Ve215 Electric Circuits

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Chapter 6

Capacitors and Inductors

6.1 Introduction

- In this chapter, we study two passive circuit elements: the capacitor and the inductor.
- Capacitors and inductors do not dissipate but store energy, which can be retrieved at a later time. For this reason, capacitors and inductors are called *storage* elements.

6.2 Capacitors

- A capacitor is a passive element designed to store energy in its electric field.
- A capacitor is typically constructed as depicted in Fig. 6.1. It consists of two conducting plates separated by an insulator (or dielectric).

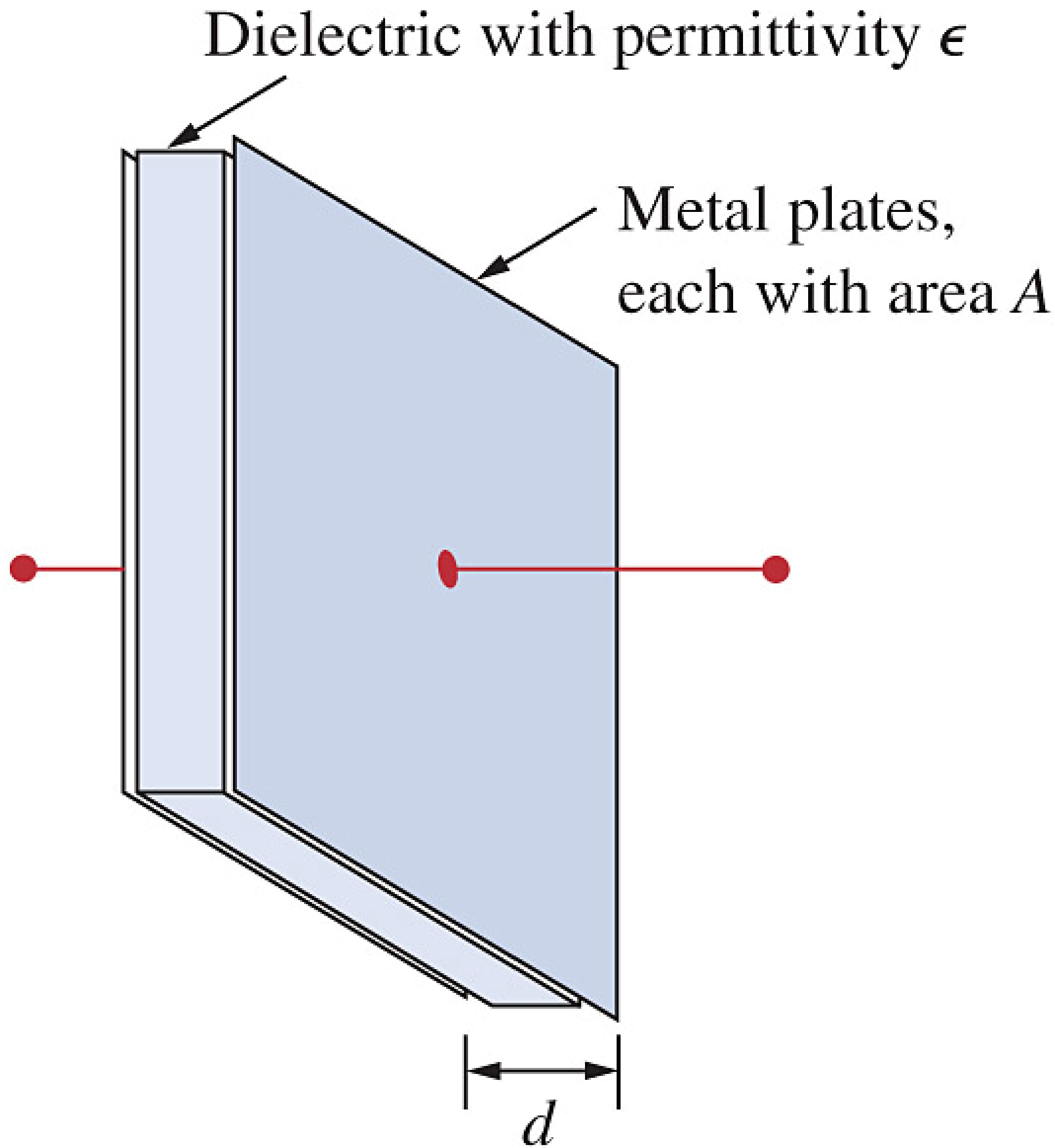


Figure 6.1 A typical capacitor.

When a voltage source v is connected to the capacitor, as in Fig. 6.2, the source deposits a positive charge q on one plate and a negative charge $-q$ on the other. The capacitor is said to store the electric charge.

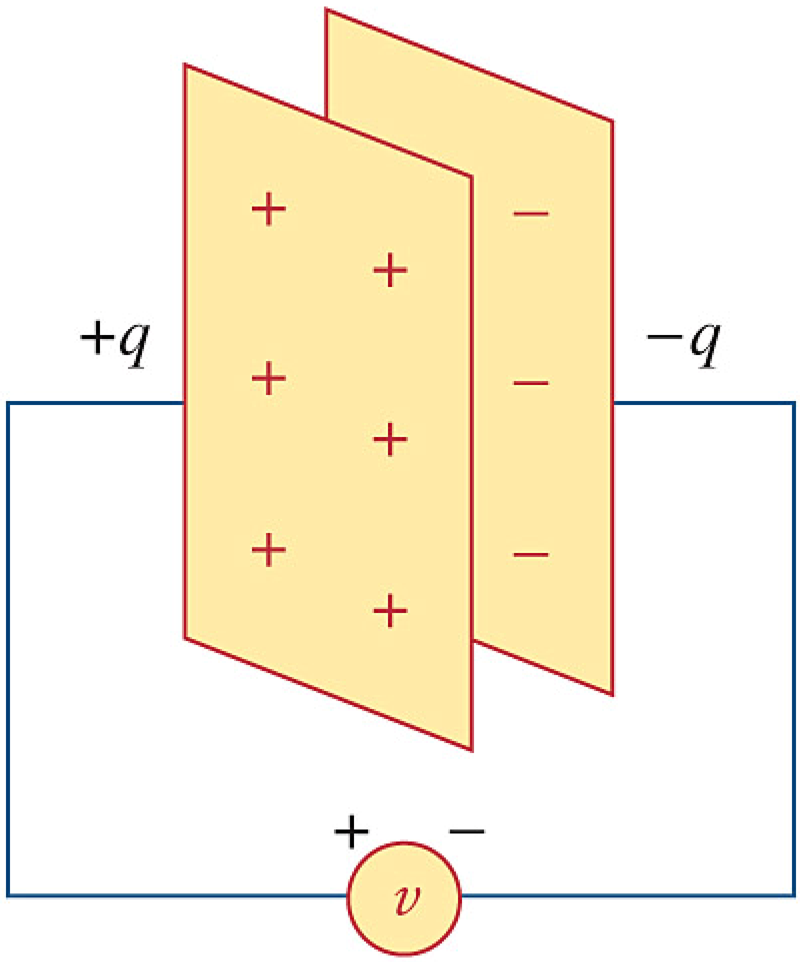


Figure 6.2 A capacitor with applied voltage v.

The amount of charge stored, represented by q, is given by

$$q = Cv$$

where C is known as the capacitance of the capacitor, measured in farads (F). If C is independent of v, the capacitor is said to be *linear*.

Figure 6.3 shows the circuit symbols for fixed and variable capacitors. Note that acccording to the passive sign convention, if $v \cdot i > 0$, the capacitor is being charged, and if $v \cdot i < 0$, the capacitor is discharging.

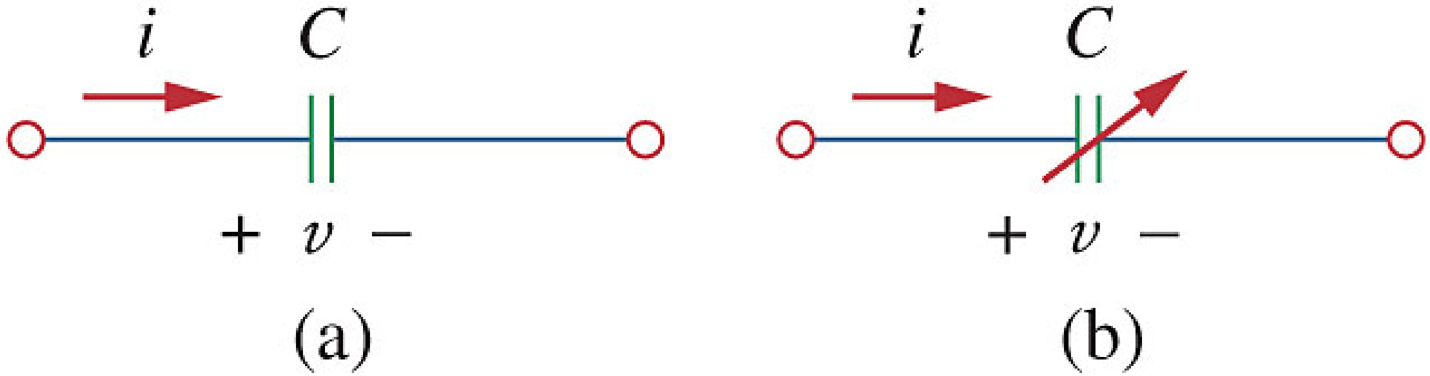


Figure 6.3 Circuit symbols for capacitors: (a) fixed capacitor, (b) variable capacitor.

The current-voltage relationship for the capacitor is given by

$$i = C \frac{dv}{dt}$$

as illustrated in Fig. 6.6.

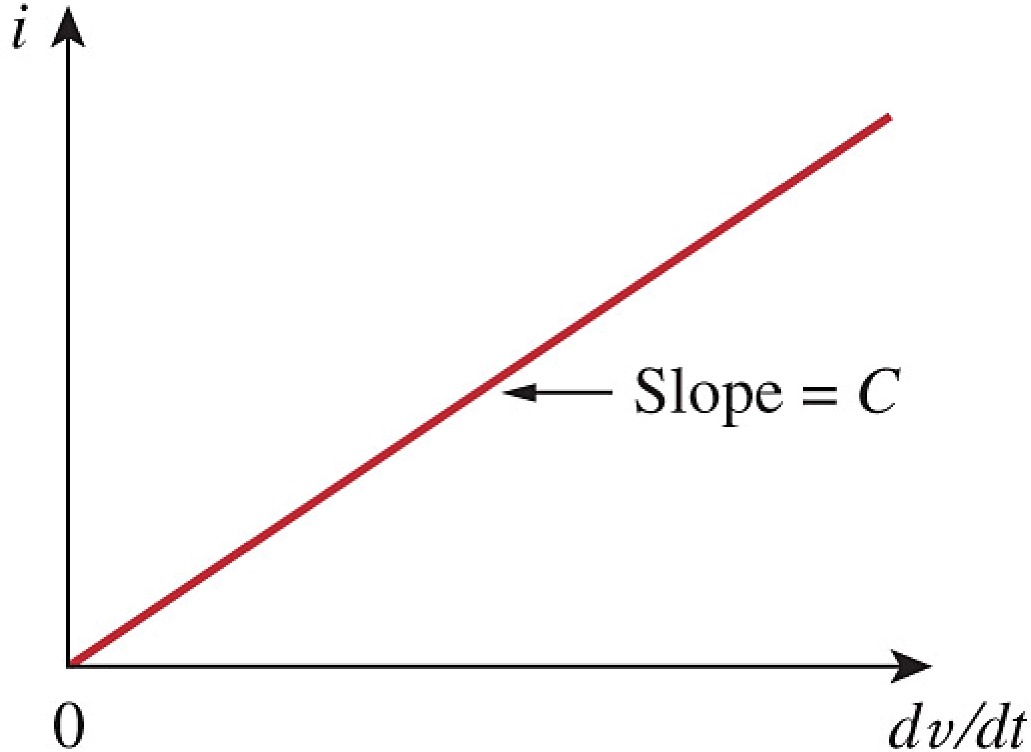


Figure 6.6 Current-voltage relationship of a capacitor.

The voltage-current relation of the capacitor is

$$v = \frac{1}{C}\int_{-\infty}^{t} i\,dt = \frac{1}{C}\int_{t_0}^{t} i\,dt + v(t_0)$$

where $v(t_0) = \frac{1}{C}\int_{-\infty}^{t_0} i\,dt = \frac{q(t_0)}{C}$ is the voltage across the capacitor at time t_0.

The instantaneous power delivered to the capacitor is

$$p = vi = v\left(C\frac{dv}{dt} \right)$$

The energy stored in the capacitor is therefore

$$w = \int_{-\infty}^{t} p\,dt = \int_{-\infty}^{t} v\left(C\frac{dv}{dt} \right) dt = C\int_{v(-\infty)}^{v} v\,dv$$

$$= \frac{1}{2}Cv^2 \Big|_{v(-\infty)}^{v} = \frac{1}{2}Cv^2$$

Capacitora have the following important properties:

1. When the voltage across a capacitor is not changing with time (i.e., dc voltage), the current through the capacitor is zero. Thus, a capacitor is an open circuit to dc.

