STATS531 HW2

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Question 2.1(A)

1. Based on the definition of ACF γ_h

   $$\gamma_h = \text{Cov}(X_n, X_{n+h}) = \text{Cov}(X_n, \phi X_{n+h-1} + \epsilon_{n+h}) = \phi\text{Cov}(X_n, X_{n+h-1}) + \text{Cov}(X_n, \epsilon_{n+h})$$

2. Since there is no correlation between ϵ and X_n, we can delete it, and form a recursive equation

   $$\gamma_h = \phi\gamma_{h-1} = \phi^k\gamma_{h-k}$$

3. Let $k = h$, we have

   $$\gamma_h = \phi^h\gamma_0$$

4. For γ_0, we have

   $$\gamma_0 = \text{Var}(X_n) = \text{Var}(X_{n-1}) = \phi^2\text{Var}(X_{n-1}) + \text{Var}(\epsilon)$$

5. Therefore, we have
   $A = \gamma_0 = \frac{\sigma^2}{1-\phi^2}$
6. Since $\lambda = \phi$, the final result is

   $$\gamma_h = \frac{\phi^h}{1-\phi^2}\sigma^2$$

Question 2.1(B)

1. By subtracting ϕX_{n-1} from both sides of the equation, we have the MA(∞) representation

   $$X_n - \phi X_{n-1} = \epsilon_n \rightarrow X_n = (1 - B\phi)^{-1}\epsilon_n$$

2. Then we can have the Taylor series expansion for $g(x) = (1 - \phi x)^{-1}$

   $$g(x) = \sum_{j=0}^{\infty} \frac{\mathrm{d}^j g(x)}{\mathrm{d}x^j}|_{x=0}\frac{x^j}{j!}$$

3. The derivative can be computed recursively as

   $$\frac{\mathrm{d}g(x)}{\mathrm{d}x} = \phi(1-\phi x)^{-2} \rightarrow \frac{\mathrm{d}^j g(x)}{\mathrm{d}x^j} = j!\phi^j(1-\phi x)^{-j-1}$$

4. Therefore, we can apply the Taylor expansion with the operator B

   $$g(B) = \sum_{j=0}^{\infty}\phi^j B^j \rightarrow X_n = \sum_{j=0}^{\infty}\phi^j\epsilon_{n-j}$$

5.
6. For MA (∞) auto covariance, we have

$$\gamma_h = \sum_{j=0}^{\infty} g_j g_{j+h} \sigma^2 = \sigma^2 \sum_{j=0}^{\infty} g_j g_{j+h} \quad \text{where} \quad g_j = \phi^j$$

7. Therefore, we have the result

$$\gamma_h = \sigma^2 \phi^h \sum_{j=0}^{\infty} \phi^{2j} = \frac{\phi^h}{1-\phi^2} \sigma^2$$

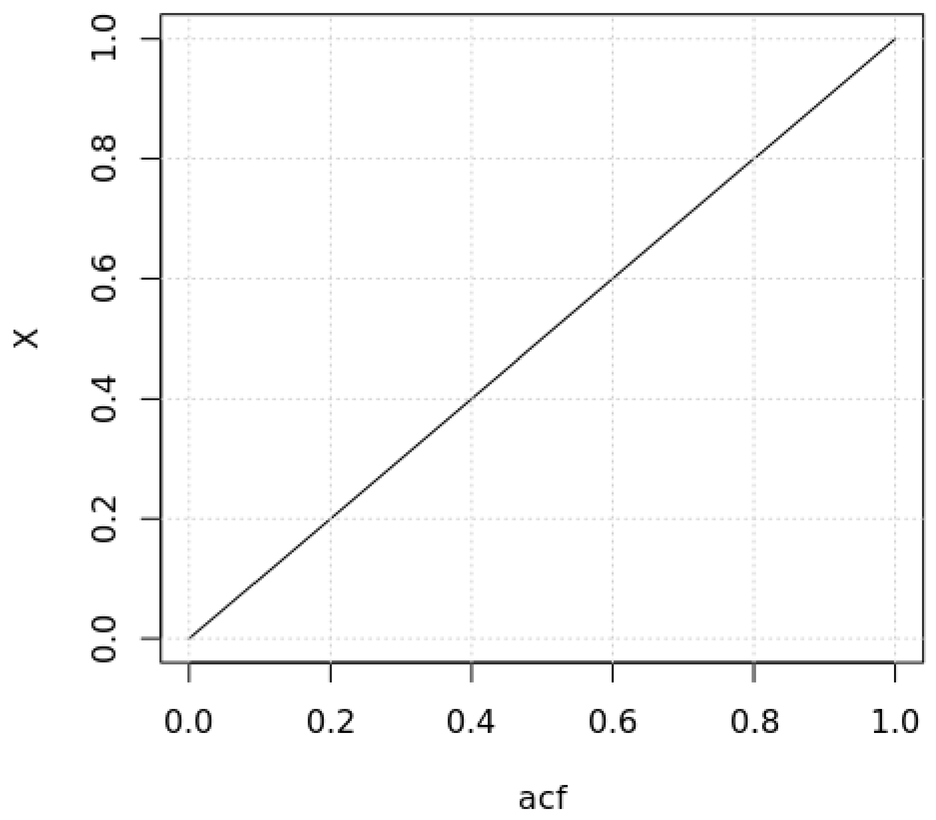
Question 2.1(C)

Based on our calculation of auto covariance, the auto correlation is

$$\rho_h = \frac{\gamma_h}{\gamma_0} = \phi^h$$

The result from RStudio, which is extremely close to my computation

```
phi <- 0.8
N <- 100
X <- numeric(N)
for(n in 1:N)X[n] <- phi ^ (n-1)
acf <- ARMAacf(ar=0.8, lag.max=N-1)
max(acf - X)
# 5.551115e-17
```



Question 2.2

1. Assume $n = m + h$, then based on the definition of $\text{Cov}(X_m, X_{m+h})$, we have

$$\gamma_h = \text{Cov}(X_m, X_{m+h}) = \text{Cov}(X_m, X_{m+h-1} + \epsilon_{m+h})$$

2. Since ϵ is i.i.d, we have

$$\gamma_{mn} = \gamma_h = \text{Cov}(X_m, X_{m+h-1}) = \text{Cov}(X_m, X_m)$$

3. Based on the result of X_m

$$\gamma_{mn} = \text{Cov}(\sum_{i=1}^{m} \epsilon_i, \sum_{i=1}^{m} \epsilon_i) = \sum_{i=1}^{m} \text{Var}(\epsilon_i) = m\sigma^2$$

4. Similarly, if $m = n + h$, we have

$$\gamma_{mn} = n\sigma^2$$

5. Therefore, the result is

$$\gamma_{mn} = \min(m, n)\sigma^2$$

Question 2.3

I've two participation records at Piazza this week.

1. Discuss the question related to burn-in
2. Post a question related the sinusoidal solution to the oscillate equation

Reference

- Taylor series - Wikipedia
- The lecture slides for chapter 3
- The lecture slides for chapter 4