Ve215 Electric Circuits

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Chapter 9

Sinusoids and Phasors

9.1 Introduction

- Circuits driven by sinusoidal current or voltage sources are called alternating current (ac) circuits.
- We now begin the analysis of ac circuits.
- We are interested in sinusoidal steady-state response of ac circuits.

9.2 Sinusoids

A sinusoid is a signal that has the form of the sine or cosine function. For example,

$$v(t) = V_m \sin(\omega t + \phi)$$

where

V_m : amplitude

ω: angular frequency

ϕ: initial phase

Let us examine the two sinusoids

$$v_1(t) = V_m \sin \omega t$$

$$v_2(t) = V_m \sin(\omega t + \phi)$$

shown in Fig. 9.2.

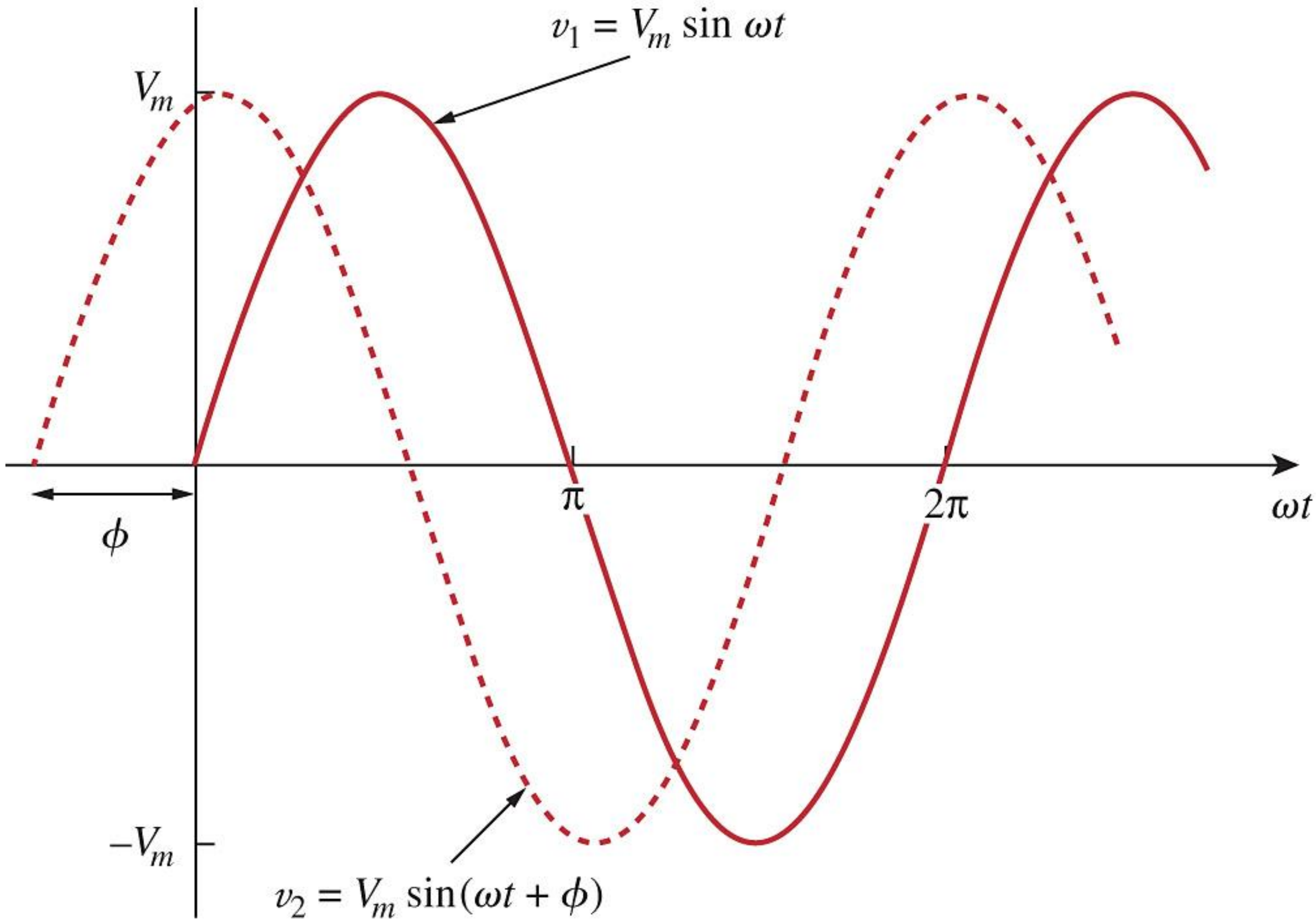


Figure 9.2 Two sinusoids with different phases.

The starting point of v_2 in Fig. 9.2 occurs first in time. Therefore, we say that v_2 leads v_1 by ϕ or that v_1 lags v_2 by ϕ. If $\phi \neq 0$, we say that v_1 and v_2 are *out of phase*. If $\phi = 0$, then v_1 and v_2 are said to be *in phase*.

9.3 Phasors

Sinusoids are easily expressed in terms of phasors, which are more convenient to work with than sine and cosine functions. A phasor is a complex number that represents the amplitude and phase of a sinusoid.

Given a sinusoid

$$v(t) = V_m \cos(\omega t + \phi)$$

$$= \mathrm{Re}\left(V_m e^{j(\omega t + \phi)}\right)$$

$$= \mathrm{Re}\left(V_m e^{j\phi} e^{j\omega t}\right)$$

$$= \mathrm{Re}\left(\dot{V} e^{j\omega t}\right)$$

where $\dot{V} = V_m e^{j\phi} = V_m \angle \phi$ is the phasor representation of the sinusoid $v(t)$.

Figure 9.7 shows the *sinor* $\dot{V}e^{j\omega t}=V_m e^{j(\omega t+\phi)}$ on the complex plane. As time increases, the sinor rotates on a circle of radius V_m at an angular velocity ω in the counterclockwise direction. We may regard $v(t)$ as the projection of the sinor on the real axis.

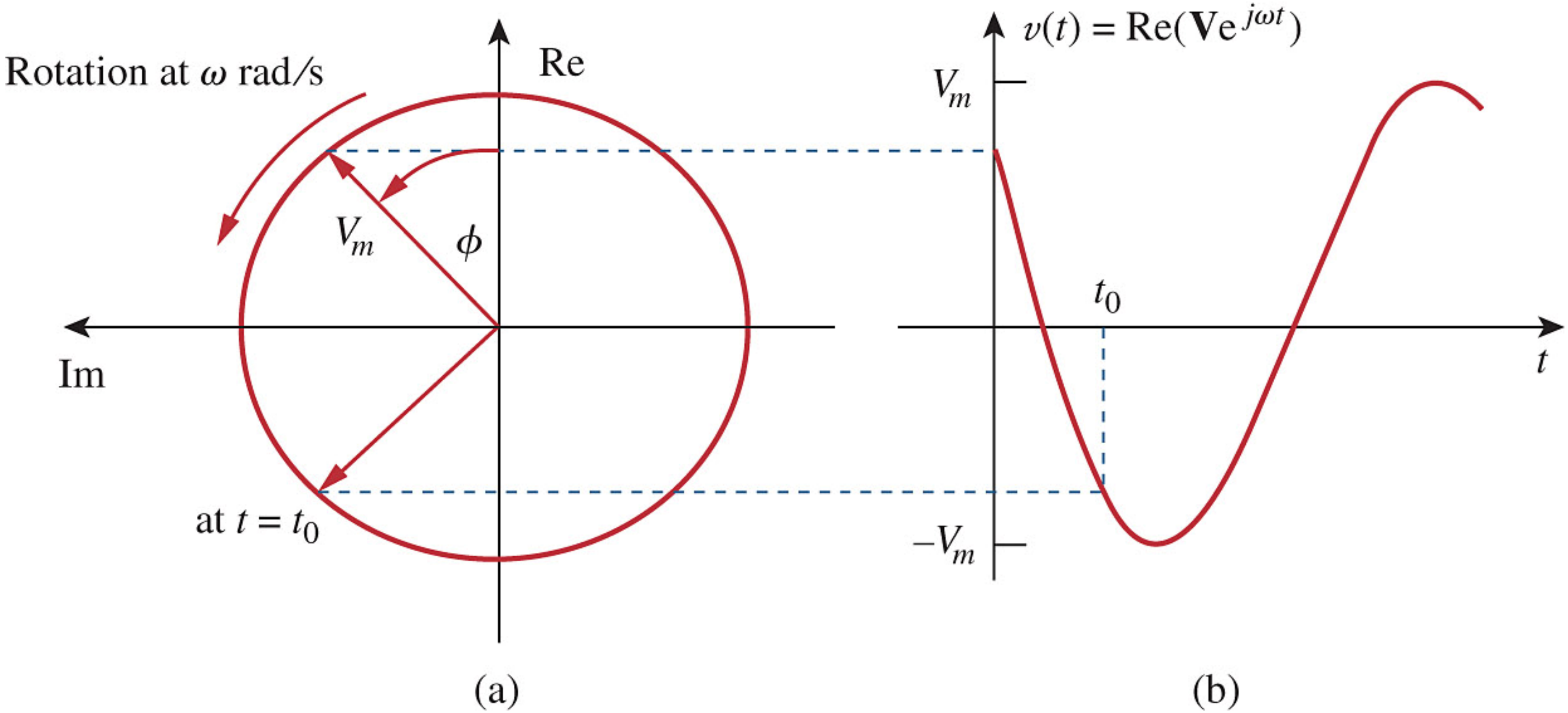


Figure 9.7 Representation of sinor: (a) sinor rotating counterclockwise, (b) its projection on the real axis, as a function of time.

