# STATS 509 Winter 2022, Solution to Homework 6

## 1

### (a)

The problem to solve here is

$$\min_{\mathbf{w}} \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w} \quad \text{such that } \mathbf{w}^\top \mathbf{1} = 1, \mathbf{w}^\top \boldsymbol{\mu} = \mu_{\text{targ}}\,, \mathbf{w} \geq -.1 * 1, -\mathbf{w} \geq -.5 * 1$$

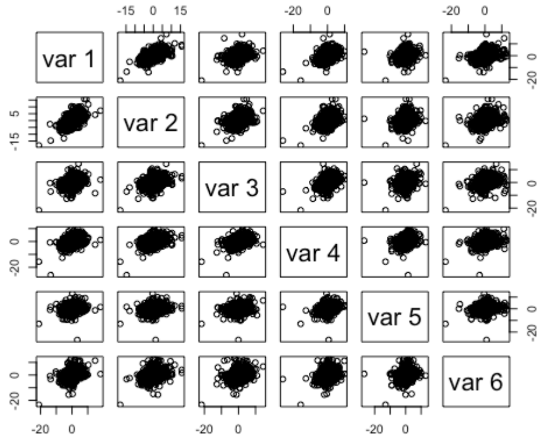
where $\mu_{\text{targ}}$ is the target mean. With the minimum return being .409 and the maximum return being .0799, we carried out the plot of the reward-risk space (i.e., plot of mean return vs. standard deviation of return, with units of percent per day) with target mean ranging from .04 to .09. Carrying this out in $R$, similar to problem in class and using quadratic programming, generated the frontier plot, and we show the efficient frontier with solid line, the minimum variance portfolio with a " +", and the tangency portfolio with "*", and showing the location of the 6 individual stocks. Note that the stocks reward/risk values are all interior to the frontier plot, as is expected. The tangency portfolio corresponded to the weights, mean, and standard deviation shown in the table below. Note that the tangency portfolio short-sells both the GM and Ford stock.

| Item | Value |
|---|---|
| Weight - GM | -0.092 |
| Weight - F | -0.003 |
| Weight - CAT | 0.336 |
| Weight - UTX | 0.385 |
| Weight - MRK | 0.320 |
| Weight - IBM | 0.055 |
| Mean $\mu_T$ | 0.073 |
| St Dev $\sigma_T$ | 1.36 |

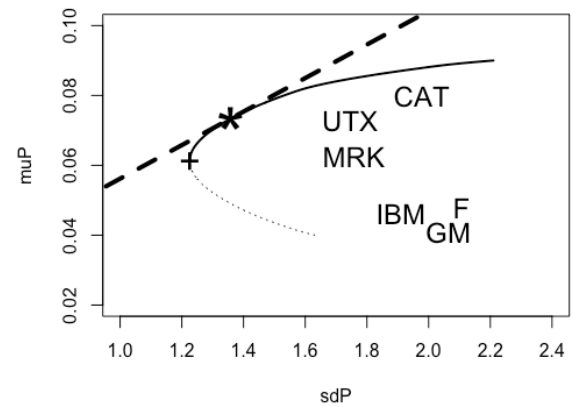
Table 1: Caption

R codes:

```
> dat = read.csv("Stock_Bond.csv", header = T)
> prices = cbind(dat$GM_AC, dat$F_AC, dat$CAT_AC, dat$UTX_AC,
+                   + dat$MRK_AC, dat$IBM_AC)
> n = dim(prices)[1]
> returns = 100 * (prices[2:n, ] / prices[1:(n-1), ] - 1)
> pairs(returns)
```




