10

Utility and decision making

Expected value: a paradox

SUPPOSE that a casino offers to keep flipping a coin until it comes up heads for the first time. If it comes up heads on the 1st flip, you win $2. If it comes up heads for the first on the 2nd flip, you win $4. On the 3rd flip, $8. Indeed, if the coin initially comes up tails $k-1$ times in a row before coming up heads on the kth flip, you will win $\$2^k$.

If you play the game, you're guaranteed to win at least a little bit of money. And if you're lucky, you might win a bundle. For example, if the coin comes up tails 9 times in a row before finally coming up heads on flip $k = 10$—unlikely, but not impossible—you'll walk away with $2^{10} = \$1024$. The question is: how much money should you rationally be willing to pay to enter the game?

If this is the first time you've encountered this hypothetical, the naïve answer may surprise you. Let's follow the reasoning of Swiss mathematician Gabriel Cramer, in his letter of 28 May 1728 written to Nicolas Bernoulli, the man who had originally posed the question:

> I know not if I deceive myself, but I believe to hold the solution of the singular case that you have proposed to Mr. de Montmort in your letter of 9 September. . . . In order to render the case more simple I will suppose that A throw in the air a piece of money, B undertakes to give him a coin, if the side of the Heads falls on the first toss, 2, if it is only the second, 4, if it is the 3rd toss, 8, if it is the 4th toss, etc. The paradox consists in this: that the calculation gives for the equivalent that A must give to B an infinite sum, which would seem absurd, since no person of good sense, would wish to give 20 coins. [1]

[1] *Correspondence of Nicolas Bernoulli concerning the St. Petersburg Game*, available at `www.cs.xu.edu/math/Sources/Montmort/stpetersburg.pdf`.

An infinite sum? To understand Cramer's point, let's compute the expected value of the game. If X is the random variable representing your potential winnings, then

$$\begin{aligned} E(X) &= \sum_{k=1}^{\infty} x_k \cdot p(X = x_k) \\ &= \sum_{k=1}^{\infty} 2^k \cdot (1/2)^k \\ &= 1(1) + 2(1/2) + 4(1/4) + \ldots \\ &= 1 + 1 + 1 + \ldots \\ &= \infty \end{aligned}$$

The expected value of the game is indeed infinite. If you are trying to maximize your expected value, you should play the game no matter how much the casino charges. Very large wins may happen with very small probability, but their potentially enormous amount compensates you for the risk. Yet as a practical matter, very few people express a desire to pay even $10 or $20 enter into the game, even though it would seem to be to their advantage to do so.

Is this behavior irrational? Not at all. Here's Cramer again, in the same letter to Bernoulli (emphasis added):

> One asks the reason for the difference between the mathematical calculation and the vulgar estimate. I believe that it comes from this: *that the mathematicians estimate money in proportion to its quantity, and men of good sense in proportion to the usage that they may make of it.* This which renders the mathematical expectation infinite, this is the prodigious sum that I am able to receive, if Heads falls only very late, the 100th or 1000th toss. Now this sum, if I reason as a sensible man, is not more for me, does not make for more pleasure for me, does not engage me more to accept the game, than if it would be only 10 or 20 million coins. [2]

[2] *ibid.*

In other words, even if the potential rewards (2, 4, 8, 16, 32 ...) don't reach any ceiling in terms of face value, they do in terms of practical value. This simple fact makes the game much less attractive.

Expected utility

CRAMER'S ANSWER is, essentially, the modern-day economist's answer: *sensible people don't keep score with money.* Rather, they keep score according to a more private, idiosyncratic scale of value. This scale need not match up in any linear way with actual money or material goods. It might also be characterized by something like a law of diminishing returns. Having $20 million might give you twice as big a bank account as having $10 million, but would it mean twice as much happiness or pleasure? For most people, the answer is no. (And if you answered yes: would $20 billion give you twice as much pleasure as $10 billion?)

This private scale of relative satisfaction is called *utility*. The hypothetical situation discussed by Bernoulli and Cramer came to be known as the *St. Petersburg paradox*,[3] and it is the origin of modern utility theory. The game is considered a paradox in the sense that naïvely attempting to maximize expected value leads to a silly recommendation: that you should be willing to pay any finite sum of money to play, however enormous—a willingness that no one exhibits in real life.

[3] So named after the 1738 presentation of the problem by Daniel Bernoulli (Nicolas's cousin) in a collection of scientific and mathematical essays called *Commentaries of the Imperial Academy of Science of Saint Petersburg*.

The paradox can be made to disappear by invoking the notion of *expected utility*. Expected utility is a mathematical concept, while a preference is a psychological concept. It therefore it takes some sort of justification if we are going to use the former to describe the latter. We now sketch the outlines of one such justification.

Subjective utilities

A utility is a real number that represents the value or desireability for all possible outcomes in life. Your utilities are defined in terms of your preferences between different outcomes. We'll use the notation "$P \succ Q$" to express mathematically the idea that you'd rather take option P than Q, regardless of whether P and Q are sure things or lotteries. The binary relation "$\succ$" is called your *preference relation*.

