

Modeling and Analysis of Time Series Data

Chapter 7: Introduction to time series analysis in the frequency domain

Edward L. Ionides

Outline

- 1 Frequency components
 - Eigenvalues and eigenvectors of a stationary covariance matrix
- 2 The Fourier transform
 - A normal approximation
- 3 The periodogram to estimate the spectral density
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 - Smoothing the periodogram
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 - Fitting an AR model to estimate the spectrum

Frequency components of a time series

- ① A time series dataset (like any other sequence of numbers) can be written as a sum of sine and cosine functions with varying frequencies.
- ② This is called the **Fourier representation** or **Fourier transform** of the data.
- ③ The coefficients corresponding to the sine and cosine at each frequency are called frequency components of the data.
- ④ Looking at which frequencies have large and small components can help to identify appropriate models.
- ⑤ Looking at the frequency components present in our models can help to assess whether they are doing a good job of describing our data.

What is the spectrum of a time series model?

- We begin by reviewing eigenvectors and eigenvalues of covariance matrices. This eigen decomposition also arises elsewhere in statistics, e.g. principle component analysis.
- A univariate time series model is a vector-valued random variable $Y_{1:N}$ which we suppose has a covariance matrix V which is an $N \times N$ matrix with entries $V_{mn} = \text{Cov}(Y_m, Y_n)$.
- V is a non-negative definite symmetric matrix, and therefore has N non-negative eigenvalues $\lambda_1, \dots, \lambda_N$ with corresponding eigenvectors u_1, \dots, u_N such that

$$Vu_n = \lambda_n u_n. \quad (1)$$

- A basic property of these eigenvectors is that they are orthogonal, i.e.,

$$u_m^T u_n = 0 \text{ if } m \neq n. \quad (2)$$

- We may work with **normalized** eigenvectors that are scaled such that $u_n^T u_n = 1$.

对称矩阵的特征向量正交

- We can also check that the components of Y in the directions of different eigenvectors are uncorrelated.
- Since $\text{Cov}(AY, BY) = A\text{Cov}(Y, Y)B^T$, we have

$$\begin{aligned}
 \text{Cov}(u_m^T Y, u_n^T Y) &= u_m^T \text{Cov}(Y, Y) u_n \\
 &= u_m^T V u_n \\
 &= \lambda_n u_m^T u_n \\
 &= \begin{cases} \lambda_n & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}
 \end{aligned}$$

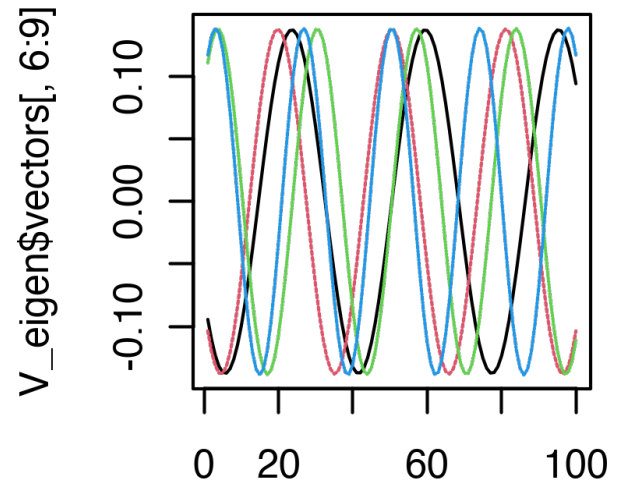
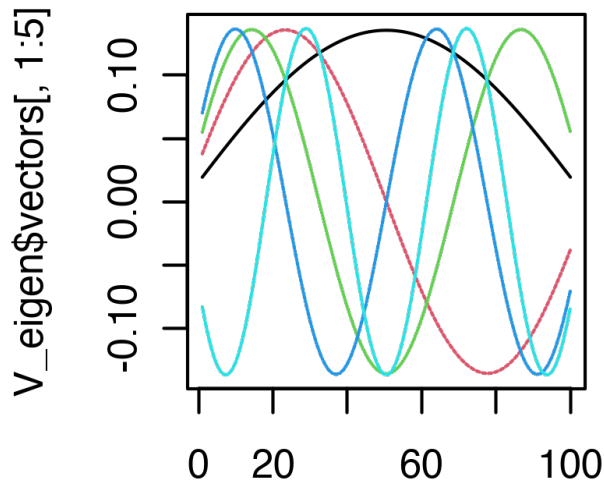
u_m, u_n 都是随机变量特征向量,经过Y
变换后就不相关了

For the last equality, we have supposed that the eigenvectors are normalized.