The value of the sinor at time $t = 0$ is the phasor $\dot{V}$. The sinor may be regarded as a rotating phasor. Thus, whenever a sinusoid is expressed as a phasor, the term $e^{j\omega t}$ is implicitly present.

Since a phasor has magnitude and phase ("direction"), it behaves as a vector. Figure 9.8 shows two phasors: $\dot{V} = V_m \angle \phi$ and $\dot{I} = I_m \angle -\theta$. Such a graphical representation of phasors is known as a *phasor diagram.*

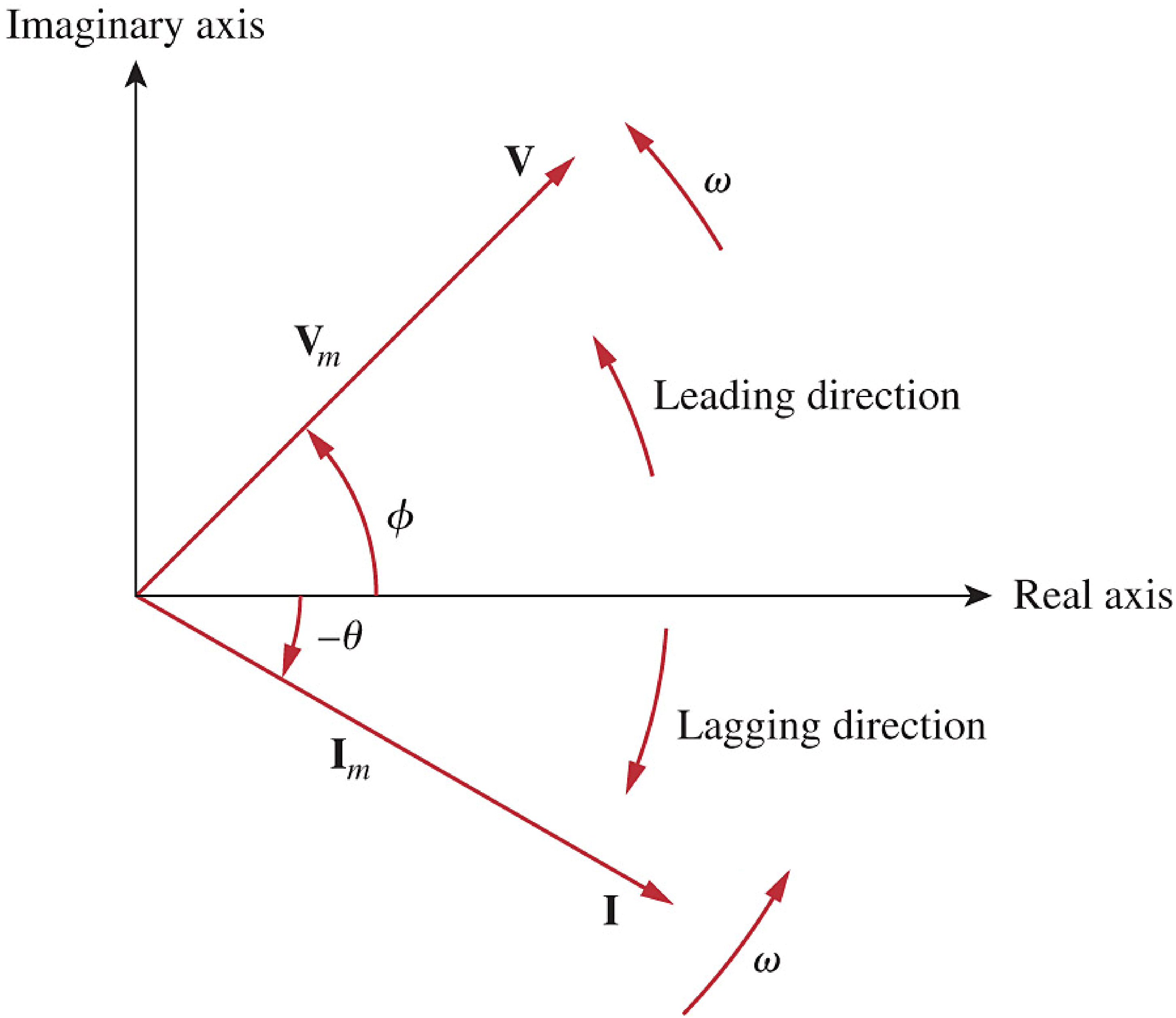


Figure 9.8 A phasor diagram.

In conclusion, a sinusoid has a time-domain representation $v(t) = V_m \cos(\omega t + \phi)$ and a phasor-domain representation $\dot{V} = V_m \angle \phi$. The phasor domain is also known as the frequency domain.

$$v(t) = V_m \cos(\omega t + \phi) \Leftrightarrow V = V_m \angle \phi$$

9.4 Phasor Relationships for Circuit Elements

We begin with the resistor. If the current through a resistor R is $i = I_m \cos(\omega t + \phi)$, the voltage across it is given by Ohm's law as

$$v = iR = RI_m \cos(\omega t + \phi)$$

The phasor form of this voltage is

$$\dot{V} = RI_m \angle \phi$$

But the phasor representation of the current is $\dot{I} = I_m \angle \phi$. Hence,

$$\dot{V} = R\dot{I}$$

showing the voltage-current relation for the resistor in the phasor domain continues to be Ohm's law, as in the time domain.

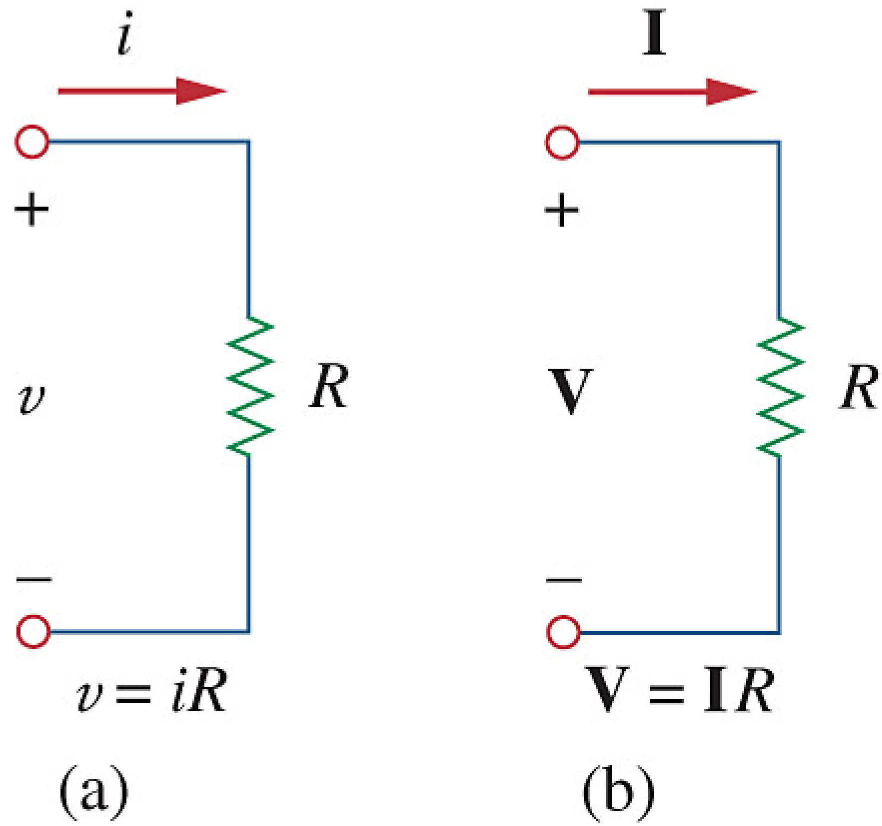


Figure 9.9 Voltage-current relations for a resistor in the (a) time domian, (b) frequency domain.