Utilities, like probabilities, are subjective. The idea is that utilities map your preferences for different outcomes to numbers, where a higher number means a more preferred outcome. That is, if you prefer option P to option Q, then your own utilities would hold that $u(P) > u(Q)$. The idea that we can assign real numbers representing the desireability of any possible outcome in life—money, love, chocolate, coming home to a snuggly kitten, *anything*—initially strikes most people as crazy. Presumably Paul McCartney would have agreed:

> I'll buy you a diamond ring my friend
> If it makes you feel all right
> I'll get you anything my friend

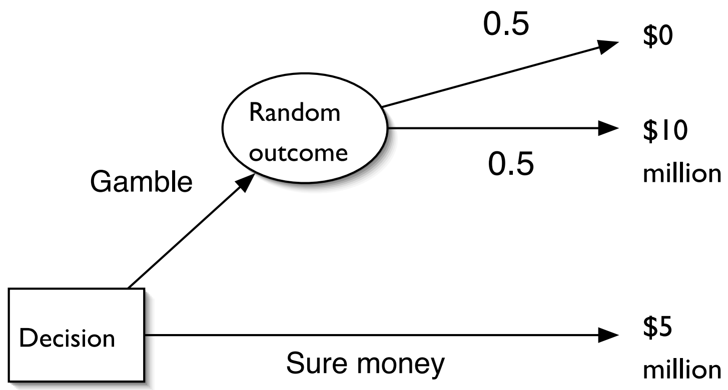


Figure 10.1: In a decision tree, all random outcomes are depicted with an oval, with the possible outcomes and their probabilities attached. These are called *stochastic nodes.* All points where you must make a decision among different courses of action are marked with rectangles. These are called *decision nodes*.

> If it makes you feel all right
> 'Cause I don't care too much for money
> For money can't buy me love.[4]

[4] "Can't Buy Me Love," composed by Paul McCartney and released by the Beatles as a single in 1964.

But it turns out that we can put this seemingly crazy idea on a firm logical foundation by assuming that your preferences follow a few simple rules. To get there, we first need a bit of vocabulary and notation.

First, we'll define a *simple lottery* as a probability distribution P with just two possible outcomes: A (which happens with probability w) and B (probability $1 - w$). For example, Figure 10.1 shows a simple *decision tree*, in which a sequence of decisions (and their probable consequences) are depicted as branches on a tree. This particular tree depicts a choice between a sure thing ($5 million) and a coin flip: heads you win $10 million, tails you win nothing. The top branch involving the coin flip is a simple lottery.

We'll define the *expected utility* of a simple lottery as the weighted average of the utilities for the possible outcomes: $E\{u(P)\} = w \cdot u(A) + (1 - w) \cdot u(B)$. In the decision tree above, the branch leading to a coin flip between $A = \$10$ million and $B = \$0$ is a simple lottery, whose expected utility is $0.5 \cdot u(0) + 0.5 \cdot u(10)$, where 10 is shorthand for $10 million.[5] If you prefer the sure $5 million branch to the coin flip, then we know that your utilities satisfy $u(5) > 0.5u(10) + 0.5u(0)$, in units of $1 million.

[5] Technically, the $5 million branch of the tree is also a simple lottery where one option is $5 million and the other option is 0. It is just an especially simple lottery, since one of these two events occurs with probability 1.

A compound lottery is a lottery with more than one possible outcome $A_1, \ldots, A_n$. We write this as $P = w_1 A_1 + \cdots w_n A_n$, where the probabilities $w_1, \ldots, w_n$ sum to 1. For a compound lottery, the definition of expected utility is extended in the obvious way:

$$E\{u(P)\} = \sum_{i=1}^{n} w_i u(A_i) \, .$$

Any compound lottery can be reduced recursively to a tree of simple lotteries. For example, the compound lottery $P = w_1 A_1 + w_2 A_2 + w_3 A_3$ is equivalent to the simple lottery

$$P = w_1 A_1 + (1 - w_1) Q \, ,$$

where Q is itself a simple lottery involving A_2 and A_3.

The von Neumann–Morgenstern theorem of expected utility

The principle of expected utility is simple: when confronted by a choice, take the option leading to the higher expected utility.

We'll now try to place this principle on a firm foundation by relating the mathematical idea of expected utility to the psychological idea of a preference for one option versus another. We will start by assuming that your preferences among lotteries obey four simple rules of rationality, which are called the *von Neumann–Morgenstern rules*. What the principle of coherence is to probability theory, these rules are to utility theory: some very simple, common-sense constraints on your preferences from which many beautiful results can be derived. The rules are:

(1) *Your preferences are complete.* That is, for any two options P and Q, you are able to state whether $P \succ Q$, $Q \succ P$, or $P \equiv Q$ (the third option indicating your indifference between the two.) You're not allowed to say, "I don't know."

(2) *Your preferences are transitive:* if $P \succ Q$ and $Q \succ R$, then $P \succ R$.

(3) *You are willing to "split the difference" between favorable and unfavorable options.* This one is often called the rule of continuity. Suppose there are three options P, and Q, and R, such that P is your favorite choice, R is your least favorite, and Q is somewhere in the middle ($P \succ Q \succ R$). If your preferences satisfy the rule of continuity, then there exists some probability w such that $Q \equiv wP + (1 - w)R$. In other words, there must be some probability w where you indifferent between the middle option Q versus a w-weighted coin flip between P (your favorite) and R (your least favorite).

