* Matrix Diagonalization
* Matrix Power
* Commutative Matrices
* Assignment.

Multiplicity of Eigenvalue:

The number of times an eigenvalue occurs as a root of the characteristics polynomial of a matrix.

$$A \in \mathbb{R}^{n \times n}$$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0 \Rightarrow \lambda = 1 \text{ (twice)}$$

multiplicity of 1 is $\underline{\underline{2}}$

Matrix Diagonalization.

Let $A \in \mathbb{R}^{n \times n}$. We say that A is diagonalizable if $P^{-1}AP$ is diagonal for some invertible matrix $P \in \mathbb{R}^{n \times n}$.

P is called the diagonalizing/transition matrix

$$P^{-1}AP = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$$

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_n \end{pmatrix}$$

Theorem: A matrix $A \in \mathbb{R}^{n \times n}$ is diagonalizable iff every eigenvalue of multiplicity "m" has exactly "m" eigenvectors.

λ_1, λ_2 (twice), λ_3
1 2 1

Eigenvector: $\lambda_1(1)$, $\lambda_2(2)$, $\lambda_3(1)$

Theorem: A matrix $A \in \mathbb{R}^{n \times n}$ with "n" distinct eigenvalues is diagonalizable

$$P^{-1} A P = D$$

P ↓ invertible matrix; D ↓ diagonal

$$D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

$$P = [v_1 \; v_2 \; \ldots \; v_n]$$

linearly independent vectors

Example:
$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$$

* Find the eigenvalues and eigenvectors of A
* Is A diagonalizable? If yes, diagonalize it.

Solution:

$\det(A - \lambda I) = 0$ Characteristics equation

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & -1 \\ 1 & 3 & -2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\big((2-\lambda)(-2-\lambda) + 3\big) = 0$$
$$(2-\lambda)(\lambda-2)(\lambda+2) + 3(2-\lambda) = 0$$
$$(2-\lambda)(\lambda^2-4) + 6 - 3\lambda = 0$$
$$2\lambda^2 - 8 - \lambda^3 + 4\lambda + 6 - 3\lambda = 0$$
$$2\lambda^2 - 2 - \lambda^3 + \lambda = 0$$
$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$
$$(\lambda-2)(\lambda^2-1) = 0 \iff \lambda = 2, 1, -1$$

Compute the eigenvectors:

For $\lambda_1 = 2$

$$(A - 2I)v = 0$$
$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & 3 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$v_1 - v_3 = 0 \iff v_1 = v_3$

$v_1 + 3v_2 - 4v_3 = 0$

$v_1 + 3v_2 - 4v_1 = 0$

$3v_2 - 3v_1 = 0$

$v_1 = v_2$

For $\lambda_1 = 2$,

$$v_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{for } v_1 \neq 0$$

Take $v_1 = 1$, $\vec{V}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda_2 = 1$

$$(A - I)\, v = 0$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$V_1 = 0$

$V_1 + V_2 - V_3 = 0 \Leftrightarrow V_2 - V_3 = 0$

$V_2 = V_3$

$3V_2 - 3V_3 = 0 \Leftrightarrow V_2 = V_3$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 3 & -3 & 0 \end{array}\right] \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - 3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$V_1 = 0$

$V_2 - V_3 = 0 \Leftrightarrow V_2 = V_3$

$$\vec{V}_2 = \begin{bmatrix} 0 \\ V_2 \\ V_2 \end{bmatrix} = \underbrace{V_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{V_2 \neq 0}$$

. Let $V_2 = 1$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda_3 = -1$, $\vec{V}_3 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

$\lambda_1 = 2, \; v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \lambda_2 = 1, \; v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \lambda_3 = -1, \; v_3 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

Is A diagonalizable? Yes

P and D ?

$$\begin{array}{c} \quad \lambda_1 \quad \lambda_2 \quad \lambda_3 \qquad \lambda_1 \; \lambda_3 \; \lambda_2 \\ \underline{D} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{array}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

$$A = PDP^{-1}$$

A is diagonalizable if $\boxed{P^{-1}AP}$ is diagonal for some invertible $P \in \mathbb{R}^{n \times n}$

$$P^{-1}AP = D$$

$$\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

A is similar B if $\exists$ an invertible matrix C such that

$$C^{-1}AC = B$$

$$2, \underbrace{-1, -1}$$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \underbrace{\lambda_1}_{1} = 2, \quad \underbrace{\lambda_2}_{2} = -1 \text{ \{twice\}}$$

This is diagonalizable!