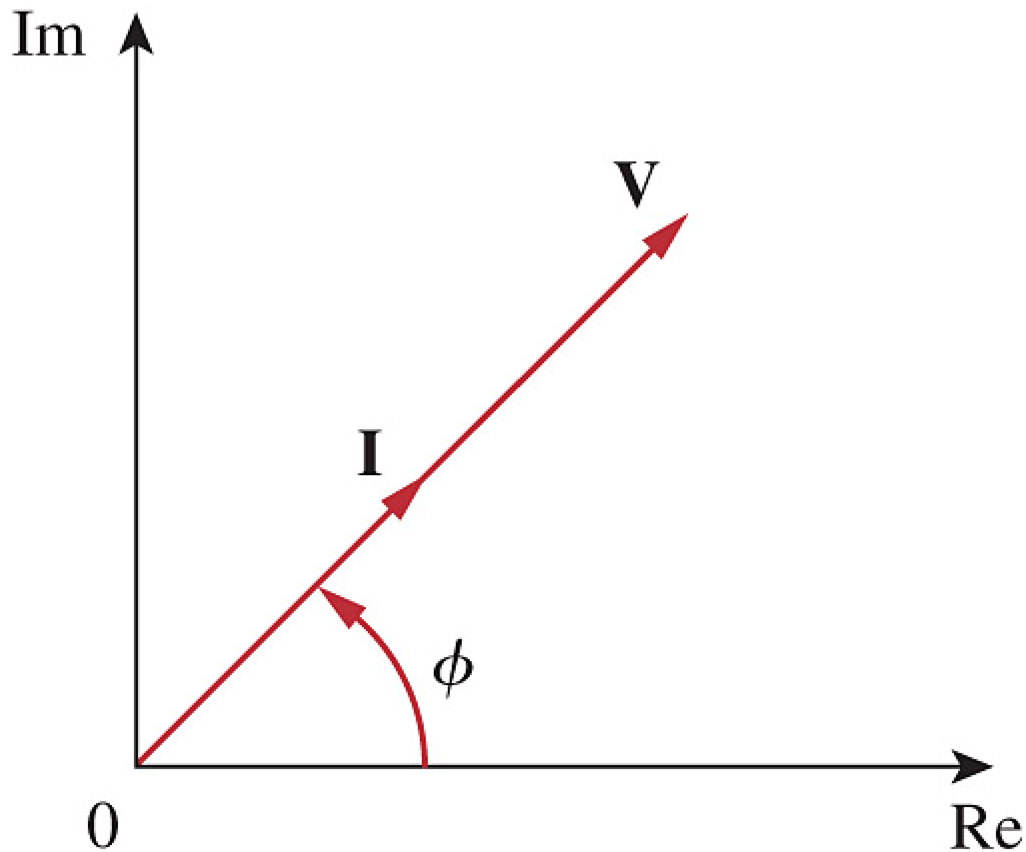


Figure 9.10 Phasor diagram for the resistor.

For the inductor L, if $i = I_m \cos(\omega t + \phi)$

$\Leftrightarrow \dot{I} = I_m \angle \phi$, then

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi)$$

$$= \omega L I_m \cos(\omega t + \phi + 90°) \Leftrightarrow$$

$$\dot{V} = \omega L I_m \angle(\phi + 90°) = j\omega L I_m \angle \phi$$

$$= j\omega L \dot{I}$$

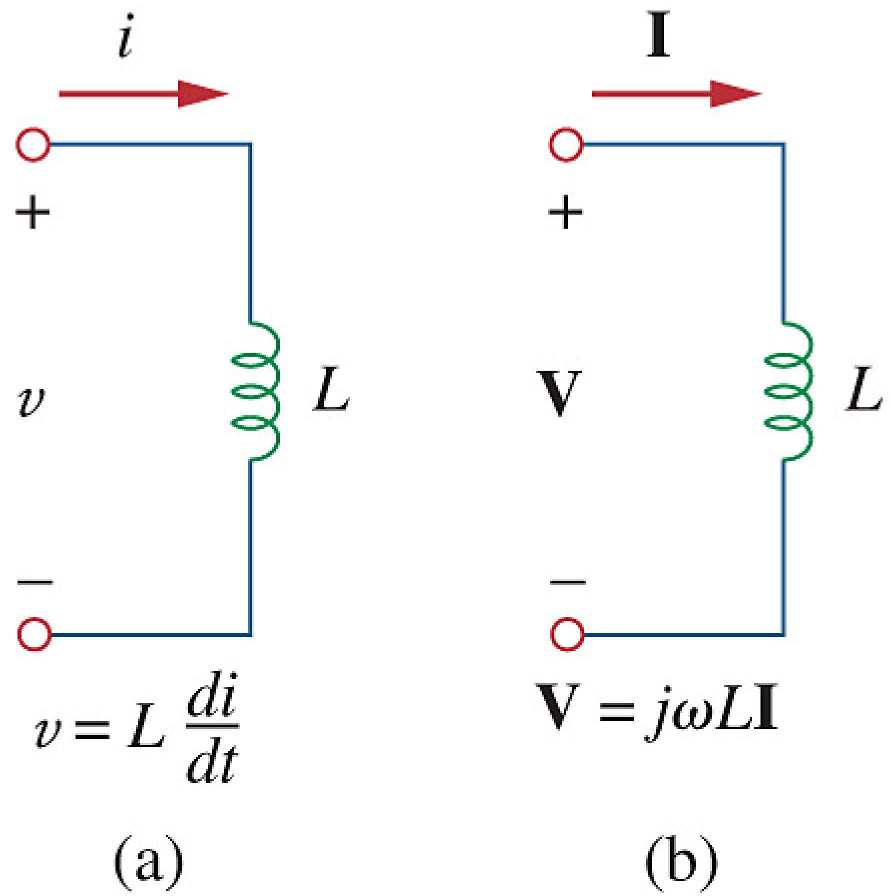


Figure 9.11 Voltage-current relations for an inductor in the (a) time domain, (b) frequency domain.

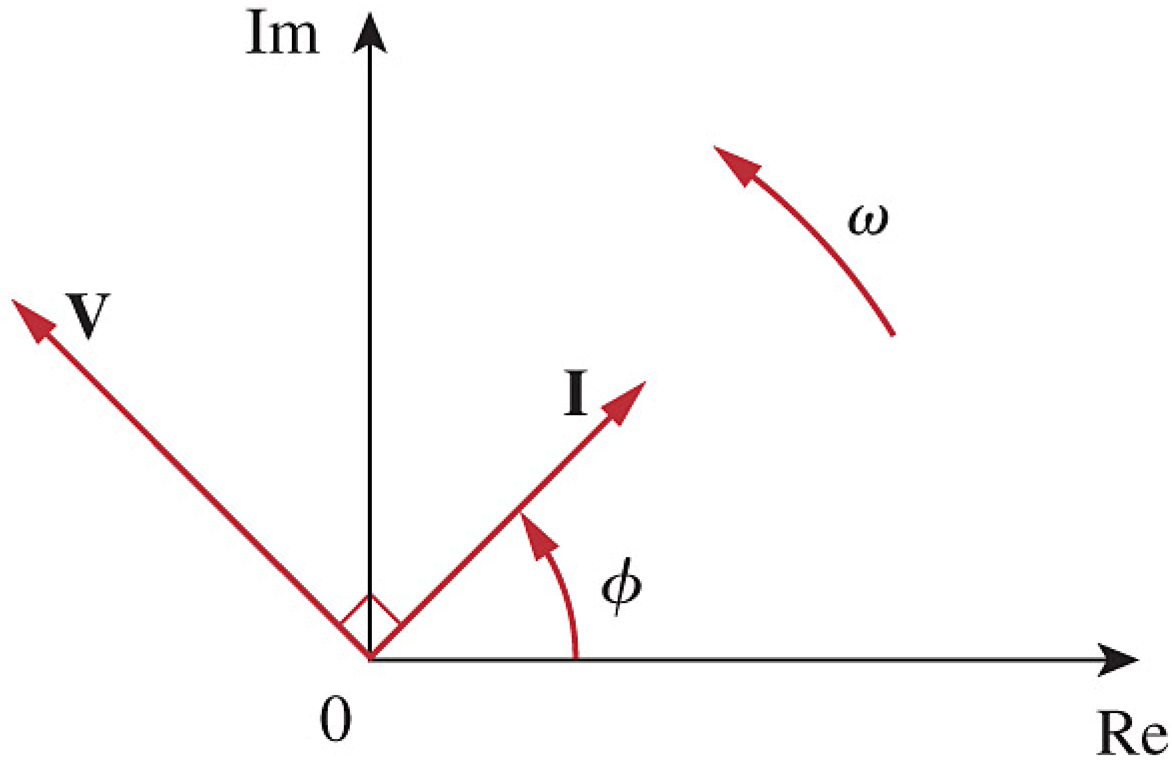


Figure 9.12 Phasor diagram for the inductor.

