Checking in

How is the class going?

- Revisit the slido?
- Check in on HW difficulty
- How to do a math/computing class
  - How to find resources
  - How to read math
  - How to take notes & why we often use hand-written lectures rather than slides

Problem Solving Algorithms/Methods

- Unclear work is typically a symptom of unclear thinking
  - It always starts unclear, and revision makes it better
- Heuristic structures can help organize our thoughts on this
  - Ex: We'll look at what they teach beginning engineers at JPL
  - Given, Find, Solve
- The more general the algorithm, probably the more useless it is
- Ex: Feynman Problem Solving Algorithm (guaranteed to work!)
  - Step 1: Write down the problem
  - Step 2: Think really hard
  - Step 3: Write down the solution
- This algorithm always works, but somehow fails to convey any useful information about actually solving the problem
- "A formal manipulator in mathematics often experiences the discomforting feeling that his pencil surpasses him in intelligence" – Howard Eves

Probability

Check in on probability HW

- Go over problems on the probability section and give a sense of how they should go, answer any questions

Linear Algebra

Determinant

Useful for many matrix operations

- Matrix inverses
- Checking if a matrix is invertible
- Calculating eigenvalues

Geometric meaning

Formula

Recap: Inverses, Eigenvalues & Eigenvectors

- Review the definitions for these
- Revisit how we calculate now that we know what the determinant is
- Characteristic polynomial

Trace (skip this?)

Check in

- How is the [homework]() going? Talk over problems
- Posted the code examples we've been using over the last two weeks [here]() (note most of these were already in the slides! But this way you can play with them in a notebook format)

What is a vector space?

A **vector space** is a set, equipped with two operations, addition and scalar multiplication:

$$v, u \in V$$

$$c \in \mathbb{R}\,(\text{or}\,\mathbb{C}, \text{etc.})$$

Vector addition: $v + u$

Scalar multiplication: $c \cdot v$

These two operations must obey the following properties:

https://en.wikipedia.org/wiki/Vector_space

Where u, v are in V, and
F = R or C (real or complex numbers)

commutativity

$u + v = v + u$ for all $u, v \in V$;

associativity

$(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and all $a, b \in \mathbf{F}$;

additive identity

there exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$;

additive inverse

for every $v \in V$, there exists $w \in V$ such that $v + w = 0$;

multiplicative identity

$1v = v$ for all $v \in V$;

distributive properties

$a(u + v) = au + av$ and $(a + b)u = au + bu$ for all $a, b \in \mathbf{F}$ and all $u, v \in V$.

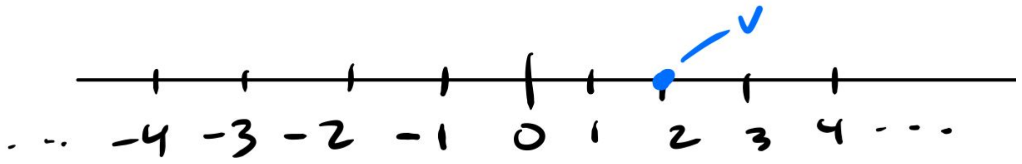
What is a vector space?

- When we talk about a vector space, we'll usually say a "vector space over R" or a "vector space over C"
  - This terminology is defining what set of numbers we will use for scalar multiplication, usually either the real (R) or complex numbers (C)
  - (Although it is possible to use other sets of numbers for scalar multiplication! The numbers we use for scalar multiplication have to form a field—ask me for more info if you're curious!)
- Vector spaces are what we've been working with this whole time! We're just formalizing them a bit.

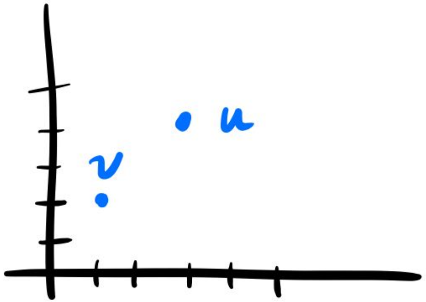
Examples

$\mathbb{R}^n$: real numbers, 2-dimensional reals, 3-dimensional reals, etc!

