Homogeneous first-order ODEs

$$\frac{dy}{dx} + 3x^2y + 4x = 0$$ Is this homogeneous ODE?

A first order ODE is said to be homogeneous, if it is of the form

$$\frac{dy}{dx} = \underbrace{f\left(\frac{y}{x}\right)}$$

$$\Leftrightarrow \frac{dy}{dx} - f\left(\frac{y}{x}\right) = 0$$ · General form of a homogeneous 1st order ODE

Example: $$\frac{dy}{dx} = \frac{3xy + y^2}{3x^2 + xy} \Longleftrightarrow \underbrace{\frac{dy}{dx} = f\left(\frac{y}{x}\right)}$$

Divide the numerator & denominator of the R.H.S by x^2

$$\frac{dy}{dx} = \frac{3\frac{y}{x} + \frac{y^2}{x^2}}{3 + \frac{y}{x}} = \frac{3\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2}{3 + \frac{y}{x}}$$

$$= f\left(\frac{y}{x}\right)$$

where $f(t) = \dfrac{3t + t^2}{3+t}$. Thus, it is homogeneous

Solution to homogeneous equations

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad \text{—} (*)$$

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Change of variable: Let $v = \frac{y}{x} \Rightarrow y = vx$

$y = vx$

v ↓ function of x

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = f(v)$$

$$v + x\frac{dv}{dx} = f(v)$$

$$x\frac{dv}{dx} = f(v) - v$$

$$x\,dv = (f(v) - v)\,dx$$

$$\left.\frac{dv}{f(v) - v} = \frac{dx}{x}.\right]$$ Variable separable.

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Example : $\frac{dy}{dx} = \frac{3xy + y^2}{3x^2 + xy} \iff \frac{dy}{dx} = \frac{3\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2}{3 + \left(\frac{y}{x}\right)}$

Let $v = y/x \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{3v + v^2}{3 + v} = \frac{(3+v)v}{3+v} = v$$

$$\Rightarrow v + x\frac{dv}{dx} = v \iff x\frac{dv}{dx} = 0$$

$\boxed{\frac{d}{dx}(K) = 0}$

$$\iff \frac{dv}{dx} = 0 \quad \text{for } x \neq 0$$

$$\int dv = \int 0 \cdot dx \iff v(x) = C, \; C \in \mathbb{R}.$$

$$\iff \frac{y}{x} = C$$

$$\iff \boxed{y = Cx} \quad \text{Solution}$$

Independent variable , dependent variable

$\frac{dy}{dx} = f(x, y)$ $\quad y \to$ dependent variable

$\frac{dx}{dy} = \cancel{?}$

Example : $\boxed{\dfrac{dy}{dx} = \dfrac{y^2+2xy}{x^2}} = \underline{\left(\dfrac{y}{x}\right)^2 + 2\left(\dfrac{y}{x}\right)}$

Is this homogeneous ?

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right)$$ By multiplying by $\frac{x^2}{x^2}$

Let $v = y/x \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$\Rightarrow v + x\frac{dv}{dx} = v^2 + 2v$$

$$x\frac{dv}{dx} = v^2 + v$$

$$\int \frac{dv}{v^2+v} = \int \frac{dx}{x}$$

$\displaystyle\int \frac{dv}{v^2+v} =$ integrand $= \dfrac{1}{v^2+v} = \dfrac{1}{v(v+1)} = \dfrac{A}{v} + \dfrac{B}{v+1}$

$$1 = A(v+1) + Bv$$

Put $v=-1$: $1 = -B \Rightarrow B = -1$

Put $v=0$: $1 = A \Rightarrow A = 1$

$$\frac{1}{v^2+v} = \frac{1}{v} - \frac{1}{v+1}$$

$$\int \frac{dv}{v^2+v} = \int \frac{1}{v}\,dv - \int \frac{1}{v+1}\,dv$$

$$= \ln|v| - \ln|v+1| = \ln\left|\frac{v}{v+1}\right|$$

$$\int \frac{dv}{v^2+v} = \int \frac{dx}{x}$$

C

$\ln C$

$\downarrow$

constant

$$\ln\left|\frac{v}{v+1}\right| = \ln|x| + \ln C$$

$$\left|\frac{v}{v+1}\right| = C|x|$$

$$\frac{v}{v+1} = Cx \iff v = Cxv + Cx$$

$$\iff v - xvC = Cx$$

$$v(1-xC) = xC$$

$$v = \frac{Cx}{1-Cx}$$

But, $v = y/x$.

