Homework 4 Solution

Problems:

1. (a) [2!] $x(t) = 2\text{rect}\left(\frac{t-2}{4}\right)$ so $\boxed{X(\omega) = 8e^{-2j\omega}\text{sinc}\left(\frac{2\omega}{\pi}\right)}$ by time scaling and shifting property.

   (b) [2!] $x(t) = e^{-3t}\text{rect}\left(\frac{t-2}{4}\right) = e^{-3t}u(t) - e^{-3(t-4)}u(t-4)e^{-12}$ so $\boxed{X(\omega) = \frac{1}{3+j\omega}\left[1 - e^{-12}e^{-4j\omega}\right]}$

   (c) [3!] $x(t) = t\text{rect}\left(\frac{t-2}{4}\right) = ty(t)$ where from part (a): $Y(\omega) = 4e^{-2j\omega}\text{sinc}\left(\frac{2\omega}{\pi}\right)$.

   Thus $X(\omega) = \frac{d}{d\omega} jY(\omega) = \frac{d}{d\omega} j2e^{-2j\omega}\frac{\sin(2\omega)}{\omega} = \boxed{\frac{e^{-4j\omega}(1+4j\omega)-1}{\omega^2}}$

   (d) [3!] $x(t) = \cos(4\pi t)y(t)$ where from part (a): $Y(\omega) = 4e^{-2j\omega}\text{sinc}\left(\frac{2\omega}{\pi}\right)$.

   Hence $X(\omega) = \frac{1}{2}(Y(\omega - 4\pi) + Y(\omega + 4\pi)) = \boxed{2e^{-2jw}\left(\text{sinc}\left(\frac{2\omega - 8\pi}{\pi}\right) + (\text{sinc}\left(\frac{2\omega + 8\pi}{\pi}\right)\right)}$

2. (a) [3!]

$$[e^{-at}\cos\omega_0 t]u(t) = \frac{1}{2}e^{-at}e^{j\omega_0 t}u(t) + \frac{1}{2}e^{-at}e^{-j\omega_0 t}u(t)$$

$$\boxed{X(j\omega) = \frac{1}{2(\alpha - j\omega_0 + j\omega)} + \frac{1}{2(\alpha + j\omega_0 + j\omega)} = \frac{\alpha + j\omega}{(\alpha + j\omega)^2 + \omega_0^2}}$$

   (b) [3!]

$$x(t) = x_1(t) + x_2(t)$$

   where,

$$x_1(t) = e^{-3t}sin(2t)u(t) \xrightarrow{\mathcal{F}} X_1(j\omega) = \frac{1/(2j)}{3 - j2 + j\omega} - \frac{1/(2j)}{3 + j2 + j\omega} = \frac{2}{(3+j\omega)^2 + 4}$$

$$x_2(t) = e^{3t}sin(2t)u(-t) \xrightarrow{\mathcal{F}} X_2(j\omega) = -X_1(-j\omega) = -\frac{1/(2j)}{3 - j2 - j\omega} + \frac{1/(2j)}{3 + j2 - j\omega} = \frac{-2}{(3-j\omega)^2 + 4}$$

   Thus,

$$\boxed{X(j\omega) = X_1(j\omega) + X_2(j\omega) = \frac{3j}{9 + (\omega+2)^2} - \frac{3j}{9 + (\omega-2)^2}}$$

   (c) [4!]
   According to the FT table,

$$e^{-t^2/4} \stackrel{\mathcal{F}}{\longleftrightarrow} 2\sqrt{\pi}e^{-\omega^2} = F_1(\omega)$$

   and also

$$(-jt)^2 e^{-t^2/4} \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{d^2}{d\omega^2}F_1(\omega) = -4\sqrt{\pi}e^{-\omega^2}(1 - 2\omega^2)$$

   Thus $F(\omega) = -\frac{d^2}{d\omega^2}F_1(\omega) = \boxed{4\sqrt{\pi}e^{-\omega^2}(1 - 2\omega^2)}$

3. [7!]

