Lecture 7: Portfolio Theory - Classical & CAPM
Statistics 509 – Winter 2022
Reference: Chapters 16-18 of Ruppert/Matteson

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Overview

- Classical portfolio theory
- Optimization of portfolios
- Capital asset pricing models – and related models
- Intro to estimation of parameters for portfolio models

Classical Portfolio Theory - 2 Risky Investments

Background

- Suppose have 2 risky investments 1 and 2, with R_1, R_2 being the relative returns with means and standard deviations of μ_1, μ_2 and σ_1, σ_2 and correlation ρ_{12}.
- A portfolio consists of investing proportions of your money into the two stocks, say w for stock 1 and $(1-w)$ for stock 2. Then the relative return for portfolio is

$$R_p = wR_1 + (1-w)R_2$$

  - Assumption: $\mu_2 < \mu_1$
- May or may not include/allow short-selling

Allowing short selling corresponds to 做空说明w可以小于0或大于1 .

Not allowing short selling corresponds to 0<=w<=1

Example. Suppose have 2 risky investments 1 and 2, with R_1, R_2 being the relative returns having variances, standard deviations of $\mu_1, \mu_2, \sigma_1, \sigma_2$ and correlation ρ_{12}, and $\mu_2 < \mu_1$ and have portfolio with $R_p = wR_1 + (1-w)R_2$.

(a) Derive general formulas for mean and variance of portfolio return R_p.

(b) Derive the minimum variance portfolio.

(c) If w_* corresponds to the minimum variance portfolio, what can you say about the portfolios with $w < w_*$ or $w > w_*$?

Answer.

(a) $\mu = w\mu_1 + (1-w)\mu_2$

$\sigma^2 = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_1\sigma_2\rho_{12}$

(b) $W_* = \dfrac{\sigma_2^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}$

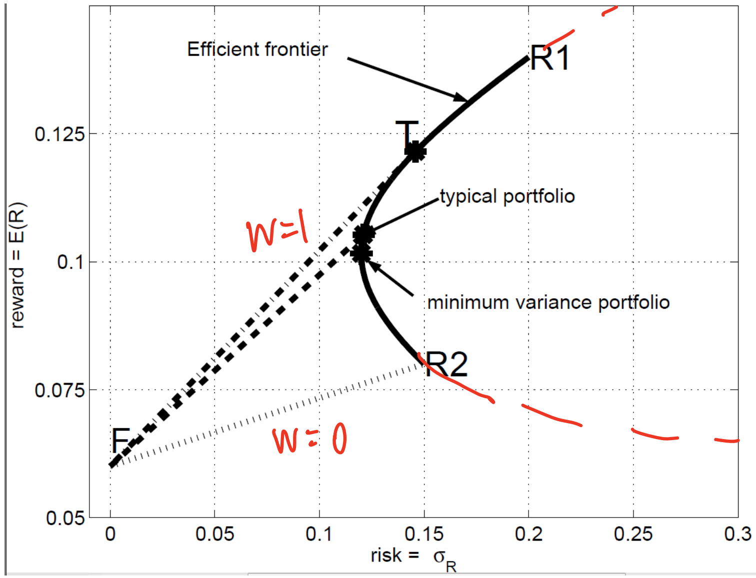
(c) $E(R_p) = w\mu_1 + (1-w)\mu_2 = \mu_2 + w(\mu_1 - \mu_2)$

Increasing w increases expected return

Answer.

Classical Portfolio Theory - 2 Risky Investments

Assumption: Assume $\mu_2 < \mu_1$



$R_w = wR_1 + (1-w)R_2$

$\mu_w = w\mu_1 + (1-w)\mu_2$

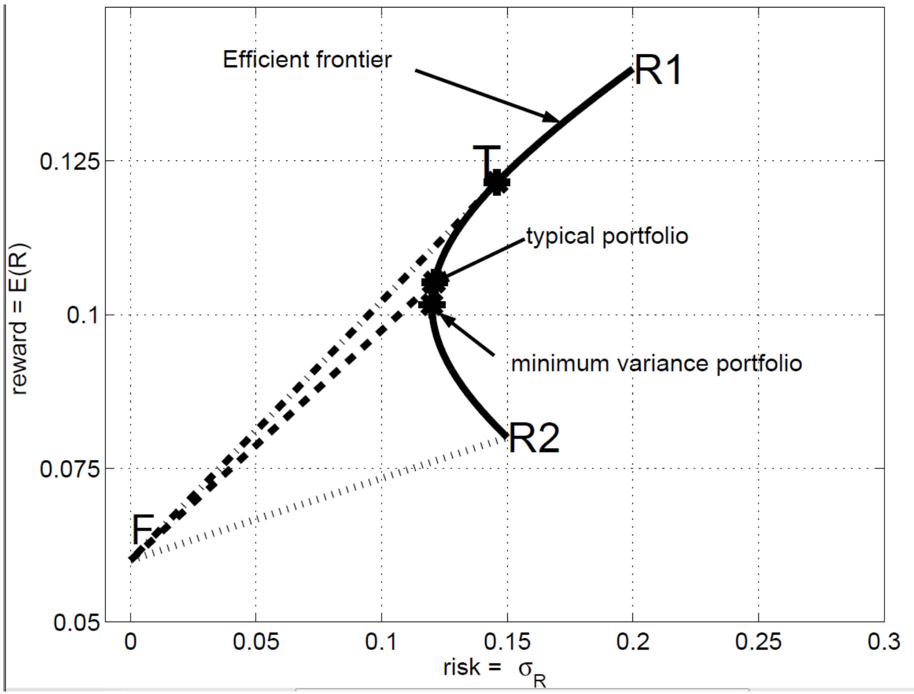
$\sigma_w = \sqrt{\mathrm{Var}\, R_w}$

- Note that any portfolio R_p can be described by the pair of mean and standard deviation, i.e., (μ_p, σ_p)
  - Goal is to maximize μ_p while minimizing the σ_p

Example. Draw in plots for portfolios with and without short-selling

Efficient Frontier of Portfolio

- The portfolios with $w \geq W^*$ are called the efficient frontier – this is because the expected return is maximized when volatility is fixed



Combining 2 Risky Assets with Fixed Asset no risk, less gain

Background: Now combine two risky assets as before with a fixed asset with mean return of μ_f and $\sigma_f = 0$. Interpretation is

Example. Suppose have portfolio with the two risky assets *and* the fixed asset, w is the total proportion in the risky assets, and R_p denotes the return of the portfolio of just the two risky assets.

(a) Derive the mean and standard deviation of the resulting portfolio return $wR_p + (1-w)R_f$. $\mu_* = \mu_f + w(\mu_p - \mu_f)$ $\sigma = w\sigma_p$

(b) Suppose that have two portfolios R_p and $R_{p'}$ and suppose that $w = \frac{\mu_* - \mu_f}{\mu_p - \mu_f}$ $\sigma = \frac{\mu_* - \mu_f}{\mu_p - \mu_f}\sigma_p$

$$\frac{E(R_p) - \mu_f}{\sigma_{R_p}} \geq \frac{E(R_{p'}) - \mu_f}{\sigma_{R_{p'}}}$$

What can you say about considering R_p vs. $R_{p'}$? Maximize Sharpe ratio

(c) What are the implications of **(b)** relative to search of the right portfolio to mix with the fixed asset?

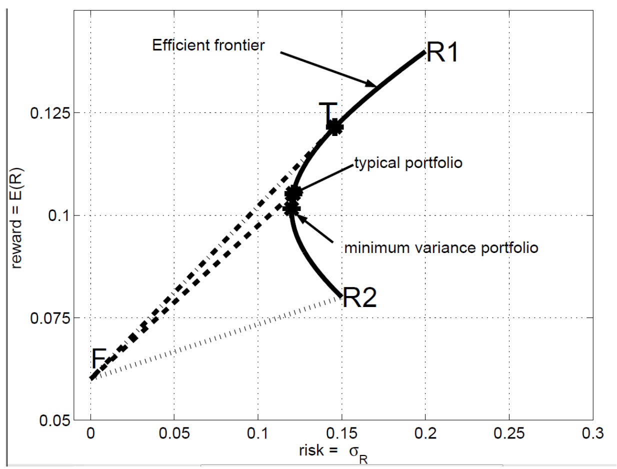
Answer. .