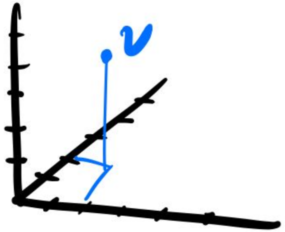
$V = \mathbb{R}$



$V = \mathbb{R}^2$



$V = \mathbb{R}^3$



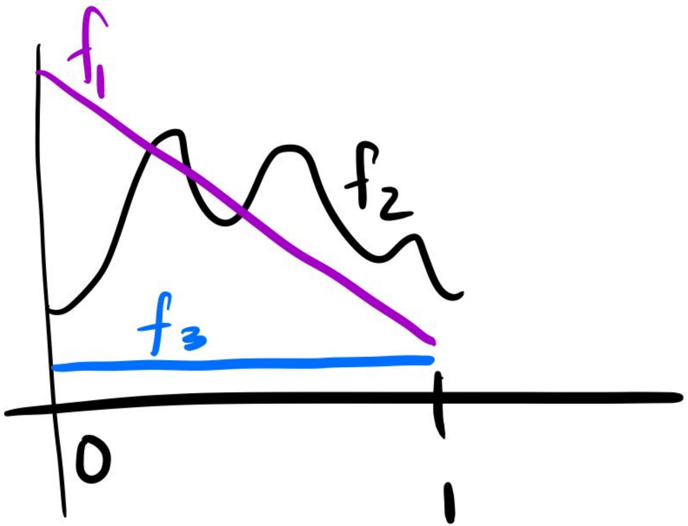
Polynomials of degree n or less

- Is this a vector space? Let's verify together!
    - Verify the properties
    - Discuss how vectors operate much the same way as R^n

- What if we don't bound the degree of n?

Continuous functions on [0,1]

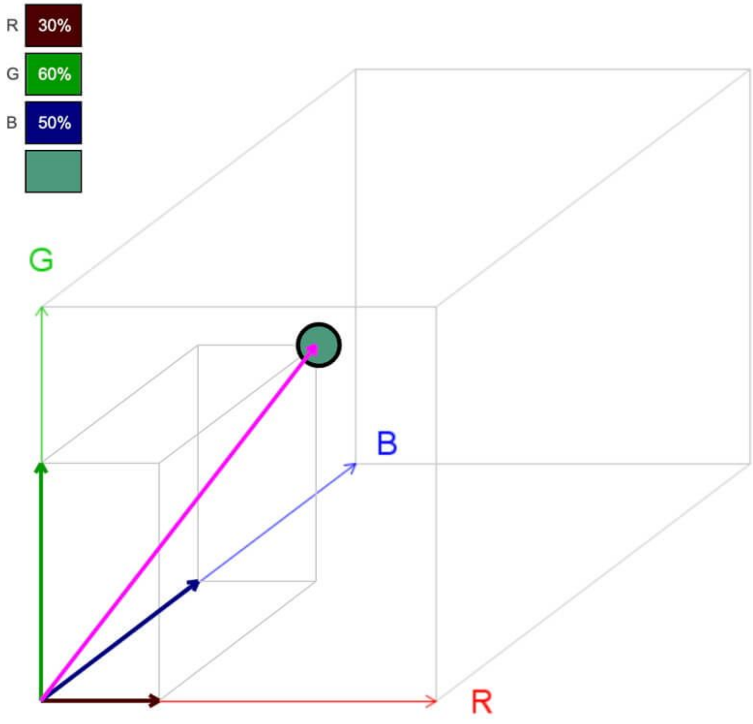
- Is this a vector space?

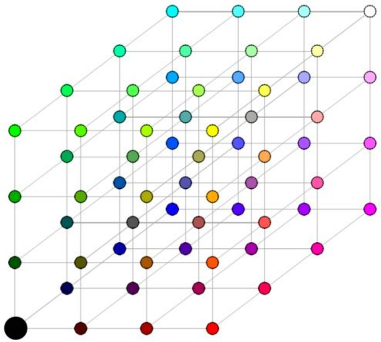


Color space!

