Assignment 3
Linear Algebra

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Exercise 1: Matrix Operations

1. Let $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$.

   We want to find all matrices $B \in M_2(\mathbb{R})$ such that $AB = BA$

   Let the matrix B be represented by $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$

   Thus we want to find $b_1, b_2, b_3, b_4 \in \mathbb{R}$ so that

$$\begin{aligned} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} &= \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \\ \Leftrightarrow \quad \begin{pmatrix} ab_1 + bb_3 & ab_2 + bb_4 \\ ab_3 & ab_4 \end{pmatrix} &= \begin{pmatrix} ab_1 & bb_1 + ab_2 \\ ab_3 & bb_3 + ab_4 \end{pmatrix} \end{aligned}$$

   In particular we want to find b_1, b_2, b_3, b_4 so that the following are true:

$$ab_1 + bb_3 = ab_1 \quad \Leftrightarrow \quad bb_3 = 0 \tag{1}$$
$$ab_2 + bb_4 = ab_2 + bb_1 \quad \Leftrightarrow \quad bb_4 = bb_1 \tag{2}$$

   From 1 since there is no original condition on the entry b of A then b_3 must equal zero to force 1 to always be true.

   Likewise from 2 it is easy to see that $b_1 = b_4$ for 2 to always be true.

   Thus our matrix B that commutes with A is of the form $B = \begin{pmatrix} b_1 & b_2 \\ 0 & b_3 \end{pmatrix}, \quad b_1, b_2 \in \mathbb{R}$

2. Consider the matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ It is clear to see that

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

   Now

$$BA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

   Thus we have matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ such that $AB = 0$ and $BA \neq 0$

3. Let $A, B \in M_n(\mathbb{R})$ be two stochastic matrices. Then we know that for every $i, j \in \{1, 2, \cdots, n\}$ fixing a column t then $\sum\limits_{i=1}^{n} (a)_{it} = 1$. That is the sum of each row in A results in 1 likewise, we have that $\sum\limits_{i=1}^{n} (b)_{it} = 1$ for some fixed column t, that is the sum of each column in B results in 1.

Consider, now, the product AB. The entry $(ab)_{ij}$ of AB is given as $\sum\limits_{k=1}^{n} a_{ik}b_{kj}$. Thus the sum of all the entries in a fixed column t of AB is given as

$$\sum_{i=1}^{n}(ab)_{it} = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{kt}$$

Notice that we can reorder sums since they are finite resulting in the following

$$\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{kt} = \sum_{k=1}^{n}\sum_{i=1}^{n} a_{ik}b_{kt} = \sum_{k=1}^{n}\left(\sum_{i=1}^{n} a_{ik}\right) b_{kt} = \sum_{k=1}^{n} b_{kt} = 1$$

Thus for stochastic matrices A, B the sum of the coefficients in each column is 1.

4. Let $A, B \in M_n(\mathbb{R})$. Assume $\mathrm{tr}(AA^T) = 0$. Now notice that for AA^T the entries on the main diagonal (a_{kk}) are of the form $a_{kk} = \sum\limits_{j=1}^{n} a_{ij}^2$ where i is a fixed row of the matrix A. That is, each element on the main diagonal is a sum of nonnegative entries which must be itself nonnegative. Then for AA^T the elements $a_{kk} \geq 0$. Now we know that $\mathrm{tr}(AA^T) = 0$. So if the sum of nonnegative entries results in zero, this means that each entry must be itself equal to zero. So for AA^T the elements $a_{kk} = 0$. But notice that $a_{kk} = \sum\limits_{j=1}^{n} a_{ij}^2$ is just the squared magnitude of a fixed row i of the matrix A and so if the squared magnitude of the vector is 0 it follows that the magnitude of the vector must itself be the 0 vector. So we have that the magnitude of each row vector of A is the zero vector thus each row in A is the zero vector and so A is the zero matrix

5. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$. Then $A^2 = \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}$ and $A^3 = \begin{pmatrix} 1 & 7 \\ 0 & 8 \end{pmatrix}$

   notice that $A^2 = \begin{pmatrix} 1 & 2^2-1 \\ 0 & 2^2 \end{pmatrix}$ and $A^3 = \begin{pmatrix} 1 & 2^3-1 \\ 0 & 2^3 \end{pmatrix}$

   Claim. For every integer $n \geq 1$,

$$A^n = \begin{pmatrix} 1 & 2^n-1 \\ 0 & 2^n \end{pmatrix}$$

   Consider the case where $n = 1$ Then we have the following

$$A = \begin{pmatrix} 1 & 2^1-1 \\ 0 & 2^1 \end{pmatrix}$$

   =

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

   Suppose that for some $n = k \geq 1$:

$$A^k = \begin{pmatrix} 1 & 2^k-1 \\ 0 & 2^k \end{pmatrix}.$$

   Then for $n = k+1$ we have

$$A^{k+1} = A^kA = \begin{pmatrix} 1 & 2^k-1 \\ 0 & 2^k \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

