STATS 509 Winter 2022, Solution to Homework 9

1

$$\begin{aligned}
\operatorname{Cov}\left(X_n, X_{n+k}\right) &= \operatorname{Cov}\left(\sum_{j=-\infty}^{\infty} \psi_j \epsilon_{n-j}, \sum_{j'=-\infty}^{\infty} \psi_{j'} \epsilon_{n+k-j'}\right) \\
&= \sum_{j=-\infty}^{\infty} \sum_{j'=-\infty}^{\infty} \psi_j \psi_{j'} \operatorname{Cov}\left(\epsilon_{n-j}, \epsilon_{n+k-j'}\right) \\
&= \sum_{j=-\infty}^{\infty} \sum_{j'=-\infty}^{\infty} \psi_j \psi_{j'} \sigma^2 1_{\{n-j=n+k-j'\}} \\
&= \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+k} \quad \text{since need } j' = k+j \text{ for equality to 1 above}
\end{aligned}$$

2

(a)

Now

$$\begin{aligned}
X_n &= \alpha_1 X_{n-1} + \theta_1 \epsilon_{n-1} + \theta_2 \epsilon_{n-2} + \epsilon_n \\
&= \alpha_1 \left(\alpha_1 X_{n-2} + \theta_1 \epsilon_{n-2} + \theta_2 \epsilon_{n-3} + \epsilon_{n-1}\right) + \theta_1 \epsilon_{n-1} + \theta_2 \epsilon_{n-2} + \epsilon_n \\
&= \epsilon_n + \left(\alpha_1 + \theta_1\right) \epsilon_{n-1} + \left(\theta_2 + \alpha_1 \theta_1\right) \epsilon_{n-2} + \alpha_1 \theta_2 \epsilon_{n-3} + \alpha_1^2 \left(\alpha_1 X_{n-3} + \theta_1 \epsilon_{n-3} + \theta_2 \epsilon_{n-4} + \epsilon_{n-2}\right) \\
&= \epsilon_n + \left(\alpha_1 + \theta_1\right) \epsilon_{n-1} + \left(\theta_2 + \alpha_1 \theta_1 + \alpha_1^2\right) \epsilon_{n-2} + \left(\alpha_1 \theta_2 + \alpha_1^2 \theta_1 + \alpha_1^3\right) \epsilon_{n-3} + \cdots \\
&= \epsilon_n + \left(\alpha_1 + \theta_1\right) \epsilon_{n-1} + \sum_{j=2}^{\infty} \alpha_1^{j-2} \left(\theta_2 + \alpha_1 \theta_1 + \alpha_1^2\right) \epsilon_{n-j}
\end{aligned}$$

so this is a linear process with

$$\psi_j = \begin{cases} 1 & j = 0 \\ \alpha_1 + \theta_1 & j = 1 \\ \alpha_1^{j-2} \left(\theta_2 + \alpha_1 \theta_1 + \alpha_1^2\right) & j \geq 2 \\ 0 & j < 0 \end{cases}$$

Also, if $|\alpha_1| < 1$, then the coefficients are absolutely summable, as

$$\sum_{j=2}^{\infty} \left| \alpha_1^{j-2} \left(\theta_2 + \alpha_1 \theta_1 + \alpha_1^2\right) \right| \leq \frac{1}{1 - |\alpha_1|} \left| \theta_2 + \alpha_1 \theta_1 + \alpha_1^2 \right|$$

(b)

By 1.

