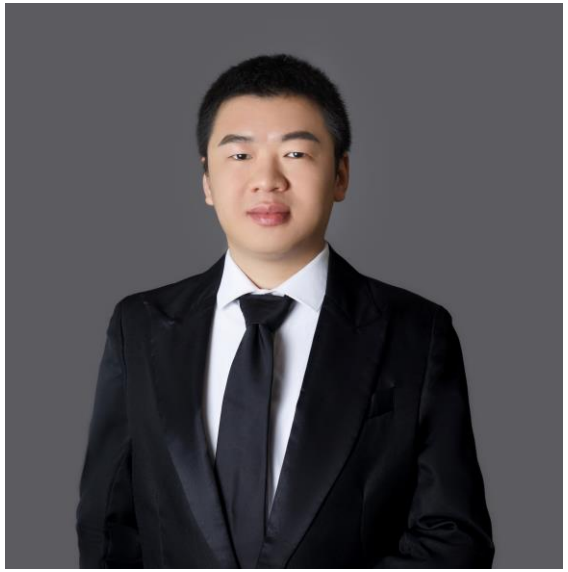
# Explicit Min-wise Hash Families with Optimal Size


Xue Chen*



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# Min-wise Hash Family

**Definition (Min-wise Hash Family [Broder, Charikar, Frieze, Mitzenmacher '00])**

- We say $\mathcal{H} = \{h : [N] \to [M]\}$ is a min-wise hash family with (multiplicative) error $\delta$, if for any $X \subseteq [N]$ and $y \in X$,

$$\Pr_{h\sim\mathcal{H}}[h(y) < \min h(X\backslash y)] := \Pr_{h\sim\mathcal{H}}\left[h(y) < \min_{x\in X\backslash y} h(x)\right] = \frac{1\pm\delta}{|X|}.$$

- Play a crucial role in the design of graph algorithms and streaming algorithms. Applications in similarity estimation [Cohen, Datar, Fujiwara, Gionis, Indyk, Motwani, Ullman, Yang '01], $\ell_0$-sampler [Cormode, Firmani '14], etc.

# $k$-min-wise Hash Family

**Definition ($k$-min-wise Hash Family [Feigenblat, Porat, Shiftan '11])**

- We say $\mathcal{H} = \{h : [N] \to [M]\}$ is a $k$-min-wise hash family with (multiplicative) error $\delta$, if for any $X \subseteq [N]$ and $Y \in \binom{X}{\leq k}$,

$$\Pr_{h\sim\mathcal{H}}[\max h(Y) < \min h(X\backslash Y)] := \Pr_{h\sim\mathcal{H}}\left[\max_{y\in Y} h(y) < \min_{x\in X\backslash Y} h(x)\right] = \frac{1\pm\delta}{\binom{|X|}{|Y|}}.$$

- Play a crucial role in the design of graph algorithms and streaming algorithms. Applications in similarity estimation [Cohen, Datar, Fujiwara, Gionis, Indyk, Motwani, Ullman, Yang '01], $\ell_0$-sampler [Cormode, Firmani '14], etc.

# Problem – A Good Min-wise Hash Family

- Two ways for evaluating the quality of a hash family:

**TIME**

- **Fast Evaluation Time:** time complexity to compute $h(x)$ from a given input $x \in [N]$ for any hash function $h \in \mathcal{H}$.

**SPACE**

- **Small size** $|\mathcal{H}|$, or equivalently seed length $= \log_2 |\mathcal{H}|$ (number of random bits used to generate a hash function).

- **Goal:** Construct an explicit ($k$-)min-wise hash family with short seed length.

**EXPLICITNESS**

- There exists an efficient ($\mathrm{poly}(N)$-time) algorithm to compute $h(x)$ from a given input $x \in [N]$ and the seed of the hash $h \in \mathcal{H}$.

# Prior Works

- [Saks, Srinivasan, Zhou, Zuckerman '00]: $O\left(\log^{3/2} N\right)$ bits for any polynomially small error.
- This was improved to $O(\log N \log\log N)$ by [Gopalan, Yehudayoff '20].

- [Indyk '01]: $O(\log(1/\delta))$-wise independence is enough to have error $\delta$. $O(\log(1/\delta) \cdot \log N)$ seed length.
- [Feigenblat, Porat, Shiftan '11]: $O(\log(1/\delta) + k \log\log(1/\delta))$-wise independence is sufficient for $k$-min-wise hash.
- [Pătraşcu, Thorup '16]: $\Omega(\log(1/\delta))$-wise independence is needed for min-wise hash.

- **Non-explicitly**, $O(\log(N/\delta))$ bits for min-wise and $O(k \log N + \log(1/\delta))$ bits for $k$-min-wise.
- **No** construction with $O(\log N)$ bits for min-wise (and $O(k \log N)$ bits for $k$-min-wise) and sub-constant error was known before.

# Our Results

- We give an explicit min-wise hash family with seed length $O(\log N)$ and error $2^{-O\left(\frac{\log N}{\log\log N}\right)}$, as well as an explicit $k$-min-wise hash family with seed length $O(k \log N)$ for any $k = \log^{O(1)} N$.

- Reduce the space complexity of algorithms that use min-wise hash families.

| Seed length with error $2^{-O\left(\frac{\log N}{\log\log N}\right)}$ | Min-wise hash | $k$-min-wise hash $\left(k = \log^{O(1)} N\right)$ |
|---|---|---|
| [Ind01] && [FPS11] | $O\left(\frac{\log^2 N}{\log\log N}\right)$ | $O\left(\frac{\log^2 N}{\log\log N} + k \log N \log\log N\right)$ |
| [SSZZ00] && [GY20] | $O(\log N \log\log N)$ | $O(k \log N \log\log N)$ |
| **Our Results** | $\boldsymbol{O(\log N)}$ | $\boldsymbol{O(k \log N)}$ |

# *Polynomially Small Error*

[Saks, Srinivasan, Zhou, Zuckerman ’00] && [Gopalan, Yehudayoff ’20]

# Hash Family as Pseudorandom Generator

**Definition (Pseudorandom Generator)**

- $G: \{0,1\}^r \to [M]^N$ is an $\varepsilon$-PRG for a family of functions $\mathcal{F} = \{f: [M]^N \to \{0,1\}\}$ if for any $f \in \mathcal{F}$,

$$\left|\mathrm{E}\left[f\left(U_{[M]^N}\right)\right] - \mathrm{E}\left[f\left(G(U_r)\right)\right]\right| \leq \varepsilon.$$

- $r$ is called the seed length of $G$.

$$\mathcal{H} = \{h: [N] \to [M]\} \leftrightarrow G: \{0,1\}^{\log_2 |\mathcal{H}|} \to [M]^N$$
$$G(s) = \left(h_s(1), h_s(2), \ldots, h_s(N)\right)$$

# Decompose into Sum of Combinatorial Rectangles

**Definition (Combinatorial Rectangle)**

- We say $f : \Sigma^d \to \{0,1\}$ is a $\Sigma^d$-combinatorial-rectangle, if there exists $A_1, \dots, A_d \subseteq \Sigma$ such that $f(x_1, \dots, x_d) = \prod_{i=1}^{d} 1(x_i \in A_i)$.

