# Ve460/Vm461 Automatic Control Systems
# Chapter 3 Block diagram and Signal Flow Graph

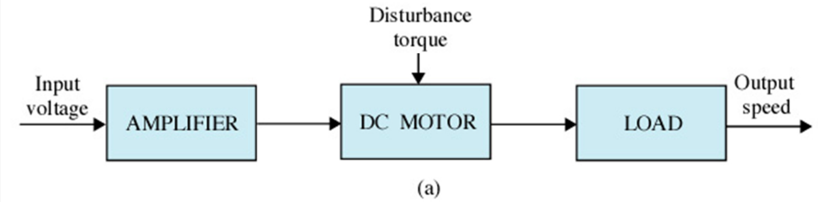
Jun Zhang

Shanghai Jiao Tong University

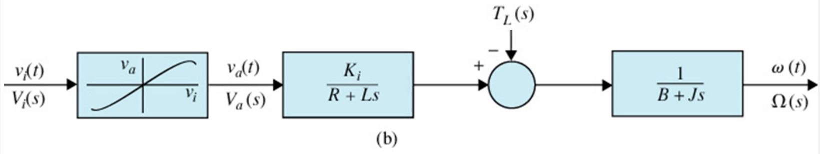
# Chapter 3 Block diagram and Signal Flow Graph

## 3-1 Block diagrams

- Used to model all types of system: plant and controller;
- Describe cause-and-effect relationship;
- Describe composition and interconnection.




(a) Block diagram of a DC-motor control system.




(b) Block diagram with TFs and amplifier characteristics.

# 3-1-1 Block Diagrams of Control Systems

## Sensing devices: addition & subtraction

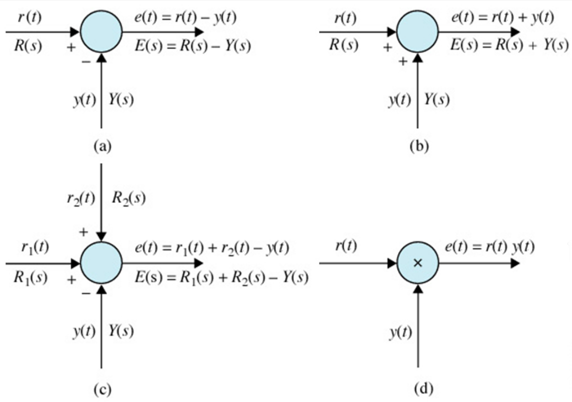
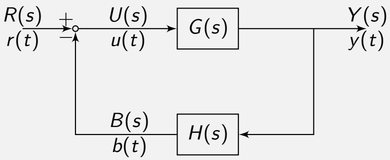



Figure: Block diagram element of typical sensing devices of control systems. (a) Subtraction; (b) Addition; (c) Addition and Subtraction; (d) Multiplication.

## A feedback control system (an *LTI* system!)




$r$: reference input

$y$: output (controlled variable)

$b$: feedback signal

$u$: actuating signal

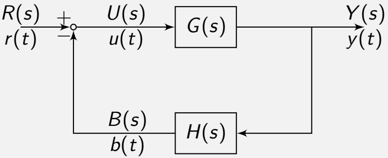
$H(s)$: feedback TF

$G(s)H(s) = L(s)$: Loop TF

$G(s)$: forward-path TF

$M(s) = Y(s)/R(s)$: closed-loop TF or system TF

## How to calculate $M(s)$?




$$\begin{cases} Y(s) = G(s)U(s) \\ B(s) = H(s)Y(s) \\ U(s) = R(s) - B(s) \end{cases}$$

$$\begin{aligned} Y(s) &= G(s)(R(s) - B(s)) \\ &= G(s)R(s) - G(s)H(s)Y(s) \end{aligned}$$

$$\Rightarrow (1 + G(s)H(s))Y(s) = G(s)R(s)$$

$$M(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

## 3-1-2 Multivariable System

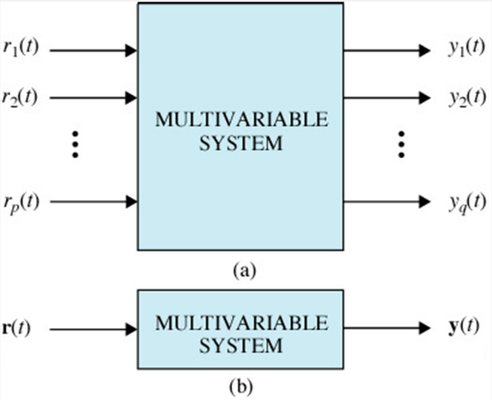
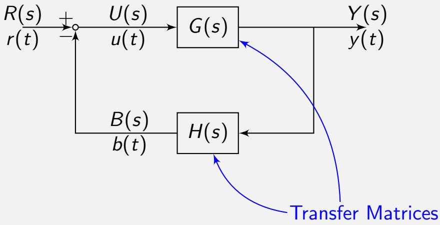



