Ve215 Electric Circuits

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Chapter 8

Second-Order Circuits

8.1 Introduction

- In this chapter, we consider circuits containing two storage elements, known as second-order circuits.
- Examples of second-order circuits are shown in Fig. 8.1.

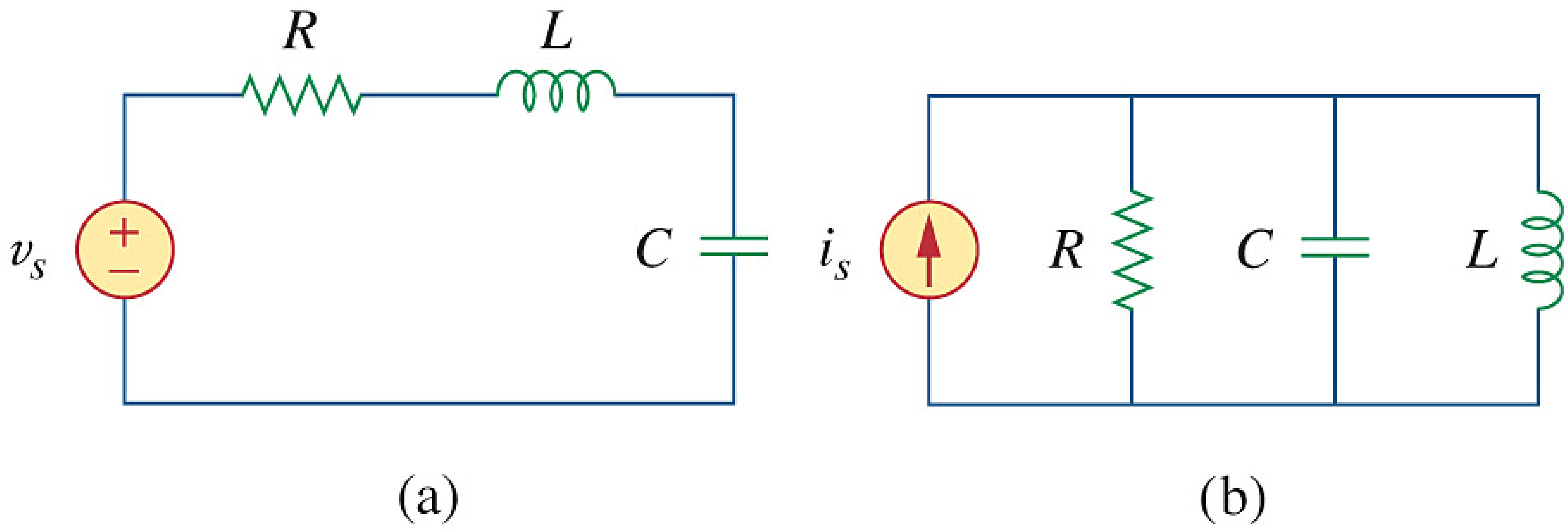


Figure 8.1 Typical examples of second-order circuits: (a) series RLC circuit, (b) parallel RLC circuit.

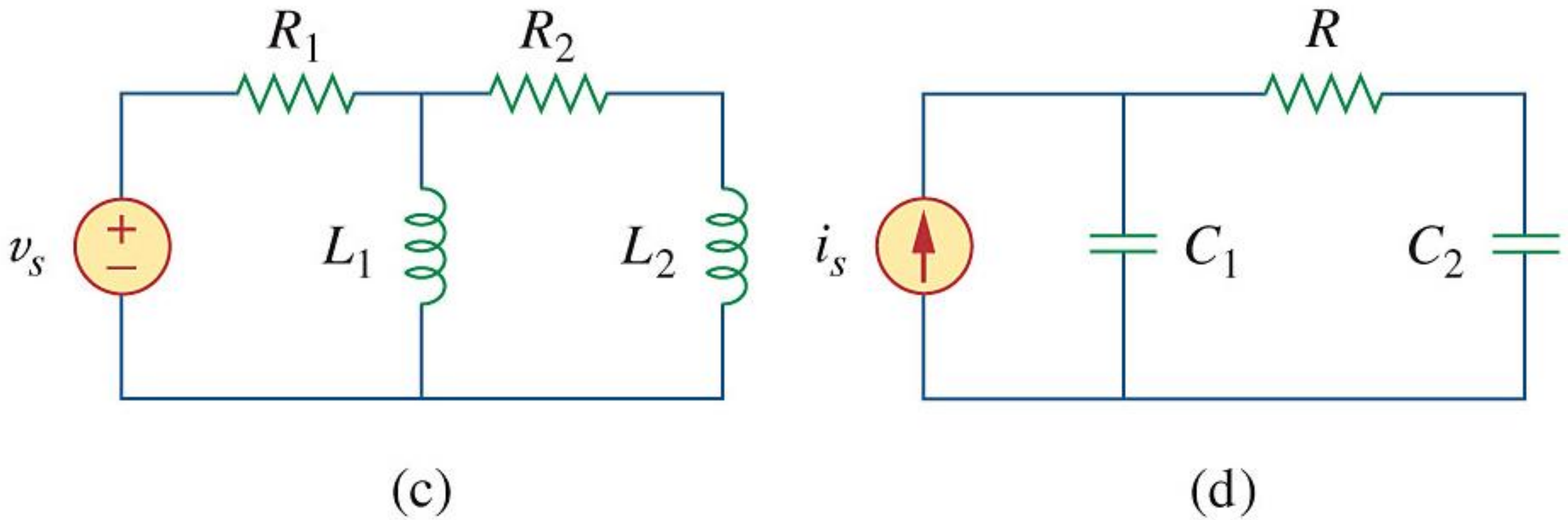


Figure 8.1 Typical examples of second-order circuits: (c) RLL circuit, (d) RCC circuit.

8.2 Finding Initial and Final Values

Example 8.1 The switch in Fig. 8.2 has been closed for a long time. It is open at $t = 0$. Find: (a) $i(0^+)$, $v(0^+)$, (b) $di(0^+)/dt$, $dv(0^+)/dt$, (c) $i(\infty)$, $v(\infty)$.

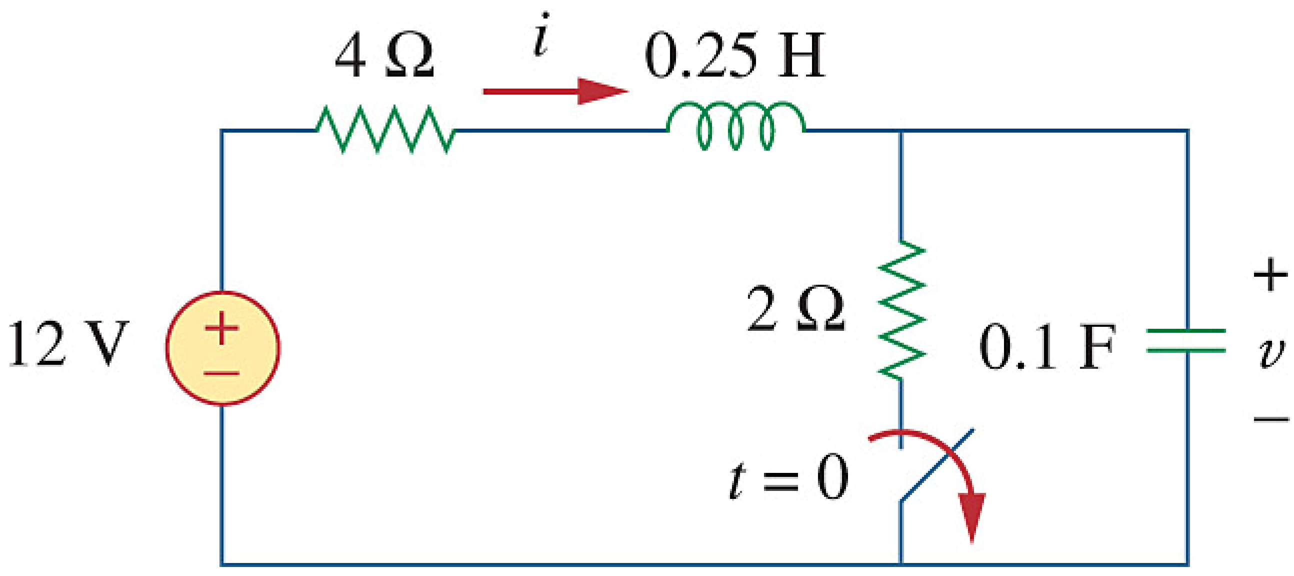


Figure 8.2

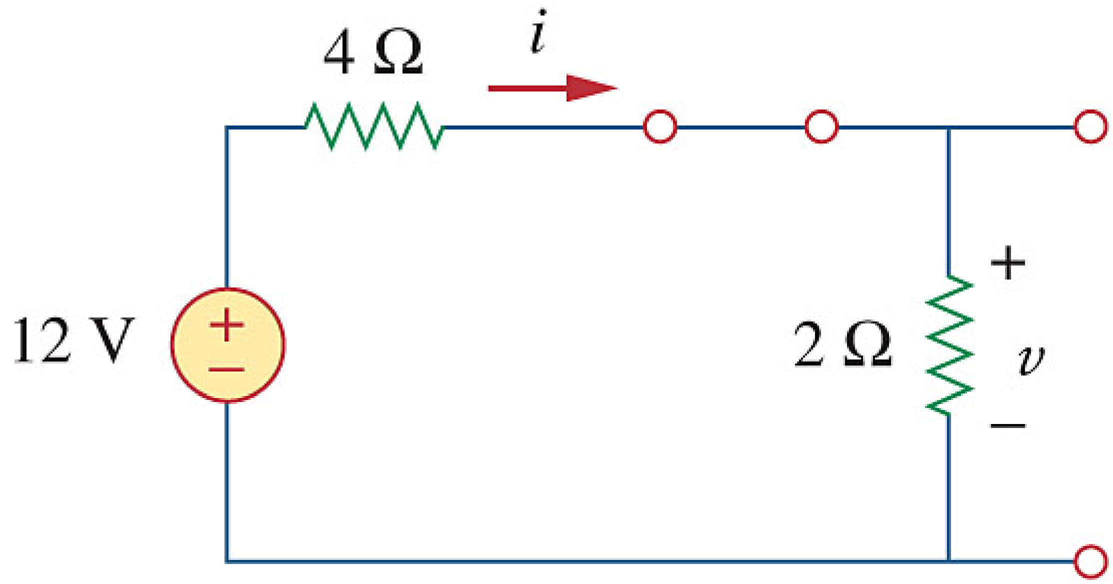


Figure 8.3 (a) Equivalent circuit of that in Fig. 8.2 for $t = 0^-$.