Sharpe Ratios

Definition. With two risky assets and one fixed asset as in previous slides, and taking R_p to be a portfolio of the risky assets, then the Sharpe ratio is defined as

$$\frac{E(R_p) - \mu_f}{\sigma_{R_p}}$$

Sharpe ratio corresponds to tangent portfolio



Example. Derive the optimal portfolio (of the 2 risky assets) to be mixed with the fixed asset? This is called the tangency portfolio.

Answer. .

$$V_1 = \mu_1 - \mu_f \qquad V_2 = \mu_2 - \mu_f$$

$$\frac{E(R_p) - \mu_p}{\sigma_p} = \frac{w_p \mu_1 + (1-w_p)\mu_2 - \mu_f}{\sqrt{w_p^2 \sigma_1^2 + (1-w_p)^2 \sigma_2^2 + 2 w_p (1-w_p) \rho \sigma_1 \sigma_2}}$$

$$W_T = \frac{V_1 \sigma_2^2 - V_2 \rho_{12} \sigma_1 \sigma_2}{V_1 \sigma_2^2 + V_2 \sigma_1^2 - (V_1 + V_2) \rho_{12} \sigma_1 \sigma_2}$$

Example: Tangent Portfolio

Example. Suppose $\mu_f = .02$, $\mu_1 = .10$ and $\mu_2 = .06$, and $\sigma_1 = .10$ and $\sigma_2 = .07$, and $\rho_{12} = .2$.

(a) Derive the minimum variance portfolio of the two risky assets and the tangency portfolio.

(b) Derive the optimal portfolio, relative to all of the assets and no short-selling, that has $\sigma_{R_p} \leq .06$.

(c) Derive the optimal investment, using all of the investments, that achieves mean return of .06.

(d) Derive the optimal investment, using only risky investments, that achieves standard deviation $\leq .07$?Repeat for $\leq .11$.

Answer.

(a) $w^* = \dfrac{\sigma_2^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2-2\rho_{12}\sigma_1\sigma_2} = 2.89$ $\quad \mu_w = 0.0787 \quad \sigma = 0.0624$

$V_1 = 0.08 \quad V_2 = 0.04 \quad w^T = 0.538 \quad \mu_T = 0.0815 \quad \sigma_T = 0.0688$

Maximize Sharpe ratio

Answer.

(b) Optimal Portfolio of risky and fixed with $\sigma \leq 0.06$

$$W = \frac{\sigma_{target}}{\sigma_T} = \frac{0.06}{0.0688} = 0.872 \text{ (fixed asset)}$$

$0.872 \times 0.538 = 0.469$ (Asset 1) 0.403 (Asset 2)

(c) We want mean return $\mu = 0.06$

$$W = \frac{\mu_{target} - \mu_f}{\mu_T - \mu_f} = \frac{0.06 - 0.02}{0.0815 - 0.02} = 0.65$$

$$\sigma = 0.65 \times 0.0689 = 0.0477$$

(d) We have two solutions for 0.07
For 0.11 we need to short selling

Answer.

Answer.

Portfolio Considerations

- For finding optimized portfolios between 2 risky assets and a fixed asset
  - Maximize mean with bound on standard deviation - this corresponds to mixing tangent portfolio with fixed asset

    $$w = \frac{\sigma_{target}}{\sigma_T}$$

  - Minimize standard deviation with bound on mean - this corresponds to

    $$w = \frac{\mu_{target} - \mu_f}{\mu_T - \mu_f}$$

Step 1: Find optimal tangent portfolio

Step 2: Find optimal mixture of tangent portfolio with fixed return asset

General Problem - N Risky Assets

Objective: Want to analyze portfolios $\sum_{i=1}^{N} w_i R_i$

- Two cases
  - Case 1: Allow arbitrary w_i with $w_i < 0$ corresponding to short
  - Case 2: Require $w_i \geq 0$, i.e., no short selling
- Ignore transaction costs for this analysis

Comment: Will start with N risky investments and then move to analyzing mixing in a risk-free asset. maximum sharpe ratio

N Risky Assets - Allowing short selling

Background: Have N risky assets with

$$\mathbf{R} = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_N \end{bmatrix}, \quad E(\mathbf{R}) = \boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{bmatrix}, \quad \text{Cov}(\mathbf{R}) = \boldsymbol{\Sigma}$$

Now weights are denoted by vector

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}$$

with requirement $\underline{\mathbf{w}^\top \mathbf{1} = 1}$. Here $\mathbf{1}$ is vector of N ones.
The portfolio is represented by $R_p = \mathbf{w}^\top \mathbf{R}$, and

$$E(R_p) = w^T R$$

$$V(R_p) = \sigma_p^2 = w^T \Sigma w$$

$$\text{Sharp} = \frac{w^T \mu - \mu_f}{\sqrt{w^T \Sigma w}}.$$

N Risky Assets - Allow short selling

Objective: Suppose have target mean μ_* – allowing short-selling, we want to minimize $\sigma_p^2 = w^T \Sigma w$

Example Write down the above in matrix form as a constrained optimization problem.

Answer. .

$$\min \; w^T \Sigma w$$

$$\text{st: } wI = 1$$

$$w^T \mu = \mu^*$$

N Risky Assets - No short selling

Objective: Suppose have target mean μ_* – we want to minimize

Example. (a) Write down the above in matrix form as a constrained optimization problem, when require that there is no short-selling.

(b) Write down the above in matrix form as a constrained optimization problem, with no short-selling, and requiring that no asset have more than 25% of the portfolio. No too much concentration

Answer. .

(a) $\min \ w^T \Sigma w$

s.t. $w^T \mu = \mu^*$

$w^T \mathbb{1} = 1$

$w \geq 0$

所有元素大于0

(b) $w \leq \frac{1}{4}$

在(a)条件上所有元素小于0.25

Quadratic Programming

R-package: quadprog

二次规划

```
Description:

This routine implements the dual method of Goldfarb and Idnani (1982, 1983)
for solving quadratic programming problems of the form
min(-d^T b + 1/2 b^T D b) with the constraints A^T b >= b_0.
```

对我们来说d永远等于0

```
Usage:  solve.QP(Dmat, dvec, Amat, bvec, meq=0, factorized=FALSE)
Arguments:

Dmat -- matrix appearing in the quadratic function to be minimized.

dvec -- vector appearing in the quadratic function to be minimized.

Amat -- matrix defining the constraints under which we want to minimize
the quadratic function.

bvec -- vector holding the values of b_0 (defaults to zero).

meq -- the first meq constraints are treated as equality constraints, all
further as inequality constraints (defaults to 0).

factorized logical flag: if TRUE, then we are passing R^(-1) (where D = R^T R)
instead of the matrix D in the argument Dmat.
```

Quadratic Programming

R-package: quadprog

`Values: a list with the following components:`

`solution -- vector containing the solution of the quadratic programming problem`

`value -- scalar, the value of the quadratic function at the solution`

`unconstrained.solution -- vector containing the unconstrained minimizer of the`

`iterations -- vector of length 2, the first component contains the number of iterations the algorithm needed, the second indicates how often constraints became inactive after becoming active first.`

`Lagrangian -- vector with the Lagragian at the solution.`

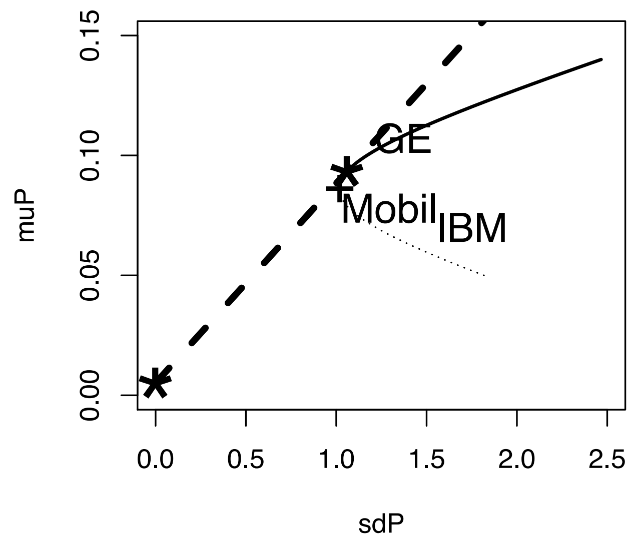
`iact -- vector with the indices of the active constraints at the solution.`

