AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES

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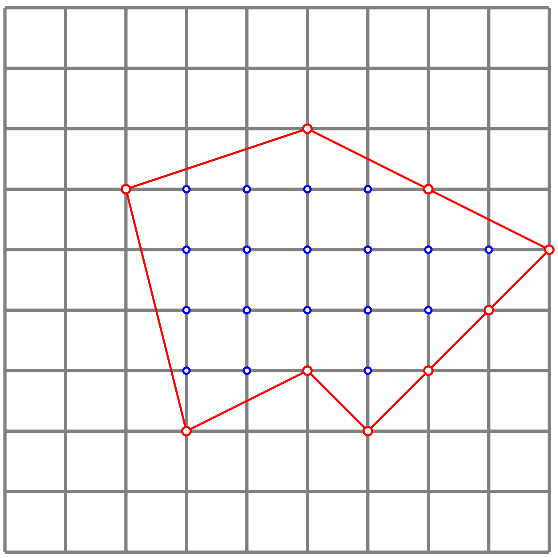
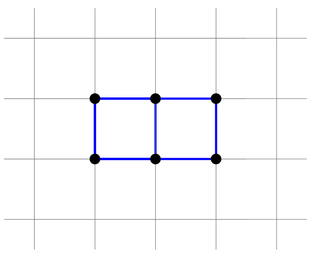
Question 2

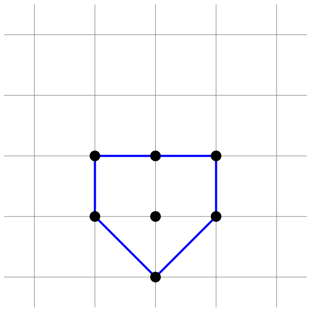
Suppose that in a gird, each square has unit area. Then from basic trigonometry, we see that a diagonal line through each square means each right triangle has area $\frac{1}{2}$. Consider the example below. The polygon has area 22.5



we want to find and prove a general formula for the area of the triangle. We explore a few possibilities:
Let $p = 0$ and $q = 6$

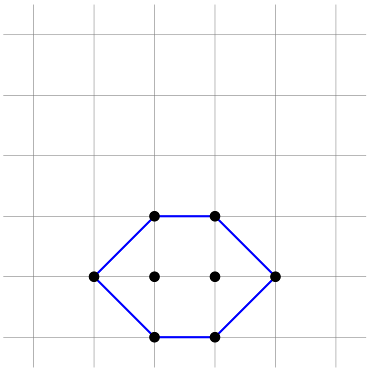


Let $p = 1$ and $q = 6$



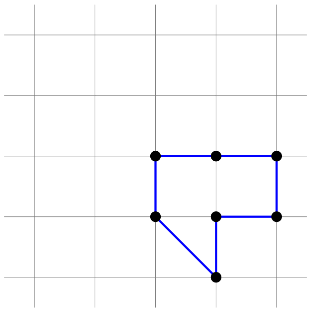
Notice that the area is 3.

Let $p = 2$ and $q = 6$. figure



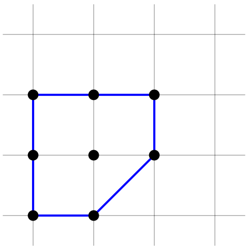
Notice that the area is 4

Let $p = 0$ and $q = 7$



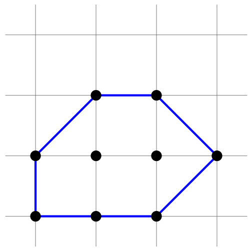
Notice that the area is 2.5.

Let $p = 1$ and $q = 7$



Notice that the area is 3.5.

Let $p = 2$ and $q = 7$.



Notice that the area is 4.5

Notice that increasing q by 1 while keeping p constant changes the total area by 0.5 whereas keeping q constant and increasing p by 1 increases the area by 1. Thus from this I claim that the total area of any given polygon; call it A is related to the internal lattice points p and boundary lattice points q by the following:

$$A = p + \frac{q}{2}$$

But notice that for the examples we exploblack this formula does not hold. For instance for $p = 1$ and $q = 6$ we get the area to be 4 from the formula, but we found that the area is 3. Again, for $p = 2$ and $q = 6$ we find that there area is 5 from the formula, but we found the area to be 4. It can, therefore, be observed that our formula so far is off by some constant C. Thus our formula is now of the form

$$A = p + \frac{q}{2} + C$$

To find the value of C we consider the following equation for $p = 1, \quad q = 6$

$$3 = 1 + \frac{6}{2} + C$$

This simplifies to $C = -1$. Thus our formula is of the form

$$A = p + \frac{q}{2} - 1$$

Testing this with our few explorations, we see that it works for all our cases.