Solution :

(a)

$$i(0^+) = i(0^-) = \frac{12}{4+2} = 2 \text{ (A)}$$

$$v(0^+) = v(0^-) = 2i(0^-) = 4 \text{ (V)}$$

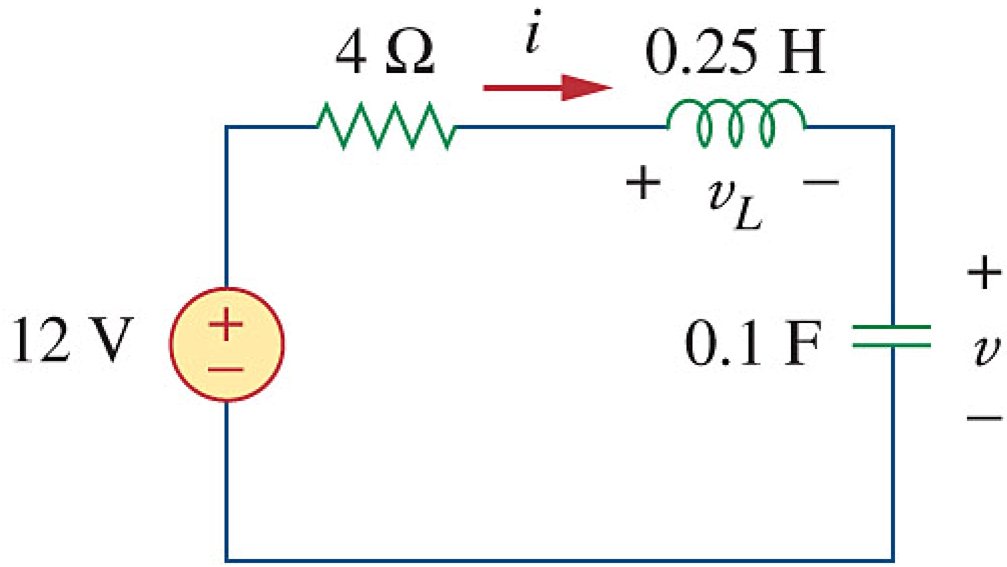


Figure 8.3 (a) Equivalent circuit of that in Fig. 8.2 for $t = 0^+$.

(b)

$$\begin{cases} i = 0.1\dfrac{dv}{dt} \\ 12 = 4i + 0.25\dfrac{di}{dt} + v \end{cases} \Rightarrow \begin{cases} \dfrac{dv}{dt} = \dfrac{i}{0.1} \\ \dfrac{di}{dt} = \dfrac{12 - 4i - v}{0.25} \end{cases}$$

$$dv(0^+)/dt = i(0^+)/0.1 = 2/0.1 = 20\ \text{(V/s)}$$

$$di(0^+)/dt = \left[12 - 4i(0^+) - v(0^+)\right]/0.25$$

$$= \left[12 - 4\times 2 - 4\right]/0.25 = 0\ \text{(A/s)}$$

(c)

$$i(\infty) = 0$$

$$v(\infty) = 12\ \text{(V)}$$

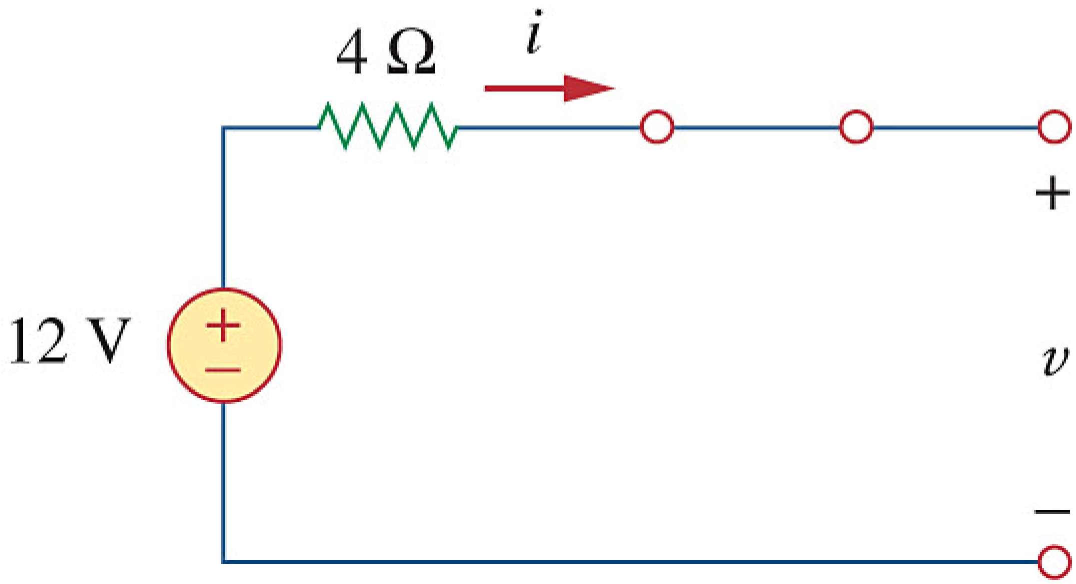


Figure 8.3 (c) Equivalent circuit of that in Fig. 8.2 for $t =$ infinity.

8.3 The Source-Free Series *RLC* Circuit

Consider the circuit shown in Fig. 8.8. The circuit is being excited by the energy initially stored in the capacitor and inductor. At $t = 0$,

$$v(0) = V_0, \quad i(0) = I_0$$

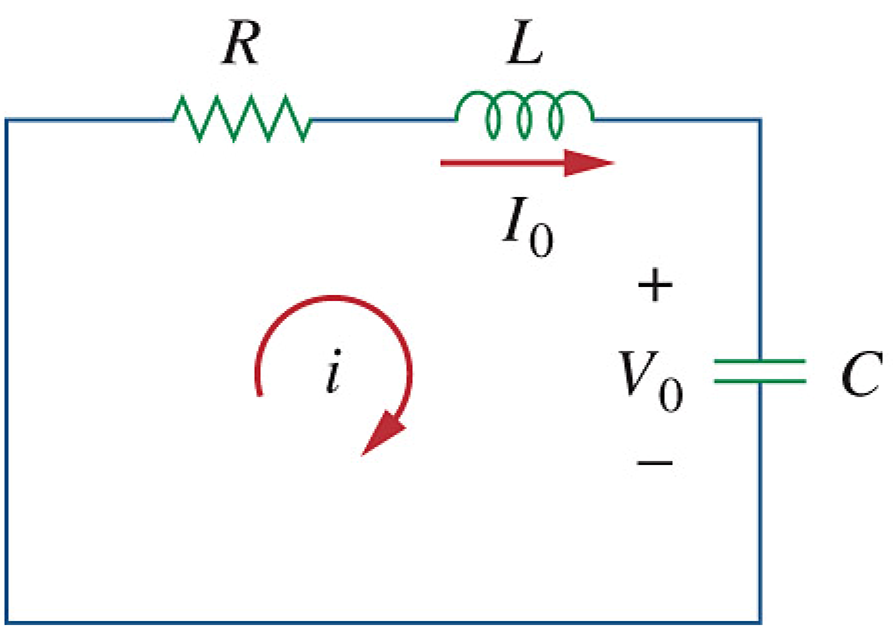


Figure 8.8 A source-free series RLC circuit.

$$iR + L\frac{di}{dt} + v = 0$$

$$iR + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i\,dt = 0$$

$$\frac{di}{dt}R + L\frac{d^2 i}{dt^2} + \frac{1}{C}i = 0$$

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{RC}i = 0$$

The initial conditions are

$$i(0^+) = i(0^-) = I_0$$

$$i'(0^+) = -\frac{1}{L}\left(i(0^+)R + v(0^+)\right)$$

$$= -\frac{1}{L}\left(i(0^-)R + v(0^-)\right)$$

$$= -\frac{1}{L}\left(I_0 R + V_0\right)$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s = \frac{-R/L \pm \sqrt{(R/L)^2 - 4\times 1\times\left(1/(LC)\right)}}{2\times 1}$$

$$= -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$= -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

where

$\alpha = \dfrac{R}{2L}$: neper frequency (damping factor), Np/s (nepers per second)

$\omega_0 = \dfrac{1}{\sqrt{LC}}$: resonant frequency (undamped natural frequency), rad/s

$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$, $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$: natural frequencies, Np/s

There are three types of solutions:

1. If $\alpha > \omega_0$, $s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$, $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$, we have the *overdamped* case,

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where

$$A_1 = \frac{i'(0^+) - s_2 i(0^+)}{s_1 - s_2}$$

$$A_2 = \frac{s_1 i(0^+) - i'(0^+)}{s_1 - s_2}$$

2. If $\alpha = \omega_0$, $s_1 = s_2 = -\alpha$, we have the *critically damped* case,

$$i(t) = (B_1 t + B_2)e^{-\alpha t}$$

where

$$B_1 = i'(0^+) + \alpha i(0^+)$$

$$B_2 = i(0^+)$$

3. If $\alpha < \omega_0$, $s_1 = -\alpha + j\omega_d$, $s_2 = -\alpha - j\omega_d$,

where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$, which is called the *damping frequency*, we have the *underdamped* case,

$$i(t) = e^{-\alpha t}(C_1 \cos \omega_d t + C_2 \sin \omega_d t)$$

where

$$C_1 = i(0^+)$$

$$C_2 = \frac{i'(0^+) + \alpha i(0^+)}{\omega_d}$$

Once the inductor current $i(t)$ is found, other circuit quantities can be found,

$$v_R(t) = i(t)R$$

$$v_L(t) = L\frac{di(t)}{dt}$$

$$v_C(t) = \frac{1}{C}\int_0^t i(t)dt + v_C(0)$$

Practice Problem 8.4 The circuit in Fig. 8.12 has reached steady state at $t = 0^-$. If the make-before-break switch moves to position b at $t = 0$, calculate $i(t)$ for $t > 0$.

