# Lecture 3: Probability Models/Distributions and Intro to Value-at-Risk (VaR)/Expected Shortfall

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# Lecture Overview

- Brief review of probability models and random variables
    - Random variables
    - Common distributions for random variables
        - Continuous - Normal, exponential, gamma, t, generalized error, and pareto/generalized pareto
- Intro to Value-at-Risk and Expected Shortfall
    - Initial Computations
- Tail probabilities

# Random Variables

**Definition (random variable).** A random variable is a function $X$ from sample space $\Omega$ to the real numbers. We write

$$\begin{aligned}
(X = x) &\quad \text{as shorthand for } \{w \in \Omega;\ X(w) = x\} \\
(a \le X \le b) &\quad \text{as shorthand for } \{w \in \Omega;\ a \le X(w) \le b\} \\
(X \le b) &\quad \text{as shorthand for } \{w \in \Omega;\ X(w) \le b\} \\
(X \ge a) &\quad \text{as shorthand for } \{w \in \Omega;\ X(w) \ge a\}
\end{aligned}$$

**Remark.** Random variables are (typically) classified as **discrete** or **continuous**

- **Discrete** corresponds to $X$ taking values within a discrete set (could be infinite number)
- **Continuous** corresponds to $X$ taking values over a continuum (e.g., an interval)

**Remark.** The discrete vs. continuous refers to the model.

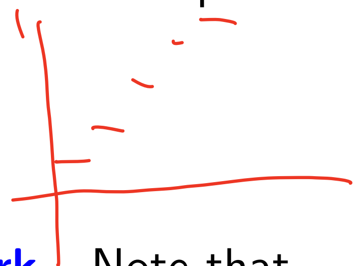
- stock/bond price
- individual/company income levels

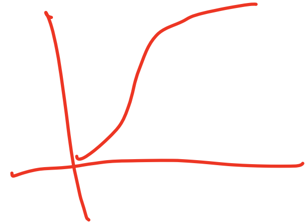
# Cumulative Distributions

**Definition.** The cumulative distribution function (cdf) of a discrete/continuous rv $X$ is always the function

$$F(x) = P(X \leq x)$$

**Picture:** Example cdf of discrete rv/cdf of continuous rv





**Remark.** Note that

- cdf $F$ is an increasing function, i.e., $F(x) \leq F(x')$ for $x < x'$.
- As $x \downarrow -\infty$, $F(x) \to 0$
- As $x \uparrow \infty$, $F(x) \to 1$

**Remark.** Suppose that $X$ is the stock price of Google at the end of this coming week. What can we say about $F(x)$ for $x < 0$.

**Answer.** 0

# Quantiles

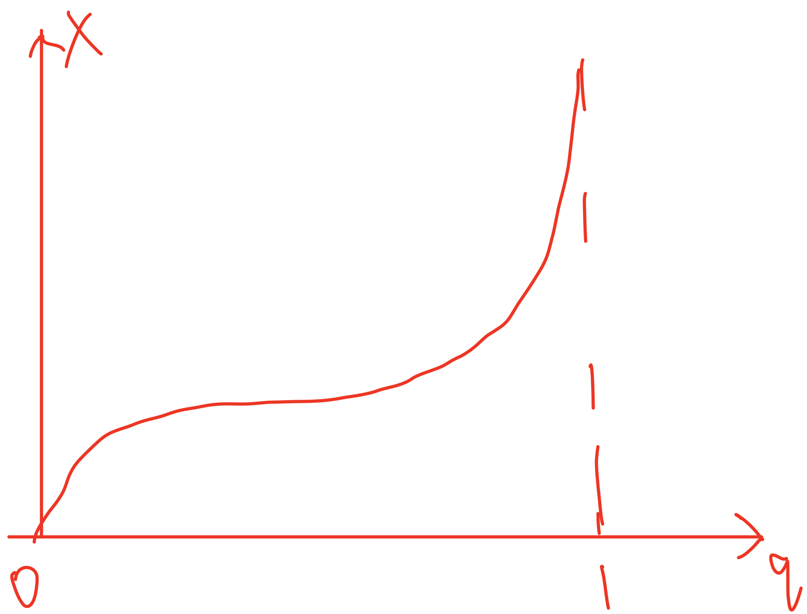
**Definition.** For rv $X$ with cdf $F$, the quantile function is sort of a generalized inverse of $F$, Specifically for $0 < q < 1$,

$$F^{-1}(q) = \inf\{x : \; q \leq P(X \leq x)\}$$

or equivalently,

$$F^{-1}(q) =$$

**Pictures**



**Example.** Suppose $X$ corresponds to the amount paid out an insurance claim (in thousands) and for positive $x > 0$ has cumulative distribution function

$$F(x) = .1 + .9(1 - e^{-x/10})$$

**(a)** What is probability of claim being between 5000 and 10000?

**(b)** What is the probability of a claim paying out zero?

**(c)** Derive the .99 quantile of the distribution of the insurance claims, and interpret this value in your own words.

**Answer**

(a): F(10000) - F(5000)

(b): F(0)

(c): v such that F(v) = 0.99

**Answer.**

# Moments of Distribution/Random Variable

**Definition.** For rv $X$ with distribution function $F$, we define **central moments** $\mu_k$ as k阶中心矩

$$\mu_k = E[\,(X - E[X])^k\,] \qquad k = 2, 3, 4, \ldots$$

**Definition.** For sample $x_1, x_2, \ldots, x_n$, define **sample mean** as

$$\overline{x} = \frac{1}{n} \sum X$$

and we define the **central sample moments** $m_k$ for this data as

$$m_k = \frac{1}{n} \sum (X_i - \bar{X})^k \qquad k = 2, 3, 4, \ldots$$

可能会biased,但最终会收敛于真实值

**Background:** Typically assuming that sample $x_1, x_2, \ldots, x_n$ corresponds to a sampling from some process/population with an underlying distribution function $F$.

