MATH1012 (Semester 1, 2024)

Practice Class 6: Improper integrals, sequences

Summary of what you have learned

- Improper integrals are defined with limits of definite integrals.
- Sometimes we need to split the domain on integration.
- Sequences, convergence, divergence, bounded, monotone convergence theorem.

Foundational questions

EXERCISE 1. *Determine if the following improper integrals are convergent and evaluate those which are convergent.*

(a) $\displaystyle\int_0^\infty \frac{1}{2x+1}dx$ (b) $\displaystyle\int_{-\infty}^0 \frac{1}{(1-3x)^2}dx$ (c) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$

(d) $\displaystyle\int_2^\infty \frac{1}{\sqrt[3]{x-2}}dx$ (e) $\displaystyle\int_{-3}^3 \frac{1}{x^2+3x+2}dx$

SOLUTION:

(a) The integrand is not defined at $x = -\frac{1}{2}$ but that is outside the bounds of this integral. Then

$$\int_0^\infty \frac{1}{2x+1}dx = \lim_{t\to\infty}\int_0^t \frac{1}{2x+1}dx$$
$$= \lim_{t\to\infty}\left[\frac{\ln(2x+1)}{2}\right]_0^t = \lim_{t\to\infty}\frac{\ln(2t+1)}{2} \to \infty,$$

so this integral is divergent. Note that you can use the substitution $u = 2x+1$ to find the antiderivative.

(b) The function is not defined at $x = \frac{1}{3}$ but that is outside the bounds of this integral. So

$$\int_{-\infty}^0 \frac{1}{(1-3x)^2}dx = \lim_{t\to-\infty}\int_t^0 \frac{1}{(1-3x)^2}dx$$
$$= \lim_{t\to-\infty}\left[\frac{1}{3(1-3x)}\right]_t^0 = \lim_{t\to-\infty}\frac{1}{3} - \frac{1}{3(1-3t)} = \frac{1}{3}.$$

This integral is convergent. Note that you can use the substitution $u = 1-3x$ to find the antiderivative.

(c) The function is not defined at $x = 1$ which is one of the bounds, so

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \lim_{t\to1^-}\int_0^t \frac{1}{\sqrt{1-x^2}}dx$$
$$= \lim_{t\to1^-}\left[\sin^{-1}(x)\right]_0^t = \lim_{t\to1^-}\sin^{-1}(t) = \frac{\pi}{2}.$$

This integral is convergent.

(d) The function is not defined at $x = 2$ which is one of the bounds, so the integral has to be split into a type I and a type II improper integral (the split at 3 is arbitrary):

$$\begin{aligned}
\int_2^\infty \frac{1}{\sqrt[3]{x-2}}dx &= \int_2^3 \frac{1}{\sqrt[3]{x-2}}dx + \int_3^\infty \frac{1}{\sqrt[3]{x-2}}dx \\
&= \lim_{t\to 2^+} \int_t^3 (x-2)^{-\frac{1}{3}}dx + \lim_{u\to\infty} \int_3^u (x-2)^{-\frac{1}{3}}dx.
\end{aligned}$$

On integration the second term is

$$\lim_{u\to\infty} \left[\frac{3}{2}(x-2)^{\frac{2}{3}}\right]_3^u = \lim_{u\to\infty} \frac{3}{2}(u-2)^{\frac{2}{3}} - \frac{3}{2}$$

which is divergent, so the original integral is divergent. (The first integral is convergent, but it doesn't matter since both integrals need to be convergent.)

(e) The denominator factors as $x^2 + 3x + 2 = (x+1)(x+2)$ so the function is not defined at $x = -1$ and $x = -2$ (it has vertical asymptotes at those points), so

$$\begin{aligned}
\int_{-3}^3 \frac{1}{x^2+3x+2}dx = \int_{-3}^{-2} \frac{1}{x^2+3x+2}dx + \int_{-2}^{-1.5} \frac{1}{x^2+3x+2}dx \\
+ \int_{-1.5}^{-1} \frac{1}{x^2+3x+2}dx + \int_{-1}^{3} \frac{3}{x^2+3x+2}dx
\end{aligned}$$

and all four terms are Type II improper integrals.

To find an antiderivative for the integrand we'll need partial fractions (from MATH1011). We get

$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$

so an antiderivative is

$$\int \frac{1}{x^2+3x+2} = \ln|x+1| - \ln|x+2| = \ln\frac{|x+1|}{|x+2|}$$

and hence

$$\begin{aligned}
\int_{-3}^{-2} \frac{1}{x^2+3x+2}dx &= \lim_{t\to -2^-} \int_{-3}^{t} \frac{1}{x^2+3x+2}dx \\
&= \lim_{t\to -2^-} \left[\ln\frac{|x+1|}{|x+2|}\right]_{-3}^t \\
&= \lim_{t\to -2^-} \ln\frac{|t+1|}{|t+2|} - \ln\frac{|2|}{|1|} \\
&= \lim_{t\to -2^-} \ln\frac{|t+1|}{|t+2|} - \ln 2
\end{aligned}$$

which diverges to $-\infty$ as $t \to -2^-$. Thus the original integral is divergent, even though the antiderivative exists at -3 and 3. So if we don't pay attention we can easily write

$$\int_{-3}^3 \frac{1}{x^2+3x+2}dx = \ln\frac{4}{5} - \ln 2 \qquad \text{which is a mistake!}$$

Exercise 2. *Find the limits (if they exist) as $n \to \infty$ of the following sequences:*

$$a_n = \frac{3n^2 - 1}{10n + 8n^2} \qquad b_n = \frac{n^{\frac{3}{4}} - 2n + 3}{n^3 + n + 1} \qquad c_n = \sin^2 n + \cos^2 n \qquad d_n = \sqrt{\frac{n+1}{n}}$$

Solution:

(a) We know that $\lim_{n\to\infty} \frac{1}{n^p} = 0$ for $p > 0$. Dividing every element of a_n by n^2 gives

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^2 - 1}{10n + 8n^2} = \lim_{n\to\infty} \frac{3 - \frac{1}{n^2}}{\frac{10}{n} + 8} = \frac{3-0}{0+8} = \frac{3}{8}$$

Note: formally we should write, using limit rules

$$\lim_{n\to\infty} \frac{3 - \frac{1}{n^2}}{\frac{10}{n} + 8} = \frac{\lim_{n\to\infty} 3 - \lim_{n\to\infty} \frac{1}{n^2}}{\lim_{n\to\infty} \frac{10}{n} + \lim_{n\to\infty} 8} = \frac{3-0}{0+8} = \frac{3}{8}$$

(b) We have

$$\begin{aligned}\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^{\frac{3}{4}} - 2n + 3}{n^3 + n + 1} &= \lim_{n\to\infty} \frac{n(1/n^{\frac{1}{4}} - 2 + 3/n)}{n^3(1 + 1/n^2 + 1/n^3)} \\ &= \lim_{n\to\infty} \frac{1/n^{\frac{1}{4}} - 2 + 3/n}{n^2(1 + 1/n^2 + 1/n^3)} \\ &= \lim_{n\to\infty} \frac{1}{n^2} \cdot \lim_{n\to\infty} \frac{1/n^{1/4} - 2 + 3/n}{1 + 1/n^2 + 1/n^3} = 0 \cdot \frac{-2}{1} = 0\end{aligned}$$

(c) As $\sin^2 n + \cos^2 n = 1$ for all n, the sequence is actually $a_n = 1$ so it converges to 1.

$$\lim_{n\to\infty} \frac{n+1}{n} = \frac{1 + \frac{1}{n}}{1} = \frac{1+0}{1} = 1 \qquad \text{and hence} \qquad \lim_{n\to\infty} d_n = \sqrt{1} = 1$$

Exercise 3. *Find the limits (if they exist) as $n \to \infty$ of the following sequences:*

$$a_n = \frac{\ln n}{n^3 + 1} \qquad b_n = (-1)^n \left(1 + \frac{1}{n}\right) \qquad c_n = 2^{\frac{1}{n}}$$

Solution:

(a) We can make use of the squeeze theorem:

$$0 \leq \frac{\ln n}{n^3 + 1} \leq \frac{\ln n}{n^3} \qquad \text{and} \qquad \lim_{n\to\infty} \frac{\ln n}{n^3} = 0 \qquad \text{and hence} \qquad \lim_{n\to\infty} a_n = 0$$

The intermediate result is Theorem 8.21 in the unit reader.

(b) Note that

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1$$

The $(-1)^n$ causes this to alternate between positive and negative values. The terms with n even approach 1 as $n \to \infty$ while the odd terms approach -1 as $n \to \infty$

(c) We have

$$\lim_{n\to\infty} \frac{1}{n} = 1 \qquad \text{and hence} \qquad \lim_{n\to\infty} c_n = 2^0 = 1$$

EXERCISE 4. *Find the limit (if it exists) as $n \to \infty$ of the sequences*

$$a_1 = 1 \quad \text{and} \quad a_n = 2 + \frac{1}{a_{n-1}} \quad \text{for} \quad n > 1$$

The sequence begins

$$1,\ 3,\ \frac{7}{3},\ \frac{17}{7},\ \frac{41}{17},\ \frac{99}{41},\ \cdots \qquad \text{which in decimals is} \qquad 1,\ 3,\ 2.333,\ 2.429,\ 2.412,\ 2.415,\ \cdots$$

SOLUTION: It looks like its converging, but to what value? Let $L = \lim_{n\to\infty} a_n$. Taking limits in the definition of the sequence gives

$$\lim_{n\to\infty} a_n = 2 + \frac{1}{\lim_{n\to\infty} a_{n-1}} \qquad \text{that is,} \qquad L = 2 + \frac{1}{L} \qquad \text{and hence} \qquad L^2 - 2L - 1 = 0$$

The quadratic formula tells us that the solution of this quadratic equation is

$$L = \frac{1 \pm \sqrt{4+4}}{2} = \frac{1 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2} = 2.414214\ldots, -0.414214\ldots$$

and we reject the second solution because a_n can never be negative, so $\lim_{n\to\infty} a_n = 1+\sqrt{2}$

EXERCISE 5. *Find the limit as $n \to \infty$ of the sequence*

$$a_1 = 1 \quad \text{and} \quad a_n = \sqrt{3 + a_{n-1}} \quad \text{for} \quad n > 1$$

SOLUTION: Let $L = \lim_{n\to\infty} a_n$. Taking limits in the definition of the sequence gives

$$\lim_{n\to\infty} a_n = \sqrt{3 + \lim_{n\to\infty} a_{n-1}} \qquad \text{that is,} \qquad L = \sqrt{3+L} \qquad \text{and hence} \qquad L^2 - L - 3 = 0$$

The quadratic formula tells us that the solution of this quadratic equation is

$$L = \frac{1 \pm \sqrt{1+12}}{2} = \frac{1 \pm \sqrt{13}}{2} = 2.303\ldots, -1.303\ldots$$

and we reject the second solution because a_n can never be negative.