Example 1 (allowing short-selling)

Background. Focus on 3 stocks GE, IBM, and Mobil (from CRSPday data set in Ecdat)

- Find efficient portfolio for 300 values from $\mu_p = .05$ to $\mu_p = .14$
- Allow arbitrary weights, as long as they add to 1

每次遍历固定收益率,找标准差最小的点



Example 1 R-code for Portfolio Optimization

$$\min \frac{1}{2} x^T D x - d^T x = \frac{1}{2} x^T (2\Sigma) x$$

$$\text{s.t. } A^T x \geq b \Rightarrow \boxed{A^T}\,[x] \geq [\,]$$

$$A^T x = b = 1$$

```
> library(Ecdat)
> library(quadprog)
> data(CRSPday)
> R = 100*CRSPday[,4:6]
> mean_vect = apply(R,2,mean)
> cov_mat = cov(R)
> sd_vect = sqrt(diag(cov_mat))
> Amat = cbind(rep(1,3),mean_vect) # set the constraints matrix
> muP = seq(.05,.14,length=300) # set of 300 possible target values
> # for the expect portfolio return
> sdP = muP # set up storage for std devs of portfolio returns
> weights = matrix(0,nrow=300,ncol=3) # storage for portfolio weights
> for (i in 1:length(muP)) # find the optimal portfolios for
+ # each target expected return
+ {
+ bvec = c(1,muP[i]) # constraint vector
+ result =
+ solve.QP(Dmat=2*cov_mat,dvec=rep(0,3),Amat=Amat,bvec=bvec,meq=2)
+ sdP[i] = sqrt(result$value)
+ weights[i,] = result$solution
+ }
```

计算均值,协方差和标准差

输入目标函数和约束矩阵

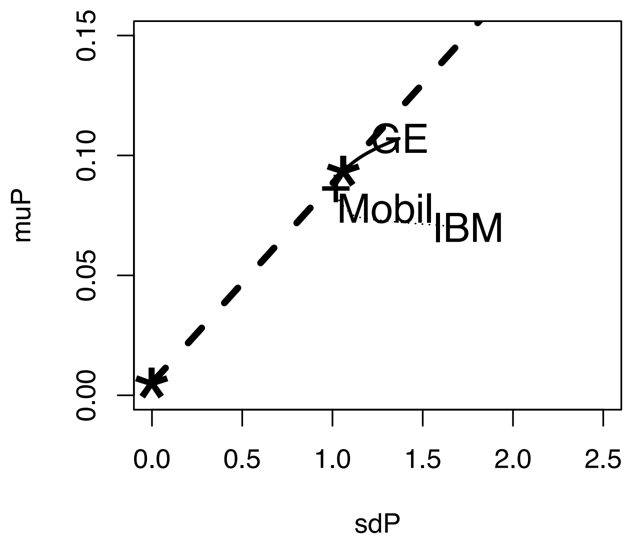
Example 1 R-code for Portfolio Optimization

```
> plot(sdP,muP,type="l",xlim=c(0,2.5),ylim=c(0,.15),lty=3) # plot
> # the efficient frontier (and inefficient portfolios
> # below the min var portfolio)
> mufree = 1.3/253 # input value of risk-free interest rate
> points(0,mufree,cex=4,pch="*") # show risk-free asset
> sharpe =( muP-mufree)/sdP # compute Sharpe's ratios
> ind = (sharpe == max(sharpe)) # Find maximum Sharpe's ratio
> options(digits=3)
> weights[ind,] # print the weights of the tangency portfolio
[1] 0.5512 0.0844 0.3645
> lines(c(0,2),mufree+c(0,2)*(muP[ind]-mufree)/sdP[ind],lwd=4,lty=2)
> # show line of optimal portfolios
> points(sdP[ind],muP[ind],cex=4,pch="*") # show tangency portfolio
> ind2 = (sdP == min(sdP)) # find the minimum variance portfolio
> points(sdP[ind2],muP[ind2],cex=2,pch="+") # show min var portfolio
> ind3 = (muP > muP[ind2])
> lines(sdP[ind3],muP[ind3],type="l",xlim=c(0,.25),
+ ylim=c(0,.3),lwd=2) # plot the efficient frontier
> text(sd_vect[1],mean_vect[1],"GE",cex=1.5)
> text(sd_vect[2],mean_vect[2],"IBM",cex=1.5)
> text(sd_vect[3],mean_vect[3],"Mobil",cex=1.5)
```

Example 2 (not allowing short-selling)

Background. Focus on 3 stocks GE, IBM, and Mobil (from CRSPday data set in Ecdat)

- Find efficient portfolio for 300 values from $\mu_p = .05$ to $\mu_p = .14$
- Allow only non-negative weights, as long as they add to 1



R-code for Portfolio Optimization

```
muP = seq(min(mean_vect)+.0001,max(mean_vect)-.0001,length=300)
Amat = cbind(rep(1,3),mean_vect,diag(1,nrow=3))
# for the expect portfolio return
sdP = muP # set up storage for std dev's of portfolio returns
weights = matrix(0,nrow=300,ncol=3) # storage for portfolio weights
for (i in 1:length(muP)) # find the optimal portfolios for
# each target expected return
{
bvec = c(1,muP[i],rep(0,3)) # constraint vector
result =
solve.QP(Dmat=2*cov_mat,dvec=rep(0,3),Amat=Amat,bvec=bvec,meq=2)
```

Note. The answer will be same as in the unconstrained case (allowing short-selling), since the weights all ended up being non-negative.

Estimation for Portfolio Optimization

前提必须是对mean和cov的估计值准确

- Estimating means/covariances for the portfolio assessment/optimization
- Issues are want recent data (small sample sizes) and accuracy (large sample sizes)

Example. Suppose that $\boldsymbol{\Sigma}$ and $\boldsymbol{\mu}$ are true covariance/mean and $\hat{\boldsymbol{\Sigma}}$ and $\hat{\boldsymbol{\mu}}$ are the estimated covariance/mean, with

$$\hat{\boldsymbol{\mu}} = \bar{X}_n = \frac{1}{n}\sum X_i$$

X_i is $n \times 1$ vector of mean

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{n}\sum (X_i - \bar{X}_n)(X_i - \bar{X}_n)^T$$

Question: How do estimated covariances/means effect estimation of tangency portfolio?

Answer. Assess via

It will introduce bias to variance

Use bootstrap

Estimation - Review of Bias, Variance, and MSE

Background. Have data (modeled as iid with pdf f_θ) $X_1, X_2, \ldots, X_n$ and want to estimate parameter θ. Have an estimator $\hat{\theta}$ – then

$$\begin{aligned}
\text{bias}(\hat{\theta}) &= E[\hat{\theta}] - \theta \\
\text{Var}(\hat{\theta}) &= E[(\hat{\theta} - E[\hat{\theta}])^2] \\
\text{MSE}(\hat{\theta}) &= \text{bias}(\hat{\theta})^2 + \text{Var}(\hat{\theta}) \\
&= (\hat{\theta} - \theta)^2
\end{aligned}$$

.

Example. Suppose $X_1, X_2, \ldots, X_n$ are iid $\mathcal{N}(\mu, 10)$, and $\overline{X}_n$ is sample mean.

(a) Derive bias, variance, and MSE of $\hat{\mu} = \overline{X}_n$.

(b) Derive bias, variance, and MSE of $\hat{\mu}_\alpha = \alpha\overline{X}_n$ with $0 < \alpha \leq 1$.

(c) Sketch a plot of MSE vs. μ for $\alpha = .8$ and $n = 4$.

(d) What are the implications of plot in **(c)** ?

Answer.

(a) bias $= 0$ $\quad$ $\text{Var} = \frac{100}{n}$ $\quad$ $\text{MSE} = \frac{100}{n}$

(b) bias $= (\alpha - 1)\mu$ $\quad$ $\text{Var} = \frac{\alpha^2 10^2}{n}$ $\quad$ $\text{MSE} = (\alpha-1)^2\mu^2 + \frac{\alpha^2 10^2}{n}$

$$\alpha_{opt} = \frac{\mu^2}{\mu^2 + \frac{10^2}{n}}$$

(c) $(0.2)^2\mu^2 + \frac{0.8^2 \cdot 10^2}{4}$

Answer

Overview of Quantifying Bias/Variance/MSE

Background. Have data (modeled as iid with pdf f_θ) $X_1, X_2, \ldots, X_n$ and want to estimate parameter θ with an estimator $\hat{\theta}$, and want to assess bias and quantify standard error.

