Assignment 2

Ordinary Differential Equations

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1 First order ODEs

A Find the general solution for each of the following first-order ODEs.

(a) $x\frac{\mathrm{dy}}{dx} + 8y = x^2e^x$

Notice that this differential equation is a linear first-order differential equation when written in the form

$$\frac{\mathrm{dy}}{\mathrm{dx}} + \frac{8y}{x} = xe^x$$

so we would use the integrating factor $I(x) = e^{\int \frac{8}{x}\,\mathrm{dx}} = e^{8\ln x} = x^8$. So our linear first-order differential equation becomes

$$\begin{aligned} x^8\left(\frac{\mathrm{dy}}{\mathrm{dx}} + \frac{8y}{x}\right) &= x^9e^x \\ \frac{d}{\mathrm{dx}}(x^8y) &= x^9e^x \\ x^8y &= \int x^9e^x\,\mathrm{dx} \end{aligned}$$

Hence $y = \frac{1}{x^8}\int x^9e^x\,\mathrm{dx}$

(b) $xy^2 - x + (x^2y + y)\,\frac{dy}{dx} = 0$

Rewriting the equation as

$$(xy^2 - x)dx + (x^2y + y)\,dy = 0$$

it is easy to see that this is an exact differential equation since

$$\frac{\partial}{\partial x}(x^2y + y) = 2xy = \frac{\partial}{\partial y}(xy^2 - x)$$

Then the solution to the differential equation looks like

$$\int (xy^2 - x)dx = \frac{x^2y^2}{2} - \frac{x^2}{2} + h(y) \tag{1}$$

differentiating 1 with respect to y gives us

$$x^2y + h'(y)$$

Comparing this with (x^2y+y) we see that $y = h'(y) \Leftrightarrow h(y) = \frac{y^2}{2} + C,\ C \in \mathbb{R}$. So the solution to the given differential equation is

$$\frac{x^2y^2}{2} - \frac{x^2}{2} + \frac{y^2}{2} + C,\ C \in \mathbb{R}$$

(c) $(y \log y)\,dx + (x - \log y)\,dy = 0$

Notice that this is not an exact differential equation since $\frac{\partial}{\partial y}(y \log y) = 1 + \log y \neq \frac{\partial}{\partial x}(x - \log y) = 1$. So we get an integrating factor $\mu(y)$ to make the differential equation exact

$$\mu(y) = e^{\left(\int \frac{\frac{\partial}{\partial x}(x - \log y) - \frac{\partial}{\partial y}(y \log y)}{y \log y}\,dy\right)} = \frac{1}{y}$$

Multiplying through the differential equation by the integrating factor we get

$$\log y\,dx + \frac{x - \log y}{y}\,dy = 0 \tag{2}$$

and now we can see that
$$\frac{\partial}{\partial y}(\log y) = \frac{1}{y} = \frac{\partial}{\partial x}\left(\frac{x - \log y}{y}\right)$$

hence equation 2 is exact.

$$\begin{aligned}
\int \log y\,dx &= x \log y + h(y) \quad \text{h is some function of } y \\
\frac{\partial}{\partial y}(x \log y + h(y)) &= \frac{x}{y} + h'(y) \\
\Leftrightarrow \quad \frac{x}{y} + h'(y) &= \frac{x - \log y}{y} \\
\Leftrightarrow \quad h'(y) &= \frac{\log y}{y} \\
\Leftrightarrow \quad h(y) &= \frac{(\log y)^2}{2}
\end{aligned}$$

Hence the solution to the differential equation is

$$f(x, y) = x \log y + \frac{(\log y)^2}{2}$$

(d) $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1 + x)e^x \sec y$

Multiplying through by $\cos(y)$ we get the following equation:

$$\cos(y)\frac{dy}{dx} - \frac{\sin(y)}{1+x} = (1 + x)e^x \tag{3}$$

Let $v = \sin(y)$ then $\frac{dv}{dx} = \cos(y)\frac{dy}{dx}$

Then 3 becomes
$$\frac{dv}{dx} - \frac{v}{1+x} = (1 + x)e^x \tag{4}$$

which is a linear first-order differential equation. Hence we get our integrating factor

$$e^{\int \frac{-1}{1+x}\,dx} = \frac{1}{1+x}$$

Multiplying both sides of 4 with the integrating factor we get

$$\begin{aligned}
\frac{d}{dx}\left(\frac{v}{1+x}\right) &= e^x \\
\Leftrightarrow \left(\frac{v}{1+x}\right) &= \int e^x \, dx \\
\Leftrightarrow \left(\frac{v}{1+x}\right) &= e^x + C \qquad\qquad\qquad C \in \mathbb{R}
\end{aligned}$$

