STATS531 HW1

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Question 1.1

1. Based on the definition of $\hat{\mu}(Y_{1:N})$ and covariance's property P1,P3, we get

$$\mathrm{Var}(\hat{\mu}(Y_{1:N})) = \mathrm{Var}(\frac{1}{N}\sum_{i=1}^{N} Y_i) = \frac{1}{N^2}\mathrm{Var}(\sum_{i=1}^{N} Y_i) = \frac{1}{N^2}\mathrm{Cov}(\sum_{i=1}^{N} Y_i, \sum_{j=1}^{N} Y_i)$$

2. Thorough property P2, P4, we can make decomposition

$$\mathrm{Cov}(\sum_{i=1}^{N} Y_i, \sum_{j=1}^{N} Y_i) = \sum_{i=1}^{N}\sum_{j=1}^{N}\mathrm{Cov}(Y_i, Y_j) = \sum_{i=1}^{N}\mathrm{Var}(Y_i) + 2\sum_{h=1}^{N-1}\sum_{i=1}^{N-h}\mathrm{Cov}(Y_i, Y_{i+h})$$

3. Since the model is covariance stationary, based on the definition of γ

$$\gamma_0 = \mathbb{E}[(Y_i - \mu_i)^2] = \mathrm{Var}(Y_i) \quad \forall i$$

$$\gamma_h = \mathbb{E}[(Y_i - \mu_i)(Y_{i+h} - \mu_{i+h})] = \mathrm{Cov}(Y_i, Y_{i+h}) \quad \forall i$$

4. Plug the γ into the first equation, we get

$$\mathrm{Var}(\hat{\mu}(Y_{1:N})) = \frac{1}{N^2}[N\gamma_0 + 2\sum_{h=1}^{N-1}(N-h)\gamma_h] = \frac{\gamma_0}{N} + \frac{2}{N^2}\sum_{h=1}^{N-1}(N-h)\gamma_h$$

Question 1.2 (A)

1. Based on the definition of γ_h, we get

$$\gamma_h = \frac{U}{V}$$
$$\frac{\partial g}{\partial U} = \frac{1}{V}$$
$$\frac{\partial g}{\partial V} = -\frac{U}{V^2}$$

2. Plug the derivatives, we get

$$Y \approx \frac{\mu_U}{\mu_V} + \frac{U}{\mu_V} - \frac{\mu_U}{\mu_V} - \frac{V\mu_U}{\mu_V^2} + \frac{\mu_U}{\mu_V} = \frac{U\mu_V - V\mu_U + \mu_U\mu_V}{\mu_V^2}$$

3. Take the mean on both side, we get

$$\mu_{\hat{\rho}_h} = \frac{\mu_U}{\mu_V} = \frac{\mathbb{E}(\sum_{n=1}^{N-h} Y_n Y_{n+h})}{\mathbb{E}(\sum_{n=1}^{N} Y_n^2)} = \frac{\sum_{n=1}^{N-h}\mathrm{Cov}(Y_n, Y_{n+h})}{N\sigma^2}$$

4. Since Y_n and Y_{n+h} are i.i.d, their covariance should be 0, therefore the mean is

$$\mu_{\hat{\rho}_h} = \frac{0}{N\sigma^2} = 0$$

5. Since the mean is 0, then the variance is

$$\sigma^2_{\hat{\rho}_h} = \mathbb{E}[\frac{(U\mu_V - V\mu_U + \mu_U\mu_V)^2}{\mu_V^4}]$$

6. Based on previous analysis, we know $\mu_U = 0$, therefore

$$\sigma^2_{\hat{\rho}_h} = \frac{\mathbb{E}[U^2]}{\mu_V^2} = \frac{\mathrm{Var}(\sum_{n=1}^{N-h} Y_n Y_{n+h})}{N^2\sigma^4}$$

7. Since Y_i, Y_j are uncorrelated and with both mean equals to 0, then

$$\mathrm{Var}(\sum_{n=1}^{N-h} Y_n Y_{n+h}) = \sum_{n=1}^{N-h} \mathrm{Var}(Y_n Y_{n+h}) = \sum_{n=1}^{N-h} \mathbb{E}(Y_n^2 Y_{n+h}^2) = (N-h)\sigma^4$$

8. Therefore, $\sigma^2_{\hat{\rho}_h} = \frac{N-h}{N^2}$, the standard deviation will be close to $\frac{1}{\sqrt{N}}$ with large N

Question 1.2 (B)

According to the definition of confidence interval, the probability that the unknown population parameter in this interval should be 95% or α used to create the interval if our null hypothesis is valid. The procedure of creating the confidence depends on our models and our assumption.

Therefore, if we observe a value out of the confidence interval, it means we're observing an extreme value during our assumption, so that we can reject the null hypothesis at α level of significance.

In this case, when we're plotting the autocorrelation function, our assumptions is that our model is well-fitting so that the there is no large autocorrelation within the noise. If we have a data point out of the interval, it implies our assumption and model may not be working well because it may ignore some correlation information contained in the noise.