Homework 3

Answers:

1. (a) It is an impulse train with a period of $T = 4$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\frac{2\pi}{T}t} = \frac{1}{4}$$

We can find that a_k is a constant.

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{4}e^{jk\omega_0 t} = \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{2}\cos(k\frac{\pi}{2}t)$$

(b) Sketch $x(t)$ and we find it can be expressed as

$$x(t) = 1 + \sum_{n=-\infty}^{\infty} rect(t - 5n - \frac{1}{2})$$

$$x(t) = 1 + \sum_{k=-\infty}^{\infty} \frac{1}{5}sinc(\frac{k}{5})e^{jk(2\pi/5)(1/2)}e^{jk(2\pi/5)t}$$

$$x(t) = \frac{6}{5} + \sum_{k=1}^{\infty} \frac{1}{5}sinc(\frac{k}{5})\cos(\frac{2\pi(t-1/2)k}{5})$$

(c) For this signal $T = 2$, $\omega_0 = \pi$

$$a_k = \frac{1}{2}\int_0^1 e^{-t}e^{-jk\omega_0 t}dt$$

$$a_k = \frac{1}{2}\int_0^1 e^{-(1+jk\pi)t}dt$$

$$a_k = \frac{1-e^{-(1+jk\pi)}}{2(1+jk\pi)}$$

$$x(t) = \frac{1-e^{-1}}{2} + \sum_{k=1}^{\infty}[\frac{1-e^{-1}\cos(k\pi)}{1+k^2\pi^2}\cos(k\pi t) + \frac{k\pi(1-e^{-1}\cos(k\pi))}{1+k^2\pi^2}\sin(k\pi t)]$$

2.

$$x(t) = a_0 + \sum_{k=1}^{\infty}(a_k e^{jk\omega_0 t} + a_{-k}e^{-jk\omega_0 t})$$

$$x(t) = a_0 + \sum_{k=1}^{\infty}[(a_k + a_{-k})\cos(k\omega_0 t) + (a_k - a_{-k})j\sin(k\omega_0 t)]$$

$$B[0] = a_0 = \frac{1}{T}\int_0^T x(t)dt$$

$$B[k] = a_k + a_{-k} = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t} + \frac{1}{T}\int_0^T x(t)e^{jk\omega_0 t} = \frac{1}{T}\int_0^T x(t)(e^{-jk\omega_0 t} + e^{jk\omega_0 t})dt$$

$$B[k] = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)dt$$

For the same reason,

$$A[k] = (a_k - a_{-k})j = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)dt$$

3. (a) If $x(t)$ is real. Then $x(t) = x(t)^*$, which implies that $a_k = a^*_{-k}$. It is not true in this case, so $x(t)$ is not real.

   (b) If $x(t)$ is even, then $x(t) = x(-t)$ and $a_k = a_{-k}$. It is true in this case, so $x(t)$ is even.

   (c) We have

$$g(t) = \frac{dx(t)}{dt} \xrightarrow{FS} b_k = jk\frac{2\pi}{T_0}a_k$$

   Therefore,

$$b_k = \begin{cases} 0 & ,k = 0 \\ -k(\frac{1}{2})^{|k|}\frac{2\pi}{T_0}, & otherwise \end{cases}$$

   Since b_k is not even, $\dfrac{dx(t)}{xt}$ is not even.

4. (a) we have

$$x(t) = \sum_{odd\ k} a_k e^{jk(2\pi/T)t}$$

   Therefore,

$$x(t + \frac{T}{2}) = \sum_{odd\ k} a_k e^{jk\pi} e^{jk(2\pi/T)t}$$

   Since $e^{jk\pi} = -1$ for odd k.

$$x(t + \frac{T}{2}) = -x(t)$$

   (b) The Fourier series coefficients of $x(t)$ are

$$a_k = \frac{1}{T}\int_0^{T/2} x(t)e^{-jk\omega_0 t}dt + \frac{1}{T}\int_{T/2}^{T} x(t)e^{-jk\omega_0 t}dt$$

$$a_k = \frac{1}{T}\int_0^{T/2} [x(t) + x(t + \frac{T}{2})e^{-jk\pi}]e^{-j\omega_0 t}dt$$

   Note that a_k evaluate to zero for even k if we have $x(t) = -x(t + \frac{T}{2})$. So it is odd harmonious.

