Ve460/Vm461 Automatic Control Systems
Chapter 2 Laplace Transform

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2-2 Laplace Transform

Laplace transform is used to solve linear ordinary differential equations. It has the following feature:

- Converting the differential equation into an algebraic equation.

  Can perform simple algebraic rules to obtain the solution in the s-domain. The final solution is obtained by taking the inverse Laplace transform.

2-2-1 Definition of the Laplace Transform

Given the real function $f(t)$ that satisfies the condition

$$\int_0^\infty \left| f(t)e^{-\sigma t} \right| dt < \infty$$

for some finite, real σ, the Laplace transform of $f(t)$ is defined as

$$F(s) = \int_{0^-}^\infty f(t)e^{-st} dt \tag{1}$$

or

$$F(s) = \text{Laplace transform of } f(t) = \mathcal{L}[f(t)]$$

The variable s is referred to as the Laplace operator, which is a complex variable; that is, $s = \sigma + j\omega$, where σ is the real component and ω is the imaginary component.

The defining integration in Eq. (1) is also known as the one-sided Laplace transform, as the integration is evaluated from $t = 0$ to ∞. This simply means that all information contained in $f(t)$ prior to $t = 0$ is ignored or considered to be zero.

Example

Let $u_s(t)$ be a unit-step function that is defined as

$$u_s(t) = \begin{cases} 1, & t \geq 0; \\ 0, & t < 0. \end{cases}$$

The Laplace transform of $u_s(t)$ can be obtained as

$$F(s) = \mathcal{L}[u_s(t)] = \int_0^\infty u_s(t) e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s}. \tag{2}$$

Eq. (2) is valid if

$$\int_0^\infty \left| u_s(t) e^{-\sigma t} \right| dt = \int_0^\infty \left| e^{-\sigma t} \right| dt < \infty,$$

which means that σ, the real part of s, must be greater than zero.

In practice, we simply refer to the Laplace transform of the unit-step function as $1/s$, and rarely do we have to be concerned with the region in the s-plane in which the transform integral converges absolutely.

Example

Consider the exponential function

$$f(t) = e^{-\alpha t}, \quad t \geq 0,$$

where α is a real constant. The Laplace transform of $f(t)$ is written

$$F(s) = \int_0^\infty e^{-\alpha t} e^{-st} dt = \left.\frac{e^{-(s+\alpha)t}}{-(s+\alpha)}\right|_0^\infty = \frac{1}{s+\alpha}.$$

Replace α by a complex number $\alpha + i\beta$, then

$$\mathcal{L}\left[e^{-(\alpha+i\beta)t}\right] = \frac{1}{s+\alpha+i\beta}$$

$$\Rightarrow \mathcal{L}\left[e^{-\alpha t}\cos\beta t - ie^{-\alpha t}\sin\beta t\right] = \frac{s+\alpha-i\beta}{(s+\alpha)^2+\beta^2}$$

$$\Rightarrow \mathcal{L}\left[e^{-\alpha t}\cos\beta t\right] = \frac{s+\alpha}{(s+\alpha)^2+\beta^2}$$

$$\text{and } \mathcal{L}\left[e^{-\alpha t}\sin\beta t\right] = \frac{\beta}{(s+\alpha)^2+\beta^2}$$

In particular

$$\mathcal{L}[\cos\beta t] = \frac{s}{s^2+\beta^2}$$

$$\mathcal{L}[\sin\beta t] = \frac{\beta}{s^2+\beta^2}$$

2-2-2 Inverse Laplace Transformation

Given the Laplace transform $F(s)$, the operation of obtaining $f(t)$ is termed the inverse Laplace transformation and is denoted by

$$f(t) = \text{Inverse Laplace Transform of } F(s) = \mathcal{L}^{-1}[F(s)]$$

The inverse Laplace transform integral is given as

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s) e^{st} ds$$

where c is a real constant that is greater than the real parts of all the singularities of $F(s)$.

