Ordinary Differential Equations (ODEs)

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Terminology

An ODE is an equation involving an unknown function of one variable, $x = x(t)$ and its derivatives.
The general form is given as:

$$F\left(t, x(t), \dot{x}(t), \ddot{x}(t), \ldots\right) = 0. \tag{1}$$

Notation:

$$\frac{dx}{dt} = x'(t) = \dot{x}(t). \tag{2}$$

- An ODE is said to be **linear** if it is linear in the unknown function and its derivatives.
- If an ODE is not linear, then it is **non-linear**.

Order and Degree

The **order** n of an ODE is the order of the highest derivative that appears in the equation.
The **degree** of an ODE is the degree of the highest derivative in the equation.

Example. What is the order and degree of the following ODEs?

1. $\dfrac{dx}{dt} + 2xy = 0$
2. $\ddot{x}(t) + (\dot{x}(t))^3 + x(t) = g(t)$
3. $(y''(x))^3 + 4xy''' - 2x^2y' = 3xy^2$

First Order Differential Equation

A first order ODE is any equation that can be written in the form

$$F(t, x, \dot{x}(t)) = 0 \tag{3}$$

Variable Separable

If a first order ODE is of the form

$$\frac{dx}{dt} = f(t) \cdot g(x), \tag{4}$$

Step 1: Separate the variable as $\dfrac{dx}{g(x)} = f(t)dt$

Step 2: Integrate both sides as $\displaystyle\int \frac{dx}{g(x)} = \int f(t)\ dt$

Variable Separable: Example

Example: Solve

$$\frac{dx}{dt} = 2tx^2$$

Solution: Separating the variables gives

$$\frac{dx}{x^2} = 2t \; dt$$

Integrating both sides, we get

$$\begin{aligned} \int \frac{dx}{x^2} &= 2t \; dt \\ -\frac{1}{x} &= t^2 + C \\ x(t) &= -\frac{1}{t^2 + C}, \quad \text{where } C \in \mathbb{R} \end{aligned}$$

Linear ODEs

A DE of the form

$$\frac{dx}{dt} + a(t)x = b(t), \tag{5}$$

is called a first order linear differential equation.

Linear ODEs: Solution

To solve this kind of ODE, we multiply both sides of (5) by the integrating factor given as

$$I(t) = e^{\int a(t)\ dt}$$

This gives

$$\left(\frac{dx}{dt} + a(t)\right) e^{\int a(t)\ dt} = b(t)e^{\int a(t)\ dt}$$

This is equivalent to

$$\frac{d}{dt}\left(x(t)e^{\int a(t)dt}\right) = b(t)e^{\int a(t)\ dt}$$

The solution after integrating both sides is

$$x(t) = \frac{1}{I(t)} \int b(t)I(t)\ dt$$

Linear ODEs: Example

Solve the DE given as

$$\frac{dx}{dt} + x = \sin(t)$$

Here, $a(t) = 1$ and $b(t) = \sin(t)$. Therefore, the integrating factor is

$$I(t) = e^{\int 1 \ dt} = e^{t}$$

Multiplying the ODE by I(t), we have

$$e^{t} \frac{dx}{dt} + e^{t} x = e^{t} \sin(t)$$

This is equivalent to

$$\frac{d}{dt} \left(x e^{t} \right) = e^{t} \sin t$$

Linear First Order ODEs: Example

This is equivalent to

$$\frac{d}{dt}\left(xe^{t}\right) = e^{t}\sin t$$

Integrating both sides gives

$$xe^{t} = \frac{1}{2}e^{t}(\sin t - \cos t) + C$$

$$\Rightarrow x(t) = \frac{1}{2}(\sin t - \cos t) + Ce^{-t}$$

Exact ODEs

An ODE of the form

$$N(x,y)\frac{dy}{dx} + M(x,y) = 0 \tag{6}$$

is said to be **exact** if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{7}$$

The solution is given as

$$\int M\ dx + \int (\text{terms of } N \text{ not containing } x)dy = c$$

Homogeneous ODEs

A differential equation of the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \tag{8}$$

is called a **first-order homogeneous ODE**.
Example: Is this an homogeneous ODE?

$$\frac{dy}{dx} = \frac{3xy + y^2}{3x^2 + xy}$$

$$\frac{dy}{dx} = \frac{3\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2}{3 + \left(\frac{y}{x}\right)}$$

Homogeneous ODEs: Solution

How can we solve this kind of ODE?

