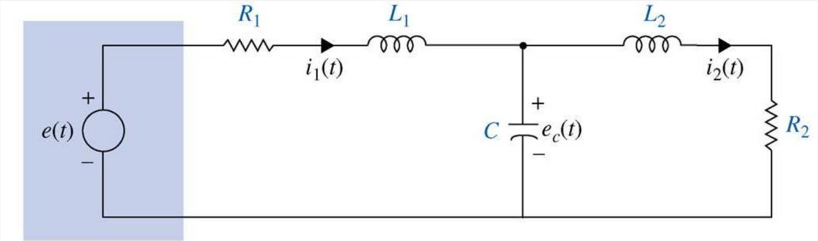


Resistor: $V = R \cdot I$

Inductor: $V = L\dfrac{di}{dt}, \ V(s) = sL \cdot I(s)$

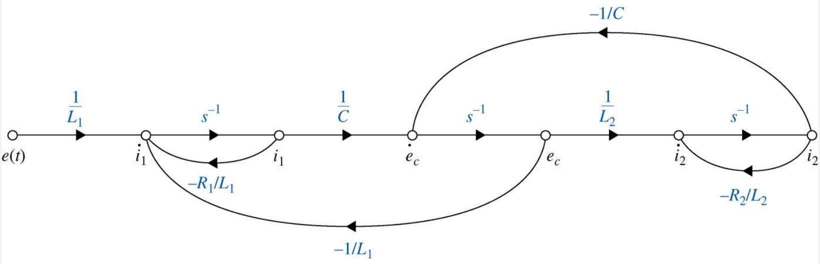
Capacitor: $i = C\dfrac{dV}{dt}, \ V(s) = \dfrac{1}{sC} \cdot I(s)$

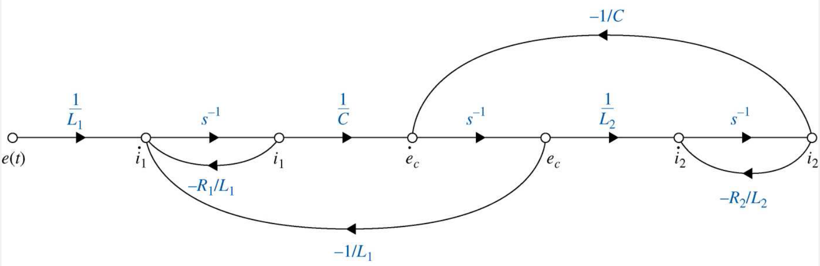
Example



$$\begin{cases} L_1 \dfrac{di_1}{dt} = -R_1 i_1 - e_c + e \\ L_2 \dfrac{di_2}{dt} = -R_2 i_2 + e_c \\ C \dfrac{de_c}{dt} = i_1 - i_2 \end{cases}$$

$$\Rightarrow \quad \frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \\ e_c \end{bmatrix} = \begin{bmatrix} -\dfrac{R_1}{L_1} & 0 & -\dfrac{1}{L_1} \\ 0 & -\dfrac{R_2}{L_2} & \dfrac{1}{L_2} \\ \dfrac{1}{C} & -\dfrac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ e_c \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L_1} \\ 0 \\ 0 \end{bmatrix} e$$





$$\Delta = 1 - \left(-\frac{1}{s}\frac{R_1}{L_1} - \frac{1}{s}\frac{R_2}{L_2} - \frac{1}{s}\frac{1}{C}\frac{1}{s}\frac{1}{L_1} - \frac{1}{s}\frac{1}{L_2}\frac{1}{s}\frac{1}{C}\right) + \left(-\frac{1}{s}\frac{R_1}{L_1}\right)\cdot\left(-\frac{1}{s}\frac{R_2}{L_2}\right) + \left(-\frac{1}{s}\frac{R_1}{L_1}\right)\cdot\left(-\frac{1}{s}\frac{1}{L_2}\frac{1}{s}\frac{1}{C}\right) + \left(-\frac{1}{s}\frac{1}{C}\frac{1}{s}\frac{1}{L_1}\right)\cdot\left(-\frac{1}{s}\frac{R_2}{L_2}\right).$$

Therefore

$$\frac{I_1(s)}{E(s)} = \frac{L_2 C s^2 + R_2 C s + 1}{D},$$

where

$$D = L_1 L_2 C s^3 + (R_1 L_2 + R_2 L_1) C s^2 + (L_1 + L_2 + R_1 R_2 C) s + R_1 + R_2.$$

Similarly,

$$\frac{I_2(s)}{E(s)} = \frac{1}{D},$$

and

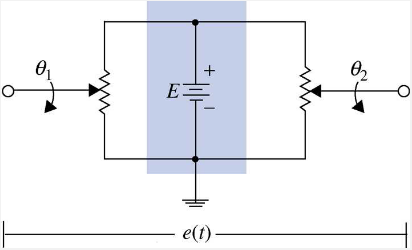
$$\frac{E_c(s)}{E(s)} = \frac{L_2 s + R_2}{D}.$$

