Ve215 Electric Circuits

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Chapter 11

AC Power Analysis

11.1 Introduction

- Our effort in ac circuit analysis so far has been focused on calculating voltage and current. Our major concern in this chapter is power analysis.

11.2 Instantaneous and Average Power

It is known that the *instantaneous* power $p(t)$ absorbed by an element is the product of the instantaneous voltage $v(t)$ across the element and the instantaneous current $i(t)$ through it.

$$p(t) = v(t)i(t)$$

Consider the instantaneous power absorbed by a circuit (or *load impedance*) under sinusoidal excitation, as shown in Fig. 11.1.

$$p(t) = v(t)i(t)$$

$$= V_m \cos(\omega t + \theta_v) \times I_m \cos(\omega t + \theta_i)$$

$$= \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$

A sketch of $p(t)$ is shown in Fig. 11.2.

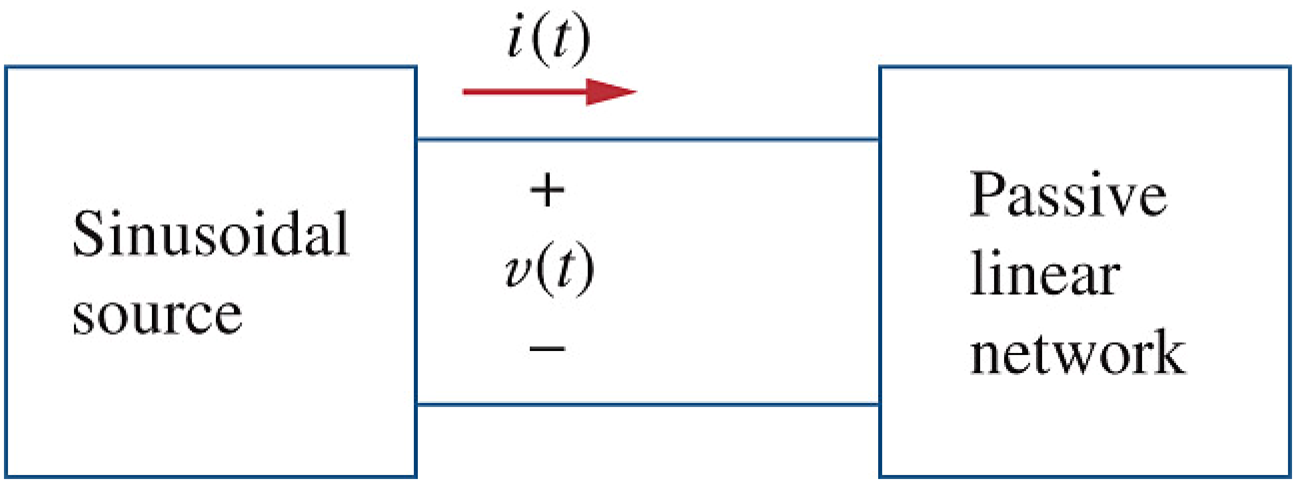

Figure 11.1 Sinusoidal source and passive linear circuit.

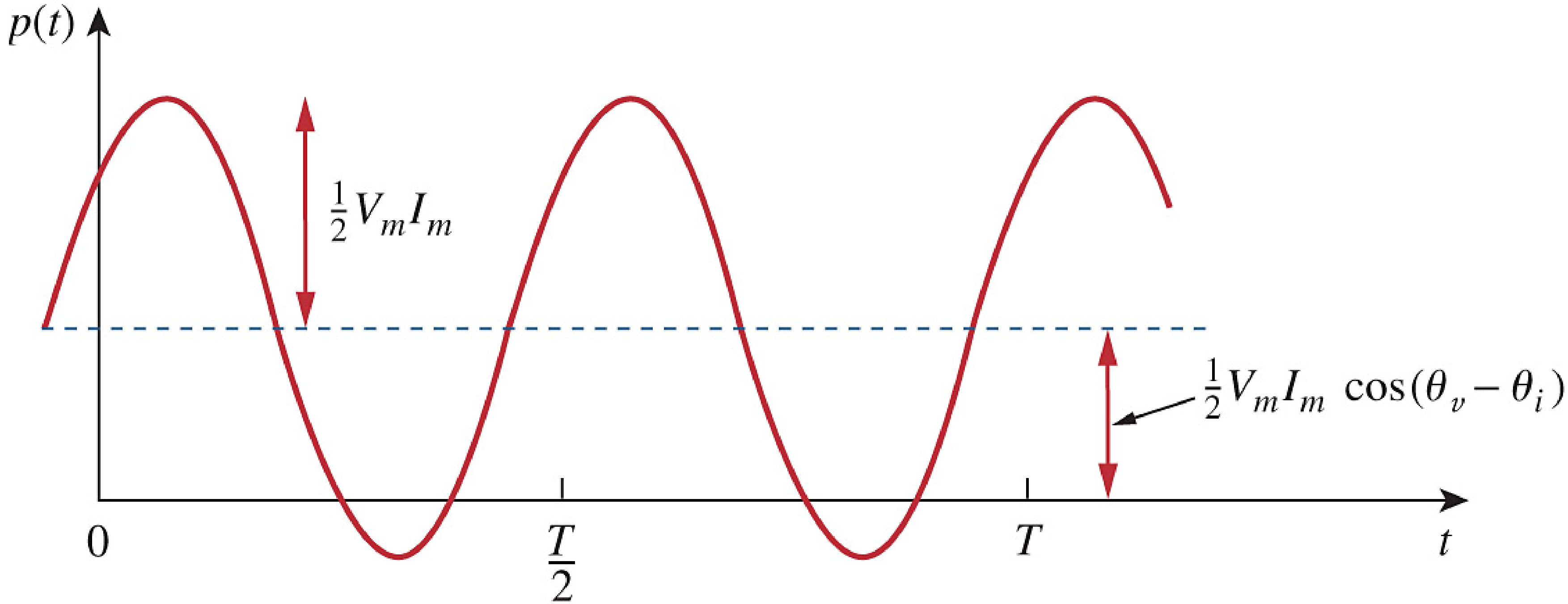

Figure 11.2 The instantaneous power absorbed by a circuit.

The *average* power, in watts, is the average of the instantaneous power over one period.

$$P = \frac{1}{T}\int_0^T p(t)dt$$

$$= \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)dt$$

$$+ \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)dt$$

$$= \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)$$

The average power P can be expressed in terms of phasors $\dot{V}$ and $\dot{I}$.

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$$

$$= \frac{1}{2} \mathrm{Re}\left(V_m I_m e^{j(\theta_v - \theta_i)} \right)$$

$$= \frac{1}{2} \mathrm{Re}\left(V_m e^{j\theta_v} I_m e^{-j\theta_i} \right)$$

$$= \frac{1}{2} \mathrm{Re}\left(\dot{V} \dot{I}^* \right)$$

When $\theta_v = \theta_i$, the voltage and current are in phase. This implies a purely resistive load *R*.

$$P = \frac{1}{2} V_m I_m = \frac{1}{2} I_m^2 R = \frac{1}{2} \frac{V_m^2}{R}$$

showing that a purely resistive load absorbs power at all times.

When $\theta_v - \theta_i = \pm 90°$, we have a purely reactive load.

$$P = \frac{1}{2} V_m I_m \cos\left(\pm 90°\right) = 0$$

showing that a purely reactive load absorbs no average power.

