Ve 216: Introduction to Signals and Systems

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Based on Lecture Notes by Prof. Jeffrey A. Fessler

Outline

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Applications of the FT

- In chapter 4 we covered all of the fundamental mathematical properties of the FT which are used in EE.
- These properties are not just “interesting math;” they are the theoretical foundation of how just about everything involving signals work, from AM radios to digital TVs to PC sound cards etc.
- Now (at last!) we can begin to use the FT tools to understand the basic principles behind some of these applications.

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Overview

- Filtering (used universally)
  convolution property
- Modulation (AM radio, digital comm (modems))
  modulation property
- Sampling (A/D converters in sound cards)
  FT of sampled signals
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Output signal spectrum

The convolution property says that if

$$x(t) \to \boxed{\text{LTI } h(t)} \to y(t) = h(t) * x(t),$$

then

$$Y(\omega) = H(\omega)X(\omega).$$

- So the output signal spectrum is the input signal spectrum multiplied by $H(\omega)$.
- Some frequencies are attenuated, some are amplified, and others are possibly unchanged.

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In signal processing, often we are interested in selecting only a certain portion of the spectrum, and eliminating other portions.

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Ideal filters

Definition

Filters that pass unchanged a certain band of frequencies (called the **passband**) while completely removing other frequency components (called the **stopband**) are called **ideal filters**.

(Picture) of $|H(\omega)|$ with passband and stopband for

1. **lowpass** filter
2. **highpass** filter
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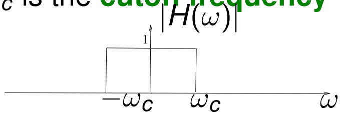
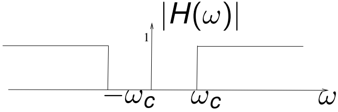
Lowpass and highpass filter

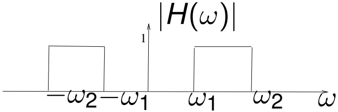
1. **lowpass** filter. ω_c is the **cutoff frequency**



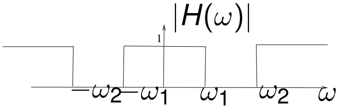
2. **highpass** filter



3. **bandpass** filter



4. **bandstop** filter



Ideal filters

These are called ideal filters because unfortunately we cannot build real devices with exactly those magnitude responses. We can come sufficiently close in practice, but not exactly.

- We will focus on causal filters, since those can be implemented in real time.
- Recall an LTI system is causal iff its impulse response

$$h(t) = 0 \text{ for } t < 0.$$

- If $h(t)$ is causal, then $H(\omega)$ cannot have any finite intervals where $H(\omega) = 0$, except if $h(t) = 0$.

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Ideal filters: example

Example

To see why, consider the ideal lowpass filter:

$$H(\omega) = \mathrm{rect}\left(\frac{\omega}{2\omega_c}\right).$$

What is the impulse response $h(t)$? Taking the inverse FT using the FT table shows

$$h(t) = \frac{\omega_c}{\pi}\mathrm{sinc}\left(\frac{\omega_c}{\pi}t\right) \xleftrightarrow{\mathcal{F}} H(\omega) = \mathrm{rect}\left(\frac{\omega}{2\omega_c}\right).$$

(Picture)(MIT, Lecture 12, p.4)

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Ideal filters: example

- It is seen that the impulse response for this idea filter begins long before the impulse occurs at $t = 0$ (theoretically, at $t = -\infty$).
- Systems such as this, which respond to an input before the input is applied, are clled noncausal systems.
- The physical existence of noncausal systems is impossible.
- Similar anayses could be used to show that ideal bandpass, ideal high-pass, and ideal bandstop filters are also physically unrealizable.
- However, the concept of noncausal systems, such as idea filters, can be useful during the inital stages of a design or analysis effor.

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Question

Why non-ideal (real) filters?

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Why non-ideal (real) filters?

