Example

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Consider the following cascade of LTI systems:

$$x(t) \rightarrow \boxed{h_1(t)} \rightarrow \boxed{h_2(t)} \rightarrow y(t),$$

where $h_1(t) = e^{-t}u(t)$ and $h_2(t) = e^{-3t}u(t)$.

1. Find the frequency response of the overall system.
2. Find the linear constant coefficient differential equation that describes this system.

(Selected from Midterm Exam 2 of Summer 2014)

Solution

1. *The overall frequency response is*

$$\boxed{H(\omega) = \frac{1}{j\omega + 1}\frac{1}{j\omega + 3}}$$

2. *The linear constant coefficient differential equation*

$$H(\omega) = \frac{1}{(j\omega + 1)(j\omega + 3)} = \frac{1}{(j\omega)^2 + 4j\omega + 3}$$

so

$$\boxed{3y(t) + 4\frac{\mathrm{d}}{\mathrm{d}t}y(t) + \frac{d^2}{dt^2}y(t) = x(t)}$$

Example (1)

Example

Show that

$$\frac{1}{a}\,\mathrm{rect}\left(\frac{t}{a}\right) * \mathrm{rect}\left(\frac{t}{a}\right) = \mathrm{tri}\left(\frac{t}{a}\right)$$

where $a > 0$.

Hint: You may use the fact that $\mathrm{rect}(t) * \mathrm{rect}(t) = \mathrm{tri}(t)$

Use the convolution definition and variable exchange

$$\mathrm{tri}(t) = \mathrm{rect}(t) * \mathrm{rect}(t)$$

$$\mathrm{tri}(t) = \int_{-\infty}^{\infty} \mathrm{rect}(\tau)\,\mathrm{rect}(t-\tau)\; d\tau$$

$$\mathrm{tri}\left(\frac{t}{a}\right) = \int_{-\infty}^{\infty} \mathrm{rect}(\tau)\,\mathrm{rect}\left(\frac{t}{a}-\tau\right)\; d\tau$$

$$\mathrm{tri}\left(\frac{t}{a}\right) = \int_{-\infty}^{\infty} \mathrm{rect}\left(a\frac{\tau}{a}\right)\mathrm{rect}\left(\frac{t}{a}-a\frac{\tau}{a}\right)\; d\tau$$

$$\mathrm{tri}\left(\frac{t}{a}\right) = \int_{-\infty}^{\infty} \mathrm{rect}\left(\frac{\tau'}{a}\right)\mathrm{rect}\left(\frac{t}{a}-\frac{\tau'}{a}\right)\frac{1}{a}\,d\tau', \quad \tau' = a\tau$$

$$\mathrm{tri}\left(\frac{t}{a}\right) = \frac{1}{a}\,\mathrm{rect}\left(\frac{t}{a}\right) * \mathrm{rect}\left(\frac{t}{a}\right)$$

Solution (2)

Use the convolution definition and integration

$$y(t) = \mathrm{rect}\left(\frac{t}{a}\right) * \mathrm{rect}\left(\frac{t}{a}\right)$$

- *For $t < -a$, the integral is 0.*
- *For* $-a < t < 0$

$$\int_{-a/2}^{t+a/2} d\tau = a + t$$

- *For* $0 < t < a$

$$\int_{t-a/2}^{a/2} d\tau = a - t$$

- *For $t > a$, the integral is 0.*
- *Combining*

$$y(t) = \begin{cases} a + t, & -a < t < 0 \\ a - t, & 0 < t < a \\ 0, & \textit{otherwise} \end{cases} = a\,\mathrm{tri}(t/a)\,.$$

Solution (3)

Show that the LHS and RHS have the same Fourier transform.

$$\mathrm{tri}(t) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{sinc}^2\left(\frac{\omega}{2\pi}\right).$$

Using FT time transform property, we have

$$\mathrm{tri}\left(\frac{t}{a}\right) \overset{\mathcal{F}}{\longleftrightarrow} a\,\mathrm{sinc}^2\left(a\frac{\omega}{2\pi}\right).$$

Using FT convolution property, LHS becomes

$$\frac{1}{a}\mathrm{rect}\left(\frac{t}{a}\right) * \mathrm{rect}\left(\frac{t}{a}\right) \overset{\mathcal{F}}{\longleftrightarrow} a\,\mathrm{sinc}^2\left(a\frac{\omega}{2\pi}\right).$$

Therefore

$$\frac{1}{a}\mathrm{rect}\left(\frac{t}{a}\right) * \mathrm{rect}\left(\frac{t}{a}\right) = \mathrm{tri}\left(\frac{t}{a}\right)$$