   Multiplying gives

$$A^{k+1} = \begin{pmatrix} 1 & 1+2^{k+1}-2 \\ 0 & 2^{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 2^{k+1}-1 \\ 0 & 2^{k+1} \end{pmatrix}$$

   Thus

$$A^{k+1} = \begin{pmatrix} 1 & 2^{k+1}-1 \\ 0 & 2^{k+1} \end{pmatrix}$$

   And so by the principle of mathematical induction, we have that for $n \geq 1$,

$$A^n = \begin{pmatrix} 1 & 2^n-1 \\ 0 & 2^n \end{pmatrix}$$

Exercise 2: Inverse of Matrices

1. Let $n \geq 1$ and $A \in M_n(\mathbb{R})$ such that $A^2 = 0$.
   (a) Since $A^2 = 0$ then $\det(A^2) = 0$. Now since $\det(\mathrm{A}^2) = (\det(\mathrm{A}))^2$ then $\det(A^2) = 0 \Leftrightarrow \det(\mathrm{A}) = 0$ thus A is invertible.
   (b) $I_n = I_n - 0 = (I_n - A^2) = (I_n + A)(I_n - A)$. Thus the matrix $I_n + A$ is invertible and its inverse is $(I_n - A)$
2. Let $A \in M_n(\mathbb{R})$ be a nilpotent matrix of order p ($A^p = 0$). Show that $I_n - A$ is invertible and find its inverse.
3. Consider the matrix
$$A = \begin{pmatrix} 1 & 1 & m \\ 1 & m & 1 \\ m & 1 & 1 \end{pmatrix}$$
. We want to find the values of m for which A is invertible, that is $\det(A) \neq 0$

   $\det(A) = (m-1) + (m-1) + m(1-m^2) = -2 + 3m - m^3$
   The roots of the determinant are $m = -2, 1$. Thus the values of m for which the matrix A is invertible are $m \in \mathbb{R} \setminus \{-2, 1\}$

Exercise 3: Eigenvalues and Eigenvectors of Matrices

1. Consider the matrix
$$A = \begin{pmatrix} 0 & 2 & -1 \\ 3 & -2 & 0 \\ -2 & 2 & 1 \end{pmatrix}$$
   (a) To determine its eigenvalues, we want to find scalars λ so that for some vectors v the following holds $Av = \lambda v$

   That is $\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{pmatrix} -\lambda & 2 & -1 \\ 3 & -(2+\lambda) & 0 \\ -2 & 2 & 1-\lambda \end{pmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= -\lambda(-(2+\lambda)(1-\lambda)) - 2(3 - 3\lambda) - 2 + 2\lambda \\ &= \lambda^3 + \lambda^2 - 10\lambda + 8 \end{aligned}$$

   Finding the roots to the characteristic equation we have that $\lambda = 2, 1, -4$

   To find the eigenvectors for $\lambda = 1$ we need to find non trivial solutions to the following system of equations $\begin{pmatrix} -1 & 2 & -1 \\ 3 & -3 & 0 \\ -2 & 2 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$

   it is easy to see that the solutions are of the form $c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad c \in \mathbb{R}$

   To find the eigenvectors for the eigenvalue $\lambda = 2$ we need to find non trivial solutions to the following system of equations $\begin{pmatrix} -2 & 2 & -1 \\ 3 & -4 & 0 \\ -2 & 2 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$ it is easy to see that the solutions are of the form $c \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} \quad c \in \mathbb{R}$

To find the eigenvectors for the eigenvalue $\lambda = -4$ we need to find non trivial solutions to the following system of equations $\begin{pmatrix} -4 & 2 & -1 \\ 3 & 2 & 0 \\ -2 & 2 & 5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$ it is easy to see that the solutions are of the form $c \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} \quad c \in \mathbb{R}$

(b) Since A is a 3×3 matrix with 3 distinct eigenvalues then A must be diagonalizable

(c) $D = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $P = \begin{pmatrix} 2 & 1 & 4 \\ -3 & 1 & 3 \\ 2 & 1 & -2 \end{pmatrix}$. Notice that $\det(P) = 2(-5) + 4(-5) = -30$ hence P is invertible. In particular $P^{-1} = \begin{pmatrix} 1/6 & -1/5 & 1/3 \\ 0 & 2/5 & 3/5 \\ 1/6 & 0 & -1/6 \end{pmatrix}$

$$PDP^{-1} = \begin{pmatrix} 2 & 1 & 4 \\ -3 & 1 & 3 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} -4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1/6 & -1/5 & 1/3 \\ 0 & 2/5 & 3/5 \\ 1/6 & 0 & -1/6 \end{pmatrix} = \begin{pmatrix} 0 & 2 & -1 \\ 3 & -2 & 0 \\ -2 & 2 & 1 \end{pmatrix} = A$$

2. Find all 2×2 matrices A such that:

$$A^3 - 3A^2 = \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}$$

Exercise 4: Matrix Powers

Consider the following matrices

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

1. $\det(P) = 2 - 1 = 1 \neq 0$ Thus matrix P is invertible.