Figure: Block diagram representations of a multivariable system

## A Multivariable Feedback Control System




$$\begin{cases} Y(s) = G(s)U(s) \\ U(s) = R(s) - B(s) \\ B(s) = H(s)Y(s) \end{cases}$$

Therefore

$$\begin{aligned} Y(s) &= G(s)R(s) - G(s)B(s) \\ &= G(s)R(s) - G(s)H(s)Y(s), \end{aligned}$$

and

$$(I + G(s)H(s))Y(s) = G(s)R(s)$$

$$M(s) = (I + G(s)H(s))^{-1}G(s).$$

## Example

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & -\frac{1}{s} \\ 2 & \frac{1}{s+2} \end{bmatrix}, \qquad H(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$I + G(s)H(s) = \begin{bmatrix} \frac{s+2}{s+1} & -\frac{1}{s} \\ 2 & \frac{s+3}{s+2} \end{bmatrix}.$$

Note that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

then

$$(I + G(s)H(s))^{-1} = \frac{1}{\Delta} \begin{bmatrix} \frac{s+3}{s+2} & \frac{1}{s} \\ -2 & \frac{s+2}{s+1} \end{bmatrix},$$

where

$$\Delta = \frac{s+2}{s+1} \cdot \frac{s+3}{s+2} + \frac{2}{s} = \frac{s^2+5s+2}{s(s+1)}.$$

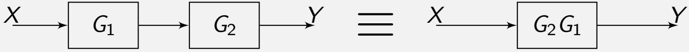
Therefore

$$\begin{aligned} M(s) &= (I + G(s)H(s))^{-1} G(s) \\ &= \frac{s(s+1)}{s^2+5s+2} \begin{bmatrix} \frac{s+3}{s+2} & \frac{1}{s} \\ -2 & \frac{s+2}{s+1} \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s} \\ 2 & \frac{2}{s} \end{bmatrix} \\ &= \frac{s(s+1)}{s^2+5s+2} \begin{bmatrix} \frac{3s^2+9s+4}{s(s+2)(s+1)} & -\frac{1}{s} \\ 2 & \frac{3s+2}{s(s+1)} \end{bmatrix}. \end{aligned}$$

