$E_1 = A \cap \bar{B} \cap \bar{C}$ $\quad$ $E_2 = A \cap (B \cup C)$

$E_1 \cap E_2 = \phi$ $\Rightarrow$ E_1 & E_2 are disjoint

$E_1 \cup E_2 = A$

For two disjoint events A and B,

$$P(A \cup B) = P(A) + P(B).$$

$$P(\underbrace{E_1 \cup E_2}_{A}) = P(E_1) + P(E_2) = P(A)$$

$$P(E_1) + P(E_2) = P(A)$$

$$P(E_2) = P(A \cap (B \cup C))$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$P(E_2) = P((A \cap B) \cup (A \cap C))$ | $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\begin{aligned} &= P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C)) \\ &= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C) \\ &= 0.2 + 0.1 - 0.05 \\ &= 0.25 \end{aligned}$$

$$\begin{aligned} P(E_1) &= P(A) - P(E_2) \\ &= 0.6 - 0.25 = 0.35 \end{aligned}$$

$P(E_1) = 0.35$, $P(E_2) = 0.25$

$E_1 = A \cap \bar{B} \cap \bar{C} \qquad E_2 = A \cap (B \cup C)$

$E_1 \cap E_2 = (\underline{A \cap \bar{B} \cap \bar{C}}) \cap (A \cap (B \cup C))$

$\bar{A} \cap \bar{B} = \overline{(A \cup B)} = A \bar{\cup} B$

$A^c \cap B^c = (A \cup B)^c$

$A^c \cup B^c = (A \cap B)^c$

$E_1 \cap E_2 = (A \cap (B \bar{\cup} C)) \cap (A \cap (B \cup C))$

$= A \cap (\underbrace{(B \bar{\cup} C)}_{\bar{X}} \cap \underbrace{(B \cup C)}_{X}) \qquad A \cap \bar{A} = \phi$

$= A \cap \phi = \phi$

$P(A_1) = 0.9, \quad P(A_2) = 0.2, \quad P(A_3) = 0.1$
$P(B|A_1) = 0.9, \quad P(B|A_2) = 0.01, \quad P(B|A_3) = 0.01$

We already know the event B – "free"
What is the probability that it is spam

$P(A_1|B) = ? \qquad P(A_i|B) = \dfrac{P(B|A_i)P(A_i)}{\sum_{j=1}^{n} P(B|A_j)P(A_j)}$

$P(A_1|B) = ?$ $\quad P(A_i|B) = \dfrac{P(B|A_i)P(A_i)}{\sum_{j=1}^{n} P(B|A_j)P(A_j)}$

$$P(A_1|B) = \frac{P(B|A_1)P(A_1)}{\sum_{j=1}^{3} P(B|A_j)P(A_j)}$$ Bayes Theorem

$$= \frac{0.9(0.7)}{0.9(0.7) + 0.01(0.2) + 0.01(0.1)}$$

$$\approx 0.995\cancel{2}6\ 06$$

$$\approx 0.995$$

Bernoulli Distribution $\quad X \sim B(1, p)$.

$\Omega = \{0, 1\}$.

$X : \Omega \to \mathbb{R} \quad \overset{\text{r.v}}{=} \quad X : \{0, 1\} \to \mathbb{R}$.

If X is a measureable function.

$X^{-1}\{[a, b]\}$ is also a borel set.

$[a, b]$ is a borel set

(1

$\boxed{r.v}$ p.d.f , c.d.f

discrete & continuous

$\Omega = \{0, 1, \dots n\}$ $\mathbb{R}$. $\rightarrow$ Exponential, Normal, Gaussian

$\downarrow$

Bernoulli, Binomial, Poisson

$F'_X(x) = f_X(x)$

$$\underline{\text{c.d.f}} = F_X(x) = \int_{x \in X} f_X(x)\,dx. = \mathbb{P}(X \leq x)$$

Exponential distribution: $x \in \mathbb{R}_+$ $(0, \infty)$

$$f_X(x) = \frac{1}{\theta} \exp\left\{-\frac{x}{\theta}\right\} \qquad \text{p.d.f}$$

$$F_X(x) = \int_0^x f_X(t)\,dt = \int_0^x \frac{1}{\theta} \exp\left\{\frac{-t}{\theta}\right\} dt$$

$$\frac{1}{\theta} \int_0^x \exp\left\{\frac{-t}{\theta}\right\} dt = \frac{1}{\theta}\left[\frac{\exp\left\{-\frac{t}{\theta}\right\}}{-1/\theta}\right]_0^x$$

