Ve215 Electric Circuits

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Chapter 14

Frequency Response

14.1 Introduction

In our sinusoidal steady-state analysis, we have learned how to find voltages and currents in a circuit with a constant frequency source. If we let the amplitude and phase angle of the sinusoidal source remain constant and vary the frequency, we obtain the circuit's *frequency response*.

The frequency response may be regarded as a complete description of the sinusoidal steady-state behavior of a circuit as a function of frequency.

14.2 Frequency Response

The frequency response $H(j\omega)$ of a circuit is the frequency-dependent ratio of a phasor output $\dot{Y}(j\omega)$ (an element voltage or current) to a phasor input $\dot{X}(j\omega)$ (source voltage or current).

$$H(j\omega) = \frac{\dot{Y}(j\omega)}{\dot{X}(j\omega)}$$

There are four types of frequency response:

$$H(j\omega) = \frac{\dot{V}_o(j\omega)}{\dot{V}_i(j\omega)} \qquad \text{(Voltage gain)}$$

$$H(j\omega) = \frac{\dot{I}_o(j\omega)}{\dot{I}_i(j\omega)} \qquad \text{(Current gain)}$$

$$H(j\omega) = \frac{\dot{V}_o(j\omega)}{\dot{I}_i(j\omega)} \qquad \text{(Transfer impedance)}$$

$$H(j\omega) = \frac{\dot{I}_o(j\omega)}{\dot{V}_i(j\omega)} \qquad \text{(Transfer admittance)}$$

The frequency response can be expressed in terms of its numerator polynomial $N(j\omega)$ and denominator polynomial $D(j\omega)$ as

$$H(j\omega) = \frac{N(j\omega)}{D(j\omega)} = \frac{\sum_{m=0}^{M} b_m (j\omega)^m}{\sum_{n=0}^{N} a_n (j\omega)^n}$$

Proof :

A linear network can be described by a linear constant-coefficient differential equation

$$\sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}$$

Transform the equation to the phasor domain,

$$\sum_{n=0}^{N} a_n (j\omega)^n \dot{Y}(j\omega) = \sum_{m=0}^{M} b_m (j\omega)^m \dot{X}(j\omega)$$

$$\dot{Y}(j\omega)\sum_{n=0}^{N} a_n (j\omega)^n = \dot{X}(j\omega)\sum_{m=0}^{M} b_m (j\omega)^m$$

$$H(j\omega) = \frac{\dot{Y}(j\omega)}{\dot{X}(j\omega)} = \frac{\sum_{m=0}^{M} b_m (j\omega)^m}{\sum_{n=0}^{N} a_n (j\omega)^n}$$

Being a complex quantity, $H(j\omega)$ has a magnitude and a phase; that is, $H(j\omega) = H\angle\phi$. The plot of H versus ω is called the *magnitude frequency response*. The plot of ϕ versus ω is called the *phase frequency response*.

Example 14.1 For the RC circuit in Fig. 14.2(a), obtain the frequency response $\dot{V}_o(j\omega)/\dot{V}_s(j\omega)$. Let $v_s = V_m \cos\omega t$.

Solution:

$$H(j\omega) = \frac{\dot{V}_o(j\omega)}{\dot{V}_s(j\omega)} = \frac{1/(j\omega C)}{R+1/(j\omega C)}$$

$$= \frac{1}{1+j\omega RC}$$

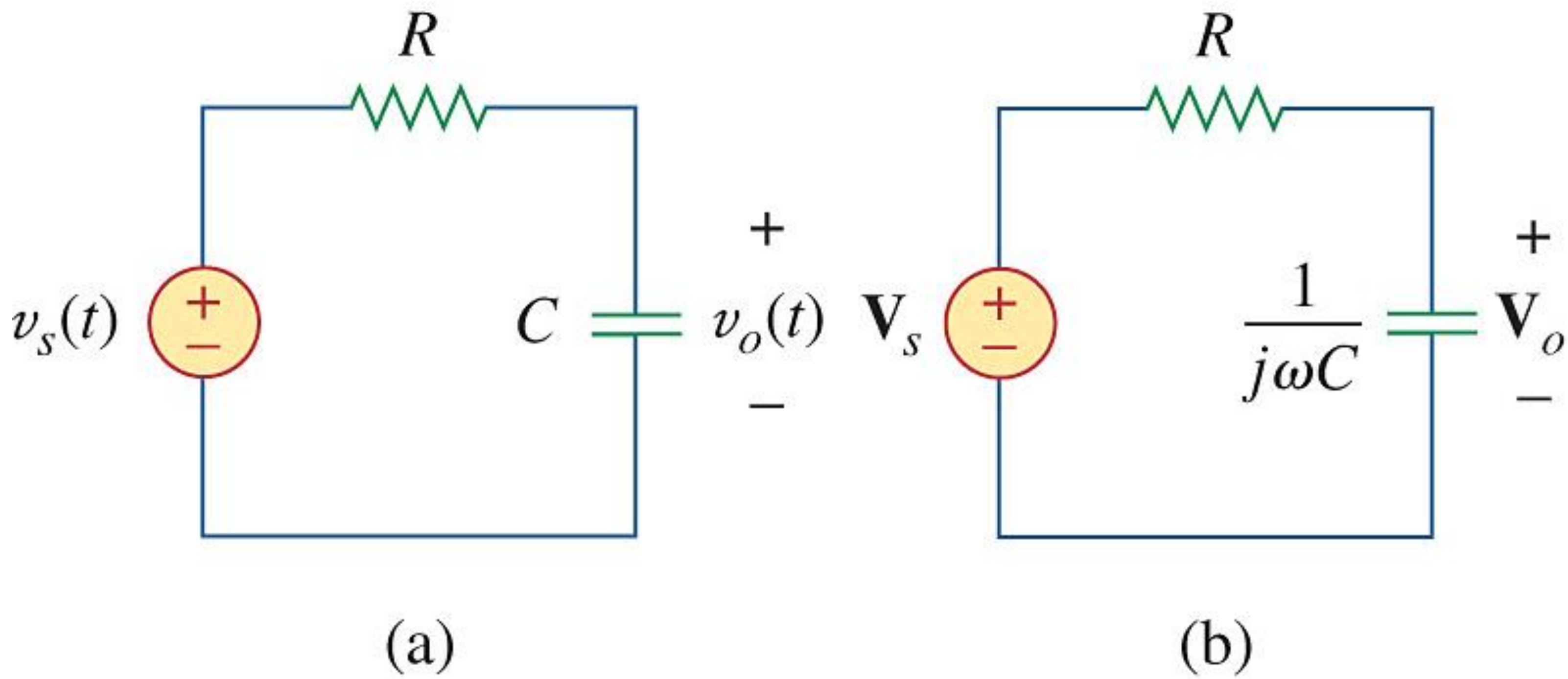


Figure 14.2

The magnitude and phase frequency responses are

$$H = \frac{1}{\sqrt{1+(\omega RC)^2}} = \frac{1}{\sqrt{1+(\omega/\omega_0)^2}}$$

$$\phi = -\tan^{-1}(\omega RC) = -\tan^{-1}(\omega/\omega_0)$$

where $\omega_o = \dfrac{1}{RC}$.

The plots of H an ϕ are shown in Fig. 14.3.

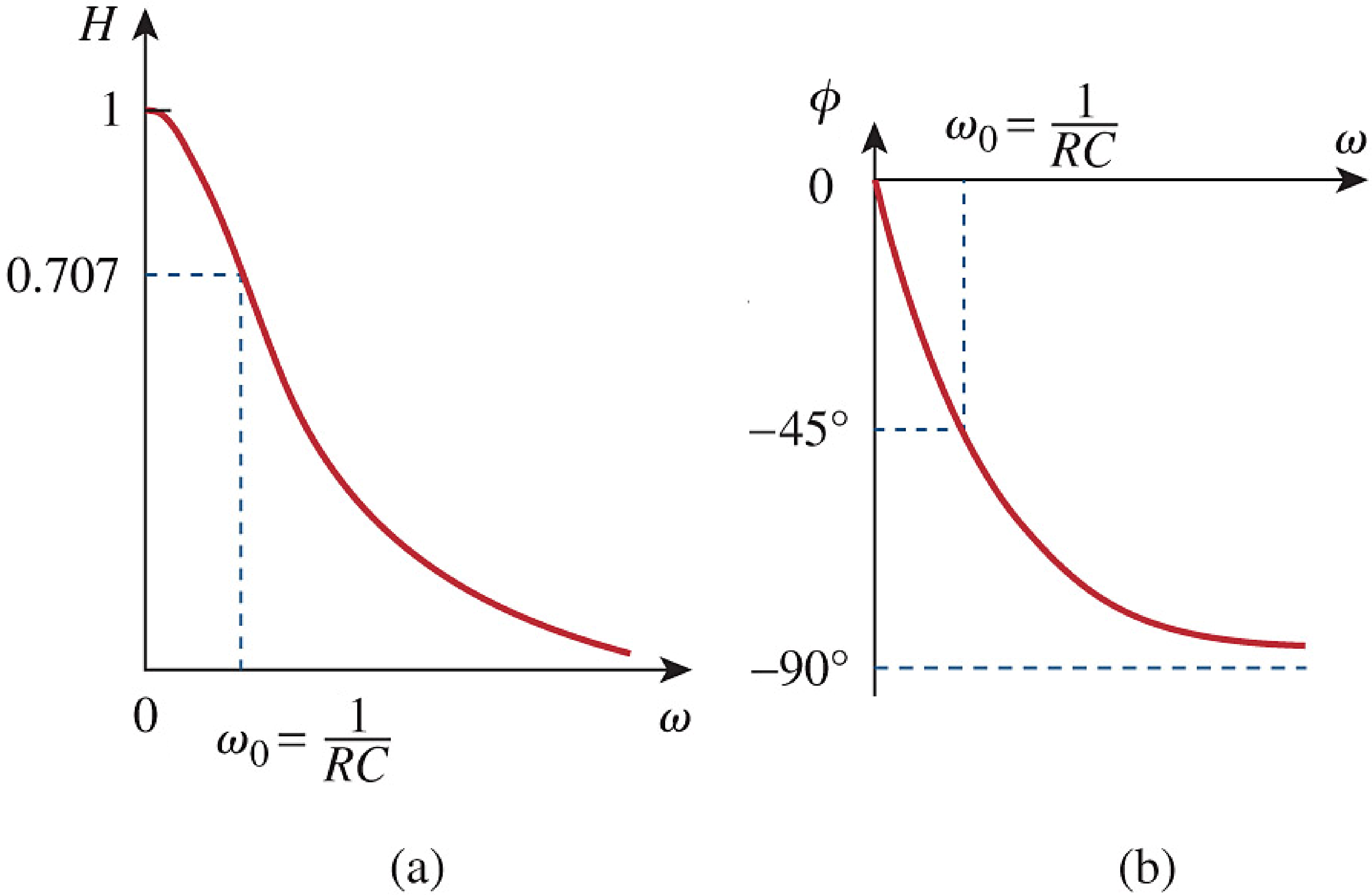


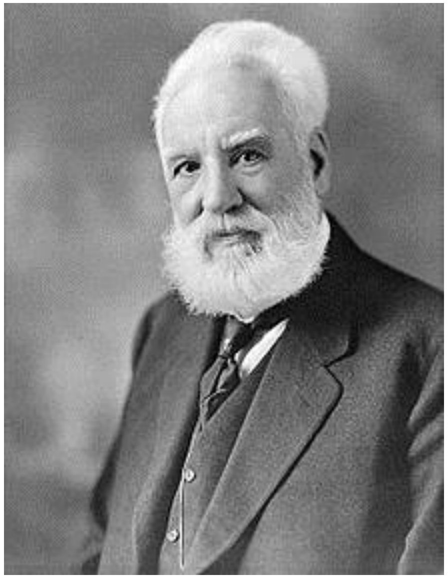
Figure 14.3 Frequency response of the RC circuit:
(a) amplitude response, (b) phase response.

14.3 The Decibel Scale

In communication systems, gain is measured in bels. Historically, the *bel* is used to measure the ratio of two levels of power or power gain G; that is,

$$G = \text{Number of bels} = \log_{10}\frac{P_2}{P_1}$$

Alexander Graham Bell (March 3, 1847 – August 2, 1922) was an eminent scientist, inventor, engineer and innovator who is credited with inventing the first practical telephone.

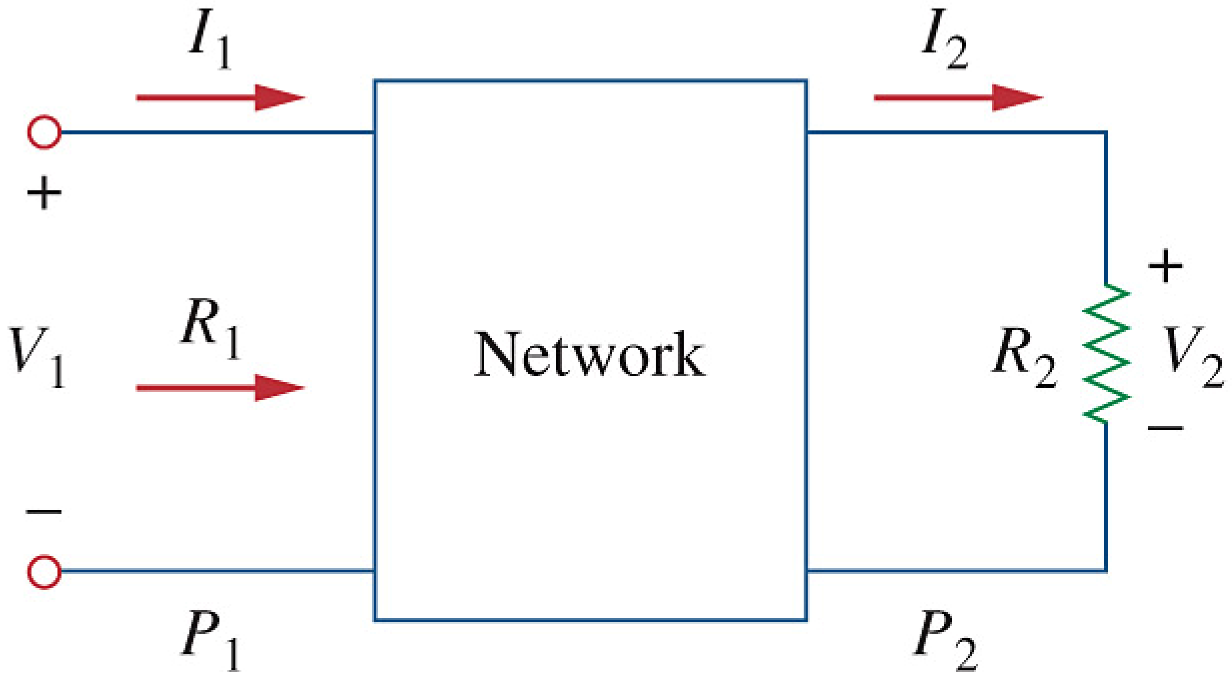


The *decibel* (dB) provides us with a unit of less magnitude. It is 1/10th of a bel and is given by

$$G_{dB} = 10 \log_{10} \frac{P_2}{P_1}$$

The gain G can be expressed in terms of voltage or current ratio. Consider the network shown in Fig. 14.8. If V_1 is the input voltage, V_2 is the output voltage, R_1 is the input resistance, and R_2 is the load resistance, then

$$G_{dB} = 10\log_{10}\frac{P_2}{P_1} = 10\log_{10}\left(\frac{V_2^2 / R_2}{V_1^2 / R_1}\right)$$



I_1
I_2
+
V_1
R_1
−
Network
+
R_2
V_2
−
P_1
P_2

For the case when $R_2 = R_1$, a condition that is often assumed when comparing voltage levels,

$$G_{dB} = 10\log_{10}\left(\frac{V_2}{V_1}\right)^2 = 20\log_{10}\frac{V_2}{V_1}$$

Similarly, for $R_1 = R_2$,

$$G_{dB} = 20\log_{10}\frac{I_2}{I_1}$$

14.4 Bode Plots

The frequency range required in plotting frequency response is often so wide that it is inconvenient to use a linear scale for the frequency axis. *Bode plots* are semilog plots in which the magnitude in decibels is plotted against the logarithm of the frequency, the phase in degrees is plotted against the logarithm of the frequency.

The frequency response may be written in factored form:

$$H(j\omega) = \frac{N(j\omega)}{D(j\omega)} = \frac{\sum_{m=0}^{M} b_m (j\omega)^m}{\sum_{n=0}^{N} a_n (j\omega)^n}$$

$$= \frac{b_M (j\omega)^M + b_{M-1}(j\omega)^{M-1} + \cdots + b_1(j\omega) + b_0}{a_N (j\omega)^N + a_{N-1}(j\omega)^{N-1} + \cdots + a_1(j\omega) + a_0}$$

$$= \frac{b_M}{a_N} \frac{(j\omega)^M + \frac{b_{M-1}}{b_M}(j\omega)^{M-1} + \cdots + \frac{b_0}{b_M}}{(j\omega)^N + \frac{a_{N-1}}{a_N}(j\omega)^{N-1} + \cdots + \frac{a_0}{a_N}}$$

$$= \frac{b_M}{a_N} \frac{(j\omega + z_1)(j\omega + z_2)\cdots(j\omega + z_M)}{(j\omega + p_1)(j\omega + p_2)\cdots(j\omega + p_N)}$$

$$= \frac{\tilde{b}\prod_{m=1}^{M}(j\omega + z_m)}{\prod_{n=1}^{N}(j\omega + p_n)}$$

All of the coefficients of $N(j\omega)$ are real, therefore the roots of $N(j\omega)=0$ must be either real or appear in complex conjugate pairs. That implies $z_m,\ m=1,2,\ldots,M$ are either real or appear in complex conjugate pairs. The same is true for $p_n,\ n=1,2,\ldots,N$.