(4) *Your preference for P or Q remains unchanged in the face of a third option.* Formally, if $P \succ Q$, w is a probability, and R is any third lottery, then

$$wP + (1-w)R \succ wQ + (1-w)R \, .$$

That is, you prefer P to Q, then you also prefer a lottery involving P to the same lottery involving Q.

The significance of these rules is that they make possible the construction of something called a *utility function*, a fact realized in 1944 by John von Neumann and Oskar Morgenstern. In their book entitled *Theory of Games and Economic Behavior*, von Neumann and Morgenstern proved the following lovely result, often called the *expected-utility theorem*:

Theorem 1 *Suppose that your preferences among outcomes satisfy Rules 1–4. Then there exists a utility function $u(A_i)$ that assigns a real number to each possible outcome A_i, and that satisfies two properties:*

1. *For any two options P and Q and any probability w,*

$$u\{wP + (1-w)Q\} = wu(P) + (1-w)u(Q) \, .$$

In words: your utility for a gamble involving P and Q, where w is the probability of getting option P, is equal to the expected utility of the gamble.

2. *For any two options P and Q, $E\{u(P)\} > E\{u(Q)\}$ if and only if $P \succ Q$. In words: if a lottery P has a higher expected utility than lottery Q, then you prefer P to Q.*

A proof of this theorem can be found in just about any textbook on utility theory, including von Neumann and Morgenstern's original work on the topic.

The von Neumann–Morgenstern theorem shows that it is both philosophically meaningful and mathematically precise to describe your preferences using real numbers called utilities, subject to the assumption that your preferences obey certain rules. These utilities are on an arbitrary scale, but can be ordered just like ordinary numbers. It is therefore sensible to compare a single individual's utilities across different outcomes (though it is not necessarily meaningful to compare utilities across different people).

You'll also notice that the theorem makes no assumption about the kind of events involved in the lotteries. They could be monetary or hedonistic, but they need not be. Indeed, they could be personally significant outcomes—like getting married, having the respect of one's peers, going for a swim in the afternoon, or pondering the life of Pierre Bezukhov in *War and Peace* over cup of tea—whose worth defies easy quantification in terms of money. Regardless of the events themselves, the von Neumann–Morgenstern result says that your utilities for these outcomes (yes, even monetary ones) can all be placed on a common scale, as long as your preferences among the events behave the basic rules of rationality (1–4).

Invoking the notion of utility resolves the St. Petersburg paradox: a person's expected utility for the bet can be small, even if the expected monetary value of the bet is infinite. Expressed mathematically, if $u(x)$ denotes a person's utility for having x dollars, having a small expected utility for the St. Petersburg game would mean that

$$E\{u(X)\} = \sum_{k=0}^{\infty} u(x_k) \cdot p(x_k) = c\,,$$

for some small number c, even though

$$E\{X\} = \sum_{k=0}^{\infty} x_k \cdot p(x_k) = \infty\,.$$

This would happen whenever the utility of a reward increases more slowly than the numerical size of a reward itself.

This leads us to the notion of *risk aversion* and a categorization of people according to their risk tolerance. Your utility function $u(x)$ marks you as **risk averse** if $\mathrm{E}\{u(X)\} \leq u\{\mathrm{E}(X)\}$; **risk neutral** if $\mathrm{E}\{u(X)\} = u\{\mathrm{E}(X)\}$; or **risk seeking** if $\mathrm{E}\{u(X)\} \geq u\{\mathrm{E}(X)\}$. (These classifications are subject to the assumption that these inequalities hold for all probabilities and random variables.) In layman's terms: a risk-averse person prefers a certain outcome over a lottery with an equivalent expected value, while a risk-seeking person prefers the opposite. Risk-averse people buy insurance; risk-seeking people go to Vegas.

Risk premium. If you are indifferent between a sure thing P and a lottery Q, then P is your *certainty equivalent* for the uncertain gamble Q. The difference between the expected value of Q and its certainty equivalent P is called a *risk premium*.

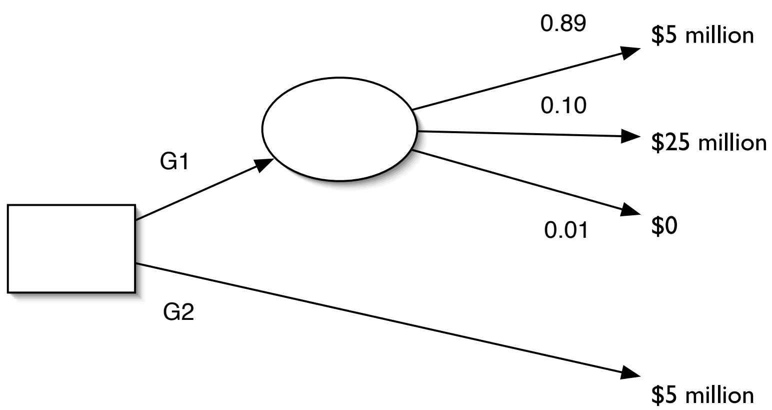
Paying a risk premium for something is like buying insurance. Let's take an example familiar to you if you've purchased liability insurance to protect yourself in case you a car accident. In 2015, the U.S. had about 1.9 million car accidents causing injury, for an average rate of about $1.9/320 \approx 0.006$ accidents per person. If we assume that each accident involved two drivers, only one of whom was at fault, then the probability of being liable for a car accident is about 0.003 in any given year for the average driver. Now suppose that the average payout is about \$25,000.[6] In that case, the average driver should expect to be liable for $0.003 \times 25000 = \$75$ worth of car accidents per year. If your actual liability insurance policy costs \$300 per year, then under these assumptions, you are paying a \$225 risk premium—that is, making an "unfavorable" (but to most people, still very sensible) trade to protect yourself against the small probability of a very large loss.