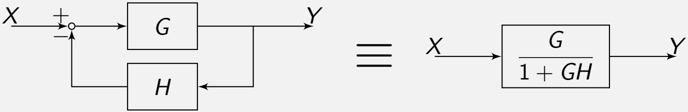
## Simplification of Block Diagram

### Simple cases

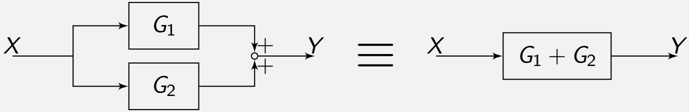
a) Combining blocks in cascade



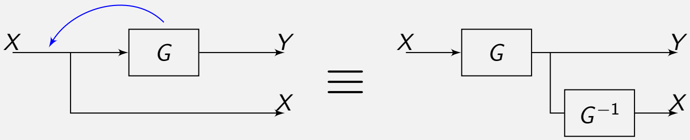

b) Elimination of feedback loop

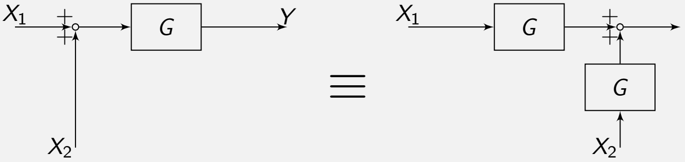



c) Combining blocks in parallel

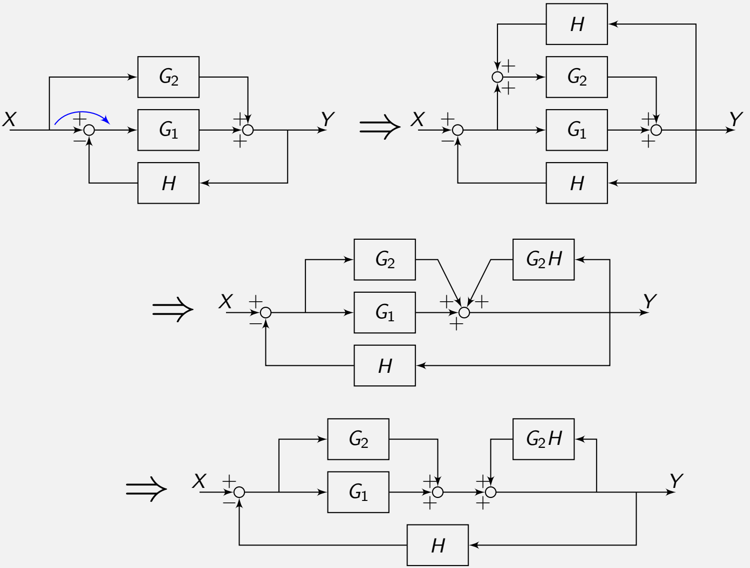


## d) Moving a block



## Example



Therefore, we have

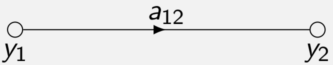
$$M(s) = \frac{(G_1 + G_2) \cdot \dfrac{1}{1 - G_2 H}}{1 + (G_1 + G_2) \cdot \dfrac{1}{1 - G_2 H} \cdot H}$$
$$= \frac{G_1 + G_2}{1 - G_2 H + (G_1 + G_2) H}$$
$$= \frac{G_1 + G_2}{1 + G_1 H}.$$

## 3-2 Signal Flow Graphs (SFGs)

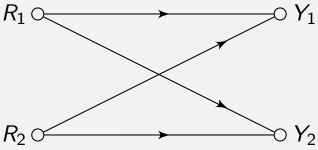
### Simplified version of block diagram

Cause-and-effect relation:

$$y_2 = a_{12} y_1$$




### Multivariable systems




$$Y_1(s) = G_{11}(s)R_1(s) + G_{21}(s)R_2(s)$$
$$Y_2(s) = G_{12}(s)R_1(s) + G_{22}(s)R_2(s)$$

Matrix representation:

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{21}(s) \\ G_{12}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} R_1(s) \\ R_2(s) \end{bmatrix}$$

Generally, we have

$$G_{ij}(s) = \frac{Y_j(s)}{R_i(s)}, \qquad Y(s) = G(s)R(s),$$

$$R(s) = \begin{bmatrix} R_1(s) \\ \vdots \\ R_p(s) \end{bmatrix}, \quad Y(s) = \begin{bmatrix} Y_1(s) \\ \vdots \\ Y_q(s) \end{bmatrix},$$

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{21}(s) & \cdots & G_{p1}(s) \\ G_{12}(s) & G_{22}(s) & \cdots & G_{p2}(s) \\ \vdots & \vdots & \ddots & \vdots \\ G_{1q}(s) & G_{2q}(s) & \cdots & G_{pq}(s) \end{bmatrix}.$$

Note that the indices here are different from a usual matrix.

## Example

$$\begin{cases} y_2 = a_{12}y_1 + a_{32}y_3 \\ y_3 = a_{23}y_2 + a_{43}y_4 \\ y_4 = a_{24}y_2 + a_{34}y_3 + a_{44}y_4 \\ y_5 = a_{25}y_2 + a_{45}y_4 \end{cases}$$

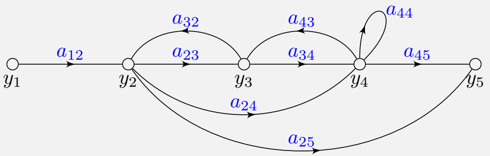


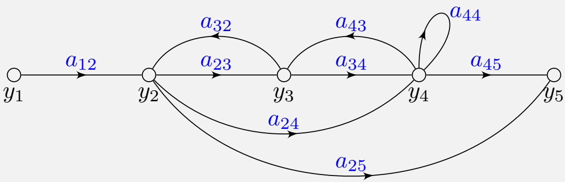

Figure: Signal Flow Graph example

### SFG basic properties

- Applies only to linear systems;
- Cause-and-effect algebraic equations.