Generally, the average power absorbed by an impedance $Z = R + jX$ is given by

$$P = \frac{1}{2}\mathrm{Re}\left(\dot{V}\dot{I}^*\right) = \frac{1}{2}\mathrm{Re}\left(\dot{I}(R + jX)\dot{I}^*\right)$$

$$= \frac{1}{2}\mathrm{Re}\left(I_m^2 R + jI_m^2 X\right)$$

$$= \frac{1}{2}I_m^2 R$$

Practice Problem 11.3 Calculate the average power absorbed by the resistor and inductor in the circuit of Fig. 11.4. Find the average power supplied by the voltage source.

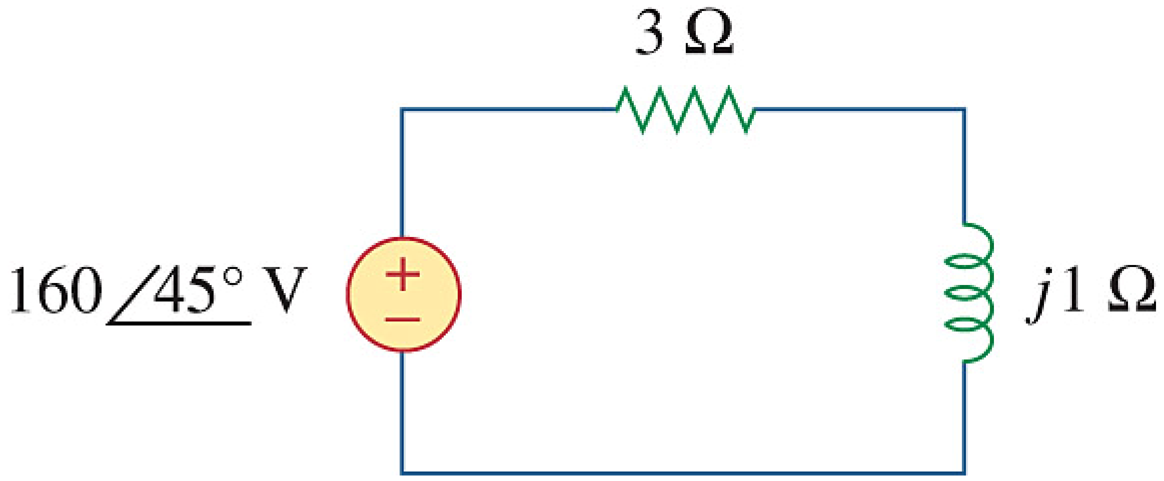

Figure 11.4

$$\dot{V} = 160\angle 45^\circ \text{ (V)}$$

$$\dot{I} = \frac{\dot{V}}{Z} = \frac{160\angle 45^\circ}{3 + j1} \approx \frac{160\angle 45^\circ}{3.1623\angle 18.43^\circ}$$

$$\approx 50.5961\angle 26.57^\circ \text{ (A)}$$

The average power absorbed by the resistor is

$$P_R = \frac{1}{2} I_m^2 R = \frac{1}{2} \times 50.5961^2 \times 3$$

$$\approx 3839.95 \text{ (W)} \approx 3.84 \text{ kW}$$

The average power absorbed by the inductor is zero, i.e.,

$$P_L = 0$$

The average power supplied by the voltage source is the same as the average power absorbed, i.e.,

$$P = 3.84 \text{ kW}$$

11.3 Maximum Average Power Transfer

Consider the circuit in Fig. 11.7, where an ac circuit is connected to a load Z_L and is represented by its Thevenin equivalent.

$$\dot{I} = \frac{\dot{V}_{Th}}{Z_{Th} + Z_L} = \frac{\dot{V}_{Th}}{(R_{Th} + jX_{Th}) + (R_L + jX_L)}$$

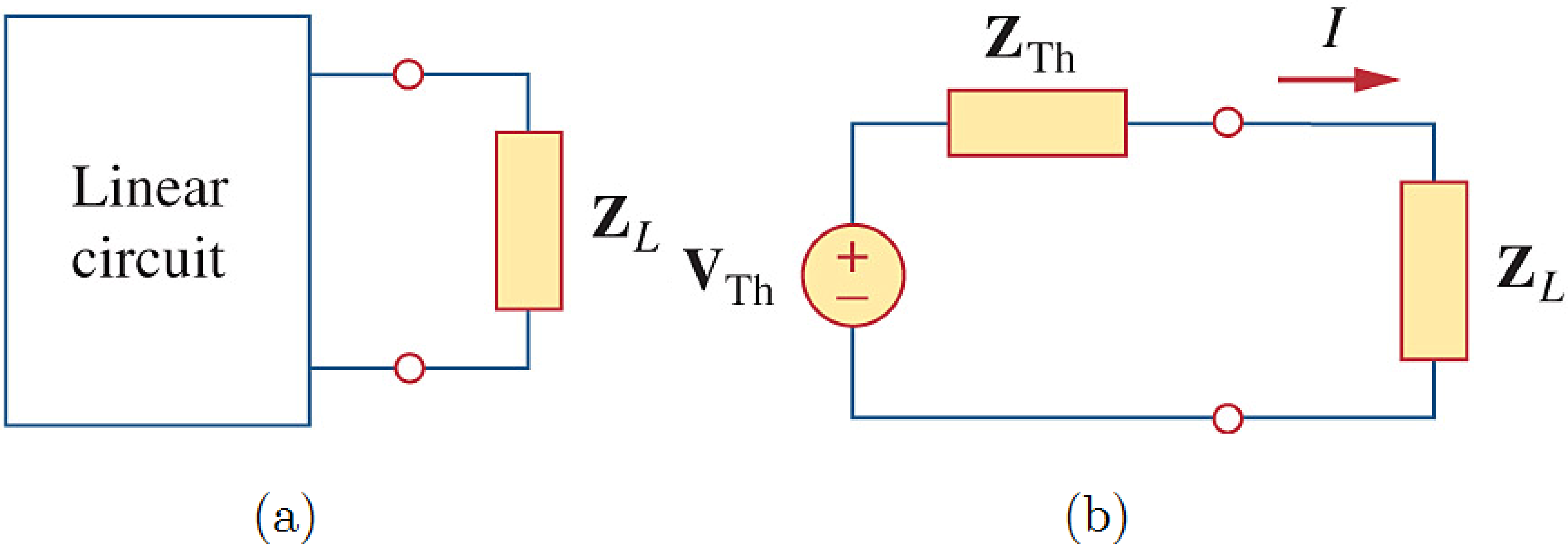

Figure 11.7 Finding the maximum average power transfer (a) circuit with a load, (b) the Thevenin equivalent.

The average power delivered to the load is

$$P = \frac{1}{2} I_m^2 R_L = \frac{1}{2} \frac{V_{Th}^2 R_L}{(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2}$$

Our objective is to adjust R_L and X_L so that P is maximum.

$$\begin{cases} \dfrac{\partial P}{\partial R_L} = \dfrac{V_{Th}^2[R_{Th}^2 - R_L^2 + (X_{Th} + X_L)^2]}{2[(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2]^2} = 0 \\ \dfrac{\partial P}{\partial X_L} = -\dfrac{V_{Th}^2 R_L (X_{Th} + X_L)}{[(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2]^2} = 0 \end{cases}$$

$$\begin{cases} R_L = R_{Th} \\ X_L = -X_{Th} \end{cases} \quad \text{or } Z_L = Z_{Th}^*$$

This result is known as the *maximum average power transfer theorem* for the sinusoidal steady state.