- Note that (analysis in [Indyk '01], [Saks, Srinivasan, Zhou, Zuckerman '00])

$$\Pr_{h\sim\mathcal{H}}[h(y) < \min h(X\backslash y)] = \sum_{\theta=1}^{M} \Pr_{h\sim\mathcal{H}}[\underbrace{h(y) = \theta \wedge \min h(X\backslash y) > \theta}]$$

$$A_y = \{\theta\}$$

$$A_x = \{\theta+1, \cdots, M\}, \forall x \in X\backslash y$$

$[M]^{|X|}$- combinatorial-rectangle

# Connection with PRG for Combinatorial Rectangles

- If $\mathcal{H}$ is an $\varepsilon$-PRG for $[M]^N$-combinatorial-rectangles, then $\forall X \subseteq [N]$ and $y \in X$,

$$\Pr_{h\sim\mathcal{H}}[h(y) < \min h(X\backslash y)] = \sum_{\theta=1}^{M} \Pr_{h\sim\mathcal{H}}[h(y) = \theta \wedge \min h(X\backslash y) > \theta]$$

$$= \sum_{\theta=1}^{M} \Pr_{h\sim U}[h(y) = \theta \wedge \min h(X\backslash y) > \theta] \pm M\varepsilon$$

$$= \Pr_{h\sim U}[h(y) < \min h(X\backslash y)] \pm M\varepsilon$$

$$= \Pr_{h\sim U}[h(y) < \min h(X\backslash y)] \cdot (1 \pm |X| \cdot M\varepsilon)$$

$\Pr_{h\sim U}[h(y) < \min h(X\backslash y)] \approx 1/|X|$ if $M = \Omega(N/\delta)$

**Conclusion**

- An $\varepsilon$-PRG for $[M]^N$-combinatorial-rectangles is a min-wise hash family with multiplicative error $\Theta(NM\varepsilon)$.

# Connection with PRG for Combinatorial Rectangles

**Theorem [Gopalan, Yehudayoff '20]**

- There exists an explicit $\varepsilon$-PRG for $[M]^N$-combinatorial-rectangles with seed length $O(\log(M \log N\ /\varepsilon) \log\log(M/\varepsilon))$.

**Corollary**

- A min-wise hash family of $O(\log N \log\log N)$ bits for any polynomially small error.

# Gap between Multiplicative and Additive Errors

$$\Pr_{h\sim\mathcal{H}}[h(y) < \min h(X\backslash y)] = \Pr_{h\sim U}[h(y) < \min h(X\backslash y)] \cdot (1 \pm \delta) \text{ if } M = \Omega(N/\delta)$$

- Min-wise hash family is a PRG with ***multiplicative*** error!
- **Problem:** additive → multiplicative, $\varepsilon \to \Theta(NM\varepsilon)$. Even for constant $\delta$, $\varepsilon = \delta/(NM)$ would be polynomially small.
- If we want shorter seed length, like $O(\log N)$, then it is not enough to apply the PRG for combinatorial rectangles directly.

# *Optimal Size && Sub-constant Error*

Our Approach [Chen, Huang, Li ’2025]

## 1st Step – Balls into Bins

- Use constant-wise independence to split $X$ into several tiny blocks s.t. each block only needs a small size family.

## 2nd Step – Recycle Randomness

- Use the Nisan-Zuckerman PRG and a special extractor to recycle randomness between blocks.

## 3rd Step – Domain Reduction

- Use PRGs for combinatorial rectangles and the special extractor to further reduce the error.

$$\Pr_{h\sim\mathcal{H}}[h(y) < \min h(X\backslash y)] = \frac{1\pm\delta}{|X|}$$

$$= \sum_{\theta=1}^{M} \Pr_{h\sim\mathcal{H}}[h(y) = \theta \wedge \min h(X\backslash y) > \theta]$$

# Two Level Hash Structure

$$h(x) = \sigma_{\varphi(x)}(x)$$

- **Balls into Bins:** small max-load, concentration.

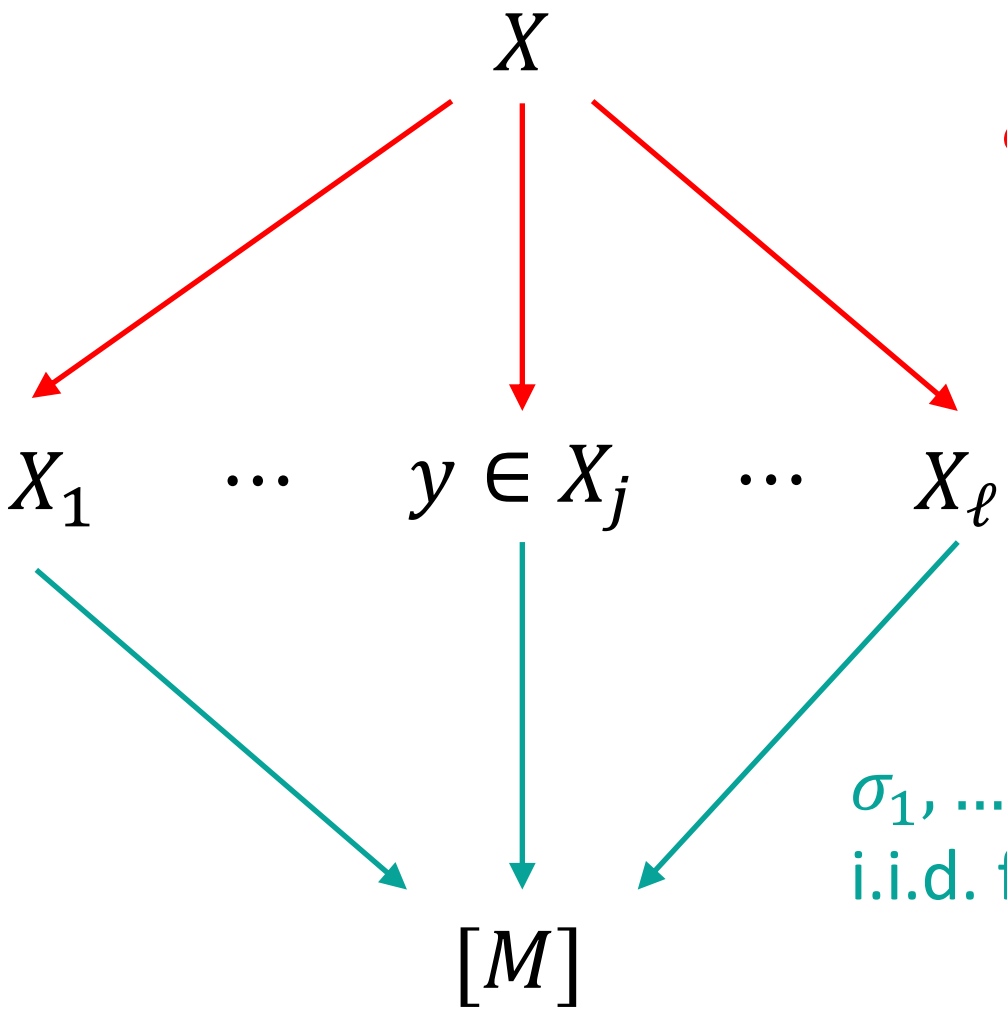
$$|X_i| \approx |X|/\ell, \forall i \in [\ell]$$

$$\max |X_i| \text{ is small}$$

- **Each Block:** Since $|X_i|$ is small, there exists $\mathcal{G} = \{\sigma: [N] \to [M]\}$ with $|\mathcal{G}| = \text{poly}(N)$, s.t.