# Summary Statistics

从量纲角度思考,必定用到2阶矩

**Background.** Suppose that $X \sim F$ and have sample $x_1, x_2, \ldots, x_n$, and central moments of $\mu_k, m_k$ for $k = 2, 3, 4$.

| Parameters/Statistics | Distn Parameter | Sample Statistic |
|---|---|---|
| Standard deviation | $\sigma = \sqrt{\mu_2}$ | $\mathrm{SD}(x) = \sqrt{m_2}$ |
| Skewness | $\frac{\mu_3}{(\mu_2)^{\frac{3}{2}}}$ | $\frac{m_3}{m_2^{\frac{3}{2}}}$ |
| (Excess) Kurtosis<br>和正态分布对比 | $\frac{\mu_4}{\mu_2^2} - 3$ | $\frac{m_4}{m_2^2} - 3$ |

**Remarks.**

- Skewness is a measure of the asymmetry of the distribution
- Kurtosis is a measure of how heavy tailness the distribution is

# Continuous Random Variables

- For financial data/statistical analysis, RVs are often modeled as taking values over a continuum
  - RVs are called continuous and
  - the distribution is characterized via a probability density function (pdf), $f(x)$,

$$f(x) \geq 0 \quad \text{for all x,} \quad \int_{-\infty}^{\infty} f(x) = 1$$

$$P(a < X < b) = \int_a^b f(x)\, dx$$

- The cdf of $X$ is continuous and is given by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)\, du$$

# Expected Value/Variance/St Dev of Continuous RVs

**Definition.** Suppose $X$ is a continuous random variable with a pdf $f$. The **expected value of $X$** is given by

$$E(X) = \int x f(x) dx \quad .$$

provided that $\int_{-\infty}^{\infty} |x| f(x)\, dx < \infty$. Otherwise we say the expectation is undefined. Provided the expected value exists, the **variance of $X$** is then defined as

$$\text{Var}(X) = E\left[(X - E(X))^2\right] = \int (x - E[x])^2 dx$$

$$E(X^2) = \int x^2 f(x) dx \quad .$$

The **standard deviation of $X$** is

$$\text{SD}(X) = \sqrt{\text{Var}\, X}$$

# Properties of Continuous RVs

Suppose $X$ is a continous rv with pdf $f$ and cdf $F$, and $a < b$ and $c$ are real numbers. Then

**(i)** $P(X = c) =$ $f(c) = F'(c)$

**(ii)**

$$\begin{aligned} P(a \leq X \leq b) &= P(a < X < b) \\ &= P(a < X \leq b) = P(a \leq X < b) \end{aligned}$$

**(iii)** Can write $P(a \leq X \leq b)$ in terms of cdf, i.e.,

$$P(a \leq X \leq b) = F(b) - F(a) \qquad .$$

# Expectation/Variance/Quantiles of Functions of RVs

**Results.** Suppose $X$ is a rv and $Y = a + bX$. Then

**(a)** If mean $E(X)$ exists, then $E(Y) =$ $a+bE[X]$ .

**(b)** If mean and variance of $X$ exists, then
Var$(a + bX) =$ $b^2 \operatorname{Var} X$ .

**(c)** If $b > 0$, quantiles are related via $y_q =$ $a+bX_q$ .

**(c')** If $b < 0$, quantiles are related via $y_q =$ $a+bX_{1-q}$ .

**(d)** If $h$ is a strictly increasing function and $Y = h(X)$, then quantiles of $Y$ are given by

$$y_q = h(X_q)$$

**(d)'** If $h$ is a strictly decreasing function and $Y = h(X)$, then quantiles of $Y$ are given by

$$y_q = h(X_{1-q})$$

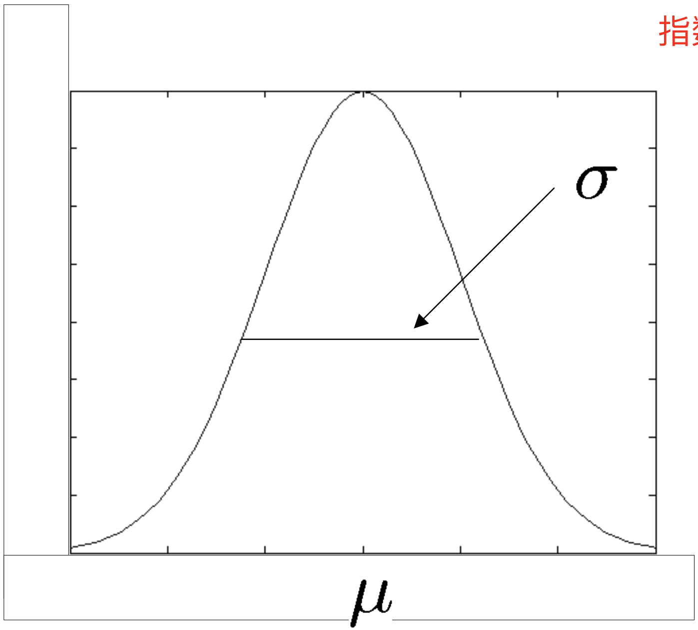
**Proof.**

# Continuous Distribution: Normal Distribution

**Definition: Normal Distribution.** A pdf $f$ is said to correspond to a normal distribution with a mean of $\mu$ and standard deviation of $\sigma$ if

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty$$

指数上越大,峰度高,less tail



# Remarks/notation for normal distributions

- If $X$ is a normal rv with parameters $\mu, \sigma^2$, write $X \sim \mathcal{N}(\mu, \sigma^2)$
- $\mathcal{N}(0,1)$ is referred to as standard normal distribution and the corresponding pdf and cdf are denoted by $\phi$ and $\Phi$, i.e.,

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\Phi(x) = \int_0^x f(x)\,dx$$

- No nice closed form for $\Phi$, but it is easily computed via R.

# More on the Normal Distribution

## Expected Value/Variance/Skewness/Kurtosis of $X \sim \mathcal{N}(\mu, \sigma^2)$

$$
\begin{aligned}
E(X) &= \mu \\
\text{Var}(X) &= \sigma^2 \\
\text{SD}(X) &= \sigma \\
\text{Skew}(X) &= \text{因为对称所以是0} \\
\text{Kurt}(X) &= 0 \quad \text{excess kurtosis}
\end{aligned}
$$

## Normal Distribution R-functions

```
dnorm(x, mean=0, sd=1) # density function
pnorm(q, mean=0, sd=1) # cdf
qnorm(p, mean=0, sd=1) # quantile
rnorm(n, mean=0, sd=1) # generates random deviates
```

- Can drop the "mean=" and "sd="