2-2-3 Important theorems of Laplace Transform

Multiplication by a constant	$\mathcal{L}[kf(t)] = kF(s)$
Sum and difference	$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$
Differentiation	$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$
	$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - f^{(n-1)}(0),$
	where $f^{(k)}(0) = \left.\frac{d^k f(t)}{dt^k}\right\rvert_{t=0}$
Integration	$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}$
	$\mathcal{L}\left[\int_0^{t_n}\int_0^{t_{n-1}}\cdots\int_0^{t_1} f(t)dt_1 dt_2\cdots dt_{n-1}\right] = \frac{F(s)}{s^n}$

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Shift in time	$\mathcal{L}[f(t-T)u_s(t-T)] = e^{-Ts}F(s)$
Initial Value Theorem	$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$
Final Value Theorem	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$ if $sF(s)$ does not have poles on or to the right of the imaginary axis in the s-plane
Complex shifting	$\mathcal{L}[e^{\mp\alpha t}f(t)] = F(s\pm\alpha)$
Real convolution	$F_1(s)F_2(s) = \mathcal{L}\left[\int_0^t f_1(\tau)f_2(t-\tau)d\tau\right] = \mathcal{L}\left[\int_0^t f_2(\tau)f_1(t-\tau)d\tau\right] = \mathcal{L}[f_1(t)*f_2(t)]$
Complex convolution	$\mathcal{L}[f_1(t)f_2(t)] = F_1(s)*F_2(s)$

Differentiation

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0).$$

Proof

$$\begin{aligned}
\mathcal{L}\left[\frac{df(t)}{dt}\right] = \int_0^\infty \frac{df(t)}{dt} e^{-st} dt = \int_0^\infty e^{-st} df(t) \\
&= f(t)e^{-st}\Big|_0^\infty - \int_0^\infty f(t)\frac{de^{-st}}{dt} dt \\
&= f(t)e^{-st}\Big|_0^\infty - \int_0^\infty f(t)(-s)e^{-st} dt \\
&= \underbrace{f(\infty)e^{-s\cdot\infty}}_{=0} - f(0) + s \cdot F(s) \\
&= sF(s) - f(0).
\end{aligned}$$

$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - f^{(n-1)}(0),$$

where $f^{(k)}(0) = \left.\dfrac{d^k f(t)}{dt^k}\right|_{t=0}$.

Proof

$$\begin{aligned}
\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] &= \mathcal{L}\left[\frac{d}{dt}\left(\frac{d^{n-1} f(t)}{dt^{n-1}}\right)\right] \\
&= s \cdot \mathcal{L}\left[\frac{d^{n-1} f(t)}{dt^{n-1}}\right] - \left.\frac{d^{n-1} f(t)}{dt^{n-1}}\right|_{t=0} \\
&= s\left[s\mathcal{L}\left[\frac{d^{n-2} f(t)}{dt^{n-2}}\right] - \left.\frac{d^{n-2} f(t)}{dt^{n-2}}\right|_{t=0}\right] - f^{(n-1)}(0) \\
&= \cdots
\end{aligned}$$

Integration

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}.$$

Proof

$$\begin{aligned}
\text{LHS} &= \int_0^\infty \int_0^t f(\tau)d\tau e^{-st}dt \\
&= \int_0^\infty \int_0^t f(\tau)d\tau \; d\left(-\frac{1}{s}e^{-st}\right) \\
&= -\frac{1}{s}e^{-st}\cdot \int_0^t f(\tau)d\tau\bigg|_{t=0}^{\infty} - \int_0^\infty \left(-\frac{1}{s}\right)e^{-st}f(t)dt \\
&= \frac{F(s)}{s}.
\end{aligned}$$

Shift in time

$$\mathcal{L}[f(t-T)u_s(t-T)] = e^{-Ts}F(s).$$

Proof

$$\int_0^\infty f(t-T)u_s(t-T)e^{-st}dt \overset{\tau=t-T}{=\!=\!=} \int_{-T}^\infty f(\tau)u_s(\tau)e^{-s(\tau+T)}d\tau$$
$$= e^{-sT}\int_0^\infty f(\tau)e^{-s\tau}d\tau$$
$$= e^{-Ts}F(s).$$

Final Value Theorem

If the Laplace transform of $f(t)$ is $F(s)$, and if $sF(s)$ does not have poles on or to the right of the imaginary axis in the s-plane

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

Proof (not rigorous)