Options for Assessing bias/variance/MSE

- Exact calculations sometimes available
- Asymptotic arguments (e.g., asymptotic unbiasedness and using inverse Fisher information for MLE)
- Bootstrap

Bootstrap: Overview

Background. Suppose data (modeled as iid with pdf f_θ) $X_1, X_2, \ldots, X_n$ and want to estimate parameter θ with an estimator $\hat{\theta}$ and want to assess bias/variance/MSE without using theory – bootstrap is an option.

- Fundamental idea: pretend the observed data is the population.
- Resample observed data (with replacement) and create multiple samples.
- From each sample, estimate parameters and assess bias and variability.

Remark.

Bootstrap: Specifics

Suppose data (modeled as iid with pdf f_θ) $X_1, X_2, \ldots, X_n$ and want to estimate parameter θ with an estimator $\hat{\theta}$.

Step 1. Carry out B resamples (B can be large) – i.e., generate sample $X^*_{b1}, \ldots, X^*_{bn}$ from $X_1, X_2, \ldots, X_n$ (with replacement).

Step 2. Generate estimate $\hat{\theta}^*_b$ from sample $X^*_{b1}, \ldots, X^*_{bn}$.

Step 3. Based on $\hat{\theta}^*_1, \hat{\theta}^*_2, \ldots, \hat{\theta}^*_B$, can assess bias, standard error, and MSE via

$$
\begin{aligned}
\text{bias}_{\text{boot}}(\hat{\theta}) &= E[\hat{\theta}^*] - \hat{\theta} \qquad . \\
s_{\text{boot}} &= \sqrt{\frac{1}{B-1}\Sigma(\hat{\theta}^*_B - \hat{\theta})^2} \\
\text{MSE}_{\text{boot}}(\hat{\theta}) &= E[(\hat{\theta}^* - \hat{\theta})^2] \\
&\approx \text{bias}^2_{\text{boot}} + S^2_{\text{boot}}
\end{aligned}
$$

Bootstrap Confidence Intervals (CI)

Approach. Suppose usual bootstrap set-up and have generated B bootstrap resampled estimates $\{\hat{\theta}_b^*\}_1^B$. Now

$$\hat{\theta} = \theta + \epsilon \quad \text{or} \quad \hat{\theta} - \theta = \epsilon$$

A 95% CI for θ is based on the distn of ϵ, i.e., .025 and .975 quantiles, q_L and q_U, CI is $[\hat{\theta} - q_u, \hat{\theta} - q_L]$

.

- But $\{\hat{\theta}_b^* - \hat{\theta}\}$ is a surrogate sample from distribution of errors ϵ, so if q_L^*, q_U^* are the quantiles of this sample, a 95% CI is
$[\hat{\theta} - q_u^*, \hat{\theta} - q_L^*]$.
- Corresponding quantiles of $\{\hat{\theta}_b^*\}_1^B$ of q_L, q_U are related as
$q_L^* = \hat{\theta} + q_L$, $q_U^* = \hat{\theta} + q_u$
- Final confidence interval is $[2\hat{\theta} - q_u, 2\hat{\theta} - q_L]$

.

Remark. The above is sometimes called the basic bootstrap CI.

Bootstrap: Example from Ruppert

- Estimating distribution of daily returns of GE, i.e. μ, σ, ν for t distribution.
- Two cases: $n = 2528$ and $n = 250$
- Used standard errors from "fitdistr" and compared

	μ	σ	ν
Estimate	0.000873	0.0112	6.34
Bootstrap mean	0.000873	0.0112	6.34
SE	0.000254	0.000259	0.73
Bootstrap SE	0.000257	0.000263	0.81

n=2528

	μ	σ	ν
Estimate	0.00142	0.01055	5.51
Bootstrap mean	0.00146	0.01067	6.81
SE	0.000767	0.000817	1.97
Bootstrap SE	0.000777	0.000849	2.99

n=250

Bootstrap在数据量小的场合nu的估计不准

Estimation of Tangency Portfolios

Background: Estimating $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ and then estimating tangency portfolio

Example: Have Monthly returns from equity markets for 10 countrie

- MSCI Hong Kong, MSCI Singapore, MSCI Brazil, MSCI Argentina, MSCI UK, MSCI Germany, MSCI Canada, MSCI France, MSCI Japan, and SP500
- Monthly returns from Jan 1988 - Jan 2002 ($n = 169$)
- Estimated mean and covariance and tangency portfolio using fixed rate of 1.3%/yr $\mu_f = 1.3 \times 10^{-2}$
- Based on estimated tangency portfolio, have estimated Sharpe Ratio = .3681 $\frac{\mu_R - \mu_f}{\sigma_R} = 0.3681$

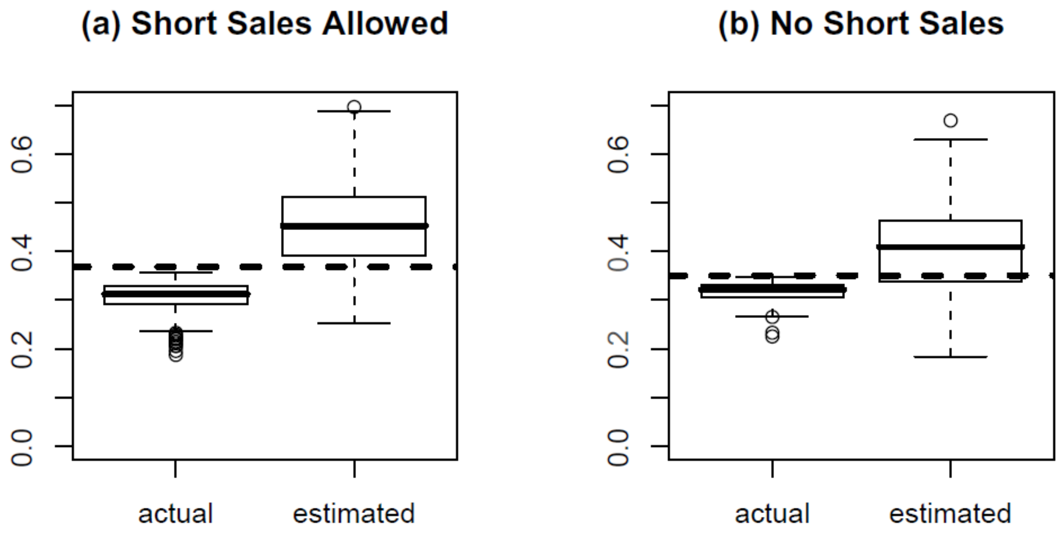
Objective: Assess bias and variability of the Sharpe Ratio estimate

Sharpe Ratio Estimation Assessment: Bootstrap Approach

Approach: Utilize a resampling of $B = 250$ – for $b = 1, 2, \ldots, 250,$

- Select 169 of the months at random (with replacement)
- Estimate $\boldsymbol{\mu}_b^*$ and $\boldsymbol{\Sigma}_b^*$
- Using $\boldsymbol{\mu}_b^*$ and $\boldsymbol{\Sigma}_b^*$, estimate the tangency portfolio and $\hat{w}_b$ associated estimated Sharpe Ratio $\hat{SR}_b^*$
- Using bootstrap estimated tangency portfolio, compute the ("actual") Sharpe Ratio from the original $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$

Results: Boxplots for "short sales" and "no short sales"



General Conclusions/Implications

Other Improvements for Portfolio Est Example

Background. Seems that "variability" in the estimate of the mean and covariance can cause bias and large variance in estimation of tangency portfolio and Sharpe ratio.

Approach: Utilize a lower MSE estimator of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, and in particular might shrink the mean estimates for each country to , i.e.,

mean of $\begin{bmatrix} Y_{1t} \\ Y_{2t} \\ \cdots \\ Y_{10t} \end{bmatrix}$

$$\hat{\mu}_i = \alpha \overline{Y}_{i\cdot} + (1-\alpha)\overline{Y}_{\cdot\cdot}$$

一部分是每个国家自己的,一部分是全球的

where

$$\overline{Y}_{i\cdot} = \frac{1}{169}\sum_{t=1}^{169} Y_{it} = \text{mean of } i^{th} \text{ country}$$

$$\overline{Y}_{\cdot\cdot} = \frac{1}{10}\sum_{i'=1}^{10} \overline{Y}_{i'} = \text{mean of all 10 countries}$$

就是整个bootstrap的均值

Example. Compute bias and variance of $\hat{\mu}_i$ in terms of true means and covariance.