$$\frac{y}{x} = \frac{Cx}{1-x} \iff y = \frac{Cx^2}{1-Cx}$$

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

$$\frac{dy}{dx} = \frac{y^2+2xy}{x^2} \quad \Big| \quad \frac{dy}{dx} = f(x,y).$$

$f(\lambda x, \lambda y) = f(x,y)$ then it is homogeneous

$$f(x,y) = \frac{y^2+2xy}{x^2}, \quad f(\lambda x, \lambda y) = \frac{(\lambda y)^2 + 2(\lambda x)(\lambda y)}{(\lambda x)^2}$$

$$f(\lambda x, \lambda y) = \frac{\lambda^2 y^2 + 2\lambda^2 xy}{\lambda^2 x^2} = \frac{\cancel{\lambda^2}(y^2+2xy)}{\cancel{\lambda^2}x^2} = f(x,y)$$

$$f(\lambda x, \lambda y) = f(x,y)$$

It is homogeneous.

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Bernoulli Equations

Are equations of the form

$$\underbrace{\frac{dy}{dx} + a(x)\,y = b(x)\,y^{n}} \qquad n \in \mathbb{R}\setminus\{1\}$$

diff $= y^n$

linear 1^{st} order: $\frac{dy}{dx} + a(x)\,y = b(x)$

Solution: $\frac{dy}{dx} + a(x)\,y = b(x)\,y^{n} \qquad \times\, y^{-n}$

$$y^{-n}\frac{dy}{dx} + a(x)\,y^{1-n} = b(x) \quad *$$

Let $v = y^{1-n}$, $\quad \frac{dv}{dx} = (1-n)\,y^{1-n-1}\cdot\frac{dy}{dx}$

$$\frac{dv}{dx} = (1-n)\,y^{-n}\frac{dy}{dx}$$

$$y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dv}{dx}$$

$$\frac{1}{1-n}\frac{dv}{dx} + a(x)\,v = b(x) \qquad \times\,(1-n)$$

$$\frac{dv}{dx} + a(x)\,v\,(1-n) = b(x)\,(1-n)$$

↳ linear first-order ODE

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v is a function of x

y is a function of x

$v = y^{1-n}$

$v(x) = (3x^2+2)^5$

$y(x) = 3x^2+2$

$\frac{dv}{dx} = 5 \cdot 6x \cdot (3x^2+2)^4$

$v = y(x)^5$

$\frac{dv}{dx} = 5 \cdot y^4 \cdot \frac{dy}{dx}$

Example: $\boxed{y' + y = y^2}$

$$\frac{dy}{dx} + y = y^2, \quad a(x) = 1, \quad b(x) = 1$$

Multiply the DE by y^{-2}

y

Linear: $\frac{dy}{dx} + a(x)y = b(x)$ $\quad y' + a(x)y = b(x)$

Bernoulli: $\underbrace{\frac{dy}{dx} + a(x)y = b(x)y^{n}}_{2,\ 3,\ 5}$ $\quad y' + a(x)y = b(x)y^n$