EXERCISE 6. *Use a sequence to find the value of the so-called* continued fraction

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}}}$$

SOLUTION: This is equivalent to finding the limit as $n \to \infty$ of the sequence

$$a_1 = 1 \quad \text{and} \quad a_n = 1 + \frac{1}{a_{n-1}} \quad \text{for} \quad n > 1$$

which we've already done (above). Hence the value of the continued fraction is

$$\lim_{n\to\infty} a_n = \phi = \frac{1+\sqrt{5}}{2}$$

Exercise 7. *Find the limit (if it exists) as $n \to \infty$ of the sequence*

$$d_n = \sqrt[n]{4^n + 5^n}$$

Hint: Try using the Squeeze theorem.

Solution: Observe that

$$5^n \leqslant 4^n + 5^n \leqslant 2 \cdot 5^n$$

and so

$$5 = \sqrt[n]{5^n} \leqslant \sqrt[n]{4^n + 5^n} \leqslant \sqrt[n]{2} \cdot \sqrt[n]{5^n} = 5 \cdot 2^{1/n}$$

and as both left and right converge to 5 (using c_n above and a limit law), then so does the middle by the Squeeze theorem.

Exercise 8. *Let $b_1 = 2$ and define a sequence b_n by*

$$b_{n+1} = \frac{b_n^2 + 1}{b_n}$$

Does this sequence converge?

Solution: No, this sequence never has a limit. Suppose it did have a limit L. Then

$$\lim_{n\to\infty} b_{n+1} = \lim_{n\to\infty} \frac{b_n^2 + 1}{b_n} = \frac{(\lim_{n\to\infty} b_n)^2 + 1}{\lim_{n\to\infty} b_n}$$

using limit laws. Hence L must satisfy

$$L = \frac{L^2 + 1}{L} \qquad \Rightarrow \qquad L^2 = L^2 + 1$$

which is clearly impossible.

Exercise 9. *Show that the following sequence is decreasing and bounded below. Determine whether or not it is convergent.*

$$(a_n)_{n=1}^{\infty} \qquad \text{where} \qquad a_n = \frac{n}{e^n}$$

Note that you are not required to find the limit.

Solution: We have

$$a_{n+1} - a_n = \frac{n+1}{e^{n+1}} - \frac{n}{e^n} = \frac{n+1-ne}{e^{n+1}} = \frac{n(1-e)+1}{e^{n+1}} < 0 \quad \text{as} \quad n(1-e) < -1 \quad \text{for all} \quad n \geqslant 1$$

and hence the sequence is decreasing.

Since n and e^n are positive for all values of n we have that 0 is a lower bound for a_n and so the sequence is bounded below.

Thus by the Monotone Sequences theorem the sequence is convergent.

Note that this sequence converges to 0 because an exponential grows much quicker than a polynomial.

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Practice Class 7: Series

Summary of what you have learned

- A series converges if the limit of its partial sums exists.
- Geometric series
$$\sum_{n=0}^{\infty} r^n$$
converge if and only if $|r| < 1$ in which case the value is $\frac{1}{1-r}$.
- p-series
$$\sum_{n=0}^{\infty} \frac{1}{n^p}$$
converge if and only if $p > 1$.
- Divergence, Integral, Comparison, Limit Comparison and Alternating Series tests.
- Conditional convergence and absolute convergence.
- **Reminder**: Do not confuse a *sequence* with a *series*.

Foundational questions

Exercise 1. Determine which of the following series converge and which diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^2+4}$ $\quad(b)$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$ $\quad(c)$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ $\quad(d)$ $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$

Solution:

(a) Since $\displaystyle\lim_{n\to\infty} \frac{n^2+1}{n^2+4} = 1 \neq 0$ this series is divergent by the divergence test.

(b) This is a p-series with $p = 4/3 > 1$ and so is convergent.

(c) This is a p-series with $p = 1/3 < 1$ and so is divergent.

(d) As $n \to \infty$, the cosine term tends to 1 hence divergent by the divergence test.

Exercise 2. Determine if the following series are convergent and find their value if they are.

(a) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{-5}{4}\right)^n$ $\quad(b)$ $\displaystyle\sum_{n=0}^{\infty} 3^{2n}2^{1-n}$ $\quad(c)$ $\displaystyle\sum_{n=3}^{\infty} \left(\frac{1}{3}\right)^n$

Solution:

(a) Geometric series with $r = -5/4$ but $|r| = 5/4 > 1$ so it is divergent.

(b) This is a geometric series in disguise:
$$\sum_{n=0}^{\infty} 3^{2n}2^{1-n} = 2\sum_{n=0}^{\infty} \left(\frac{9}{2}\right)^n \qquad \text{but } \frac{9}{2} > 1 \text{ so divergent.}$$

(c) Here $r = \frac{1}{3}$ so $|r| < 1$ and the series is convergent. The sum is
$$\sum_{n=3}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{3^3}\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{3^3}\left(\frac{1}{1-\frac{1}{3}}\right) = \frac{1}{18}$$

EXERCISE 3. Determine if the following series are convergent and find their value if they are.

$$(a)\ \sum_{n=1}^{\infty}\frac{5^n}{3^n+4^n} \qquad (b)\ \sum_{n=1}^{\infty}\frac{3^n+4^n}{5^n} \qquad (c)\ \sum_{n=2}^{\infty}\frac{n\ln n}{n^2-1} \qquad (d)\ \sum_{n=1}^{\infty}\frac{n\ln n}{n^2+1}$$

SOLUTION:

(a) The terms do not converge to 0. Indeed

$$\lim_{n\to\infty}\frac{5^n}{3^n+4^n}=\frac{1}{(3/5)^n+(4/5)^n}=\infty$$

and hence the series diverges by the divergence test.

(b) Observe that

$$\sum_{n=1}^{\infty}\frac{3^n+4^n}{5^n} = \sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n+\left(\frac{4}{5}\right)^n = \sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n+\sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n$$

and both of these series are geometric series with common ratio less than 1. Hence, by the series laws, the original series is convergent. Moreover

$$\sum_{n=1}^{\infty}\frac{3^n+4^n}{5^n} = \frac{3}{5}\cdot\frac{1}{1-\frac{3}{5}}+\frac{4}{5}\cdot\frac{1}{1-\frac{4}{5}} = \frac{3}{5}\cdot\frac{5}{2}+\frac{4}{5}\cdot 5 = \frac{11}{2}$$

(c) Note that for $n\geq 3$, $\ln n>1$ so $n\ln n>n$ and so $n\ln n/n^2>1/n$. It follows that

$$\frac{n\ln n}{n^2-1} \geqslant \frac{n\ln n}{n^2} > \frac{1}{n} > 0, \qquad n\geq 3$$

and therefore by comparison with the harmonic series, this series is divergent.

(d) Here we need the limit comparison test with the harmonic series where

$$a_n=\frac{n\ln n}{n^2+1}\geq 0, \qquad b_n=\frac{1}{n}>0$$

We compute that

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^2\ln n}{n^2+1} = \lim_{n\to\infty}\frac{\ln n}{1+\frac{1}{n^2}} = \infty$$

and since the harmonic series is divergent then so is this series.

EXERCISE 4. Use either the comparison test or the limit comparison test to determine if the following series converge:

$$(a)\ \sum_{n=1}^{\infty}\frac{\cos^2(n)}{n^2} \qquad (b)\ \sum_{n=1}^{\infty}\frac{4}{n} \qquad (c)\ \sum_{n=1}^{\infty}\frac{n^2+2n+1}{n^3+n+2}$$

SOLUTION:

(a) Observe that

$$0 \leqslant \cos^2(n) \leqslant 1 \qquad \Rightarrow \qquad 0 \leqslant \frac{\cos^2(n)}{n^2} \leqslant \frac{1}{n^2}$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent (it is a p-series with $p=2>1$) it follows from the comparison test that $\sum_{n=1}^{\infty}\frac{\cos^2(n)}{n^2}$ is also convergent.

(b) Note that
$$0 \quad \leqslant \quad \frac{1}{n} \quad \leqslant \quad \frac{4}{n} \qquad\qquad\qquad n \geqslant 1.$$
Since $\sum_{n=1}^{\infty} 1/n$ is divergent (it is the harmonic series), it follows from the comparison test that $\sum_{n=1}^{\infty} 4/n$ is also divergent.
Note: Here you could also have used the limit comparison test (or a series law.)

(c) We will compare the series with the harmonic series $\sum_{n=1}^{\infty} b_n$ with $b_n = \frac{1}{n}$. We have
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(n^2+2n+1)n}{n^3+n+2} \quad = \quad \lim_{n\to\infty} \frac{\cancel{n^3}(1+2/n+1/n^2)}{\cancel{n^3}(1+1/n^2+2/n^3)} = 1$$
and since $\sum_{n=1}^{\infty} b_n$ is divergent, it follows that $\sum_{n=1}^{\infty} a_n$ is also divergent.

Exercise 5. We wish to decide if the series
$$\sum_{n=1}^{\infty} n e^{-n^2}$$
is convergent or not.

(a) What continuous, positive function $f(x)$ would be needed to apply the integral test here?
(b) Confirm that your function is decreasing (using regular calculus).
(c) Now apply the integral test and decide whether the series converges.

