Explicit Min-wise Hash Families with Optimal Size

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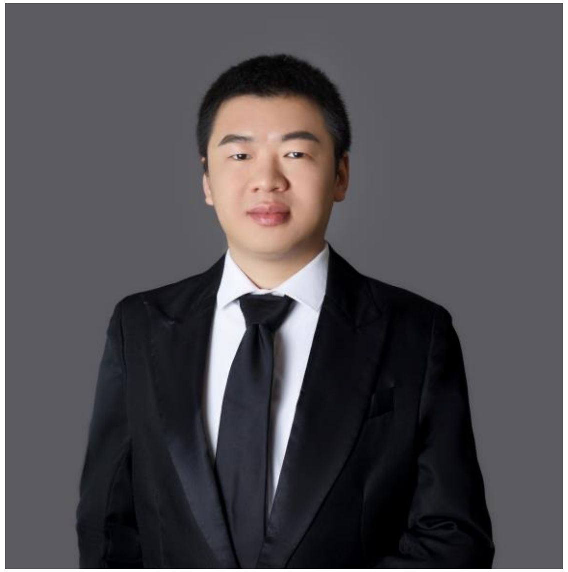
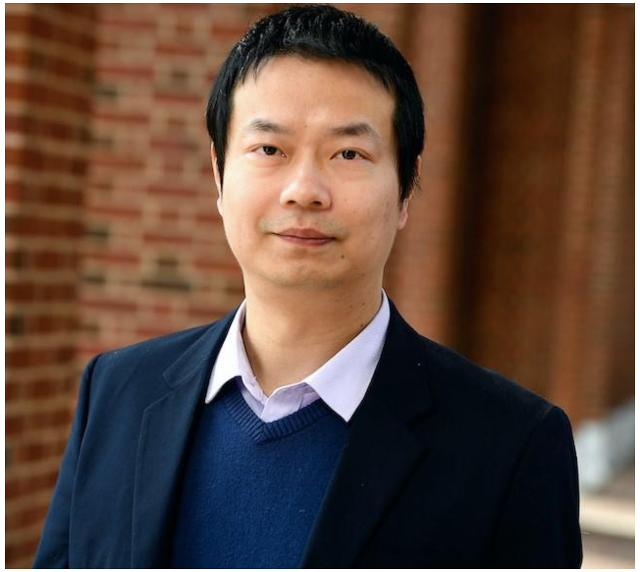


MAIN SPAN 4200 FEET
LENGTH OF ONE CABLE 7650 FT. (2331.7 M)
DIAMETER OF ONE CABLE 36 IN. (92.4 CM)
WIRES IN EACH CABLE 27,572
TOTAL WIRE USED 80,000 MILES (128,748 KM)
WEIGHT OF CABLE 24,500 TONS (22,226 M. TONS)
Cable Contractor : John A. Roebling's Sons Co.
Trenton & Roebling, New Jersey
Cal



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Min-wise Hash Family

Definition (Min-wise Hash Family [Broder, Charikar, Frieze, Mitzenmacher ’00])

- We say $\mathcal{H} = \{h : [N] \to [M]\}$ is a min-wise hash family with (multiplicative) error δ, if for any $X \subseteq [N]$ and $y \in X$,

$$\Pr_{h\sim\mathcal{H}}[h(y) < \min h(X \backslash y)] := \Pr_{h\sim\mathcal{H}}\left[h(y) < \min_{x\in X\backslash y} h(x)\right] = \frac{1 \pm \delta}{|X|}.$$

- Play a crucial role in the design of graph algorithms and streaming algorithms. Applications in similarity estimation [Cohen, Datar, Fujiwara, Gionis, Indyk, Motwani, Ullman, Yang ’01], ℓ_0-sampler [Cormode, Firmani ’14], etc.

k-min-wise Hash Family

Definition (k-min-wise Hash Family [Feigenblat, Porat, Shiftan '11])

- We say $\mathcal{H} = \{h : [N] \to [M]\}$ is a k-min-wise hash family with (multiplicative) error δ, if for any $X \subseteq [N]$ and $Y \in \binom{X}{\leq k}$,

$$\Pr_{h \sim \mathcal{H}}[\max h(Y) < \min h(X \backslash Y)] := \Pr_{h \sim \mathcal{H}}\left[\max_{y \in Y} h(y) < \min_{x \in X \backslash Y} h(x)\right] = \frac{1 \pm \delta}{\binom{|X|}{|Y|}}.$$

- Play a crucial role in the design of graph algorithms and streaming algorithms. Applications in similarity estimation [Cohen, Datar, Fujiwara, Gionis, Indyk, Motwani, Ullman, Yang '01], ℓ_0-sampler [Cormode, Firmani '14], etc.

Problem – A Good Min-wise Hash Family

- Two ways for evaluating the quality of a hash family:

Time

- **Fast Evaluation Time:** time complexity to compute $h(x)$ from a given input $x \in [N]$ for any hash function $h \in \mathcal{H}$.

Space

- **Small size $|\boldsymbol{\mathcal{H}}|$**, or equivalently $\text{seed length} = \log_2 |\mathcal{H}|$ (number of random bits used to generate a hash function).

- **Goal:** Construct an explicit (k-)min-wise hash family with short seed length.

Explicitness

- There exists an efficient ($\text{poly}(N)$-time) algorithm to compute $h(x)$ from a given input $x \in [N]$ and the seed of the hash $h \in \mathcal{H}$.

Prior Works

- [Saks, Srinivasan, Zhou, Zuckerman ’00]: $O\left(\log^{3/2} N\right)$ bits for any polynomially small error.
- This was improved to $O(\log N \log\log N)$ by [Gopalan, Yehudayoff ’20].

- [Indyk ’01]: $O(\log(1/\delta))$-wise independence is enough to have error δ. $O(\log(1/\delta) \cdot \log N)$ seed length.
- [Feigenblat, Porat, Shiftan ’11]: $O(\log(1/\delta) + k \log\log(1/\delta))$-wise independence is sufficient for k-min-wise hash.
- [Pătraşcu, Thorup ’16]: $\Omega(\log(1/\delta))$-wise independence is needed for min-wise hash.

- **Non-explicitly**, $O(\log(N/\delta))$ bits for min-wise and $O(k \log N + \log(1/\delta))$ bits for k-min-wise.
- **No** construction with $O(\log N)$ bits for min-wise (and $O(k \log N)$ bits for k-min-wise) and sub-constant error was known before.

Our Results

- We give an explicit min-wise hash family with seed length $O(\log N)$ and error $2^{-O\left(\frac{\log N}{\log\log N}\right)}$, as well as an explicit k-min-wise hash family with seed length $O(k \log N)$ for any $k = \log^{O(1)} N$.

- Reduce the space complexity of algorithms that use min-wise hash families.

