MATH1012 Mathematical Theory and Methods

Week 11

Laplace Transforms

Let $f(t)$ be defined for all $t \geq 0$. The *Laplace transform* of $f(t)$ is the function

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)\,dt$$

for all $s \in \mathbb{R}$ for which the improper integral is convergent.

$f(t) \longrightarrow \boxed{\text{L.T.}} \longrightarrow F(s)$

We use the notation

- $F(s)$ or just $\mathcal{L}(f)$ for $\mathcal{L}(f)(s)$
- t is the variable for $f(t)$ and s is the variable for $F(s)$

Don't confuse $F(s)$ as the Laplace transform of a function $f(t)$ with the notation for the antiderivative, often denoted $F(t)$.

Improper integral

Recall that

$$\int_0^\infty e^{-st} f(t)dt = \lim_{\alpha \to \infty} \int_0^\alpha e^{-st} f(t)dt$$

if this limit exists.

$$f(t) \leftarrow \boxed{\text{I.L.T.}} \leftarrow F(s)$$

If $F(s)$ is the Laplace transform of $f(t)$, then $f(t)$ is the inverse Laplace transform of $F(s)$, and we write

$$F(s) = \mathcal{L}\{f(t)\} \longleftrightarrow f(t) = \mathcal{L}^{-1}\{F(s)\}$$

There is no formula to compute the inverse Laplace transform. Instead we refer to a table of Laplace transforms.

Example

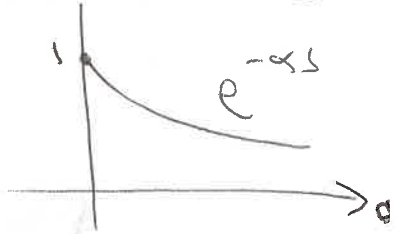
Find the Laplace transform of $f(t) = 1$

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}\cdot 1 \cdot dt = \int_0^\infty e^{-st}\,dt$$

$$= \lim_{\alpha\to\infty}\int_0^\alpha e^{-st}\,dt = \lim_{\alpha\to\infty}\left[\frac{e^{-st}}{-s}\right]_{0=t}^{\alpha=t}$$

$$\boxed{s > 0}$$

$$= \lim_{\alpha\to\infty}\left[-\frac{e^{-\alpha s}}{s} + \frac{e^{0}}{s}\right]$$



$$= \lim_{\alpha\to\infty}\left(\frac{1}{s} - \frac{e^{-\alpha s}}{s}\right)$$

$$= \frac{1}{s} - \frac{0}{s} = \frac{1}{s}$$

So, $\boxed{\mathcal{L}\{1\} = \frac{1}{s}}$

More complicated example

Find the Laplace transform of $\sin(\omega t)$

$$\int_a^b u'(t)\,v(t)\,dt = \Big[u(t)v(t)\Big]_a^b - \int_a^b u(t)\,v'(t)\,dt$$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty \underbrace{e^{-st}}_{u'} \underbrace{\sin(wt)}_{v}\,dt$$

$$= \left[\frac{e^{-st}}{-s}\sin(wt)\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} w\cos(wt)\,dt$$

$$= (0-0) + \frac{w}{s}\int_0^\infty \underbrace{e^{-st}}_{u'} \underbrace{\cos(wt)}_{v}\,dt$$

$$= \frac{w}{s}\left[\frac{e^{-st}}{-s}\cos(wt)\right]_0^\infty - \frac{w}{s}\int_0^\infty -\frac{e^{-st}}{-s} w\sin(wt)\,dt$$

$$= \frac{w}{s^2} - \frac{w^2}{s^2}\int_0^\infty e^{-st}\sin(wt)\,dt$$

Hence $\boxed{F(s) = \frac{w}{s^2} - \frac{w^2}{s^2}F(s)}$ and we can solve this for $F(s)$:

$$s^2 F(s) = w - w^2 F(s) \Rightarrow (s^2 + w^2)F(s) = w$$

$$\Rightarrow F(s) = \frac{w}{s^2+w^2} \quad \text{where } f(t) = \sin(wt)$$

Linearity of the Laplace transform

Let $f(t)$ and $g(t)$ be functions. If

$$F(s) = \mathcal{L}(f)(s) \qquad G(s) = \mathcal{L}(g)(s)$$

then for any constants $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ we have

$$\mathcal{L}\left(\alpha f + \beta g\right)(s) = \alpha\mathcal{L}(f)(s) + \beta\mathcal{L}(g)(s)$$

Example: The Laplace transform of $h(t) = \sin(2t) - 3\cos(2t)$ is

$$H(s) = \mathcal{L}\{1 \cdot \sin(2t) - 3\cos(2t)\}$$

$$= 1 \cdot \mathcal{L}\{\sin(2t)\} - 3 \cdot \mathcal{L}\{\cos(2t)\}$$

$$= 1 \cdot \frac{2}{s^2+4} - 3 \cdot \frac{s}{s^2+4}$$

$$= \frac{2 - 3s}{s^2+4}$$

$$\mathcal{L}\{\sin(wt)\} = \frac{w}{s^2+w^2}$$

$$\mathcal{L}\{\cos(wt)\} = \frac{s}{s^2+w^2}$$

Linearity of the inverse Laplace transform

Let $F(s)$ and $G(s)$ be Laplace transforms. If

$$f(t) = \mathcal{L}^{-1}(F)(t) \qquad g(t) = \mathcal{L}^{-1}(G)(t)$$

then for any constants $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ we have

$$\mathcal{L}^{-1}(\alpha F + \beta G)(t) = \alpha \mathcal{L}^{-1}(F)(t) + \beta \mathcal{L}^{-1}(G)(t)$$