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = \int_0^\infty \frac{df(t)}{dt} e^{-st} dt = sF(s) - f(0)$$

$$\lim_{s\to 0} \int_0^\infty \frac{df(t)}{dt} e^{-st} dt = \lim_{s\to 0} sF(s) - f(0)$$

$$\text{LHS} = \int_0^\infty \frac{df(t)}{dt} dt = \lim_{t\to\infty} f(t) - f(0)$$

$$\Rightarrow \quad \lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t)$$

Initial Value Theorem is similar

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

$$\begin{aligned}
\lim_{s\to\infty}\left(sF(s) - f(0)\right) &= \lim_{s\to\infty}\mathcal{L}\left[\frac{df(t)}{dt}\right] \\
&= \lim_{s\to\infty}\int_0^\infty \frac{df(t)}{dt}e^{-st}dt \\
&= \int_0^\infty \frac{df(t)}{dt}e^{-t\infty}dt = 0
\end{aligned}$$

$$\Rightarrow \quad \lim_{s\to\infty} sF(s) = f(0)$$

Example

Consider the function

$$F(s) = \frac{5}{s(s^2 + s + 2)}.$$

Because $sF(s)$ is analytic on the imaginary axis and in the Right Half Plane (RHP), the Final Value Theorem may be applied. Using Final Value Theorem, we have

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) = \lim_{s \to 0} \frac{5}{s^2 + s + 2} = \frac{5}{2}.$$

Example

Consider the function

$$F(s) = \frac{\omega}{s^2 + \omega^2},$$

which is the Laplace transform of $f(t) = \sin \omega t$.

Because the function $sF(s)$ has two poles on the imaginary axis of the s-plane, the Final Value Theorem cannot be applied in this case. In other words, although the Final Value Theorem would yield a value of zero as the final value of $f(t)$, the result is erroneous.

Rational Transfer Function

$$a_n \frac{d^n y(t)}{dt^n} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + \cdots + b_1 \frac{du(t)}{dt} + b_0 u(t).$$

With zero initial state, we have

$$a_n s^n Y(s) + \cdots + a_1 s Y(s) + a_0 Y(s) = b_m s^m U(s) + \cdots + b_1 s U(s) + b_0 U(s)$$

$$\Rightarrow \quad \frac{Y(s)}{U(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0}.$$

2-3 Inverse Laplace Transform by Partial Fraction Expansion

Consider the Inverse Laplace Transform of a rational function in s:

$$G(s) = \frac{Q(s)}{P(s)},$$

where $P(s)$ and $Q(s)$ are (coprime) polynomials of s.

It is assumed that the order of $P(s)$ in s is greater than that of $Q(s)$. The polynomial may be written

$$P(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0,$$

where a_0, a_1, $\cdots$, a_{n-1} are real coefficients. The methods of partial fraction expansion will be given depending on the poles of $G(s)$:

1. Simple poles;
2. Multiple-order poles;
3. Complex conjugate poles.

Case 1: $G(s)$ has simple poles

If all the poles of $G(s)$ are simple and real, we have

$$G(s) = \frac{Q(s)}{P(s)} = \frac{Q(s)}{(s+s_1)(s+s_2)\cdots(s+s_n)}, \tag{3}$$

where $s_1 \neq s_2 \neq \cdots \neq s_n$. Applying the partial fraction expansion, Eq. (3) can be written as

$$G(s) = \frac{K_{s_1}}{s+s_1} + \frac{K_{s_2}}{s+s_2} + \cdots + \frac{K_{s_n}}{s+s_n}. \tag{4}$$

The coefficient K_{s_i}, $i = 1, \cdots, n$, can be determined by multiplying both sides of Eq. (4) by the factor $(s+s_i)$ and then setting s equal to $-s_i$. For instance, to find K_{s_1}, we have

$$K_{s_1} = \left[(s+s_1)\frac{Q(s)}{P(s)}\right]\bigg|_{s=-s_1} = \frac{Q(-s_1)}{(s_2-s_1)(s_3-s_1)\cdots(s_n-s_1)}.$$

