MATH1012 (Semester 2, 2024)

Practice Class 1: Systems of linear equations (SLE)

Summary of what you have learnt

Solving a system of linear equations in several variables using augmented matrices.
Gaussian elimination and back-substitution, or Gauss-Jordan elimination.
How to ecognise when a matrix is in row echelon form or reduced row-echelon form.
How to identify when an SLE has zero, one or infinitely many solutions, and
in the "infinite solutions" case, express the solution set parametrically using the
non-basic variables as parameters.

Foundation Questions

EXERCISE 1. Which of the following four equations are *linear*?

$x^2 + 2y + 7z = 3,\qquad \sqrt{2}x - 5^2 y = 0,\qquad x^2 - 23x = 10,\qquad x + 2y = 3z.$

SOLUTION: An equation is linear if every term is a number, or a number multiplied by a variable. So the second and fourth are linear. The values $\sqrt{2}$ and 5^2 are just numbers. The first and third ones are not linear because they involve x^2.

EXERCISE 2. What is the augmented matrix of the following system of linear equations?

$$\begin{aligned} 3x + y + z - t &= 2 \\ x + y &= 3 \\ y - 2t &= 7 \end{aligned}$$

SOLUTION: We must line up the coefficients with the right variable:

$$\left[\begin{array}{cccr|c} 3 & 1 & 1 & -1 & 2 \\ 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 & 7 \end{array}\right]$$

EXERCISE 3. A system of linear equations has the following matrix after it has been reduced to reduced row-echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

How many solutions does this SLE have? Write out the solution set.

SOLUTION: The third coordinate does not appear in any of the equations, so x_3 will remain free so there are infinitely many solutions. More precisely,

$$S = \{(3, 6, x_3, -1) \mid x_3 \in \mathbb{R}\}.$$

Exercise 4. Which of the following matrices are in reduced row-echelon form, row-echelon form but not reduced, neither?

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 5 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Solution: The first matrix is neither since the leading entry of R_2 is in the same column as the leading entry of R_1.
The second is in row echelon form but not reduced since a leading entry is 5 instead of 1.
The third matrix is in reduced row echelon form.
The fourth matrix is in row echelon form but not reduced since the leading entry in R2 is not the only non-zero entry in that column.
The fifth matrix is in reduced row echelon form.
The sixth matrix is neither since there is a zero row above a nonzero row.

Exercise 5. Write down the solution sets for each of the following SLEs:

$$\begin{aligned} 5x + 2y &= 1 \\ -10x - 4y &= 0 \end{aligned} \qquad \begin{aligned} 5x + 2y &= 1 \\ -10x - 3y &= 1 \end{aligned} \qquad \begin{aligned} 5x + 2y &= 1 \\ -10x - 4y &= -2 \end{aligned}$$

Solution: After performing the elementary row operation $R_2 \leftarrow R_2 + 2R_1$ (on each of the three systems) we have matrices in row-echelon form:

$$\left[\begin{array}{cc|c} 5 & 2 & 1 \\ 0 & 0 & 2 \end{array}\right] \qquad \left[\begin{array}{cc|c} 5 & 2 & 1 \\ 0 & 1 & 3 \end{array}\right] \qquad \left[\begin{array}{cc|c} 5 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

The first SLE has no solutions as it is impossible for $0x + 0y = 2$, and so it is *inconsistent*.
The second SLE has the unique solution $y = 3$, $x = -1$ obtained from back-substitution, and so the solution set is

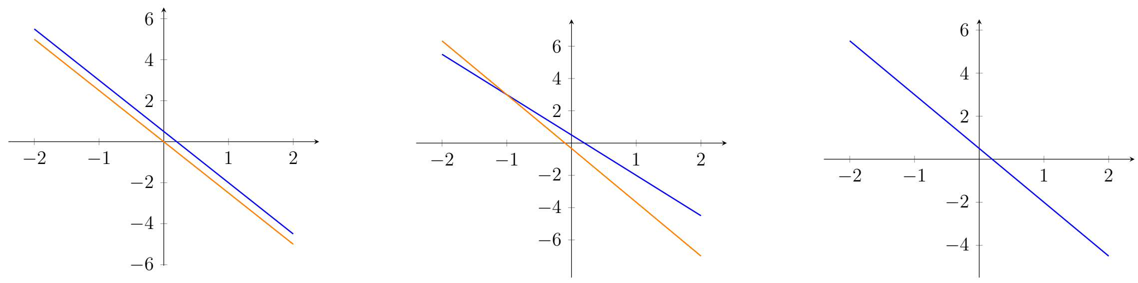
$$S = \{(-1, 3)\}$$

The third SLE has y as a free parameter and then $x = \frac{1-2y}{5}$, and so

$$S = \left\{ \left(\frac{1-2y}{5}, y \right) : y \in \mathbb{R} \right\}$$

Exercise 6. For each of the systems of the previous question, draw a diagram in $\mathbb{R}^2$ showing the points satisfied by each individual linear equation (e.g. $5x + 2y = 1$ is a particular line in the plane) making sure that it is consistent with your answers to the previous question.

Solution:



Exercise 7. Suppose that a given system of linear equations has 3 equations in 4 variables. Is it possible to say anything about whether this system can have zero, one or infinitely many solutions without more information?

Solution: The system is either inconsistent or has infinitely many solutions. It cannot have a unique solution, for if it is consistent then there is at least one non-basic variable.

Exercise 8. Do any two of the following SLEs have the same solution set?

$$\begin{aligned} x + 2y &= 5 \\ 2x - 3y &= -4 \end{aligned} \qquad \begin{aligned} 3x - 4y &= 2 \\ x - 3y &= -1 \end{aligned} \qquad \begin{aligned} 5x + y &= 7 \\ 6x - 2y &= 2 \end{aligned}$$

Solution: The first one has a unique solution $(x, y) = (1, 2)$, which does not satisfy the equation $3x - 4y = 2$ but does satisfy the two equations in the third SLE. As the third SLE also has a unique solution (because the two lines defined by the two linear equations are not parallel), this system is identical to the first.

Exercise 9. Suppose that an SLE has an augmented matrix that has been reduced to reduced row echelon form

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & k & k-1 \end{array}\right]$$

where k is some (real) number whose value you do not know. For which values of k does the system have no solutions, one solution, infinitely many solutions?

