STATS531 HW1

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Question 1.1

1. Based on the definition of $\hat{\mu}(Y_{1:N})$ and covariance's property P1,P3, we get

$$\mathrm{Var}(\hat{\mu}(Y_{1:N})) = \mathrm{Var}(\frac{1}{N}\sum_{i=1}^{N} Y_i) = \frac{1}{N^2}\mathrm{Var}(\sum_{i=1}^{N} Y_i) = \frac{1}{N^2}\mathrm{Cov}(\sum_{i=1}^{N} Y_i, \sum_{j=1}^{N} Y_i)$$

2. Thorough property P2, P4, we can make decomposition

$$\mathrm{Cov}(\sum_{i=1}^{N} Y_i, \sum_{j=1}^{N} Y_i) = \sum_{i=1}^{N}\sum_{j=1}^{N}\mathrm{Cov}(Y_i, Y_j) = \sum_{i=1}^{N}\mathrm{Var}(Y_i) + 2\sum_{h=1}^{N-1}\sum_{i=1}^{N-h}\mathrm{Cov}(Y_i, Y_{i+h})$$

3. Since the model is covariance stationary, based on the definition of γ

$$\gamma_0 = \mathbb{E}[(Y_i - \mu_i)^2] = \mathrm{Var}(Y_i) \quad \forall i$$

$$\gamma_h = \mathbb{E}[(Y_i - \mu_i)(Y_{i+h} - \mu_{i+h})] = \mathrm{Cov}(Y_i, Y_{i+h}) \quad \forall i$$

4. Plug the γ into the first equation, we get

$$\mathrm{Var}(\hat{\mu}(Y_{1:N})) = \frac{1}{N^2}[N\gamma_0 + 2\sum_{h=1}^{N-1}(N-h)\gamma_h] = \frac{\gamma_0}{N} + \frac{2}{N^2}\sum_{h=1}^{N-1}(N-h)\gamma_h$$

Question 1.2 (A)

1. Based on the definition of γ_h, we get

$$\hat{\rho}_h = \frac{U}{V}$$
$$\frac{\partial g}{\partial U} = \frac{1}{V}$$
$$\frac{\partial g}{\partial V} = -\frac{U}{V^2}$$

2. Plug the derivatives, we get

$$Y \approx \frac{\mu_U}{\mu_V} + \frac{U}{\mu_V} - \frac{\mu_U}{\mu_V} - \frac{V\mu_U}{\mu_V^2} + \frac{\mu_U}{\mu_V} = \frac{U\mu_V - V\mu_U + \mu_U\mu_V}{\mu_V^2}$$

3. Take the mean on both side, we get

$$\mu_{\hat{\rho}_h} = \frac{\mu_U}{\mu_V} = \frac{\mathbb{E}(\frac{1}{N}\sum_{n=1}^{N-h} Y_n Y_{n+h})}{\mathbb{E}(\frac{1}{N}\sum_{n=1}^{N} Y_n^2)} = \frac{\sum_{n=1}^{N-h}\mathrm{Cov}(Y_n, Y_{n+h})}{N\sigma^2}$$

4. Since Y_n and Y_{n+h} are i.i.d, their covariance should be 0, therefore the mean is

$$\mu_{\hat{\rho}_h} = \frac{0}{N\sigma^2} = 0$$

5. Since the mean is 0, then the variance is

$$\sigma^2_{\hat{\rho}_h} = \mathbb{E}[\frac{(U\mu_V - V\mu_U + \mu_U\mu_V)^2}{\mu_V^4}]$$

6. Based on previous analysis, we know $\mu_U = 0$, therefore

$$\sigma^2_{\hat{\rho}_h} = \frac{\mathbb{E}[U^2]}{\mu_V^2} = \frac{\mathrm{Var}(\sum_{n=1}^{N-h} Y_n Y_{n+h})}{N^2\sigma^4}$$

7. Since Y_i, Y_j are uncorrelated and with both mean equals to 0, then

$$\mathrm{Var}(\sum_{n=1}^{N-h} Y_n Y_{n+h}) = \sum_{n=1}^{N-h} \mathrm{Var}(Y_n Y_{n+h}) = \sum_{n=1}^{N-h} \mathbb{E}(Y_n^2 Y_{n+h}^2) = (N-h)\sigma^4$$

8. Therefore, $\sigma^2_{\hat{\rho}_h} = \frac{N-h}{N^2}$, the standard deviation will be close to $\frac{1}{\sqrt{N}}$ with large N

Question 1.2 (B)

According to the definition of confidence interval, the probability that the unknown population parameter in this interval should be 95% or α used to create the interval if our null hypothesis is valid. The procedure of creating the confidence depends on our models and our assumption.

Therefore, if we observe a value out of the confidence interval, it means we're observing an extreme value during our assumption, so that we can reject the null hypothesis at α level of significance.

In this case, when we're plotting the autocorrelation function, our assumptions is that our model is well-fitting so that the there is no large autocorrelation within the noise, which doesn't depend on the data. If we have a data point out of the interval, it implies our assumption and model may not be working well because it may ignore some correlation information contained in the noise.

Question 1.3

I've two participation records at Piazza this week.

1. Post a question about a specific example of generalized least squares
2. Answer a question related the arrangement of reference in our homework

Reference

- The courses slides about chapter 2
- My previous learning materials about probability and statistics