Answer. .

Answer.

$$E[\hat{\mu}_i] = E[\alpha \bar{Y}_{i\cdot} + (1-\alpha)\bar{Y}_{\cdot\cdot}] = \alpha\mu_i + (1-\alpha)\bar{\mu}$$

$$\begin{aligned}\mathrm{Bias}(\hat{\mu}_i) &= E[\hat{\mu}_i] - \mu_i \\ &= (\alpha-1)\mu_i + (1-\alpha)\bar{\mu} \\ &= (1-\alpha)(\bar{\mu}-\mu_i)\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(\hat{\mu}_i) &= \frac{\alpha}{169}\sum_{t=1}^{169} Y_{it} + \frac{(1-\alpha)}{10\times 169}\sum_{t=1}^{169}\sum_{i=1}^{169} Y_{it} \\ &= \frac{1}{169} V^T \Sigma V\end{aligned}$$

Answer.

Bootstrap Assessment of Alternative Estimator

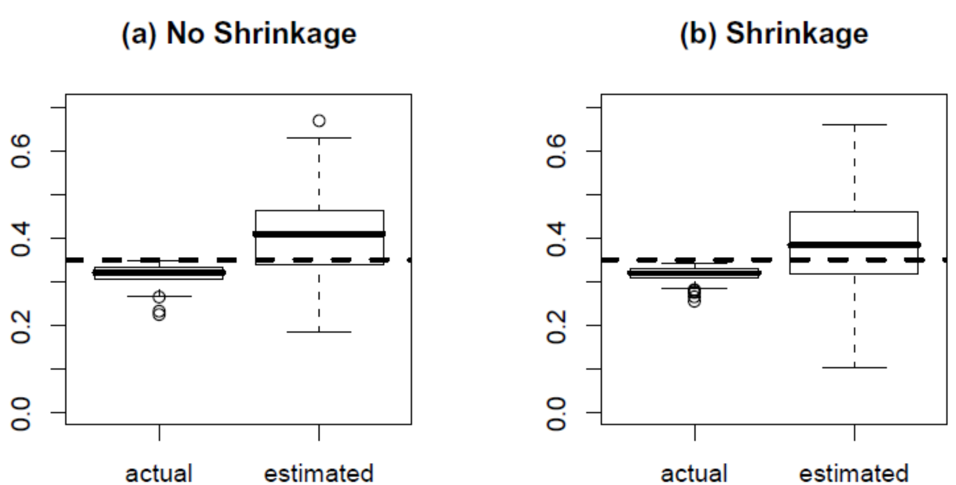
Remark. For estimators

$$\hat{\mu}_i = \alpha \overline{Y}_i + (1-\alpha)\overline{Y}.$$

the effect of α is:

- As α increases, bias decrease, variance increase
- As α decreases, bias increase, variance decrease

Bootstrap Results



shrink配合bootstrap估计更准确

Capital Market Line

Definition. For a portfolio R, the **excess expected return** is

$$\mu_R - \mu_f$$

where μ_f is fixed rate of the return. The equation that governs the relation between the performance and the risk of an "efficient" portfolio, expressed as excess expected return, is given by

$$\mu_R = \mu_f + \frac{\mu_M - \mu_f}{\sigma_M}\sigma_R$$

Note that is equivalent to

$$\frac{\mu_R - \mu_f}{\sigma_R} = \frac{\mu_M - \mu_f}{\sigma_M} .$$

i.e., the Sharpe Ratio . are always the same.

- This was what we showed for mixing risk-free asset with portfolio of risky assets and finding that the best is always to mix the tangency portfolio . with the risk-free asset

Figure Depicting the CML

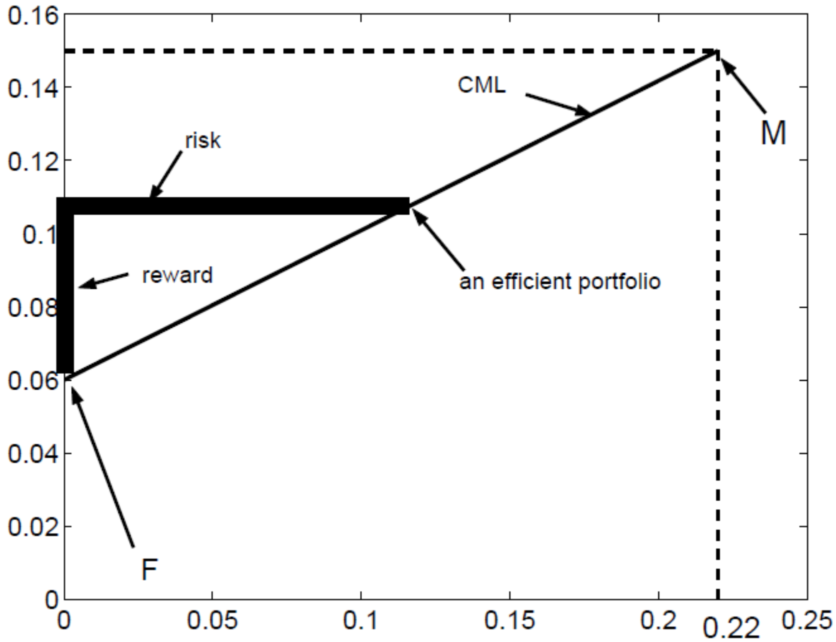


Fig. 16.1. *CML when* $\mu_f = 0.06$, $\mu_M = 0.15$, *and* $\sigma_M = 0.22$. *All efficient portfolios are on the line connecting the risk-free asset (F) and the market portfolio (M). Therefore, the reward-to-risk ratio is the same for all efficient portfolios, including the market portfolio. This fact is illustrated by the thick lines, whose lengths are the risk and reward for a typical efficient portfolio.*

Implications of CML

Recall: Suppose that have parameters of the risk-free asset (μ_f) and efficient market portfolio (μ_M and σ_M). The optimal portfolio in terms of mixing the risk-free asset with the optimal portfolio has two cases:

Achieve Performance μ_R**:** $w = \frac{\mu_R - \mu_f}{\mu_m - \mu_f}$

Achieve Risk σ_R**:** $w = \frac{\sigma_R}{\sigma_w}$

Answer. .

$$R = (1-w)R_M + wR_f$$

两个w相等可以得到夏普比率相等

More General Criteria for Portfolio Selection

Problem. Assuming the CML framework, suppose firm has amount of capital C and desires a portfolio that maximizes the expected return subject to being $100(1-\alpha)\%$ confident that the loss is at most M. Derive an approach to solving this problem (it will involve an optimization). Be explicit on how it relates to the probability distribution of the efficient portfolio.

Answer. .

给定收益率分布计算资产权重

$$P\left[(1-w)R_f + wR_m \le -\frac{M}{C}\right]$$

$$\alpha = P\left[R_m \le \frac{-M/C-(1-w)R_f}{w}\right]$$

$$w = \frac{M/C + R_f}{R_f - \pi_\alpha} \qquad \pi_\alpha = \text{quantile of distribution of R}$$

Other Market Lines

Background. Suppose have a arbitrary risky asset i, and want to know how it relates to another market portfolio

- efficient tangency portfolio
- another portfolio/index such as SP500, NASDAQ, DJIA

Definition. The beta, denoted by β_R of the risky asset portfolio R is defined as

$$\boxed{\beta_R = \frac{\text{Cov}(R, R_M)}{\sigma_M^2}}$$

and the Security Market Line (SML) is defined as

$$\mu_R - \mu_f = \beta_R(\mu_M - \mu_f)$$

Exercise. Show that the above is simply the CML equation if the portfolio is one of the efficient portfolio mixtures. *Note:* The SML is defined for all risky assets.