Solution:
(a) The function $f(x) = xe^{-x^2}$ is positive and continuous on $[1, \infty)$.
(b) TIts derivative is $f'(x) = e^{-x^2}(1-2x^2)$, which is zero only at $x = \pm 1/\sqrt{2}$ and so $f(x)$ is decreasing for $x \geqslant 1/\sqrt{2} \approx 0.7$, which is the case since $x \geqslant 1$.
(c) The improper integral is
$$\lim_{t\to\infty} \int_1^t xe^{-x^2} dx \quad = \quad \lim_{t\to\infty} \left[\frac{-e^{-x^2}}{2}\right]_1^t \quad = \quad \lim_{t\to\infty} \frac{e^{-1}-e^{-t^2}}{2} \quad = \quad \frac{e^{-1}}{2}$$
The improper integral is convergent and so the series is convergent (but we don't know what the sum is equal to).

Exercise 6. In lectures, we saw that $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ is divergent. What about $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$? Hint
$$f(x) = \frac{1}{x(\ln x)^2} \qquad \Rightarrow \qquad f'(x) = -\frac{1}{\ln x}$$

Solution: This function meets the requirements for the integral test. It is continuous, positive and decreasing on $[2, \infty)$. To prove it is decreasing, notice that both x and $\ln(x)$ grow with x, so $x(\ln x)^2$ increases with x and hence $f(x)$ decreases with x. We have
$$\int_2^t \frac{1}{x(\ln x)^2} dx \quad = \quad \left[\frac{-1}{\ln x}\right]_2^t = \frac{-1}{\ln t} + \frac{1}{\ln 2} \quad = \quad \frac{1}{\ln 2}$$
Therefore the series is convergent, but we don't know its value.

Exercise 7. Determine if the following alternating series converge:

$$(a)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^4+7} \quad (b)\sum_{n=1}^{\infty}\frac{(-2)^n}{3^n} \quad (c)\sum_{n=1}^{\infty}\frac{(-3)^n}{2^n} \quad (d)\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{5n^2-2n+1} \quad (e)\sum_{n=1}^{\infty}(-1)^n\frac{4n^2}{n^3+9}$$

Solution:

(a) Yes by the alternating series test, terms converge to 0 and decrease from a_1 onwards.
(b) This is a geometric series $1+r+r^2+\cdots$ with $r=-2/3$, but without the first term 1, and so the value is $1/(1-r)-1=-2/5$ (and hence it converges).
(c) Terms do not converge to 0, hence this series is divergent by the test for divergence.
(d) Terms do not converge to 0, hence this series is divergent by the test for divergence.
(e) Terms eventually converge to 0, but we need to show that they are eventually decreasing (because they start off increasing). To do that we find where the derivative of the function $4x^2/(x^3+9)$ is negative. We have

$$f(x)=\frac{4x^2}{x^3+9} \qquad \Rightarrow \qquad f'(x)=\frac{x(72-4x^3)}{(x^3+9)^2}$$

The denominator is a square (always positive) so we are only intereted in the numerator $x(72-4x^3)$. This is zero at $x=0$ (not relevant) and $x=\sqrt[3]{72/4}=\sqrt[3]{18}\approx 2.62$ and hence a_3 is the largest term and from then onwards the terms decrease.

Exercise 8. For all the convergent series of Exercise 7, determine whether each series is absolutely convergent or conditionally convergent.

Solution:

(a) We need to determine if the series $\sum_{n=1}^{\infty}\frac{1}{n^4+7}$ is convergent. Using the comparison test with $\sum_{n=1}^{\infty}\frac{1}{n^4}$, which is a convergent geometric p-series, we have

$$0 \leqslant \frac{1}{n^4+7} \leqslant \frac{1}{n^4}$$

so $\sum_{n=1}^{\infty}\frac{1}{n^4+7}$ is convergent, and hence $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^4+7}$ is absolutely convergent.

(b) We need to determine if $\sum_{n=1}^{\infty}\frac{2^n}{3^n}$ is convergent. This is a geometric series with $r=2/3$ so it is convergent and hence $\sum_{n=1}^{\infty}\frac{(-2)^n}{3^n}$ is absolutely convergent.

(e) We need to determine if $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{4n^2}{n^3+9}$ is convergent. Looking at the dominant terms, we see that this behaves like the harmonic series so we apply the limit comparison term with $b_n=\frac{1}{n}$. We see that

$$\frac{a_n}{b_n} = \frac{4n^2.n}{n^3+9} = \frac{4n^3}{n^3+9} = \frac{4}{1+\frac{9}{n^3}} \to 4 \quad \text{as} \quad n\to\infty$$

However, $\sum_{n=1}^{\infty}b_n$ is divergent and hence $\sum_{n=1}^{\infty}a_n$ is divergent. We therefore have that $\sum_{n=1}^{\infty}(-1)^n\frac{4n^2}{n^3+9}$ is conditionally convergent.

Conceptual understanding

EXERCISE 9. If the n-th partial sum s_n of a series $\sum_{i=1}^{\infty} a_i$ is $s_n = \sum_{i=1}^{n} a_i = \frac{n+1}{n+2}$, find a_i as a function of i and determine the sum of the series (if it is convergent).

SOLUTION: We can answer the second part of the question first. The sum is

$$\lim_{n\to\infty} s_n = 1.$$

Now

$$a_i = s_i - s_{i-1} = \frac{i+1}{i+2} - \frac{i}{i+1} = \frac{1}{(i+1)(i+2)}, \qquad i \geq 2$$

and $a_1 = s_1 = \frac{2}{3}$. Note that $1/((i+1)(i+2)) = 1/(i+1) - 1/(i+2)$ but the following is NONSENSE:

$$\sum_{n=1}^{\infty} \frac{1}{(i+1)(i+2)} = \sum_{i=1}^{\infty} \frac{1}{i+1} - \sum_{i=1}^{\infty} \frac{1}{i+2} = \infty - \infty = 0$$

We can't do arithmetic with infinities.

EXERCISE 10. Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

is convergent and find its value. Hint: use the fact that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

SOLUTION: We can use the comparison test with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Since

$$0 \quad < \quad \frac{1}{n(n+1)} \quad = \quad \frac{1}{n^2+n} \quad \leqslant \quad \frac{1}{n^2}$$

and $\sum_{n=1}^{\infty} 1/n^2$ converges (it is a p-series with $p = 2 > 1$), the comparison test says that our series is also convergent. Note that the limit comparison test with the same series also works. Now to determine the sum we establish a formula for the partial sum s_n.

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} \cdots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$$

We see that nearly all of the terms cancel out. Hence

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \quad = \quad \lim_{n\to\infty} s_n \quad = \quad \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) \quad = \quad 1$$

EXERCISE 11. Express $3.14141414141414\ldots$ as a ratio of integers.

SOLUTION: This number can be written as $3 + \frac{14}{100} + \frac{14}{10000} + \cdots$ For each term we divide by 100 so this gives us a geometric series:

$$3.14141414141414\ldots = 3 + \frac{14}{100} \sum_{n=0}^{\infty} \left(\frac{1}{100}\right)^n = 3 + \frac{14}{100} \cdot \frac{1}{1 - \frac{1}{100}} = 3 + \frac{14}{100} \cdot \frac{100}{99} = 3 + \frac{14}{99} = \frac{311}{99}$$

MATH1012 (Semester 1, 2024)

Practice Class 8: Ratio test and power series

Summary of what you have learned

- Conditional convergence and absolute convergence
- Ratio test
- Power series
- radius of convergence
- Taylor series
- **Reminder**: Do not confuse a *sequence* with a *series*

Foundational questions

EXERCISE 1. Use the ratio test to determine if the following series are convergent.

$$(a) \quad \sum_{n=1}^{\infty} \frac{4^n}{n!} \qquad\qquad (b) \quad \sum_{n=1}^{\infty} \frac{n^2}{(2n-1)!}$$

SOLUTION:

(a) Here we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{4^{n+1}}{(n+1)!}}{\frac{4^n}{n!}} = \frac{4^{n+1}n!}{(n+1)!4^n} = \frac{4}{n+1} \;\to\; 0 \quad \text{as} \quad n \to \infty$$

and hence the series is convergent.

(b) Here we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{(n+1)^2}{(2n+1)!}}{\frac{n^2}{(2n-1)!}} = \frac{(n+1)^2(2n-1)!}{n^2(2n+1)!} = \frac{\left(1+\frac{1}{n}\right)^2}{4n^2+2n} \;\to\; 0 \text{ as } n \to \infty$$

and hence the series is convergent.

In each case we can immediately drop the absolute value bars because every factor in the ratio expressions is always positive.

EXERCISE 2. Use the ratio test to determine if the following series is absolutely convergent.

$$\sum_{n=1}^{\infty} \frac{n^7}{(-7)^n}$$

SOLUTION: We have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^7}{(-7)^{n+1}}}{\frac{n^7}{(-7)^n}}\right| = \lim_{n\to\infty}\frac{(n+1)^7}{7n^7} = \lim_{n\to\infty}\frac{1}{7}\left(1+\frac{1}{n}\right)^7 = \frac{1}{7} < 1$$

so the ratio test tells us that the series is absolutely convergent.

Exercise 3. Determine the values of x for which the following power series is absolutely convergent, conditionally convergent and divergent.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{3^n x^n}{\sqrt{n}} = \sum_{n=1}^{\infty} b_n x^n$$

Solution: We'll use the ratio test. We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{3^{n+1}|x|^{n+1}\sqrt{n}}{\sqrt{n+1}\,3^n|x|^n} = \lim_{n\to\infty} \frac{3|x|\sqrt{n}}{\sqrt{n+1}} = 3|x| \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{\frac{1}{2}} = 3|x|$$

Hence the power series is absolutely convergent for $3|x| < 1$, that is for $-\frac{1}{3} < x < \frac{1}{3}$. It is divergent for all $x > \frac{1}{3}$ and $x < -\frac{1}{3}$. Alternatively we can compute the radius of convergence:

$$R = \lim_{n\to\infty} \left|\frac{b_n}{b_{n+1}}\right| = \lim_{n\to\infty} \frac{\sqrt{n+1}\;3^n}{3^{n+1}\sqrt{n}} = \frac{1}{3}$$

and we know the power series is centred around 0. What about $x = -\frac{1}{3}$ and $x = \frac{1}{3}$? When $x = \frac{1}{3}$ the series is

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \qquad = \qquad \sum_{n=1}^{\infty} \frac{1}{n^p} \qquad \text{with} \qquad p = \frac{1}{2}$$

This p-series has $p < 1$ and is hence divergent. When $x = -\frac{1}{3}$ the series is $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$. This is an alternating series and $\frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$ so the series is convergent at $x = -\frac{1}{3}$. However, it is only conditionally convergent because $\sum_{n=1}^{\infty} |\frac{(-1)^n}{\sqrt{n}}| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, divergent.