The maximum average power is

$$P_{\max} = \frac{V_{Th}^2}{8R_{Th}}$$

When the load is purely resistive, the condition for maximum average power transfer is

$$R_L = \sqrt{R_{Th}^2 + X_{Th}^2}$$

and the maximum average power is

$$P_{\max} = \frac{V_{Th}^2}{4\left(R_{Th} + \sqrt{R_{Th}^2 + X_{Th}^2}\right)}$$

Practice Problem 11.5 For the circuit shown in Fig. 11.10, find the load impedance Z_L that absorbs the maximum average power. Calculate that maximum average power.

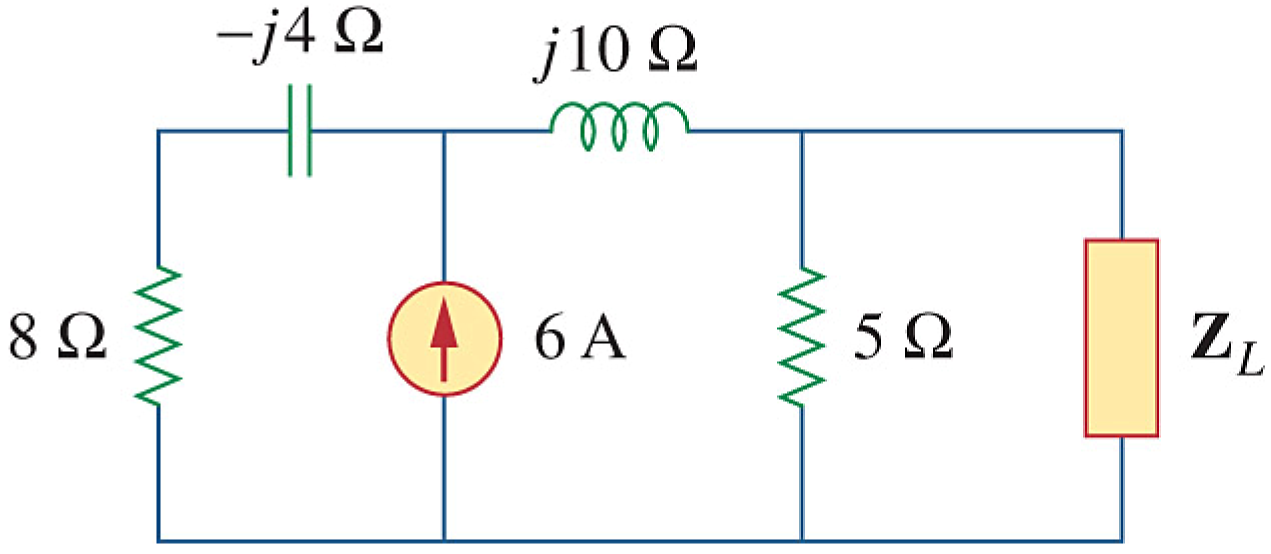

Figure 11.10

Solution :

$$Z_{Th} = 5 \parallel (j10 - j4 + 8) = 5 \parallel (8 + j6)$$

$$= \frac{5 \times (8 + j6)}{5 + (8 + j6)} = \frac{140 + j30}{41}$$

$$\approx 3.4146 + j0.7317\ (\Omega)$$

$$Z_L = Z_{Th}^* = 3.4146 - j0.7317\ \Omega$$

$$\dot{V}_{Th} = 6\times[(5+j10)\,\|\,(8-j4)]\times\frac{5}{5+j10}$$

$$= 6\times\frac{80+j60}{13+j6}\times\frac{5}{5+j10}$$

$$\approx 6\times\frac{100\angle 36.87^\circ}{14.3178\angle 24.78^\circ}\times\frac{5}{11.1803\angle 63.43^\circ}$$

$$\approx 18.7409\angle -51.34^\circ\ \text{(V)}$$

$$P_{\max} = \frac{V_{Th}^2}{8R_{Th}} = \frac{18.7409^2}{8\times 3.4146} \approx 12.86\ \text{(W)}$$

11.4 Effective or RMS Value

The *effective value* of an ac current i is the dc current I_{eff} that delivers the same average power to a resistor as the ac current .

In Fig. 11.13, the circuit in (a) is ac while that of (b) is dc. Our objective is to find I_{eff} that will transfer the same power to resistor R as the current i. The power absorbed by the resistor is

$$P = \frac{1}{T}\int_0^T i^2 R dt = \left(\frac{1}{T}\int_0^T i^2 dt\right) R = I_{eff}^2 R$$

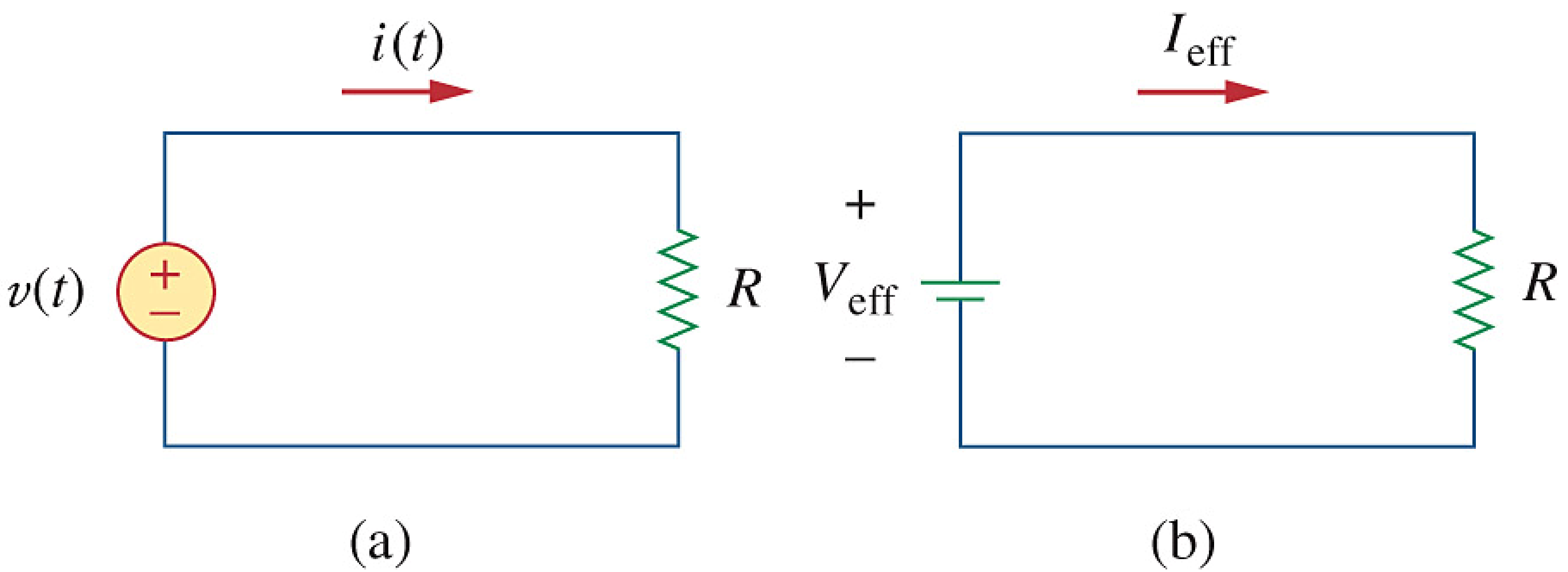

Figure 11.13 Finding the effective current: (a) ac circuit, (b) dc circuit.