(i) [4!] From the FT table, $\frac{1}{\pi}\text{rect}\left(\frac{t}{\pi}\right) \overset{\mathcal{F}}{\longleftrightarrow} \text{sinc}\left(\frac{\omega}{2}\right)$ and $\frac{1}{\pi}\text{tri}\left(\frac{t}{\pi}\right) \overset{\mathcal{F}}{\longleftrightarrow} \text{sinc}^2\left(\frac{\omega}{2}\right)$.

Hence $F(\omega) = \text{sinc}^3(\omega/2) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{\pi}\text{rect}\left(\frac{t}{\pi}\right) * \frac{1}{\pi}\text{tri}\left(\frac{t}{\pi}\right)$.

(ii) [3!] By using graphical convolution,

$$\text{rect}\left(\frac{t}{\pi}\right) * \text{tri}\left(\frac{t}{\pi}\right) = \begin{cases} \frac{1}{2\pi}\left(\frac{3}{2}\pi - t\right)^2 & \frac{1}{2}\pi < t < \frac{3}{2}\pi \\ \frac{1}{2\pi}\left(\frac{3}{2}\pi + t\right)^2 & -\frac{3}{2}\pi < t < -\frac{1}{2}\pi \\ \frac{3}{4}\pi - \frac{t^2}{\pi} & -\frac{1}{2}\pi < t < \frac{1}{2}\pi \\ 0 & \text{otherwise} \end{cases}$$

and it follows that

$$\boxed{f(t) = \begin{cases} \frac{1}{2\pi^3}\left(\frac{3}{2}\pi - |t|\right)^2 & \frac{1}{2}\pi < |t| < \frac{3}{2}\pi \\ \frac{3}{4\pi} - \frac{t^2}{\pi^3} & -\frac{1}{2}\pi < t < \frac{1}{2}\pi \\ 0 & \text{otherwise} \end{cases}}$$

4. [8!]

$$f_e(t) = \frac{1}{2}[f(t) + f(-t)] \overset{\mathcal{F}}{\longleftrightarrow} F_e(\omega) = \frac{1}{2}[F(\omega) + F(-\omega)]$$
$$f_o(t) = \frac{1}{2}[f(t) - f(-t)] \overset{\mathcal{F}}{\longleftrightarrow} F_e(\omega) = \frac{1}{2}[F(\omega) - F(-\omega)]$$

Since $f(t)$ is real, $F(-\omega) = F^*(-\omega)$, and it follows that

$$f_e(t) \overset{\mathcal{F}}{\longleftrightarrow} F_e(\omega) = \frac{1}{2}[F(\omega) + F^*(\omega)] = \text{real}\{F(\omega)\}$$

$$f_o(t) \overset{\mathcal{F}}{\longleftrightarrow} F_e(\omega) = \frac{1}{2}[F(\omega) - F^*(\omega)] = j\text{imag}\{F(\omega)\}$$

5. [10!]

i) Method 1. The generator signal(the time-limited signal when n=0)is

$$g(t) = x(t)\text{rect}(t/6) = 2\delta(t) - \delta(t-2) - \delta(t+2)$$

so its FT is

$$G(\omega) = 2 - e^{j2\omega} - e^{-j2\omega} = 2(1 - \cos 2\omega)$$

Thus the FT of the period signal $x(t)$ is

$$X(\omega) = \sum_{k=-\infty}^{\infty} c_k 2\pi\delta(\omega - k\omega_0)$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T_0} G(\omega)|_{\omega=k\omega_0} 2\pi\delta(\omega - k\omega_0)$$

$$= \sum_{k=-\infty}^{\infty} \omega_0 G(k\omega_0)\delta(\omega - k\omega_0)$$

$$\boxed{= \sum_{k=-\infty}^{\infty} (\pi/3)2(1 - \cos 2k\pi/3)\delta(\omega - k\pi/3)}$$

$$= \sum_{k=-\infty}^{\infty} a_k\delta(\omega - k\pi/3)$$

where $a_0 = 0, a_{\pm1} = \pi, a_{\pm2} = \pi, a_{\pm3} = 0...$

ii) Method 2.

