Math 2700B Assignment 1
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- Show all the steps of your calculations and justify any statements made. However, do not show any rough work that isn't needed to justify your answers.
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- You should do the work on your own. Read the course syllabus for the rules about scholastic offenses, which include sharing solutions with others, uploading material to a website, viewing material of others or on a website (even if you don't use it), etc. The penalty for cheating on homework will be a grade of 0 on the homework set as well as a penalty of negative 5% on the overall course grade.

Problem 1

[1] We aim to show that $V = \mathbb{R}^2$, equipped with the operations

1. **Vector Addition** $\oplus$:

$$(x_1, y_1) \oplus (x_2, y_2) = \big(x_1 + x_2,\ y_1 + y_2 + 1\big),$$

2. **Scalar Multiplication** $\odot$:

$$a \odot (x, y) = \big(ax,\ ay + a - 1\big),$$

satisfy all the axioms of a real vector space over $\mathbb{R}$.

Proof

1. $(V, \oplus)$ is an abelian group.
2. $\odot : \mathbb{R} \times V \to V$ satisfies the usual scalar-multiplication axioms:
   (a) $a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v})$,
   (b) $(a + b) \odot \mathbf{u} = (a \odot \mathbf{u}) \oplus (b \odot \mathbf{u})$,
   (c) $(ab) \odot \mathbf{u} = a \odot (b \odot \mathbf{u})$,
   (d) $1 \odot \mathbf{u} = \mathbf{u}$.

We set $V = \mathbb{R}^2$ with the operations:

- **Vector addition** $\oplus$:

$$(x_1, y_1) \ \oplus\ (x_2, y_2) \ =\ \big(x_1 + x_2,\ y_1 + y_2 \ +\ 1\big),$$

- **Scalar multiplication** $\odot$:

$$a \ \odot\ (x, y) \ =\ \big(a\,x,\ a\,y \ +\ a \ -\ 1\big),$$

where $a \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$.

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1. $(V, \oplus)$ **is an Abelian Group**

We need to check the following group axioms:

1. **Associativity** of $\oplus$:
$$\big((x_1, y_1) \oplus (x_2, y_2)\big) \oplus (x_3, y_3) \;=\; (x_1, y_1) \oplus \big((x_2, y_2) \oplus (x_3, y_3)\big).$$
Proof:
- **LHS**:
$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2,\ y_1 + y_2 + 1),$$
so
$$\big((x_1, y_1) \oplus (x_2, y_2)\big) \oplus (x_3, y_3) = (x_1 + x_2,\ y_1 + y_2 + 1) \oplus (x_3, y_3).$$
By definition of $\oplus$,
$$= \Big((x_1 + x_2) + x_3,\ (y_1 + y_2 + 1) + y_3 + 1\Big) = (x_1 + x_2 + x_3,\ y_1 + y_2 + y_3 + 2).$$
- **RHS**:
$$(x_2, y_2) \oplus (x_3, y_3) = (x_2 + x_3,\ y_2 + y_3 + 1),$$
so
$$(x_1, y_1) \oplus \big((x_2, y_2) \oplus (x_3, y_3)\big) = (x_1, y_1) \oplus (x_2 + x_3,\ y_2 + y_3 + 1).$$
Hence
$$= \Big(x_1 + (x_2 + x_3),\ y_1 + (y_2 + y_3 + 1) + 1\Big) = (x_1 + x_2 + x_3,\ y_1 + y_2 + y_3 + 2).$$
Both sides agree. Thus, $\oplus$ is associative.

2. **Commutativity** of $\oplus$:
$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2,\ y_1 + y_2 + 1) = (x_2 + x_1,\ y_2 + y_1 + 1) = (x_2, y_2) \oplus (x_1, y_1).$$

3. **Existence of a Zero (Identity) Element $\mathbf{0}$**:
We want $(x, y) \oplus \mathbf{0} = (x, y)$. By inspection, set
$$\mathbf{0} = (0,\ -1).$$
Indeed,
$$(x, y) \oplus (0, -1) = (x + 0,\ y + (-1) + 1) = (x, y).$$

4. **Existence of Additive Inverses**:
We want $(x, y) \oplus (u, v) = \mathbf{0} = (0, -1)$. From
$$(x + u,\ y + v + 1) = (0,\ -1),$$
it follows that $u = -x$ and $y + v + 1 = -1$, so $v = -2 - y$. Hence, the inverse of (x, y) is
$$-(x, y) = (-x,\ -y - 2).$$

Thus, $(V, \oplus)$ is indeed an abelian group.

