Assignment 1

Propositional Logic and Proofs

Pearl Kuuridong

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1 Propositional Logic

Question 1.1

Let $p, q,$ and r be propositions.

a) Suppose you have a compound statement

$$S : [(p \vee q) \wedge (r \vee \neg q)] \Rightarrow (p \vee r)$$

Then the truth table for S is given below:

p	q	r	$p \vee q$	$\neg q$	$r \vee \neg q$	$(p \vee q) \wedge (r \vee \neg q)$	$p \vee r$	$[(p \vee q) \wedge (r \vee \neg q)] \Rightarrow (p \vee r)$
T	T	T	T	F	T	T	T	T
T	T	F	T	F	F	F	T	T
T	F	T	T	T	T	T	T	T
T	F	F	T	T	T	T	T	T
F	T	T	T	F	T	T	T	T
F	T	F	T	F	F	F	F	T
F	F	T	F	T	T	F	T	T
F	F	F	F	T	T	F	F	T

Notice that the statement S is always true thus S is a tautology.

b) Suppose you have a compound statement

$$S : [(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$$

Then the truth table for S is given below:

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r)$	$p \Rightarrow r$	$[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Likewise, since the compound statement S is always true then S is a tautology.

c) Let S be the compound statement

$$S : [\neg q \wedge (p \Rightarrow q)] \Rightarrow \neg p.$$

Then the truth table for S is given below:

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$	$\neg q \wedge (p \Rightarrow q)$	$[\neg q \wedge (p \Rightarrow q)] \Rightarrow \neg p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

Hence since the compound statement S is always true then S is a tautology.

d) Given the compound statement

$$S : \neg q \wedge [(p \Rightarrow q) \Leftrightarrow (\neg p \vee r)].$$

Then the truth table for S is given below:

p	q	r	$\neg q$	$\neg p$	$q \vee r$	$p \Rightarrow q$	$(\neg p \Rightarrow q) \Rightarrow (q \vee r)$	$\neg q \wedge [(p \Rightarrow q) \Leftrightarrow (\neg p \vee r)]$
T	T	T	F	F	T	T	T	F
T	T	F	F	F	F	T	F	F
T	F	T	T	F	T	F	F	F
T	F	F	T	F	F	F	T	T
F	T	T	F	T	T	T	T	F
F	T	F	F	T	T	T	T	F
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

In this case, the truth values for S change with changing input of $p, q,$ and r. Hence S is a contingency.

Question 1.2

a) Let p be the statement

$$p : \text{The number of elements in } \{0\} \text{ is } 0.$$

Then p is a false statement since the cardinality of $\{0\}$ is 1. That is the set contains a single element (the number 0).

b) Consider the statement

$$p : \forall x \in \mathbb{Q}, x^2 - 4 = 0.$$

This statement is false. This is because there exists a number $2 \in \mathbb{Q}$ for which

$$x^2 - 4 = 2^2 - 4 = 4 - 4 = 0$$

Hence p is a false statement. The negation of p is given by the statement

$$\neg p : \exists x \in \mathbb{Q}, x^2 - 4 = 0.$$

c) Let p and q be the statements:

p : Today is sunny
q : It is a good day for walking on the beach

and consider the conditional statement $S : p \Rightarrow q$. Then

(a) The converse of S is given by the statement $q \Rightarrow p$. That is the statement *If it is a good day for walking on the beach then today is sunny*)

(b) The inverse of S is given by the statement $\neg p \Rightarrow \neg q$. That is the statement *If today is not sunny then it is not a good day for walking on the beach*)

(c) The contrapositive of S is given by $\neg q \Rightarrow \neg p$. That is the statement *If it is not a good day for walking on the beach then today is not sunny*)

