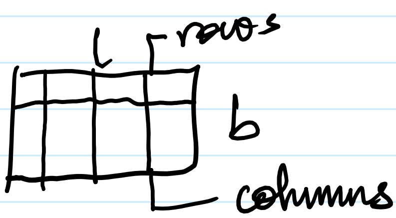
Week 3: Linear Algebra

Matrices, Algebra, Inverse, Eigenvalues/vectors, Diagonalization

Matrices

A matrix is a rectangular array of numbers



$A \in \mathbb{R}^{m \times n}$

A - m rows
n columns.

$m = n$, then we have a square matrix

Example: $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ $m = 2$, $n = 2$ $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{pmatrix}$ $m=2$, $n=3$

Combination of column vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$
collection

collection of row vectors $[2 \quad 3]$ & $[1 \quad 2]$

Algebra of matrices.

1. Matrix Addition.

$$A = (a)_{ij} = [a]_{ij}$$

$i \downarrow$ row, $j \downarrow$ columns

$$A = (a)_{ij}, \quad B = (b)_{ij}$$

Note: The two matrices should have same no of rows and columns.

$$A + B = (a_{ij} + b_{ij})$$

Ex:
$$\underset{3\times 3}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}} + \underset{3\times 3}{\begin{pmatrix} 2 & 0 & 1 \\ 4 & 1 & 5 \\ 6 & 2 & 0 \end{pmatrix}} = \begin{pmatrix} 3 & 2 & 4 \\ 8 & 6 & 11 \\ 13 & 10 & 9 \end{pmatrix}$$

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2. Scalar Multiplication.

This is the multiplication of a matrix by a scalar, $\lambda \in \mathbb{F} (\mathbb{R}, \mathbb{C}, \dots)$

A scaler, λ, matrix A

$$\lambda A = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \dots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \dots & \lambda a_{2n} \\ \vdots & & & \\ \lambda a_{m1} & \lambda a_{m2} & \dots & \lambda a_{mn} \end{pmatrix}$$

Example: $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ $\lambda = 5$

$$\lambda A = 5 \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 10 & 5 \end{pmatrix}$$

Properties

1. Distributive over addition:
$$c(A+B) = cA + cB$$
2. $(cd)A = c(dA)$
3. $0 \cdot A = 0$

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Matrix Multiplication

Note: The number of column of the first matrix should be the same as the no of rows of the second matrix

$A \in \mathbb{R}^{m \times \underline{n}}$, $B \in \mathbb{R}^{\underline{n} \times p}$ $AB \in \mathbb{R}^{m \times p}$

$A = (a)_{ij}$ (b_{ij})

$$(AB)_{ij} = \sum_{k=1}^{n} \underbrace{a_{ik} b_{kj}}$$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & & \dots & b_{2p} \\ \vdots & & & \\ b_{n1} & & \dots & b_{np} \end{pmatrix}$$

$$\begin{pmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{pmatrix} \times \begin{pmatrix} b_1 & b_2 & \dots & b_p \end{pmatrix}$$

$$= \begin{pmatrix} a_1^T b_1 & a_2^T b_2 & \dots & a_m^T b_p \end{pmatrix}$$

$$AB \neq BA$$

Example: $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 6 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 2 \\ 1 & 1 \\ 3 & 2 \end{pmatrix}$

$A \in \mathbb{R}^{2\times 3}$ $B \in \mathbb{R}^{3\times 2}$

$AB \in \mathbb{R}^{2\times 2}$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 6 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 11 & 10 \\ 19 & 17 \end{pmatrix}$$

$$(1\ 2\ 3) \cdot (0\ 1\ 3) = (1\times 0) + (2\times 1) + (3\times 3)$$
$$= 0 + 2 + 9 = 11$$

$$(1\ 2\ 3) \cdot (2\ 1\ 2) = (1\times 2) + (2\times 1) + (3\times 2) = 10$$

Transpose of a matrix

Let $A \in \mathbb{R}^{m\times n}$, The transpose of A denoted by $A^T \in \mathbb{R}^{n\times m}$ is defined as

$$(A^T)_{ij} = (a)_{ji} = (A)_{ji}$$

$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ $A^T = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$