Solution: If $k = 0$ the system is inconsisent. If $k \neq 0$ the system has unique solution

$$S = \left\{ \left(2, \frac{k-1}{k} \right) : k \in \mathbb{R},\ k \neq 0 \right\}$$

Exercise 10. A mixture problem. A farmer has two types of cows, Jersey and Illawarra. Jersey cows produce milk with 4.8% milk fat and Illawarra cows produce milk with 3.6% milk fat. She wishes to make a mixture of the two milk types to get 120 litres of milk with 4% milk fat. How many litres of each type of milk does she need to use?

Solution: Let J be the number of litres of Jersey milk and I be the number of litres of Illawarra milk. Then we must have $J + I = 120$. We must also have

$$\frac{4.8}{100} J + \frac{3.6}{120} I = \frac{4}{100} \cdot 120 \qquad \Rightarrow \qquad 48J + 36I = 4800$$

The corresponding augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 120 \\ 48 & 36 & 4800 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 120 \\ 12 & 9 & 1200 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 120 \\ 0 & -3 & -240 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 120 \\ 0 & 1 & 80 \end{array}\right]$$

So $I = 80$, and hence $J = 40$. That is, she should mix 40 litres of milk from the Jersey cows with 80 litres of milk from the Illawarra cows.

Exercise 11. Use Gauss-Jordan elimination to solve the following system:

$$\begin{array}{rcrcrcr} -4x_1 & + & 2x_2 & - & 2x_3 & = & 2 \\ x_1 & - & 3x_2 & - & 2x_3 & = & -3 \\ 2x_1 & - & x_2 & + & 3x_3 & = & 3 \end{array}$$

Solution: The augmented matrix of this system is

$$\left[\begin{array}{rrr|r} -4 & 2 & -2 & 2 \\ 1 & -3 & -2 & -3 \\ 2 & -1 & 3 & 3 \end{array}\right].$$

The Gauss-Jordan algorithm calls for dividing the first row by -4, but in order to avoid fraction we will first swap the first two rows.

$$\left[\begin{array}{rrr|r} 1 & -3 & -2 & -3 \\ -4 & 2 & -2 & 2 \\ 2 & -1 & 3 & 3 \end{array}\right] \begin{array}{l} R_1 \leftrightarrow R_2 \\ R_2 \leftrightarrow R_1 \\ \end{array}$$

Then we pivot on the first entry to put zeroes in the first column

$$\left[\begin{array}{rrr|r} 1 & -3 & -2 & -3 \\ 0 & -10 & -10 & -10 \\ 0 & 5 & 7 & 9 \end{array}\right] \begin{array}{l} \\ R_2 \leftarrow R_2 + 4R_1 \\ R_3 \leftarrow R_3 - 2R_1 \end{array}$$

We now make the second pivot equal to 1 and use it to zero out the rest of the second column.

$$\left[\begin{array}{rrr|r} 1 & -3 & -2 & -3 \\ 0 & 1 & 1 & 1 \\ 0 & 5 & 7 & 9 \end{array}\right] \; R_2 \leftarrow -R_2/10 \; \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \end{array}\right] \begin{array}{l} R_1 \leftarrow R_1 + 3R_2 \\ \\ R_3 \leftarrow R_3 - 5R_2 \end{array}$$

We now make the third pivot equal to 1 and use it to zero out the rest of the third column.

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \; R_3 \leftarrow R_3/2 \quad \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 \leftarrow R_1 - R_3 \\ R_2 \leftarrow R_2 - R_3 \\ \end{array}$$

We now easily read off the solution: $x_1 = -2$, $x_2 = -1$, $x_3 = 2$. We can also write the solution set as $S = \{(-2, -1, 2)\}$.

Conceptual questions

Exercise 12. Suppose that an SLE has the following augmented matrix

$$\left[\begin{array}{rrrr|r} 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & -1 & 0 & 1 & 2 \end{array}\right].$$

Is this matrix in row echelon form? What is the solution of this system? (Hint: You do not need to do any row operations to find the solutions.)

Solution: This matrix is not in row echelon form as the leading entries are in positions 1,2,2 respectively. However, if we choose x_2 as a free variable, we immediately see that this system has solution set $\{(1 - 2x_2, x_2, 2 - x_2, 2 + x_2) \mid x_2 \in \mathbb{R}\}$. In this case we see that if we "forgot" about the second column then the matrix would be in reduced row echelon form, so that allows us to solve the system quickly. This usually won't happen.

EXERCISE 13. Consider the following system of linear equations, with parameter k:

$$\begin{array}{rcrcrcr} -4x & + & 2y & & & = & 2 \\ 4x & - & 3y & - & 2z & = & -3 \\ 2x & - & y & + & (k-k^2)z & = & -k \end{array}$$

Determine the solution set of this system, depending on the value of k. In particular determine for which value(s) of k the system has: (i) a unique solution, (ii) infinitely many solutions, (iii) no solution. Advice: don't split into cases until you really need to.

SOLUTION: For the augmented matrix of the system we have

$$\left[\begin{array}{ccc|c} -4 & 2 & 0 & 2 \\ 4 & -3 & -2 & -3 \\ 2 & -1 & k(1-k) & -k \end{array}\right].$$

We get the row echelon form by performing some simple EROs:

$$\left[\begin{array}{ccc|c} -4 & 2 & 0 & 2 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & k(1-k) & 1-k \end{array}\right] \begin{array}{l} \\ R_2 \leftarrow R_2 + R_1 \\ R_3 \leftarrow R_3 + \frac{1}{2}R_1 \end{array}$$

If $k = 0$ the system is inconsistent so there is no solution.
If $k = 1$ there are infinitely many solutions as the third row becomes all zeros. In this case we do a few more EROs to get the reduced row echelon form.

$$\left[\begin{array}{ccc|c} -4 & 2 & 0 & 2 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -4 & 0 & -4 & 0 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 \leftarrow R_1 + 2R_2 \\ \\ \\ \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 \leftarrow -R_1/4 \\ R_2 \leftarrow -R_2 \\ \\ \end{array}$$