2. The voltage on a capacitor must be continuous. In other words, the voltage on a capacitor cannot change abruptly. (A discontinuous change in voltage requires an infinite current, which is physically impossible.)

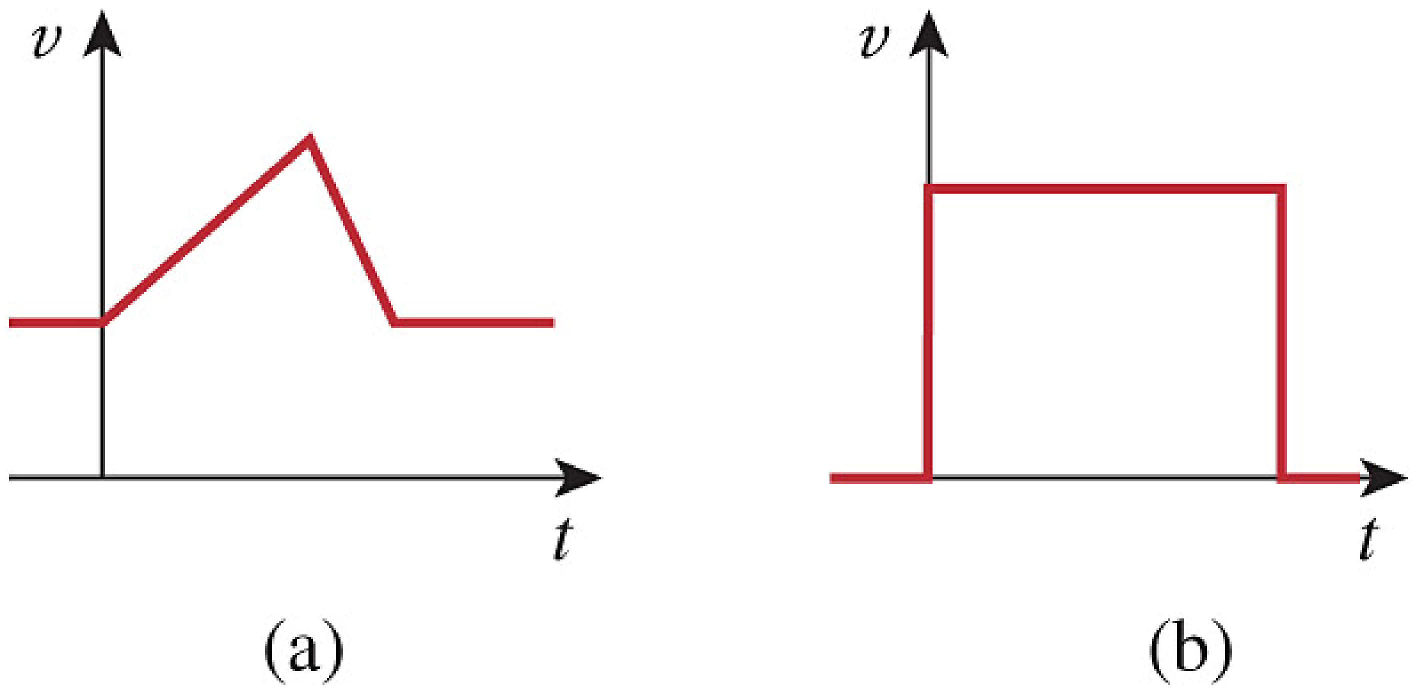


Figure 6.7 Voltage across a capacitor:
(a) possible, (b) impossible.

3. The ideal capacitor does not dissipate energy. It takes power from the circuit when storing energy in its electric field and returns previously stored energy when delivering power to the circuit.

4. A real capacitor has a large leakage resistance, as shown in Fig. 6.8.

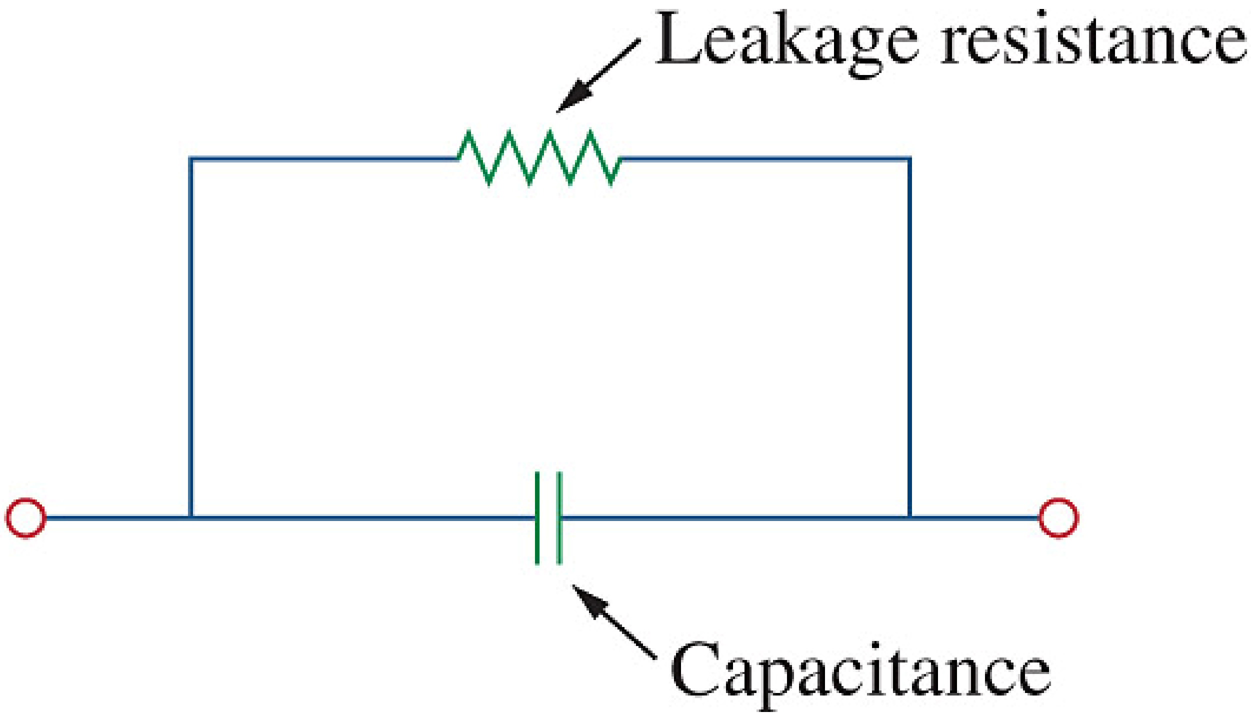


Figure 6.8 Circuit model of a nonideal capacitor.

Example 6.3 Determine the voltage across a 2-μF capacitor if the current through it is

$$i(t) = 6e^{-3000t} \text{ mA}, \quad t \geq 0$$

Assume the initial capacitor voltage is zero.

Solution :

$$v = \frac{1}{C}\int_0^t i\,dt + v(0) = \frac{1}{C}\int_0^t i\,dt$$

$$= \frac{1}{2\times10^{-6}}\int_0^t 6e^{-3000t}\times10^{-3}\,dt$$

$$= \frac{3\times10^{4}}{-3000}e^{-3000t}\Big|_0^t = 1 - e^{-3000t}\ \text{(V)}$$

Example 6.5 Obtain the energy stored in each capacitor in Fig. 6.12(a) under dc conditions.

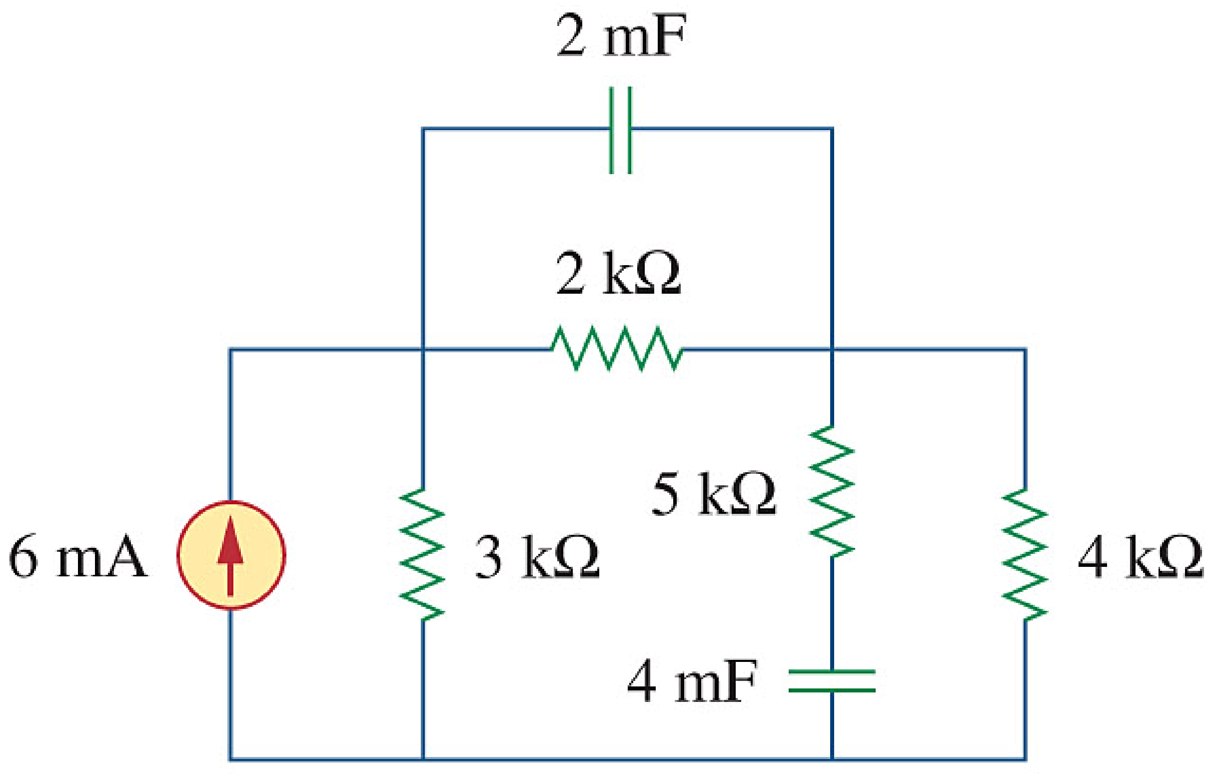


Figure 6.12(a)

Solution : We replace each capacitor with an open circuit, as shown in Fig. 6.12(b).

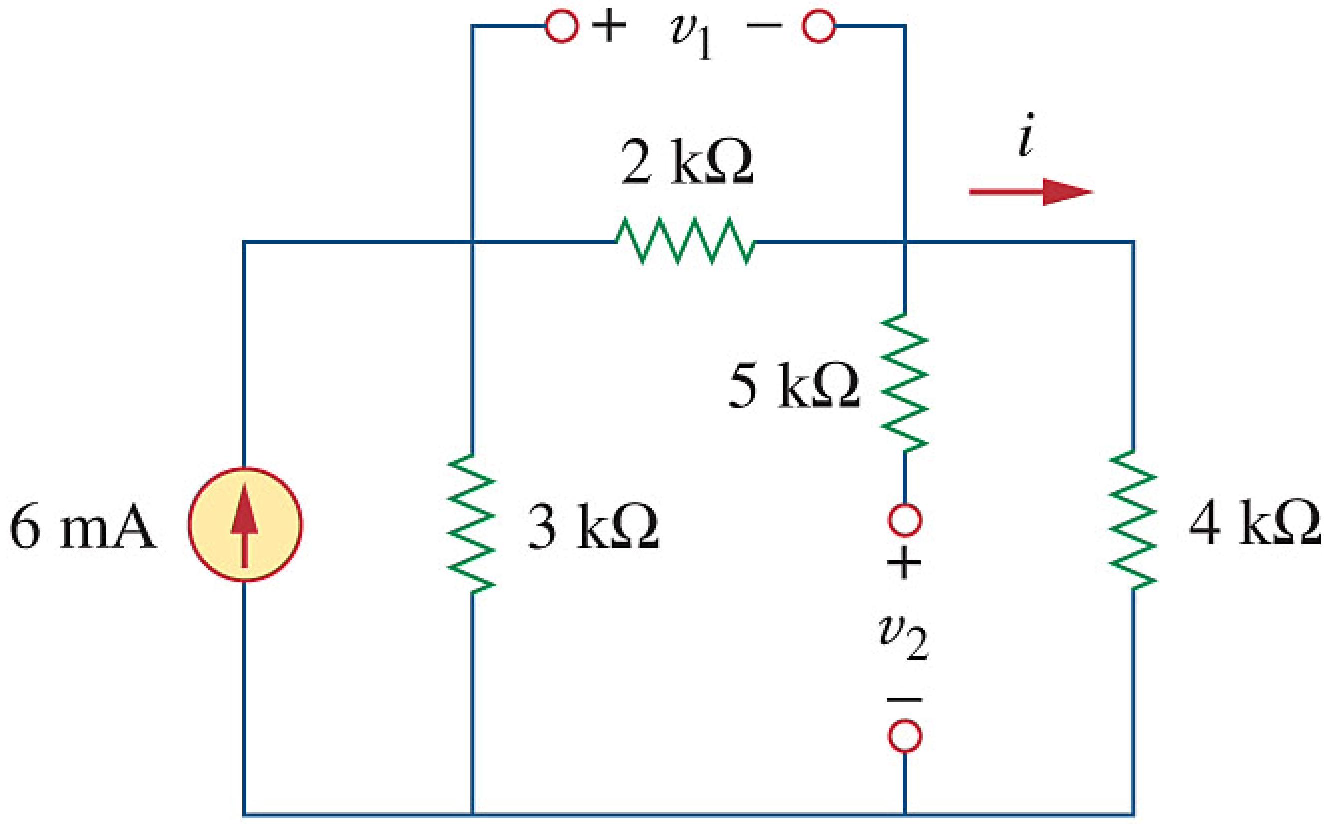


Figure 6.12(b)

$$i = \frac{3}{3+2+4} \times 6 = 2 \text{ (mA)}$$

$$v_1 = 2i = 2 \times 2 = 4 \text{ (V)}$$

$$v_2 = 4i = 4 \times 2 = 8 \text{ (V)}$$

$$w_1 = \frac{1}{2} C_1 v_1^2 = \frac{1}{2} \times 2 \times 10^{-3} \times 4^2 = 0.016 \text{ (J)}$$

$$w_2 = \frac{1}{2} C_2 v_2^2 = \frac{1}{2} \times 4 \times 10^{-3} \times 8^2 = 0.128 \text{ (J)}$$

6.3 Series and Parallel Capacitor

- The series-parallel combination technique can be extended to series-parallel connections of capacitors.