If z_m is real, we write

$$j\omega + z_m = \begin{cases} z_m(1 + j\omega / z_m), & z_m \neq 0 \\ j\omega, & z_m = 0 \end{cases}$$

If z_m is comlex, we groupe factors $(j\omega + z_m)$ and $(j\omega + z_m^*)$ into real-valued quadratic factor

$$(j\omega + z_m)(j\omega + z_m^*) = (j\omega)^2 + (z_m + z_m^*)j\omega + z_m z_m^*$$

$$= (j\omega)^2 + 2\,\mathrm{Re}(z_m)(j\omega) + |z_m|^2$$

$$= |z_m|^2 \left[1 + \frac{2\,\mathrm{Re}(z_m)(j\omega)}{|z_m|^2} + \frac{(j\omega)^2}{|z_m|^2} \right]$$

$$= |z_m|^2 \left[1 + \frac{2\,\mathrm{Re}(z_m)}{|z_m|} \left(\frac{j\omega}{|z_m|} \right) + \left(\frac{j\omega}{|z_m|} \right)^2 \right]$$

$$= \omega_k^2 \left[1 + 2\zeta_k (j\omega / \omega_k) + (j\omega / \omega_k)^2 \right]$$

where $\omega_k = |z_k|$ and $\zeta_k = \mathrm{Re}(z_m) / |z_m|$. It is evident that $0 \le |\zeta_k| \le 1$.

So, $H(j\omega)$ may be represented in the *standard form*:

$$H(j\omega) = K(j\omega)^{\pm 1} \times$$

$$\frac{(1 + j\omega / z_1)[1 + j2\zeta_1(j\omega / \omega_k) + (j\omega / \omega_k)^2]\cdots}{(1 + j\omega / p_1)[1 + j2\zeta_2(j\omega / \omega_n) + (j\omega / \omega_n)^2]\cdots}$$

We will now plot straight-line-segment approximations (called *asymptotes*) associated with the factors in $H(j\omega)$.

(1) For the gain K,

$$H_{dB} = 20\log_{10}|K|$$

$$\phi = \begin{cases} 0^\circ, & K > 0 \\ 180^\circ, & K < 0 \end{cases}$$

as shown in Fig. 14.9.

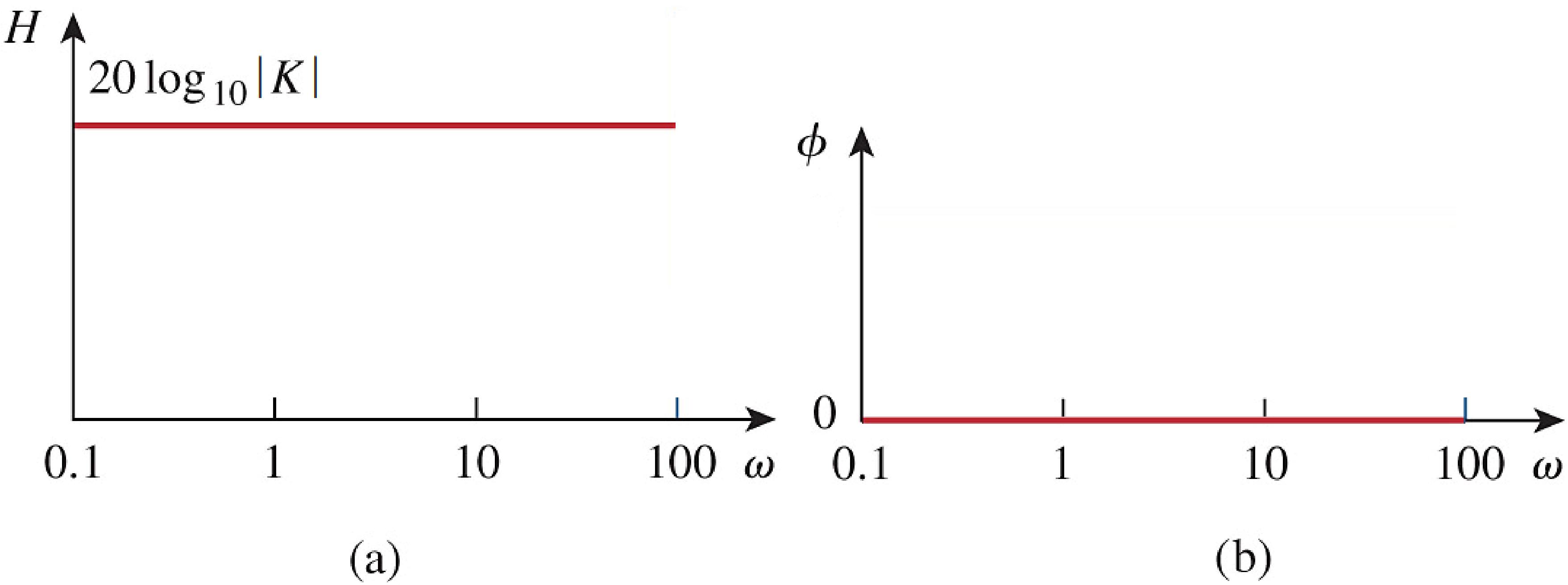


Figure 14.9 Bode plots for gain K: (a) magnitude plot, (b) phase plot.

(2) For $j\omega$,

$$H_{dB} = 20\log_{10}|j\omega| = 20\log_{10}\omega$$

$$\phi = 90°$$

These are shown in Fig. 14.10, where we notice that the slope of the magnitude plot is 20 dB/decade, where the word *decade* means a group or series of ten.

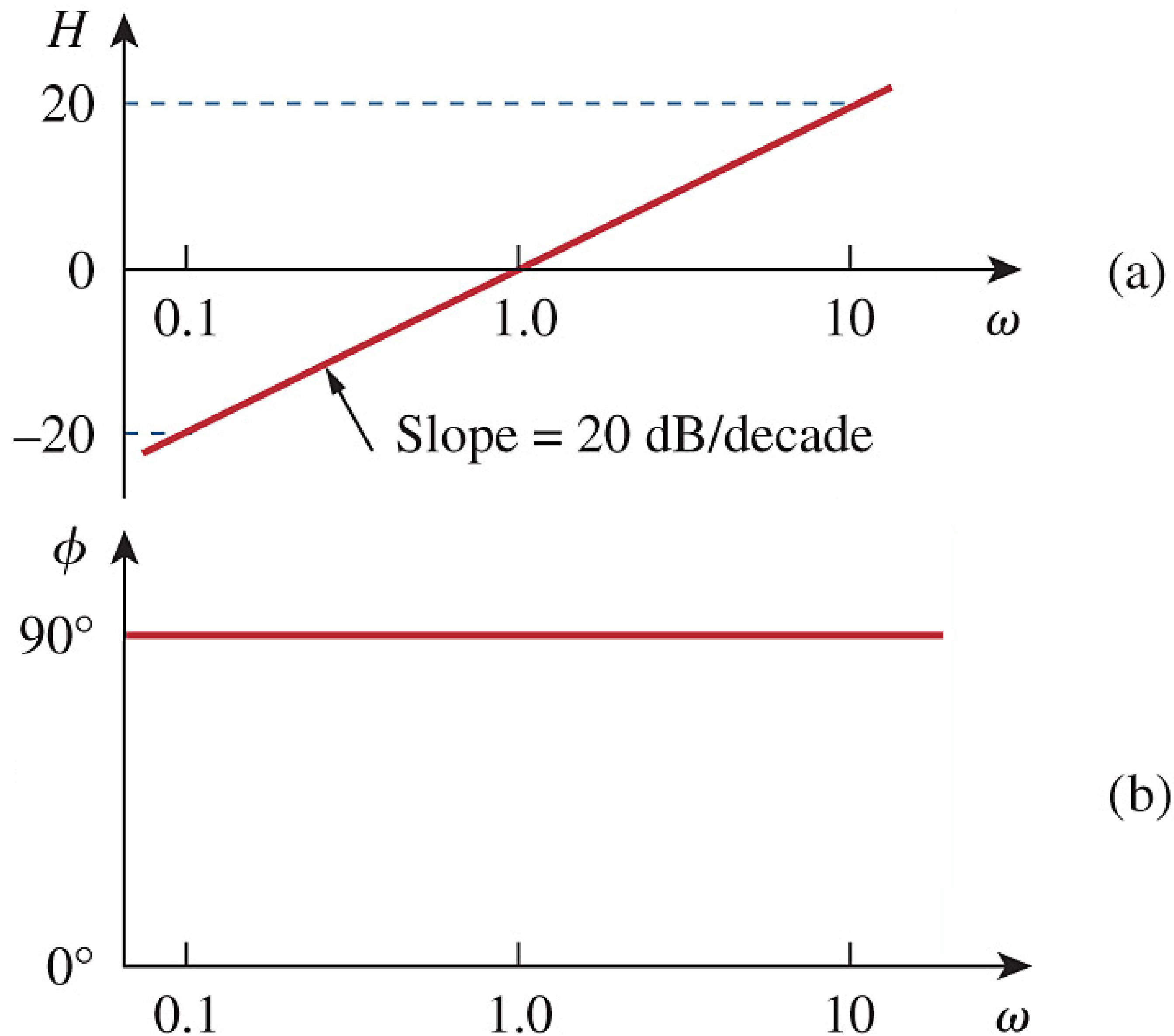


Figure 14.10 Bode plots for $j\omega$.

The Bode plots for $(j\omega)^{-1}$ are similar except that the slope of the magnitude plot is -20 dB/decade while the phase is $-90°$. In general, for $(j\omega)^N$, where N is an interger, the magnitude plot will have a slope of $20N$ dB/decade, while the phase is $90N°$.

(3) For $(1 + j\omega / z_1)$,

$$H_{dB} = 20\log_{10}\left|1 + j\omega / z_1\right|$$

$$\phi = \tan^{-1}(\omega / z_1)$$

We notice that

$$\begin{cases} H_{dB} = 20\log_{10} 1 = 0 \\ \phi = 0^\circ \end{cases}, \ \omega \to 0$$

$$\begin{cases} H_{dB} = 20\log_{10}(\omega / z_1) \\ \phi = 90^\circ \end{cases}, \ \omega \to \infty$$

As a straight-line approximation, we let

$$H_{dB} = \begin{cases} 0, & \omega \le z_1 \\ 20\log_{10}(\omega / z_1), & \omega \ge z_1 \end{cases}$$

$$\phi = \begin{cases} 0^\circ, & \omega \le 0.1z_1 \\ 45^\circ + 45^\circ \log_{10}(\omega / z_1), & 0.1z_1 \le \omega \le 10z_1 \\ 90^\circ, & \omega \ge 10z_1 \end{cases}$$

as shown in Fig. 14.11. The frequency $\omega =$ z_1 is called the *corner frequency* or *break frequency*.

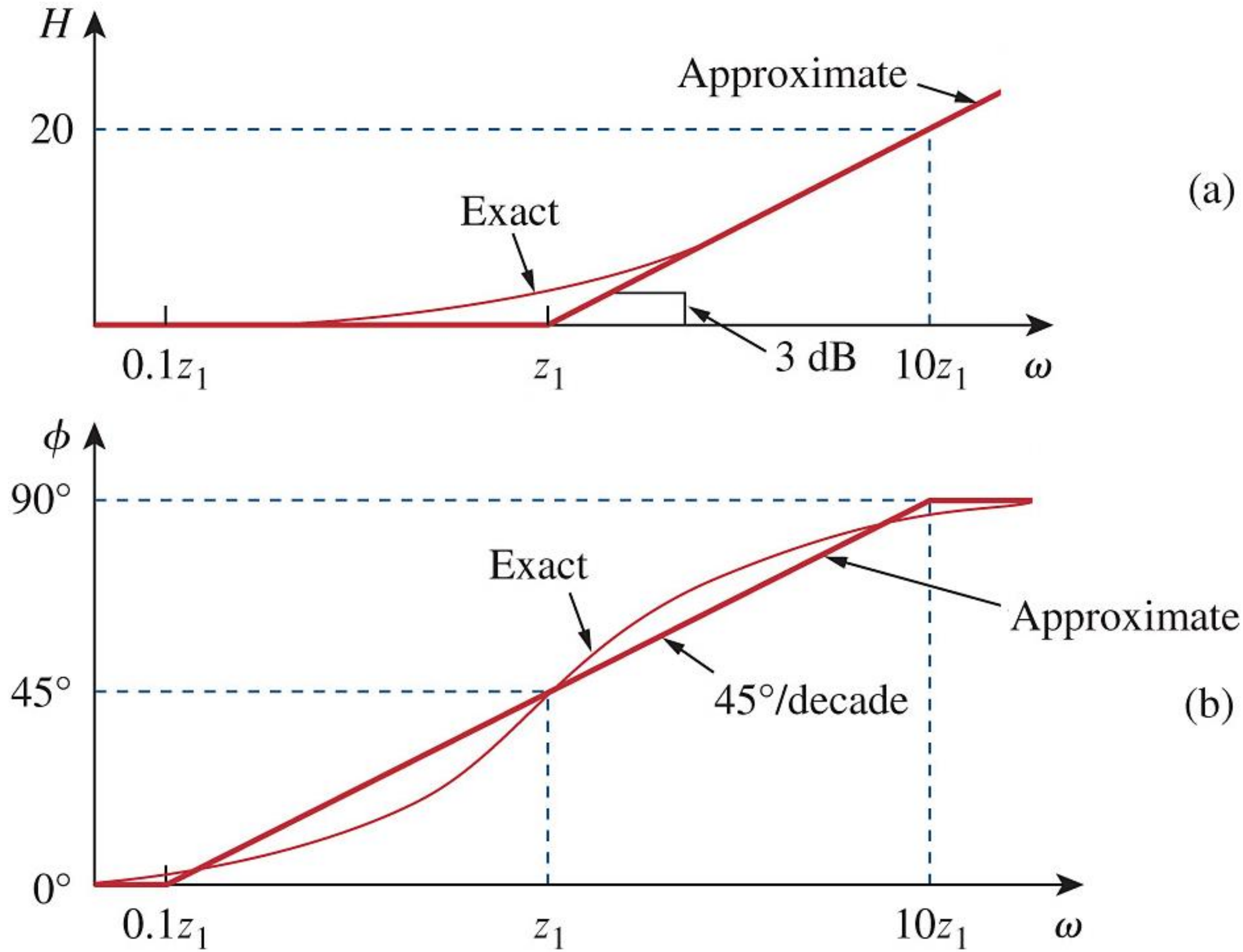


Figure 14.11 Bode plots for $1+j\omega/z_1$: (a) magnitude plot, (b) phase plot.

The Bode plots for $1/(1+j\omega/p_1)$ are similar to those in Fig. 14.11 except that the corner frequency is at $\omega = p_1$, the magnitude has a slope of -20 dB/decade, and the phase has a slope of $-45°$ per decade.

In general, for $(1+j\omega/z_1)^N$, where N is an interger,

$$H_{dB}=\begin{cases}0, & \omega\le z_1\\ 20N\log_{10}(\omega/z_1), & \omega\ge z_1\end{cases}$$

$$\phi=\begin{cases}0^\circ, & \omega\le 0.1z_1\\ 45N^\circ+45N^\circ\log_{10}(\omega/z_1), & 0.1z_1\le\omega\le 10z_1\\ 90N^\circ, & \omega\ge 10z_1\end{cases}$$

(4) For $1/[1+2\zeta_2(j\omega/\omega_n)+(j\omega/\omega_n)^2]$,

$$H_{dB} = -20\log_{10}\left|1+2\zeta_2(j\omega/\omega_n)+(j\omega/\omega_n)^2\right|$$

$$\phi = -\tan^{-1}\left(\frac{2\zeta_2\omega/\omega_n}{1-\omega^2/\omega_n^2}\right)$$

We notice that

$$\left\{\begin{array}{l} H_{dB} = -20\log_{10}1 = 0 \\ \phi = 0^\circ \end{array}\right., \ \omega \to 0$$

$$\left\{\begin{array}{l} H_{dB} = -40\log_{10}(\omega/\omega_n) \\ \phi = -180^\circ \end{array}\right., \ \omega \to \infty$$

As a straight-line approximation, we let

$$H_{dB} = \begin{cases} 0, & \omega \le \omega_n \\ -40\log_{10}(\omega/\omega_n), & \omega \ge \omega_n \end{cases}$$

$$\phi = \begin{cases} 0^\circ, & \omega \le 0.1\omega_n \\ -90^\circ - 90^\circ\log_{10}(\omega/\omega_n), & 0.1\omega_n \le \omega \le 10\omega_n \\ -180^\circ, & \omega \ge 10\omega_n \end{cases}$$

as shown in Fig. 14.12. The frequency $\omega = \omega_n$ is called the *corner frequency* or *break frequency*.