# Plots/Histograms for Normal Distn

## R-code and Output Plots

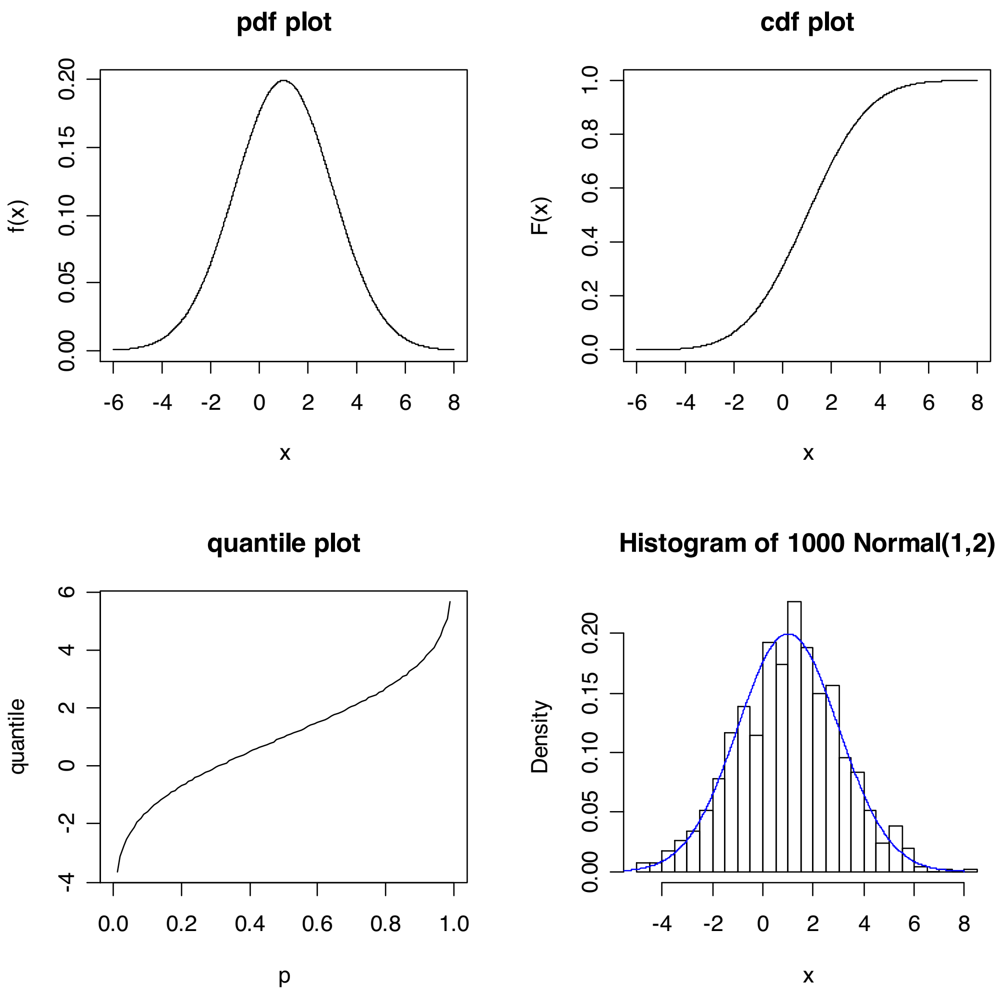
```
## Plots for Normal Case ##

mu <- 1
sigma <- 2
x <- seq(-6,8, by = .01) # vector of x-values
p <- seq(0,1, by = .01)  # vector of probabilities
n <- 1000

dnormo <- dnorm(x, mu, sigma) # pdf values
pnormo <- pnorm(x, mu, sigma) # cdf values
qnormo <- qnorm(p, mu, sigma) # quantiles
rnormo <- rnorm(n, mu, sigma) # random deviates

# Doing 3 plots and histogram
windows()
par(mfrow=c(2,2)) # setting up for a 2 x 2 arrangement of subplots
plot(x,dnormo,xlab='x',ylab='f(x)',type='l',main='pdf plot')
plot(x,pnormo,xlab='x',ylab='F(x)',type='l',main='cdf plot')
plot(p,qnormo,xlab='p',ylab='quantile',type='l',main='quantile plot')
hist(rnormo,xlab='x',breaks=25,main='Histogram of 1000 Normal(1,2)',freq=FALSE)
par(col="blue")
lines(x,dnormo)
```

# Output Plots from R-code



# Example.

**Example** Suppose $X \sim \mathcal{N}(2,2)$ and $Y = e^X$ – find .99-quantile of $Y$.

**Answer.**

$$\frac{\ln y - 2}{\sqrt{2}} \sim (0,1)$$

$$y = e^{2+\sqrt{2}\,\Phi(0.99)}$$

**Example** Suppose log-return $\tilde{R} \sim \mathcal{N}(0,(.02)^2)$. Derive .01-quantile of log-return and the .01-quantile of the return.

**Answer.**

$$\frac{R}{0.02} \sim (0,1) \quad R = 0.02\,\Phi(0.01)$$

$$r_t = e^R - 1 \qquad \frac{\ln r + 1}{0.02} \sim N(0,1) \quad r = e^{0.02\Phi(0.01)} - 1$$

# Continuous Distribution: Exponential

**Definition (exponential distribution).** A pdf $f$ is said to correspond to an exponential distribution with parameter $\lambda$ if it is given by

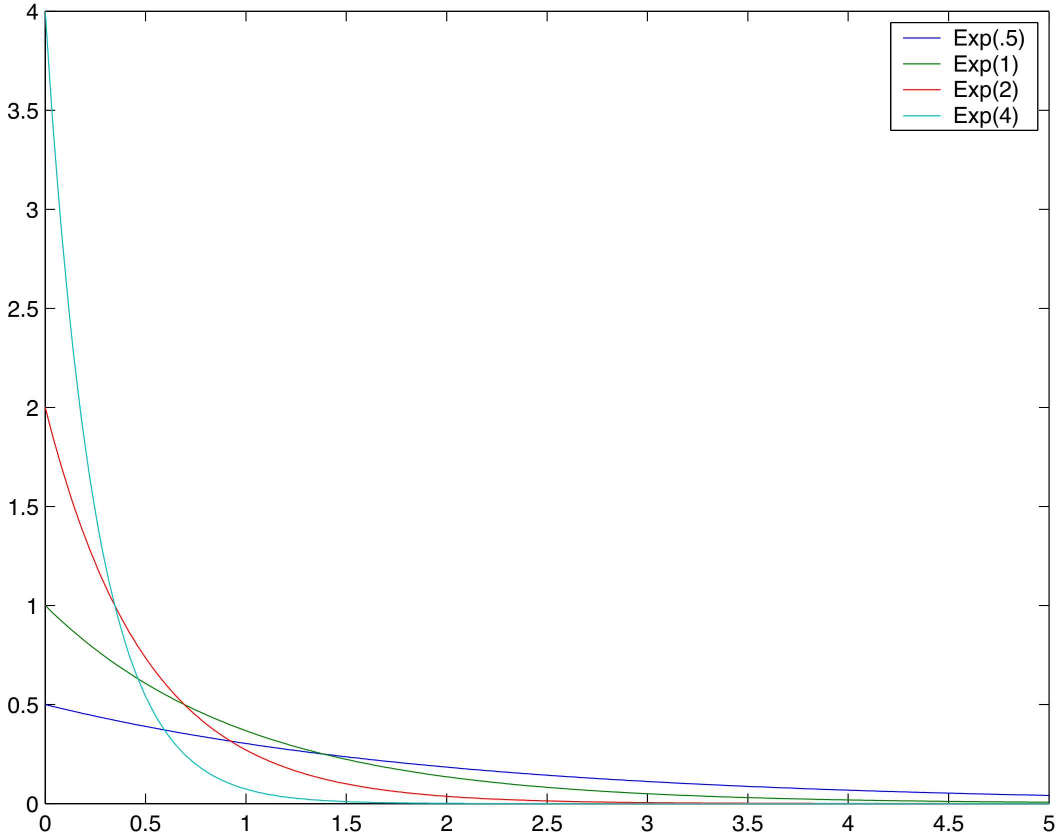
$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The cdf for $x \geq 0$ is given by

