Solution for Homework 1

1. (a) $\boxed{x(t) = |\cos(t)|}$

   (b)

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

$$= \boxed{\begin{cases} 0 & t < -\pi, \\ -\sin(t) & -\pi \leqslant t \leqslant -\pi/2, \\ 2 + \sin(t) & -\pi/2 < t \leqslant \pi/2, \\ 4 - \sin(t) & \pi/2 < t \leqslant \pi, \\ 4 & \pi < t. \end{cases}}$$

   The sketch is on the last page.

2.

$$A = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{0}^{T} e^{-t}dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} \left(1 - e^{-T}\right)$$

$$= \boxed{0}$$

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{0}^{T} e^{-2t}dt$$

$$= \lim_{T \to \infty} \frac{1}{4T} \left(1 - e^{-2T}\right)$$

$$= \boxed{0}$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{0}^{\infty} e^{-2t}dt$$

$$= \boxed{\frac{1}{2}}$$

   It is an energy signal.

3. (a) If T_1/T_2 is rational, then there exists some integers n_1 and n_2 such that $T_1/T_2 = n_2/n_1$. Let $T = n_1 T_1 = n_2 T_2$. Then

$$x(t + T) = x_1(t + T) + x_2(t + T) = x_1(t + n_1 T_1) + x_2(t + n_2 T_2) = x_1(t) + x_2(t) = x(t)$$

   Thus $x(t)$ is periodic with period T.

(b) Similarly, if T_1/T_2 is rational, then there exists some integers n_1 and n_2 such that $T_1/T_2 = n_2/n_1$. Let $T = n_1 T_1 = n_2 T_2$. Then

$$x(t+T) = x_1(t+T)x_2(t+T) = x_1(t+n_1 T_1)x_2(t+n_2 T_2) = x_1(t)x_2(t) = x(t)$$

Thus $x(t)$ is periodic with period T. Moreover, if we take n_1 and n_2 to be co-prime, then T is the least common multiple of T_1 and T_2.

(c)   i. The periods of $\sin(\pi t/3)$, $\cos(\pi t/4)$, $\sin(\pi t/5)$, and $\sin(\pi t/2)$ are 6, 8, 10, and 4, respectively. Then least common multiple is 120, so one period of $x(t)$ is $\boxed{120}$.

   ii. The ratio of two periods is not rational, so the sum of the two signals is not periodic.

4. (a) The system is $\boxed{\text{linear}}$ and $\boxed{\text{stable}}$.

$$y_1(t) = x_1(t-2) + x_1(2-t)$$

$$y_2(t) = x_2(t-2) + x_2(2-t)$$

$$x(t) = \alpha x_1(t) + \beta x_2(t)$$

$$y(t) = x(t-2) + x(2-t)$$

$$y(t) = \alpha x_1(t-2) + \beta x_2(t-2) + \alpha x_1(2-t) + \beta x_2(2-t)$$

$$y(t) = \alpha(x_1(t-2) + x_1(2-t)) + \beta(x_2(t-2) + x_2(2-t)) = \alpha y_1(t) + \beta y_2(t)$$

It is obviously $\boxed{\text{not Memoryless or Causal}}$.
It is $\boxed{\text{not Time-invariant}}$,

$$y_d(t) = x(t-2-d) + x(2-t-d) = x((t-d)-2) + x(2-(t+d) \neq y(t-d)$$

(b) The system is $\boxed{\text{Linear}}$, $\boxed{\text{Stable}}$.
It is $\boxed{\text{not time-invariant}}$ because,

$$y_d(t) = x(\frac{t}{3} - d) = x(\frac{t-3d}{3}) = y(t-3d)$$

it is $\boxed{\text{not Memoryless or Causal}}$, consider $t = \pm 1$, when $t = -1$, the output depend on future input and when $t = 1$, the output depend on past input.

(c) The system is $\boxed{\text{Memoryless}}$ thus $\boxed{\text{Causal}}$. It is also $\boxed{\text{Stable}}$ because the absolute value of the output of cos function is bounded at 1.
It is $\boxed{\text{Time invariant}}$,

$$y_d(t) = \cos(x(t-d)) = y(t-d)$$

It is $\boxed{\text{not linear}}$,

$$y_1(t) = \cos(x_1(t))$$

$$y_2(t) = \cos(x_2(t))$$

$$x(t) = \alpha x_1(t) + \beta x_2(t)$$

$$y(t) = \cos(x(t)) = \cos(\alpha x_1(t))\cos(\beta x_2(t)) - \sin(\alpha x_1(t))\sin(\beta x_2(t)) \neq \alpha y_1(t) + \beta y_2(t)$$