For the capacitor C, if $v = V_m \cos(\omega t + \phi)$

$\Leftrightarrow \dot{V} = V_m \angle \phi$, then

$$i = C\frac{dv}{dt} = -\omega C V_m \sin(\omega t + \phi)$$

$$= \omega C V_m \cos(\omega t + \phi + 90^\circ) \Leftrightarrow$$

$$\dot{I} = \omega C V_m \angle(\phi + 90^\circ) = j\omega C V_m \angle \phi$$

$$= j\omega C \dot{V}$$

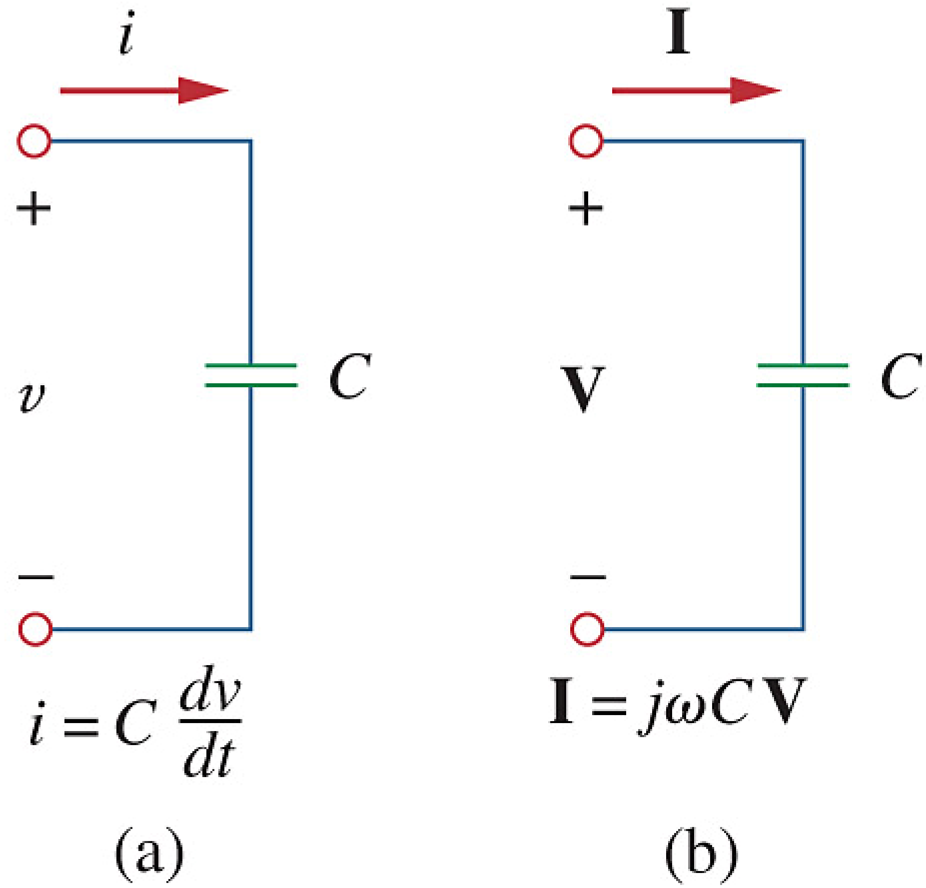


Figure 9.13 Voltage-current relations for a capacitor in the (a) time domain, (b) frequency domain.

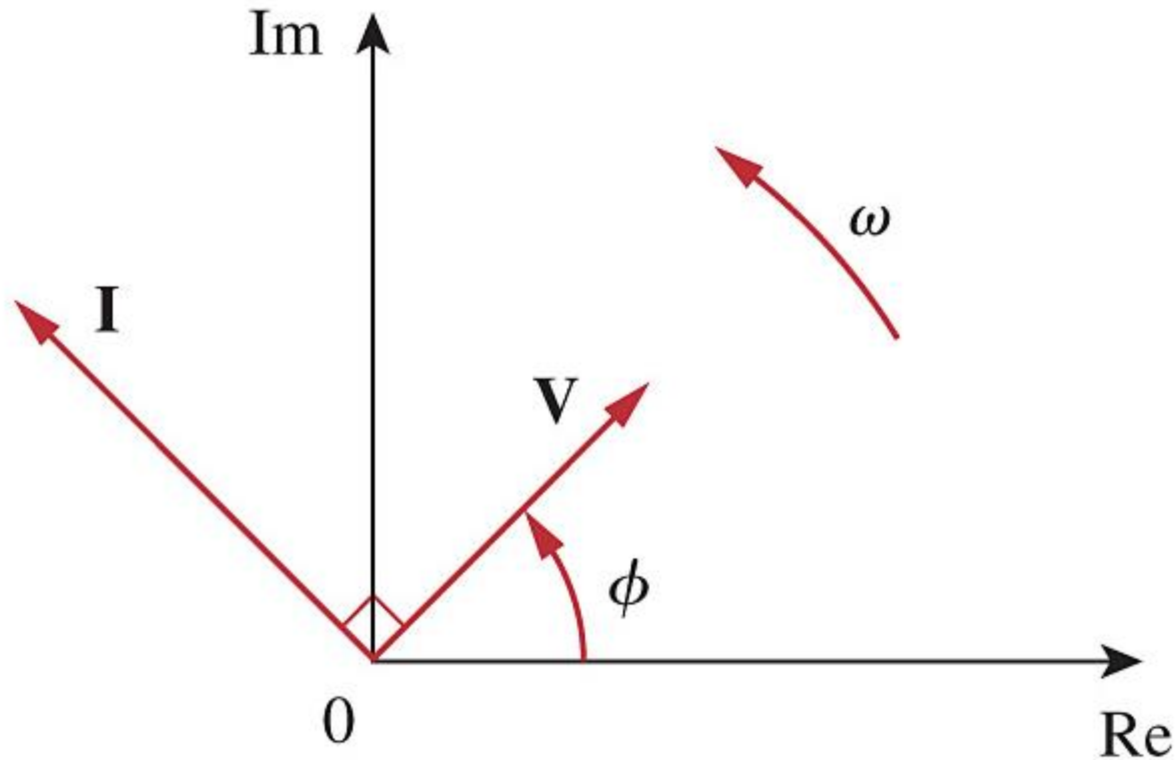


Figure 9.14 Phasor diagram for the capacitor.

TABLE 9.2

Summary of voltage - current relationships

Element	Time domain	Frequency domain
R	$v = Ri$	$\dot{V} = R\dot{I}$
L	$v = L\dfrac{di}{dt}$	$\dot{V} = j\omega L\dot{I}$
C	$i = C\dfrac{dv}{dt}$	$\dot{V} = \dfrac{1}{j\omega C}\dot{I}$

9.5 Impedance and Admittance

The impedance Z of a circuit is the ratio of the phasor voltage $\dot{V}$ to the phasor current $\dot{I}$, measured in ohms (Ω).

$$Z = \frac{\dot{V}}{\dot{I}}$$

For the three passive elements, we have $Z = R$, $Z = j\omega L$, and $Z = 1/j\omega C$, respectively.

The admittance Y of a circuit is the ratio of the phasor current $\dot{I}$ to the phasor voltage $\dot{V}$, measured in siemens (S).

$$Y = \frac{\dot{I}}{\dot{V}} = \frac{1}{Z}$$

For the three passive elements, we have $Y = 1/R$, $Y = 1/j\omega L$, and $Y = j\omega C$, respectively.

The Ohm's law in phasor form is

$$\dot{V} = Z\dot{I} \text{ or } \dot{I} = Y\dot{V}$$

As a complex quantity, the impedence may be expressed in rectangular form or polar form

$$Z = R + jX = |Z| \angle \theta$$

where

R : resistance

X : reactance

If $X > 0,$ we say that the impedance is inductive or lagging since current lags voltage; If $X < 0,$ we say that the impedance is capacitive or leading because current leads voltage.

The impedance, resistance, and reactance are all measured in ohms.

The admittance can be written as

$$Y = G + jB$$

where

G : conductance

B : susceptance

The admittance, conductance, and susceptance are all measured in siemens.

9.6 Kirchhoff's Laws in the Frequency Domain

For KVL, Let $v_1, v_2, \ldots, v_n$ be the voltages around a closed loop. Then

$$\sum_{i=1}^{n} v_i = 0 \qquad (9.51)$$

In sinusoidal steady state,

$$v_i = V_{mi}\cos(\omega t + \phi_i) = \mathrm{Re}\left(\dot{V}_i e^{j\omega t}\right)$$

That implies the sinor is zero, that is,

$$\left(\sum_{i=1}^{n} \dot{V}_i\right) e^{j\omega t} = 0$$

but $e^{j\omega t} \neq 0$,

$$\sum_{i=1}^{n} \dot{V}_i = 0$$

indicating that KVL holds for phasors.

For KCL, if $i_1, i_2, \ldots, i_n$ are the currents leaving or entering a closed surface in a circuit at time t, and $\dot{I}_1, \dot{I}_2, \ldots, \dot{I}_n$ are the phasor forms of $i_1, i_2, \ldots, i_n$, then

$$\sum_{i=1}^{n} i_i = 0 \Rightarrow \sum_{i=1}^{n} \dot{I}_i = 0$$

Since basic circuit laws, Kirchoff's and Ohm's, hold in phasor domain, it is not difficult to analyze ac circuits.

9.7 Impedance Combinations

For the N series-connected impedances shown in Fig. 9.18, the equivalent impedance at the input terminals is

$$Z_{eq} = \frac{\dot{V}}{\dot{I}} = \frac{\sum_{i=1}^{N} \dot{V}_i}{\dot{I}} = \sum_{i=1}^{N} \frac{\dot{V}_i}{\dot{I}} = \sum_{i=1}^{N} Z_i$$

