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Problem Statement

Imagine a 3×3 array of squares. The challenge is to put the integers $1, 2, \ldots 9$, one in each square, so that each row and each column adds up to the same number. The best magic squares not only have all the rows and all the columns summing to the same number, but also the diagonals sum to the same number. How about 5×5 magic squares? Or even more...

Introduction

Magic squares have their origins way back in history and their introduction is attributed to the ancient Chinese [6]. It was believed, by the ancient Chinese, that a magic square (a 3×3 magic square) was marked on the shell of a tortoise that emerged from the 'Lo River'. The nature of the square, with row values, column values, and diagonal values summing to the same number, led them to believe that the square had a 'magical' significance. Thus the name 'magic squares'.

Throughout history, this object has fascinated mathematicians. From the ancient Chinese to modern researchers, every generation of mathematicians has tried to answer some question or the other related to magic squares (For instance, the construction of a 4×4 magic square using distinct positive cubed integers or proving that it is impossible is still an open problem [1]).

In this report we shall explore two of the many different classifications of magic squares that exist. We shall also look at some algorithms for constructing magic squares and finally, we shall discuss some generalizations of magic squares to other fields.

Some Basic Definitions About Magic Squares

1. A *magic square* A, with magic constant S is an $n \times n$ matrix which is to be filled with numbers in a way that:
    - The sum of all entries in each row equals S.
    - The sum of all entries in each column equals S.

- The sum of all entries in both diagonals equals S.

2. The *order* of a magic square A is a natural number n that denotes the number of rows and columns in A.

3. A *normal magic square* also known as a *natural magic square* of order n is a magic square that is filled with consecutive integers $1, 2, 3, \ldots, n^2$.

4. The *magic constant* S is the value to which the values in each row, column, and/or diagonal must sum up to. For a normal magic square K of order m, the magic constant equals $\dfrac{m(m^2+1)}{2}$. This must be so since we are finding the total sum from 1 to m^2 of the values to be arranged into the matrix. The total sum of these numbers is given by

$$\sum_{i=1}^{m^2} i = \frac{m^2(m^2+1)}{2}.$$

Since this sum $\dfrac{m^2(m^2+1)}{2}$ must be spread, considering (without loss of generality) the rows of the matrix, among the m different columns then we get that each column must have the sum

$$\frac{m^2(m^2+1)}{2} \cdot \frac{1}{m} = \frac{m(m^2+1)}{2}.$$

Thus for natural magic squares of order m the magic constant is given as $\dfrac{m(m^2+1)}{2}$.

Now that we have seen some basic definitions related to magic squares, we can now explore the different types of magic squares that exist. Different types of magic squares exist. Including, but not limited to, panmagic squares, compact magic squares, and associative magic squares [1]. However for the purposes of this report, we shall only make a distinction between two main types of magic squares:

1. **Perfect magic squares**: A normal magic square where the values in each row, column, and both diagonals sum up to the magic constant.

2. **Semi-magic squares**: A normal magic square where values in each row and each column, but not the diagonal values, sum up to the magic constant.

Having made a distinction between the two main types of magic squares, we shall discuss some of the algorithms that exist for constructing perfect magic squares.

Algorithms for Finding the Best Magic Squares

Many methods for constructing the perfect magic square exist however, no one method exists for constructing a magic square of an arbitrary order [4]. This is to say that the method for constructing the perfect magic squares depends on the order of the magic square. Based on this, there are three main distinctions of the order of a magic square:

1. **Odd magic squares**: A magic square whose order is an odd number. For example 3×3 magic squares, 5×5 magic squares and so on.

2. **Doubly Even magic squares**: A magic square whose order is in the equivalence class of $0 \mod 4$. For example 4×4 magic squares, 8×8 magic squares, etc.

3. **Singly Even magic squares**: A magic square whose order is in the equivalence class of $2 \mod 4$. For example 6×6 magic squares, 10×10 magic squares, etc.

In the following subsections, we shall discuss one method each for constructing magic squares based on the above classification.

Siamese Algorithm for Odd Magic Squares

The Siamese algorithm for constructing the perfect odd-order magic squares is attributed to French mathematician and diplomat Simon de la Loubère who developed this algorithm in 1693 while serving as an ambassador to Siam (present day Thailand) from the court of Louis XIV [2]. The method summarized below:

1. Starting from the top row and center column, place the number 1 in that cell.

2. For the remaining numbers, move one cell to the right and one cell upwards and fill that cell with the next number in the sequence.

   - When the upward move takes you out of the grid, move to the cell in the bottom row of the same column.
   - When the rightward move takes you out of the grid, move to the cell in the first column of the same row.

3. When the upward-rightward move takes you to a cell that is already filled, backtrack to the most recently filled cell and then move one cell downward and fill that cell with the next number in the sequence.