Example

Consider the function

$$G(s) = \frac{5s+3}{(s+1)(s+2)(s+3)} = \frac{5s+3}{s^3+6s^2+11s+6},$$

which can be written in the partial fraction expansion form:

$$G(s) = \frac{K_{-1}}{s+1} + \frac{K_{-2}}{s+2} + \frac{K_{-3}}{s+3}.$$

The coefficients K_{-1}, K_{-2}, and K_{-3} are determined as follows:

$$K_{-1} = [(s+1)G(s)]\Big|_{s=-1} = \frac{5s+3}{(s+2)(s+3)}\Big|_{s=-1}$$

$$= \frac{5(-1)+3}{(2-1)(3-1)} = -1,$$

$$K_{-2} = [(s+2)G(s)]\Big|_{s=-2} = \frac{5s+3}{(s+1)(s+3)}\Big|_{s=-2}$$
$$= \frac{5(-2)+3}{(1-2)(3-2)} = 7,$$
$$K_{-3} = [(s+3)G(s)]\Big|_{s=-3} = \frac{5s+3}{(s+1)(s+2)}\Big|_{s=-3}$$
$$= \frac{5(-3)+3}{(1-3)(2-3)} = -6.$$

Thus, we have

$$G(s) = \frac{-1}{s+1} + \frac{7}{s+2} - \frac{6}{s+3},$$

and then

$$g(t) = -e^{-t} + 7e^{-2t} - 6e^{-3t}.$$

Case 2: $G(s)$ has multiple-order poles

If r of the n poles of $G(s)$ are identical, we say that the pole at $s = -s_i$ is of multiplicity r. In this case, $G(s)$ can be written as

$$G(s) = \frac{Q(s)}{P(s)} = \frac{Q(s)}{(s+s_1)(s+s_2)\cdots(s+s_{n-r})(s+s_i)^r},$$

$i \neq 1, 2, \cdots, n-r$. Then $G(s)$ can be expanded as

$$\begin{aligned} G(s) =& \underbrace{\frac{K_{s_1}}{s+s_1} + \frac{K_{s_2}}{s+s_2} + \cdots + \frac{K_{s_{n-r}}}{s+s_{n-r}}}_{|\longleftarrow \quad n-r \text{ terms of simple poles} \quad \longrightarrow|} \\ &+ \underbrace{\frac{A_1}{s+s_i} + \frac{A_2}{(s+s_i)^2} + \cdots + \frac{A_r}{(s+s_i)^r}}_{|\longleftarrow \quad r \text{ terms of repeated poles} \quad \longrightarrow|}. \end{aligned}$$

Then $(n-r)$ coefficients, K_{s_1}, K_{s_2}, $\cdots$, $K_{s_{n-r}}$, which correspond to simple poles, may be evaluated by the method described in Case 1. The determination of the coefficients that correspond to the multiple-order poles is described as follows:

$$
\begin{aligned}
A_r &= \left.\left[(s+s_i)^r G(s)\right]\right|_{s=-s_i}, \\
A_{r-1} &= \frac{d}{ds}\left.\left[(s+s_i)^r G(s)\right]\right|_{s=-s_i}, \\
A_{r-2} &= \frac{1}{2!}\frac{d^2}{ds^2}\left.\left[(s+s_i)^r G(s)\right]\right|_{s=-s_i}, \\
&\vdots \\
A_1 &= \frac{1}{(r-1)!}\frac{d^{r-1}}{ds^{r-1}}\left.\left[(s+s_i)^r G(s)\right]\right|_{s=-s_i}.
\end{aligned}
$$

Example

Consider the function

$$G(s) = \frac{1}{s(s+1)^3(s+2)} = \frac{1}{s^5 + 5s^4 + 9s^3 + 7s^2 + 2s}.$$

We can write $G(s)$ as

$$G(s) = \frac{K_0}{s} + \frac{K_{-2}}{s+2} + \frac{A_1}{s+1} + \frac{A_2}{(s+1)^2} + \frac{A_3}{(s+1)^3}.$$

The coefficients corresponding to the simple poles are

$$K_0 = [sG(s)]\Big|_{s=0} = \frac{1}{(s+1)^3(s+2)}\Big|_{s=0} = \frac{1}{2},$$

$$K_{-2} = [(s+2)G(s)]\Big|_{s=-2} = \frac{1}{s(s+1)^3}\Big|_{s=-2} = \frac{1}{2}.$$