Exercise 4. Determine the values of x for which the following power series are convergent.

$$(i) \quad \sum_{n=1}^{\infty} n!(x-1)^n \qquad\qquad (ii) \quad \sum_{n=1}^{\infty} \frac{n(x-a)^n}{c^n}$$

where a is any real number and $c > 0$.

Solution: (i) The radius of convergence is

$$R = \lim_{n\to\infty} \left|\frac{b_n}{b_{n+1}}\right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

and we know the power series is centred around 1. So the series is convergent at $x = 1$ and divergent everywhere else.

(ii) The radius of convergence is

$$R = \lim_{n\to\infty} \left|\frac{b_n}{b_{n+1}}\right| = \lim_{n\to\infty} \frac{c^{n+1} n}{(n+1)c^n} = \lim_{n\to\infty} c\left(\frac{n}{n+1}\right) = c$$

Hence the series is convergent on $(a-c, a+c)$ and divergent on $(-\infty, a-c)$ and $(a+c, \infty)$. In addition,

$$x = a - c \quad \to \quad \sum_{n=1}^{\infty} (-1)^n n \qquad \text{and} \qquad x = a + c \quad \to \quad \sum_{n=1}^{\infty} n$$

and both are divergent by the divergence test. Hence convergence if $x \in (a-c, a+c)$.

Exercise 5. Consider the function

$$f(x) = \frac{1}{(1+x)^2}$$

(i) Determine the Taylor series of f about 0 (in other words, its MacLaurin Series).
(ii) Substitute $x = 1$ into f and its Taylor series.
(iii) Determine the values of x for which the Taylor series is convergent.
(iv) Reconcile your answers in part (b).

Solution:

(i) The required derivatives are

$$f'(x) = \frac{-2}{(1+x)^3}, \qquad f''(x) = \frac{6}{(1+x)^4}, \qquad f'''(x) = \frac{-24}{(1+x)^5},$$

and continuing we see that

$$f^{(n)}(x) = \frac{(-1)^n(n+1)!}{(1+x)^{n+2}} \qquad \to \qquad f^{(n)}(0) = (-1)^n(n+1)!$$

Hence the Taylor series of $f(x)$ is

$$T(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)!}{n!} x^n = \sum_{n=0}^{\infty} (-1)^n(n+1)x^n$$

(ii) We have

$$f(1) = 1/4 \qquad \text{and} \qquad T(1) = 1 - 2 + 3 - 4 + 5 - 6 \ldots$$

(iii) The radius of convergence of the Taylor series is

$$R = \lim_{n\to\infty} \left| \frac{b_n}{b_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^n(n+1)!}{n!}}{\frac{(-1)^{n+1}(n+2)!}{(n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n+2} \right| = 1$$

Hence absolutely convergent for all x such that $|x| < 1$, and divergent for $|x| > 1$. In addition,

$$x = -1 \quad \to \quad \sum_{n=0}^{\infty} (n+1) \qquad \text{and} \qquad x = 1 \quad \to \quad \sum_{n=0}^{\infty} (-1)^n(n+1),$$

both of which are divergent.

(iv) The Taylor series for $f(x)$ is convergent if and only if $|x| < 1$. In which case we can't guarantee that it will be equal to f at $x = 1$, and of course it clearly isn't.

Exercise 6. Knowing that $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x, find the MacLaurin series for $\cos(x)$ and determine for which x it converges.

Solution: By the theorem on differentiation of Taylor series, we can just differentiate term-by-term to get the power series of $\cos(x)$:

$$\cos(x) = (\sin(x))' = \sum_{n=0}^{\infty} (-1)^n \frac{(x^{2n+1})'}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Moreover the radius of convergence is the same as for $\sin(x)$: it converges everywhere.
All this can also be checked the "traditional way", computing the derivatives to find the terms and then computing R directly.

EXERCISE 7.

(i) Find the third degree Taylor polynomial $T_{3,0}(x)$ of the function $g(x) = e^{2x}$ about 0.
(ii) Give an upper bound of the error in using $T_{3,0}(x)$ to approximate $g(x)$ on $[-0.5, 0.5]$.
(iii) Find the radius of convergence of the Taylor series for $g(x)$ about 0.

SOLUTION:

(a) The nth derivative of the function is $g^{(n)}(x) = 2^n e^{2x}$ so

$$T_{3,0}(x) = 1 + 2x + 4\frac{x^2}{2} + 8\frac{x^3}{6} = 1 + 2x + 2x^2 + \frac{4}{3}x^3.$$

(b) The error is given by $|g^{(4)}(z)x^4/4!|$, that is, $|16e^{2z}x^4/24|$ for some value of $z \in [-0.5, 0.5]$. This has its maximum value when $x = 1/2$ and $z = 1/2$, that is,

$$|16e^{2z}\frac{x^4}{24}| \leqslant \frac{2e}{3}0.5^4 \approx 0.1133.$$

(c) Here

$$b_n = \frac{2^n}{n!} \qquad \Rightarrow \qquad \left|\frac{b_n}{b_{n+1}}\right| = \frac{2^n(n+1)!}{n!2^{n+1}} = \frac{n+1}{2} \qquad \to \qquad R = \infty \quad \text{as} \quad n \to \infty$$

Therefore the Taylor series converges for all x.

Conceptual understanding

EXERCISE 8. Explain why the ratio test gives no additional information (over the simpler tests) for any series of the form

$$\sum_{n=1}^{\infty} \frac{\text{polynomial in } n}{\text{polynomial in } n}$$

Use these examples to illustrate your argument

$$\sum_{n=1}^{\infty} \frac{n^2+3}{2n^2-5} \qquad\qquad \sum_{n=1}^{\infty} \frac{n}{n^2-2}$$

SOLUTION: As n tends to infinity, the highest power of n (in both the numerator and denominator) dominates and so a_n will behave like Cn^k for some integer k. For example, in the first example a_n behaves like $1/2$ as n grows and in the second example a_n behaves like $1/n$ as n grows.

When we consider the ratio of successive terms:

$$\frac{a_{n+1}}{a_n} \quad \text{will behave like} \quad \frac{C(n+1)^k}{Cn^k} = \left(\frac{n+1}{n}\right)^k \quad \to \quad 1^k = 1 \quad \text{as} \quad n \to \infty$$

So either the terms do not converge to 0 in which case the divergence test tells us that the sequence diverges, or the terms do converge to 0 and we only get the $L = 1$ case for the ratio test (that is, no information).

Exercise 9. Determine the values of x for which the power series $\sum_{n=1}^{\infty} \frac{x^{2n}}{n}$ is convergent.

Solution: The series is $\sum_{n=1}^{\infty} \frac{x^{2n}}{n} = x^2 + \frac{x^4}{2} + \frac{x^6}{3} + \cdots$ but alternate coefficients are zero so $\lim_{n\to\infty} |b_n/b_{n+1}|$ does not exist and we have to use the ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x|^{2n+2} n}{(n+1)\ |x|^{2n}} = \lim_{n\to\infty} |x|^2 \left(\frac{n}{n+1} \right) = |x|^2$$

so the power series is absolutely convergent if $|x|^2 < 1$, that is for x in the range $-1 < x < 1$. It is divergent for all $x > 1$ and $x < -1$. It remains to test the series at $x = 1$ and $x = -1$. In both cases, the series is the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ so is divergent.

Exercise 10. Recall the geometric series (which is also a Taylor series)

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \qquad \text{with radius of convergence } R = 1$$

(i) Differentiate both sides and then multiply by x. What is the radius of convergence of the new series? Deduce the value of $\sum \frac{n}{2^n}$.

(ii) Differentiate both sides of the expression from Part (i) and then multiply by x. What is the radius of convergence of the new series? Deduce the value of $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

(iii) Generalise your results to find formula for $\sum_{n=1}^{\infty} \frac{n}{a^n}$ and $\sum_{n=1}^{\infty} \frac{n^2}{a^n}$ for any $a > 1$.

Solution:

(i) We get

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1} = \sum_{n=1}^{\infty} n x^{n-1} \qquad \text{and hence} \qquad \frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n$$

The radius of convergence is $R = \lim_{n\to\infty} |b_n/b_{n+1}| = \lim_{n\to\infty} n/(n+1) = 1$.
Taking $x = \frac{1}{2}$, which is within the radius of convergence, we get

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{\frac{1}{2}}{\left(1 - \frac{1}{2}\right)^2} = 2$$

(ii) Differentiating both sides of the previous result we get

$$\frac{x+1}{(1-x)^3} = \sum_{n=1}^{\infty} n^2 x^{n-1} \qquad \text{and hence} \qquad \frac{x^2+x}{(1-x)^3} = \sum_{n=1}^{\infty} n^2 x^n$$

The radius of convergence is $R = \lim_{n\to\infty} |b_n/b_{n+1}| = \lim_{n\to\infty} n^2/(n+1)^2 = 1$.
aking $x = \frac{1}{2}$, which is within the radius of convergence, we get

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{\left(\frac{1}{2}\right)^2 + \frac{1}{2}}{\left(1 - \frac{1}{2}\right)^3} = 6$$

(iii) The series $\sum_{n=1}^{\infty} \frac{n}{a^n}$ and $\sum_{n=1}^{\infty} \frac{n^2}{a^n}$ are those in parts (i) and (i), evaluated at $x = \frac{1}{a} < 1$, which is within the radius of convergence for both, so

$$\sum_{n=1}^{\infty} \frac{n}{a^n} = \frac{\frac{1}{a}}{(1-\frac{1}{a})^2} = \frac{a}{(a-1)^2} \qquad \sum_{n=1}^{\infty} \frac{n^2}{a^n} = \frac{\frac{1}{a^2} + \frac{1}{a}}{(1-\frac{1}{a})^3} = \frac{a+a^2}{(a-1)^3}$$

MATH1012 (Semester 1, 2024)