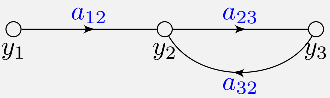
- Node: variable;
- Signal flows along the direction of arrow;
- $y_k \rightarrow y_j$: arrow represents dependence;
- $a_{kj}$: gain.

## 3-2-3 Definition of SFG Terms



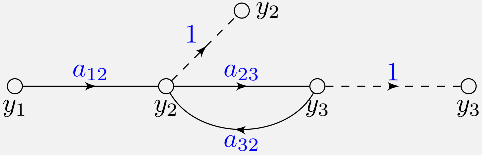

- Input node: has only outgoing branch, $y_1$;
- Output node: has only incoming branch, $y_5$.

Some SFG does not have output node, *e.g.*,




(a) Original signal flow graph

Can add a new node to make an output node:



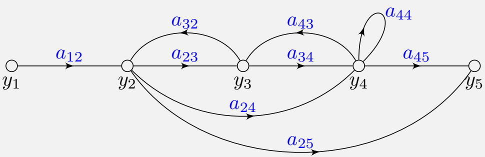

(b) Modified signal flow graph

### Definition of SFG Terms

- Path: continuous succession of branches traversed in the same direction;
- Path gain: product of branch gains along a path;
- Forward path: a path from an input node to an output node, and no node traversed more than once.

## Definition of SFG Terms

- Forward path gain: path gain of a forward path, *e.g.*, between $y_1$ and $y_5$:



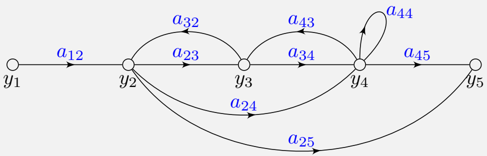

$$M_1 : y_1 \xrightarrow{a_{12}} y_2 \xrightarrow{a_{23}} y_3 \xrightarrow{a_{34}} y_4 \xrightarrow{a_{45}} y_5$$

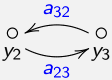
$$M_2 : y_1 \xrightarrow{a_{12}} y_2 \xrightarrow[a_{25}]{} y_5$$

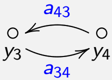
$$M_3 : y_1 \xrightarrow{a_{12}} y_2 \xrightarrow[a_{24}]{} y_4 \xrightarrow{a_{45}} y_5$$

$$M_1 = a_{45}a_{34}a_{23}a_{12}, \quad M_2 = a_{25}a_{12}, \quad M_3 = a_{45}a_{24}a_{12}.$$

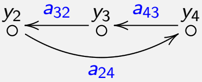
- Loop: a path originates and terminates on the same node and no other node encountered more than once;
- Loop gain: path gain of a loop, *e.g.*,




$L_1$ : 

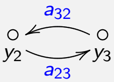

$L_2$ : 


$L_3$ : 


$L_4$ : 


$$L_1 = a_{23}a_{32},\ L_2 = a_{34}a_{43},\ L_3 = a_{44},\ L_4 = a_{24}a_{43}a_{32}.$$

- Non-touching Loops: loops do not share a common node, *e.g.*,




and

## 3-2-4 SFG algebra

1. Value of node $= \sum$ all signals entering the node:

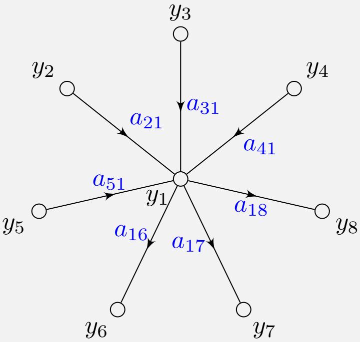
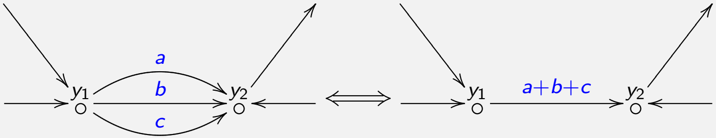



Figure: $y_1 = a_{21}y_2 + a_{31}y_3 + a_{41}y_4 + a_{51}y_5$.