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2. $\odot$ Satisfies the Scalar Multiplication Axioms

Let $a, b \in \mathbb{R}$ and $(x_1, y_1), (x_2, y_2) \in V$. We check the usual four conditions:

(1) Distributivity of Scalar Multiplication over Vector Addition

Claim:

$$a \odot \big((x_1, y_1) \oplus (x_2, y_2)\big) = \big(a \odot (x_1, y_1)\big) \oplus \big(a \odot (x_2, y_2)\big).$$

Proof:
- **LHS**:

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2,\ y_1 + y_2 + 1),$$

so

$$a \odot \big((x_1, y_1) \oplus (x_2, y_2)\big) = a \odot (x_1 + x_2,\ y_1 + y_2 + 1).$$

By definition of $\odot$,

$$= \Big(a(x_1 + x_2),\ a(y_1 + y_2 + 1) + a - 1\Big).$$

Expand in the second coordinate:

$$a(y_1 + y_2 + 1) + a - 1 = ay_1 + ay_2 + a + a - 1 = ay_1 + ay_2 + 2a - 1.$$

So,

$$\text{LHS} = \big(ax_1 + ax_2,\ ay_1 + ay_2 + 2a - 1\big).$$

- **RHS**:

$$a \odot (x_1, y_1) = (ax_1,\ ay_1 + a - 1),$$
$$a \odot (x_2, y_2) = (ax_2,\ ay_2 + a - 1).$$

Then,

$$\big(a \odot (x_1, y_1)\big) \oplus \big(a \odot (x_2, y_2)\big) = (ax_1,\ ay_1 + a - 1) \oplus (ax_2,\ ay_2 + a - 1).$$

By definition of $\oplus$,

$$= \Big(ax_1 + ax_2,\ (ay_1 + a - 1) + (ay_2 + a - 1) + 1\Big).$$

Combine like terms in the second coordinate:

$$= \Big(ax_1 + ax_2,\ ay_1 + ay_2 + 2a - 1\Big).$$

This agrees exactly with the LHS. Hence, distributivity over $\oplus$ holds.

(2) Distributivity of Scalar Multiplication over Real-Field Addition

Claim:
$$(a+b) \odot (x,y) = \big(a \odot (x,y)\big) \oplus \big(b \odot (x,y)\big).$$

Proof:
- **LHS**:
$$(a+b) \odot (x,y) = \big((a+b)x,\ (a+b)y + (a+b) - 1\big).$$
- **RHS**:
$$a \odot (x,y) = (ax,\ ay + a - 1),$$
$$b \odot (x,y) = (bx,\ by + b - 1).$$

Then,
$$(a \odot (x,y)) \oplus (b \odot (x,y)) = (ax,\ ay + a - 1) \oplus (bx,\ by + b - 1).$$
By definition of $\oplus$,
$$= \Big(ax + bx,\ (ay + a - 1) + (by + b - 1) + 1\Big).$$
Combine the second coordinate carefully:
$$= \Big((a+b)x,\ (a+b)y + (a+b) - 1\Big).$$
This matches the LHS exactly. Hence, scalar-addition distributivity holds.

(3) Compatibility of Scalar Multiplication with Real-Field Multiplication

Claim:
$$(ab) \odot (x,y) = a \odot \big(b \odot (x,y)\big).$$

Proof:
- **LHS**:
$$(ab) \odot (x,y) = \Big((ab)x,\ (ab)y + (ab) - 1\Big).$$
- **RHS**: First compute $b \odot (x,y)$:
$$b \odot (x,y) = (bx,\ by + b - 1).$$

Then,
$$a \odot \big(b \odot (x,y)\big) = a \odot (bx,\ by + b - 1) = \Big(a(bx),\ a(by + b - 1) + a - 1\Big).$$
Simplify the second coordinate:
$$a(by + b - 1) + a - 1 = aby + ab - a + a - 1 = aby + ab - 1.$$
So, the RHS becomes
$$(abx,\ aby + ab - 1).$$
Since $abx = (ab)x$ and $aby + ab - 1 = (ab)y + (ab) - 1$, both sides agree.