The effective value of the current i is

$$I_{eff} = \sqrt{\frac{1}{T}\int_0^T i^2 dt}$$

The effective value of the voltage v is found in the same way as current, i.e.,

$$V_{eff} = \sqrt{\frac{1}{T}\int_0^T v^2 dt}$$

This indicates that the effective value is the (square) *root* of the *mean* of the *square* of the periodic signal. Thus, the effective value is often known as the *root-mean-square* value, or *rms* value for short; and we write

$$I_{eff} = I_{rms}, \quad V_{eff} = V_{rms}$$

Practice Problem 11.8 Find the rms value of the full-wave rectified sine wave in Fig. 11.17. Calculate the average power dissipated in a 6-Ω resistor.

Solution :

$$V_{rms} = \sqrt{\frac{1}{\pi}\int_0^{\pi} (10\sin t)^2 \, dt}$$

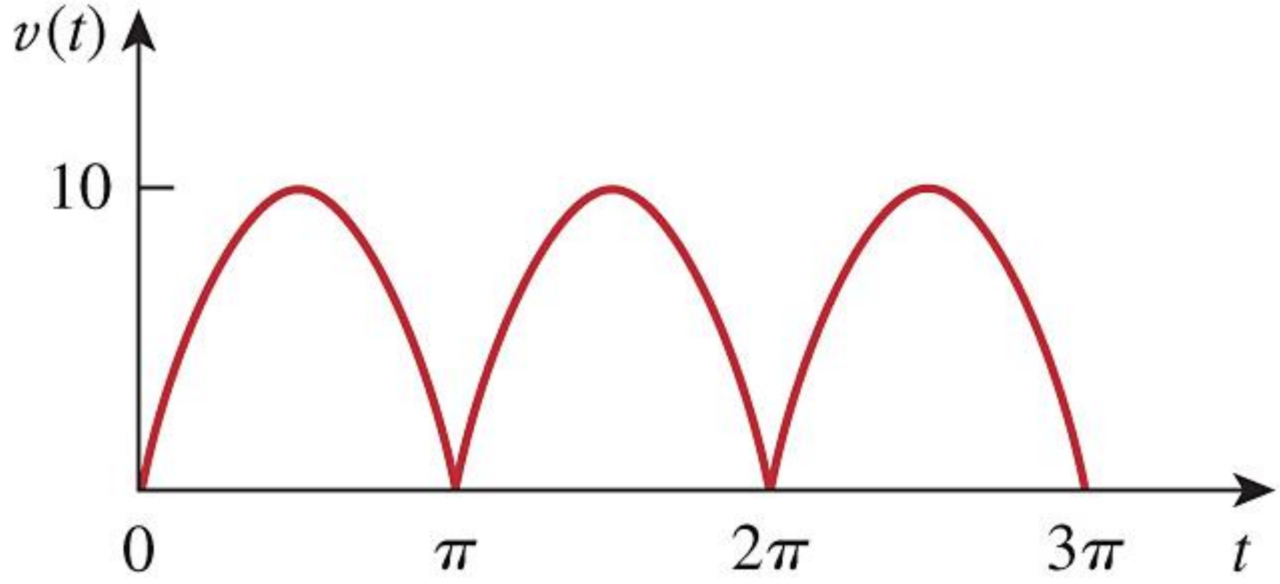

Figure 11.17

$$V_{rms} = \sqrt{\frac{1}{\pi}\int_0^{\pi}(10\sin t)^2\,dt}$$

$$= 10\sqrt{\frac{1}{\pi}\int_0^{\pi}\frac{1-\cos 2t}{2}\,dt} = 5\sqrt{2} \approx 7.07\ (\mathrm{V})$$

$$P = \frac{V_{rms}^2}{R} = \frac{\left(5\sqrt{2}\right)^2}{6} \approx 8.33\ (\mathrm{W})$$

For the sinusoid $i(t) = I_m \cos \omega t$,

$$I_{rms} = \sqrt{\frac{1}{T}\int_0^T I_m^2 \cos^2 \omega t dt}$$

$$= \sqrt{\frac{I_m^2}{T}\int_0^T \frac{1+\cos 2\omega t}{2} dt} = \frac{I_m}{\sqrt{2}}$$

Similarly, for $v(t) = V_m \cos \omega t$,

$$V_{rms} = \frac{V_m}{\sqrt{2}}$$

The average power

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$$

$$= \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos(\theta_v - \theta_i) = V_{rms} I_{rms} \cos(\theta_v - \theta_i)$$

The average power absorbed by a resistor

$$P = I_{rms}^2 R = \frac{V_{rms}^2}{R}$$

Define

$$\dot{V}_{rms} = V_{rms}\angle\theta_v \text{ and } \dot{I}_{rms} = I_{rms}\angle\theta_i$$

$$\dot{V}_{rms} = V_{rms}\angle\theta_v = \frac{V_m}{\sqrt{2}}\angle\theta_v = \frac{\dot{V}}{\sqrt{2}}$$

$$\dot{I}_{rms} = I_{rms}\angle\theta_i = \frac{I_m}{\sqrt{2}}\angle\theta_i = \frac{\dot{I}}{\sqrt{2}}$$

The load impedance

$$Z=\frac{\dot{V}}{\dot{I}}=\frac{\dot{V}_{rms}}{\dot{I}_{rms}}$$

since

$$\frac{\dot{V}}{\dot{I}}=\frac{V_m\angle\theta_v}{I_m\angle\theta_i}=\frac{V_m}{I_m}\angle(\theta_v-\theta_i)=\frac{V_{rms}}{I_{rms}}\angle(\theta_v-\theta_i)$$

$$=\frac{V_{rms}\angle\theta_v}{I_{rms}\angle\theta_i}=\frac{\dot{V}_{rms}}{\dot{I}_{rms}}$$

11.5 Apparent Power and Power Factor

In the expression

$$P = V_{rms} I_{rms} \cos(\theta_v - \theta_i)$$

The product $V_{rms} I_{rms}$ is known as the *apparent power* ($|S|$). The factor $\cos(\theta_v - \theta_i)$ is called the *power factor* (pf or $\cos\theta$). The angle $\theta_v - \theta_i$ is called the *power factor angle* (θ).

The apparent power is so called because it seems apparent that the power should be the voltage-current product, by analogy with dc resistive circuits. It is measured in volt-amperes or VA to distinguish it from the average or *real* power, which is measured in watts.

The power factor is dimensionless, since it is the ratio of the average power to the apparent power.

$$\text{pf} = \cos\theta = \frac{P}{|S|}$$

The power factor angle is equal to the angle of the load impedance.

$$Z = \frac{\dot{V}}{\dot{I}} = \frac{V_m \angle \theta_v}{I_m \angle \theta_i} = \frac{V_m}{I_m} \angle(\theta_v - \theta_i) \text{ or}$$

$$= \frac{V_{rms}}{I_{rms}} \angle(\theta_v - \theta_i) = \frac{V_{rms} \angle \theta_v}{I_{rms} \angle \theta_i} = \frac{\dot{V}_{rms}}{\dot{I}_{rms}}$$

The value of pf ranges between zero and unity. For a purely resistive load, $\theta_v - \theta_i = 0$, pf $= 1$, implifying that $P = |S|$. For a purely reactive load, $\theta_v - \theta_i = \pm 90°$, pf $= 0$, implifying that $P = 0$. In between these two extreme cases, pf is said to be leading $(\theta_v - \theta_i < 0)$ or lagging $(\theta_v - \theta_i > 0)$.

