Chapter 2

2.1

1.1 Show that $\mathcal{A}$ has the right universal property. Let $\mathcal{G}$ be any sheaf and let $\mathcal{F}$ be the presheaf $U \mapsto A$, and suppose $\varphi : \mathcal{F} \to \mathcal{G}$. Let $f \in \mathcal{A}(U)$, i.e. $f : U \to A$ is a continuous map. Write $U = \coprod V_\alpha$ with V_α the connected components of U so $f(V_\alpha) = a_\alpha \in A$. Then we get $b_\alpha = \varphi_{V_\alpha}(a_\alpha)$ since $\mathcal{F}(U) = A$ for any U, and since $\mathcal{G}$ is a sheaf we obtain $b \in \mathcal{G}(U)$. We define $\psi : \mathcal{A} \to \mathcal{G}$ by $\psi_U(f) = b$. This map has the right properties.

1.2 a) Observe $(\ker\varphi)_P = \varinjlim_{U \ni P}(\ker\varphi)(U) = \varinjlim_{U \ni P}\ker\varphi_U$ is a subgroup of $\mathcal{F}_P$, as is $\ker\varphi_P$, so we show equality inside $\mathcal{F}_P$. For $x \in (\ker\varphi)_P$ pick (U, y) representing x, with $y \in \ker\varphi_U$. Then the image of y in $\mathcal{F}_P$, i.e. x, is mapped to zero by φ_P. Conversely, if $x \in \ker\varphi_P$ there exist (U, y) with $y \in \mathcal{F}(U)$ and $\varphi_U(y) = 0$ so $x \in (\ker\varphi)_P$. For $\operatorname{im}\varphi$ one proceeds similarly, noting only that $(\operatorname{im}\varphi)_P = \varinjlim_{U \ni P}\operatorname{im}\varphi_U$ since the presheaf "$\operatorname{im}\varphi$" and the sheaf $\operatorname{im}\varphi$ have the same stalks at every point.
b) The morphism φ is inj. resp surj. iff $(\ker\varphi)_P = 0$ resp. $(\operatorname{im}\varphi)_P = \mathcal{G}_P$ for all P. By part a), this holds iff $\ker\varphi_P = 0$ resp. $\operatorname{im}\varphi_P = \mathcal{G}_P$ for all P, that is, iff φ_P is inj. resp. surj.
c) We have $\operatorname{im}\varphi^{i-1} = \ker\varphi^i$ iff $\operatorname{im}\varphi_P^{i-1} = (\operatorname{im}\varphi^{i-1})_P = (\ker\varphi^i)_P = \ker\varphi_P^i$.

1.3 a) By 1.2, φ is surjective iff φ_P is surj. for all P, that is, iff for all U and all $s \in \mathcal{G}(U)$ there exist $(U_i, t_i) \in \mathcal{F}_P$ with $t_i \in \mathcal{F}(U_i)$ such that $\varphi_{U_i}(t_i)\big|_P = s_P$, or shrinking U_i if need be, iff for all $P \in U$ we have $\varphi_{U_i}(t_i) = s\big|_{U_i}$ with each U_i a nbd of P. The U_i cover U.
b) Let $\mathcal{F}_\infty$ be the sheaf of holomorphic fns on $\mathbf{CP}^1$ vanishing at ∞ and $\mathcal{F}_0$ the sheaf of holo. fns. vanishing at 0. Let $\mathcal{G}$ be the sheaf of holo. fns. and $\varphi : \mathcal{F}_\infty \oplus \mathcal{F}_0 \to \mathcal{G}$ be given over an open U by $(f_1, f_2) \mapsto f_1 + f_2$. This is a surjective map of sheaves since it is obviously surjective on every neighborhood not containing both $\infty, 0$. However, the map on global sections is not surjective: any holo. function on $\mathbf{CP}^1$ is constant, so the global sections of $\mathcal{F}_\infty$ and $\mathcal{F}_0$ are just $\{0\}$, while the global sections of $\mathcal{G}$ are $\mathbf{C}$.

1.4 a) If φ_U is injective for all U then $\varphi_P : \mathcal{F}_P \to \mathcal{G}_P$ is injective for all P, and since $\mathcal{F}_P^+ = \mathcal{F}_P$, $\mathcal{G}_P^+ = \mathcal{G}_P$ and $\varphi_P^+ = \varphi_P$, we see that φ^+ is injective by 1.2.
b) Since $\operatorname{im}\varphi(U) \hookrightarrow \mathcal{G}(U)$ for all U is injective (the map being just inclusion) we see that the induced map $\operatorname{im}\varphi \to \mathcal{G}^+ = \mathcal{G}$ is injective by the above, so $\operatorname{im}\varphi$ is a subsheaf of $\mathcal{G}$.

1.5 Reduce to the corresponding statement for abelian groups by Prop 1.1: φ is an isom. iff φ_P is an isom. for all P, iff φ_P is inj. and surj. for all P, iff φ is inj. and surj. by 1.2 b).

1.6 a) The natural map is $\mathcal{F}(U) \to \mathcal{F}(U)/\mathcal{F}'(U) \to (\mathcal{F}/\mathcal{F}')(U)$, and we may check surjectivity on stalks by 1.2. But $\mathcal{F}_P \to \mathcal{F}_P/\mathcal{F}'_P$ is clearly surjective for all P. The sequences $0 \to \mathcal{F}'_P \to \mathcal{F}_P/\mathcal{F}'_P \to 0 \to$ induced by $0 \to \mathcal{F}'(U) \to \mathcal{F}(U)/\mathcal{F}'(U) \to 0$ are all exact, so the corresponding sequence of sheaves is exact, that is, $\ker(\mathcal{F} \to \mathcal{F}/\mathcal{F}') = \mathcal{F}'$.

b) By 1.4 it suffices to show that $\varphi : \mathcal{F}' \to \operatorname{im}\varphi$ is surjective. Checking on stalks and using 1.2 reduces this to the surjectivity of $\mathcal{F}'_P \to \operatorname{im}\varphi_P$, which is a tautology. By abuse, we now consider $\mathcal{F}'$ as a subsheaf of $\mathcal{F}$. We claim that the map $\mathcal{F}(U) \to \mathcal{F}''(U)$ kills $\mathcal{F}'(U)$ for each U. Indeed, any $s \in \mathcal{F}'(U)$ has image that is zero in every stalk, and hence must be zero by the sheaf axioms. Thus we obtain a map $\mathcal{F}(U)/\mathcal{F}'(U) \to F''(U)$ induced by $\mathcal{F} \to \mathcal{F}''$, which gives a morphism $\mathcal{F}/\mathcal{F}' \to \mathcal{F}''$ that is an isomorphism on stalks by 1.2 and the given exact sequence.

1.7 a) There is a natural map $\mathcal{F} \to \operatorname{im}\varphi$ induced by $\mathcal{F}(U) \xrightarrow{\varphi_U} \operatorname{im}\varphi_U \to (\operatorname{im}\varphi)(U)$, and on stalks we have the exact sequences $0 \to \ker\varphi_P \to \mathcal{F}_P \to \operatorname{im}\varphi_P \to 0$ where we have used 1.2 again. Thus by 1.2, the sequence $0 \to \ker\varphi \to \mathcal{F} \to \operatorname{im}\varphi \to 0$ is exact, so 1.6 b) yields the result.
b) Similarly, the map $\mathcal{G}(U) \to \mathcal{G}(U)/\operatorname{im}\varphi(U) \to (\operatorname{coker}\varphi)(U)$ gives a map $\mathcal{G} \to \operatorname{coker}\varphi$, and identifying $\operatorname{im}\varphi$ as a subsheaf of $\mathcal{G}$ by 1.4 b), we see that $0 \to \operatorname{im}\varphi \to \mathcal{G} \to \operatorname{coker}\varphi \to 0$ is exact on stalks, so is exact by 1.2. Now use 1.6 b) again.

1.8 It suffices to show exactness at $\Gamma(U, \mathcal{F})$ as injectivity holds since $\mathcal{F}'(U) \to \mathcal{F}(U)$ is injecitve for all U iff $\mathcal{F}' \to \mathcal{F}$ is injective. Let $\phi : \mathcal{F} \to \mathcal{F}''$ and $\psi : \mathcal{F}' \to \mathcal{F}$. Then for $s \in \mathcal{F}'(U)$ we have

$$(\phi_U \circ \psi_U(s))_P = \varinjlim_{U \supseteq V \ni P} (\phi_U \circ \psi_U(s))\big|_V = \varinjlim_{U \supseteq V \ni P} (\phi_V \circ \psi_V(s\big|_V)) = \phi_P \circ \psi_P(s_P) = 0,$$

so since $\mathcal{F}''$ is a sheaf we have $\phi_U \circ \psi_U = 0$. Conversely, suppose $s \in \mathcal{F}(U)$ has $\phi_U(s) = 0$. Since the sequence of stalks at P is exact, for each $P \in U$ we have $t_P = (V_i, t_i)$ with $t_i \in \mathcal{F}'(V_i)$ such that $\psi_P(t_p) = s_P$. Shrinking V_i if need be, we may suppose $\psi_{V_i}(t_i) = s\big|_{V_i}$. It follows that $\psi_{V_i \cap V_j}(t_i\big|_{V_j \cap V_i}) = \psi_{V_i \cap V_j}(t_j\big|_{V_i \cap V_j})$ as both are equal to $s\big|_{V_i \cap V_j}$. Since ψ_V is injective for all V, we have the compatibility condition on the t_i to ensure (observe the V_i cover U) that they glue to a section $t \in \mathcal{F}'(U)$. Checking on stalks shows that $\psi_U(t) = s$ and left exactness of $\Gamma(U, \bullet)$ follows.

1.9 That $\mathcal{F} \oplus \mathcal{G}$ is a sheaf is obvious. Moreover, if $f : \mathcal{F} \to \mathcal{H}$ and $g : \mathcal{G} \to \mathcal{H}$ are morphisms of sheaves, then for every U we have maps of abelian groups $\mathcal{F}(U) \to \mathcal{H}(U)$ and $\mathcal{G}(U) \to \mathcal{H}(U)$ compatible with restriction. By the universal property of direct sum in the category of abelian groups, we get unique homomorphisms of ab. gpg. $\mathcal{F}(U) \oplus \mathcal{G}(U) \to \mathcal{H}(U)$ for all U, and these are evidently compatible with restriction because restriction is a homomorphism (in particular on $\mathcal{H}$). This gives a morphism $\mathcal{F} \oplus \mathcal{G} \to \mathcal{H}$, which must also be unique.

If $f : \mathcal{H} \to \mathcal{F}$ and $g : \mathcal{H} \to \mathcal{G}$ are two morphisms, then for all U we have a unique morphism $\mathcal{H}(U) \to \mathcal{F}(U) \oplus \mathcal{G}(U)$ (implicitly using that direct sum and product of finitely many gps. are isomorphic in category of ab. gps.) and thus a unique morphism $\mathcal{H} \to \mathcal{F} \oplus \mathcal{G}$, which is compatible with restriction because f, g are morphisms of sheaves and hence themselves compatible with restriction.

1.10 By the universal property of direct limit in the category of ab. gps., there is a unique morphism of presheaves "$\varinjlim \mathcal{F}_i$" $\to \mathcal{G}$ having the desired properties. Now use the universal property of the sheafification $\varinjlim \mathcal{F}_i$ of "$\varinjlim \mathcal{F}_i$".

1.11 Let $s_j \in \varinjlim_i \mathcal{F}_i(V_j)$ be sections compatible on overlaps with V_j covering $U \subset X$. Since X is noetherian, there are finitely many indices $j_1, \ldots, j_n$ such that $V_j \subseteq \bigcup_{k=1}^n V_{j_k}$ for all j. Thus, to glue the s_j we need only glue $s_{j_1}, \ldots, s_{j_n}$. By the definition of a directed system, there is an $N > 0$ such that $s_{j_k} \in \mathcal{F}_N(V_{j_k})$ for all k and since $\mathcal{F}_N$ is a sheaf and the V_{j_k} cover U, we obtain a section $s \in \varinjlim_i \mathcal{F}_i$ agreeing with s_j over V_j for all j; i.e. $U \mapsto \varinjlim_i \mathcal{F}_i(U)$ is a sheaf (as the other sheaf axiom follows by an almost identical argument).

1.12 Let $s_j = \{s_j^i\}_i \in \varprojlim_i \mathcal{F}_i(V_j)$ be compatible sections, where V_j is a cover of U. Since each $\mathcal{F}_i$ is a sheaf, the $\{s_j^i\}$ glue to give $s^i \in \mathcal{F}_i(U)$ for each i. If $\phi_{i'i} : \mathcal{F}_{i'} \to \mathcal{F}_i$ then we have $\phi_{i'i}(s_{i'})_P = (\phi_{i'i})_P((s_{i'})_P) = (s_i)_P$

since $\phi_{i'i}\big|_{V_j}(s_j^{i'}) = s_j^i$ for all j and $P \in V_j$ for some j. Thus we conclude that the s^i are compatible with the given maps defining the inverse system so we have an element $s \in \varprojlim_i \mathcal{F}_i(U)$ restricting to s_j over each V_j.

Suppose that $f_i : \mathcal{G} \to \mathcal{F}_i$ is a collection of morphisms, compatible with the inverse system morphisms. Define $f : \mathcal{G}(U) \to \varprojlim_i \mathcal{F}_i(U)$ by $s \mapsto \{f_i(U)\}_i$. The compatibility of the f_i with the inverse system morphisms ensure that this is an element of $\varprojlim_i \mathcal{F}_i(U)$, and compatibility with the restriction morphisms is clear (as the f_i are sheaf morphisms). Thus we obtain $f : \mathcal{G} \to \varprojlim_i \mathcal{F}_i$. Uniqueness follows easily as $\pi_i \circ f = f_i$ where $\pi_i : \varprojlim_i \mathcal{F}_i \to \mathcal{F}_i$ is projection onto the i th component.

1.14 The complement of $\operatorname{Supp} s$ is the set $S = \{P \in U \big| s_P = 0\}$. If $s_P = 0$ then there is a nbd. V_P of P such that $s\big|_{V_P} = 0$, and hence for all $Q \in V_P$ we have $s_Q = 0$ since the restriction maps are gp. homs. so map 0 to 0. Thus, $V_P \subseteq S$ which shows that S is open and hence $\operatorname{Supp} s$ is closed.

1.15 Let $f, g : \mathcal{F}\big|_U \to \mathcal{G}\big|_U$. Define $f + g : \mathcal{F}\big|_U \to \mathcal{G}\big|_U$ by $(f+g)_V(s) = f_V(s) + g_V(s)$. Observe that this is a morphism of sheaves $\mathcal{F}\big|_U \to \mathcal{G}\big|_U$ since the restriction maps are homomorphisms of abelian groups. Suppose that $f_i : \mathcal{F}\big|_{V_i} \to \mathcal{G}\big|_{V_i}$ are compatible morphisms, with V_i a cover of U. Define $f\mathcal{F}\big|_U \to \mathcal{G}\big|_U$ as follows: any $V \subseteq U$ is cover by $W_i = V \cap V_i$. For $s \in \mathcal{F}(V \cap U)$ let $s_i = s\big|_{W_i}$ and put $t_i = f_i\big|_{W_i}(s_i)$. Since the f_i are compatible on overlaps, so are the t_i, which therefore glue to $t \in \mathcal{G}(V \cap U)$. We set $f_V(s) = t$. We must check that this is compatible with the restriction maps: if $V' \subset V$ then $f_V(s\big|_{V'})$ is the glueing of $f_i\big|_{V' \cap W_i}(s_i\big|_{V' \cap W_i}) = t_i\big|_{V' \cap W_i}$ since the f_i are morphisms. Since $(t\big|_{V'})\big|_{W_i} = t_i$, we obtain the desired compatibility.

1.16 a) Since X is irreducible, it consists of one connected component, and hence any $U \subseteq X$ is connected. If $V \subset U$ and $f : V \to A$ is cts. then $f(V) = a$ for some a. We extend f to $\widetilde{f} : U \to A$ by definining $\widetilde{f}(U) = a$.
b) By 1.8 we need only show that $\mathcal{F}(U) \to \mathcal{F}''(U)$ is surjective. Let $s \in \mathcal{F}''(U)$. By 1.3, there is a cover $\{U_i\}_{i \in I}$ of U and sections $t_i \in \mathcal{F}(U_i)$ with $t_i \mapsto s\big|_{U_i}$. Consider the set S of pairs (J, z) with $J \subseteq I$ and $z \in \mathcal{F}(\cup_{j \in J} U_j)$ with $z \mapsto s\big|_{\cup_{j \in J} U_j}$. We order S by $(J, z) \leq (J', z')$ iff $J \subseteq J'$ and $z'\big|_{\cup_{j \in J} U_j} = z$. The set S is nonempty as for any fixed index $j_0 \in I$ we have $(\{j_0\}, t_{j_0})$. Moreover, any chain of S is bounded above by the sheaf axiom, so by Zorn's lemma, S has a maximal element (I_0, z). If $I_0 \neq I$, pick $i \in I \setminus I_0$, set $V = \cup_{j \in I_0} U_j$ and let $t_i \in \mathcal{F}(U_i)$ be as above. Since $x = z\big|_{V \cap U_i} - t_i\big|_{V \cap U_i} \mapsto 0 \in \mathcal{F}''(V \cap U_i)$, there exists $v_i \in \mathcal{F}'(V \cap U_i)$ mapping to x. Since $\mathcal{F}'$ is flasque, we may lift v_i to $w_i \in \mathcal{F}'(U_i)$ and define $t_i' = t_i + w_i$. Then z, t_i' are compatible sections and glue to $t \in \mathcal{F}(V \cup U_i)$. Clearly $t \mapsto s\big|_{V \cup U_i}$. Since I_0 was chosen to be maximal, we have $i \in I_0$ so $I = I_0$.
c) By part b) for $V \subseteq U$ we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}'(U) & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{F}''(U) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}'(V) & \longrightarrow & \mathcal{F}(V) & \longrightarrow & \mathcal{F}''(V) & \longrightarrow & 0
\end{array}$$

in which the first two vertical arrows are surjective. It follows that the third is also, and hence that $\mathcal{F}''$ is flasque.
d) This amounts to the fact that $V \subseteq U$ implies $f^{-1}(V) \subseteq f^{-1}(U)$.
e) The sheaf $\mathcal{G}$ is flasque since if $V \subseteq U$ then any $s : V \to \cup_{P \in V} \mathcal{F}_P$ may be extended to $\widetilde{s} : U \to \cup_{P \in U} \mathcal{F}_P$ by

$$\widetilde{s}(P) = \begin{cases} s(P) & \text{if } P \in V \\ 0 & \text{otherwise} \end{cases}.$$

Define $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ by $s \mapsto \big(P \mapsto s_P\big)$. This is injective since $s = 0$ iff $s_P = 0$ for all P iff $P \mapsto s_P$ is the zero map.

1.17 Suppose $Q \in \overline{P}$. Then every open set containing Q contains P so $\varinjlim_{U \ni Q} i_P(A)(U) = \varinjlim_{U \ni P} A = A$. If $Q \notin \overline{P}$ then there exists an open U containing Q and not P. For any open V containing Q the set $V \cap U$ is open, contains Q and not P. Since $\varinjlim_{V \ni Q} i_P(A)(V) = \varinjlim_{V \ni Q} i_P(V \cap U) = 0$ we conclude that $i_P(A)_Q = 0$ for such Q. Observe that

$$i_*(A)(U) = A(i^{-1}(U)) = \begin{cases} A & \text{if } P \in U \\ 0 & \text{otherwise} \end{cases} = i_P(A)(U).$$

1.18 Define $\varphi_U : \varinjlim_{V \supseteq f(U)} \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U)$ by the collection of maps $\mathrm{res}_{f^{-1}(V),U}$ for all V occurring in the direct limit (observe $V \supseteq f(U)$ implies $f^{-1}(V) \supseteq U$). The universal property of sheafification then gives a map $\varphi^+ : f^{-1}f_*\mathcal{F} \to \mathcal{F}$ that is functorial in $\mathcal{F}$ (essentialy because φ is just a collection of restriction maps). Since f^{-1} is a functor from sheaves on Y to sheaves on X, this gives a map $\tau : \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}) \to \mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F})$ determined by $g \mapsto \varphi^+ \circ (f^{-1}g)$.

Now define $\psi_U : \mathcal{G}(U) \to \lim_{V \supseteq f(f^{-1}(U))} \mathcal{G}(V)$ by inclusion of $\mathcal{G}(U)$ as a term in the direct limit (observe that $U \supseteq f(f^{-1}(U))$). Composing ψ with the map to the sheafification yields a map $\psi^+ : \mathcal{G} \to f_*f^{-1}\mathcal{G}$, functorial in $\mathcal{G}$ (again, roughly because ψ is defined by the identity map). Thus we can define $\sigma : \mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) \to \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$ by $g \mapsto (f_*g) \circ \psi$.

It remains to check that $\sigma \circ \tau = \mathrm{id}_{\mathrm{Hom}_Y}$ and $\tau \circ \sigma = \mathrm{id}_{\mathrm{Hom}_X}$. Perhaps a stalk calculation?