$$\Pr_{\sigma_i \sim \mathcal{G}}[\min \sigma_i(X_i) > \theta]$$

$$\approx \Pr_{h \sim U}[\min h(X_i) > \theta]$$




$\varphi: [N] \to [\ell]$

Constant-wise independence

$\sigma_1, \ldots, \sigma_\ell: [N] \to [M]$

i.i.d. from $\mathcal{G}$

# Split into Several Small Blocks

$$h(x) = \sigma_{\varphi(x)}(x)$$

$$\Pr_{h\sim\mathcal{H}}[h(y) = \theta \wedge \min h(X\backslash y) > \theta]$$

$$\| \qquad \varphi\colon [N] \to [\ell]$$

$$\Pr_{\sigma_1\sim\mathcal{G}}[\min \sigma_1(X_1) > \theta] \cdots \Pr_{\sigma_j\sim\mathcal{G}}\left[\sigma_j(y) = \theta \wedge \min \sigma_j\left(X_j\backslash y\right) > \theta\right] \cdots \Pr_{\sigma_\ell\sim\mathcal{G}}[\min \sigma_\ell(X_\ell) > \theta]$$

$$\approx$$

$$\Pr_{\sigma_i\sim\mathcal{G}}[\min \sigma_i(X_i) > \theta] \approx \Pr_{h\sim U}[\min h(X_i) > \theta]$$

! Require many random bits.

$\sigma_1, \ldots, \sigma_\ell\colon [N] \to [M]$ i.i.d. from $\mathcal{G}$

$$\Pr_{h\sim U}[\min h(X_1) > \theta] \cdots \Pr_{h\sim U}\left[h(y) = \theta \wedge \min h\left(X_j\backslash y\right) > \theta\right] \cdots \Pr_{h\sim U}[\min h(X_\ell) > \theta]$$

$$= \Pr_{h\sim U}[h(y) = \theta \wedge \min h(X\backslash y) > \theta]$$

# Read-once Branching Program

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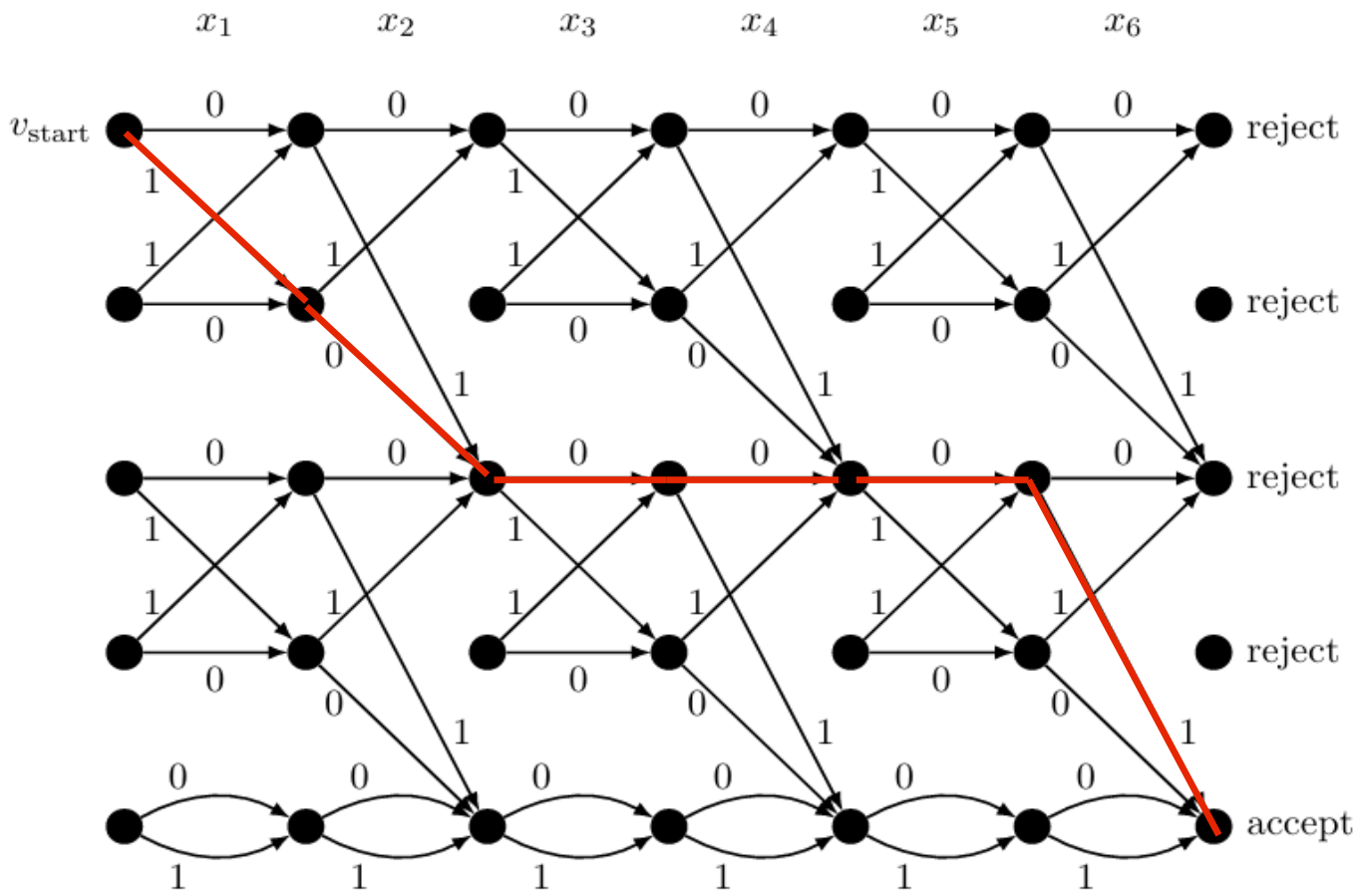
Alphabet: $\{0,1\}$

$(x_1, x_2, x_3, x_4, x_5, x_6) = (1,0,0,0,0,1)$
$\mathrm{MAJ}(x_1 \oplus x_2, x_3 \oplus x_4, x_5 \oplus x_6) = 1$