Thus, compatibility with field multiplication holds.

(4) Multiplying by 1 Acts as the Identity

Claim:

$$1 \odot (x, y) = (x, y).$$

Proof: By definition of $\odot$,

$$1 \odot (x, y) = \big(1 \cdot x,\ 1 \cdot y + 1 - 1\big) = (x,\ y).$$

So, the unit scalar $1 \in \mathbb{R}$ acts as the identity on vectors.

$\square$

Conclusion

All eight vector-space axioms (the abelian-group axioms for $\oplus$ and the four scalar-multiplication axioms) are satisfied by $\big(\mathbb{R}^2, \oplus, \odot\big)$. Therefore, $V = \mathbb{R}^2$ with these custom operations is indeed a real vector space.

Problem 2

We want to determine if each given subset $W \subseteq V$ is a subspace of the vector space V. Subset W of a vector space V over a field F is a **subspace** if and only if:

(a) $V = F(\mathbb{R}, \mathbb{R}),\ F = \mathbb{R}$
$W = \{f \mid f(x) \geq 0,\ \forall x \in \mathbb{R}\}$

(b) $V = F(\mathbb{R}, \mathbb{R}),\ F = \mathbb{R}$
$W = \{f \mid f(x+y) = f(x) + f(y),\ \forall x, y \in \mathbb{R}\}$

Solution

1. W is **nonempty**, and in particular, contains the 0-vector (i.e., the zero function when V is a function space).

2. W is **closed under addition**: for any $u, v \in W$, we have $u + v \in W$.

3. W is **closed under scalar multiplication**: for any $u \in W$ and any scalar $\alpha \in F$, we have $\alpha u \in W$.

(a) $V = F(\mathbb{R}, \mathbb{R}), \quad W = \{f : \mathbb{R} \to \mathbb{R} \mid f(x) \geq 0,\ \forall x \in \mathbb{R}\}$

1. **Nonempty and contains the zero function.**
   The zero function $f_0(x) = 0$ for all x satisfies $f_0(x) \geq 0$. Hence, $f_0 \in W$, so W is nonempty.

2. **Closed under addition.**
   Let $f, g \in W$. Then for all $x \in \mathbb{R}$,
   $$f(x) \geq 0 \quad \text{and} \quad g(x) \geq 0.$$
   Therefore,
   $$(f + g)(x) = f(x) + g(x) \geq 0 + 0 = 0.$$
   Thus, $f + g \in W$, and W is closed under addition.

3. **Closed under scalar multiplication.**
   Consider a nonzero scalar $\alpha \in \mathbb{R}$ and a function $f \in W$. If $\alpha < 0$, then for any x where $f(x) > 0$,
   $$(\alpha f)(x) = \alpha \cdot f(x) < 0.$$
   This violates the condition $(\alpha f)(x) \geq 0$. Therefore, $\alpha f \notin W$ when $\alpha < 0$.

Conclusion: Since W is not closed under scalar multiplication by negative scalars, W is **not** a subspace of V.

(b) $V = F(\mathbb{R}, \mathbb{R}), \quad W = \{f : \mathbb{R} \to \mathbb{R} \mid f(x+y) = f(x) + f(y), \ \forall x, y \in \mathbb{R}\}$ These functions are known as **additive functions**.

1. **Nonempty and contains the zero function.**
   The zero function f_0 satisfies

   $$f_0(x+y) = 0 = 0 + 0 = f_0(x) + f_0(y).$$

   Hence, $f_0 \in W$, so W is nonempty.

2. **Closed under addition.**
   Let $f, g \in W$. Then for all $x, y \in \mathbb{R}$,

   $$f(x+y) = f(x) + f(y) \quad \text{and} \quad g(x+y) = g(x) + g(y).$$

   Therefore,

   $$(f+g)(x+y) = f(x+y)+g(x+y) = [f(x)+f(y)]+[g(x)+g(y)] = [f(x)+g(x)]+[f(y)+g(y)] = (f+g)(x$$

   Thus, $f + g \in W$, and W is closed under addition.