Solution:

$$i(0^+) = i(0^-) = \frac{50}{10} = 5 \text{ (A)}$$

$$v(0^+) = v(0^-) = 0 \text{ (V)}$$

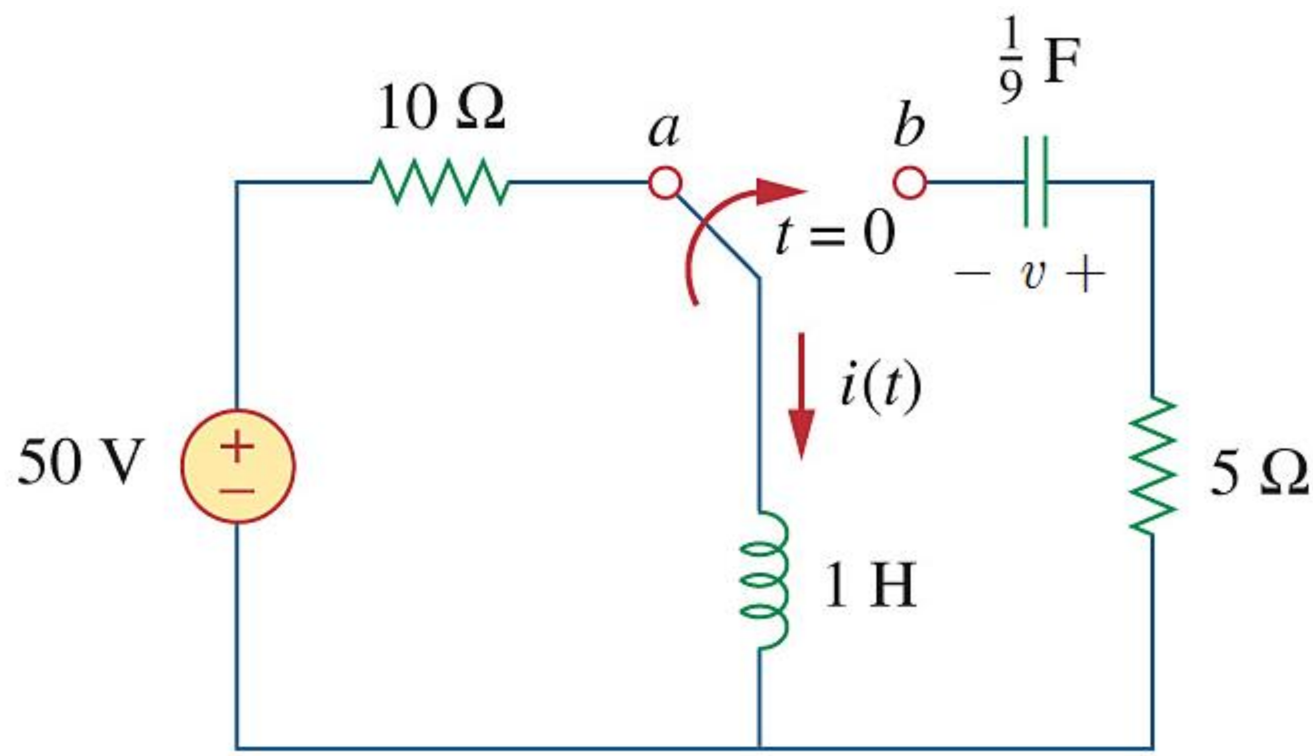


Figure 8.12

$$1\times\frac{di(t)}{dt}+i(t)\times 5+v(t)=0,\ i(t)=\frac{1}{9}\frac{dv(t)}{dt}$$

$$i'(0^+)=-5i(0^+)-v(0^+)=-5\times 5-0$$

$$=-25\ (\mathrm{A/s})$$

$$1\times\frac{d^2i(t)}{dt^2}+\frac{di(t)}{dt}\times 5+\frac{dv(t)}{dt}=0$$

$$1\times\frac{d^2i(t)}{dt^2}+\frac{di(t)}{dt}\times 5+\frac{1}{1/9}i(t)=0$$

$$\frac{d^2i(t)}{dt^2}+5\frac{di(t)}{dt}+9i(t)=0$$

$$s=\frac{-5\pm\sqrt{5^2-4\times1\times9}}{2\times1}=\frac{-5\pm j\sqrt{11}}{2}$$

$$i(t)=e^{-2.5t}\left(A_1\cos\frac{\sqrt{11}}{2}t+A_2\sin\frac{\sqrt{11}}{2}t\right)$$

$$i(0^+)=A_1\Rightarrow A_1=i(0^+)=5$$

$$i'(0^+)=-2.5A_1+\frac{\sqrt{11}}{2}A_2\Rightarrow A_2=\frac{i'(0^+)+2.5A_1}{\sqrt{11}/2}$$

$$=\frac{-25+2.5\times5}{\sqrt{11}/2}=-\frac{25}{\sqrt{11}}$$

$$i(t)\approx e^{-2.5t}(5\cos1.6583t-7.5378\sin1.6583t)\ (\mathrm{A})$$

8.3 The Source-Free Parallel *RLC* Circuit

Consider the circuit shown in Fig. 8.13. The circuit is being excited by the energy initially stored in the capacitor and inductor. At $t = 0$,

$$v(0) = V_0, \quad i(0) = I_0$$

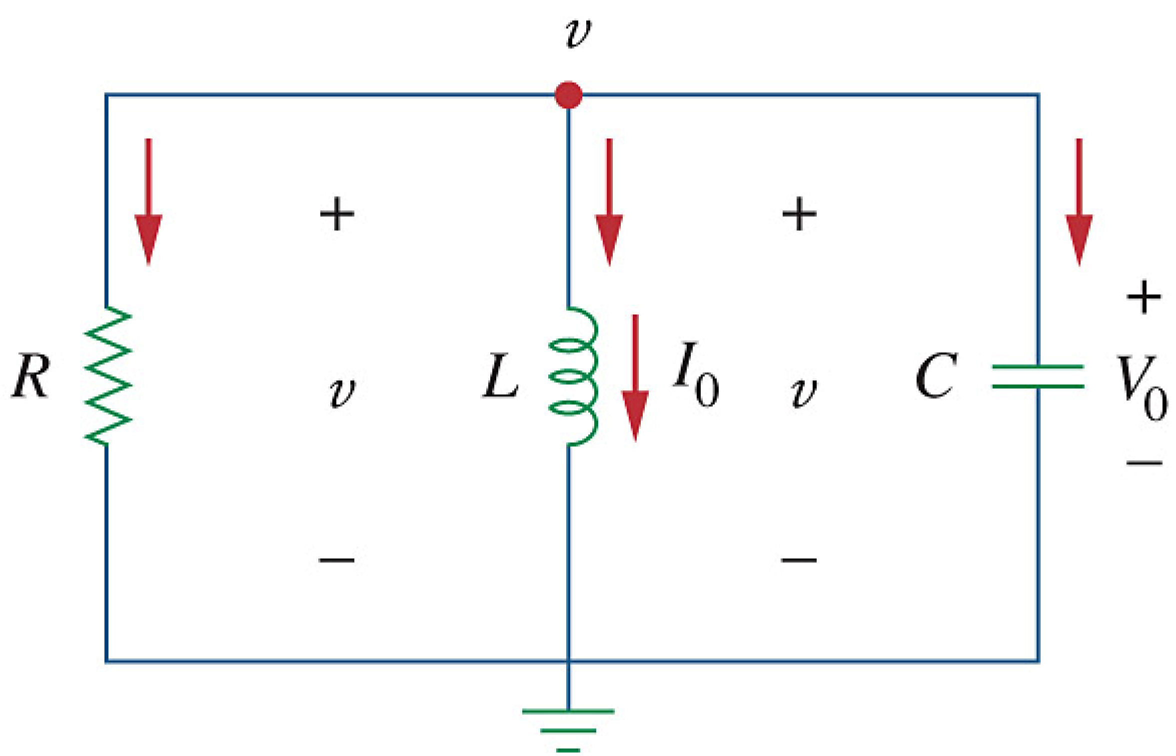


Figure 8.13 A source-free parallel RLC circuit.

$$\frac{v}{R} + i + C\frac{dv}{dt} = 0$$

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v\,dt + C\frac{dv}{dt} = 0$$

$$\frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v + C\frac{d^2v}{dt^2} = 0$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

The initial conditions are

$$v(0^+) = v(0^-) = V_0$$

$$v'(0^+) = -\frac{1}{C}\left(v(0^+)/R + i(0^+)\right)$$

$$= -\frac{1}{C}\left(v(0^-)/R + i(0^-)\right)$$

$$= -\frac{1}{C}\left(V_0/R + I_0\right)$$

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

$$s = \frac{-1/(RC) \pm \sqrt{1/(RC)^2 - 4 \times 1 \times \left(1/(LC)\right)}}{2 \times 1}$$

$$= -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$= -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

where

$$\alpha = \frac{1}{2RC}: \text{ neper frequency (damping factor),}$$

Np/s (nepers per second)

$$\omega_0 = \frac{1}{\sqrt{LC}}: \text{resonant frequency (undamped}$$

natural frequency), rad/s

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \; s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}:$$

natural frequencies, Np/s

There are three types of solutions:

1. If $\alpha > \omega_0$, $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$, we have the *overdamped* case,

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

2. If $\alpha = \omega_0$, $s_1 = s_2 = -\alpha$, we have the *critically damped* case,

$$v(t) = (B_1 t + B_2) e^{-\alpha t}$$

3. If $\alpha < \omega_0$, $s_1 = -\alpha + j\omega_d$, $s_2 = -\alpha - j\omega_d$, we have the *underdamped* case,

$$v(t) = e^{-\alpha t}(C_1 \cos \omega_d t + C_2 \sin \omega_d t)$$

Once the capacitor voltage $v(t)$ is found, other circuit quantities can be found,

$$i_R(t) = \frac{v(t)}{R}$$

$$i_L(t) = \frac{1}{L}\int_0^t v(t)dt + i_L(0)$$

$$i_C(t) = C\frac{di(t)}{dt}$$

Example 8.6 Find $v(t)$ for $t > 0$ in the *RLC* circuit of Fig. 8.15.

