Lie Groups

Graduate Course INTR-6000P

Week 3 - Lecture 6

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Egomotion Estimation

Visual Odometry (VO):

Estimate the vehicle's 3D pose (rotation + translation) over time using cameras.

Euler angles (Roll, Pitch, Yaw).

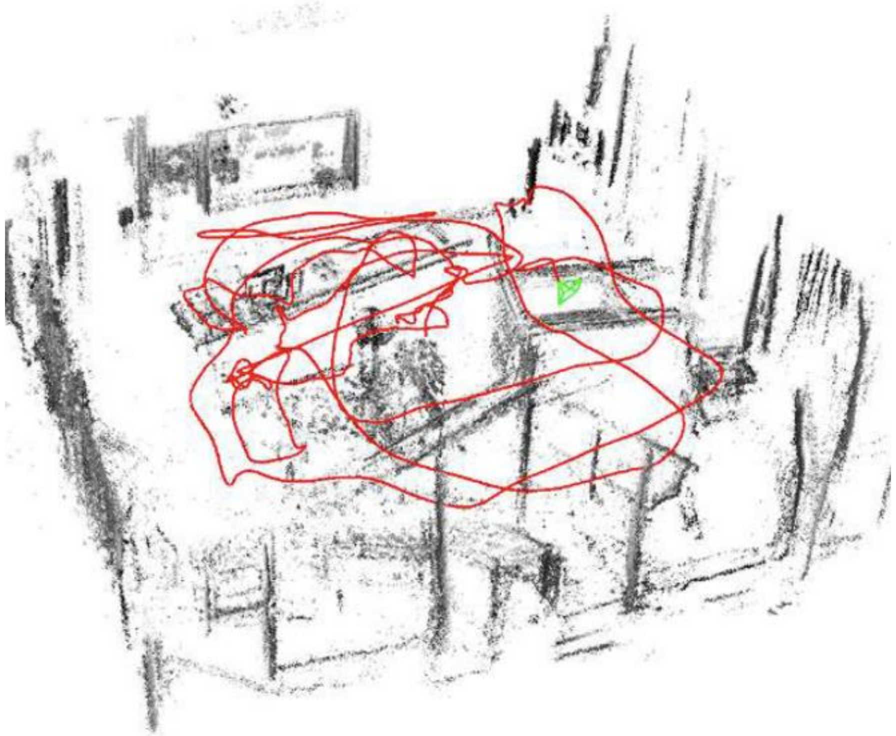
Why not to use Euler angles in motion estimation?

Problem 1: Gimbal Lock: The loss of a degree of freedom, causing numerical instability.