$$= -\exp\left\{\frac{-t}{\theta}\right\}\Big|_0^x$$

$$= -\exp\left\{-\frac{x}{\theta}\right\} + 1$$

$$F_X(x) = 1 - \exp\left\{-\frac{x}{\theta}\right\}$$

Bernoulli Distribution

$$p^{x}(1-p)^{1-x}$$

$$P(X=x) = \sum_{k=0}^{x} p^{k}(1-p)^{1-k}$$

Example :

$$P(X=x) = \binom{n}{x} p^{x}(1-p)^{n-x}$$

$$Y = \frac{a^{X}}{2^{n}}, \quad a \in \mathbb{R} \quad . \text{ Find } E(Y).$$

$p = 1/2$

$$E(g(x)) = \sum_{x \in V} g(x) P(X=x) \qquad x = \{0, 1, \ldots, n\}$$

$$E(Y) = E\left(\frac{a^{X}}{2^{n}}\right) = \frac{1}{2^{n}} E(a^{X})$$

$$= \frac{1}{2^{n}} \sum_{x=0}^{n} a^{x} P(X=x)$$

$$= \frac{1}{2^{n}} \sum_{x=0}^{n} a^{x} \binom{n}{x} \left(\frac{1}{2}\right)^{x} \underbrace{\left(1-\frac{1}{2}\right)}^{n-x} : \left(\frac{1}{2}\right)^{x+n-\cancel{x}}$$

$$= \frac{1}{2^{n}} \sum_{x=0}^{n} a^{x} \binom{n}{x} \left(\frac{1}{2}\right)^{n} = \frac{1}{2^{n}} \cdot \frac{1}{2^{n}} \underbrace{\sum_{x=0}^{n} \binom{n}{x} a^{x}}_{(1+a)^{n}}$$

$$E(Y) = \frac{1}{4^{n}} (1+a)^{n}$$

Moment Generating function (M.G.F)

$$M_X(t) = E(e^{tX})$$

$$E(X^q) = \frac{d^q}{dt^q} M_X(t) \Big|_{t=0}$$

$$E(X) = \frac{d}{dt} M_X(t) \Big|_{t=0}$$

$$\underbrace{Var(X) = E(X^2) - (E(X))^2}$$

$$E(X^2) = \frac{d^2}{dt^2} M_X(t) \Big|_{t=0}$$

Example: Find the Variance of Bernoulli Distribution.

$$P(X=x) = p^x (1-p)^{1-x} \qquad x \in \{0,1\}$$

$$E(X) = \sum_{x=0}^{1} x\, P(X=x)$$

$$= 0 \cdot P(X=0) + 1 \cdot P(X=1)$$

$$= P(X=1)$$

$$= p^1 \cdot (1-p)^{1-1} = p\,(1-p)^0$$

$$E(X) = p$$

$$E(X^2) = \sum_{x=0}^{1} x^2 P(X=x) = 0^2 \cdot P(X=0) + 1^2 \cdot P(X=1)$$

$$= P(X=1) = p$$

$$Var(X) = E(X^2) - (E(X))^2 = p - p^2 = p(1-p)$$

$$M_X(t) = E(e^{tX})$$
$$= \sum_{x=0}^{1} e^{tx} P(X=x)$$
$$= e^{t\cdot 0} P(X=0) + e^{t\cdot 1} P(X=1)$$
$$= P(X=0) + e^t P(X=1)$$
$$= p^0(1-p)^1 + e^t p^1 (1-p)^0$$
$$= 1-p + e^t p$$

$$E(X) = \frac{d}{dt} M_X(t) \Big|_{t=0} = \frac{d}{dt}\left(1-p+e^t p\right)_{t=0}$$
$$= e^t p \Big|_{t=0} = e^0 \cdot p = p$$

$$E(X^2) = \frac{d^2}{dt^2} M_X(t) \Big|_{t=0} = e^t p \Big|_{t=0} = p$$

$$Var(X) = p - p^2 = p(1-p)$$

Let $X_1, X_2, \dots, X_n$ be n identical independent ~~r~~.v's

$$X = \sum_{i=1}^{n} X_i$$

$$\Big| \, M_X(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

$$M_X(t) = E(e^{tX}) = E\left(e^{t\sum_{i=1}^{n} X_i}\right)$$
$$= E\left[\prod_{i=1}^{n} e^{tX_i}\right) = \prod_{i=1}^{n} \underbrace{E(e^{tX_i})}_{M_{X_i}(t)}$$

If X and Y are independent, then $E(XY) = E(X) \cdot E(Y)$