- If we knew V , we could convert the model to a representation where the observable random variables are uncorrelated.
- Transforming the data into its components in the directions of the eigenvectors of the model allows us to use an uncorrelated model. In the Gaussian case, we have independence.

Eigenvectors for the covariance matrix of an AR(1) model with $N = 100$ and $\phi = 0.8$

```
N <- 100; phi <- 0.8; sigma <- 1
V <- matrix(NA,N,N)
for(m in 1:N) for(n in 1:N) V[m,n]<-sigma^2*phi^abs(m-n)/(1-phi^2)
V_eigen <- eigen(V,symmetric=TRUE)
matplot(V_eigen$vectors[,1:5],type="l")
matplot(V_eigen$vectors[,6:9],type="l")
```



Eigenvalues for the covariance matrix of an AR(1) model with $N = 100$ and $\phi = 0.8$

- We see that the eigenvectors, plotted as functions of time, look like sine wave oscillations.
- The eigenvalues are

```
round(V_eigen$values[1:9],2)
[1] 24.59 23.44 21.73 19.70 17.57 15.51 13.61 11.91 10.42
```

- We see that the eigenvalues are decreasing. For this model, the components of $Y_{1:N}$ with highest variance correspond to long-period oscillations.
- Are the sinusoidal eigenvectors a special feature of this particular time series model, or something more general?

特征值越大,特征向量的频率越低

The eigenvectors for a long stationary time series model

- Suppose $\{Y_n, -\infty < n < \infty\}$ has a stationary autocovariance function γ_h . Write Γ for the infinite array with entries 自方差矩阵

$$\Gamma_{m,n} = \gamma_{m-n} \quad \text{for all integers } m \text{ and } n. \quad (3)$$

- To focus on concepts over technical details, we assume infinite sums converge and order of summation can be exchanged, so infinite arrays behave like finite matrices.
- An eigenvector for Γ is a sequence $u = \{u_n, -\infty < n < \infty\}$ with corresponding eigenvalue λ such that

$$\Gamma u = \lambda u, \quad \text{n,m是index(行号)} \quad (4)$$

or, writing out the matrix multiplication explicitly,

$$\sum_{n=-\infty}^{\infty} \Gamma_{m,n} u_n = \lambda u_m \quad \text{for all } m. \quad (5)$$

- We look for a sinusoidal solution, $u_n = e^{2\pi i \omega n}$, where ω is cycles per unit time. omega是频率,特征向量都是三角函数正交基

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \Gamma_{m,n} u_n &= \sum_{n=-\infty}^{\infty} \gamma_{m-n} u_n \\
&= \sum_{h=-\infty}^{\infty} \gamma_h u_{m-h} \quad \text{setting } h = m - n \\
&= \sum_{h=-\infty}^{\infty} \gamma_h e^{2\pi i \omega (m-h)} \quad \text{特征值是ACF的傅里叶变换} \\
&= e^{2\pi i \omega m} \sum_{h=-\infty}^{\infty} \gamma_h e^{-2\pi i \omega h} \\
&= u_m \lambda(\omega) \quad \text{for } \lambda(\omega) = \sum_{h=-\infty}^{\infty} \gamma_h e^{-2\pi i \omega h}
\end{aligned}$$

Question 7.1. Why does this calculation show that $u_n(\omega) = e^{2\pi i \omega n}$ is an eigenvector for Γ for any choice of ω .

This argument holds for any ω and matches the definition of eigenvector and eigenvalues for matrices, ignoring issues related to infinite sums.

