Math Assignment 3: Linear Algebra

August 13, 2025

Exercise 1: Matrix Operations (25pts)

1. Let $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$. Find all matrices $B \in M_2(\mathbb{R})$ that commute with A. (5pts)
2. Find matrices $A, B \in M_2(\mathbb{R})$ such that $AB = 0$ and $BA \neq 0$. (5pts)
3. Let $A, B \in M_n(\mathbb{R})$ be two stochastic matrices. Show that the sum of the coefficients in every column of AB is equal to 1. (5pts)
4. Let $A, B \in M_n(\mathbb{R})$. Assume $\mathrm{tr}(AA^T) = 0$. What can you say about A? (5pts)
5. Compute A^2 and A^3 for $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ and use induction to find A^n. (5pts)

Exercise 2: Inverse of Matrices (15pts)

1. Let $n \geq 1$ and $A \in M_n(\mathbb{R})$ such that $A^2 = 0$. (5pts)
   (a) Prove that A is not invertible.
   (b) Prove that $I_n + A$ is invertible.
2. Let $A \in M_n(\mathbb{R})$ be a nilpotent matrix of order p ($A^p = 0$). Show that $I_n - A$ is invertible and find its inverse. (5pts)
3. Determine values of m for which matrix $A = \begin{pmatrix} 1 & 1 & m \\ 1 & m & 1 \\ m & 1 & 1 \end{pmatrix}$ is invertible. (5pts)

Exercise 3: Eigenvalues and Eigenvectors of Matrices (25pts)

1. For matrix $A = \begin{pmatrix} 0 & 2 & -1 \\ 3 & -2 & 0 \\ -2 & 2 & 1 \end{pmatrix}$:
   (a) Determine eigenvalues and eigenvectors. (10pts)
   (b) Show that A is diagonalizable. (5pts)

(c) Determine the invertible matrix P, such that $A = PDP^{-1}$, when$D$ is the diagonal matrix. (5pts)

2. Find all 2×2 matrices A such that: (5pts)

$$A^3 - 3A^2 = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}$$

Exercise 4: Matrix Powers (35pts)

In this exercise, we will calculate the power of a matrix using a method that can be generalized. We will then apply this result to find the general term of two recursive sequences. In the following, let

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

1. Justify that matrix P is invertible and calculate P^{-1}. (5pts)
2. Calculate PDP^{-1}. (5pts)
3. Compute D^2 and D^3. Conjecture a formula for D^n and prove it. (10pts)
4. Prove that, for all $n \geq 1$, $A^n = PD^nP^{-1}$. (10pts)
5. Hence deduce the value of the matrix A^n, for all $n \geq 1$. (5pts)

Bonus Questions (20pts)

1. For any integer $n \geq 2$ and two $n \times n$ matrices with real entries A, B that satisfy the equation: (10pts)

$$A^{-1} + B^{-1} = (A + B)^{-1}$$

   Prove that $\det(A) = \det(B)$.
   Does the same conclusion follow if the matrices A and B had Complex entries?

2. Let A and B be $n \times n$ matrices with complex entries satisfying $AB^2 = A - B$. (10pts)
   (a) Prove that $I + B$ is invertible.
   (b) $AB = BA$