The equivalent capacitance of N parallel-connected capacitors is the sum of the individual capacitances:

$$C_{eq} = C_1 + C_2 + \cdots + C_N$$

Proof :

$$i = \sum_{k=1}^{N} i_k = \sum_{k=1}^{N} C_k \frac{dv}{dt} = \left(\sum_{k=1}^{N} C_k \right) \frac{dv}{dt} = C_{eq} \frac{dv}{dt}$$

$$C_{eq} = \sum_{k=1}^{N} C_k$$

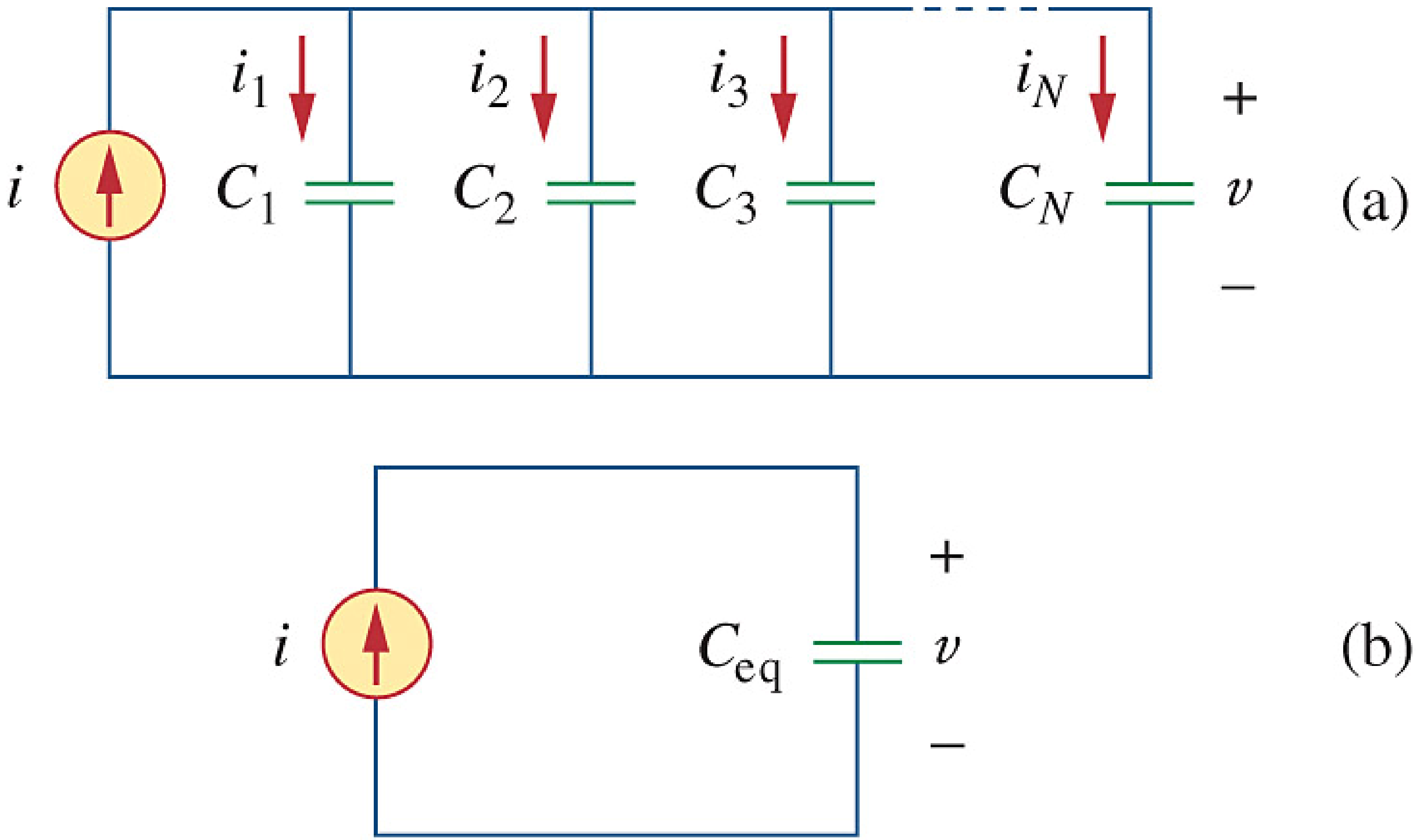


Figure 6.14 (a) Parallel-connected N capacitors, (b) equivalent circuit for the parallel capacitors.

The equivalent capacitance of N series-connected capacitors is the reciprocal of the sum of the reciprocals of the individual capacitances:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}$$

The initial voltage across C_{eq} is the sum of the individual initial voltages:

$$v(t_0) = v_1(t_0) + v_2(t_0) + \cdots + v_N(t_0)$$

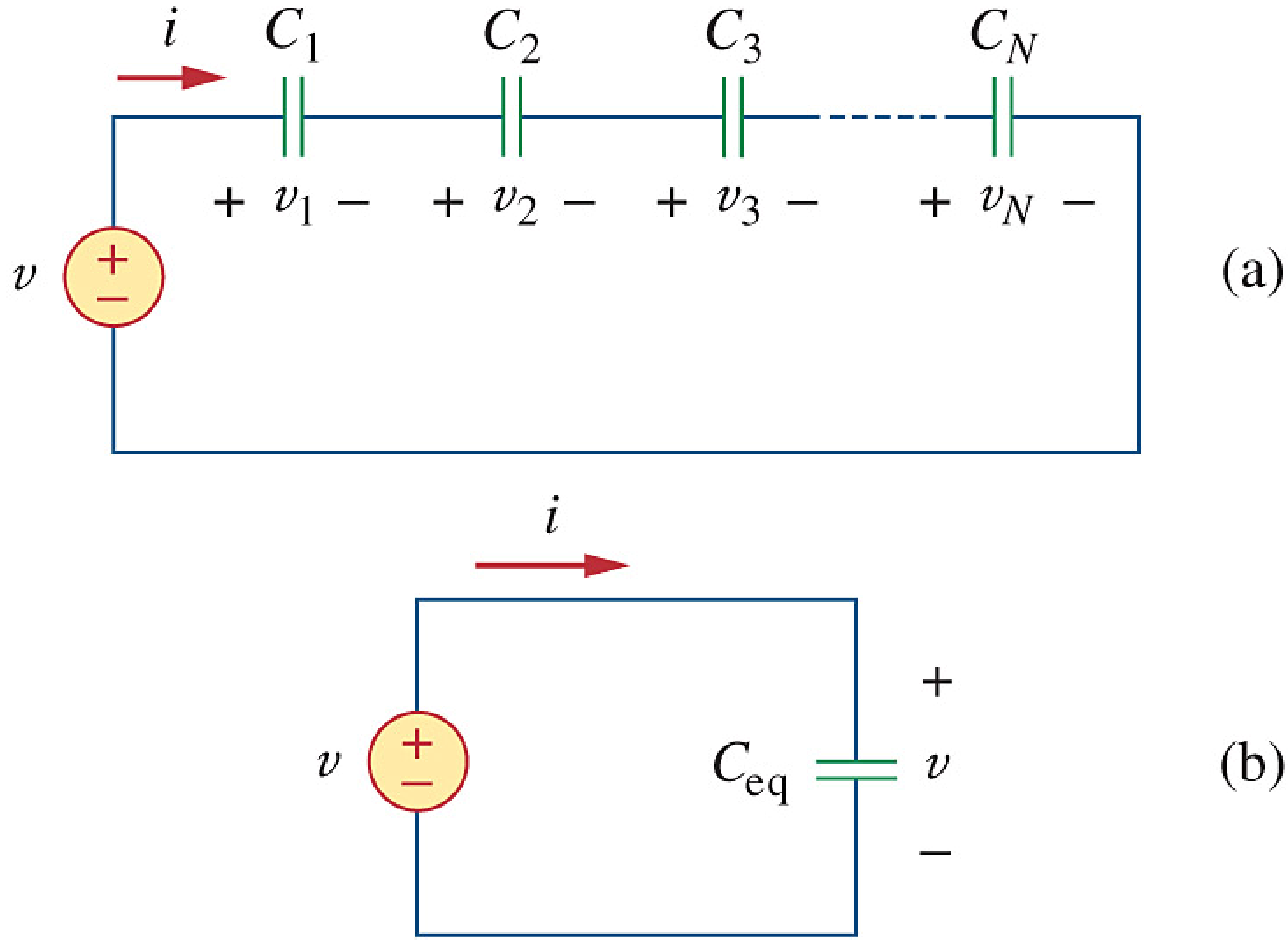


Figure 6.15 (a) Series-connected N capacitors, (b) equivalen ciecuit for the series capacitors.

Proof :

$$v = \sum_{k=1}^{N} v_k = \sum_{k=1}^{N} \left(\frac{1}{C_k} \int_{t_0}^{t} i dt + v_k(t_0) \right)$$

$$= \left(\sum_{k=1}^{N} \frac{1}{C_k} \right) \int_{t_0}^{t} i dt + \sum_{k=1}^{N} v_k(t_0)$$

$$= \frac{1}{C_{eq}} \int_{t_0}^{t} i dt + v(t_0)$$

$$\frac{1}{C_{eq}} = \sum_{k=1}^{N} \frac{1}{C_k}, \; v(t_0) = \sum_{k=1}^{N} v_k(t_0)$$

Example 6.6 Find the equivalent capacitance seen between terminals *a* and *b* of the circuit in Fig. 6.16.

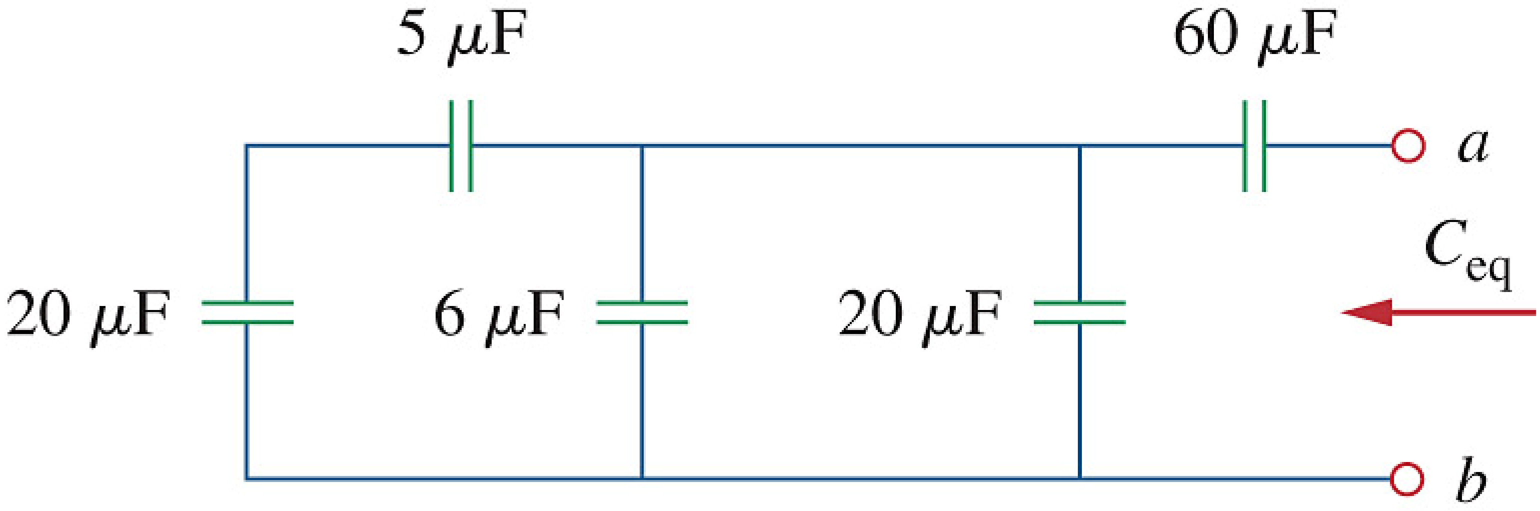


Figure 6.16

Solution :

$$\frac{1}{C_{eq}} = \frac{1}{60} + \frac{1}{6 + 20 + \frac{1}{\frac{1}{5} + \frac{1}{20}}} = \frac{1}{60} + \frac{1}{30} = \frac{1}{20}$$

$$C_{eq} = 20\ (\mu\text{F})$$

6.4 Inductors

- An inductor is a passive element designed to store energy in its magnetic field.
- An inductor is usually formed into a cylindrical coil with many turns of conducting wire, as shown in Fig. 6.21.