$$\begin{aligned}
\gamma_X(0) &= \sum_{j=0}^{\infty} \psi_j^2 \\
&= 1 + \left(\alpha_1 + \theta_1\right)^2 + \sum_{j=2}^{\infty} \left(\alpha_1^2\right)^{j-1} \left(\theta_2 + \alpha_1 \theta_1 + \alpha_1\right)^2 \\
&= 1 + \left(\alpha_1 + \theta_1\right)^2 + \left(\theta_2 + \alpha_1 \theta_1 + \alpha_1\right)^2 \cdot \frac{1}{1 - \alpha_1^2} \\
\gamma_X(1) &= \sum_{j=0}^{\infty} \psi_j \psi_{j+1} \\
&= 1 \cdot \left(\alpha_1 + \theta_1\right) + \left(\alpha_1 + \theta_1\right) \left(\theta_2 + \alpha_1 \theta_1 + \alpha_1^2\right) + \left(\theta_2 + \alpha_1 \theta_1 + \alpha_1^2\right)^2 \sum_{j=2}^{\infty} \alpha_1^{j-2} \alpha_1^{j+1-2} \\
&= \left(\alpha_1 + \theta_1\right) \left(1 + \theta_2 + \alpha_1 \theta_1 + \alpha_1^2\right) + \left(\theta_2 + \alpha_1 \theta_1 + \alpha_1^2\right)^2 \cdot \frac{\alpha_1}{1 - \alpha_1^2} \\
\gamma_X(2) &= 1 \cdot \left(\theta_2 + \alpha_1 \theta_1 + \alpha_1^2\right) + \left(\alpha_1 + \theta_1\right) \left(\theta_2 + \alpha_1 \theta_1 + \alpha_1^2\right) \alpha_1 + \left(\theta_2 + \alpha_1 \theta_1 + \alpha_1^2\right)^2 \sum_{j=2}^{\infty} \alpha_1^{j-2} \alpha_1^{j+2-2} \\
&= \left(1 + \alpha_1^2 + \alpha_1 \theta_1\right) \left(\theta_2 + \alpha_1 \theta_1 + \alpha_1^2\right) \alpha_1 + \left(\theta_2 + \alpha_1 \theta_1 + \alpha_1^2\right)^2 \cdot \frac{\alpha_1^2}{1 - \alpha_1^2}
\end{aligned}$$

and based on induction, can see that $\gamma_X(k)$

$$= 1 \cdot \left(\theta_2 + \alpha_1\theta_1 + \alpha_1^2\right)\alpha_1^{k-2} + (\alpha_1 + \theta_1)\left(\theta_2 + \alpha_1\theta_1 + \alpha_1^2\right)\alpha_1^{k-1} + \left(\theta_2 + \alpha_1\theta_1 + \alpha_1^2\right)^2 \sum_{j=2}^{\infty} \alpha_1^{j-2}\alpha_1^{j+k-2}$$

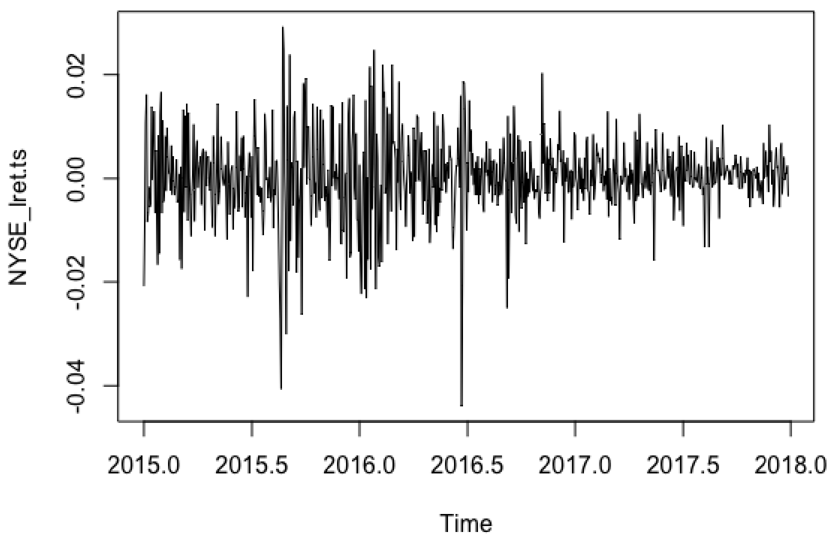
$$= \left(1 + \alpha_1^2 + \alpha_1\theta_1\right)\left(\theta_2 + \alpha_1\theta_1 + \alpha_1^2\right)\alpha_1^{k-2} + \left(\theta_2 + \alpha_1\theta_1 + \alpha_1^2\right)^2 \cdot \frac{\alpha_1^k}{1 - \alpha_1^2}, \quad k \geq 2$$

3

a

The plot is shown as follows. It appears that the volatility is high before 2016.5 and relatively small after then.

```
> X = read.csv("NYA-2015-2017.csv",header=TRUE)
> NYSE_lret = diff(log(X$AdjClose))
> NYSE_lret.ts <- ts(data=NYSE_lret,start=c(2015,1),frequency=252,names=c('logret'))
> # a
> plot(NYSE_lret.ts)
```



b

The estimates and standard errors are summarized as follows:

Parameters	Estimates	Standard Error
α_0	2.926e-06	6.321e-07
α_1	.1657	.0204
β_1	.7875	.0273

Codes are as follows:

```
> library(tseries)
> NYSE_garchest11 = garch(NYSE_lret.ts, order = c(1, 1), coef = NULL, itmax = 200, eps = NULL)
```

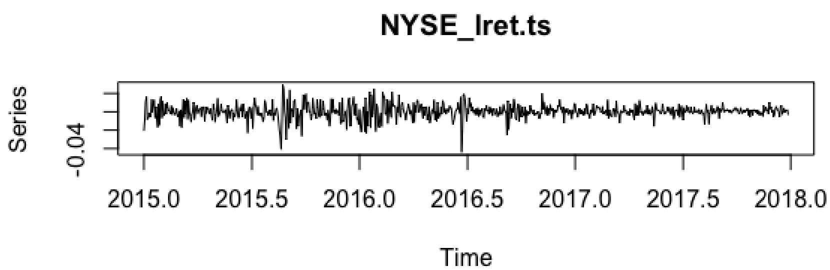
```
***** ESTIMATION WITH ANALYTICAL GRADIENT *****


    I     INITIAL X(I)        D(I)

    1     5.631230e-05     1.000e+00
    2     5.000000e-02     1.000e+00
    3     5.000000e-02     1.000e+00

   IT   NF      F         RELDF    PRELDF    RELDX   STPPAR   D*STEP   NPRELDF
    0    1 -3.290e+03
    1    7 -3.290e+03  4.52e-05  1.70e-03  1.0e-04  5.6e+10  1.0e-05  4.75e+07
    2    8 -3.291e+03  4.42e-04  5.64e-04  5.0e-05  2.1e+00  5.0e-06  1.28e+01
    3    9 -3.291e+03  3.77e-06  3.29e-06  5.0e-05  2.0e+00  5.0e-06  1.31e+01
    4   17 -3.306e+03  4.31e-03  6.14e-03  4.1e-01  2.0e+00  6.9e-02  1.31e+01
    5   20 -3.328e+03  6.74e-03  6.02e-03  7.1e-01  1.9e+00  2.8e-01  8.04e-01
    6   22 -3.335e+03  2.07e-03  1.96e-03  7.7e-02  2.0e+00  5.5e-02  3.21e+01
    7   25 -3.361e+03  7.87e-03  7.08e-03  2.2e-01  2.0e+00  2.2e-01  1.01e+02
    8   27 -3.367e+03  1.59e-03  1.66e-03  3.5e-02  2.0e+00  4.4e-02  1.05e+04
    9   28 -3.371e+03  1.39e-03  2.51e-03  6.3e-02  2.0e+00  8.8e-02  1.95e+02
   10   34 -3.371e+03  1.98e-05  4.79e-04  6.0e-07  9.6e+00  8.8e-07  6.48e+00
   11   35 -3.372e+03  1.23e-04  1.46e-04  3.0e-07  2.0e+00  4.4e-07  3.75e-02
   12   36 -3.372e+03  8.44e-07  7.63e-07  3.0e-07  2.0e+00  4.4e-07  1.14e-02
   13   37 -3.372e+03  1.10e-08  1.25e-08  3.0e-07  2.0e+00  4.4e-07  9.97e-03
   14   44 -3.372e+03  1.68e-06  2.64e-06  1.2e-03  2.0e+00  1.8e-03  9.98e-03
   15   47 -3.372e+03  1.23e-04  6.69e-05  8.6e-03  0.0e+00  1.6e-02  6.69e-05
   16   48 -3.373e+03  2.13e-04  2.64e-04  3.2e-02  2.8e-01  6.5e-02  2.78e-04
   17   49 -3.373e+03  9.66e-06  1.69e-05  2.5e-03  0.0e+00  4.3e-03  1.69e-05
   18   50 -3.373e+03  2.26e-06  2.20e-06  1.2e-03  0.0e+00  2.0e-03  2.20e-06
   19   51 -3.373e+03  5.93e-09  9.17e-09  1.4e-04  0.0e+00  2.5e-04  9.17e-09
   20   52 -3.373e+03  9.04e-10  1.10e-09  2.3e-05  0.0e+00  3.7e-05  1.10e-09
   21   53 -3.373e+03  2.89e-11  6.80e-12  5.7e-06  0.0e+00  1.1e-05  6.80e-12
   22   54 -3.373e+03 -3.20e-12  2.32e-13  1.0e-06  0.0e+00  2.0e-06  2.32e-13

***** RELATIVE FUNCTION CONVERGENCE *****

FUNCTION    -3.373048e+03   RELDX        1.019e-06
FUNC. EVALS      54         GRAD. EVALS      22
PRELDF       2.322e-13      NPRELDF      2.322e-13

    I      FINAL X(I)        D(I)          G(I)

    1    2.926505e-06     1.000e+00     1.753e+01
```