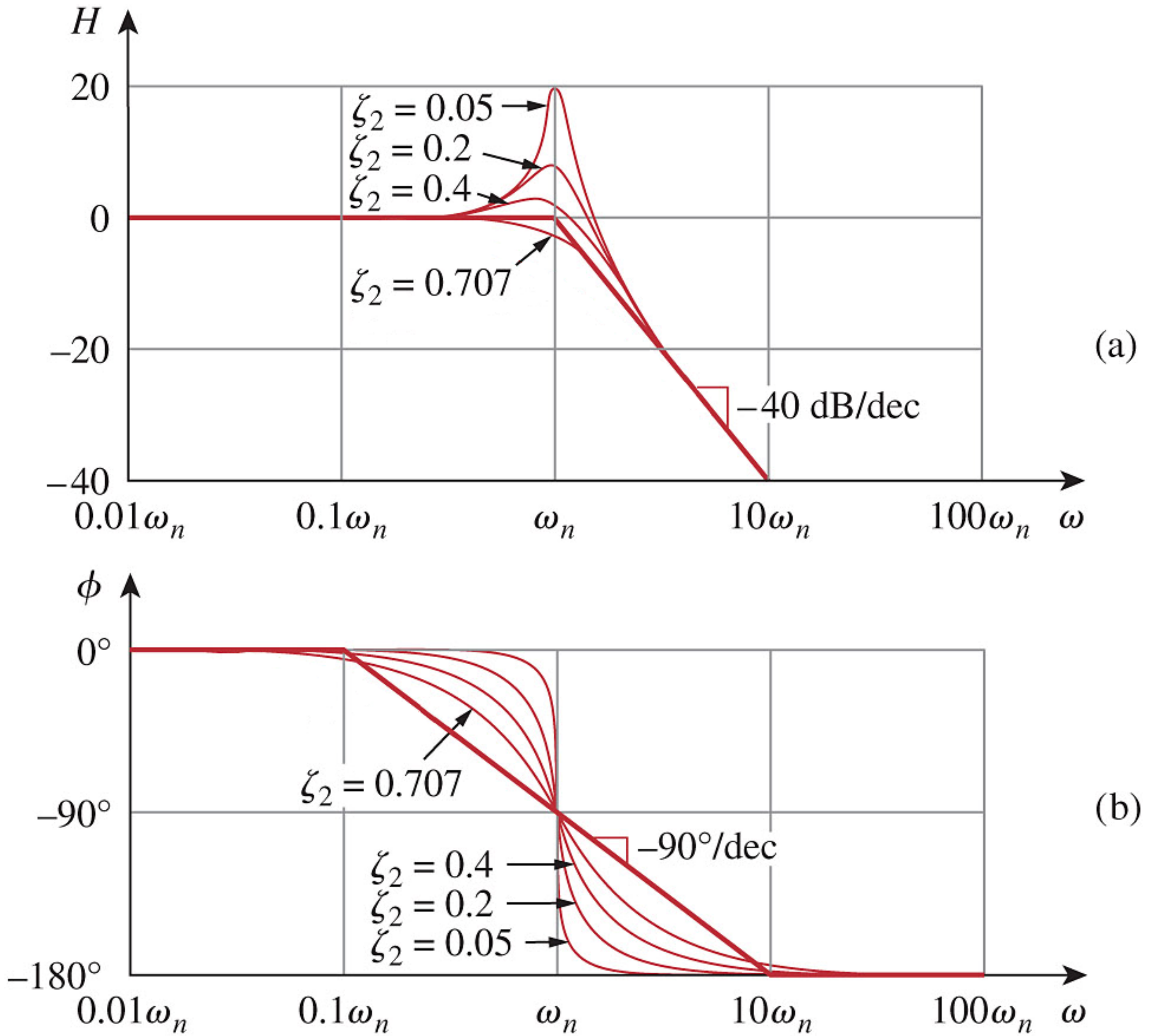


Figure 14.12 Bode plots of $1/[1+2\zeta_2(j\omega/\omega_n)+(j\omega/\omega_n)^2]$: (a) magnitude plot, (b) phase plot.

The Bode plots for $[1+2\zeta_1(j\omega/\omega_k)+(j\omega/\omega_k)^2]$ are similar to those in Fig. 14.12 except that the corner frequency is at $\omega=\omega_k$, the magnitude has a slope of 40 dB/decade, and the phase has a slope of 90° per decade.

Practice Problem 14.3 Draw the Bode plots for

$$H(j\omega) = \frac{5(j\omega + 2)}{j\omega(j\omega + 10)}$$

Solution :

$$H(j\omega) = \frac{(1 + j\omega / 2)}{j\omega(1 + j\omega / 10)}$$

$$H_{dB} = 20\log_{10}\left|\frac{(1 + j\omega / 2)}{j\omega(1 + j\omega / 10)}\right|$$

$$= 20\log_{10}|1 + j\omega/2| + 20\log_{10}\frac{1}{|j\omega|}$$

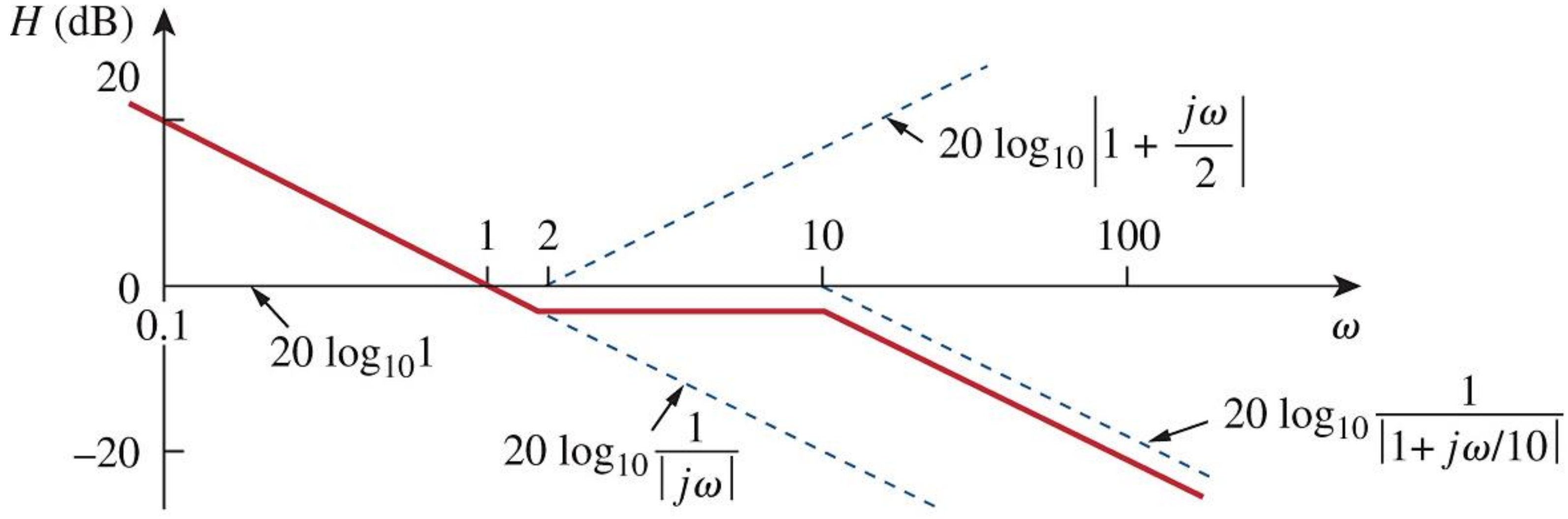
$$+20\log_{10}\frac{1}{|1 + j\omega/10|}$$

$$= 20\log_{10}|1 + j\omega/2| - 20\log_{10}|j\omega|$$

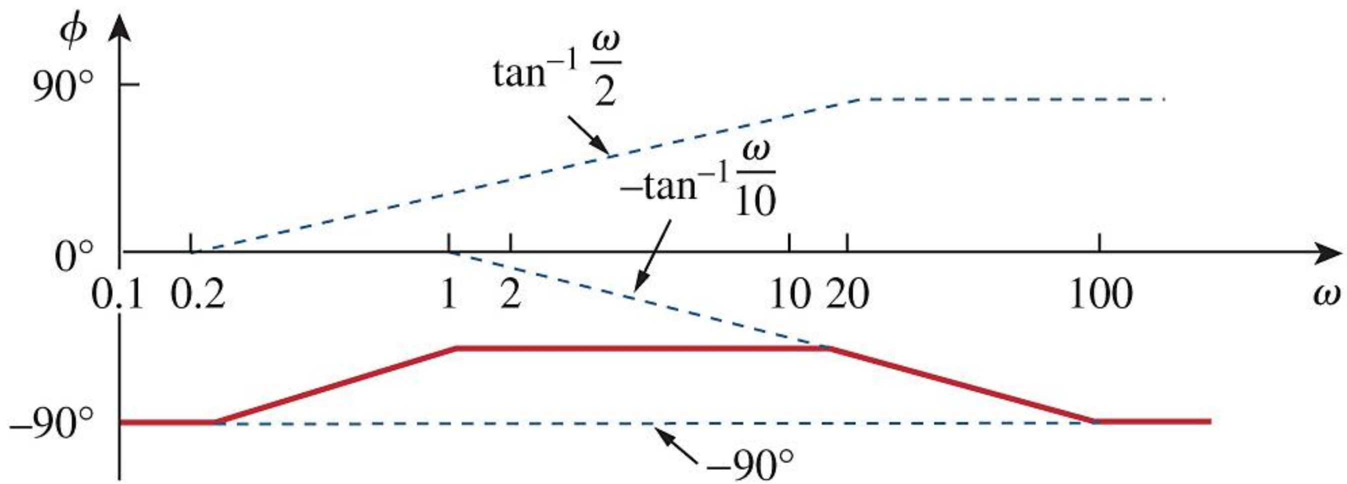
$$-20\log_{10}|1 + j\omega/10|$$

$$\phi = \tan^{-1}\left(\omega/2\right) - 90° - 2\tan^{-1}(\omega/10)$$

The Bode plots are in Fig. 14.14.



(a)



(b)

Figure 14.14

Example 14.4 Obtain the Bode plots for

$$H(j\omega) = \frac{j\omega + 10}{j\omega(j\omega + 5)^2}$$

Solution :

$$H(j\omega) = \frac{0.4(1 + j\omega/10)}{j\omega(1 + j\omega/5)^2}$$

$$H_{dB} = 20\log_{10}\left|\frac{0.4(1 + j\omega/10)}{j\omega(1 + j\omega/5)^2}\right|$$

$$= 20\log_{10} 0.4 + 20\log_{10}\left|1 + j\omega/10\right| +$$

$$-20\log_{10}\left|j\omega\right| - 40\log_{10}\left|1 + j\omega/5\right|$$

$$\phi = 0° + \tan^{-1}\left(\omega/10\right) - 90° - 2\tan^{-1}(\omega/5)$$

The Bode plots are in Fig. 14.15.

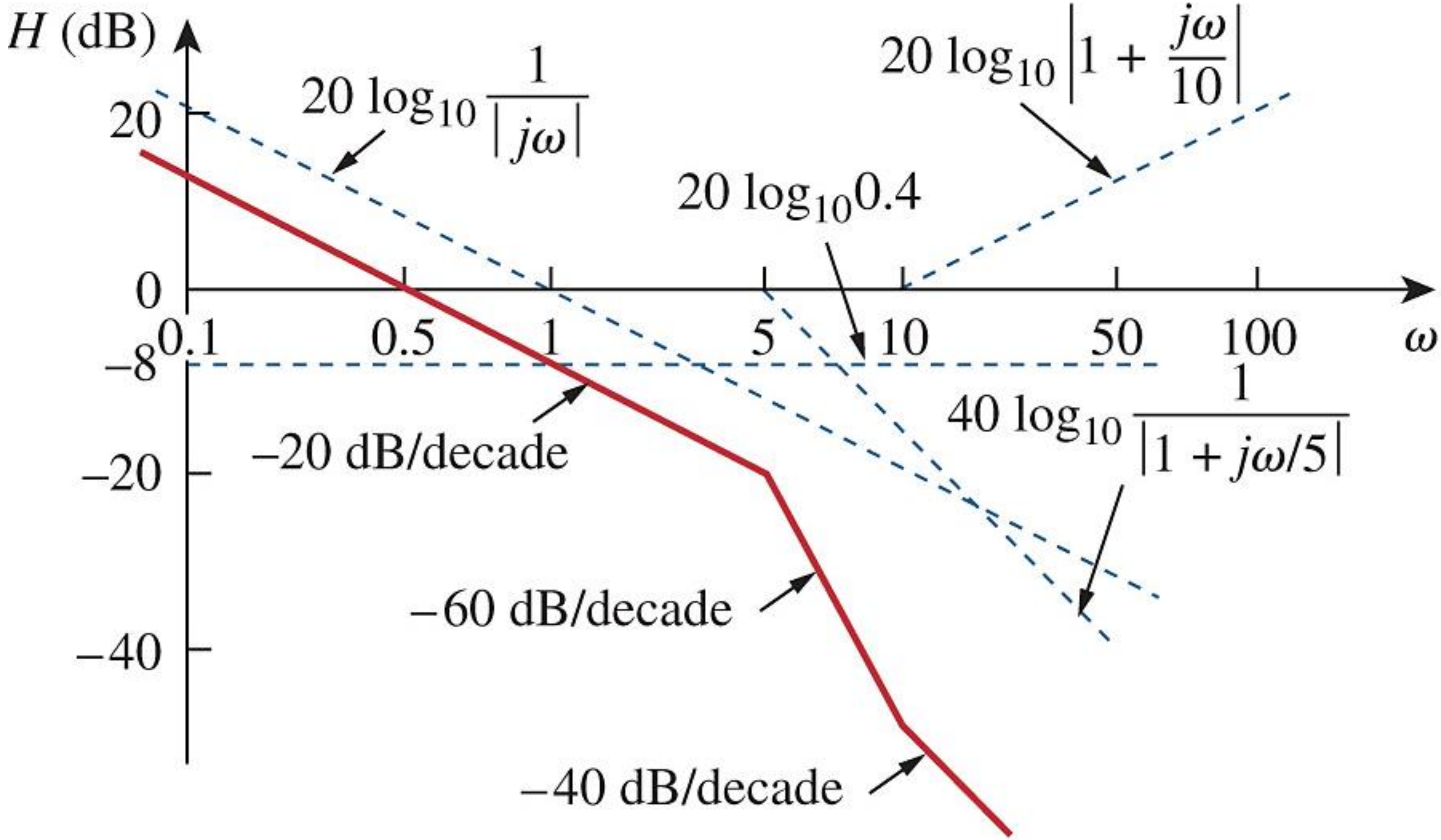


Figure 14.15(a)

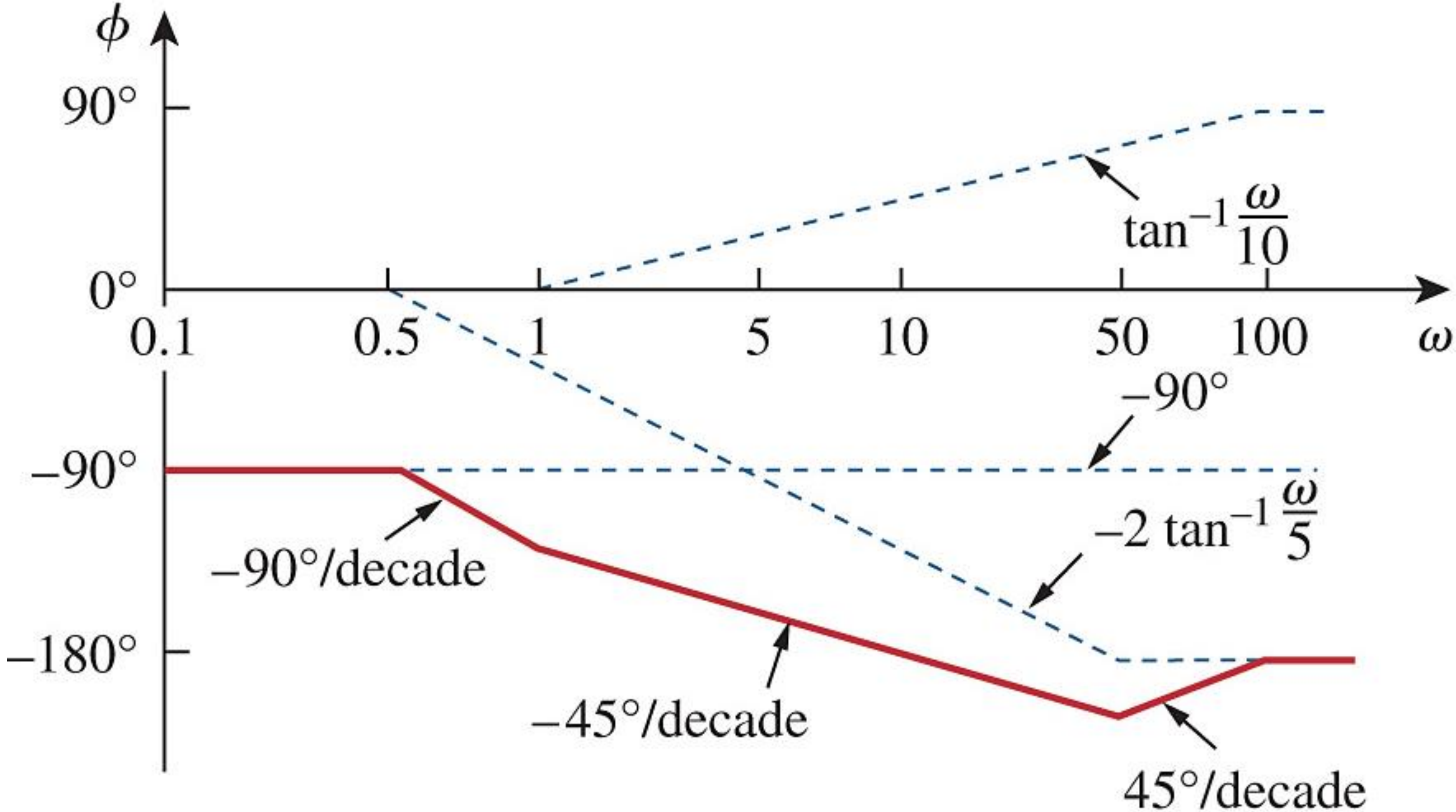


Figure 14.15(b)

Example 14.5 Draw the Bode plots for

$$H(j\omega) = \frac{j\omega + 1}{(j\omega)^2 + 12(j\omega) + 100}$$

Solution :

$$H(j\omega) = \frac{0.01(1 + j\omega/1)}{1 + 0.12(j\omega) + (j\omega/10)^2}$$

$$= \frac{0.01(1 + j\omega/1)}{1 + 2 \times 0.6(j\omega/10) + (j\omega/10)^2}$$

$$H_{dB} = 20\log_{10}\left|\frac{0.01(1+j\omega/1)}{1+2\times 0.6(j\omega/10)+(j\omega/10)^2}\right|$$

$$= 20\log_{10} 0.01 + 20\log_{10}\left|1+j\omega/1\right|$$

$$-20\log_{10}\left|1+2\times 0.6(j\omega/10)+(j\omega/10)^2\right|$$

$$\phi = 0° + \tan^{-1}\left(\omega/1\right) - \tan^{-1}\left[\frac{2\times 0.6(\omega/10)}{1-(\omega/10)^2}\right]$$

The Bode plots are in Fig. 14.17.