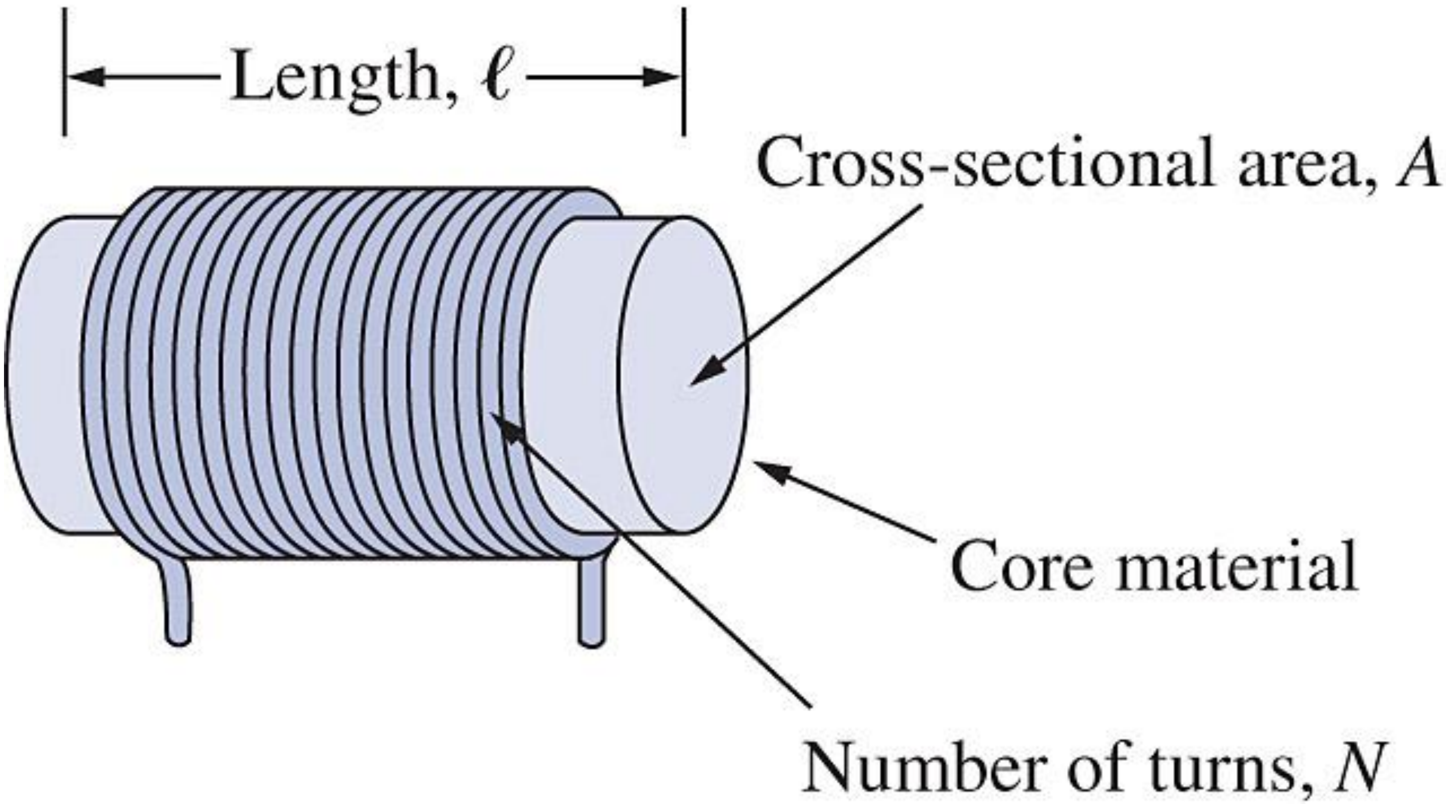


Figure 6.21 Typical form of an inductor.

When a current source i is connected to the inductor, the source sets up a magnetic field. The magnetic flux linkage in the inductor, represented by ψ, is given by

$$\psi = Li$$

where L is known as the inductance of the inductor, measured in henrys (H). If L is independent of i, the inductor is said to be linear.

The circuit symbols for inductors are shown in Figure 6.23, following the passive sign convention.

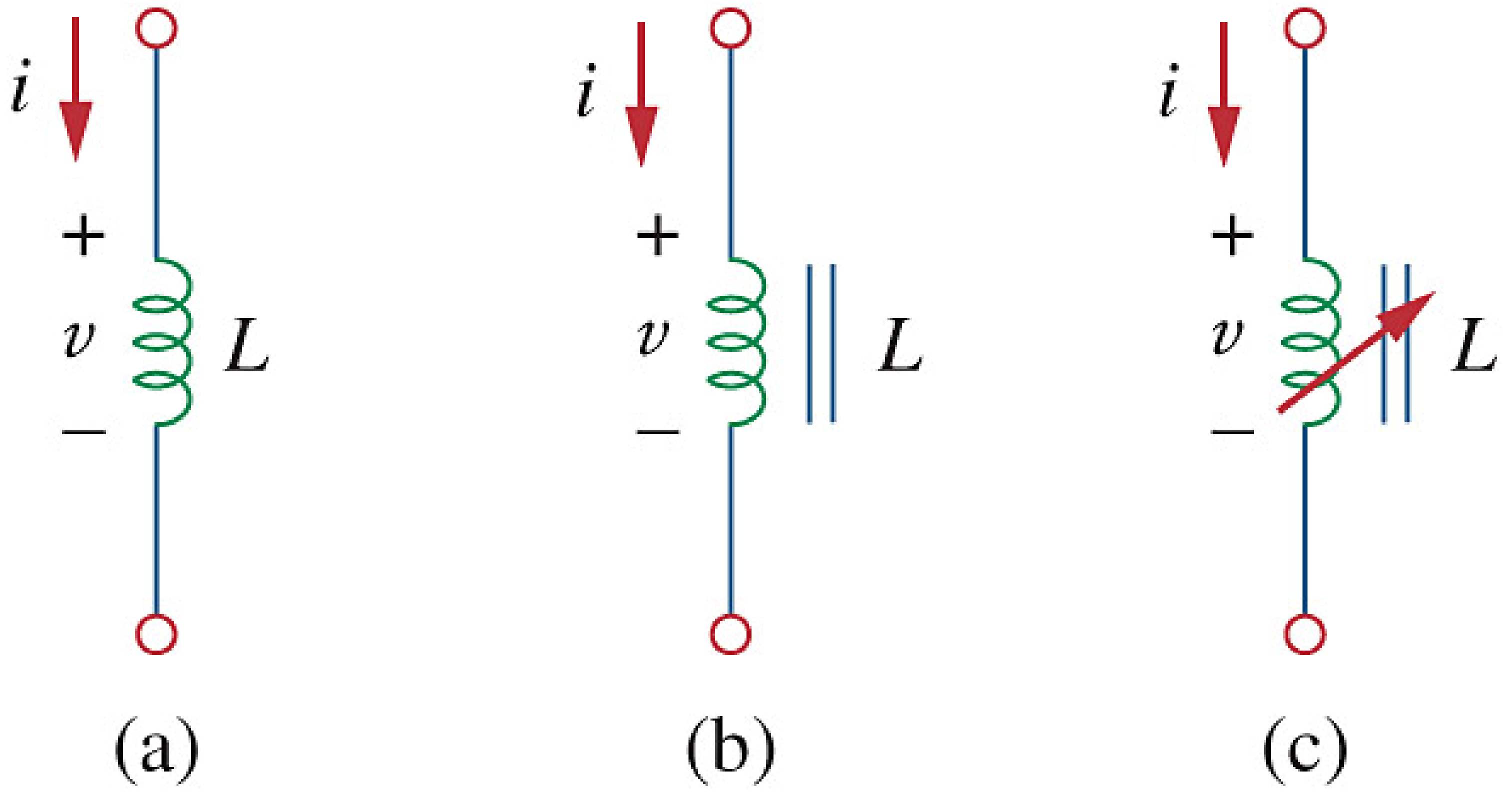


Figure 6.23 Circuit symbols for inductors: air-core, (b) iron-core, (c) variable iron-core.

The voltage-current relationship for the inductor is given by

$$v = L\frac{di}{dt}$$

as illustrated in Fig. 6.24.

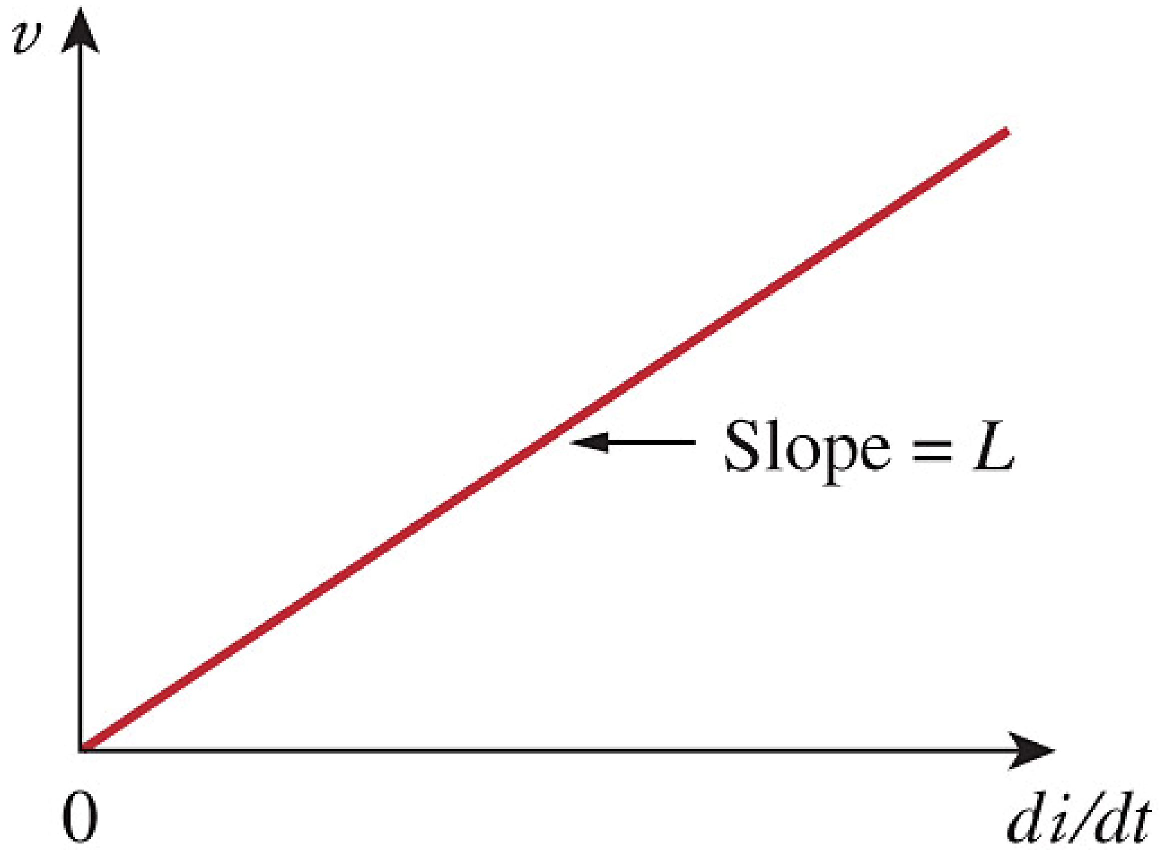


Figure 6.24 Voltage-current relationship of an inductor.

The current-voltage relation of the inductor is

$$i = \frac{1}{L}\int_{-\infty}^{t} v dt = \frac{1}{L}\int_{t_0}^{t} v dt + i(t_0)$$

where $i(t_0) = \frac{1}{L}\int_{-\infty}^{t_0} v dt = \frac{\psi(t_0)}{C}$ is the current through the inductor at time t_0.