Example: The inverse Laplace transform of $H(s) = \frac{s+4}{s^2+9}$ is

$$h(t) = \mathcal{L}^{-1}(H(s)) = \mathcal{L}^{-1}\left(\frac{s+4}{s^2+9}\right) = \mathcal{L}^{-1}\left(\frac{s}{s^2+9} + \frac{4}{s^2+9}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right) + \frac{4}{3}\cdot\mathcal{L}^{-1}\left(\frac{3}{s^2+9}\right)$$

$$= \cos(3t) + \frac{4}{3}\sin(3t)$$

Partial fractions

Suppose that $F(s) = \frac{P(s)}{Q(s)}$ with $P(s)$ and $Q(s)$ polynomials in s. Then we can compute the Inverse Laplace transform $f(t)$ of $F(s)$ using partial fractions.

Example: Determine the inverse Laplace transform of

$$F(s) = \frac{2s+3}{(s-1)(s+2)(s-3)}$$

"Cover-up method"

We can compute the partial fraction decomposition of

$$\frac{2s+3}{(s-1)(s+2)(s-3)} = \frac{-\frac{5}{6}}{s-1} + \frac{-\frac{1}{15}}{s+2} + \frac{\frac{9}{10}}{s-3}$$

"Finger method"

Hence

$$= -\frac{5}{6}\left(\frac{1}{s-1}\right) - \frac{1}{15}\left(\frac{1}{s+2}\right) + \frac{9}{10}\left(\frac{1}{s-3}\right)$$

$$\boxed{f(t) = -\frac{5}{6}e^{t} - \frac{1}{15}e^{-2t} + \frac{9}{10}e^{3t}}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^n}\right\} = e^{at}\frac{t^{n-1}}{(n-1)!}$$

Example

Determine the inverse Laplace Transform of

$$F(s) = \frac{2s+3}{(s-1)^3}$$

We can computed the partial fraction decomposition of

$$F(s) = \frac{A}{(s-1)^3} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^1}$$

So $$\frac{2s+3}{(s-1)^3} = \frac{A + (s-1)B + (s-1)^2 C}{(s-1)^3}$$

Equating numerators gives

$$2s+3 = A + sB - B + s^2C - 2sC + C$$

$$\Rightarrow 2s+3 = (A-B+C) + (B-2C)s + Cs^2$$

$$\Rightarrow C = 0 \Rightarrow B - 2C = 2 \Rightarrow B = 2 \text{ and } A - B + C = 3 \Rightarrow A = 5$$

So $\boxed{F(s) = \frac{5}{(s-1)^3} + \frac{2}{(s-1)^2}} \Rightarrow \boxed{f(t) = 5e^t\frac{t^2}{2} + 2e^t\frac{t^1}{1}}$

Laplace transform of the derivative

If $f(t)$ is differentiable and has Laplace transform $F(s)$ then

$$\mathcal{L}(f') = sF(s) - f(0)$$

This result is a clue that Laplace transforms might be useful for solving differential equations.

$$\int uv'\,dt = [uv] - \int u'v\,dt$$

Proof

$$\mathcal{L}(f') = \int_0^\infty \underbrace{e^{-st}}_{u} \underbrace{f'(t)}_{v'}\,dt$$

$$= \left[e^{-st} f(t)\right]_0^\infty + \int_0^\infty s\,e^{-st} f(t)\,dt$$

$$= e^{-st} f(t)\Big|_{t=\infty} - f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= 0 - f(0) + sF(s)$$

So $\mathcal{L}(f') = sF(s) - f(0)$

Laplace transforms of higher derivatives

$$\mathcal{L}(f'') = s^2 F(s) - s f(0) - f'(0)$$

$$\mathcal{L}(f''') = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$$

The general formula is

$$\mathcal{L}(f^{(n)}) = s^n F(s) - s^{n-1} f(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

$$\mathcal{L}(n!) = s^{n} F(s) - 0 - 0 - \ldots - 0$$

We see that $\mathcal{L}(f^{(n)})$ involves no derivatives of $F(s)$. Its simply a multiple s^n of $F(s)$ plus a polynomial of degree $n-1$ in s.