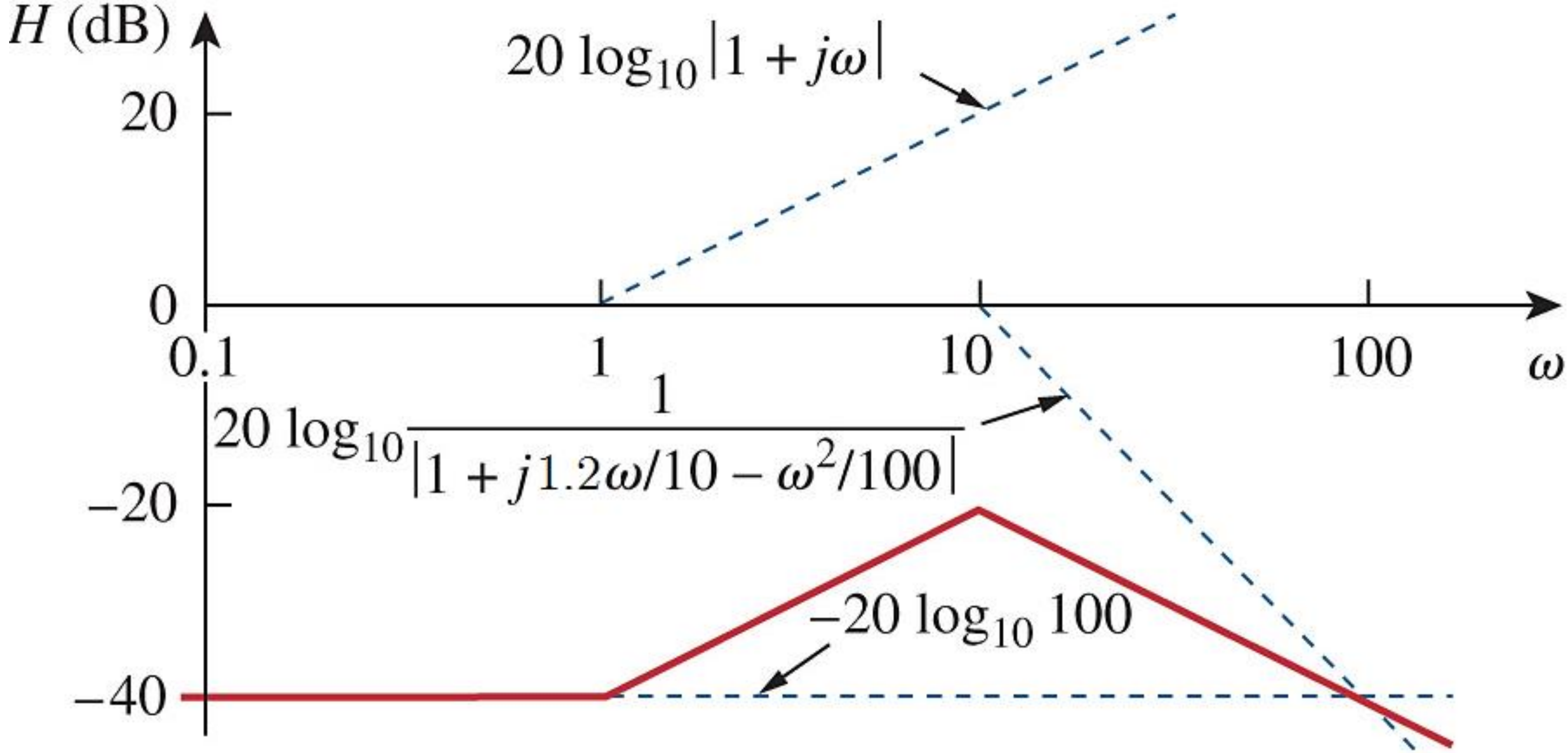


Figure 14.17(a)

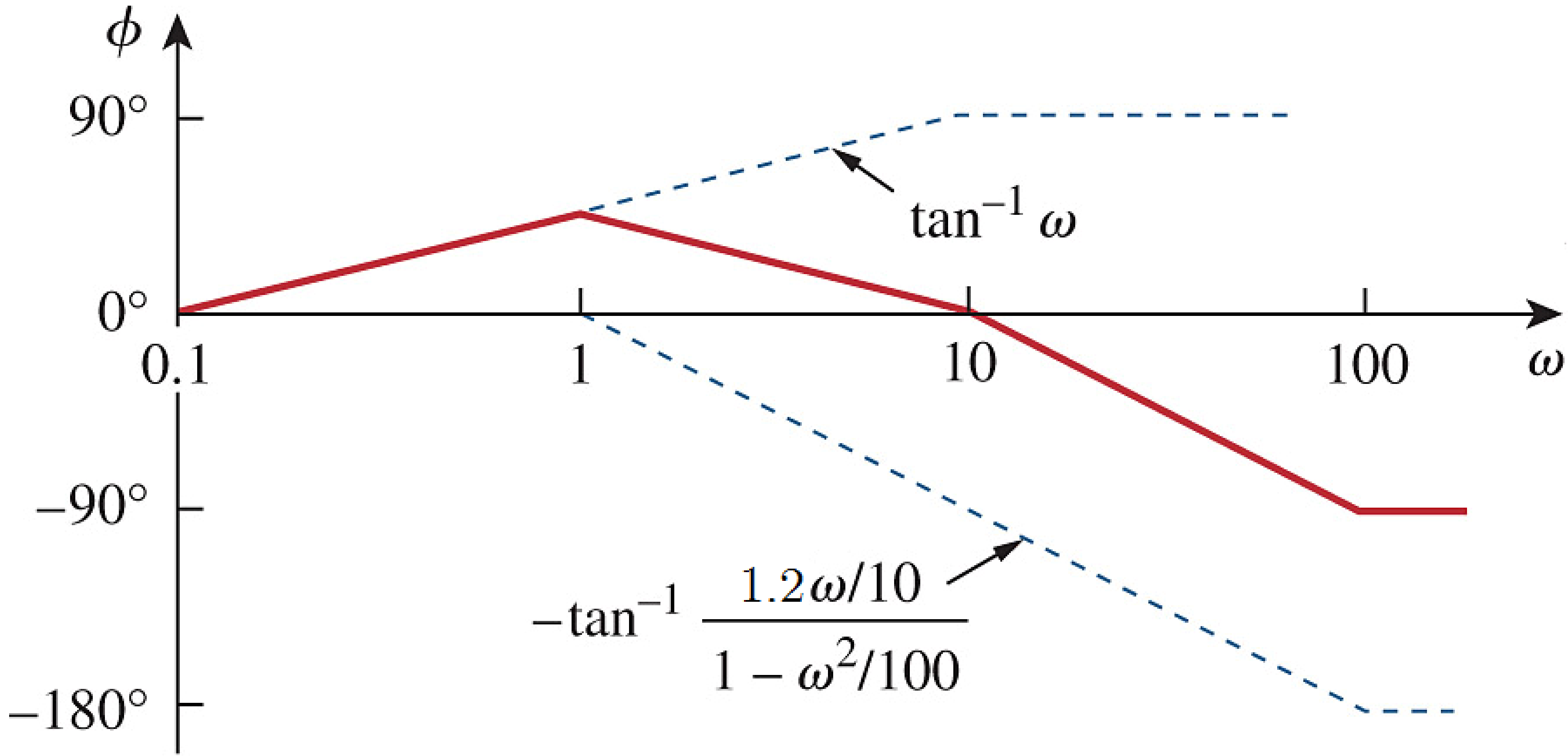


Figure 14.17(b)

14.5 Series Resonance

Resonance occurs in any circuits that has at least one inductor and one capacitor. Consider the series *RLC* circuit shown in Fig. 14.21. The input impedance is

$$Z = R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

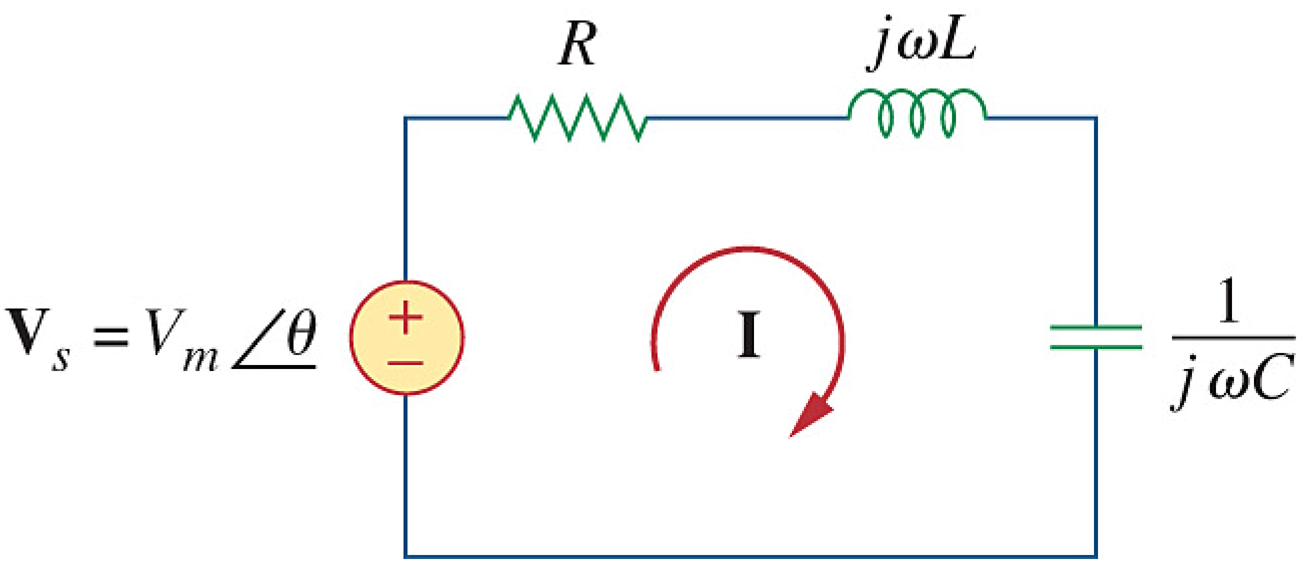


Figure 14.21 The series resonant circuit.

Resonance results when $\operatorname{Im}(Z) = 0,$ that is, the capacitive and inductive reactances are equal in magnitude. The value of ω that satisfies this condition is called the *resonant frequency* ω_0. Thus, the resonance condition is

$$\omega_0 L = \frac{1}{\omega_0 C}$$

$$\omega_0 = 2\pi f_0 = \frac{1}{\sqrt{LC}}$$

Note that at resonance:

1. The impedance is purely resistive, thus, $Z = R$. In other words, the LC series combination acts like a short circuit, and the entire voltage is across R.

2. The voltage $\dot{V}_s$ and the current $\dot{I}$ are in phase.

3. The magnitude of the current is maximum.

Proof :

The circuit's cuurent magnitude

$$I = \left|\frac{\dot{V}_s}{Z}\right| = \left|\frac{V_m\angle\theta}{R + j(\omega L - \frac{1}{\omega C})}\right|$$

$$= \frac{V_m}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} \le \frac{V_m}{R} = I_{\max}$$

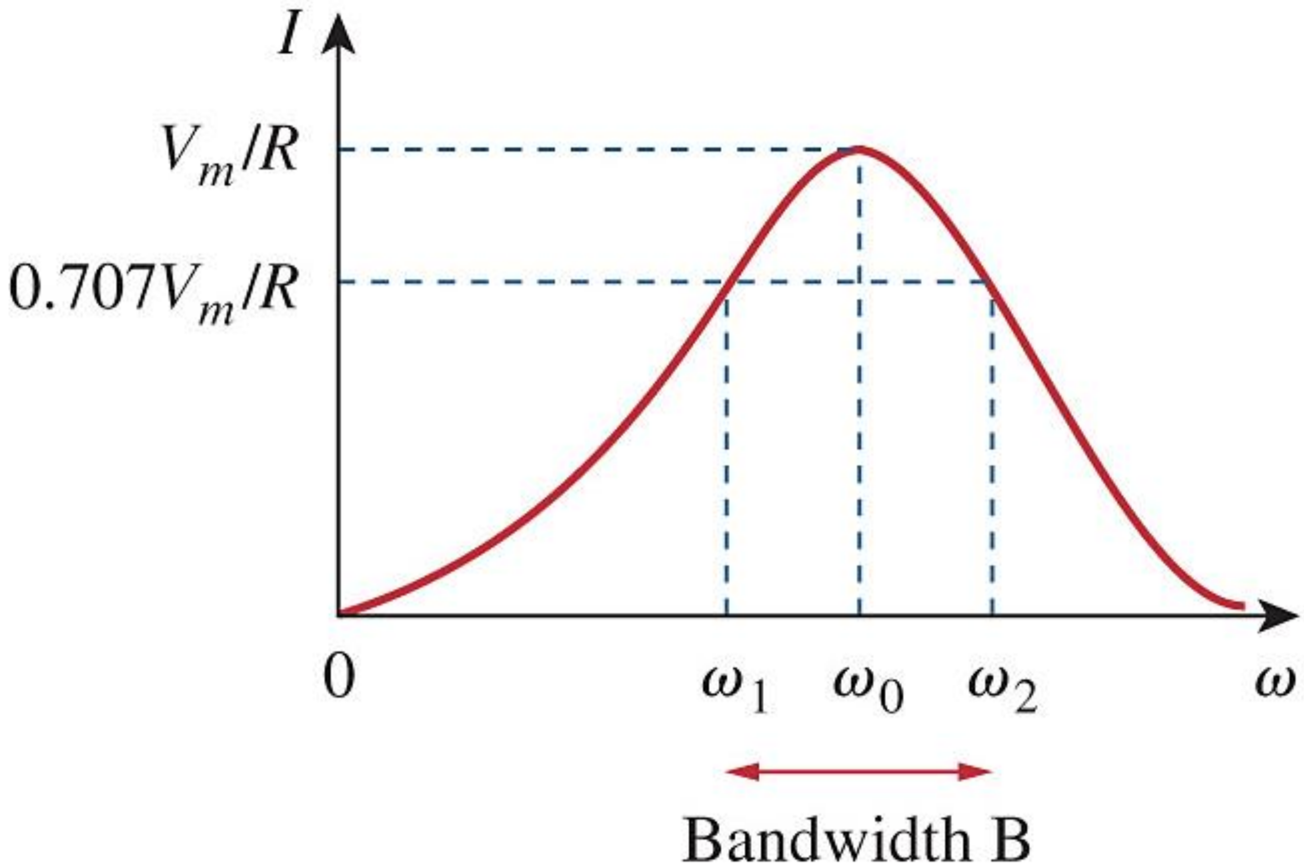


Figure 14.22 The current amplitude versus frequency for the series resonant circuit of Fig. 14.21.

The average power dissipated by the *RLC* circuit is

$$P(\omega)=\frac{1}{2}I^2R=\frac{1}{2}\frac{V_m^2}{R^2+\left(\omega L-1/(\omega C)\right)^2}R$$

The highest power dissipated occurs at resonance,

$$P(\omega_0)=\frac{1}{2}I_{\max}^2R=\frac{1}{2}\frac{V_m^2}{R}$$

At certain frequencies $\omega = \omega_1, \omega_2$, the dissipated power is half the maximum value; that is,

$$P(\omega_1) = P(\omega_2) = \frac{1}{2} P(\omega_0) = \frac{V_m^2}{4R}$$

Hence, ω_1 and ω_2 are called the *half* - *power frequencies*. $I(\omega_1) = I(\omega_2) = \dfrac{I_{\max}}{\sqrt{2}}$.

The half-power frequencies are obtained by solving the equation

$$\frac{1}{2}\frac{V_m^2}{R^2+\left(\omega L-1/(\omega C)\right)^2}R=\frac{V_m^2}{4R}$$

$$\left(\omega L-1/(\omega C)\right)^2=R^2$$

$$\omega L-1/(\omega C)=\pm R$$

$$LC\omega^2\mp RC\omega-1=0$$

$$\omega = \frac{\pm RC \pm \sqrt{(RC)^2 + 4LC}}{2LC}$$

$$= \pm \frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

ω_1 and ω_2 are positive,

$$\omega = \pm \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

We can relate the half-power frequencies and the resonant frequency,

$$\omega_1\omega_2 = \frac{1}{LC} = \omega_0^2 \Rightarrow \omega_0 = \sqrt{\omega_1\omega_2}$$

showing that the resonant frequency is the geometric mean of the half-power frequencies. Notice that ω_1 and ω_2 are in general not symmetrical around ω_0 because the frequency response is not generally symmetrical. However, if the frequency axis is a logarithm, we have $\log_{10}\omega_0 = (\log_{10}\omega_1 + \log_{10}\omega_2)/2$.

The *half - power bandwidth* is defined as the difference between the two half-power frequencies,

$$B = \omega_2 - \omega_1 = \frac{R}{L}$$

The "sharpness" of the resonance is measured quantitatively by the *quality factor Q*.

The quality factor Q can be defined by

$$Q = 2\pi \frac{E_s}{E_d}$$

where E_s is the peak energy stored in the circuit and E_d is the energy dissipated in one period at resonance.

$$Q = 2\pi \frac{\frac{1}{2} L I_{\max}^2}{\frac{1}{2} I_{\max}^2 R (1/\ f_0)} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC}$$

The relationship between B and Q is

$$B = \frac{R}{L} = \frac{\omega_0}{Q} \Rightarrow Q = \frac{\omega_0}{B}$$

Thus, the quality factor of an RLC circuit can be defined as the ratio of its resonant frequency to its bandwidth.

As illustrated in Fig. 14.23, the higher the value of Q, the more selective the circuit is. The *selectivity* of an RLC circuit is the ability of the circuit to respond to a certain frequency and discriminate against all other frequencies.