- The eigenvalue at frequency ω is

$$\lambda(\omega) = \sum_{h=-\infty}^{\infty} \gamma_h e^{-2\pi i \omega h} \quad (6)$$

- Viewed as a function of ω , this is called the **spectral density function**.
- $\lambda(\omega)$ is the **Fourier transform** of γ_h .
- An integral version of (6) is used in applied math and engineering:

$$\lambda(\omega) = \int_{-\infty}^{\infty} \gamma(x) e^{-2\pi i \omega x} dx. \quad (7)$$

- We obtain (6) from (7) when $\gamma(h)$ has a point mass γ_h when h is an integer, and $\gamma(x) = 0$ for non-integer x .

- It was convenient to do this calculation with complex exponentials.

However, writing

把特征向量的实部和虚部拆分

$$u_n = e^{2\pi i \omega n} = \cos(2\pi \omega n) + i \sin(2\pi \omega n), \quad (8)$$

and noting that γ_h is real, we see that the real and imaginary parts of $\lambda(\omega) = \sum_{h=-\infty}^{\infty} \gamma_h e^{-2\pi i \omega h}$ give us two real eigenvectors, $\cos(2\pi \omega n)$ and $\sin(2\pi \omega n)$. 传统的傅里叶基

Question 7.2. Review: how would you demonstrate the correctness of the identity $e^{2\pi i \omega} = \cos(2\pi \omega) + i \sin(2\pi \omega)$. Taylor series approximation

$$e^{2\pi i \omega} = \sum_{k=0}^{\infty} \frac{(2\pi i \omega)^k}{k!} = 1 + 2\pi i \omega - \frac{(2\pi \omega)^2}{2!} - \frac{(2\pi \omega)^3}{3!} i + \frac{(2\pi \omega)^4}{4!}$$

$$\cos 2\pi \omega = 1 - \frac{(2\pi \omega)^2}{2!} + \frac{(2\pi \omega)^4}{4!} \dots$$

$$i \sin 2\pi \omega = i \left[2\pi \omega - \frac{(2\pi \omega)^3}{3!} \dots \right]$$

- Assuming that this computation for an infinite sum represents a limit of increasing dimension for finite matrices, we have found that the eigenvectors for any long, stationary time series model are approximately sinusoidal. N维空间, 一共N个特征向量
- For the finite time series situation, we only expect N eigenvectors for a time series of length N . We have one eigenvector for $\omega = 0$, two eigenvectors corresponding to sine and cosine functions with frequency

$$\omega_n = n/N, \text{ for } 0 < n < N/2, \quad (9)$$

and, if N is even, a final eigenvector with frequency

$$\omega_{(N/2)} = 1/2. \quad \text{cos } \pi k, \text{ sin } \pi k \quad (10)$$

- These sine and cosine vectors are the **Fourier basis**.
- The time series $y_{1:N}^*$ is the **time domain** representation of the data. Transforming to the Fourier basis gives the **frequency domain** representation.

Frequency components and the Fourier transform

- The **frequency components** of $Y_{1:N}$ are the components in the directions of these eigenvectors, given by

把k理解成t, omega理解成n

$$C_n = \frac{1}{\sqrt{N}} \sum_{k=1}^N Y_k \cos(2\pi\omega_n k) \text{ for } 0 \leq n \leq N/2,$$

$$S_n = \frac{1}{\sqrt{N}} \sum_{k=1}^N Y_k \sin(2\pi\omega_n k) \text{ for } 1 \leq n \leq N/2.$$

$\int_0^1 Y(tN) \cos 2\pi n t dt = \frac{\langle Y, \cos 2\pi n t \rangle}{\langle \cos 2\pi n t, \cos 2\pi n t \rangle}$

- Similarly, the **frequency components** of data $y_{1:N}^*$ are

$$c_n = \frac{1}{\sqrt{N}} \sum_{k=1}^N y_k^* \cos(2\pi\omega_n k) \text{ for } 0 \leq n \leq N/2,$$

$$s_n = \frac{1}{\sqrt{N}} \sum_{k=1}^N y_k^* \sin(2\pi\omega_n k) \text{ for } 1 \leq n \leq N/2.$$

- The frequency components of the data can be written as real and imaginary parts of the **discrete Fourier transform**,

$$\begin{aligned}
 d_n &= \frac{1}{\sqrt{N}} \sum_{k=1}^N y_k^* e^{-2\pi i n k / N} \approx \int_0^1 y(t) e^{-2\pi i n t} dt \\
 &= c_n - i s_n
 \end{aligned}$$