Practice Problem 11.9 Obtain the power factor and the apparent power of a load whose impedance is $Z = 60 + j40\ \Omega$ when the applied voltage is $v(t) = 160\cos(377t + 10°)$ V.

Solution :

$$Z = 60 + j40 \approx 72.1110\angle 33.69°\ (\Omega)$$

$$\theta = 33.69^\circ$$

$$\text{pf} = \cos\theta = \cos 33.69^\circ \approx 0.8321 \text{ lagging}$$

$$\dot{I} = \frac{\dot{V}}{Z} = \frac{160\angle 10^\circ}{72.1110\angle 33.69^\circ}$$

$$\approx 2.2188\angle -23.69^\circ \text{ (A)}$$

$$|S| = \frac{1}{2} V_m I_m = \frac{1}{2} \times 160 \times 2.2188$$

$$\approx 177.50 \text{ (VA)}$$

11.6 Complex Power

Introducing the concept of *complex power* enables us to express power relations as simply as possible.

In Fig. 11.20, the complex power absorbed by the load is

$$S = \frac{1}{2}\dot{V}\dot{I}^{*} = \dot{V}_{rms}\dot{I}^{*}_{rms}$$

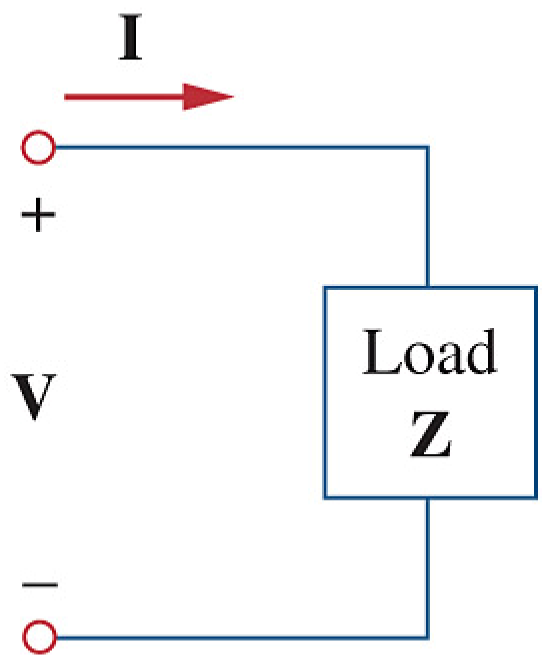

Figure 11.20 The voltage and current phasors associated with a load.

$$S = \dot{V}_{rms} \dot{I}_{rms}^{*} = \left(V_{rms} \angle \theta_v\right)\left(I_{rms} \angle \theta_i\right)^{*}$$

$$= V_{rms} I_{rms} \angle(\theta_v - \theta_i)$$

$$S = V_{rms} I_{rms} \angle(\theta_v - \theta_i) = \left|S\right| \angle(\theta_v - \theta_i)$$

$$
\begin{aligned}
S &= V_{rms} I_{rms} \angle(\theta_v - \theta_i) \\
&= V_{rms} I_{rms} \cos(\theta_v - \theta_i) + jV_{rms} I_{rms} \sin(\theta_v - \theta_i) \\
&= \mathrm{Re}\left(S\right) + j\,\mathrm{Im}\left(S\right) \\
&= P + jQ
\end{aligned}
$$

P is the average or real power in watts.

$$P = \mathrm{Re}(S) = I_{rms}^2 R = V_{rms} I_{rms} \cos(\theta_v - \theta_i)$$

$$= |S| \cos(\theta_v - \theta_i)$$

Q is the reactive or quadrature power, in *volt-amperes reactive* (VAR).

$$Q = \mathrm{Im}(S) = I_{rms}^2 X = V_{rms} I_{rms} \sin(\theta_v - \theta_i)$$

$$= |S| \sin(\theta_v - \theta_i)$$

The reactive power Q represents a lossless power exchange between the load and the source. Notice that

1. $Q = 0$ for resistive loads (unity pf)
2. $Q < 0$ for capacitive loads (leading pf)
3. $Q > 0$ for inductive loads (lagging pf)

$$S = \dot{V}_{rms}\dot{I}_{rms}^{*} = \dot{V}_{rms}\left(\frac{\dot{V}_{rms}}{Z}\right)^{*} = \frac{V_{rms}^{2}}{Z^{*}}$$

$$S = \dot{V}_{rms}\dot{I}_{rms}^{*} = \left(\dot{I}_{rms}Z\right)\dot{I}_{rms}^{*} = I_{rms}^{2}Z$$

$$S = I_{rms}^{2}Z = I_{rms}^{2}(R + jX) = I_{rms}^{2}R + jI_{rms}^{2}X$$

$$P = I_{rms}^{2}R$$

$$Q = I_{rms}^{2}X$$

It is a standard practice to represent S, P, and Q in the form of a triangle, known as the power triangle, shown in Fig. 11.21(a). This is similar to the impedance triangle showing the relationship between Z, R, and X, illustrated in Fig. 11.21(b).

