Instructions for GradeScope:

You may handwrite your answers or type them. Only type solutions if you are able to type the appropriate symbols. If **typed**, use at least a 12pt font, with **one question per page**. You can then directly submit the pdf file to GradeScope.

If **handwritten on a tablet**:

- You can save your handwritten solutions to pdf files and upload the pdf files to the GradeScope website.
- You then do not have to scan your solutions.

If **handwritten on paper**:

- Don't erase or cross out more than a word or two.
- Only write on one side of the page. Use dark, large, neat writing.
- If you have a real scanner, great! Most people will scan their solutions using their phone. Use one of the recommended apps: For iOS, use Scannable by Evernote or Genius Scan. For Android, use Genius Scan. Do *not* just take regular photos. Read the pages at the end of this document for GradeScope's instructions.
- When scanning, have good lighting and try to avoid shadows.
- It is best to have **one question per scan**. If the solution is short, **fold the page in half** and scan just the half it is on, so there isn't so much blank space. Or, you can scan and then crop the scan. You don't need to scan parts (a), (b), etc. separately or put them on separate pages.
- It works well to scan each question separately and produce one pdf file with one question per page. Most scanning apps will automatically combine your images into one pdf file.
- You must **check the quality** of your scans afterwards, and rescan if needed.

You must access Gradescope by clicking on the link on the course OWL Brightspace page. You do not need to create a Gradescope account or use a course access code.

You can **resubmit** your work any number of times until the deadline.

Don't forget to accurately **match questions to pages**. If you do this incorrectly, the grader will not see your solution and will give you zero.

See the GradeScope help website for lots of information: `https://help.gradescope.com/`
Select "Student Center" and then either "Scanning Work on a Mobile Device" or "Submitting an Assignment".

Instructions for writing solutions:

- Homework is graded both on **correctness** and on **presentation/style**.
- **Show all the steps of your calculations and justify any statements made.** However, do not show any rough work that isn't needed to justify your answers.
- Do not cross out or erase more than a word or two. If you write each final solution on a new page, it's easy to start over on a fresh page when you've made a large error.
- You should do the work **on your own**. Read the course syllabus for the rules about scholastic offences, which include sharing solutions with others, uploading material to a website, viewing material of others or on a website (even if you don't use it), etc. The penalty for cheating on homework will be a grade of **0** on the homework set as well as a penalty of **negative 5%** on the overall course grade.

1. Prove that $V = \mathbb{R}^2$ with addition defined as

$$(x_1, y_1) \oplus (x_2, y_2) \equiv (x_1 + x_2, y_1 + y_2 + 1), (x_1, y_1), (x_2, y_2) \in V$$

and scalar multiplication defined as

$$a \odot (x, y) \equiv (ax, ay + a - 1), (x, y) \in V, a \in \mathbb{R}$$

is a real vector space under these operations.

2. Which of the following sets W are subspaces of the given vector space V over the field F? Support your answer.

   (a) $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$, $F = \mathbb{R}$
   $$W = \{f | f(x) \geq 0, \text{ for all } x \in \mathbb{R}\}$$

   (b) $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$, $F = \mathbb{R}$
   $$W = \{f | f(x + y) = f(x) + f(y) \text{ for all } x, y \in \mathbb{R}\}$$

3. Let V be an F vector space. Let W_1 and W_2 be subspaces of V. A vector space W is called the *direct sum* of W_1 and W_2 if $W_1 \cap W_2 = \{0\}$ and $W = W_1 + W_2$ where

$$W_1 + W_2 = \{\mathbf{w}_1 + \mathbf{w}_2 | \mathbf{w}_1 \in W_1, \mathbf{w}_2 \in W_2\}$$

We denote this direct sum by $W = W_1 \oplus W_2$.

   (a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

   (b) Prove that any subspace of V that contains both W_1 and W_2 must contain $W_1 + W_2$.

   (c) Show that a vector space W is a direct sum of subspaces W_1 and W_2 if and only if each vector in W can be written uniquely as $\mathbf{x}_1 + \mathbf{x}_2$ where $\mathbf{x}_1 \in W_1, \mathbf{x}_2 \in W_2$.

4. A function $g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ is called an *even function* if $g(-t) = g(t)$ for all $t \in \mathbb{R}$ and is called an *odd function* if $g(-t) = -g(t)$.

   (a) Prove that $\mathcal{F}_+(\mathbb{R}, \mathbb{R})$, the set of even functions in $\mathcal{F}(\mathbb{R}, \mathbb{R})$ and $\mathcal{F}_-(\mathbb{R}, \mathbb{R})$, the set of odd functions in $\mathcal{F}(\mathbb{R}, \mathbb{R})$ are both subspaces of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

   (b) Prove that $\mathcal{F}(\mathbb{R}, \mathbb{R}) = \mathcal{F}_+(\mathbb{R}, \mathbb{R}) \oplus \mathcal{F}_-(\mathbb{R}, \mathbb{R})$.

5. (a) Are the following subsets of V linearly independent or linearly dependent? If dependent, find a linear dependence relation.

   - $V = \mathcal{F}([0, 1], \mathbb{R}); S = \{\frac{1}{x^2+x-6}, \frac{1}{x^2-5x+6}, \frac{1}{x^2-9}\}$
   - $V = \mathcal{F}(\mathbb{R}, \mathbb{R}); S = \{x, e^x, e^{2x}\}$.

(b) Let

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

be matrices in $M_{22}(\mathbb{R})$. Show that

$$M_{22}(\mathbb{R}) = \mathrm{Span}_{\mathbb{R}}\{\mathrm{E}_{11}, \mathrm{E}_{12}, \mathrm{E}_{21}, \mathrm{E}_{22}\} = \mathrm{Span}_{\mathbb{R}}\{\mathrm{E}_{12}+\mathrm{E}_{21}+\mathrm{E}_{22}, \mathrm{E}_{11}+\mathrm{E}_{22}, \mathrm{E}_{12}+\mathrm{E}_{21}, \mathrm{E}_{11}+\mathrm{E}_{12}+\mathrm{E}_{22}\}$$