$$x(t) = 2x_1(t) - x_2(t) - x_3(t)$$

$$x_1(t) = \sum_{n=-\infty}^{\infty} \delta(t - T_0 n)$$

$$x_2(t) = \sum_{n=-\infty}^{\infty} \delta(t - T_0 n - 2)$$

$$x_3(t) = \sum_{n=-\infty}^{\infty} \delta(t - T_0 n + 2)$$

Where $T_0 = 6$. The FS of $x_1(t)$ is(See Lecture 11, p91-p94)

$$x_1(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{jk\omega_0 t}$$

And by time shift property of FS we can get,

$$x_2(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{-jk\omega_0 2} e^{jk\omega_0 t}$$

$$x_3(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{jk\omega_0 2} e^{jk\omega_0 t}$$

By linear property of FS,

$$x(t) = 2\sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{jk\omega_0 t} - \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{-jk\omega_0 2} e^{jk\omega_0 t} - \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{jk\omega_0 2} e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} \frac{2}{T_0}(1 - \cos(k\omega_0 2))e^{jk\omega_0 t}$$

The FT of $x(t)$ is

$$x(t) \xrightarrow{\mathcal{F}} X(\omega) = \sum_{k=-\infty}^{\infty} \frac{2}{T_0}(1-\cos(k\omega_0 2))\delta(\omega - k\omega_0)$$

$$\boxed{= \sum_{k=-\infty}^{\infty} (\pi/3)2(1-\cos 2k\pi/3)\delta(\omega - k\pi/3)}$$

$$= \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\pi/3)$$

where $a_0 = 0, a_{\pm 1} = \pi, a_{\pm 2} = \pi, a_{\pm 3} = 0...$

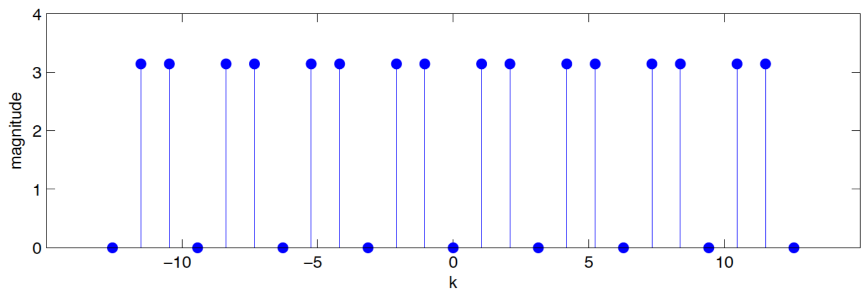

Figure 1: Problem 5

6. [10!]

(a) [4!]

$$X(j\omega) = \frac{e^{(4\omega+\pi/3)j} + e^{-(4\omega+\pi/3)j}}{2}$$

$$\boxed{x(t) = \frac{1}{2}e^{-j\pi t/12}\delta(t+4) + \frac{1}{2}e^{j\pi t/12}\delta(t-4)}$$

(b) [6!] From the given figure we can get

$$X(j\omega) = \begin{cases} \omega e^{-3j\omega} & , |\omega| < 1 \\ 0 & , |\omega| > 1 \end{cases}$$

$$\boxed{x(t) = \frac{\cos(t-3) - 1 - (t-3)sin(t-3)}{\pi(t-3)^2}}$$

7. [15!] From the figure we can get the frequency response of the lowpass differentiator is,

$$H(j\omega) = \begin{cases} \frac{j\omega}{3\pi} & , -3\pi \le \omega \le 3\pi \\ 0 & , otherwise \end{cases}$$