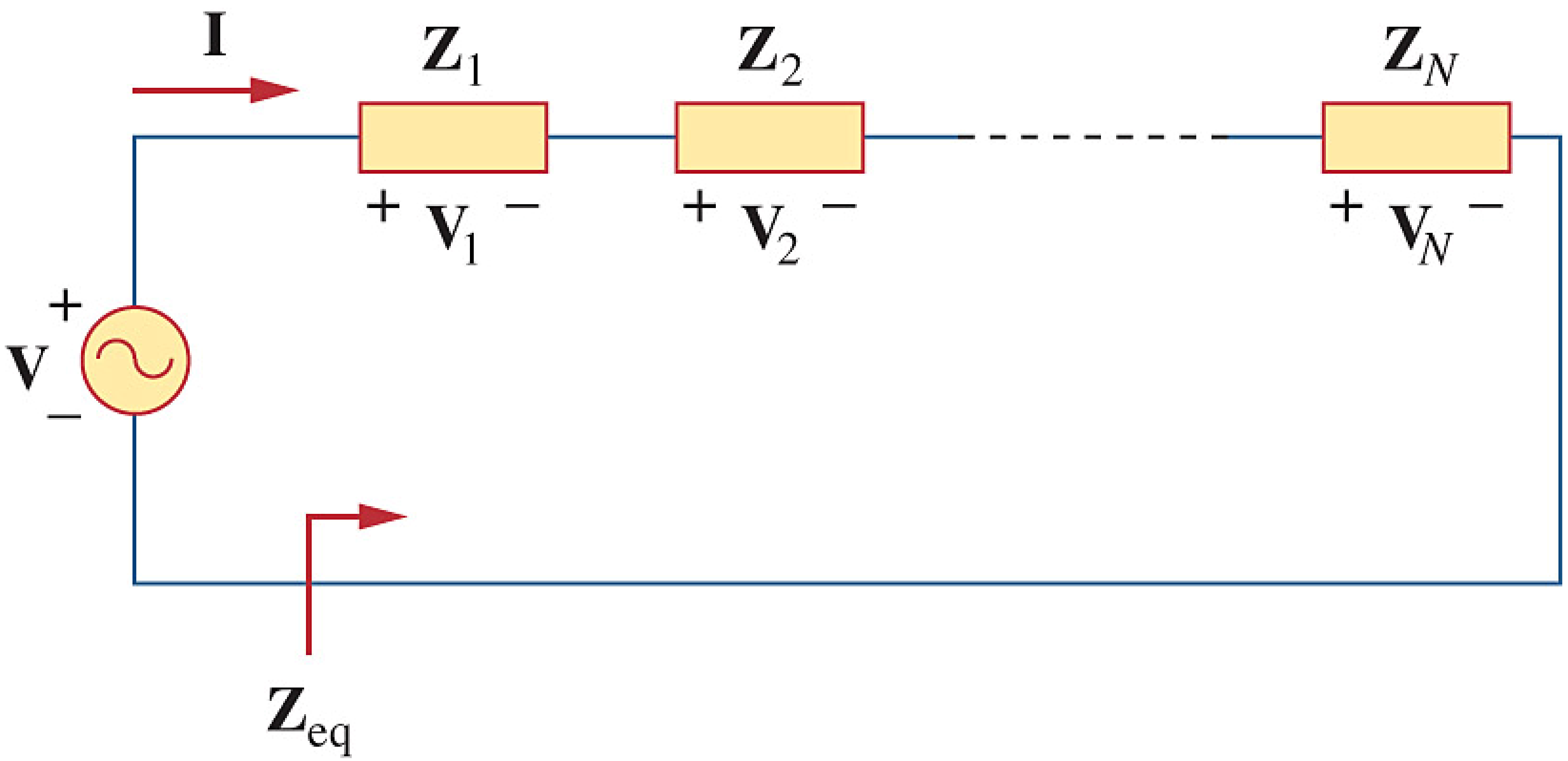


Figure 9.18 N impedances in series.

For the N parallel-connected impedances shown in Fig. 9.20, the equivalent admittance at the input terminals is

$$Y_{eq} = \frac{\dot{I}}{\dot{V}} = \frac{\sum_{i=1}^{N} \dot{I}_i}{\dot{V}} = \sum_{i=1}^{N} \frac{\dot{I}_i}{\dot{V}} = \sum_{i=1}^{N} Y_i$$

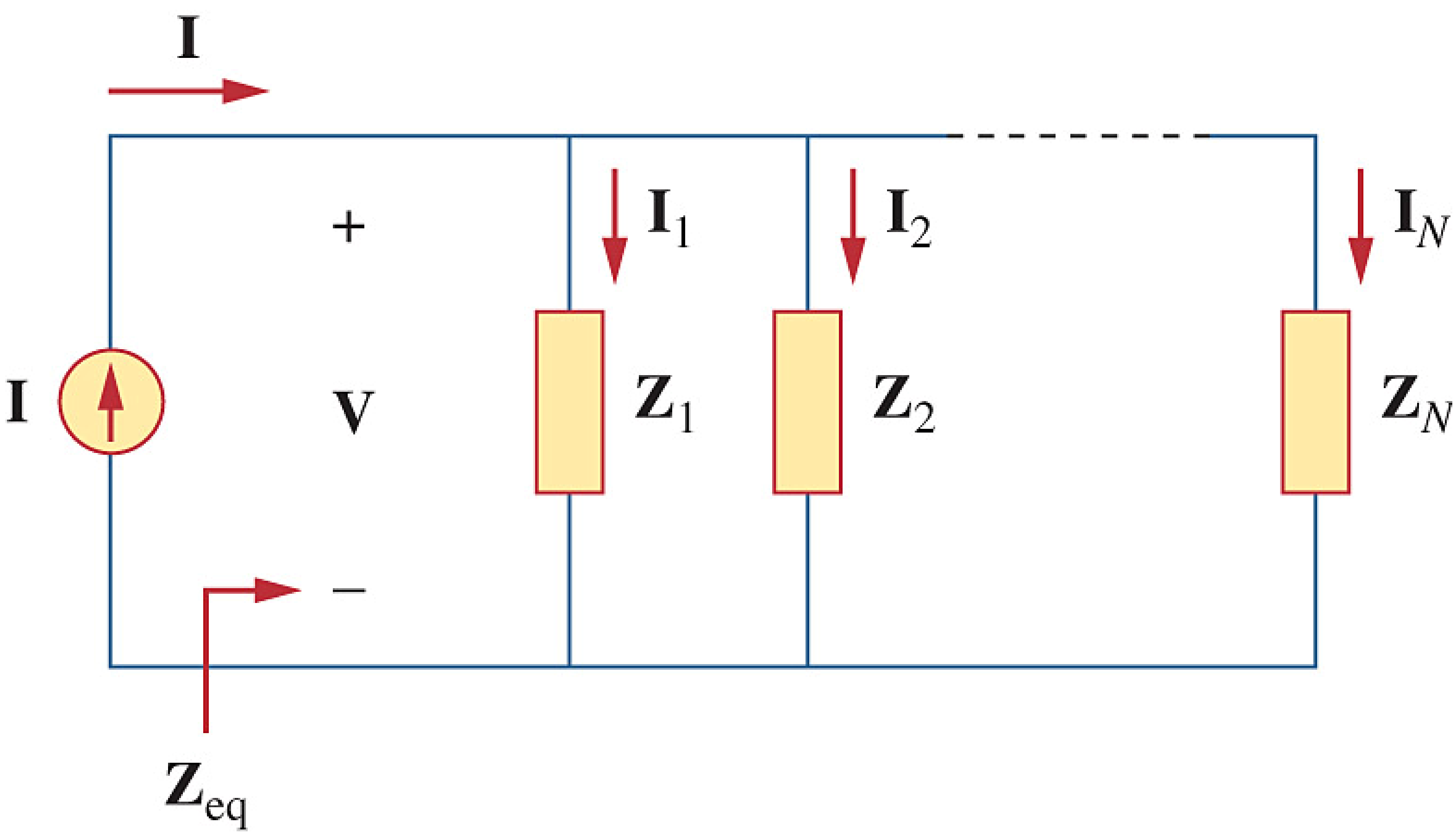


Figure 9.20 N impedances in parallel.

The delta-to-wye and wye-to-delta transformations that we applied to resistive circuits are also valid for impedances. With reference to Fig. 9.22, the conversion formulas are as follows:

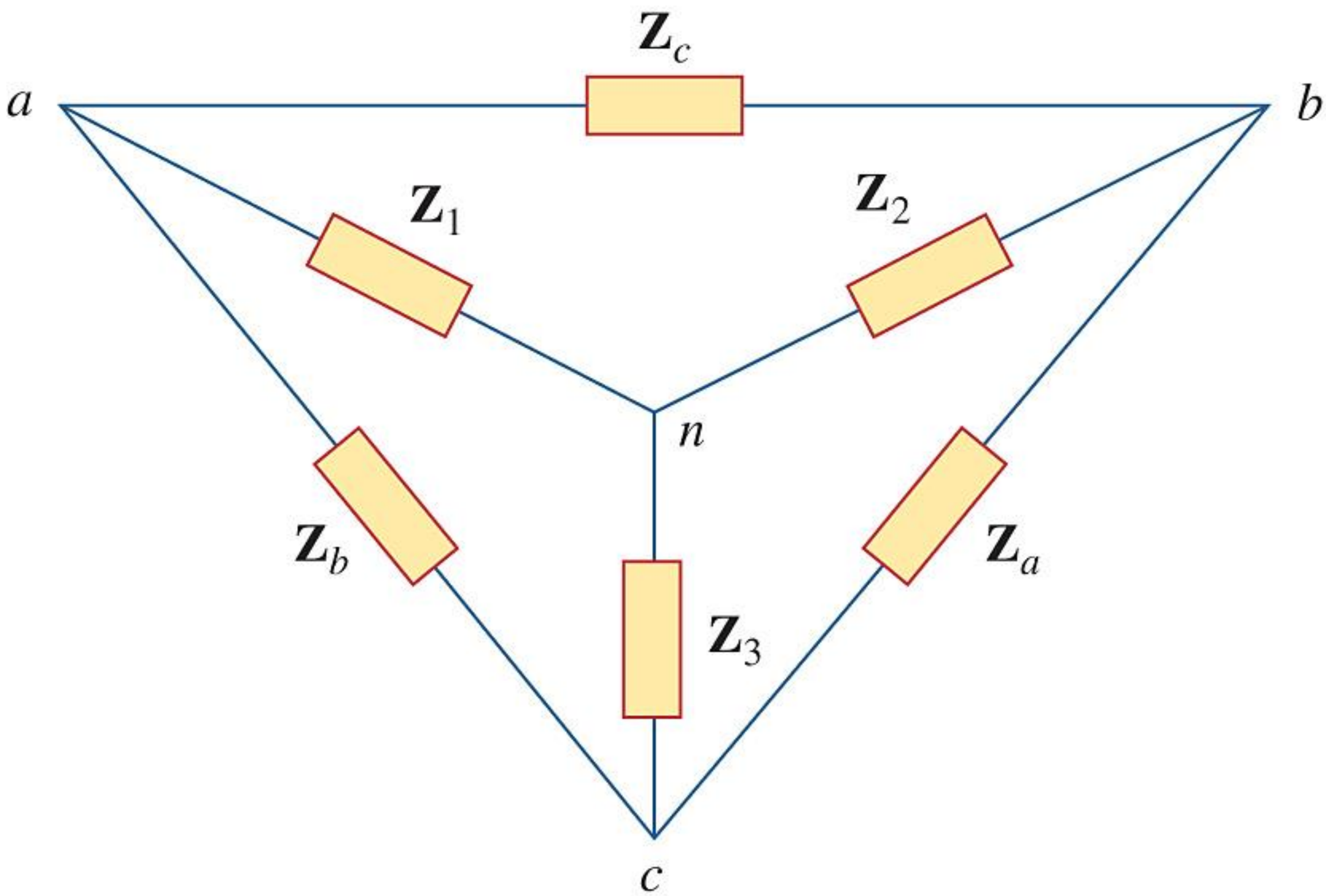


Figure 9.22 Superimposed wye and delta networks.

Y-Δ conversion:

$$Z_a = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_1}$$

$$Z_b = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_2}$$

$$Z_c = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_3}$$

Δ-Y conversion:

$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c}$$

$$Z_2 = \frac{Z_c Z_a}{Z_a + Z_b + Z_c}$$

$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c}$$

Practice Problem 9.10 Find the input impedance of the circuit in Fig. 9.24 at $\omega = 10$ rad/s.

Solution :

8-H inductor: $Z_1 = j10 \times 8 = j80\ (\Omega)$

0.5-mF capacitor: $Z_2 = \dfrac{1}{j10 \times (0.5 \times 10^{-3})}$

$= -j200\ (\Omega)$

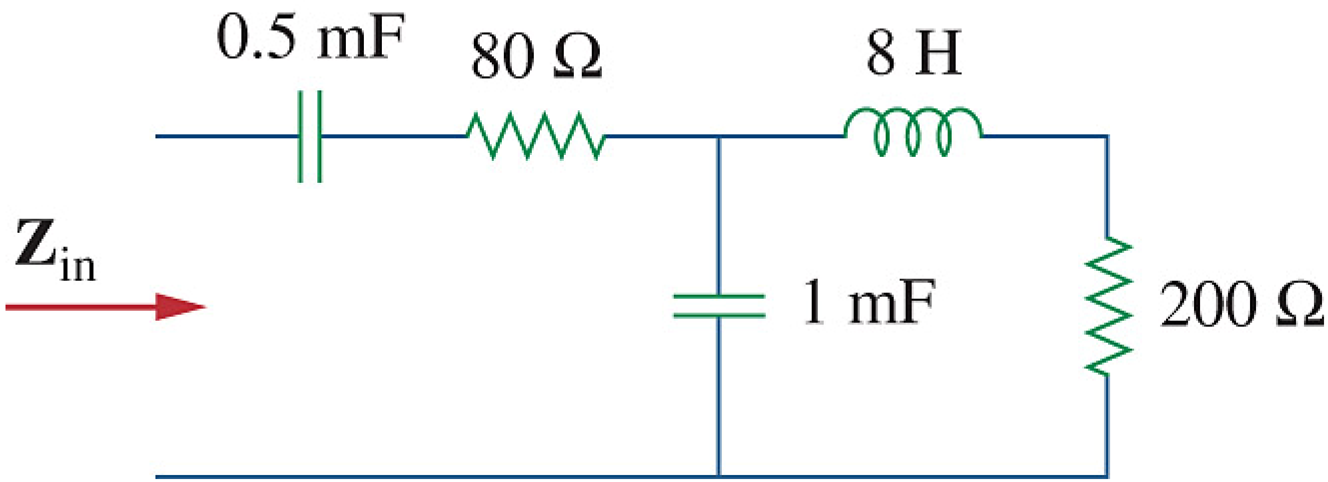


Figure 9.24

1-mF capacitor: $Z_3 = \dfrac{1}{j10 \times (1 \times 10^{-3})}$

$= -j100\ (\Omega)$

$Z_{in} = Z_2 + 80 + Z_3 \,||\, (Z_1 + 200)$

where

$Z_3 \,||\, (Z_1 + 200) = (-j100) \,||\, (j80 + 200)$

$= \dfrac{(-j100) \times (j80 + 200)}{(-j100) + (j80 + 200)}$

$$= \frac{(-j100)\times(200+j80)}{200-j20}$$

$$\approx \frac{(100\angle-90°)\times 215.4066\angle 21.80°}{200.9975\angle-5.71°}$$

$$\approx 107.1688\angle-62.49°$$

$$\approx 49.5016-j95.0512\ (\Omega)$$

$$Z_{in} = -j200+80+49.5016-j95.0512$$

$$\approx 129.50-j295.05\ (\Omega)$$

Practice Problem 9.11 Calculate v_o in the circuit of Fig. 9.27.

Solution :

0.5-H inductor: $Z_1 = j10 \times 0.5 = j5\ (\Omega)$

$\frac{1}{20}$-F capacitor: $Z_2 = \dfrac{1}{j10 \times (1/20)}$

$= -j2\ (\Omega)$

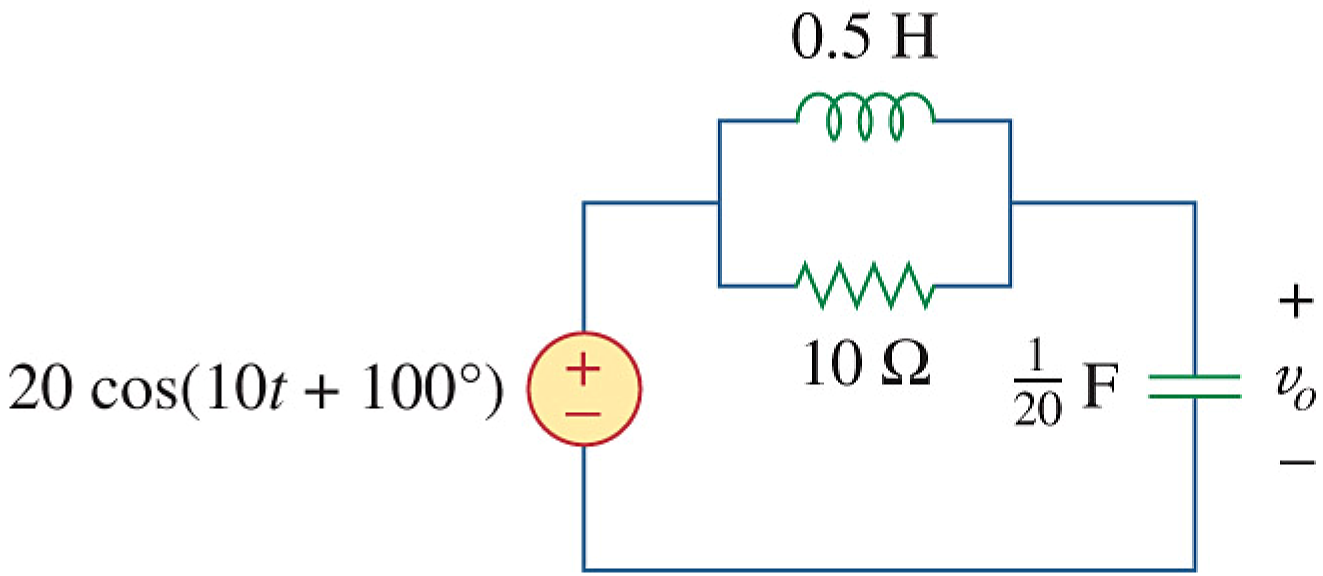


Figure 9.27

$$20\cos(10t+100°)\ \text{V}:\ 20\angle 100°\ \text{V}$$

$$\dot{V}_o = 20\angle 100° \times \frac{-j2}{-j2+10 \parallel j5}$$

$$10 \parallel j5 = \frac{10\times j5}{10+j5} = \frac{j10}{2+j} = 2+j4$$

$$\dot{V}_o = 20\angle 100° \times \frac{-j2}{2+j2} = 10\sqrt{2}\angle -35°\ (\text{V})$$

$$v_o(t) = 10\sqrt{2}\cos(10t-35°)\ (\text{V})$$

9.8 Applications

Phase-Shifters In Fig. 9.31(a),

$$\dot{V}_o = \dot{V}_i \frac{R}{R + 1/(j\omega C)} = \dot{V}_i \frac{R}{R - j(1/\omega C)}$$

$$= \dot{V}_i \frac{R}{\sqrt{R^2 + (1/\omega C)^2} \angle - \tan^{-1}(1/(\omega RC))}$$

$\dot{V}_o$ leads $\dot{V}_i$ by $\theta = \tan^{-1}(1/(\omega RC))$,

$0° < \theta < 90°$, as shown in Fig. 9.32(a).