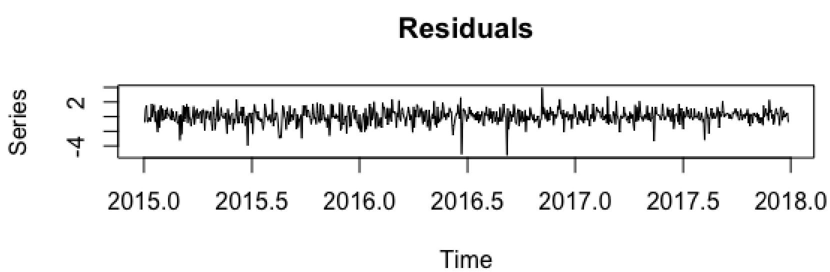
```
     2    1.659208e-01     1.000e+00     6.821e-04
     3    7.875203e-01     1.000e+00    -9.926e-05

> summary(NYSE_garchest11)

Call:
garch(x = NYSE_lret.ts, order = c(1, 1), coef = NULL, itmax = 200,     eps = NULL)

Model:
GARCH(1,1)

Residuals:
     Min       1Q   Median       3Q      Max
-5.25462 -0.48218  0.04271  0.63034  3.89839

Coefficient(s):
    Estimate  Std. Error  t value Pr(>|t|)
a0 2.926e-06   6.321e-07    4.630 3.66e-06 ***
a1 1.659e-01   2.035e-02    8.153 4.44e-16 ***
b1 7.875e-01   2.730e-02   28.851  < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Diagnostic Tests:
Jarque Bera Test

data:  Residuals
X-squared = 238, df = 2, p-value < 2.2e-16


Box-Ljung test

data:  Squared.Residuals
X-squared = 0.53557, df = 1, p-value = 0.4643
```