2 Proof, Sets and Functions

Question 2.1

a) Let A and B be any two sets, and $\emptyset$ be the empty set. Suppose that $A \times B = \emptyset$ and that $A \neq \emptyset$ and $B \neq \emptyset$. Then by definition, there exists at least one element $a_1 \in A$ and another element $b_1 \in B$. Also from the definition of the cross product of sets, we know that $(a_1, b_1) \in A \times B$ and so $A \times B \neq \emptyset$. But this conclusion contradicts our assumption that $A \times B = \emptyset$. Hence our assumption that $A \neq \emptyset$ and $B \neq \emptyset$. is false and so either $A = \emptyset$ or $B = \emptyset$.

b) Let a be an integer, and consider the following consecutive integers: $a, a+1, a+2$. Without loss of generality, suppose, also, that a is non-negative. That is $a \geq 0$. Consider the case where $a = 0$. Then we have that

$$0^3 + 1^3 + 2^3 = 9 = 9 \cdot 1$$

That is, the sum, S, of the cubes of three consecutive integers $0, 1, 2$ can be written as a multiple of 9. Hence S is divisible by 9.
Now assume that for some $a = k$ the following is true:

$$k^3 + (k+1)^3 + (k+2)^3$$

is divisible by 9. This means that there exists some $t \in \mathbb{Z}$ such that

$$k^3 + (k+1)^3 + (k+2)^3 = 9t \tag{1}$$

Now consider the case where $a = (k+1)$. Consider the sum

$$(k+1)^3 + ((k+1)+1)^3 + ((k+1)+2)^3 = (k+1)^3 + (k+2)^3 + (k+3)^3 \tag{2}$$

From 1 we have that $(k+1)^3 + (k+2)^3 = 9t - k^3$ for $t \in \mathbb{Z}$ hence 2 becomes

$$\begin{aligned}
(k+1)^3 + ((k+1)+1)^3 + ((k+1)+2)^3 &= (k+1)^3 + (k+2)^3 + (k+3)^3 \\
&= 9t - k^3 + (k+3)^3 \\
&= 9t - k^3 + k^3 + 9k^2 + 27k + 27 \\
&= 9t + 9k^2 + 27k + 27 \\
&= 9(t + k^2 + 3k + 3)
\end{aligned}$$

Now since $\mathbb{Z}$ is closed under the usual operations of multiplication and addition, then $(t + k^2 + 3k + 3) \in \mathbb{Z}$ and so $9(t + k^2 + 3k + 3) = 9 \cdot t_0$ for $t_0 = (t + k^2 + 3k + 3) \in \mathbb{Z}$. So the sum $(k+1)^3 + ((k+1)+1)^3 + ((k+1)+2)^3 = (k+1)^3 + (k+2)^3 + (k+3)^3$ is divisible by 9.

Hence, by the principle of mathematical induction, the sum of the cubes of 3 consecutive non negative integers is divisible by 9. Notice that for $a < 0$ the same argument holds since choosing

$a \geq 0$ does not impose any restrictions on a so that $a < 0$ fails at any point in the argument.

Thus the sum of the cubes of 3 consecutive integers is always divisible by 9.

c) Let a be an arbitrary rational number and b be an arbitrary irrational number.

- Suppose that the sum of a and b is a rational number. Then there exists some integers $x, y \in \mathbb{Z}$ with $y \neq 0$ such that
$$a + b = \frac{x}{y}.$$
This would mean that we can write b as
$$b = \frac{x}{y} - a = \frac{x - ay}{y}$$
since $\mathbb{Z}$ is closed under the operations of the usual multiplication and subtraction (inverse of addition), then $x - ay \in \mathbb{Z}$. That is to say, there exists integers t_0 and y such that $b = \frac{t_0}{y}$. But this conclusion contradicts our assumption that b is an irrational number and so the sum $a + b$ must be irrational.