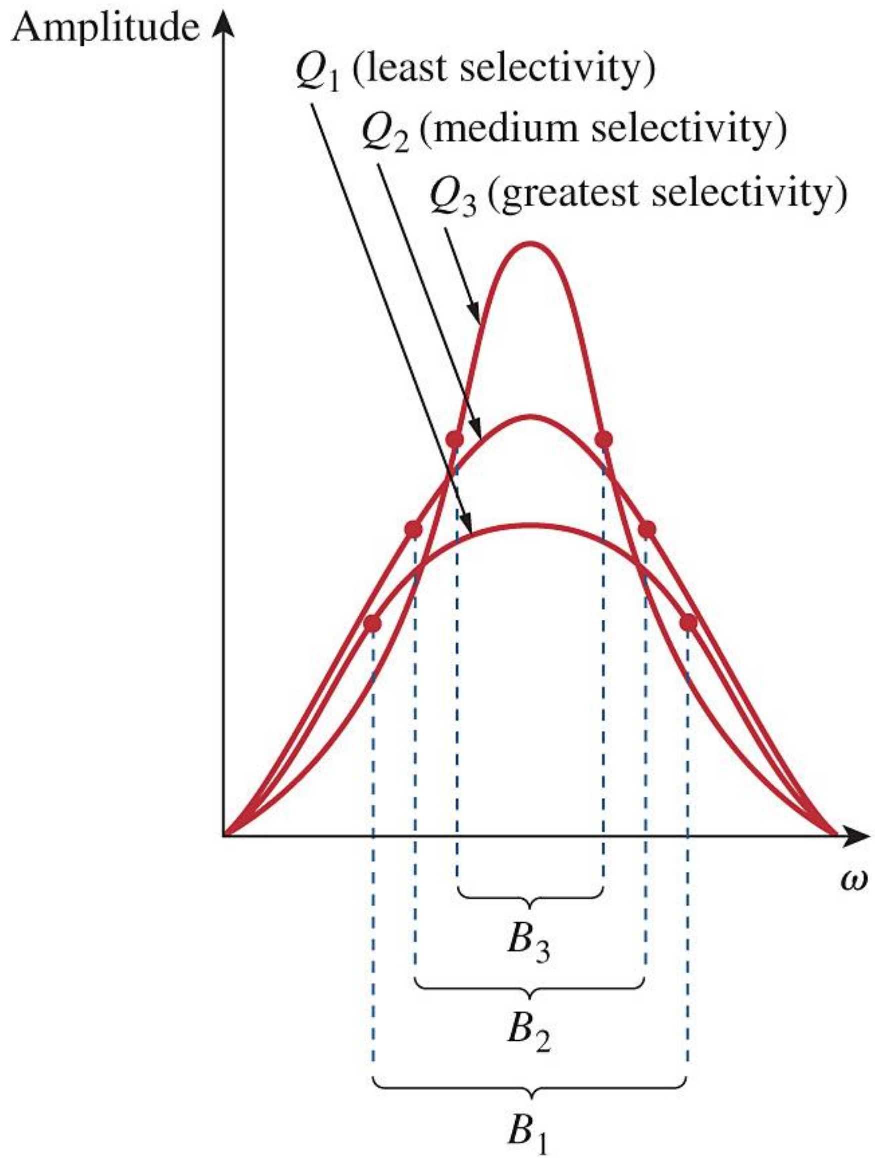


Figure 14.23 The higher the circuit Q, the smaller the bandwidth.

A resonant circuit is designed to operate at or near its resonant frequency. It is said to be a *high - Q circuit* when $Q \geq 10$. For high-Q circuits,

$$\omega_{1,2} = \mp \frac{\omega_0}{2Q} + \omega_0 \sqrt{\left(\frac{1}{2Q}\right)^2 + 1} \approx \mp \frac{\omega_0}{2Q} + \omega_0$$

$$= \mp \frac{B}{2} + \omega_0$$

The inductor and capacitor voltages can be much more than the source voltage at resonance.

$$V_L = \frac{V_m}{R}\omega_0 L = \frac{V_m}{R}\frac{1}{\omega_0 C} = V_C$$

For high-Q circuits, $V_L = V_C = QV_m \gg V_m$.

Practice Problem 14.7 A series-connected circuit has $R = 4\ \Omega$ and $L = 25$ mH. (a) Calculate the value of C that will produce a quality factor of 50. (b) Find ω_1, ω_2, and B. (c) Determine the average power dissipated at $\omega = \omega_0$, ω_1, ω_2. Take $V_m = 100$ V.

Solution :

$$\text{(a) } Q = \frac{\omega_0 L}{R}$$

$$\omega_0 = \frac{QR}{L} = \frac{50 \times 4}{25 \times 10^{-3}} = 8000 \text{ (rad/s)}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$C = \frac{1}{\omega_0^2 L} = \frac{1}{8000^2 \times 25 \times 10^{-3}}$$

$$= 6.25 \times 10^{-7} \text{ (F)} = 0.625\ \mu\text{F}$$

(b)

$$\frac{R}{2L} = \frac{4}{2\times 25\times 10^{-3}} = 80 \text{ (rad)}$$

$$\sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$= \sqrt{80^2 + 8000^2}$$

$$\approx 8000.40 \text{ (rad)}$$

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$= -80 + 8000.40 = 7920.40 \text{ (rad/s)}$$

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$= 80 + 8000.40 = 8080.40 \text{ (rad/s)}$$

$$B = \omega_2 - \omega_1 = 8080.40 - 7920.40$$

$$= 160 \text{ (rad/s)}$$

(c)

$$P(\omega_0) = \frac{1}{2}\frac{V_m^2}{R} = \frac{1}{2}\times\frac{100^2}{4} = 1250\ (\mathrm{W})$$

$$P(\omega_1) = P(\omega_2) = \frac{1}{2}P(\omega_0)$$

$$= \frac{1}{2}\times 1250$$

$$= 625\ (\mathrm{W})$$

14.6 Parallel Resonance

The parallel *RLC* circuit in Fig. 14.25 is the dual of the series *RLC* circuit. So we will avoid needless repetition.

$$Y = \frac{1}{R} + j\omega C + \frac{1}{j\omega L} = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

Resonance occurs when $\mathrm{Im}(Y) = 0$,

$$\omega_0 C - \frac{1}{\omega_0 L} = 0 \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$$

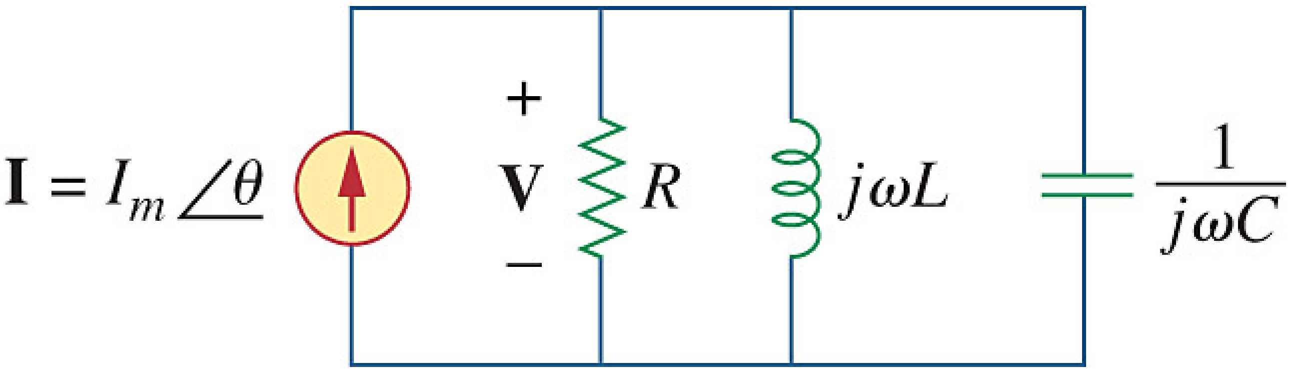


Figure 14.25 The parallel resonant circuit.

The magnitude of voltage $\dot{V}$ is sketched in Fig. 14.26 as a function of frequency. Notice that at resonance, the parallel *LC* combination acts like an open circuit, so that the entire current flows through *R*. Also, the inductor and capacitor currents can be much more than the source current at resonance.

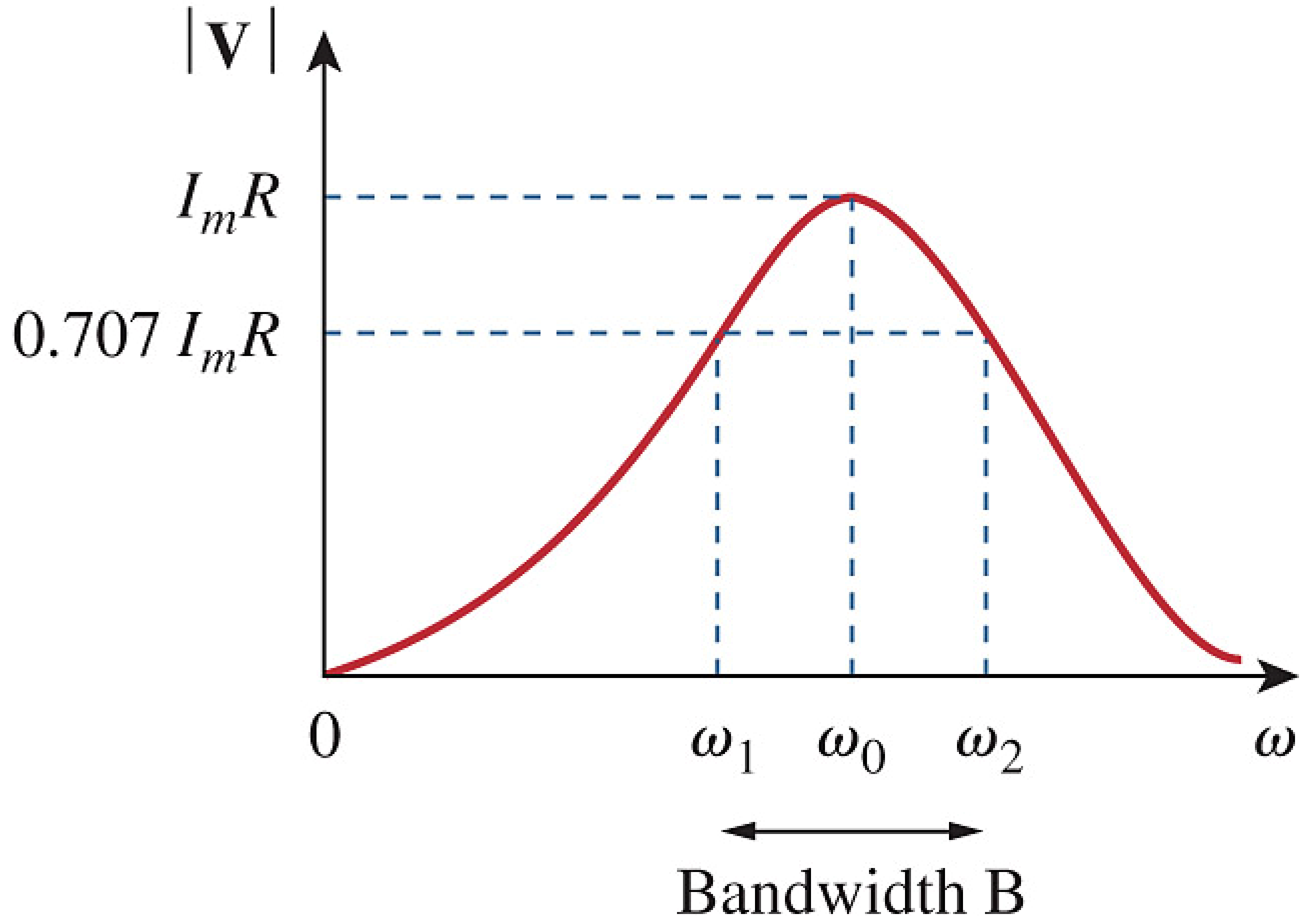


Figure 14.26 The voltage amplitude versus frequency for the parallel resonant circuit of Fig. 14.25.

By exploiting duality, we have

$$\omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$B = \omega_2 - \omega_1 = \frac{1}{RC}$$

$$Q = \frac{\omega_0}{B} = \omega_0 RC = \frac{R}{\omega_0 L}$$

Practice Problem 14.8 A parallel resonant circuit has $R = 100\ \text{k}\Omega$ and $L = 20$ mH, and $C = 5$ nF. Calculate $\omega_0,\ \omega_1,\ \omega_2,\ Q,$ and B.

Solution :

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{20\times10^{-3}\times5\times10^{-9}}}$$

$$= 10^5\ \text{(rad/s)}$$

$$\frac{1}{2RC} = \frac{1}{2\times100\times10^{3}\times5\times10^{-9}} = 1000\ \text{(rad/s)}$$

$$\sqrt{\left(\frac{1}{2RC}\right)^2+\frac{1}{LC}}=\sqrt{1000^2+(10^5)^2}$$

$$\approx 100{,}005.00\ \text{(rad/s)}$$

$$\omega_1=-\frac{1}{2RC}+\sqrt{\left(\frac{1}{2RC}\right)^2+\frac{1}{LC}}$$

$$=-1000+100{,}005.00=99{,}005.00\ \text{(rad/s)}$$

$$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$= 1000 + 100,005.00 = 101,005.00 \text{ (rad/s)}$$

$$B = \omega_2 - \omega_1 = 101,005.00 - 99,005.00$$

$$= 2000 \text{ (rad/s)}$$

$$Q = \frac{\omega_0}{B} = \frac{10^5}{2000} = 50$$

Practice Problem 14.9 Calculate the resonant frequency of the circuit in Fig. 14.29.

Solution :

$$Z = j\omega L + R \,||\, \frac{1}{j\omega C} = j\omega L + \frac{R}{1 + j\omega RC}$$

$$= j\omega L + \frac{R - j\omega R^2 C}{1 + (\omega RC)^2}$$

$$= \frac{R}{1 + (\omega RC)^2} + j\left(\omega L - \frac{\omega R^2 C}{1 + (\omega RC)^2} \right)$$

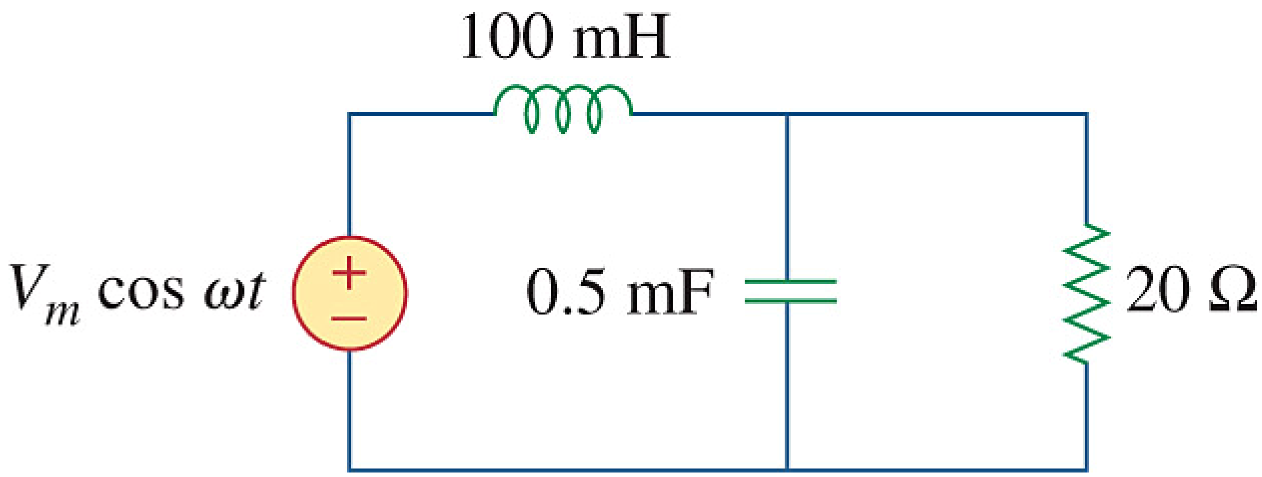


Figure 14.29

When $\omega = \omega_0$, $\mathrm{Im}(Z) = 0$,

$$\omega_0 L - \frac{\omega_0 R^2 C}{1 + (\omega_0 RC)^2} = 0$$

$$\omega_0 = \sqrt{\frac{1}{LC} - \frac{1}{(RC)^2}}$$

$$= \sqrt{\frac{1}{100 \times 10^{-3} \times 0.5 \times 10^{-3}} - \frac{1}{(20 \times 0.5 \times 10^{-3})^2}}$$

$$= 100 \text{ (rad/s)}$$

14.7 Passive Filters

A *filter* is a circuit that is designed to pass signals with desired frequencies and reject or attenuate others.

A filter is a *passive filter* if it consists of only passive elements *R*, *L*, and *C*. It is said to be an *active filter* if it consists of active elements in addition to passive elements.

As shown in Fig. 14.30, there are four types of filters whether passive or active:

1. A *lowpass filter* passes low frequencies and rejects high frequencies, as shown ideally in Fig. 14.30(a).
2. A *highpass filter* passes high frequencies and rejects low frequencies, as shown ideally in Fig. 14.30(b).