- *The above $h(t)$ is noncausal!, so no real system can implement it.*
- *The ease of implementation. In general, the more precisely we try to approximate or implement an ideal filter, the more complicated or costly the implementation becomes, whether in terms of components such as resistors, capacitors, and operational amplifiers.*
- *Many applications do not need very sharp cutoff frequencies. Video(MIT, Lecture 9, 31:40)*

Outline

Real filters

(Picture)(MIT, Lecture 12-5)

- A gradual transition region from passband with a cutoff frequency ω_p to stopband with a cutoff frequency of ω_s.
- A deviation from unity of $[1-\delta_1, 1+\delta_1]$ where $0 < \delta_1 \ll 1$ is allowed in the passband.
- A deviation of $0 < \delta_2 \ll 1$ from zero is allowed in the stopband.

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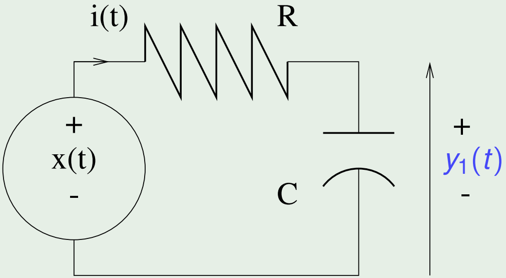
Lowpass filter example (1)

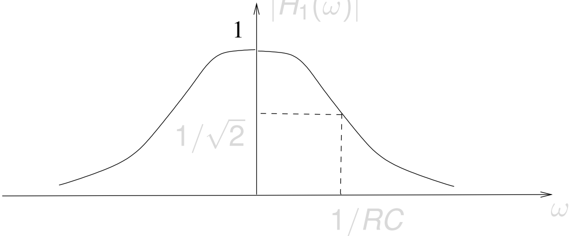
Example

Consider the following basic RC circuit, where the input signal $x(t)$ is the voltage source, and the output signal $y_1(t)$ is the voltage across the capacitor. Find the frequency response.



Lowpass filter example (1)

$$H_1(\omega) = \frac{Y_1(\omega)}{X(\omega)} = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R} = \frac{1}{1 + RCj\omega}$$



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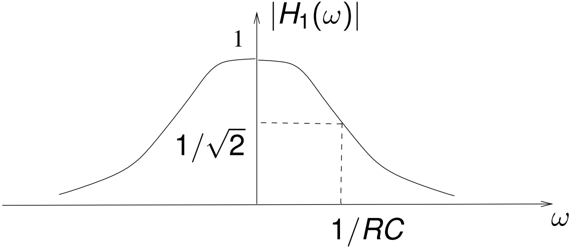

Highpass filter example (1)

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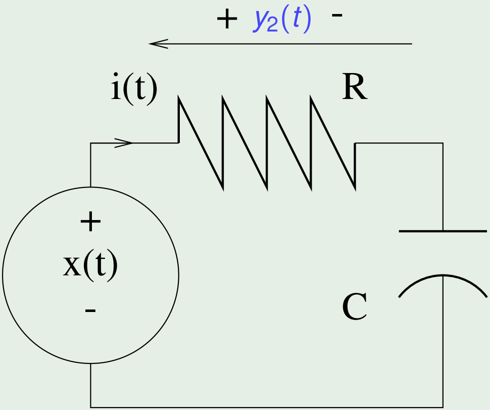
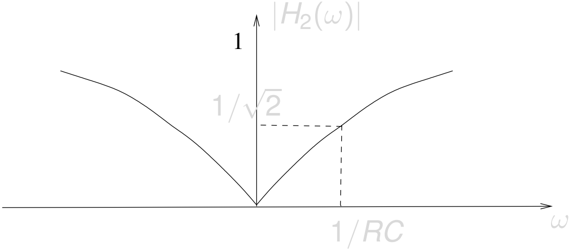


Highpass filter example (2)

$$H_2(\omega) = \frac{Y_2(\omega)}{X(\omega)} = \frac{R}{\frac{1}{j\omega C} + R} = \frac{RCj\omega}{1 + RCj\omega}$$