(d) it is $\boxed{\text{not Memoryless, Causal, Stable or Time invariant}}$. But it is $\boxed{\text{Linear}}$. It is $\boxed{\text{not Time invariant}}$ because,

$$y_d(t) = \int_{-\infty}^{t/2} x(\tau - d)d\tau = \int_{-\infty}^{t/2-d} x(\tau - d)d(\tau - d) = \int_{-\infty}^{(t-2d)/2} x(s)ds = y(t-2d)$$

(e) It is $\boxed{\text{not memoryless}}$ by the definition of derivative,

$$\frac{dx(t)}{dt} = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{x(t) - x(t - \Delta t)}{\Delta t}$$

It is $\boxed{\text{Time invariant}}$ and $\boxed{\text{Linear}}$.
For causality, both yes and no are OK. It is causal practically, but if you consider the mathematical requirement that left and right derivatives must be defined and match for $x(t)$ to be differentiable, it is $\boxed{\text{not causal}}$
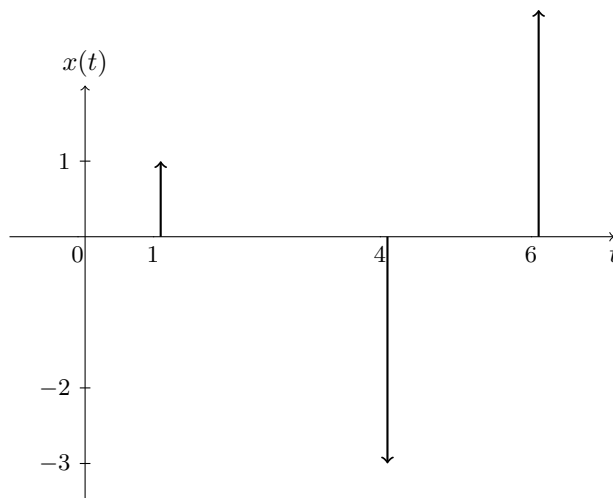It is $\boxed{\text{not stable}}$.

5. (a)

$$x(t) = \boxed{\text{u(t-1) - 3u(t-4)u(6-t)}} \text{ or } \boxed{\text{u(t-1)-3u(t-4)+3u(t-6)}}$$

   (b)

$$\frac{dx(t)}{dt} = \boxed{\delta(t-1) - 3\delta(t-4) + 3\delta(t-6)}$$

6. (a) $\boxed{\text{No}}$. Causal system's output depends only on values of the input at present and in the past. Look at the second pair, during the period (0, 1), if the system is causal, the output during this period should remain 0. Because the input left to 1 are all the same(all 0).

   (b) $\boxed{\text{No}}$. Look at the first pair and the third pair. $x_2(t) = x_1(t - 1)$, but $y_2(t) \neq y_1(t - 1)$, so it is not time invariant.

   (c) $\boxed{\text{No}}$. We already proved that the system is not causal in (a), so it is not memoryless, either.

   (d) Use the linearity of the system, cascade the input $x_1(t)$ and $x_3(t)$ to form $x_4(t)$. Then we can get the output $y_4(t)$ based on $y_1(t), y_3(t)$.
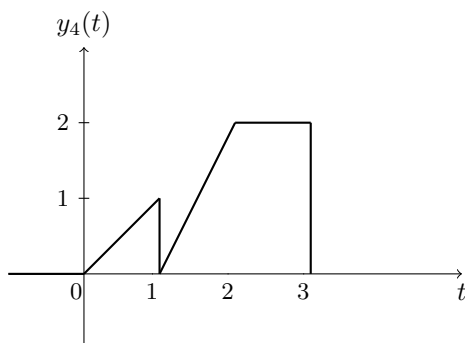
7. (a) The sketch is shown below.

$y_4(t)$

0.25

    0   1   2 t
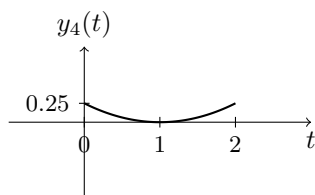
(b) The value of rect function at $t = -1/2$ or $t = 1/2$ can be 1, 0 or 1/2.
By the scaling property of $\delta(t)$, $x(t) = \delta(t - \frac{1}{2}) + \delta(t - 2) - \frac{1}{3}\delta(t - 4/3)$. $s(1/2) = \frac{1}{16}$, $s(2) = \frac{1}{4}$
or 0 or $\frac{1}{8}$, $s(4/3) = \frac{1}{36}$. By Shifting property, $\int_{-\infty}^{\infty} s(t)x(t)dt = s(1/2) + s(2) - \frac{1}{3}s(4/3) =$

$\boxed{\frac{131}{432} \text{ or } \frac{23}{432} \text{ or } \frac{77}{432}}$