Answer. . $R = (1-w)R_f + wR_m$ $\quad \beta_R = \frac{\text{Cov}(R, R_m)}{\sigma_m^2} = \frac{w\sigma_m^2}{\sigma_m^2} = w$

Problem. Suppose that have risky asset, i, that is part of risky assets portfolio. Then show that the beta, β_i, satisfies that

$$\mu_i - \mu_f = \beta_i(\mu_M - \mu_f)$$

where R_M is the tangency portfolio with mean μ_M

Answer. .

即使i不在tangency portfolio中公式也成立

$R_p = wR_i + (1-w)R_T$

$\sigma_p(w), \mu_p(w)$ starts at R_i when $w=1$

goes to σ_T, μ_T at $w=0$

slope of tangent line

$$\left.\frac{\frac{\partial \sigma_p(w)}{\partial f}}{\frac{\partial \mu(w)}{\partial w}}\right|_{w=0} = \frac{\mu_i - \mu_f}{\sigma_T}$$

Answer.

$$\mu_p = w\mu_i + (1-w)\mu_T$$

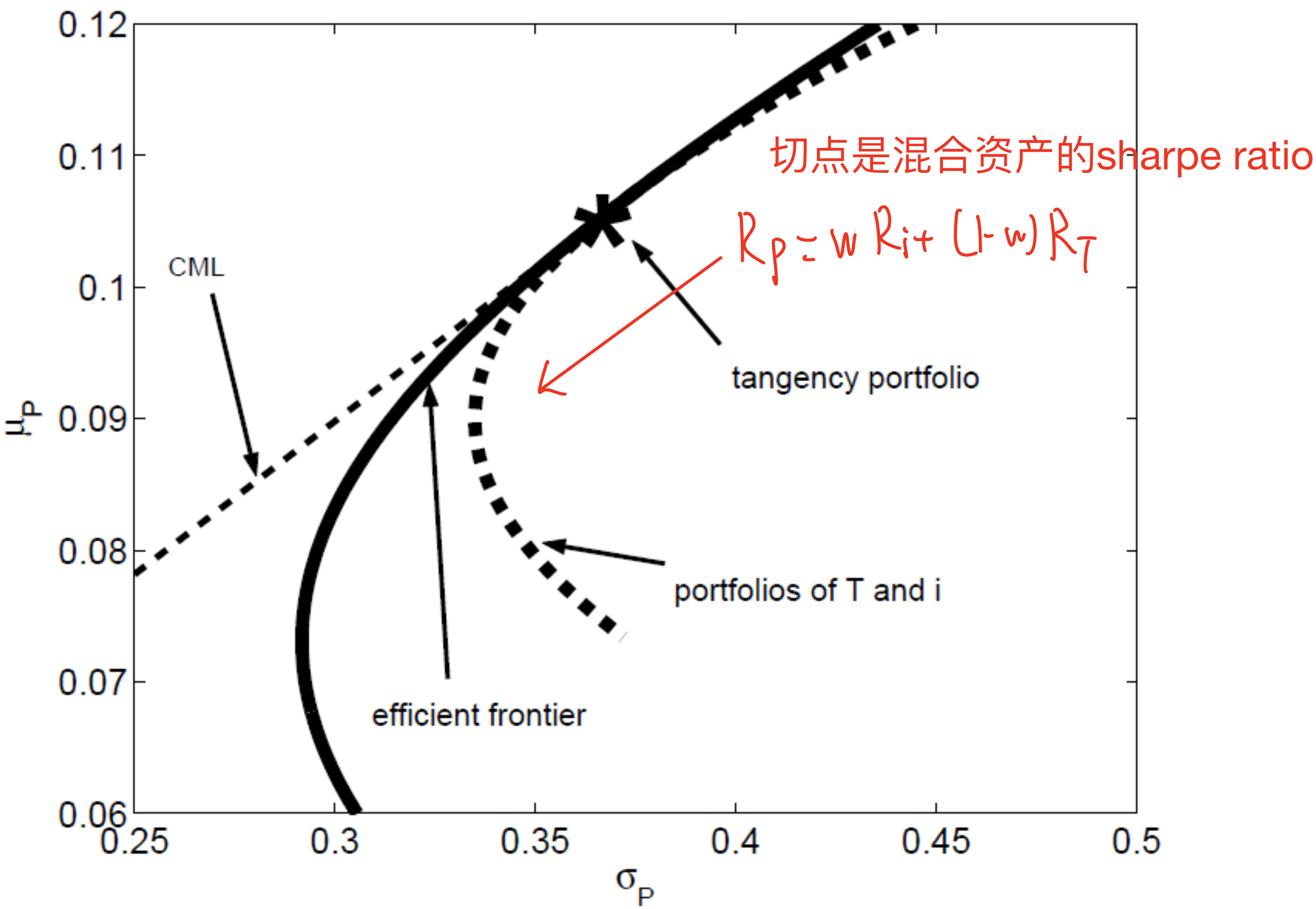
$$\sigma_p = \sqrt{w^2\sigma_i^2 + (1-w)^2\sigma_T^2 + 2w(1-w)\,Cov(R_i, R_T)}$$

$$\frac{d\mu_p}{dw} = \mu_i - \mu_T \qquad \frac{d\sigma_p}{dw}$$

$$\frac{\frac{d\mu_p}{dw}}{\frac{d\sigma_p}{dw}} = \frac{\mu_T - \mu_f}{\sigma_T} \Rightarrow \mu_i - \mu_f = \frac{Cov(R, R_m)}{\sigma_m^2}(\mu_m - \mu_f)$$

Answer.

Background for the SML of Risky Asset



Security Characteristic Line - Linear Prediction

Note: The security market line formula is

$$\mu_R - \mu_f = \beta_R(\mu_M - \mu_f)$$

with

$$\beta_R = \frac{\text{Cov}(R, R_M)}{\sigma_M^2}$$

Related Problem. Derive the optimal linear predictor of $R - \mu_f$ in terms of $R_M - \mu_f$.

Answer. Predict $(R-\mu_f)$ from $(R_m - \mu_f)$

$R-\mu_f = a + \beta(R_m - \mu_f)$

Optimized linear predictor minimize $E[R-\mu_f-(R-\mu_f)]^2$

Security Characteristic Line - Regression

Related Regression Model. Consider asset R_j and the linear regression model of

$$R_{jt} - \mu_f = \alpha_j + \beta_j(R_{Mt} - \mu_f) + \epsilon_{jt}$$

where ϵ_{jt} is mean-zero with variance $\sigma^2_{\epsilon,j}$, and independent of R_{Mt}. Assuming stationarity of the time series, answer the following.

(a) Derive the variance of R_{jt} in terms of model mean/variance of R_{Mt} and noise process ϵ_{jt}.

(b) Derive the covariance between R_{jt} and R_{mt}.

(c) Suppose for two assets $R_{jt}, R_{j't}$, the errors of ϵjt and $\epsilon_{j't}$ are independent of each other. Derive covariance between R_{jt} and $R_{j't}$.

(d) What are the implications of **(c)** in terms of the tangency portfolio?

Answer. .

(a) $Var[R_{jt}] = \sigma_{\epsilon_j}^2 + \beta_j^2 \sigma_m^2$

(b) $Cov[R_{jt}, R_{mt}] = \beta_j \sigma_m^2$

(c) $Cov[R_{jt}, R_{j't}] = \beta_j \beta_{j'} \sigma_m^2$

Implications of Regression Model

Regression Model. Asset R_j and the linear regression model of

$$R_{jt} - \mu_f = \alpha_j + \beta_j(R_{Mt} - \mu_f) + \epsilon_{jt}$$

- Total variance/risk can be decomposed into two terms:
  - Variance $\beta_j^2\sigma_M^2$, referred to as market systematic risk
  - Variance σ_ϵ^2, referred to as non-market risk
- Implications of parameters:
  - $\alpha_j \neq 0$ implies mispricing relative to CAPM - why??
  - $\beta_j >=< 1$ is related to "risk"
    - aggressive, moderate, and not aggressive
- From time series data, estimate of $\hat{\beta}_j$ is given by

$$\boxed{\hat{\beta}_j = \frac{\sum_{t=1}^{n}(R_{jt} - \overline{R_{j\cdot}})(R_{Mt} - \overline{R_{M\cdot}})}{\sum_{t=1}^{n}(R_{Mt} - \overline{R_{M\cdot}})^2}.}$$

Table of betas relative to S&P500 - Book

Table 16.1. Selected stocks and in which industries they are. Betas are given for each stock (Stock's β) and its industry (Ind's β). Betas taken from the Salomon, Smith, Barney website between February 27 and March 5, 2001.