$B = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 1 & 3 \end{pmatrix} \in \mathbb{R}^{3\times 2}$

$B^T = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 4 & 3 \end{pmatrix} \in \mathbb{R}^{3\times 2}$

Properties

* $(A^T)^T = A$ * $(A+B)^T = A^T + B^T$

$A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ $A+B = \begin{pmatrix} 3 & 1 \\ 3 & 6 \end{pmatrix}$

$A^T = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ $B^T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ $A^T + B^T = \begin{pmatrix} 3 & 3 \\ 1 & 6 \end{pmatrix}$

$(A+B)^T = \begin{pmatrix} 3 & 1 \\ 3 & 6 \end{pmatrix}$

* $(cA)^T = cA^T$ * $(AB)^T = B^T A^T$

Trace of a matrix.

Sum of the main diagonal entries

$$tr(A) = \sum_{i=j}^{n} a_{ii}$$

* $tr(A+B) = tr(A) + tr(B)$
* $tr(A^T) = tr(A)$ * $tr(AB) = tr(BA)$
* $tr(cA) = c\,tr(A)$

Determinant of a matrix.

What is the determinant of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$|A| = ad - bc$$

Determinant is a property of a square matrix

Determinant of a 3x3 matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$(-1)^{i+j} \det(A_{ij})$$

Example: $\det\begin{pmatrix} \overset{+}{1} & \overset{-}{2} & \overset{+}{1} \\ 2 & 4 & 2 \\ 0 & 1 & 2 \end{pmatrix} = ?$

$$1\begin{vmatrix} 4 & 2 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 0 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix}$$

$$1(8-2) - 2(4-0) + 1(2-0) = 6 - 8 + 2$$
$$= 0$$

Properties

1. $\det(I_n) = 1$
2. $\det(A^T) = \det(A)$
3. $\det(AB) = \det(A)\det(B)$
4. $\det(cA) = c^n \det(A)$
5. $\det(A^{-1}) = 1/\det(A)$

2. $\det(A) = \det(A)$ 5. $\det(A^{-1}) = 1/\det(A)$

Inverse of a matrix.

Let $A \in \mathbb{R}^{n \times n}$. Then the inverse of A denoted by A^{-1} is a matrix such that

$$AA^{-1} = A^{-1}A = I_n$$

A is invertible

* A is invertible iff $\det(A) \neq 0$
* If A is not invertible (singular), then $\det(A) = 0$
* The inverse of a matrix is unique $\exists!\ A^{-1}$ for A

Example: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$ Provided that $\det(A) \neq 0$

Ex: $A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$, $\det(A) = 4$

$$A^{-1} = \frac{1}{4} \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ -1/4 & 1/2 \end{pmatrix}$$

$$AA^{-1} = \frac{1}{4} \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{Adj}(A)$$

$$\operatorname{Adj}(A) = (\text{Co factor of } A)^T$$

$$(-1)^{i+j} \det(A_{ij})$$

$$\begin{array}{c|c} A & I \\ \downarrow & \downarrow \\ I & A^{-1} \end{array}$$

row elementary operations

Special Matrices.

$$A = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{pmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{pmatrix} \end{array} \qquad A^{-1} = ?$$

$$(-1)^{i+j} \det(A_{11})$$

$$E_{11} = (-1)^{1+1} \det\begin{pmatrix} 4 & -1 \\ 3 & 0 \end{pmatrix} = 3$$

$$E_{12} = (-1)^{1+2} \det\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = -1$$

$$E_{13} = (-1)^{1+3} \det\begin{pmatrix} 1 & 4 \\ 1 & 3 \end{pmatrix} = -1$$

$$E_{21} = (-1)^{2+1} \det\begin{pmatrix} 7 & 1 \\ 3 & 0 \end{pmatrix} = 3, \quad E_{22} = \det\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = -1$$

$$E_{23} = -\det\begin{pmatrix} 2 & 7 \\ 1 & 3 \end{pmatrix} = 1, \quad E_{31} = \det\begin{pmatrix} 7 & 1 \\ 4 & -1 \end{pmatrix} = -11$$