$$\begin{aligned} F(x) &= \int_0^x f(u)\,du \\ &= \int_0^x \lambda e^{-\lambda u}\,du = 1 - e^{-\lambda x} \quad . \end{aligned}$$

and $F(x) = 0$ for $x < 0$. A rv $X$ with the above pdf

- is an exponential rv, and write $X \sim \text{Exp}(\lambda)$




**Plots of Exponential pdf's**

# More on Exponential Distribution

## Expected Value/Variance of $X \sim \text{Exp}(\lambda)$

$$
\begin{aligned}
E(X) &= \lambda^{-1} \\
\text{Var}(X) &= \lambda^{-2} \\
\text{SD}(X) &= \lambda^{-1}
\end{aligned}
$$

## Exponential Distribution R-functions

- Note that rate $= \lambda$ R中把lambda叫作rate,来源于泊松分布

```
dexp(x, rate=a) # density function
pexp(q, rate=a) # cdf
qexp(p, rate=a) # quantile
rexp(n, rate=a) # generates random deviates
```

# Continuous Distribution: Double Exponential

**Definition (double exponential distribution).** A pdf $f$ is said to correspond to a double exponential distribution with mean $\mu$ and parameter $\lambda$ if it is given by

$$\boxed{f(x) = \frac{\lambda}{2} e^{-\lambda|x-\mu|}}$$

The cdf is given by

$$F(x) = \begin{cases} \frac{1}{2} e^{-\lambda|x-\mu|} & x < \mu \\ 1 - \frac{1}{2} e^{-\lambda|x-\mu|} & x \geq \mu \end{cases}$$

本质上通过指数分布转换

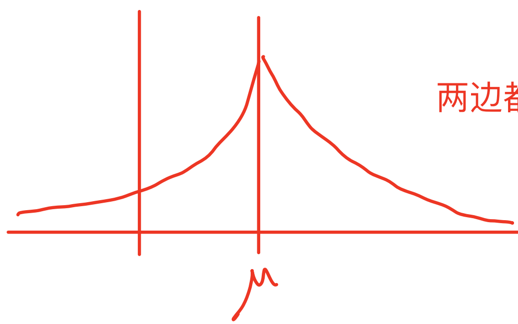
**Remark.** For rv $X$ with above pdf, write $X \sim \text{DExp}(\mu, \lambda)$.

**Pictures**

$X \sim Dexp(0,1)$

$Z = \{\frac{1}{2}, -\frac{1}{2}\}$

$Y = \mu + \sigma Z X$




两边都是半个lambda分布

# Expected Value/Variance of $X \sim \text{DExp}(\mu, \lambda)$

$$
\begin{aligned}
E(X) &= \mu \\
\text{Var}(X) &= \frac{2}{\lambda^2} \\
\text{SD}(X) &= \frac{\sqrt{2}}{\lambda} \\
\text{Skew}(X) &= 0 \\
\text{Kurt}(X) &= 3
\end{aligned}
$$

(两倍正态分布)

**Double exponential Distribution R-functions**

- These are in startup.R on Canvas – put startup.R in R work directory
- Need to do command of `source('startup.R')`

```
ddexp(x,mu,lambda) # density function
pdexp(x,mu,lambda) # cdf
qdexp(p,mu,lambda) # quantile
rdexp(n,mu,lambda) # generates random deviates
```

# Example.

**Example** Suppose log-return $\tilde{R}$ is double exponential with mean of 0 and standard deviation of .02. Derive the .01-quantile of log-return and the .01-quantile of the return.

**Answer.** $\tilde{R} \sim Dexp(0, .02)$

$\lambda = \frac{\sqrt{2}}{0.02}$ $\quad \frac{1}{2}e^{-\lambda x} = 0.01$ $\quad q_{\tilde{R}} = -\frac{0.02}{\sqrt{2}} \ln 0.02$

$q_R = e^{q_{\tilde{R}}} - 1$

# Generalized Error Distributions

**Definition.** A rv $X$ has a Generalized Error Distribution with parameter $\nu$ if 只有一个参数,关注x的幂

$$f_{ged,\nu}(x) = \kappa_\nu e^{-\frac{1}{2}\left|\frac{x}{\lambda_\nu}\right|^\nu}$$

- $\kappa_\nu$ and $\lambda_\nu$ are constants determined by $\nu$ and $\text{Var}(X) = 1$ (for more details consult section 5.6 in Ruppert)
- for $\nu = 2$ have 正态分布 and $\nu = 1$ have 双指数分布
- can generalize to location-scale family, by

$$Y = \mu + \lambda X$$

nu越小,tail越heavy

- Now have 3 parameters of $\mu, \lambda, \nu$
- $\mu$ is the mean and $\lambda^2$ is the variance
- Notation of $Y \sim \text{GED}(\mu, \lambda^2, \nu)$

# More on $GED$-Distribution

## Exp Value/Variance/St Dev/Skewness/Kurtosis of GED-RVs

For $X \sim \text{GED}(\mu, \lambda^2, \nu)$

$$
\begin{aligned}
E(X) &= \mu \\
\text{Var}(X) &= \lambda^2 \\
\text{SD}(X) &= \lambda \\
\text{Skew}(X) &= 0 \\
\text{Kurt}(X) &= \frac{\Gamma(\frac{5}{\nu})\Gamma(1/\nu)}{(\Gamma(3/\nu))^2}
\end{aligned}
$$

## GED-Distribution R-functions

```
dged(x, mean = 0, sd = 1, nu = 2) # density function
pged(q, mean = 0, sd = 1, nu = 2) # cdf
qged(p, mean = 0, sd = 1, nu = 2) # quantile
rged(n, mean = 0, sd = 1, nu = 2) # generates random deviat
```