Seed length with error $2^{-O\left(\frac{\log N}{\log\log N}\right)}$	Min-wise hash	k-min-wise hash $\left(k = \log^{O(1)} N\right)$
[Ind01] && [FPS11]	$O\left(\frac{\log^2 N}{\log\log N}\right)$	$O\left(\frac{\log^2 N}{\log\log N} + k \log N \log\log N\right)$
[SSZZ00] && [GY20]	$O(\log N \log\log N)$	$O(k \log N \log\log N)$
Our Results	$\boldsymbol{O(\log N)}$	$\boldsymbol{O(k \log N)}$

Polynomially Small Error

[Saks, Srinivasan, Zhou, Zuckerman ’00] && [Gopalan, Yehudayoff ’20]

Hash Family as Pseudorandom Generator

Definition (Pseudorandom Generator)

- $G: \{0,1\}^r \to [M]^N$ is an ε-PRG for a family of functions $\mathcal{F} = \left\{f: [M]^N \to \{0,1\}\right\}$ if for any $f \in \mathcal{F}$,

$$\left|\mathrm{E}\left[f\left(U_{[M]^N}\right)\right] - \mathrm{E}[f(G(U_r))]\right| \le \varepsilon.$$

- r is called the seed length of G.

$$\mathcal{H} = \{h: [N] \to [M]\} \leftrightarrow G: \{0,1\}^{\log_2 |\mathcal{H}|} \to [M]^N$$
$$G(s) = \left(h_s(1), h_s(2), \dots, h_s(N)\right)$$

Decompose into Sum of Combinatorial Rectangles

Definition (Combinatorial Rectangle)

- We say $f : \Sigma^d \to \{0,1\}$ is a Σ^d-combinatorial-rectangle, if there exists $A_1, \dots, A_d \subseteq \Sigma$ such that $f(x_1, \dots, x_d) = \prod_{i=1}^{d} 1(x_i \in A_i)$.

- Note that (analysis in [Indyk ’01], [Saks, Srinivasan, Zhou, Zuckerman ’00])

$$\Pr_{h\sim\mathcal{H}}[h(y) < \min h(X\backslash y)] = \sum_{\theta=1}^{M} \Pr_{h\sim\mathcal{H}}\underbrace{[h(y) = \theta \wedge \min h(X\backslash y) > \theta]}_{\substack{A_y = \{\theta\} \\ A_x = \{\theta+1, \cdots, M\}, \forall x \in X\backslash y \\ [M]^{|X|}\text{- combinatorial-rectangle}}}$$

Connection with PRG for Combinatorial Rectangles

- If $\mathcal{H}$ is an ε-PRG for $[M]^N$-combinatorial-rectangles, then $\forall X \subseteq [N]$ and $y \in X$,

$$\begin{aligned}
\Pr_{h\sim\mathcal{H}}[h(y) < \min h(X\backslash y)] &= \sum_{\theta=1}^{M} \Pr_{h\sim\mathcal{H}}[h(y) = \theta \wedge \min h(X\backslash y) > \theta] \\
&= \sum_{\theta=1}^{M} \Pr_{h\sim U}[h(y) = \theta \wedge \min h(X\backslash y) > \theta] \pm M\varepsilon \\
&= \Pr_{h\sim U}[h(y) < \min h(X\backslash y)] \pm M\varepsilon \\
&= \Pr_{h\sim U}[h(y) < \min h(X\backslash y)] \cdot (1 \pm |X| \cdot M\varepsilon)
\end{aligned}$$

$$\boxed{\begin{gathered}\Pr_{h\sim U}[h(y) < \min h(X\backslash y)] \approx 1/|X| \\ \text{if } M = \Omega(N/\delta)\end{gathered}}$$

Conclusion

- An ε-PRG for $[M]^N$-combinatorial-rectangles is a min-wise hash family with multiplicative error $\Theta(NM\varepsilon)$.

Connection with PRG for Combinatorial Rectangles

Theorem [Gopalan, Yehudayoff '20]

- There exists an explicit ε-PRG for $[M]^N$-combinatorial-rectangles with seed length $O(\log(M \log N / \varepsilon) \log\log(M/\varepsilon))$.

Corollary

- A min-wise hash family of $O(\log N \log\log N)$ bits for any polynomially small error.

Gap between Multiplicative and Additive Errors

$$\Pr_{h\sim\mathcal{H}}[h(y) < \min h(X\backslash y)] = \Pr_{h\sim U}[h(y) < \min h(X\backslash y)] \cdot (1 \pm \delta) \text{ if } M = \Omega(N/\delta)$$

- Min-wise hash family is a PRG with ***multiplicative*** error!

- **Problem:** additive → multiplicative, $\varepsilon \rightarrow \Theta(NM\varepsilon)$. Even for constant δ, $\varepsilon = \delta/(NM)$ would be polynomially small.

- If we want shorter seed length, like $O(\log N)$, then it is not enough to apply the PRG for combinatorial rectangles directly.

Optimal Size && Sub-constant Error

Our Approach [Chen, Huang, Li ’2025]

1st Step – Balls into Bins

- Use constant-wise independence to split X into several tiny blocks s.t. each block only needs a small size family.

2nd Step – Recycle Randomness

- Use the Nisan-Zuckerman PRG and a special extractor to recycle randomness between blocks.

3rd Step – Domain Reduction

- Use PRGs for combinatorial rectangles and the special extractor to further reduce the error.

$$\Pr_{h\sim\mathcal{H}}[h(y) < \min h(X\backslash y)] = \frac{1 \pm \delta}{|X|}$$

$$= \sum_{\theta=1}^{M} \Pr_{h\sim\mathcal{H}}[h(y) = \theta \wedge \min h(X\backslash y) > \theta]$$

Two Level Hash Structure

$h(x) = \sigma_{\varphi(x)}(x)$

- **Balls into Bins:** small max-load, concentration.

  $|X_i| \approx |X|/\ell, \forall i \in [\ell]$

  $\max |X_i|$ is small

- **Each Block:** Since $|X_i|$ is small, there exists $\mathcal{G} = \{\sigma: [N] \to [M]\}$ with $|\mathcal{G}| = \mathrm{poly}(N)$, s.t.

  $$\Pr_{\sigma_i \sim \mathcal{G}}[\min \sigma_i(X_i) > \theta] \approx \Pr_{h \sim U}[\min h(X_i) > \theta]$$

$$\begin{tikzcd}
 & & X \arrow[dll] \arrow[d] \arrow[drr] & & \\
X_1 \arrow[drr] & \cdots & y \in X_j \arrow[d] & \cdots & X_\ell \arrow[dll] \\
 & & {[M]} & &
\end{tikzcd}$$

$\varphi: [N] \to [\ell]$

Constant-wise independence

$\sigma_1, \dots, \sigma_\ell: [N] \to [M]$

i.i.d. from $\mathcal{G}$

Split into Several Small Blocks

$$h(x) = \sigma_{\varphi(x)}(x)$$

$$\Pr_{h\sim\mathcal{H}}[h(y) = \theta \wedge \min h(X\backslash y) > \theta]$$

$$\|$$

$$\varphi\colon [N] \to [\ell]$$

$$\Pr_{\sigma_1\sim\mathcal{G}}[\min \sigma_1(X_1) > \theta] \cdots \Pr_{\sigma_j\sim\mathcal{G}}\left[\sigma_j(y) = \theta \wedge \min \sigma_j\left(X_j\backslash y\right) > \theta\right] \cdots \Pr_{\sigma_\ell\sim\mathcal{G}}[\min \sigma_\ell(X_\ell) > \theta]$$

$$\Pr_{\sigma_i\sim\mathcal{G}}[\min \sigma_i(X_i) > \theta] \approx \Pr_{h\sim U}[\min h(X_i) > \theta]$$

$$\wr\wr$$

! Require many random bits.