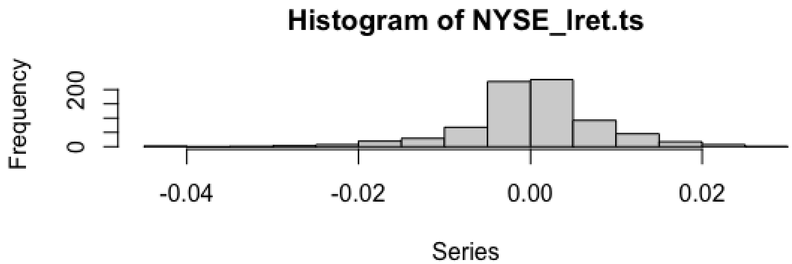
c

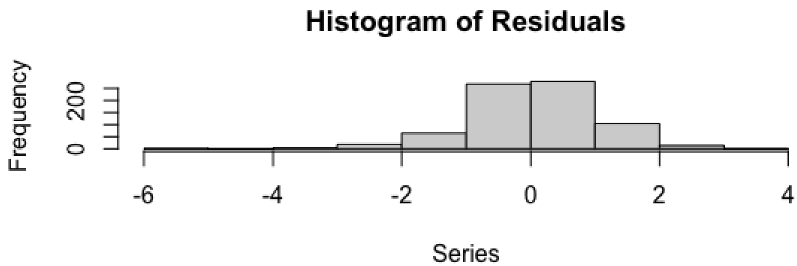
For the diagnostic tests, the Jarque-Bera Test had a p-value that was very small, suggesting that the residuals do not follow a normal distribution. Before declaring this to be true, we need to realize that we have a lot of data and so we want to look at the QQ plots before making a final assessment of the approximate normality/non-normality of the residuals. The Box-Ljung test gives a p-value of 0.4643, which is not too bad considering the amount of data. This p-value suggests that we have done a reasonable job of explaining the volatility with the GARCH$(1,1)$ model. We show the diagnostic plots of the GARCH analysis below. Based on these plots, we see a little bit of assymmetry in the histogram in the residuals - this may be the cause of the low p-value of the Jarque-Bera Test. The assymmetry suggests that one might want to use a the E-GARCH model which utilizes a different distribution on volatility depending on whether the log-returns are positive or negative. The QQ plot for the residuals, from the output generated by plotting the output from garch, looks reasonable and suggests that, based on the tails of the empirical distribution for the residuals, the normal distribution seems to be a reasonable approximation. But this plot is small, and on doing a more "blown-up" QQ plot, we see that the residuals appear to have a heavier tails than normal, and this is particularly true on the lower tail. Also appears from the more detailed QQ plot, that there might be some asymmetries. The plot of the auto-correlation function of the squared values of the original log-return data has a significant correlation implying that there is a temporal correlation of the volatility (justifying trying to utilize the GARCH$(1, 1)$ model for this data). The resultant autocorrelation plot of the squared residuals seems to show that none of the autocorrelations are significantly above the 95% confidence bands - this suggests the resultant squared residuals does not appear to have significant correlations of non-zero lags and gives us some confidence that the GARCH model has done a reasonable job of modeling the volatility.