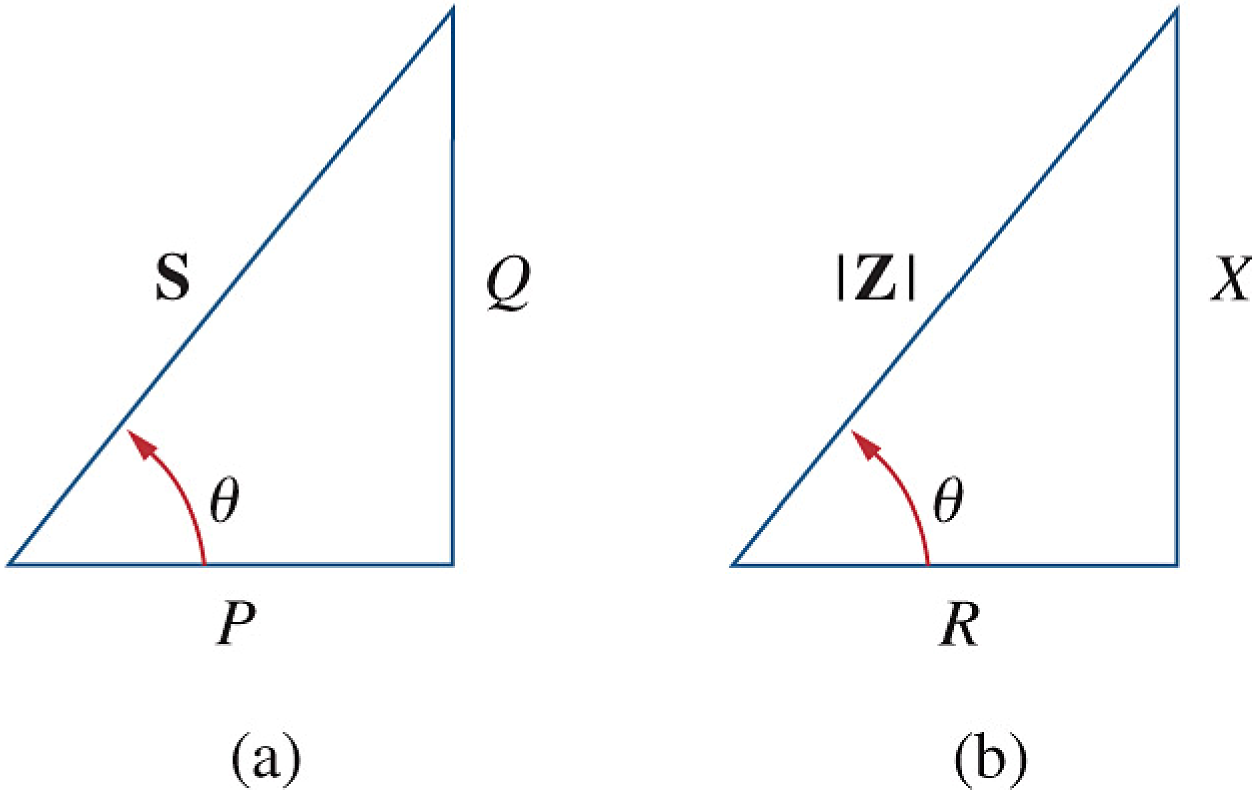

Figure 11.21 (a) Power triangle, (b) impedance triangle.

In conclusion, the complex power S contains all power information of a load. The real part of S is the real power P; its imaginary part is the reactive power Q; its magnitude is the apparent power $|S|$; and the cosine of its phase angle is the power factor pf.

Practice Problem 11.11 For a load, $\dot{V}_{rms} = 110\angle 85°$ V, $\dot{I}_{rms} = 0.4\angle 15°$ A. Determine: (a) the complex and apparent powers, (b) the real and reactive powers, and the power factor and the load impedance.

Solution :

$$S = \dot{V}_{rms}\dot{I}^{*}_{rms} = 110\angle 85° \times \left(0.4\angle 15°\right)^{*}$$

$$= 44\angle 70^\circ \text{ (VA)}$$

$$|S| = 44 \text{ VA}$$

$$P = \mathrm{Re}(S) = 44\cos 70^\circ \approx 15.05 \text{ (W)}$$

$$Q = \mathrm{Im}(S) = 44\sin 70^\circ \approx 41.35 \text{ (VAR)}$$

$$\text{pf} = \cos 70^\circ \approx 0.3420 \text{ lagging}$$

$$Z = \frac{\dot{V}_{rms}}{\dot{I}_{rms}} = \frac{110\angle 85^\circ}{0.4\angle 15^\circ} = 275\angle 70^\circ$$

$$\approx 94.06 + j258.42 \text{ }(\Omega)$$

11.7 Conservation of AC Power

The Principle of Conservation of AC Power : The complex, real, and reactive powers of the source equal the respective sums of the complex, real, and reactive powers of the individual loads.

FROM NOW ON, UNLESS OTHERWISE SPECIFIED, ALL VALUES OF VOLTAGES AND CURRENTS WILL BE ASSUMED TO BE *RMS* VALUES.

Practice Problem 11.13 In the circuit in Fig. 11.25, the 60-Ω resistor absorbs an average power of 240 W. Find $\dot{V}$ and the complex power of each branch of the circuit. What is the overall complex power of the circuit? (Assume the current through the 60-Ω resistor has no phase shift.)

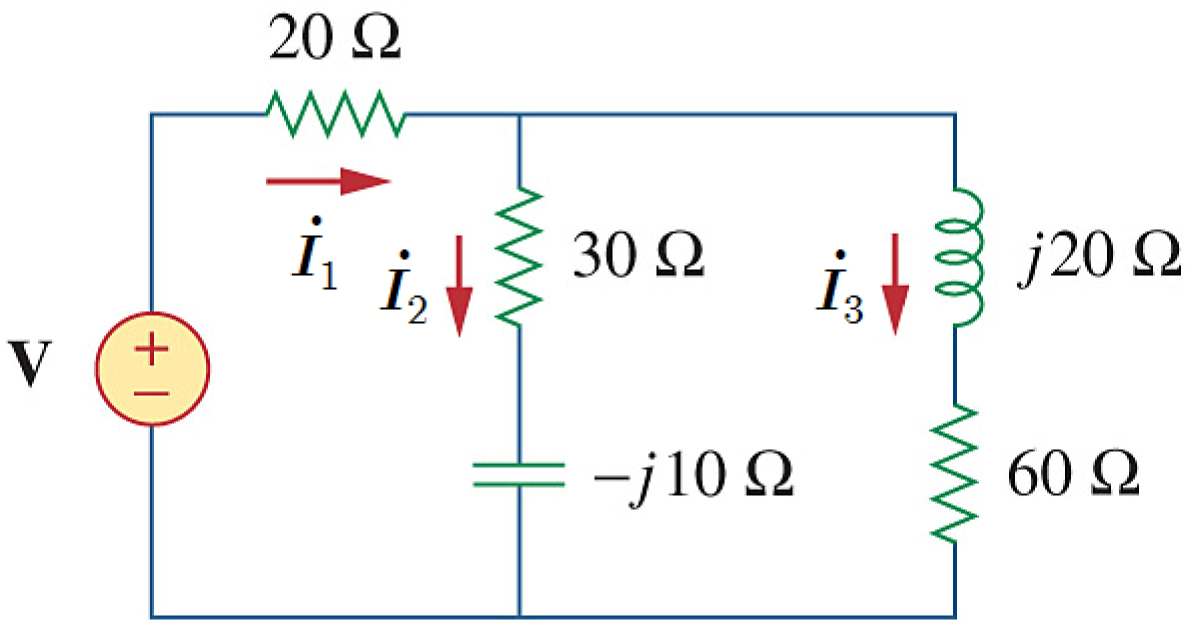

Figure 11.25

Solution :

$$I_3 = \sqrt{\frac{240}{60}} = 2 \text{ (A)}$$

$$\dot{I}_3 = 2\angle 0^\circ \text{ (A)}$$

$$\dot{V}_3 = \dot{I}_3 Z_3 = 2\times(60 + j20)$$

$$= 120 + j40 \text{ (V)}$$

$$\dot{I}_2 = \frac{\dot{V}_3}{Z_2} = \frac{120 + j40}{30 - j10} = 3.2 + j2.4 \text{ (A)}$$

$$\dot{I}_1 = \dot{I}_2 + \dot{I}_3 = (3.2 + j2.4) + 2$$

$$= 5.2 + j2.4 \text{ (A)}$$

$$\dot{V} = \dot{I}_1 Z_1 + \dot{V}_3$$

$$= (5.2 + j2.4) \times 20 + (120 + j40)$$

$$= 224 + j88 \text{ (V)}$$

$$S_1 = I_1^2 Z_1 = (5.2^2 + 2.4^2) \times 20 = 656 \text{ (VA)}$$

$$S_2 = \dot{V}_3 \dot{I}_2^* = (120 + j40)(3.2 - j2.4)$$

$$= 480 - j160 \text{ (VA)}$$

$$S_3 = \dot{V}_3 \dot{I}_3^* = (120 + j40) \times 2$$

$$= 240 + j80 \text{ (VA)}$$

$$S = S_1 + S_2 + S_3 = 656 + (480 - j160) +$$

$$(240 + j80) = 1376 - j80 \text{ (VA)}$$

Practice Problem 11.14 Two loads connected in parallel are respectively 2 kW at a pf of 0.75 leading and 4 kW at a pf of 0.95 lagging. Calculate the pf of the two loads. Find the complex power provided by the source.

