SOLUTIONS TO STATS 509 PROBLEM SET 2

1.

(a). The return has distribution $R \sim \text{DExp}(\mu, \lambda)$. We know that minus relative VaR at level q is equal to the q-quantile of R. Thus

$$q = F_R(-\widetilde{\text{VaR}}).$$

Since $q < 0.5$ so that $-\widetilde{VaR}_q < \mu$ (since the distribution of R is symmetric around its mean μ). Then we have need to solve

$$q = \frac{1}{2}\mathsf{exp}\left(-\lambda\left|-\widetilde{\text{VaR}}_q - \mu\right|\right) = \frac{1}{2}\mathsf{exp}\left(\lambda\left(-\mu - \widetilde{VaR}_q\right)\right),$$

resulting in $\widetilde{VaR}_q = -\mu - \lambda^{-1}\log(2q)$.

(b). Note that $2q < 1$ so that $\log p < 0$ and $-\mu - \lambda^{-1}\log(2q) > -\mu$, thus

$$\begin{aligned}
\Theta_q(x) &= \text{P}(X \le x | X > \widetilde{VaR}_q) \\
&= \frac{\text{P}(\widetilde{VaR}_q < X \le x)}{\text{P}(X > \widetilde{VaR}_q)}, \quad x > \widetilde{VaR}_q \\
&= \frac{1 - \frac{1}{2}e^{-\lambda|x+\mu|} - F_R(\widetilde{VaR}_q)}{q}, \quad x > \widetilde{VaR}_q \\
&= \frac{1 - \frac{1}{2}e^{-\lambda|x+\mu|} - (1-q)}{q}, \quad x > \widetilde{VaR}_q \\
&= 1 - \frac{\frac{1}{2}e^{-\lambda|x+\mu|}}{q}, \quad x > \widetilde{VaR}_q
\end{aligned}$$

So

$$\begin{aligned}
\text{E}S_q &= \frac{1}{q}\int_{\widetilde{VaR}_q}^{\infty} xf(x)dx \\
&= \frac{1}{q}\int_{\widetilde{VaR}_q}^{\infty} \frac{\lambda x}{2}e^{-\lambda(x+\mu)}dx \\
&= \frac{1}{q}\int_{-\lambda^{-1}\log(2q)}^{\infty} \frac{\lambda(y-\mu)}{2}e^{-\lambda y}dy \\
&= \frac{1}{q}\left(\frac{\mu}{2}e^{-\lambda y}\Big|_{-\lambda^{-1}\log(2q)}^{\infty}\right) + \frac{1}{q}\int_{-\lambda^{-1}\log(2q)}^{\infty} \frac{\lambda y}{2}e^{-\lambda y}dy \\
&= -\mu - \frac{y}{2q}e^{-\lambda y}\Big|_{-\lambda^{-1}\log(2q)}^{\infty} + \int_{-\lambda^{-1}\log(2q)}^{\infty} \frac{1}{2q}e^{-\lambda y}dy \\
&= -\mu - \frac{\log(2q)}{\lambda} + \frac{1}{\lambda} \\
&= \widetilde{VaR}_q + \frac{1}{\lambda}
\end{aligned}$$

(c). The codes and results are below:

```
> Xday = read.csv("Nasdaq_daily_Jan1_2019-Dec31_2021.csv",header=TRUE)
> NASDAQday <- Xday$Adj.Close
> length(NASDAQday)
[1] 756
> returns <- NASDAQday[2:756] / NASDAQday[1:755] - 1
> # estimate mu and lambda
> mu <- mean(returns)
> lambda <- sqrt(2 / var(returns))
> q = 0.01
> # VaR
> rVaR = - mu - lambda^{-1} * log(2 * q)
> VaR = 10000000 * rVaR
> VaR
[1] 443278.8
> # Expected shortfall
> ES = (- mu - log(2 * q) / lambda + 1 / lambda) * 10000000
> ES
[1] 553367.8
```

For VaR, we need $443,279 amount of capital to cover a loss in a portfolio. The expected shortfall is a significantly higher value of $553,368 which indicates a reasonably high risk.