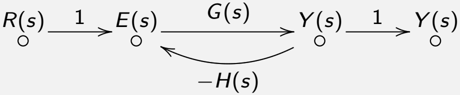
2. Value of node is transmitted through all branches leaving the node, *e.g.*, $y_6 = a_{16}y_1$, $y_7 = a_{17}y_1$, $y_8 = a_{18}y_1$.

3. Parallel branches in the same direction.




4. Series connection of unidirectional branches

## Example: Feedback Control System




$$Y(s) = G(s) \cdot E(s) = G(s)(R(s) - H(s) \cdot Y(s))$$

$$\therefore (1 + G(s)H(s))Y(s) = G(s) \cdot R(s)$$

$$\therefore \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

## 3-2-6 Gain Formula for SFG (Mason's formula)

SFG with $N$ forward paths and $K$ loops:

$$M = \frac{y_{\text{out}}}{y_{\text{in}}} = \sum_{k=1}^{N} \frac{M_k \Delta_k}{\Delta} \text{ : gain between } y_{\text{in}} \text{ and } y_{\text{out}},$$

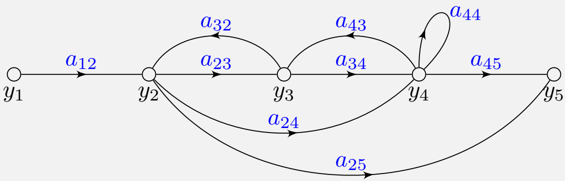
where

$\Delta$ : 1 − (sum of the gains of all individual loops)
+ (sum of products of gains of all possible combinations of TWO non-touching loops)
− (sum of products of gains of all possible combinations of THREE non-touching loops) $\cdots$

$M_k$ : gain of the $k$-th forward path between $y_{\text{in}}$ and $y_{\text{out}}$,

$\Delta_k$ : the $\Delta$ for that part of the SFG that is non-touching with $k$th forward path.

## Example

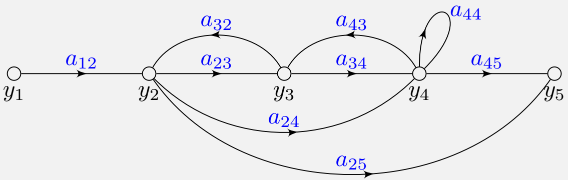



1. Forward Path Gain $M_k$:

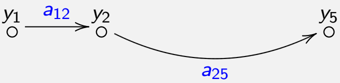
❶ $M_1$:

$$y_1 \xrightarrow{a_{12}} y_2 \xrightarrow{a_{23}} y_3 \xrightarrow{a_{34}} y_4 \xrightarrow{a_{45}} y_5$$

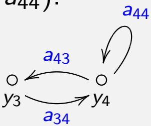
$M_1 = a_{12}a_{23}a_{34}a_{45}$; Non-touching part: none $\Longrightarrow \Delta_1 = 1$.

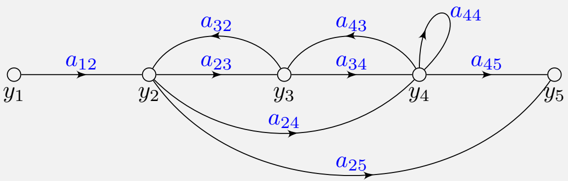



2. $M_2$:

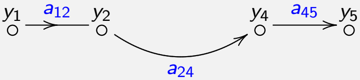



$M_2 = a_{12}a_{25}$. Non-touching part:
$\Longrightarrow \Delta_2 = 1 - (a_{43}a_{34} + a_{44})$.

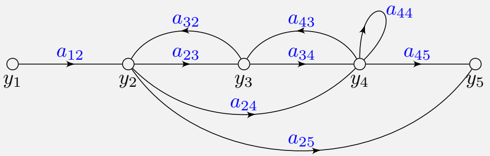





3. $M_3$:



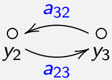

$M_3 = a_{45}a_{24}a_{12}$. Non-touching part: $\Longrightarrow \Delta_3 = 1$.