Solution :

$$v(0^+) = v(0^-) = 40 \times \frac{50}{30+50} = 25 \text{ (V)}$$

$$i(0^+) = i(0^-) = -\frac{40}{20+50} = -0.5 \text{ (A)}$$

$$v'(0^+) = -\frac{1}{C}\left(v(0^+)/R + i(0^+)\right)$$

$$= -\frac{1}{20\times 10^{-6}}\left(25/50 + (-0.5)\right) = 0 \text{ (V/s)}$$

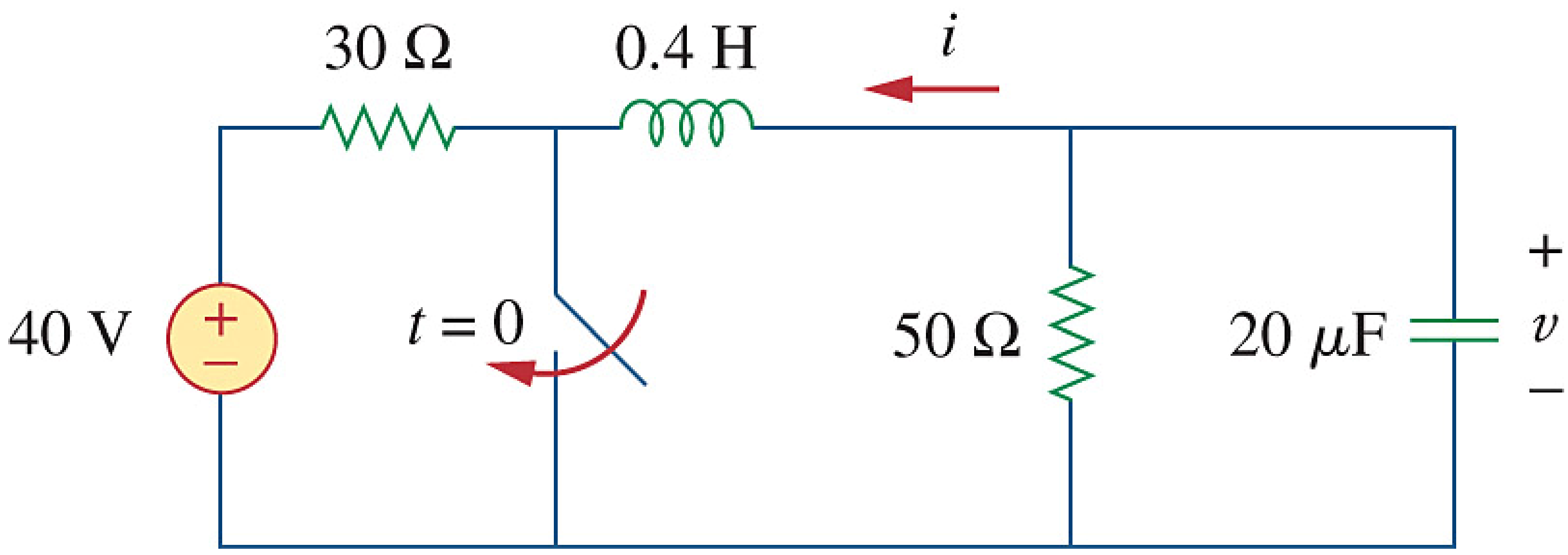


Figure 8.15

$$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{1}{LC}v=0$$

$$\frac{1}{LC}=\frac{1}{0.4\times 20\times 10^{-6}}=125000$$

$$\frac{1}{RC}=\frac{1}{50\times 20\times 10^{-6}}=1000$$

$$\frac{d^2v}{dt^2}+1000\frac{dv}{dt}+125000v=0$$

$$s_{1,2}=\frac{-1000\pm\sqrt{1000^2-4\times 1\times 125000}}{2\times 1}$$

$$s_1 \approx -146.4466,\ s_2 \approx -853.5534$$

$$v(t) = A_1 e^{-146.4466t} + A_2 e^{-853.5534t}$$

$$v(0^+) = A_1 + A_2 = 25$$

$$v'(0^+) = -146.4466 A_1 - 853.5534 A_2 = 0$$

$$A_1 \approx 30.1777,\ A_2 \approx -5.1777$$

$$v(t) \approx 30.18 e^{-146.45t} - 5.18 e^{-853.55t} \ \text{(V)}$$

8.5 Series *RLC* Circuit with Step Input

Consider the circuit in Fig. 8.18. For $t > 0$,

$$\begin{cases} V_s = iR + L\dfrac{di}{dt} + v \\ i = C\dfrac{dv}{dt} \end{cases}$$

$$LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v = V_s$$

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{LC}V_s$$

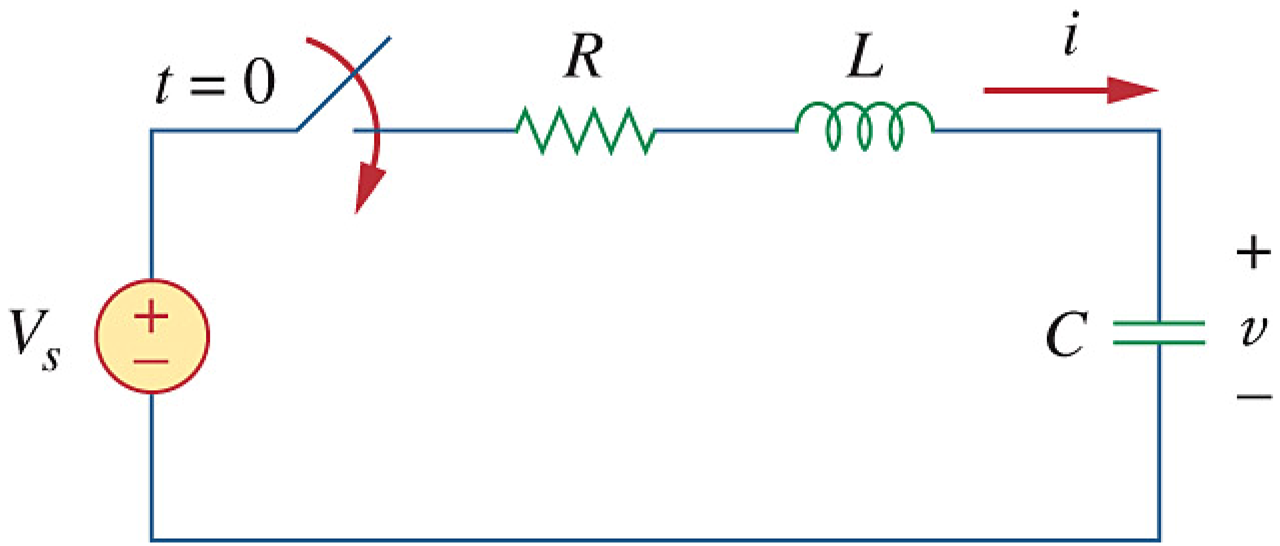


Figure 8.18 Step voltage applied to a series RLC circuit.

It can be shown that the solution has three possible forms:

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + V_s$$

(Overdamped)

$$v(t) = (A_1 + A_2 t)e^{-\alpha t} + V_s$$

(Critically damped)

$$v(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t) + V_s$$

(Underdamped)

Example 8.7 For the circuit in Fig. 8.19, find $v(t)$ for $t > 0$. Consider these cases: $R = 5\ \Omega$, $R = 4\ \Omega$, $R = 1\ \Omega$.

Solution :