Solution :

$$P = P_1 + P_2 = 2 + 4 = 6\ (\text{kW})$$

$$\text{pf}_1 = \cos\theta_1 = 0.75 \text{ leading} \Rightarrow$$

$$\theta_1 = -\cos^{-1}(0.75) \approx -41.41^\circ$$

$$\text{pf}_2 = \cos\theta_2 = 0.95 \text{ lagging} \Rightarrow$$

$$\theta_2 = \cos^{-1}(0.95) \approx 18.19^\circ$$

$$Q = Q_1 + Q_2 = P_1 \tan\theta_1 + P_2 \tan\theta_2$$

$$= 2 \times \tan(-41.41°) + 4 \times \tan 18.19°$$

$$\approx -0.4495 \text{ (kVAR)}$$

$$S = P + jQ = 6 - j0.4495 \text{ (kVA)}$$

$$|S| \approx 6.0168 \text{ (kVA)}$$

$$\text{pf} = \frac{P}{|S|} = \frac{6}{6.0168} \approx 0.9972 \text{ leading}$$

11.8 Power Factor Correction

Most domestic and industrial loads are inductive and operate at a low lagging power factor.

The power factor of an inductive load can be improved by deliberately installing a capacitor in parallel with the load, as shown in Fig. 11.27.

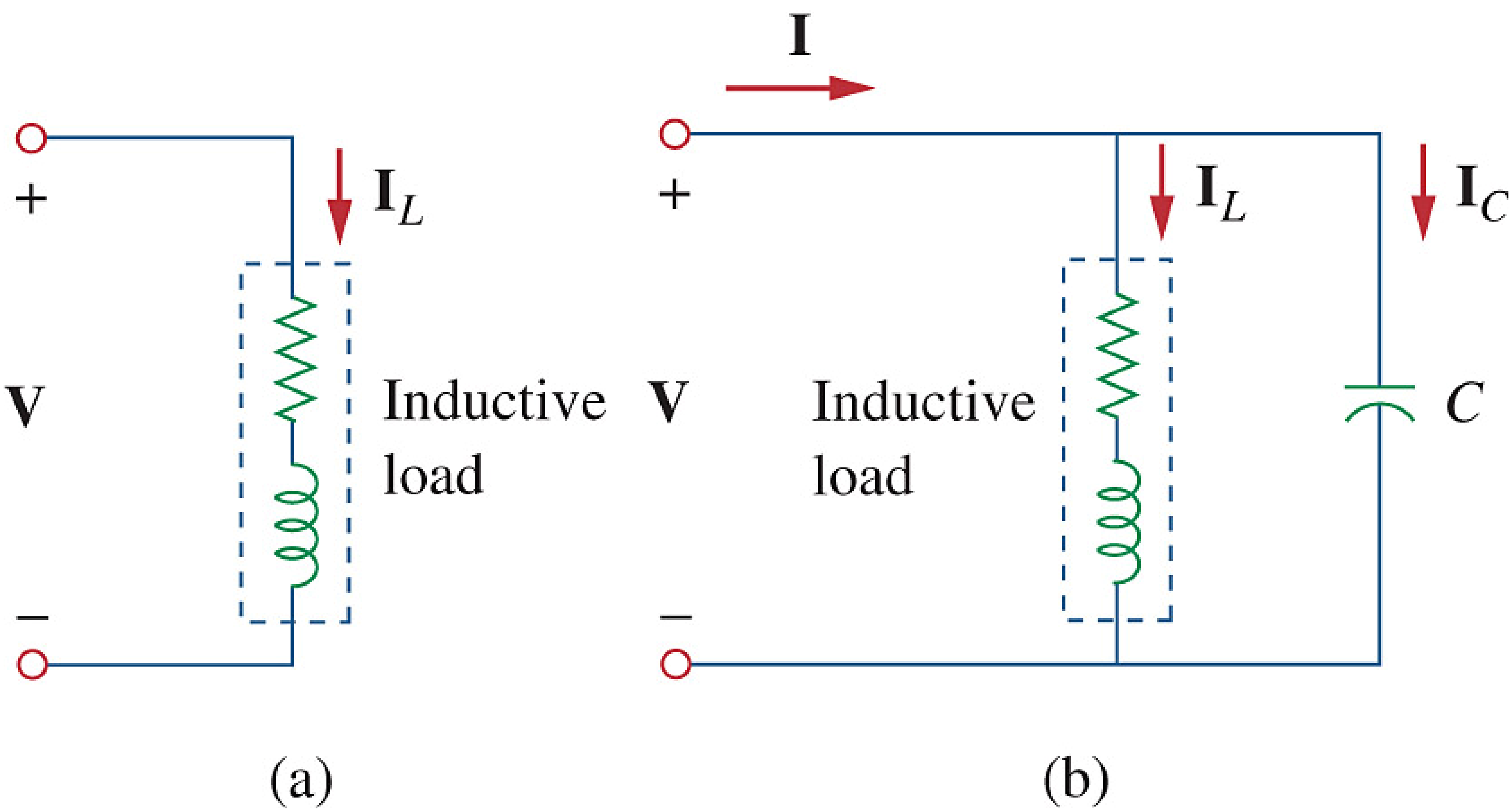

Figure 11.27 Power factor correction: (a) original inductive load, (b) inductive load with improved power factor.

The process of increasing the power factor of a load is known as *power factor correction.*

Figure 11.28 shows the phasor diagram of the currents involved. It is evident that adding the capacitor has caused the power factor to reduce from θ_1 to θ_2, thereby increasing the power factor.