Using z as free parameter gives the one-parameter solution set $\{(-z, 1-2z, z) \mid z \in \mathbb{R}\}$.
Finally assume $k \neq 0$ and $k \neq 1$. Then the system becomes:

$$\left[\begin{array}{ccc|c} -4 & 2 & 0 & 2 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 1 & 1/k \end{array}\right] R_3 \leftarrow R_3/\left(k(1-k)\right) \quad \to \left[\begin{array}{ccc|c} -4 & 2 & 0 & 2 \\ 0 & -1 & 0 & -1+2/k \\ 0 & 0 & 1 & 1/k \end{array}\right] R_2 \leftarrow R_2 + 2R_3$$

$$\to \left[\begin{array}{ccc|c} -4 & 0 & 0 & 4/k \\ 0 & -1 & 0 & -1+2/k \\ 0 & 0 & 1 & 1/k \end{array}\right] R_1 \leftarrow R_1 + 2R_2 \quad \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/k \\ 0 & 1 & 0 & 1-2/k \\ 0 & 0 & 1 & 1/k \end{array}\right] \begin{array}{l} R_1 \leftarrow -R_1/4 \\ R_2 \leftarrow -R_2 \\ \\ \end{array}$$

Hence there is a unique solution $\left(-\frac{1}{k}, 1-\frac{2}{k}, \frac{1}{k}\right)$. To summarise: the system has
(i) a unique solution $\left(-\frac{1}{k}, 1-\frac{2}{k}, \frac{1}{k}\right)$ when $k \notin \{0,1\}$,
(ii) infinitely many solutions $\{(-z, 1-2z, z) \mid z \in \mathbb{R}\}$ when $k = 1$,
(iii) no solution when $k = 0$.

EXERCISE 14. Given a system of three linear equations in three variables, each equation defines a plane in $\mathbb{R}^3$. If two of the planes are parallel then there are no solutions to the SLE. Find a configuration of the planes with no two parallel that also implies that there is no solution.

SOLUTION: No common intersection to the three planes; a "toblerone" configuration.

MATH1012 (Semester 1, 2024)

Practical Class 2:
Subspaces, spanning sets and linear independence

Summary of what you have learnt

A *subspace* of $\mathbb{R}^n$ is a set S of vectors satisfying:
(S1) $\mathbf{0} \in S$
(S2) If $\boldsymbol{u}$, $\boldsymbol{v} \in S$, then $\boldsymbol{u}+\boldsymbol{v} \in S$
(S3) If $\boldsymbol{u} \in S$ and $\alpha \in \mathbb{R}$, then $\alpha\boldsymbol{u} \in S$

The *span* of a set of vectors is the set of *all possible linear combinations* of these vectors.
A *spanning set* for a subspace is a set of vectors whose span is equal to the subspace.
A set $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}$ of vectors if *linearly independent* if the only solution to the equation

$$\lambda_1\boldsymbol{v_1} + \lambda_2\boldsymbol{v_2} + \cdots + \lambda_k\boldsymbol{v_k} = \mathbf{0}$$

in the variables λ_1, λ_2, …, λ_k is the *trivial solution* $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$.

Foundation questions

Exercise 1. Which of the following sets of vectors in $\mathbb{R}^2$ are *closed under vector addition*?
(1) $S_1 = \{(x, y) : x \geqslant 0,\ y \geqslant 0\}$
(2) $S_2 = \{(x, y) : xy \geqslant 0\}$
(3) $S_3 = \{(x, y) : x + 2y = 0\}$
(4) $S_4 = \{(x, y) : x^2 - y^2 > 0\}$

Solution:
(1) The set S_1 is closed, as adding two vectors with both coordinates positive gives another vector with both coordinates positive.
(2) The set S_2 is not closed, because $(-2, -2) + (1, 3) = (-1, 1)$ which is not in S_2.
(3) The set S_3 is closed because any vector has the form $(a, -a/2)$ and if we add two such vectors together we get

$$(a, -a/2) + (b, -b/2) = (a + b, -a/2 - b/2) = (a + b, -(a + b)/2) \quad \text{satisfying } x + 2y = 0.$$

(4) The set S_4 is not closed as $(-2, -1) \in S_4$, $(2, -1) \in S_4$, but $(0, -2) \notin S_4$.

Exercise 2. For each of the sets of vectors in the previous question, determine whether or not it is *closed under scalar multiplication*?

Solution:
(1) The set S_1 is not closed because $(1, 1) \in S_1$, but $-1(1, 1) = (-1, -1) \notin S_1$.
(2) The set S_2 is closed, because if $xy \geqslant 0$, then $(\alpha x)(\alpha y) = \alpha^2 xy \geqslant 0$.
(3) The set S_3 is closed because if we multiply $(a, -a/2)$ and by a scalar α we get

$$\alpha(a, -a/2) = (\alpha a, \alpha(-a/2)) = (\alpha a, -(\alpha a)/2)) \qquad \text{satisfying } x + 2y = 0.$$

(4) The set S_4 is not closed because $(2, 1) \in S_4$, but $0(2, 1) = (0, 0)$ is not in S_4.

Exercise 3. Suppose that a *subspace* of $\mathbb{R}^2$ contains the vectors $(1, 0)$ and $(0, 1)$. Show that it contains every vector in $\mathbb{R}^2$.

Solution: If the subspace contains $(1, 0)$ then it must also contain $(a, 0)$ for any $a \in \mathbb{R}$ (closed under scalar multiplication), and if it contains $(0, 1)$ then for the same reason it contains $(0, b)$ for any $b \in \mathbb{R}$. But as it closed under vector addition, this means it contains $(a, 0) + (0, b) = (a, b)$ for any a, b. This is every vector in $\mathbb{R}^2$.

Exercise 4. Are the following sets of vectors in $\mathbb{R}^3$ subspaces or not?

(a) The plane $x = y$, that is, the set $\{(x, y, z) \in \mathbb{R}^3 \mid x = y\}$.
(b) The plane $x = y - 1$.
(c) The union of the planes $x = 0$ and $y = 0$.
(d) The set of vectors all of whose coordinates are integers.

Solution:

(a) Yes. This is a plane through the origin, so it is a subspace. This can also be proved by showing the 3 conditions for a vector space are satisfied.
(b) The plane $x = y - 1$ is not a subspace. Indeed (S1) is not satisfied, since $0 \neq 0 - 1$, and so $\mathbf{0} = (0, 0, 0)$ is not in the plane $x = y - 1$.
(c) Let S be the union of the planes $x = 0$ and $y = 0$. Then S is not a subspace. Indeed (S2) is not satisfied: $(1, 0, 0) \in S$ (in the plane $y = 0$) and $(0, 1, 0) \in S$ (in the plane $x = 0$) but $(1, 0, 0) + (0, 1, 0) = (1, 1, 0) \notin S$.
(d) Thieset is not a subspace. Indeed (S3) is not satisfied: $(1, 0, 0)$ is in the set and $\frac{1}{2} \in \mathbb{R}$, but $\frac{1}{2}(1, 0, 0) = (\frac{1}{2}, 0, 0)$ is not in the set.