The instantaneous power delivered to the inductor is

$$p = vi = \left(L\frac{di}{dt} \right) i$$

The energy stored in the inductor is therefore

$$w = \int_{-\infty}^{t} p\,dt = \int_{-\infty}^{t} \left(L \frac{di}{dt} \right) i\,dt = L \int_{i(-\infty)}^{i} i\,di$$

$$= \frac{1}{2} L i^2 \Big|_{i(-\infty)}^{i} = \frac{1}{2} L i^2$$

Inductor have the following important properties:

1. When the current through an inductor is not changing with time (i.e., dc current), the voltage across the inductor is zero. Thus, an inductor is a short circuit to dc.

2. The current in an inductor must be continuous. In other words, the current in an inductor cannot change abruptly. (A discontinuous change in current requires an infinite voltage, which is physically impossible.)

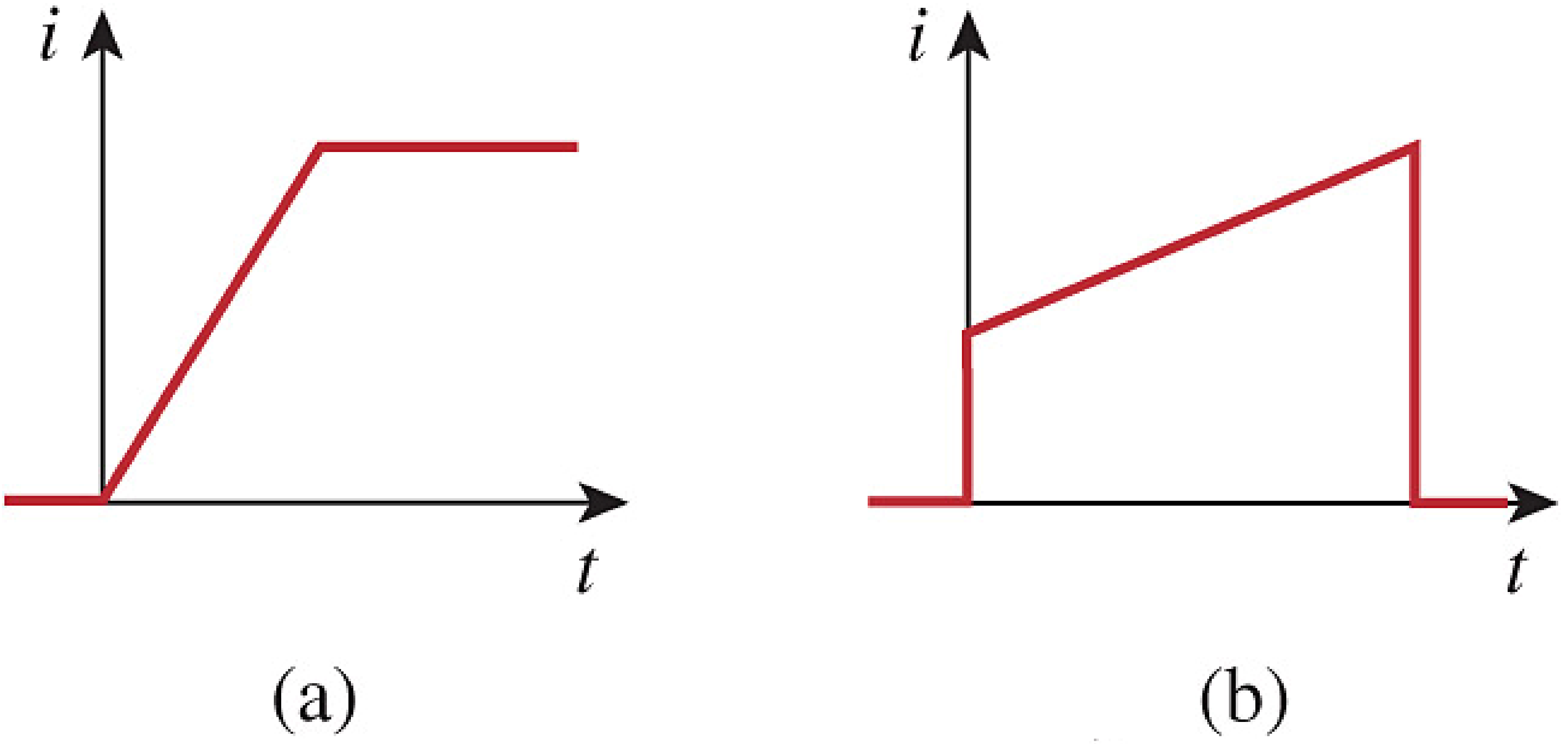


Figure 6.25 Current through an inductor: (a) possible, (b) impossible.

3. The ideal inductor does not dissipate energy. It takes power from the circuit when storing energy in its magnetic field and returns previously stored energy when delivering power to the circuit.

4. A real inductor has a significant *winding resistance* and a small *winding capacitance*, as shown in Fig. 6.26.

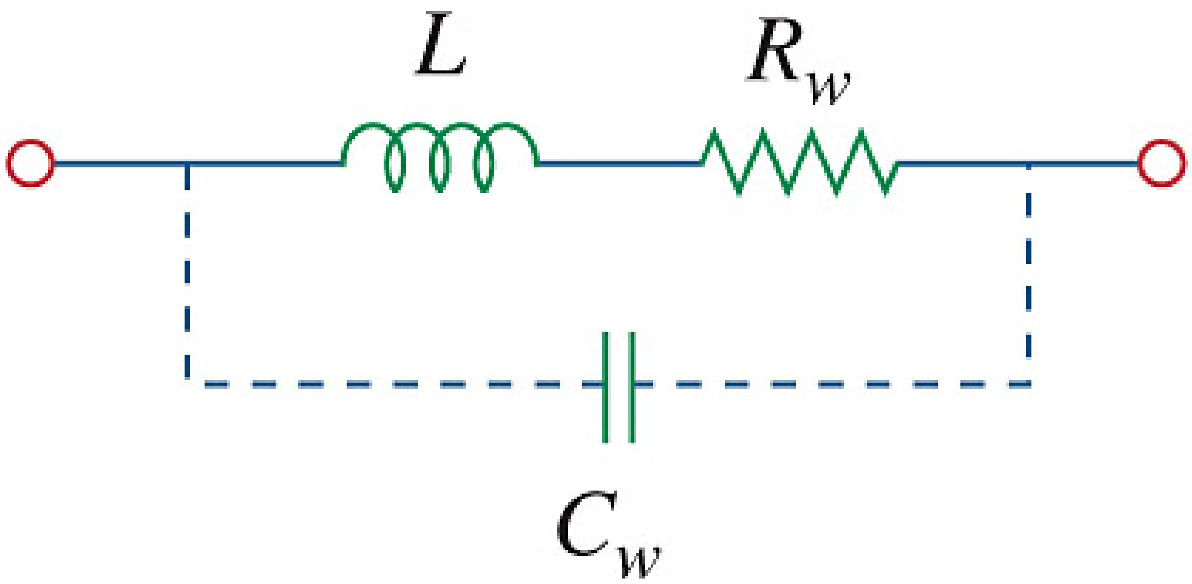


Figure 6.26 Circuit model for a practical inductor.

Example 6.8 The current through a 0.1-H inductor is

$$i(t) = 10te^{-5t} \text{ A}$$

Find the voltage across the inductor and the energy stored in it.

Solution :

$$v = L\frac{di}{dt} = 0.1\frac{d}{dt}\left(10te^{-5t}\right) = \frac{d}{dt}\left(te^{-5t}\right)$$

$$= e^{-5t} + t\left(-5te^{-5t}\right) = e^{-5t}\left(1-5t\right)\ \text{(V)}$$

$$w = \frac{1}{2}Li^2 = \frac{1}{2}\times 0.1\times\left(10te^{-5t}\right)^2$$

$$= 5t^2e^{-10t}\ \text{(J)}$$

Example 6.10 Consider the circuit in Fig. 6.27. Under dc conditions, find (a) i, v_C, and i_L, (b) the energy stored in the capacitor and inductor.

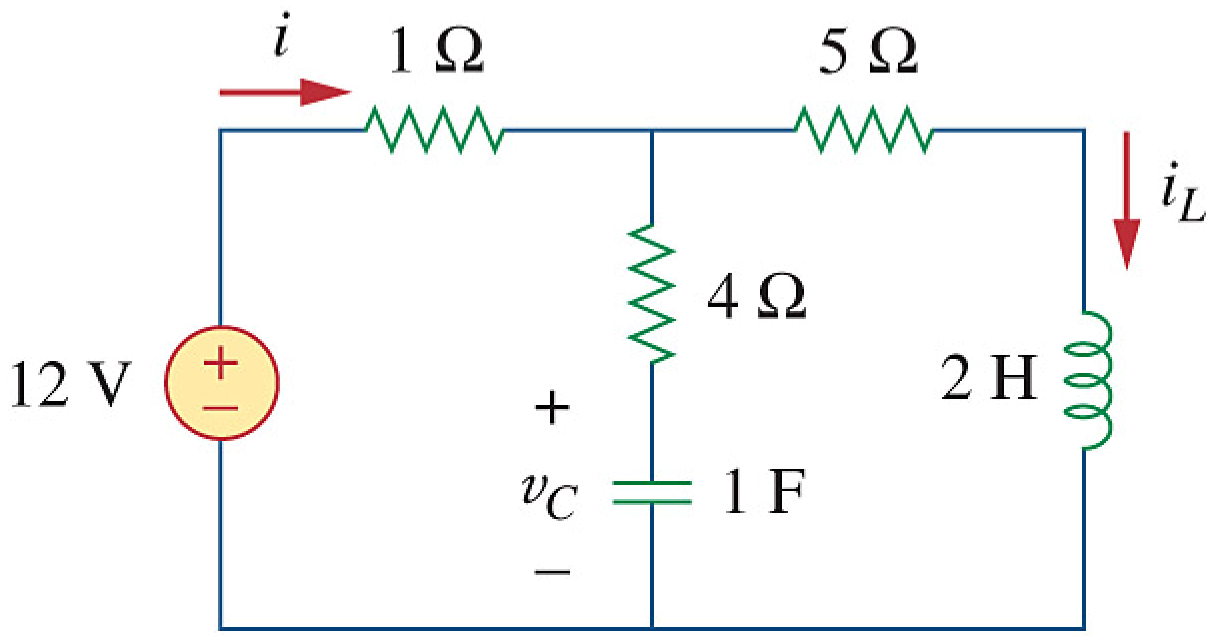


Figure 6.27(a)

Solution : We replace the capacitor with an open circuit and the inductor with a short circuit, as shown in Fig. 6.27(b).

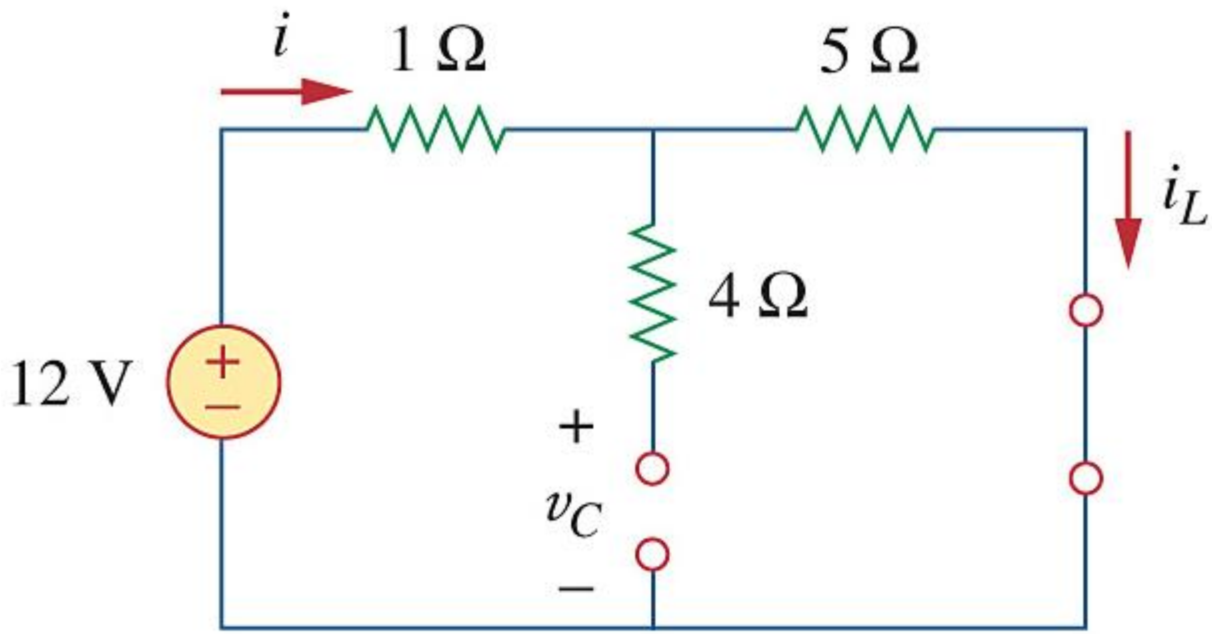


Figure 6.27(b)

(a)

$$i = i_L = \frac{12}{1+5} = 2 \text{ (A)}$$

$$v_c = 5i_L = 5\times 2 = 10 \text{ (V)}$$

(b)

$$w_C = \frac{1}{2}Cv_C^2 = \frac{1}{2}\times 1\times 10^2 = 50 \text{ (J)}$$

$$w_L = \frac{1}{2}Li_L^2 = \frac{1}{2}\times 2\times 2^2 = 4 \text{ (J)}$$

6.5 Series and Parallel Inductor

- The series-parallel combination technique can be extended to series-parallel connections of inductors.