- The normalizing constant of $1/\sqrt{N}$ is convenient for a central limit theorem.
- Various choices about signs and factors of 2π , $\sqrt{2\pi}$ and \sqrt{N} can be made in the definition of the Fourier transform. For example, the `fft` command in R does not include this constant.
- `fft` is an implementation of the fast Fourier transform algorithm, which enables computation of all the frequency components with order $N \log(N)$ computation. Computing the frequency components may appear to require a matrix multiplication involving order N^3 additions and multiplications. When $N = 10^5$ or $N = 10^6$ this difference becomes important!

- The first frequency component, C_0 , is a special case, since it has mean $\mu = \mathbb{E}[Y_n]$ whereas the other components have mean zero.
- In practice, we subtract a mean before computing the frequency components, which is equivalent to removing the frequency component for frequency zero.
- The frequency components $(C_{0:N/2}, S_{1:N/2})$ are asymptotically uncorrelated. They are constructed as a sum of a large number of terms, with the usual $1/\sqrt{N}$ scaling for a central limit theorem. So, it may not be surprising that a central limit theorem applies, giving asymptotic justification for the following normal approximation.
- Moving to the frequency domain (i.e., transforming the data to its frequency components) has **decorrelated** the data. Statistical techniques based on assumptions of independence are appropriate when applied to frequency components.

Normal approximation for the frequency components

- $(C_{1:N/2}, S_{1:N/2})$ are approximately independent, mean zero, Normal random variables with

$$\text{Var}(C_n) = \text{Var}(S_n) \approx \underbrace{(1/2)}_{\mathcal{N}(0, \frac{\lambda(\omega_n)}{2})} \lambda(\omega_n). \quad (11)$$

- C_0/\sqrt{N} is approximately Normal, mean μ , independent of $(C_{1:N/2}, S_{1:N/2})$, with

$$\text{Var}(C_0/\sqrt{N}) \approx \lambda(0)/N. \quad (12)$$

- It follows from the normal approximation that, for $1 \leq n \leq N/2$,

$$\boxed{C_n^2 + S_n^2 \approx \lambda(\omega_n) \frac{\chi_2^2}{2}}, \quad (13)$$

where χ_2^2 is a chi-squared random variable on two degrees of freedom.

- Taking logs, we have

$$\log(C_n^2 + S_n^2) \approx \log \lambda(\omega_n) + \underbrace{\log(\chi_2^2/2)}_{\text{Noise}}. \quad (14)$$

The periodogram

- The chi-squared property in (13) motivates the **periodogram**,

$$I_n = c_n^2 + s_n^2 = |d_n|^2 \quad \text{因为有noise所以不一定准} \quad (15)$$

as an estimator of the spectral density.

- From (14), $\log I_n$ is as an estimator of the log spectral density with a convenient statistical property: asymptotically independent, identically distributed errors at each Fourier frequency.
- Therefore, a signal-plus-white-noise model is appropriate for estimating the log spectral density using the log periodogram.
- The periodogram is an **inconsistent estimator** of the spectrum. We can smooth the periodogram to borrow strength between nearby frequencies. **local average = smooth**

Interpreting the spectral density as a power spectrum

- The power of a wave is proportional to the square of its amplitude.
- The spectral density gives the mean square amplitude of the components at each frequency, and therefore gives the expected power.
- The spectral density function can therefore be called the **power spectrum**.