Example: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 1$

$\lambda = \underline{1}$ (multiplicity $= 2$)

$(A - I)v = 0$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$v_2 = 0$
v_1 is a free variable
$v_1 \in \mathbb{R} \setminus \{0\}$

$$v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_1 = 1 \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$P^{-1}AP = D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Assume A is diagonalizable for contradiction sake.

$$\underline{PP^{-1}}AP = PI \qquad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$IAP = P$$

$$AP = P \qquad ?$$

$$APP^{-1} = PP^{-1} \Leftrightarrow AI = I \Leftrightarrow A = I$$

Contradiction!

Hence, A is not diagonalizable

Matrix Power.

Matrix power is denoted by A^K for $K \in \mathbb{N}$.

$$A^K = \underbrace{A \circ A \circ A \circ \dots \circ A}_{K \text{ times}}$$

$A^2 = A \circ A$, $A^3 = A^2 \cdot A$
$= A \circ A \circ A$

A^{10} is tiring!

Theorem: If $A = PDP^{-1}$ where P is invertible and D is a diagonal matrix, then

$$\underline{A^K = PD^K P^{-1}}, \text{ for } K = 1, 2, \dots \in \mathbb{N}$$

Proof sketch

$$A = PDP^{-1}$$

$$\begin{aligned} A^2 &= A \cdot A \\ &= (PDP^{-1})(PDP^{-1}) \\ &= PD(P^{-1}P)DP^{-1} \\ &= PDDP^{-1} \\ &= PD^2P^{-1} \end{aligned} \iff A^2 = PD^2P^{-1}$$

PDP^{-1}

$\cancel{K}+1$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$$

$$A^{10} = PD^{10}P^{-1} = P\begin{bmatrix} 2^{10} & 0 & 0 \\ 0 & 1^{10} & 0 \\ 0 & 0 & (-1)^{10} \end{bmatrix}$$

Power of a diagonal matrix.

Let $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

$$D^k = \mathrm{diag}(\lambda_1^k, \lambda_2^k, \lambda_3^k, \ldots, \lambda_n^k)$$

$$A^{10} = P \begin{bmatrix} 2^{10} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} P^{-1}$$

$\boxed{\text{Commutative Matrices}}$

For matrix multiplication, $AB \neq BA$.

Two matrices A and B are called commutative matrices if

$$AB = BA$$

A "commutes" with B or B "commutes" with A

Examples: (1) Zero matrix, O

$$A \cdot O = O \cdot A = O$$

(2) Identity matrix I

$$A \cdot I = I \cdot A = A$$

3. Two diagonal matrices.

$A = \mathrm{diag}(a_1, a_2, \dots, a_n)$, $B = \mathrm{diag}(b_1, b_2, \dots, b_n)$

$AB = \mathrm{diag}(a_1b_1, a_2b_2, \dots, a_nb_n) = BA$

$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$ $B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$ $AB = \begin{pmatrix} a_1b_1 & 0 \\ 0 & a_2b_2 \end{pmatrix}$

$BA = \begin{pmatrix} a_1b_1 & 0 \\ 0 & a_2b_2 \end{pmatrix}$

4. Let $B = A^{-1}$, then

$AA^{-1} = A^{-1}A = I$

5. Diagonalizable matrices with same eigenbasis

$A = PD_AP^{-1}$ $B = PD_BP^{-1}$

$AB = BA$

$AB = PD_AP^{-1}PD_BP^{-1}$

$= PD_AD_BP^{-1}$ $= PD_BD_AP^{-1}$

$= PD_BID_AP^{-1}$ $I = P^{-1}P$

$= (PD_BP^{-1})(PD_AP^{-1})$

$= BA$

BA

$\Rightarrow AB = BA$

A Find $B \in \mathbb{R}^{2\times 2}$ $(M_2(\mathbb{R}))$ that commutes with A

$$AB = BA$$

$$A\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} A$$

True

Stochastic matrix: Matrix whereby its row summed up to 1

$$\sum_{j=1}^{n} a_{ij} = 1 \quad \text{for all } 1 \le i \le n$$

$$\begin{pmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{pmatrix} \begin{matrix} = 1 \\ = 1 \end{matrix}$$

$a_{ij} \geq 0$

$0 \le a_{ij} \le 1$

St Doubling stochastic matrix

A nilpotent matrix of order p is a matrix such that

$$A^p = 0 \quad \text{and} \quad A, A^{p-1} \neq 0$$

$$A, A^2, \dots, A^{p-1} \neq 0,\ A^p = 0,\ A^{p+1}, \dots, \dots = 0$$

Thursday, August 14, 2025 4:13 PM

timothyolusola98@gmail.com

timothyolusola98_m3

stephen.timothy@gmail.co

stephen.timothy_m4.pdf
.ipynb