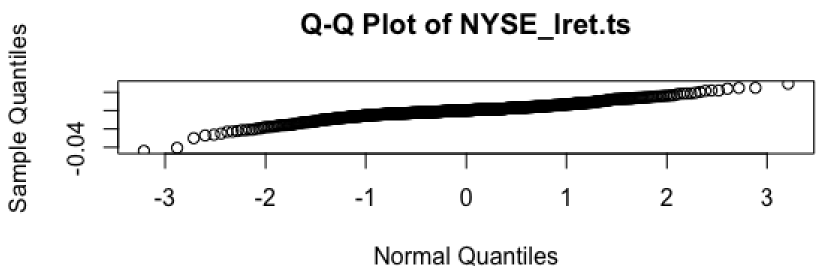
```
plot(NYSE_garchest11)
```

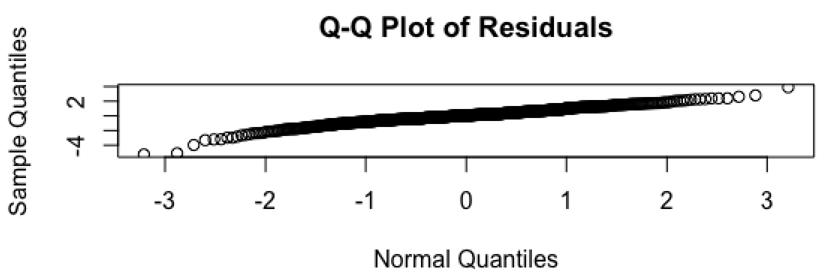


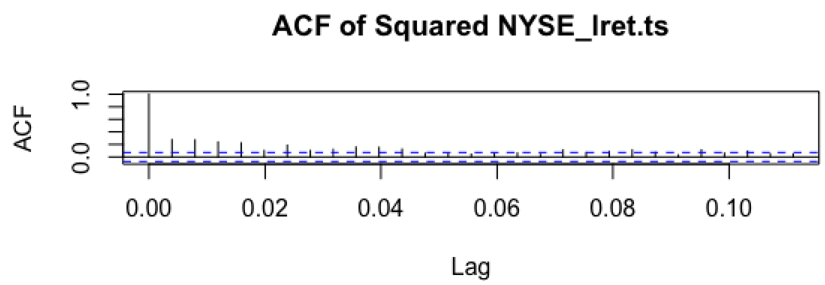


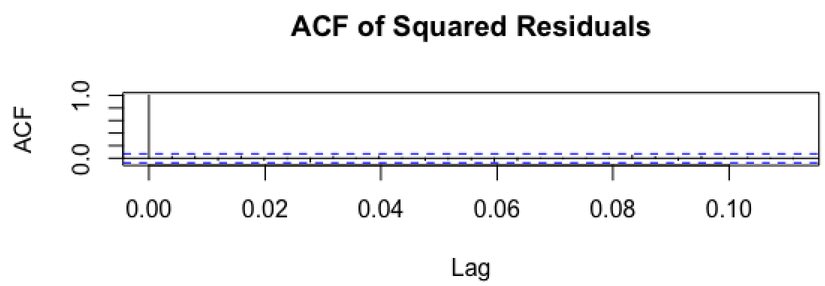

Histogram of NYSE_lret.ts
Frequency
0
200
-0.04
-0.02
0.00
0.02
Series


Histogram of Residuals
Frequency
0
200
-6
-4
-2
0
2
4
Series

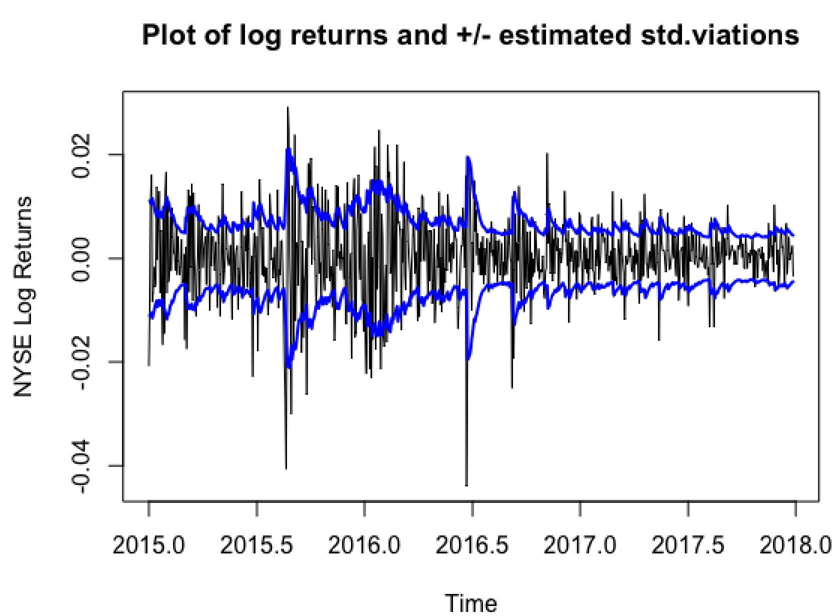

Q-Q Plot of NYSE_lret.ts
Sample Quantiles
-0.04
-3
-2
-1
0
1
2
3
Normal Quantiles