5. (a) $x(t) = \mathrm{LC}\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + \mathrm{RC}\frac{\mathrm{d}y(t)}{\mathrm{d}t} + y(t)$, so the diffeq is given by $\boxed{\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + \frac{\mathrm{d}y(t)}{\mathrm{d}t} + y(t) = x(t)}$.

   (b) The output of $e^{j\omega t}$ should be in the form $H(j\omega)e^{j\omega t}$, where $H(j\omega)$ have nothing to do with time t. Hence $\frac{d^2}{dt^2}(H(j\omega)e^{j\omega t}) + \frac{d}{dt}(H(j\omega)e^{j\omega t}) + H(j\omega)e^{j\omega t} = e^{j\omega t}$. From which we have $H(j\omega) = \boxed{\frac{1}{-\omega^2 + j\omega + 1}}$.

   (c) $H(s) = \boxed{\frac{1}{s^2+s+1}}$.

   (d) $|H(j\omega)| = \left(\frac{1}{(1-\omega^2)^2+\omega^2}\right)^{1/2}$. Use Matlab `ezplot('1/((1-x^ 2)^ 2 + x^ 2) ^  (1/2)', [0 6])` to plot this function of ω:

   (e) Sample code:

```
num = [1];
den = [1 1 1];
x = linspace(0, 6);
H = freqs(num, den, x);
plot(x, abs(H));
xlabel('Angluar Freqs w (rad/s)'),
ylabel('Magnitude Response |H(jw)|'),
title('Hw6-3e')
```

   And the output figure:

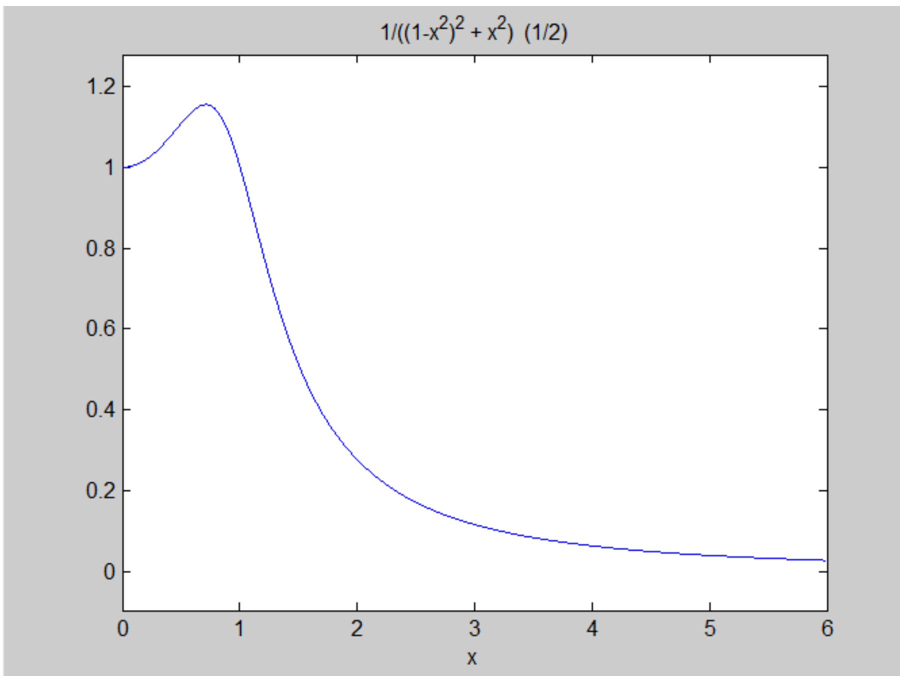

Figure 1: HW3-5d

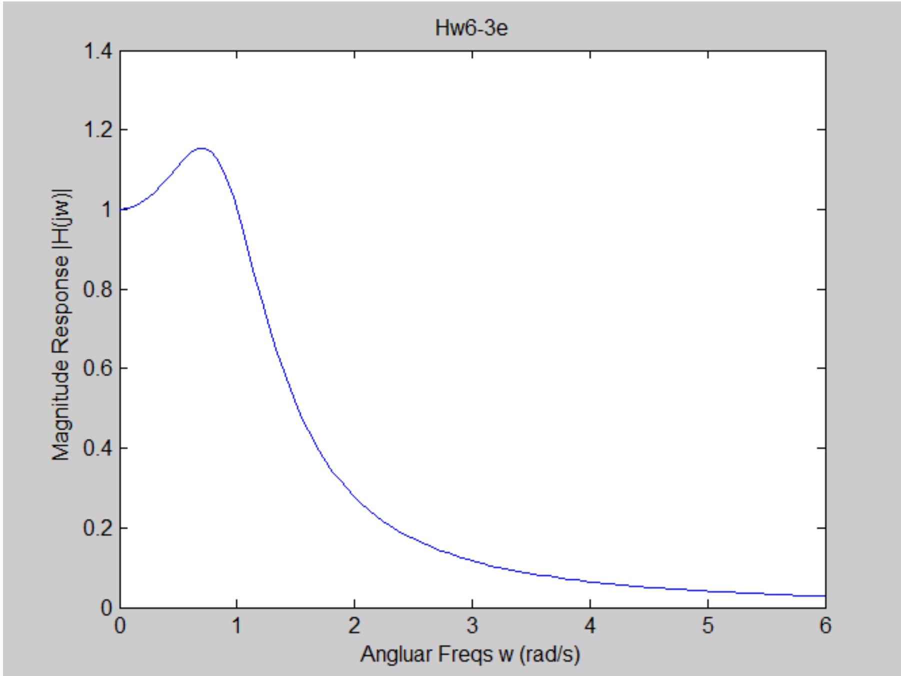

Figure 2: HW3-5e

We can see that the figure is identical to part (d), as expected.

(f) $T_0 = 2\pi$ and $\omega_0 = 1$. And $x(t) = \boxed{1 + \frac{1}{2j}(e^{jt} - e^{-jt}) + \frac{1}{2j}(e^{4jt} - e^{-4jt})}$. Note that this already implies $c_1 = c_4 = 1/(2j)$, $c_{-1} = c_{-4} = -1/(2j)$ and $c_0 = 1$. After calculating the $|c_k|^2$'s, we use the following code to generate a PDS:

```
x = [-4 -1 0 1 4];
y = [0.25 0.25 1 0.25 0.25];
stem(x,y,'filled');
axis([-5 5 0 1.5]);
xlabel('Angular frequncy \omega (k\omega_0)'),
ylabel('Component power |c_k|^2'),
title('Power Density Spectrum (Hw6-3f)')
```

(g) By using computer, we get $\frac{1}{2\pi}\int_0^{2\pi} |x(t)|^2 = 2$. And we have $\sum |c_k|^2 = 1+0.25+0.25+0.25+0.25 = 2$. Hence the Parseval's relation is verified.

(h) $y(t) = 1 * H(0) + \frac{1}{2j}(H(1)e^{jt} - H(-1)e^{-jt} + H(4)e^{j4t} - H(-4)e^{-j4t}) = 1 + \frac{1}{2j}(\frac{1}{j}e^{jt} - \frac{1}{-j}e^{-jt} + \frac{1}{-15+j4}e^{j4t} - \frac{1}{-15-j4}e^{-j4t}) = \boxed{1 - \cos(t) - \frac{4}{241}\cos(4t) - \frac{15}{241}\sin(4t)}$. The code for PDS is very similar to part (f), and the figure is shown below: (*Note that the stem at* $\pm$ *4 should have height 1/964.*)

(i) The high frequency component (4 rad/s sine wave) is attenuated. The circuit serves as a $\boxed{\text{lowpass}}$ filter.