The equivalent inductance of N series-connected inductors is the sum of the individual inductances:

$$L_{eq} = L_1 + L_2 + \cdots + L_N$$

Proof :

$$v = \sum_{k=1}^{N} v_k = \sum_{k=1}^{N} L_k \frac{di}{dt} = \left(\sum_{k=1}^{N} L_k \right) \frac{di}{dt} = L_{eq} \frac{di}{dt}$$

$$L_{eq} = \sum_{k=1}^{N} L_k$$

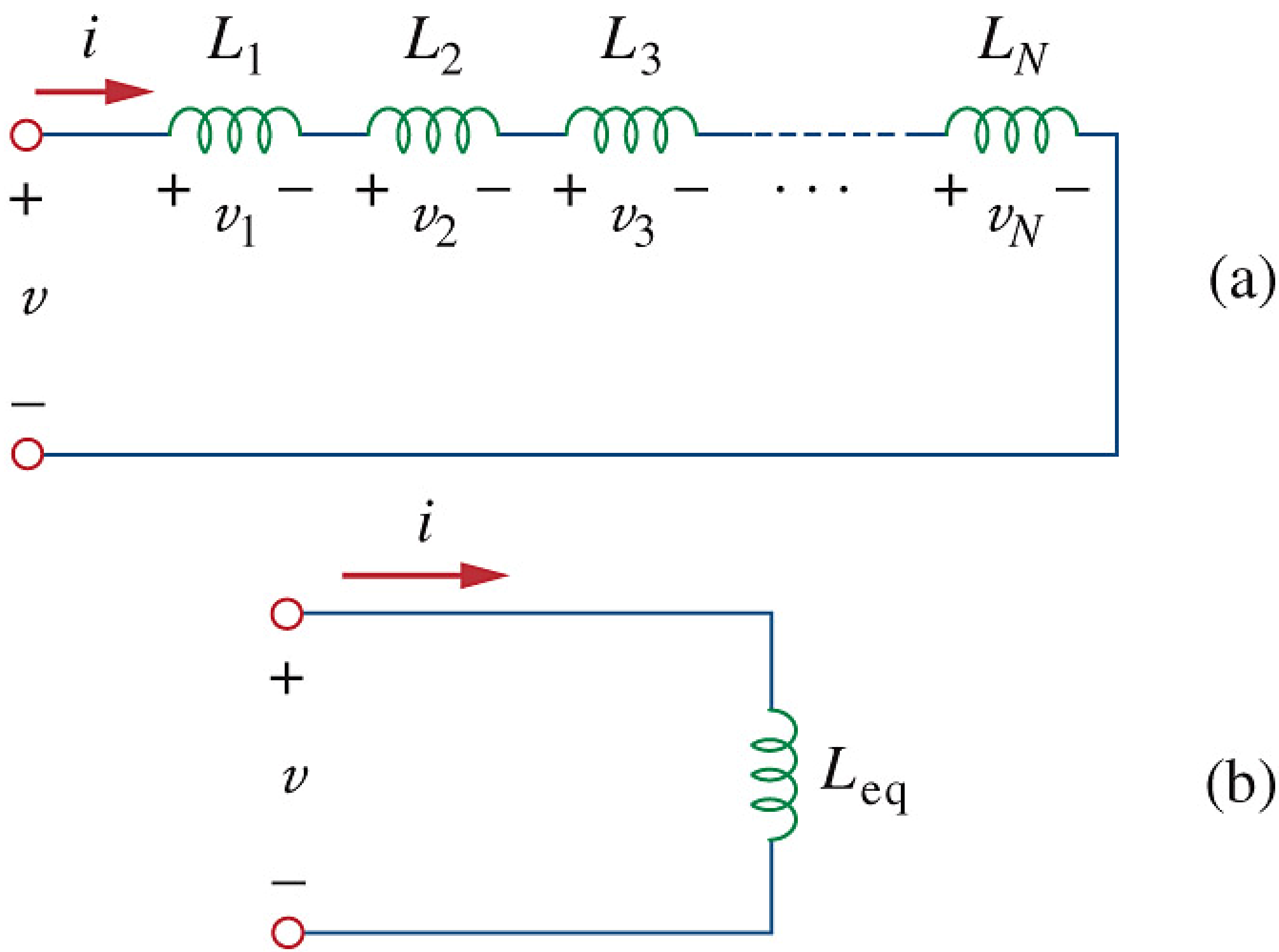


Figure 6.29 (a) A series connection of N inductors, (b) equivalent circuit for the series inductors.

The equivalent inductance of N parallel-connected inductors is the reciprocal of the sum of the reciprocals of the individual inductances:

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}$$

The initial current through L_{eq} is the sum of the individual initial currents:

$$i(t_0) = i_1(t_0) + i_2(t_0) + \cdots + i_N(t_0)$$

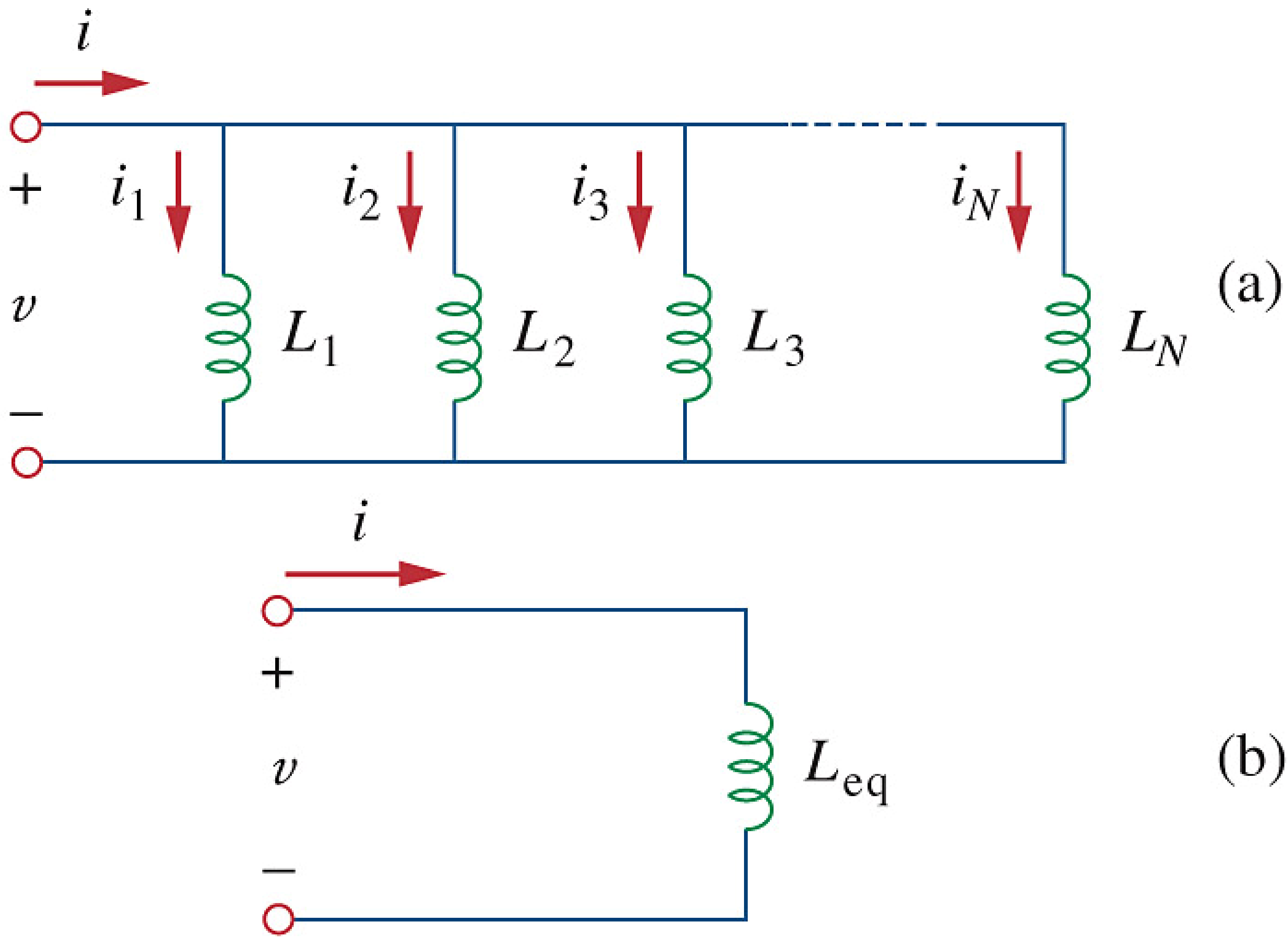


Figure 6.30 A parallel connection of N inductors, (b) equivalent circuit for the parallel inductors.

Proof :

$$i = \sum_{k=1}^{N} i_k = \sum_{k=1}^{N} \left(\frac{1}{L_k} \int_{t_0}^{t} v dt + i_k(t_0) \right)$$

$$= \left(\sum_{k=1}^{N} \frac{1}{L_k} \right) \int_{t_0}^{t} v dt + \sum_{k=1}^{N} i_k(t_0)$$

$$= \frac{1}{L_{eq}} \int_{t_0}^{t} v dt + i(t_0)$$

$$\frac{1}{L_{eq}} = \sum_{k=1}^{N} \frac{1}{L_k},\ i(t_0) = \sum_{k=1}^{N} i_k(t_0)$$

Practice Problem 6.11 Calculate the equivalent inductance for the inductive ladder network in Fig. 6.32.

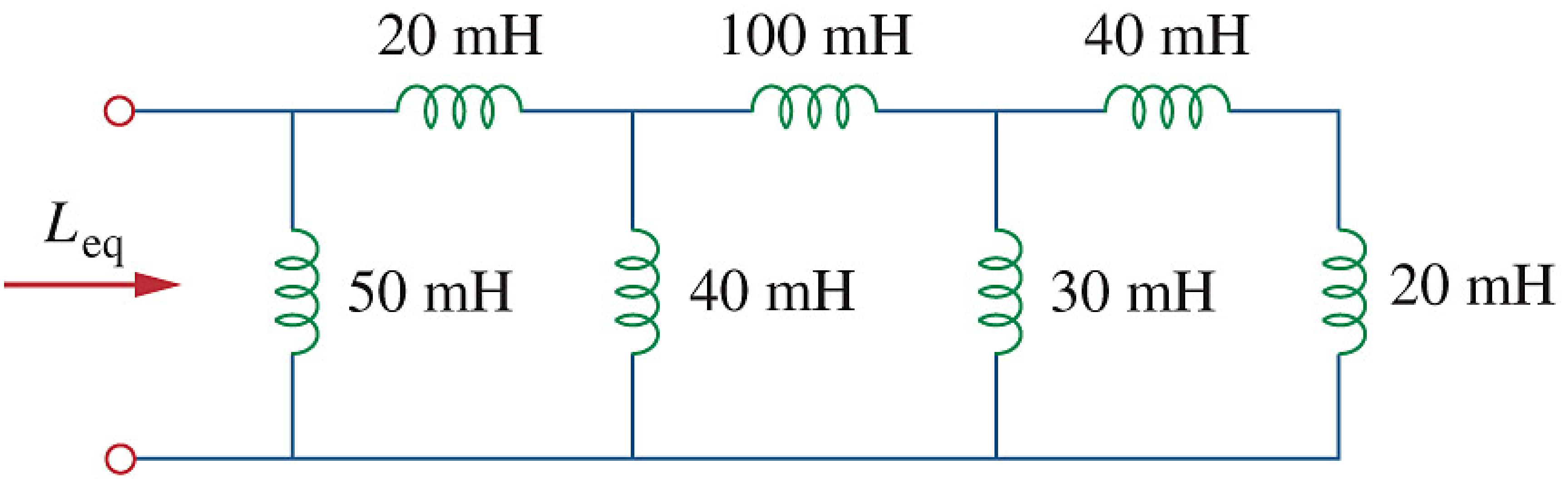


Figure 6.32

Solution :

$$\frac{1}{L_{eq}} = \frac{1}{50} + \cfrac{1}{20 + \cfrac{1}{\cfrac{1}{40} + \cfrac{1}{100 + \cfrac{1}{\cfrac{1}{30} + \cfrac{1}{40+20}}}}}$$

$$L_{eq} = 25 \text{ (mH)}$$

6.6 Applications

- Circuit elements such as resistors and capacitors are commercially available in either discrete form or integrated-circuit form. Inductors usually come in discrete form and tend to be more bulky and expensive.