$\sigma_1, \ldots, \sigma_\ell\colon [N] \to [M]$

i.i.d. from $\mathcal{G}$

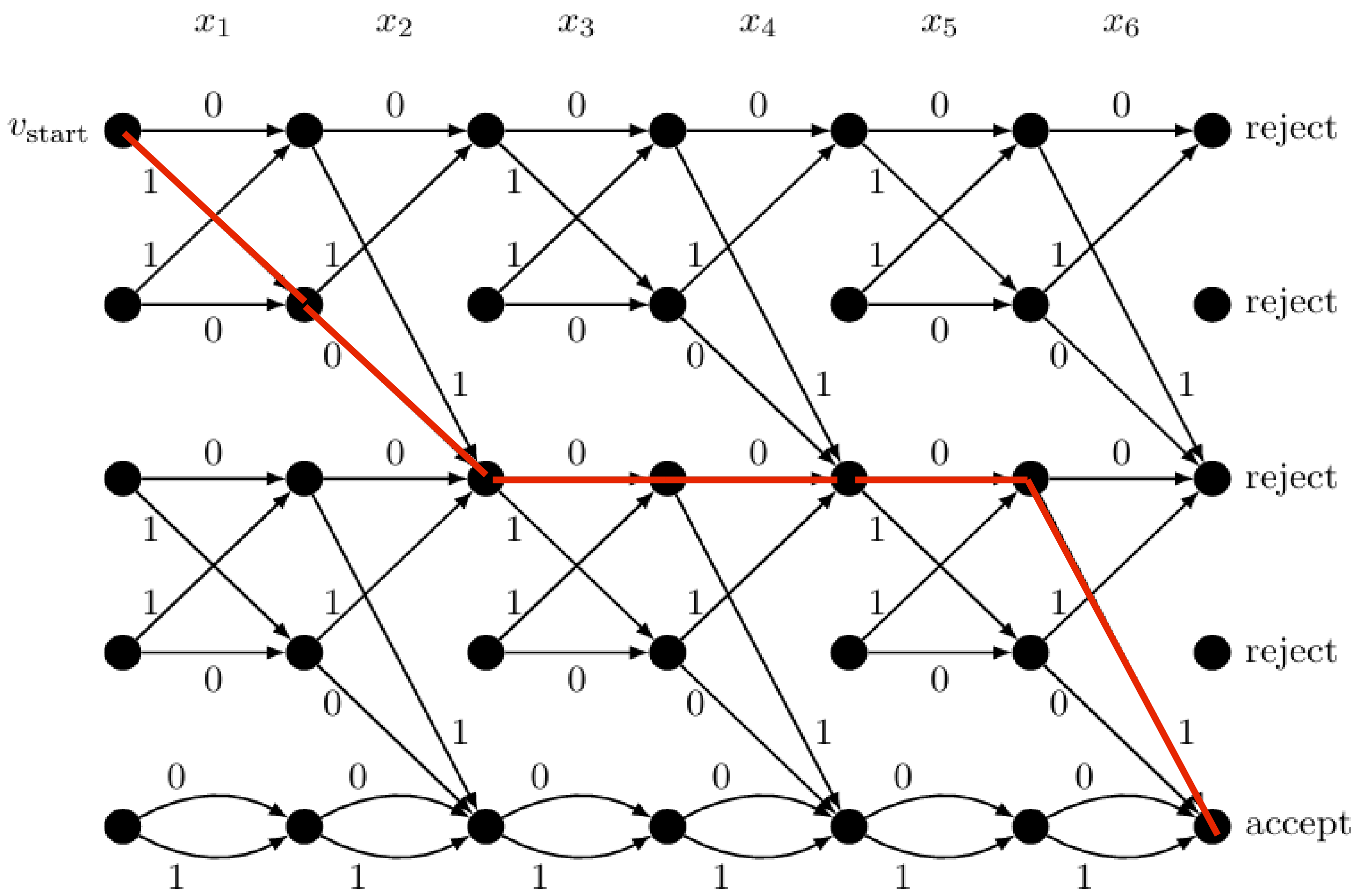
$$\Pr_{h\sim U}[\min h(X_1) > \theta] \cdots \Pr_{h\sim U}\left[h(y) = \theta \wedge \min h\left(X_j\backslash y\right) > \theta\right] \cdots \Pr_{h\sim U}[\min h(X_\ell) > \theta]$$

$$= \Pr_{h\sim U}[h(y) = \theta \wedge \min h(X\backslash y) > \theta]$$

Read-once Branching Program

Width: 5
Length: 6
Alphabet: {0,1}

$(x_1, x_2, x_3, x_4, x_5, x_6) = (1,0,0,0,0,1)$
$\text{MAJ}(x_1 \oplus x_2, x_3 \oplus x_4, x_5 \oplus x_6) = 1$

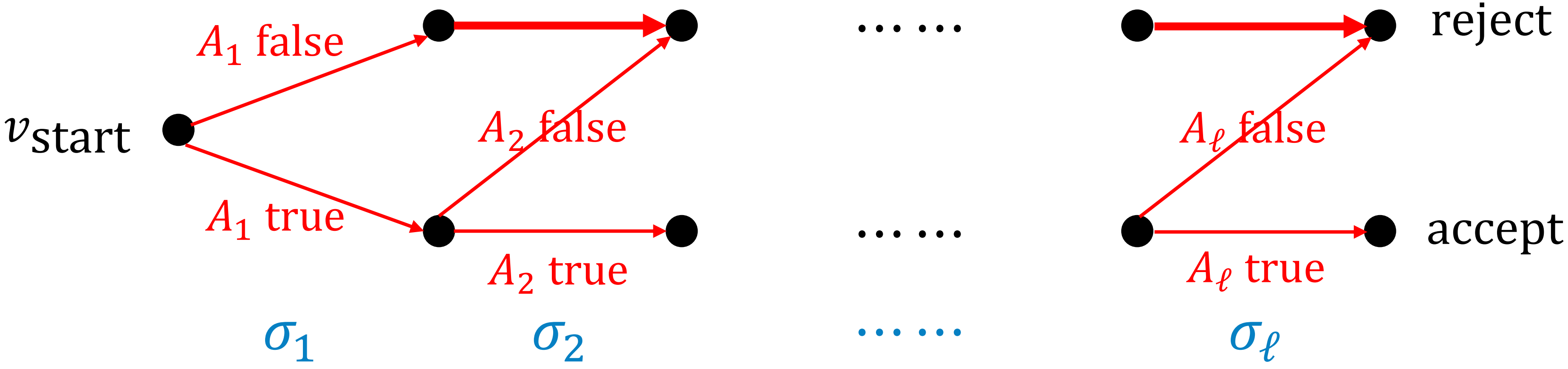


$\text{MAJ}(x_1 \oplus x_2, x_3 \oplus x_4, x_5 \oplus x_6)$

Recycle Randomness

$$\underbrace{(\min \sigma_1(X_1) > \theta)}_{\text{Event } A_1} \wedge \cdots \wedge \underbrace{\left(\sigma_j(y) = \theta \wedge \min \sigma_j\left(X_j \backslash y\right) > \theta\right)}_{\text{Event } A_j} \wedge \cdots \wedge \underbrace{(\min \sigma_\ell(X_\ell) > \theta)}_{\text{Event } A_\ell}$$