$$i(0^+) = i(0^-) = \frac{24}{R+1}$$

$$v(0^+) = v(0^-) = 24 \times \frac{1}{R+1}$$

$$i(t) = 0.25\frac{dv(t)}{dt} \Rightarrow v'(0^+) = \frac{1}{0.25}i(0^+)$$

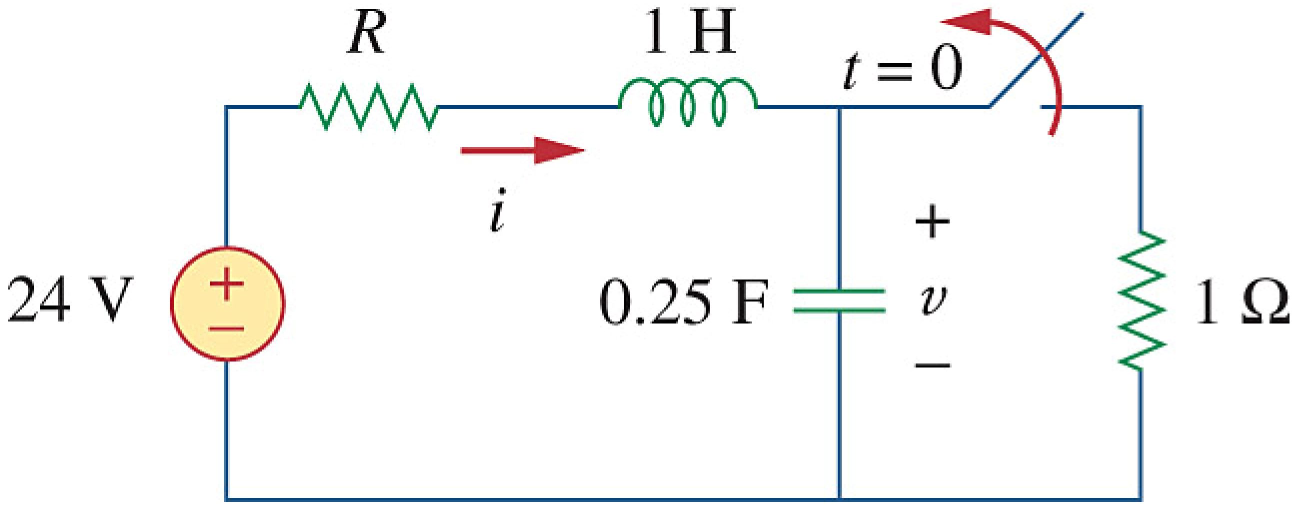


Figure 8.19

$$\frac{d^2v}{dt^2}+\frac{R}{L}\frac{dv}{dt}+\frac{1}{LC}v=\frac{1}{LC}V_s$$

(a) $R = 5\ \Omega$

$i(0^+) = 4\ \text{A},\ v(0^+) = 4\ \text{V},\ v'(0^+) = 16\ \text{V/s}$

$$\frac{d^2v}{dt^2}+5\frac{dv}{dt}+4v=96$$

$$s^2+5s+4=0 \Rightarrow s_1=-1,\ s_2=-4$$

$$v_n(t)=A_1e^{-t}+A_2e^{-4t}$$

$$v_f(t) = B \Rightarrow B = 96/4 = 24$$

$$v(t) = v_n(t) + v_f(t) = A_1 e^{-t} + A_2 e^{-4t} + 24$$

$$v(0^+) = A_1 + A_2 + 24 = 4$$

$$v'(0^+) = -A_1 - 4A_2 = 16$$

$$A_1 = -\frac{64}{3}, A_2 = \frac{4}{3}$$

$$v(t) = -\frac{64}{3}e^{-t} + \frac{4}{3}e^{-4t} + 24 \text{ (V)}$$

$$i(t) = \frac{16}{3}e^{-t} - \frac{4}{3}e^{-4t} \text{ (A)}$$

(b) $R = 4\ \Omega$

$$i(0^+) = 4.8 \text{ A}, v(0^+) = 4.8 \text{ V}, v'(0^+) = 19.2 \text{ V/s}$$

$$\frac{d^2v}{dt^2} + 4\frac{dv}{dt} + 4v = 96$$

$$s^2 + 4s + 4 = 0 \Rightarrow s_1 = s_2 = -2$$

$$v_n(t) = (A_1 + A_2 t)e^{-2t}$$

$$v_f(t) = B \Rightarrow B = 96/4 = 24$$

$$v(t) = v_n(t) + v_f(t) = (A_1 + A_2 t)e^{-2t} + 24$$

$$v(0^+) = A_1 + 24 = 4.8$$

$$v'(0^+) = A_2 - 2A_1 = 19.2$$

$$A_1 = A_2 = -19.2$$

$$v(t) = (-19.2 - 19.2t)e^{-2t} + 24 \text{ (V)}$$

$$i(t) = 4.8(1 + 2t)e^{-2t} \text{ (A)}$$

(c) $R = 1\ \Omega$

$$i(0^+) = 12 \text{ A}, v(0^+) = 12 \text{ V}, v'(0^+) = 48 \text{ V/s}$$

$$\frac{d^2v}{dt^2} + \frac{dv}{dt} + 4v = 96$$

$$s^2 + s + 4 = 0 \Rightarrow s_{1,2} = -\frac{1}{2} \pm j\frac{\sqrt{15}}{2}$$

$$v_n(t) = e^{-t/2}(A_1 \cos\frac{\sqrt{15}}{2}t + A_2 \sin\frac{\sqrt{15}}{2}t)$$

$$v_f(t) = B \Rightarrow B = 96/4 = 24$$

$$v(t) = v_n(t) + v_f(t)$$

$$= e^{-t/2}(A_1 \cos\frac{\sqrt{15}}{2}t + A_2 \sin\frac{\sqrt{15}}{2}t) + 24$$

$$v(0^+) = A_1 + 24 = 12$$

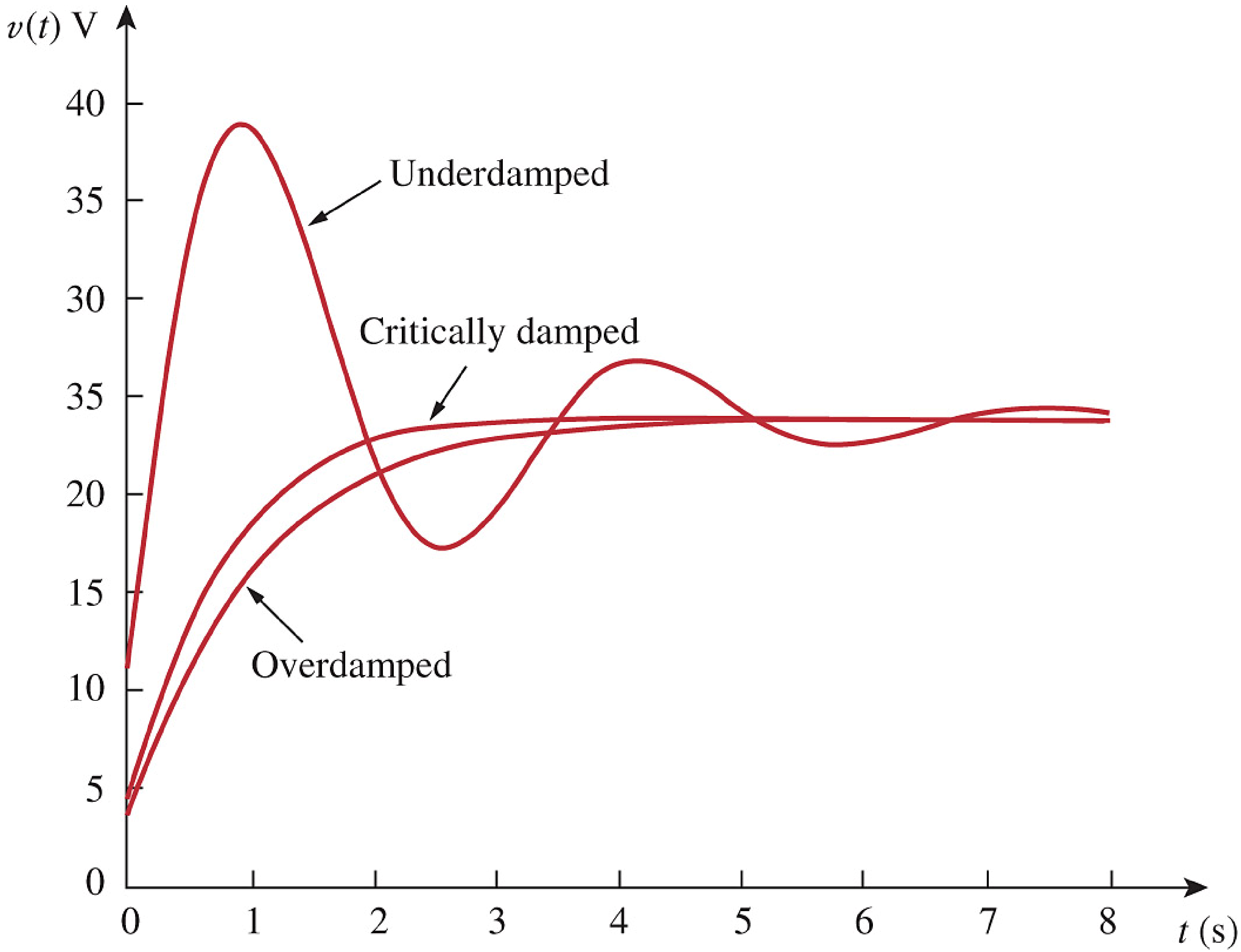
$$v'(0^+) = -\frac{1}{2}A_1 + \frac{\sqrt{15}}{2}A_2 = 48$$

$$A_1 = -12,\ A_2 = \frac{84}{\sqrt{15}} \approx 21.689$$

$$v(t) = e^{-t/2}(-12\cos\frac{\sqrt{15}}{2}t + \frac{84}{\sqrt{15}}\sin\frac{\sqrt{15}}{2}t) + 24 \text{ (V)}$$

$$i(t) = e^{-t/2}(12\cos\frac{\sqrt{15}}{2}t + \frac{12}{\sqrt{15}}\sin\frac{\sqrt{15}}{2}t) \text{ (A)}$$

Figure 8.20 plots the responses for the three cases. From this figure, we observe that the critically damped response approaches the step input of 24 V the fastest.



v(t) V
40
35
30
35
20
15
10
5
0
Underdamped
Critically damped
Overdamped
0
1
2
3
4
5
6
7
8
t (s)

Figure 8.20

8.6 Parallel *RLC* Circuit with Step Input

Consider the circuit in Fig. 8.22. We want to find i due to a sudden application of a dc current. For $t > 0$,

$$I_s = \frac{v}{R} + i + C\frac{dv}{dt}, \quad v = L\frac{di}{dt}$$

$$LC\frac{d^2 i}{dt^2} + \frac{L}{R}\frac{di}{dt} + i = I_s$$

$$\frac{d^2 i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{1}{LC}i = \frac{1}{LC}I_s$$