- Now, in addition, assume that $a \neq 0$ and that the product $a \cdot b$ is rational. Then there exists some $x, y \in \mathbb{Z}$ with $y \neq 0$ such that $a \cdot b = \frac{x}{y}$. That is to say b can be written as $b = \frac{x}{ay}$. Again $ay \in \mathbb{Z}$ since $\mathbb{Z}$ is closed under the operation of integer multiplication. That is there exist integers $x, t_0 = ay, t_0 \neq 0$ such that $b = \frac{x}{t_0}$. But this contradicts our assumption that b is an irrational number. Hence the product $a \cdot b$ must be irrational.

d) Let a, b, x, y be integers. Suppose that $ax + by = 1$ and that the $\gcd(a, b) = d$ for some $d \in \mathbb{Z}$ then there exists some integers m, n so that $a = dm$ and $b = dn$. So we have that $ax + by = dmx + dny = 1$ that is $d(mx + ny) = 1$ and so $d \mid 1$ now since 1 has only one divisor, which is itself then $d = 1$. Hence $\gcd(a, b) = 1$.

Question 2.2

a) Consider the statement
$$S : 1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \cdots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}.$$
We want to show that S is true for all $n \in \mathbb{N}$. Consider the case when $n = 1$. Then we have that $1 = 4 - 3 = 4 - \left(\frac{1+2}{2^{1-1}}\right)$. Thus we can see that $S(1)$ holds. Now assume that $S(k)$ is true for some $k \in \mathbb{N}$. Now consider the case where $n = k + 1$. Then $S(k + 1)$ becomes
$$\begin{aligned}
1 + 2\left(\frac{1}{2}\right) + \cdots + k\left(\frac{1}{2}\right)^{k-1} + (k+1)\left(\frac{1}{2}\right)^k &= 4 - \frac{k+2}{2^{k-1}} + (k+1)\left(\frac{1}{2}\right)^k \\
&= 4 + \left(\frac{-2(k+2) + k + 1}{2^k}\right) \\
&= 4 + \left(\frac{-k-3}{2^k}\right) \\
&= 4 - \left(\frac{(k+1) + 2}{2^k}\right)
\end{aligned}$$
That is $S(k + 1)$ is true. Thus by the principle of mathematical induction, we can say that
$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \cdots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}$$
is true for all $n \in \mathbb{N}$

b) We want to show that

$$\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}, \text{ for any integer } n \geq 1$$

Consider the case where $n = 1$. Then we have

$$\sum_{r=1}^{1} r^3 = 1^3 = 1 = \frac{1^2(1+1)^2}{4}$$

And so the statement holds true for $n = 1$. Assume that, for $n = k$, the statement holds true. That is

$$\sum_{r=1}^{k} r^3 = \frac{k^2(k+1)^2}{4} \tag{3}$$

and consider the case where $n = (k+1)$. Then we have that

$$\sum_{r=1}^{k+1} r^3 = \sum_{r=1}^{k} r^3 + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3 (\text{from } 3)$$

$$= (k+1)^2 \left[\frac{k^2}{4} + (k+1)\right]$$

$$= (k+1)^2 \left[\frac{k^2+4k+4}{4}\right]$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$$= \frac{(k+1)^2((k+1)+1)^2}{4}$$

That is

$$\sum_{r=1}^{k+1} r^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$$

for some $n = k+1$ given that the statement

$$\sum_{r=1}^{k} r^3 = \frac{k^2(k+1)^2}{4}$$

is true for some $k \in \mathbb{N}$. And so by the principle of mathematical induction, we have that the statement

$$\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$$

holds true for any integer $n \geq 1$

c) Let n be a positive integer. Consider the statement

$$S : 11^{n+1} + 12^{2n-1} \text{ is divisible by } 133 \text{ for all positive integers } n.$$

We want to show that S is true.

Consider the case where $n = 1$ then we have that $S(1) : 11^2 + 12 = 133(1)$. So for $n = 1, 11^{1+1} + 12^{2(1)-1}$ is divisible by 133.