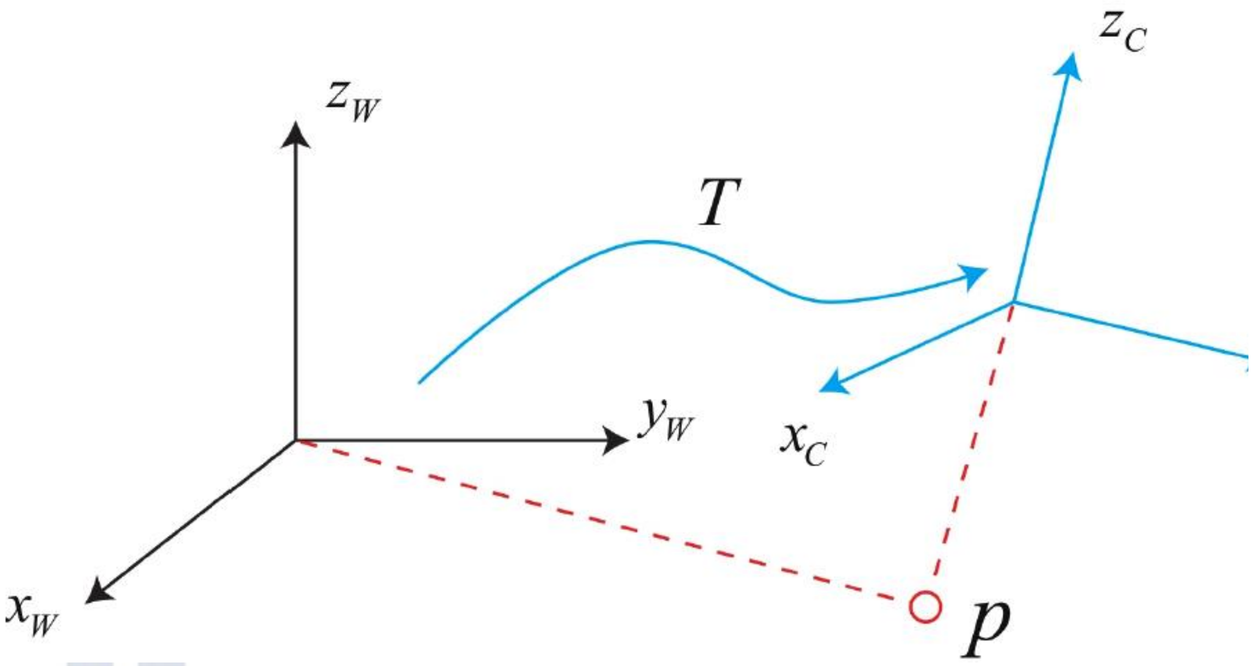
Problem 2: Non-Uniqueness: Multiple angle values can represent the same rotation.

Problem 3: Non-Euclidean Space: You cannot simply add two rotations or interpolate between them linearly. (10º, 20º, 30º) + (10º, 20º, 30º) ≠ (20º, 40º, 60º).

We need a better representation for rotation and pose.



Camera Motion



$$\begin{bmatrix} \boldsymbol{a}' \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{a} \\ 1 \end{bmatrix} \triangleq \boldsymbol{T} \begin{bmatrix} \boldsymbol{a} \\ 1 \end{bmatrix}.$$

$$\mathrm{SE}(3) = \left\{ \boldsymbol{T} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} | \boldsymbol{R} \in \mathrm{SO}(3), \boldsymbol{t} \in \mathbb{R}^3 \right\}$$

Group

A *group* $\mathcal{G}$ is a (finite or infinite) set of elements together with a binary *group operation* $\otimes$ that satisfies the following conditions:

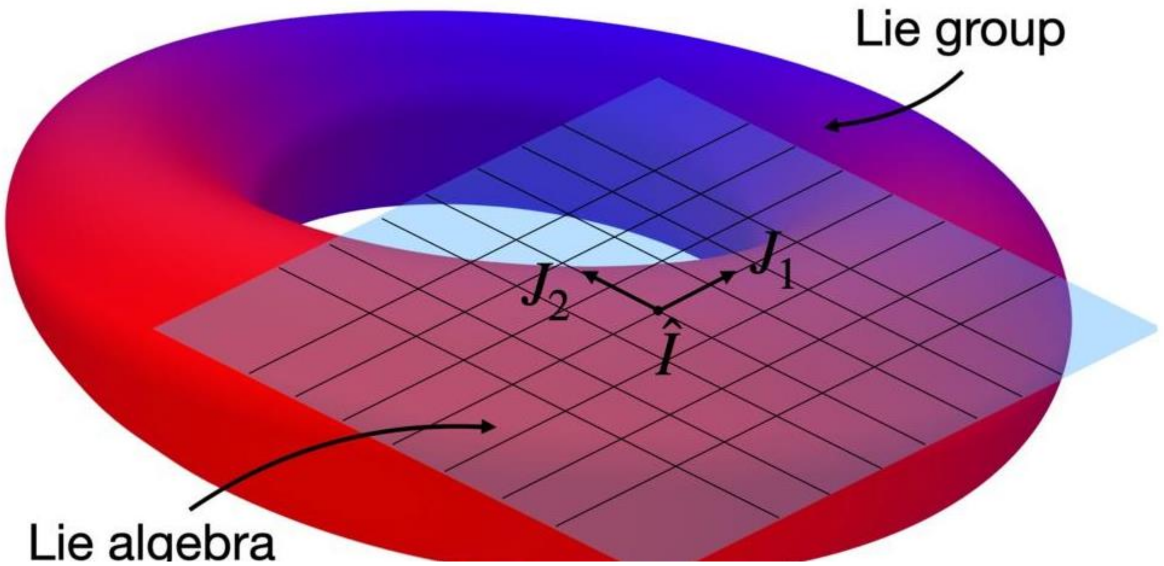
- *closure*: for any $A, B \in \mathcal{G}$, it holds $A \otimes B \in \mathcal{G}$;
- *associativity*: for any $A, B, C \in \mathcal{G}$, it holds $(A \otimes B) \otimes C = A \otimes (B \otimes C)$;
- *identity element*: there exists an identity element $I \in \mathcal{G}$ such that $A \otimes I = I \otimes A = A$ for any $A \in \mathcal{G}$;
- *inverse*: for any $A \in \mathcal{G}$ there exist an inverse element A^{-1} such that $A \otimes A^{-1} = A^{-1} \otimes A = I$.