1.19 a) If $P \in Z$, every open $V \ni P$ in Z is of the form $U \cap Z$ for some open U in X. Thus, $(i_*\mathcal{F})_P = \varinjlim_{X \supseteq U \ni P} \mathcal{F}(U \cap Z) = \varinjlim_{Z \supseteq V \ni P} \mathcal{F}(V) = \mathcal{F}_P$. If $P \notin Z$ then since Z is closed there exists an open $U \ni P$ with $U \cap Z = 0$. Now $(i_*\mathcal{F})_P = \varinjlim_{X \supseteq V \ni P} \mathcal{F}(V \cap Z) = \varinjlim_{X \supseteq V \ni P} \mathcal{F}(V \cap U \cap Z) = 0$.
b) Recall that the stalk of a presheaf is the stalk of its sheafification at any point. If $P \in U$ then $\varinjlim_{V \ni P} j_!(V) = \varinjlim_{V \ni P} j_!(U \cap V) = \varinjlim_{W \ni P} \mathcal{F}(W) = \mathcal{F}_P$. If $P \notin U$ then no open set containing P is contained in U, so $\varinjlim_{V \ni P} j_!(V) = 0$. If $\mathcal{G}$ is any sheaf on X with $\mathcal{G}_P = \mathcal{F}_P$ for $P \in U$ and $\mathcal{G}_P = 0$ otherwise, and the restriction of $\mathcal{G}$ to U is $\mathcal{F}$, then we have a map $\mathcal{G} \to j_!\mathcal{F}$ given by composing $\mathcal{G}(V) \xrightarrow{\mathrm{res}} \mathcal{G}(U \cap V) = \mathcal{F}(U \cap V)$ with the sheafification map to $j_!\mathcal{F}$. The condition on stalks shows that this map is an isomorphism on all stalks, hence an isomorphism of sheaves.
c) If $V \subseteq U$ we map $\mathcal{F}(V) \to \mathcal{F}(V)$ by the identity; otherwise we use the zero map. This gives a morphism $j_!(\mathcal{F}|_U) \to \mathcal{F}$. We use the map $\mathcal{F} \to i_*i^{-1}\mathcal{F} = i_*(\mathcal{F}|_Z)$ described in 1.18. The sequence we obtain is exact on stalks (two cases: $P \in Z$ or $P \in U$) and hence exact.

2.2

2.1 Define $\varphi : \operatorname{Spec} A_f \to D(f)$ by the map $\mathfrak{p} \mapsto \mathfrak{p} \cap A = i^{-1}(\mathfrak{p})$ induced by the map $i : A \to A_f$. One checks easily that φ is a homeomorphism with $\varphi^{-1}D(f) \to \operatorname{Spec} A_f$ given by $\mathfrak{p} \mapsto \mathfrak{p}^e$ (by commutative algebra there is a 1-1 corr. between primes of A_f and primes of A not containing (f)). For example, φ takes the basic open set $D(g/f^r)$ to $D(fg)$ and so is an open map. We define $\varphi^\# : \mathcal{O}_{\operatorname{Spec} A}\big|_{D(f)} \to \varphi_*\mathcal{O}_{\operatorname{Spec} A_f}$ on basic opens $D(g) \subseteq \operatorname{Spec} A$ by $\mathcal{O}_{\operatorname{Spec} A}\big|_{D(f)}(D(g)) = A_{fg} \xrightarrow{\mathrm{id}} (A_f)_{g/1} = \mathcal{O}_{\operatorname{Spec} A_f}(\varphi^{-1}(D(g)))$. This gives a well defined sheaf map since every point has a basic open nbd.

2.2 Let $P \in U$ and $V \ni P$ be a nbd. of P in X such that $(V, \mathcal{O}_X\big|_V)$ is affine, say isomorphic to $\operatorname{Spec} R$. There exists a basic open nbd. $D(f) \subseteq U \cap V$ containing P that is open in V. By 2.1, the locally ringed space $(D(f), (\mathcal{O}_X\big|_V)\big|_{D(f)})$ is isomorphic to $\operatorname{Spec} R_f$ and is hence affine. Thus $(U, \mathcal{O}_X\big|_U)$ is a scheme.

2.3 a) Suppose $\mathcal{O}_P$ reduced for all P and let $s \in \mathcal{O}_X(U)$ be nilpotent. Then $s^n = 0$ in every stalk, so $s_P = 0$ for all $P \in U$ hence $s = 0$ by the sheaf axiom. Conversely, suppose $\mathcal{O}_X(U)$ reduced for all U. If $s \in \mathcal{O}_P$ is

nilpotent, pick U, s' representing s so $(s')^n = 0$ in s_P hence there is a nbd. V of P with $(s'\big|_{U\cap V})^n = 0$ (as restriction is a homomorphism) hence $s'\big|_{U\cap V} = 0$ since $\mathcal{O}_X(U\cap V)$ is reduced, whence $s = 0$ in s_P.
b) Let $P \in X$ and let $U \ni P$ be such that $(\varphi, \varphi^\#) : (U, \mathcal{O}_X\big|_U) \simeq \operatorname{Spec} R$ is affine. Then $(U, (\mathcal{O}_X)_{\text{red}}\big|_U) \simeq \operatorname{Spec} R_{\text{red}}$. Indeed, the topological spaces $\operatorname{Spec} R$ and $\operatorname{Spec} R_{\text{red}}$ are homeomorphic (as every prime contains the nilradical) via the surjection $R \to R/N$. There is a natural map $\psi : \mathcal{O}_{\operatorname{Spec} R} \to \mathcal{O}_{\operatorname{Spec} R_{\text{red}}}$ defined on the basic open $D(f)$ by the quotient map $R_f \to (R_f)_{\text{red}} = (R_{\text{red}})_f$ (since localization commutes with quotients). Then the map $(\psi \circ \varphi^\#)_V : \mathcal{O}_X(V) \to \mathcal{O}_{\operatorname{Spec} R_{\text{red}}}(\varphi^{-1}(V))$ is a map to a reduced ring, and therefore factors through $\mathcal{O}_V(V)_{\text{red}}$. The universal property of sheafification then yields a map $(\mathcal{O}_X)_{\text{red}} \to \varphi_* \mathcal{O}_{\operatorname{Spec} R_{\text{red}}}$ that is an isomorphism on stalks, hence an isomorphism.
c) This amounts to the commutative algebra statement that any map from a ring R to a reduced ring S factors uniquely through R_{red} and the fact that the push forward of a reduced sheaf is reduced.

2.4 The result holds when X is affine. In general, cover X by affines U_i and cover the double overlaps $U_i \cap U_j$ by affines U_{ijk}. Then we have an exact sequence of rings

$$0 \longrightarrow \Gamma(X, \mathcal{O}_X) \xrightarrow{g} \prod_i \Gamma(U_i, \mathcal{O}_{U_i}) \xrightarrow{f} \prod_{i,j,k} \Gamma(U_{ijk}, \mathcal{O}_{U_{ijk}})$$

where g is given by $s \mapsto \prod_i s\big|_{U_i}$ and f is given by $\prod_i s_i \mapsto \prod_{i,j,k} (s_i - s_j)\big|_{U_{ijk}}$. The functor $\operatorname{Hom}(A, \bullet)$ is left exact, so we obtain the exact sequence

$$0 \longrightarrow \operatorname{Hom}(A, \Gamma(X, \mathcal{O}_X)) \longrightarrow \prod_i \operatorname{Hom}(A, \Gamma(U_i, \mathcal{O}_{U_i})) \longrightarrow \prod_{i,j,k} \operatorname{Hom}(A, \Gamma(U_{ijk}, \mathcal{O}_{U_{ijk}}))$$

Meanwhile, since to give a morphism $X \to \operatorname{Spec} A$ is to give morphisms $U_i \to \operatorname{Spec} A$ that agree on the coverings of double overlaps, we have the exact sequence of sets

$$0 \longrightarrow \operatorname{Hom}(X, \operatorname{Spec} A) \longrightarrow \prod_i \operatorname{Hom}(U_i, \operatorname{Spec} A) \longrightarrow \prod_{i,j,k} \operatorname{Hom}(U_{ijk}, \operatorname{Spec} A)$$

(where "kernel" is interpreted in the appropriate sense, i.e. in the category of sets). Piecing these sequences together, we have

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}(A, \Gamma(X, \mathcal{O}_X)) \arrow[r] \arrow[d, leftrightarrow] & \prod_i \operatorname{Hom}(A, \Gamma(U_i, \mathcal{O}_{U_i})) \arrow[r] \arrow[d, leftrightarrow] & \prod_{i,j,k} \operatorname{Hom}(A, \Gamma(U_{ijk}, \mathcal{O}_{U_{ijk}})) \arrow[d, leftrightarrow] \\
0 \arrow[r] & \operatorname{Hom}(X, \operatorname{Spec} A) \arrow[r] & \prod_i \operatorname{Hom}(U_i, \operatorname{Spec} A) \arrow[r] & \prod_{i,j,k} \operatorname{Hom}(U_{ijk}, \operatorname{Spec} A)
\end{tikzcd},$$

where the second two vertical arrows are bijections. Thus the first is also.

2.5 The closed points of $\operatorname{Spec} \mathbf{Z}$ are the prime ideals (p) with $p \in \mathbf{Z}$ not equal to 0. There is a single open point (the generic point), namely (0). The basic open sets are of the form $D(n)$ with $n \in \mathbf{Z}$ and consist of those primes not dividing n. The local ring at the closed point (p) is $\mathbf{Z}_{(p)}$ and the residue field is $\mathbf{F}_p$ while the local ring and residue field at (0) are both $\mathbf{Q}$. Since there is a unique homomorphism from $\mathbf{Z}$ to any ring defined by $1 \mapsto 1$, 2.4 implies that every scheme X admits a unique morphism to $\operatorname{Spec} \mathbf{Z}$.

2.6 Let R be the zero ring. Then $\operatorname{Spec} R = \{\emptyset\}$ and $\mathcal{O}_{\operatorname{Spec} R}(\emptyset) = R = 0$. There is a unique morphism $f : \operatorname{Spec} R \to X$ to any scheme X defined by inclusion on topological spaces and $\mathcal{O}_X \to f_* \mathcal{O}_{\operatorname{Spec} R}$ given by the identically zero morphism.

2.7 Given a morphism $(\varphi, \varphi^\#) \operatorname{Spec} K \to X$ we obtain a point $x = \varphi((0)) \in X$ and a sheaf morphism $\varphi^\# : \mathcal{O}_X \to \varphi_* \mathcal{O}_{\operatorname{Spec} K}$, which gives a *local* map of local rings $\varphi^\#_x : \mathcal{O}_{X,x} \to \mathcal{O}_{\operatorname{Spec} K,(0)} = K$; in particular $m_x \mapsto 0$ so we obtain a morphism $k(x) \to K$ which must be injective since $k(x)$ is a field. Conversely, given a point $x \in X$ and an inclusion $k(x) \hookrightarrow K$ we define $\phi : \operatorname{Spec} K \to X$ by $\phi((0)) = x$ and $\varphi_U^{\#} : \mathcal{O}_X(U) \to \mathcal{O}_{\operatorname{Spec} K}(\varphi^{-1}(U))$ for each open U in X by $\mathcal{O}_X(U) \to \mathcal{O}_{X,x} \to k(x) \hookrightarrow K$, where the first arrow is the zero map when $x \notin U$ and is inclusion into the direct limit when $x \in U$.

2.8 Suppose given a map $\varphi : \operatorname{Spec} k[\epsilon]/\epsilon^2 = \{(\epsilon)\} \to X$ as schemes over k. This gives a point $\varphi(\epsilon) = x \in X$ together with the diagram of local rings

$$\begin{tikzcd} \mathcal{O}_{X,x} \arrow[rr] & & k[\epsilon]/\epsilon^2 \\ & k \arrow[ul] \arrow[ur] & \end{tikzcd}$$

(since $(0) \in \operatorname{Spec} k \mapsto (\epsilon) \in \operatorname{Spec} k[\epsilon]/\epsilon^2$). Since the map $\varphi_x : \mathcal{O}_{X,x} \to k[\epsilon]/\epsilon^2$ is *local*, we have $\varphi_x(m_x) \subseteq (\epsilon)$, from which we deduce that the map $k(x) \to k[\epsilon]/\epsilon \simeq k$ is an isomorphism. Since $\epsilon^2 = 0$, we define $\psi : m_x \to k$ by $\psi(z) = \varphi_x(z)/\epsilon$ and note that this is well defined, k-linear (by the commutativity of the above diagram), and kills m_x^2 so yields an element of $\operatorname{Hom}(m_x/m_x^2, k)$.

Conversely, suppose given a point $x \in X$ with residue field k and a k-linear $\psi : m_x \to k$ killing m_x^2. We define $\varphi_x : \mathcal{O}_{X,x} \to k[\epsilon]/\epsilon^2$ by $\varphi(\alpha + z) = \alpha + \psi(z)\epsilon$, where $\alpha \in k$ and $z \in m_x$ (using that $\mathcal{O}_{X,x}/m_x \simeq k$). One checks using the linearity of ψ and the fact that $\epsilon^2 = 0$ and that ψ kills m_x^2 that φ_x is a local homomorphism with the above diagram commuting. Finally, define a map $\varphi : \operatorname{Spec} k[\epsilon]/\epsilon^2 \to X$ by $\varphi((\epsilon)) = x$ and $\varphi^{\#} : \mathcal{O}_X \to \mathcal{O}_{\operatorname{Spec} k[\epsilon]/\epsilon^2}$ by the map $\mathcal{O}_X(U) \to \mathcal{O}_{X,x} \xrightarrow{\varphi_x} k[\epsilon]/\epsilon^2$ where the first arrow is 0 if $x \notin U$ and is inclusion into the direct limit otherwise. One easily checks that this gives a map of sheaves.

2.9 Suppose ζ_1, ζ_2 are two generic points of Z. Then since $\overline{\zeta_i} = Z$, an open set contains ζ_1 iff it contains ζ_2. Letting $U = \operatorname{Spec} R$ be an affine nbd. of ζ_1, we identify $\zeta_i = p_i \in \operatorname{Spec} R$. Since $p_2 \in \overline{p_1}$, we have $p_2 \supseteq p_1$ and vice versa, so $\zeta_1 = \zeta_2$. This settles uniqueness. As for existence, let Z be irreducible, nonempty, and closed and let $U \subseteq Z$ be a (nonempty) open affine. Then $U = \operatorname{Spec} R$ is dense in Z and irreducible. By Zorn's lemma, R has minimal primes, and the irreducibility of $\operatorname{Spec} R$ implies R has a unique minimal prime, $\zeta \in U$. It follows that $\overline{\zeta} = U$ (closure in U) and since U is dense in Z, we have $\overline{\zeta} = Z$ (closure in Z, hence also X as Z is closed).

2.10 Observe that $\mathbf{R}[x]$ is a PID. The prime ideals of $\mathbf{R}[x]$ fall into three types:

1. The generic (and open) point (0), with residue field $\mathbf{R}(x)$.
2. Closed points of the form $(x - \alpha)$ with $\alpha \in \mathbf{R}$. The residue field in each case is $\mathbf{R}$.
3. Closed points of the form $(x^2 + \alpha x + \beta)$, with residue field $\mathbf{C}$.

As a set, $\operatorname{Spec} \mathbf{R}[x] = \mathbf{R} \cup (\mathbf{C} - \mathbf{R}) = \mathbf{C}$ with the bijection sending $z \in \mathbf{C}$ to $((x - z)(x - \overline{z}))$ if $z \notin \mathbf{R}$ and to $(x - z)$ if $z \in \mathbf{R}$.

2.11 Again, $\mathbf{F}_p[x]$ is a PID, and the points of $\operatorname{Spec} \mathbf{F}_p[x]$ are

1. The generic point (0) with residue field $\mathbf{F}_p(x)$.
2. Closed points of the form (f) with f a monic irreducible polynomial of degree $d \geq 1$. The residue field is $\mathbf{F}_{p^d}$.

Since $x^{p^n} - x$ is the product of all monic irreducibles (over $\mathbf{F}_p$) of degree $d|n$ we get the formula

$$\#\{\text{monic degree } d \text{ irreducibles}\} = \sum_{d|n} \mu(d) p^{n/d}$$

by möbius inversion.

2.12

2.13 a) Let U be an open set of X that is not quasi-compact and $\{U_i\}$ an open cover of U that does not admit a finite subcover. Using the axiom of choice, we can find a sequence of U_i, say U_{i_j} for $1 \leq j$ with $U_{i_{j+1}}$ not contained in $V_j := \cup_{1 \leq k \leq j} U_{i_k}$. Then the V_j form an ascending nonterminating chain of open subsets of X. Conversely, suppose that X is noetherian and let $\{U_i\}$ be an open cover of the open set $U \subseteq X$. Let S be the set of finite unions of the U_i, partially ordered by inclusion. Every chain evidently has an upper bound, so by Zorn's lemma S has a maximal element, which is easily seen to be U, viz. U is quasi-compact.
b) Let U_i be an open covering of $X = \operatorname{Spec} R$. Covering each U_i by basic opens, we may suppose that the U_i are all basic, $U_i = D(f_i)$. Then the f_i generate the unit ideal, so in particular, finitely many of them do, giving a finite subcover. Let $R = k[x_i]$ for $1 \leq i$ be a polynomial ring in infinitely many variables and consider the set $U = \sup D(x_i)$. This is not quasi-compact, so by a) $\operatorname{Spec} R$ is not noetherian.
c) Let U be an open set and U_i an open cover, which we may assume to be basic ($U_i = D(f_i)$)as above. If A is noetherian, then the ideal (f_i) is generated by finitely many of the f_i, say $f_i : i \in J$ with $\#J < \infty$. Then U_i with $i \in J$ is a finite subcover.
d) Take $A = k[x_i]/(x_i^2)$ for $1 \leq i$. Evidently A is not noetherian, but the space $\operatorname{Spec} A$ consists of a single point (0).

2.14 a) Recall that the set $\operatorname{Proj} S$ consists of those homogenous prime ideals that do not contain the irrelevant ideal S_+. If every element S_+ is nilpotent then S_+ is contained in the nilradical and hence in every prime of S, so $\operatorname{Proj} S$ is empty. Conversely, if $\operatorname{Proj} S = \emptyset$ then every homogenous prime ideal of S contains S_+, so S_+ is contained in the intersection of all homogenous prime ideals and is therefore contained in the nilradical.
b) Let $p \in U$. Then there is some $s \in S_+$ with $\varphi(s) \notin p$. The set $D(\varphi(s))$ is an open nbd of p in U. The morphism $f : U \to \operatorname{Proj} S$ is given by contraction. Each homogenous prime of $\operatorname{Proj} T$ contracts to a homogenous prime of S, and when $p \in U$, the contraction $\varphi^{-1}(p)$ does not contain all of S_+ so is in $\operatorname{Proj} S$. Continuity follows since contraction preserves containment relations, and the sheaf map is given by.....
c) Since φ is graded, we have $\varphi(S_+) \subseteq T_+$. If $\varphi_d : S_d \to T_d$ is an isomorphism for all $d \geq d_0$ then every prime that does not contain T_+ cannot contain $\varphi(S_+)$: indeed, if there is some $t \in T_r$ not in p then $t^k \notin p$ for all k and since $r > 1$ we may choose k so that $rk > d_0$, thus producing $t^k = \varphi(s)$ for some $s \in S_+$ so p does not contain $\varphi(S_+)$, thus giving $\operatorname{Proj} T \subseteq U$ so in fact equality holds.

We now show that $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is an isomorphism. Let $\{t_i\}$ generate T_+, so $\{D_+(t_i)\}$ is a cover of $\operatorname{Proj} T$. Then $\{D_+(t_i^{d_0})\}$ is also a cover of $\operatorname{Proj} T$. Put $s_i = \varphi^{-1}(t_i^{d_0})$ (here we use that φ_d is an isomorphism for all $d \geq d_0$). Then $f_i = f\big|_{D_+(t_i)} \to D_+(s_i)$ is a morphism of affine schemes (as $D_+(t_i) \simeq \operatorname{Spec} T_{(t_i)}$ and $D_+(s_i) \simeq \operatorname{Spec} S_{(s_i)}$) corresponding to the ring homomorphism $\varphi_i : S_{(s_i)} \to T_{(t_i)}$ induced by φ. But φ_i is an isomorphism since s_i has degree at least d_0, and φ_d is an isomorphism for all $d \geq d_0$. Thus, f_i is an isomorphism. Now the $D_+(s_i)$ cover $\operatorname{Proj} S$ since any $p \in ProjS$ fails to contain some s_i (otherwise the contraction of p, which is in $\operatorname{Proj} T$, contains t_i for all i and hence T_+, a contradiction), so to show that f is an isomorphism we need only show it is injective (since a bijective local isomorphism is an isomorphism and surjectivity follows from the fact that the $D_+(s_i)$ cover $\operatorname{Proj} S$). If $f(x) = f(y)$ then $\varphi^{-1}(x) = \varphi^{-1}(y) = p \in \operatorname{Proj} S$, from which it follows that $x_d = y_d$ for all $d \geq d_0$ (here $x_d = x \cap T_d$). Pick $z \in x$ not in y and let $s \in S_+$ be such that $s \notin y$ (recall $S_+ \not\subseteq y$). Then $s^{d_0} z \in x$ is of degree at least d_0 and so is in y. Since y is prime, this implies that $z \in y$ so f is injective, hence an isomorphism.