6. The sample code and the three plots are given below:

```
close all;
clear all;

[num, den] = besself(5,1e4);
f = linspace(0, 2e4, 1e5);
H = freqs(num, den, f);
```

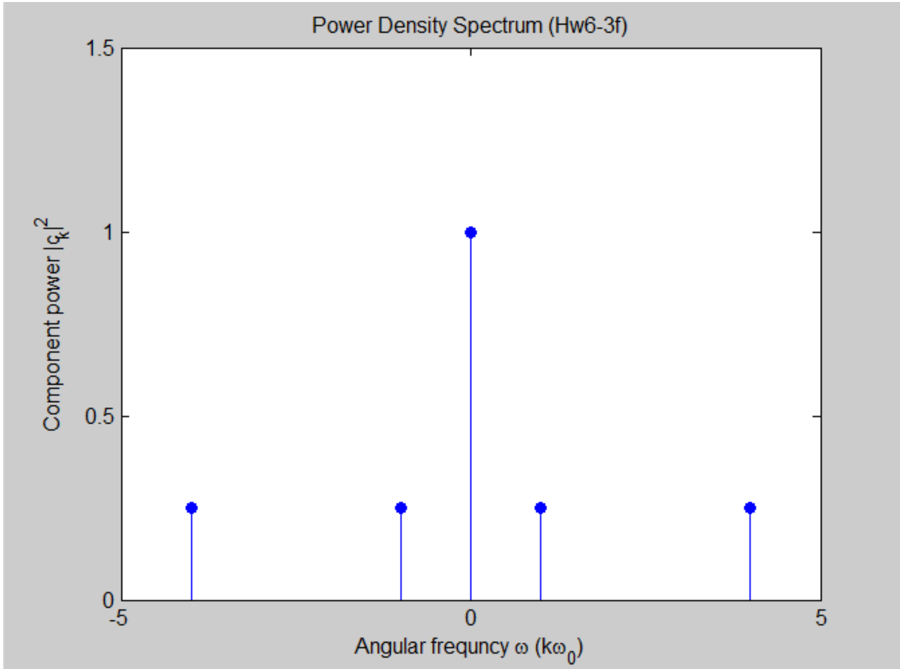

Figure 3: HW3-5f

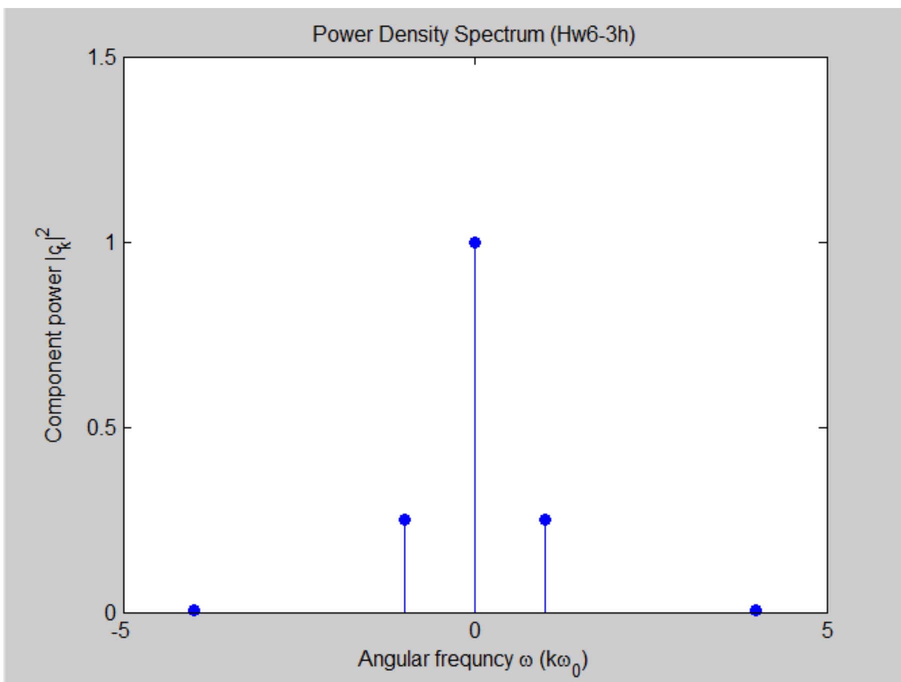

Figure 4: HW3-5h

```
figure, plot(f, abs(H));

[u, t] = gensig('square',0.002, 1, 1e-4);
figure, plot(t,u);
axis([0 0.01 -1 2]);

sys = tf(num, den);
y = lsim(sys, u, t);
```

```
figure, plot(t, y);
axis([0 0.01 -1 2]);
```