Question 7.3. Consider the AR(1) model, $\phi(B)Y_n = \epsilon_n$ with $\phi(B) = 1 - \phi_1 B$ and $\epsilon_n \sim \text{WN}(\sigma^2)$, i.e., white noise with variance σ^2 . Show that the spectrum of Y is

$$\lambda(\omega) = \frac{\sigma^2}{|\phi(e^{2\pi i \omega})|^2} = \frac{\sigma^2}{1 + \phi_1^2 - 2\phi_1 \cos(2\pi \omega)}. \quad (16)$$

1. set $v = 2\pi \omega$, $\lambda(\omega) = \sum_{h=-\infty}^{\infty} \gamma_h e^{-ivh}$ 看HW2算ACF

2. $\gamma_h = \frac{\phi_1^{|h|} \sigma^2}{1 - \phi_1^2} \rightarrow \lambda(\omega) = \frac{\sigma^2}{1 - \phi_1^2} \left(\sum_{h=0}^{\infty} e^{-ivh} \phi_1^h + \sum_{h=0}^{\infty} e^{ivh} \phi_1^h - 1 \right)$

3. $\lambda(\omega) = \frac{\sigma^2}{1 - \phi_1^2} \left(\frac{1}{1 - \phi_1 e^{-iv}} + \frac{1}{1 - \phi_1 e^{iv}} - 1 \right)$

4. $\lambda(\omega) = \frac{\sigma^2}{1 - \phi_1^2} \frac{1 - \phi_1^2}{(1 - \phi_1 e^{iv})^2} = \frac{\sigma^2}{|\phi(e^{iv})|^2}$

ARMA models have a rational spectrum

- The calculation for the AR(1) model generalizes. We give the result without proof.
- Let Y_n be an ARMA(p,q) model, $\phi(B)Y_n = \psi(B)\epsilon_n$ with $\epsilon_n \sim \text{WN}(\sigma^2)$. The spectrum of Y is

$$\lambda(\omega) = \sigma^2 \left| \frac{\psi(e^{2\pi i\omega})}{\phi(e^{2\pi i\omega})} \right|^2. \quad (17)$$

- The so-called **rational spectrum** of ARMA models is computationally convenient.
- A stationary, causal ARMA model cannot have roots on the unit circle. If a root approaches the unit circle, the denominator in (17) becomes close to zero. **特征值会爆炸**
- The special case of $\phi(x) = \psi(x) = 1$ gives $\lambda(\omega) = \sigma^2$. **White noise has a constant spectrum**, matching the analogy that white light has uniform intensity across the visible light spectrum.

Michigan winters revisited: Frequency domain methods

```
y <- read.table(file="ann_arbor_weather.csv",header=TRUE)
head(y[,1:9],3)
```

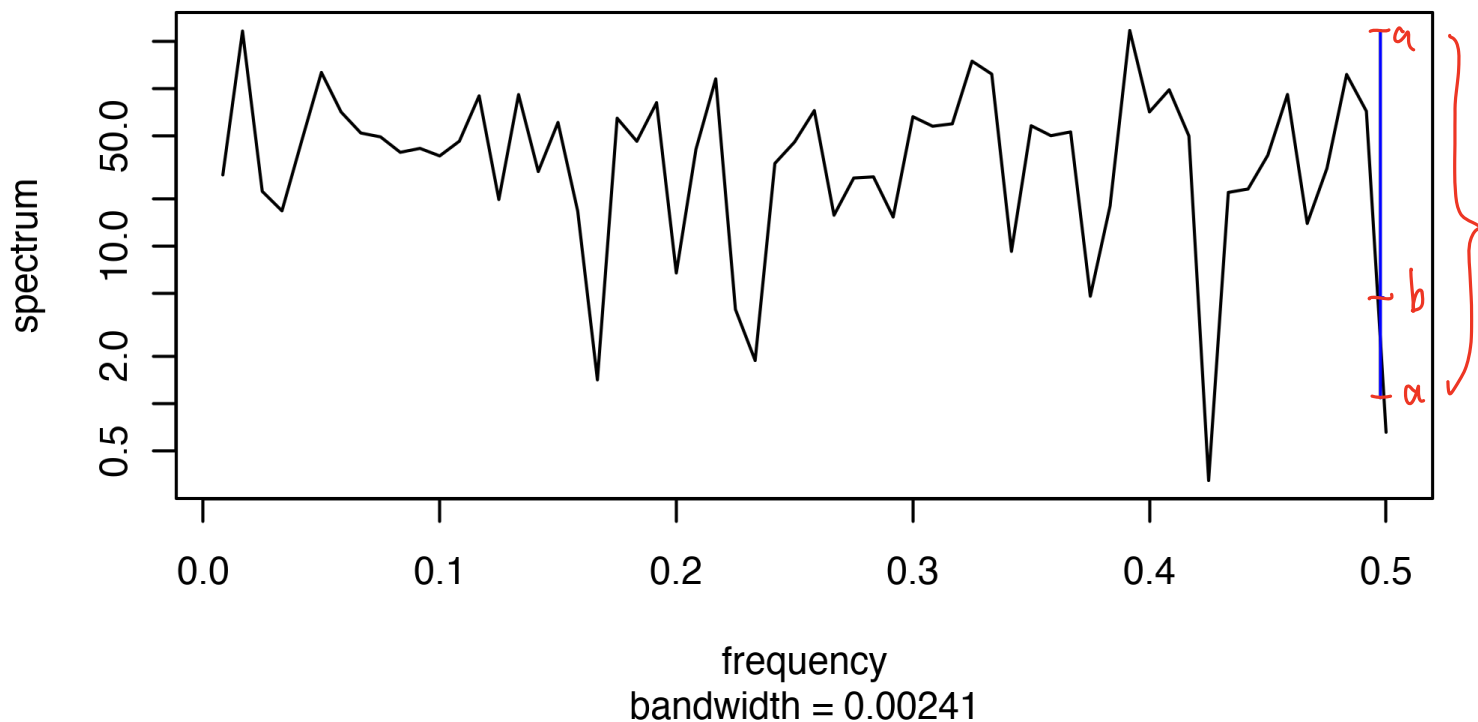
Year	Low	High	Hi_min	Lo_max	Avg_min	Avg_max	Mean	Precip
1900	-7	50	36	12	18	34.7	26.3	1.06
1901	-7	48	37	20	17	31.8	24.4	1.45
1902	-4	41	27	11	15	30.4	22.7	0.60