Exercise 5. Give a *geometric description* of the following sets of vectors:

(a) The span of $\{(1, 0, 1), (1, 0, 2)\}$
(b) The span of $\{(1, 0, 0), (0, 1, 0), (1, 1, 1), (2, 2, 1)\}$

Solution:

(a) This is the xz-plane. Each of the two vectors is in that plane. We must show any such vector $(x, 0, z)$ can be written as a linear combination of $(1, 0, 1)$ and $(1, 0, 2)$:

$$(x, 0, z) = \alpha(1, 0, 1) + \beta(1, 0, 2),$$

This SLE for α and β has augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & x \\ 0 & 0 & 0 \\ 1 & 2 & z \end{array}\right] \quad \to \quad \left[\begin{array}{cc|r} 1 & 0 & 2x - z \\ 0 & 1 & z - x \\ 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} \text{doing the EROs: } R2 \leftrightarrow R3 \\ \text{followed by } R2 \leftarrow R2 - R1 \\ \text{followed by } R1 \leftarrow R1 - R2 \end{array}$$

Hence $\alpha = 2x - z$ and $\beta = z - x$, so we can write

$$(x, 0, z) = (2x - z)(1, 0, 1) + (z - x)(1, 0, 2).$$

(b) The third is the whole of $\mathbb{R}^3$. Using a similar method as in (b) we solve an SLE to get

$$(x, y, z) = (x - z)(1, 0, 0) + (y - z)(0, 1, 0) + z(1, 1, 1) + 0(2, 2, 1).$$

Exercise 6. Which of the following sets of vectors are *linearly independent*? (Try to spot a dependency, and if that fails then use the *linear independence test.*)

$$\begin{aligned}
S_1 &= \{(0,1,1),(1,0,1),(1,1,0)\}\\
S_2 &= \{(0,0,1),(0,1,1),(1,1,1)\}\\
S_3 &= \{(1,1,2),(1,2,1),(3,4,5)\}\\
S_4 &= \{(-1,1,1),(1,-1,1),(1,1,-1)\}\\
S_5 &= \{(0,1,-1),(0,1,1),(0,-1,1)\}\\
S_6 &= \{(1,0,1),(0,0,0),(0,1,0)\}
\end{aligned}$$

Solution: The sets S_1, S_2, S_4 are linearly independent, while S_3, S_5 and S_6 are linearly dependent. All of these can be answered using the linear independence test. For example, for S_3 we need to solve the vector equation

$$\lambda_1(1,1,2)+\lambda_2(1,2,1)+\lambda_3(3,4,5)=(0,0,0)$$

By considering each co-ordinate position in turn, we get a system of 3 linear equations

$$\begin{aligned}
1\lambda_1+1\lambda_2+3\lambda_3&=0\\
1\lambda_1+2\lambda_2+4\lambda_3&=0\\
2\lambda_1+1\lambda_2+5\lambda_3&=0
\end{aligned}$$

The augmented matrix of this system of linear equations is

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0\\ 1 & 2 & 4 & 0\\ 2 & 1 & 5 & 0 \end{array}\right]$$

Performing elementary row operations we get

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0\\ 0 & 1 & 1 & 0\\ 0 & -1 & -1 & 0 \end{array}\right] \quad \begin{array}{l} \\ R2 \leftarrow R2-R1\\ R3 \leftarrow R3-2R1 \end{array}$$

and then

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} \\ \\ R3 \leftarrow R3+R2 \end{array}$$

which is now in row-echelon form. Immediately we see that there are two basic variables (λ_1, λ_2) and one free parameter (λ_3) and therefore there are infinitely many solutions to the SLE. By the linear independence test, these vectors form a *dependent set*.
To find a specific linear equation proving dependency, we can just choose any convenient value of the free variable. So if $\lambda_3=1$, then by back-substitution we get $\lambda_2=-1$ and $\lambda_1=-2$ and therefore

$$-2(1,1,2)-1(1,2,1)+1(3,4,5)=(0,0,0)$$

as required.
For S_6 we immediately see that it is dependent as it contains the zero vector, and $0(1,0,1)+1(0,0,0)+0(0,1,0)=(0,0,0)$ is a non-trivial solution to the independence test.
For S_5 we immediately see that $1(0,1,-1)+0(0,1,1)+1(0,-1,1)=(0,0,0)$, so S_5 is linearly dependent.
For S_1, S_2, S_4, we need to solve a homogeneous SLE and find only the trivial solution. This shows these sets are linearly independent.

EXERCISE 7. Find a vector that is not in the span of the set S_5 (from Exercise 6). Can you identify *all* the vectors that are not in the span of S_5? Now find a vector that is not in the span of S_3.

SOLUTION: Every vector in the span of S_5 has first coordinate equal to 0, and therefore this span does not contain $(1,0,0)$. Indeed, the vectors not in S_5 are exactly the vectors with first coefficient non-zero.
For S_3 we see that $(3,4,5) = (1,2,1)+2(1,1,2)$ and so the third vector is redundant. Hence the span of S_3 is equal to the span of $\{(1,2,1),(1,1,2)\}$ which gives a plane defined by the equation $z = 3x - y$. Consequently any vector (x,y,z) which does not satisfy this equation will not be in the span of S_3. For example, the vector $(1,1,1)$ does not lie in the span of S_3 since it does not satisfy this equation and equivalently cannot be made from a linear combination of $\{(1,2,1),(1,1,2)\}$. To see this, observe that there is no solution to

$$a(1,2,1) + b(1,1,2) = (1,1,1)$$

because the first coordinate gives us $a+b=1$, the second $2a+b=1$. This has a unique solution $a=0$, $b=1$ but this does not satisfy the requirements for the third coordinate.