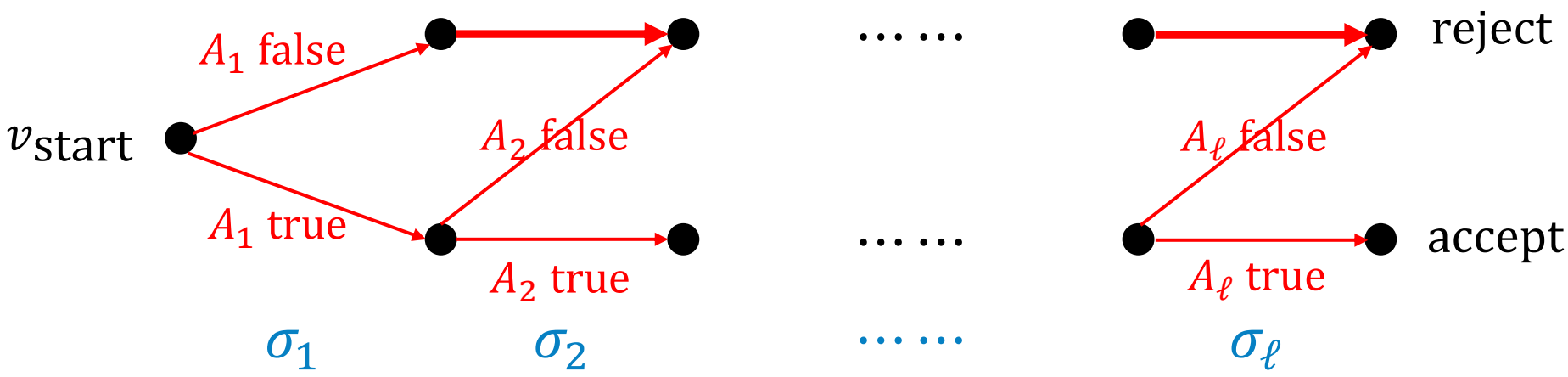
$\mathrm{MAJ}(x_1 \oplus x_2, x_3 \oplus x_4, x_5 \oplus x_6)$



# Recycle Randomness

$$\underbrace{(\min \sigma_1(X_1) > \theta)}_{\text{Event } A_1} \wedge \cdots \wedge \underbrace{\left(\sigma_j(y) = \theta \wedge \min \sigma_j\left(X_j \backslash y\right) > \theta\right)}_{\text{Event } A_j} \wedge \cdots \wedge \underbrace{(\min \sigma_\ell(X_\ell) > \theta)}_{\text{Event } A_\ell}$$

$\sigma_1, \ldots, \sigma_\ell$ i.i.d. from $\mathcal{G}$

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Alphabet: $\mathcal{G}$

# Recycle Randomness

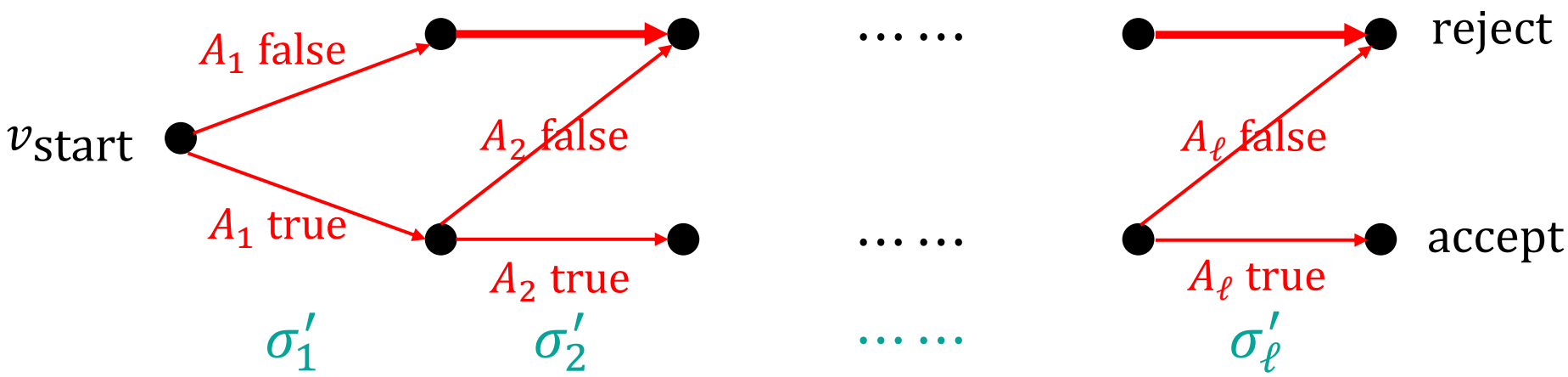
$$\underbrace{(\min \sigma_1(X_1) > \theta)}_{\text{Event } A_1} \wedge \cdots \wedge \underbrace{\left(\sigma_j(y) = \theta \wedge \min \sigma_j\left(X_j \backslash y\right) > \theta\right)}_{\text{Event } A_j} \wedge \cdots \wedge \underbrace{(\min \sigma_\ell(X_\ell) > \theta)}_{\text{Event } A_\ell}$$

$\sigma_1, \ldots, \sigma_\ell$ i.i.d. from $\mathcal{G}$

≈

$$(\min \sigma_1'(X_1) > \theta) \wedge \cdots \wedge \left(\sigma_j'(y) = \theta \wedge \min \sigma_j'\left(X_j \backslash y\right) > \theta\right) \wedge \cdots \wedge (\min \sigma_\ell'(X_\ell) > \theta)$$

$\sigma_1', \ldots, \sigma_\ell'$ are correlated

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# Nisan-Zuckerman PRG

**Definition (Extractor)**

- $\mathrm{Ext}: \{0,1\}^p \times \{0,1\}^d \to \{0,1\}^q$ is a $(k, \varepsilon)$-extractor, if for any random source $X$ over $\{0,1\}^p$ with min-entropy $H_\infty(X) \geq k$, it holds that $\mathrm{Ext}(X, U_d)$ is $\varepsilon$-close to $U_q$.

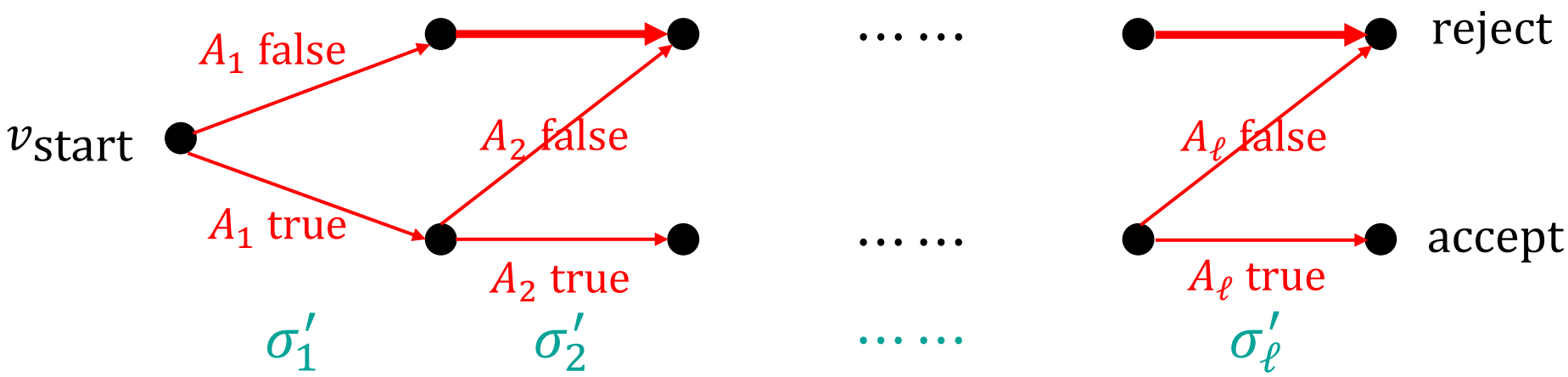
- Nisan-Zuckerman PRG:

$$\mathrm{NZPRG}(w, s_1, \dots, s_\ell) = \big(\mathrm{Ext}(w, s_1), \dots, \mathrm{Ext}(w, s_\ell)\big) \in (\{0,1\}^q)^\ell,$$
$$w \sim U_p, s_1, \dots, s_\ell \text{ i.i.d. from } U_d.$$

