Assignment 1

Propositional Logic and Proofs

Evariste Migabo, Emmanuel Ansah, Miarisoa Elalie

August 1, 2025

1 Propositional Logic

Use the uploaded materials in the google classroom as a guide in this section.

Question 1.1 [10 pts]

Using truth tables, show whether or not the following statements are tautologies, contradictions, or contingencies.

(a) $[(p \vee q) \wedge (r \vee \neg q)] \Rightarrow (p \vee r)$

(b) $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$

(c) $[\neg q \wedge (p \Rightarrow q)] \Rightarrow \neg p$

(d) $\neg q \wedge [(p \Rightarrow q) \Leftrightarrow (\neg p \vee r)]$

Question 1.2 [20 pts]

(a) The statement p is given by:
p : The number of elements in $\{0\}$ is 0.
What is the truth value of p? Justify your answer.

(b) State the truth value of the following statement and justify your answer. Write its negation.

$$\forall x \in \mathbb{Q},\ x^2 - 4 \neq 0.$$

(c) Let
p : Today is sunny.
q : It is a good day for walking on the beach.
Write the **converse**, **contrapositive**, and **inverse** of the conditional statement $p \Rightarrow q$.

2 Proof, Sets and Functions

Refer to the uploaded materials in the google classroom as a guide in this section.

Question 2.1 [20 pts]

(a) Let A and B be any two sets and $\emptyset$ the empty set. Show that if $A \times B = \emptyset$ then $A = \emptyset$ or $B = \emptyset$.

(b) Prove that the sum of three consecutive cubes is divisible by 9.

(c) Show that the sum and product of a rational number (non-zero for multiplication) and an irrational number are irrational numbers.

(d) Suppose x, y, a and b are integers. Prove that if $ax + by = 1$ then $\gcd(a, b) = 1$.

(e) Let $p > 3$ be a prime and $a \in \mathbb{N}$, find the

$$\gcd(a+1, \frac{a^p + 1}{a+1})$$

Question 2.2 [15 pts]

Prove the following by using mathematical induction.

(a) $1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \cdots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}$

(b) $\displaystyle\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$, for any integer $n \geq 1$.

(c) For all positive integer n, $11^{n+1} + 12^{2n-1}$ is divisible by 133.

Question 2.3 [15 pts]

(a) Let n be an odd integer. Show that $n^2 + n$ is an even integer.

(b) Let $a, b, c \in \mathbb{Z}$. Show that if $a \mid b$ and $b \mid c$, then $a \mid c$.

(c) Provide a counterexample to disprove the following statement:
For all integers $a, b, c \in \mathbb{Z}$, if $a \mid bc$, then $a \mid b$ or $a \mid c$. Now, which condition can you impose so that the statement becomes true?

Question 2.4 [20 pts]

1. Let f, g and h be the real-valued functions on $\mathbb{R}$ defined by:

$$f(x) := \frac{1}{1+x^2}, \quad g(x) := |x|, \quad \text{and} \quad h(x) := x + 1.$$

   Write the formulas for the compositions $f \circ g$, $f \circ h$, $g \circ h$, $g \circ f$, $h \circ g$, $h \circ f$ and examine which of these functions are injective, surjective, bijective and equal.

   Bonus: [**30 pts**] Sketch each of the functions.