Homework 2 Solutions

1. .

(a) Correct statement: If $y(t) = h(t) * x(t)$ then $y(t-3) = h(t) * x(t-3)$.
Proof: $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$ so $y(t-3) = \int_{-\infty}^{\infty} h(\tau)x(t-3-\tau)\,d\tau = h(t) * x(t-3)$.

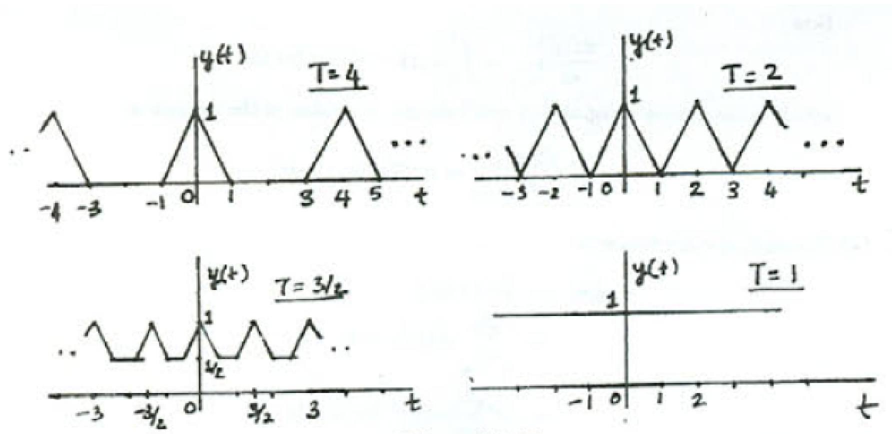
(b) Incorrect statement: If $y(t) = h(t) * x(t)$ then $y(t-3) \overset{not!}{=} h(t-3) * x(t-3)$. Example: $h(t) = x(t) = \delta(t)$. Then $y(t) = \delta(t)$ so $y(t-3) = \delta(t-3)$. But $h(t-3) * x(t-3) = \delta(t-3) * \delta(t-3) = \delta(t-6) \neq \delta(t-3)$.
(c) Correct statement: If $y(t) = h(t) \cdot x(t)$ then $y(t-3) = h(t-3) \cdot x(t-3)$, by definition of the product of two signals. But $y(t-3) \neq h(t) \cdot x(t-3)$ again by considering $h(t) = x(t) = \delta(t)$.

2. .

(a) $h(t-\tau)$ is nonzero over $t-a < \tau < t-b$ and $x(\tau)$ is nonzero over $c < \tau < d$. The integral of their product is nonzero when $t-a > c$ and $t-b < d$, so that the intervals overlap. Thus $\boxed{a+c < t < b+d}$ is the range of values of t for which $y(t)$ is possibly nonzero.
(b) $a = 1, b = 3, c = -5, d = -1$ so $\boxed{-4 < t < +2.}$
For $t-1 > -5$ but $t-3 < -5$, $y(t) = \int_{-5}^{t-1} 1\,dt = t+4$. For $t-3 > -5$ but $t-1 < -1$, $y(t) = \int_{t-3}^{t-1} 1\,dt = 2$.

For $t-3 < -1$ but $t-1 > -1$, $y(t) = \int_{t-3}^{-1} 1\,dt = 2-t$. So $\boxed{y(t) = \begin{cases} t+4, & -4 < t < -2 \\ 2, & -2 < t < 0 \\ 2-t, & 0 < t < 2 \\ 0, & \text{otherwise.} \end{cases}}$

3. The sketches are shown below:

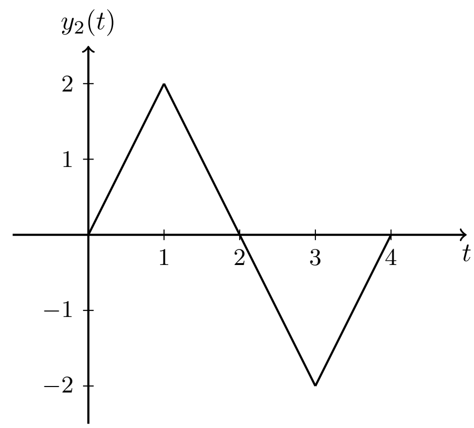


4. (a) $\int_{-\infty}^{\infty} y(t)dt = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right]dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau)x(t-\tau)dtd\tau = \int_{-\infty}^{\infty} h(\tau)\left[\int_{-\infty}^{\infty} x(t-\tau)dt\right]d\tau = \int_{-\infty}^{\infty} h(\tau)\left[\int_{-\infty}^{\infty} x(t^*)dt^*\right]d\tau = \left[\int_{-\infty}^{\infty} x(t)dt\right]\left[\int_{-\infty}^{\infty} h(\tau)d\tau\right]$.

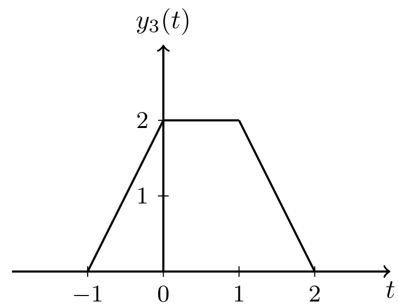
   (b) $\frac{d}{dt}y(t) = \frac{d}{dt}\left[\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right] = \int_{-\infty}^{\infty} h(\tau)\frac{d}{dt}x(t-\tau)d\tau = h(t) * (\frac{d}{dt}x(t))$.
   The other part of the statement was very similar.