(a) Since $x(t) = \cos(2\pi t + \theta)$, $X(j\omega) = e^{-j\omega\theta/2\pi}\pi[\delta(\omega - 2\pi) + \delta(\omega + 2\pi)]$. It is zero outside the region $-3\pi \le \omega \le 3\pi$. Thus,

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{j\omega}{3\pi}X(j\omega)$$

$$\boxed{y(t) = \frac{1}{3\pi}\frac{dx(t)}{dt} = -\frac{2}{3}\sin(2\pi t + \theta)}$$

(b) Since $x(t) = cos(4\pi t + \theta)$, $X(j\omega) = e^{-j\omega\theta/4\pi}\pi[\delta(\omega - 4\pi) + \delta(\omega + 4\pi)]$. The nonzero portions of $X(j\omega)$ lie outside the region $-3\pi \le \omega \le 3\pi$. This implies,

$$Y(j\omega) = X(j\omega)H(j\omega) = 0$$

$$\boxed{y(t) = 0}$$

(c) The Fourier series coefficients of the signal $x(t)$ are given by

$$c_k = \frac{1}{1}\int_0^{0.5} sin(2\pi t)e^{-jk2\pi t}dt$$

Also we have

$$X(j\omega) = 2\pi\sum_{k=-\infty}^{\infty} a_k\delta(\omega - k\omega_0)$$

Where $\omega_0 = 2\pi$. Thus, we can see that only the portions when $k = 0, \pm 1$ lie in the region $-3\pi \le \omega \le 3\pi$.

$$c_0 = \frac{1}{\pi}, c_1 = c_{-1}^* = \frac{-1}{4j}$$

The portion that can pass the filter is $x_{lp}(t) = \frac{1}{\pi} + \frac{1}{2}\sin(2\pi t)$. Finally, $\boxed{y(t) = \frac{1}{3\pi}\frac{dx_{lp}(t)}{dt} = \frac{1}{3}\cos(2\pi t)}$

8. [10!]

$$x(t) = t\text{sinc}^2(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = j\frac{d}{d\omega}\text{tri}\left(\frac{\omega}{2\pi}\right) = \pm j\frac{1}{2\pi}\text{rect}\left(\frac{\omega}{4\pi}\right)$$

So $E = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-2\pi}^{2\pi}\frac{1}{4\pi^2}d\omega = \boxed{\frac{1}{2\pi^2}}$

9. (a) [10!]

$$H(s) = \frac{s^2+s+1}{(s^2+6s+25)(s^2+2)} = \frac{s^2+s+1}{(s+2)(s+3-4j)(s+3+4j)} = \frac{r_1}{s+2} + \frac{r_2}{s+3-4j} + \frac{r_2^*}{s+3+4j}$$

with $r_1 = \left.\frac{s^2+s+1}{s^2+6s+25}\right|_{s=-2} = \frac{3}{17}$ and $r_2 = \left.\frac{s^2+s+1}{(s+3+4j)(s+2)}\right|_{s=-3+4j} = \frac{7}{17} + \frac{71}{136}j = 0.665e^{j0.903}$.

Thus

$$H(j\omega) = \frac{3/17}{j\omega+2} + \frac{0.665e^{j0.903}}{j\omega+3-4j} + \frac{0.665e^{-j0.903}}{j\omega+3+4j}$$

Taking the inverse FT yields

$$\boxed{h(t) = \frac{3}{17}e^{-2t}u(t) + 1.33e^{-3t}\cos(4t + 0.903)u(t)}$$

(b) [10!] Expanding the denominator polynomial:

$$\frac{Y(\omega)}{X(\omega)} = \frac{(j\omega)^2 + j\omega + 1}{(j\omega)^3 + 8(j\omega)^2 + 37j\omega + 50}$$

so by cross multiplying

$$\boxed{50y(t) + 37\frac{d}{dt}y(t) + 8\frac{d^2}{dt^2}y(t) + \frac{d^3}{dt^3}y(t) = x(t) + \frac{d}{dt}x(t) + \frac{d^2}{dt^2}x(t)}$$