That is

$$v = (1+x)(e^x + C)$$

Since $v = \sin(y)$ we have the solution to the differential equation to be

$$\sin(y) = (1+x)(e^x + C) \quad C \in \mathbb{R}$$

(e) $x^3(\frac{dy}{dx} - x) = y^2$

This differential equation can be rewritten as

$$\frac{dy}{dx} = \frac{y^2}{x^3} + x$$

Consider the substitution $y = x^2 v$ then $\frac{dy}{dx} = 2xv + \frac{dv}{dx}x^2$

So the original differential equation becomes

$$\begin{aligned}
2xv + \frac{dv}{dx}x^2 &= \frac{x^4v^2}{x^3} + x \\
\Leftrightarrow \quad \frac{dv}{dx} &= \frac{xv^2 + x - 2xv}{x^2} \\
\Leftrightarrow \quad \frac{dv}{dx} &= \frac{v^2 - 2v + 1}{x} \\
\Leftrightarrow \quad \int \frac{dv}{(v-1)^2} &= \int \frac{dx}{x} \\
\Leftrightarrow \quad -(v-1)^{-1} &= \ln x + C \quad C \in \mathbb{R} \\
\Leftrightarrow \quad v &= 1 - \frac{1}{\ln x + C}
\end{aligned}$$

Hence the solution to the differential equation is

$$y = x^2 - \frac{x^2}{\ln x + C} \quad C \in \mathbb{R}$$

B Solve the following IVPs

(a) $y' = 3x^2\left(1 - e^{-y}\right), \quad y(0) = 1$

Since the right hand side involves the product of a function of x only and a function of y only it is easy to see that this is a separable linear first order differential equation. Thus

$$\begin{aligned}
\frac{1}{1-e^{-y}}dy &= 3x^2 dx \\
\Leftrightarrow \quad \int \frac{1}{1-e^{-y}}dy &= \int 3x^2 dx \\
\Leftrightarrow \quad \ln(e^y - 1) &= x^3 + C \quad C \in \mathbb{R} \\
\Leftrightarrow \quad e^y &= Ae^{x^3} + 1 \quad A = e^C \\
\Leftrightarrow \quad y &= \ln \mid Ae^{x^3} + 1 \mid
\end{aligned}$$

But we know that $y(0) = 1$ Thus

$$1 = \ln \mid A + 1 \mid \Leftrightarrow A = e - 1$$

So the solution to the differential equation is

$$y = \ln \mid (e-1)e^{x^3} + 1 \mid$$

(b) $\dot{x}(t) = \cos\big(x(t)\big), \quad x(0) = 0$

Notice that this is a variable separable first order DE hence we have that

$$\begin{aligned} \int \sec(x)dx &= \int dt = t + C \quad C \in \mathbb{R} \\ \ln \mid \sec x + \tan x \mid &= t + C \\ \sec x + \tan x &= Ae^t \quad A = e^C \\ C &= 1 \quad \text{from the initial condition given} \end{aligned}$$

Thus the solution to the IVP is

$$\sec x + \tan x = e^t$$

(c) $\frac{dx}{dt} + x = x^4 \quad x(0) = 1$

The differential equation given can be rewritten as $\frac{dx}{dt} = x^4 - x$ which tells us that this is a separable first order differential equation. So we have the following

$$\begin{aligned} \frac{1}{x^4 - x}dx &= dt \\ \Leftrightarrow \int \frac{1}{x^4 - x}dx &= \int dt \end{aligned}$$

Applying the method of partial fractions to the left hand side we get

$$\begin{aligned} \int \frac{-1}{x}dx + \int \frac{x^2}{x^3 - 1}dx &= t + C \quad C \in \mathbb{R} \\ \Leftrightarrow \ln \mid \frac{1}{x} \mid + \frac{1}{3}\ln \mid x^3 - 1 \mid &= t + C \\ \Leftrightarrow \frac{x^3 - 1}{x^3} &= e^{3t+3C} \\ \Leftrightarrow \frac{x^3 - 1}{x^3} &= Ae^{3t} \quad A = e^{3C} \\ \Leftrightarrow x^3 &= \frac{1}{1 - Ae^{3t}} \end{aligned}$$

For the initial condition given, we know have that

$$1 = \frac{1}{1 - A} \Leftrightarrow A = 0$$

Thus the solution to the differential equation given is the function $x \equiv 1$.