4-5 Sensors & Encoders in Control System

4-5-1 Potentiometer

Input: Linear or rotational displacement

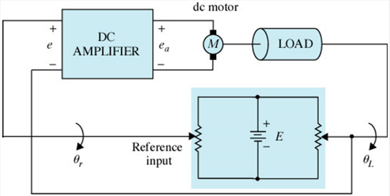
Output: Voltage proportional to input displacement



$$e(t) = K_S(\theta_1(t) - \theta_2(t))$$

K_S: constant

Can be used, *e.g.*, in DC motor control system for position feedback.



θ_r: reference input;
θ_L: input;
e_a: armature voltage of a DC motor.

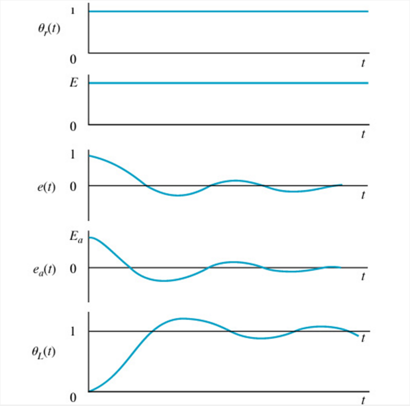
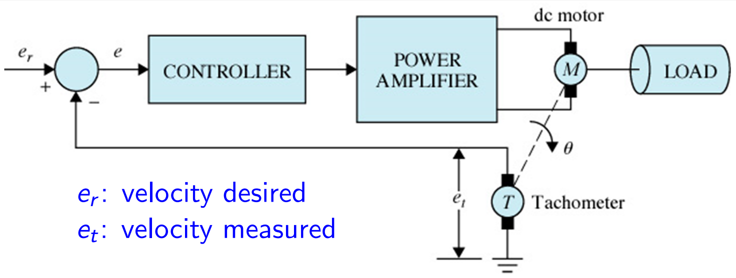


Figure: Typical waveforms of signals

4-5-2 Tachometers

Convert mechanical energy into electrical energy

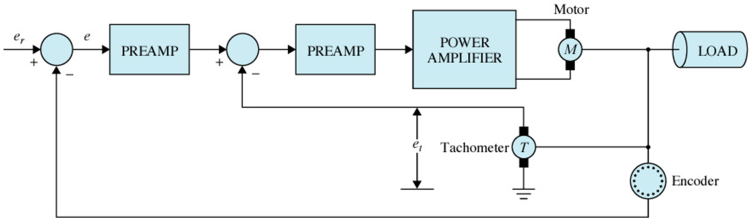
- Output: Voltage proportional to angular velocity input;
- Used as: Velocity indicator



e_r: velocity desired

e_t: velocity measured

- Velocity Control: accuracy of tachometer is highly critical



- Position Control
  Tachometer: velocity feedback
  $\longrightarrow$ to improve stability or damping, accuracy of tachometer is not that critical.

- Another usage: visual speed readout of a rotating shaft

Mathematical model

$$e_t(t) = K_t \frac{d\theta(t)}{dt} = K_t \omega(t),$$

where

$e_t(t)$: output voltage,

$\theta(t)$: rotor displacement (in radians),

$\omega(t)$: rotor velocity (in rad/sec),

K_t: tachometer constant (in V/(rad/sec)).

$$\Rightarrow \quad \frac{E_t(s)}{\Theta(s)} = K_t s.$$

4-5-3 Incremental Encoder

Convert linear / rotary displacement into digitally coded / pulse signals

- Absolute encoder: output a distinct digital code indicating each particular position within the range (does not need knowledge of previous positioning);
- Incremental encoder: cyclical, provides a pulse for each increment.