Q-Q Plot of Residuals
Sample Quantiles
-4
2
-3
-2
-1
0
1
2
3
Normal Quantiles

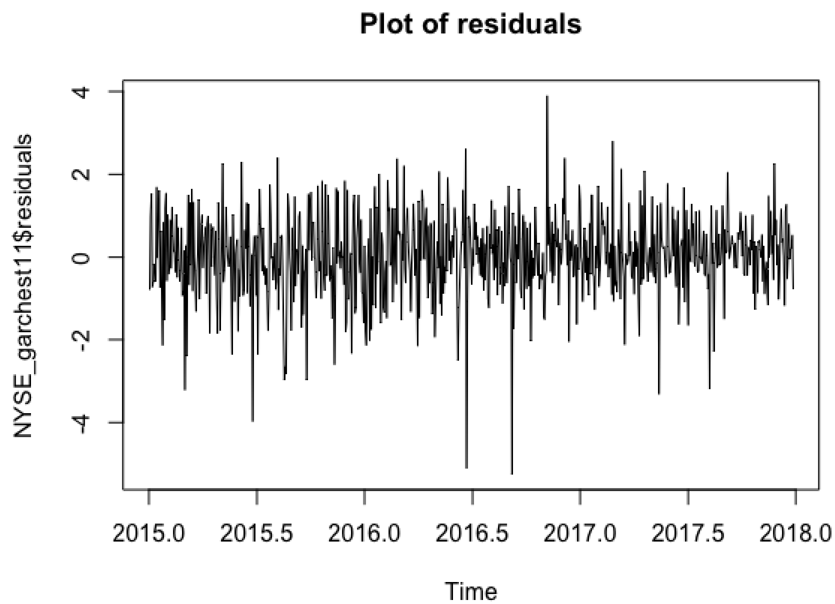


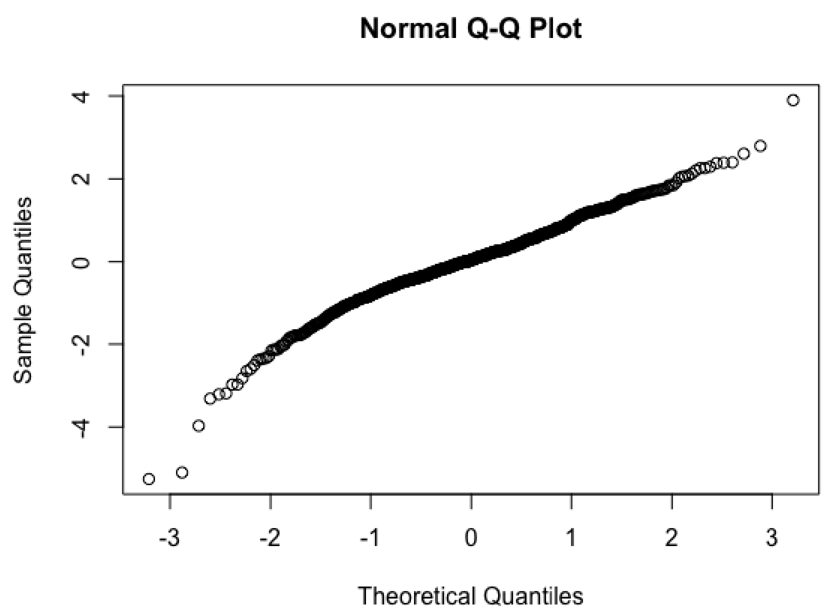



```
> u = predict(NYSE_garchest11)
> plot(NYSE_lret.ts,type='l',xlab='Time',ylab='NYSE Log Returns',
+       main='Plot of log returns and +/- estimated std.viations')
> lines(u[,2],lwd=2,lty=1,col='blue')
> lines(u[,1],lwd=2,lty=1,col='blue')
> legend(1948,.06, c("Log Returns","+/- sigma"), lty=c(1,1),lwd=c(1,2),col=c("black","blue"))
```



```
> plot(NYSE_garchest11$residuals,main='Plot of residuals')
> qqnorm(NYSE_garchest11$residuals)
```