$\sigma_1, \ldots, \sigma_\ell$ i.i.d. from $\mathcal{G}$



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Recycle Randomness

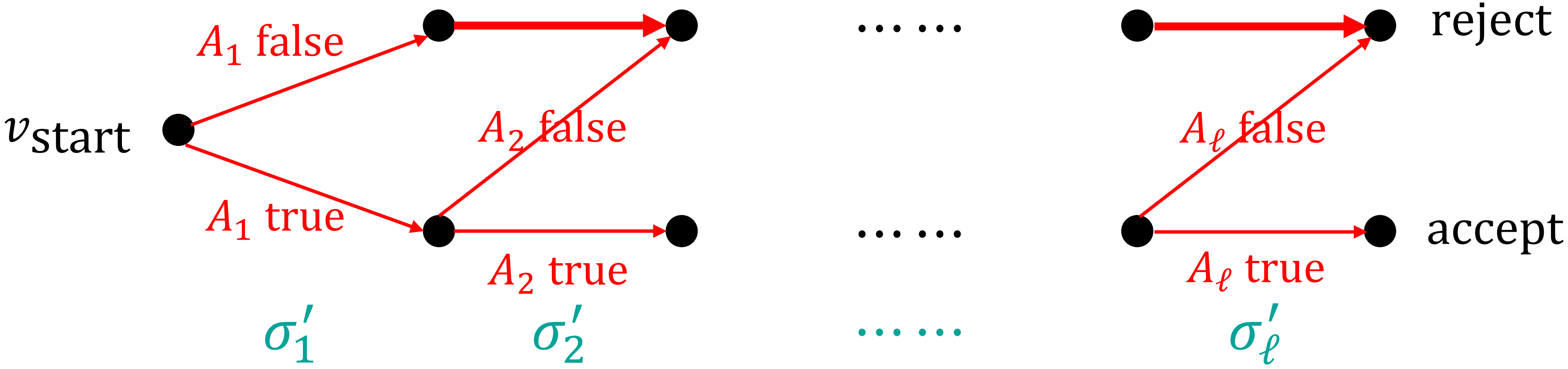
$$\underbrace{(\min \sigma_1(X_1) > \theta)}_{\text{Event } A_1} \wedge \cdots \wedge \underbrace{\left(\sigma_j(y) = \theta \wedge \min \sigma_j\left(X_j \backslash y\right) > \theta\right)}_{\text{Event } A_j} \wedge \cdots \wedge \underbrace{(\min \sigma_\ell(X_\ell) > \theta)}_{\text{Event } A_\ell}$$

$\sigma_1, \ldots, \sigma_\ell$ i.i.d. from $\mathcal{G}$

$$\wr$$

$$(\min \sigma'_1(X_1) > \theta) \wedge \cdots \wedge \left(\sigma'_j(y) = \theta \wedge \min \sigma'_j\left(X_j \backslash y\right) > \theta\right) \wedge \cdots \wedge (\min \sigma'_\ell(X_\ell) > \theta)$$

$\sigma'_1, \ldots, \sigma'_\ell$ are correlated



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Nisan-Zuckerman PRG

Definition (Extractor)

- $\mathrm{Ext}:\ \{0,1\}^p \times \{0,1\}^d \to \{0,1\}^q$ is a (k,ε)-extractor, if for any random source X over $\{0,1\}^p$ with min-entropy $H_\infty(X) \geq k$, it holds that $\mathrm{Ext}(X, U_d)$ is ε-close to U_q.

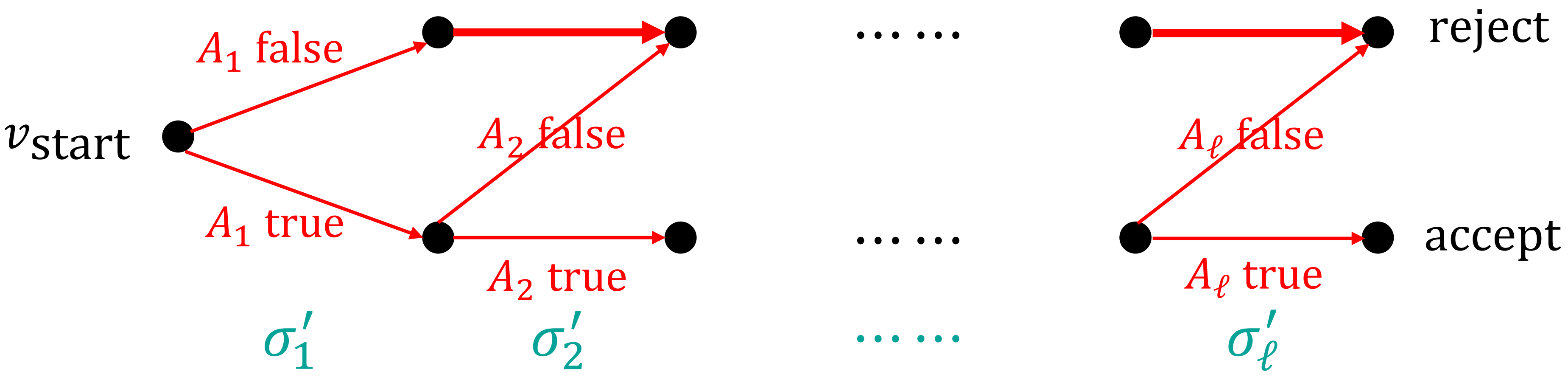
- Nisan-Zuckerman PRG:

$$\mathrm{NZPRG}(w, s_1, \dots, s_\ell) = \big(\mathrm{Ext}(w, s_1), \dots, \mathrm{Ext}(w, s_\ell)\big) \in (\{0,1\}^q)^\ell,$$
$$w \sim U_p, s_1, \dots, s_\ell \text{ i.i.d. from } U_d.$$

Nisan-Zuckerman PRG

$$\underbrace{\left(\min \sigma_1'(X_1) > \theta\right)}_{\text{Event } A_1} \wedge \cdots \wedge \underbrace{\left(\sigma_j'(y) = \theta \wedge \min \sigma_j'\left(X_j \backslash y\right) > \theta\right)}_{\text{Event } A_j} \wedge \cdots \wedge \underbrace{\left(\min \sigma_\ell'(X_\ell) > \theta\right)}_{\text{Event } A_\ell}$$

$$\boxed{\begin{gathered}\text{Ext}: \{0,1\}^p \times \{0,1\}^d \to \mathcal{G} \\ \sigma_1' = \text{Ext}(w, s_1), \ldots, \sigma_\ell' = \text{Ext}(w, s_\ell) \\ w \sim U_p, s_1, \ldots, s_\ell \text{ i.i.d. from } U_d\end{gathered}}$$



