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ODES - Ordinary Differential Equations.

ODEs are equations involving an unknown function of one variable and its derivatives.

$$F(t, x(t), \dot{x}(t), \ldots) = 0$$

$x(t) \to$ unknown function
$t \to$ independent variable

Notation:

$$\frac{dy}{dx} = y'(x) = \dot{y}(x)$$

Linear ODEs: If it is linear in the unknown function, $x(t)$ and its derivative $x'(t), x''(t), \ldots$

Otherwise, it is <u>non-linear</u>.

1. $y'' + y' + 2xy = x^3 + 2x$ $\qquad 2x \cdot y$
   Linear $\qquad$ linear w.r.t y

2. $(\cos x) y' - y^2 = x^3$: y^2 makes it non-linear

Order and degree of an ODE

$$x''(t) + (x'(t))^3 = t^2 + 2$$: order $= 2$, degree $= 1$

$$\left.\begin{aligned} F(t, x(t), \dot{x}(t), \ldots) &= 0 \\ x(t_0) &= x_0, \\ x'(t_0) &= x'_0 \\ &\vdots \\ x^{(n-1)}(t_0) &= x_0^{(n-1)} \end{aligned}\right\}$$ Initial Value Problem

IVP

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Hadamard's Principle of wellposedness.

1. solution exists
2. solution is unique
3. solution depends continuously on the given initial data.

First order ODEs

$$F(t, x(t), \dot{x}(t)) = 0 \qquad 1^{st} \text{ order ODE}$$

1. Variable Separable ODEs

$$\frac{dx}{dt} = f(t) \cdot g(x)$$

$$\frac{dx}{g(x)} = f(t)\,dt \underset{\text{integrating both sides}}{\Rightarrow} \int \frac{dx}{g(x)} = \int f(t)\,dt$$

Example: Solve $\frac{dx}{dt} = 2x^2 t$.

$$\frac{dx}{x^2} = 2t\,dt \Rightarrow \int \frac{dx}{x^2} = \int 2t\,dt$$

$$\Rightarrow -\frac{1}{x} = t^2 + C$$

$$\Rightarrow x = -\frac{1}{t^2+C}, \quad C \in \mathbb{R}$$

$\rightarrow$ General solution

Is this DE well-posed? No!

To solve that, we impose an initial condition

$$\begin{cases} \frac{dx}{dt} = 2x^2 t, \quad x(0) = 1 \end{cases} \qquad \text{IVP}$$

$$x(t) = -\frac{1}{t^2+C}$$

$$x(0) = -\frac{1}{0^2+C} \Leftrightarrow 1 = \frac{-1}{C} \Leftrightarrow C = -1$$

$$x(t) = -\frac{1}{t^2-1} = \frac{1}{1-t^2}$$

Example: Solve $\frac{dy}{dx} = \frac{x^2}{y}$, $y(1) = 1$

$$\int y\,dy = \int x^2\,dx$$

$$\frac{y^2}{2} = \frac{x^3}{3} + C$$

$$\frac{[y(1)]^2}{2} = \frac{1^3}{3} + C \Leftrightarrow \frac{1}{2} = \frac{1}{3} + C \Leftrightarrow C = \frac{1}{6}$$

$$\frac{y^2}{2} = \frac{x^3}{3} + \frac{1}{6}$$

$$\Rightarrow y^2 = \frac{2x^3+1}{3}$$

§ Linear 1^{st} order ODEs.

A first order ODE is said to be <u>linear</u> if it is of the form

$$\frac{dx}{dt} + a(t)x = b(t) \quad \text{——} (*)$$

Ex: $x' + \frac{1}{t}x = t^2$, $a(t) = \frac{1}{t}$, $b(t) = t^2$

To solve this kind of ODE, we apply integrating factor

$$I(t) = e^{\int a(t)\,dt}$$

Multiply $(*)$ by $I(t)$

$$e^{\int a(t)dt}\left(\frac{dx}{dt} + a(t)x\right) = b(t)e^{\int a(t)dt}$$

$$\int \frac{d}{dt}\left(x e^{\int a(t)\,dt}\right) = \int b(t)e^{\int a(t)dt}\,dt$$

$$\frac{dx}{dt}e^{\int a(t)dt} + a(t)xe^{\int a(t)dt}$$

$$x e^{\int a(t)dt} = \int b(t)e^{\int a(t)dt}\,dt$$

$$x(t) = e^{-\int a(t)dt}\int b(t)e^{\int a(t)dt}\,dt.$$

Example: $\frac{dx}{dt} + x = \sin(t)$.

$$a(t) = 1, \quad b(t) = \sin(t)$$

$$I(t) = e^{\int a(t)\,dt} = e^{\int 1 \cdot dt} = e^{t}.$$

Multiply the DE by $I(t)$

$$e^t\left(\frac{dx}{dt} + x\right) = e^t \sin(t)$$

$$e^t\frac{dx}{dt} + xe^t$$

$$e^t\frac{dx}{dt} + x \cdot \frac{d}{dt}(e^t)$$

$$\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dt}(e^t \cdot x) = \frac{d}{dt}(xe^t)$$

$$\int \frac{d}{dt}(xe^t) = \int e^t \sin t\,dt$$

$$xe^t = \int e^t \sin t\,dt$$

I.B.P: $\int u\,dv = uv - \int v\,du$.