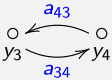

## Loop gain $L_k$

1. Loop:

$L_1$ :

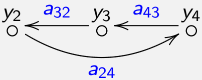



$L_2$ :




$L_3$ :


$L_4$ :

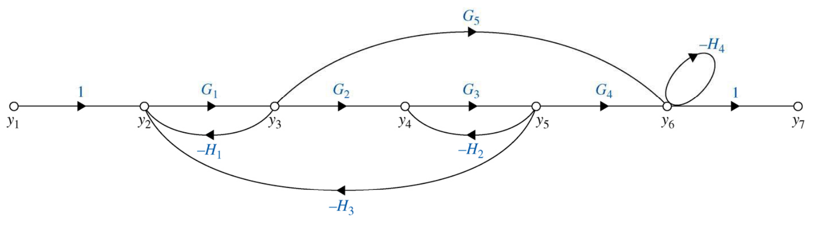



$$L_1 = a_{23}a_{32}, L_2 = a_{34}a_{43}, L_3 = a_{44}, L_4 = a_{24}a_{43}a_{32}.$$

2. Two non-touching loops: $L_1$ and $L_3$, $(a_{23}a_{32})(a_{44})$.
3. No three non-touching loop.

$$
\begin{aligned}
\Delta &= 1-(L_1+L_2+L_3+L_4)+L_1L_3 \\
&= 1-(a_{23}a_{32}+a_{34}a_{43}+a_{44}+a_{24}a_{43}a_{32})+a_{23}a_{32}a_{44}; \\
\Delta_1 &= \Delta_3 = 1; \\
\Delta_2 &= 1-(a_{34}a_{43}+a_{44}); \\
\therefore \frac{y_5}{y_1} &= \frac{M_1\Delta_1+M_2\Delta_2+M_3\Delta_3}{\Delta} \\
&= \frac{a_{12}a_{23}a_{34}a_{45}+a_{12}a_{25}(1-a_{34}a_{43}-a_{44})+a_{12}a_{24}a_{45}}{\Delta}.
\end{aligned}
$$

## 3-2-7 Gain formula between output nodes & non-input nodes



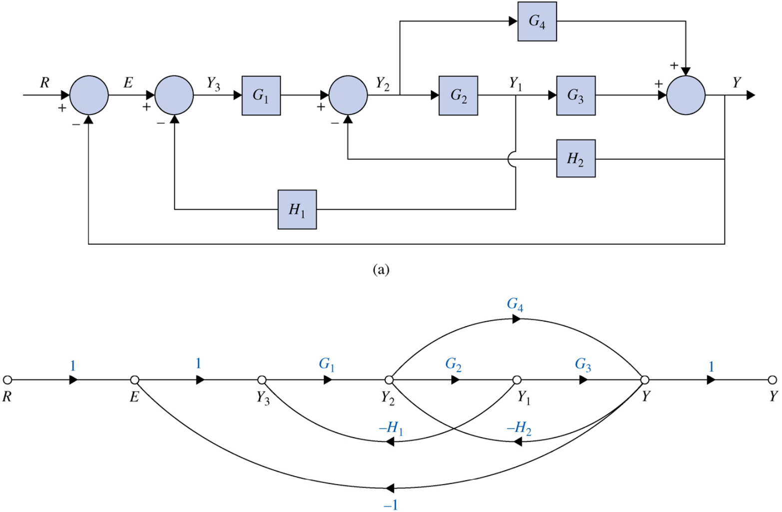

What if we want $\dfrac{y_7}{y_2}$, where $y_2$ is not an input?

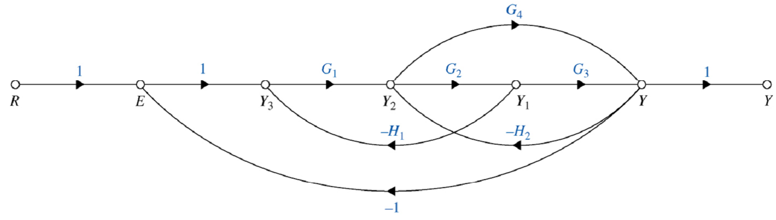
$$\frac{y_7}{y_2} = \frac{y_7/y_1}{y_2/y_1} = \frac{\sum M_k \Delta_k|_{\text{from } y_1 \text{ to } y_7}/\Delta}{\sum M_k \Delta_k|_{\text{from } y_1 \text{ to } y_2}/\Delta},$$

$\Delta$ is independent of the inputs and outputs

$$\therefore \frac{y_7}{y_2} = \frac{y_7/y_1}{y_2/y_1} = \frac{\sum M_k \Delta_k|_{\text{from } y_1 \text{ to } y_7}}{\sum M_k \Delta_k|_{\text{from } y_1 \text{ to } y_2}}$$