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Alphabet: $\mathcal{G}$

Analysis

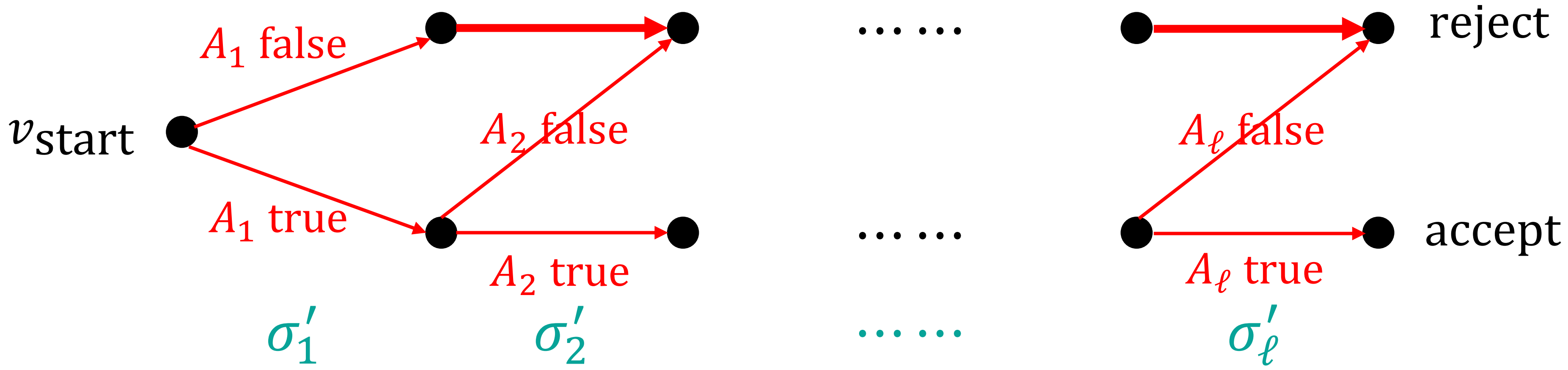
$$\text{Ext}: \{0,1\}^p \times \{0,1\}^d \to \mathcal{G}$$
$$\sigma_1' = \text{Ext}(w, s_1), \dots, \sigma_\ell' = \text{Ext}(w, s_\ell)$$
$$w \sim U_p, s_1, \dots, s_\ell \text{ i.i.d. from } U_d$$

$$P_1[\text{NZ}]$$
$$\wr$$
$$(P_1[U] \pm \text{ExtErr})$$

$$\xleftarrow{\quad} \quad \begin{aligned} P_1[\text{NZ}] &= P_1[\mathcal{G}] \pm \text{ExtErr} \\ &\approx P_1[U] \pm \text{ExtErr} \end{aligned}$$

$$H_\infty(w = U_p) \geq k$$
$$\Rightarrow \text{Ext}(w, s_1) \approx_{\text{ExtErr}} \mathcal{G}$$

$$P_i[\mathcal{G}] \approx P_i[U]$$
$$P_i[\text{NZ}] \coloneqq \Pr_{\sigma_i' : \text{NZPRG}}[A_i]$$
$$P_i[\mathcal{G}] = \Pr_{\sigma_i \sim \mathcal{G}}[A_i]$$
$$P_i[U] \coloneqq \Pr_{h \sim U}[A_i]$$



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Alphabet: $\mathcal{G}$

Analysis

$$P_1[\mathrm{NZ}]\cdot P_2[\mathrm{NZ}]$$

$$\wr$$

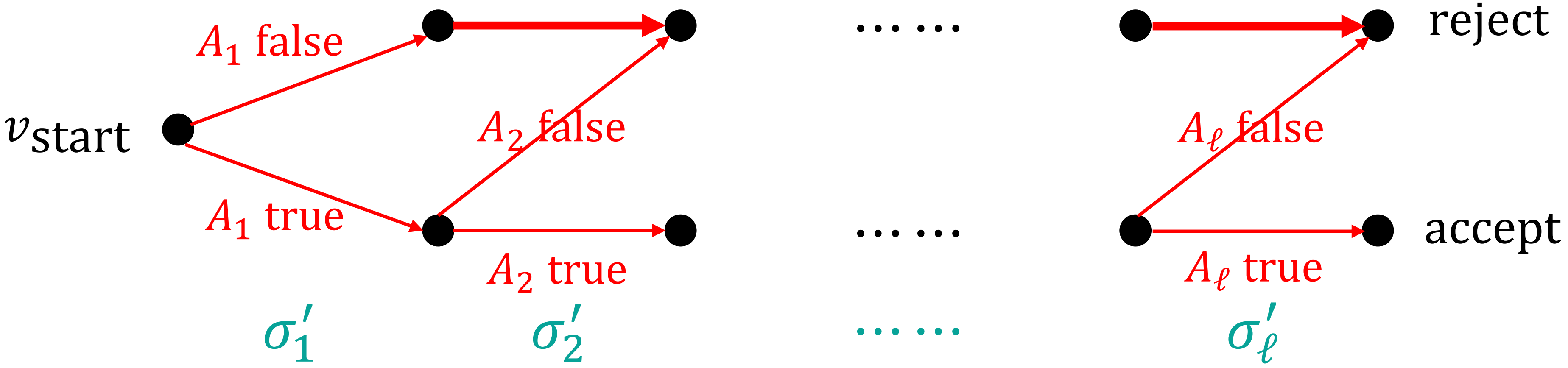
$$(P_1[U]\pm\mathrm{ExtErr})\cdot(P_2[U]\pm\mathrm{ExtErr})$$

$$\boxed{\begin{gathered}H_\infty(w\mid A_1)\geq k\\ \Rightarrow \mathrm{Ext}(w\mid A_1,s_2)\approx_{\mathrm{ExtErr}}\mathcal{G}\end{gathered}}$$

$$\boxed{\Pr[A_1A_2]=\Pr[A_1]\cdot\Pr[A_2\mid A_1]}$$

$$\boxed{\begin{gathered}\mathrm{Ext}\colon\{0,1\}^p\times\{0,1\}^d\to\mathcal{G}\\ \sigma_1'=\mathrm{Ext}(w,s_1),\dots,\sigma_\ell'=\mathrm{Ext}(w,s_\ell)\\ w\sim U_p, s_1,\dots,s_\ell \text{ i.i.d. from } U_d\end{gathered}}$$

$$\boxed{\begin{gathered}P_i[\mathcal{G}]\approx P_i[U]\\ P_i[\mathrm{NZ}]\coloneqq\Pr_{\sigma_i':\mathrm{NZPRG}}[A_i]\\ P_i[\mathcal{G}]=\Pr_{\sigma_i\sim\mathcal{G}}[A_i]\\ P_i[U]\coloneqq\Pr_{h\sim U}[A_i]\end{gathered}}$$