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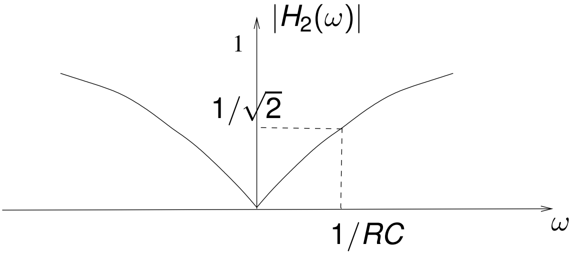

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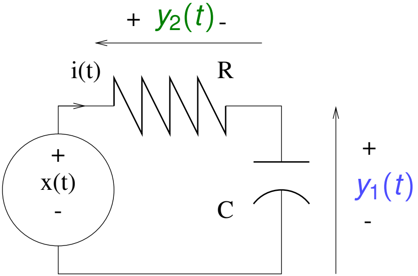

$$y_2(t) = x(t) - y_1(t) \Longrightarrow Y_2(\omega) = X(\omega) - Y_1(\omega)$$

$$\Longrightarrow \frac{Y_2(\omega)}{X(\omega)} = 1 - \frac{Y_1(\omega)}{X(\omega)} \Longrightarrow H_2(\omega) = 1 - H_1(\omega)$$

Verify this relation:

$$\frac{1}{1 + RCj\omega} + \frac{RCj\omega}{1 + RCj\omega} = 1$$

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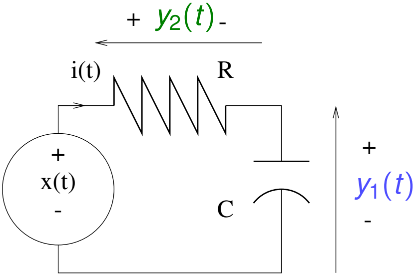

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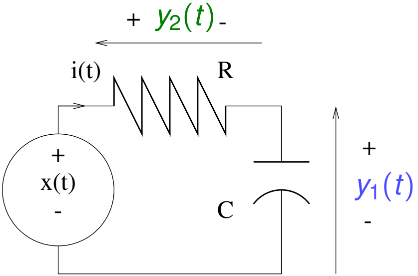

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Phase and magnitude relation

From the convolution property for CT FT, the FT of the output of an LTI system is related to the FT $X(\omega)$ to the system by the equation:

$$Y(\omega) = H(\omega)X(\omega) \Longrightarrow |Y(\omega)|e^{j\angle Y(\omega)} = |H(\omega)|e^{j\angle H(\omega)}|X(\omega)|e^{j\angle X(\omega)}$$

Phase: addition

$$\angle Y(\omega) = \angle H(\omega) + \angle X(\omega)$$

Magnitude: multiplication

$$|Y(\omega)| = |H(\omega)||X(\omega)|$$

Magnitude on a logarithmic scale: addition

$$\log|Y(\omega)| = \log|H(\omega)| + \log|X(\omega)|$$

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Phase and magnitude plot

- If we have a graph of the log magnitude and phase of $X(\omega)$ and $H(\omega)$, the plot of $Y(\omega)$ is obtained by adding the log-magnitude plots and by adding the phase plots.
- The logarithmic scale allows detail to be displayed over a wider dynamic range.
- We can obtain plots of the log magnitude and phase of the overall frequency response of cascaded systems by adding the corresponding plots for each of the component systems. *e.g.*,

$$H(\omega) = H_1(\omega)H_2(\omega)H_3(\omega).$$

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The specific logarithmic amplitude scale used is in units of $20\log_{10}(\cdot)$, referred to as **decibels** (abbreviated **dB**).

magnitude	1	10	0.1	2
dB	0	20	-20	6

Definition

Plots of $20\log_{10}|H(\omega)|$ and $\angle H(\omega)$ versus $\log_{10}\omega$ are referred to as **bode plots**.