Scatter Plots




Plot of Reward-Risk plot with tangency portfolio min var portfolio, and efficient frontier.

```
> mean_vect = colMeans(returns)
> cov_mat = cov(returns)
> sd_vect = sqrt(diag(cov_mat))
> min(mean_vect)
[1] 0.0408973
> max(mean_vect)
[1] 0.07989876
> Amat = cbind(rep(1,6),mean_vect,diag(1,nrow=6),diag(1,nrow=6)) # set the constraints matrix
> muP = seq(.04,.09,length=300) # set of 300 possible target values
> # for the expect portfolio return
> sdP = muP # set up storage for std devs of portfolio returns
> weights = matrix(0,nrow=300,ncol=6) # storage for portfolio weights
> for (i in 1:length(muP)){ # find the optimal portfolios for each target expected return
+   bvec = c(1,muP[i],rep(-.1,6),rep(-.5,6)) # constraint vector
+   result = solve.QP(Dmat=2*cov_mat,dvec=rep(0,6),Amat=Amat,bvec=bvec,meq=2)
+   sdP[i] = sqrt(result$value)
+   weights[i,] = result$solution
+ }
> plot(sdP,muP,type="l",xlim=c(1.0,2.4),ylim=c(.02,.10),lty=3) # plot
> # the efficient frontier (and inefficient portfolios
> # below the min var portfolio)
> mufree = 3/365 # input value of risk-free interest rate
> points(0,mufree,cex=4,pch="*") # show risk-free asset
> sharpe =( muP-mufree)/sdP # compute Sharpes ratios
> ind = (sharpe == max(sharpe)) # Find maximum Sharpe's ratio
> options(digits=3)
> weights[ind,] # print the weights of the tangency portfolio
```

```
[1] -0.09214 -0.00325  0.33646  0.38453  0.31961  0.05478
> muT = muP[ind]
> sigmaT = sdP[ind]
> c(muT,sigmaT)
[1] 0.0734 1.3598
> lines(c(0,2),mufree+c(0,2)*(muP[ind]-mufree)/sdP[ind],lwd=4,lty=2)
> # show line of optimal portfolios
> points(sdP[ind],muP[ind],cex=4,pch="*") # show tangency portfolio
> ind2 = (sdP == min(sdP)) # find the minimum variance portfolio
> points(sdP[ind2],muP[ind2],cex=2,pch="+") # show min var portfolio
> ind3 = (muP > muP[ind2])
> lines(sdP[ind3],muP[ind3],type="l",lwd=2) # plot the efficient frontier
> text(sd_vect[1],mean_vect[1],"GM",cex=1.5)
> text(sd_vect[2],mean_vect[2],"F",cex=1.5)
> text(sd_vect[3],mean_vect[3],"CAT",cex=1.5)
> text(sd_vect[4],mean_vect[4],"UTX",cex=1.5)
> text(sd_vect[5],mean_vect[5],"MRK",cex=1.5)
> text(sd_vect[6],mean_vect[6],"IBM",cex=1.5)
```

## (b)

Now the optimum portfolio is to choose portfolio consisting of the tangency portfolio and the fixed return, with the appropriate weight so as to have expected return of .07. Letting $w$ be proportion for tangency portfolio, we have

$$E\left(R_P\right) = w\mu_T + (1-w) * \frac{3}{365}$$

so that setting the above equal to .07, we have

$$w = \frac{.07 - \frac{3}{365}}{\mu_T - \frac{3}{365}} = \frac{.07 - \frac{3}{365}}{0.073 - \frac{3}{365}} = .954$$

and the standard deviation is

$$\sigma_P = .954 * 1.36 = 1.30$$

and the weights for the stocks are .954 times the weights for the tangency porfolio. So final portfolio is shown below, and the R-code for generating this is shown after that.

| Item | Value |
|---|---|
| Weight - GM | $-0.088$ |
| Weight - F | $-0.003$ |
| Weight - CAT | 0.321 |
| Weight - UTX | 0.367 |
| Weight - MRK | 0.305 |
| Weight - IBM | 0.052 |
| Weight - Risk-Free | .046 |
| Mean $\mu_P$ | 0.07 |
| St Dev $\sigma_P$ | 1.30 |

R codes:

```
> w = (.07-3/365)/(0.073-3/365)
> w
[1] 0.954
> sigma_P = .954*1.36
> sigma_P
[1] 1.3
> weights[ind,]*.954
[1] -0.0879 -0.0031 0.3210 0.3668 0.3049 0.0523
```

### (c)

Now the equations that must be satisfied are that for each of the bonds,

$$\mu_B - \frac{3}{365} = \beta_B \left( \mu_T - \frac{3}{365} \right) = \beta_B \left( .0734 - \frac{3}{365} \right)$$

where $\mu_B$ is the expected return of the individual bond, and $\beta_B$ is given as

$$\beta_B = \frac{\text{Cov}\left(R_B, R_T\right)}{\sigma_T^2} = \frac{\left[w_T^\top \hat{\Sigma}_R\right]_i}{\sigma_T^2}$$

where $i$ is the index corresponding to the bond and $\hat{\Sigma}$ is the estimated covariance matrix and $w_T$ are the weights corresponding to the tangent portfolio.

### (d)

To check this we need to verify that the above equation is true-we compute the LHS of (1) as the excess return of all the 2 bonds - that is given in $R$-commands below. We then computed the RHS of the (1) as

$$\text{Cov}\left(R, w_T^\top R\right) = \hat{\Sigma}_R w_T$$

and that is shown in R-commands as well. As we can see they do match as suggested by the theory presented in class.
R codes:

```
> # Problem 2 - (b)
> wT = weights[ind,]
> Cov_R_RT = cov_mat%*%wT
> # LHS of SML eqn
> (mean_vect - 3/365)
[1] 0.0327 0.0396 0.0717 0.0646 0.0542 0.0379
> # RHS of SML eqn
> ((muT-3/365)/sigmaT^2)*t(Cov_R_RT)
[,1] [,2] [,3] [,4] [,5] [,6]
[1,] 0.0326 0.0396 0.0717 0.0646 0.0542 0.0379
```

## 2

### Problem 6 on page 513 in Ruppert/Matteson.

#### (a)

Now

$$\beta_A = \frac{\text{Cov}\left(R_A, R_M\right)}{\sigma_M^2} = \frac{.0165}{(.11)^2} = 1.36$$

#### (b)

The expected return on $R_A$ is given as

$$\mu_A = \mu_f + \beta_A\left(\mu_M - \mu_f\right) = .04 + 1.36 * (.12 - .04) = 0.149.$$

#### (c)

Now variance due to the market is given as

$$(1.36)^2 * (.11)^2 = .0224$$

and the total variance is given as .0250, so the proportion is

$$(.0224/.0250) = 0.896$$

or 89.6% of the variance of the return of stock $A$ is market risk.