(d) $y'x^6 = 1 - y', \quad y(0) = 1$

Rewriting the equation as $\frac{dy}{dx} = \frac{1}{x^6+1}$ it is easy to see that this is a separable first order differential equation which gives us the following

$$\int dy = \int \frac{1}{x^6 + 1}dx$$

2 Second order ODEs

A Find the general solution to the following ODEs.

(a) $(1+x^2)y'' + 2xy' + \frac{1}{1+x^2}y = 0$

(b) $y^{(iv)} + 9y'' = t$

Solving the associated homogeneous equation

$$\frac{d^4y}{dt^4} + 9\frac{d^2y}{dt^2} = 0$$

Assume that the solution to the differential equation is of the form $y = e^{rt}$ then

$$y'' = r^2 e^{rt}$$

$$y^{(iv)} = r^4 e^{rt}$$

Plugging that into the homogeneous equation we get:

$$\begin{aligned} r^4 e^{rt} + 9r^2 e^{rt} &= 0 \\ \Leftrightarrow e^{rt}(r^4 + 9r^2) &= 0 \\ \Leftrightarrow r^2(r^2+9) &= 0 \end{aligned}$$

$$\Leftrightarrow r_1 = 0 \quad \text{and} \quad r_2 = 3i \quad \text{and} \quad r_3 = -3i \quad i = \sqrt{-1}$$

So the homogeneous solution to the differential equation is of the form

$$y_h = c_1 + c_2 + c_3 \cos 3t + c_4 \sin 3t$$

Solving for the particular equation: Assume $y_p = At^3 + Bt^2 + Ct + D$ is a particular solution to the DE then

$$\begin{aligned} y'' &= 6At + 2B \\ y^{(iv)} &= 0 \end{aligned}$$

So the non homogeneous DE becomes

$$\begin{aligned} 0 + 9(6At + 2B) &= t \\ \Leftrightarrow \quad 54At + 18B &= t \end{aligned}$$

$$\Leftrightarrow \quad A = \frac{1}{54}, \quad B = 0$$

Hence $y_p = \frac{t^3}{54} + Ct + D$

So the solution to the differential equation becomes

$$y(t) = c_1 + c_2 t + c_3 \cos 3t + c_4 \sin 3t + \frac{t^3}{54}$$

B Solve the following IVPs

(a) $y'' - 8y' + 15y = 2e^{3t}, \quad y(0) = y'(0) = 0$

Solving for the associated homogeneous part, $y'' - 8y' + 15y = 0$, suppose $y = e^{rt}$ is a solution then

$$
\begin{aligned}
r^2e^{rt} - 8re^{rt} + 15e^{rt} &= 0 \\
\Leftrightarrow \quad e^{rt}(r^2 - 8r + 15) &= 0 \\
\Leftrightarrow \quad (r^2 - 8r + 15) &= 0 \\
\Leftrightarrow \quad r = 5 \quad \text{or} \quad r &= 3
\end{aligned}
$$

Hence the associated homogeneous equation is

$$y_h = C_1e^{5t} + C_2e^{3t}$$

We assume that the particular equation is of the form Ate^{2t} since there exists a homogeneous solution with e^{3t}

From this we have

$$
\begin{aligned}
9Ate^{3t} + 6Ae^{3t} - 24Ate^{3t} - 8Ae^{3t} + 15Ate^{3t} &= 3e^{3t} \\
\Leftrightarrow \quad -2Ae^{3t} &= 2e^{3t} \\
\Leftrightarrow \quad A &= -1
\end{aligned}
$$

Hence

$$y_p = -te^{3t}$$

So the family of solutions to the differential equation is

$$y(t) = C_1e^{5t} + C_2e^{3t} - te^{3t}$$

From the inital conditions given we know that $C_1 + C_2 = 0$ and $5C_1 + 3C_2 = 1$ thus

$$C_1 = \frac{1}{2}, C_2 = \frac{-1}{2}$$

So the solution to the IVP is the equation

$$\frac{1}{2}e^{5t} - \frac{1}{2}e^{3t} - te^{3t}$$

3 What is Mathematics without proofs?

1. Consider the following differential equation

$$\dot{x}(t) = \sqrt{x(t)} \quad x(0) = 0$$

Notice that this differential equation has a non-trivial solution $x(t) \equiv 0 \quad \forall t$. But the differential equation is also variable separable and so we can solve it by algebraic means to get a solution $x(t)$. That is to say

$$
\begin{aligned}
\int \frac{1}{\sqrt{x}}dx &= \int dt \\
\sqrt{x} &= \frac{t+C}{2} \quad C \in \mathbb{R} \\
x &= \left(\frac{t+C}{2}\right)^2 \quad C \in \mathbb{R} \\
x &= \frac{t^2}{4} \quad \text{from the initial conditions given}
\end{aligned}
$$

So we have that there exists 2 different solutions to the IVP given, hence the IVP does not have a unique solution.