$y' + \quad y = y^2$

$n \in \mathbb{R} \setminus \{1\}$

$$y' + x^2 y = e^x x^2 y^4$$

$$y' + a(x)y = b(x)y^4 \quad \text{Bernoulli}$$

$$y' + y = y^{2}$$

Multiply the DE by y^{-2}

$$\underline{y^{-2}y^{(1)}} + y^{1-2} = 1 \iff y^{-2}y' + y^{-1} = 1$$

Let $v = y^{-1} \Rightarrow \frac{dv}{dx} = -y^{-2}y^{(1)} \Rightarrow y^{-2}y^{(1)} = \underline{-v^{(1)}}$

$$y^{1-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dv}{dx} \qquad n = 2$$

$$= \frac{1}{1-2}\frac{dv}{dx} = -\frac{dv}{dx}$$

$$-v^{(1)} + v = 1 \iff v^{(1)} - v = -1$$

$$\frac{dv}{dx} - \cancel{w}\, v = -1 \quad \text{—}(**)$$

$a(x) = -1$, $b(x) = -1$

$$\frac{dv}{dx} + a(x)v = b(x)$$

$a(x) = -1,\ b(x) = -1$

$$I.F = e^{\int a(x)\,dx} = e^{\int -1 \cdot dx} = e^{-x}$$

Multiply $(**)$ by e^{-x}

$$e^{-x}\left(\frac{dv}{dx} - v\right) = -e^{-x}$$

$$\frac{d}{dx}\left(ve^{-x}\right) = -e^{-x}$$

$$ve^{-x} = -\int e^{-x}\,dx$$

$$ve^{-x} = -\left[-e^{-x}\right] + c$$

$$ve^{-x} = e^{-x} + c \iff v = 1 + ce^{x}$$

But, $v = y^{-1} = \frac{1}{y}$

$$\frac{1}{y} = 1 + ce^{x} \iff y(x) = \frac{1}{1+ce^{x}}$$

Second order ODEs.

$$F\left(x, y(x), \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$$

$$y'' + 2xy' + 3x = 0 \qquad 2^{nd} \text{ order}$$

Second order ODEs with constant coefficients.

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)\, y = f(x)$$

(General form of second-order ODE)

If $f(x) = 0$, Homogeneous 2^{nd} order ODE
If $f(x) \neq 0$, Inhomogeneous 2^{nd} order ODE

$$\left\{ \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0 \right. \qquad \{ \text{Homogeneous}$$

$a_1, a_0 \in \mathbb{R}$ {constants}

Solution: Let the solution be $y = e^{rx}$

$$\frac{dy}{dx} = re^{rx}, \quad \frac{d^2y}{dx^2} = r^2 e^{rx}$$

$$r^2 e^{rx} + a_1 r e^{rx} + a_0 e^{rx} = 0$$

$$(r^2 + a_1 r + a_0)e^{rx} = 0, \quad e^{rx} \neq 0$$

$$\boxed{r^2 + a_1 r + a_0 = 0}$$ Characteristics Auxilliary equation

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$$r^2 + a_1 r + a_0 = 0 \quad , \quad r_1 \lessgtr r_2$$

1. two real distinct roots $r_1 \neq r_2$, $r_1, r_2 \in \mathbb{R}$
2. equal real roots $r_1 = r_2$, $r_1, r_2 \in \mathbb{R}$
3. Complex roots $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, $\alpha, \beta \in \mathbb{R}$

Case 1: Two real distinct roots

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} \qquad \{e^{r_1 x}, e^{r_2 x}\}$$

Case 2: Equal real root $r_1 = r_2$

$$y(x) = c_1 e^{r_1 x} + c_2 x e^{r_2 x} \qquad \{e^{r_1 x}, x e^{r_1 x}\}$$

Case 3: $r = \alpha \pm i\beta$, $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$

$$y(x) = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$
$$= e^{\alpha x}\left(c_1 \cos\beta x + c_2 \sin\beta x\right)$$

$$y(x) = c_1 e^{r_1 x} + c_2 \cancel{x} e^{r_1 x} = (c_1 + c_2) e^{r_1 x} = \underline{A} e^{r_1 x}$$

$$= (c_1 + x c_2) e^{r_1 x}$$

Example: 1. $y'' - 2y' + y = 0$

2. $y'' + 9y' + 20y = 0$

3. $y'' + y = 0$

1. $y'' - 2y' + y = 0$

Characteristic equation: $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r = 1$ twice $\quad r_1 = 1, \; r_2 = 1$