EXERCISE 8. The following set of vectors is in $\mathbb{R}^4$. Decide whether it is *linearly independent* or not, and also whether it is a *spanning set* for $\mathbb{R}^4$ or not.

$$S = \{\ (1,-2,-1,1),\ (2,1,3,2),\ (2,1,4,2),\ (0,1,2,0)\ \}$$

SOLUTION: To check linear independence it is necessary to use the linear independence test. Consider the vector equation

$$\lambda_1(1,-2,-1,1) + \lambda_2(2,1,3,2) + \lambda_3(2,1,4,2) + \lambda_4(0,1,2,0) = (0,0,0,0).$$

If this has only the trivial solution $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$, then S is linearly independent; otherwise the set is dependent. The augmented matrix for this homogeneous system is

$$\left[\begin{array}{rrrr|r} 1 & 2 & 2 & 0 & 0 \\ -2 & 1 & 1 & 1 & 0 \\ -1 & 3 & 4 & 2 & 0 \\ 1 & 2 & 2 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 2 & 2 & 0 & 0 \\ 0 & 5 & 5 & 1 & 0 \\ 0 & 5 & 6 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 2 & 2 & 0 & 0 \\ 0 & 5 & 5 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since there is a column without leading coefficent (the last one) in this echelon form matrix, there is a free variable (λ_4) and so there are nontrivial solutions, and S is linearly dependent.
To check if it is a spanning set, we must check if any vector (x,y,z,u) can be written as a linear combination of the vectors in the set, that is, does the following vector equation (with unknowns $\lambda_1, \lambda_2, \lambda_3, \lambda_4$) have a solution for every x, y, z, u:

$$\lambda_1(1,-2,-1,1) + \lambda_2(2,1,3,2) + \lambda_3(2,1,4,2) + \lambda_4(0,1,2,0) = (x,y,z,u).$$

This translates into a system with the following augmented matrix:

$$\left[\begin{array}{rrrr|c} 1 & 2 & 2 & 0 & x \\ -2 & 1 & 1 & 1 & y \\ -1 & 3 & 4 & 2 & z \\ 1 & 2 & 2 & 0 & u \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 2 & 2 & 0 & x \\ 0 & 5 & 5 & 1 & y+2x \\ 0 & 5 & 6 & 2 & z+x \\ 0 & 0 & 0 & 0 & u-x \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 2 & 2 & 0 & x \\ 0 & 5 & 5 & 1 & y+2x \\ 0 & 0 & 1 & 1 & z-x-y \\ 0 & 0 & 0 & 0 & u-x \end{array}\right]$$

We see this system is inconsistent when $u \neq x$, so the set does not span the whole of $\mathbb{R}^4$.

Exercise 9. The following set of vectors is in $\mathbb{R}^4$. Decide whether it is *linearly independent* or not, and also whether it is a *spanning set* for $\mathbb{R}^4$ or not.
$S = \{(1,-2,-1,1),(2,1,3,2),(2,1,4,2),(0,1,2,1)\}$

Solution: To check if S is linearly independent we solve the following system:

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 0 & 0 \\ -2 & 1 & 1 & 1 & 0 \\ -1 & 3 & 4 & 2 & 0 \\ 1 & 2 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 2 & 0 & 0 \\ 0 & 5 & 5 & 1 & 0 \\ 0 & 5 & 6 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 2 & 0 & 0 \\ 0 & 5 & 5 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

The only solution is the zero vector, since there are only basic variables and no free variables, so S is linearly independent. To check if S span $\mathbb{R}^4$ we solve the following system

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 0 & x \\ -2 & 1 & 1 & 1 & y \\ -1 & 3 & 4 & 2 & z \\ 1 & 2 & 2 & 1 & u \end{array}\right] \sim \left[\begin{array}{cccc|r} 1 & 2 & 2 & 0 & x \\ 0 & 5 & 5 & 1 & y+2x \\ 0 & 5 & 6 & 2 & z+x \\ 0 & 0 & 0 & 1 & u-x \end{array}\right] \sim \left[\begin{array}{cccc|r} 1 & 2 & 2 & 0 & x \\ 0 & 5 & 5 & 1 & y+2x \\ 0 & 0 & 1 & 1 & z-x-y \\ 0 & 0 & 0 & 1 & u-x \end{array}\right]$$

which does have a unique solution so S spans $\mathbb{R}^4$. We can find an explicit linear combination by doing back substitution or by getting the system into reduced row echelon form. We get

$$(-\frac{1}{5}x - 4z + \frac{2}{5}u - \frac{2}{5}y)(1,-2,-1,1) + (\frac{6}{5}y + \frac{4}{5}u + \frac{3}{5}x - z)(2,1,3,2)$$
$$+(z-u-y)(2,1,4,2) + (u-x)(0,1,2,0) = (x,y,z,u).$$

Exercise 10. Let $S = \{(0,1,1),(1,0,1),(1,1,0)\}$. Find a linear combination of the vectors in S that is equal to $(2,0,0)$. Do the same for $(0,2,0)$ and $(0,0,2)$. Can you use your results to show that an arbitrary vector (x,y,z) is a linear combination of the vectors in S? What does this show about S?

Solution: We have

$$\begin{aligned}(2,0,0) &= -1(0,1,1) + 1(1,0,1) + 1(1,1,0)\\ (0,2,0) &= 1(0,1,1) - 1(1,0,1) + 1(1,1,0)\\ (0,0,2) &= 1(0,1,1) + 1(1,0,1) - 1(1,1,0)\end{aligned}$$

and so

$$\begin{aligned}(x,0,0) &= -x/2(0,1,1) + x/2(1,0,1) + x/2(1,1,0)\\ (0,y,0) &= y/2(0,1,1) - y/2(1,0,1) + y/2(1,1,0)\\ (0,0,z) &= z/2(0,1,1) + z/2(1,0,1) + -z/2(1,1,0)\end{aligned}$$

Adding these three equations gives

$$(x,y,z) = \left(\frac{-x+y+z}{2}\right)(0,1,1) + \left(\frac{x-y+z}{2}\right)(1,0,1) + \left(\frac{x+y-z}{2}\right)(1,1,0)$$

Therefore S is a spanning set for $\mathbb{R}^3$ as every vector in $\mathbb{R}^3$ is a linear combination of the vectors in S.

Conceptual Questions

EXERCISE 11. Are the following sets of vectors of $\mathbb{R}^n$ subspaces (where $n \geq 2$) or not? For any that are subspaces, find a basis for the subspace.