- There are several applications in which inductors have no practical substitute. They are routinely used in relays, delays, sensing devices, pick-up heads, telephone circuits, radio and TV receivers, power supplies, electric motors, microphones, and loudspeakers, to mention a few.

- The capacity to store energy makes capacitors and inductors useful as temporary voltage sources and current sources, respectively.
- Capacitors and inductors are frequency sensitive. This property makes them useful for frequency discrimination.

- The property of opposing any abrupt change in voltage makes capacitors useful for glitch (or spike) elimination. The property of opposing any abrupt change in current makes inductors useful for spark (or arc) suppression. This property also makes capacitors and inductors useful for converting pulsating dc supply into relatively smooth dc supply.

An *integrator* is an op amp circuit whose output is proportional to the integral of the input. For the circuit in Fig. 6.35,

$$v_o = -\frac{1}{RC}\int_0^t v_i dt + v_o(0)$$

If $v_o(0) = 0$, then

$$v_o = -\frac{1}{RC}\int_0^t v_i dt$$

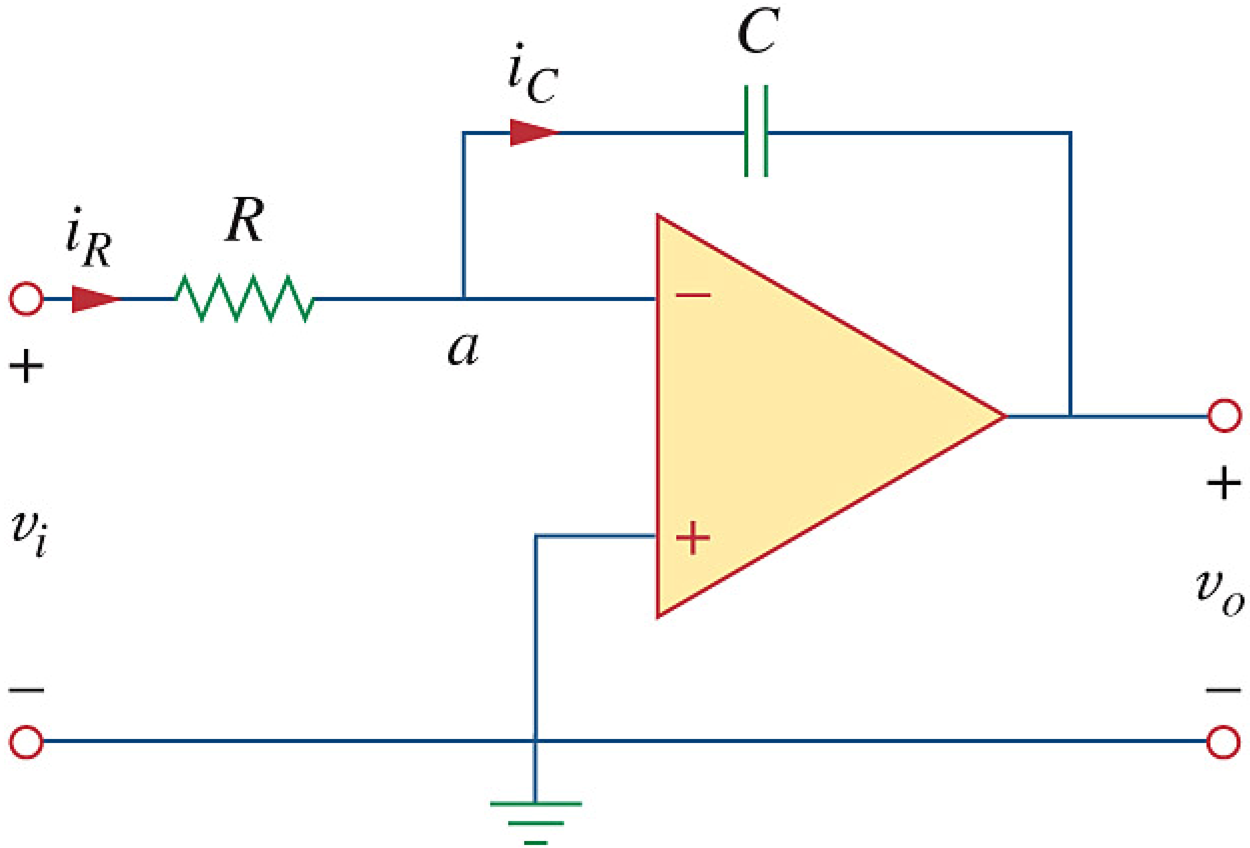


Figure 6.35(b) An integrator circuit.

Proof :

$$\begin{cases} i_R = i_C \\ i_R = \dfrac{v_i}{R}, i_C = -C\dfrac{dv_o}{dt} \end{cases} \Rightarrow \frac{dv_o}{dt} = -\frac{v_i}{RC}$$

$$v_o - v_o(0) = -\frac{1}{RC}\int_0^t v_i dt$$

$$v_o = -\frac{1}{RC}\int_0^t v_i dt + v_o(0)$$

A *differentiator* is an op amp circuit whose output is proportional to the derivative of the input. For the circuit in Fig. 6.37,

$$v_o = -RC\frac{dv_i}{dt}$$

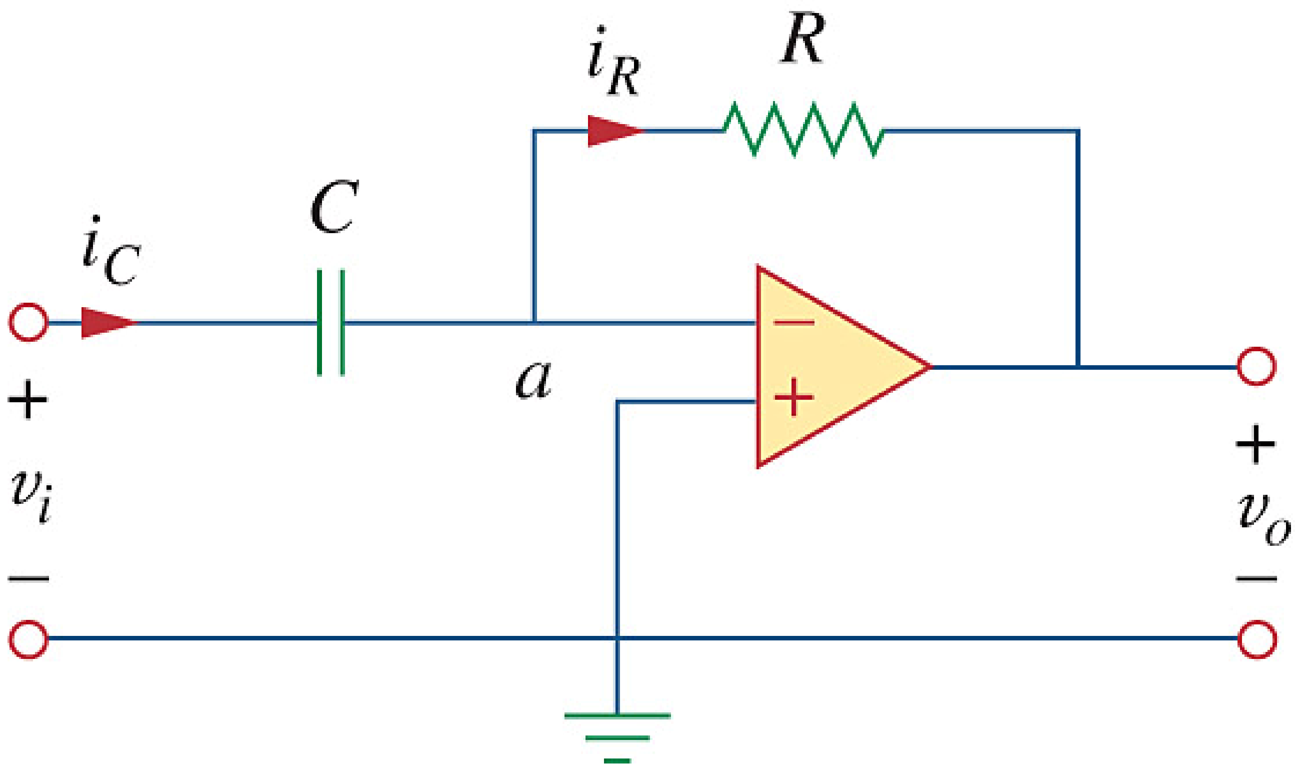


Figure 6.37 A differentiator circuit.

Proof :

$$\begin{cases} i_C = i_R \\ i_C = C\dfrac{dv_i}{dt}, \; i_R = -\dfrac{v_o}{R} \end{cases} \Rightarrow v_o = -RC\dfrac{dv_i}{dt}$$

It is better not to use differentiator. If the input signal is contaminated by noise, taking derivative will magnify the noise.

Op amps were initially developed for electronic *analog computers*. Analog computers can be programmed to slove differential equations.

The analog computer consists of three types of op amp circuits: integrator circuits, summing amplifiers, and inverting or noninverting amplifiers. Example 6.15 illustrates how to build an analog computer.

Example 6.15 Design an analog computer circuit to solve the differential equation

$$\frac{d^2 v_o}{dt^2} + 2\frac{dv_o}{dt} + v_o = 10\sin 5t, \quad t > 0$$

subject to $v_o(0) = -4,\ v_o'(0) = 1,$ where the prime refers to the time derivative.

Solution : For $t > 0$,

$$\frac{d^2 v_o}{dt^2} = 10\sin 5t - 2\frac{dv_o}{dt} - v_o$$

$$\frac{dv_o}{dt} = \int_0^t \left(10\sin 5t - 2\frac{dv_o}{dt} - v_o \right) dt + v_o'$$

We implement this equation using the "summing integrator" in Fig. 6.40(a).

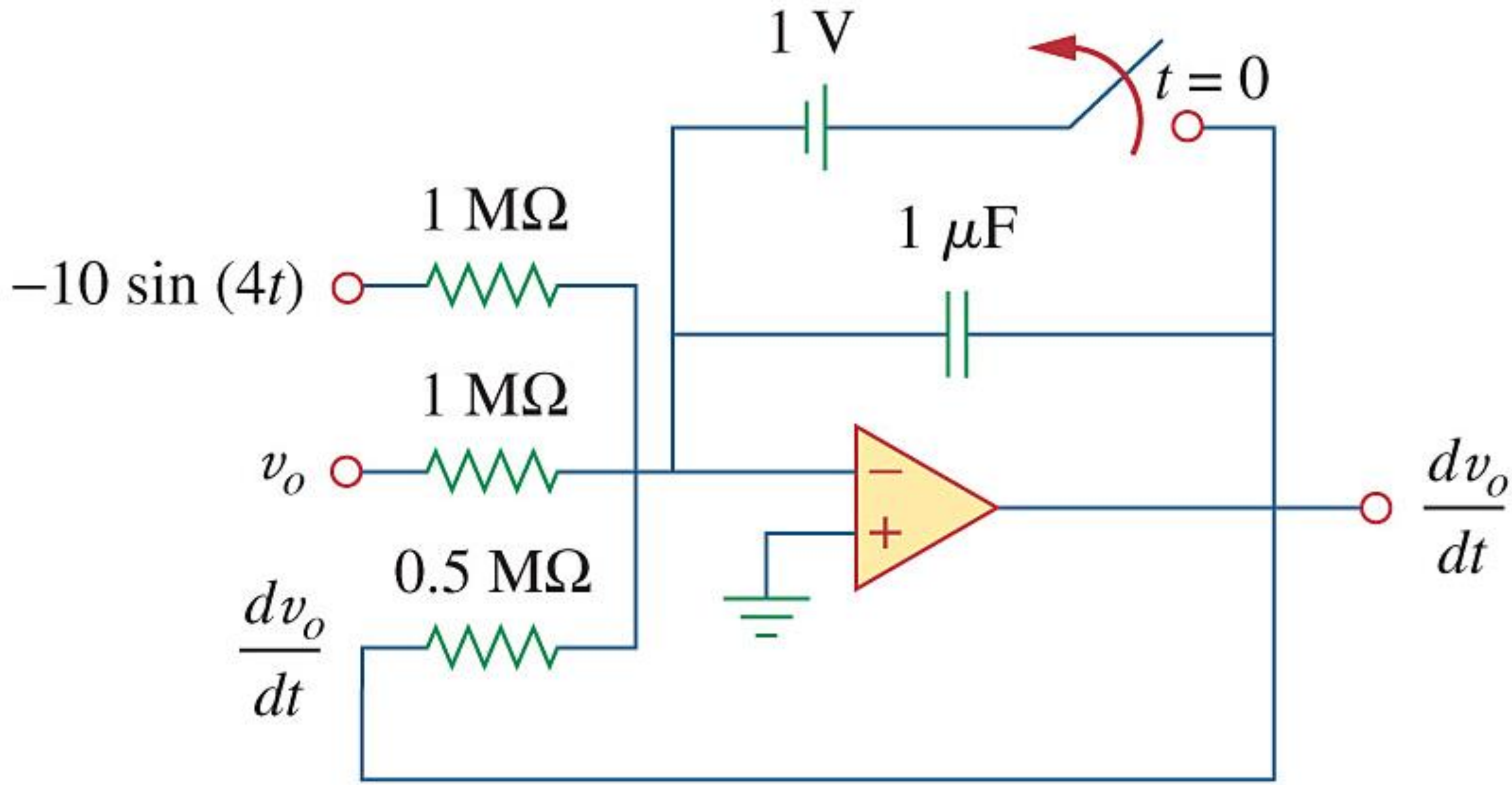


Figure 6.40(a) The "summing integrator" circuit.

$$v_o = \int_0^t \left(\frac{dv_o}{dt} \right) dt + v_o(0)$$

This is implemented with the circuit in Fig. 6.40(b). The complete circuit is shown in Fig. 6.40(c).