General solution: $y(x) = C_1 e^{1.x} + x C_2 e^{1.x}$

$y(x) = (C_1 + x C_2) e^x$

2. $y'' + 9y' + 20y = 0$

$r^2 + 9r + 20 = 0 \iff (r+4)(r+5) = 0$

$\Rightarrow r_1 = -4, \; r_2 = -5$

$y(x) = C_1 e^{-4x} + C_2 e^{-5x}$

3. $y'' + y = 0$

$\alpha \pm i\beta$

$r^2 + 1 = 0 \iff r^2 = -1$

$0 \pm i1$

$\iff r = \pm\sqrt{-1} = \pm i$

$\Rightarrow \alpha = 0$

$\beta = 1$

$r = \alpha \pm i\beta \quad , \alpha = 0, \; \beta = 1$

$y(x) = e^{0.x}(C_1 \cos x + C_2 \sin x) = C_1 \cos x + C_2 \sin x$

Inhomogeneous case

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = f(x)$$

where $f(x) \neq 0$

$$y'' + 2y' + y = x^2$$

Solution steps

1. Find the solution to the homogeneous part.

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0$$

y_h

2. Find the particular integral (solution

y_p

3. $y(x) = y_h + y_p$

Method of undetermined Coefficients

1. Assume the particular solution to take the form of the inhomogeneous part of DE: $f(x)$

2. Compute $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and substitute to the DE

3. Get your y_p

$$\underbrace{\frac{d^2y}{dx^2}} + a_1 \underbrace{\frac{dy}{dx}} + a_0 y = \underbrace{f(x)}$$

Assume y_p is the general form of $f(x)$

$f(x)$	General form
$5x,\ 2x+3,$	$y_p(x) = ax + b$
$x^2,\ 5x^2+1,\ 3x^2+2x+3,$	$y_p(x) = \underset{\smile}{a}x^2 + \underset{\smile}{b}x + \underset{\smile}{c}$
$e^{5x},\ e^{\beta x}$	$\underset{\smile}{A}e^{\beta x}$
$\sin\beta x,\ \cos\beta x$	$y_p(x) = \alpha(\sin\beta x + \cos\beta x)$
$e^{5x}(\sin 3x)$	$y_p(x) = Ae^{5x}(\sin 3x + \cos 3x)$

$y_h(x) = c_1 \underbrace{e^{x}} + c_2 e^{2x}$

$y'' + 2y' + 3y = 5\underbrace{e^{x}}$

$y_p(x) = xAe^{x}$

$= Axe^{x}$

Example: $y'' + 3y' + 2 = 5t + 3$

$$y'' + 3y' + 2y$$

$$y_h = C_1 e^{x} + C_2 x e^{x}$$

$$y_p(x) = Ax^2 e^{x}$$

$$y'' + 3y' + 2y = \underline{5x + 3}$$

1. $y'' + 3y' + 2y = 0$

$r^2 + 3r + 2 = 0 \Leftrightarrow (r+1)(r+2) = 0$

$r = -1, \; r = -2$

$y_h(x) = c_1 e^{-x} + c_2 e^{-2x}$

2. Assume, $y_p(x) = ax + b$

$y_p'(x) = a \quad , \quad y_p''(x) = 0$

$0 + 3a + 2(ax + b) = 5x + 3$

$3a + 2ax + 2b = 5x + 3$

$2ax + (3a + 2b) = 5x + 3$

$2a = 5 \quad , \quad 3a + 2b = 3$

$a = 5/2 \quad , \quad \frac{15}{2} + 2b = 3 \Rightarrow 2b = 3 - \frac{15}{2}$

$2b = -\frac{9}{2} \Rightarrow b = -9/4$

$y_p(x) = \frac{5}{2}x - \frac{9}{4}$

$y(x) = c_1 e^{-x} + c_2 e^{-2x} + \frac{5}{2}x - \frac{9}{4}$ //