Example

A neat way to find the Laplace transform of $f(t) = t^n$, where n a positive integer is to observe that

$$f^{(n)}(t) = n! = c \qquad (*)$$

and that

$$f(0) = 0, \quad f'(0) = 0, \quad \cdots \quad f^{(n-1)}(0) = 0$$

Taking Laplace transforms of both side of (*) gives

$$s^n F(s) \overset{+0}{=} \frac{n!}{s} \qquad \Rightarrow \qquad \boxed{F(s) = \mathcal{L}(t^n) = \frac{n!}{s^{n+1}}}$$

$f(t) = t^4$

$f'(t) = 4t^3$

$f''(t) = 4.3t^2$

$f'''(t) = 4.3.2t$

$f''''(t) = 4.3.2.1$

We have used the fact that the LT of a constant c is

$$\mathcal{L}(c) = \mathcal{L}(c1) = c\mathcal{L}(1) = c\left(\frac{1}{s}\right) = \frac{c}{s}$$

Example

$f'(t) = ae^{at}$ and $f(0) = 1$

A neat way to find the Laplace transform of $f(t) = e^{at}$ is to recall that it satisfies the differential equation

$$\frac{df}{dt} = af \qquad (*)$$

Taking Laplace transforms of both side gives

$$\mathcal{L}(f') = sF(s) - f(0)$$

L.T. of $(*)$ $\Rightarrow \quad sF(s) - 1 = aF$

$$\Rightarrow \quad (s-a)F(s) = 1$$

$$\Rightarrow \quad F(s) = \frac{1}{s-a}$$

EXAMPLE

A neat way to find the Laplace transform of $\overset{\overset{f(t)}{\shortparallel}}{\cos(\omega t)}$ is to recall that it satisfies the differential equation

$$\frac{d^2 f}{dt^2} + \omega^2 f = 0$$

$f' = -w \sin(wt)$

$f'' = -w^2 \cos(wt) = -w^2 f$

Taking Laplace transforms of both side gives

$f(0) = 1$

$f'(0) = 0$

L.T. of both sides gives

$$s^2 F(s) - sf(0) - f'(0) + w^2 F(s) = 0$$

$$\Rightarrow \quad s^2 F(s) - s - 0 + w^2 F(s) = 0$$

$$\Rightarrow \quad (s^2 + w^2) F(s) = s \quad \Rightarrow \quad F(s) = \frac{s}{s^2 + w^2}$$

$$f(t) = \sin(wt) \quad \Rightarrow \quad s^2 F(s) - 0 - w + w^2 F(s) = 0$$

$$\Rightarrow \quad F(s) = \frac{w}{s^2 + w^2}$$

Laplace transform of a definite integral

$$\mathcal{L}\left(\int_0^t f(u)\,du\right) = \frac{F(s)}{s}$$

This can be useful in helping to find inverse transforms of functions which have a factor s appearing in the denominator.

Sketch of proof

If $g(t) = \int_0^t f(u)\,du$, then $g'(t) =$ $f(t)$ and $g(0) =$ $\int_0^0 f(u)\,du = 0$

Applying the derivative result to $g(t)$ gives

$$\mathcal{L}(g') = s\mathcal{L}(g)(s) - g(0)$$

$$\Rightarrow \quad \mathcal{L}(f) = s\,\mathcal{L}\left(\int_0^t f(u)\,du\right) - 0$$

$$\Rightarrow \quad F(s) = s\,\mathcal{L}(\quad)$$

$$\Rightarrow \quad \frac{F(s)}{s} = \mathcal{L}(\quad)$$

We can write the integral result as

$$\mathcal{L}^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(u)\,du$$

Example: To find $\mathcal{L}^{-1}\left(\frac{1}{s(s+3)}\right)$

Let $\boxed{F(s) = \frac{1}{s+3}}$ then $\boxed{f(t) = e^{-3t}}$ $\rightarrow$ $f(u) = e^{-3u}$

and $\mathcal{L}^{-1}\left(\frac{F(s)}{s}\right) = \mathcal{L}^{-1}\left(\frac{1}{s(s+3)}\right)$

$$= \int_0^t f(u)\,du = \int_0^t e^{-3u}\,du$$

$$= \left[\frac{e^{-3u}}{-3}\right]_{0=u}^{t=u} = \left[-\frac{1}{3}e^{-3t} + \frac{1}{3}\right]$$

$$= \frac{1-e^{-3t}}{3}$$

Using Laplace transforms to solve DEs

We can use Laplace transforms to solve linear differential equations. We'll concentrate on the second-order constant coefficient differential equation with initial conditions:

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t) \qquad t \geq 0$$

$$y(0) = y_0 \qquad \frac{dy}{dx}(0) = v_0$$

We start by letting

$$Y(s) = \mathcal{L}(y) \qquad G(s) = \mathcal{L}(g)$$

and take Laplace transforms of the above system using linearity and the derivative formulae. We get

$$s^2Y(s) - sy_0 - v_0 + p(sY(s) - y_0) + qY(s) = G(s)$$

Now we solve for $Y(s)$ and invert this to get $y(t)$.

Example

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$y'' + y' - 2y = 9e^{3t} \qquad y(0) = 0 \qquad y'(0) = 2$$

LT: $\underbrace{s^2 Y(s) - s y(0) - y'(0)} + \underbrace{sY - y(0)} - 2Y(s) = \mathcal{L}\{9e^{3t}\}$

$$s^2 Y(s) - 2 + sY(s) - 2Y(s) = 9 \cdot \frac{1}{s-3}$$

so $(s^2 + s - 2)\, Y(s) = 2 + \frac{9}{s-3} = \frac{2s - 6 + 9}{s-3} = \frac{2s+3}{s-3}$