[6] It's very hard to get data on average insurance payouts, so that \$25,000 figure is just an assumption made for the purpose of illustrating the idea of a risk premium.

The Allais paradox

The von Neumann–Morgenstern rules strike most people as eminently sensible requirements of a rational actor. Nonetheless, they have some teeth. Indeed, it is quite common for people's declared preferences to be inconsistent with one or more of the axioms.

One such example is the *Allais paradox*. Take this lottery:



The expected value of G_1 is

$$E(G_1) = 0.89 \cdot 5 + 0.1 \cdot 25 + 0.01 \cdot 0 \approx 6.95$$

in units of \$1 million, which is clearly higher than the expected value of a guaranteed \$5 million. Yet many people prefer G_2, reasoning that \$5 million is a life-changing amount of money, and that the risk of getting nothing—even if only 1%—simply isn't worth the 10% shot at \$25 million. This is perfectly sensible from an expected-utility point of view. Invoking the von Neumann–Morgenstern rules, for a rational agent to choose G_2 over G_1 simply means that her utilies are such that $E\{u(G_2)\} > E\{u(G_1)\}$. In turn, this means that, in multiples of \$1 million,

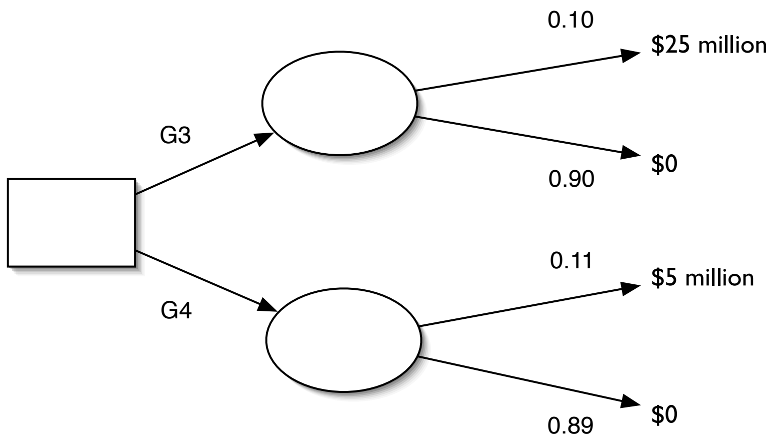
$$u(5) > 0.89u(5) + 0.1u(25) + 0.01u(0)\,.$$

If we assume that $u(0) = 0$, then this is equivalent to having utilities $u(5)$ and $u(25)$ such that

$$u(5) > \left(\frac{0.10}{0.11}\right) u(25) \approx 0.909u(25)\,. \tag{10.1}$$

That is to say: as long as your utility for \$5 million is greater than 90.9% of your utility for \$25 million, then G_2 will maximize your expected utility, and the bottom branch of the tree is the rational choice.

So far so good. But what about this lottery?



Many people state a strong preference for G_3 here, reasoning that the probability of winning G_3 is only slightly smaller, but the reward much larger, than for G_4. This is also perfectly sensible from an expected-utility point of view, as long as

$$0.11u(5) < 0.10u(25)\,,$$

or equivalently, that

$$u(5) < \left(\frac{0.10}{0.11}\right) u(25) \approx 0.909u(25)\,. \tag{10.2}$$

This set of preferences is also fine, on its own. But it flatly contradicts the constraint on $u(25)$ and $u(5)$ implied by the choice $G_2 \succ G_1$, in Equation (10.1).

It is therefore impossible to hold a consistent set of utilities for \$5 million and \$25 million such that $G_2 \succ G_1$ and $G_3 \succ G_4$ are both true. Even though your utilities are subjective, the notion of expected utility requires that either G_2 or G_3 must be the "wrong" choice, no matter what your utilities are. Someone who takes both of these gambles must be violating one of the four rules.

Are real people actually utility maximizers?

No. The Allais paradox is just one of many examples where many people's declared preferences put them at odds with the consistency requirements of formal utility. Simply put: in practice, most people don't behave like utility maximizers at all. The principle of expected utility is a prescription of how rational people ought to behave, not a description of how real people actually do behave. For a wonderful discussion of this, try reading up on the work of Daniel Kahneman and Amos Tversky, who won a Nobel Prize in economics for their work characterizing the cognitive underpinnings—the biases, the fears, the hasty generalizations, the faulty heuristics—for many human "errors" of precisely this type. In particular, Kahnemann's book "Thinking, Fast and Slow" is a popular and highly accessible introduction to the topic.

Decision trees

One of the most basic uses of utility assignments is in solving a decision problem using a decision tree, of which we've seen a few simple examples. In a decision tree, all random outcomes are depicted with an oval, with the possible outcomes and their probabilities attached. These are called *stochastic nodes.* All points where you must make a decision among different courses of action are marked with rectangles. These are called *decision nodes*. The tree will also show the costs and benefits (measured in utility, or often just dollars) as you go down each branch of the tree.[7]

[7] Of course they don't have to be ovals and rectangles; this is just convention.