- Above is from package fGarch

# Continuous Distribution: Gamma

**Definition (gamma distribution)** A pdf $f$ is said to correspond to a gamma distribution with parameters $\alpha, \lambda > 0$ if

$$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\Gamma(\alpha)$ is the gamma function

$$\Gamma(\alpha) = \int_0^\infty x^{(\alpha-1)} e^{-x}\, dx.$$

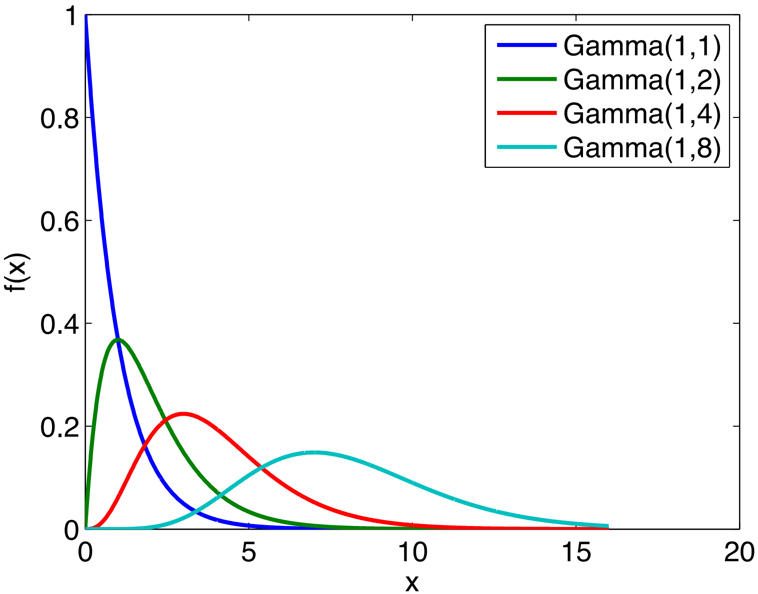
1. $\lambda$ is called the scale parameter
2. $\alpha$ is called the shape parameter alpha=1就是指数分布

If $X$ is a random variable with above pdf, write $X \sim \text{Gamma}(\lambda, \alpha)$.

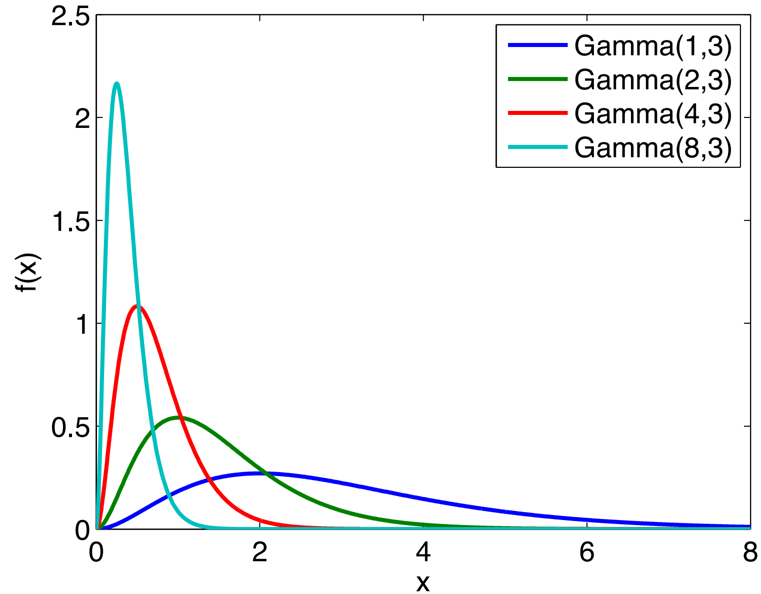
# Plots of Gamma pdf's

若干个指数分布的和

$\lambda = 1,\ \alpha = 1, 2, 4, 8$




$\alpha = 3,\ \lambda = 1, 2, 4, 8$



# More on Gamma Distribution

**Expected Value/Variance/St Dev of Gamma RVs**

For $X \sim \text{Gamma}(\lambda, \alpha)$

$$\begin{aligned} E(X) &= \frac{\alpha}{\lambda} \\ \text{Var}(X) &= \frac{\alpha}{\lambda^2} \\ \text{SD}(X) &= \frac{\sqrt{\alpha}}{\lambda} \end{aligned}$$

**Gamma Distribution R-functions**

```
dgamma(x, shape, rate=a) # density function
pgamma(q, shape, rate=a) # cdf
qgamma(p, shape, rate=a) # quantile
rgamma(n, shape, rate=a) # generates random deviates
```

- Can drop the “rate=”

# Properties/attributes of Gamma distribution

- **Flexible distribution – overall shape**
  - exponential is a special case with $\alpha = 1$
    - relies on $\Gamma(1) = 1$ which is easy to verify
- **Sum of iid exponential rvs is gamma**
  If $X_1, X_2, \ldots, X_n$ are iid Exp($\lambda$), then the sum

$$Y = \sum_{i=1}^{n} X_i \sim \text{Gamma}(\lambda, n) \quad .$$

- **Sum of iid gamma rvs is gamma**
  If $X_1, X_2, \ldots, X_n$ are iid Gamma($\lambda, \alpha$), then alpha需要相同

$$Y = \sum_{i=1}^{n} X_i \sim \text{Gamma}(\lambda, n\alpha) \quad .$$

- **Chi-square** with $\nu$ degrees of freedom is Gamma $\left(\frac{1}{2}, \frac{\nu}{2}\right)$ and corresponds to distribution of $\sum_{i=1}^{\nu} Z_i^2$ when $Z_1, Z_2, \ldots, Z_\nu$ are iid $\mathcal{N}(0, 1)$.

# Continuous Distribution: $t$-distribution

**Definition.** Suppose $Z \sim \mathcal{N}(0,1)$ and $W \sim \chi^2_\nu$ are independent. Then $X = \frac{Z}{\sqrt{\frac{W}{\nu}}}$ has a $t$-distribution with $\nu$ degrees of freedom, and has a pdf given by

多项式级的tail,因为当x变大,
分母会成为 $K x^{(\nu+1)}$

$$f_{t,\nu}(x) = \left[\frac{\Gamma\left\{\frac{\nu+1}{2}\right\}}{\sqrt{\pi\nu}\,\Gamma\left(\frac{\nu}{2}\right)}\right] \frac{1}{\left\{1 + \left(\frac{x^2}{\nu}\right)\right\}^{\frac{\nu+1}{2}}}$$

可以不是整数

- General $t$ distribution with parameter $\nu > 0$ - denote by $t_\nu$
- Scaled $t$-distribution for $Y = \mu + \lambda X$, where $X \sim t_\nu$ and $\lambda > 0$. Notation is $Y \sim t_\nu(\mu, \lambda^2)$ and this is classical $t$ lambda不是标准差
- There is a standardized $t$ and need to be careful (see textbook), in particular $t_\nu^{std}(\mu, \sigma^2)$ corresponds to re-scaled t-distn so as to have mean 0 and variance of $\sigma^2$ for $\nu > 2$