Assume that for some $n = k \in \mathbb{N}$

$$S(k) : 11^{k+1} + 12^{2k-1} \text{ is divisible by } 133 \tag{4}$$

is true. This means that there exists some $t \in \mathbb{Z}$ such that $11^{k+1} + 12^{2k-1} = 133t$ which is equivalent to $11^k = \frac{133(12t)-12^{2k}}{132}$

Now consider the case where $n = (k+1)$. We want to check that $S(k+1)$ is true. Notice that

$$\begin{aligned}
11^{(k+1)+1} + 12^{2(k+1)-1} &= 11^{k+2} + 12^{2k+1} \\
&= 11^2(11^k) + 12(12^{2k}) \\
&= 11^2\left(\frac{133(12t) - 12^{2k}}{132}\right) + 12(12^{2k}) \\
&= 11\left(\frac{133(12t) - 12^{2k}}{12}\right) + 12(12^{2k}) \\
&= \frac{11(133(12t) - 12^{2k}) + 144(12^{2k})}{12} \\
&= \frac{133(132t) - 11(12^{2k}) + 144(12^{2k})}{12} \\
&= \frac{133(132t) + 12^{2k}(144 - 11)}{12} \\
&= \frac{133(132t) + 133(12^{2k})}{12} \\
&= 133\left(\frac{132t + 12^{2k}}{12}\right) \\
\\
&= 133(11t + 12^{2k-1}) = 133t_0 \qquad (t_0 = 11t + 12^{2k-1})
\end{aligned}$$

since $k \in \mathbb{Z}$ and $k > 0$ then $t_0 = 11t + 12^{2k-1} \in \mathbb{Z}$. Thus by the principle of mathematical induction the statement S is true for all positive integers n.

Question 2.3

a) Let n be an odd integer. Then there exists an integer $k \in \mathbb{Z}$ such that $n = 2k+1$. Consider the expression $n^2 + n$. Then since $n = 2k+1$ this expression becomes

$$\begin{aligned}
n^2 + n &= (2k+1)^2 + (2k+1) \\
&= 4k^2 + 4k + 1 + 2k + 1 \\
&= 4k^2 + 6k + 2 \\
&= 2(2k^2 + 3k + 1) \\
&= 2t_0 \qquad (t_0 = 2k^2 + 3k + 1)
\end{aligned}$$

Now since $k \in \mathbb{Z}$ then $t_0 = 2k^2 + 3k + 1 \in \mathbb{Z}$. Now since there exists an integer, t_0 for which $n^2 + n = 2t_0$ then $n^2 + n$ is an even integer.

b) Let $a, b, c \in \mathbb{Z}$. Suppose that $a \mid b$ then there exists some $t_1 \in \mathbb{Z}$ such that $b = at_1$. Suppose also that $b \mid c$ then there exists some $t_2 \in \mathbb{Z}$ such that $c = bt_2$. That means

$$\begin{aligned}
c &= (at_1)t_2 \\
&= a(t_1t_2) \qquad \text{(integer multiplication is associative)} \\
&= at_0 \qquad (t_0 = t_1t_2)
\end{aligned}$$

Since both $t_1, t_2 \in \mathbb{Z}$ then $t_0 = t_1 t_2 \in \mathbb{Z}$. That is there exists some integer such that $c = at_0$ hence $a \mid c$.

c) Consider the following statement:

$$\text{for all integers } a, b, c \in \mathbb{Z} \text{ if } a \mid bc \text{ then } a \mid b \text{ or } a \mid c$$

Consider the following example $a = 4, b = 2, c = 6$. Notice that $4 \mid (6 \cdot 2)$ but $4 \nmid 2$ and $4 \nmid 6$ hence the statement

$$\text{for all integers } a, b, c \in \mathbb{Z} \text{ if } a \mid bc \text{ then } a \mid b \text{ or } a \mid c$$

is false. We can rephrase the statement as

$$\text{for all integers } a, b, c \in \mathbb{Z} \text{ with } a \text{ prime if } a \mid bc \text{ then } a \mid b \text{ or } a \mid c$$

Which makes the statement true.

Question 2.4

1. Let f, g and h be real-valued functions on $\mathbb{R}$ defined by:

$$f(x) := \frac{1}{1+x^2},\ g(x) : |x|, \text{ and } h(x) := x + 1$$

- Then the function $f \circ g$ is given as $\frac{1}{1+x^2}$.