This is best understood by example. Imagine you live in a house located in a region prone to mud slides—say, coastal California. The question at hand is whether to buy "insurance," in the form of a retaining wall around the house. You make the following estimates:

- Each rainy season there is a 1% chance of a mud slide.
- A mud slide, if it were to occur, would do \$1 million in dam-

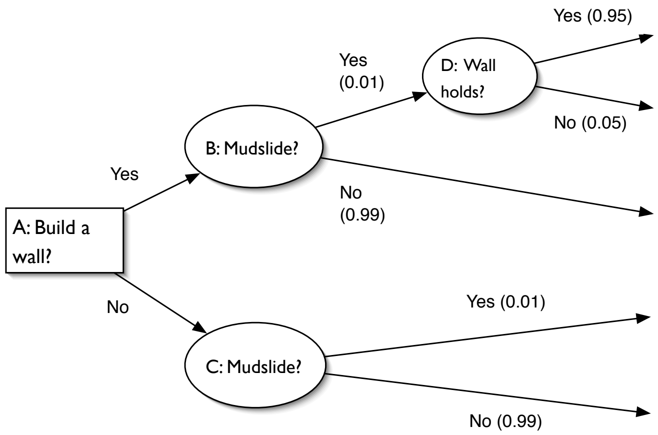


Figure 10.2: Decision tree for the mudslide example.

age to your house.

- You have the option of building a retaining wall that would likely save your house in the event of a devastating mud slide. The wall costs $40,000 to build. If the slide occurs, you believe the wall will hold with a 95% probability.

The corresponding decision tree is shown in Figure 10.2.

There are four basic rules in using a decision tree to solve a problem:

(1) Ensure that the tree includes all relevant decision and stochastic nodes, in temporal order from left to right.

(2) For every stochastic node, compute the expected utility at that node. In many cases, utility is measured in dollars, but in general it need not be.

(3) For every decision node, choose the course of action with the highest expected utility. This value then becomes the expected utility of that node.

(4) Apply rules (2)-(3) from *right to left* (i.e. backwards in time).

Let's follow these steps here, treating expected utility as if it were measured in dollars (i.e. for a risk-neutral person). We've already taken care of step 1: draw a complete tree. For step 2, we must first calculate the gains and losses associated with each terminal node. To do this, simply add up the gains and losses as you proceed down each branch of the tree. Then we must use these outcomes to calculate the expected value of each stochastic node, working from right to left, as shown in Figure 10.3.

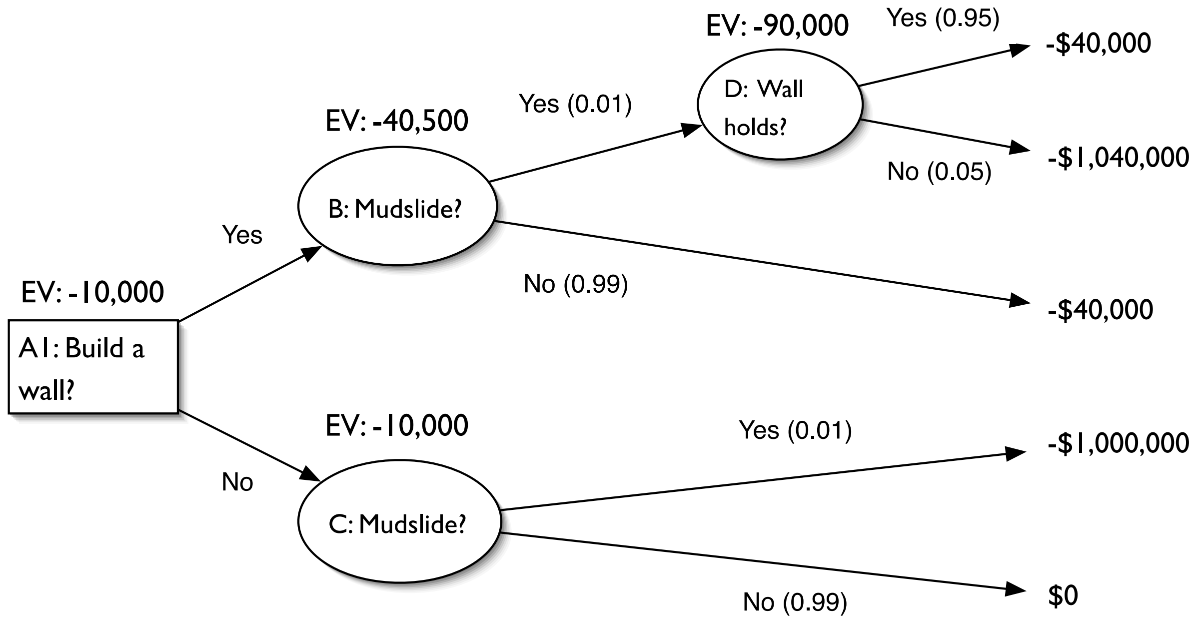


Figure 10.3: Each node has been labeled with its expected utility (here measured in dollars).