Practice Class 9: Fourier series

Summary of what you have learned

- If $f(t+P) = f(t)$ for all $t \in \mathbb{R}$ then $f(t)$ has *period* P
- We say that function f is P-periodic
- For a $2L$-periodic function $f(t)$, the infinite series

$$FS_f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right)$$

  is called the *Fourier series of* f and the constants a_0, a_n, b_n are called the *Fourier coefficients of* f.
- The Fourier coefficients are calculated using *Euler's formulae*:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t) \cos\left(\frac{n\pi t}{L}\right)dt,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t) \sin\left(\frac{n\pi t}{L}\right)dt.$$

- Periodic extensions.
- Odd and even functions
- Half-range expansions
- Fourier cosine series for even functions and Fourier sine series for odd functions

To solve the problems in this practical class, you might need the following integrals, where n is a positive integer and α is any non-zero real number.

$$\int t\cos(nt)dt = \frac{t}{n}\sin(nt) + \frac{1}{n^2}\cos(nt) + C, \qquad \int \sin(\alpha t)dt = -\frac{\cos(\alpha t)}{\alpha} + C,$$

$$\int t\sin(nt)dt = -\frac{t}{n}\cos(nt) + \frac{1}{n^2}\sin(nt) + C, \qquad \int \cos(\alpha t)dt = \frac{\sin(\alpha t)}{\alpha} + C.$$

In particular,

$$\int_0^{\pi} t\cos(nt)dt = \frac{\cos(n\pi) - 1}{n^2}, \qquad \int_0^{\pi} t\sin(nt)dt = \frac{-\pi\cos(n\pi)}{n}$$

Note the following properties of odd and even functions

$$E + E = E \qquad O + O = O \qquad E + O = \text{neither}$$

$$EE = E \qquad OO = E \qquad EO = O$$

$$\int_{-L}^{L} E(t)dt = 2\int_0^{L} E(t)dt \qquad \int_{-L}^{L} O(t)dt = 0$$

Foundational questions

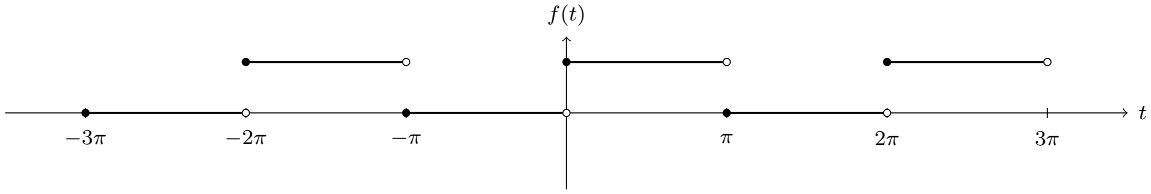
EXERCISE 1. Let f be the 2π-periodic function defined by

$$f(t) = \begin{cases} 0, & -\pi \le t < 0, \\ 1, & 0 \le t < \pi, \end{cases} \qquad \text{and} \qquad f(t+2\pi) = f(t) \text{ for all } t.$$

(i) Draw the graph of this function on the domain $[-3\pi, 3\pi]$.
(ii) Find the Fourier coefficients of f and hence write the Fourier series of f.
(iii) Draw the graph of this Fourier series on the domain $[-3\pi, 3\pi]$.

SOLUTION:

(i) The function is a square wave with the following graph on $[-3\pi, 3\pi]$:



(ii) Using Euler's formulae:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{1}{\pi}\int_0^{\pi} dt = 1.$$

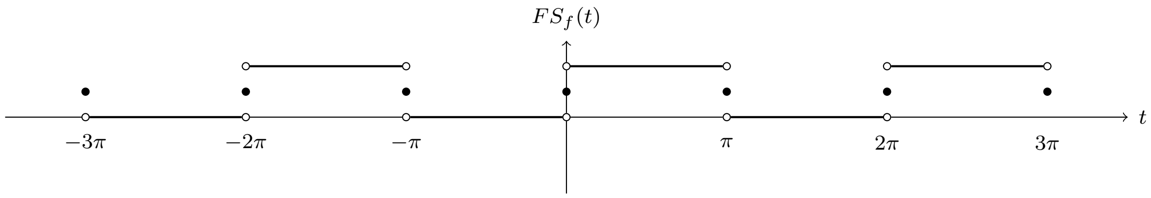
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt = \frac{1}{\pi}\int_0^{\pi}\cos(nt)dt = \frac{1}{\pi}\left[\frac{\sin(nt)}{n}\right]_0^{\pi} = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt = \frac{1}{\pi}\int_0^{\pi}\sin(nt)dt = \left[-\frac{\cos(nt)}{n\pi}\right]_0^{\pi} = \frac{1-\cos(n\pi)}{n\pi} = \begin{cases} 0, & n \text{ even}, \\ \frac{2}{n\pi}, & n \text{ odd}. \end{cases}$$

Hence the Fourier series of f is

$$\begin{aligned} FS_f(t) &= \frac{1}{2} + \sum_{\text{odd } n=1}^{\infty} \frac{2}{n\pi}\sin(nt) \\ &= \frac{1}{2} + \frac{2}{\pi}\left(\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \dots\right) \end{aligned}$$

(iii) The graph of this Fourier series is



Note that we usually omit the white circles when plotting Fourier series.

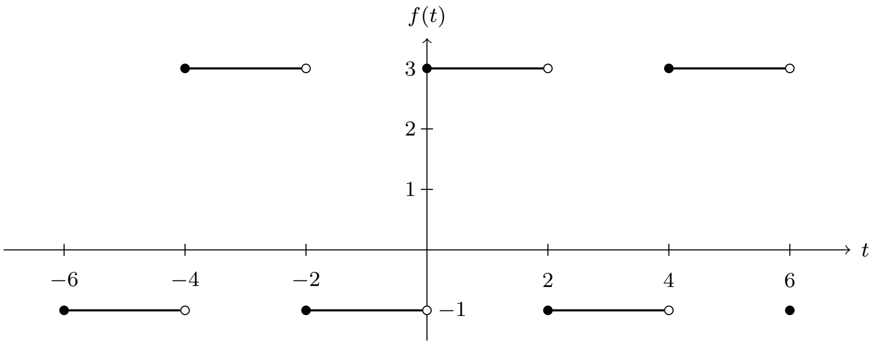
Exercise 2. Let f be the 4-periodic function defined by

$$f(t) = \begin{cases} -1, & -2 \le t < 0, \\ 3, & 0 \le t < 2, \end{cases} \qquad \text{and} \qquad f(t+4) = f(t) \text{ for all } t.$$

(i) Draw the graph of this function on the domain $[-6, 6]$.
(ii) Find the Fourier coefficients of f.

Solution:

(i) The function is another square wave with the following graph on $[-6, 6]$:



(ii) Using Euler's formulae:

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(t)dt = \frac{1}{2}\int_{-2}^{0}(-1)dt + \frac{1}{2}\int_{0}^{2}(3)dt = -1 + 3 = 2.$$

$$\begin{aligned} a_n &= \frac{1}{2}\int_{-2}^{2} f(t)\cos\left(\frac{n\pi t}{2}\right) dt \\ &= \frac{1}{2}\left(\int_{-2}^{0}(-1)\cos\left(\frac{n\pi t}{2}\right) dt + \int_{0}^{2}(3)\cos\left(\frac{n\pi t}{2}\right) dt\right) = 0 \text{ for all } n > 0. \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{2}\int_{-2}^{2} f(t)\sin\left(\frac{n\pi t}{2}\right) dt \\ &= \frac{1}{2}\left(\int_{-2}^{0}(-1)\sin\left(\frac{n\pi t}{2}\right) dt + \int_{0}^{2}(3)\sin\left(\frac{n\pi t}{2}\right) dt\right) \\ &= \frac{4}{n\pi}(1 - (-1)^n) = \begin{cases} 0, & n \text{ even}, \\ \frac{8}{n\pi}, & n \text{ odd}. \end{cases} \end{aligned}$$

Hence the Fourier series of f is

$$\begin{aligned} FS_f(t) &= 1 + \sum_{n=1}^{\infty} \frac{4}{n\pi}(1 - (-1)^n)\sin\left(\frac{n\pi t}{2}\right) \\ &= \frac{1}{2} + \frac{8}{\pi}\left(\sin\left(\frac{\pi t}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi t}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi t}{2}\right) + \dots\right) \end{aligned}$$

Note that $f(t)$ is actually a vertically raised odd function so it must be that $a_n = 0$. Furthermore, the raise is by one unit so $a_0/2 = 1 \ \rightarrow\ a_0 = 2$.

Exercise 3. Decide whether each of the following functions is even, odd, neither or both.

$$a(t) = \cos^3(t), \qquad b(t) = \sin(t^2), \qquad c(t) = 0, \qquad d(t) = \cos(t) + \sin(t), \qquad e(t) = 2^t - 2^{-t}.$$

Solution:

$$a(-t) = \cos^3(-t) = (\cos(-t))^3 = (\cos(t))^3 = \cos^3(t) = a(t) \implies a(t) \text{ is even}$$
$$b(-t) = \sin((-t)^2) = \sin(t^2) = b(t) \implies b(t) \text{ is even}$$
$$c(-t) = 0 = c(t) = -c(t) \implies c(t) \text{ is both}$$
$$d(-t) = \cos(-t) + \sin(-t) = \cos(t) - \sin(t) \text{ so } d(-t) \neq \pm d(t) \implies d(t) \text{ is neither}$$
$$e(-t) = 2^{-t} - 2^t = -(2^t - 2^{-t}) = -e(t) \implies e(t) \text{ is odd}$$

Exercise 4. Find the Fourier series of the odd expansion of the function

$$f(t) = \pi^2 - t^2, \quad 0 \leqslant t < \pi$$

and sketch its periodic extension on $[-3\pi, 3\pi]$

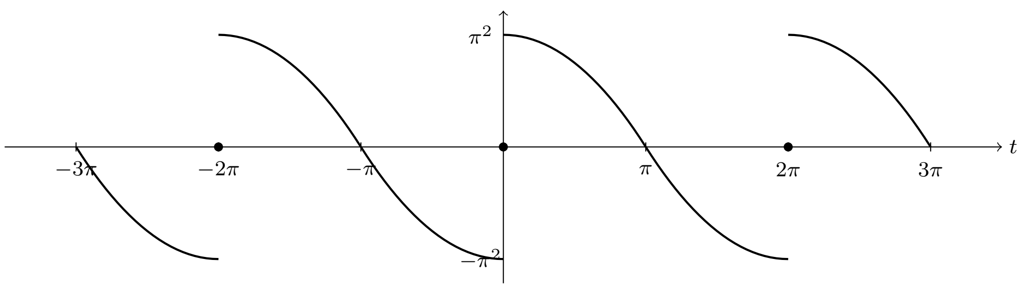
$$\text{Note that} \quad \int_0^\pi t^2 \sin(nt)dt = \frac{-2 + (-1)^n(2 - n^2\pi^2)}{n^3}$$