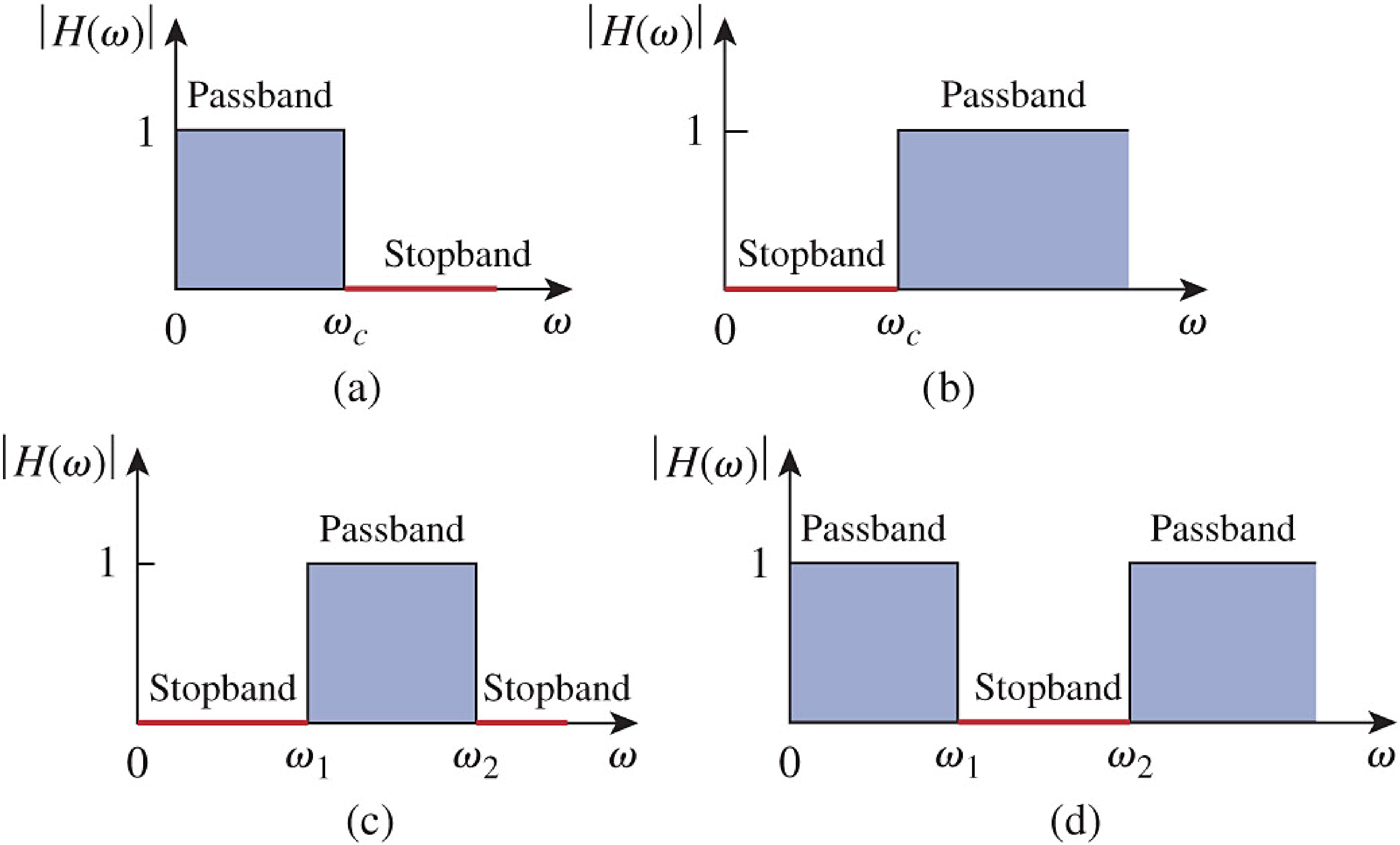


Figure 14.30 Ideal frequency response of four types of filter: (a) lowpass filter, (b) highpass filter, (c) bandpass filter, (d) bandstop filter.

3. A *bandpass filter* passes frequencies within a frequency band and blocks frequencies outside the band, as shown ideally in Fig. 14.30(c).

2. A *bandstop filter* (or *bandreject filter*, or *notch filter*) passes frequencies outside a frequency band and blocks frequencies within the band, as shown ideally in Fig. 14.30(d).

Lowpass Filter Figure 14.31 shows a lowpass filter. The frequency response is

$$H(j\omega) = \frac{\dot{V}_o(j\omega)}{\dot{V}_i(j\omega)} = \frac{1}{1 + j\omega RC}$$

Figure 14.32 shows the plot of $\left|H(j\omega)\right|$, along with the ideal characteristic.

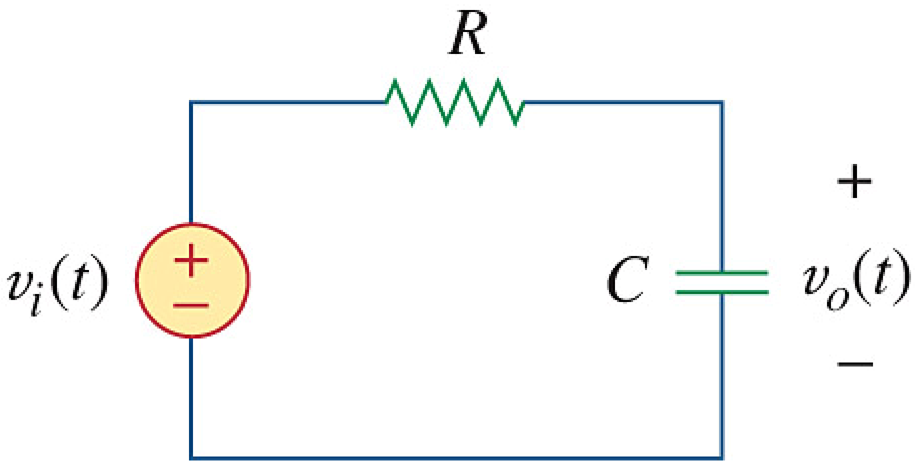


Figure 14.32 A lowpass filter.

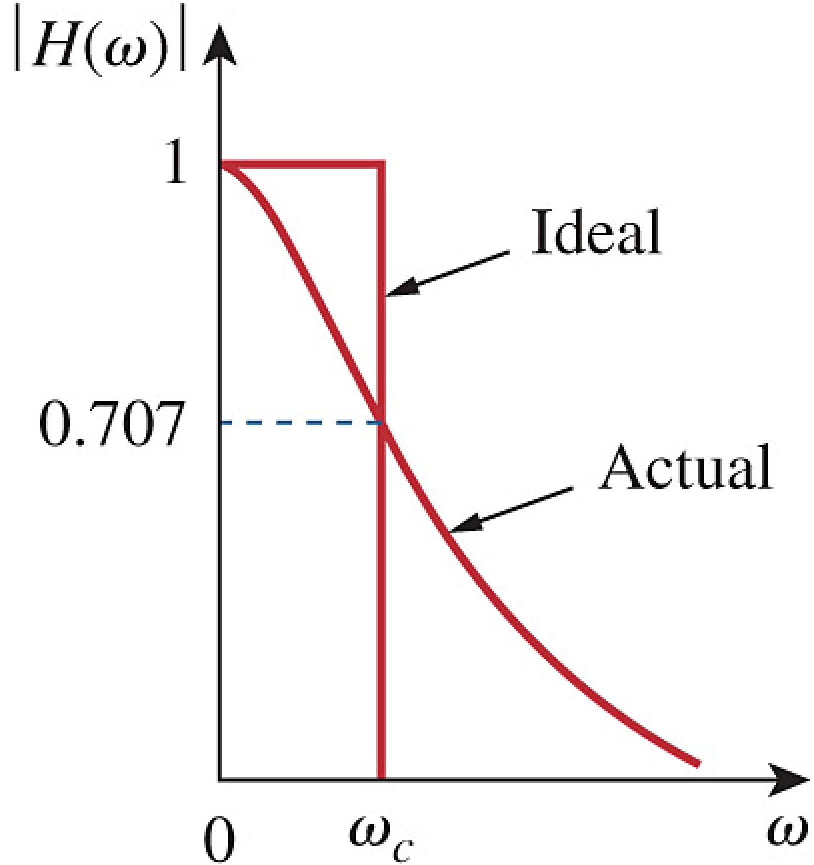


Figure 14.32 Ideal and actual frequency response of a lowpass filter.

The half-power frequency, which is equivalent to the corner frequency on the Bode plots but in the context of filters is usually known as the *cutoff frequency* ω_c, is obtained by setting $|H(j\omega)| = 1/\sqrt{2}$,

$$|H(j\omega_c)| = \left|\frac{1}{1 + j\omega_c RC}\right| = \frac{1}{\sqrt{2}} \Rightarrow \omega_c = \frac{1}{RC}$$

Highpass Filter Figure 14.33 shows a highpass filter. The frequency response is

$$H(j\omega) = \frac{\dot{V}_o(j\omega)}{\dot{V}_i(j\omega)} = \frac{j\omega RC}{1 + j\omega RC}$$

Figure 14.34 shows the plot of $\left|H(j\omega)\right|$.

Again, the cutoff frequency is $\omega_c = \dfrac{1}{RC}$.

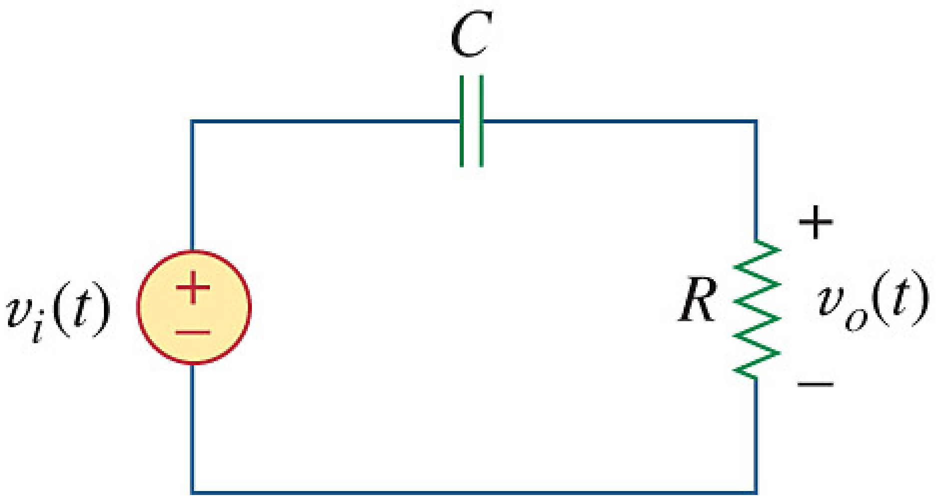


Figure 14.34 A highpass filter.

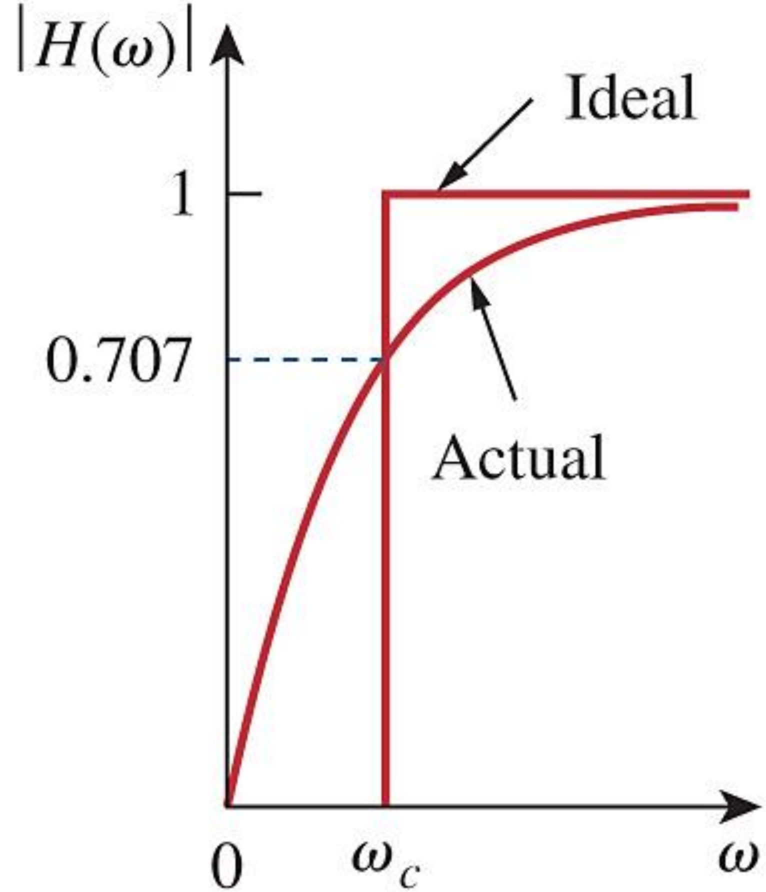


Figure 14.34 Ideal and actual frequency response of a highpass filter.

Bandpass Filter Figure 14.35 shows a bandpass filter. The frequency response is

$$H(j\omega)=\frac{\dot{V}_o(j\omega)}{\dot{V}_i(j\omega)}=\frac{R}{R+j\left(\omega L-\dfrac{1}{\omega C}\right)}$$

Figure 14.36 shows the plot of $\left|H(j\omega)\right|$. ω_0 is called the *center frequency*. ω_1 is called the *lower cutoff frequency*. ω_2 is called the *upper cutoff frequency*.

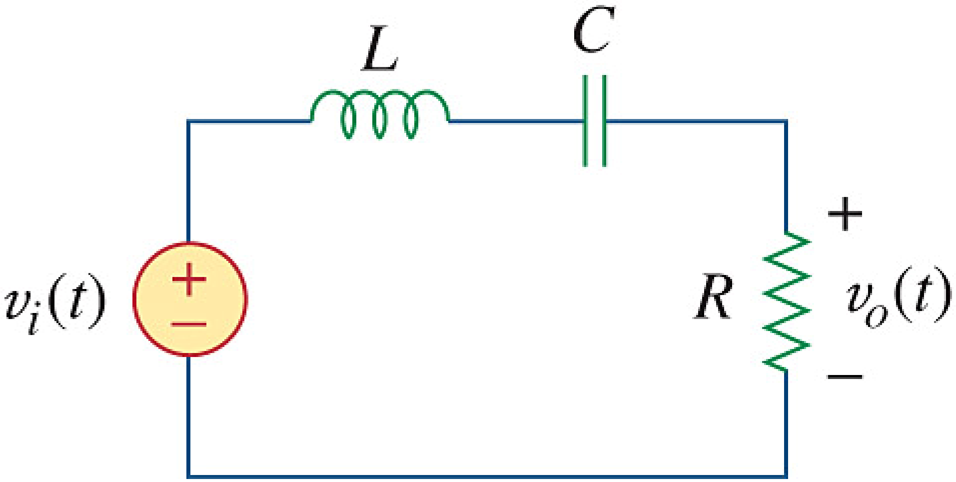


Figure 14.35 A bandpass filter.

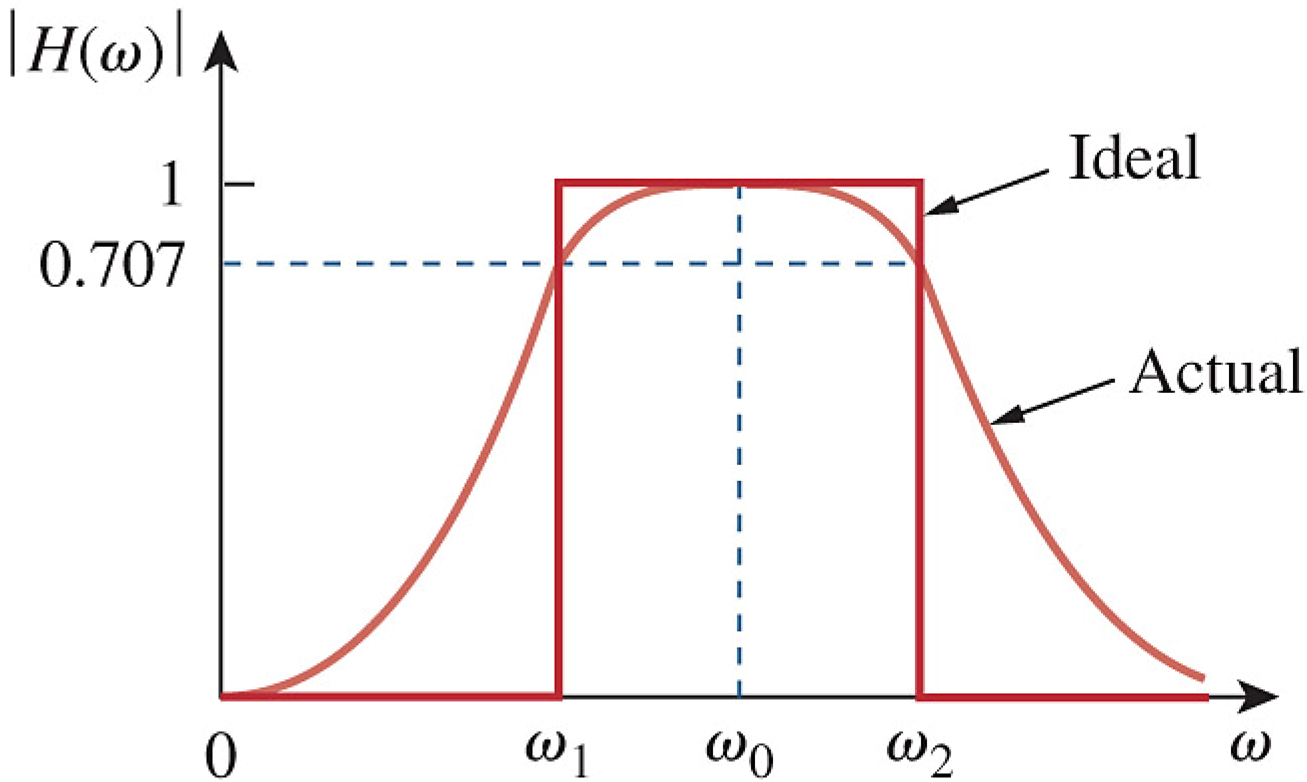


Figure 14.36 Ideal and actual frequency response of a bandpass filter.

Bandstop Filter Figure 14.37 shows a bandstop filter. The frequency response is

$$H(j\omega) = \frac{\dot{V}_o(j\omega)}{\dot{V}_i(j\omega)} = \frac{j[\omega L - 1/(\omega C)]}{R + j[\omega L - 1/(\omega C)]}$$

Figure 14.38 shows the plot of $|H(j\omega)|$.

Again, ω_0 is the center frequency. ω_1 and ω_2 are the cutoff frequencies.