To do this, first we worked out the expected value for node D, ignoring the nodes earlier in the tree:

$$E(D) = 0.95 \cdot (-40,000) + 0.05 \cdot (-1,040,000) = -90,000\,.$$

Then we fed this into the calculation for the expected value of node B:

$$E(B) = 0.01 \cdot E(D) + 0.99 \cdot (-40,000) = -40,500\,.$$

And finally, we calculated the expected value of node C:

$$E(C) = 0.90 \cdot (0) + 0.01 \cdot (-1,000,000) = -10,000\,.$$

As for step 3: from a pure expected-value calculation, it looks as though the right choice at node A is not to build a wall. The expected value of this choice is $-\$10,000$, and so this becomes the value associated with decision node A. Since this is the left-most node on the tree, this is the expected value of the entire tree—that is, how much you expect to gain (or, since this is a negative number, lose) from even being in this situation to begin with.

Of course, if you are highly risk averse, you might rationally take the other decision, instead. This would be the right choice if your utilities satisfy

$$0.9995 \cdot u(-4) + 0.0005u(-104) > 0.01u(-100) \tag{10.3}$$

in units of $10,000, in which case the expected utility of the top branch would be higher than that of of the bottom branch. This inequality might easily describe your utilities if you find the prospect of losing your home so unacceptable that you judge it worth $40,000 to reduce the risk of this outcome by a factor of 20—that is,

$$\frac{P(\text{Lose your house} \mid \text{Don't build a wall})}{P(\text{Lose your house} \mid \text{Build a wall})} = \frac{0.01}{0.0005} = 20\,.$$

Indeed, this kind of utility calculation is behind every rational decision to buy insurance, whether in the form of a retaining wall or, more traditionally, by paying an insurance company to cover your losses in the event of a disaster. If you buy insurance in order to reduce your exposure to unacceptably large losses, you agree to pay the insurance company a risk premium—that is, an amount that exceeds the expected value of the loss itself.

What if nobody were willing to pay a premium in the insurance market, and instead were only willing to buy protection if the policy costs were equal to the expected value of the coverage? In that case, the insurance industry would cease to exist. It would be impossible for insurance companies to cover their administrative costs, much less make a profit, since in the long run they would lose at least as much in payouts as they took in on every insurance policy. An insurance industry can only exist in a society whose members are, on average, risk averse, and therefore willing to pay a risk premium.

Suppose that, in the example above, you decide to build the wall because your utilities satisfy Equation (10.3). How much of a risk premium are you paying? This is the difference between the "fair value" (which balances the expected values of the top and bottom branches) and the face value ($40,000) of the wall.

The fair value of the wall is a number x such that

$$0.9995x + 0.0005 \cdot (100 + x) = 0.01 \cdot 100\,,$$

again expressed in units of $10,000. The value of x that satisifes this equation will lead to Nodes B and C having the same expected loss. After some algebra, this becomes

$$x = 0.0095 \cdot 100 = 0.95\,,$$

or \$9500 (check this yourself). If you decide to build the wall at a face value of \$40,000, then you are paying a risk premium of \$30,500—which, at more than three times the fair value, is a very high markup.

Always keep in mind that "fair value" is a subjective judgment. If you believe that the probability of a mudslide is higher than 1%, or that the probability of the wall holding is higher than 95%, then your subjective assessment of the wall's fair value will change.

The value of perfect information

What if you knew an expert—say, a hydrologist or a geologist—who could come to your house, run some tests, and determine whether your property was going to suffer a mudslide or not? If the test is 100% accurate, and there is no possibility of a misdiagnosis, then it is referred to as *perfect information*.

Such information would be valuable to you, and it's clear that you'd accept it if it were free, since it might change your opinion on whether to build a wall. It's equally clear that you wouldn't pay more than \$40,000 for it, since the wall costs that much anyway. But would you pay \$3,000 for it? Remember, you don't know what the test will say, and you have to pay for it before you get the answer.

The key concept here is: information has utility, and *this utility is equal to the expected improvement in utility from possessing the information.* To calculate this, we have to rearrange the decision-tree to reflect the receipt of the new information, and compare the new expected utility with the old. Let T denote a positive test and $\sim T$ a negative test. Since we're assuming that the information is 100% accurate, then a positive test means that your property is guaranteed to suffer a mudslide, while a negative test means that there is no possibility of a mudslide.

You believe that, since there is an overall 1% chance in your area of a mudslide, that there is also a 1% chance that the expert will come back with a positive test for your property. Therefore the tree becomes what we see in Figure 10.4. Remember that, in solving the tree, you should work right to left. At every stochastic node, compute the expected value (or utility). At every decision node, choose the branch that maximizes expected value (or utility). Remember, if you want to compute the fair value of the information, don't include its \$3,000 cost in the expected-value calculations at every node.

Why not include the price of the information in the expected value calculations? The long answer is: you certainly could, and then your expected value might go up or down as a result of having the information *and paying for it*. You could then change the assumed price of the information in your calculations, in an attempt to find the price that made the new expected value precisely equal to the old. This is an equally valid way of proceeding. But it's more direct to assume that you had the information for free, and then to compute how much this would improve your expected value. This improvement is the fair value of the information. If the fair value is higher than the price offered, then the information is clearly a good deal.