# Nisan-Zuckerman PRG

$$\underbrace{(\min \sigma_1'(X_1) > \theta)}_{\text{Event } A_1} \wedge \cdots \wedge \underbrace{\left(\sigma_j'(y) = \theta \wedge \min \sigma_j'\left(X_j \backslash y\right) > \theta\right)}_{\text{Event } A_j} \wedge \cdots \wedge \underbrace{(\min \sigma_\ell'(X_\ell) > \theta)}_{\text{Event } A_\ell}$$

$$\text{Ext}: \{0,1\}^p \times \{0,1\}^d \to \mathcal{G}$$
$$\sigma_1' = \text{Ext}(w, s_1), \ldots, \sigma_\ell' = \text{Ext}(w, s_\ell)$$
$$w \sim U_p, s_1, \ldots, s_\ell \text{ i.i.d. from } U_d$$

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# Analysis

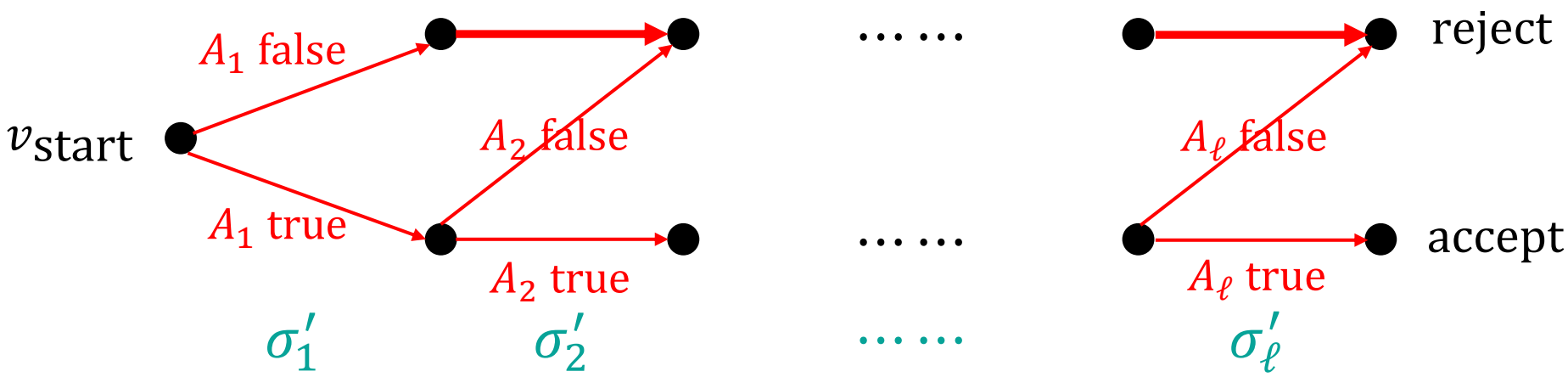
$$P_1[\mathrm{NZ}]$$
$$\approx$$
$$(P_1[U] \pm \mathrm{ExtErr})$$

$$P_1[\mathrm{NZ}] = P_1[\mathcal{G}] \pm \mathrm{ExtErr}$$
$$\approx P_1[U] \pm \mathrm{ExtErr}$$

$$H_\infty(w = U_p) \geq k$$
$$\Rightarrow \mathrm{Ext}(w, s_1) \approx_{\mathrm{ExtErr}} \mathcal{G}$$

$$\mathrm{Ext}: \{0,1\}^p \times \{0,1\}^d \to \mathcal{G}$$
$$\sigma'_1 = \mathrm{Ext}(w, s_1), \ldots, \sigma'_\ell = \mathrm{Ext}(w, s_\ell)$$
$$w \sim U_p, s_1, \ldots, s_\ell \text{ i.i.d. from } U_d$$

$$P_i[\mathcal{G}] \approx P_i[U]$$
$$P_i[\mathrm{NZ}] := \Pr_{\sigma'_i:\mathrm{NZPRG}}[A_i]$$
$$P_i[\mathcal{G}] = \Pr_{\sigma_i \sim \mathcal{G}}[A_i]$$
$$P_i[U] := \Pr_{h \sim U}[A_i]$$

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Length: $\ell$
Alphabet: $\mathcal{G}$

# Analysis

$$P_1[\mathrm{NZ}] \cdot P_2[\mathrm{NZ}]$$

$$\approx$$

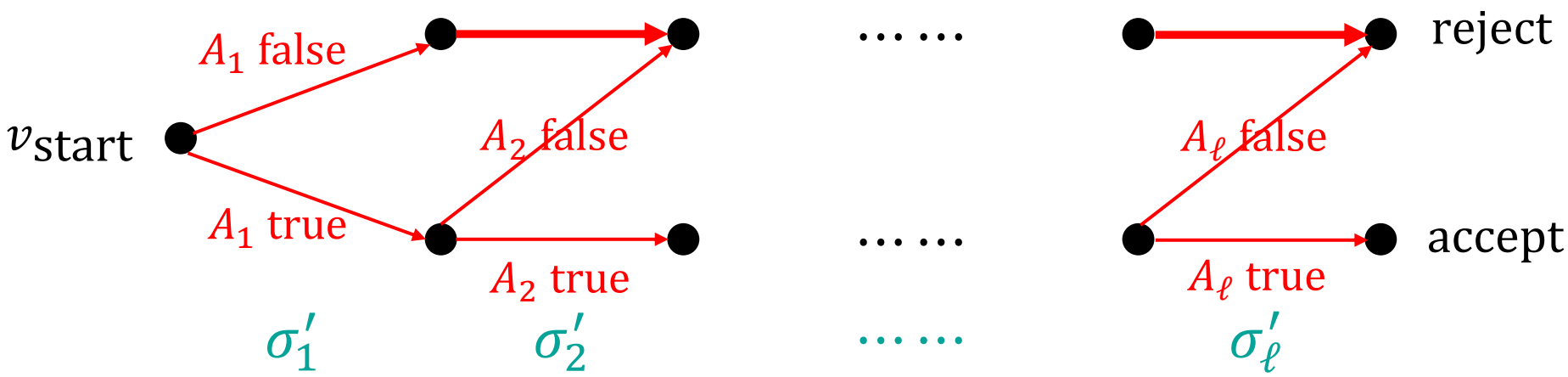
$$(P_1[U] \pm \mathrm{ExtErr}) \cdot (P_2[U] \pm \mathrm{ExtErr})$$

$$H_\infty(w \mid A_1) \geq k \Rightarrow \mathrm{Ext}(w \mid A_1, s_2) \approx_{\mathrm{ExtErr}} \mathcal{G}$$

$$\mathrm{Pr}[A_1 A_2] = \mathrm{Pr}[A_1] \cdot \mathrm{Pr}[A_2 \mid A_1]$$

$$\mathrm{Ext}: \{0,1\}^p \times \{0,1\}^d \to \mathcal{G}$$
$$\sigma'_1 = \mathrm{Ext}(w, s_1), \ldots, \sigma'_\ell = \mathrm{Ext}(w, s_\ell)$$
$$w \sim U_p, s_1, \ldots, s_\ell \text{ i.i.d. from } U_d$$

$$P_i[\mathcal{G}] \approx P_i[U]$$
$$P_i[\mathrm{NZ}] \coloneqq \Pr_{\sigma'_i : \mathrm{NZPRG}}[A_i]$$
$$P_i[\mathcal{G}] = \Pr_{\sigma_i \sim \mathcal{G}}[A_i]$$
$$P_i[U] \coloneqq \Pr_{h \sim U}[A_i]$$