## 3-2-8 Gain Formula for Block Diagram

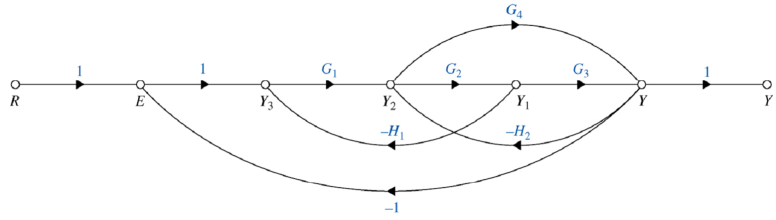





$$M_1: \quad R \to E \to Y_3 \to Y_2 \to Y_1 \to Y \qquad M_1 = G_1 G_2 G_3$$
$$M_2: \quad R \to E \to Y_3 \to Y_2 \to Y \qquad M_2 = G_1 G_4$$

Note that neither path has non-touching loops.

$$\begin{aligned}\Delta =& 1 - (\overbrace{G_1 G_2(-H_1)}^{Y_3\text{-}Y_2\text{-}Y_1\text{-}Y_3} + \overbrace{G_4(-H_2)}^{Y_2\text{-}Y\text{-}Y_2} + \overbrace{(G_2 G_3)(-H_2)}^{Y_2\text{-}Y_1\text{-}Y\text{-}Y_2} \\ &+ \underbrace{G_1 G_2 G_3(-1)}_{E\text{-}Y_3\text{-}Y_2\text{-}Y_1\text{-}Y\text{-}E} + \underbrace{1 \cdot G_1 \cdot G_4 \cdot (-1))}_{E\text{-}Y_3\text{-}Y_2\text{-}Y\text{-}E} \\ =& 1 + G_1 G_2 H_1 + G_4 H_2 + G_2 H_3 H_2 + G_1 G_2 G_3 + G_1 G_4\end{aligned}$$




$$\therefore \frac{Y(s)}{R(s)} = \frac{M_1 + M_2}{\Delta} = \frac{G_1 G_2 G_3 + G_1 G_4}{\Delta}$$

$$\frac{E(s)}{R(s)} = \frac{1 - (\overbrace{G_1 G_2(-H_1)}^{Y_3\text{-}Y_2\text{-}Y_1\text{-}Y_3} + \overbrace{G_4(-H_2)}^{Y_2\text{-}Y\text{-}Y_2} + \overbrace{(G_2 G_3)(-H_2)}^{Y_2\text{-}Y_1\text{-}Y\text{-}Y_2})}{\Delta}$$

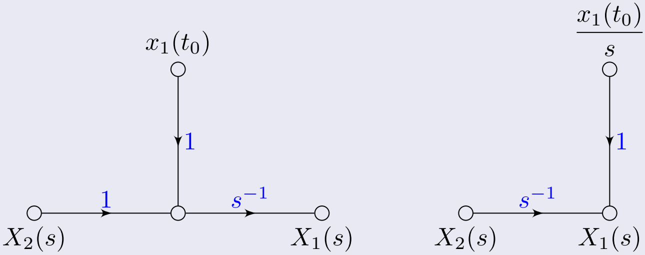
$$= \frac{1 + G_1 G_2 H_1 + G_4 H_2 + G_2 G_3 H_2}{\Delta}$$

$$\therefore \frac{Y(s)}{E(s)} = \frac{G_1 G_2 G_3 + G_1 G_4}{1 + G_1 G_2 H_1 + G_4 H_2 + G_2 G_3 H_2}$$

## 3-3 State Diagram

Use SFG to describe state equation and differential equation. Consider

$$\frac{dx_1(t)}{dt} = x_2(t),$$
$$sX_1(s) - x_1(t_0) = X_2(s),$$
$$\therefore X_1(s) = \frac{X_2(s)}{s} + \frac{x_1(t_0)}{s}.$$



Now consider a differential equation

$$\frac{d^n y(t)}{dt^n} + a_n \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_2 \frac{dy(t)}{dt} + a_1 y(t) = r(t)$$

$$\Rightarrow \quad \frac{d^n y(t)}{dt^n} = -a_n \frac{d^{n-1} y(t)}{dt^{n-1}} - \cdots - a_2 \frac{dy(t)}{dt} - a_1 y(t) + r(t)$$