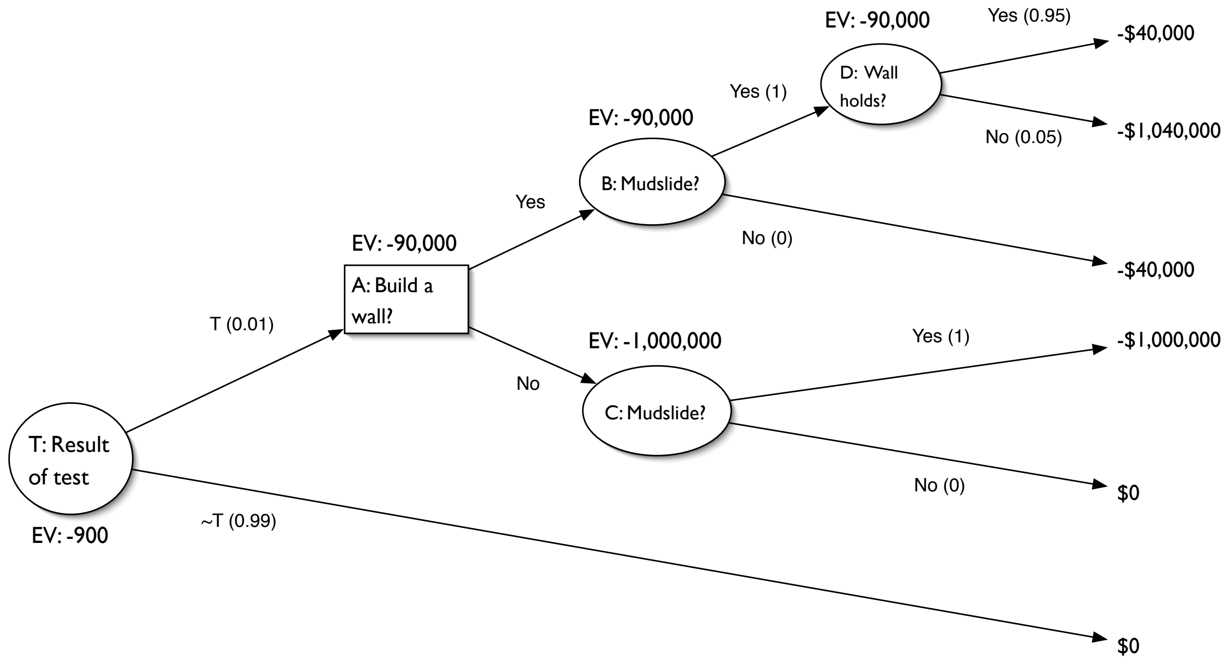


Figure 10.4: The bottom half of the decision tree is clipped here, with the understanding that you clearly shouldn't build a wall if the test comes back negative.

Clearly, the smart decision depends on the result of the test, which is a random outcome. Notice how the information affects the conditional probabilities at Nodes B and C, and therefore changes the expected value of the "build a wall?" decision node.

So is the information worth $3,000? Yes. Previously, the expected value for the whole tree was -$10,000; now it's -$900. The fair value of the information is equal to the improvement in expected value, or $9100. The $3000 test would therefore be a bargain even at three times the price.

The value of imperfect information

Unfortunately, the above calculations are a bit divored from the real world, where no information is truly perfect. More realistically we should assume *imperfect information*, in which the accuracy of the test is not 100%. Imperfect information is always worth less than perfect information, but it still has value.

Suppose, for example, that the hydrologist's test had the following properties, where T denotes a positive test and M denotes the