$u = \sin t$, $dv = e^t\,dt$, $v = \int e^t\,dt = e^t$.

$$\int e^t \sin t\,dt = e^t \sin t - \int e^t \cos t\,dt$$

$$= e^t \sin t - \left[e^t \cos t + \int e^t \sin t\right]$$

$$2\int e^t \sin t\,dt = e^t(\sin t - \cos t)$$

$$\int e^t \sin t\,dt = \frac{1}{2}e^t(\sin t - \cos t) + C$$

$$xe^t = \frac{1}{2}e^t(\sin t - \cos t) + C$$

$$x(t) = \frac{1}{2}(\sin t - \cos t) + Ce^{-t}$$

Exact ODEs

A first order ODE of the form

$$N(x,y)\frac{dy}{dx} + M(x,y) = 0 \Leftrightarrow M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial z}$$

$$(2xy + 3)\,dx + (x^2 + 2y)\,dy = 0$$

Is it exact?

$M(x,y) = 2xy + 3$, $N(x,y) = x^2 + 2y$

$\frac{\partial M}{\partial y} = 2x$, $= \frac{\partial N}{\partial x} = 2x$

Thus, the DE is exact!

Solution of an exact ODE.

$F(x,y) = C$, where $\frac{\partial F}{\partial x} = M(x,y)$, $\frac{\partial F}{\partial y} = N(x,y)$

$$\frac{\partial F}{\partial x} = M(x,y) \Leftrightarrow F(x,y) = \int M(x,y)\,dz + G(y).$$

$\frac{\partial F}{\partial y} = N(x,y)$ — From here, we can get $G(y)$.

Example: $(2xy+3)\,dx + (x^2+2y)\,dy = 0$

$M(x,y) = 2xy+3, \quad N(x,y) = x^2+2y$

$\exists$ $F(x,y)$ such that $\frac{\partial F}{\partial x} = M(x,y)$ and $\frac{\partial F}{\partial y} = N(x,y)$.

$$\frac{\partial F}{\partial x} = 2xy+3$$

$$F(x,y) = \int (2xy+3)\,dx$$

$$F(x,y) = x^2y + 3x + G(y) \qquad \frac{d}{dx}G(y) = 0$$

Differentiate $F(x,y)$ w.r.t y.

$$\frac{\partial F}{\partial y} = x^2 + G'(y) \iff N(x,y) = x^2 + G'(y)$$

$$x^2+2y = x^2 + G'(y)$$

$$G'(y) = 2y$$

$$\int G'(y) = \int 2y\,dy \iff G(y) = y^2$$

$$F(x,y) = x^2y + 3x + y^2$$

Solution: $x^2y + 3x + y^2 = C, \qquad C \in \mathbb{R}$

Summary:

$$\int M(x,y)\,dx + \int (\text{Term of } N \text{ without } x)\,dy = C$$

$$\int (2xy+3)\,dx + \int 2y\,dy = C$$

$$x^2y + 3x + y^2 = \underline{\underline{C}}.$$

Example: $(x^2+y^2+x)\,dx + xy\,dy = 0$

is it exact? $\quad \dfrac{2y-y}{xy} = \dfrac{y}{xy} = \dfrac{1}{x}$

1. If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of x only, say $g(x)$

$$I.F = I(x) = e^{\int g(x)\,dx}.$$

2. If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ is a function of y only, say $f(y)$

$$I(y) = e^{\int f(y)\,dy}$$

3. If $M(x,y) = y\,f(xy)$, $\quad N(x,y) = x\,g(xy)$.

$$I.F = \frac{1}{Mx - Ny}$$

Back to the example:

$$(x^2+y^2+x)\,dx + xy\,dy = 0$$

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{2y-y}{xy} = \frac{1}{x}, \; I(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x.$$

$$(x^3+xy^2+x^2)\,dx + x^2y\,dy = 0$$

$$2xy = 2xy \quad \text{It is exact!}$$

solution: $\displaystyle\int (x^3+xy^2+x^2)\,dx = C \iff \frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} + C$ //

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Thursday: $\begin{cases} \text{Homogeneous 1st order ODE} \\ \text{Bernoulli equation} \\ \text{Second order ODEs} \end{cases}$ } change of variables

Example for the 3rd case of Non-exact

$$y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$$

This is not exact!

But,

$$M(x,y) = y(xy + 2(xy)^2) = y\,f(xy) \qquad f(t) = t + 2t^2, \quad f(xy) = xy + 2(xy)^2$$

$$N(x,y) = x(xy - (xy)^2) = x\,g(xy) \qquad g(t) = t - t^2, \quad g(xy) = xy - (xy)^2$$

$$I.F = \frac{1}{Mx - Ny} = \frac{1}{xy(xy + 2(xy)^2) - xy(xy - (xy)^2)} = \frac{1}{3(xy)^2}$$

Multiply the ODE by $\frac{1}{3(xy)^2}$ to make it exact.