- We have to deal with the NA measurement for 1955. A simple approach is to replace the NA by the mean.
- What other approaches can you think of for dealing with this missing observation?
- What are the strengths and weaknesses of these approaches?

```
low <- y$Low
low[is.na(low)] <- mean(low, na.rm=TRUE)
```

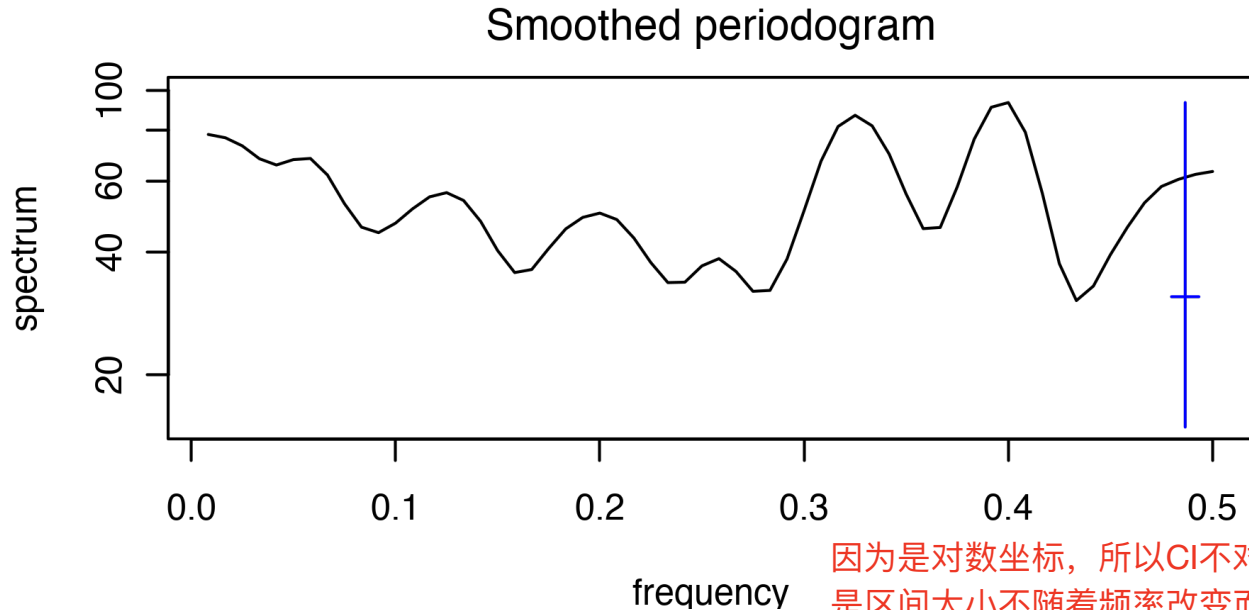
```
spectrum(low, main="Unsmoothed periodogram")
```