Lie Group

A group that is also a smooth manifold, where the group operations (composition and inversion) are smooth.

A continuous set of transformations that you can:

- Compose: $T1 \circ T2$ is also a transformation.
- Invert: T^{-1} exists and is also a transformation.
- It's smooth.



Lie Groups. A group $\mathcal{G}$ is said to be a *Lie group* embedded in $\mathbb{R}^N$ if:

- $\mathcal{G}$ is a manifold in $\mathbb{R}^N$;
- the group operations (composition and inverse) are smooth (infinitely differentiable)

Optimization on Manifolds

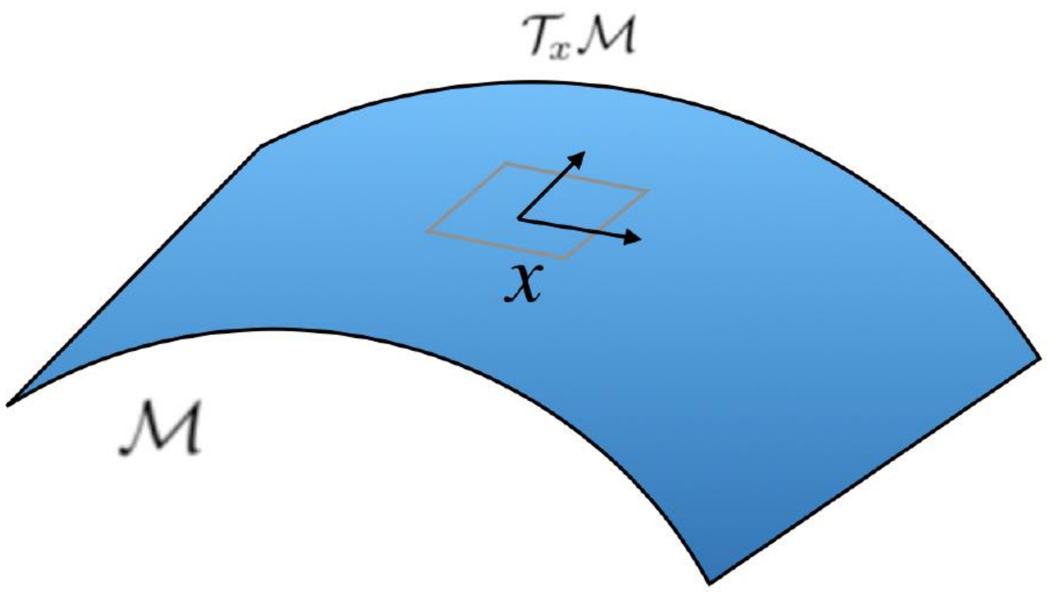
Represent camera pose as an element of a smooth, continuous manifold (e.g., a Lie group).