3. **Closed under scalar multiplication.**
   Let $f \in W$ and $\alpha \in \mathbb{R}$. Then for all $x, y \in \mathbb{R}$,

   $$(\alpha f)(x+y) = \alpha f(x+y) = \alpha[f(x) + f(y)] = \alpha f(x) + \alpha f(y) = (\alpha f)(x) + (\alpha f)(y).$$

   Hence, $\alpha f \in W$, and W is closed under scalar multiplication.

Conclusion: Since W is nonempty and closed under both addition and scalar multiplication, W **is** a subspace of V.

Final Answer

(a) The set $W = \{f \mid f(x) \geq 0 \text{ for all } x \in \mathbb{R}\}$ is **not** a subspace of V because it is not closed under scalar multiplication by negative scalars.

(b) The set $W = \{f \mid f(x+y) = f(x) + f(y) \text{ for all } x, y \in \mathbb{R}\}$ **is** a subspace of V as it contains the zero function and is closed under both addition and scalar multiplication.

Problem 3

(a) **Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.**

Proof. To show that $W_1 + W_2$ is a subspace of V, we need to verify three properties:

1. **Non-emptiness:** The zero vector 0 is in W_1 and W_2 since they are subspaces. Thus, $0 = 0 + 0 \in W_1 + W_2$.
2. **Closure under addition:** Let $u, v \in W_1 + W_2$. By definition, there exist vectors $w_1, w_1' \in W_1$ and $w_2, w_2' \in W_2$ such that
$$u = w_1 + w_2 \quad \text{and} \quad v = w_1' + w_2'.$$
Then,
$$u + v = (w_1 + w_2) + (w_1' + w_2') = (w_1 + w_1') + (w_2 + w_2').$$
Since W_1 and W_2 are subspaces, $w_1 + w_1' \in W_1$ and $w_2 + w_2' \in W_2$. Therefore, $u + v \in W_1 + W_2$.
3. **Closure under scalar multiplication:** Let $u \in W_1 + W_2$ and $\alpha \in F$. Then $u = w_1 + w_2$ for some $w_1 \in W_1$ and $w_2 \in W_2$. Thus,
$$\alpha u = \alpha(w_1 + w_2) = \alpha w_1 + \alpha w_2.$$
Since W_1 and W_2 are subspaces, $\alpha w_1 \in W_1$ and $\alpha w_2 \in W_2$. Hence, $\alpha u \in W_1 + W_2$.

Additionally, $W_1 + W_2$ contains both W_1 and W_2:

$$\text{For } w_1 \in W_1, \quad w_1 = w_1 + 0 \in W_1 + W_2.$$

$$\text{For } w_2 \in W_2, \quad w_2 = 0 + w_2 \in W_1 + W_2.$$

Therefore, $W_1 + W_2$ is a subspace of V containing both W_1 and W_2. □

(b) **Prove that any subspace of V that contains both W_1 and W_2 must contain $W_1 + W_2$.**

Proof. Let U be a subspace of V such that $W_1 \subseteq U$ and $W_2 \subseteq U$. We aim to show that $W_1 + W_2 \subseteq U$.

Take any vector $w \in W_1 + W_2$. By definition, there exist $w_1 \in W_1$ and $w_2 \in W_2$ such that
$$w = w_1 + w_2.$$
Since $W_1 \subseteq U$ and $W_2 \subseteq U$, both w_1 and w_2 are in U. Because U is a subspace, it is closed under addition, hence
$$w = w_1 + w_2 \in U.$$
Since w was arbitrary, $W_1 + W_2 \subseteq U$. □

(c) **Show that a vector space W is a direct sum of subspaces W_1 and W_2 if and only if each vector in W can be written uniquely as $x_1 + x_2$ where $x_1 \in W_1$ and $x_2 \in W_2$.**

Proof. We need to prove both directions of the equivalence.