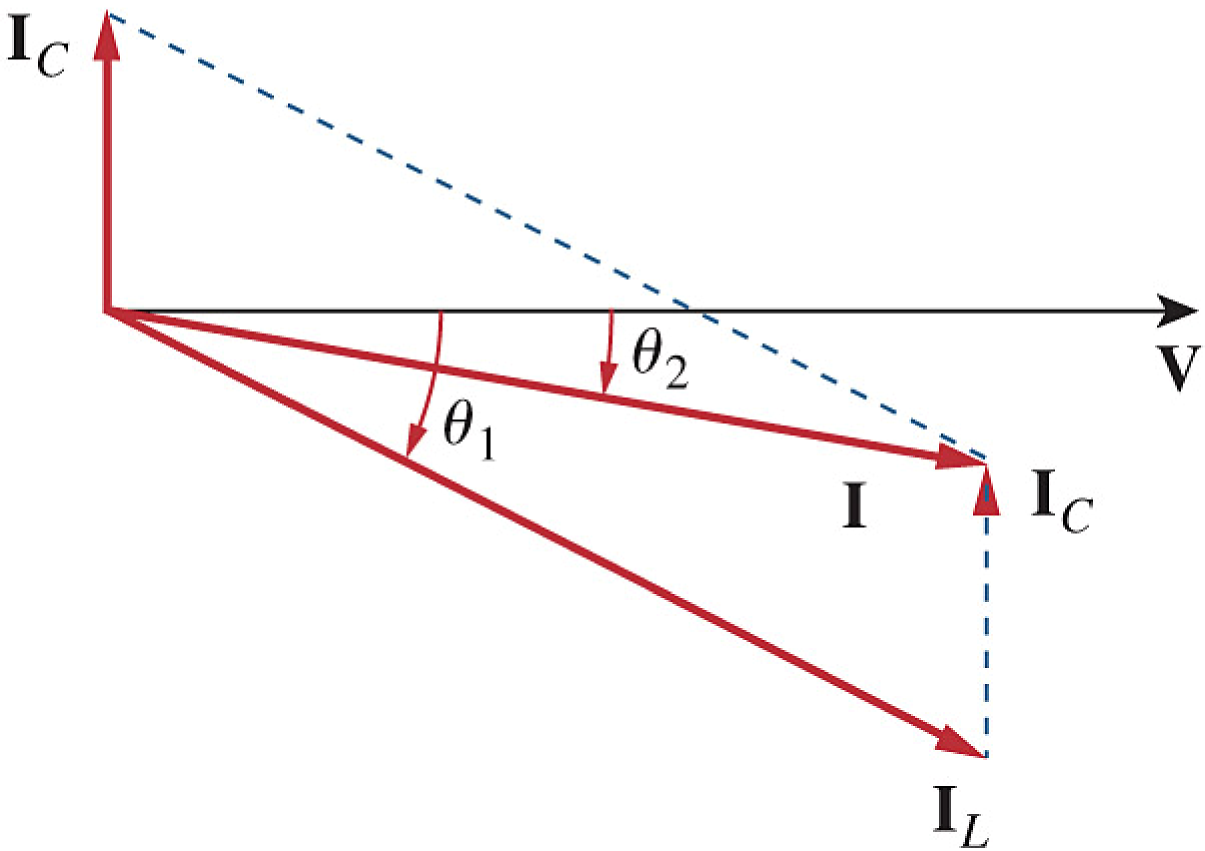

Figure 11.28 Phasor diagram showing the effect of adding a capacitor in parallel with the inductive load.

We also notice that with the same supplied voltage, the circuit in Fig. 11.27(b) draws smaller current than the current drawn by the circuit in Fig. 11.27(a), that is, $\dot{I} < \dot{I}_L$, resulting in decreased power losses.

$$I^2 R < I_L^2 R$$

where R is the resistance of the power line.

Consider the power triangle in Fig. 11.29.

$$P = |S_1|\cos\theta_1 = |S_2|\cos\theta_2$$

$$Q_1 = |S_1|\sin\theta_1 = P\tan\theta_1$$

$$Q_2 = |S_2|\sin\theta_2 = P\tan\theta_2$$

$Q_2 < Q_1$ is due to the shunt capacitor, i.e.,

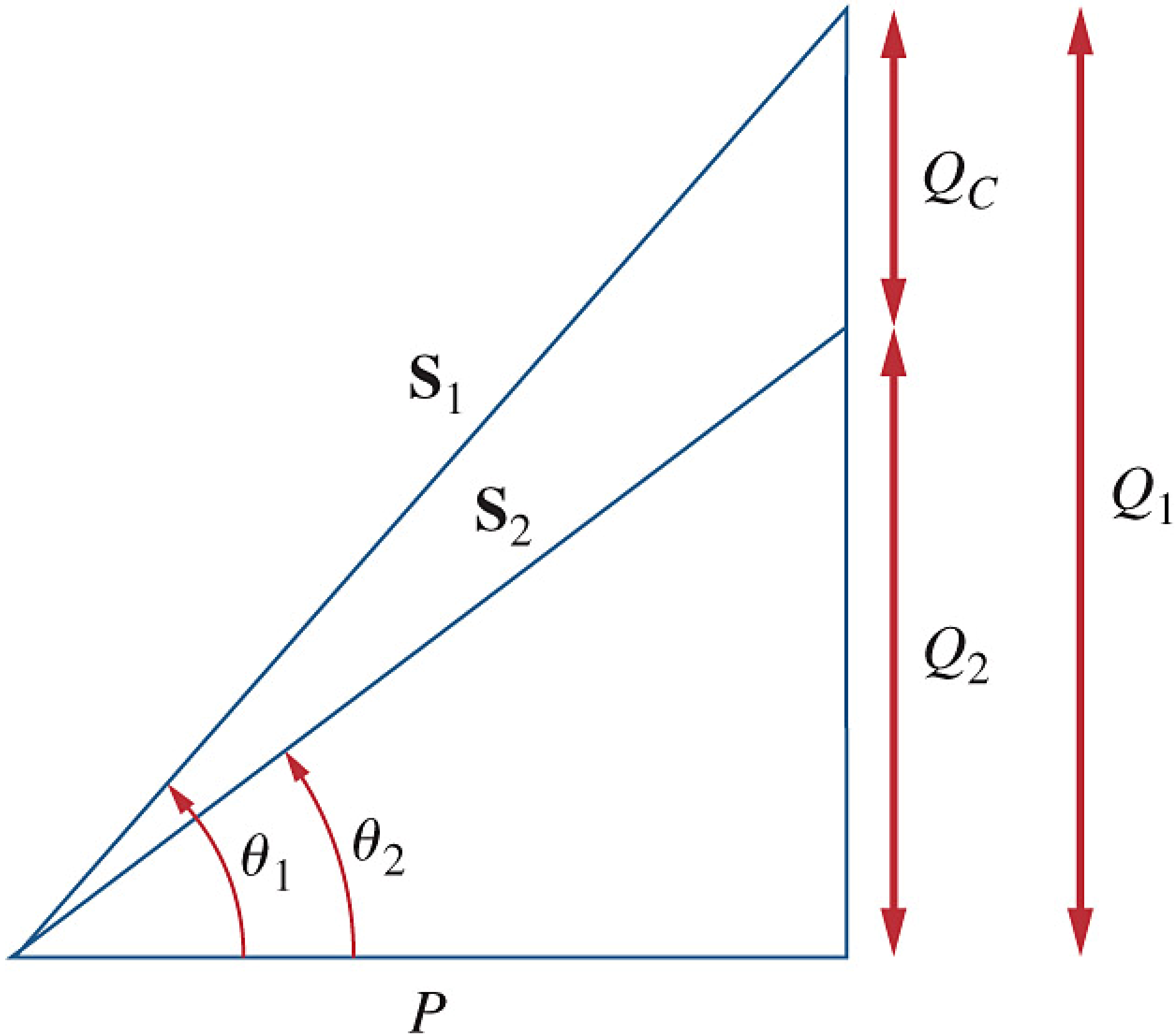

Figure 11.29 Power triangle illustrating power factor correction.

$$Q_2 = Q_1 + Q_C \text{ or } Q_C = Q_2 - Q_1$$

$$\text{But } Q_C = I_C^2 X_C = \frac{V^2}{X_C} \text{ and } X_C = -\frac{1}{\omega C}$$

The value of the required shunt capacitance C is determined by

$$C = \frac{P(\tan\theta_1 - \tan\theta_2)}{\omega V^2}$$

Example 11.15 When connected to a 120-V (rms), 60-Hz power line, a load absorbs 4 kW at a lagging power factor of 0.8. Find the value of capacitance necessary to raise the pf to 0.95.

Solution :

$$\cos\theta_1 = 0.8 \Rightarrow \tan\theta_1 = \frac{\sqrt{1-0.8^2}}{0.8} = 0.75$$

$$\cos\theta_2 = 0.95 \Rightarrow \tan\theta_2 = \frac{\sqrt{1-0.95^2}}{0.95} \approx 0.3287$$

$$C = \frac{P(\tan\theta_1 - \tan\theta_2)}{\omega V^2}$$

$$= \frac{4000\times(0.75-0.3287)}{(2\pi\times 60)\times 120^2} \approx 3.1043\times 10^{-4}\ \text{(F)}$$