   (c) $h(t) = \frac{d}{dt}s(t)$. Sketch $s(t)$ and we easily get from graph that $h(t) = \boxed{3\delta(t) - \delta(t-2) - \mathrm{rect}(\frac{t-1}{2})}$.

5. (a) $x_2(t) = x_1(t) - x_1(t-2) \Rightarrow y_2(t) = y_1(t) - y_1(t-2)$

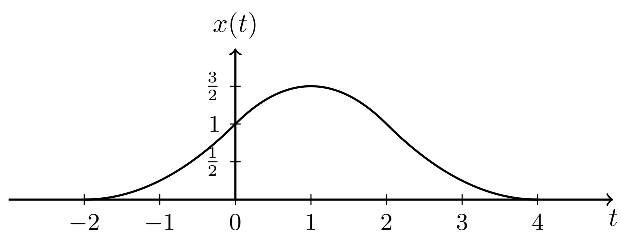


(b) $x_3(t) = x_1(t) + x_1(t+1) \Rightarrow y_3(t) = y_1(t) + y_1(t+1)$



6. $x(\tau) = (1 - |\tau/2|)\text{rect}(\tau/4)$, so graphically,

$$x(t) = \begin{cases} \int_{-2}^{t}(1+\tau/2)d\tau & -2 < t < 0 \\ \int_{t-2}^{0}(1+\tau/2)d\tau + \int_0^t(1-\tau/2)d\tau & 0 < t < 2 \\ \int_{t-2}^{2}(1-\tau/2)d\tau & 2 < t < 4 \\ 0 & \text{otherwise} \end{cases} = \boxed{\begin{cases} 1+t+t^2/4 & -2 < t < 0 \\ 1+t-t^2/2 & 0 < t < 2 \\ 4-2t+t^2/4 & 2 < t < 4 \\ 0 & \text{otherwise} \end{cases}}$$



7. (a) $y(t) = \int_{-\infty}^{t}(t-\tau)e^{-(t-\tau)}x(\tau)d\tau = \int_{-\infty}^{\infty}(t-\tau)u(t-\tau)e^{-(t-\tau)}x(\tau)d\tau \Rightarrow \boxed{h(t) = te^{-t}u(t)}$

$\boxed{\text{Causal}}$: $h(t) = 0$ for $t < 0$.

$\boxed{\text{Stable}}$: $\int_{-\infty}^{\infty}|h(t)|dt = \int_0^{\infty} te^{-t}dt = -(t+1)e^{-t}\big|_0^{\infty} = 1$.

$\boxed{\text{Dynamic}}$: $h(t) \neq 0$ for $t > 0$.

(b) $y(t) = \int_{t-1}^{t+1} e^{-2(t-\tau)}x(\tau)d\tau = \int_{-\infty}^{\infty} \text{rect}\left(\frac{t-\tau}{2}\right) e^{-2(t-\tau)}x(\tau)d\tau \Rightarrow \boxed{h(t) = \text{rect}(t/2)e^{-2t}}$

$\boxed{\text{Non-causal}}$: $h(t) \neq 0$ for $t < 0$.

$\boxed{\text{Stable}}$: $\int_{-\infty}^{\infty}|h(t)|dt = \int_{-1}^{1} e^{-2t}dt = \frac{e^2-e^{-2}}{2}$.

$\boxed{\text{Dynamic}}$: $h(t) \neq 0$ for $t > 0$.

8.
$$\frac{dx(t)}{dt} = -6e^{-3t}u(t-1) + 2e^{-3t}\delta(t-1) = -3x(t) + 2e^{-3}\delta(t-1) \to -3y(t) + e^{-2t}u(t)$$

Thus we know
$$2e^{-3}\delta(t-1) \to e^{-2t}u(t) \quad \Rightarrow \quad \delta(t) \to \frac{1}{2}e^{-2t+1}u(t+1)$$

and it follows $\boxed{h(t) = \frac{1}{2}e^{-2t+1}u(t+1)}$.

9. Plug $y(t) = Ce^{st}$ into the equation, and we get $s = -10$. Hence the homogeneous solution is
$$y_h(t) = Ce^{-10t}$$

For the particular solution, when $t > 0$
$$y_p(t) = P_0 x(t) + P_1 \frac{d}{dt}x(t) + \cdots = P_0$$

Thus
$$y(t) = y_h(t) + t_p(t) = Ce^{-10t} + P_0 \qquad \text{for } t > 0$$

Plugging into the differential equation, we get
$$-10Ce^{-10t} + 10Ce^{-10t} + 10P_0 = 2 \Rightarrow P_0 = \frac{1}{5}$$

Moreover, with the initial condition, we have
$$y(0) = C + P_0 = 1 \Rightarrow C = \frac{4}{5}$$

Thus $\boxed{y(t) = \left(\frac{4}{5}e^{-10t} + \frac{1}{5}\right)u(t)}$.