(a) $S_1 = \{(x_1, x_2, \dots, x_n) \mid x_1 + x_2 + \dots + x_{n-1} = x_n\}$.
(b) $S_2 = \{(x_1, x_2, \dots, x_n) \mid x_1 + x_2 + \dots + x_{n-1} \geq x_n\}$.

SOLUTION:

(a) The set S_1 is a subspace, which we prove by going through the three conditions:

(S1) *Does S_1 contain the zero vector?*
(S2) *Is S_1 closed under vector addition?*
(S3) *Is S_1 closed under scalar multiplication?*

Suppose that $\boldsymbol{u} = (u_1, u_2, \dots, u_n) \in S_1$ and $\boldsymbol{v} = (v_1, v_2, \dots, v_n) \in S_1$ and $\alpha \in \mathbb{R}$. Then

(S1) We observe that

$$0 + 0 + \dots + 0 = 0 \qquad \checkmark$$

(S2) We are told that

$$u_1 + u_2 + \dots + u_{n-1} = u_n \quad \text{and} \quad v_1 + v_2 + \dots + v_{n-1} = v_n$$

Now consider the sum

$$\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$$

Test it for membership in S. We have

$$\begin{aligned} &(u_1 + v_1) + (u_2 + v_2) + \dots + (u_{n-1} + v_{n-1}) \\ =&(u_1 + u_2 + \dots + u_{n-1}) + (v_1 + v_2 + \dots + v_{n-1}) \\ =&u_n + v_n \qquad \checkmark \end{aligned}$$

(S3) Consider the vector

$$\alpha\boldsymbol{u} = (\alpha u_1, \alpha u_2, \dots, \alpha u_n)$$

Test it for membership in S_1. We have

$$\begin{aligned} &\alpha u_1 + \alpha u_2 + \dots + \alpha u_{n-1} \\ =&\alpha(u_1 + u_2 + \dots + u_{n-1}) \\ =&\alpha u_n \qquad \checkmark \end{aligned}$$

Therefore S_1 is a subspace of $\mathbb{R}^n$. To find a basis note that we can write

$$\begin{aligned} \boldsymbol{u} &= (\ u_1,\ u_2,\ \dots,\ u_{n-1},\ u_1 + u_2 + \dots + u_{n-1}\ ) \\ &= u_1(1, 0, \dots, 0, 1) + u_2(0, 1, 0, \dots, 0, 1) + \dots + u_{n-1}(0, 0, \dots, 0, 1, 1) \end{aligned}$$

So a spanning set is $\{e_i + e_n \mid 1 \leqslant i < n\}$. We now prove it is linearly independent. Suppose that $\sum_{i=1}^{n-1} \lambda_i(e_i + e_n) = \mathbf{0}$. This is equivalent to

$$(\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_1 + \lambda_2 + \dots + \lambda_{n-1}) = (0, 0, \dots, 0)$$

so $\lambda_1 = \lambda_2 = \dots = \lambda_{n-1} = 0$. Thus it is a linearly independent set and hence a basis.

(b) The set S_2 is not a subspace. Indeed (S3) is not satisfied. For example,

$(1, 0, \dots, 0) \in S_2$ (since $1 + 0 + \dots + 0 \geq 0$) and $-1 \in \mathbb{R}$,

but

$-1(1, 0, \dots, 0) = (-1, 0, \dots, 0) \notin S_2$ (since $(-1) + 0 + \dots + 0 < 0$).

Exercise 12. Let S_1 and S_2 be subspaces of the vector space $\mathbb{R}^n$.
(a) Is $S_1 \cap S_2$ a subspace of $\mathbb{R}^n$?
(b) What about $S_1 \cup S_2$?
Remember that $A \cap B$ is the *intersection* of two sets, while $A \cup B$ is the *union* of two sets.

Solution:
(a) Yes, $S_1 \cap S_2$ is a subspace, which is shown by verifying the three subspace conditions:
  (a) (S1) *Does S contain the zero vector?*
  Yes, the zero vector is in S_1 (because S_1 is a subspace) and the zero vector is in S_2 (because S_2 is a subspace), so the zero vector is in $S_1 \cap S_2$.
  (b) (S2) *Is S closed under vector addition?*
  Yes. Suppose $\boldsymbol{u}, \boldsymbol{v} \in S_1 \cap S_2$. Then $\boldsymbol{u} + \boldsymbol{v} \in S_1$ (because S_1 is a subspace), but also $\boldsymbol{u} + \boldsymbol{v} \in S_2$ (because S_2 is a subspace) and so $\boldsymbol{u} + \boldsymbol{v} \in S_1 \cap S_2$.
  (c) (S3) *Is S closed under scalar multiplication?*
  Yes. Suppose $\boldsymbol{u} \in S_1 \cap S_2$ and $\alpha \in \mathbb{R}$. Then $\alpha\boldsymbol{u} \in S_1$ (because S_1 is a subspace), but also $\alpha\boldsymbol{u} \in S_2$ (because S_2 is a subspace) and so $\alpha\boldsymbol{u} \in S_1 \cap S_2$.
(b) The union of two subspaces is not necessarily a subspace. Counter-example:
$S_1 = \{(x, 0) : x \in \mathbb{R}\}$ and $S_2 = \{(0, y) : y \in \mathbb{R}\}$. Then $(1, 0) \in S_1 \cup S_2$, $(0, 1) \in S_1 \cup S_2$ but $(1, 1) \notin S_1 \cup S_2$.

Exercise 13. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

which is in row echelon form. (Note, this is not the augmented matrix of an SLE.)
(a) Show that the rows of A are linearly independent.
(b) Show that the nonzero rows of any matrix in row echelon form are linearly independent.