Stock (symbol)	Industry	Stock's β	Ind's β
Celanese (CZ)	Synthetics	0.13	0.86
General Mills (GIS)	Food—major diversif	0.29	0.39
Kellogg (K)	Food—major, diversif	0.30	0.39
Proctor & Gamble (PG)	Cleaning prod	0.35	0.40
Exxon-Mobil (XOM)	Oil/gas	0.39	0.56
7-Eleven (SE)	Grocery stores	0.55	0.38
Merck (Mrk)	Major drug manuf	0.56	0.62
McDonalds (MCD)	Restaurants	0.71	0.63
McGraw-Hill (MHP)	Pub—books	0.87	0.77
Ford (F)	Auto	0.89	1.00
Aetna (AET)	Health care plans	1.11	0.98
General Motors (GM)	Major auto manuf	1.11	1.09
AT&T (T)	Long dist carrier	1.19	1.34
General Electric (GE)	Conglomerates	1.22	0.99
Genentech (DNA)	Biotech	1.43	0.69
Microsoft (MSFT)	Software applic.	1.77	1.72
Cree (Cree)	Semicond equip	2.16	2.30
Amazon (AMZN)	Net soft & serv	2.99	2.46
Doubleclick (Dclk)	Net soft & serv	4.06	2.46

Table of betas - ABG-Analytics Webpage

echnolog × | Stock Betas List for S ×

www.abg-analytics.com/stock-betas.shtml#.UWw6pbW85rM

, place your bookmarks here on the bookmarks bar. Import bookmarks now...

List of US Stock Betas for Large-Cap Stocks

We update the report below at the end of each week. Links on individual stocks take you to detail on each. Sign up to receive **Update Notifications.**

We provide two stock beta estimates for nearly 100 US large-cap stocks. The first is a long-term average over 250 weeks. The second and more novel beta estimate is a time-varying beta which reflects recent market conditions and stock price behavior.

Changes over time in the characteristics of a company which affect the way the its stock price covaries with the overall market become reflected in the time-varying beta estimates. As a result, long-term and time-varying betas can differ.

Stock Beta Estimates as of

Ticker	Company	Beta Estimate (long-term average)	Beta Estimate (current time-varying estimate)
AAPL	Apple Inc	1.16	1.08
ABT	Abbott Laboratories	0.41	0.41
ACN	Accenture Ltd	0.85	1.14
AEP	American Electric Power	0.62	0.58
ALL	Allstate Corporation	1.26	0.98
AMGN	Amgen Inc	0.53	0.53
AMZN	Amazon.Com Inc	1.09	0.95
APA	Apache Corp	1.40	1.55
APC	Anadarko Petroleum Corp	1.50	1.64
AXP	American Express Co	1.62	1.17
BA	Boeing Co	1.26	1.13
BAC	Bank Of America Corp	2.39	1.95
BAX	Baxter International Inc	0.51	0.82
BHI	Baker Hughes Inc	1.62	1.62
BK	The Bank of New York Mellon Corporation	1.41	1.38
BMY	Bristol-Myers Squibb Co	0.59	0.54
BRKB	Berkshire Hathaway Cl B	0.74	0.71
C	Citigroup	2.67	2.01
CAT	Caterpillar Inc	1.69	1.51
CL	Colgate-Palmolive Co	0.40	0.40
CMCSA	Comcast Corp Cl A	1.08	1.10
COF	Capital One Financial Cp	1.89	1.49

Example Problem from Ruppert Suppose that the risk-free interest rate is 0.023, that the expected return on the market portfolio is $\mu_M = 0.10$, and that the volatility of the market portfolio is $\sigma_M = 0.12$.

(a) What is the expected return on an efficient portfolio with $\sigma_R = 0.05$?

夏普率相等

(b) Stock A returns have a covariance of 0.004 with market returns. What is the beta of Stock A?

(c) Stock B has beta equal to 1.5 and $\sigma_\epsilon = 0.08$. Stock C has beta equal to 1.8 and $\sigma_\epsilon = 0.10$. What is the expected return of a portfolio that is one-half Stock B and one-half Stock C?

(d) For part **(c)** , what is the volatility of a portfolio that is one-half Stock B and one-half Stock C? Assume that the ϵ's of Stocks B and C are independent.

Answer.

(a) $$\frac{\mu_R-\mu_f}{\sigma_R}=\frac{\mu_m-\mu_f}{\sigma_m} \Rightarrow \sigma_R=0.055$$

(b) $$\beta_A=\frac{\mathrm{Cov}[R_A, R_m]}{\sigma_m^2}=\frac{0.004}{(0.12)^2}=0.26$$

(c) $$R_B-\mu_f=\beta_B(R_m-\mu_f)+\epsilon_B$$

$$R_C-\mu_f=\beta_C(R_m-\mu_f)+\epsilon_C$$

$$E\left[\frac{1}{2}R_B+\frac{1}{2}R_C\right]=\frac{1}{2}[2\mu_f+2.3(\mu_m-\mu_f)]=0.15$$

(d) $$\mathrm{VaR}\left[\frac{1}{2}(\beta_B R_m+\epsilon_B+\beta_C R_m+\epsilon_C)\right]$$

$$=\frac{1}{2^2}\left[(\beta_B+\beta_C)^2\sigma_m^2+\sigma_{\epsilon_B}^2+\sigma_{\epsilon_C}^2\right]$$

Answer.

Factor Models

CAPM Regression Model

$$R_{jt} - \mu_f = \beta_{oj} + \beta_{1j}(R_{Mt} - \mu_f) + \epsilon_{jt}$$

- Excess return of asset can be predicted/modeled as linear regression on excess return of the market

General Regression Model

$$R_{jt} - \mu_f = \beta_{oj} + \beta_{1j}F_{1t} + \cdots + \beta_{pj}F_{pt} + \epsilon_{jt}$$

通过不同因子来拟合收益率

- $\{F_{jt}\}_{j=1}^{p}$ are factors - examples are:
  - excess returns on market portfolio (e.g., CAPM)
  - inflation rate
  - interest rate spreads
  - returns on specific stock portfolios (e.g., high book equity to market equity ratio - BE/ME)
  - difference between returns on two portfolios (e.g., high BE/ME to low BE/ME)

Fama-French 3-factor Model

- Example: Fama & French (1993)
  - Common risk factors in the returns on stocks and bonds
  - Multi-factor model: For j^{th} risky asset, have model

$$R_{jt}-R_{ft} = \beta_{j0}+\beta_{j1}\cdot(R_{Mt}-R_{ft})+\beta_{j2}\cdot SMB_t+\beta_{j3}\cdot HML_t+\epsilon_{jt}$$

Here have times $t = 1, 2, \ldots T$.

  - SMB: "small (market capitalization) minus big"
    - excess returns of small caps over big caps
  - HML: "high (book-to-price ratio) minus low"
    - excess returns of value stocks over growth stocks
  - $\{\epsilon_{jt}\}_{t=1}^{T}$

三因子模型
- 小市值
- 高PE
- 市场收益率

Fama & French Example 1

- Monthly data from 1926 to 2002
- Factors – Mkt-RF, SMB, HML, RF
- Average value weighted returns for industries
  - Manufacturing, Utilities, Retail, Financials, Other

```
> industry <- read.table("C:/data/ff_industry.txt",
    skip=12, col.names=c("date", "manuf",
    "utils", "shops", "money", "other"))
> factors <- read.table("C:/data/ff_factors.txt",
    skip=4, col.names=c("date", "mkt", "smb",
    "hml", "rf"))
> mkt <- factors$mkt
> smb <- factors$smb
> hml <- factors$hml
## Compute excess rate of return
> utils <- industry$utils - factors$rf
```

```
## Use the lm() function
> fitted.model <- lm(utils ~ mkt + smb + hml)
> summary(fitted.model)
Coefficients:
             Estimate Std.Error t value Pr(>|t|)
(Intercept) -0.05456   0.11569  -0.472 0.637347
 mkt         0.80621   0.02227  36.195  < 2e-16
 smb        -0.13725   0.03574  -3.840 0.000132
 hml         0.29893   0.03224   9.271  < 2e-16
---
Residual standard error:
  3.468 on 914 degrees of freedom
Multiple R-Squared: 0.6451
Adjusted R-squared: 0.644
F-statistic: 553.8 on 3 and 914 DF
p-value: < 2.2e-16
```

```
## Carry out an F-test
> h0 <- lm(utils ~ mkt)
> h0a <- lm(utils ~ mkt + smb + hml)
> anova(h0, h0a)
Analysis of Variance Table
Model 1: utils ~ mkt
Model 2: utils ~ mkt + smb + hml
  Res.Df     RSS  Df Sum of Sq      F    Pr(>F)
1    916 12185.2
2    914 10994.4   2    1190.7 49.495 < 2.2e-16
```

```
## Confidence Intervals
> conf <- confint(fitted.model, level=0.95)
> conf
                  2.5 %      97.5 %
(Intercept) -0.2816054  0.17249336
mkt          0.7624943  0.84992301
smb         -0.2073969 -0.06709673
hml          0.2356505  0.36220986
```