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Alphabet: $\mathcal{G}$

Analysis

$$\mathrm{Ext}\colon \{0,1\}^p \times \{0,1\}^d \to \mathcal{G}$$
$$\sigma'_1 = \mathrm{Ext}(w, s_1), \ldots, \sigma'_\ell = \mathrm{Ext}(w, s_\ell)$$
$$w \sim U_p, s_1, \ldots, s_\ell \text{ i.i.d. from } U_d$$

$$P_1[\mathrm{NZ}] \cdot P_2[\mathrm{NZ}] \cdots P_\ell[\mathrm{NZ}]$$

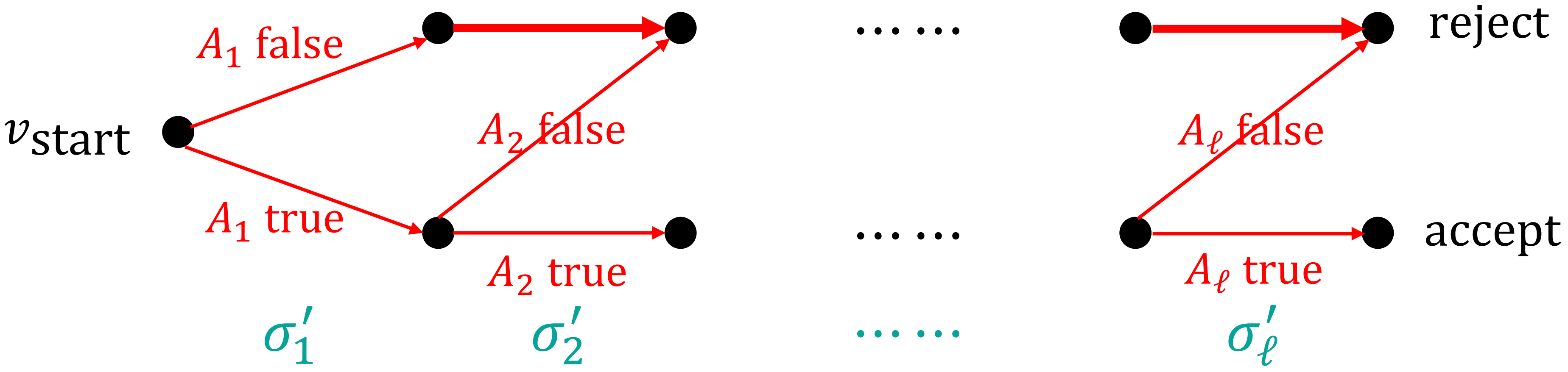
$$\wr\wr$$

$$(P_1[U] \pm \mathrm{ExtErr}) \cdot (P_2[U] \pm \mathrm{ExtErr}) \cdots (P_\ell[U] \pm \mathrm{ExtErr})$$

$$P_i[\mathcal{G}] \approx P_i[U]$$
$$P_i[\mathrm{NZ}] \coloneqq \Pr_{\sigma'_i:\mathrm{NZPRG}}[A_i]$$
$$P_i[\mathcal{G}] = \Pr_{\sigma_i \sim \mathcal{G}}[A_i]$$
$$P_i[U] \coloneqq \Pr_{h \sim U}[A_i]$$

$$H_\infty(w \mid A_1 \cdots A_{\ell-1}) \geq k$$
$$\Rightarrow \mathrm{Ext}(w \mid A_1 \cdots A_{\ell-1}, s_\ell) \approx_{\mathrm{ExtErr}} \mathcal{G}$$

$$\boxed{\Pr[A_1 \cdots A_\ell] = \Pr[A_1 \cdots A_{\ell-1}] \cdot \Pr[A_\ell \mid A_1 \cdots A_{\ell-1}]}$$



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Alphabet: $\mathcal{G}$

One bin is Sensitive to Multiplicative Error

$h(x) = \sigma'_{\varphi(x)}(x)$

$$\Pr_{h \sim \mathcal{H}}[h(y) = \theta \wedge \min h(X \backslash y) > \theta]$$

$$\|$$

$\varphi: [N] \to [\ell]$

$P_i[\mathcal{G}] \approx P_i[U]$

$$P_i[\mathrm{NZ}] = \Pr_{\sigma'_i:\mathrm{NZPRG}}[A_i]$$

$$P_i[\mathcal{G}] := \Pr_{\sigma_i \sim \mathcal{G}}[A_i]$$

$$P_i[U] := \Pr_{h \sim U}[A_i]$$

$$P_1[\mathrm{NZ}] \cdots P_j[\mathrm{NZ}] \cdots P_\ell[\mathrm{NZ}]$$

$$\approx$$

$\sigma'_1, \ldots, \sigma'_\ell: [N] \to [M]$ from NZPRG, $\mathrm{ExtErr} = N^{-o(1)}$

? We hope that this special bin does not need to pay for the error from extractor.

$$(P_1[U] \pm \mathrm{ExtErr}) \cdots (P_j[U] \cancel{\pm \mathrm{ExtErr}}) \cdots (P_\ell[U] \pm \mathrm{ExtErr})$$

$P_j[U] = \Pr_{h \sim U}[h(y) = \theta \wedge \min h(X_j \backslash y) > \theta] \le 1/M \implies P_j[U] \pm \mathrm{ExtErr} = P_j[U] \cdot (1 \pm M \cdot \mathrm{ExtErr})$

! $P_j[U]$ is very small, and it makes this bin very sensitive to multiplicative error.

Change the Order of Inputs

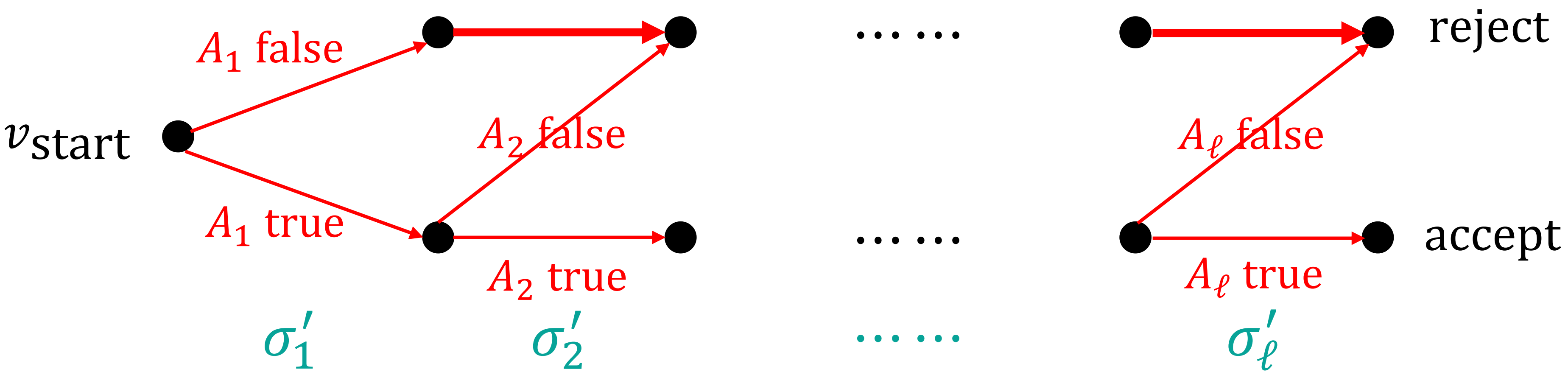
$$\underbrace{(\min \sigma_1'(X_1) > \theta)}_{\text{Event } A_1} \wedge \cdots \wedge \underbrace{\left(\sigma_j'(y) = \theta \wedge \min \sigma_j'\left(X_j \backslash y\right) > \theta\right)}_{\text{Event } A_j} \wedge \cdots \wedge \underbrace{(\min \sigma_\ell'(X_\ell) > \theta)}_{\text{Event } A_\ell}$$

Ext: $\{0,1\}^p \times \{0,1\}^d \to \mathcal{G}$

$\sigma_1' = \text{Ext}(w, s_1), \ldots, \sigma_\ell' = \text{Ext}(w, s_\ell)$

$w \sim U_p, s_1, \ldots, s_\ell$ i.i.d. from U_d

★ $s_1, \ldots s_\ell$ (or correspondingly $\sigma_1', \ldots, \sigma_\ell'$) are symmetric.



