Exercise 1:

a) $T(N) = T(\frac{N}{2}) + N^2 : T(1)$ and $N = 2^k$

N^2 $\quad$ $IC = \sum_{i=0}^{k-1} \frac{N^2}{4^i}$

$\downarrow$

$\frac{N^2}{4}$ $\quad$ $= N^2 \sum_{i=0}^{k-1} \frac{1}{4^i}$ ← geometric

$\downarrow$

$\frac{N^2}{4^2}$ $\quad$ $= N^2 \frac{1}{1-\frac{1}{4}}$

$\downarrow$ $\quad$ $= \frac{4}{3} N^2$

$\frac{N^2}{4^k}$

$\vdots$

(1)

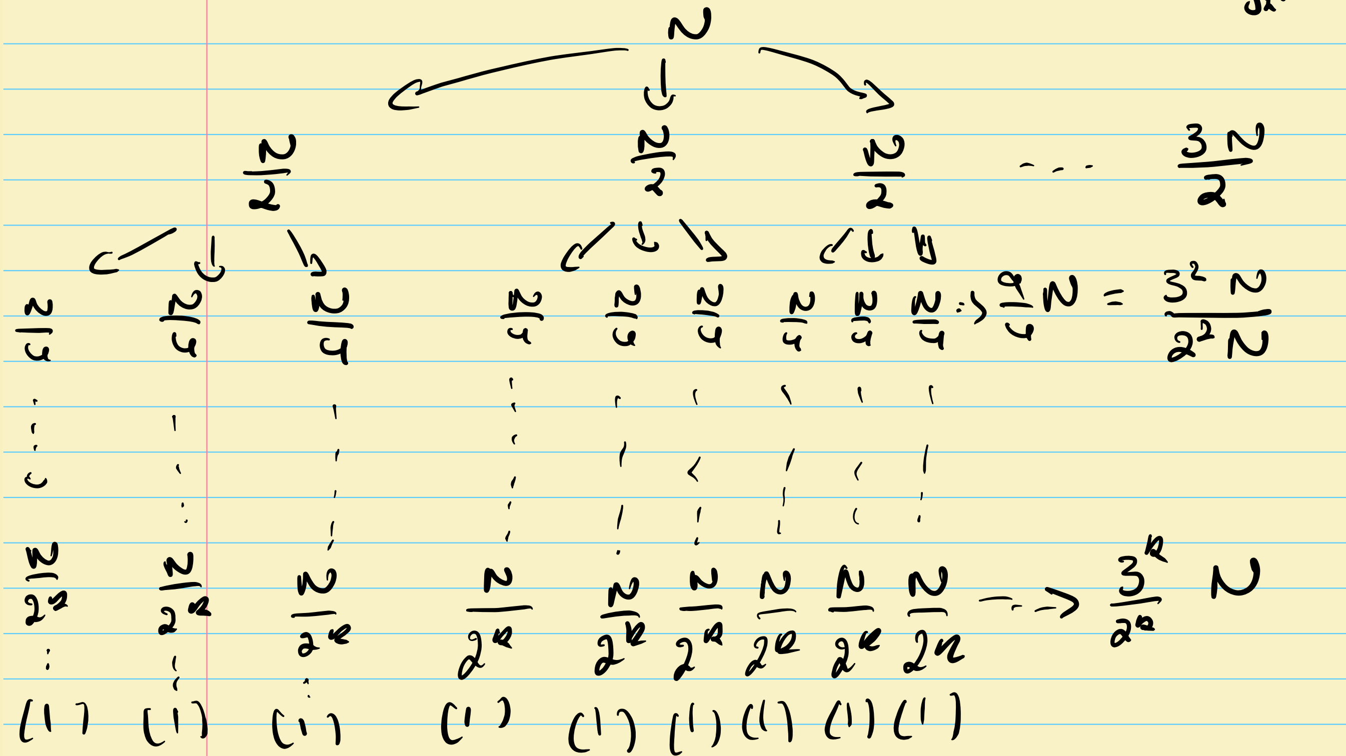
$LC = 1^k \cdot 1 = 1$

Total cost $= LC + IC$

$= \frac{4}{3} N^2 + 1 \Rightarrow O(N^2)$

b) $T(N) = 3\,T(N/2) + N;\quad T(1) = 1 \;\&\; N = 2^k$

$k = \log_2 n$



$$IC = \sum_{i=0}^{k-1} \frac{3^i}{2^i} N$$

$$= N \sum_{i=0}^{k-1} \left(\frac{3}{2}\right)^i$$

$$= N \cdot \left(\frac{n^k - 1}{n - 1} \right)$$

$$= N \cdot \frac{\frac{3}{2}^{k} - 1}{\frac{3}{2} - 1}$$

$$= 2N \cdot \log_{3/2} n - 2N$$

$$LC = 3^{k} \cdot 1 = 3^{\log n} = N^{\log 3}$$

$$\text{total cost} = 2N\log_{3/2} N - 2N + N^{\log 3}$$

$$\Rightarrow O(N^{\log 3})$$

c) $T(N) = 3T(N/2) + N^2$; $T(1) = 1$ & $N = 2^k$, $k = \log N$

$$N^2$$

$$\frac{N^2}{4} \quad \frac{N^2}{4} \quad \frac{N^2}{4} \qquad \frac{3N^2}{4} \Rightarrow \frac{3N^2}{4}$$