2.16 a) We have $U \cap X_f = \{x \in \operatorname{Spec} B \big| f_x = \overline{f}_x \notin m_x \subseteq \mathcal{O}_{X,x}\}$ (where $\overline{f}_x = f_x$ since $\overline{f}$ and f agree in a nbd. of x). Putting $x = p \subseteq B$ this is $\{p \in \operatorname{Spec} B \big| \overline{f}_p \notin pB_p\}$ and it is easy to verify that $\overline{f}_p \notin pB_p$ iff $\overline{f} \notin p$, so $U \cap X_f = D(\overline{f})$. Hence X_f intersects every open affine (hence every open) in a (union of) basic opens, and so must be open (for $x \in X_f$ pick an affine open nbd. U of x in X so $U \cap X_f$ is open in U hence in X, and is contained in X_f).

b) Cover X by affines $U_i = \operatorname{Spec} B_i$ for $i = 1 \dots n$ and let $a_i = a\big|_{U_i}$ and $f_i = f\big|_{U_i}$. By part a), $U_i \cap X_f = D(f_i)$, so $a_i = 0$ on $D(f_i)$, i.e. $a_i = 0$ in $(B_i)_{f_i}$. Thus there is $r_i \geq 0$ such that $a_i f_i^{r_i} = 0$ in B_i. Letting $N = \max_{1 \leq i \leq n} r_i$ we find that the global section af^N restricts to 0 on each U_i and is therefore 0.
c) Let $U_i = \operatorname{Spec} B_i$ for $1 \leq i \leq n$ be a finite affine cover of X. Put $b\big|_{X_f \cap U_i} = b_i/\overline{f_i}^{d_i} \in (B_i)_{f_i}$ with $b_i \in B_i$. Set $d = \sum_i d_i$ (finite) and $b_i' = f^{d-d_i} b_i \in \Gamma(U_i, \mathcal{O}_X)$. Since $b_i'\big|_{X_f} = f^{d-d_i} f^{d_i} b$ we see that $(b_i' - b_j')\big|_{U_i \cap U_j \cap X_f} = 0$ so by part b), for each pair i, j there is an integer d_{ij} with $f^{d_{ij}}(b_{i'} - b_{j'}) = 0$ as an element of $\Gamma(U_i \cap U_j, \mathcal{O}_X)$. Letting $D = \max_{i,j} d_{ij}$ (finite since the double overlaps have finite affine covers by hypothesis) we find that $f^D b_i' \in \Gamma(U_i, \mathcal{O}_X)$ are compatible on double overlaps, so give an element $a \in \Gamma(X, \mathcal{O}_X)$. By construction, $a\big|_{X_f \cap U_i} = f^D b_i' = f^{D+d} b$ so in particular $a\big|_{X_f} = f^{D+d} b$ by the sheaf axiom.
d) Define $\varphi : A_f \to \Gamma(X_f, \mathcal{O}_{X_f})$ by $a/f^n \mapsto a\big|_{X_f} / f^n\big|_{X_f}$. (Observe that $f\big|_{U_i \cap X_f} \in \Gamma(U_i \cap X_f, \mathcal{O}_{X_f}) = (B_i)_f$ is a unit for all i and hence $f\big|_{X_f} \in \Gamma(X_f, \mathcal{O}_{X_f})$ is a unit). The map is a homomorphism since restriction is, and is injective since otherwise $f^k a = 0$ for some k by part b), so $a/f^n = 0$ in A_f. Surjectivity is part c) above.

2.17 a) Let $g_i : U_i \to f^{-1}(U_i)$ be the inverse to $f\big|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$. Observe that $g_i . g_j$ agree on $U_i \cap U_j$ (because $f\big|_{U_i \cap U_j} \circ g_i\big|_{U_i \cap U_j} = f\big|_{U_i \cap U_j} \circ g_j\big|_{U_i \cap U_j} = \mathrm{id}_{U_i \cap U_j}$ and $f\big|_{U_i \cap U_j}$ is an isomorphism, so we can "cancel" the f from both sides). Thus we can glue the g_i to get a morphism $g : Y \to X$ that is locally—hence globally—inverse to f.
b) By 2.4, we have a morphism $f : X \to \operatorname{Spec} A$ corresponding to the identity ring homomorphism $A \to \Gamma(X, \mathcal{O}_X)$. By 2.16 d) we have $\Gamma(X_f, \mathcal{O}_{X_f}) \simeq A_f$, and the isomorphism makes the diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}"] \arrow[d] & \Gamma(X, \mathcal{O}_X) \arrow[d, "\mathrm{res}"] \\ A_f \arrow[r, "\sim"] & \Gamma(X_f, \mathcal{O}_{X_f}) \end{tikzcd}$$

commute, from which it follows that $f\big|_{X_f} : X_f \to \operatorname{Spec} A_f$ is an isomorphism for each $f = f_i$. Since the f_i generate the unit ideal, $\operatorname{Spec} A_{f_i}$ covers $\operatorname{Spec} A$. Thus the hypothesis of part a) are satisfied, so f is an isomorphism and X is affine. The converse follows from the quasi-compactness of an affine scheme; see 2.13 b).

2.18 a) $D(f)$ is empty iff f is contained in the intersection of all primes; i.e. iff f is nilpotent.
b) If the map of sheaves $\mathcal{O}_X \to f_* \mathcal{O}_Y$ is injective then $A = \mathcal{O}_X(X) \to \mathcal{O}_Y(f^{-1}(X)) = \mathcal{O}_Y(Y) = B$ is injective. Conversely, if $A \to B$ is injective, then $A_f \to B_{\varphi(f)}$ is injective for all $f \in A$ (if $a/f^k \mapsto 0$ then $\varphi(f)^m \varphi(a) = \varphi(f^m a) = 0$ so $f^m a = 0$ since φ is injective). This shows that the map $f^\# : \mathcal{O}_X(D(f)) \to \mathcal{O}_Y(f^{-1}(D(f)))$ is injective for all f (where we use that $f^{-1}(D(f)) = D(\varphi(f))$). Thus the map of sheaves is injective since it is injective on a base of opens of X, and hence injective on every stalk.

Let $f \in A$. Then if $D(f)$ is nonempty, there exists $q \in \operatorname{Spec} B$ with $f \notin \varphi^{-1}(q)$, or what is the same, every nonempty $D(f) \subseteq \operatorname{Spec} A$ contains some $f(q)$. Indeed, if $f \in \varphi^{-1}(q)$ for all $q \in \operatorname{Spec} B$ then $\varphi(f) \in q$ for all q, and is hence nilpotent. As φ is an injective homomorphism, it follows that f is nilpotent and hence by a) that $D(f)$ is empty. Thus $f(Y)$ is dense.
c) If φ is surjective, then $A/\ker\varphi \simeq B$ so their spectra are homeomorphic, and $\operatorname{Spec} A/\ker\varphi = V(\ker\varphi) \subseteq \operatorname{Spec} A$ is a closed subset. Now let $s \in (f_* \mathcal{O}_Y)_p$ be represented by $\widetilde{s} \in \mathcal{O}_Y(f^{-1}(U))$. Shrinking if necessary, we may suppose $U = D(f)$ is basic, so $f^{-1}(U) = D(\varphi(f))$ as above. Thus, $\widetilde{s} \in B_{\varphi(f)}$. Since $A \to B$ is surjective, so is $A_f \to B_{\varphi f}$, so there exists $t \in \mathcal{O}_X(U)$ mapping to $\widetilde{s}$. It follows that the induced maps on stalks are all surjective, so the sheaf map is surjective.
d) Let $X' = \operatorname{Spec} A/\ker\varphi$. Then we have $Y \xrightarrow{\psi} X' \xrightarrow{\phi} X$, where ϕ is a homeomorphism onto a closed subset by c) and $\psi(Y)$ is dense in X' by b). Since the composite is f (this is just the fact that the ring map $A \to B$

factors through $A/\ker\varphi$) and $f(Y)$ is homeomorphic to a closed subset of X, we conclude that $\psi(Y) \subseteq X'$ is dense and closed, so that $\psi(Y) = X'$. Moreover, since f, ϕ are homeomorphisms, so is ψ. We claim that $\psi^\#$ is an isomorphism. It is injective by b) and surjective since $f^\#$ is surjective and the map $f^\# : \mathcal{O}_X \to f_*(\mathcal{O}_Y)$ factors as $\mathcal{O}_X \xrightarrow{\phi^\#} \phi_*\mathcal{O}_{X'} \xrightarrow{\phi_*(\psi^\#)} \phi_*\psi * \mathcal{O}_Y = f_*\mathcal{O}_Y$ as $f = \phi \circ \psi$. Hence it is an isomorphism by 1.5. It follows that $\operatorname{Spec} B \simeq \operatorname{Spec} A/\ker\varphi$ from which we conclude that $\varphi : A/\ker\varphi \to B$ is an isomorphism, i.e. that φ is surjective.

2.19 ((i) $\Longrightarrow$ (ii)) Suppose $\operatorname{Spec} A = V(I) \cup V(J)$ is disconnected. Pick a non-nilpotent, nonunit $e_1 \in I$ (if I consists only of nilpotents then the disconnect is trivial). Then $e_2 = 1 - e_1 \in J$, and hence e_1e_2 is in every prime ideal and so nilpotent. That is, e_1, e_2 as elements of A_{red} are orthogonal idempotents. We claim that such idempotents can be lifted to A. Indeed, let N denote the nilradical of A. Let e_1' be any lift of e_1 to A and put $e_2' = (1 - e_1')$. Then $e_1'e_2' = n$ is nilpotent, so for some j we have ${e_1'}^j {e_2'}^j = 0$. Set $\tilde{e}_1 = {e_1'}^j$ and $\tilde{e}_2 = {e_2'}^j$. Observe that $\tilde{e}_1 \equiv e_1' \mod N$ and $\tilde{e}_2 \equiv e_2' \mod N$ since $e_1' \equiv {e_1'}^2$, $e_2' \equiv {e_2'}^2 \mod N$. Moreover, we have $\tilde{e}_1 + \tilde{e}_2 \equiv 1 \mod N$ since $e_1' + e_2' \equiv 1 \mod N$. It follows that $\tilde{e}_1 + \tilde{e}_2$ is a unit, say with inverse $a \in A$. Clearly $a \equiv 1 \mod N$. We now put $e_1^* = a\tilde{e}_1$ and $e_2^* = a\tilde{e}_2$. Then $e_1^* + e_2^* = a(\tilde{e}_1 + \tilde{e}_2) = 1$ and $e_1^*e_2^* = a^2\tilde{e}_1\tilde{e}_2 = 0$. Since $a \equiv 1 \mod N$ we have $e_1^* \equiv \tilde{e}_1 \equiv e_1' \mod N$ so that e_1^* lifts e_1 and similarly e_2^* lifts e_2.
((ii) $\Longrightarrow$ (i)) Since $e_1e_2 = 0$ we see that $\operatorname{Spec} R = V(e_1) \cup V(e_2)$. Moreover, $V(e_1) \cap V(e_2) = \emptyset$ since no prime can contain $1 = e_1 + e_2$.
((ii) $\Longrightarrow$ (iii)) Suppose that A has orthogonal idempotents e_1, e_2. Define $\varphi : Ae_1 \times Ae_2$ by $(u, v) \mapsto u + v$. One checks this is a homomorphism. It is injective since if $re_1 = se_2$ then $re_1^2 = re_1 = se_1e_2 = 0$. It is surjective since $r = r(e_1 + e_2) = re_1 + re_2$.
((iii) $\Longrightarrow$ (ii)) If $A \simeq A_1 \times A_2$ then $e_1 = (1, 0)$ and $e_2 = (0, 1)$ are orthogonal idempotents.

2.3

Nike's trick: We will use the following "trick" repeatedly. Let $\operatorname{Spec} R, \operatorname{Spec} R' \subseteq X$ be affine opens with $x \in \operatorname{Spec} R \cap \operatorname{Spec} R'$. Then there is an affine neighborhood U of x that is basic open in both $\operatorname{Spec} R$ and $\operatorname{Spec} R'$.

3.1 The "if" direction is obvious. Conversely, let $\operatorname{Spec} B_i$ be an open affine cover of Y with $f^{-1}(\operatorname{Spec} B_i)$ covered by $\operatorname{Spec} A_{ij}$ with A_{ij} a finitely generated B_i-algebra for all j. Observe that $f^{-1}(\operatorname{Spec}(B_i)_b)$ is covered by $\operatorname{Spec}(A_{ij})_b$ (if $f(x) \in \operatorname{Spec}(B_i)_b$ then $f(x)$ is a prime of B_i not containing b so x is a prime of some $\operatorname{Spec} A_{ij}$ not containing the image of b under the algebra map $B_i \to A_{ij}$) and that $(A_{ij})_b$ is a finitely generated $(B_i)_b$-algebra. Thus, the hypotheses are inherited by basic opens of U_i.

Now let $\operatorname{Spec} B \subseteq Y$ be arbitrary. By "Nike's trick," there exists a cover of $\operatorname{Spec} B$ by affines that are basic open in *both* $\operatorname{Spec} B$ and $\operatorname{Spec} B_i$ (for varying i). This allows us to reduce to the case that $Y = \operatorname{Spec} B$ is affine, with the same hypotheses as above, and we need only show that any affine in the $f^{-1}(Y) = X$ is a finitely generated B-algebra.

Thus, let $\operatorname{Spec} B_{b_i}$ be our cover of $\operatorname{Spec} B$ (constructed above) by basic opens with $f^{-1}(\operatorname{Spec} B_{b_i})$ covered by $\operatorname{Spec} A_{ij}$ and A_{ij} a finitely generated $B_{b_i} = B[1/b_i]$-algebra, and hence a finitely generated B-algebra. Let $\operatorname{Spec} A \subseteq X$ be arbitrary. By the Nike trick again, there is a cover $\operatorname{Spec} A_{a_k}$ of $\operatorname{Spec} A$ with each $\operatorname{Spec} A_{a_k}$ basic open in *both* $\operatorname{Spec} A$ and $\operatorname{Spec} A_{ij}$ (for varying i, j). Each A_{a_k} is isomorphic to a localization of some A_{ij} and is therefore a finitely generated A_{ij}-algebra, and hence also a finitely generated B-algebra.

Thus, we are reduced to the following problem: A is a ring with (a_k) generating the unit ideal, and each A_{a_k} is a finitely generated B-algebra, and we must show that A is a f.g. B-algebra. We may clearly assume that there are only finitely many a_k and that we have $x_1, \ldots, x_n \in A$ with $\sum x_k a_k = 1$. Let $A_{a_k} = B[y_{k1}/\alpha_k^N, y_{k2}/\alpha_k^N, \ldots, y_{km_k}/\alpha_k^N]$ with $y_{kl} \in A$ for all k, l. Put $A' = B[x_k, a_k, y_{kl}]$ with k, l running over all possible indices (a *finite* set!). Then A' is obviously a B-subalgebra of A. Moreover, we have $A_{a_k} = A'_{a_k}$ for all k. For any $p \in \operatorname{Spec} A$ choose $a_k \notin p$ (since (a_k) is the unit ideal this is possible). Then

A'_p is a further localization of $A'_{a_k} = A_{a_k}$, from which we conclude that $A'_p = A_p$ for all $p \in \operatorname{Spec} A$. Thus $A = A'$ is a finitely generated B-algebra.

3.2 As in 3.1, one direction is trivial. For the converse, let $\operatorname{Spec} B \subseteq Y$ be arbitrary and suppose we have a cover $\operatorname{Spec} B_i$ of Y with $f^{-1}(\operatorname{Spec} B_i)$ quasi-compact. Suppose $f^{-1}(\operatorname{Spec} B_i)$ is covered by $\{\operatorname{Spec} A_{ij}\}_{j\in J_i}$ with $\#J_i < \infty$ for each i. Then we can cover $\operatorname{Spec} B$ by *finitely* many opens of the form $\operatorname{Spec}(B_i)_{b_i}$, say for $i \in I$, and we have $f^{-1}(\operatorname{Spec}(B_i)_{b_i}) = \cup_{j\in J_i} \operatorname{Spec}(A_{ij})_{b_i}$ so $f^{-1}(\operatorname{Spec}(B_i)_{b_i})$ is quasicompact. It follows that $f^{-1}(\operatorname{Spec} B) = \cup_{i\in I,\ j\in J_i}(A_{ij})_{b_i}$, and since $\#I < \infty$ and $\#J_i < \infty$ for all $i \in I$, we conclude that $f^{-1}(\operatorname{Spec} B)$ is quasicompact.

3.3 a) If f is of finite type then it is clearly of locally finite type and quasi-compact. Conversely, if f is q-compact and locally of finite type, then we have a covering of Y by affines $\operatorname{Spec} B_i$ with $f^{-1}(\operatorname{Spec} B_i)$ covered by $\operatorname{Spec} A_{ij}$ with A_{ij} finitely generated B_i-algebras. By 3.2, $f^{-1}(\operatorname{Spec} B_i)$ is quasi-compact, so finitely many of the $\operatorname{Spec} A_{ij}$ will do; i.e. f is of finite type.
b) By a) f is f.t. iff. it is locally f.t. and q-compact. Now apply 3.1 and 3.2.
c) This was done in 3.1.

3.4 The "if" direction is obvious. Conversely, let $\operatorname{Spec} B_i$ be an open affine cover of Y with $f^{-1}(\operatorname{Spec} B_i) = \operatorname{Spec} A_i$ with A_i a finitely B_i-module. for all i. Observe that $f^{-1}(\operatorname{Spec}(B_i)_b) = \operatorname{Spec}(A_i)_b$ and that $(A_i)_b$ is a finite $(B_i)_b$-module so the hypotheses are inherited by basic opens of $\operatorname{Spec} B_i$.

Now let $\operatorname{Spec} B \subseteq Y$ be arbitrary. By "Nike's trick," there exists a *finite* cover of $\operatorname{Spec} B$ by affines that are basic open in *both* $\operatorname{Spec} B$ and $\operatorname{Spec} B_i$ (for varying i). This allows us to reduce to the case that $Y = \operatorname{Spec} B$ is affine with a covering $\operatorname{Spec} B_{b_i}$, by finitely many *basic* opens and $f^{-1}(\operatorname{Spec} B_{b_i}) = \operatorname{Spec} C_i$ affine with C_i a finite B_{b_i}-module. We need only show that $f^{-1}(Y) = X$ is affine, equal to $\operatorname{Spec} A$, with A a finite B-module.

By 2.4, we have a ring homomorphism $B \to \Gamma(X, \mathcal{O}_X) = A$ corresponding to the map $f : X \to Y = \operatorname{Spec} B$. Moreover, we see that the b_i generate the unit ideal in A because they do in B, and that $f^{-1}(\operatorname{Spec} B_{b_i}) = X_{b_i} = \operatorname{Spec} C_i$ is affine and gives a finite cover of X. By 2.17, $X = \operatorname{Spec} A$ is affine.

It remains to show that A is a finite B-module. We now know that $X_{b_i} = \operatorname{Spec} A_{b_i} = \operatorname{Spec} C_i$, so $A_{b_i} \simeq C_i$. That is, we are reduced to the following problem: A is a B-algebra and $b_i \in B$ is a finite collection generating the unit ideal such that A_{b_i} is a finite B_{b_i}-module, and we wish to conclude that A is a finite B-module. Let $\{z_{ij}\}$ for $1 \le j \le m_i$ generate A_{b_i} as a B_{b_i}-module, where by clearing denominators we may suppose that $z_{ij} \in A$. Any $a \in A$ can be written $a = \sum \beta_{ij} z_{ij}$, where $\beta_{ij} \in B_{b_i}$. Since there are only finitely many β_{ij}, we can find some N so that $b_i^N \beta_{ij} = \gamma_{ij} \in B$ for all i, j. Then $b_i^N a = \sum \gamma_{ij} z_{ij}$ for all i. Since b_i generate the unit ideal, so do b_i^N, so we have $\sum x_i b_i^N = 1$ for $x_i \in B$. Thus, putting $\mu_{ij} = x_i \gamma_{ij}$ we have $\mu_{ij} \in B$ and $a = \sum_{i,j} \mu_{ij} z_{ij}$. Thus A is a finite B-module.

3.5 a) One reduces immediately to the affine case, $\operatorname{Spec} B \to \operatorname{Spec} A$ with B integral over A. The fact that there are only finitely many primes of B lying over any given prime of A is standard Commutative Algebra.
b) This is the "going up" theorem from Commutative Algebra.
c) Take $k[x] \to k[x, y]/(xy - 1) \times k$ with the map $p(x) \mapsto (p(x), p(0))$. The corresponding map of spectra is surjective, finite type, and quasi-finite. However, $k[x, y]/(xy - 1) \times k$ is not a finite $k[x]$-module.