Solution:
(a) Suppose that $\lambda_1(2, 1, 1, 0) + \lambda_2(0, 0, 1, 1) + \lambda_3(0, 0, 0, 3) = (0, 0, 0, 0)$. Then we get the system of equations

$$\begin{aligned} 2\lambda_1 \quad & &&= 0 \\ \lambda_1 \quad & &&= 0 \\ \lambda_1 \quad &+\lambda_2 &&= 0 \\ &\lambda_2 \quad +3\lambda_3 &&= 0 \end{aligned}$$

It follows that $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and so the rows of A are linearly independent.
(b) Let B be a matrix in row echelon form with nonzero rows $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$. Suppose that

$$\lambda_1\boldsymbol{v}_1 + \lambda_2\boldsymbol{v}_2 + \cdots + \lambda_k\boldsymbol{v}_k = \mathbf{0}$$

Suppose that the leading entry of the first row is in the r^{th} column. Then comparing the r^{th} coordinates of the vector equation we get the equation $\lambda_1 a = 0$ where a is the r^{th} coordinate of $\boldsymbol{v}_1$. Since $a \neq 0$ we deduce that $\lambda_1 = 0$. Next consider the coordinate corresponding to the column of the leading entry of the second row. This will give us an equation $b\lambda_1 + c\lambda_2 = 0$ where $c \neq 0$ is the leading entry of the second row and b is the entry above it in the first row. Since $\lambda_1 = 0$ we deduce that $\lambda_2 = 0$. Continuing in this fashion we see that the only solution is $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$ and so the rows are linearly independent.

MATH1012 (Semester 1, 2024)

Practice Class 3: Bases, matrix algebra, row space, column space and null space

Summary of what you have learnt

A *basis* for a subspace V is a *linearly independent spanning set* for V. A vector has unique *coordinates* with respect to a basis. The number of elements in a basis is called the *dimension* of the subspace.
Let A be an $m \times n$ matrix.

(1) The row space of A is the subspace of $\mathbb{R}^n$ that is *spanned by the rows* of A.
The column space of A is the subspace of $\mathbb{R}^m$ that is *spanned by the columns* of A.
The row space and column space have the same dimension, called the *rank* of A.
(2) The null space of a matrix is the subspace $\{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} = \boldsymbol{0}\}$, its dimension is called the *nullity* of A.
(3) The Rank-Nullity theorem says that $\operatorname{rank}(A) + \operatorname{nullity}(A) = n$.

Foundations

Exercise 1. For each of the following three *homogeneous* systems of linear equations, find a *basis* for the solution space.

$$\begin{aligned} 5x + 2y &= 0 \\ -10x - 4y &= 0 \end{aligned} \qquad \begin{aligned} x + y + z &= 0 \\ 2x + y + 3z &= 0 \\ 4x + 3y + 5z &= 0 \end{aligned} \qquad \begin{aligned} a + b + 3c + d &= 0 \\ a + b + c + 2d &= 0 \end{aligned}$$

Solution: In all three cases, we need to reduce the augmented matrix to reduced row echelon form first.
The first system has reduced row echelon form

$$\left[\begin{array}{cc|c} 1 & 2/5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore y is a free variable, and $x = -2y/5$. Hence the solution set is

$$S = \{(-2y/5, y) : y \in \mathbb{R}\}$$

which can be rewritten as

$$S = \{y(-2/5, 1) : y \in \mathbb{R}\}$$

and so as every solution is a multiple of $(-2/5, 1)$, a basis for the solution space is just $\{(-2/5, 1)\}$. To avoid fractions, we can also take $\{(-2, 5)\}$ as a basis.
Row reduction (Gauss-Jordan) on the second system produces

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore z is a free variable, $y = z$ and $x = -2z$ and so the solution space is

$$S = \{(-2z, z, z) : z \in \mathbb{R}\} = \{z(-2, 1, 1) : z \in \mathbb{R}\}$$

and therefore a basis is $\{(-2, 1, 1)\}$.
For the third system, the reduced row echelon form of the augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 5/2 & 0 \\ 0 & 0 & 1 & -1/2 & 0 \end{array}\right]$$

Therefore b and d are free variables, and we get that $c = d/2$ and $a = -b - 5d/2$. So the solution space is given by

$$S = \{(-b - 5d/2, b, d/2, d) : b, d \in \mathbb{R}\}$$

Here there are two free variables, so this can be rewritten as

$$S = \{b(-1, 1, 0, 0) + d(-5/2, 0, 1/2, 1) : b, d \in \mathbb{R}\}$$

and so a basis for S is $\{(-1, 1, 0, 0), (-5/2, 0, 1/2, 1)\}$. To avoid fractions we can also take as basis $\{(-1, 1, 0, 0), (-5, 0, 1, 2)\}$.

Exercise 2. What is the *dimension* of the solution space of each of the systems of linear equations from the previous question.

Solution: The dimension is the size of the basis, which is the number of free variables. So 1, 1 and 2 respectively.

Exercise 3. For each of the following subspaces V of $\mathbb{R}^4$ and for the given basis $\mathcal{B}$, determine the coordinates of the vector $\boldsymbol{v}$ of V.

(1) $V = \{(x_1, x_2, x_3, x_4) | x_1 + 2x_2 = 0\}$; $\mathcal{B} = \{(-2, 1, 0, 0), (0, 0, 1, 0), (0, 0, 1, 1)\}$ and $\boldsymbol{v} = (4, -2, 5, 6)$.
(2) $V = \text{span}(\mathcal{B})$ where $\mathcal{B} = \{(-1, 1, 0, 0), (0, 1, 2, 0)\}$ and $\boldsymbol{v} = (4, -2, 4, 0)$.
(3) $V = \mathbb{R}^4$; $\mathcal{B} = \{(0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 0, 0, 0)\}$ and $\boldsymbol{v} = (4, -2, 5, 6)$.

Solution: For each question we need to write $\boldsymbol{v}$ as a linear combination of the basis vectors, and the coefficients (in the right order) will yield the coordinates. Note the solution will always be unique. If it is not either you made a mistake solving the system or $\mathcal{B}$ was not a basis.

(1) $(4, -2, 5, 6) = \alpha_1(-2, 1, 0, 0) + \alpha_2(0, 0, 1, 0) + \alpha_3(0, 0, 1, 1)$ is a system with augmented matrix

$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & 4 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

which has unique solution $\alpha_1 = -2, \alpha_2 = -1, \alpha_3 = 6$. So the coordinates are $(\boldsymbol{v})_{]\mathcal{B}} = (-2, -1, 6)$.

(2) Similarly we get the system

$$\left[\begin{array}{cc|c} -1 & 0 & 4 \\ 1 & 1 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

and so $(\boldsymbol{v})_{\mathcal{B}} = (-4, 2)$.

(3) We immediately see $(4, -2, 5, 6) = -2(0, 1, 0, 0) + 5(0, 0, 1, 0) + 6(0, 0, 0, 1) + 4(1, 0, 0, 0)\}$ and so $(\boldsymbol{v})_{\mathcal{B}} = (-2, 5, 6, 4)$.