Hence $\boxed{(s-1)(s+2)\, Y(s) = \frac{2s+3}{s-3}}$

$$Y(s) = \frac{2s+3}{(s-1)(s+2)(s-3)} = -\frac{5/6}{s-1} - \frac{1/15}{s+2} + \frac{9/10}{s-3}$$

and hence

$$y(t) = -\frac{5}{6}e^{t} - \frac{1}{15}e^{-2t} + \frac{9}{10}e^{3t}$$

Example

$\mathcal{L}\{\cos wt\} = \dfrac{s}{s^2+w^2}$ $\mathcal{L}\{\sin(wt)\} = \dfrac{w}{s^2+w^2}$

$$y'' + 4y = \sin t \qquad y(0) = 1 \qquad y'(0) = 0$$

$$s^2 Y(s) - s - 0 + 4Y(s) = \frac{1}{s^2+1}$$

So $$(s^2+4)Y(s) = \frac{1}{s^2+1} + s$$

Hence

$$\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$Y(s) = \frac{1}{(s^2+1)(s^2+4)} + \frac{s}{s^2+4} = \frac{1/3}{s^2+1} - \frac{1/3}{s^2+4} + \frac{s}{s^2+4}$$

and hence

$$y(t) = \frac{1}{3}\sin(t) - \frac{1}{6}\sin(2t) + \cos(2t)$$

The derivative of the Laplace transform

Suppose $F(s)$ is the Laplace transform of $f(t)$. Then

$$F'(s) = \mathcal{L}\left(-tf(t)\right) \qquad \text{or} \qquad -F'(s) = \mathcal{L}\left(tf(t)\right)$$

and conversely

$$\mathcal{L}^{-1}(-F'(s)) = tf(t)$$

Example: To find the Laplace Transform of te^{at}, let $f(t) = e^{at}$ and recall that

$$\mathcal{L}(f(t)) = \mathcal{L}(e^{at}) = F(s) = \frac{1}{s-a}$$

and hence

$$\mathcal{L}(te^{at}) = \mathcal{L}(tf(t)) = -F'(s) = -\frac{d}{ds}\left(\frac{1}{s-a}\right) = \frac{1}{(s-a)^2}$$

Example: To find the Laplace Transform of $t\sin(\omega t)$ we let $f(t) = \sin(\omega t)$ and recall that

$$\mathcal{L}(f(t)) = \mathcal{L}(\sin(\omega t)) = F(s) = \frac{\omega}{s^2+\omega^2}$$

and hence

$$\mathcal{L}(t\sin(\omega t)) = -F'(s) = -\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right) = \frac{2\omega s}{(s^2+\omega^2)^2}$$

which is essentially the last row of the Laplace transforms table:

$$\mathcal{L}\left(\frac{t\sin(\omega t)}{2\omega}\right) = \frac{s}{(s^2+\omega^2)^2}$$

Applying the result $\mathcal{L}\left(t f(t)\right) = -F'(s)$ multiple times, we get

$$\boxed{\mathcal{L}\left(t^n f(t)\right) = (-1)^n F^{(n)}(s)}$$

The s-shift theorem

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}(e^{at}f(t)) = F(s-a)$$

Proof

$$\mathcal{L}\left(e^{at}f(t)\right) = \int_0^\infty e^{-st}e^{at}f(t)\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a)$$

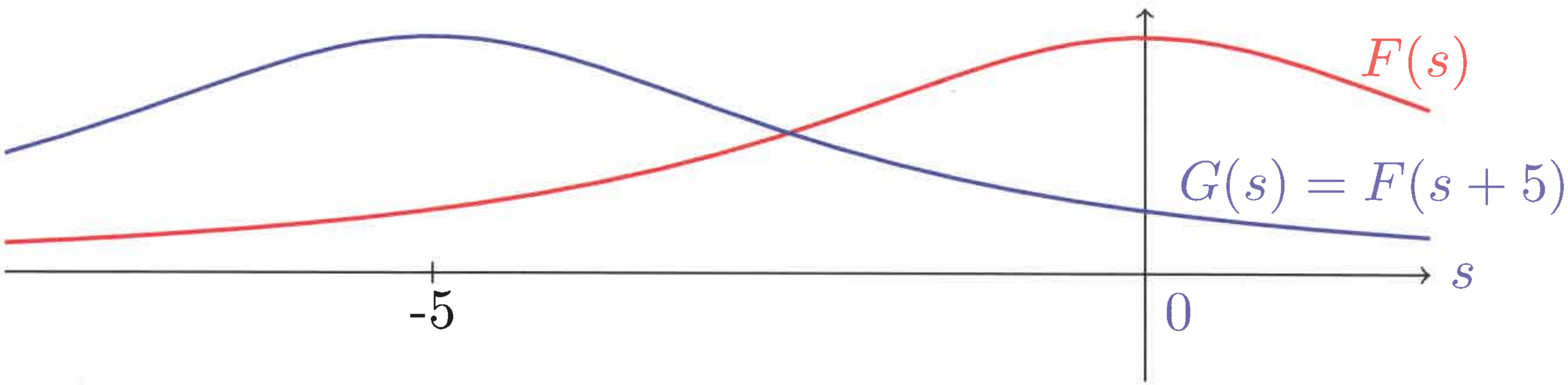
Example

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+10s+34}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s+5)^2+9}\right) = e^{-5t}\frac{\sin(3t)}{3}$$

"completing the square"

since $\mathcal{L}^{-1}\left(\frac{1}{s^2+9}\right) = \frac{\sin(3t)}{3}$

The graph of $G(s) = \frac{1}{(s+5)^2+9}$ is that of $F(s) = \frac{1}{s^2+9}$ but shifted by 5 to the left.



