Exercise 1:

a) $O(N^3)$

b) $O(N^2)$

c) $O(N^2)$

d) $O(N^2 \lg N)$

e) $O(N)$

f) $O(N^2)$

g) $O(N^2)$

h) $O(N!)$

i) $O(1)$

j) $O(1)$

Exercise 2:

a) $T(N) = T\left(\frac{N}{2}\right) + N^2$: $N = 2^k$
$T(1) = 1$

- $T\left(\frac{N}{2}\right) = T\left(\frac{N}{4}\right) + \left(\frac{N}{2}\right)^2$

$\Rightarrow T(N) = T\left(\frac{N}{4}\right) + \left(\frac{N}{2}\right)^2 + N^2$

$T\left(\frac{N}{4}\right) = T\left(\frac{N}{8}\right) + \left(\frac{N}{4}\right)^2$

$\Rightarrow T(N) = T\left(\frac{N}{8}\right) + \left(\frac{N}{4}\right)^2 + \left(\frac{N}{2}\right)^2 + N^2$

$\Rightarrow T(N) = T\left(\frac{N}{2^k}\right) + \left(\frac{N}{2^{k-1}}\right)^2 + \left(\frac{N}{4}\right)^2 + \left(\frac{N}{2}\right)^2 + N^2$

$= T(1) + N^2\left(1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \dots + \left(\frac{1}{2^{k-1}}\right)^2\right)$

$= T(1) + N^2 \sum_{i=0}^{k-1} \left(\frac{1}{2^i}\right)^2$

$= T(1) + N^2 \sum_{i=0}^{k-1} \frac{1}{2^{2i}}$ let $x = \frac{1}{4}$

$$= T(1) + N^2 \sum_{i=0}^{k-1} x^i \quad ; \text{ but } x < 1$$

$$= T(1) + N^2 \left(\frac{1}{1-x} \right)$$

$$= T(1) + \frac{4}{3} N^2$$

$$= 1 + \frac{4}{3} N^2 \Rightarrow O(N^2)$$

b) $T(N) = T\left(\frac{N}{a}\right) + 1$; for $N \geq a$

$N = a^k$

$T\left(\frac{N}{a}\right) = T\left(\frac{N}{a^2}\right) + 1$

$\Rightarrow T(N) = T\left(\frac{N}{a^2}\right) + 1 + 1$

$T\left(\frac{N}{a^2}\right) = T\left(\frac{N}{a^3}\right) + 1$

$\Rightarrow T(N) = T\left(\frac{N}{a^3}\right) + 1 + 1 + 1$

$T(N) = T\left(\frac{N}{a^k}\right) + \underbrace{1 + \cdots + 1}_{k}$

$T(N) = T(1) + k$

$T(N) = 1 + \log_a n$

$\Rightarrow O(\lg n)$

c) $T(N) = N^2 + 8T(N/2)$; $N = 2^k$, $T(N) = 1$, $N \geq 2$

$= 8T(N/2) + N^2$

$T(N/2) = 8T\left(\frac{N}{4}\right) + \frac{N^2}{4}$

$= 8\left[8T\left(\frac{N}{4}\right) + \frac{N^2}{4}\right] + N^2$

$= 8^2 T\left(\frac{N}{4}\right) + 2N^2 + N^2$

$T\left(\frac{N}{4}\right) = 8T\left(\frac{N}{8}\right) + \frac{N}{16}$

$= 8^2\left[8T\left(\frac{N}{8}\right) + \frac{N}{16}\right] + 2N^2 + N^2$

$= 8^3 T\left(\frac{N}{8}\right) + 4N^2 + 2N^2 + N^2$

$= 8^k T\left(\frac{N}{2^k}\right) + 2^{k-1}N^2 + \ldots + 2^2N^2 + 2N^2 + N^2$

$= 8^k T(1) + \sum_{i=0}^{k-1} 2^i N^2$

$$= 8^k T(1) + N^2 \sum_{i=0}^{k-1} 2^i$$

$$\sum_{i=0}^{k-1} 2^i = \frac{2^{k-1+1} - 1}{2-1} = 2^k - 1$$

$$= 2^{\log_2 n} - 1 \Rightarrow n^{\cancel{\log_2 2}} - 1$$

$$\Rightarrow 8^{\log_2 n} T(1) + N^2 (n-1)$$

$$= n^3 - n^2 + n^{\log_2 8} + 1$$

$$= 2n^3 - n^2 + 1$$

$$\Rightarrow O(n^3)$$

Exercise 3:

$$T(n) = T(n-1) + 1 + T(n-1)$$

$$= 2T(n-1) + 1 \qquad\qquad T(0) = 1$$

$$T(n-1) = 2T(n-2) + 1$$

$$T(n) = 2[2T(n-2) + 1] + 1$$

$$= 4T(n-2) + 2 + 1$$

$$T(n-2) = 2T(n-3) + 1$$

$$T(n) = 8T(n-3) + 4 + 2 + 1$$

$$T(n) = 8T(n-3) + 7$$

$$\Rightarrow T(n) = 2^n T(n-n) + (2^n - 1)$$

$$= 2^n (T0) + 2^n - 1$$

$$= 2 \cdot 2^n - 1$$

$$= 2^{n+1} - 1$$

$$\Rightarrow O(2^N)$$