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Alphabet: $\mathcal{G}$

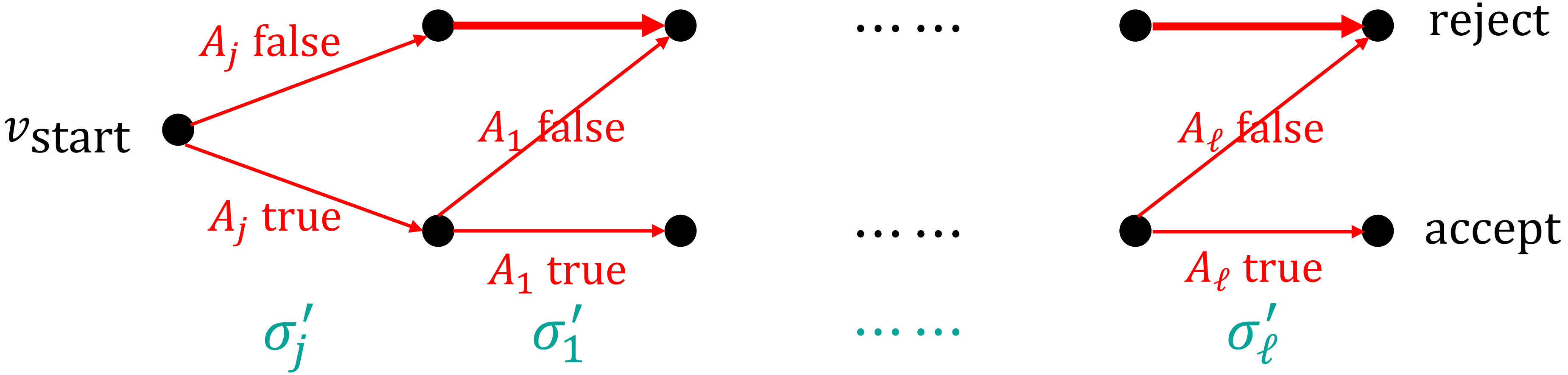
Change the Order of Inputs

$$\underbrace{\left(\sigma_j'(y) = \theta \wedge \min \sigma_j'(X_j \backslash y) > \theta\right)}_{\text{Event } A_j} \wedge \underbrace{\left(\min \sigma_1'(X_1) > \theta\right)}_{\text{Event } A_1} \wedge \cdots \wedge \underbrace{\left(\min \sigma_\ell'(X_\ell) > \theta\right)}_{\text{Event } A_\ell}$$

$\text{Ext}: \{0,1\}^p \times \{0,1\}^d \to \mathcal{G}$

$\sigma_j' = \text{Ext}(w, s_j), \sigma_1' = \text{Ext}(w, s_1), \dots, \sigma_\ell' = \text{Ext}(w, s_\ell)$

$w \sim U_p, s_j, s_1, \dots, s_\ell$ i.i.d. from U_d



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Alphabet: $\mathcal{G}$

Change the Order of Inputs

$$P_j[\mathrm{NZ}] \cdot P_1[\mathrm{NZ}] \cdots P_\ell[\mathrm{NZ}]$$

$$\wr$$

$$(P_j[U] \pm \mathrm{ExtErr}) \cdot (P_1[U] \pm \mathrm{ExtErr}) \cdots (P_\ell[U] \pm \mathrm{ExtErr})$$

$\mathrm{Ext}\colon \{0,1\}^p \times \{0,1\}^d \to \mathcal{G}$

$\sigma_j' = \mathrm{Ext}(w, s_j), \sigma_1' = \mathrm{Ext}(w, s_1), \ldots, \sigma_\ell' = \mathrm{Ext}(w, s_\ell)$

$w \sim U_p, s_j, s_1, \ldots, s_\ell$ i.i.d. from U_d

$H_\infty(w = U_p) = p \geq k$

$\Rightarrow \mathrm{Ext}(w, s_j) \approx_{\mathrm{ExtErr}} \mathcal{G}$

★ w has no entropy loss at the beginning.

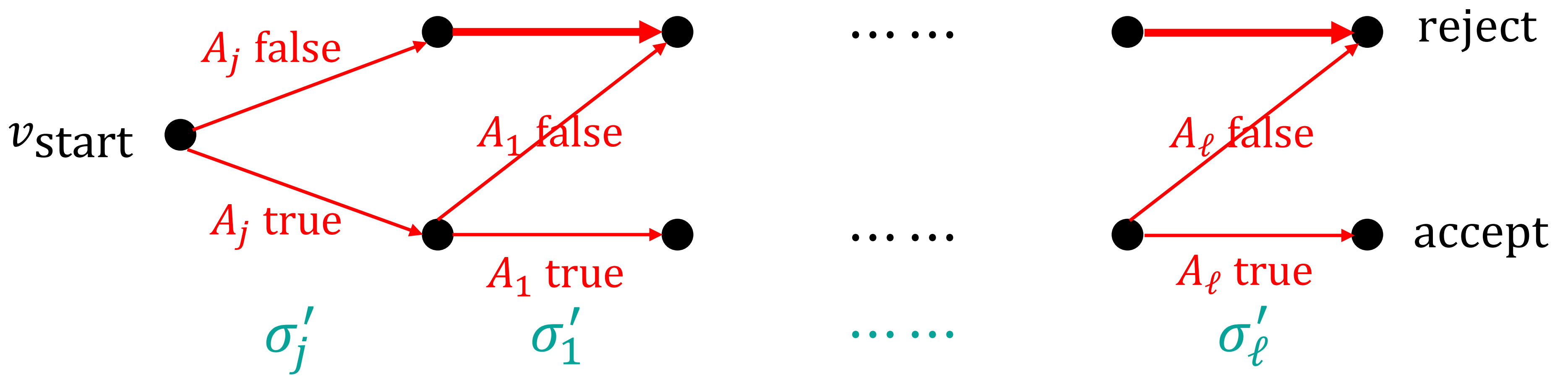
? We expect strong properties for $\mathrm{Ext}(w, s_j)$ than other $\mathrm{Ext}(w, s_i)$.