3.6 Let $U = \operatorname{Spec} A$ be any affine open. Since X is irreducible, $\overline{xi} = X$ so $\xi \in U$. We claim that ξ is a minimal prime of A. Indeed, is $\zeta \in \operatorname{Spec} A$ is contained in ξ, then $X \supseteq \overline{\zeta} \supseteq \overline{\xi} = X$, so we have equality, and the uniqueness of generic point (2.9) gives $\zeta = \xi$. But $\mathcal{O}_X(U) = A$ is an integral domain as X is integral, and there is a unique minimal prime (0) of any domain. It follows that $\mathcal{O}_\xi = \lim_{U\ni\xi} \mathcal{O}_X(U) = \lim_{f\in A-\{0\}} \mathcal{O}_X(D(f)) = \lim_{f\notin A} A_f = A_{(0)} = \operatorname{Frac} A$.

3.7 Let ξ_X, ξ_Y be the generic points of X, Y resp. If U is an open set of Y not containing $y = f(\xi_X)$ then $f^{-1}(U)$ is an open set of X not containing ξ_X, and so must be empty. But then $U \cap f(X) = \emptyset$ and f is not dominant. We conclude that every open set of Y contains $f(\xi_X)$, and hence that $f(\xi_X) = \xi_Y$. We thus have a local map of local rings $f^{\#}_{\xi_Y} : \mathcal{O}_{\xi_Y} \to \mathcal{O}_{\xi_X}$ which is injective as $\mathcal{O}_{\xi_Y} = K(Y)$ is a field, i.e. $K(X)$ is a field extension of $K(Y)$. We claim that it is an *algebraic* field extension. Indeed, If $\operatorname{Spec} B$ is any affine open in Y and $\operatorname{Spec} A$ is any affine open in $f^{-1}(\operatorname{Spec} B)$, then we have a ring homomorphism $\varphi : B \to A$ corresponding to $f : \operatorname{Spec} A \to \operatorname{Spec} B$, and since f is generically finite, there are only finitely many primes of A lying over $(0) \in B$. If $\operatorname{Frac} A = K(X)$ is transcendental over $\operatorname{Frac} B = K(Y)$ then A is transcendental over B, and there are infinitely many primes of A lying over $(0) \in \operatorname{Spec} B$, contradicting the generic finiteness assumption. Hence $K(X)$ is an algebraic, finitely generated (since f is of finite type) $K(Y)$-algebra, and is therefore a finite extension of fields. It follows that there exists some $b \in B$ with A finitely generated as a B_b-module. Thus, we may shrink $\operatorname{Spec} B$ (if necessary) to obtain a cover of $f^{-1}(\operatorname{Spec} B)$ by affines $\operatorname{Spec} A_i$ for $1 \leq i \leq n$ with each A_i a finite B-module. Now for each $i < n$ there exists $a_i \in A_i$ such that $\operatorname{Spec}(A_i)_{a_i} \subseteq \operatorname{Spec} A_n$, and since A_i is an integral B-extension, a_i satisfies a monic polynomial $\sum_k \beta_{ik} a_i^k = 0$ with $\beta_{ik} \in B$ and $\beta_{i0} \neq 0$. Let $b = \prod_{1 \leq i < n} \beta_{i0}$, so $b \neq 0$, and every prime of A_i containing a_i contains b for all $i < n$, that is, $\operatorname{Spec}(A_i)_b \subseteq \operatorname{Spec}(A_i)_{a_i}$ for all $i < n$. It follows that $f^{-1}(B_b) = \cup_{i<n} \operatorname{Spec}(A_i)_b \cup \operatorname{Spec}(A_n)_b = \operatorname{Spec}(A_n)_b$ is affine, and since A_n is a finite B-module, $(A_n)_b$ is a finite B_b-module. As X, Y are integral (hence irreducible), $U = \operatorname{Spec} B_b$ and $f^{-1}(U) = \operatorname{Spec}(A_n)_b$ are dense in Y, X respectively, and $f : f^{-1}(U) \to U$ is finite.

3.8 This is a standard application of **The Fourfold Way**:

We wish to construct an X-scheme $P(X) \to X$ for every scheme X with $P(X)$ having some universal property $\mathcal{P}$ in some subcategory of the category of X-schemes.

1. Prove that if $P(X) \to X$ exists for a single X, then the open subscheme in $P(X)$ that lies over an open subscheme U of X satisfies the universal property to be $P(U)$; in particular, existence for X implies existence for any open subscheme of X.

2. Suppose the problem can be solved locally on a single X. That is, assume there is an open cover $\{U_i\}$ of X such that $P(U_i) \to U_i$ exists. Now consider P_{ij}, the part of $P(U_i)$ that lies over $U_{ij} = U_i \cap U_j = U_{ji}$. Notice that P_{ij} and P_{ji} both satisfy the same universal property in the category of U_{ij}-schemes, by step 1 (applied to the scheme U_{ij}), so they are *uniquely* U_{ij}-isomorphic, and the uniqueness ensures triple-overlap consistency when comparing the various triples of isomorphisms we get over triple overlaps.

3. Using step 2, we may (for X and $\{U_i\}$ as in step 2) glue the $P(U_i)$ to make an X-scheme. Now this glued X-scheme restricts over U_i to give the universal thing $P(U_i) \to U_i$, and so one then can usually exploit this to prove that the glued thing over X does in fact satisfy the universal property to serve as the desired $P(X) \to X$.

4. By steps 1–3, the existence problem for $P(X)$ for any particular X is "local on the base" (i.e., suffices to solve the problem for the constituents of an open covering of X), and of course the uniqueness aspect is generally OK by whatever universal property is to be satisfied by the construction. Now we may suppose X is affine and we perhaps make an explicit construction in this case, and thereby solve the global problem in view of the preceding considerations.

Suppose that $\pi : \widetilde{X} \to X$ exists and let $U \subseteq X$ be open. We wish to show that $\pi^{-1}(U) \simeq \widetilde{U}$. The scheme $\pi^{-1}(U)$ is normal because it is a subscheme of $\widetilde{X}$ and normality is a local property. It is integral because it is reduced (again a local property) and irreducible (because it is an open subscheme of an irreducible scheme). Moreover, if $Z \to U$ is any normal integral U-scheme with dominant structure map, then Z becomes an X-scheme with dominant structure map (since U is dense in X), and hence factors uniquely through $\widetilde{X}$, and it is clear that the image lies in $p^{-1}(U)$, so $p^{-1}(U)$ has the right universal property and is thus (isomorphic to) $\widetilde{U}$.

Next we show that it suffices to solve the existence of normalization locally on X. Indeed, let $\{X_i\}$ be an open cover of X and let $\widetilde{X}_i$ be the normalization of X_i. By 1), we get the normalization of $X_{ij} = X_i \cap X_j$ in two different ways: one as a subscheme of $\widetilde{X}_i$ and the other as a subscheme of $\widetilde{X}_j$. The uniqueness (which is an immediate consequence of the universal property) yields an isomorphism ϕ_{ij} identifying these two constructions. Moreover, uniqueness up to unique isomorphism ensures that $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ (triple overlap compatibility), so that we can glue the $\widetilde{X}_i$ to obtain a scheme $\pi : \widetilde{X} \to X$.

We wish to show that $\widetilde{X}$ has the required universal property. Over each X_i, the scheme $\widetilde{X}$ restricts to $\widetilde{X}_i$ so given an integral normal scheme $\phi : Z \to X$ with dominant structure map, we obtain unique maps $\phi_i : \phi^{-1}(X_i) \to X_i$ (which must be dominant by irreducibility considerations) and hence unique factorizations $\psi_i : \phi^{-1}(X_i) \to \widetilde{X}_i$. The uniqueness of these maps ensures compatibility on overlaps, so we may glue them to show that $Z \to X$ factors uniquely through $\widetilde{X}$. Moreover, $\widetilde{X}$ is normal and reduced since $\widetilde{X}_i$ is for each i (and these are local properties) and is irreducible since the $\widetilde{X}_i$ are irreducible and all intersect pairwise (because $\widetilde{X}_i = \pi^{-1}(X_i)$ and the X_i all intersect as X is irreducible).

We have reduced the existence of normalization to the case of affine $X = \operatorname{Spec} A$, with A a domain. Let B be the integral closure of A in $\operatorname{Frac} A$. Then every localization of B is integrally closed, so $\operatorname{Spec} B$ is normal, and since $A \to B$ is injective, the map $\operatorname{Spec} B \to \operatorname{Spec} A$ is dominant (2.18). If $Z \to \operatorname{Spec} A$ is a dominant map with Z an integral normal scheme, then we have an injective (converse to 2.18 a)) map $A \to \Gamma(Z, \mathcal{O}_Z)$ by 2.4. Since $\Gamma(Z, \mathcal{O}_Z)$ is normal (as all localizations at prime ideals are) the morphism $A \to \Gamma(Z, \mathcal{O}_Z)$ factors through B so by 2.4 again, the morphism $Z \to \operatorname{Spec} A$ factors through $\operatorname{Spec} B$. This gives the normalization for affines, and we are done.

Finally, if X is finite type over a field, then we have a cover $X_i = \operatorname{Spec} A_i$ by affines with $\pi^{-1}(X_i) = \operatorname{Spec} B_i$ with B_i integral over A_i, so by Theorem 3.9A of Chapter I, B_i is a finite A_i-module and π is finite.

3.9 The fiber product $X = \mathbf{A}^1_k \times \mathbf{A}^1_k$ has the following universal property: to give a k-morphism $Y \to X$ (with Y a scheme over k) is to give k-morphisms $\phi_1, \phi_2 : Y \to \mathbf{A}^1_k$, that is, to give two k-algebra homomorphisms $k[x] \to \Gamma(Y, \mathcal{O}_Y)$. Thus, $\operatorname{Hom}(Y, X) \simeq \Gamma(Y, \mathcal{O}_Y)^2$ is a bijection. Since this is the universal property of $\mathbf{A}^2_k$, we conclude that $X \simeq \mathbf{A}^2_k$. We could also observe that $X = \operatorname{Spec}(k[x] \otimes_k k[y]) \simeq \operatorname{Spec} k[x, y]$. The underlying point-set of the product has points that correspond to irreducible curves in $\mathbf{A}^2_k$ (even if k is algebraically closed). For example, we might take $xy - 1$. If such a point p were to correspond to an element of the product set $\operatorname{sp}(\mathbf{A}^1_k) \times_{\operatorname{sp}(k)} \operatorname{sp}(\mathbf{A}^1_k)$ it would have to be the point $(i_1^{-1}(p), i_2^{-1}(p))$ where $i_1 : k[x] \to k[x, y]$ and $i_2 : k[y] \to k[x, y]$ are the natural inclusions. But for such p, the contraction of p via i_j is (0) for $j = 1, 2$, so the sets are not equal (Really I should give a cardinality argument).

b) Since $k(s) = S^{-1}k[s]$ and $k(t) = T^{-1}k[t]$ where S, T are the multiplicative subsets of $k[s], k[t]$ consisting of all nonzero elements, we have $k(s) \otimes_k k(t) = S^{-1}k[s] \otimes_k T^{-1}k[t] = (ST)^{-1}k[s] \otimes_k k[t] = U^{-1}k[s, t]$, where U is the multiplicative subset of $k[s, t]$ consisting of all nonzero polynomials $P(s)Q(t)$. Since $k[s, t]$ and $k[s]$ are finitely generated k-algebras, they are Jacobson rings, so maximal ideals contract to maximal ideals under the inclusion $k[s] \hookrightarrow k[s, t]$ (see Eisenbud's Commutative Algebra, Theorem 4.19). Thus, every maximal ideal m of $k[s, t]$ contains some nonzero $P(s)$. It follows that the expansion of m under $k[s, t] \to U^{-1}k[s, t]$ is the unit ideal. We conclude that the prime ideals of $U^{-1}k[s, t]$ correspond to the height-1 prime ideals of $k[s, t]$ *not* of the form $P(s)Q(t)$. That is, the prime ideals of $U^{-1}k[s, t]$ are principal, generated by some irreducible $g \in k[s, t] \setminus (k[s] \cup k[t])$. There are infinitely many such irreducibles, and it follows that $\operatorname{Spec}(U^{-1}k[s, t]) \simeq \operatorname{Spec} k(s) \times_k \operatorname{Spec} k(t)$ has infinitely many points. Moreover, $\operatorname{Spec} k(s) \times_k \operatorname{Spec} k(t)$ is 1-dimensional as it is the spectrum of a ring in which every prime is principal.

3.10 a) We claim that the first projection $p : X \times_Y \operatorname{Spec} k(y) \to X$ induces a homeomorphism $X_y \to f^{-1}(y)$. Letting $\operatorname{Spec} A$ be any affine nbd. of $y \in Y$, we see from the universal properties of fiber product that $X_y = (X \times_Y \operatorname{Spec} A) \times_{\operatorname{Spec} A} \operatorname{Spec} k(y) = (f^{-1}(\operatorname{Spec} A))_y$, so we may suppose $Y = \operatorname{Spec} A$ is affine. For any open subset $U \subseteq X$ we have $p^{-1}(U) = U \times_Y \operatorname{Spec} k(y)$ (by universal properties). Moreover, if we can show that $p : p^{-1}(U) \to U$ induces a homeomorphism $U_y \to f^{-1}(y) \cap U$ then we can use universal properties to glue and obtain the desired result (sketchy). Thus, we reduce to the case that $X = \operatorname{Spec} B$ is affine. Let $p \in \operatorname{Spec} A = Y$ be the point y and let $g = f_Y^{\#} : A \to B$ be the ring map corresponding to $f : X \to Y$ making B into an A-algebra. We have ring maps $B \xrightarrow{\phi} B \otimes_A A_p \xrightarrow{\psi} B \otimes_A k(p)$, and the projection p is the map $\operatorname{Spec}(B \otimes_A k(p)) \to \operatorname{Spec} B$ corresponding to $\psi \circ \phi$. We already know that $\operatorname{Spec} \phi$, $\operatorname{Spec} \psi$ are continuous.

Now ψ is surjective, so by 2.18 c), $\operatorname{Spec}\psi : X_y \to \operatorname{Spec}(B \otimes_A A_p)$ is a homeomorphism onto the closed subset $V(\ker\psi)$. We claim that $\operatorname{Spec}\phi$ is a homeomorphism onto the set $\{q \in \operatorname{Spec} B : q \cap S = \emptyset\}$, where $S = g(A - p)$. ($\operatorname{Spec}\phi$ is closed since $\operatorname{Spec}\phi(V(I)) = V(\phi^{-1}(I)) \cap (\operatorname{Spec}\phi)(\operatorname{Spec} B)$ is a closed subset of the image of $\operatorname{Spec}\phi$, which are those primes of B not meeting S). Therefore, p is the composition of two homeomorphisms and hence a homeomorphism. Since $\ker\psi = pB$, we see that p is a homeomorphism onto the set of primes $q \in \operatorname{Spec} B$ such that $q \supseteq pB = g(p)B$ and $q \cap S = q \cap g(A - p) = 0$, that is, $g^{-1}(q) \supseteq p$ and $g^{-1}(q) \subseteq p$, i.e. $g^{-1}(q) = p$.

b) Assuming k to be algebraically closed, the fiber over $y = (s - a) \in \operatorname{Spec} k[s]$ consists (by part a)) of those primes in $k[s,t]/(s - t^2)$ contracting to $(s - a)$, that is $(s - a, t - \sqrt{a})$ and $(s - a, t + \sqrt{a})$. The residue field at each point is k (defined by mapping $s \to a$ and $t \to \pm\sqrt{a}$). If $a = 0$, the fiber over $y = (s)$ is the scheme $\operatorname{Spec}(k[s,t]/(s - t^2) \otimes_{k[s]} k[s]_{(s)}/(s)) = \operatorname{Spec}(k[s,t]/(s - t^2) \otimes_{k[s]} k)$ where the map $k[s] \to k$ sends s to 0, so s acts on the left of the tensor product as 0. Thus, $k[s,t]/(s-t^2) \otimes_{k[s]} k \simeq k[t]/t^2$ and the fiber is the one-point non-reduced scheme $\operatorname{Spec}(k[t]/t^2)$. The prime ideals of $k[s,t]/(s - t^2)$ that contract to $(0) \subseteq k[s]$ are those prime ideals of $k[s,t]$ containing $(s - t^2)$ that contain no polynomials in s. The only such ideal is $(s - t^2)$ (see 3.9 b)). In other words, $k[s,t]/(s - t^2) \otimes_{k[s]} k(s) \simeq k(s)[t]/(s - t^2)$ (a field) so the fiber is a one-point scheme with residue field a degree-2 extension of $k(s)$.

3.12 a) Since φ is surjective and degree-preserving, we have $\varphi(S_+) = T_+$ so $U = \operatorname{Proj} T$. As φ is surjective, we have $S/\ker\varphi \simeq T$ so since there is a 1-1 inclusion-preserving correspondence between homogeneous prime ideals of S that contain $\ker\varphi$ and homogeneous prime ideals of $S/\ker\varphi$, we conclude that $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is a homeomorphism onto $V(\ker\varphi)$ ($\ker\varphi$ is a homogeneous ideal). We need only remark that $f^\# : \mathcal{O}_{\operatorname{Proj} S} \to f_*\mathcal{O}_{\operatorname{Proj} T}$ is surjective. But this follows from the fact that $\mathcal{O}_{\operatorname{Proj} S}(D_+(f)) = S_{(f)} \to T_{\varphi(f)} = \mathcal{O}_{\operatorname{Proj} T}(f^{-1}(D_+(f))$ is surjective for any $f \in S$ since $S \to T$ is surjective (equivalently $S_{\varphi^{-1}(p)} \to T_p$ is surjective for any prime $p \in \operatorname{Proj} T$, so the sheaf map is surjective on stalks).

b) Observe that $(S/I)_d \simeq (S/I')_d$ for all $d \geq d_0$. By 2.14 c), the morphism $f : \operatorname{Proj}(S/I) \to \operatorname{Proj}(S/I')$ associated to $S/I' \to S/I$ is an isomorphism that is evidently compatible with the closed immersions $\operatorname{Proj}(S/I) \to \operatorname{Proj} S$ and $\operatorname{Proj}(S/I') \to \operatorname{Proj} S$ since the ring maps are.

3.13 a) Let $f : X \to Y$ be a closed immersion. Observe that for $U \subseteq Y$ open, the map $f : f^{-1}(U) \to U$ is a closed immersion (indeed, it is a homeomorphism onto the closed subset $f(X) \cap U$ of U and the sheaf map $\mathcal{O}_Y\big|_U \to f_*\mathcal{O}_X\big|_{f^{-1}(U)}$ is surjective since it is on stalks). Thus, being a closed immersion is local on the base. We have already seen that being a finity-type morphism is local on the base (3.1, 3.3), so we reduce to the case $Y = \operatorname{Spec} A$ whence $X \simeq \operatorname{Spec} A/I$. Since A/I is clearly a finitely generated A-algebra, we conclude that f is finite-type.

b) By 3.3 a) it will suffice to show that f is locally of finite type. Let $f : X \to Y$ be an isomorphism onto $U \subseteq Y$ and let $\operatorname{Spec} A$ be any affine open of Y. Then $f^{-1}(\operatorname{Spec} A) = f^{-1}(U \cap \operatorname{Spec} A)$, and we may cover $U \cap \operatorname{Spec} A$ by open affines $\operatorname{Spec} A_{a_i}$. Since $f : X \to U$ is an isomorphism, $f^{-1}(\operatorname{Spec} A)$ is covered by open affines isomorphic to $\operatorname{Spec} A_{a_i}$, and each A_{a_i} is a finitely generated A-algebra.

c) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ with f, g finite type, and let $\{\operatorname{Spec} A_i\}$ be a covering of Z with $g^{-1}(\operatorname{Spec} A_i) = \cup_{j=1}^{n_j} \operatorname{Spec} B_{ij}$ and B_{ij} a finitely generated A_i-algebra. By 3.1, $f^{-1}(\operatorname{Spec} B_{ij}) = \cup_{k=1}^{m_{ij}} \operatorname{Spec} C_{ijk}$ with C_{ijk} a finite B_{ij}-algebra, and hence a finite A_i-algebra. Thus, $(g \circ f)^{-1}(\operatorname{Spec} A_i) = \cup_{j,k} \operatorname{Spec} C_{ijk}$ is a finite cover with C_{ijk} a finitely generated A_i-algebra for all i, j, k. Thus $g \circ f$ is of finite type.

d) Let $f : X \to S$ be an S-scheme and $S' \to S$ a base change. Because finite type is local on the base, we may assume $S = \operatorname{Spec} A$ and $S' = \operatorname{Spec} B$ are affine. Let $X = f^{-1}(\operatorname{Spec} A)$ be covered by $\operatorname{Spec} C_i$, with C_i a finite A-algebra. Then $(f')^{-1}(S') = X \times_S S'$ is covered by $\operatorname{Spec}(B \otimes_A C_i)$, and since C_i is a f.g. A-algebra, $B \otimes_A C_i$ is a f.g. B-algebra.

e) We may assume $S = \operatorname{Spec} A$ is affine. Let $\operatorname{Spec} B_i$ and $\operatorname{Spec} B_j'$ be finite covers of X, Y with B_i, B_j' finite A-algebras. Then $\operatorname{Spec}(B_i \otimes_A B_j')$ is a finite cover of $X \times_S Y$ and $B_i \otimes_A B_j$ is a finite A-algebra.

f) Again, we need only check that f is locally finite type. Cover Z by open affines $\operatorname{Spec} C_i$ with $(g \circ f)^{-1}(\operatorname{Spec} C_i)$ covered by finitely many $\operatorname{Spec} A_{ij} \subseteq X$. Now let $\operatorname{Spec} B_{ik}$ be a cover of $g^{-1}(\operatorname{Spec} A_i)$ and observe that $f^{-1}(\operatorname{Spec} B_{ik})$ is covered by a collection of the $\operatorname{Spec} A_{ij}$, so we have ring maps $C_i \to B_{ik} \to A_{ij}$

such that A_{ij} is finite type over C_i, from which we conclude that A_{ij} is finite type over B_{ik} and hence that f is locally of finite type.
g) Let $\operatorname{Spec} A_i$ be a finite cover of Y with A_i noetherian. By 3.1, $f^{-1}(A_i)$ can be covered by finitely many $\operatorname{Spec} B_{ij}$ with B_{ij} a finite A_i-algebra. Since each A_i is noetherian, so are all the B_{ij} and $\operatorname{Spec} B_{ij}$ is a finite cover of X with B_{ij} noetherian.