How do we do calculus (like gradient descent) on these curved spaces?

using Lie Group :

Perturb: Make small changes not on the manifold itself, but in the flat tangent space (Lie algebra).

Update: "Map" this small change back onto the manifold.



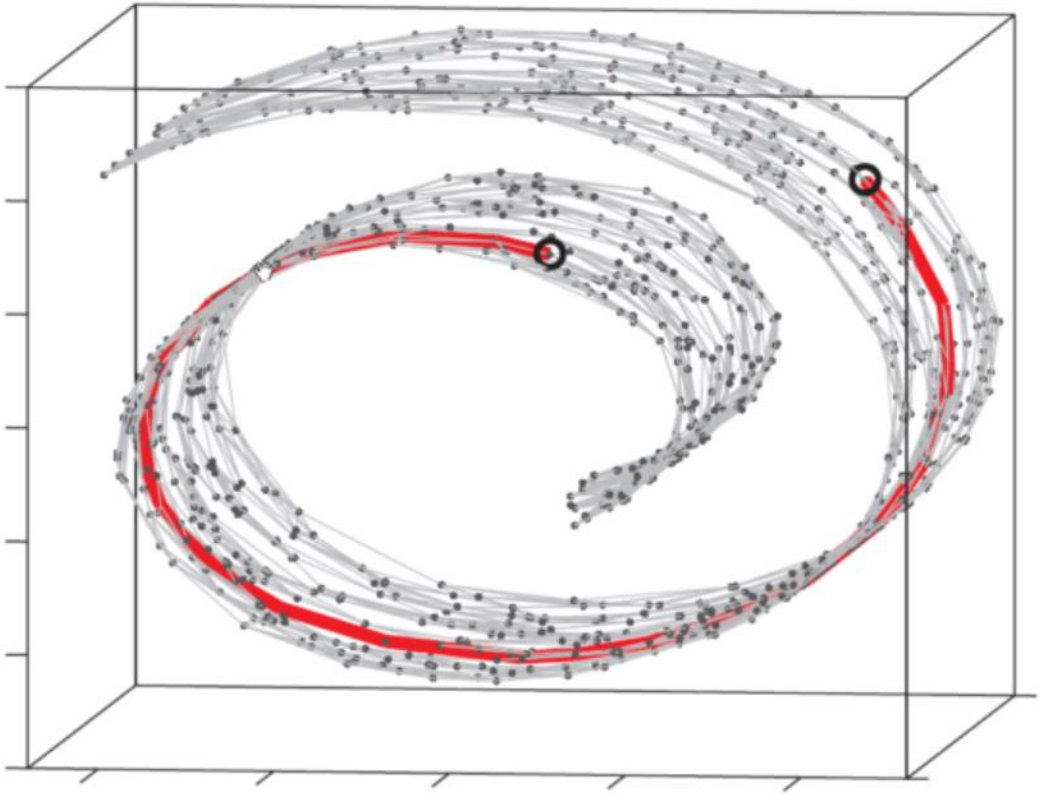
Manifold

A space that locally looks like Euclidean space (is "flat") but is globally curved.

For example,

The surface of the Earth (2D manifold).

The set of all possible 3D rotations (a 3D manifold).



Lie Algebra

Lie Algebra (g): The space of all possible "instantaneous changes" or velocities you can apply to the group element at the identity.

Vector Space. We can add elements [$\delta\phi_1 + \delta\phi_2$] and multiply by scalars [$a * \delta\phi$] without any problems.