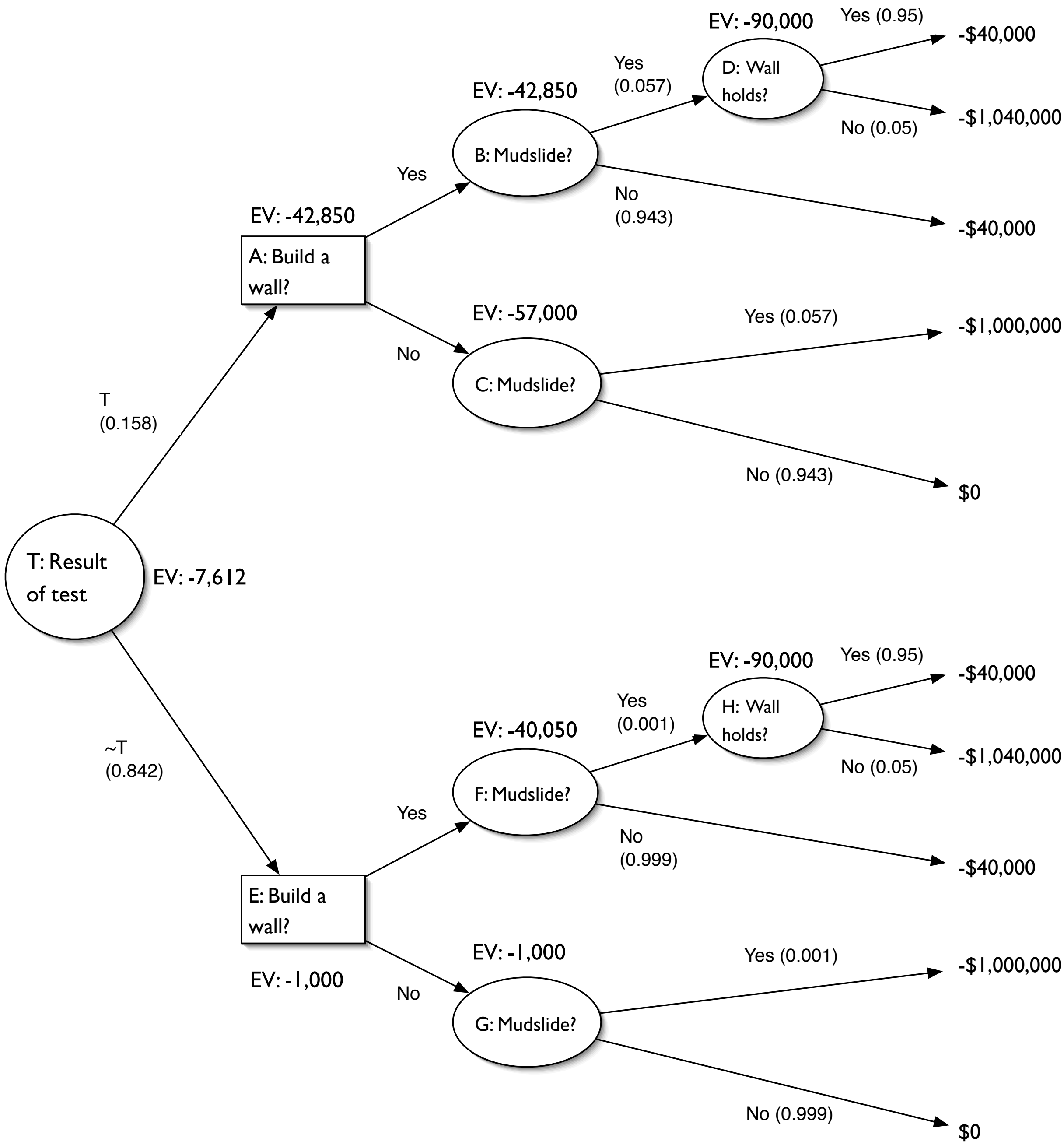


Figure 10.5: The decision tree for determining the value of the imperfect test.

event that a mudslide will occur at your house:

$$\begin{aligned} P(T \mid M) &= 0.9 \\ P(\sim T \mid \sim M) &= 0.85 \end{aligned}$$

The test, in other words, is 90% accurate at forecasting mudslides that are going to occur, and 85% accurate at forecasting "no mudslide" situations.

In the previous example, we assumed that both of these probabilities were equal to 1, and concluded that the test was a bargain at $3000. Do the imperfections in the test change our opinion? To answer, we must see how much out expected value improves, assuming that we possessed the information.

In order to determine the value of the information provided by this test, we must use the rules of probability theory in order to compute several important quantities. First, we need the marginal probabilities of a positive test (T) and a negative test ($\sim T$). Before these were simply 1% and 99%, respectively, but now it's not so simple, since we have to account for the possibility of a false positive or false negative. Using the rules of addition and multiplication, we find that the probability of a positive test is:

$$\begin{aligned} P(T) &= P(T, M) + P(T, \sim M) \\ &= P(T \mid M) \cdot P(M) + P(T \mid \sim M) \cdot P(\sim M) \\ &= 0.9 \cdot 0.01 + 0.15 \cdot 0.99 \\ &= 0.1575 \end{aligned}$$

Therefore, it must be the case that $P(\sim T) = 0.8425$. This will give us the correct probabilities at node T in the new decision tree.

Second, we also need $P(M \mid T)$, or the probability of a mudslide occuring, given that a positive test occurs. Using Bayes' rule, this is

$$\begin{aligned} P(M \mid T) &= \frac{P(M) \cdot P(T \mid M)}{P(T)} \\ &= \frac{0.01 \cdot 0.9}{0.1575} \\ &\approx 0.057\,, \end{aligned}$$

or roughly 6%. The reason that this is so low is that the base rate of mudslides—that is, the prior probability $P(M)$ in Bayes' rule—is so small, and the test isn't all that accurate.

Finally, we also need $P(\sim M \mid\sim T)$, or the probability of a mudslide not occuring, given that a negative test occurs. Again using Bayes' rule, we find that

$$\begin{aligned} P(\sim M \mid\sim T) &= \frac{P(\sim M) \cdot P(\sim T \mid\sim M)}{P(\sim T)} \\ &= \frac{0.99 \cdot 0.85}{0.8425} \\ &\approx 0.999\,, \end{aligned}$$

or almost 100%. The chance that a mudslide will still occur despite a negative test result is extremely small.

If we plug these new probabilities in to the relevant portions of the tree, we get the result shown in Figure 10.5. Notice how the probability of a mudslide depends upon the result of the test, and is therefore different at Node B (0.057) than at Node F (0.001). This difference reflects the numbers we just calculated using Bayes' rule. Notice, too, how the probabilities at Node T are different than before, to reflect the 0.1575 marginal probability of seeing a positive test. Compare this to a 0.01 probability before, when the test was perfect and the likelihood of a false positive was zero.

The new expected value is -\$7,612. The imperfect test improves our expected value by only \$2,388, a bit shy of the \$3,000 asking price. Of course, your utilities don't reflect the pure expected values, then the test might still be a bargain. You'd be paying a \$612 risk premium in order to reduce the chance of losing your house by a factor of

$$\frac{P(\text{Lose your house} \mid \text{Don't buy the test, don't build a wall})}{P(\text{Lose your house} \mid \text{Buy the test, maximize EV thereafter})} = \frac{0.0100}{0.0013} \approx 8\,.$$

If you are risk averse, you might like this trade.