$$E_{32} = -\det\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} = 3, \quad E_{33} = \det\begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix} = 1$$

$$C(A) = \begin{pmatrix} 3 & -1 & -1 \\ 3 & -1 & 1 \\ -11 & 3 & 1 \end{pmatrix}, \quad \mathrm{Adj}(A) = \begin{pmatrix} 3 & 3 & -11 \\ -1 & -1 & 3 \\ -1 & 1 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \mathrm{Adj}(A)$$

$$\det(A) = 2(3) + 7(-1) + 1(-1) = -2$$

$$A^{-1} = -\frac{1}{2} \begin{pmatrix} 3 & 3 & -11 \\ -1 & -1 & 3 \\ -1 & 1 & 1 \end{pmatrix} //$$

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Special matrices

1. Diagonal matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = 2 \qquad \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & & 1 \end{pmatrix} = 0$$

$\text{diag}(2, 1)$ $\qquad$ $\text{diag}(3, 0, 1)$

determinant $= \prod_{i=1}^{n} a_{ii}$

2. Identity, I_n

3. Orthogonal matrix

If $A^T A = A A^T = I_n$, then A is an orthogonal matrix.

4. Symmetric matrix

If $A^T = A$, A is symmetric

5. Skew-symmetric matrix: If $A^T = -A$

Eigenvalue and Eigenvectors

Let $A \in \mathbb{R}^{n \times n}$.

A non-zero vector v is an <u>eigenvector</u> of A if

$$A v = \lambda v$$

The scalar λ is the corresponding <u>eigenvalue</u>

Steps to finding eigenvalues and eigenvectors.

1. Solve the characteristics equation

$$|A - I\lambda| = 0 , \quad \det(A - I\lambda) = 0$$

2. The non-zero solution to the equation

$$(A - I\lambda)\underline{v} = \underline{0}$$ is the eigenvector

Example: $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$

1. $\det(A - I\lambda) = 0$

$$A - I\lambda = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{pmatrix}$$

$$\det(A - I\lambda) = (1-\lambda)(1-\lambda) - 4$$

$$(1-\lambda)^2 - 4 = 0 \iff (1-\lambda-2)(1-\lambda+2) = 0$$

$$(-\lambda-1) = 0 \text{ or } 3-\lambda = 0$$

$$\lambda = -1 \text{ or } \lambda = 3 , \quad \lambda_1 = -1 , \lambda_2 = 3$$

To solve for the eigenvectors

For $\lambda_1 = -1$

$$(A - I\lambda)v = 0$$

$$\begin{pmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\left(\begin{array}{cc|c} 2 & 4 & 0 \\ 1 & 2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right) \xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_1 + 2v_2 = 0$$

$$\boxed{v_1 = -2v_2}, \quad v_2 \neq 0$$

Let $v_2 = 1 \Rightarrow v_1 = -2$

For $\lambda = -1$, the corresponding eigenvector $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

For $\lambda = 3$, $\begin{pmatrix} 1-3 & 4 \\ 1 & 1-3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Leftrightarrow \left(\begin{array}{cc|c} -2 & 4 & 0 \\ 1 & -2 & 0 \end{array}\right)$

$$\left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow v_1 - 2v_2 = 0$$

$$\Rightarrow \boxed{v_1 = 2v_2} \quad v_2 \neq 0$$

$v_2 = \frac{1}{2}v_1$

~~Eat~~ Let $v_2 = 1$, $\Rightarrow v_1 = 2$ $\quad v_2 = 2$, $v_1 = 4$

For $\lambda = 3$, the corresponding eigenvector is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Eigenvalues : -1 , 3 $\quad A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$

Trace of a matrix is defined as

$$tr(A) = \sum_{i=1}^{n} \lambda_i = -1 + 3 = 2$$

Determinant of a matrix is

$$det(A) = \prod_{i=1}^{n} \lambda_i = -1 \times 3 = -3$$

$$\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$$

$$tr(A) = 1 + 1 = 2$$

$$det(A) = (1 \times 1) - (4 \times 1) = -3$$