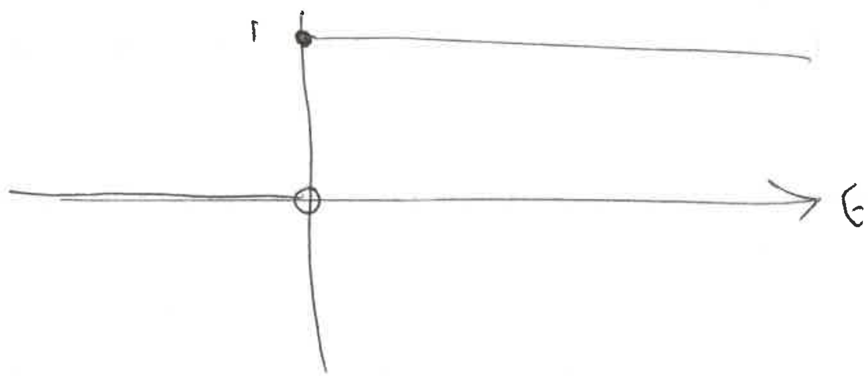
In general, if $G(s) = F(s-a)$ the graph of $G(s)$ is the graph of $F(s)$ shifted to the right (left) by a, if $a > 0$ (or $a < 0$).

Heaviside function

The *Heaviside function* $H(t)$ is defined to be

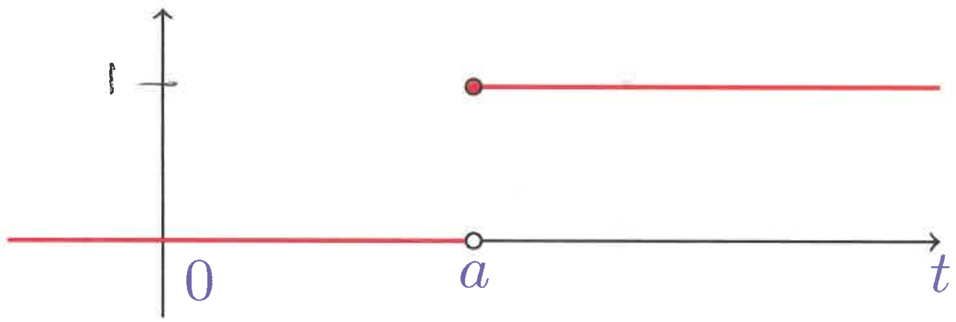
$$H(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

This function is sometimes called the *unit step function* and denoted $u(t)$.



For any $a \in \mathbb{R}$ the graph of $H(t-a)$ is obtained from the graph of $H(t)$ by shifting a units (to the right if $a > 0$ and to the left if $a < 0$), that is:

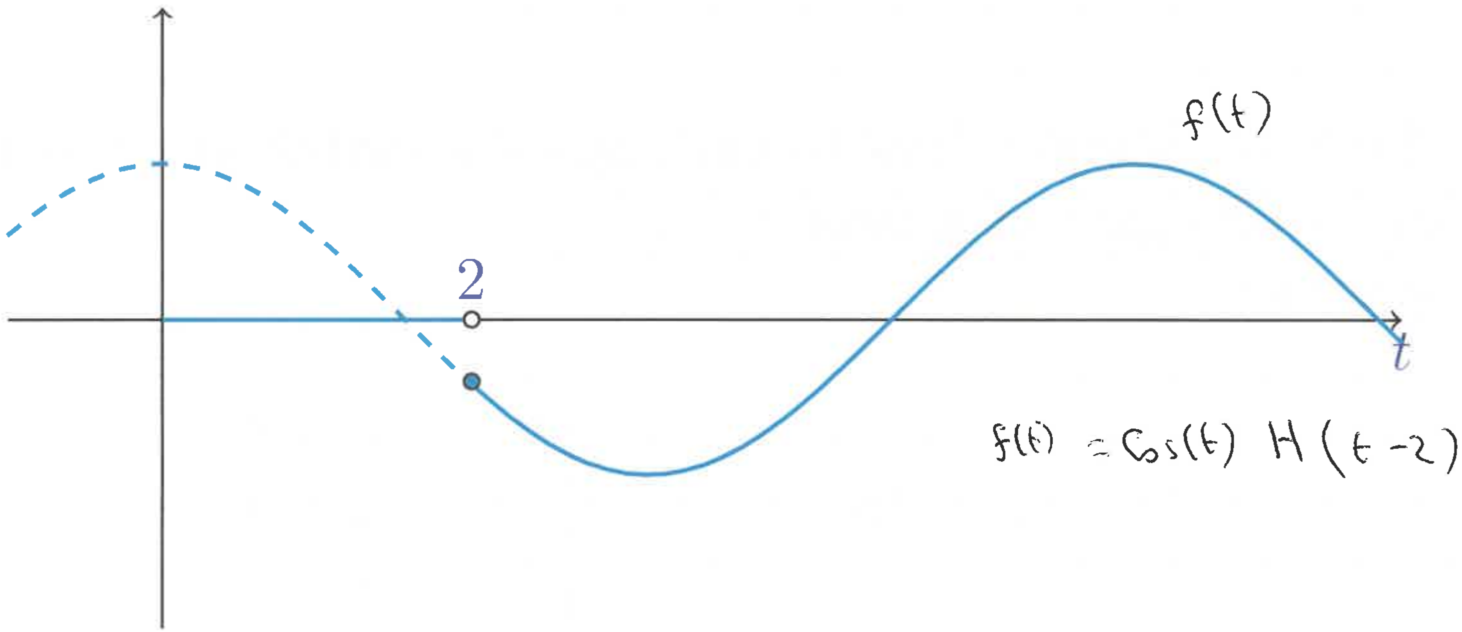
$$H(t-a) = \begin{cases} 0, & t < a, \\ 1, & t \geq a. \end{cases}$$