(Lie algebra of SO(3)). The vector space corresponding to the Lie algebra of SO(3) is:

$$\mathrm{so}(3) = \left\{ \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix} : \boldsymbol{\phi} = [\phi_1\ \phi_2\ \phi_3]^\mathsf{T} \in \mathbb{R}^3 \right\} \tag{4.1}$$

(Lie algebra of SE(3)). The vector space corresponding to the Lie algebra of SE(3) is:

$$\mathrm{se}(3) = \left\{ \begin{bmatrix} \boldsymbol{\phi}^\wedge & \boldsymbol{\rho} \\ \mathbf{0}_3^\mathsf{T} & 0 \end{bmatrix} : \boldsymbol{\rho}, \boldsymbol{\phi} \in \mathbb{R}^3 \right\} \tag{4.3}$$

Exponential and Logarithm Map

The Exponential Map (exp)

exp : g (Lie algebra) → G (Lie group)

Takes a small twist ξ in the algebra and produces a finite transformation T in the group. It "wraps" the tangent vector onto the manifold.

Following a straight line (the tangent vector) on a curved map for a fixed time gives you a point on the Earth (the group element).

The Logarithm Map (log)

log : G → g

The inverse of exp. Finds the tangent vector that leads from the identity to a given group element T.

SO(3): The Special Orthogonal Group

The group of all proper 3D rotations.

Representation: 3x3 matrices where $R^T R = I$ and $\det(R) = +1$.

Its Lie Algebra: so(3)

Elements: 3x3 skew-symmetric matrices $[\omega]_\times$.

Isomorphism to $\mathbb{R}^3$: We represent a tangent vector as a simple 3-vector $\omega = (\omega_1, \omega_2, \omega_3)$. This is the axis-angle representation.

Exponential Map: Rodrigues' formula: $R = \exp([\omega]_\times) = I + \sin(\theta)/\theta\, [\omega]_\times + (1-\cos(\theta))/\theta^2\, [\omega]_\times^2$ where $\theta = ||\omega||$.

The Lie Groups in Visual Navigation

SE(3): The Special Euclidean Group

The group of rigid-body transformations (rotation + translation). This is the camera pose.

Representation: 4x4 matrices: T = [R t; 0 1]

Its Lie Algebra: se(3)

Elements: 4x4 matrices representing a "twist" $\xi = (\rho, \omega)$, containing linear (ρ) and angular (ω) velocity.

Exponential Map: Has a closed-form solution involving SO(3)'s exp.

The Lie Groups in Visual Navigation

Sim(3): The Similarity Group

Rigid transformation + a scale factor s.

T = [sR t; 0 1]

In monocular VO, the scale is unobservable from geometry alone (scale-drift problem).

Loop closures in monocular SLAM require optimizing in Sim(3) to align maps with different scales, making them scale-aware.

Optimization on the Manifold

Step 1: Parameterization

- Represent camera pose as $T \in SE(3)$.
- Represent the update as a small twist $\delta\xi \in se(3)$ in the tangent space.

Step 2: Define the Update Rule

- $T\_new = T\_old \circ \exp(\delta\xi)$
- This is a "right-multiply" update. (Can also be left-multiply).

Step 3: Compute the Derivative

- We compute the derivative of the error E with respect to the tangent space vector $\delta\xi$.
- This gives us a gradient g or a Jacobian J telling us which direction in se(3) to move to reduce error.

Step 4: Iterate

- Use Gauss-Newton/Levenberg-Marquardt in the tangent space to solve for the best $\delta\xi^*$.
- Update the pose: $T \leftarrow T \circ \exp(\delta\xi^*)$.
- Repeat until convergence.

Advantages of Lie Groups

1) No Singularities: We avoid Gimbal lock entirely.

2) Minimal Parametrization: $\delta\xi$ is always a 6-vector, no redundant parameters.

3) Theoretical Consistency: The math is correct on the curved manifold.

4) Standard Practice: This is the foundation of modern libraries like g2o, GTSAM, OKVIS, and VINS-Mono.

Summary

- Lie groups (SO(3), SE(3)) provide a singularity-free way to represent pose.
- The Lie algebra is the tangent space where we can do calculus.
- The exp and log maps connect the group and the algebra.
- Visual navigation optimizes pose by iteratively updating in the tangent space and mapping back to the manifold.
- This is not just abstract math—it's the engine inside every serious VO/SLAM system.