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# Analysis

$$P_1[\mathrm{NZ}] \cdot P_2[\mathrm{NZ}] \cdots P_\ell[\mathrm{NZ}]$$

$$\approx$$

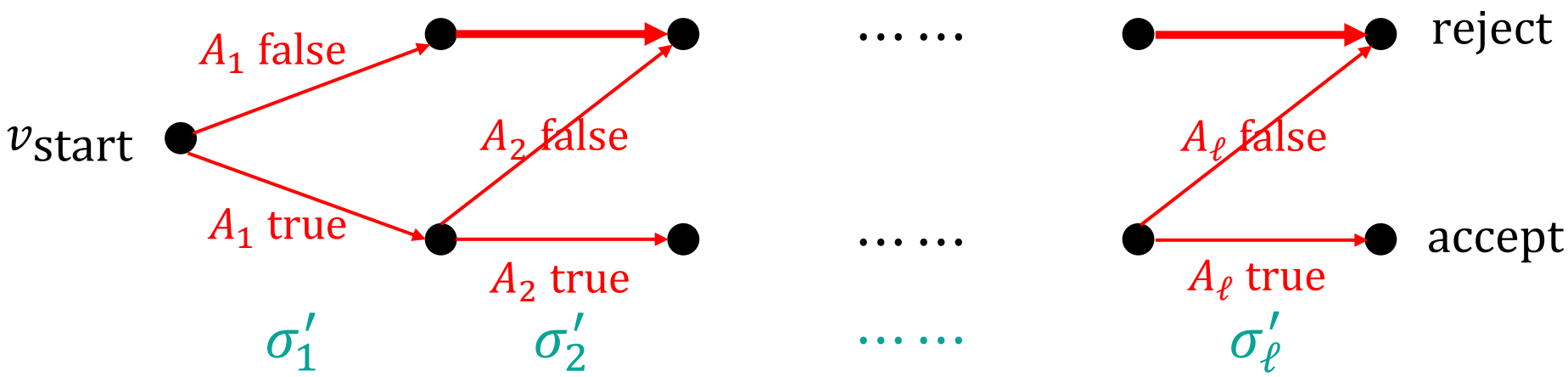
$$(P_1[U] \pm \mathrm{ExtErr}) \cdot (P_2[U] \pm \mathrm{ExtErr}) \cdots (P_\ell[U] \pm \mathrm{ExtErr})$$

$$\mathrm{Ext}: \{0,1\}^p \times \{0,1\}^d \to \mathcal{G}$$
$$\sigma'_1 = \mathrm{Ext}(w, s_1), \ldots, \sigma'_\ell = \mathrm{Ext}(w, s_\ell)$$
$$w \sim U_p, s_1, \ldots, s_\ell \text{ i.i.d. from } U_d$$

$$H_\infty(w \mid A_1 \cdots A_{\ell-1}) \geq k$$
$$\Rightarrow \mathrm{Ext}(w \mid A_1 \cdots A_{\ell-1}, s_\ell) \approx_{\mathrm{ExtErr}} \mathcal{G}$$

$$P_i[\mathcal{G}] \approx P_i[U]$$
$$P_i[\mathrm{NZ}] \coloneqq \Pr_{\sigma'_i:\mathrm{NZPRG}}[A_i]$$
$$P_i[\mathcal{G}] = \Pr_{\sigma_i \sim \mathcal{G}}[A_i]$$
$$P_i[U] \coloneqq \Pr_{h \sim U}[A_i]$$

$$\Pr[A_1 \cdots A_\ell] = \Pr[A_1 \cdots A_{\ell-1}] \cdot \Pr[A_\ell \mid A_1 \cdots A_{\ell-1}]$$

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# One bin is Sensitive to Multiplicative Error

$$h(x) = \sigma'_{\varphi(x)}(x)$$

$$\Pr_{h\sim\mathcal{H}}[h(y) = \theta \wedge \min h(X\backslash y) > \theta]$$

$$\| \qquad\qquad \varphi\colon [N] \to [\ell]$$

$$P_1[\mathrm{NZ}] \cdots P_j[\mathrm{NZ}] \cdots P_\ell[\mathrm{NZ}]$$

$$\approx \qquad\qquad \sigma'_1, \ldots, \sigma'_\ell\colon [N] \to [M] \text{ from NZPRG, ExtErr} = N^{-o(1)}$$

$P_i[\mathcal{G}] \approx P_i[U]$

$P_i[\mathrm{NZ}] = \Pr_{\sigma'_i:\mathrm{NZPRG}}[A_i]$

$P_i[\mathcal{G}] \coloneqq \Pr_{\sigma_i\sim\mathcal{G}}[A_i]$

$P_i[U] \coloneqq \Pr_{h\sim U}[A_i]$

? We hope that this special bin does not need to pay for the error from extractor.

$$(P_1[U] \pm \mathrm{ExtErr}) \cdots (P_j[U] \pm \mathrm{ExtErr}) \cdots (P_\ell[U] \pm \mathrm{ExtErr})$$

$$P_j[U] = \Pr_{h\sim U}[h(y) = \theta \wedge \min h(X_j\backslash y) > \theta] \le 1/M \implies P_j[U] \pm \mathrm{ExtErr} = P_j[U] \cdot (1 \pm M \cdot \mathrm{ExtErr})$$