($\Rightarrow$) Assume $W = W_1 \oplus W_2$.

By definition of direct sum:

$$W = W_1 + W_2 \quad \text{and} \quad W_1 \cap W_2 = \{0\}.$$

- **Existence:** Every $w \in W$ can be written as $w = x_1 + x_2$ for some $x_1 \in W_1$ and $x_2 \in W_2$. - **Uniqueness:** Suppose $w = x_1 + x_2 = y_1 + y_2$ with $x_1, y_1 \in W_1$ and $x_2, y_2 \in W_2$. Then,

$$x_1 - y_1 = y_2 - x_2.$$

The left side $x_1 - y_1$ is in W_1, and the right side $y_2 - x_2$ is in W_2. Therefore,

$$x_1 - y_1 \in W_1 \cap W_2 = \{0\}.$$

Hence, $x_1 = y_1$ and consequently $x_2 = y_2$. This proves uniqueness.

($\Leftarrow$) Assume every $w \in W$ can be written uniquely as $w = x_1 + x_2$ with $x_1 \in W_1$ and $x_2 \in W_2$.

- **Sum Equals W:** By assumption, every $w \in W$ is in $W_1 + W_2$, hence $W = W_1 + W_2$.
- **Intersection is Trivial:** Suppose $v \in W_1 \cap W_2$. Then v can be written as

$$v = v + 0 = 0 + v,$$

where $v \in W_1$ and $0 \in W_2$, and $0 \in W_1$ and $v \in W_2$. By uniqueness,

$$v = 0.$$

Therefore, $W_1 \cap W_2 = \{0\}$.

Combining these, $W = W_1 \oplus W_2$. □

Conclusion

We have established that:

- $W_1 + W_2$ is indeed a subspace containing both W_1 and W_2.
- Any subspace containing W_1 and W_2 must also contain $W_1 + W_2$.
- The direct sum $W = W_1 \oplus W_2$ is characterized by the unique decomposition of each vector in W as a sum of vectors from W_1 and W_2.

□

Problem 4

We work over the vector space $F(\mathbb{R}, \mathbb{R})$ of all real-valued functions on $\mathbb{R}$. Recall that:

- A function g is **even** if $g(-t) = g(t)$ for all $t \in \mathbb{R}$.
- A function g is **odd** if $g(-t) = -g(t)$ for all $t \in \mathbb{R}$.

We denote

$$F^+(\mathbb{R}, \mathbb{R}) = \{\, g \in F(\mathbb{R}, \mathbb{R}) : g(-t) = g(t) \,\},$$
$$F^-(\mathbb{R}, \mathbb{R}) = \{\, g \in F(\mathbb{R}, \mathbb{R}) : g(-t) = -g(t) \,\}.$$

(a) **Proving that $F^+(\mathbb{R}, \mathbb{R})$ and $F^-(\mathbb{R}, \mathbb{R})$ are Subspaces**

To show that each set is a subspace of $F(\mathbb{R}, \mathbb{R})$, we verify the following three properties:

1. **Zero Function is in Each Subspace**
   - **Even Subspace F^+:** The zero function 0 satisfies
     $$0(-t) = 0 = 0(t),$$
     so it is even.
   - **Odd Subspace F^-:** The zero function also satisfies
     $$0(-t) = 0 = -0(t),$$
     so it is odd.
2. **Closed Under Addition**
   - **Even Subspace F^+:** Let $f, g \in F^+(\mathbb{R}, \mathbb{R})$. Then
     $$f(-t) = f(t), \quad g(-t) = g(t) \quad \text{for all } t \in \mathbb{R}.$$
     Consider $(f + g)$. We have
     $$(f + g)(-t) = f(-t) + g(-t) = f(t) + g(t) = (f + g)(t).$$
     Thus $f + g$ is even, so $f + g \in F^+$.
   - **Odd Subspace F^-:** Let $f, g \in F^-(\mathbb{R}, \mathbb{R})$. Then
     $$f(-t) = -f(t), \quad g(-t) = -g(t) \quad \text{for all } t \in \mathbb{R}.$$
     Consider $(f + g)$. We have
     $$(f + g)(-t) = f(-t) + g(-t) = -f(t) - g(t) = -(f + g)(t).$$
     Thus $f + g$ is odd, so $f + g \in F^-$.