Unsmoothed periodogram 蓝线是通过卡方分布构造的置信区间,没有平滑的置信区间很大



- To smooth, we use the default periodogram smoother in R

```
spectrum(low, spans=c(3,5,3), main="Smoothed periodogram",  
ylim=c(15,100))
```

 This determines the smoothing

- The bar is a 95% pointwise confidence interval which we can slide to any frequency of interest.
- The chi-squared property (14) means this CI is the same width for each frequency, on the log scale. Note it is asymmetric.

span(3,5,3)的含义是什么

Question 7.4. What is the default periodogram smoother in R?

Each span is a window width for a rectangular moving average smoother

Question 7.5. How should we use it?

A smoother of this kind is a convolution


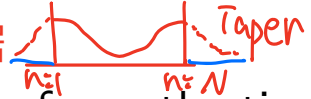
A convolution of pdf is the sum of two independent random variables.

By CLT, the combination of these random variables approximately follows normal distribution.滑动窗口的和服从正态分布

Question 7.6. Why is that default chosen?

Historical reasons. Rectangular smoothing is easier to calculate efficiently. The main reason to do this nowadays is to conform to standard statistical practice.

More details on computing and smoothing the periodogram

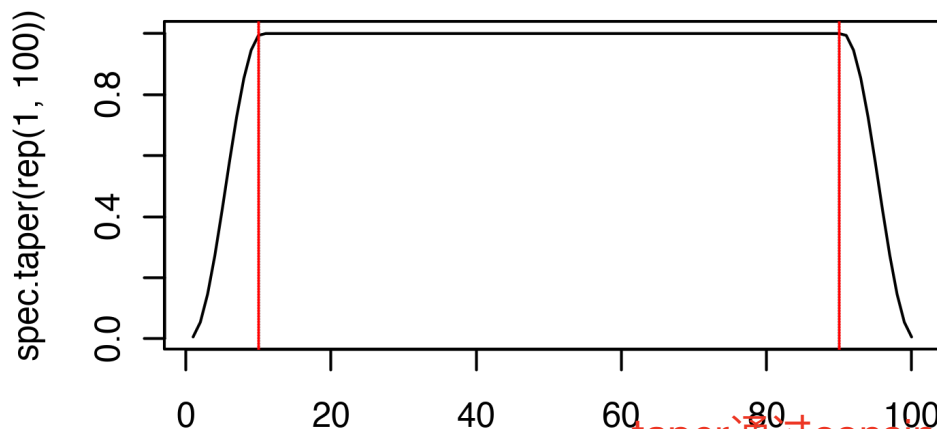
- To see what R actually does to compute and smooth the periodogram, type `?spectrum`. At the very least, subtract a mean. We want to look at data well-modeled on stationary
- This will lead you to type `?spec.pgram`.
- You will see that, by default, R removes a linear trend, fitted by least squares. This may often be a sensible thing to do. Why?  希望stationary
- You will see that R then multiplies the data by a quantity called a **taper**, computed by `spec.taper`.
- The taper smooths the ends of the time series and removes high-frequency artifacts arising from an abrupt start and end to the time series. taper会平滑初始和结尾的时间序列数据 
- Formally, from the perspective of the Fourier transform, the time series takes the value zero outside the observed time points $1:N$. The sudden jump to and from zero at the start and end produces unwanted effects in the frequency domain.