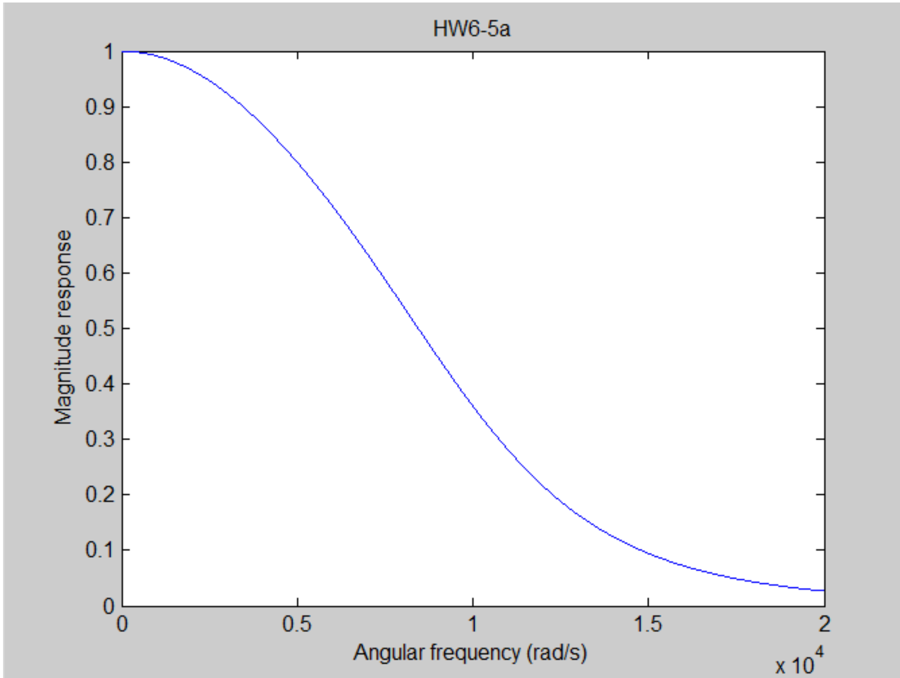

Figure 5: HW3-6a

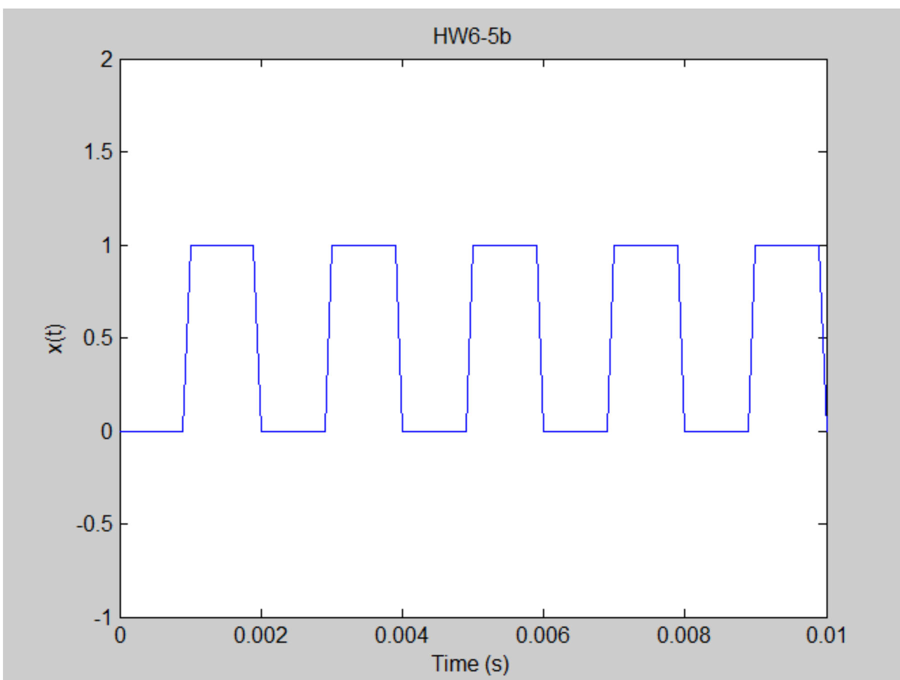

Figure 6: HW3-6b

7. If the FS coefficients of $x(t)$ are periodic with period N, then $a_k = a_{k-N}$ for all k. This implies that $x(t) = x(t)e^{j(2\pi/T)Nt}$ for all t. This is possible only if $x(t)$ is zero for all t other than when $(2\pi/T)Nt = 2\pi k$, where $k \in \mathbb{Z}$. Therefore, $x(t)$ must be in the form $x(t) = \sum_{k=-\infty}^{\infty} g[k]\delta(t - kT/N)$. On the other hand,

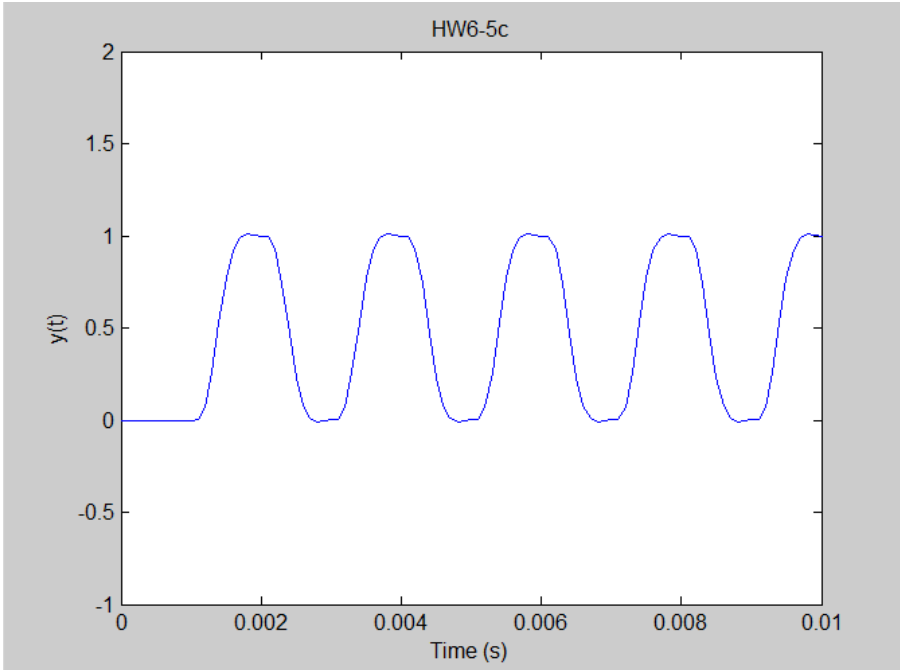

Figure 7: HW3-6c

because $x(t + NT/N) = x(t + T) = x(t)$ from the periodicity of $x(t)$, we easily obtain $g[k + N] = g[k]$ for all k.

8. .

$\sin^5 x = (1/(2j))^5 (e^{jx} - e^{-jx})^5 = \frac{1}{32j}(e^{j5x} - 5e^{j3x} + 10e^{jx} - 10e^{-jx} + 5e^{-j3x} - e^{-j5x}) = \frac{10\sin x - 5\sin(3x) + \sin(5x)}{16}$.
Thus $y(t) = 7\left[\sin 3t + \frac{10b}{16}\sin 3t - \frac{5b}{16}\sin 9t + \frac{b}{16}\sin 15t\right]$. So $c_1 = 7(1 + 10b/16)$.
Thus THD $= 1 - \frac{(1+10b/16)^2}{(1+10b/16)^2+(5b/16)^2+(b/16)^2} 100\% = \boxed{0.0899\%}$ for $b = 0.10$.