4. Repeat this process until all cells are filled.

	1	

	1	
		2

	1	
3		
		2

	1	
3		
4		2

	1	
3	5	
4		2

	1	6
3	5	
4		2

	1	6
3	5	7
4		2

8	1	6
3	5	7
4		2

8	1	6
3	5	7
4	9	2

Figure 1: Illustration of Siamese Algorithm for 3×3 magic squares

Notice that if we number the rows and columns in the square form 0 to $n-1$ (bottom to top for rows and left to right for columns) then move where we wrap around when we move out of the grid is equivalent to counting in modular arithmetic (specifically mod n). Thus each row can be seen as an equivalence class of mod n. Likewise each column can be seen as an equivalence class of mod n.
It can be shown that this method always works for odd n since we are increasing both the row number r and column number c by $r+1 \mod n$ and $c+1 \mod n$. Thus we are guaranteed that each number in the sequence gets a unique cell (since we have to move downward when the cell is already filled) and that numbers in the same equivalence class mod n are put in different rows and different columns [5].

Complementation Algorithm for Doubly Even Magic Squares

The earliest known solutions for doubly even magic squares can be traced to the Indian mathematicians. The method is summarized below:

Given an $n \times n$ magic square:

1. Fill in the numbers $1, 2, \ldots, n^2$ in the magic square starting from the topmost, leftmost cell and filling in subsequent numbers row-wise from left to right. Do this until all n^2 numbers are filled into the grid.
2. If $n > 4$ then divide the $n \times n$ grid into 4 $(n/2) \times (n/2)$ sub magic squares. Else move to the next step.
3. Draw lines along the main diagonal and the anti diagonal in each sub-magic square.
4. For each element x that does not fall on any of the diagonals, replace the value of x by $(n^2+1) - x$

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

Figure 2: Illustration of Complementation Algorithm for 4×4 Magic Squares

Strachey-Conway Algorithm for Singly Even Magic Squares

The Strachey-Conway algorithm for odd magic squares, also known as the LUX algorithm is attributed to both Ralph Strachey and John Horton Conway. The method is as follows:

1. Divide the grid into four quadrants of $(\frac{n}{2}) \times (\frac{n}{2})$ sub magic squares.
2. Apply Siamese Algorithm for Odd Magic Squares to each of these submagic squares in the following order:

- Use Siamese Algorithm for Odd Magic Squares with the numbers 1 to $n^2/4$ in the top right quadrant.
- Use Siamese Algorithm for Odd Magic Squares with the numbers $(n^2/4)+1$ to $n^2/2$ in the bottom right quadrant
- Use Siamese Algorithm for Odd Magic Squares with the numbers $(n^2/2)+1$ to $(3n^2)/4$ in the top right quadrant
- Use Siamese Algorithm for Odd Magic Squares with the numbers $(3n^2/4)+1$ to $n^2)$ in the bottom left quadrant

3. Identify the first $\lfloor(n/4)\rfloor$ columns to the left and the first $\lfloor(n/4)\rfloor-1$ columns to the right of this matrix.

4. For the columns to the left that were identified in both quadrants, on the $((n/4)+1)$th row from the top of the quadrant, skip the first column and identify next $\lfloor(n/4)\rfloor$ columns.

5. For the cells that have been identified on both sides of the grid, move the bottom cells to the top and then the top cells to the bottom.

8	1	6	26	19	24
3	5	7	21	23	25
4	9	2	22	27	20
35	28	33	17	10	15
30	32	34	12	14	16
31	36	29	13	18	11

35	1	6	26	19	24
3	32	7	21	23	25
31	9	2	22	27	20
8	28	33	17	10	15
30	5	34	12	14	16
4	36	29	13	18	11

Figure 3: Illustration of the Strachey-Conway Algorithm for 6×6 Magic Square

Conclusion

In this report, we discussed magic squares and a few of the algorithms available for constructing the best magic squares. We began with the classical 3×3 magic square, which has roots in ancient China and discussed its historical significance. We then examined the key definitions, types, and the magic constant that governs all magic squares.
We demonstrated how the order of a magic square dictates the choice of algorithm for constructing perfect squares. The Siamese method provides an elegant procedure for odd-order magic squares, the complementation algorithm works effectively for doubly even squares, and the Strachey-Conway algorithm handles singly even squares.
While no universal method exists for all orders, the combination of these algorithms allows us to construct magic squares of any given order systematically.

However, it should be noted that while the study of magic squares may seem purely recreational at first glance, it has applications in other disciplines such as physics [1], computer science [3], among others. For instance, in cryptography, magic squares can be used as a substitute for the cipher text. The proposed method for encryption and decryption is by constructing magic squares with Narayana's folding method and Knight's move method (methods not discussed in this report) [3].

Finally, despite centuries of study, several open problems remain. For example, the complete enumeration of the number of magic squares of order $n \geq 6$ is still unresolved, and new methods continue to be explored. These open questions ensure that magic squares remain not only a classical recreational pursuit but also a fertile ground for ongoing mathematical research.

References

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