3.14 Let $U \subseteq X$ be open and let $U_i = \operatorname{Spec} A_i$ be an affine cover of X. We claim that x closed in U implies x closed in U_i for all $U_i \ni x$. Indeed, pick a basic open $\operatorname{Spec} B$ inside $U_i \cap U$ containing x so the inclusion $U_i \cap U \hookrightarrow U_i$ gives a ring homomorphism $A_i \to B$. Both A_i and B are finitely generated k-algebras, hence Jacobson rings, so maximal ideals contract to maximal ideals (c.f. 3.9). In particular, since x is closed in $U_i \cap U$, its image in U_i is closed. Since x is closed in each U_i and the U_i cover X, we conclude that x is closed in X.

Thus it suffices to prove that every nonempty basic open subset of an affine $\operatorname{Spec} A$ contains a maximal ideal of A. But if $f \in A$ is in every maximal ideal then it is nilpotent (as A is Jacobson) hence $D(f)$ is empty.

As an example where this fails, let R be any local domain (for example, $\mathbf{Z}_{(3)}$). Then if $f \in m \setminus \{0\}$, the set $D(f)$ is nonempty and open, and contains no closed points.

3.15 a) We have $(iii) \implies (i) \implies (ii)$, so we show $(ii) \implies (i) \implies (iii)$. We claim that if K/k is purely inseparable, then X irreducible implies X_K irreducible. Indeed, if X_K is not irreducible, then there is an open affine subset $U_K \subset X_K$ with $\Gamma(U_K, \mathcal{O}_{X_K})$ not a domain (take U_K to be the union of two disjoint open affines $V_1, V_2 \subseteq X_K$). Since $\operatorname{Spec} A$ is homeomorphic to $\operatorname{Spec} A_{\mathrm{red}}$, it will suffice to show that if A is a domain so is $A \otimes_k K$ for any purely inseparable extension K/k. But $A \otimes_k K$ having a zero-divisor is equivalent to a system of equations with coefficients in k having a solution over K. We may suppose that K/k is finite since any element of $A \otimes_k K$ is contained in $A \otimes_k L$ with L a finite extension of k. Thus, there exists $q = (\operatorname{char} k)^r$ for some r such that $K^q \subseteq k$. Letting $\{f_i\}$ be our system of equations with a solution in K, we see that $\{f_i^q\}$ has s solution in k, and since $x \mapsto x^q$ is injective, we obtain a zero-divisor in A, contradicting our assumption.

We now prove that if k'/k is an extension of fields with k algebraically closed then $\{f_j\}$ with $f_j \in k[X_1, \ldots, X_n]$ has a solution over k iff it has one over k'. One direction is clear. For the opposite direction, we prove the contrapositive. By the Nullstellensatz (crucially using that k is algebraically closed) if $\{f_j\}$ has no solution over k then the f_j generate the unit ideal of $k[X_1, \ldots, X_n]$, and hence they also generate the unit ideal of $k'[X_1, \ldots, X_n]$ and therefore have no solution over k'.

Lastly, we show that $(i) \implies (iii)$. We may suppose that K is an extension of $\overline{k}$. Then having a zero divisor in $A \otimes_k K$ is equivalent to a system of equations with coefficients in $\overline{k}$ having a solution over K, and by the above result, such a system also has a solution over $\overline{k}$ whence $A \otimes_k \overline{k}$ is not a domain.
b) Obviously $(iii) \implies (i) \implies (ii)$, so it will suffice to prove $(ii) \implies (i) \implies (iii)$. We claim that if K/k is *separable* then X reduced implies X_K reduced. Indeed, we may suppose that $X = \operatorname{Spec} A$ is affine (for it suffices to show that U_K is reduced for every affine $U \subseteq X$) and we must therefore show that $\Gamma(U_K) = A \otimes_k K$ is a reduced ring given that A is reduced. We may suppose that A is a domain: indeed, $A \hookrightarrow \prod A/p_i$, so $A \otimes_k K \hookrightarrow \prod A/p_i \otimes kK$ as K/k is flat, the product being over all minimal primes of A, and a product of rings is reduced iff each factor is. (We have also tacitly used that tensor product commutes with finite direct products= finite direct sums, which employs the finite type hypothesis, i.e. that A has finitely many minimal primes.) Since $A \hookrightarrow \operatorname{Frac}(A)$, it will suffice to show that $F \otimes_k K$ is reduced for every extension field F/k. We may replace K by a *finite* extension L/k since every element of $F \otimes_k K$ is contained in $F \otimes_k L$ for some finite L (depending on the element). But then $L \simeq k[T]/(f)$ with $f \in k[T]$ a *separable* polynomial, so that $F \otimes_k L \simeq F[t]/(f)$ and $f \in F[t]$ is still separable, so $F[t]/(f)$ is reduced. Since $\overline{k}/k_p$ is a separable extension, we have shown $(ii) \implies (i)$.

Now we show that if $A \otimes_k \overline{k}$ is reduced, then $A \otimes_k K$ is reduced for any extension K/k of fields. We reduce at once (by a further field extension if necessary) to the case that K is an extension of $\overline{k}$. But then

having a nilpotent element of $A \otimes_k K$ is equivalent to giving a system of equations with coefficients in $\overline{k}$ that have a solution over K, and by the result in part a), must therefore have a solution over $\overline{k}$.
c) Let $k = \mathbf{F}_p(T)$ and $K = \mathbf{F}_p(T^{1/p})$. Then $\operatorname{Spec} K$ is reduced but not geometrically reduced as $\operatorname{Spec} K \times_k \operatorname{Spec} K = \operatorname{Spec}(K \otimes_k K)$ and $K \otimes_k K$ has the nonzero nilpotent $x = 1 \otimes T^{1/p} - T^{1/p} \otimes 1$ with $x^p = 0$. Similarly, $X = \operatorname{Spec} \mathbf{R}[x]/(x^2+1) \simeq \operatorname{Spec} \mathbf{C}$ is irreducible (it is a single point) but not geometrically irreducible as $X_{\mathbf{C}} = \operatorname{Spec} \mathbf{C}[x]/(x^2+1) = \operatorname{Spec}(\mathbf{C} \oplus \mathbf{C}) = \operatorname{Spec} \mathbf{C} \coprod \operatorname{Spec} \mathbf{C}$.

2.4

4.1 Let $f : X \to Y$ be finite. Since properness is local on the base (cor 4.8), we may assume $Y = \operatorname{Spec} A$ and $f^{-1}(Y) = X = \operatorname{Spec} B$ with B a finite A-module (since f is finite; c.f. 3.4). By Prop. 4.1, $f : X \to Y$ is separated and it is of finite type since it is finite. We need to check that $f' : X \times_Y Y' \to Y'$ is closed for all $Y' \to Y$. Since this is local on Y, we may suppose that $Y' = \operatorname{Spec} C$ is affine. We are reduced to showing that $\operatorname{Spec} B \otimes_A C \to \operatorname{Spec} C$ induced by the map $C \to B \otimes_A C$ to the second factor is closed. But $B \otimes_A C$ is integral over C as B is integral over A (generated as a C module by $g_i \otimes 1$ with g_i a finite set of A-module generators of B), so by 3.5 b), f' is closed and f is universally closed. Thus, f is proper.

4.2 Let $h = (f, g) : X \to Y \times_S Y$. Observe that $(f, f) = \Delta \circ f : X \to Y \times_S Y$ agrees with h on the open dense subset U Since Y is separated, $\Delta : Y \to Y \times_S Y$ is a closed immersion, so Δ^Y is closed. Thus, $h^{-1}(\Delta(Y))$ is a closed subset of X containing U (since $h\big|_U = \Delta \circ f\big|_U$), and since U is dense, $h^{-1}(\Delta(Y)) = X$, so $h(X) \subseteq \Delta(Y)$ so $f = g$ on X.

To check the sheaf maps are equal, we may suppose $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are affine (since equality of sheaf maps can be checked locally) and we have maps $f^\#, g^\# : A \to B$. For $a \in A$ consider $b = f^\#(a) - g^\#(a)$. Observe that $b\big|_U = 0$ so $V(b) \subseteq X$ contains the dense open U and hence $V(b) = X$. Thus, b is nilpotent. Since X is reduced, $b = 0$ and we are done.

If X is nonreduced, this can fail. For example, take $X = \operatorname{Spec} \mathbf{Z}[x]/(x^2, xp)$ for a prime p. We define $\phi_i : X \to X$ for $i = 1, 2$ by $x \mapsto x$, $x \mapsto 0$ respectively. The open set $U = X_(p) = \operatorname{Spec} \mathbf{Z}[1/p]$ is dense in X since X is irreducible (the unique minimal prime is (x)) and $\phi_1 = \phi_2$ on U. However, $\phi_1 \neq \phi_2$ as they are not induced by the same ring map.

Similarly, the result can fail for Y not separated. Take Y to be the affine line with doubled origin and let $\phi_1 : Y \to Y$ be the identity map and $\phi_2 : Y \to Y$ the map that switches the two copies of $\mathbf{A}^1$. Then $\phi_1 = \phi_2$ on the dense open U consisting of Y minus the two origins, but not on all of Y.

4.3 Let $f : X \to S = \operatorname{Spec} A$ be separated, and $U = \operatorname{Spec} B$, $V = \operatorname{Spec} B'$ affine opens in X. Then $\Delta : X \to X \times_S X$ is a closed immersion, and by universal properties of the fiber product, we have $U \cap V = \Delta^{-1}(U \times_S V)$. Since being a closed immersion is local, $\Delta : U \cap V \to U \times_S V = \operatorname{Spec}(B \otimes_A B')$ is a closed immersion, so in particular by 3.11 b), $\Delta(U \cap V)$ is affine, which implies that $U \cap V$ is affine as Δ is a homeomorphism.

As an example when this fails if X is nonseparated, let X be the affine plane with the origin doubled (over an algebraically closed field k) and U, V the two copies of $\mathbf{A}_k^2$. Then U, V are open affines, but their intersection is isomorphic to $\mathbf{A}_k^2$ with the origin deleted, which is not affine.

4.4 Let $\pi_X : X \to S$ and $\pi_Y : Y \to S$ be the structure maps, and $\pi_Z : Z \to S = \pi_Z\big|_Z$. Since $\pi_Y \circ f|_Z = \pi_Z$ and π_Z is proper by hypothesis and π_Y is separated, Prop. 4.8 tells us that $f\big|_Z : Z \to f(Z)$ is proper. Replace X by Z and Y by $f(Z)$ so that we have $f : X \to Y$ a surjective S-morphism and π_X proper; we wish to show π_Y is proper. Since $X \times_Y Y \times_S T = X \times_S T$ for any S-scheme T, we see that the morphism $f \times \mathrm{id} : X \times_S T \to Y \times_S T$ is the base change of $f : X \to Y$ and $X \times_S T \to T$ is the base change of $X \to S$. Thus, since properness and surjectivity are stable under base change (see below), we may replace X by $X \times_S T$, Y by $Y \times_S T$, S by T, and f by $f \times \mathrm{id}$ to reduce showing that f is universally closed to just showing that it is closed. But if $W \subseteq Y$ is closed, then since $f : X \to Y$ is surjective, we have $W = f(f^{-1}(W))$ and hence $\pi_Y(W) = \pi_Y \circ f(f^{-1}(W)) = \pi_X(f^{-1}(W))$ and this is closed since π_X is proper (hence closed) and f is continuous so $f^{-1}(W)$ is closed.

To prove that surjectivity is stable under base change, observe that if $f : X \to S$ is a surjective map and $\pi : S' \to S$ any base change, then $(f \times 1)^{-1}(s')$ is homeomorphic to $X \times_S S' \times_{S'} \operatorname{Spec} k(s') = X \times_S \operatorname{Spec} k(s') = X \times_S \pi(s')$, which is homeomorphic to $f^{-1}(\pi(s'))$ and must therefore be nonempty as a set, since f is surjective (we have used 3.10).

4.6 Let $f : \operatorname{Spec} B = X \to Y = \operatorname{Spec} A$ be proper, and let $\varphi : A \to B$ be the associated ring map. Since X, Y are varieties, A and B are domains of finite type over an algebraically closed field k. Let $K = \operatorname{Frac} B$ and R be any valuation ring of K containing $\varphi(A)$. The valuative criterion of properness ensures the existence of a unique map $\operatorname{Spec} R \to \operatorname{Spec} B$ making the diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} A
\end{tikzcd}$$

commute. In other words, there is a unique map of rings $B \to R$ making

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & R \arrow[d, hook] \\
B \arrow[r, hook] \arrow[ur, dashed] & K
\end{tikzcd}$$

commute, and we easily see this map is injective. Thus, B is contained in every valuation ring of K containing A, so by 4.11 A, it is contained in the integral closure of A in K and is hence integral over A. By 3.4, we conclude that $f : X \to Y$ is finite.

4.8 d) If $\pi_X : X \to Z$ and $\pi_Y : Y \to Z$ have $\mathcal{P}$ then since $X \times_Z Y \to Y$ is the base change $Y \to Z$ of $X \to Z$, we see that $X \times_Z Y \to Y$ has $\mathcal{P}$. SInce $X \times_Z Y \to Z$ is the composition $X \times_Z Y \to Y \xrightarrow{\pi_Y} Z$ of two morphisms having $\mathcal{P}$, it also has $\mathcal{P}$.
e) The morphism $\Gamma_f : X \to X \times_Z Y$ is the base change of $\Delta : Y \to Y \times_Z Y$ by $f \times \mathrm{id} : X \times_Z Y \to Y \times_Z Y$, and since Y is separated, Δ is a closed immersion. Since closed immersions are stable under base change, Γ_f is also a closed immersion, hence has $\mathcal{P}$. Now $g \circ f : X \to Z$ has $\mathcal{P}$ so the base change $X \times_Z Y \to Y$ by $Y \to Z$ also has $\mathcal{P}$. But f factors as $X \xrightarrow{\Gamma_f} X \times_Z Y \to Y$ and so is the composition of two morphisms having $\mathcal{P}$ and therefore has $\mathcal{P}$.
f) The morphism $X_{\text{red}} \to X$ is a closed immersion, hence has $\mathcal{P}$. Then the composite $X_{\text{red}} \to X \to Y$ has $\mathcal{P}$ and factors as $X_{\text{red}} \to Y_{\text{red}} \to Y$ by 2.3 c). Since $Y_{\text{red}} \to Y$ is separated (use the valuative criterion: $T = \operatorname{Spec} R$ is reduced for any valuation ring R as valuation rings are domains, so the map $\operatorname{Spec} R \to Y$ factors uniquely as $\operatorname{Spec} R \to Y_{\text{red}} \to Y$ by 2.3) we conclude by e) that $X_{\text{red}} \to Y_{\text{red}}$ has $\mathcal{P}$.

2.5

5.1 a) Define a map $\varphi_U : \mathcal{E}(U) \to \check{\check{\mathcal{E}}}(U) = \operatorname{Hom}(\mathscr{H}om(\mathcal{E}, \mathcal{O}_X)\big|_U, \mathcal{O}_X\big|_U)$ by sending $e \in \mathcal{E}(U)$ to the collection of maps $\{e_V\}_V : \operatorname{Hom}(\mathcal{E}, \mathcal{O}_X)(U \cap V) \to \mathcal{O}_X(U \cap V)$, with $e_V(\sigma) = \sigma_{U \cap V}(e\big|_{U \cap V})$, where $\sigma : \mathcal{E}\big|_{U \cap V} \to \mathcal{O}_X\big|_{U \cap V}$. One checks that the stalk $\mathscr{H}om(\mathcal{E}, \mathcal{O}_X)_P$ is $\operatorname{Hom}(\mathcal{E}_P, \mathcal{O}_{X,P})$ (because $\mathcal{E}$ is coherent; see, for example, Serre, "Faisceux Algébraiques Cohérents." In general, the canonical map $\operatorname{Hom}(\mathcal{E}, \mathcal{O}_X)_P \to \operatorname{Hom}(\mathcal{E}_P, \mathcal{O}_{X,P})$ is not an isomorphism.), and that the morphism we have defined induces the stalk morphism $\mathcal{E}_P \to \operatorname{Hom}(\operatorname{Hom}(\mathcal{E}_P, \mathcal{O}_{X,P}), \mathcal{O}_{X,P})$ given by $e_P \mapsto (\sigma_P \mapsto \sigma_P(e_P))$. Since $\mathcal{E}$ is locally free of finite rank, the stalk $\mathcal{E}_P$ is free of finite rank, and the stalk map is the *canonical* isomorphism of a free module of finite rank with its double dual. Since all of the induced stalk maps are isomorphisms, we have $\check{\check{\mathcal{E}}} \simeq \mathcal{E}$.

b) Define $\varphi_U : \operatorname{Hom}(\mathcal{E}\big|_U, \mathcal{O}_X\big|_U) \otimes \mathcal{F}(U) \to \operatorname{Hom}(\mathcal{E}\big|_U, \mathcal{F}\big|_U)$ by $(\varphi_U(\psi \otimes f))_V(e) = \psi_V(e) \cdot f\big|_V \in \mathcal{F}(U \cap V)$, where $\psi = \{\psi_V\}$ and $e \in \mathcal{E}(U \cap V)$. We extend this definition linearly. Observe that the restriction maps are compatible and that the φ_U glue to give φ, since they are all canonically defined. On stalks, φ_P is the map $\operatorname{Hom}(\mathcal{E}_P, \mathcal{O}_{X,P}) \otimes \mathcal{F}_P \to \operatorname{Hom}(\mathcal{E}_P, \mathcal{F}_P)$ given by $\psi \otimes f \mapsto (e \mapsto \psi(e) \cdot n)$, which is an isomorphism of $\mathcal{O}_{X,P}$-modules since $\mathcal{E}_P$ is free (this is standard commutative algebra). Thus the map φ is an isomorphism.
c) Let $\varphi : \mathcal{E} \otimes \mathcal{F} \to \mathcal{G}$ and define $F(\varphi) : \mathcal{F} \to \mathscr{H}om(\mathcal{E}, \mathcal{G})$ by $F(\varphi)_V(f) = \{\sigma_W\}$ where $\sigma_W : \mathcal{E}\big|_V(W) \to \mathcal{G}\big|_V(W)$ is $e \mapsto \varphi_{V\cap W}(\theta^+(e \otimes f\big|_{W \cap V}))$, with $\theta^+ : \mathcal{E}(U) \otimes \mathcal{F}(U) \to (\mathcal{E} \otimes \mathcal{F})(U)$ the sheafification map. It is easily checked that $F(\varphi)$ is a map of sheaves of modules, so F gives a map $\operatorname{Hom}(\mathcal{E} \otimes \mathcal{F}, \mathcal{G}) \to \operatorname{Hom}(\mathcal{F}, \mathscr{H}om(\mathcal{E}, \mathcal{G}))$. We claim that F is injective. Indeed, if $F(\varphi) = 0$ then $\varphi_{V\cap W}(\theta^+(e \otimes f)) = 0$ for all open V, W and $e \otimes f \in \mathcal{E}(V \cap W) \otimes \mathcal{F}(V \cap W)$, which implies that φ is the zero map. Moreover, F is surjective as given $\psi : \mathcal{F} \to \mathscr{H}om(\mathcal{E}, \mathcal{G})$ we define $\varphi_U : \mathcal{E}(U) \otimes \mathcal{F}(U) \to \mathcal{G}(U)$ by $\varphi_U(e \otimes f) = (\psi_U(f))_U(e) \in \mathcal{G}(U)$. It is clear this defines morphism of sheaves (using the universal property of sheafification) $\mathcal{E} \otimes \mathcal{F} \to \mathcal{G}$ such that $F(\varphi) = \psi$. Surjectivity follows.
d) Let's try to be slick about this: We have the identification $f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E} \simeq f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \check{\check{\mathcal{E}}}$ by part a) and $f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \check{\check{\mathcal{E}}} \simeq \mathscr{H}om_{\mathcal{O}_Y}(\check{\mathcal{E}}, f_*\mathcal{F})$ by part b). Using the *canonical* isomorphism

$$\operatorname{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \simeq \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}),$$

by patching together over opens we obtain an isomorphism of sheaves of $\mathcal{O}_Y$-modules

$$\mathscr{H}om_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}) \simeq f_*\mathscr{H}om_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}).$$

Now let $\mathcal{G} = \check{\mathcal{E}}$ and combine with the above to obtain

$$f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E} \simeq \mathscr{H}om_{\mathcal{O}_Y}(\check{\mathcal{E}}, f_*\mathcal{F}) \simeq f_*\mathscr{H}om_{\mathcal{O}_X}(f^*(\check{E}), \mathcal{F}) \simeq f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \check{f^*\check{E}}),$$

where we have used b) again. Now we need only realize the isomorphism $\check{f^*\check{E}} \simeq f^*\check{\check{\mathcal{E}}}$ of sheaves on X and use part a). Observe that the two duals on the LHS are *different*: the inner one is $\mathscr{H}om(\bullet, \mathcal{O}_Y)$ while the outer one is $\mathscr{H}om(\bullet, \mathcal{O}_X)$.