Note that

$$g(t)H(t-a) = \begin{cases} 0, & t < a, \\ g(t), & t \geq a. \end{cases}$$

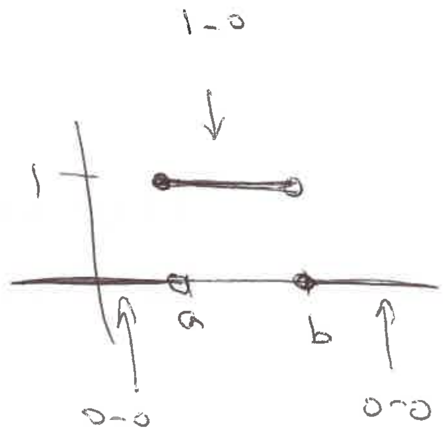
Thus multiplying $g(t)$ by $H(t-a)$ has the effect of suppressing the function until time $t = a$ and then activating it.



$a = 2$ and $g(t) = \cos t$

The pulse function

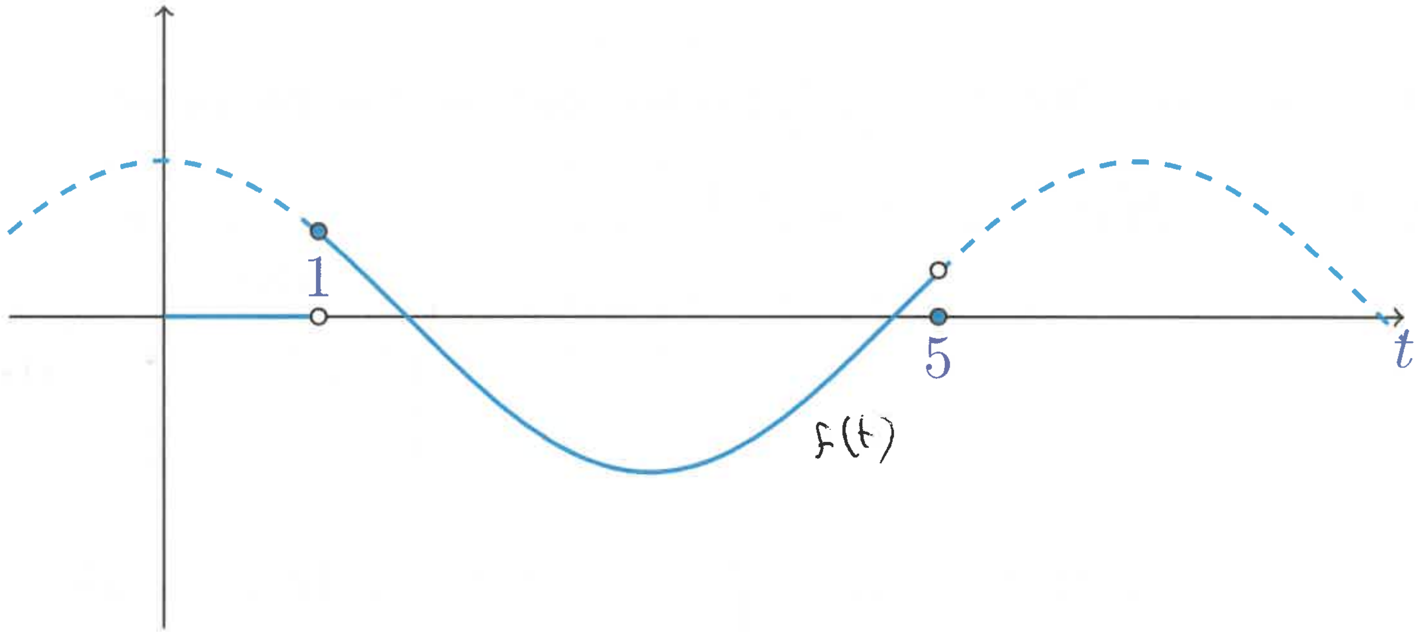
$$H(t-a) - H(t-b) = \begin{cases} 0, & t < a, \\ 1, & a \le t < b, \\ 0, & b \le t. \end{cases}$$



This function is equivalent to turning on a switch at $t = a$ then turning it off again at a later $t = b$.
We note that

$$g(t)[H(t-a) - H(t-b)] = \begin{cases} 0, & t < a, \\ g(t), & a \le t < b, \\ 0, & b \le t. \end{cases}$$

$H(t-a)$ is sometimes denoted by $u(t-a)$ or $u_a(t)$.