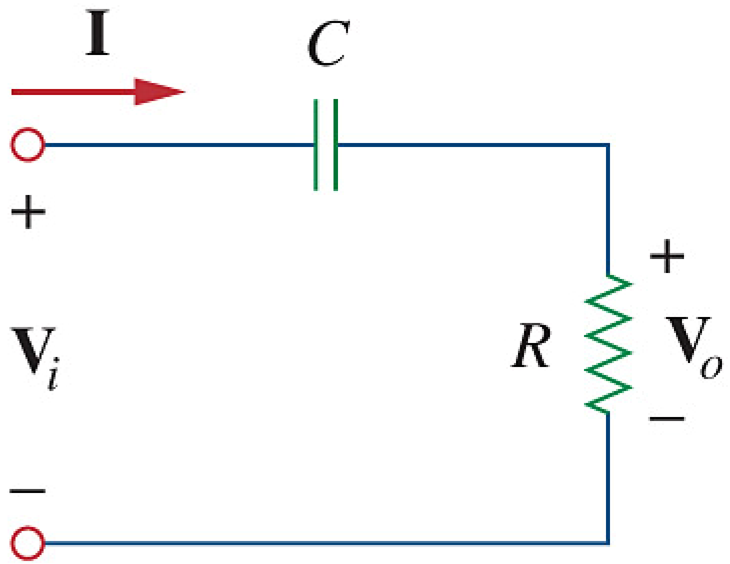


Figure 9.31(a) Series RC shift circuit: leading output.

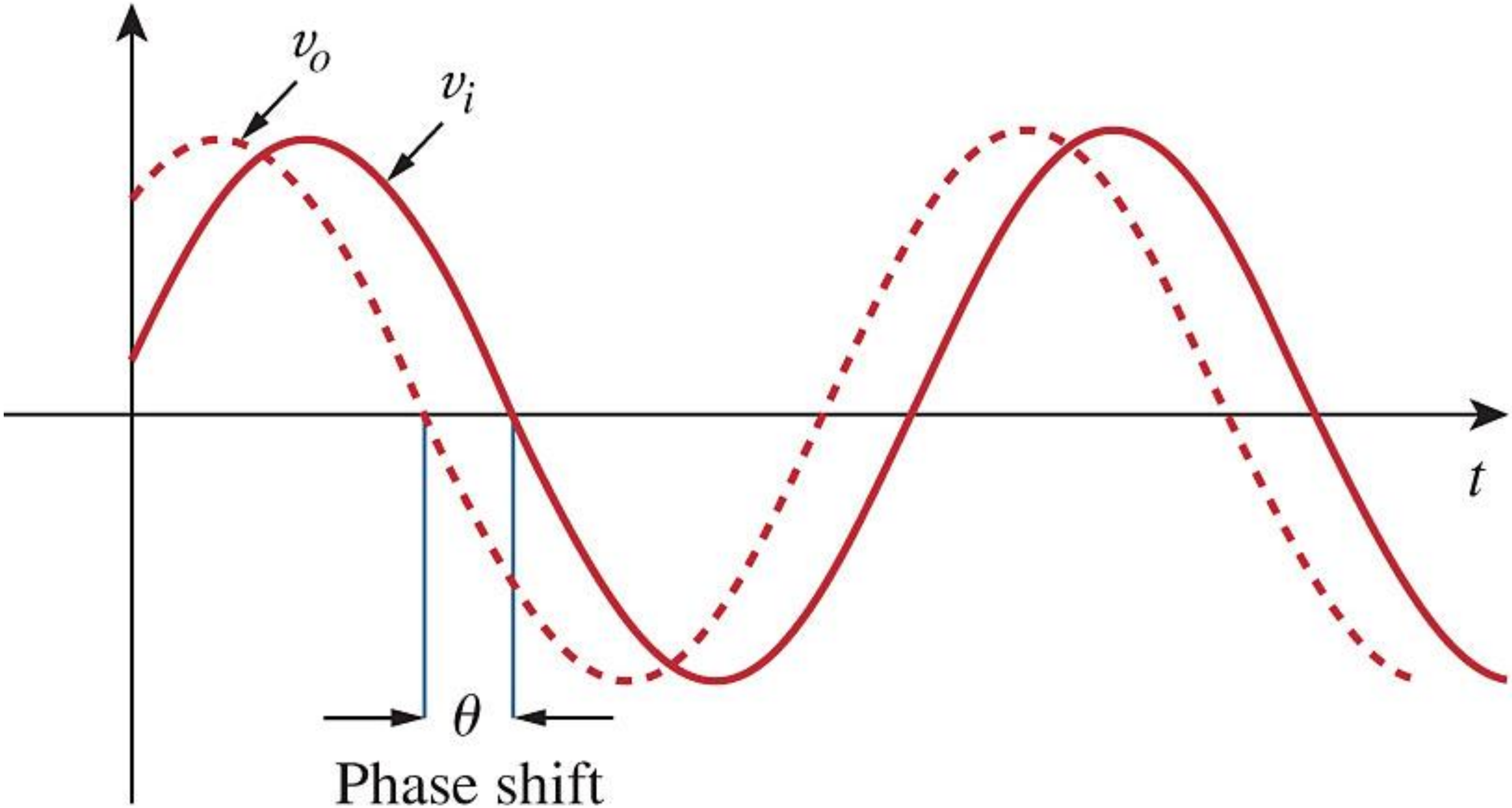


Figure 9.32(a) Phase shift in RC circuits: leading output.

In Fig. 9.31(b),

$$\dot{V}_o = \dot{V}_i \frac{1/(j\omega C)}{R + 1/(j\omega C)} = \dot{V}_i \frac{1}{1 + j\omega RC}$$

$$= \dot{V}_i \frac{1}{\sqrt{1 + (\omega RC)^2} \angle \tan^{-1}(\omega RC)}$$

$\dot{V}_o$ lags $\dot{V}_i$ by $\theta = \tan^{-1}(\omega RC)$, $0° < \theta < 90°$,

as shown in Fig. 9.32(b).

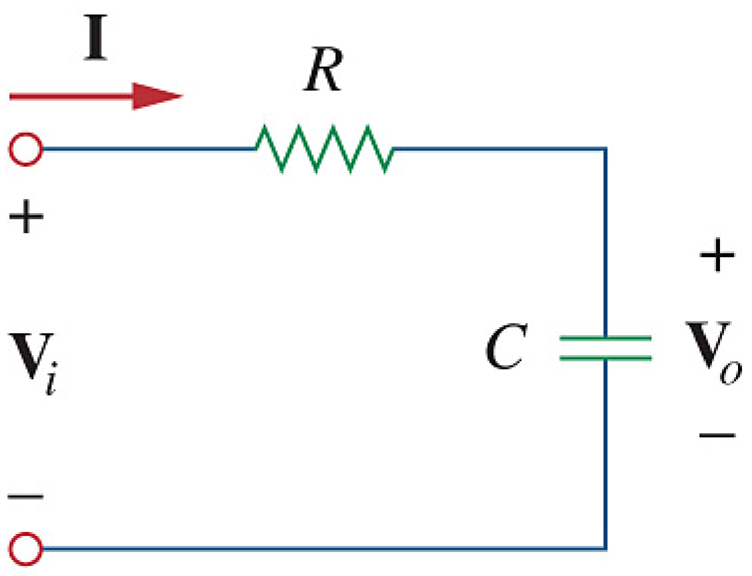


Figure 9.31(b) Series RC shift circuit: lagging output.