5.2 a) As R is a dvr, there are two open sets: X and $\{0\} = D(m)$ for any $m \in P$ where P is the unique nonzero prime ideal of R. As such, to give a sheaf of modules on X is to give a R-module M and a $R_m = K$-module (vector space) L such that the restriction map $M \to L$ is compatible with the module structures $R \to \operatorname{End} M$ and $K \to \operatorname{End} L$, i.e. such that we have a homomorphism of K-vector spaces $M \otimes_R K \to L$.

b) The sheaf given above is quasi-coherent iff, by 5.4, it is of the form $\widetilde{M}$, in which case we must have $M_m = M \otimes_R K = L$, with the map given above (coming from restriction) providing an isomorphism.

5.3 Let $\varphi : M \to \Gamma(X, \mathcal{F})$ be an A-module homomorphism and for any $f \in A$ define $\psi_{D(f)} : M_f \to \mathcal{F}(D(f))$ by $\psi_{D(f)}(m/f^n) = (1/f^n) \cdot \varphi(m)\big|_{D(f)}$. Observe this is well defined since $\mathcal{F}(D(f))$ is an A_f-module, and that the $\psi_{D(f)}$ patch together to give a morphism $\psi : \widetilde{M} \to \mathcal{F}$ (as is easily checked by looking at the double-overlap condition). Given $\psi \in \operatorname{Hom}(\widetilde{M}, \mathcal{F})$ we define $\varphi : M \to \Gamma(X, \mathcal{F})$ by $\varphi = \psi_X$. It is clear that these two constructions provide inverses to eachother, so provide the desired isomorphism.

5.4 If $\mathcal{F}$ is quasi-coherent, then every point $x \in X$ has an affine nbd. $\operatorname{Spec} A = U$ with $\mathcal{F}\big|_U \simeq \widetilde{M}$ for some A-module M. Then M has presentation $A^I \to A^J \to M$; applying $\widetilde{}$ and recalling that this is an exact functor that commutes with arbitrary direct sum and that $\widetilde{A} = \mathcal{O}_X\big|_U$, we obtain the exact sequence of sheaves of modules

$$(\mathcal{O}_X\big|_U)^I \to (\mathcal{O}_X\big|_U)^J \to \mathcal{F}\big|_U \to 0.$$

Observe that if $\mathcal{F}$ is coherent and X is noetherian, then A is a noetherian ring and M is a finitely generated—hence finitely presented—module. We may therefore take the index sets I and J to be finite in this case.

Conversely, suppose that $\mathcal{F}\big|_U$ is the cokernel of the morphism

$$\widetilde{A}^n = (\mathcal{O}_X\big|_U)^I \xrightarrow{\varphi} (\mathcal{O}_X\big|_U)^J.$$

Applying 5.3 to this morphism, we obtain an A module homomorphism $\psi : A^I \to A^J$ with $\widetilde{\psi} = \varphi$. Thus, letting $M = \operatorname{coker}\psi$ and applying $\widetilde{\ }$ to the exact sequence $A^I \xrightarrow{\psi} A^J \to M \to 0$, we obtain an exact sequence of sheaves

$$(\mathcal{O}_X\big|_U)^I \xrightarrow{\varphi} (\mathcal{O}_X\big|_U)^J \to \widetilde{M} \to 0.$$

Now use the snake lemma to obtain $\widetilde{M} \simeq \mathcal{F}\big|_U$. If I, J are finite index sets and A is noetherian, then M is finitely generated and hence $\mathcal{F}\big|_U$ is coherent.

5.5 a) Let $X = \operatorname{Spec} k[x,y]$ and $Y = \operatorname{Spec} k[x]$ with $f : X \to Y$ induced by the inclusion $k[x] \hookrightarrow k[x,y]$. Then $f_*\mathcal{O}_X$ is not a coherent $\mathcal{O}_Y$-module. Indeed, $k[x,y]$ is not a finitely generated $k[x]$-module; now use Prop. 5.4.
b) Let $f : X \to Y$ be a closed immersion, and $U = \operatorname{Spec} A$ an affine subset of Y, and $V = f^{-1}(U)$. Then $f(V) = U \cap f(X)$ is a closed subset of U since $f(X)$ is closed in Y, so by Corollary 5.10 we have $f(V) \simeq \operatorname{Spec} A/I$ for some ideal I of A. Since $f : V \to f(V)$ is a homeomorphism, we conclude that $V = \operatorname{Spec} B$ is affine and since $f^\#$ is surjective, that $A/I \to B$ is surjective. Hence $A \to B$ is surjective and B is a finite A-module (generated by 1); thus f is finite.
c) Let $f : X \to Y$ be finite and $\mathcal{F}$ a coherent sheaf on X. By Prop. 5.4, $f_*\mathcal{F}$ is coherent on Y iff for any affine $U = \operatorname{Spec} A$, the restricted sheaf $f_*\mathcal{F}\big|_U$ is $\widetilde{M}$ for some finite A-module M. Since f is finite, $f^{-1}(U) = \operatorname{Spec} B$ with B a finite A-module, and since $\mathcal{F}$ is coherent, we have $\mathcal{F}\big|_{f^{-1}(U)} \simeq \widetilde{M}$ for some finite B-module M. Prop 5.2 (d) says that $f_*\mathcal{F}\big|_U \simeq \widetilde{({}_AM)}$. Since B is a finite A-module and M is a finite B-module, M is a finite A-module and hence $f_*\mathcal{F}$ is coherent.

5.6 a) By definition, $\operatorname{Supp} m = \{x \in \operatorname{Spec} A : m_x \neq 0\} = \{p \subseteq A : m_p \neq 0\}$. But $m_p = 0$ iff there exists $f \notin p$ with $fm = 0$, that is, iff $\operatorname{Ann} m \subsetneq p$. Hence $\operatorname{Supp} m = V(\operatorname{Ann} m)$.
b) Recall $\operatorname{Supp} \mathcal{F} = \{x \in X : \mathcal{F}_x \neq 0\} = \cup_{m \in M} V(\operatorname{Ann} m)$ by part a). Since M is finitely generated, say by $m_1, \ldots, m_n$, we have $\cup_{m\in M} V(\operatorname{Ann} m) = \cup_{i=1}^n V(\operatorname{Ann} m_i) = V(\cap_{i=1}^n \operatorname{Ann} m_i) = V(\operatorname{Ann} M)$. (where did I use the hypothesis that A is noetherian?
c) Let $U = \operatorname{Spec} A$ be any open affine subset of X. By Prop. 5.4, we have $\mathcal{F}\big|_U = \widetilde{M}$ with M a finitely generated A-module. By part b), $\operatorname{Supp} \mathcal{F} \cap U = \operatorname{Supp} \mathcal{F}\big|_U = V(\operatorname{Ann} M)$ is closed in U. It follows that $\operatorname{Supp} \mathcal{F}$ is closed (take U_i a finite affine cover of X since X is noetherian; then $\operatorname{Supp} \mathcal{F} = \cup \operatorname{Supp} \mathcal{F} \cap U_i$ is closed). Perhaps this only shows locally closed?
d) By 1.20, we have the exact sequence of sheaves on X

$$0 \to \mathscr{H}^0_Z(\mathcal{F}) \to \mathcal{F} \to j_*(\mathcal{F}\big|_U),$$

where $j_* : U = X - Z \to X$ is the inclusion. By Prop. 5.8, $j_*(\mathcal{F}\big|_U)$ is quasi-coherent since X is noetherian. By Prop. 5.7, $\mathscr{H}^0_Z$ is quasi-coherent since it is the kernel of a morphism of quasi-coherent sheaves. Now

$$\begin{aligned}\Gamma_Z(\mathcal{F}) &= \{m \in M : \operatorname{Supp} m \subseteq V(\mathfrak{a})\} = \{m \in M : V(\operatorname{Ann} m) \subseteq V(\mathfrak{a})\} \\ &= \{m \in M : \mathfrak{a} \subseteq \operatorname{Rad}(\operatorname{Ann} m)\} = \{m \in M : \mathfrak{a}^n m = 0 \text{ for some } n\}\end{aligned}$$

since A is noetherian, and hence $\mathfrak{a}$ is finitely generated. Thus, $\Gamma_Z(\mathcal{F}) = \Gamma_{\mathfrak{a}}(M)$. But since $\mathscr{H}^0_Z(\mathcal{F})$ is quasi-coherent and $\Gamma_Z(\mathcal{F}) \simeq \Gamma_{\mathfrak{a}}(M)$, Cor. 5.5 allows us to conclude that $\mathscr{H}^0_Z(\mathcal{F}) \simeq \widetilde{\Gamma_{\mathfrak{a}}(M)}$.
e) Let $U = \operatorname{Spec} A$ be any affine. Then $Z \cap U$ is closed in U so is isomorphic to $\operatorname{Spec} A/\mathfrak{a}$ for some ideal $\mathfrak{a}$ (by Cor. 5.10), i.e. $Z \cap U = V(\mathfrak{a})$. If $\mathcal{F}$ is quasi-coherent, we have $\mathcal{F}\big|_U \simeq \widetilde{M}$ by Prop. 5.4, and by part d) we have $\mathscr{H}^0_Z(\mathcal{F})\big|_U \simeq \widetilde{\Gamma_{\mathfrak{a}}(M)}$. It follows that $\mathscr{H}^0_Z(\mathcal{F})$ is quasi-coherent. If $\mathcal{F}$ is coherent, then M is a finitely

generated A module, so $\Gamma_{\mathfrak{a}}(M)$ is a finitely generated A-module since A is noetherian so it is a submodule of a noetherian module.

5.7 a) Let $U = \operatorname{Spec} A$ be a nbd. of x with $\mathcal{F}|_U \simeq \widetilde{M}$, with M a finite A-module, generated by $m_1, \ldots, m_n$. Then the images of m_i in $\mathcal{F}_x$ generate $\mathcal{F}_x \simeq M_x$ as an A_x-module. Renumbering if necessary, we may assume that (the images of) $m_1, \ldots, m_v$ generate $\mathcal{F}_x$ freely, and we replace M by the submodule generated by the m_i. Because X is noetherian, A is noetherian and M is finitely presented, so let n_j for $1 \le j \le m$ be a basis for the module of relations. Since the images of m_i in $\mathcal{F}_x$ generate freely, the image of each n_i in $\mathcal{F}_x$ is zero, so there is an open set V_i such that $n_i\big|_{V_i} = 0$ for each i. Let $V = \cap V_i$; since there are finitely many n_i, V is a nbd of x which we may take to be a basic affine open contained inside U. Then $\mathcal{F}\big|_V$ is generated freely by the global sections m_i.
b) One direction is obvious, and the converse is part a).
c) Suppose first that $\mathcal{F}$ is locally free of rank 1. Then by 5.1 b) we have $\mathcal{F} \otimes_{\mathcal{O}_X} \check{\mathcal{F}} \simeq \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$. We define an isomorphism $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \simeq \mathcal{O}_X$ as follows: cover X by open affines U_i with $\mathcal{F}\big|_{U_i} \simeq \mathcal{O}_X$ so $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})(U_i) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X\big|_{U_i}, \mathcal{O}_X\big|_{U_i})$, and we let $\varphi_{U_i}(\psi) = \psi_{U_i}(1)$ with $\psi : \mathcal{O}_X\big|_{U_i} \to \mathcal{O}_X\big|_{U_i}$. This gives a map $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \to \mathcal{O}_X$ that is an isomorphism on each U_i, hence an isomorphism and $\mathcal{F}$ is invertible. One easily checks that $\check{\mathcal{F}}$ is coherent by looking locally and translating it into a question about modules.

Conversely, suppose there exists $\mathcal{G}$ such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \simeq \mathcal{O}_X$. Then $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x \simeq \mathcal{O}_{X,x}$ for all $x \in X$, so by part b) it is enough to show that if M, N are A-modules with (A, m, k) a local ring (here I am using that $\mathcal{F}, \mathcal{G}$ are coherent) and $M \otimes_A N \simeq A$ then $M \simeq A$ and $N \simeq A$. Indeed, we have an isomorphism (this is tricky commutative algebra) $M/mM \otimes_k N/nM \simeq (M \otimes_A \otimes N) \otimes_A k \simeq k$, so M/mM (and N/mN) has rank 1; by Nakayama's lemma, M is a rank 1 A-module. Let $a \in \operatorname{Ann} M$. Then a annihilates A since $M \otimes_A N \simeq A$, and in particular, $a \cdot 1 = 0$ so $a = 0$ and M is free of rank 1.

5.8 a) We show the complement is open. Let $x \in X$ satisfy $\varphi(x) = k < n$ and choose a nbd. $U = \operatorname{Spec} A$ of x with $\mathcal{F}\big|_U \simeq \widetilde{M}$ with M a finitely generated A-module, generated by $m_1, \ldots, m_r$. Since $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x) \simeq M_p/pM_p$ where $p \in \operatorname{Spec} A$ corresponds to $x \in X$, we may take $u_i \in M$ for $1 \le i \le k$ with images a generating set of M_p/pM_p as a $k(x)$ vector-space. NAK implies that the images of the u_i generate M_P as an A_p-module. Writing $m_j = \sum a_{ij}/f_{ij} u_i$ for each j (inside M_p) and $f = \prod_{i,j} f_{ij}$, we see that $p \in D(f)$ and if $q \in D(f)$ then $m_j \in M_q$ can be written as an A_q-linear combination of the (images of) u_i. Since the m_j generate M as an A-module, their images generate M_q as an A_q-module, so the images of u_i for $1 \le i \le k$ generate M_q as an A_q module whence $\varphi(q) \le k < n$. Observe that by taking $n = k$ we have shown that the sets $\{x : \varphi(x) < n\}$ and $\{x : \varphi(x) \le n\}$ are open (which follows anyway from the fact that $\mathbf{Z}$ is discrete).
b) If $\mathcal{F}$ is locally free, then $U = \varphi^{-1}(n) = \{x : \varphi(x) = n\}$ is open. Indeed, let $x \in U$. By 5.7 a), there is a nbd V of X with $\mathcal{F}\big|_V$ free, necessarily of rank n, so for every $y \in V$ we have $\varphi(y) = n$ so U is open. Since $\varphi(x) \ge 0$, we may find $x \in X$ with $\varphi(x) = n \ge 0$ minimal. Then $\{x : \varphi(x) > n\} = \cup_{k>n} \varphi^{-1}(k)$ is open by the above and closed by part a). and since X is connected it is either empty or all of X. The latter possibility is ruled out since we have one point x with $\varphi(x) = n$. Thus $\varphi(y) \le n$ for all $y \in X$, and since n was chosen minimally, we conclude that $\varphi(y) = n$ for all $y \in X$.
c) Let $U = \operatorname{Spec} A$ be a nbd. of $x \in X$ with $\varphi(x) = n$ and $\mathcal{F}\big|_U = \widetilde{M}$. By 5.7 b) it will suffice to show that M_p is a free A_p-module for all p. Pick $m_1, \ldots, m_n \in M$ whose images are a basis of M_p/pM_p as an A_p/p-vector space. By NAK, $m_1, \ldots, m_n$ generate M_p as an A_p-module, and thus also generate M_q as an A_q-module for any $q \subseteq p$; since $\varphi(q) = \varphi(p)$, we must have that the images of $m_1, \ldots, m_n$ in M_q/qM_q are linearly independent over A_q/q for all $q \subseteq p$. Thus, if $\sum a_i m_i = 0$ is any relation with $a_i \in A_p$, then we have $a_i = 0$ in A_q/q for all $q \subseteq p$, or what is the same thing, that $a_i \in \cap_{q \subseteq p} q$. But as X is reduced, A_p is reduced, so $\cap_{q \subseteq p} q = 0$ so $a_i = 0$ and the m_i are linearly independent over A_p so M_p is a free A_p-module.

5.10 a) Let $s = \sum f_j$ be in $\overline{I}$ with $f_j \in S_j$. Then $x_i^n s = \sum x_i^n f_j \in I$ so since I is homogeneous, $x_i^n f_j \in I$ for all i, j and hence $\overline{I}$ is homogeneous.

b) Two ideals I, J define the same closed subscheme iff $\widetilde{I} = \widetilde{J}$, as subsheaves of $\mathcal{O}_X$. That is, iff $I_{(x_i)} = \widetilde{I}(D_+(x_i)) = \widetilde{J}(D_+(x_i)) = J_{(x_i)}$. Now $I_{(x_i)} = J_{(x_i)}$ for all i iff for any $s \in \overline{I}$, there exists an integer m (which we can take to be positive) such that $x_i^m s \in J$, that is, $s \in \overline{J}$. Indeed, $s \in \overline{I}$ iff. $x_i^n s \in I$ for some n and all i iff $s \in I_{x_i}$ for all i. This is the case iff there exists $m \in \mathbf{Z}$ such that $x_i^m s \in I_{(x_i)}$. Thus, $I_{(x_i)} = J_{(x_i)}$ iff $x_i^k s \in J$ for some $k > 0$, i.e. iff $s \in \overline{J}$. Interchanging the roles of I, J gives the desired result.
c) Observe that

$$\Gamma(X, \mathscr{I}_Y(n)) = \{s \in \Gamma(X, \mathcal{O}_X(n)) = S_n : s_p = 0 \text{ for all } p \in Y\}.$$

Thus, if $s \in \overline{\Gamma_*(\mathscr{I}_Y)}$ then there exists $m > 0$ with $(x_i^m s)_p = 0$ in S_p for all $p \in Y$, or ehat is the same thing, there exists $f_i \in S - p$ with $f_i x_i^m s = 0$ in S. Since x_i generate S_+, given any $p \in Y$, we can find i such that $x_i \notin p$ and hence $f_i x_i^m \in S - p$, whence $s_p = 0$ so $s \in \Gamma_*(\mathscr{I}_Y)$.
d) This follows immediately from a)–c).

5.11 We assume S, T are generated by S_1, T_1 over A. Let S_1 be generated by $s_1, \dots, s_a$ over A and T_1 by $t_1, \dots, t_b$ over A. We claim there is an isomorphism of rings (*not* a graded isomorphism)

$$S_{(s_i)} \otimes_A T_{(t_j)} \simeq (\oplus_{d \geq 0} S_d \otimes T_d)_{s_i \otimes t_j}.$$

Indeed, $S_{(s_i)} \otimes_A T_{(t_j)} = \oplus_{m,n} (S_m)_{(s_i)} \otimes (T_n)_{(t_j)}$ and $(\oplus_{d \geq 0} S_d \otimes T_d)_{s_i \otimes t_j} = \oplus_{d \geq 0} (S_d)_{(s_i)} \otimes (T_d)_{(t_j)}$, and the map $(S_m)_{(s_i)} \otimes (T_n)_{(t_j)} \to (S_n)_{(s_i)} \otimes (T_n)_{(t_j)}$ is defined (say for $n \geq m$) by

$$\frac{s}{s_i^m} \otimes \frac{t}{t_j^n} \mapsto \frac{s \cdot s_i^{n-m}}{s_i^n} \otimes \frac{t}{t_j^n}$$

on simple tensors and extended by A-linearity. This gives the claimed isomorphism since the two sides of the map are already *equal* inside $S_{(s_i)} \otimes T_{(t_j)}$. We thus have an isomorphism of affine schemes

$$D_+(s_i) \times_A D_+(t_j) \simeq \operatorname{Spec}(S_{(s_i)} \otimes T_{(t_j)}) \xleftarrow{\sim} \operatorname{Spec}((S \times_A T)_{s_i \otimes t_j}) \simeq D_+(s_i \otimes t_j),$$

and these glue in the evident manner to give an isomorphism $\operatorname{Proj}(S \times_A T) \to \operatorname{Proj} S \times_A \operatorname{Proj} T$.

