STATS 509 HW2

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1. Problem 1

(a)

Based on the definition of $\mathrm{P}[R_t \le -\tilde{\mathrm{VaR}}] = q$, and

$$F_{\text{dexp}}(x) = \begin{cases} \frac{1}{2}e^{\lambda(x-\mu)} & x < \mu \\ 1 - \frac{1}{2}e^{\lambda(\mu-x)} & x \ge \mu \end{cases}$$

We have

$$\tilde{\mathrm{VaR}}_q = -\frac{1}{\lambda}\ln 2q - \mu$$

(b)

We can change the the definition of ES_q to be based on the return where $ES_q = -\frac{1}{q}\int_{x \le -\mathrm{VaR}_q} xf(x)\mathrm{d}x$,

Since $-\tilde{\mathrm{VaR}}_q \le \mu$, we have

$$ES_q = -\frac{\lambda}{2q}\int_{-\infty}^{-\tilde{\mathrm{VaR}}_q} xe^{\lambda(x-\mu)}\mathrm{d}x = -\frac{e^{-\lambda\mu}\lambda}{2q}\int_{-\infty}^{-\tilde{\mathrm{VaR}}_q} xe^{\lambda x}\mathrm{d}x$$

For the integral, we have

$$\int_{-\infty}^{-\tilde{\mathrm{VaR}}_q} xe^{\lambda x}\mathrm{d}x = -\frac{1+\lambda\tilde{\mathrm{VaR}}_q}{\lambda^2}e^{-\lambda\tilde{\mathrm{VaR}}_q} = \frac{\lambda\mu + \ln 2q - 1}{\lambda^2}e^{\ln 2q + \lambda\mu}$$

Plugging the integral, we have

$$ES_q = \frac{1 - ln2q}{\lambda} - \mu$$

(c)

First we calculate the mean and deviation of the sample, and use them to estimate mu and lambda of the distribution

```
df <- read.csv("Nasdaq.csv", header=TRUE)
LogReturn <- diff(log(df$Adj.Close))
mu = mean(LogReturn)
std = sd(logReturn)
lambda = (2 ** 0.5) / sd(sd)
```

Therefore we get $\mu = 0.001138137$ and $\lambda = 90.28484$

Since $q = 0.01 \le \frac{1}{2}e^{-2\mu\lambda} \approx 0.4$, we have

```
if(0.5 * exp(-2 * mu * lambda)>0.1){
    VaR = 1e7 * (-log(0.02) / lambda - mu)
    ES = 1e7 * ((1-log(0.02)) / lambda - mu)
}
```

- $\mathrm{VaR}_q \approx 421916.5$
- $ES_q \approx 532677.1$

The Value at Risk means at probability less than 1% of suffering a loss greater than 421916.5

(d)

For normal distribution, we have

```
VaR = 1e7 * -qnorm(p=0.01, mean=mean(logReturn), sd = sd(logReturn))
ES = 1e7 * (mu + std * dnorm(qnorm(0.01))/ 0.01)
```

- $\mathrm{VaR}_q \approx 353015.7$
- $ES_q \approx 406095.4$

I think result from double exponential distribution is more accurate since we have a high kurtosis result from last homework, meaning the return have a heavier tail than normal distribution and a higher risk.

2. Problem 2

(a)

Based on the definition of Value of Risk, we have

$$P[R_t < -\tilde{\mathrm{VaR}}_t] = P[X > \tilde{\mathrm{VaR}}_q] = 1 - P[X \leq \tilde{\mathrm{VaR}}] = q$$

Therefore, the $\tilde{\mathrm{VaR}}_q$ is the $1-q$ quantile of the loss

Since $P(X \leq \mu) = 0.9$, then $1 - (1 + \frac{\xi(x-\mu)}{\sigma})^{-1/\xi} = (1-q-0.9)/(1-0.9) = 1-10q$

Therefore $\tilde{\mathrm{VaR}}_q = \mu - \frac{\sigma}{\xi} + \frac{\sigma}{\xi(10q)^\xi} = \mu - \frac{\sigma}{\xi} + \frac{\sigma}{\xi 0.1^\xi}$

(b)

Based on the definition of shortfall,

$$F_{X_q} = P[X \leq x | X \geq \tilde{\mathrm{Var}}_q]$$

When $x < \tilde{\mathrm{Var}}_q$, $F_{X_q} = 0$

otherwise

$$F_{X_q}(x) = \frac{F_X(x) - F_X(\tilde{\mathrm{Var}}_q)}{1 - F_X(\tilde{\mathrm{Var}}_q)} = \frac{F_X(x) - (1-q)}{q} = 1 - \frac{1}{q}(1 - F_X(x))$$

Since we know $\tilde{\mathrm{VaR}}_q > \mu$, we can plug in the distribution of $F_X(x) = 0.9 + 0.1[1 - (1 + \frac{\xi(x-\mu)}{\sigma})^{-1/\xi}]$

$$F_{X_q}(x) = 1 - \frac{1}{10q}(1 + \frac{\xi(x-\mu)}{\sigma})^{-1/\xi} = 1 - [(10q)^\xi + \frac{(10q)^\xi \xi(x-\mu)}{\sigma}]^{-1/\xi}$$

We can reorganize the CDF to update μ' and σ'

$$F_{X_q}(x) = 1 - [\frac{\sigma + \xi(x-\mu)}{\sigma/(10q)^\xi}]^{-1/\xi} = 1 - (1 + \frac{\xi[x - \mu + \sigma/\xi - \sigma/(\xi(10q)^\xi)]}{\sigma/(10q)^\xi})^{-1/\xi}$$

Therefore we have the new parameters $\xi' = \xi, \sigma' = \frac{\sigma}{(10q)^\xi}, \mu' = \mu - \frac{\sigma}{\xi} + \frac{\sigma}{\xi(10q)^\xi} = \tilde{\mathrm{VaR}}_q$

(c)

Based on the expectation of generalized Pareto distribution

$$E(X) = \int_{-\infty}^{\infty} xf(x)\mathrm{d}(x)$$

Since $f(x) = 0 \quad \forall x \leq \mu$, then

$$E(x) = \int_{\mu}^{\infty} xf(x)\mathrm{d}(x) = \int_{\tilde{\mathrm{Var}}_q}^{\infty} xf(x)\mathrm{d}x = qES_q$$

Therefore, we have

$$ES_q = qE(x) = q\mu - \frac{q\sigma}{\xi} + \frac{q\sigma}{\xi(10q)^\xi} + \frac{q\sigma}{(1-\xi)(10q)^\xi}$$