$P_i[\mathcal{G}] \approx P_i[U]$

$P_i[\mathrm{NZ}] \coloneqq \Pr_{\sigma_i'\colon\mathrm{NZPRG}}[A_i]$

$P_i[\mathcal{G}] = \Pr_{\sigma_i \sim \mathcal{G}}[A_i]$

$P_i[U] \coloneqq \Pr_{h \sim U}[A_i]$



Width: 2

Length: ℓ

Alphabet: $\mathcal{G}$

Change the Order of Inputs

$$P_j[\mathrm{NZ}] \cdot P_1[\mathrm{NZ}] \cdots P_\ell[\mathrm{NZ}]$$

$$\wr\wr$$

$$(P_j[U] \cancel{\pm \mathrm{ExtErr}}) \cdot (P_1[U] \pm \mathrm{ExtErr}) \cdots (P_\ell[U] \pm \mathrm{ExtErr})$$

$\mathrm{Ext}\colon \{0,1\}^p \times \{0,1\}^d \to \mathcal{G}$

$\sigma'_j = \mathrm{Ext}(w, s_j), \sigma'_1 = \mathrm{Ext}(w, s_1), \ldots, \sigma'_\ell = \mathrm{Ext}(w, s_\ell)$

$w \sim U_p, s_j, s_1, \ldots, s_\ell$ i.i.d. from U_d

$H_\infty(w = U_p) = p \geq k$

$\Rightarrow$ ~~$\mathrm{Ext}(w, s_j) \approx_{\mathrm{ExtErr}} \mathcal{G}$~~

$\mathrm{Ext}(w, s_j) = \mathcal{G}$

★ w has no entropy loss at the beginning.

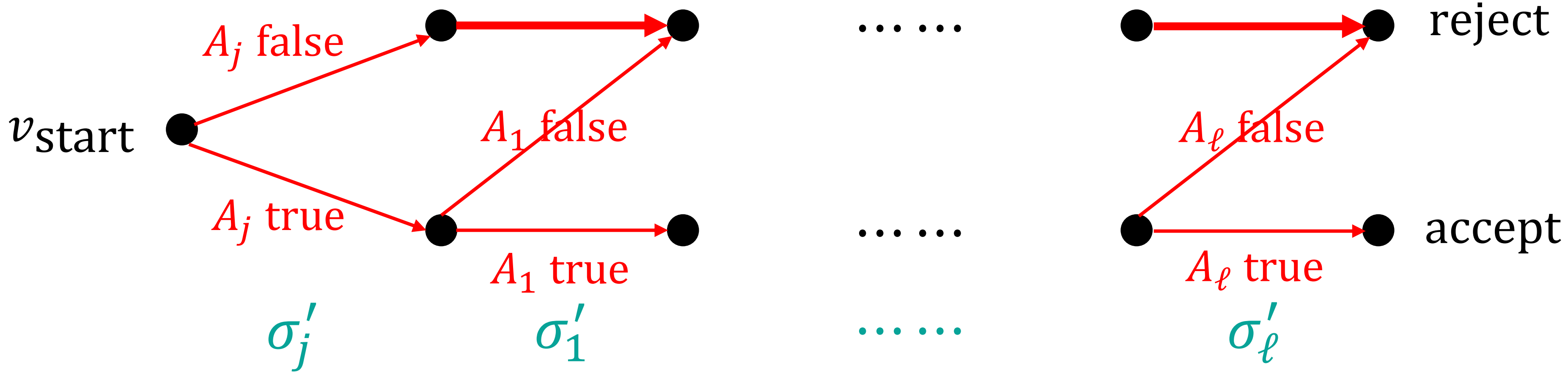
? We expect strong properties for $\mathrm{Ext}(w, s_j)$ than other $\mathrm{Ext}(w, s_i)$.

$P_i[\mathcal{G}] \approx P_i[U]$

$P_i[\mathrm{NZ}] \coloneqq \Pr_{\sigma'_i:\mathrm{NZPRG}}[A_i]$

$P_i[\mathcal{G}] = \Pr_{\sigma_i \sim \mathcal{G}}[A_i]$

$P_i[U] \coloneqq \Pr_{h \sim U}[A_i]$



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Length: ℓ

Alphabet: $\mathcal{G}$

Special Extractor

Lemma

- Given any p and $k < p$, for any error ε, there exists an explicit (k, ε)-extractor $\mathrm{Ext}: \{0,1\}^p \times \{0,1\}^d \to \{0,1\}^q$ with $q = k/2$ and $d = O(\log(p/\varepsilon))$. And Ext satisfies an extra property: $\boxed{\mathrm{Ext}(U_p, s) = U_q}$ for any fixed seed s.

$$\sigma_j' = \mathrm{Ext}\left(w, s_j\right) = \mathrm{Ext}(U_p, U_d) \sim \mathcal{G} \implies X_j \text{ has no error from extractor}$$

$$\begin{gathered} P_j[\mathrm{NZ}] \cdot P_1[\mathrm{NZ}] \cdots P_\ell[\mathrm{NZ}] \\ \wr\wr \\ \boldsymbol{P_j[U]} \cdot (P_1[U] \pm \mathrm{ExtErr}) \cdots (P_\ell[U] \pm \mathrm{ExtErr}) \end{gathered}$$

Domain Reduction

$$\mathrm{NZPRG}(w, s_1, \ldots, s_\ell) = \left(\mathrm{Ext}(w, s_1), \ldots, \mathrm{Ext}(w, s_\ell)\right)$$

- Use a PRG for $\left(\{0,1\}^d\right)^\ell$-combinatorial rectangles to generate $s_1, \ldots, s_\ell$, instead of i.i.d. sampling.

- Also need the special property of our extractor.

- Help us to further reduce the error from $1/\log^{O(1)} N$ to $2^{-O\left(\frac{\log N}{\log\log N}\right)}$.

Summary

Main Theorem 1 (Optimal Size && Sub-constant Error, Min-wise Hash)

- There exists an explicit min-wise hash family with seed length $O(\log N)$ and error $2^{-O\left(\frac{\log N}{\log \log N}\right)}$.

Main Theorem 2 (Optimal Size && Sub-constant Error, k-min-wise Hash)

- There exists an explicit k-min-wise hash family with seed length $O(k \log N)$ and error $2^{-O\left(\frac{\log N}{\log \log N}\right)}$, for any $k = \log^{O(1)} N$.

Main Technical Lemma (Extractor with Extra Properties)

- Given any p and $k < p$, for any error ε, there exists an explicit (k, ε)-extractor $\mathrm{Ext}\colon \{0,1\}^p \times \{0,1\}^d \to \{0,1\}^q$ with $q = k/2$ and $d = O(\log(p/\varepsilon))$. Moreover, Ext satisfies an extra property: $\mathrm{Ext}(U_p, s) = U_q$ for any fixed seed s.

Open Problems

Smaller error with optimal seed length?

Extending the result for larger k (like $\sqrt{N}$) on k-min-wise hash?

Faster evaluation time with optimal seed length?

Lower bound on t-wise independence for k-min-wise hash?

More applications of the powerful Nisan-Zuckerman framework?

Thank you for listening!