# Assignment 2
Ordinary Differential Equations

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## Exercise 1: Matrix Operations

1. Let $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$.

   We want to find all matrices $B \in M_2(\mathbb{R})$ such that $AB = BA$

   Let the matrix $B$ be represented by $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$

   Thus we want to find $b_1, b_2, b_3, b_4 \in \mathbb{R}$ so that

$$\begin{aligned} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} &= \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \\ \Leftrightarrow \quad \begin{pmatrix} ab_1 + bb_3 & ab_2 + bb_4 \\ ab_3 & ab_4 \end{pmatrix} &= \begin{pmatrix} ab_1 & bb_1 + ab_2 \\ ab_3 & bb_3 + ab_4 \end{pmatrix} \end{aligned}$$

   In particular we want to find $b_1, b_2, b_3, b_4$ so that the following are true:

$$ab_1 + bb_3 = ab_1 \quad \Leftrightarrow \quad bb_3 = 0 \tag{1}$$

$$ab_2 + bb_4 = ab_2 + bb_1 \quad \Leftrightarrow \quad bb_4 = bb_1 \tag{2}$$

   From 1 since there is no original condition on the entry $b$ of $A$ then $b_3$ must equal zero to force 1 to always be true.

   Likewise from 2 it is easy to see that $b_1 = b_4$ for 2 to always be true.

   Thus our matrix $B$ that commutes with $A$ is of the form $B = \begin{pmatrix} b_1 & b_2 \\ 0 & b_3 \end{pmatrix}, \quad b_1, b_2 \in \mathbb{R}$

2. We want to find $2 \times 2$ matrices $A, B$ with real entries such that $AB = 0$ and $BA \neq 0$.

   Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$. thus we want

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

   In particular we want the following to be true

$$\langle a_1, a_2 \rangle \cdot \langle b_1, b_3 \rangle = 0 \tag{3}$$

$$\langle a_1, a_2 \rangle \cdot \langle b_2, b_4 \rangle = 0 \tag{4}$$

$$\langle a_3, a_4 \rangle \cdot \langle b_1, b_3 \rangle = 0 \tag{5}$$

$$\langle a_3, a_4 \rangle \cdot \langle b_2, b_4 \rangle = 0 \tag{6}$$

   Since the vectors in equations 3 and 4 are in $\mathbb{R}^2$ then $\langle b_1, b_3 \rangle$ must be in the same direction as $\langle b_2, b_4 \rangle$ for 3 and 4 to be true simultaneously. Similarly, for equations 3 and 5 to be true simultaneously, since the vectors in those equations are from $\mathbb{R}^2$ then $\langle a_1, a_2 \rangle$ must also be in the same direction as $\langle a_3, a_4 \rangle$

Thus we have the following

$$\begin{aligned} \langle a_3, a_4 \rangle &= \lambda_1 \langle a_1, a_2 \rangle \quad \lambda_1 \in \mathbb{R} \\ \langle b_2, b_4 \rangle &= \lambda_2 \langle b_1, b_3 \rangle \quad \lambda_2 \in \mathbb{R} \end{aligned}$$

also for equation 3 to hold then $\langle a_1, a_2 \rangle = \gamma \langle -b_3, b_1 \rangle \quad \gamma \in \mathbb{R}$. Thus we have the following

$$\begin{aligned} \langle a_1, a_2 \rangle &= \gamma \langle -b_3, b_1 \rangle \\ \langle b_2, b_4 \rangle &= \lambda_2 \langle b_1, b_3 \rangle \\ \langle a_3, a_4 \rangle &= \gamma_1 \langle -b_3, b_1 \rangle \quad \gamma, \gamma_1, \lambda_2 \in \mathbb{R} \end{aligned}$$

Thus we have that $A = \begin{pmatrix} -\gamma b_3 & \gamma b_1 \\ -\gamma_1 b_3 & \gamma_1 b_1 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & \lambda_2 b_1 \\ b_3 & \lambda_2 b_3 \end{pmatrix}$ thus we have that

$$AB = \begin{pmatrix} -\gamma b_3 & \gamma b_1 \\ -\gamma_1 b_3 & \gamma_1 b_1 \end{pmatrix} \begin{pmatrix} b_1 & \lambda_2 b_1 \\ b_3 & \lambda_2 b_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

We have another condition that our choice of entries for $A$ and $B$ must satisfy. In particular we want the following to hold

$$BA = \begin{pmatrix} b_1 & \lambda_2 b_1 \\ b_3 & \lambda_2 b_3 \end{pmatrix} \begin{pmatrix} -\gamma b_3 & \gamma b_1 \\ -\gamma_1 b_3 & \gamma_1 b_1 \end{pmatrix} = \begin{pmatrix} (-\gamma - \gamma_1 \lambda_2) b_1 b_3 & (\gamma + \lambda_2 \gamma_1) b_1^2 \\ (-\gamma - \lambda_2 \gamma_1) b_3^2 & (\gamma + \gamma_1 \lambda_2) b_1 b_3 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

It, thus, follows that

$$\begin{aligned} -\gamma &\neq \lambda_2 \gamma_1 \\ \gamma &\neq -\gamma_1 \lambda_2 \end{aligned}$$