5.15 a) By Prop 5.4, any quasi-coherent $\mathcal{F}$ has the form $\mathcal{F} = \widetilde{M}$ for some A-module M (with $\operatorname{Spec} A = X$ and A noetherian). Then the natural map $\varinjlim_\alpha M_\alpha \to M$ is an isomorphism, where $\{M_\alpha\}_\alpha$ are the finitely generated submodules of M. Since the $\widetilde{\ }$ operation commutes with direct limit (because tensor product does) and is exact, we see that the natural map $\varinjlim \widetilde{M_\alpha} \to \widetilde{M}$ is an isomorphism.
b) The sheaf $i_*\mathcal{F}$ is q-coh. by 5.8 c) since X is noetherian, so part a) applies. We claim that there is a coherent subsheaf $\mathcal{F}_\alpha$ of $i_*\mathcal{F}$ with $\mathcal{F}_a\big|_U = \mathcal{F}$. Indeed, the sheaves $\mathcal{F}_\alpha\big|_U$ are all subsheaves of the coherent sheaf $\mathcal{F}$, so any chain of these sheaves is bounded above (by a coherent sheaf!). Zorn's lemma yields a maximal subsheaf $\mathcal{F}_a\big|_U \subseteq \mathcal{F}$ and we claim equality holds. If not, there is some point P with $(\mathcal{F}_a)_P \neq \mathcal{F}_P$ and hence an open set V contained in U with $\mathcal{F}_a(V) \subsetneq \mathcal{F}(V)$, and we can take V small enough so that $\mathcal{F}\big| = \widetilde{M}$ and $\mathcal{F}_a\big|_V = \widetilde{N}$ with $N \subsetneq M$ and N, M f.g. $\mathcal{O}(V)$-modules. Pick $m \in M \setminus N$ and let N' be the $\mathcal{O}(V)$-module generated by m, N. Then $\widetilde{N'}$ is a coherent subsheaf of $\mathcal{F}$ strictly containing $\mathcal{F}_a$ as a subsheaf, contradicting the maximality of $\mathcal{F}_a$.
c)

2.7

7.1 Passing to stalks, we are reduced to the following: if $\varphi : M \to N$ is a surjective map of free A-modules of rank 1 (with (A, m, k) local) then it is an isomorphism. By tensoring $M \to N \to 0$ with k, we see that $\varphi \otimes 1$ is a surjective map of k vector spaces of the same dimension, hence an isomorphism. Thus, if $x \in \ker \varphi$

then $x \otimes 1 = 0$, or what is the same, $x \in M_{\text{tors}}$. But since M is free, we conclude that $\ker \varphi = 0$ so φ is an isomorphism. By identifying M, N with A (thought of as an A-module) One could also use a different result (cf. Matsumura, Th. 2.4) which states that for any ring A and any finite A-module M, any surjective $f : M \to M$ is an isomorphism.

7.2

Chapter 3

3.2

2.3 a) Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be exact. We know that $0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'')$ is exact, so it follows that $\Gamma_Y(X, \mathcal{F}') \hookrightarrow \Gamma(X, \mathcal{F})$. The image is contained in the subgroup $\Gamma_Y(X, \mathcal{F})$ because the sequence on stalks $0 \to \mathcal{F}'_P \to \mathcal{F}_P \to \mathcal{F}''_P$ is exact for all P, so $s \in \Gamma_Y(X, \mathcal{F}')$ has nonzero stalk at P iff its image in $\Gamma(X, \mathcal{F})$ has nonzero stalk at P. For this reason, the map $\Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'')$ induces $\Gamma_Y(X, \mathcal{F}) \to \Gamma_Y(X, \mathcal{F}'')$. Now it is clear that the map $\Gamma_Y(X, \mathcal{F}') \to \Gamma_Y(X, \mathcal{F}) \to \Gamma_Y(X, \mathcal{F}'')$ is the zero map as it is induced by the map of usual global sections. It therefore remains to show that is $s \in \Gamma_Y(X, \mathcal{F})$ maps to 0 in $\Gamma_Y(X, \mathcal{F}'')$ then it is in $\Gamma_Y(X, \mathcal{F}')$. We know that it is in the image of $\Gamma(X, \mathcal{F}')$, and checking the sequence on stalks shows that it is in $\Gamma_Y(X, \mathcal{F}')$ as required.
b)

3.3

3.1 If $X = \operatorname{Spec} A$ then $X_{\text{red}} = \operatorname{Spec} A/N$ is affine, where N is the nilradical. Conversely, suppose that X_{red} is affine, and let $\mathscr{N}$ be the sheaf of nilpotents on X and $\mathcal{F}$ any quasi-coherent sheaf on X. Then for any j we have an exact sequence

$$0 \to \mathscr{N}^{j+1} \cdot \mathcal{F} \to \mathscr{N}^j \cdot \mathcal{F} \to \mathscr{N}^j \cdot \mathcal{F}/\mathscr{N}^{j+1} \cdot \mathcal{F} \to 0.$$

Observe that $\mathscr{N}^j \cdot \mathcal{F}/\mathscr{N}^{j+1} \cdot \mathcal{F}$ is a q-coh. sheaf of $\mathcal{O}_X/\mathscr{N}$-modules, i.e. a q-coh sheaf of modules on X_{red}. Using Serre's criterion, we may suppose that $H^i(X, \mathscr{N}^j \cdot \mathcal{F}/\mathscr{N}^{j+1} \cdot \mathcal{F}) = 0$ for all $i \geq 1$, and the long exact sequence of cohomology associated to the short exact sequence above yields an isomorphism $H^i(X, \mathscr{N}^j \cdot \mathcal{F}) \simeq H^i(X, \mathscr{N}^{j+1} \cdot \mathcal{F})$ for all $j \geq 0$ and all $i \geq 2$, and a surjection $H^1(X, \mathscr{N}^{j+1} \cdot \mathcal{F}) \to H^1(X, \mathscr{N}^j \cdot \mathcal{F})$. Since X is noetherian, we may cover it by finitely many affines $\operatorname{Spec} A_i$ with A_i noetherian and $\mathscr{N}^j\big|_{\operatorname{Spec} A_i} = \widetilde{N_i^j}$ with N_i the module of nilpotents on A_i. By the Noetherian hypothesis, we can find j_i such that $N_i^{j_i} = 0$ for each i and choosing $j = \max_i j_i$, we find that $\mathscr{N}^j$ is the zero sheaf on X and hence all the cohomology vanishes. Thus, using the isomorphisms above and our surjection on H^1's, we conclude that $H^i(X, \mathcal{F}) = 0$ for all $i > 0$ and any q-coh. $\mathcal{F}$, and hence that X is affine,

3.2 Since X is noetherian, there are finitely many irreducible components, say Y_i for $q \leq i \leq n$. Let $\mathscr{I}_i$ be the ideal sheaf defining Y_i and filter a q-coh sheaf $\mathcal{F}$ on X as

$$\mathcal{F} \supset \mathscr{I}_1 \cdot \mathcal{F} \supset \mathscr{I}_1 \mathscr{I}_2 \cdot \mathcal{F} \supset \cdots \supset \mathscr{I}_1 \mathscr{I}_2 \cdots \mathscr{I}_n \cdot \mathcal{F}.$$

Breaking into short exact sequences

$$0 \to \mathscr{I}_1 \cdots \mathscr{I}_k \cdot \mathcal{F} \to \mathscr{I}_1 \cdots \mathscr{I}_{k-1} \cdot \mathcal{F} \to \mathscr{I}_1 \cdots \mathscr{I}_{k-1} \cdot \mathcal{F}/\mathscr{I}_1 \cdots \mathscr{I}_k \cdot \mathcal{F} \to 0,$$

and the quotient sheaf $\mathscr{I}_1 \cdots \mathscr{I}_{k-1} \cdot \mathcal{F} / \mathscr{I}_1 \cdots \mathscr{I}_k \cdot \mathcal{F}$ is a q-coh. sheaf of $\mathcal{O}_X/I_k$-modules, i.e. a q-coh sheaf on Y_k. The long exact cohomology sequences and the assumption that all the irreducible components are affine yields isomorphisms

$$H^i(X, \mathcal{F}) \simeq H^i(X, \mathscr{I}_1 \cdots \mathscr{I}_n \cdot \mathcal{F})$$

for all $i > 1$ and a surjection $H^1(X, \mathscr{I}_1 \cdots \mathscr{I}_n \cdot \mathcal{F}) \twoheadrightarrow H^1(X, \mathcal{F})$. However, $\mathscr{I}_1 \cdots \mathscr{I}_n$ is the zero sheaf on X because its support consists of those points $P \in X$ not contained in any irreducible component. We conclude by Serre's criterion that X is affine. The converse follows from the fact that a closed subscheme of an affine scheme is affine.

3.3 a) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. Since $\Gamma_a(M')$ is a submodule of M', we have $\Gamma_a(M') \hookrightarrow \Gamma_a(M)$. Moreover, since $\phi : M' \to M$ is a homomorphism of A-modules, we have $\phi(a^n m') = a^n \phi(m')$ so if $m' \in \Gamma_a(M')$ then $\phi(m') \in \Gamma_a(M)$. For the same reason, the image of $\Gamma_a(M)$ under $M \to M''$ is contained in $\Gamma_a(M'')$, and since $M' \to M \to M''$ is the zero map, its restriction to $\Gamma_a(M') \to \Gamma_a(M'')$ is zero. It remains to show that if $m \in \Gamma_a(M)$ maps to zero in $\Gamma_a(M'')$ then it is in the image of $\Gamma_a(M') \to \Gamma_a(M)$. But if $m \mapsto 0$ then there is $m' \in M'$ with $\phi(m') = m$. But then for some n, $\phi(a^n m') = a^n m = 0$ and since ϕ is injective, we conclude that $m' \in \Gamma_a(M')$.
b) Let $0 \to M \to I^\cdot$ be an injective resolution of M. Then $0 \to \widetilde{M} \to \widetilde{I^\cdot}$ is a flasque resolution of $\widetilde{M}$, so it will suffice to show that $\Gamma_a(M) \simeq \Gamma_Y(\operatorname{Spec} A, \widetilde{M})$ for any A-module M. Let $f_1, \ldots, f_s$ generate a so $X - Y$ is covered by $D(f_i)$. Then an element $m \in \Gamma_Y(\operatorname{Spec} A, \widetilde{M})$ is just an element $m \in M$ such that $m_P = 0$ for all $P \notin Y$. Equivalently, we must have $m \mapsto 0$ in M_{f_i} for all i since the $D(f_i)$ cover $X - Y$. Thus, there is some n such that $f_i^n m = 0$ for all i and hence $m \in \Gamma_a(M)$. In the reverse direction, if $m \in \Gamma_a(M)$ then $m \in \Gamma(\operatorname{Spec} A, \widetilde{M})$ and $m \mapsto 0$ in M_{f_i} so $m \in \Gamma_Y(\operatorname{Spec} A, \widetilde{M})$.
c) It suffices to show that $c \in H_a^i(M)$ is killed by a^n for some n. But $H_a^i(M)$ is a quotient of a submodule of $\Gamma_a(I)$ for some I, so pick $m \in \Gamma_a(I)$ lifting c. Then by definition, there exists n with $a^n m = 0$ so the same is true of c.

3.7 a) Pick generators $f_1, \ldots, f_s$ of $\mathfrak{a}$ and observe that $U = \cup D(f_i)$. Define

$$\operatorname{Hom}_A(\mathfrak{a}^n, M) \to \Gamma(U, \widetilde{M}) = \{(\alpha_i) \in \prod M_{f_i} : \alpha_i = \alpha_j \in M_{f_i f_j}\}$$

by $\varphi \mapsto \left(\frac{\varphi(f_i^n)}{f_i^n}\right)_{i=1}^s$.

In the reverse direction, suppose that $\left(\frac{m_i}{f_i^{n_i}}\right)_{i=1}^s \in \Gamma(U, \widetilde{M})$. Let $m = \max n_i$ and define $\varphi \in \operatorname{Hom}(\mathfrak{a}^n, M)$ for any $n \geq ms$ as follows: let $f_1^{j_1} \cdots f_s^{j_s} \in \mathfrak{a}^n$ and let i_0 be the index for which j_{i_0} is maximal, so $j_{i_0} \geq m$. Then define (to start off with)

$$\psi(f_1^{j_1} \cdots f_s^{j_s}) = f_1^{j_1} \cdots \widehat{f_{i_0}^{j_{i_0}}} \cdots f_s^{j_s} \cdot f_{i_0}^{j_{i_0} - n_{i_0}} m_{i_0}.$$

The problem is that this may not be well defined. However, observe that since $m_i f_j^{n_j} - m_j f_i^{n_i}$ is killed by a power of $f_i f_j$ (cf. II,5.14), there is some large N such that $\phi := (f_1 \cdots f_s)^N \psi$ *is* well defined.

It is not hard to check that this is well defined (using the compatibility properties of the local sections $m_i/f_i^{n_1}$) and inverse to the map defined above, so we have the claimed isomorphism.
b) When M is injective, any section $s \in \Gamma(U, \widetilde{M})$ gives a homomorphism $\mathfrak{a}^n \to M$ for some n, which extends to a morphism $A \to M$ as M is injective and gives a section $\widetilde{s} \in \Gamma(X, \widetilde{M})$ restricting to s. In other words, $\Gamma(X, \widetilde{M}) \to \Gamma(U, \widetilde{M})$ is surjective and $\widetilde{M}$ is flasque.

3.4

4.1 By II, 5.8, $f_* \mathcal{F}$ is a quasi-coherent sheaf on Y. Let U be an affine covering of Y. Then since f is affine, $f^{-1}U$ is an affine covering of X, so by Theorem 4.5, we have natural isomorphisms $\widehat{H}^p(f^{-1}U, \mathcal{F}) \simeq H^p(X, \mathcal{F})$

and $\widehat{H}^p(U, f_*\mathcal{F}) \simeq H^p(Y, f_*\mathcal{F})$ for all $p \geq 0$. But the Cech complexes for $\widehat{H}^p(f^{-1}U, \mathcal{F})$ and $\widehat{H}^p(U, f_*\mathcal{F})$ are, respectively,

$$\prod_{i_0<\cdots<i_p} \mathcal{F}(f^*U_{i_0\ldots i_p}), \qquad \prod_{i_0<\cdots<i_p} \mathcal{F}(f^{-1}U_{i_0\ldots i_p}),$$

where

$$f^*U_{i_0\ldots i_p} := \bigcap_{j=0}^{p} f^{-1}U_{i_j} = f^{-1}U_{i_0\ldots i_p},$$

so the Cech complexes are isomorphic, whence the cohomology groups are isomorphic.

4.2 a) We show that $\mathscr{M} = \mathcal{O}_X$ fits the bill. Indeed, let $\operatorname{Spec} A \ni y$ be an open affine nbd of $y \in Y$, where y is the generic point. Then $\operatorname{Spec} B = f^{-1}(\operatorname{Spec} A)$ is an open affine nbd of the generic point $x \in X$, since f is finite, hence affine. Moreover, as f is finite, $A \to B$ makes B a finite A-module. Since X is affine, the open sets X_g for $g \in \Gamma(X, \mathcal{O}_X)$ form a basis of opens of X, so let $g \in \Gamma(X, \mathcal{O}_X)$ be such that $x \in X_g \subseteq \operatorname{Spec} B$. Put $K = \operatorname{Frac} B$ and $k = \operatorname{Frac} A$ (X, Y are assumed integral!). Then since B is A-finite, it follows that K/k is finite, and we may pick a basis $s_i, \ldots, s_m \in B$ for K/k. Now $s_i\big|_{X_g} \in \Gamma(X_g, \mathcal{O}_X)$, so by II, Lemma 5.3 (b) there exists some $n > 0$ such that $x_i := g^n s_i\big|_{X_g}$ is a *global* section of $\mathcal{O}_X$, and hence an element of $\Gamma(Y, f_*\mathcal{O}_X)$. The x_i then give a morphism $\mathcal{O}_Y^m \to f_*\mathcal{O}_X$ defined by $(t_i) \mapsto \sum x_i t_i$. This is an isomorphism at the generic point of Y since the map $k^m \to K$ defined by $(t_i) \mapsto \sum t_i s_i$ is an isomorphism.

b) Now let $\mathcal{F}$ be any coherent sheaf on Y and apply $\mathscr{H}om(\cdot, \mathcal{F})$ to $\mathcal{O}_Y^m \to f_*\mathcal{O}_X$ to get a map $\beta : \mathscr{H}om(f_*\mathcal{O}_X, \mathcal{F}) \to \mathscr{H}om(\mathcal{O}_Y^m, \mathcal{F})$. Observe that $\mathscr{H}om(f_*\mathcal{O}_X, \mathcal{F})$ is both a $\mathcal{O}_Y$-module, *and* a $f_*\mathcal{O}_X$-module (via inner composition), so Ex. 5.17 (e) gives a coherent sheaf $\mathcal{G}$ on X with $f_*\mathcal{G} \simeq \mathscr{H}om(f_*\mathcal{O}_X, \mathcal{F})$. The map $\beta : f_*\mathcal{G} \to \mathcal{F}^m$ is an isomorphism at y since α is.

c) Since f is finite, $f_*\mathcal{G}$ is coherent by Ex. 5.5, so $\ker\beta$ and $\operatorname{coker}\beta$ are coherent by Prop. 5.7. It follows that $Y_1 := \operatorname{Supp}\ker\beta$ and $Y_2 := \operatorname{Supp}\operatorname{coker}\beta$ are closed (let $\mathcal{F}$ be any coherent sheaf and suppose $\mathcal{F}_P = 0$. Pick an affine nbd $\operatorname{Spec} A$ of P with $\mathcal{F}\big|_{\operatorname{Spec} A} = \widetilde{M}$ and let M be generated by $m_1, \ldots, m_r$ over A. Then $(m_i)_P = 0$ so we have opens U_i with $m_i\big|_{U_i} = 0$. Then $V = U \cap U_1 \cap \cdots \cap U_r$ is an open nbd of P with $\mathcal{F}\big|_V = 0$). Moreover, since β is an isomorphism at y, we have $y \notin Y_i$ for $i = 1, 2$. As $f^{-1}(Y_i)$ is closed and X is affine, it is affine, and $f : f^{-1}(Y_i) \to Y_i$ is then a finite morphism of integral noetherian schemes, so we assume (using Noetherian induction) that this implies that Y_1, Y_2 are affine, and we must show that Y is affine. Let $j_i : Y_i \to Y$ be the inclusion. Then since $\operatorname{Supp}\ker\beta = Y_1$ we have $(j_1)_*(j_1)^*\ker\beta = \ker\beta$ and similarly for $\operatorname{coker}\beta$; we may then apply Lemma 2.10 to conclude that

$$H^p(Y, \ker\beta) = H^p(Y_1, (j_1)^*\ker\beta) = 0$$

since $(j_i)^*\ker\beta$ is coherent (Prop. 5.8) and Y_1 is affine by hypothesis (using Serre's criterion Thm. 3.7). Similarly, $H^p(Y, \operatorname{coker}\beta) = 0$. But we have the exact sequence

$$0 \to \ker\beta \to f_*\mathcal{G} \to \mathcal{F}^m \to \operatorname{coker}\beta \to 0$$

which gives two short exact sequences

$$0 \to \ker\beta \to f_*\mathcal{G} \to \operatorname{im}\beta \to 0$$

and

$$0 \to \operatorname{im}\beta \to \mathcal{F}^m \to \operatorname{coker}\beta \to 0.$$

Taking cohomology and using the fact that $H^p(Y, \ker\beta) = H^p(Y, \operatorname{coker}\beta) = 0$ for all $p > 0$ (by Noetherian induction hypothesis) yields an isomorphism

$$H^p(Y, f_*\mathcal{G}) \simeq H^p(Y, \mathcal{F}^m) = H^p(Y, \mathcal{F})^m.$$

By Ex. 4.1, using the fact that f is affine, we have $H^p(X, \mathcal{G}) = H^p(Y, f_*\mathcal{G})$. Finally, applying Serre's criterion and the hypothesis that X is affine yields $H^p(Y, \mathcal{F}) = 0$. Since $\mathcal{F}$ was arbitrary, we again apply Serre's crit. to conclude that Y is affine. By Noetherian induction, we are done in the case that X, Y are integral. Ex. 3.1 and 3.2 allow us to immediately reduce to this case.