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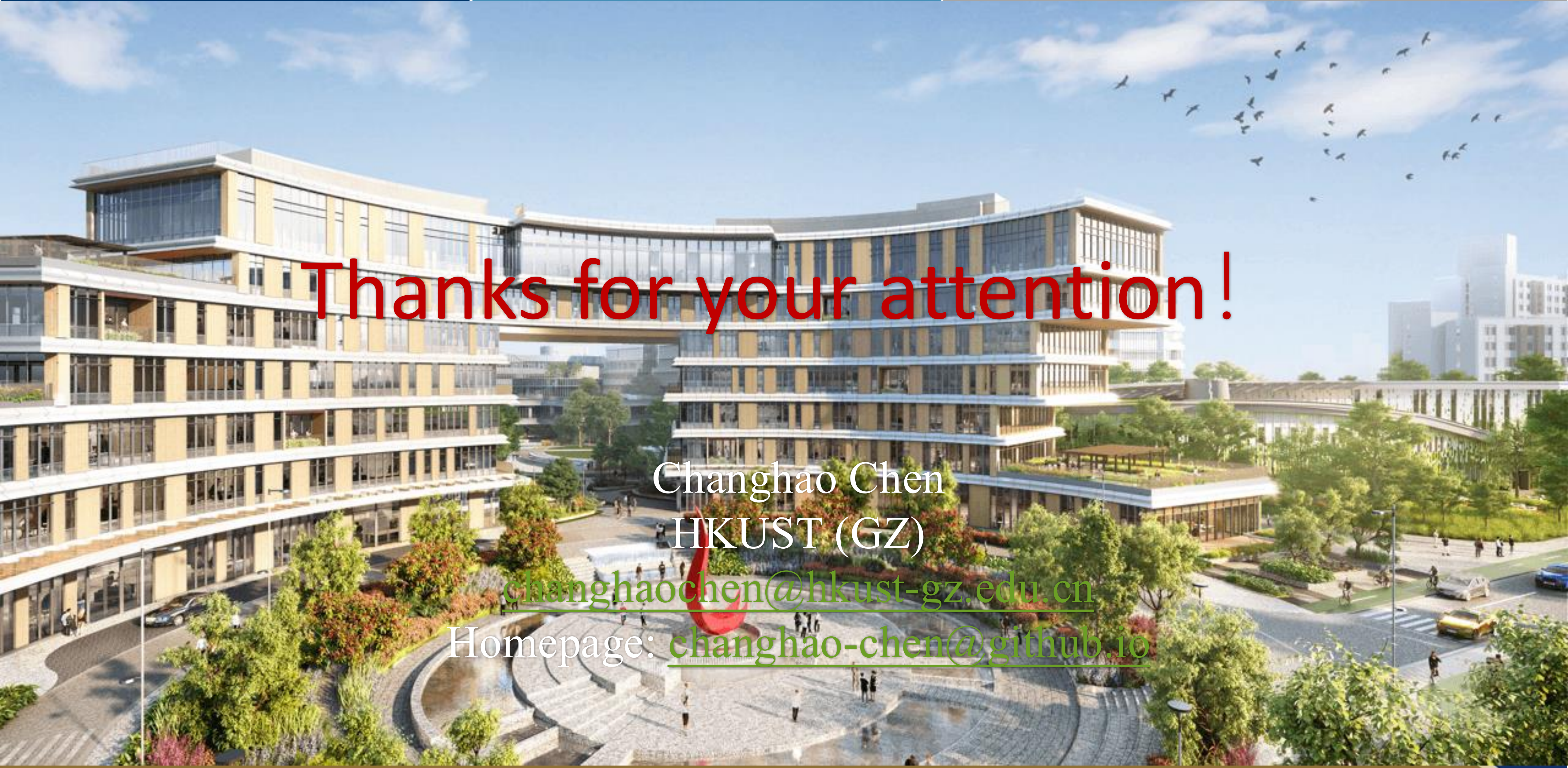

Thanks for your attention！

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