3. **Closed Under Scalar Multiplication**
   - **Even Subspace F^+:** Let $f \in F^+(\mathbb{R}, \mathbb{R})$ and $\alpha \in \mathbb{R}$. Then

   $$f(-t) = f(t).$$

   Consider αf. We have

   $$(\alpha f)(-t) = \alpha f(-t) = \alpha f(t) = (\alpha f)(t).$$

   Hence αf is even, so $\alpha f \in F^+$.
   - **Odd Subspace F^-:** Let $f \in F^-(\mathbb{R}, \mathbb{R})$ and $\alpha \in \mathbb{R}$. Then

   $$f(-t) = -f(t).$$

   Consider αf. We have

   $$(\alpha f)(-t) = \alpha f(-t) = \alpha(-f(t)) = -\alpha f(t) = -(\alpha f)(t).$$

   Hence αf is odd, so $\alpha f \in F^-$.

Since all three properties are satisfied for both $F^+(\mathbb{R}, \mathbb{R})$ and $F^-(\mathbb{R}, \mathbb{R})$, both are indeed **subspaces** of $F(\mathbb{R}, \mathbb{R})$.

$\square$

(b) **Proving that $F(\mathbb{R}, \mathbb{R}) = F^+(\mathbb{R}, \mathbb{R}) \oplus F^-(\mathbb{R}, \mathbb{R})$**

To establish that

$$F(\mathbb{R}, \mathbb{R}) = F^+(\mathbb{R}, \mathbb{R}) \oplus F^-(\mathbb{R}, \mathbb{R}),$$

we need to verify two conditions:

1. **Every Function Decomposes into an Even and an Odd Part**
   Given any $h \in F(\mathbb{R}, \mathbb{R})$, define

   $$h^+(t) = \frac{1}{2}\big(h(t) + h(-t)\big), \quad h^-(t) = \frac{1}{2}\big(h(t) - h(-t)\big).$$

   - **h^+ is Even:**

   $$h^+(-t) = \frac{1}{2}\big(h(-t) + h(t)\big) = \frac{1}{2}\big(h(t) + h(-t)\big) = h^+(t).$$

   - **h^- is Odd:**

   $$h^-(-t) = \frac{1}{2}\big(h(-t) - h(t)\big) = -\frac{1}{2}\big(h(t) - h(-t)\big) = -h^-(t).$$

   Moreover,

   $$h(t) = \frac{1}{2}\big(h(t) + h(-t)\big) + \frac{1}{2}\big(h(t) - h(-t)\big) = h^+(t) + h^-(t).$$

   Hence every $h \in F(\mathbb{R}, \mathbb{R})$ can be written as the sum of an even function h^+ and an odd function h^-, i.e., $h = h^+ + h^-$ with $h^+ \in F^+$ and $h^- \in F^-$.

2. **Intersection is Only the Zero Function**
   Suppose a function f is both even and odd. Then:

$$f(-t) = f(t) \quad \text{(even)},$$

$$f(-t) = -f(t) \quad \text{(odd)}.$$

   Combining these equations, we get

$$f(t) = -f(t),$$

   which implies

$$f(t) = 0 \quad \text{for all } t \in \mathbb{R}.$$

   Therefore, the only function in $F^+(\mathbb{R}, \mathbb{R}) \cap F^-(\mathbb{R}, \mathbb{R})$ is the zero function.

Since both conditions are satisfied, we conclude that

$$F(\mathbb{R}, \mathbb{R}) = F^+(\mathbb{R}, \mathbb{R}) \oplus F^-(\mathbb{R}, \mathbb{R}).$$

□

Problem 5 (a)

Part 1

Let

$$f(x) = \frac{1}{x^2 + x - 6}, \quad g(x) = \frac{1}{x^2 - 5x + 6}, \quad h(x) = \frac{1}{x^2 - 9},$$

and consider the vector space

$$V = F([0,1], \mathbb{R}).$$

We wish to show that the set

$$S = \{f(x),\, g(x),\, h(x)\}$$

is linearly independent.