The default taper in R smooths the first and last $p = 0.1$ fraction of the time points, by modifying the detrended data $y_{1:N}^*$ to tapered version $z_{1:N}$ defined by

$$z_n = \begin{cases} y_n^* (1 - \cos(\pi n / Np)) / 2 & \text{if } 1 \leq n < Np \\ y_n^* & \text{if } Np \leq n \leq N(1 - p) \\ y_n^* (1 - \cos(\pi [N + 1 - n] / Np)) / 2 & \text{if } N(1 - p) < n \leq N \end{cases}$$

```
plot(spec.taper(rep(1,100)),type="l",
     main="Default taper in R, for a time series of length 100")
abline(v=c(10,90),lty="dotted",col="red")
```

Default taper in R, for a time series of length 100



taper通过cosine来平滑数据

Spectral density estimation by fitting a model

Another standard way to estimate the spectrum is to fit an AR(p) model with p selected by AIC.

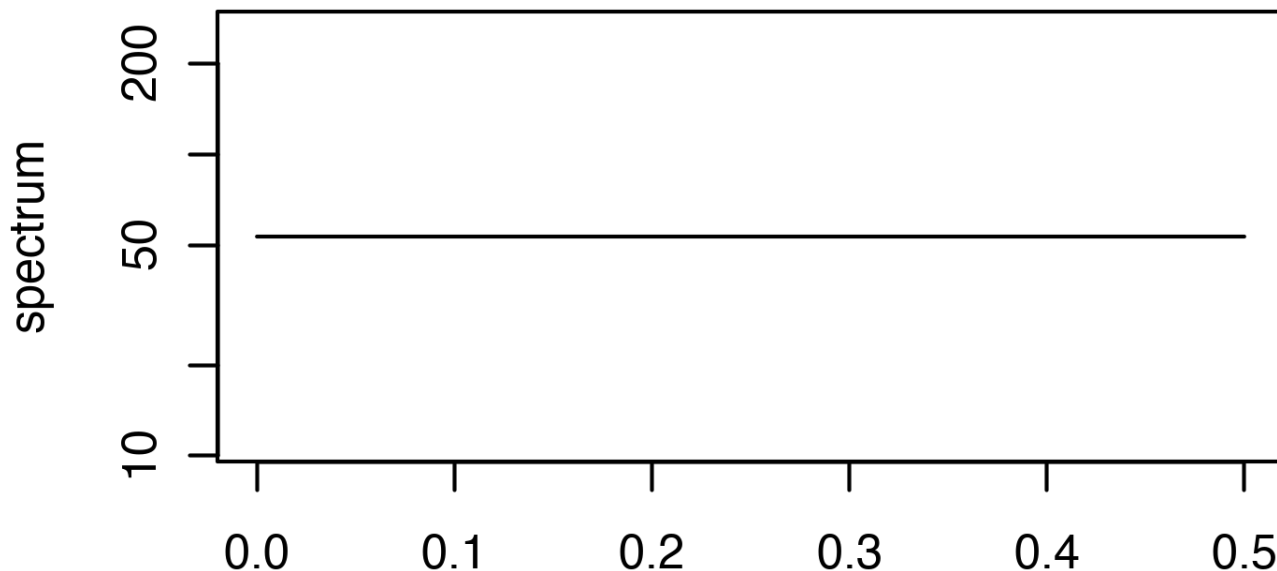
通常用这个方法估计spectrum

相比ARIMA,使用AR(p)可以移除numerical instability

```
spectrum(low,method="ar",  
main="Spectrum estimated via AR model picked by AIC")
```

但如果数据服从MA分布,则可能估计效率很低(因为参数太多了)

Spectrum estimated via AR model picked by AIC



Units of frequency and period

- When we call ω the frequency in cycles per unit time, we really mean **cycles per unit observation**.
- Suppose the time series consists of equally spaced observations, with $t_n - t_{n-1} = \Delta$ years. Then, the frequency is ω/Δ **cycles per year**.
- The **period** of an oscillation is the time for one cycle,

注意单位的转换

$$\text{period} = \frac{1}{\text{frequency}}. \quad (18)$$

- When the observation intervals have a time unit (years, seconds, etc) we usually use that unit for the period, and its inverse for the frequency.

Further reading

- Sections 4.1 to 4.3 of Shumway and Stoffer (2017) cover similar topics to this chapter.

References and Acknowledgements

Shumway RH, Stoffer DS (2017). *Time Series Analysis and its Applications: With R Examples*. Springer.

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