The coefficients for the pole $s = 1$ can be obtained:

$$A_3 = \left[(s+1)^3 G(s)\right]\Big|_{s=-1} = \frac{1}{s(s+2)}\Big|_{s=-1} = -1,$$

$$A_2 = \frac{d}{ds}\left[(s+1)^3 G(s)\right]\Big|_{s=-1} = \frac{d}{ds}\left[\frac{1}{s(s+2)}\right]\Big|_{s=-1}$$

$$= \frac{-2s-2}{s^2(s+2)^2}\Big|_{s=-1} = 0,$$

$$A_1 = \frac{d^2}{ds^2}\left[(s+1)^3 G(s)\right]\Big|_{s=-1} = \frac{1}{2}\frac{d^2}{ds^2}\left[\frac{1}{s(s+2)}\right]\Big|_{s=-1}$$

$$= \frac{1}{2}\frac{6s^2+12s+8}{s^3(s+2)^3}\Big|_{s=-1} = \frac{3(-1)^2+6(-1)+4}{(-1)^3(-1+2)^3} = -1.$$

The complete partial fraction expansion is

$$G(s) = \frac{1}{2s} + \frac{1}{2(s+2)} - \frac{1}{s+1} - \frac{1}{(s+1)^3},$$

and then

$$g(t) = \frac{1}{2}u_s(t) + \frac{1}{2}e^{-2t} - e^{-t} - \frac{1}{2}t^2e^{-t}.$$

Case 3: $G(s)$ has complex conjugate poles

The partial fraction expansion of $G(s)$ is valid also for simple complex conjugate poles.

Because complex conjugate poles are more difficult to handle and are of special interest in control system studies, they deserve special treatment here.

Suppose that $G(s)$ contains a pair of complex poles:

$$s = -\sigma + j\omega \quad \text{and} \quad s = -\sigma - j\omega.$$

The corresponding coefficients can be found by:

$$K_{-\sigma+j\omega} = (s + \sigma - j\omega)G(s)\Big|_{s=-\sigma+j\omega},$$

$$K_{-\sigma-j\omega} = (s + \sigma + j\omega)G(s)\Big|_{s=-\sigma-j\omega}.$$

Example

Consider the second-order prototype function

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Let us assume that $\zeta < 1$ so that the poles of $G(s)$ are complex. Then, $G(s)$ is expanded as follows:

$$G(s) = \frac{K_{-\sigma+j\omega}}{s+\sigma-j\omega} + \frac{K_{-\sigma-j\omega}}{s+\sigma+j\omega},$$

where

$$\sigma = \zeta\omega_n,$$

and

$$\omega = \omega_n\sqrt{1-\zeta^2}.$$

The coefficients are then determined as

$$
\begin{aligned}
K_{-\sigma+j\omega} &= (s+\sigma-j\omega)G(s)\Big|_{s=-\sigma+j\omega} \\
&= \frac{\omega_n^2}{s+\sigma+j\omega}\Bigg|_{s=-\sigma+j\omega} = \frac{\omega_n^2}{2j\omega}, \\
K_{-\sigma-j\omega} &= (s+\sigma+j\omega)G(s)\Big|_{s=-\sigma-j\omega} \\
&= \frac{\omega_n^2}{s+\sigma-j\omega}\Bigg|_{s=-\sigma-j\omega} = -\frac{\omega_n^2}{2j\omega}.
\end{aligned}
$$

The complete partial fraction expansion is thus

$$
G(s) = \frac{\omega_n^2}{2j\omega}\left[\frac{1}{s+\sigma-j\omega} - \frac{1}{s+\sigma+j\omega}\right].
$$

Taking the inverse Laplace transform yields

$$
\begin{aligned}
g(t) &= \frac{\omega_n^2}{2j\omega} e^{-\sigma t}(e^{j\omega t} - e^{-j\omega t}) \\
&= \frac{\omega_n^2}{\omega} e^{-\sigma t} \cdot \sin \omega t \\
&= \frac{\omega_n^2}{\omega_n\sqrt{1-\zeta^2}} e^{-\sigma t} \cdot \sin \omega_n \sqrt{1-\zeta^2}\, t,
\end{aligned}
$$

or,

$$
g(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t, \quad t \geq 0.
$$