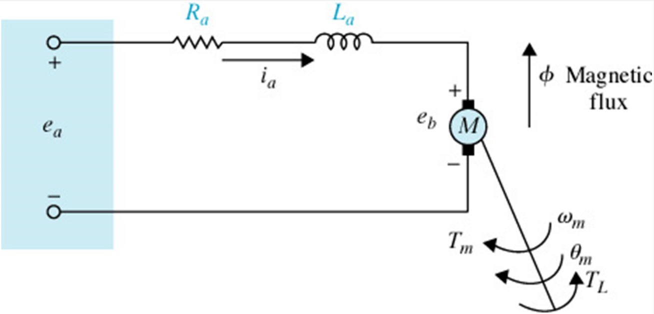
4-6 DC Motors – widely used as mover in industry

Mathematical modeling

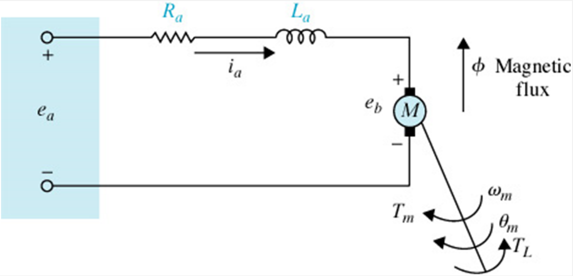
- Convert electric energy into mechanical energy: the torque T_m on the motor shaft $\propto$ field flux ϕ and armature current i_a:

$$\underbrace{T_m}_{\text{motor torque}} = \underbrace{K_m}_{\text{proportional constant}} \cdot \underbrace{\phi}_{\text{magnetic flux}} \cdot \underbrace{i_a}_{\text{armature current}}$$

- Back emf: when conductor moves, a voltage is generated that opposes the current flow, proportional to shaft velocity.



$i_a(t)$ = armature current	L_a = armature inductance
R_a = armature resistance	e_a = applied voltage
$e_b(t)$ = back emf	K_b = back-emf constant
T_L = load torque	ϕ = magnetic flux in the air gap
T_m = motor torque	$\omega_m(t)$ = rotor angular velocity
$\theta_m(t)$ = rotor displacement	J_m = rotor inertia
K_i = torque constant	B_m = viscous-friction coefficient

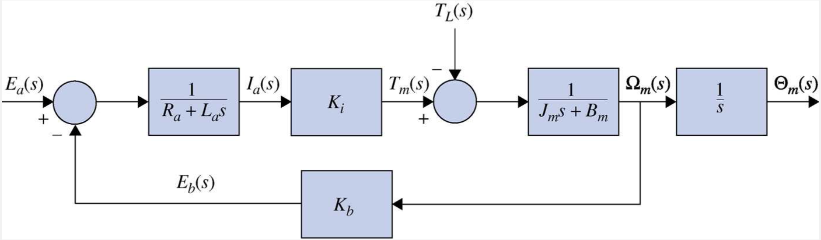


When ϕ is constant, $T_m = K_m\phi \cdot i_a = K_i i_a$,

$$
\begin{cases}
L_a \dfrac{di_a}{dt} = e_a - R_a i_a - e_b, \\
e_b(t) = K_b \dfrac{d\theta_m(t)}{dt} = K_b\omega_m(t), \quad \text{Back emf} \\
T_m = K_i i_a, \\
J_m \dfrac{d^2\theta_m(t)}{dt^2} = T_m(t) - T_L(t) - B_m \dfrac{d\theta_m(t)}{dt}
\end{cases}
$$

$$\begin{cases} L_a \dfrac{di_a}{dt} = e_a - R_a i_a - K_b \omega_m, \\ J_m \dfrac{d^2\theta_m(t)}{dt^2} = K_i i_a - T_L - B_m \omega_m \end{cases}$$

$$\therefore \frac{d}{dt}\begin{bmatrix} i_a \\ \omega_m \\ \theta_m \end{bmatrix} = \begin{bmatrix} -\dfrac{R_a}{L_a} & -\dfrac{K_b}{L_a} & 0 \\ \dfrac{K_i}{J_m} & -\dfrac{B_m}{J_m} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_a \\ \omega_m \\ \theta_m \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L_a} & 0 \\ 0 & -\dfrac{1}{J_m} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_a \\ T_L \end{bmatrix}$$



$$\begin{aligned}\frac{\Theta_m(s)}{E_a(s)} &= \frac{K_i}{L_a J_m s^3 + (R_a J_m + B_m L_a)s^2 + (K_b K_i + R_a R_m)s} \\ &= \frac{K_i}{s\left[L_a J_m s^2 + (R_a J_m + B_m L_a)s + K_b K_i + R_a R_m\right]}\end{aligned}$$

- Essentially an integrator
- A built-in feedback loop

4-7 Linearization of Nonlinear Systems

Motivation

Consider a nonlinear system represented by vector-matrix state equation:

$$\frac{dx(t)}{dt} = f(x(t), u(t)).$$

State vector: $x(t) \in \mathbb{R}^n \rightarrow n \times 1$ vector;

Input vector: $u(t) \in \mathbb{R}^p \rightarrow p \times 1$ vector;

Vector field: $f(x(t), u(t)) \in \mathbb{R}^n$.

Linearization: expanding f into a Taylor series about a nominal operating point or trajectory, discarding higher order terms.