- The function $f \circ h$ is given as $\frac{1}{x^2+2x+2}$

- The function $g \circ h$ is given as $|x + 1|$

- The function $g \circ f$ is given as $\frac{1}{1+x^2}$

- The function $h \circ g$ is given as $|x| + 1$

- The function $h \circ f$ is given as $\frac{2+x^2}{1+x^2}$

Notice that $f \circ g(x) = g \circ f(x) = f(x) = \frac{1}{1+x^2}$. So the functions $f \circ g(x), g \circ f, f$ are equal

We will now check for injectivity and surjectivity for $f, g,$ and h

Consider the function f and consider the real numbers $1, -1$ notice that $f(1) = \frac{1}{2}$ and $f(-1) = \frac{1}{2}$. So f is not injective. Notice, also, that $1 + x^2 \geq 1$ for all x so $0 < \frac{1}{1+x^2} \leq 1$ so f is not surjective when it has the codomain $\mathbb{R}$. Since f is not injective then f is not bijective.

Consider the function g. Notice that $g(1) = g(-1)$ hence g is not injective. Similarly, notice that $g(x) \geq 0$ for all x. So g is not surjective. Since g is not injective then g is not bijective

Finally consider the function h. Consider some real numbers $a, b \in \mathbb{R}$ and assume that $h(a) = h(b)$. Then $a+1 = b+1 \Rightarrow a = b$. Hence h is an injective function. Also consider the number $y \in \mathbb{R}$ in the codomain of h. Then there is a number $y-1 \in \mathbb{R}$ in the domain of h so that $h(y-1) = y-1+1 = y$. Hence h is also a surjective function. Since h is both injective and surjective then h is a bijective function.

Now since g is not injective then $f \circ g$ is also not injective. Since there exists some $-1, 1 \in \mathbb{R}$ for which $f \circ g(1) = f(1) = \frac{1}{2} = f(1) = f \circ g(-1)$. Since f is not surjective then $f \circ g$ is also not surjective. since there exists no real number $x \in \mathbb{R}$ for which $f \circ g(x) = -1$. Since $f \circ g$ is neither injective nor surjective then $f \circ g$ is not bijective

For the function $f \circ h$ consider the real numbers $0, -2 \in \mathbb{R}$. Notice that $f \circ h(0) = f(1) = \frac{1}{2} = f(-1) = f \circ h(-2)$. So $f \circ h$ is not injective. Now since f is not surjective then $f \circ h$ is not surjective since there exists a value $-1 \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$ $f \circ h(x) \neq -1$. Hence $f \circ h$ is not bijective

For the function $g \circ h$ since g is not injective, even though h is injective, the composition $g \circ h$ is not injective. Again consider the real numbers $0, -2$. Then $g \circ h(0) = g(1) = 1 = g(-1) = g \circ h(-2)$. Since g is not surjective then $g \circ h$ is also not surjective. That is for all $x \in \mathbb{R}, g \circ h(x) \neq -1$. So $g \circ h$ is not bijective.

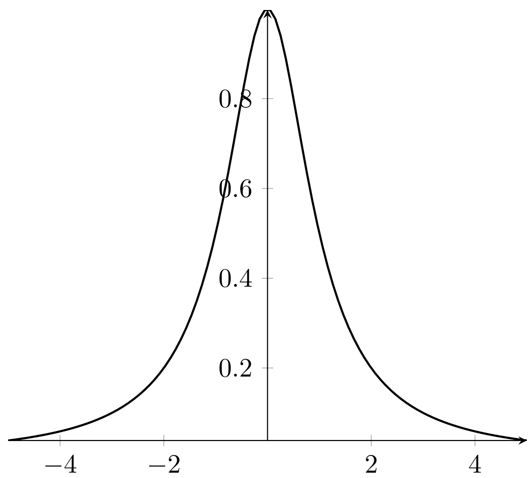
For the function $g \circ f$ since f is not injective then $g \circ f$ is also not injective. Consider $1, -1$ then $g \circ f(1) = g(1/2) = \frac{1}{2} = g(1/2) = g \circ f(-1)$. Similarly, since g is not surjective then $g \circ f$ is also not surjective. That is for all $x \in \mathbb{R}, g \circ f(x) \neq -1$. So $g \circ f$ is not bijective.