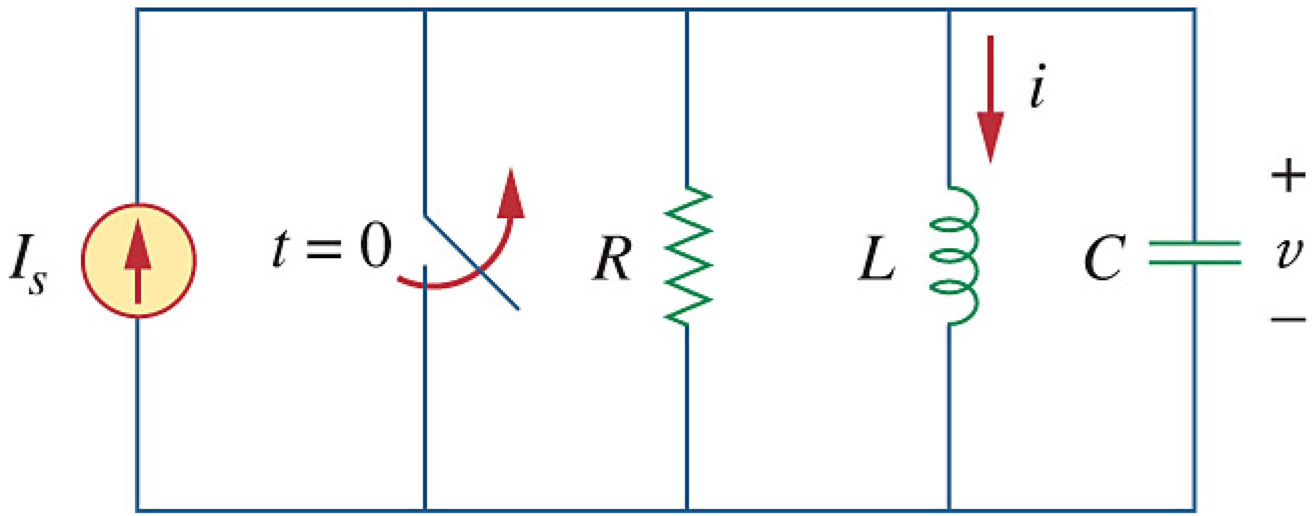


Figure 8.22 Parallel RLC circuit with an applied current.

It can be shown that the solution has three possible forms:

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + I_s$$

(Overdamped)

$$i(t) = (A_1 + A_2 t)e^{-\alpha t} + I_s$$

(Critically damped)

$$i(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t) + I_s$$

(Underdamped)

Practice Problem 8.8 Find $i(t)$ and $v(t)$ for $t > 0$ in the circuit of Fig. 8.24.

Solution :

$$i(0^+) = i(0^-) = 0$$

$$v(0^+) = v(0^-) = 0$$

$$v(0^+) = 5\frac{di(0^+)}{dt} \Rightarrow i'(0^+) = \frac{1}{5}v(0^+) = 0$$

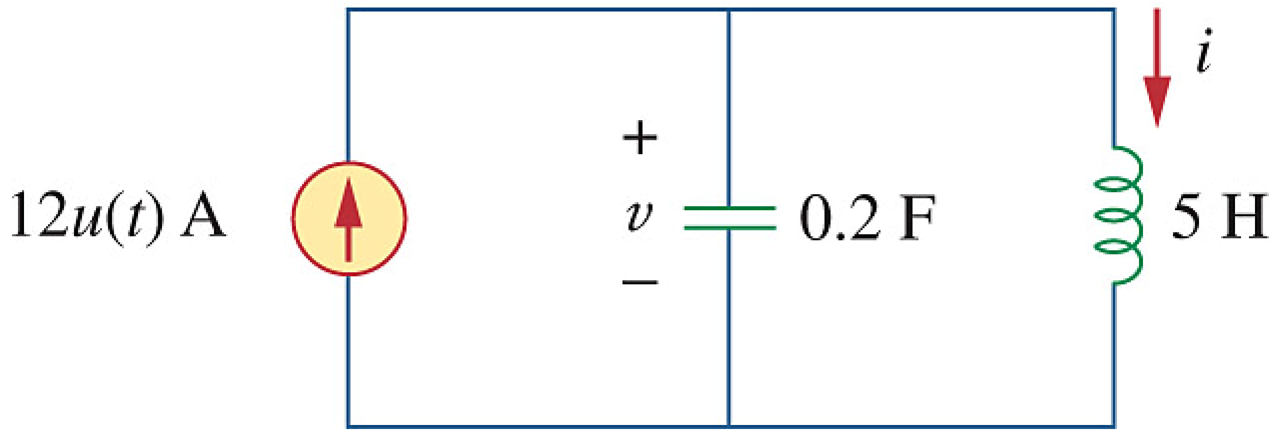


Figure 8.24 An LC circuit.

$$12 = 0.2\frac{dv}{dt} + i, \quad v = 5\frac{di}{dt}$$

$$\frac{d^2 i}{dt^2} + i = 12$$

$$s^2 + 1 \Rightarrow s_{1,2} = \pm j$$

$$i_n(t) = A_1 \cos t + A_2 \sin t$$

$$i_p(t) = 12$$

$$i(t) = i_n(t) + i_p(t) = A_1 \cos t + A_2 \sin t + 12$$

$$i(0^+) = A_1 + 12 = 0$$

$$i'(0^+) = -A_2 = 0$$

$$A_1 = -12,\ A_2 = 0$$

$$i(t) = -12\cos t + 12 = 12(1 - \cos t)\ \text{(A)}$$

$$v(t) = 5\frac{di(t)}{dt} = 60\sin t\ \text{(V)}$$

8.7 General Second-Order Circuits

Practice Problem 8.10 For $t > 0$, obtain $v_o(t)$ in the circuit of Fig. 8.32. (*Hint :* First find v_1 and v_2.)

Solution :

$$v_1(0^+) = v_2(0^+) = 0$$

$$\frac{20 - v_1(0^+)}{1} = \frac{1}{2}\frac{dv_1(0^+)}{dt} + \frac{v_1(0^+) - v_2(0^+)}{1}$$

$$v_1'(0^+) = 2[20 - 2v_1(0^+) + v_2(0^+)] = 40 \text{ (V/s)}$$

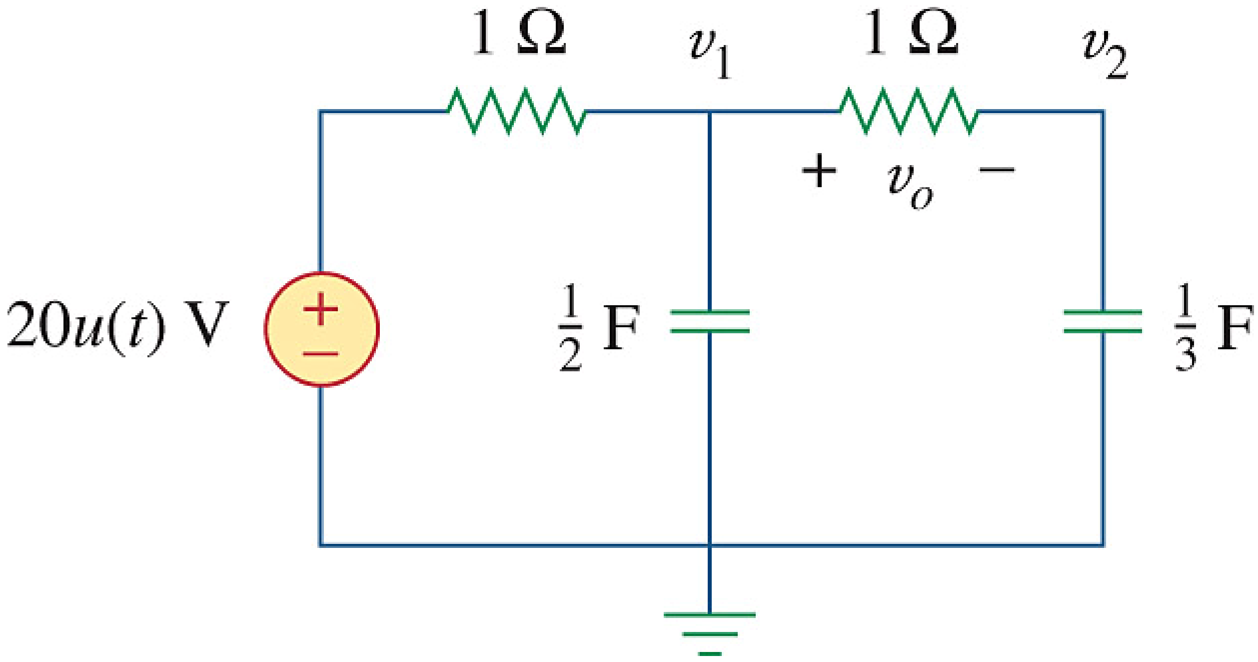


Figure 8.32 An RCC circuit.

$$v_1(\infty) = v_2(\infty) = 20 \text{ (V)}$$

$$\frac{20 - v_1}{1} = \frac{1}{2}\frac{dv_1}{dt} + \frac{v_1 - v_2}{1}, \quad \frac{v_1 - v_2}{1} = \frac{1}{3}\frac{dv_2}{dt}$$

$$\frac{d^2 v_1}{dt^2} + 7\frac{dv_1}{dt} + 6v_1 = 120$$

$$s^2 + 7s + 6 = 0 \Rightarrow s_1 = -1,\ s_2 = -6$$

$$v_{1h}(t) = A_1 e^{-t} + A_2 e^{-6t}$$

$$v_{1p}(t) = 20$$

$$v_1(t) = A_1 e^{-t} + A_2 e^{-6t} + 20$$

$$v_1(0^+) = A_1 + A_2 + 20 = 0$$

$$v_1'(0^+) = -A_1 - 6A_2 = 40$$

$$A_1 = -16,\ A_2 = -4$$

$$v_1(t) = -16e^{-t} - 4e^{-6t} + 20$$

$$v_2(t) = -24e^{-t} + 4e^{-6t} + 20$$

$$v_o(t) = v_1(t) - v_2(t) = 8e^{-t} - 8e^{-6t}\ \text{(V)}$$

8.10 Duality

- The concept of duality is a time-saving, effort-effective measure of solving circuit problems.
- Two circuits are said to be duals of one another if they are described by the same characteristic equations with dual pairs interchanged.
- Dual pairs are shown in Table 8.1.