### Problem 11 on page 514 in Ruppert/Matteson with the adjustments

#### (a)

For the weights we have

$$R_p = 0.3R_1 + 0.3R_2 + 0.4R_3$$

and so

$$\begin{aligned} R_p - \mu_f &= (0.3\beta_1 + 0.3\beta_2 + 0.4\beta_3) * (R_M - \mu_f) + (0.3\epsilon_1 + 0.3\epsilon_2 + 0.4\epsilon_3) \\ &= \tilde{\beta} * (R_M - \mu_f) + \tilde{\epsilon} \end{aligned}$$

where

$$\tilde{\beta} = 0.3\beta_1 + 0.3\beta_2 + 0.4\beta_3 = 0.79$$

and

$$\tilde{\epsilon} = 0.3\epsilon_1 + 0.3\epsilon_2 + 0.4\epsilon_3$$

and so the beta is .79.

#### (b)

The variance of the excess return is given by

$$\begin{aligned} \text{Var}\left(R_p - \mu_f\right) &= \text{Var}\left(.79 * (R_M - \mu_f)\right) + \text{Var}(\tilde{\epsilon}) \\ &= .79^2 * .02 + (0.09 * .012 + 0.09 * .025 + 0.16 * .015) = 0.0182 \end{aligned}$$

**(c)**

The proportion of risk of asset 1 due to market risk is

$$\frac{.79^2 * .02}{.79^2 * .02 + .012} = .51$$

so just bigger than 50%.

## 3

### (a)

Now the mean function is

$$\begin{aligned}\mu_X(n) = E\left(X_n\right) &= E\left(U\delta_n + \epsilon_n\right) \\ &= E(U)E\left(\delta_n\right) + E\left(\epsilon_n\right) \\ &= \frac{1}{2}\cdot 1 + 0 = \frac{1}{2}\end{aligned}$$

The autocovariance function is given by

$$\begin{aligned}\Sigma_X(n,n) &= \operatorname{Var}\left(X_n\right) \\ &= \operatorname{Var}\left(U\delta_n + \epsilon_n\right) \\ &= \operatorname{Var}\left(U\delta_n\right) + \operatorname{Var}\left(\epsilon_n\right) \quad \text{by independence of } U\delta_n, \epsilon_n \\ &= \mathbb{E}(U^2\delta_n^2) - (\mathbb{E}(U\delta_n))^2 + \frac{1}{4} \\ &= \mathbb{E}(U^2)\mathbb{E}(\delta_n^2) - (\mathbb{E}(U)\mathbb{E}(\delta_n))^2 + \frac{1}{4} \\ &= \mathbb{E}(U^2)(\operatorname{var}(\delta_n) + \mathbb{E}^2(\delta_n)) - \left(\frac{1}{2}\right)^2 + \frac{1}{4} \\ &= \frac{1}{3}(1+1) = \frac{2}{3}\end{aligned}$$

and for $m \neq n$,

$$\begin{aligned}\Sigma_X(m,n) &= \operatorname{Cov}\left(X_m, X_n\right) \\ &= \operatorname{Cov}\left(U\delta_m + \epsilon_m, U\delta_n + \epsilon_n\right) \\ &= \operatorname{Cov}\left(U\delta_m, U\delta_n\right) \\ &= \mathbb{E}(U^2\delta_m\delta_n) - \mathbb{E}(U\delta_m)\mathbb{E}(U\delta_n) \\ &= \mathbb{E}(U^2)\mathbb{E}(\delta_m)\mathbb{E}(\delta_n) - \mathbb{E}(U)\mathbb{E}(\delta_m)\mathbb{E}(U)\mathbb{E}(\delta_n) \\ &= \frac{1}{3}\cdot 1\cdot 1 - \frac{1}{2}\cdot 1\cdot\frac{1}{2}\cdot 1 \\ &= \frac{1}{12}.\end{aligned}$$

Thus this is a weakly stationary process with $\mu_X(n) = \frac{1}{2}$ for all $n$ and

$$\gamma_X(n) = \begin{cases}\frac{2}{3} & n = 0 \\ \frac{1}{12} & n \neq 0\end{cases}.$$

### (b)

As stated in class a stationary ergodic process satisfies that the autocovariance function converges to 0 as lag difference goes to infinity, but that is not the case for this process, so this process cannot be ergodic.