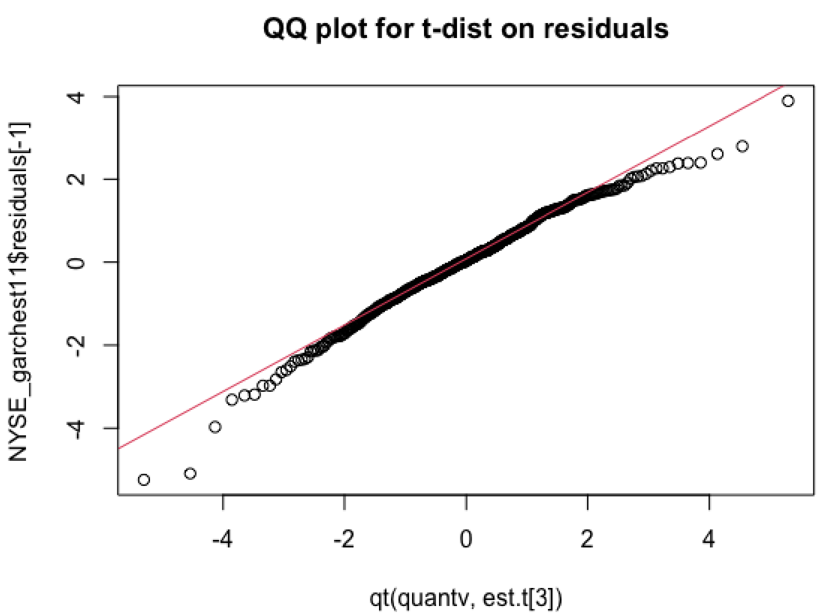




d

The main deficiency of the GARCH$(1, 1)$ model as estimate/fitted seems to be the lack of the fit of the residuals to the normal distribution, as seen by the QQ plot. Would try and fit a t-distribution, and that may help. Specifically, we did do an estimate of the t-distribution relative to the residuals. The results were that we had a shape parameter estimate of 5.3, and estimates for location and scale being .058 and .794, respectively. In addition to the estimation, we carried out a QQ-plot relative to this distribution. For this since for a fixed shape parameter, the t-distribution is closed under changes in location and scale, it is sufficient to plot empirical quantiles vs the theoretical quantiles of the t distribution parameters of location and scale being 0 and 1, and the shape parameter being 5.3, and the QQ plot is shown below. Based on the QQ plot, it appears to provide a better fit, but the log-returns are still heavier tailed than this t-distribution on the lower side, and it is lightertailed than this t-distribution on the upper/positive side. Suggest some sort of assymetric model (such as upcoming APARCH) might be useful.

```
> library('MASS')
> est.t = as.numeric(fitdistr(NYSE_garchest11$residuals[-1],"t")$estimate)
Warning message:
In dt((x - m)/s, df, log = TRUE) : NaNs produced
> est.t
[1] 0.05815966 0.79401129 5.31659664
> N = length(NYSE_garchest11$residuals[-1])
> quantv = (1/(N+1))*seq(1,N,1)
> qqplot(qt(quantv, est.t[3]),NYSE_garchest11$residuals[-1],
+          main='QQ plot for t-dist on residuals')
> qqline(NYSE_garchest11$residuals[-1], distribution = function(p) qt(p,est.t[3]),
+          prob = c(0.1, 0.9), col = 2)
```



e and f

For e, the normal distributional model of the errors, we simply are modeling X_{n+1}, conditional on the data as being $\mathcal{N}\left(0, \hat{\sigma}_{n+1}^2\right)$, and to compute the estimate $\hat{\sigma}_{n+1}$, just utilize the GARCH$(1,1)$ model and the estimates of the parameters, so that

$$\hat{\sigma}_{n+1} = \sqrt{\hat{\alpha}_o + \hat{\beta}_1 \hat{\sigma}_n^2 + \hat{\alpha}_1 X_n^2}.$$

Now the estimates $\hat{\sigma}_n$ are derived for all of the time points corresponding to the original process via the output from the predict command - that is why the first value in the output from predict is NA. So the last value is $\hat{\sigma}_n$ and not $\hat{\sigma}_{n+1}$, and so we need to compute this latter value via R. Utilizing the final value from the predict command and the above, we generated the $\hat{\sigma}_{n+1}$, and then compute the relative VaR for the normal distribution on the GARCH innovations to be .0177 - that is given in the table below.

For f, the t-distribution, we utilize the shape parameter and the scale parameter to model the innovations process, i.e., the model for X_{n+1}, conditional on the past, is that of a t-distribution with shape parameter 9.8 and overall scale parameter of $\hat{\sigma}_{n+1} \cdot \hat{\lambda}$ where $\hat{\lambda}$ is the estimated shape parameter. The resultant relative VaR is .0141 for this case t-distributed innovations. Note that we assumed a mean of 0 , even though the estimated location parameter for the t distribution was not 0 . Part of the reasoning here is that there is a slight positive mean in the original residuals It would probably be better to have included a non-zero mean in the original GARCH model, but the "garch" command does not have this capability.

Distribution	Relative VaR
Normal	0.0117
t	0.0141

```
> # e and f
> pred = predict(NYSE_garchest11)
> sig_p1 = sqrt(NYSE_garchest11$coef[1]+NYSE_garchest11$coef[3]*pred[754]*pred[754]
+                   +NYSE_garchest11$coef[2]*NYSE_lret.ts[754]*NYSE_lret.ts[754])
> sig_p1
         a0
0.004542183
> VaRt_norm = -sig_p1*qnorm(.005,0,1)
> VaRt_norm
        a0
0.01169989
> VaRt_t = -sig_p1*est.t[2]*qt(.005,est.t[3])
> VaRt_t
        a0
0.01410645
```

g

On the highest estimated conditional volatility day, the VaR at q = 0.005 is about .0544, assuming normal errors. Assuming t-distributed errors, the VaR at the same q on the same day is about .0826.

```
> sn_max <- max(NYSE_garchest11$fitted.values[2:754,1])
> -sn_max * qnorm(0.005)
[1] 0.05438817
> -sn_max*qt(0.005, df=5.31)
[1] 0.08263634
```