Step 1.1. Factor the Denominators. Observe that

$$\begin{aligned} x^2 + x - 6 &= (x-2)(x+3), \\ x^2 - 5x + 6 &= (x-2)(x-3), \\ x^2 - 9 &= (x-3)(x+3). \end{aligned}$$

Since $x - 2$ and $x - 3$ are negative on $[0,1]$ and $x + 3 > 0$ on $[0,1]$, none of these factors vanish on the interval.

Step 1.2. Assume a Linear Combination Equals the Zero Function. Suppose there exist constants a, b, and c (not all zero) such that

$$a\, f(x) + b\, g(x) + c\, h(x) = 0 \quad \text{for all } x \in [0,1].$$

That is,

$$a \frac{1}{(x-2)(x+3)} + b \frac{1}{(x-2)(x-3)} + c \frac{1}{(x-3)(x+3)} = 0.$$

Step 1.3. Clear the Denominators. Multiply the above equation by the common denominator $(x-2)(x-3)(x+3)$ (which is never zero on $[0,1]$) to obtain

$$a\,(x-3)(x+3) + b\,(x-2)(x+3) + c\,(x-2)(x-3) = 0 \quad \text{for all } x.$$

Computing the products gives:

$$\begin{aligned} (x-3)(x+3) &= x^2 - 9, \\ (x-2)(x+3) &= x^2 + x - 6, \\ (x-2)(x-3) &= x^2 - 5x + 6. \end{aligned}$$

Thus, the equation becomes

$$a(x^2 - 9) + b(x^2 + x - 6) + c(x^2 - 5x + 6) = 0 \quad \text{for all } x.$$

Step 1.4. Equate Coefficients. Collect like terms:

$$(a + b + c)x^2 + (b - 5c)x + (-9a - 6b + 6c) = 0.$$

For this polynomial to be identically zero, each coefficient must vanish:

$$\begin{cases} a+b+c=0, \\ b-5c=0, \\ -9a-6b+6c=0. \end{cases}$$

From $b-5c=0$ we have

$$b=5c.$$

Substitute into the first equation:

$$a+5c+c=a+6c=0 \quad \Longrightarrow \quad a=-6c.$$

Substitute $a=-6c$ and $b=5c$ into the third equation:

$$-9(-6c)-6(5c)+6c=54c-30c+6c=30c=0.$$

Thus, $c=0$, and consequently, $b=0$ and $a=0$.

Since the only solution is $a=b=c=0$, the functions f, g, and h are **linearly independent**.

Part 2

Let

$$V=F(\mathbb{R},\mathbb{R}) \quad \text{and} \quad S=\{x,\, e^x,\, e^{2x}\}.$$

Assume there exist constants a, b, and c such that

$$a\,x+b\,e^x+c\,e^{2x}=0 \quad \text{for all } x\in\mathbb{R}.$$

We wish to show that $a=b=c=0$.

A common strategy is to evaluate the function and its derivatives at a convenient point (say, $x=0$).

1. **At $x=0$:**
$$a\cdot 0+b\,e^0+c\,e^0=b+c=0 \quad \Longrightarrow \quad b+c=0.$$

2. **Differentiate once:**
$$\frac{d}{dx}\left(a\,x+b\,e^x+c\,e^{2x}\right)=a+b\,e^x+2c\,e^{2x}=0 \quad \text{for all } x.$$
Evaluate at $x=0$:
$$a+b+2c=0.$$

3. **Differentiate a second time:**
$$\frac{d^2}{dx^2}\left(a\,x+b\,e^x+c\,e^{2x}\right)=b\,e^x+4c\,e^{2x}=0 \quad \text{for all } x.$$
Evaluate at $x=0$:
$$b+4c=0.$$

Thus, we have the system:

$$\begin{cases} b + c = 0, \\ a + b + 2c = 0, \\ b + 4c = 0. \end{cases}$$

From the first equation, $b = -c$. Substituting into the third equation:

$$-c + 4c = 3c = 0 \quad \Longrightarrow \quad c = 0.$$

Then $b = -c = 0$. Finally, substitute into the second equation:

$$a + 0 + 0 = 0 \quad \Longrightarrow \quad a = 0.$$

Thus, $a = b = c = 0$, and the set $\{x, e^x, e^{2x}\}$ is **linearly independent**.