For the function $h \circ g$ since g is not injective then $h \circ g$ is also not injective. Consider $1, -1$ then we have that $h \circ g(1) = h(1) = 2 = h(1) = h \circ g(-1)$. Consider $-1 \in \mathbb{R}$ then for all $x \in \mathbb{R}, h \circ g(x) \neq -1$ so $h \circ g$ is not surjective. Hence $h \circ g$ is not bijective.

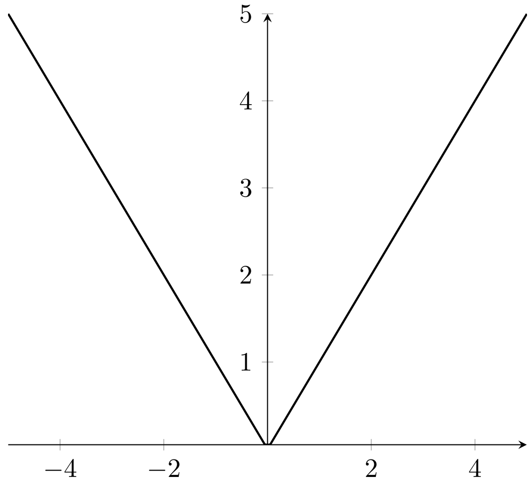
For the function $h \circ f$ since f is not injective then $h \circ f$ is also not injective. Consider $1, -1$ then $h \circ f(1) = h(1/2) = \frac{3}{2} = h(1/2) = h \circ f(-1)$. Consider the number $-1 \in \mathbb{R}$ since $h \circ f$ is a quotient of two positive values then $h \circ f(x) \neq -1$. So $h \circ f$ is not bijective.

BONUS

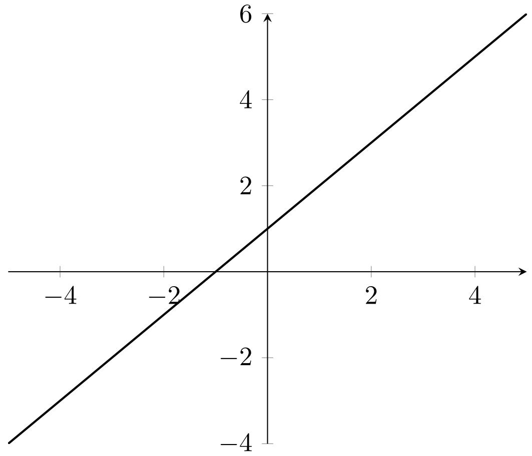
The plot for $f = \frac{1}{1+x^2}$ is given as