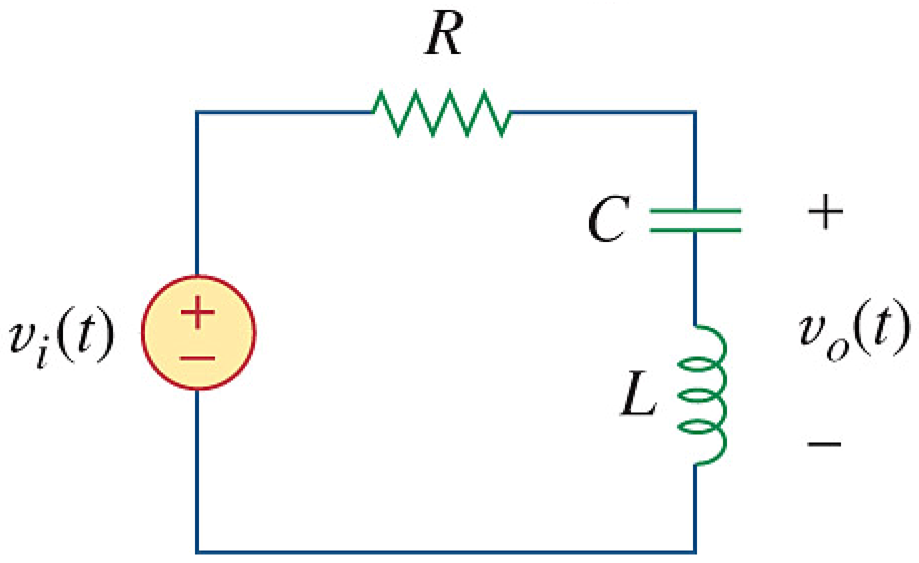


Figure 14.37 A bandstop filter.

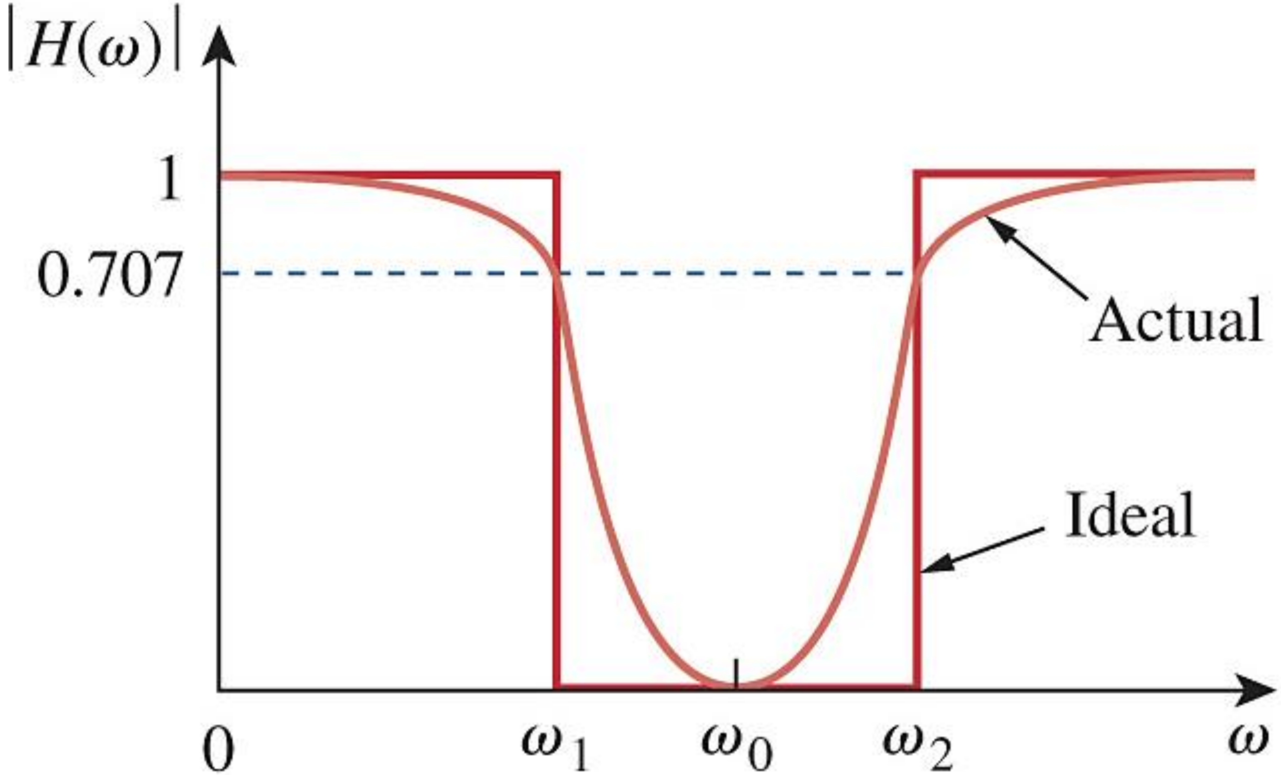


Figure 14.38 Ideal and actual frequency response of a bandstop filter.

Practice Problem 14.10 For the circuit in Fig 14.40, obtain the frequency response $\dot{V}_o(j\omega)/\dot{V}_i(j\omega)$. Identify the type of filter the circuit represents and determine the cutoff frequency. Take $R_1 = R_2 = 100\ \Omega$, $L = 2$ mH.

Solution :

$$H(j\omega) = \frac{\dot{V}_o(j\omega)}{\dot{V}_i(j\omega)} = \frac{R_2 \,||\, (j\omega L)}{R_1 + R_2 \,||\, (j\omega L)}$$

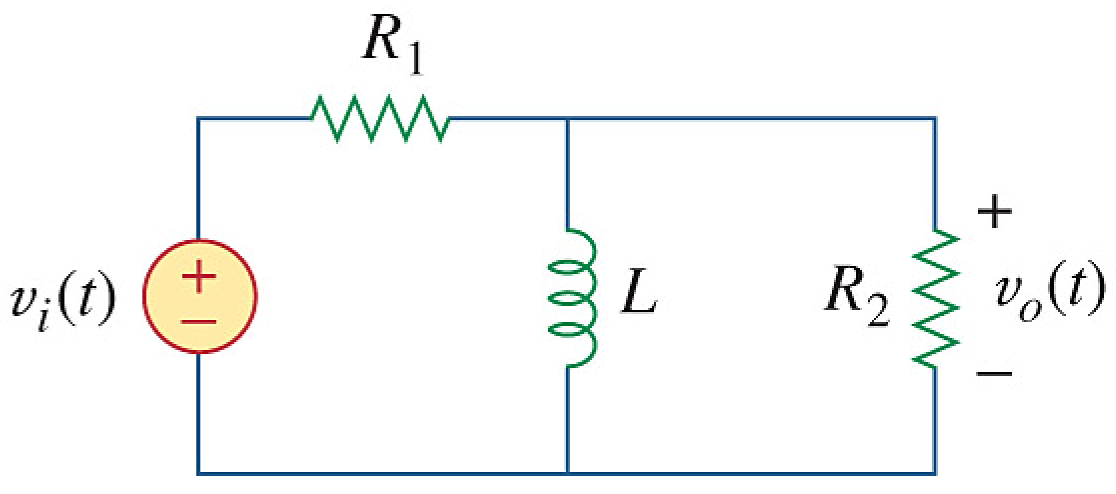


Figure 14.40

$$= \frac{j\omega R_2 L}{R_1 R_2 + j\omega(R_1 + R_2)L}$$

$$= \frac{R_2}{R_1 + R_2} \frac{j\omega L}{\frac{R_1 R_2}{R_1 + R_2} + j\omega L}$$

$$= \frac{R_2}{R_1 + R_2} \frac{j\omega L}{R_1 \| R_2 + j\omega L}$$

$$= \frac{R_2}{R_1 + R_2} \frac{j\omega L / (R_1 \| R_2)}{1 + j\omega L / (R_1 \| R_2)}$$

$$= \frac{R_2}{R_1 + R_2} \frac{j\omega / \omega_c}{1 + j\omega / \omega_c}$$

It is a highpass filter with a cutoff frequency of

$$\omega_c = \frac{R_1 \| R_2}{L} = \frac{100 \| 100}{2 \times 10^{-3}}$$

$$= 25{,}000 \text{ (rad/s)}$$

Question : What is the magnitude Bode Plot?

14.8 Active Filters

Active filters consists of combinations of resistors, capacitors, and op amps. They offer some advantages over passive filters: First, they are often samller and less expensive. Second, they can provide gain. Third, they can be combined with buffer amplifiers to isolate each stage of the filter from source and load impedance effects.

However, active filters are less reliable and less stable. The practical limit of most active filters is about 100 kHz – most active filters operate well below that frequency.

First - Order Lowpass Filter Figure 14.42 shows a first-order op amp circuit. It is used as an active low-pass filter. The frequency response is

$$H(j\omega)=\frac{\dot{V}_o(j\omega)}{\dot{V}_i(j\omega)}=-\frac{R_f \,||\, [1/(j\omega C_f)]}{R_i}$$

$$=-\frac{R_f}{R_i}\frac{1}{1+j\omega R_f C_f}=-\frac{R_f}{R_i}\frac{1}{1+j\omega/\omega_c}$$

$$\omega_c=1/(R_f C_f)$$

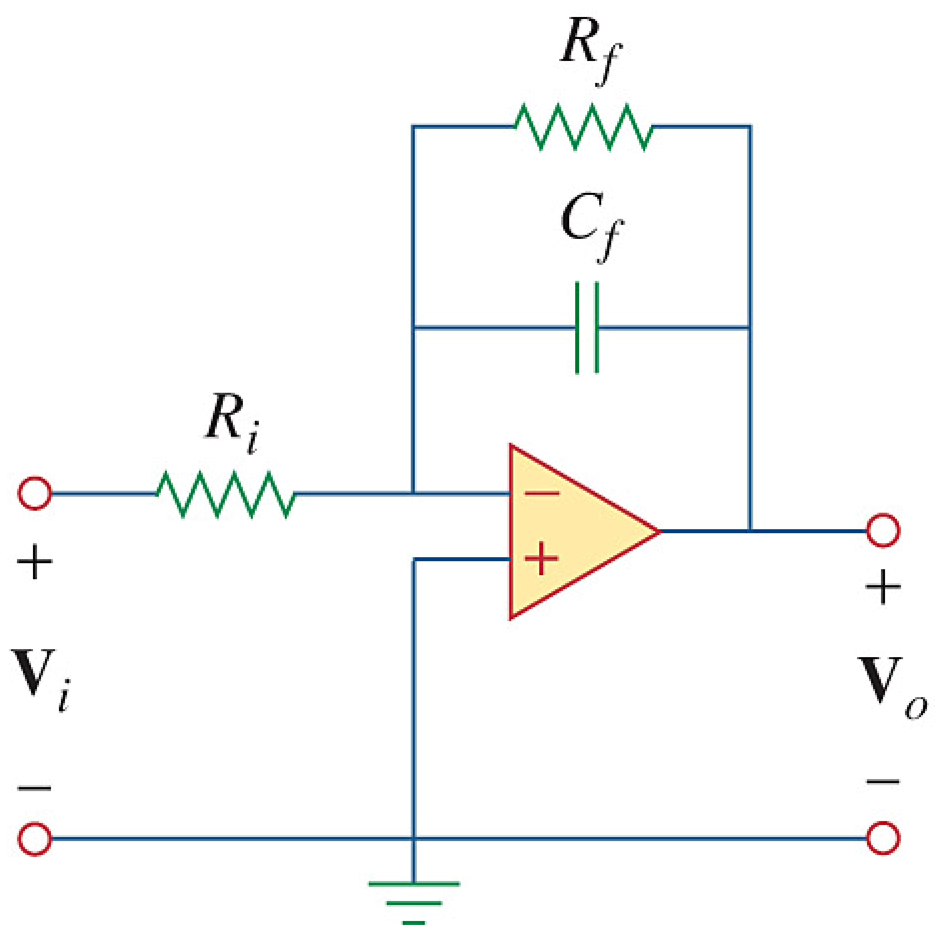


Figure 14.42 Active first-order lowpass filter.

First - Order Highpass Filter Figure 14.43 shows an active first-order high-pass filter. The frequency response is

$$H(j\omega) = \frac{\dot{V}_o(j\omega)}{\dot{V}_i(j\omega)} = -\frac{R_f}{R_i + 1/(j\omega C_i)}$$

$$= -\frac{R_f}{R_i}\frac{1}{1+1/(j\omega R_i C_i)} = -\frac{R_f}{R_i}\frac{j\omega/\omega_c}{1+j\omega/\omega_c}$$

$$\omega_c = 1/(R_i C_i)$$

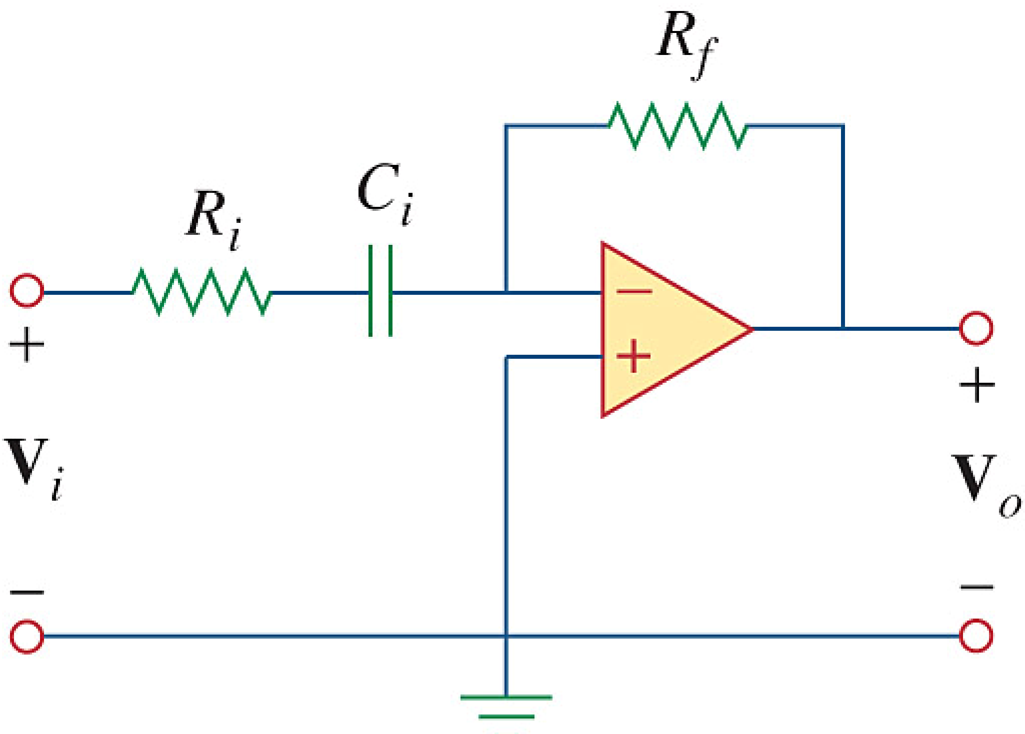


Figure 14.43 Active first-order highpass filter.

Bandpass Filter By cascading a unity-gain lowpass filter, a unity-gain highpass filter, and an inverter with gain $-R_f / R_i$, as shown in Fig. 14.44(a), we can construct a bandpass filter whose frequency response is that in Fig. 14.44(b).

Question : Why?

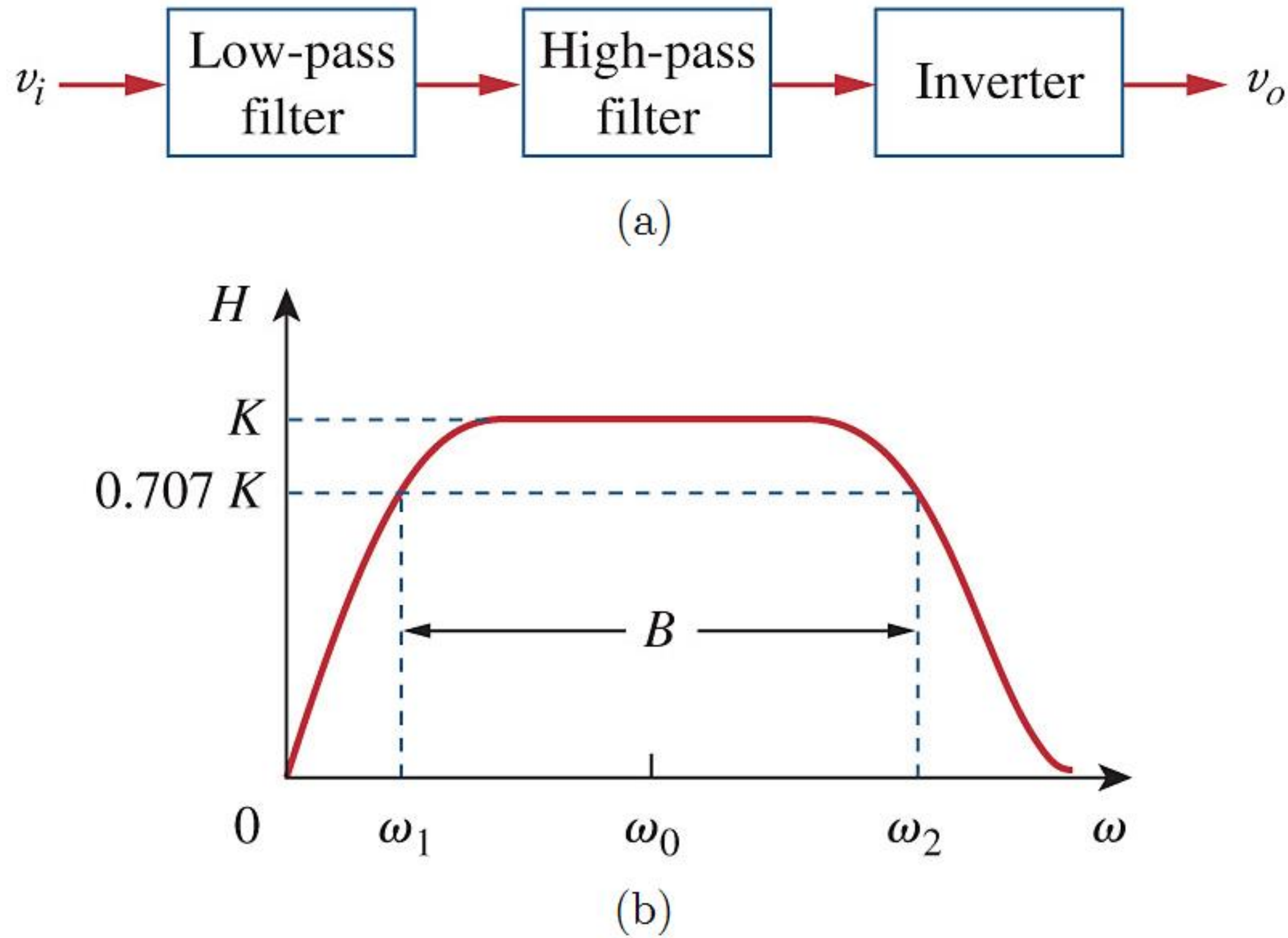


Figure 14.44 Active bandpass filter: (a) block diagram, (b) frequency response.