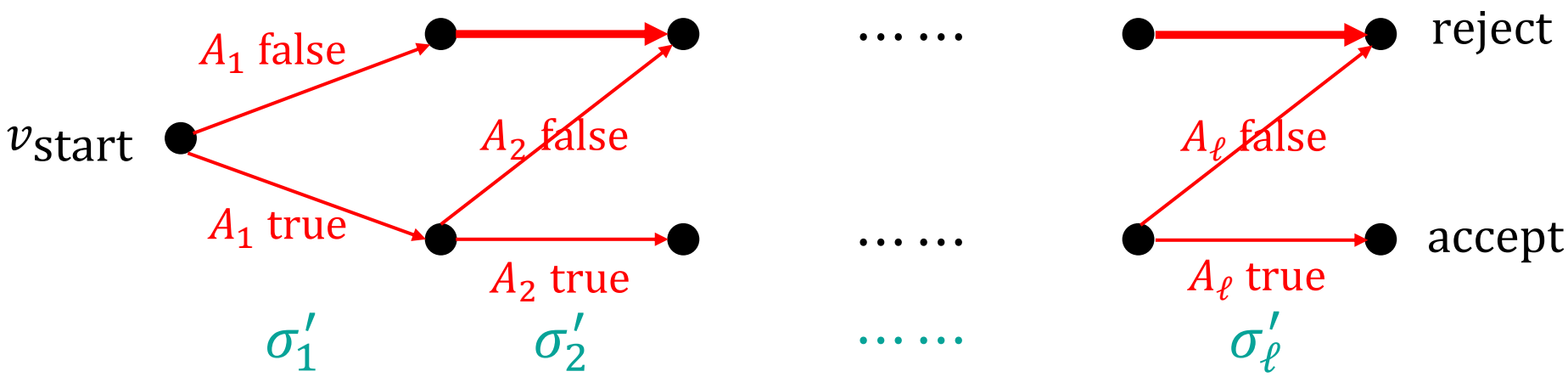
! $P_j[U]$ is very small, and it makes this bin very sensitive to multiplicative error.

# Change the Order of Inputs

$$\underbrace{(\min \sigma_1'(X_1) > \theta)}_{\text{Event } A_1} \wedge \cdots \wedge \underbrace{\left(\sigma_j'(y) = \theta \wedge \min \sigma_j'\left(X_j \backslash y\right) > \theta\right)}_{\text{Event } A_j} \wedge \cdots \wedge \underbrace{(\min \sigma_\ell'(X_\ell) > \theta)}_{\text{Event } A_\ell}$$

$$\text{Ext}: \{0,1\}^p \times \{0,1\}^d \to \mathcal{G}$$
$$\sigma_1' = \text{Ext}(w, s_1), \ldots, \sigma_\ell' = \text{Ext}(w, s_\ell)$$
$$w \sim U_p, s_1, \ldots, s_\ell \text{ i.i.d. from } U_d$$

★ $s_1, \ldots s_\ell$ (or correspondingly $\sigma_1', \ldots, \sigma_\ell'$) are symmetric.

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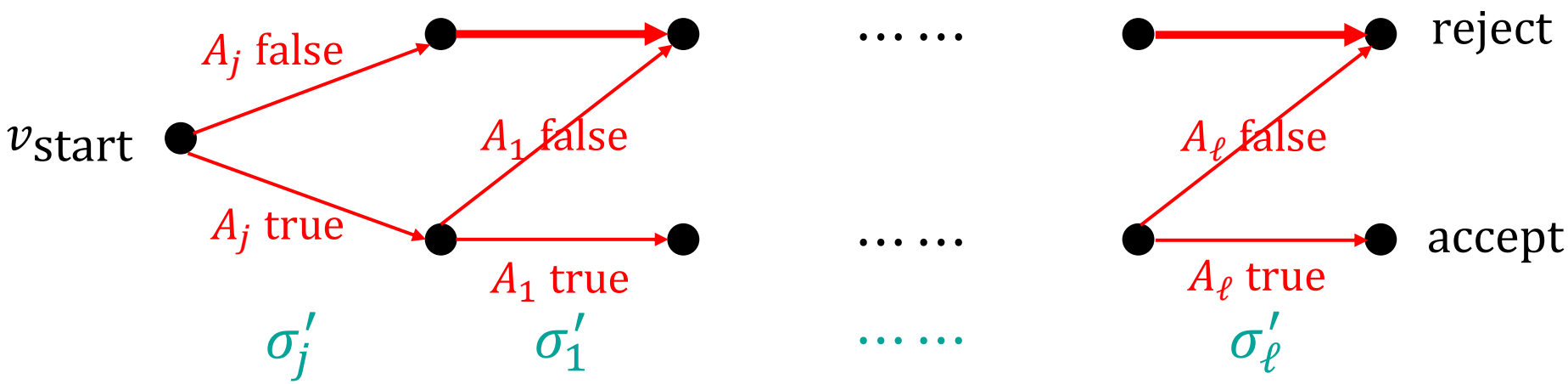
# Change the Order of Inputs

$$\underbrace{\left(\sigma_j'(y) = \theta \wedge \min \sigma_j'\left(X_j \backslash y\right) > \theta\right)}_{\text{Event } A_j} \wedge \underbrace{\left(\min \sigma_1'(X_1) > \theta\right)}_{\text{Event } A_1} \wedge \cdots \wedge \underbrace{\left(\min \sigma_\ell'(X_\ell) > \theta\right)}_{\text{Event } A_\ell}$$

$$\text{Ext}: \{0,1\}^p \times \{0,1\}^d \to \mathcal{G}$$
$$\sigma_j' = \text{Ext}(w, s_j), \sigma_1' = \text{Ext}(w, s_1), \ldots, \sigma_\ell' = \text{Ext}(w, s_\ell)$$
$$w \sim U_p, s_j, s_1, \ldots, s_\ell \text{ i.i.d. from } U_d$$

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Alphabet: $\mathcal{G}$

# Change the Order of Inputs

$$P_j[\mathrm{NZ}] \cdot P_1[\mathrm{NZ}] \cdots P_\ell[\mathrm{NZ}]$$

$$\approx$$

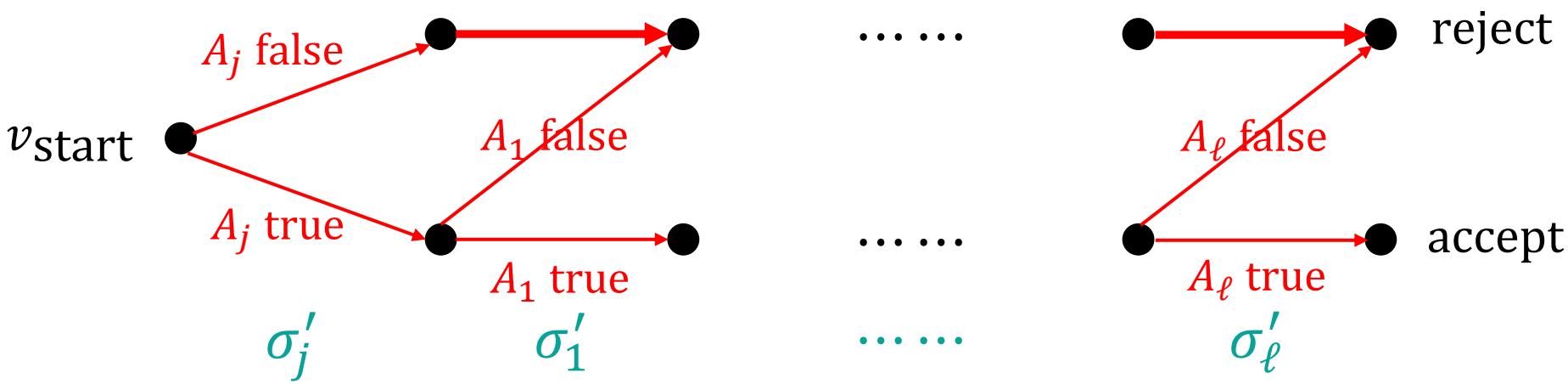
$$(P_j[U] \pm \mathrm{ExtErr}) \cdot (P_1[U] \pm \mathrm{ExtErr}) \cdots (P_\ell[U] \pm \mathrm{ExtErr})$$

$$\mathrm{Ext}: \{0,1\}^p \times \{0,1\}^d \to \mathcal{G}$$
$$\sigma'_j = \mathrm{Ext}(w, s_j), \sigma'_1 = \mathrm{Ext}(w, s_1), \dots, \sigma'_\ell = \mathrm{Ext}(w, s_\ell)$$
$$w \sim U_p, s_j, s_1, \dots, s_\ell \text{ i.i.d. from } U_d$$