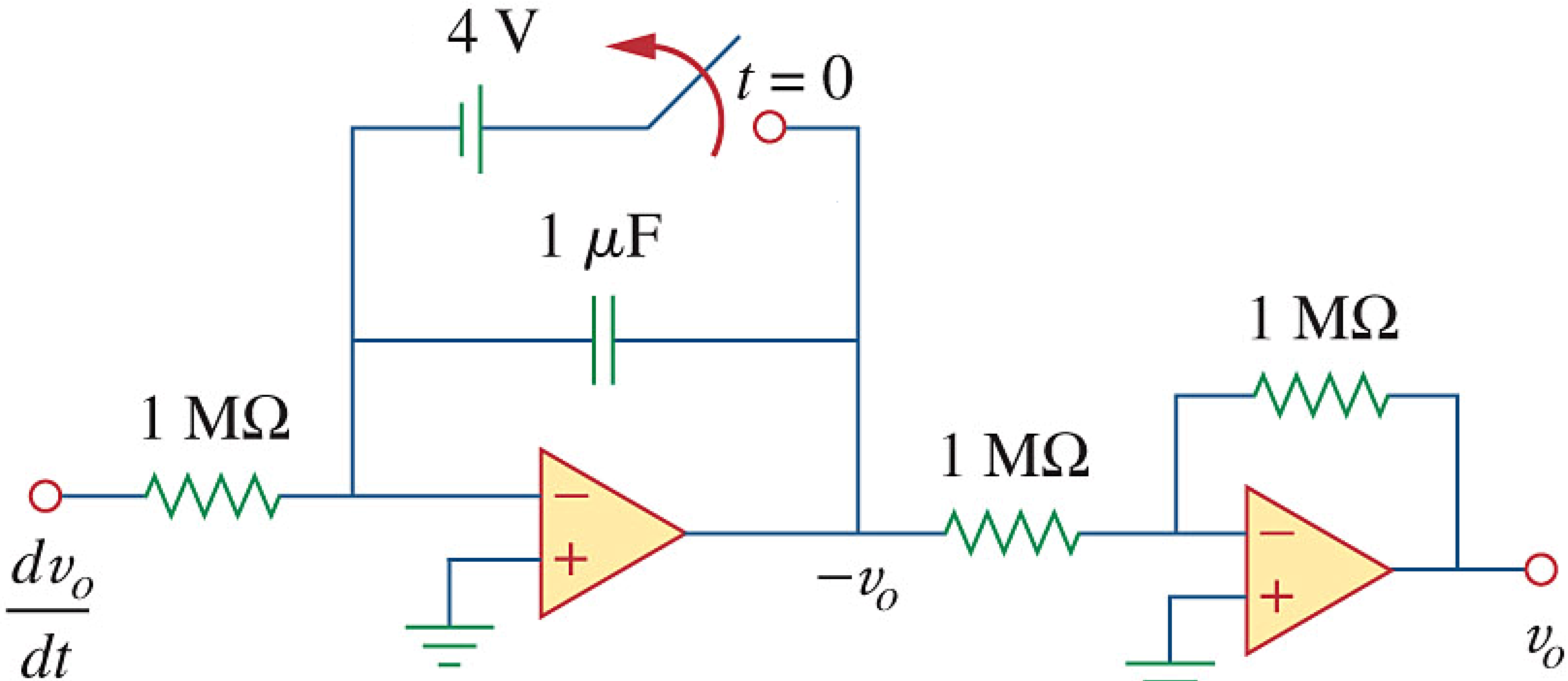


Figure 6.40(b) The cascading of an integrator and an inverting amplifier.

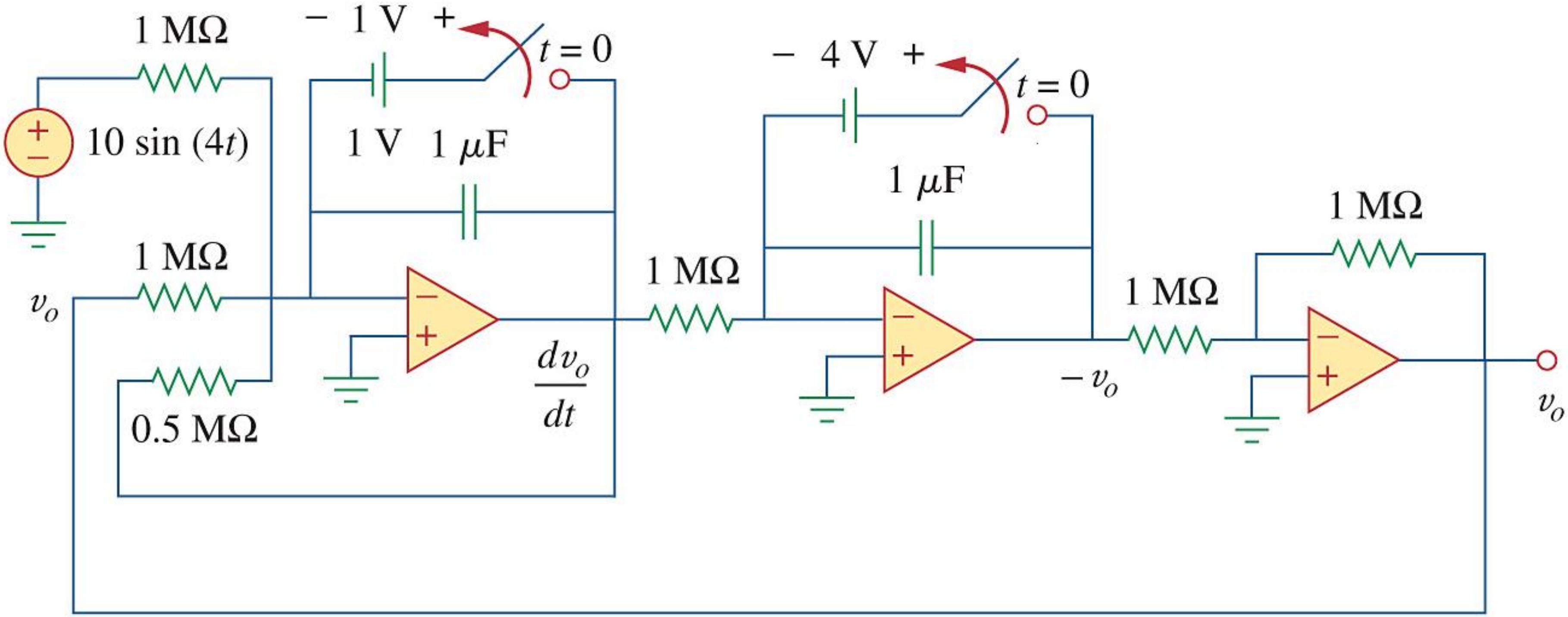

1 MΩ
− 1 V +
t = 0
1 V
1 μF
10 sin (4t)
1 MΩ
v_o
0.5 MΩ
dv_o/dt
1 MΩ
− 4 V +
t = 0
1 μF
1 MΩ
−v_o
1 MΩ
v_o

Figure 6.40(c)

Practice Problem 6.15 Design an analog computer circuit to solve the differential equation

$$\frac{d^2v_o}{dt^2} + 3\frac{dv_o}{dt} + 2v_o = 4\cos 105t, \quad t > 0$$

subject to $v_o(0) = 2,\ v_o'(0) = 0.$

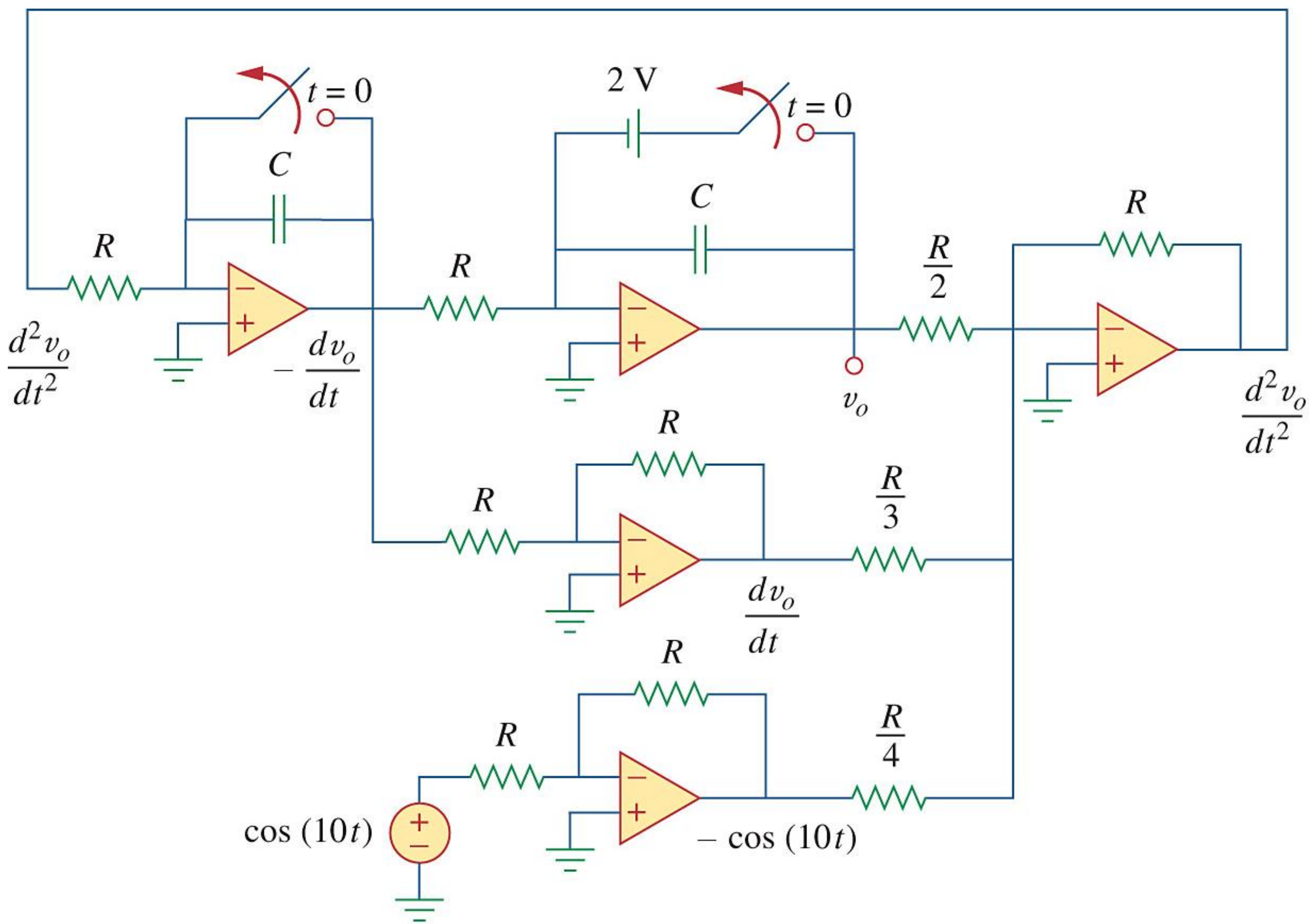
Solution :

$$\frac{d^2v_o}{dt^2} = -3\frac{dv_o}{dt} - 2v_o + 4\cos 105t$$

$$\frac{dv_o}{dt} = \int_0^t \left(\frac{d^2v_o}{dt^2} \right) dt + v'(0)$$

$$v_o = \int_0^t \left(\frac{dv_o}{dt} \right) dt + v(0)$$

The circuit is shown in Fig. 6.41, where $RC = 1$ s.



$t = 0$
2 V
$t = 0$
C
C
R
R
R
$\frac{R}{2}$
$\frac{d^2 v_o}{dt^2}$
$-\frac{dv_o}{dt}$
v_o
$\frac{d^2 v_o}{dt^2}$
R
R
$\frac{R}{3}$
$\frac{dv_o}{dt}$
R
R
$\frac{R}{4}$
cos (10t)
– cos (10t)

Figure 6.41