Problem 5 (b)

We are given the matrices

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

which form a basis for $M_{2\times 2}(\mathbb{R})$ since

$$M_{2\times 2}(\mathbb{R}) = \text{Span}\{E_{11}, E_{12}, E_{21}, E_{22}\}.$$

We must show that

$$M_{2\times 2}(\mathbb{R}) = \text{Span}\{A, B, C, D\},$$

where

$$\begin{aligned} A &= E_{12} + E_{21} + E_{22}, \\ B &= E_{11} + E_{22}, \\ C &= E_{12} + E_{21}, \\ D &= E_{11} + E_{12} + E_{22}. \end{aligned}$$

Step 2.1. Express the Standard Basis Matrices in Terms of A, B, C, D.
Since each of A, B, C, and D is a linear combination of the E_{ij}'s, it suffices to show that every E_{ij} can be written as a linear combination of A, B, C, and D.

1. **Expressing E_{11}:** Notice that

$$B = E_{11} + E_{22} \quad \text{and} \quad D = E_{11} + E_{12} + E_{22}.$$

Thus,

$$D - B = (E_{11} + E_{12} + E_{22}) - (E_{11} + E_{22}) = E_{12}.$$

Also, observe that

$$C = E_{12} + E_{21} \quad \Longrightarrow \quad E_{21} = C - E_{12} = C - (D - B).$$

A convenient expression for E_{11} is obtained by noting that

$$E_{11} = B + C - A.$$

Verification:

$$B + C - A = (E_{11} + E_{22}) + (E_{12} + E_{21}) - (E_{12} + E_{21} + E_{22}) = E_{11}.$$

2. **Expressing E_{12}:** As already found,

$$E_{12} = D - B.$$

3. **Expressing E_{21}:** Since

$$C = E_{12} + E_{21},$$

and we have $E_{12} = D - B$, it follows that

$$E_{21} = C - (D - B) = C - D + B.$$

4. **Expressing E_{22}:** From

$$A = E_{12} + E_{21} + E_{22},$$

and noting that $C = E_{12} + E_{21}$, we have

$$E_{22} = A - C.$$

Step 2.2. Conclude the Spanning. Since every standard basis element E_{11}, E_{12}, E_{21}, and E_{22} can be written as a linear combination of A, B, C, and D, it follows that

$$\text{Span}\{A, B, C, D\} \supset \text{Span}\{E_{11}, E_{12}, E_{21}, E_{22}\} = M_{2\times 2}(\mathbb{R}).$$

Conversely, each of A, B, C, and D is clearly a linear combination of the E_{ij}'s, so

$$\text{Span}\{A, B, C, D\} \subset M_{2\times 2}(\mathbb{R}).$$

Thus, we conclude that

$$\text{Span}\{A, B, C, D\} = M_{2\times 2}(\mathbb{R}).$$

Final Answers

- **(a)**
  - The set
    $$\left\{\frac{1}{x^2+x-6}, \frac{1}{x^2-5x+6}, \frac{1}{x^2-9}\right\}$$
    in $F([0,1],\mathbb{R})$ is **linearly independent**.
  - The set $\{x, e^x, e^{2x}\}$ in $F(\mathbb{R},\mathbb{R})$ is **linearly independent**.

- **(b)** We have shown that

  $M_{2\times 2}(\mathbb{R}) = \operatorname{Span}\{E_{11}, E_{12}, E_{21}, E_{22}\} = \operatorname{Span}\{E_{12}+E_{21}+E_{22},\ E_{11}+E_{22},\ E_{12}+E_{21},\ E_{11}+E_{12}+E_{22}\}.$

  In particular, by expressing each standard basis element in terms of the latter set, we conclude that the four given matrices span $M_{2\times 2}(\mathbb{R})$.

References

[1] ProofWiki Contributors. *Real Vector Space is Vector Space*. ProofWiki. `https://proofwiki.org/wiki/Real_Vector_Space_is_Vector_Space`. Accessed January 28, 2025.