$a = 1$ and $b = 5$ and $g(t) = \cos t$

$\cos(t)\left[H(t-1) - H(t-5)\right]$

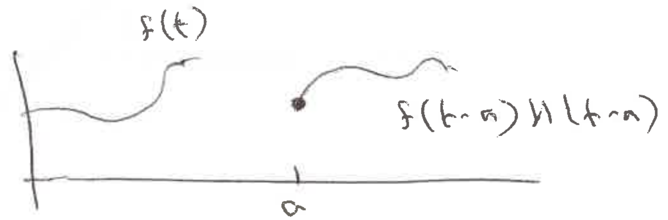
The Heaviside shift Theorem

Because the Heaviside function is quite important in real problems it is helpful to note the result of the following theorem:

If $F(s) = \mathcal{L}(f(t))$ for $s > \gamma$, then for any $a \geq 0$ we have

$$\boxed{\mathcal{L}[f(t-a)H(t-a)] = e^{-as}F(s)}$$

$s > \gamma$



Proof

$$\mathcal{L}\left(f(t-a)H(t-a)\right) = \int_0^\infty e^{-st} f(t-a)H(t-a)dt$$

$$= \int_a^\infty e^{-st} f(t-a)dt = \int_0^\infty \overset{e^{-us}e^{-as}}{e^{-s(u+a)}} f(u)du$$

$$= e^{-as}\int_0^\infty e^{-su} f(u)du \equiv e^{-as}\int_0^\infty e^{-st} f(t)dt = e^{-as}F(s)$$

where we have let $t = u + a \;\rightarrow\; dt = du$, and $t = a \;\Rightarrow\; u = 0$

EXAMPLES

$$\mathcal{L}\left[f(t-a)H(t-a)\right] = e^{-as}F(s)$$

$\mathcal{L}(H(t-a)) = ?$

Let $f(t) = 1 \rightarrow F(s) = \frac{1}{s} \Rightarrow \mathcal{L}\{H(t-a)\} = \frac{e^{-as}}{s}$

$\mathcal{L}((t-a)H(t-a)) = ?$

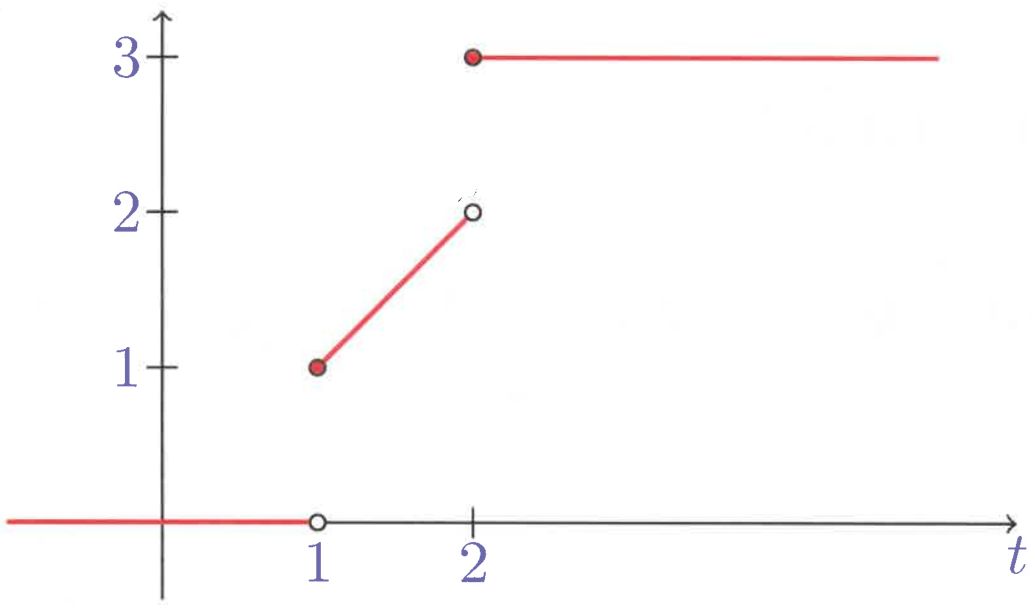
Let $f(t) = t$. Then $F(s) = \frac{1}{s^2} \Rightarrow \mathcal{L}\{(t-a)H(t-a)\}$

$= \frac{e^{-as}}{s^2}$

Example

Consider the function

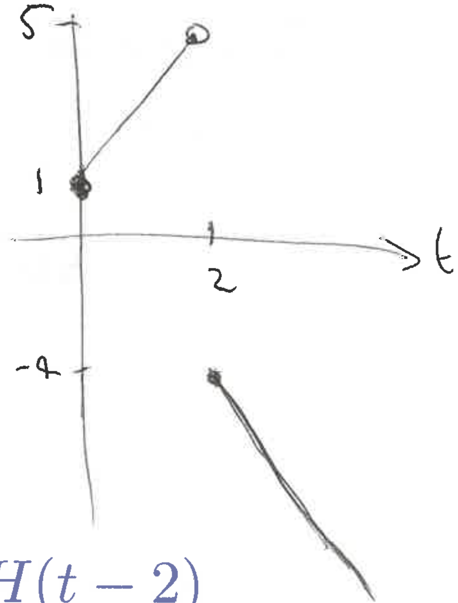
$$f(t) = \begin{cases} 0, & t < 1, \\ t, & 1 \le t < 2, \\ 3, & t \ge 2. \end{cases}$$