   P^{-1} is the matrix such that $PP^{-1} = I$. Let $P^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

   $$PP^{-1} = \begin{pmatrix} a+c & b+d \\ a+2c & b+2d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

   it follows that

   $$a = 2, \quad b = -1, \quad c = -1, \quad d = 1$$

   thus $P^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

2. $PDP^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} = A$

3. $D^2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

   $D^3 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix}$

   Claim: $D^n = \begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix}$ Consider the base case $n = 1$ we have that $D = \begin{pmatrix} 1 & 0 \\ 0 & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Now consider the case where $n = k, \quad k \in \mathbb{N}$ and $k \geq 1$ and assume that $D^k = \begin{pmatrix} 1 & 0 \\ 0 & 2^k \end{pmatrix}$ Now consider the case where $n = k+1$ then

$$D^{k+1} = D^k D = \begin{pmatrix} 1 & 0 \\ 0 & 2^k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2^{k+1} \end{pmatrix}$$

Hence by the principle of mathematical induction we have that $D^n = \begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix}$

4. We want to show that for all $n \geq 1 \quad A^n = PD^nP^{-1}$.

   Consider our base case $n = 1$ then we have that

$$PDP^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} = A$$

   Now suppose that for some $k \in \mathbb{N}, \quad k \geq 1$, $A^k = PD^kP^{-1}$. Now consider A^{k+1}. Then we have that

$$A^{k+1} = A^k A = (PD^kP^{-1})(PDP^{-1})$$

   then by the associativity of matrix multiplication we get

$$\begin{aligned} A^{k+1} &= A^k A \\ &= (PD^kP^{-1})(PDP^{-1}) \\ &= PD^k(P^{-1}P)DP^{-1} \\ &= PD^k(I)DP^{-1} \\ &= PD^kDP^{-1} \\ &= PD^{k+1}P^{-1} \end{aligned}$$

   Thus by the principle of mathematical induction we have that for all $n \geq 1, \quad A^{k+1} = PD^{k+1}P^{-1}$

5.

$$\begin{aligned} A^n = PD^nP^{-1} &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 2^{n+1} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 2(1-2^n) & -1+2^{n+1} \end{pmatrix} \end{aligned}$$

Bonus Questions

1. Let n be any integer such that $n \geq 2$ and A, B be two $n \times n$ matrices with real entries that satisfy the equation:

$$A^{-1} + B^{-1} = (A+B)^{-1}$$

   Now we know that

$$\begin{aligned} &(A+B)(A+B)^{-1} = I_n \\ \Leftrightarrow &(A+B)(A^{-1}+B^{-1}) = I_n \\ \Leftrightarrow &AA^{-1} + AB^{-1} + BA^{-1} + BB^{-1} = I_n \\ \Leftrightarrow &AB^{-1} + BA^{-1} + I_n = 0 \end{aligned}$$

Multiplying through by $(A^{-1}B)$ on the left we get the following

$$(A^{-1}B)^2 + I_n + (A^{-1}B) = 0$$

Let $C = A^{-1}B$ then

$$C^2 + C + I_n = 0$$

Multiplying through by $(C - I_n)$ we get

$$C^3 - I_n = 0$$

Thus $C^3 = I_n$. Then it follows that $\det(\mathrm{C}^3) = 1$. But $\det(\mathrm{C}^3) = (\det(\mathrm{C}))^3$ and so we can conclude that $\det(\mathrm{C}) = 1$ but $C = A^{-1}B$ so $\det(\mathrm{C}) = \frac{\det(\mathrm{B})}{\det(\mathrm{A})} = 1$ hence it follows that $\det(\mathrm{A}) = \det(\mathrm{B})$
If the matrices had complex entries we could not have concluded in the same manner since any cube root of unity would have satisfied $\left(\frac{\det(\mathrm{B})}{\det(\mathrm{A})}\right)^3 = 1$

2. Let A and B be $n \times n$ matrices with complex entries satisfying $AB^2 = A - B$.
   (a) Prove that $I + B$ is invertible.
   (b) $AB = BA$