Solution: Because the expansion will be an odd function then $a_0 = 0$, $a_n = 0$ and

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^\pi f(t)\sin(nt)dt \\
&= \frac{2}{\pi}\int_0^\pi \pi^2\sin(nt)dt - \frac{2}{\pi}\int_0^\pi t^2\sin(nt)dt \\
&= \frac{2\pi}{n}(1 - (-1)^n) - \frac{-4 + (-1)^n(4 - 2n^2\pi^2)}{\pi n^3} \\
&= \frac{2\pi}{n} - \cancel{\frac{2\pi}{n}(-1)^n} + \frac{4}{\pi n^3} - \frac{4(-1)^n}{\pi n^3} + \cancel{\frac{2\pi}{n}(-1)^n} \\
&= \frac{2\pi}{n} + \frac{4[1 - (-1)^n]}{\pi n^3} \\
&= \frac{2\pi}{n} + \begin{cases} \frac{8}{\pi n^3}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}
\end{aligned}$$

Hence the Fourier series is

$$FS_f(t) = \sum_{\text{odd } n=1}^{\infty} \left(\frac{2\pi}{n} + \frac{8}{\pi n^3}\right)\sin(nt) \;=\; \left(\frac{2\pi^2 + 8}{\pi}\right)\sin(t) + \left(\frac{18\pi^2 + 8}{27\pi}\right)\sin(3t) + \cdots$$



In this graph, the vertical access has been shrunk by a factor of four.
Note that we usually omit the white circles when plotting Fourier series.

Exercise 5. Find the Fourier series of the even expansion of the function

$$f(t) = t, \quad 0 \leqslant t < \pi$$

and sketch its periodic extension on $[-3\pi, 3\pi]$

Solution: Because the expansion will be an odd function then $b_n = 0$ and

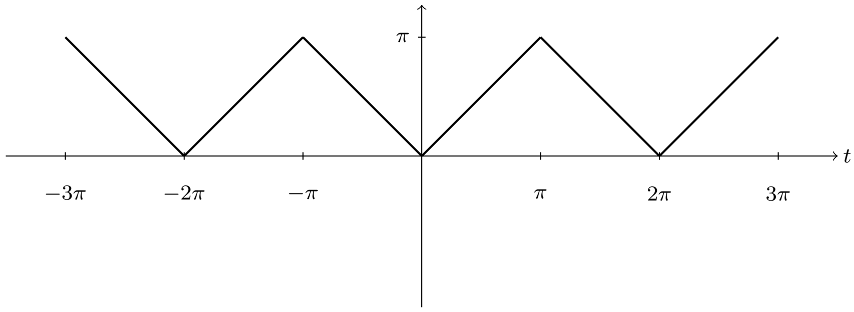
$$a_0 = \frac{1}{\pi}\int_0^\pi f(t)dt = \frac{1}{\pi}\int_0^\pi t dt = \frac{\pi}{2}$$

and

$$a_n = \frac{2}{\pi}\int_0^\pi t\cos(nt)dt = \frac{\cos(n\pi) - 1}{n^2} = \begin{cases} -\frac{4}{\pi n^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

Hence the Fourier series is

$$FS_f(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1 \text{ odd}}^{\infty} \frac{\cos(nt)}{n^2} = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos t + \frac{\cos(3t)}{9} + \cdots\right]$$



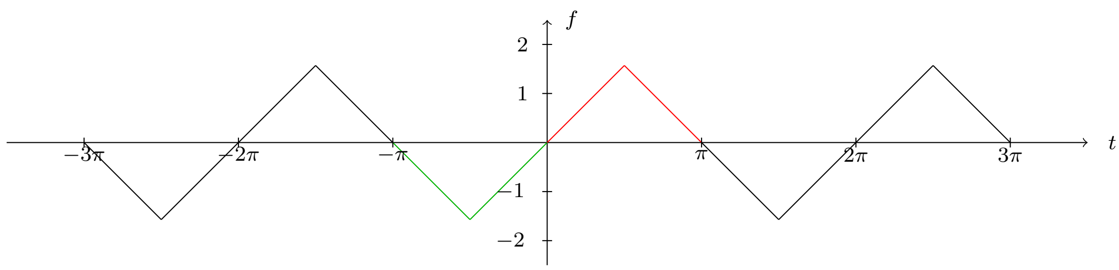
Exercise 6. Consider the function

$$f(t) = \begin{cases} t, & 0 \leq t \leq \pi/2, \\ \pi - t, & \pi/2 < t \leq \pi. \end{cases}$$

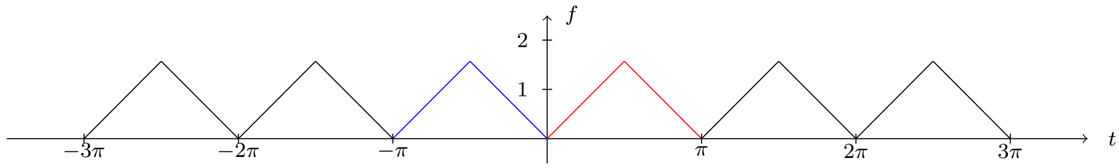
(i) Sketch the periodic extension of the odd half-range expansion of $f(t)$ on $[-3\pi, 3\pi]$.
(ii) Sketch the periodic extension of the even half-range expansion of $f(t)$ on $[-3\pi, 3\pi]$.

Solution:

(i) The graph of the odd half-range expansion on $[-3\pi, 3\pi]$ is



(ii) The graph of the even half-range expansion on $[-3\pi, 3\pi]$ is



Conceptual understanding

EXERCISE 7. Deduce the Fourier series of the following 2π-periodic functions (without using Euler's formulae, if possible).

$$a(t) = -1, \qquad b(t) = 3\sin(t), \qquad c(t) = 4\cos^2(t), \qquad d(t) = \cos(t)\sin(t).$$

SOLUTION:

The first two are already components of the general Fourier series so

$$FS_a(t) = -1, \qquad FS_b(t) = 3\sin(t).$$

Recall that $\cos(2t) = 2\cos^2(t) - 1$ and so

$$FS_c(t) = 2 + 2\cos(2t).$$

Recall that $2\cos(t)\sin(t) = \sin(2t)$ and so

$$FS_d(t) = \frac{1}{2}\sin(2t).$$

EXERCISE 8. Show that the product of an even function and an odd function is an odd function.

SOLUTION: Let $f(t)$ be an even function, $g(t)$ odd and $h(t) = f(t)g(t)$. Then

$$h(-t) = f(-t)g(-t) = \big(f(t)\big)\big(-g(t)\big) = -f(t)g(t) = -h(t)$$

so the product of an even function and an odd function is an odd function.

EXERCISE 9. Show that any function $f(t)$ can be written as the sum of an even function and an odd function.

SOLUTION: We seek to write

$$f(t) = g(t) + h(t) \qquad \text{where} \qquad g(t) = g(-t) \qquad \text{and} \qquad h(t) = -h(-t)$$

This works if we define

$$g(t) = \frac{f(t) + f(-t)}{2} \qquad h(t) = \frac{f(t) - f(-t)}{2}$$

Alternatively, consider the Fourier series of the function:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) \quad \text{is even} \qquad \text{and} \qquad \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right) \quad \text{is odd}$$

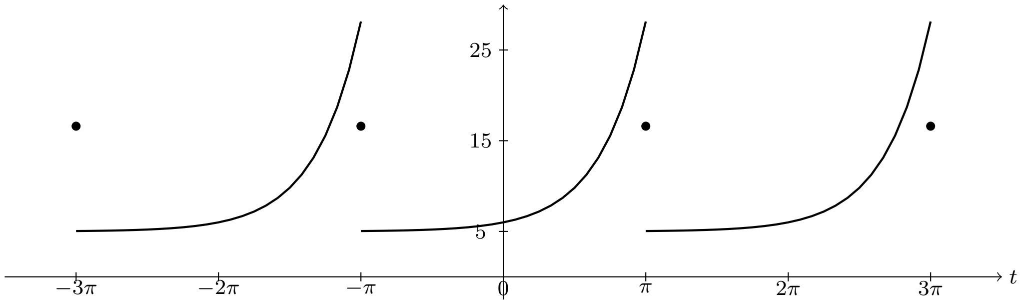
EXERCISE 10. Show that differentiation switches parity. That is, differentiating an odd function results in an even function, and vice versa. For example, $\sin(t) \to \cos(t)$ and $t^2 \to 2t$

SOLUTION: Let $f(t)$ be an *odd* function, so $f(-t) = -f(t)$, and consider $g(t) = f'(t)$. Then

$$g(-t) = \left.\frac{df(t)}{dt}\right|_{t \to -t} = \frac{df(-t)}{d(-t)} = -\frac{df(-t)}{dt} = \frac{d(-f(-t))}{dt} = \frac{df(t)}{dt} = \frac{df}{dt} = g(t)$$

and hence $g(t)$ is an *even* function. The vice versa proof is very similar.

EXERCISE 11. The graph of the periodic extension of the function $f(t) = 5 + e^t$ on $-\pi \leqslant t < \pi$ drawn on the domain $[-3\pi\; 3\pi]$ is



Its Fourier series is

$$FS_f(t) = 5 + \frac{c}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n c}{n^2+1}\cos(nt) + \sum_{n=1}^{\infty} \frac{n(-1)^n c}{n^2+1}\sin(nt), \qquad c = \frac{e^{\pi} - e^{-\pi}}{\pi}.$$

Sketch on the domain $[-3\pi\; 3\pi]$ the graphs of the Fourier series

$$(i) \quad FS_g(t) = 5 + \frac{c}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n c}{n^2+1}\cos(nt)$$

$$(ii) \quad FS_h(t) = 5 + \sum_{n=1}^{\infty} \frac{n(-1)^n c}{n^2+1}\sin(nt)$$

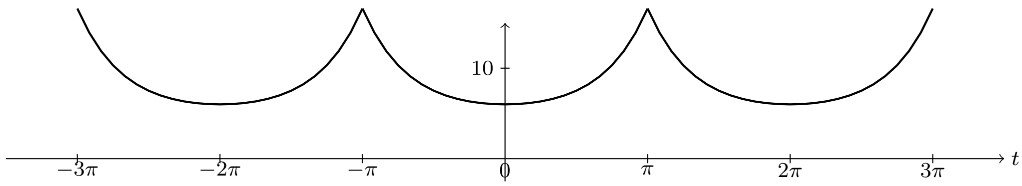
Hint: Recall that $\cos(-t) = \cos(t)$ and $\sin(-t) = -\sin t$.