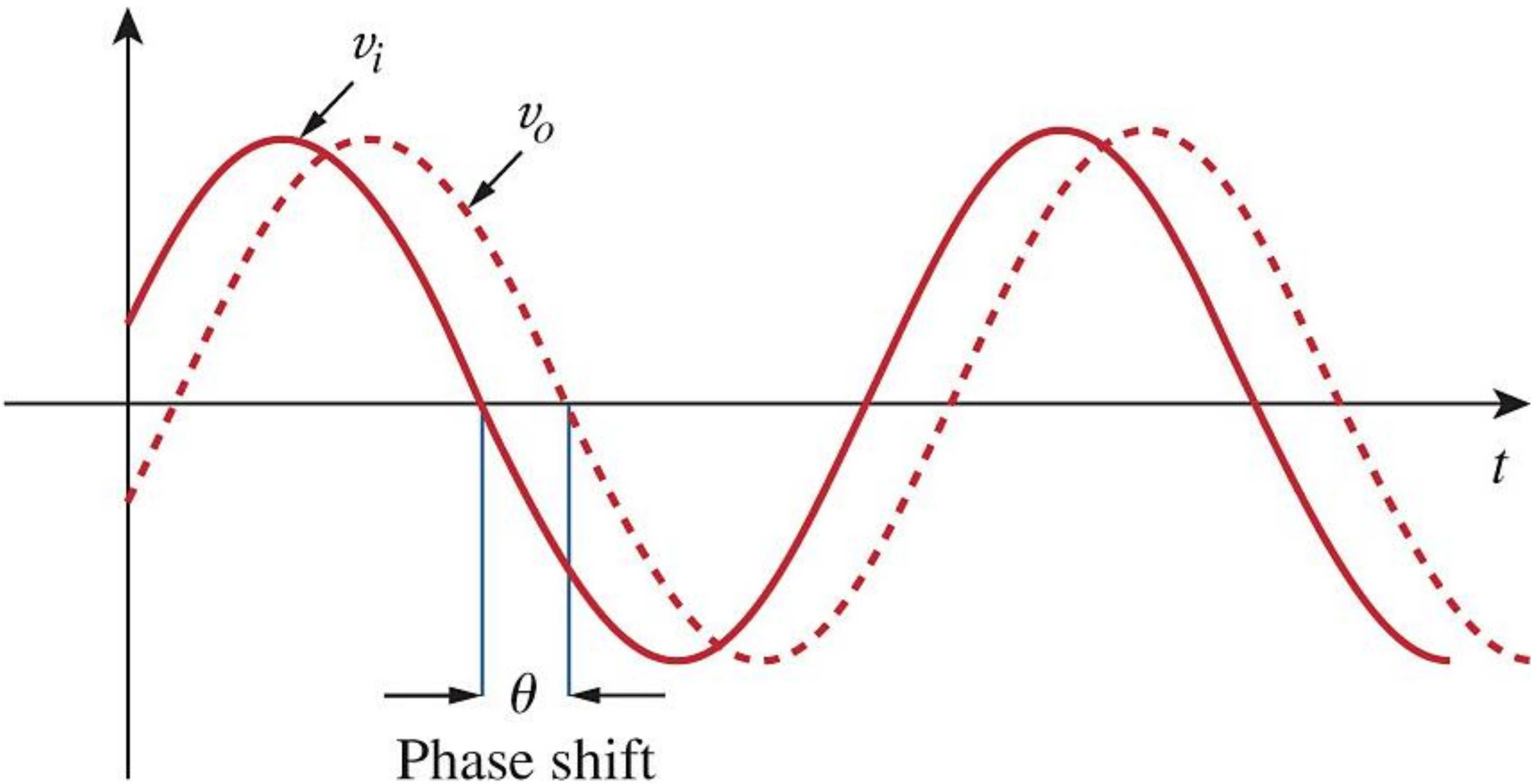


Figure 9.32(b) Phase shift in RC circuits: lagging output.

Practice Problem 9.13 Design an *RC* circuit to provide a phase shift of 90° leading.

Solution :

We need two stages, with each stage providing a phase shift of 45°.

Select $R_1 = R_2 = 20\ \Omega$,

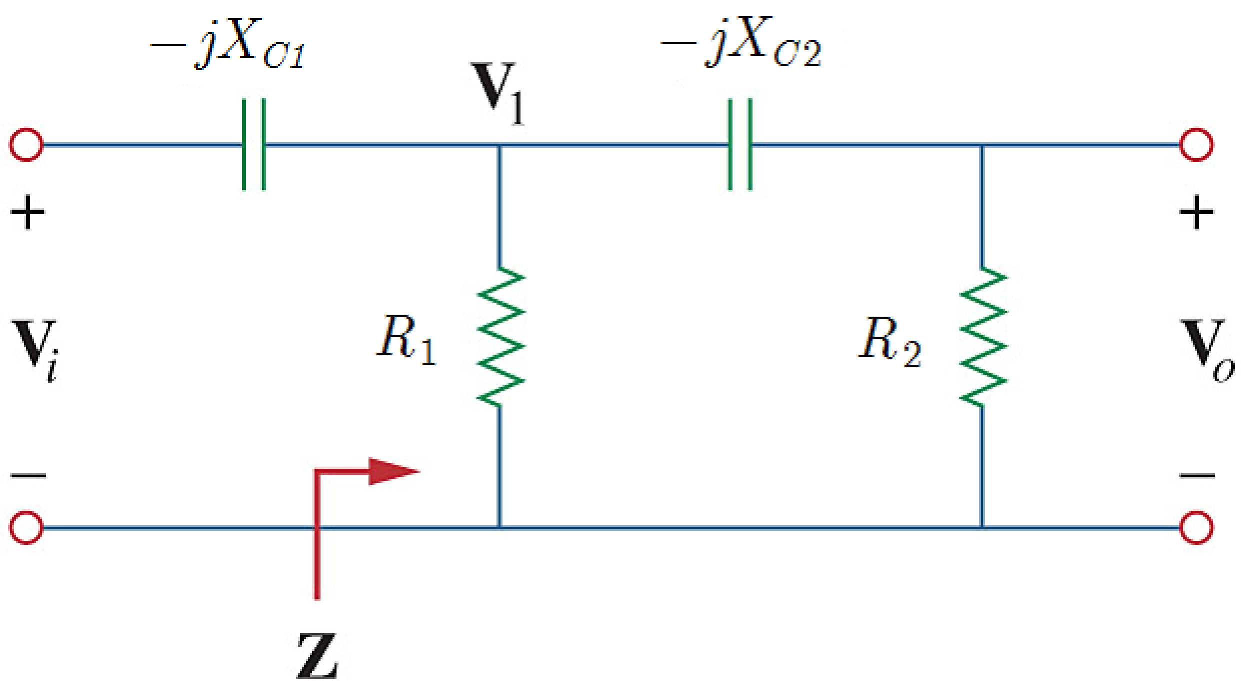


Figure 9.33 An RC phase shift circuit with 90° leading phase shift; for Example 9.13.

$$\dot{V}_o = \dot{V}_1 \frac{20}{20 - jX_{C2}} = \dot{V}_1 \frac{20(20 + jX_{C2})}{20^2 + X_{C2}^2}$$

If $X_{C2} = 20\ \Omega,$ then the second stage produces a $45°$ phase shift.

$$Z = 20 \,||\, (20 - j20) = \frac{20 \times (20 - j20)}{20 + (20 - j20)}$$

$$= 12 - j4\ (\Omega)$$

$$\dot{V}_1 = \dot{V}_i \frac{Z}{-jX_{C1} + Z} = \dot{V}_i \frac{12 - j4}{12 - j(4 + X_{C1})}$$

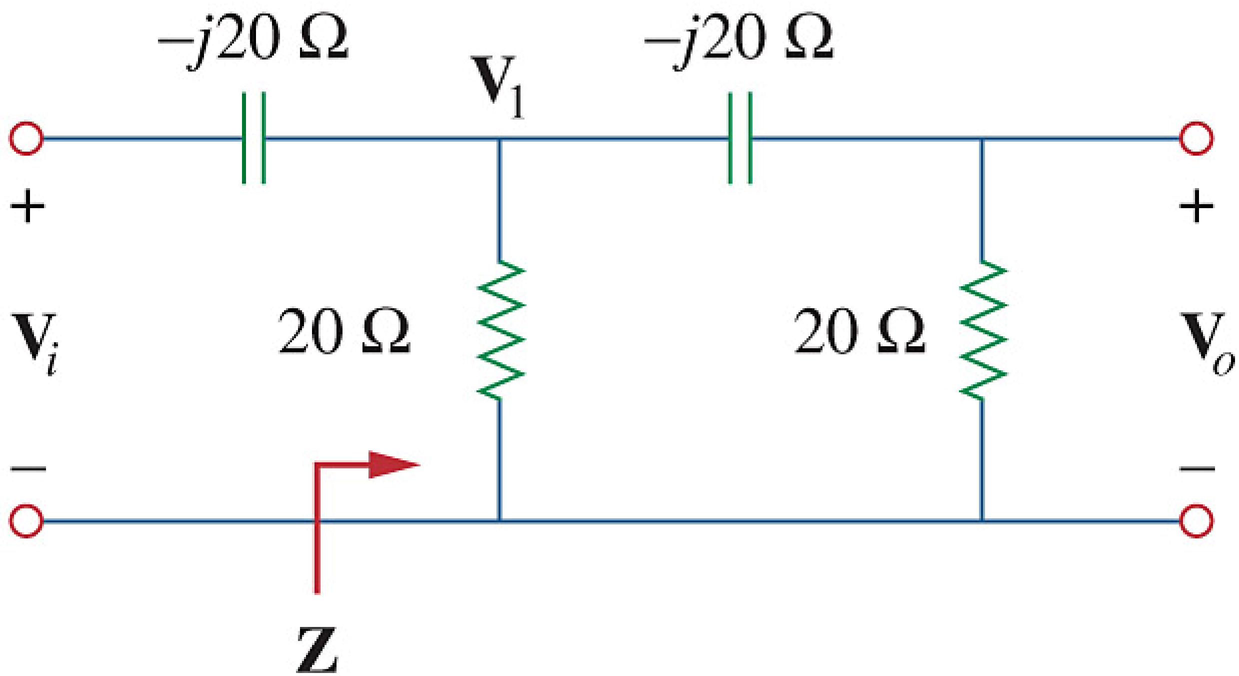
$$= \dot{V}_i \frac{(12 - j4)(12 + j(4 + X_{C1}))}{12^2 + (4 + X_{C1})^2}$$

$$= \dot{V}_i \frac{(160 + 4X_{C1}) + j(12X_{C1})}{12^2 + (4 + X_{C1})^2}$$

For the first stage to produce another $45°$,

we require $160 + 4X_{C1} = 12X_{C1}$, i.e.,

$X_{C1} = 20\ \Omega$.



$-j20\ \Omega$
$\mathbf{V}_1$
$-j20\ \Omega$
+
+
$\mathbf{V}_i$
$20\ \Omega$
$20\ \Omega$
$\mathbf{V}_o$
−
−
$\mathbf{Z}$

Figure 9.33