2. Suppose that there exists continuous functions $a_0, a_1 : \mathbb{R} \to \mathbb{R}$ so that the differential equation $\ddot{x}(t) + a_1\dot{x}(t) + a_0x(t) = 0$ has a solution $x(t) = t^2$. Then this solution must satisfy the differential equation so that we have
$$2 + 2a_1t + a_0t^2 = 0$$
this means in particular for $t = 0$ we must have that $2+0+0 = 0$ which does not make sense hence no such functions $a_0, a_1 : \mathbb{R} \to \mathbb{R}$ exist so that the differential equation $\ddot{x}(t) + a_1\dot{x}(t) + a_0x(t) = 0$ has a solution $x(t) = t^2$

4 Let's look at a real world application

Assume that there is a community of N members with I infected and U uninfected individuals so
$$U + I = N$$
define the following ratios
$$x = \frac{I}{N} \quad y = \frac{U}{N} \tag{5}$$
Assume that N is constant and very large that both x and y may be considered as continuous variables. So we have $x, y \in [0, 1]$ and
$$x + y = 1$$
Let time be represented by t and the rate of spread of the disease as $\frac{dx}{dt}$ then we have that
$$\frac{dx}{dt} = \beta xy \tag{6}$$
where β is a real and positive constant of proportionality.

1. From the equations 5 we have that
$$y = 1 - x$$
plugging that into equation 6 we get
$$\frac{dx}{dt} = \beta x(1 - x) \tag{7}$$

2. It is easy to see that 7 is a separable first order DE so we have
$$\begin{aligned}
\int \frac{dx}{x(1-x)} &= \beta t + C \quad C \in \mathbb{R} \\
\int \frac{dx}{(1-x)} + \int \frac{dx}{x} &= \beta t + C \\
\ln \left| \frac{x}{(1-x)} \right| &= \beta t + C \\
\frac{x}{(1-x)} &= Ae^{\beta t} \quad A = e^C \\
x &= \frac{Ae^{\beta t}}{1 - Ae^{\beta t}}
\end{aligned}$$
By the initial condition we have $A = \frac{x_0}{1+x_0}$ so the solution to the differential equation is
$$x(t) = \frac{x_0Ae^{\beta t}}{x_0Ae^{\beta t} + x_0 + 1}$$

3. Rewriting $x(t)$ we have

$$\frac{1}{1+\frac{1}{Ae^{\beta t}}+\frac{1}{x_0 Ae^{\beta t}}}$$

so as $t \to \infty$ then $\frac{1}{Ae^{\beta t}} \to 0, \frac{1}{x_0 Ae^{\beta t}} \to 0$ so $x(t) \to 1$. This means that eventually everyone in the population gets infected by the disease.

4. No the model is not realistic. It does not account for the immunity of some people to some diseases and the fact that even as people are being infected people are also recovering.

Some freebies...

(a) Consider the DE

$$f''(x) + f(-x) = x + \cos x, \forall x \in \mathbb{R}$$

We know that every function can be decomposed into a sum of even and odd functions thus $f(-x) = f_e(x) - f_o(x)$ and $f(x) = f_e(x)f_o(x)$ where $f_e(x), f_o(x)$ are the even and odd decompositions, respecitvely, of the function $f(x)$. Then the DE becomes

$$f_e''(x) + f_o''(x) + f_e(x) - f_o(x) = \cos x + x$$

Thus we have that since the sum of any two even functions is even and the sum of any two odd functions is odd, coupled with the fact that the second derivative of any even function is even, and the second derivative of any odd function is odd then

$$f_e''(x) + f_e(x) = \cos(x)$$

and

$$f_o''(x) - f_o(x) = x$$

since $f(x) = x$ is an odd function and $f(x) = cos(x)$ is an even function.