SOLUTION: Observe that

$$FS_f(-t) = 5 + \frac{c}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n c}{n^2+1}\cos(nt) - \sum_{n=1}^{\infty} \frac{n(-1)^n c}{n^2+1}\sin(nt)$$

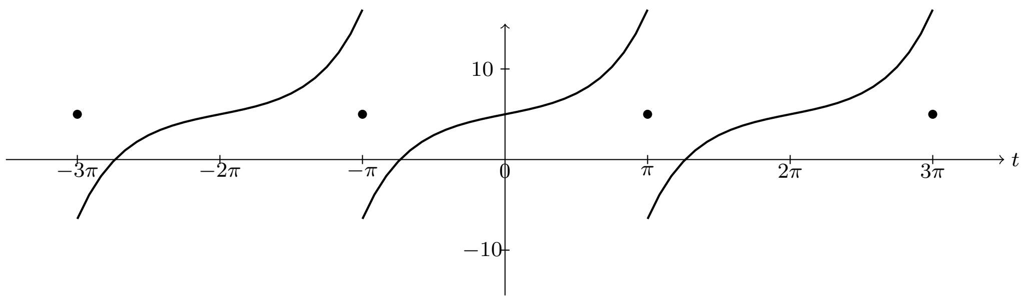
(i) $FS_g(t)$ is the Fourier series of the periodic extension of the even function

$$g(t) = 5 + \frac{f(t)+f(-t)}{2} = 5 + \frac{e^t + e^{-t}}{2}, \qquad -\pi \leqslant t < \pi$$



(ii) $FS_h(t)$ is the Fourier series of the periodic extension of the odd function

$$h(t) = 5 + \frac{f(t)-f(-t)}{2} = 5 + \frac{e^t - e^{-t}}{2}, \qquad -\pi \leqslant t < \pi$$



MATH1012 (Semester 1, 2024)

Practice Class 10: Differential equations

Summary of what you have learned

- Verifying solutions of differential equations
- First-order separable differential equations
- First-order linear differential equations
- Second-order linear constant coefficient differential equations

Foundational questions

Exercise 1. Show that $y(x) = xe^x$ is a solution of the differential equation

$$\frac{dy}{dx} - \frac{1}{x}y = xe^x.$$

Solution: Here we have

$$\frac{d}{dx}(xe^x) = e^x + xe^x$$

hence

$$\frac{dy}{dx} - \frac{1}{x}y = (e^x + xe^x) - \frac{1}{x}(xe^x) = xe^x$$

as required.

Exercise 2. Solve the following separable first-order differential equations

(i) $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{y}}$ $\quad(ii)$ $\dfrac{df}{dt} = 3t^2f^2$ $\quad(iii)$ $\dfrac{dy}{dx} - y = 5$

Solution: (i) Rearrange to get $\sqrt{y}dy = xdx$ and integrate both sides to get

$$\frac{2}{3}y^{3/2} = \frac{1}{2}x^2 + C \qquad \Rightarrow \qquad y^{3/2} = \frac{3}{4}x^2 + \frac{3}{2}C \qquad \Rightarrow \qquad y(x) = \left(\frac{3}{4}x^2 + C\right)^{2/3}.$$

(ii) Rearrange to get $df/f^2 = 3t^2dt$ and integrate both sides to get

$$-\frac{1}{f} = t^3 + C \qquad \Rightarrow \qquad f(t) = -\frac{1}{t^3 + C}.$$

(ii) We have

$$\frac{dy}{dx} - y = 5 \quad \Rightarrow \quad \frac{dy}{dx} = 5 + y \quad \Rightarrow \quad \int \frac{dy}{5+y} = \int dx$$

and hence

$$\ln(5+y) = x + \ln(C) \quad \Rightarrow \quad 5 + y = Ce^x \quad \Rightarrow \quad y(x) = Ce^x - 5.$$

Note: For the integration constant we can write *any* expression involving C. In part (iii) it's best to choose $\ln(C)$ instead of just C. We try to choose an expression which will make the final answer look clean. We usually use ln(C) when the integals result in ln.

Exercise 3. Use the integrating factor method to solve the linear differential equations

$$(i) \quad \frac{dy}{dx} + \frac{y}{x} = \frac{\sin x}{x} \qquad\qquad (ii) \quad \frac{dh}{dt} - 2h = t^2$$

Solution: (*i*) The function of x in front of y is $\frac{1}{x}$ and so the integrating factor is

$$I(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x.$$

Now multiplying both sides of our differential equation by x we get

$$x\frac{dy}{dx} + y = \sin x \qquad \text{which we can write as} \qquad \frac{d}{dx}(xy) = \sin x.$$

Integrating both sides with respect to x yields $xy = -\cos x + C$ for some constant C, so

$$y(x) = -\frac{\cos x}{x} + \frac{C}{x}.$$

(*ii*) The function of t in front of h is $f(t) = -2$ and so the integrating factor is

$$I(t) = e^{\int -2dt} = e^{-2t}.$$

Now multiplying both sides of our differential equation by t we get

$$e^{-2t}\frac{dh}{dt} - 2e^{-2t}h = t^2e^{-2t} \qquad \text{which we can write as} \qquad \frac{d}{dt}\left[e^{-2t}h\right] = e^{-2t}t^2.$$

Integrating both sides with respect to t yields

$$e^{-2t}h = C - \frac{1}{4}(2t^2 + 2t + 1)e^{-2t} \qquad \Rightarrow \qquad h(t) = Ce^{2t} - \frac{2t^2 + 2t + 1}{4}$$

for some constant C

Exercise 4. Write down the most general form of the particular solution that you would use to solve the following differential equations:

(1) $$\frac{d^2y}{dx^2} + 4y = 3x^3 - 5x + 1$$

Solution: We try $y_p(x) = A_3x^3 + A_2x^2 + A_1x + A_0$.

(2) $$\frac{d^2y}{dx^2} - 9y = e^{5x}$$

Solution: We try $y(x) = Ae^{5x}$.

(3) $$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 5y = e^{2x} + 4x$$

Solution: We try $y(x) = Be^{2x} + A_1x + A_0$.

(4) $$4\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 3\sin 2x$$

Solution: We try $y(x) = A\cos 2x + B\sin 2x$.

EXERCISE 5. Find the general solution of the following second-order differential equations:

(1) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} - 4y = 0$

SOLUTION: The characteristic equation is $m^2 - 3m - 4 = 0$, which has solutions $m = -1, 4$ and so the general solution is

$$y(x) = C_1 e^{-x} + C_2 e^{4x}.$$

(2) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 13y = 0$

SOLUTION: The characteristic equation is $m^2 - 4m + 13 = 0$, having solutions

$$m = \frac{4 \pm \sqrt{16 - 52}}{2} = 2 \pm 3i$$

and so the general solution is

$$y(x) = C_1 e^{2x} \sin 3x + C_2 e^{2x} \cos 3x.$$

(3) $\dfrac{d^2y}{dx^2} - 9\dfrac{dy}{dx} + 9y = 0$

SOLUTION: The characteristic equation is $m^2 - 9m + 9 = 0$, which has solution $m = 3$ and so the general solution is

$$y(x) = C_1 e^{3x} + C_2 x e^{3x}.$$

EXERCISE 6. Use the method of undetermined coefficients to solve the differential equation

$$\frac{d^2P}{dt^2} + 9P = e^{2t} + 1.$$

SOLUTION: The characteristic equation is $m^2 + 9 = 0$ which has solutions $m = \pm 3i$ so the complementary function (the solution to the corresponding homogeneous equation) is

$$P_c(t) = C_1 e^{0t} \sin 3t + C_2 e^{0t} \cos 3t = C_1 \sin 3t + C_2 \cos 3t$$

For a particular solution we try

$$P_p(t) = Ae^{2t} + B \qquad \to \qquad \frac{dP_p}{dt} = 2Ae^{2t} \qquad \to \qquad \frac{d^2P_p}{dt^2} = 4Ae^{2t}$$

So to be a solution we require that

$$4Ae^{2t} + 9(Ae^{2t} + B) = e^{2t} + 1 \qquad \Rightarrow \qquad 13Ae^{2t} + 9B = e^{2t} + 1$$

Equating the constant terms we get $9B = 1$ and so $B = 1/9$. Then equating the e^{2t} terms we get $13Ae^{2t} = e^{2t}$ and so $A = 1/13$. Hence the general solution is

$$P(t) = C_1 \sin(3t) + C_2 \cos(3t) + \frac{1}{13}e^{2t} + \frac{1}{9}.$$

EXERCISE 7. Solve the following problems:

(1)

$$\frac{d^2y}{dx^2} + 4y = 0, \qquad y(0) = 3, \qquad y'(0) = 2.$$

SOLUTION: The characteristic equation is $m^2+4=0$ having solutions $m = \pm 2i$ so the complementary function is

$$y(x) = A\sin 2x + B\cos 2x \qquad \rightarrow \qquad y'(x) = 2A\cos 2x - 2B\sin 2x.$$

The initial condition $y(0) = 3$ implies $B = 3$. Next, $y'(0) = 2$ gives $A = 1$. Thus, the solution to the initial value problem is

$$y(x) = \cos 2x - 3\sin 2x.$$

(2)

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0, \qquad y(0) = 3, \qquad y(1) = 0.$$