已知方差,调整v,来获得标准化的t分布

# More on $t$-Distribution

## Exp Value/Variance/St Dev/Skewness/Kurtosis of $t$-RVs

For $X \sim t_\nu(\mu, \lambda^2)$

$$\begin{aligned} E(X) &= \mu \\ \text{Var}(X) &= \lambda^2 \frac{\nu}{\nu-2} \\ \text{SD}(X) &= \lambda\sqrt{\frac{\nu}{\nu-2}} \\ \text{Skew}(X) &= 0 \\ \text{Kurt}(X) &= \frac{6}{\nu-4} \end{aligned}$$

需要v>2

需要v>4

## $t$-Distribution R-functions

都是standardize的t

```
dt(x, df) # density function
pt(q, df) # cdf
qt(p, df) # quantile
rt(n, df) # generates random deviates
```

R语言用的不是standardized

- Invoke "help(TDist)" in R for more information

# Heavy-Tailed Distributions

**Remark.** Often the rvs $X$ will represent

- Stock price
- Bond price
- Currency exchange rate
- Insurance claims
- Aggregate price of stocks
  - SP500
  - Russell 2000
  - Dow Jones Industrial Average

收益率:
f_r(x)=f_x(-x)
F_r(x)=1-F_x(-x)

**Remark.** From the viewpoint of risk, focus is often on the tail probabilities – for example if $X$ is your loss (negative return), interested in

$$P(X > x) = 1 - F(x).$$

- The above is called a tail probability

**Pictures**

# Overview of Value-at-Risk - VaR

**Example. Value-at-Risk VaR** The basic idea of VaR is that it helps to quantify the amount of capital needed for covering a loss in a portfolio. Consider the following:

Var说明了在给定概率下,可能亏损的最大值
估计loss的分布,再进行投资

$$
\begin{aligned}
P_t &= \text{value of portfolio at time } t \\
P_{t+\triangle t} &= \text{value of portfolio at time } t+\triangle t \\
R_t = \frac{P_{t+\triangle t} - P_t}{P_t} &= \text{net return at time } t+\triangle t \\
q &= \text{(small) probability of not covering losses}
\end{aligned}
$$

也是t时刻的风险

The Value-at-Risk at time $t$, $\text{VaR}_t$, is defined by

$$P\left(P_{t+\triangle t} - P_t + \text{VaR}_t < 0\right) = P\left(R_t + \tilde{\text{VaR}}_t < 0\right) = q$$

取决于value

$= P(R_t < -\tilde{\text{VaR}}_t)$ 取决于收益率

where

VaR只是R的q-quantile

$$\tilde{\text{VaR}}_t = \frac{\text{VaR}_t}{P_t} = \text{relative Value-at-Risk}$$

# More on Value-at-Risk - VaR

**Remark.** By previous slide,

**(i)** $-\text{VaR}_t$ is the $q$-quantile of the distn of raw return $P_{t+\triangle t} - P_t$

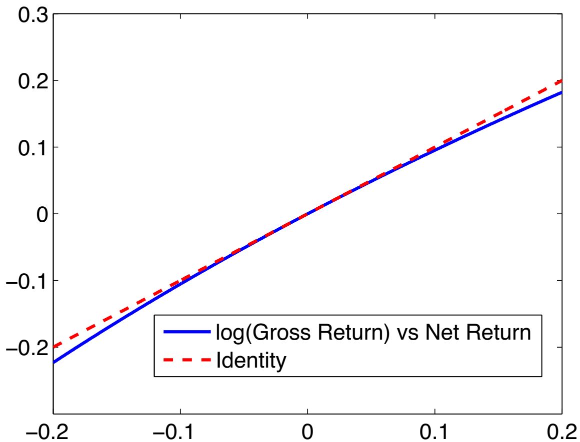
**(ii)** $-\tilde{\text{VaR}}_t$ is the $q$-quantile of the distn of return $R_t$

**Remark.** For this class sometimes consider log-returns, i.e.,

$$\tilde{R}_t \equiv \log\left[\frac{P_{t+\triangle t}}{P_t}\right] \approx R_t$$

where the approximation is justified when $R_t$ is relatively close to 0. Can always convert quantiles for $\tilde{R}$ to quantiles of $R$ easily.

**Log(Gross Return) vs Net Return**




**Formulas**

$r_t = \log 1 + R_t$

$R_t = e^{r_t} - 1$

# Notation/Shortfall Distribution

- Given a target probability $q$, and if time $t$ is fixed, we may drop $t$ and write $\mathrm{VaR}_q$

这里的X是loss

- Suppose $X$ represents loss (negative return). Then given a level $q$, the CDF of the shortfall distribution is defined by

$$\Theta_q(x) = P(X \le x | X > VaR_q) = \frac{P[VaR < X \le x]}{P[VaR < x]} = \frac{F(x) - F(VaR_q)}{1 - F(VaR_q)} = \frac{F(x) - (1-q)}{q}$$

- Expected shortfall

$$ES_q = E(X|X > VaR_q) = \frac{1}{q} \int_{x > VaR_q} x f(x)\, dx$$

- Risk Analysis focused on estimation of $VaR_q$ and $ES_q$.

pdf: $\frac{f(x)}{q}$

用VaR转换表示shortall分布

**Example.** Suppose portfolio value is currently 100 million dollars and $\alpha = .01$.

**(a)** Suppose that distribution of return $R_t$ is normal with a mean of .02 and a standard deviation of .03. Derive VaR and V$\tilde{a}$R.

**(b)** Suppose that distribution of return $R_t$ is DExp with a mean of .02 and a standard deviation of .03. Derive VaR and V$\tilde{a}$R.