TABLE 8.1 Dual Pairs

Resistance	Conductance
Inductance	Capacitance
Voltage	Current
Voltage source	Current source
Node	Mesh
Series path	Parallel path
Open circuit	Short circuit
KVL	KCL
Thevenin	Norton

Given a palnar circuit, we construct the dual circuit by taking the following steps:

1. Place a node at the center of each mesh of the given circuit. Place the reference node of the dual circuit outside the given circuit.
2. Draw lines between the nodes such that each line across an element. Replace the element by its dual.

3. To determine the polarity of voltage sources and direction of current sources, follow this rule: A voltage source that produces a positive (clockwise) mesh current has as its dual a current source whose reference direction is from the ground to the nonreference node.

In case of doubt, one may verify the dual circuit by writing the nodal or mesh equations.

Example 8.14 Construct the dual of the circuit in Fig. 8.44.

Solution : See Fig. 8.45.

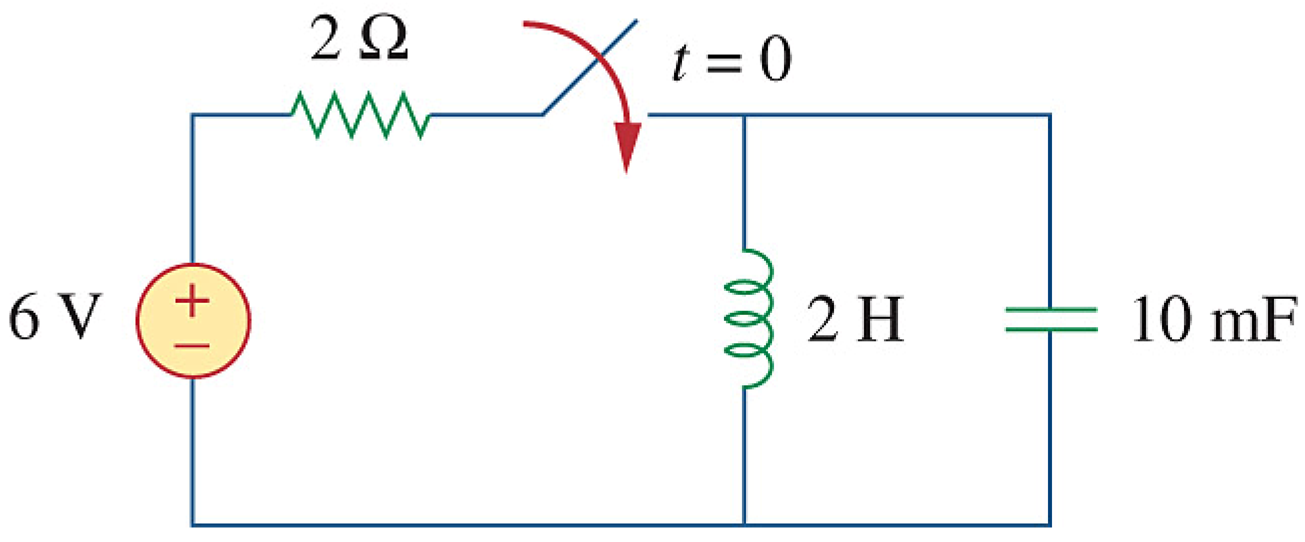


Figure 8.44

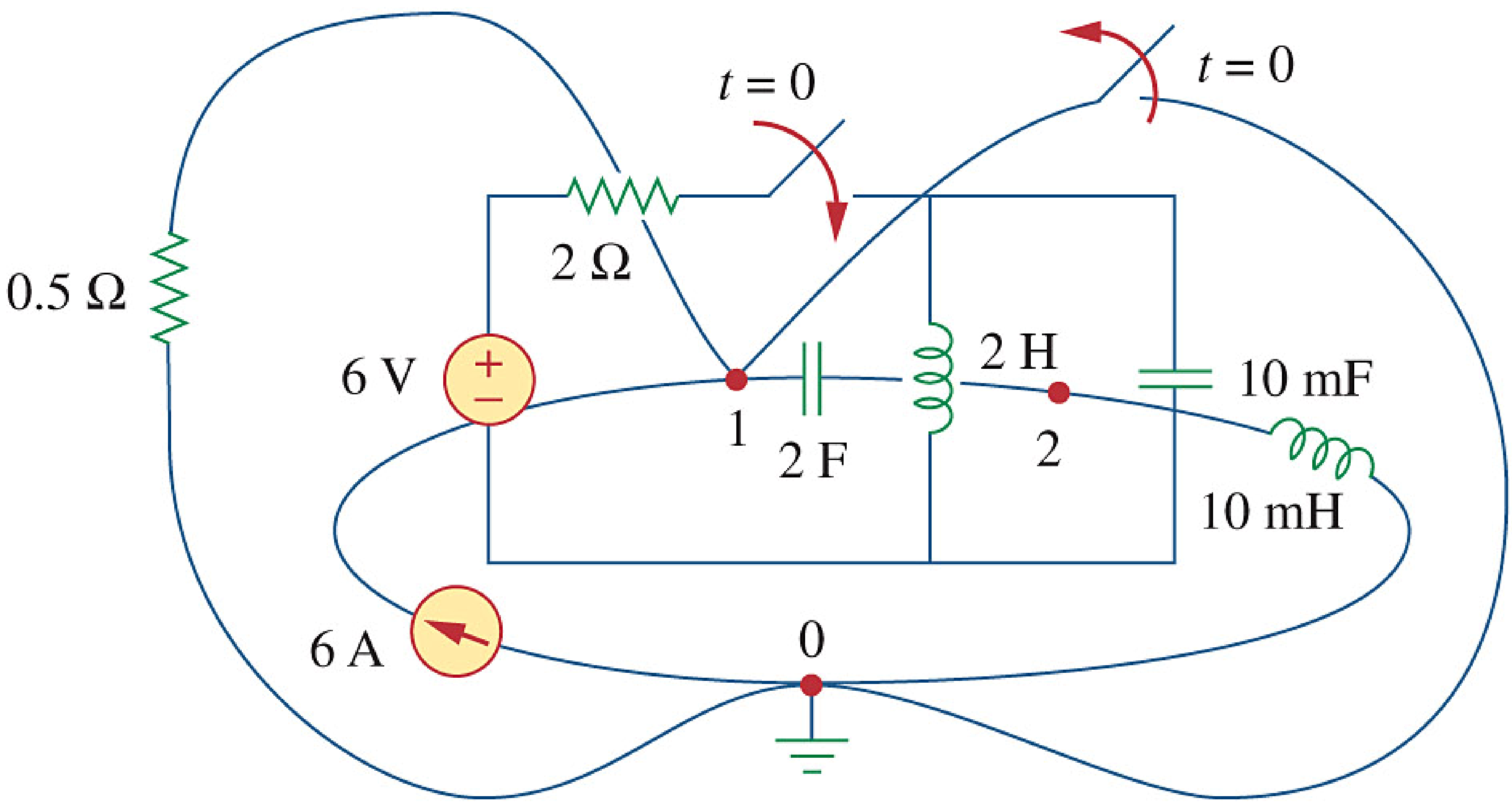


Figure 8.45(a) Construction of the dual circuit of Fig. 8.44.

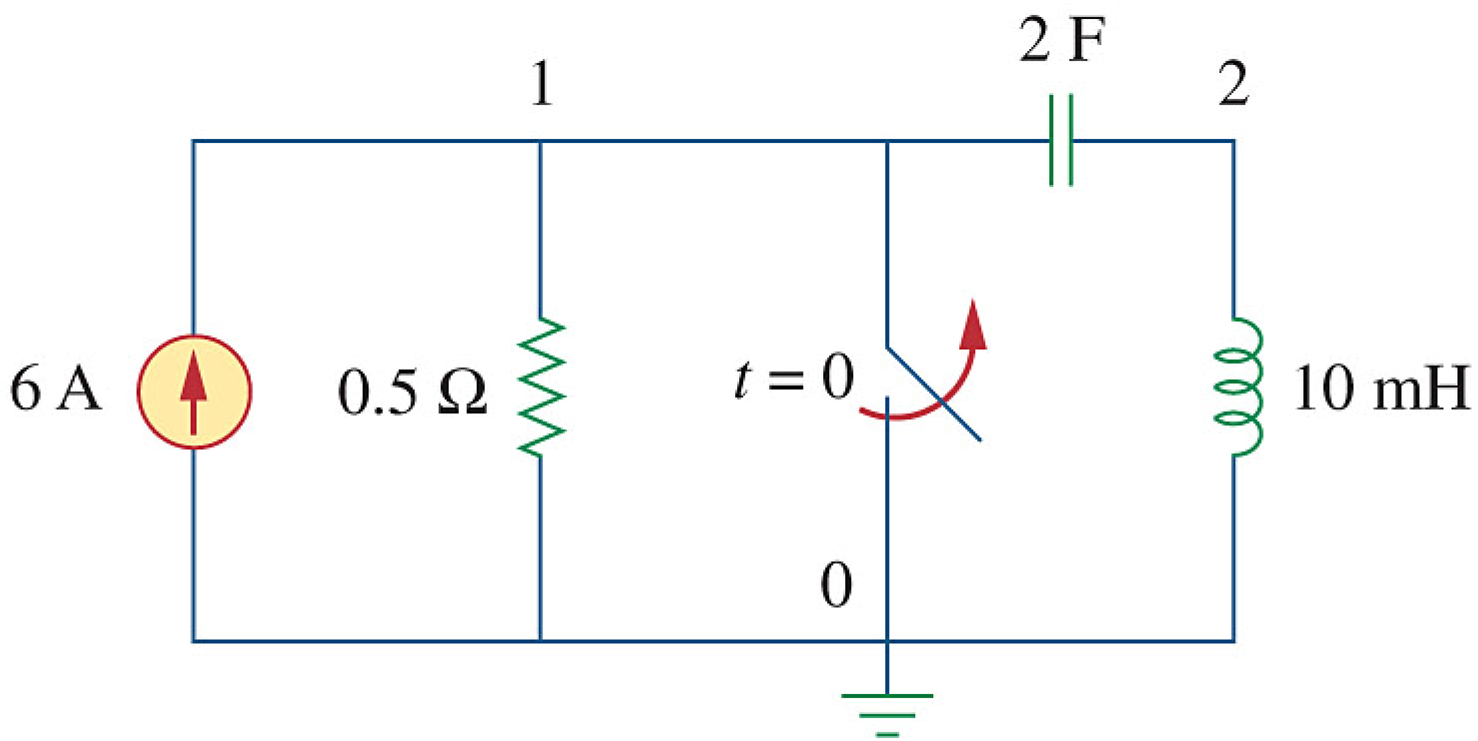


Figure 8.45(b) Dual circuit redrawn.