4.3 We cover U by the open affines $D(x) = \operatorname{Spec} k[x, 1/x, y]$ and $D(y) = k[y, 1/y, x]$. We have seen that $\Gamma(U, \mathcal{O}_X) = k[x, y, 1/x, 1/y]$. Thus, we have the Cech complex

$$k[x, 1/x, y] \oplus k[y, 1/y, x] \xrightarrow{d:(f,g)\mapsto f-g} k[x, y, 1/x, 1/y] \to 0,$$

so $\widehat{H}^1(U, \mathcal{O}_X) = k[x, y, 1/x, 1/y]/\operatorname{im} d$. But the image of d is just $k[x, y]$, so $\widehat{H}^1$ is the k-vector space spanned by $\{x^i y^j : i, j < 0\}$, and in particular is infinite-dimensional, so by Serre's criterion, we see that U is not affine.

4.5 We prove that $\operatorname{Pic} X \simeq \varinjlim_U \widehat{H}^1(U, \mathcal{O}_X^\times)$. Indeed, let $\mathcal{F}$ be an invertible sheaf, and let U be any cover consisting of affines U_i with $\phi_i : \mathcal{O}_{U_i} \xrightarrow{\sim} \mathcal{F}\big|_{U_i}$. Then $\phi_i^{-1} \circ \phi_j : \mathcal{O}_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{O}_{U_i \cap U_j}$ is an isomorphism, so gives an element $s_{ij} \in \mathcal{O}(U_i \cap U_j)^\times$. Moreover, we have $s_{jk} \cdot s_{ik}^{-1} \cdot s_{ij} = 1$ since it comes from the isomorphism $\phi_j^{-1}\phi_k \circ \phi_k^{-1}\phi_i \circ \phi_i^{-1}\phi_j = \mathrm{id}$, so we obtain an element of $\ker\big(\mathcal{O}_X^\times(U_i \cap U_j) \to \mathcal{O}_X^\times(U_i \cap U_j \cap U_k)\big)$, i.e. of $\varinjlim_U \widehat{H}^1(U, \mathcal{O}_X^\times)$ (observe from that once the isomorphisms ϕ_i have been fixed, the element of $\widehat{H}^1$ defined behaves well under refinement of open covers). The map $\operatorname{Pic} X \to \widehat{H}^1(U, \mathcal{O}_X^\times)$ thus defined is surjective, as given any cocycle representing an element of $\widehat{H}^1$, we obtain an invertible sheaf (by using the cocycle to glue together the sheaves $\mathcal{O}_{U_i}$ just as above) that maps to it. The map is injective because if the cohomology class obtained is zero, then the isomorphisms $\phi_i : \mathcal{O}_{U_i} \xrightarrow{\sim} \mathcal{F}\big|_{U_i}$ are multiplication by elements $s_i \in \mathcal{O}_{U_i}^\times$, so in fact $\mathcal{F} \simeq \mathcal{O}_X$ as an $\mathcal{O}_X$-module. Using the isomorphism $\widehat{H}^1(U, \mathcal{O}_X^\times) \simeq H^1(X, \mathcal{O}_X^\times)$ of Ex. 4.4 completes the proof.

4.7 We have

$$\mathcal{O}_X(V) = k[x_0/x_2, x_1/x_2]/(f(x_0/x_2, x_1/x_2, 1)) = k[u, v]/(f(u, v, 1))$$

and

$$\mathcal{O}_X(U) = k[x_0/x_1, x_2/x_1]/(f(x_0/x_1, 1, x_2/x_1)) = k[u/v, 1/v]/(f(u/v, 1, 1/v))$$

and $\mathcal{O}_X(U \cap V) = k[u, v, 1/v]/(f(u, v, 1))$. Thus, the image $\mathcal{O}_X(U) \oplus \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ consists of all pairs $(\alpha, \beta) \in k[u, v] \oplus k[u/v, 1/v]$ modulo $f(u, v, 1)$ that are equal. Hence the image is spanned by $u^i v^j$ with $i \geq 0$, $j \in \mathbf{Z}$ and $-j \geq i$ if $j < 0$ (otherwise no restriction) , so H^1 as a k-vector space is spanned by the monomials $u^i v^j$ with $0 < -j < i$. The relation $f(u, v, 1) = 1$ gives a linear dependence on u^d in terms of such monomials, so a basis for H^1 is $\{u^i/v^j : 0 < j < i < d\}$. Thus, $\dim_k H^1 = (d-1)(d-2)/2$. Now H^0 consists of those $(\alpha, \beta) \in k[u, v] \oplus k[u/v, 1/v]$ that are equal modulo $f(u, v, 1)$. By considering denominators, we see that this consists of the constants k, so $\dim_k H^0 = 1$.

4.11 The proof is almost identical to that of 4.5. We have $\widehat{H}^0(U, \mathcal{F}) = \Gamma(X, \mathcal{F})$ for *any* open covering $\mathcal{F}$ as the proof of 4.1 clearly shows. In general, imbed $\mathcal{F}$ in a flasque, q-coh sheaf $\mathcal{G}$ and let $\mathcal{R}$ be the quotient. Because the intersections $U_{i_0 \ldots i_p}$ have no nontrivial cohomology (for the sheaf $\mathcal{F}$), we have exact sequences

$$0 \to \mathcal{F}(U_{i_0 \ldots i_p}) \to \mathcal{G}(U_{i_0 \ldots i_p}) \to \mathcal{R}(U_{i_0 \ldots i_p}) \to 0.$$

The proof now follows that of 4.5 verbatim.

3.5

5.1 Observe first that the definition of χ makes sense by Theorem 5.2. The short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ gives a long exact sequence of k-vector spaces:

$$0 \to H^0(X, \mathcal{F}') \to \cdots \to H^i(X, \mathcal{F}) \to H^i(X, \mathcal{F}'') \to H^{i+1}(X, \mathcal{F}') \to \cdots ,$$

and this leads to the formula required formula.

5.7 We use Prop. 5.3.
a) By II, Cor. 4.8, the closed immersion i is proper, so by Caution 5.8.1, for any coherent sheaf $\mathcal{F}$ on Y, the sheaf $i_*\mathcal{F}$ is coherent on X. Thus let $\mathscr{L}$ be ample on X. Then then for every coherent $\mathcal{F}$ on Y, $i_*\mathcal{F}$ is coherent on X, so by 5.3 there exists n_0 s.t. for all $i > 0$ and all $n > n_0$ we have $H^i(X, i_*\mathcal{F} \otimes \mathscr{L}^n) = 0$. There is a natural surjective map $i_*\mathcal{F} \otimes \mathscr{L}^n \to i_*(\mathcal{F} \otimes i^*\mathscr{L})$ by II, Ex. 1.19, and thus a surjective map $H^i(X, i_*\mathcal{F} \otimes \mathscr{L}^n) \to H^i(X, i_*(\mathcal{F} \otimes i^*\mathscr{L}^n)) = H^i(Y, \mathcal{F} \otimes i^*\mathscr{L}^n)$ by III, Lemma 2.10. We thus conclude that $H^i(Y, \mathcal{F} \otimes i^*\mathscr{L}^n) = 0$ for all $i > 0$ and all $n > n_0$, so by applying 5.3 again, we see that $i^*\mathscr{L}$ is ample on Y.
b) Let $\mathcal{F}$ be a coherent sheaf on X and consider the filtration from III, Ex. 3.1. We thus obtain exact sequences

$$0 \to \mathscr{N}^{i+1}\mathcal{F} \otimes \mathscr{L}^n \to \mathscr{N}^i\mathcal{F} \otimes \mathscr{L}^n \to \mathscr{N}^i\mathcal{F}/\mathscr{N}^{i+1}\mathcal{F} \otimes \mathscr{L}^n \to .$$

Taking cohomology and using the isomorphism

$$\mathscr{N}^i\mathcal{F}/\mathscr{N}^{i+1}\mathcal{F} \otimes_{\mathcal{O}_X} \mathscr{L}^n \simeq \mathscr{N}^i\mathcal{F}/\mathscr{N}^{i+1}\mathcal{F} \otimes_{\mathcal{O}_{X_{\text{red}}}} (\mathcal{O}_{X_{\text{red}}} \otimes \mathscr{L})^n,$$

we see by 5.3 that if $\mathscr{L} \otimes \mathcal{O}_{X_{\text{red}}}$ is ample on X_{red} then for any $i > 1$ and any coherent $\mathcal{F}$ on X there exists n_0 such that for all $n > n_0$ and all $j \geq 1$ we have isomorphisms $H^i(X, \mathcal{F} \otimes \mathscr{L}^n) \simeq H^i(X, \mathscr{N}^j\mathcal{F} \otimes \mathscr{L}^n)$. Taking j large enough so $\mathscr{N}^j$ is the zero sheaf and considering the surjective maps $H^1(X, \mathscr{N}^j\mathcal{F} \otimes \mathscr{L}^n) \simeq H^1(X, \mathcal{F} \otimes \mathscr{L}^n)$, we conclude (again by 5.3) that $\mathscr{L}$ is ample on X. For the converse, we observe that the natural map $i : X_{\text{red}} \to X$ is a closed immersion, with $i^*\mathscr{L} = \mathscr{L} \otimes \mathcal{O}_{X_{\text{red}}}$. Thus, part a) yields the converse.
c) As in b), part a) yields one direction ($X_i \to X$ is a closed immersion). For the other direction, we let $\mathcal{F}$ be a coherent sheaf on X and filter $\mathcal{F}$ as in Ex. 3.2, so as to obtain exact sequences

$$0 \to \mathscr{I}_1 \cdots \mathscr{I}_k \cdot \mathcal{F} \otimes \mathscr{L}^n \to \mathscr{I}_1 \cdots \mathscr{I}_{k-1} \cdot \mathcal{F} \otimes \mathscr{L}^n \to \mathscr{I}_1 \cdots \mathscr{I}_{k-1} \cdot \mathcal{F}/\mathscr{I}_1 \cdots \mathscr{I}_k \cdot \mathcal{F} \otimes \mathscr{L}^n \to 0,$$

with $\mathscr{I}_i$ the ideal sheaf of X_i. Now

$$\mathscr{I}_1 \cdots \mathscr{I}_{k-1} \cdot \mathcal{F}/\mathscr{I}_1 \cdots \mathscr{I}_k \cdot \mathcal{F} \otimes_{\mathcal{O}_X} \mathscr{L}^n \simeq \mathscr{I}_1 \cdots \mathscr{I}_{k-1} \cdot \mathcal{F}/\mathscr{I}_1 \cdots \mathscr{I}_k \cdot \mathcal{F} \otimes_{\mathcal{O}_{X_k}} (\mathcal{O}_{X_k} \otimes_{\mathcal{O}_X} \mathscr{L})^n$$

since $\mathscr{I}_1 \cdots \mathscr{I}_{k-1} \cdot \mathcal{F}/\mathscr{I}_1 \cdots \mathscr{I}_k \cdot \mathcal{F}$ is a sheaf of $\mathcal{O}_X/\mathscr{I}_k = \mathcal{O}_{X_k}$-modules. The long exact cohomology sequences and the hypothesis that $\mathscr{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_i}$ is ample on X_i yields, via 5.3, an n_0 such that for all $i > 0$ and all $n > n_0$ there are isomorphisms

$$H^i(X, \mathcal{F} \otimes \mathscr{L}^n) \simeq H^i(X, \mathscr{I}_1 \cdots \mathscr{I}_n \cdot \mathcal{F} \otimes \mathscr{L}^n)$$

for all $i > 1$ and a surjection $H^1(X, \mathscr{I}_1 \cdots \mathscr{I}_n \cdot \mathcal{F} \otimes \mathscr{L}^n) \twoheadrightarrow H^1(X, \mathcal{F} \otimes \mathscr{L}^n)$. However, $\mathscr{I}_1 \cdots \mathscr{I}_n$ is the zero sheaf on X as we saw in Ex. 3.2, so we conclude that $H^i(X, \mathcal{F} \otimes \mathscr{L}^n) = 0$ for all $i > 0$ and all $n > n_0$, so that $\mathscr{L}$ is ample by 5.3.
d) Parts b) and c) allow us at once to reduce to the case of X and Y integral. Let $\mathcal{F}$ be any coherent sheaf on Y. Then we have seen in the proof of Ex. 4.2 that there is a sheaf $\mathcal{G}$ on X and a morphism $\beta : f_*\mathcal{G} \to \mathcal{F}^r$ that is an isomorphism at the generic point of Y. Proceeding as in Ex. 4.2 by Noetherian induction and using the long exact cohomology sequences associated to

$$0 \to \ker \beta \otimes \mathscr{L}^{nr} \to f_*\mathcal{G} \otimes \mathscr{L}^{nr} \to \operatorname{im} \beta \otimes \mathscr{L}^{nr} \to 0$$

and

$$0 \to \operatorname{im}\beta \otimes \mathscr{L}^{nr} \to (\mathcal{F} \otimes \mathscr{L}^n)^r \to \operatorname{coker}\beta \otimes \mathscr{L}^{nr} \to 0$$

we conclude that $H^i(Y, (\mathcal{F} \otimes \mathscr{L}^n)^r) \simeq H^i(Y, f_*\mathcal{G} \otimes \mathscr{L}^{nr})$. Now the natural map $\mathscr{L} \to f_*f^*\mathscr{L}$ yields a map

$$H^i(Y, f_*\mathcal{G} \otimes \mathscr{L}^{nr}) \to H^i(Y, f_*(\mathcal{G} \otimes (f^*\mathscr{L})^{nr})),$$

and by Ex. 4.1, we have

$$H^i(Y, f_*(\mathcal{G} \otimes (f^*\mathscr{L})^{nr})) = H^i(X, \mathcal{G} \otimes (f^*\mathscr{L})^{nr}).$$

HMMMMMMMMMMM????

5.10 Let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X. Then we have short exact sequences

$$0 \to \mathcal{F}^i/\ker\alpha_i \otimes \mathcal{O}_X(n) \xrightarrow{\alpha_i} \mathcal{F}^{i+1}(n) \xrightarrow{\alpha_{i+1}} \operatorname{im}\alpha_{i+1} \otimes \mathcal{O}_X(n) \to 0$$

and

$$0 \to \operatorname{im}\alpha_i \otimes \mathcal{O}_X(n) \to \mathcal{F}^{i+1}(n) \to \operatorname{coker}\alpha_i \otimes \mathcal{O}_X(n) \to 0$$

for all $i \geq 1$ and all $n > 0$. Exactness implies that $\operatorname{coker}\alpha_i = \mathcal{F}^{i+1}/\operatorname{im}\alpha_i = \mathcal{F}^{i+1}/\ker\alpha_{i+1}$. Since the $\mathcal{F}^i$ are coherent, all the sheaves in the exact sequences above are coherent. Now use 5.3 to find n_i such that for all $n > n_i$ and all j we have

$$H^i(\operatorname{im}\alpha_i \otimes \mathcal{O}_X(n)) = \mathcal{F}^i/\ker\alpha_i \otimes \mathcal{O}_X(n) = 0$$

and let $N = \max_i n_i$. Then for all $n > N$ we have exact sequences (from the long exact sequences of cohomology and the fact that we have forced all the H^1's to vanish)

$$0 \to \Gamma(X, \mathcal{F}^i/\ker\alpha_i \otimes \mathcal{O}_X(n)) \to \Gamma(X, \mathcal{F}^{i+1}(n)) \to \Gamma(X, \operatorname{im}\alpha_{i+1} \otimes \mathcal{O}_X(n)) \to 0$$

and

$$0 \to \Gamma(X, \operatorname{im}\alpha_i \otimes \mathcal{O}_X(n)) \to \Gamma(X, \mathcal{F}^{i+1}(n)) \to \Gamma(X, \operatorname{coker}\alpha_i \otimes \mathcal{O}_X(n)) \to 0.$$

Using the fact that $\operatorname{coker}\alpha_i = \mathcal{F}^{i+1}/\operatorname{im}\alpha_i = \mathcal{F}^{i+1}/\ker\alpha_{i+1}$ as noted above and splicing these exact sequences back together shows that for all $n > N$ we have an exact sequence

$$\Gamma(X, \mathcal{F}^1(n)) \to \Gamma(X, \mathcal{F}^2(n)) \to \ldots \to \Gamma(X, \mathcal{F}^r(n)),$$

as desired.

Chapter 4

4.1

1.1 Consider the divisor nP for a positive integer n. We have

$$l(nP) - l(K - nP) = n + 1 - g,$$

so choosing $n > g$ and observing that $l(K - nP) \geq 0$, we conclude that $l(nP) > 1$, which is to say that $\dim_k H^0(X, \mathscr{L}(nP)) = \dim_k \Gamma(X, \mathscr{L}(nP)) > 1$, so there is a nonconstant rational function $f \in \mathscr{L}(nP)$. Then f is regular everywhere but P, where it must have a pole.

1.2 We first claim that there is a function f_i with a pole at P_i and nonvanishing at P_j for all j. Indeed, choose n sufficiently large so

$$l(nP_i - \sum_{j \neq i} P_j) = n + 1 - g - (r - 1)$$

and

$$l(nP_i - \sum_{j \neq i,k} P_j) = n + 1 - g - (r - 1) + 1.$$

Since we have the obvious containment

$$H^0(X, \mathscr{L}(nP_i - \sum_{j \neq i} P_j)) \subseteq H^0(X, \mathscr{L}(nP_i - \sum_{j \neq i,k} P_j)),$$

dimension considerations show that there is a function $g_{i,k}$ with a pole at P_i and vanishing at P_j for $j \neq k$, but nonvanishing for $j = k$. Then the function $f_i := \sum_k g_{i,k}$ has a pole at P_i and $f(P_j) = g_{i,j}(P_j) \neq 0$. We now let $F = \prod_i f_i$. Then F has a pole at P_i for all i because there can be no cancellation by our construction.

1.3 The hypotheses allow us to embed X in a proper curve $\widetilde{X}$ and we let $S = \widetilde{X} \setminus X = \{P_1, \ldots, P_r\}$ (as it is a closed subset and nonempty since X is not proper). By 1.2, we have a function f with poles at P_i and regular elsewhere, so $f : \widetilde{X} \to \mathbf{P}^1$ satisfies $f^{-1}(\mathbf{A}^1) = X$. By II 6.8, as f is nonconstant and $\widetilde{X}$ is proper, we conclude that f is a finite morphism, and in particular affine. Hence $f^{-1}(\mathbf{A}^1) = X$ is affine.

1.4 Let X be a separated one-dimensional scheme over k none of whose irreducible components are proper. By III Ex. 3.1 we know that X is affine iff X_{red} is affine, so we may assume X reduced. Similarly, III Ex. 3.2 allows us to suppose that X is irreducible, hence integral. Now let $f : \widetilde{X} \to X$ be the normalization. It is a finite surjective morphism with $\widetilde{X}$ an integral, normal, separated, one-dimensional scheme over k, hence by I 6.2A it is regular also. If X is not proper, neither is $\widetilde{X}$, as the image of a proper scheme under a finite morphism is again proper by II, Ex. 4.4. So assume $\widetilde{X}$ is not proper. Then we may apply 1.3 to conclude that if $\widetilde{X}$ is affine. Finally, we apply III, Ex. 4.2 (as f is finite) and conclude that X is affine.

1.5 Let D be effective, so we have the containment $H^0(X, \mathscr{L}(K-D)) \subseteq H^0(X, \mathscr{L}(K))$, with equality holding iff $\deg D = 0$ or $g = 0$. Then $l(D) - 1 = \deg D + l(K - D) - g \leq \deg D$, with equality iff $\deg D = 0$ (i.e. $D = 0$ since D is effective) of $g = 0$.

1.6 It follows from II, 6.9 that $\deg f = \deg(f)_\infty$, the degree of the divisor of poles of f. Let P be any point of X. Then $l((g+1)P) = 2 + l(K - (g+1)P) > 1$, so there exists a function f with $(g+1)P - (f)_\infty$ effective, or what is the same, a morphism $f : X \to \mathbf{P}^1$ of degree $\deg f \leq g + 1$.

1.7 The only non-obvious part is that $|K|$ has no base points. If P is a base point of $|K|$ then every effective divisor linearly equivalent to K has support containing P, so every $f \in H^0(X, \mathscr{L}(K))$ has a zero at P. In other words, the containment $H^0(X, \mathscr{L}((K - P)) \subseteq H^0(X, \mathscr{L}(K))$ is an equality, so $l(K - P) = l(K) = 2$. Then $l(P) = l(K - P) = 2$ by Riemann-Roch. We conclude that there is a function f with a simple pole at P, so this defines a morphism $X \to \mathbf{P}^1$ of degree 1, (as in Ex. 1.6) which must be an isomorphism, contradicting the assumption that $g = 2$. Thus $|K|$ is base-point free, so we use II, 7.8.1 to get a morphism $f : X \to \mathbf{P}^1$ of degree $\deg K = 2$.

1.10 Following the proof of 1.3.7, it is enough to show that for any divisor D of degree 0 supported in X_{reg} there exists a point $P \in X_{\text{reg}}$ with $D \sim P - P_0$. Using Ex. 1.9 and the hypothesis that $p_a = 1$, we have

$$l(D + P_0) - l(K - D - P_0) = 1,$$

and as $K - D - P_0$ has degree -1, there exists a function f with $(f) + D + P_0 \geq 0$, and comparing degrees shows that we must have$(f) + D + P_0 = P$ for some point P, necessarily in X_{reg} as D is supported in X_{reg}.