To prove that this formula for A can be generalized to all p internal lattice points and q boundary lattice points we consider the following 3 cases:

Case 1 Rectangle:
For a rectangle, we know that given length and width as l and w then we have that the area of a rectangle is given as $A = l \cdot w$ and the perimeter of a rectangle is given as $A = 2(l + w)$. Notice that the top and bottom of a triangle both have $(l + 1)$ lattice points (including the corners) and the sides have $(w + 1)$ lattice (also including corners) points each. Thus the total number of lattice points is $2(w + 1) + 2(l + 1) - 4$ and so the total number of boundary lattice points is given as $q = 2(w + l)$. Now to find the relationship between the interior lattice points

and the length and width we notice that for each of the $(l-1)$ columns, we have $(w-1)$ rows. Thus the number of interior lattice points is given as $p=(l-1)(w-1)$. Given that the area of a rectangle with length l and width w is $A=lw$ then we want to check that our formula gives that. So plugging in our expressions for p and q into the formula we found, we have that $A=(l-1)(w-1)+(l+w)-1=lw$. Thus for the base case of a rectangle, our formula holds.

Case 2 Right Triangle:
Notice that for a right triangle, we can always find some rectangle that contains it. Let b, h be the breadth and height of the rectangle containing the right triangle. This rectangle has $(b+1)(h+1)$ total lattice points and some of these lie on the boundary of the right triangle it contains. Let r be the number of lattice points lying on the hypotenuse of the right triangle, excluding the vertices (since those are counted in the base and the height of the triangle). Then the total number of boundary points is given by $q=b+h+r+1$ and the number of interior lattice points is given as $p=\frac{(b-1)(h-1)-r}{2}$, that is half the number of interiror points in the triangle, minus the points on the hypotenuse excluding the vertices. So we have $q=b+h+r+1$ and $p=\dfrac{(b-1)(h-1)-r}{2}$. Checking with the formula we have, we get that

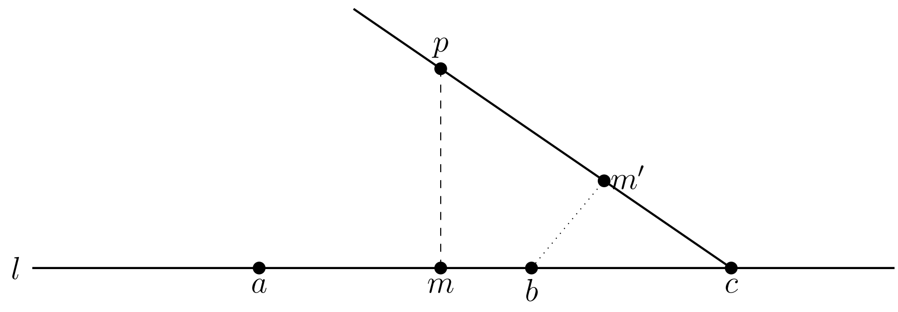
$$A=\frac{(b-1)(h-1)-r}{2}+\frac{b+h+r+1}{2}-1=\frac{bh}{2}.$$

Which is exactly the area of a triangle. Thus our formula holds for a triangle

Question 4

Suppose that there are n points in the plane which are not collinear. Suppose that some line exists that passes through at least 3 of the points. Consider the following sets P the sets of points in the plane, and L the set of the lines defined by the points in this plane. Then both P and L are finite sets since the number of points on we're considering are finite. Now consider the set $A=\{(d(p,l)\mid l\in L, p\notin L\}$, $d(p,l)$ is the distance from the point p to the line l. Then since P and L are finite, then A is also finite. So by the extremal principle there should exist a minimum element of A and a maximum element of A.

Now let $x=\min A$. That is there exist a point $p\in P$ and a line $l\in L$ such that $d(p,l)=x$. Then from our hypothesis, l contains at least 3 points of P. Now consider the perpendicular line from p to the line l and let y_0 be the point where the perpendicular touches l. Then by the pigeonhole principle, and from our hypothesis that l must have 3 points from P then we can find 1 points a that is on the one side of l after it has been divided by the perpendicular line from the point p and two other points b, c that are on the other side of l. Then the pair of points p, c also define another line l'. Now since p, c are both points in the plane then $p,c\in P$. Now consider the point $b\in l$. Notice that $b\notin l'$ so $d(b,l')\in A$ and consider the perpendicular from b to l'. Let the point of intersection of the perpendicular from b to l' be m'.



Notice that pmc and $bm'c$ are both right triangles. Now they have to be similar since they both have a point in common at c, thus angle $m'bc$ must be less than angle mpc. Notice that the length of bc is at most the length of mc and length of mc is less tahn pc since the line segment pc forms the hypotenuse of the larger right triangle. Thus the lenght of line segment bc is less than the line segment pc. This implies triangle pmc is larger than $bm'c$ thus the length of line bm' is less than the length of line pm. But $d(b, s) \in A$ where s is the line formed by pc. Which is a contradiction to the claim that the minimum $\min A$ is $y_0 = d(p, l)$. Thus the set A has to be empty since we cannot have such a contradiction, and thus L must also be empty. Hence we have shown that for n points in the plane that are not collinear, there is a line that passes through just two of these points.

For the case where we have infinitely many points we cannot guarantee the finiteness of the set $A = \{(d(p, l) \mid l \in L, p \notin L\}$, $d(p, l)$ is the distance from the point p to the line l and so the arugment for getting a minimum $\min A$ does not hold since we cannot apply the extremal principles. In particular, consider the points $(1, 2)$ and $(2, 1)$ in the plane. Then the line they form $y = -x + 3$ contains infinitely many points on the plane. Thus This phenomenon does not hold for infinitely many points in the plane.