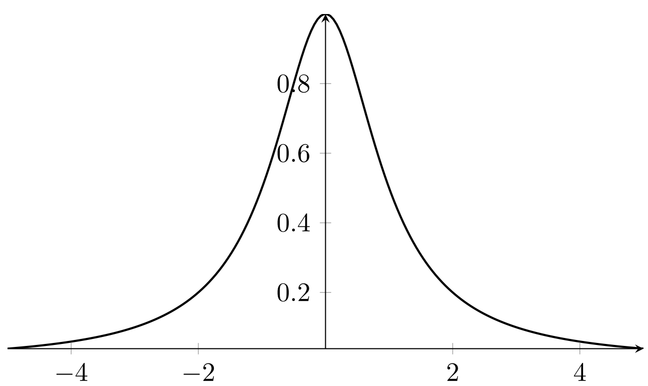
The plot for $g = |x|$ is given as



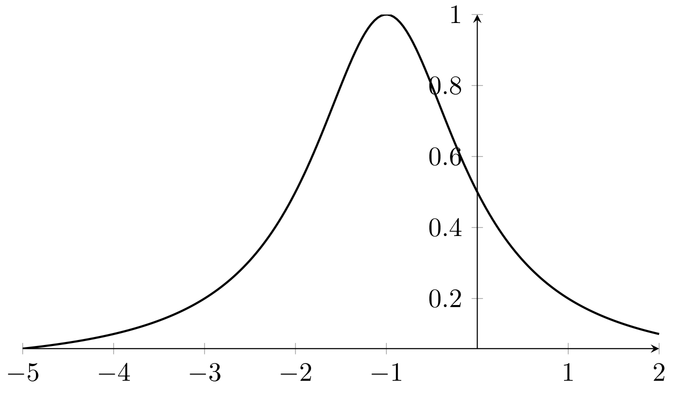
The plot for $h = x + 1$ is given as



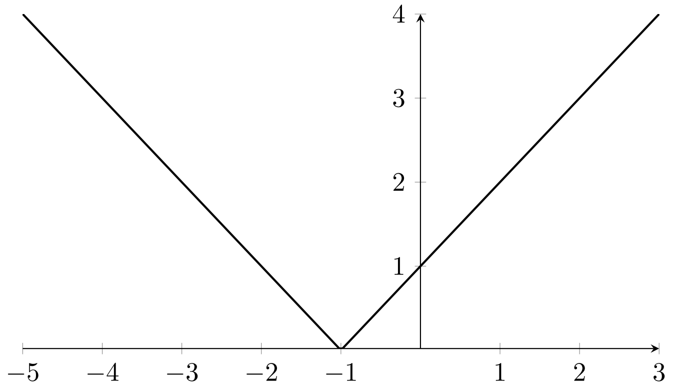
The plot for $f \circ g(x) = \frac{1}{1+x^2}$ is given as



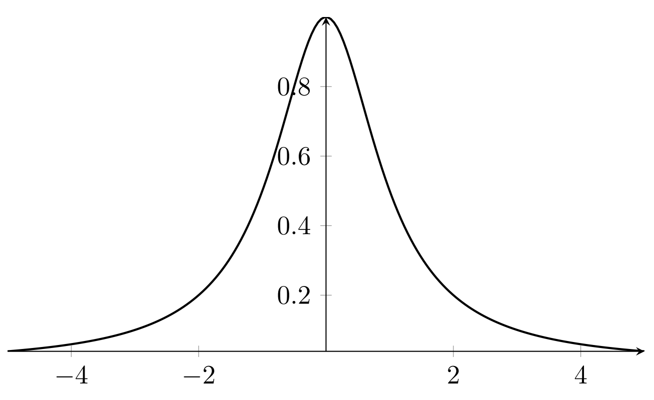
The plot for $f \circ h(x) = \frac{1}{x^2+2x+2}$ is given as



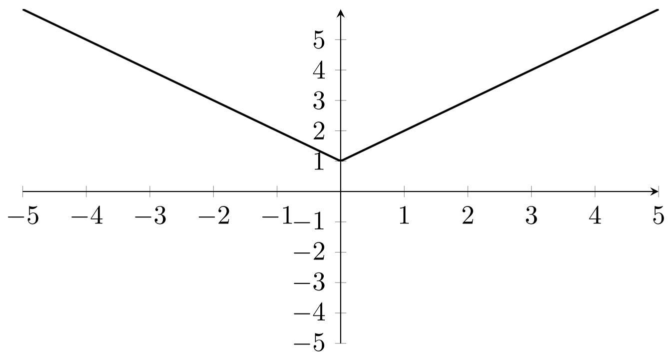
The plot for $g \circ h(x) = |x+1|$ is given as



The plot for $g \circ f(x) = \frac{1}{1+x^2}$ is given as



The plot for $h \circ g(x) = |x| + 1$ is given as



The plot for $h \circ f(x) = \frac{2+x^2}{1+x^2}$ is given as