(Picture)(MIT, Lecture 12-7)

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- Recall if $h(t)$ is real

$$H(\omega) = H^*(-\omega)$$

$$|H(\omega)| = |H(-\omega)|, \quad \angle H(\omega) = -\angle H(-\omega)$$

- The plots for $\omega < 0$ are superfluous and can be obtained immediately from the plots for $\omega \geq 0$.
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Recording control room example

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Graphic equalizer

Definition

A **graphic equalizer** is a high-fidelity audio control that allows the user to see graphically and control individually a number of different frequency bands in a stereophonic system.

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In sound recording and reproduction, **equalization** is the process commonly used to alter the frequency response of an audio system using linear filters.

Video (MIT, Lecture 12, 19:40-27:25min)

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RMS bandwidth

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Root mean-squared bandwidth or **RMS bandwidth** is defined as

$$\omega_{\text{rms}} \triangleq \sqrt{\frac{\int_{-\infty}^{\infty} \omega^2 |X(\omega)|^2 \, d\omega}{\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega}}$$

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RMS bandwidth: solution

By the FT table

$$x(t) = e^{-(t/t_0)^2} \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = t_0\sqrt{\pi}e^{-(\omega t_0/2)^2}$$

$$\omega_{\mathrm{rms}}^2 = \frac{\int_{-\infty}^{\infty} \omega^2 |X(\omega)|^2 \, d\omega}{\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega} = \frac{\int_{-\infty}^{\infty} \omega^2 (t_0\sqrt{\pi}e^{-(\omega t_0/2)^2})^2 \, d\omega}{\int_{-\infty}^{\infty} (t_0\sqrt{\pi}e^{-(\omega t_0/2)^2})^2 \, d\omega}$$

$$= \frac{t_0^{-1} \int_{-\infty}^{\infty} x^2 (\sqrt{\pi}e^{-(x/2)^2})^2 \, dx}{t_0 \int_{-\infty}^{\infty} (\sqrt{\pi}e^{-(x/2)^2})^2 \, dx} \quad (x = \omega t_0, \; d\omega = dx/t_0)$$

$$= \frac{1}{t_0^2} \frac{\pi^{3/2}\sqrt{2}}{\pi^{3/2}\sqrt{2}} = \frac{1}{t_0^2}$$

Thus

$$\boxed{\omega_{\mathrm{rms}} = \frac{1}{t_0}}$$

RMS bandwidth: solution

By the FT table

$$x(t) = e^{-(t/t_0)^2} \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = t_0\sqrt{\pi}e^{-(\omega t_0/2)^2}$$

$$\omega_{\text{rms}}^2 = \frac{\int_{-\infty}^{\infty} \omega^2 |X(\omega)|^2 \, d\omega}{\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega} = \frac{\int_{-\infty}^{\infty} \omega^2 (t_0\sqrt{\pi}e^{-(\omega t_0/2)^2})^2 \, d\omega}{\int_{-\infty}^{\infty} (t_0\sqrt{\pi}e^{-(\omega t_0/2)^2})^2 \, d\omega}$$

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$$= \frac{1}{t_0^2} \frac{\pi^{3/2}\sqrt{2}}{\pi^{3/2}\sqrt{2}} = \frac{1}{t_0^2}$$

Thus

$$\boxed{\omega_{\text{rms}} = \frac{1}{t_0}}$$

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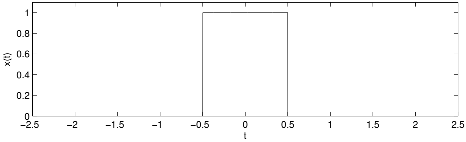
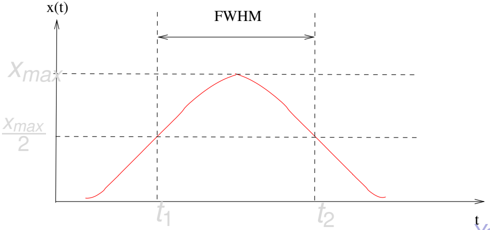
Time duration definitions

- For time-limited signal, **absolute time duration**

$$\tau = t_2 - t_1 .$$



- For non-time-limited signals, an frequently-used measure is the **full-width at half-maximum** or **FWHM** duration.