$H_\infty(w = U_p) = p \geq k$
$\Rightarrow \mathrm{Ext}(w, s_j) \approx_{\mathrm{ExtErr}} \mathcal{G}$

★ $w$ has no entropy loss at the beginning.
? We expect strong properties for $\mathrm{Ext}(w, s_j)$ than other $\mathrm{Ext}(w, s_i)$.

$$P_i[\mathcal{G}] \approx P_i[U]$$
$$P_i[\mathrm{NZ}] \coloneqq \Pr_{\sigma'_i:\mathrm{NZPRG}}[A_i]$$
$$P_i[\mathcal{G}] = \Pr_{\sigma_i \sim \mathcal{G}}[A_i]$$
$$P_i[U] \coloneqq \Pr_{h \sim U}[A_i]$$




Width: 2
Length: $\ell$
Alphabet: $\mathcal{G}$

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$\sigma'_j = \mathrm{Ext}(w, s_j), \sigma'_1 = \mathrm{Ext}(w, s_1), \ldots, \sigma'_\ell = \mathrm{Ext}(w, s_\ell)$

$w \sim U_p, s_j, s_1, \ldots, s_\ell$ i.i.d. from $U_d$

$H_\infty(w = U_p) = p \geq k$

$\Rightarrow \mathrm{Ext}(w, s_j) \approx_{\mathrm{ExtErr}} \mathcal{G}$

$\mathrm{Ext}(w, s_j) = \mathcal{G}$

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$P_i[\mathcal{G}] \approx P_i[U]$

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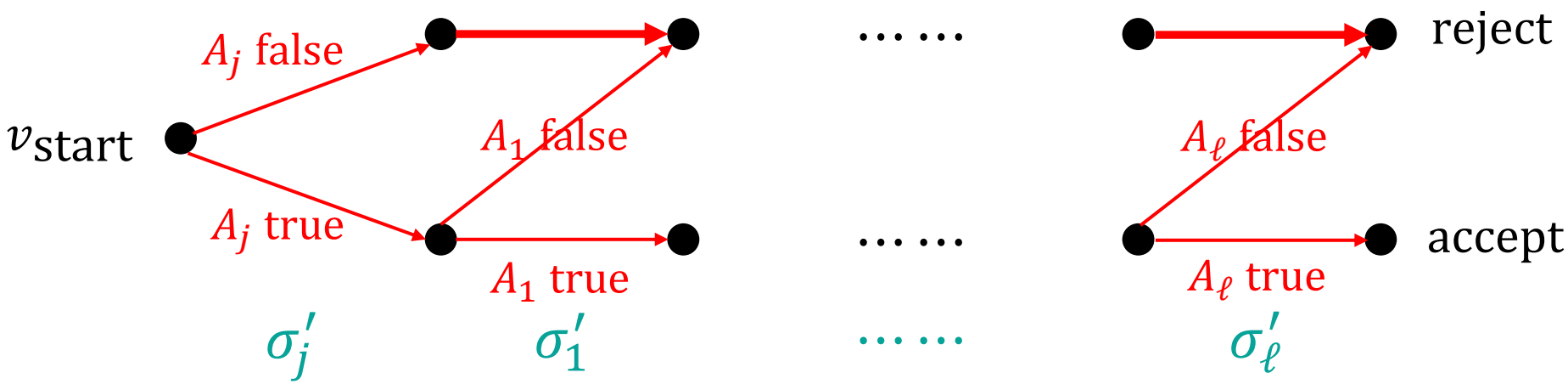


Width: 2

Length: $\ell$

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# Special Extractor

**Lemma**

- Given any $p$ and $k < p$, for any error $\varepsilon$, there exists an explicit $(k, \varepsilon)$-extractor $\mathrm{Ext}: \{0,1\}^p \times \{0,1\}^d \to \{0,1\}^q$ with $q = k/2$ and $d = O(\log(p/\varepsilon))$. And $\mathrm{Ext}$ satisfies an extra property: $\mathrm{Ext}(U_p, s) = U_q$ for any fixed seed $s$.

$$\sigma_j' = \mathrm{Ext}(w, s_j) = \mathrm{Ext}(U_p, U_d) \approx \mathcal{G} \Longrightarrow X_j \text{ has no error from extractor}$$

$$P_j[\mathrm{NZ}] \cdot P_1[\mathrm{NZ}] \cdots P_\ell[\mathrm{NZ}]$$

$$\approx$$

$$\boldsymbol{P_j[U]} \cdot (P_1[U] \pm \mathrm{ExtErr}) \cdots (P_\ell[U] \pm \mathrm{ExtErr})$$

# Domain Reduction

$$\mathrm{NZPRG}(w, s_1, \dots, s_\ell) = \big(\mathrm{Ext}(w, s_1), \dots, \mathrm{Ext}(w, s_\ell)\big)$$

- Use a PRG for $\left(\{0,1\}^d\right)^\ell$-combinatorial rectangles to generate $s_1, \dots, s_\ell$, instead of i.i.d. sampling.
- Also need the special property of our extractor.
- Help us to further reduce the error from $1/\log^{O(1)} N$ to $2^{-O\left(\frac{\log N}{\log\log N}\right)}$.

# Summary

**Main Theorem 1 (Optimal Size && Sub-constant Error, Min-wise Hash)**

- There exists an explicit min-wise hash family with seed length $O(\log N)$ and error $2^{-O\left(\frac{\log N}{\log\log N}\right)}$.

**Main Theorem 2 (Optimal Size && Sub-constant Error, $k$-min-wise Hash)**

- There exists an explicit $k$-min-wise hash family with seed length $O(k \log N)$ and error $2^{-O\left(\frac{\log N}{\log\log N}\right)}$, for any $k = \log^{O(1)} N$.

**Main Technical Lemma (Extractor with Extra Properties)**

- Given any $p$ and $k < p$, for any error $\varepsilon$, there exists an explicit $(k, \varepsilon)$-extractor $\mathrm{Ext}: \{0,1\}^p \times \{0,1\}^d \to \{0,1\}^q$ with $q = k/2$ and $d = O(\log(p/\varepsilon))$. Moreover, $\mathrm{Ext}$ satisfies an extra property: $\mathrm{Ext}(U_p, s) = U_q$ for any fixed seed $s$.

# Open Problems

Smaller error with optimal seed length?

Extending the result for larger $k$ (like $\sqrt{N}$) on $k$-min-wise hash?

Faster evaluation time with optimal seed length?

Lower bound on $t$-wise independence for $k$-min-wise hash?

More applications of the powerful Nisan-Zuckerman framework?

Thank you for listening!