Fama & French Example 2

- CRSPmon in R's Ecdat package – GE, IBM, Mobil Monthly data from Jan69-Dec98
- Factors – Mkt.RF, SMB, HML, RF from Kenneth French website

```
http://mba.tuck.dartmouth.edu/pages/faculty/ken.french
-- Data Library
-- Fama/French North American Factors
```

Fama & French Example 2

Results from R

```
Call:
lm(formula = cbind(ge, ibm, mobil) ~ Mkt.RF + SMB + HML)

Coefficients:
             ge       ibm      mobil
(Intercept)   0.3443   0.1460   0.1635
Mkt.RF        1.1407   0.8114   0.9867
SMB          -0.3719  -0.3125  -0.3753
HML           0.0095  -0.2983   0.3725
```

Correlation of Residuals

```
> cor_res = cor(fit$residuals)
> cor_res
```

```
             ge       ibm    mobil
 ge     1.000000  0.070824 -0.25401
 ibm    0.070824  1.000000 -0.10153
 mobil -0.254012 -0.101532  1.00000
```

Implications

Factor Models for Multiple Risky Assets

Background: Suppose that have n assets with n different regressions – can write the n models in matrix form as

$$\mathbf{R}_t = \boldsymbol{\beta}_o + \boldsymbol{\beta}^\top \mathbf{F}_t + \boldsymbol{\epsilon}_t$$

where

每个标的的因子值

$$\mathbf{R}_t = \begin{bmatrix} R_{1t} - \mu_f \\ R_{2t} - \mu_f \\ \vdots \\ R_{nt} - \mu_f \end{bmatrix}, \quad \boldsymbol{\epsilon}_t = \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \\ \vdots \\ \epsilon_{nt} \end{bmatrix}, \quad \mathbf{F}_t = \begin{bmatrix} F_{1t} \\ F_{2t} \\ \vdots \\ F_{pt} \end{bmatrix}$$

$$\boldsymbol{\beta}_o = \begin{bmatrix} \beta_{o1} \\ \beta_{o2} \\ \vdots \\ \beta_{on} \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_{11} & \cdots & \beta_{1j} & \cdots & \beta_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \beta_{p1} & \cdots & \beta_{pj} & \cdots & \beta_{pn} \end{bmatrix}$$

Factor Models - Covariance Estimation

Background: With previous model for n assets with n different regressions – matrix form is

$$\mathbf{R}_t = \boldsymbol{\beta}_o + \boldsymbol{\beta}^\top \mathbf{F}_t + \boldsymbol{\epsilon}_t$$

- Covariance model for errors of

$$\boldsymbol{\Sigma}_\epsilon = \text{diag}[\sigma^2_{\epsilon 1}, \sigma^2_{\epsilon 2}, \ldots, \sigma^2_{\epsilon n}]$$

- Covariance model for $\mathbf{R}_t$ is given by

$$\boldsymbol{\Sigma}_\mathbf{R} = \boldsymbol{\beta}^\top \boldsymbol{\Sigma}_\mathbf{F} \boldsymbol{\beta} + \boldsymbol{\Sigma}_\epsilon$$

i.e.,

$$\boxed{\begin{aligned} \text{Var}(R_j) &= \boldsymbol{\beta}_j^\top \boldsymbol{\Sigma}_\mathbf{F} \boldsymbol{\beta}_j + \sigma^2_{\epsilon j} \\ \text{Cov}(R_j, R_{j'}) &= \boldsymbol{\beta}_j^\top \boldsymbol{\Sigma}_\mathbf{F} \boldsymbol{\beta}_{j'} \end{aligned}}$$

Critical Assumptions.

Factor Models - Covariance Estimation

Background: Factors model of $\mathbf{\Sigma_R} = \boldsymbol{\beta}^\top \mathbf{\Sigma_F} \boldsymbol{\beta} + \mathbf{\Sigma}_\epsilon$.

Alternative Model/Approach for Estimating Covariance

Step 1: For $j = 1, 2, \ldots, n$ do regressions and generate coefficients estimates $\{\hat{\beta}_{ij}\}_{i=0}^{p}$ along with $\hat{\sigma}^2_{\epsilon j}$.

Step 2: Estimate $\Sigma_{\mathbf{F}}$ from time series data for factors.

Step 3: Use above equations to estimate final covariance matrix

Remark. Possible accuracy improvements due to

Method	# Params	$n = 200,\ p = 5$
Estimate $\mathbf{\Sigma}_R$ directly	.	
Estimate $\mathbf{\Sigma}_R$ via factors model	.	

Implications/Issues

Fama & French Example 2 - revisited

- CRSPmon in R's Ecdat package – GE, IBM, Mobil Monthly data from Jan69-Dec98
- Factors – Mkt.RF, SMB, HML, RF from Kenneth French website

The estimate of Σ_F is the sample covariance matrix of the

	Mkt.RF	SMB	HML
Mkt.RF	21.1507	4.2326	-5.1045
SMB	4.2326	8.1811	-1.0760
HML	-5.1045	-1.0760	7.1797

The estimate of β is the matrix of regression coefficients (with cepts):

	Mkt.RF	SMB	HML
ge	1.14071	-0.37193	0.009503
ibm	0.81145	-0.31250	-0.298302
mobil	0.98672	-0.37530	0.372520

The estimate of Σ_ϵ is the diagonal matrix of residual error MS

	[,1]	[,2]	[,3]
[1,]	16.077	0.000	0.000
[2,]	0.000	31.263	0.000
[3,]	0.000	0.000	27.432

Therefore, the estimate of $\beta^{\mathsf{T}}\Sigma_F\beta$ is

	ge	ibm	mobil
ge	24.960	19.303	19.544
ibm	19.303	15.488	14.467
mobil	19.544	14.467	16.155

and the estimate of $\beta^{\mathsf{T}}\Sigma_F\beta + \Sigma_\epsilon$ is

	ge	ibm	mobil
ge	41.036	19.303	19.544
ibm	19.303	46.752	14.467
mobil	19.544	14.467	43.587

For comparison, the sample covariance matrix of t

	ge	ibm	mobil
ge	40.902	20.878	14.255
ibm	20.878	46.491	11.518
mobil	14.255	11.518	43.357

Statistical Factor Models - Overview

Basic model is

$$\mathbf{R}_t = \boldsymbol{\beta}_o + \boldsymbol{\beta}^\top \mathbf{F}_t + \boldsymbol{\epsilon}_t$$

Model Assumptions: The factors are not observable (latent) - motivation is dimension reduction + possible interpretability

对因子协方差进行PCA降维，
提取有意义的因子

$$\boldsymbol{\Sigma}_{\mathbf{R}} = \boldsymbol{\beta}^\top \boldsymbol{\Sigma}_{\mathbf{F}} \boldsymbol{\beta} + \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}$$

- For identifiability, need to assume something for $\boldsymbol{\Sigma}_{\mathbf{F}}$ and $\boldsymbol{\beta}$ – standard to assume that $\boldsymbol{\Sigma} = \mathbf{I}$ and the model becomes

$$\boldsymbol{\Sigma}_{\mathbf{R}} = \boldsymbol{\beta}^\top \boldsymbol{\beta} + \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}$$

- $\boldsymbol{\beta}$ is still non-identifiable since
  - Some standard ways to induce identifability (e.g., varimax, cf. Ruppert)
- Can use R program `factanal` (cf. Ruppert)