**(c)** Derive the shortfall distribution and expected shortfall for parts **(a)** and **(b)** .

**Answer.**

注意负号

(a) V$\tilde{a}$R = -qnorm(p=0.01, $\mu$=0.02, sd=0.03)

VaR = 1e8 V$\tilde{a}$R

(b) V$\tilde{a}$R = -qdexp

**Answer.**

(c) $F(x) = \dfrac{F_{N(0.02,\,0.03)} - 0.99}{0.01}$

$f(x) = \dfrac{f_{N(0.02,\,0.03)}}{0.01}$

$E[\mu + \sigma Z] = \mu + \sigma E[Z \mid Z \geq z_{1-q}]$

$= \mu + \sigma \frac{1}{q} \int_{z \geq z_{1-q}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz$

$= \mu + \frac{\sigma}{q} \exp\left(-\frac{z_{1-q}}{2}\right)$

## Answer.

**Answer.**

# Tail probabilities

1-cdf: 大于x的概率

- For double-exponential distribution with mean $\mu$ and scale parameter $\lambda$, tail probability is

$$1 - F(x) = \frac{1}{2} e^{-\lambda |x-\mu|}$$

- For normal distribution, tail probability is

$$\begin{aligned} 1 - F(x) &= \int_x^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\ &\sim \frac{\sigma}{\sqrt{2\pi x}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{as } x \to \infty \end{aligned}$$

where the similar ($\sim$) notation here means that

ratio converges to 1,
因此不是heavy tail

$$\frac{1 - F(x)}{\frac{\sigma}{\sqrt{2\pi x}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}} \to 1 \quad \text{as } x \to \infty.$$

**Remark.** In both cases, the tail probabilities go to 0 exponentially fast. There are cases where the tail probabilities go to 0 slower than exponential.

# Background: Similarity of Tail Probabilities

**Remarks.**

- For double exponential, normal distribution, and GED, the tail probabilities go to 0 exponentially fast.
- A number of models for financial data involve tail probabilities that go to 0 slower than exponential
  - In some widely utilized models, tail probabilities go to 0 inversely related to a polynomial – example would be

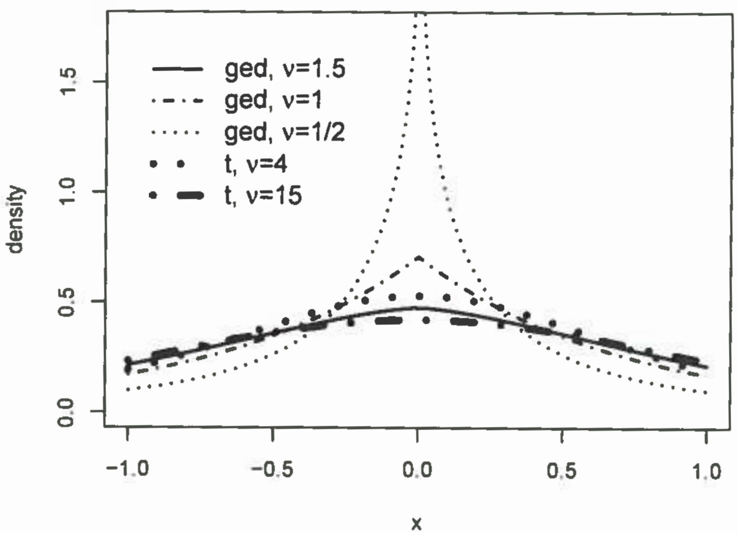
$$\boxed{1 - F(x) \sim \frac{1}{x^p} \quad \text{as } x \to \infty}$$

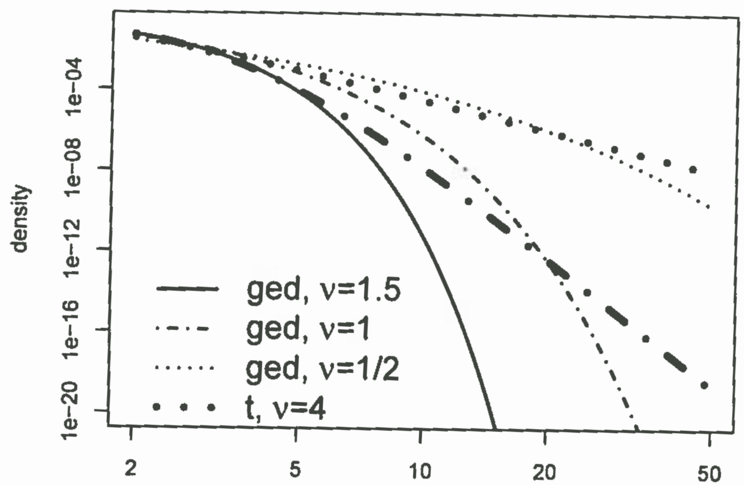
where $p > 0$

**Remark.** Want to investigate for two different cdfs $F$ and $G$ whether tail probabilities are similar, i.e., $(1 - F(x)) \sim (1 - G(x))$. To do this we need a definition of functions being similar at $\infty$.

# Comparison of GED with $t$-distribution

t分布是厚尾的




**Implications for fitting tails vs. main body of distributions**

# Continuous Distribution: Pareto

**Definition: Pareto Distribution.** A pdf $f$ is said to correspond to a Pareto distribution with location parameter of $\mu$ and shape parameter $a > 0$ if


$$f(x) = \frac{a\mu^a}{x^{a+1}} \qquad x > \mu$$

The cdf is given by

$$F(x) = \int_{-\infty}^{x} f(u)\,du$$

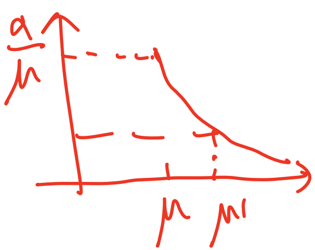
$$= \begin{cases} 1-\left(\frac{\mu}{x}\right)^a & x \geq \mu \\ 0 & x < \mu \end{cases}$$

多项式tail

...

**Notation:** If $X$ has Pareto distribution with parameters of $\mu, a$, write $X \sim \text{PD}(\mu, a)$.

**Pictures:**




$P[X \leq \mu \mid X \geq \mu']$ $\mu \geq \mu'$

conditional distribution
也是Pareto

# Continuous Distribution: Generalized Pareto

- There is a **Generalized Pareto distribution**
- The parameterization of the generalized Pareto distribution is not a simple "generalization" of Pareto

**Definition: Generalized Pareto Distribution.** The pdf $f$ for the Generalized Pareto distribution with location parameter of $\mu$, shape parameter $\xi > 0$, and scale parameter $\sigma > 0$ is

$$f(x) = \frac{1}{\sigma}\left(1 + \frac{\xi(x-\mu)}{\sigma}\right)^{\left(-\frac{1}{\xi}-1\right)} \qquad x \geq \mu$$

The cdf is given by

$$F(x) = \begin{cases} 1-\left(1+\frac{\xi(x-\mu)}{\sigma}\right)^{-\frac{1}{\xi}} & x \geq \mu \\ 0 & x < \mu \end{cases}$$

**Notation:** If $X$ has generalized Pareto distribution with $\sigma = \frac{\mu}{\alpha}$ $\xi = \frac{1}{\alpha}$ parameters of $\mu, \xi, \sigma$, write $X \sim \text{GPD}(\mu, \xi, \sigma)$.