Solving the even part of the second order DE we first find the homogeneous equation. Suppose the homogeneous equation is of the form

$$f_e^h(x) = e^{rx}$$

then

$$\begin{aligned} r^2 e^{rx} + e^{rx} &= 0 \\ \Leftrightarrow \quad r^2 + 1 &= 0 \\ \Leftrightarrow \quad r &= \pm i \end{aligned}$$

so the general form for the homogeneous part is

$$f_e^h(x) = C_1 \cos x + C_2 \sin x$$

Let the particular solution have the general form

$$f_e^p(x) = ax \sin x$$

Then we have that

$$\begin{aligned} 2a\cos x - ax\sin x + ax \sin x &= \cos x \quad \text{from } f_e''(x) + f_e(x) = \cos(x) \\ \Leftrightarrow \quad 2a \cos x &= \cos x \\ \Leftrightarrow \quad a &= 1/2 \end{aligned}$$

So the solution to the even part of the differential equation is

$$f_e(x) = C_1 \cos x + C_2 \sin x + \frac{x \sin x}{2}$$

Now since the function is an even function then $C_2 = 0$ so that we $\sin x$ which is an odd function gets eliminated and we get the general solution to the even part being

$$f_e(x) = C_1 \cos x + \frac{x \sin x}{2}$$

Now for the odd part

$$f_o''(x) - f_o(x) = x$$

we solve for the homogeneous and particular parts. For the homogeneous part assume the solution is of the form $f_o^h = e^{rx}$ then

$$\begin{aligned} r^2 e^{rx} - e^{rx} &= 0 \\ \Leftrightarrow \quad r^2 - 1 &= 0 \\ r &= \pm 1 \end{aligned}$$

so the homogeneous solution has the general form

$$f_o^h(x) = Ae^x + Be^{-x}$$

Now for the particular solution suppose it is of the general form $f_o^p(x) = mx + c$ then from the odd differential equation we have that

$$-mx - c = x$$

Hence $c = 0$ and $m = -1$. Thus the general solution to the odd part of the differential equation is of the form

$$f_o(x) = Ae^x + Be^{-x} - x$$

Now since the odd part must be an odd function then $f_o(-x)$ must equal $-f_o(x)$ thus $Be^x + Ae^{-x} + x$ must equal $-Ae^x - Be^{-x} + x$ so we have that $A = B = 0$ so the general solution to the odd part of the differential equation is

$$f_o(x) = -x$$

So the general solution to the DE is

$$f(x) = C_1 \cos x + \frac{x \sin x}{2} - x \quad C_1 \in \mathbb{R}$$

(b) Consider the IVP

$$y' = \max\{1, y^2\}, \quad y(0) = 0$$

rewriting it we get

$$\frac{dy}{dx} = \begin{cases} 1, & \mid y \mid \leq 1 \\ y^2, & \mid y \mid > 1 \end{cases}$$

Considering the case where $\mid y \mid \leq 1$ we have that

$$\begin{aligned} &\frac{dy}{dx} = 1 \Leftrightarrow y = x + C \quad C \in \mathbb{R} \\ &C = 0 \quad \text{Using the initial condition y(0)=0} \end{aligned}$$

hence

$$y = x \quad \text{for } \mid x \mid \leq 1 \tag{8}$$

For the case where $\mid y \mid > 1$ we have

$$\begin{aligned}
&\frac{dy}{dx} = y^2 \quad \text{variable separable} \\
\Leftrightarrow \quad &\int \frac{dy}{y^2} = x + C \quad C \in \mathbb{R} \\
\Leftrightarrow \quad &\frac{-1}{y} = x + C \\
\Leftrightarrow \quad &y = \frac{-1}{x + C}
\end{aligned}$$

Now since we have that $y = x$ for $\mid y \mid \leq 1$ then $y = x$ for $\mid x \mid \leq 1$. To find the constant C for the case where $\mid y \mid > 1$ we consider 2 cases.

When $y < -1$ since the solution to a first order DE must be continuous from the definition of the first derivative then we have that since

$$y(-1) = -1 \text{ from } 8$$

then

$$\begin{aligned}
-1 &= \frac{-1}{-1 + C} \\
1 - C &= -1 \Leftrightarrow C = 2
\end{aligned}$$

Hence for $y < -1$ we have that

$$y = -\frac{1}{x + 2}$$

Similarly when $y > 1$ then we know from 8 that $y(1) = 1$. The continuity of the solution then means that

$$\begin{aligned}
1 &= \frac{-1}{1 + C} \\
1 + C &= -1 \Leftrightarrow C = -2
\end{aligned}$$

Hence for $y > 1$ we have that

$$y = \frac{1}{2 - x}$$

and so the solution to the differential equation is given as

$$y = \begin{cases} \frac{-1}{x+2} & x < -1 \\ x & \mid x \mid \leq 1 \\ \frac{1}{2-x} & x > 1 \end{cases}$$