Linearization

Consider a nominal trajectory (equilibrium):

$$\dot{x}_0(t) = f(x_0(t), u_0(t)).$$

Then,

$$\begin{aligned}\dot{x}_i(t) &= f_i(x(t), u(t)) \\ &= f_i(x_0(t), u_0(t)) + \sum_{j=1}^{n} \left.\frac{\partial f_i(x,u)}{\partial x_j}\right|_{(x_0,u_0)} (x_j - x_{0j}) \\ &\quad + \sum_{j=1}^{p} \left.\frac{\partial f_i(x,u)}{\partial u_j}\right|_{(x_0,u_0)} (u_j - u_{0j}),\end{aligned}$$

where $i = 1, \cdots, n$. Let $\Delta x_i = x_i - x_{0i}$, $\Delta u_i = u_i - u_{0i}$:

$$\Delta\dot{x}_i = \dot{x}_i - \dot{x}_{0i} = \dot{x}_i(t) - f_i(x_0(t), u_0(t)).$$

We then have

$$\Delta \dot{x}_i = \sum_{j=1}^{n} \left.\frac{\partial f_i(x,u)}{\partial x_j}\right|_{(x_0,u_0)} \Delta x_j + \sum_{j=1}^{p} \left.\frac{\partial f_i(x,u)}{\partial u_j}\right|_{(x_0,u_0)} \Delta u_j$$

$$= \begin{bmatrix} \frac{\partial f_i}{\partial x_1} & \cdots & \frac{\partial f_i}{\partial x_n} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{bmatrix} + \begin{bmatrix} \frac{\partial f_i}{\partial u_1} & \cdots & \frac{\partial f_i}{\partial u_p} \end{bmatrix} \begin{bmatrix} \Delta u_1 \\ \vdots \\ \Delta u_p \end{bmatrix}$$

In vector-matrix form

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \\ \vdots \\ \Delta\dot{x}_n \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}}_{A \text{ (Jacobian Matrix)}} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{bmatrix}$$

$$+ \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_p} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_p} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_p} \end{bmatrix}}_{B} \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \\ \vdots \\ \Delta u_p, \end{bmatrix}$$

or

$$\Delta\dot{x} = A \cdot \Delta x + B \cdot \Delta u.$$

Example

$$\begin{cases} \dot{x}_1(t) = \dfrac{-1}{x_2^2(t)}, \\ \dot{x}_2(t) = u(t) \cdot x_1(t). \end{cases}$$

Consider a nominal trajectory $(x_{01}(t), x_{02}(t))$ starting from $x_1(0) = x_2(0) = 1$ and $u(t) = 0$. First, we solve the nominal trajectory.

$$u(t) = 0 \quad \Rightarrow \quad \dot{x}_2(t) = 0 \quad \Rightarrow \quad x_2(t) = \text{const} = x_2(0) = 1.$$

Therefore,

$$\dot{x}_1(t) = -1 \quad \Rightarrow \quad x_1(t) = -t + x_1(0) = -t + 1.$$

The nominal trajectory is then

$$\begin{cases} x_{01}(t) = -t + 1, \\ x_{02}(t) = 1. \end{cases}$$

The Jacobian matrix can be obtained as:

$$\frac{\partial f_1}{\partial x_1} = 0, \qquad \frac{\partial f_1}{\partial x_2} = \frac{2}{x_2^3(t)}, \quad \frac{\partial f_1}{\partial u} = 0,$$

$$\frac{\partial f_2}{\partial x_1} = u(t), \quad \frac{\partial f_2}{\partial x_2} = 0, \qquad \frac{\partial f_2}{\partial u} = x_1(t).$$

We then get

$$\begin{bmatrix} \Delta \dot{x}_1 \\ \Delta \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{2}{x_2^3(t)} \\ u_0(t) & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ x_{01}(t) \end{bmatrix} \Delta u$$

$$= \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1-t \end{bmatrix} \Delta u.$$

4-8 Time Delay

Often seen in hydraulic system and computer control

Let $b(t) = y(t - T_d)$, $B(s) = e^{-T_d s} Y(s)$. Then

$$\frac{B(s)}{Y(s)} = e^{-T_d s}.$$

This is difficult to handle.

We can then approximate it by rational functions:

$$e^{-T_d s} \approx 1 - T_d s + \frac{T_d^2 s^2}{2}$$
$$\approx \frac{1}{1 + T_d s + \frac{T_d^2 s^2}{2}}.$$

However, this is not valid when T_d is large. A better one is Padé approximation:

$$e^{-T_d s} \approx \frac{1 - T_d s/2}{1 + T_d s/2}.$$

A zero in RHP may result in a small negative undershoot in step response.