If we define $v = \frac{y}{x}$, then $y = vx$, and thus

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Hence, equation (8) becomes

$$x\frac{dv}{dx} = f(u) - u, \tag{9}$$

which is separable.

Homogeneous ODEs: Example

In our example, we obtain the following equation for v:

$$x\frac{dv}{dx} = 0 \qquad \Rightarrow v = c$$

The general solution is thus given by

$$y(x) = cx$$

Bernoulli Equations

Bernoulli equations are of the form

$$\frac{dy}{dx} + a(x)y = b(x)y^n, \tag{10}$$

where $n \in \mathbb{R} \backslash \{1\}$.
By change of variable, we can reduce equation (10) to the linear form.

Bernoulli Equations: Solution

On dividing (10) by y^n, we get

$$y^{-n}\frac{dy}{dx} + y^{1-n}a(x) = b(x) \tag{11}$$

Substitute $v = y^{1-n}$, so that

$$y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dv}{dx}$$

Therefore, (11) becomes

$$\frac{1}{1-n}\frac{dv}{dx} + a(x)v = b(x)$$

This is nothing but a linear equation and can be solved easily using the method previously discussed.

Bernoulli Equations: Example

Solve the ODE given below

$$y' + y = y^4.$$

Solution. We start by dividing through by y^4, and this gives

$$y^{-4}y' + y^{-3} = 1$$

Put $v = y^{-4}$

$$-\frac{1}{3}v' + v = 1$$

This is a linear equation and be solved easily.

Second Order Linear ODEs with Constant Coefficients

The general form is given as

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0y = f(x), \qquad a_0, a_1 \in \mathbb{R}. \tag{12}$$

- If $f(x) = 0$, the ODE is said to be **homogeneous**. Otherwise, it is **inhomogeneous**

Second Order ODEs: Solution

The method of solving this kind of ODE is summarized below.

- Let the solution be of the form $y(x) = e^{rx}$, where $r \in \mathbb{R}$ that would be determined.
- Substitute $y(x)$ into the ODE to get the **characteristics** equation

$$r^2 + a_1 r + a_0 = 0 \tag{13}$$

- Depending on the nature of the roots of the characteristics equation, we obtain the fundamental set of solutions according to Table (1). The general solution is

$$c_1 y_1(x) + c_2 y_2(x), \qquad c_1, c_2 \in \mathbb{R}$$

Second Order Linear ODEs with Constant Coefficients

Table: Fundamental sets of solution for equation (12)

Roots	Fundamental set of solutions
r_1, r_2 real, distinct	$y_1(x) = e^{r_1 x}, \quad y_2(x) = e^{r_2 x}$
$r_1 = r_2$, real	$y_1(x) = e^{r_1 x}, \quad y_2(x) = xe^{r_2 x}$
$r = \alpha \pm i\beta$	$y_1(x) = e^{\alpha x}\cos(\beta x), \quad y_2(x) = e^{\alpha x}\sin(\beta x)$

Second Order Linear ODEs with Constant Coefficients

Solve the following second order ODEs.

1. $y'' - 8y' + 15y = 0$
2. $\ddot{x}(t) + 4\dot{x}(t) + 5y = 0$

Solution to (1):
The characteristics equation is

$$r^2 - 8r + 15 = 0$$

Its roots are $r_1 = 3, r_2 = 5$.
Hence, the general solution is

$$y(t) = c_1 e^{3t} + c_2 e^{5t}$$

Solution to (2):

$$\ddot{x} + 4\dot{x} + 5x = 0$$

$$r^2 + 4r + 5 = 0$$

$$r = -2 \pm i$$

$$x(t) = e^{-2t}(A\cos t + B\sin t)$$