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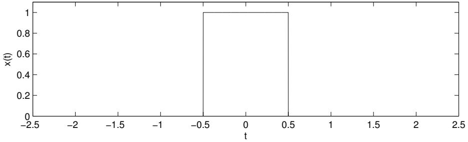

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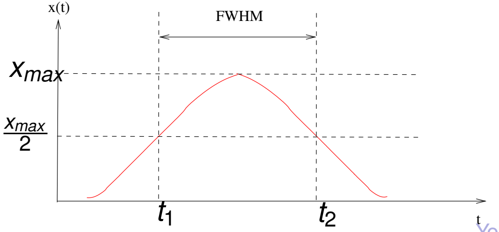

RMS time duration

Definition

Root mean-squared time duration or **RMS time duration** is defined as

$$\tau_{\mathrm{rms}} \triangleq \sqrt{\frac{\int_{-\infty}^{\infty} t^2 |x(t)|^2 \, dt}{\int_{-\infty}^{\infty} |x(t)|^2 \, dt}}$$

Example

Find the RMS time duration of the Gaussian signal

$$x(t) = e^{-(t/t_0)^2}, \quad \text{for } t_0 > 0$$

RMS time duration: solution

$$\tau_{\text{rms}}^2 = \frac{\int_{-\infty}^{\infty} t^2 |x(t)|^2 \, dt}{\int_{-\infty}^{\infty} |x(t)|^2 \, dt} = \frac{\int_{-\infty}^{\infty} t^2 (e^{-(t/t_0)^2})^2 \, dt}{\int_{-\infty}^{\infty} (e^{-(t/t_0)^2})^2 \, dt}$$

$$= \frac{t_0^3 \int_{-\infty}^{\infty} x^2 (e^{-x^2})^2 \, dx}{t_0 \int_{-\infty}^{\infty} (e^{-x^2})^2 dx} \quad (x = t/t_0, \; dt = t_0 dx)$$

$$= t_0^2 \frac{\sqrt{2\pi}/8}{\sqrt{\pi/2}} = \frac{t_0^2}{4}$$

Thus

$$\boxed{\tau_{\text{rms}} = \frac{t_0}{2}}$$

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Time-bandwidth product

Using the Cauchy-Schwarz inequality, one can show (for signals satisfying certain regularity conditions):

$$\boxed{\omega_{\mathrm{rms}} \tau_{\mathrm{rms}} \geq \frac{1}{2}.}$$

The time duration of a signal and its frequency bandwidth are inversely related. A signal cannot be localized in both time and in frequency..

Extreme cases:

- impulse function: narrow in time, wide spectrum
- sinusoid signal: periodic (infinite duration) in time, narrow spectrum
- gaussian signal: time-bandwidth product is as narrow as possible

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Time-bandwidth product: example

Example

Compute the time-bandwidth product of

$$x(t) = \mathrm{tri}(t)$$

Time-bandwidth product: solution (1)

$$x(t) = \text{tri}(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega) = \text{sinc}^2(\omega/2)$$

$$\omega_{\text{rms}} = \frac{\int_0^\infty \omega^2 \left[\text{sinc}^2(\omega/2)\right]^2 d\omega}{\int_0^\infty \left[\text{sinc}^2(\omega/2)\right]^2 d\omega} = \frac{2\pi}{2\pi/3} = 3$$

so

$$\omega_{\text{rms}} = \sqrt{3} \approx 1.73$$

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Time-bandwidth product: solution (2)

$$\tau_{\mathrm{rms}}^2 = \frac{\int_0^1 t^2(1-t)^2\,dt}{\int_0^1 (1-t)^2\,dt} = \frac{1/30}{1/3} = \frac{1}{10}$$

so

$$\tau_{\mathrm{rms}} = 1/\sqrt{10} \approx 0.32$$

Thus

$$\omega_{\mathrm{rms}}\tau_{\mathrm{rms}} = \sqrt{3/10} \approx \boxed{0.548} \geq 1/2$$

but fairly close.

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Outline

Summary

- ideal filters
- real filters
- bode plots
- bandwidth
- time-bandwidth product