FM3817B Assignment 1

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Released: January 14, 2025

All questions are finalized.

Due: January 28, 2025, 11:55 pm

General information Full scores in total: 60 points (pts) with 8 bonus points. Question(s) with bonus points: If you do it correctly, you will get the bonus pts (until reach the full scores). Otherwise, you will not lose any points. Hence, feel free to try it out and play with them! Enjoy.

More questions will be added progressively as the course advances. This assignment will be finalized by January 21, 2025. Students are strongly encouraged to begin working on it early, rather than waiting until the last day.

Submission rule Submissions must be done via Gradescope. You must carefully assign pages to their corresponding questions. You will lose a typical number of points in each case below:

a. Submission not in PDF format.
b. Questions with no pages assigned to them.

Please submit a single PDF file (you can create it electronically using word, latex, etc.). Here are other ways to achieve this:

a. If you write your derivation on papers, you can scan them into a pdf file
b. If they are images, paste images to a word document then save as a pdf file.

If you have difficulty in formatting your submission, please turn to TA office hours as soon as possible.

Important rule It is your responsibility to allocate some time to format your submission prior to the due time and ensure that you can work with Gradescope properly. Late submissions are NOT accepted. Please turn to the course outline for more details.

Other policies You are allowed to discuss with your classmates. However, each student must submit their *own work*. Scholastic offences are taken seriously, and students are directed to read the appropriate policy, specifically, the definition of what constitutes a Scholastic Offence, at this link. Please turn to the course outline for more details on policies related to assignments.

Question 1 (6 pts)**.** Prospect theory is a Nobel-winning theory in behavioral economics. More details can be found here. In this question, we are going to investigate an interesting utility function playing a crucial role in prospect theory:

$$U(x) = \begin{cases} x^\alpha & x \geq 0 \\ -\lambda(-x)^\alpha & x < 0 \end{cases},$$

where x represents gain or loss and the parameters are set as $\alpha = 0.9$ and $\lambda = 2$. Please answer the following questions using the analytical tools you have learned from this course so far:

a. Compute $U'(x)$ for $x > 0$ and $x < 0$. Does this function increase or decrease on $x > 0$? How about $x < 0$?
b. Compute $U''(x)$ for $x > 0$ and $x < 0$. Is this a convex or concave function on $x > 0$? How about $x < 0$?
c. Is U continuous at $x = 0$? (Yes or No) Is U differentiable at $x = 0$? (Yes or No)

Question 2 (14 pts)**.** Consider a set $S = [1, 2) \cup (2, 3]$ and answer the following questions in detail: [1]

a. Is S a closed set? (Yes or No) Why?
b. Is S an open set? (Yes or No) Why? (Please state two *different* pieces of evidence for this question.)
c. Is S a convex set? (Yes or No) Why?
d. If your answer in the previous question is No, what is the *simplest* way to modify the set S such that the new one S^* is convex? [2]
e. For this new set S^*, show that it is a convex set.

Question 3 (12 pts)**.** Consider the function $f : [-8, 5] \to \mathbb{R}$,

$$f(x) = 2x^3 + 3x^2 - 72x + 40.$$

a. State the set of interior points and boundary points for the domain of f.
b. Find all the critical points of f. (Show your steps.)
c. Find all the local min and local max of f. (Show your steps.)
d. Find the global min and max of f. (Show your steps.)
e. Sketch the graph of f on its domain and clearly mark the critical points, boundary points and global max (min) on the graph.