Example 8.15 Obtain the dual of the circuit in Fig. 8.48.

Solution : See Fig. 8.49.

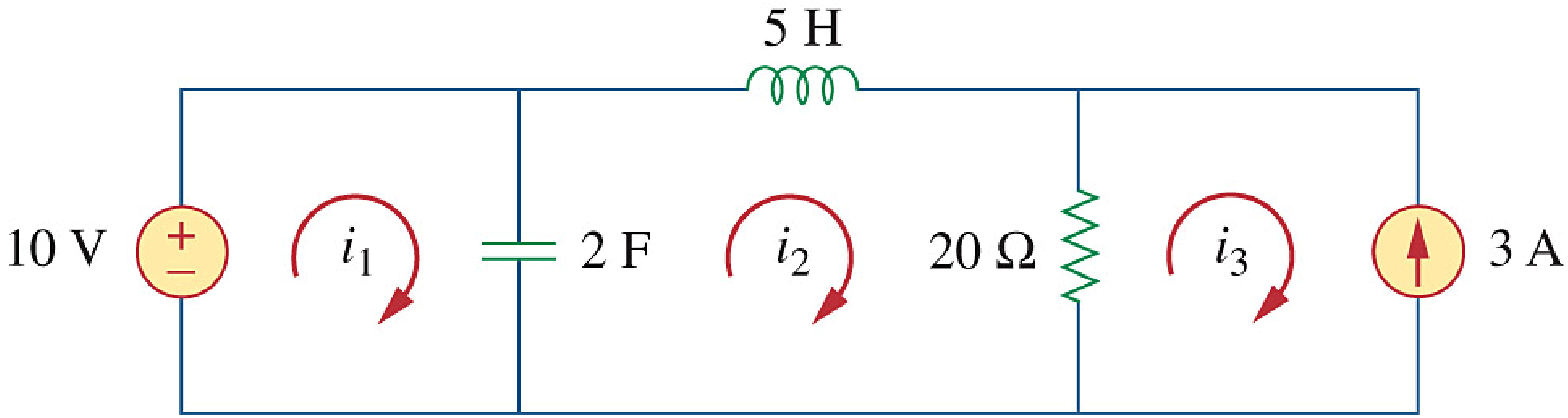


Figure 8.48

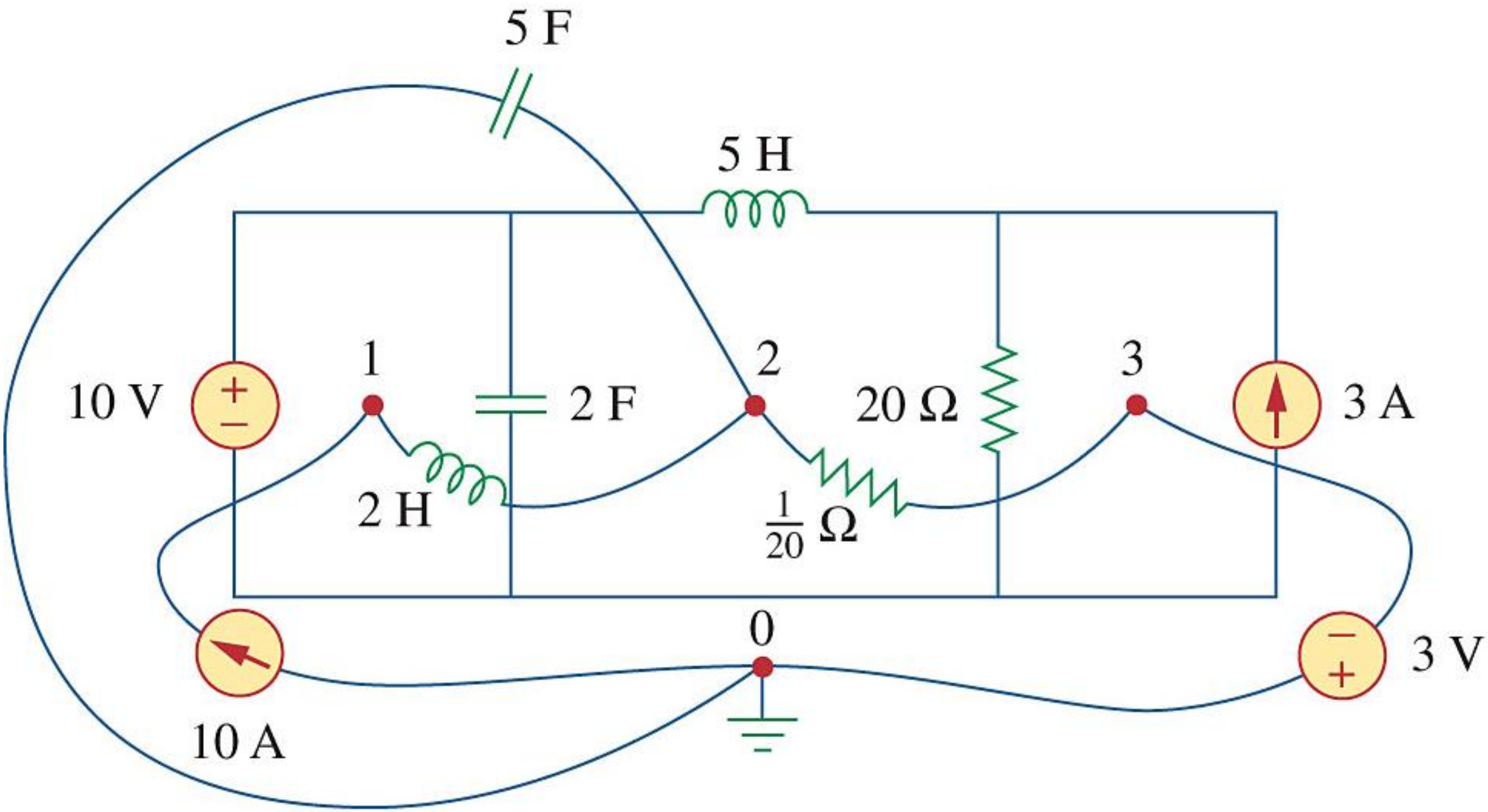


Figure 8.49(a) Construction of the dual circuit of Fig. 8.48.

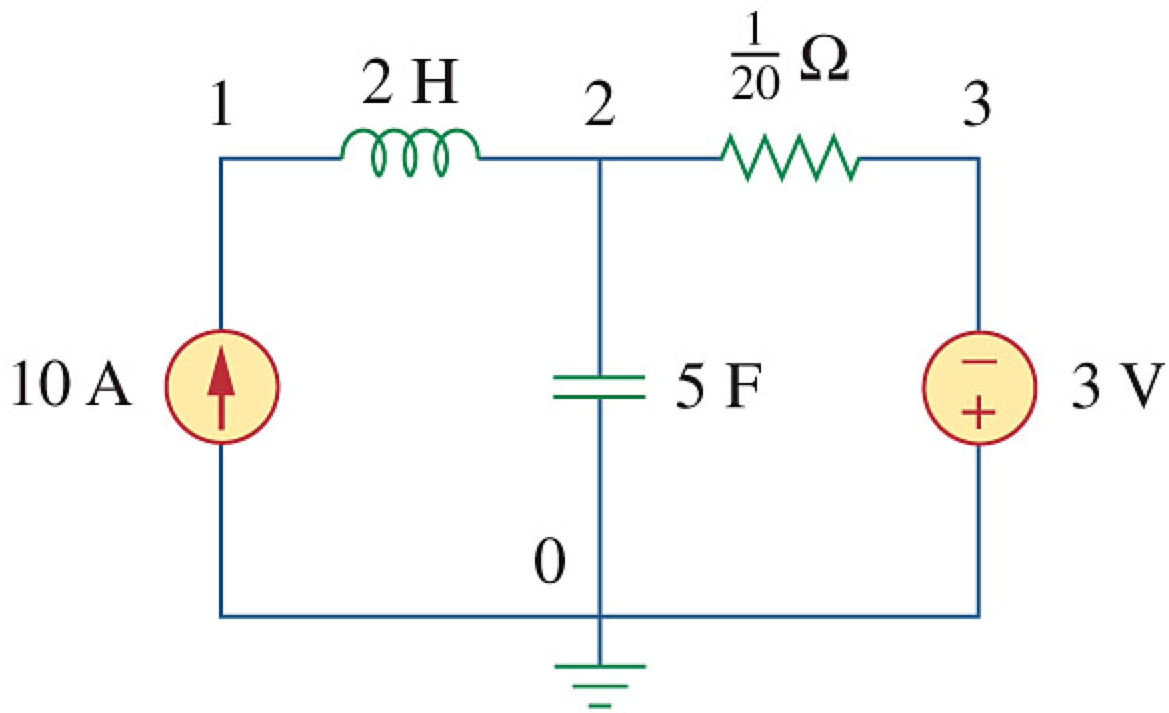


Figure 8.49(b) Dual circuit redrawn.