# More on the Generalized Pareto Distn

$\xi$ 越大tail越heavy, tail很light的时候可以达到double exponential

**Remark.** The tail probabilities for $X \sim \text{GPD}(\mu, \boxed{\xi}, \sigma)$ are

$$1 - F(x) = \left(1 + \frac{\xi(x-\mu)}{\sigma}\right)^{-1/\xi} \quad x \to \infty$$

$$\sim \left[\frac{\xi(x-\mu)}{\sigma}\right]^{-\frac{1}{\xi}} \sim \left[\frac{\xi x}{\sigma}\right]^{-\frac{1}{\xi}} = \left(\frac{\xi}{\sigma}\right)^{-\frac{1}{\xi}} \underbrace{\left(\frac{1}{x}\right)^{\frac{1}{\xi}}}_{\text{多项式Tail}}$$

**Relationship be Pareto/Generalized Pareto** It can be shown that the Pareto distribution $\text{PD}(\mu, a)$ and the generalized Pareto distn $\text{GPD}(\mu, \xi, \sigma)$ are equal when

$$\xi = \frac{1}{\alpha} \text{ and } \quad \sigma = \frac{\mu}{\alpha}$$

# Generalized Pareto Distribution R-functions

- Must install fExtremes package
- Then do command of `library(fExtremes)`

```
dgpd(x, xi = 1, mu = 0, beta = 1, log = FALSE)  # density function
pgpd(q, xi = 1, mu = 0, beta = 1, lower.tail = TRUE) # cdf
qgpd(p, xi = 1, mu = 0, beta = 1, lower.tail = TRUE) # quantile
rgpd(n, xi = 1, mu = 0, beta = 1) # generates random deviates
```

- Note that "beta" is same as our "sigma"

# Limit of Generalized Pareto - $\xi \to 0$

**Remark.** The tail distribution of GPD$(\mu, \xi, \sigma)$ satisfies

$$(1 - F(x)) \sim \left(\frac{\xi}{\sigma}\right)^{-\frac{1}{\xi}} \cdot \frac{1}{x^{\frac{1}{\xi}}}$$

- As $\xi \to 0$, tails becoming lighter and lighter as polynomial power $\frac{1}{\xi}$ increases
- For $\mu = 0$, the limit of the Generalized Pareto distribution as $\xi \downarrow 0$ is the exponential distribution – derived from simple limit result in mathematics that

$$\boxed{\lim_{M \to \infty} \left(1 + \frac{\beta}{M}\right)^{M} = e^{\beta}}$$

**Remark** Assuming that $\mu = 0$, can show that

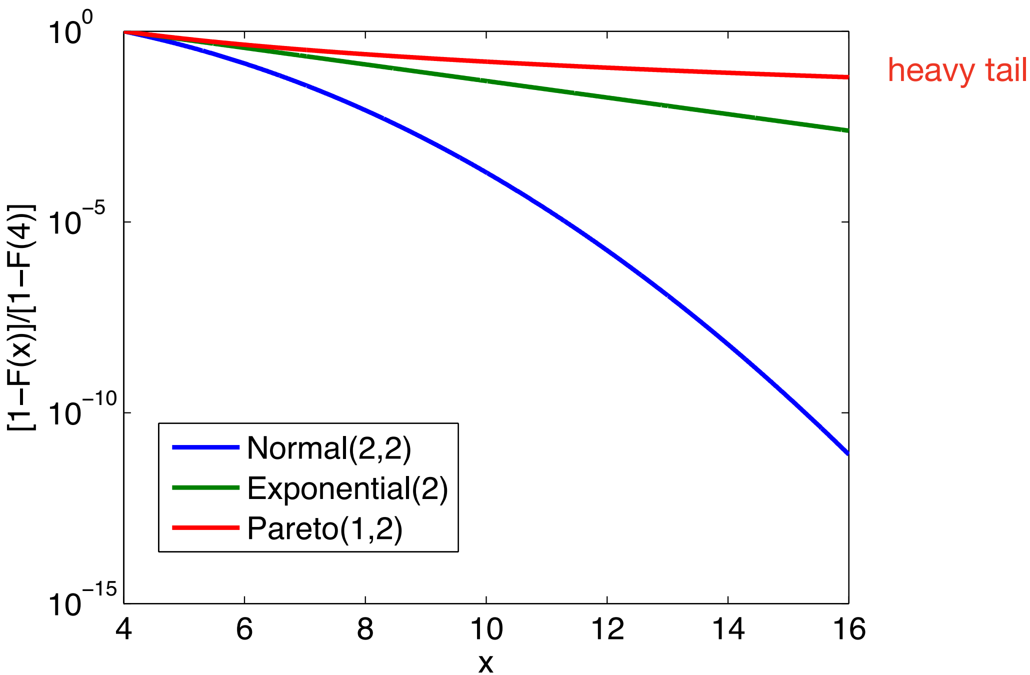
$$\lim_{\xi \to 0} (1 - F(x)) = \lim_{\varepsilon \to 0} \left(1 + \frac{\varepsilon x}{6}\right)^{-1/\varepsilon} = e^{-\frac{x}{6}}$$

i.e., tail distribution is like an . Exp

# Example: Plots of Tail Probabilities

**Remark.** Below is plot of tail probabilities for normal, exponential, and Pareto

- Normalized to starting point of $(1 - F(4))$, i.e., plotting conditional probability of $P(X \geq x | X \geq 4)$



# Pareto Tail Distributions

**Remark.** Often people are most interested in the tails of a distribution (related to risk).

**Definition.** A rv $X$ is said to have a Pareto (right) tail distribution if

$$1 - F(x) \sim \frac{1}{Cx^{a}}$$

多项式tail

**Remark.** Often people are most interested if distribution $X$ looks Pareto/Generalized Pareto in the "tail" part of the distribution

- Willing to emphasize less the fit of the distribution in the middle

Extreme Value Theory
(We don't have a lot of data for prediction)

**Example** Suppose

$$F(x) = 1 - \frac{e^{\frac{1}{x^2}}}{e \cdot x^2} \qquad x > 1$$

$x \to \infty \to 1 - F(x) = \frac{1}{ex^2}$

What is the approximate tail distribution?

**Answer.** Pareto Tail

# Transformations of Random Variables

**Linear Transformations of Normal RV**

**Theorem.** Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$, $b$ is any real number, $a > 0$, and $Y = aX + b$. The $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

**Corollary.** Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$ and

$$Z = \frac{X - \mu}{\sigma}$$

Then $Z$ is standard normal, i.e., $Z \sim N(0,1)$ .

**Linear Transformations - Uniform/Double Exp/Pareto RVs**

**Remark.** There are similar results for uniform, double exponential, GED, $t$, and Pareto rvs. a>0