Example

Find the Laplace transform of

$$f(t) = \begin{cases} 2t+1, & 0 \le t < 2, \\ 2-3t, & 2 \le t. \end{cases}$$



In step-function notation

$$\begin{aligned} f(t) &= (2t+1)\left[H(t) - H(t-2)\right] + (2-3t)H(t-2) \\ &= (2t+1)H(t) + (1-5t)H(t-2) \end{aligned}$$

$$= 2t\,H(t) + H(t) + (1 - 5(t-2) - 10)\,H(t-2)$$

$$= 2t\,H(t) + H(t) - 9\,H(t-2) - 5(t-2)\,H(t-2)$$

$$= 2\,\frac{1}{s^2} + \frac{1}{s} - 9\,\frac{e^{-2s}}{s} - 5\,\frac{e^{-2s}}{s^2}$$

To express $f(t)$ as a function using Heaviside functions, and find the Laplace transform of $f(t)$, we see that

$$f(t) = t\left[H(t-1) - H(t-2)\right] + 3H(t-2)$$

$$= tH(t-1) - (t-3)H(t-2)$$

$$= (t-1)H(t-1) + H(t-1) - (t-2)H(t-2) + H(t-2)$$

We know that

$$\mathcal{L}(H(t-a)) = \frac{e^{-as}}{s} \qquad \mathcal{L}((t-a)H(t-a)) = \frac{e^{-as}}{s^2}$$

and hence

$$F(s) = \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s^2} + \frac{e^{-2s}}{s}$$

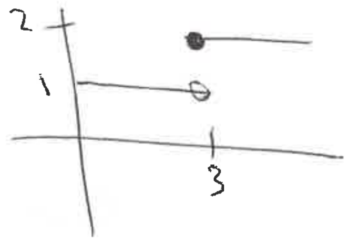
Example

Use Laplace transforms to solve the initial value problem

$$y'' + 3y' + 2y = f(t) \qquad y(0) = 0 \qquad y'(0) = 1$$

where

$$f(t) = \begin{cases} 1, & 0 \le t < 3, \\ 2, & 3 \le t. \end{cases} = 1 + H(t-3)$$



Taking Laplace transforms gives

$$s^2 Y(s) - s y(0) - y'(0) + 3(sY(s) - y_0) + 2Y(s) = \frac{1}{s} + \frac{e^{-3s}}{s}$$

$$\Rightarrow \quad s^2 Y(s) - 1 + 3(sY(s) - 0) + 2Y(s) = \frac{1}{s} + \frac{e^{-3s}}{s}$$

$$\Rightarrow \quad (s^2 + 3s + 2)Y(s) = 1 + \frac{1}{s} + \frac{e^{-3s}}{s}$$

Hence

$$(\underbrace{s^2 + 3s + 2}_{(s+1)(s+2)})Y(s) = \overbrace{1 + \frac{1}{s}}^{\frac{s+1}{s}} + \frac{e^{-3s}}{s}$$

Noting that $s^2 + 3s + 2 = (s+1)(s+2)$ and solving this equation for $Y(s)$ we get

$$Y(s) = \frac{1}{s(s+2)} + \frac{e^{-3s}}{s(s+1)(s+2)}$$

Now we need to invert this transform. Partial fractions gives

$$F(s) = \frac{1}{s(s+2)} = \frac{1/2}{s} + \frac{-1/2}{s+2} \quad \Rightarrow \quad f(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$$

and

$$G(s) = \frac{1}{s(s+1)(s+2)} = \frac{1/2}{s} + \frac{-1}{s+1} + \frac{1/2}{s+2}$$

$$\Rightarrow \quad g(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

That is,

$$y(t) = \underbrace{\frac{1}{2} - \frac{1}{2}e^{-2t}}_{f(t)} + H(t-3)\left[\underbrace{\frac{1}{2} - e^{3-t} + \frac{1}{2}e^{2(3-t)}}_{g(t-3)}\right]$$

Written as a piecewise function this is

$$y(t) = \begin{cases} \frac{1}{2} - \frac{1}{2}e^{-2t}, & 0 \leq t < 3 \\ 1 - \frac{1}{2}e^{-2t} - e^{3-t} + \frac{1}{2}e^{2(3-t)}, & t \geq 3 \end{cases}$$

Note that this function is continuous because

$$y(3^-) = \frac{1 - e^{-6}}{2} \qquad\qquad y(3^+) = \frac{1 - e^{-6}}{2}$$

The first derivative y' is also continuous, as

$$y'(3^-) = e^{-6} \qquad\qquad y'(3^+) = e^{-6}$$

However, the second derivative is not, as

$$y''(3^-) = -2e^{-6} \qquad\qquad y''(3^+) = -2e^{-6} + 1$$

The graphs of $f(t)$, $y(t)$, $y'(t)$ and $y''(t)$ are