$$\frac{N^2}{16} \quad \frac{N^2}{16} \quad \frac{N^2}{16} \quad \frac{N^2}{16} \quad \frac{N^2}{16} \quad \frac{N^2}{16} \quad \frac{N^2}{16} \quad \frac{N^2}{16} \quad \frac{N^2}{16} \Rightarrow \frac{9N^2}{16} \Rightarrow \frac{3^2N^2}{4^2}$$

$$\vdots$$

$$(1) \quad (1) \quad (1) \quad (1) \quad (1) \quad (1) \quad (1) \quad (1) \quad (1) \qquad \frac{3^k N^2}{4^k}$$

$$TC = \sum_{i=0}^{k-1} \frac{3^i}{4^i} N^2$$

$$= N^2 \sum_{i=0}^{k-1} \left(\frac{3}{4}\right)^i$$

$$= N^2 \left(\frac{1}{1-\frac{3}{4}}\right) = 4N^2$$

$LC = 3^{k} \cdot 1 = 3^{\log n} \cdot 1 = n^{\log 3}$

Total Cost $= N^{\log 3} + 4N^2$

$\Rightarrow O(N^2)$

Exercise 2:

a) $T(N) = T(N/2) + N^2$; $T(1)$ and $N = 2^k$

$\ast$ $T(N) \leq c \cdot N^2$ for some positive cte c to start at n_0

Assume that $\ast$ holds for $\frac{N}{2}$ i.e.

$$T(\tfrac{N}{2}) \leq c \cdot \frac{N^2}{2^2}$$

$$\begin{aligned} T(n) &= T(N/2) + N^2 \\ &\leq c \cdot \frac{N^2}{2^2} + N^2 \\ &\leq c \cdot N^2 \cdot \frac{1}{2^2} + N^2 \\ &\leq \underbrace{\cancel{N^2}\left(\frac{c}{2^2} + 1\right) \leq c\cancel{N^2}} \end{aligned}$$

$$\frac{c}{2^2} + 1 \leq c$$

$$\frac{c}{2^2} - c \leq -1$$

$$\frac{c - 4c}{4} \leq -1$$

$$-\frac{3c}{4} \leq -1$$

$$c \geq \frac{4}{3}$$

$\Rightarrow$ $T(n) \leq cN^2$ for any $c \geq \frac{4}{3}$

$\Rightarrow$ $T(n) \in O(N^2)$

b) $T(N) = 3T(N/2) + N$; $T(1) = 1$ and $N = 2^k$

Prove $T(N) \leq c \cdot N^{\log 3} - dN$ for some positive cte c starting at n_0

we assume that ⊛ holds for $N/2$ i.e

$$T(N/2) = c \cdot \frac{N^{\log 3}}{3} - d\frac{N}{2}$$

$$T(N) = 3T\left(\frac{N}{2}\right) + N$$

$$\leq 3\left[c \cdot \frac{N^{\log 3}}{3} - d\frac{N}{2}\right] + N$$

$$\leq c \cdot N^{\log 3} - \frac{3dN}{2} + N$$

$$\leq c \cdot N^{\log 3} + N\left(-\frac{3d}{2} + 1\right)$$

$$\Rightarrow \cancel{c \cdot N^{\log 3}} + \cancel{N}\left(-\frac{3d}{2} + 1\right) \leq \cancel{c \cdot N^{\log 3}} - d\cancel{N}$$

$$-\frac{3d}{2} + 1 \leq -d$$

$$-\frac{3d}{2} + d \leq -1$$

$$-\frac{1}{2} d \leq -1$$

$$\frac{1}{2} d \geq 1$$

$$d \geq 2$$

$$\Rightarrow T(N) \leq C \cdot N^{\log 3} - dN$$

for any positive constant c

and for $d \geq 2$

$$\Rightarrow T(N) = O(N^{\log 3}).$$

c) $T(N) = 3T(N/2) + N^2$; $T(1) = 1$ and $N = 2^k$

prove: ⊗ $T(N) \leq C \cdot N^2$ for some positive cte C starting at N_0

we assume ⊗ holds true for $N/2$:

$$T\left(\frac{N}{2}\right) \leq C \cdot \frac{N^2}{2^2}$$

$$T(N) = 3T\left(\frac{N}{2}\right) + N^2$$

$$\leq 3\left[C \cdot \frac{N^2}{2^2}\right] + N^2$$

$$\leq 3C\frac{N^2}{4} + N^2$$

$$\leq N^2\left[\frac{3C}{4} + 1\right]$$

$$\Rightarrow \cancel{N^2}\left[\frac{3C}{4} + 1\right] \leq C \cdot \cancel{N^2}$$

$$\frac{3C}{4} - C \leq -1$$

$$-\frac{1}{4}C \leq -1$$

$$C \geq 4$$

$\Rightarrow T(N) \leq c\, N^2$

for any $c \geq 4$

$\Rightarrow T(N) = O(N^2).$