(d).

```
> # estimate mu and sigma
> mu = mean(returns)
> sigma = sqrt(var(returns))
> # VaR
> VaR = - 10000000 * qnorm(q, mean=mu, sd=sigma)
> VaR
[1] 349579.6
> # ES
> pi_z = qnorm(1 - q)
> ES = (- mu + sigma/(sqrt(2*pi)*q))* exp(- pi_z ^ 2 / 2) * 10000000
> ES
[1] 414103.2
```

With the normal model, for VaR, we need $349,580 amount of capital to cover a loss in a portfolio. The expected shortfall is a significantly higher value of $414,103 which indicates a moderate risk that is significantly less than the double exponential model. But based on results from Homework 1 showing that the kurtosis of this data being 12.38, this is clearly not normally distributed and is going be closer to double exponential so that is likely to be more accurate. But given that the kurtosis of double exponential, it suggests that even the double exponential model will be significantly off.

2.

(a). Note that for $x > \mu$,

$$\begin{aligned} 1 - F_X(x) &= P(X > x) \\ &= P(X > x | X > \mu) P(X > \mu) \\ &= 0.1 \left(1 + \frac{\xi(x - \mu)}{\sigma} \right)^{-\frac{1}{\xi}} \end{aligned}$$

Solving

$$q = 1 - F_X(\widetilde{VaR}) = 0.1 \left(1 + \frac{\xi(\widetilde{VaR} - \mu)}{\sigma} \right)^{-\frac{1}{\xi}}$$

gives

$$\widetilde{VaR} = \mu + \frac{\sigma \left((10q)^{-\xi} - 1 \right)}{\xi}.$$

(b). For $x \geq \widetilde{VaR}$

$$
\begin{aligned}
P(X \leq x | X \geq \widetilde{VaR}) &= \frac{P(\widetilde{VaR} \leq X \leq x)}{P(X \geq \widetilde{VaR})} \\
&= \frac{P(\widetilde{VaR} \leq X \leq x | X \geq \mu)}{P(X \geq \widetilde{VaR} | X \geq \mu)} \\
&= \frac{P(X \leq x | X \geq \mu) - P(\widetilde{VaR} \leq x | X \geq \mu)}{P(X \geq \widetilde{VaR} | X \geq \mu)} \\
&= \frac{1 - \left(1 + \frac{\xi(x-\mu)}{\sigma}\right)^{-\frac{1}{\xi}} - \left(1 - \left(1 + \frac{\xi(\widetilde{VaR}-\mu)}{\sigma}\right)^{-\frac{1}{\xi}}\right)}{\left(1 + \frac{\xi(\widetilde{VaR}-\mu)}{\sigma}\right)^{-\frac{1}{\xi}}} \\
&= \frac{\left(1 + \frac{\xi(\widetilde{VaR}-\mu)}{\sigma}\right)^{-\frac{1}{\xi}} - \left(1 + \frac{\xi(x-\mu)}{\sigma}\right)^{-\frac{1}{\xi}}}{\left(1 + \frac{\xi(\widetilde{VaR}-\mu)}{\sigma}\right)^{-\frac{1}{\xi}}} \\
&= 1 - \left(\frac{1 + \frac{\xi(x-\mu)}{\sigma}}{1 + \frac{\xi(\widetilde{VaR}-\mu)}{\sigma}}\right)^{-\frac{1}{\xi}} \\
&= 1 - \left(\frac{1 + \frac{\xi(x-\widetilde{VaR}+\widetilde{VaR}-\mu)}{\sigma}}{1 + \frac{\xi(\widetilde{VaR}-\mu)}{\sigma}}\right)^{-\frac{1}{\xi}} \\
&= 1 - \left(1 + \frac{\frac{\xi(x-\widetilde{VaR})}{\sigma}}{1 + \frac{\xi(\widetilde{VaR}-\mu)}{\sigma}}\right)^{-\frac{1}{\xi}} \\
&= 1 - \left(1 + \frac{\xi(x-\widetilde{VaR})}{\sigma + \xi(\widetilde{VaR}-\mu)}\right)^{-\frac{1}{\xi}}
\end{aligned}
$$

So that the shortfall distribution is still a Generalized Pareto with (ξ', σ', μ'):

$$
\begin{aligned}
\xi' &= \xi \\
\sigma' &= \sigma + \xi(\widetilde{VaR} - \mu) \\
\mu' &= \widetilde{VaR}.
\end{aligned}
$$

(c). From (b), the expected shortfall is

$$
ES = \widetilde{VaR} + \frac{\sigma + \xi(\widetilde{VaR} - \mu)}{1 - \xi}
$$

where $\widetilde{VaR}$ is given in **(a)**.