SOLUTION: The characteristic equation is $m^2 - 4m + 4 = 0$ having solution $m = 2$ so the complementary function is

$$y(t) = Ae^{2t} + Bte^{2t}.$$

Next, $y(0) = 3$ gives $A = 3$, while $y(1) = 0$ implies that $Ae^2 + Be^2 = 0$. Thus, $B = -A = -3$. Hence the solution to the boundary value problem is

$$y(t) = 3e^{2t} - 3te^{2t}.$$

(3)

$$\frac{d^2x}{dt^2} - 4x = 0, \qquad x'(0) = 0, \qquad x(1) = 1.$$

SOLUTION: The characteristic equation is $m^2-4=0$ having solutions $m = \pm 2$ so the complementary function is

$$x(t) = Ae^{2t} + Be^{-2t} \qquad \rightarrow \qquad x'(t) = 2Ae^{2t} - 2Be^{-2t}.$$

The boundary conditions imply that

$$2A - 2B = 0 \quad \text{and} \quad Ae^2 + Be^{-2} = 1 \qquad \rightarrow \qquad B = A \quad \text{and} \quad A = \frac{1}{e^2 + e^{-2}}$$

Thus, the solution to the boundary value problem is

$$x(t) = \frac{e^{2t} + e^{-2t}}{e^2 + e^{-2}}.$$

Exercise 8. Use the method of undetermined coefficients to solve

$$\frac{d^2P}{dr^2} - 3\frac{dP}{dr} + 2P = \sin(r), \qquad P(0) = 1, \qquad P'(0) = -1.$$

Solution: The characteristic equation is $m^2 - 3m + 2 = 0$ which has solution $m = 1, 2$. So the complementary function is $P_c(r) = C_1e^{2r} + C_2e^r$. For a particular solution we try

$$P_p(r) = A\sin(r) + B\cos(r).$$

We have

$$\frac{dP_p}{dr} = A\cos(r) - B\sin(r) \qquad \text{and} \qquad \frac{d^2P_p}{dr^2} = -A\sin(r) - B\cos(r).$$

So to be a solution we require that

$$-A\sin(r) - B\cos(r) - 3(A\cos(r) - B\sin(r)) + 2(A\sin(r) + B\cos(r)) = \sin(r)$$

and so we require that

$$(A + 3B)\sin(r) + (B - 3A)\cos(r) = \sin(r).$$

Equating the coefficients of the $\sin(r)$ terms gives $A + 3B = 1$ and equating the coefficients of the $\cos(r)$ terms gives $R - 3B = 0$. Hence $A = 1/10$ and $B = 3/10$, and the general solution is

$$P(r) = C_1e^{2r} + C_2e^r + \frac{1}{10}\sin(r) + \frac{3}{10}\cos(r)$$

$$\rightarrow \qquad P'(t) = 2C_1e^{2r} + C_2e^r + \frac{1}{10}\cos(r) - \frac{3}{10}\sin(r).$$

The initial conditions require that

$$C_1 + C_2 + \frac{3}{10} = 1 \qquad \text{and} \qquad 2C_1 + C_2 + \frac{1}{10} = -1$$

This coupled pair of equations has solution $C_1 = -9/5$ and $C_2 = 5/2$. Hence

$$P(r) = \frac{-9}{5}e^{2r} + \frac{5}{2}e^r + \frac{1}{10}\sin(r) + \frac{3}{10}\cos(r).$$

Exercise 9. Find the general solution of the second-order differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 6e^x$$

Solution: The characteristic equation is $m^2 - 2m + 1 = 0$, which has solution $m = 1$ and so the complementary function is $y_c(x) = C_1e^x + C_2xe^x$. We note that this contains the right-hand side so we'd usually try $y_p(x) = Axe^x$. However, this is also in $y_c(x)$ so we try

$$y_p(x) = Ax^2e^x \quad \rightarrow \quad y_p'(x) = A(x^2 + 2x)e^x \quad \rightarrow \quad y_p''(x) = A(x^2 + 4x + 2)e^x.$$

Substitution into the differential equation gives

$$A([\cancel{x^2} + \cancel{4x} + 2] - 2[\cancel{x^2} + \cancel{2x}] + \cancel{x^2})e^x = 6e^x \qquad \Rightarrow \qquad A = 3$$

Hence the general solution is

$$y(x) = C_1e^x + C_2xe^x + 3x^2e^x.$$

Conceptual understanding

EXERCISE 10. Solve the initial value problem

$$\frac{d^2y}{dt^2} + 4y = f(t) \quad \text{where} \quad f(t) = \begin{cases} -\frac{\pi}{2}, & -\pi \le t < 0 \\ \frac{\pi}{2}, & 0 \le t < \pi \end{cases} \quad \text{and} \quad f(t + 2\pi) = f(t) \text{ for all } t.$$

Hint: You'll need to combine your knowledge of differential equations and Fourier series.

SOLUTION: We can calculate that the Fourier series or $f(t)$ is

$$FS_f(t) = \sum_{\text{odd } n=1}^{\infty} \frac{1}{n}\sin(nt) = \sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots$$

We'll solve the differential equation with $f(t)$ replaced by it's Fourier series $FS_f(t)$, that is

$$\frac{d^2y}{dt^2} + 4y = FS_f(t).$$

The corresponding homogeneous differential equation is $y'' + 4y = 0$ which is easily solved to yield the complementary function

$$y_c(t) = C_1\sin(2t) + C_2\cos(2t).$$

Based on what we know about sin/cos right hand sides we should try a collection of sin *and* cos terms corresponding to the sin terms on the right-hand side. That is,

$$y_p(t) = \sum_{\text{odd } n=1}^{\infty} b_n\sin(nt) + \sum_{\text{odd } n=1}^{\infty} a_n\cos(nt).$$

Substitution into the differential equation yields

$$\sum_{\text{odd } n=1}^{\infty}(-n^2b_n)\sin(nt) + \sum_{\text{odd } n=1}^{\infty}(-n^2a_n)\cos(nt) + 4\sum_{\text{odd } n=1}^{\infty} b_n\sin(nt) + 4\sum_{\text{odd } n=1}^{\infty} a_n\cos(nt)$$

$$= \sum_{\text{odd } n=1}^{\infty}\frac{1}{n}\sin(nt)$$

which we can write as

$$\sum_{\text{odd } n=1}^{\infty}(-n^2b_n + 4b_n)\sin(nt) + \sum_{\text{odd } n=1}^{\infty}(-n^2a_n + 4a_n)\cos(nt)$$

$$= \sum_{\text{odd } n=1}^{\infty}\frac{1}{n}\sin(nt) + \sum_{\text{odd } n=1}^{\infty} 0\cos(nt).$$

Hence it must be the case that $(4 - n^2)a_n = 0 \ \rightarrow \ a_n = 0$ and

$$(4 - n^2)b_n = \frac{1}{n} \qquad \Rightarrow \qquad b_n = \frac{1}{n(4 - n^2)}.$$

Hence the general solution is

$$y(t) = C_1\sin(2t) + C_2\cos(2t) + \sum_{\text{odd } n=1}^{\infty}\frac{1}{n(4 - n^2)}\sin(nt).$$

We could have saved ourselves some writing by noting that $f(t)$ is odd and recalling that $y''(t)$ has the same parity as $y(t)$, and hence $y(t)$ *must* be odd so no cosine terms needed.

Exercise 11. Our results for second order linear differential equations carry over to those of higher order. Solve the initial value problem

$$\frac{d^4y}{dx^4} - y = x^2, \qquad y(0) = 2, \qquad \frac{dy}{dx}(0) = 0, \qquad \frac{d^2y}{dx^2}(0) = 0, \qquad \frac{d^3y}{dx^3}(0) = 0$$

Solution: The characteristic equation is

$$m^4 - 1 = 0 \quad \Rightarrow \quad (m^2-1)(m^2+1) = 0 \quad \Rightarrow \quad m = -1, 1, -i, i$$

so the complementary function is

$$y_c(x) = C_1 e^{-x} + C_2 e^x + C_3 \cos x + C_4 \sin x$$

For the particular solution we try $y_p(x) = Ax^2 + Bx + C$ and substitution gives

$$0 - Ax^2 - Bx - C = x^2 \quad \Rightarrow \quad A = -1,\ B = 0,\ C = 0 \quad \Rightarrow \quad y_p(x) = -x^2$$

and so the general solution is

$$y(x) = C_1 e^{-x} + C_2 e^x + C_3 \cos x + C_4 \sin x - x^2$$

The initial conditions yield four equations for the four unknown C_1, C_2, C_3, C_4. These are easily solved and we arrive at the solution

$$y(x) = e^{-x} + e^x - x^2$$

Exercise 12. Solve the boundary value problems

$$(i) \quad \frac{d^2y}{dx^2} + 4y = 0, \qquad y(0) = 1, \qquad y\left(\frac{\pi}{4}\right) = 3$$

$$(ii) \quad \frac{d^2y}{dx^2} + 4y = 0, \qquad y(0) = 1, \qquad y\left(\frac{\pi}{2}\right) = 3$$

$$(iii) \quad \frac{d^2y}{dx^2} + 4y = 0, \qquad y(0) = 1, \qquad y\left(\frac{\pi}{2}\right) = -1$$

Solution: In all three cases, the characteristic equation is $m^2 + 4 = 0$ which has solutions $m = -2i, 2i$ and hence the general solution is

$$y(x) = C_1 \cos 2x + C_2 \sin 2x$$

(i) The boundary conditions yield:

$$y(0) = 1 \ \rightarrow \ C_1 = 1 \quad \text{and} \quad y\left(\frac{\pi}{4}\right) = 3 \ \rightarrow \ C_2 = 3 \quad \Rightarrow \quad y(x) = \cos 2x + 3 \sin 2x$$

(ii) The boundary conditions yield:

$$y(0) = 1 \ \rightarrow \ C_1 = 1 \quad \text{and} \quad y\left(\frac{\pi}{2}\right) = 3 \ \rightarrow \ C_1 = -3$$

which are contradictory and hence a solution to the problem *does not exist* (ii) The boundary conditions yield:

$$y(0) = 1 \ \rightarrow \ C_1 = 1 \quad \text{and} \quad y\left(\frac{\pi}{2}\right) = -1 \ \rightarrow \ C_1 = 1 \quad \Rightarrow \quad y(x) = \cos 2x + C_2 \sin 2x$$

and we have an *infinite number* of solutions to the problem. Any value for C_2 is valid.

Boundary value problems are much weirder than initial value problems