Question 4 (8 pts)**.** For each of the following functions, find the critical point(s) and classify them as local max, local min, saddle point or inconclusive.

a. $f(x, y) = x^4 + x^2 - 6xy + 3y^2$
b. $f(x, y) = x^2 - 6xy + 2y^2 + 10x + 2y - 5$
c. $f(x, y) = xy^2 + x^3y - xy$
d. $f(x, y) = 3x^4 + 3x^2y - y^3$

Question 5 (8 pts)**.** Which of the critical points in Question 4 are global max (min)? Show your steps. [3]

Question 6 (6 pts)**.** In economic and finance, a famous class of utility functions is called Cobb-Douglas utility function. Such a function is in the form $U(x, y) = kx^ay^b$ with $a, b \in (0, 1)$ and $k > 0$. Answer the following questions:

a. Derive the Jacobian of U.
b. Derive the Hessian of U.
c. Find the optimal commodity bundle (x, y) that can maximize the Cobb-Douglas utility $U(x, y) = 2x^{1/2}y^{1/2}$ on the budget set $x + 2y = 100$.[4]

Question 7 (Bonus 4 pts)**.** Suppose $f(x)$ and $g(x)$ are both convex functions. Let $h(x) := f(g(x))$. Can we say $h(x)$ must be a convex function? (If yes, show your proof. If not, give a counter example.)

Question 8 (6 pts)**.** As discussed in the first lecture, please find a real-world example [5] involving optimization (different from the two examples mentioned in the lecture: portfolio optimization and regression, and also different from those provided in this assignment). Present your example by addressing the following components:

a. (6 pts) Clearly describe the problem's background and outline its key components, including:
    a. What is the objective that needs to be optimized? (In other words, what does the objective function represent?) Why is this optimization necessary?
    b. What are the decision variables? If known, briefly explain how each variable impacts the objective.
    c. Are there any constraints in this problem? If so, describe them. If no constraints exist, explain why.
b. (Bonus 4 pts) Simplify the problem and write it in the general form of an optimization problem. [6]

Notes

[1]It is important to carefully review the related concepts before digging into this question.

[2]Hint: there is a unique answer here involving only a *single* point.

[3] This question is designed as a slightly challenging one for students to further explore unconstrained optimization in an *unbounded* domain. In the course, we mainly focus on a bounded domain where we may have those *boundary points*. However, for unbounded domain, after we get the critical points, we also need to check the behavior of the function $f(x, y)$ once x, y goes to infinity in certain direction (which are called *extreme cases*). Here are some hints:

- A straightforward way: check all the critical points and the extreme cases (when $x, y \to \pm\infty$, including only one of them goes to ∞). Find the answer by a simple comparison.
- For instance, to check what happens when $x, y \to +\infty$, a simple way (not intended to be an exhaustive way) to check is to consider $x = y$ (then let $x \to +\infty$). Similarly, to check the case $x \to +\infty$ and $y \to -\infty$, consider $x = -y$ (then let $x \to +\infty$). We only need to check the leading terms (the terms with the highest order): e.g. x^3 explodes faster than x^2 as x goes to infinity.
- A saddle point cannot be a global max or min (think about the reasons).
- If there is no global max and min (e.g. you have found one extreme case where $f(x, y)$ goes to $+\infty$ and another extreme case goes to $-\infty$), simply state it.
- If needed, to double-check your results (no need to present in the submission, just for your own checking), visualize the function use any softwares or online tools to directly see the function. (We do not want to make the discussions too complicated here. Such a visualization can give us some guidance in checking cases.)

[4]There is no need to worry about the non-negativity constraints because they are implicitly assumed in the expression $x^{1/2}y^{1/2}$. Only need to focus on the first-order (partial) derivatives. There is no need to check the condition (for Lagrangian method) for now. The goal of this question is to help you get familiar with the procedure of Lagrangian method. Once you find the critical point, you can claim that it is the optimal one for now. We will get into more details in later parts of our course.

[5]As discussed in the lecture, your example does not need to come from an area with an obvious or ongoing optimization process. Instead, it can fall into one of the following categories:

- An area that, to the best of your knowledge, has not yet involved optimization, but you believe could benefit from applying an optimization-style approach.
- An area that does not appear to involve optimization (e.g., it only presents a final product or result), but you suspect that optimization has already taken place in a hidden or prior process.

There are no strict rules for this question. As long as your example is clearly and consistently described with all key components laid out, you will get the points.

[6]Hint: Refer to the Week 1 summary for guidance on structuring an optimization problem.