The actual construction of the bandpass filter is shown in Fig. 14.45. Its frequency response is

$$H(j\omega) = \frac{\dot{V}_o(j\omega)}{\dot{V}_i(j\omega)}$$

$$= \left(-\frac{R}{R}\frac{1}{1+j\omega RC_1}\right)\left(-\frac{R}{R}\frac{j\omega RC_2}{1+j\omega RC_2}\right)\left(-\frac{R_f}{R_i}\right)$$

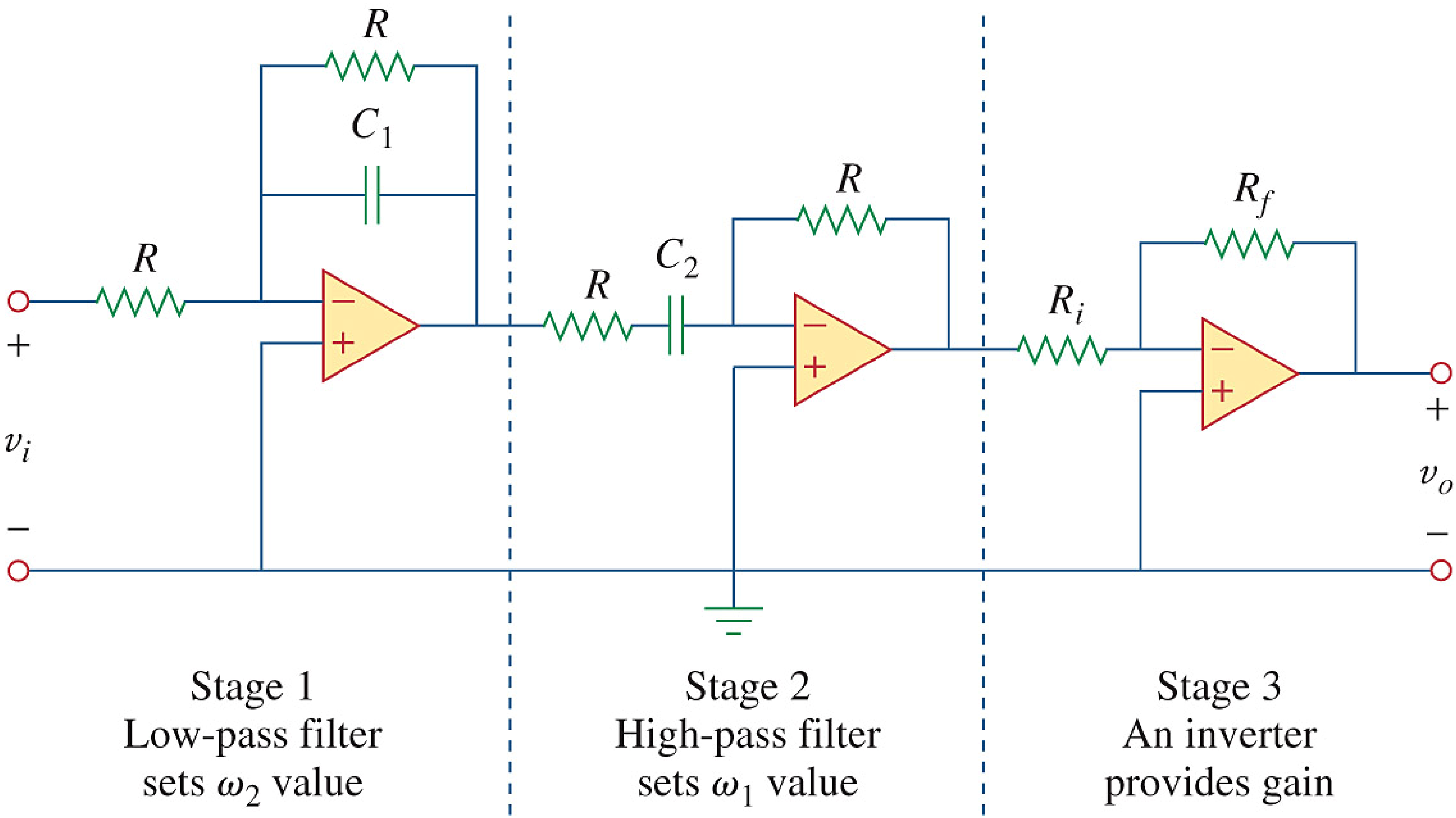


Figure 14.45 Active bandpass filter.

$$= -\frac{R_f}{R_i}\frac{1}{1+j\omega RC_1}\frac{j\omega RC_2}{1+j\omega RC_2}$$

$$= -\frac{R_f}{R_i}\frac{1}{1+j\omega/\omega_2}\frac{j\omega/\omega_1}{1+j\omega/\omega_1}$$

$$= -\frac{R_f}{R_i}\frac{j\omega/\omega_1}{(1+j\omega/\omega_1)(1+j\omega/\omega_2)}$$

The lowpass section sets the upper cutoff frequency as $\omega_2 = 1/(RC_1)$. The highpass section sets the lower cutoff frequency as $\omega_1 = 1/(RC_2)$.

Queation : What is the magnitude Bode plot for the bandpass filter?

The center frequency of the bandpass filter is

$$\omega_0 = \sqrt{\omega_1 \omega_2}$$

The bandwidth of the bandpass filter is

$$B = \omega_2 - \omega_2$$

The *passband gain* K is

$$K = \left| H(j\omega_0) \right|$$

$$= \left| -\frac{R_f}{R_i} \frac{j\omega_0 / \omega_1}{(1 + j\omega_0 / \omega_1)(1 + j\omega_0 / \omega_2)} \right|$$

$$= \frac{R_f}{R_i} \frac{\omega_0 / \omega_1}{\sqrt{1 + (\omega_0 / \omega_1)^2}\sqrt{1 + (\omega_0 / \omega_2)^2}}$$

$$= \frac{R_f}{R_i} \frac{\sqrt{\omega_2 / \omega_1}}{\sqrt{1 + \omega_2 / \omega_1}\sqrt{1 + \omega_1 / \omega_2}} = \frac{R_f}{R_i} \frac{\omega_2}{\omega_1 + \omega_2}$$

Bandreject Filter A bandreject filter can be constructed by parallel combination of a lowpass filter and a highpass filter and an adder, as shwon in Fig. 14.46(a). The frequency response is shown in Fig. 14.46 (b).

Question : Why?

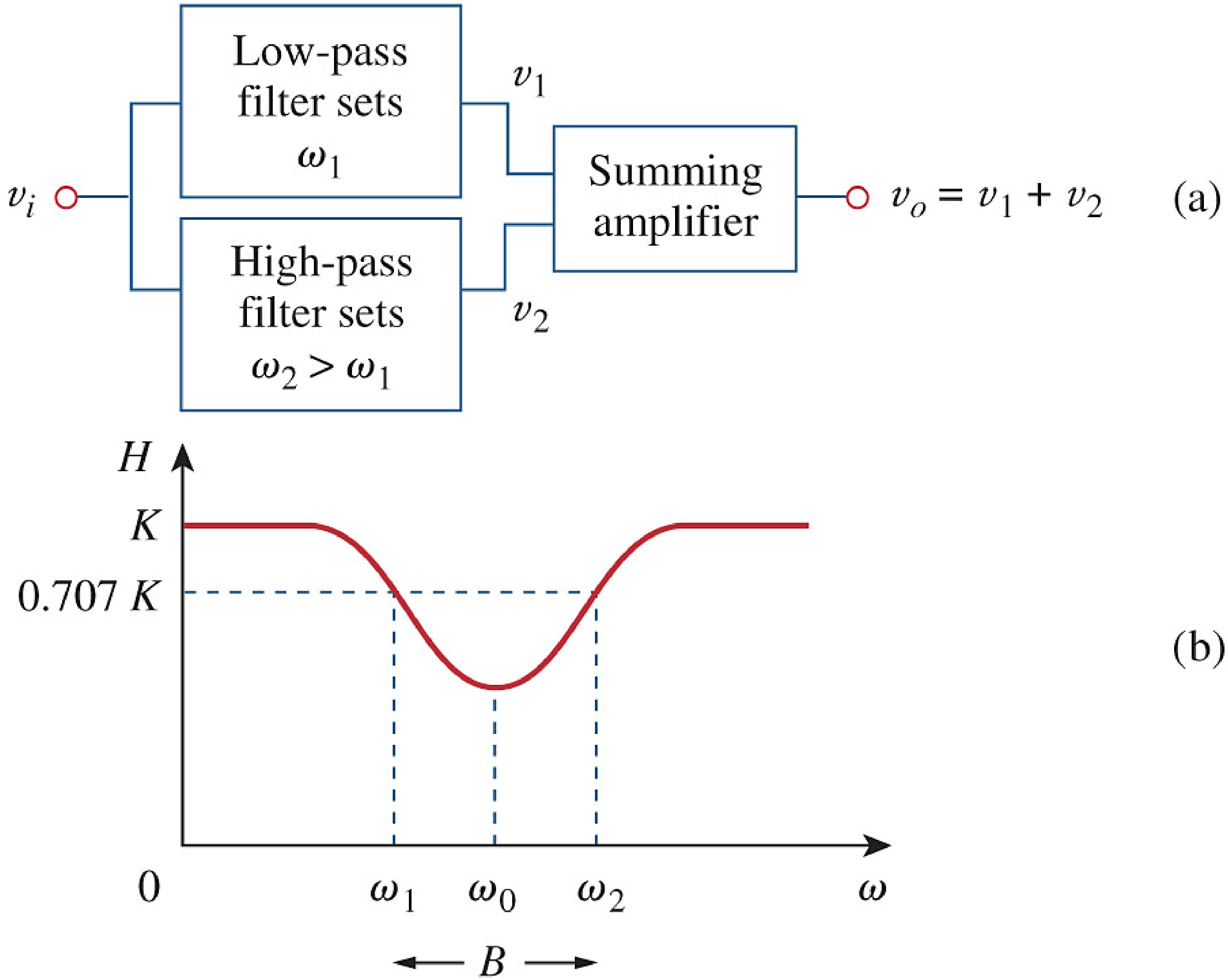


Figure 14.46 Active bandreject filter: (a) block diagram, (b) frequency response.

The actual construction of the bandreject filter is shown in Fig. 14.46. Its frequency response is

$$H(j\omega) = \frac{\dot{V}_o(j\omega)}{\dot{V}_i(j\omega)}$$

$$= \frac{R_f}{R_i}\left(\frac{1}{1+j\omega RC_1} + \frac{j\omega RC_2}{1+j\omega RC_2}\right)$$

Let

$\omega_1 = 1/(RC_1),\ \omega_2 = 1/(RC_2).$

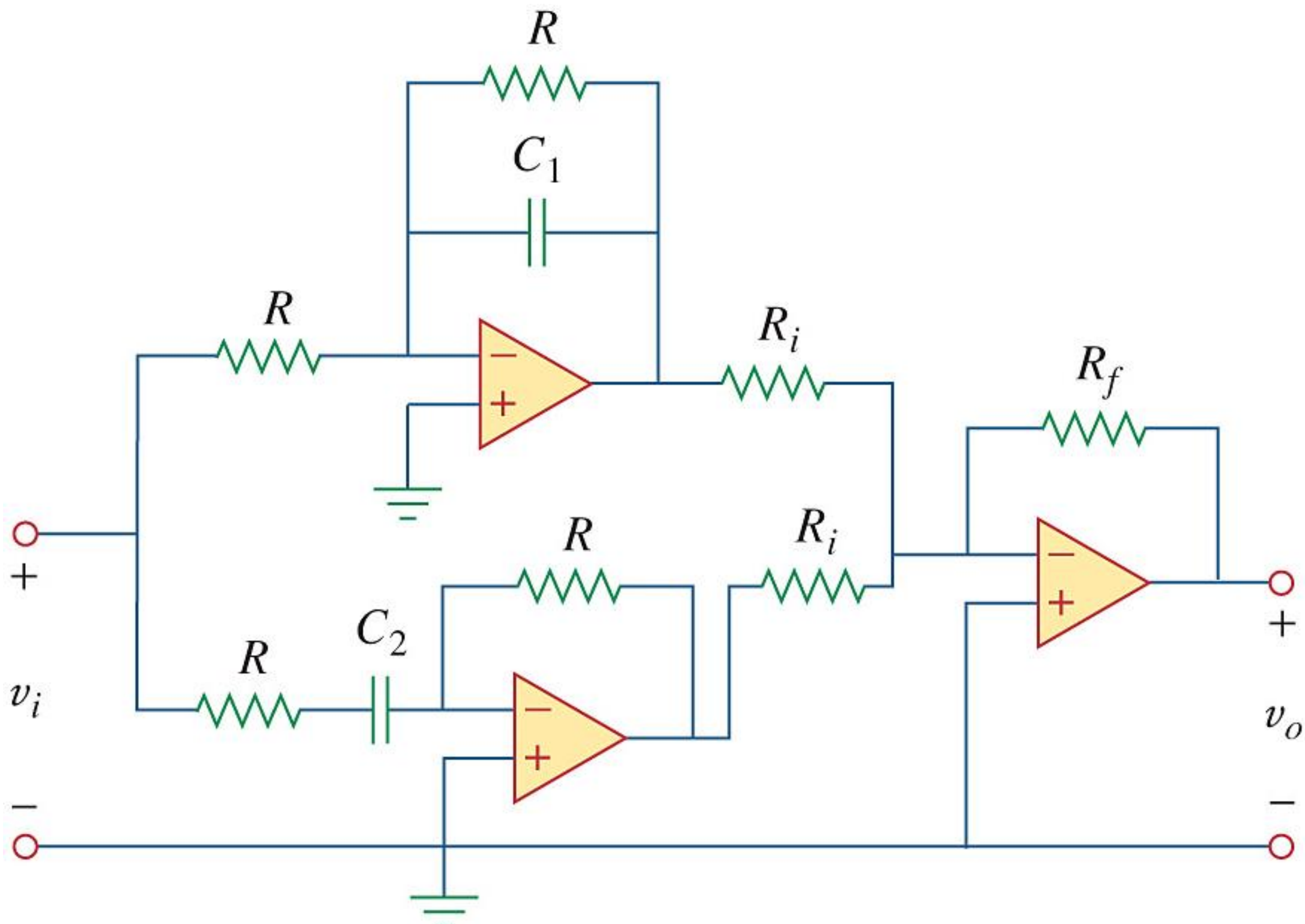


Figure 14.47 Active bandreject filter.

$$H(j\omega)=\frac{R_f}{R_i}\left(\frac{1}{1+j\omega/\omega_1}+\frac{j\omega/\omega_2}{1+j\omega/\omega_2}\right)$$

$$=\frac{R_f}{R_i}\frac{1+2(j\omega)/\omega_2+(j\omega)^2/(\omega_1\omega_2)}{(1+j\omega/\omega_1)(1+j\omega/\omega_2)}$$

$$=K\frac{1+2\zeta(j\omega)/\omega_0+(j\omega/\omega_0)^2}{(1+j\omega/\omega_1)(1+j\omega/\omega_2)}$$

$$K=\frac{R_f}{R_i},\ \omega_0=\sqrt{\omega_1\omega_2},\ \zeta=\sqrt{\frac{\omega_1}{\omega_2}}$$

Question : What is the magnitude Bode plot?

Example 14.13 Design a bandpass filter in the form of Fig. 14.45 to pass frequencies between 250 Hz and 3,000 Hz and with $K = 10$. Select $R = 20\ \text{k}\Omega$.

Solution :

$$\omega_1 = 2\pi f_1 = 2\pi \times 250 = 500\pi \text{ (rad/s)}$$

$$\omega_2 = 2\pi f_2 = 2\pi \times 3{,}000 = 6{,}000\pi \text{ (rad/s)}$$

$$\omega_1 = \frac{1}{RC_2} \Rightarrow C_2 = \frac{1}{\omega_1 R} = \frac{1}{500\pi \times 20 \times 10^3}$$

$$\approx 3.1831 \times 10^{-8}\ \text{(F)} \approx 31.83\ \text{nF}$$

$$\omega_2 = \frac{1}{RC_1} \Rightarrow C_1 = \frac{1}{\omega_2 R} = \frac{1}{6{,}000\pi \times 20 \times 10^3}$$

$$\approx 2.6526 \times 10^{-9}\ \text{(F)} \approx 2.65\ \text{nF}$$

$$K = \frac{R_f}{R_i} \frac{\omega_2}{\omega_1 + \omega_2} \Rightarrow \frac{R_f}{R_i} = K \frac{\omega_1 + \omega_2}{\omega_2}$$

$$= 10 \times \frac{500\pi + 6,000\pi}{6,000\pi} \approx 10.8333$$

If we select $R_i = 10$ kΩ, then $R_f \approx 10.83$ kΩ.