- Think about red-green-blue (RGB) triplets as vectors!
  - Whether on the 0-1 range, 0-255 range, or as hex codes
  - Can we add these?
  - Can we scalar multiply, by say, real numbers?
- We can also think about the CMY color space this way too
- A [fun app to explore this](#)

- Is this a vector space?





coolors.com, https://sciencepickle.com/earth-systems/vectors-and-forces/color-vectors/

Subspaces

A **subspace** is a subset of a vector space that is also itself a vector space! (using the same addition and scalar multiplication)

A subset U of V is called a **subspace** of V if U satisfies:

- Additive identity: the zero vector is contained in U
- Closed under addition: if u, v are vectors in U, then the vector u+v is also in U
- Closed under scalar multiplication: if a is a scalar (e.g. in R or C), and u is a vector in U, then au is also a vector in U

Examples

Which of the following are subspaces of R^3? And why?

a) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 + 2x_2 + 3x_3 = 0\}$

b) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 x_2 x_3 = 12\}$

c) The set of vectors with integer components (i.e.

$$\{(x_1, x_2, x_3) | x_1, x_2, x_3 \in \mathbb{Z}\}$$

(where Z is the positive and negative whole numbers and zero)

Span and linear independence

A **linear combination** of a set of vectors $\{v_1, v_2, \ldots, v_n\}$ is a new vector of the form:

$$a_1 v_1 + a_2 v_2 + \cdots a_n v_n$$

Where the a_i are scalars (e.g. numbers in R or C)

Note! Linear combinations are things you can make from the set $\{v_1, v_2, \ldots, v_n\}$ using the operations of the vector space! I.e. by adding the vectors and scalar multiplying them.

For this reason we often say a linear combination like the one above can be **generated** by the set of vectors $\{v_1, v_2, \ldots, v_n\}$

Span

The **span** of a set of vectors $\{v_1, v_2, \ldots, v_n\}$ is the set of all vectors generated by that set—in other words, the span is all linear combinations of the vectors in our starting set. Written mathematically:

$$\{a_1 v_1 + \cdots a_n v_n | a_1, \ldots, a_n \in F\}$$

(where F is usually either R or C)

If our whole vector space V can be spanned by $\{v_1, v_2, \ldots, v_n\}$, we say that this set **spans** (or generates) V. A vector space is called **finite dimensional** if there's a finite set of vectors that span it.

Independence

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ is called **linearly independent** if the only choice of $a_1, \ldots, a_n$ that makes

$$a_1 v_1 + \cdots a_n v_n = 0$$

is $a_1 = \cdots = a_n = 0$.

If we can write $a_1 v_1 + \cdots a_n v_n = 0$ without making all the $a_i = 0$, then the set of vectors is called **linearly dependent**.

Why is this important? This is capturing whether the set of vectors is in some sense redundant. If it's linearly dependent, then anything in the span of can be written in more than one way!

Examples

- The cardinal direction vectors
- {(2,3,1), (1,-1,2), (7,3,8)}

- Try some more! Come up with some different sets of vectors and see if they are independent

- If your set is linearly dependent, then at least one vector in the set can be generated by the remaining vectors (i.e. we could ditch at least one vector and still have the same span!)

Bases

- A **basis** is a set of vectors that is both linearly independent and spans the vector space
- In other words, a basis is a generating set for the vector space that is minimal (i.e. there is no redundancy in the list of vectors used to generate the vector space)

- The cardinal direction vectors are called the **standard basis**
- Any generating set (spanning set) for a vector space can be reduced to form a basis (i.e. we can drop vectors to form a basis)
- The **dimension** of a vector space is the length of a basis for that vector space

What is a matrix? (again)

Matrices move vectors around the vector space, or even take vectors from one vector space to another!

Note: matrices don’t have to ‘fill’ the entire space they map to!

Talk over:

- Domain
- Range (also called image or column space)
- Nullspace or kernel
- 1-to-1 (injective) and onto (surjective)
- Rank of a matrix

Teeny intro to PCA

We will do more on this next week! But wanted to start to introduce principal component analysis (PCA)