Exercise 4. (i) Let W be the plane in $\mathbb{R}^3$ defined by the equation

$$4x + y - 3z = 0.$$

Find a basis B for this subspace of $\mathbb{R}^3$.

SOLUTION: There are lots of possible bases, we will display one. We have $W = \{(x, 3z - 4x, z) | x, z \in \mathbb{R}\}$. Observe that $(x, 3z - 4x, z) = x(1, -4, 0) + z(0, 3, 1)$. So $\{(1, -4, 0), (0, 3, 1)\}$ spans W and is linearly independent: it is a basis B.

(ii) Show that the vector $\boldsymbol{v} = (2, 1, 3)$ lies in W and express it in terms of the basis B.

SOLUTION: We check that the vector $\boldsymbol{v}$ satisfies the defining equation for W: $4.2 + 1 - 3.3 = 0$. Thus we can write $\boldsymbol{v} = \alpha(1, -4, 0) + \beta(0, 3, 1)$, with $\alpha = 2$ and $\beta = 3$. So $[\boldsymbol{v}]_B = (2, 3)$. Note that the answer will depend on the choice of basis B.

(iii) If $(\boldsymbol{w})_B = (-2, 4)$ then what is $(\boldsymbol{w})_S$ (where S is the standard basis of $\mathbb{R}^3$)?

SOLUTION: $\boldsymbol{w} = -2(1, -4, 0) + 4(0, 3, 1) = (-2, 20, 4)$, so $(\boldsymbol{w})_S = (-2, 20, 4)$. Note that the answer you get here for $(\boldsymbol{w})_S$ will depend on your choice of basis B.

EXERCISE 5. Find a basis for the subspace

$$S_1 = \{(x, y, z) : x + y + z = 0\}$$

of $\mathbb{R}^3$ and hence determine its dimension.

SOLUTION: Note that S_1 is a plane containing $\mathbf{0}$, which is 2-dimensional and spanned by any two vectors in the plane that are not scalar multiples of each other, say $(-1, 1, 0)$ and $(-1, 0, 1)$. Thus $\{(-1, 1, 0), (-1, 0, 1)\}$ is a basis. There are many other possible correct answers.

EXERCISE 6. For each of the following matrices, find a basis for its row space and a basis for its column space.

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 2 & 5 & 12 \end{bmatrix}$$

SOLUTION: First we tackle the *row spaces*:

Elementary row operations do not alter the row space of a matrix, and so we can reduce each matrix to row echelon form. For the first two matrices, the only ERO needed is $R_2 \leftarrow R_2 - 2R_1$, while for the third matrix the required EROs are $R_3 \leftarrow R_3 - 2R_1$ and then $R_3 \leftarrow R_3 - R_2$. After this, the three row echelon form matrices are:

$$\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

The non-zero rows of the matrix in row-echelon form are a basis for the rowspace, and so the rowspaces have bases given by

$$B_1 = \{(1, 3)\} \qquad B_2 = \{(1, 1), (0, -3)\} \qquad B_3 = \{(1, 2, 4), (0, 1, 3), (0, 0, 1)\}$$

Other method: start with the set of row vectors, check for linear independence. If linearly independent, we have a basis. Otherwise remove a dependent vector and start again.
Next we tackle the *column spaces*: While elementary row operations do not change the row

space, they *do* change the column space. So we cannot use the matrices in row echelon form directly. However we can use some of the information obtained from the first part of the question.
The rank of the first matrix is 1, and so its column space has dimension 1 and so any non-zero column will be a basis. The second matrix has rank 2, and so its columns are linearly independent and therefore form a basis for the column space. The third matrix has rank 3 and therefore its columns are also independent.
Hence bases for the three column spaces are

$$C_1 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}, \quad C_2 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \quad C_3 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 12 \end{bmatrix} \right\}$$

Note these bases are not unique. If you found another basis with the same number of elements, it could be a perfectly correct answer.
Other method: transpose the matrix, then find the row space of that new matrix using row operations.

EXERCISE 7. Find a basis for the row space and a basis for the column space for the matrix

$$A = \begin{bmatrix} 1 & -1 & 3 & 0 \\ 5 & -4 & -4 & 1 \\ 7 & -6 & 2 & 1 \end{bmatrix}$$

Then determine rank(A) and nullity(A). (Hint: You do not need to determine the nullspace to find nullity(A).)

SOLUTION:

$$A = \begin{bmatrix} 1 & -1 & 3 & 0 \\ 5 & -4 & -4 & 1 \\ 7 & -6 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 1 & -19 & 1 \\ 0 & 1 & -19 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 1 & -19 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, $\{(1, -1, 3, 0), (0, 1, -19, 1)\}$ is a basis for the row space of A. Hence rank(A) = 2. By Theorem 3.12 in the lecture notes, the dimension of the column space is two, so any two linearly independent columns of A will form a basis for the column space of A. For example, the first two columns of A form a basis for the column space. By the Rank-Nullity Theorem, nullity(A) = $\#columns$ − rank(A) = 4 − 2 = 2.

EXERCISE 8. Find a basis for the null space of the matrix

$$A = \begin{bmatrix} 1 & -1 & 3 & 0 \\ 5 & -4 & -4 & 1 \\ 7 & -6 & 2 & 1 \end{bmatrix}$$

SOLUTION: We must solve the SLE given by $A\boldsymbol{x} = 0$, where $\boldsymbol{x}$ is a column vector of 3 variables. From the previous question we know the row echelon form of the augmented matrix and we can easily get the reduced row echelon form.

$$\left[\begin{array}{cccc|c} 1 & -1 & 3 & 0 & 0 \\ 0 & 1 & -19 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -16 & 1 & 0 \\ 0 & 1 & -19 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore the solution space has x_3, x_4 as free variables, and we get $x_1 = 16x_3 - x_4, x_2 = 19x_3 - x_4$. Thus the solution space is equal to

$$S = \{(16x_3 - x_4, 19x_3 - x_4, x_3, x_4) : x_3, x_4 \in \mathbb{R}\}$$

and hence it is a 2-dimensional space with basis $\{(16, 19, 1, 0), (-1, -1, 0, 1)\}$.
Remark: from the row echelon form, we could have just chosen x_2 and x_3 as free variables, to get the basis $\{(1, 1, 0, -1), (-3, 0, 1, 19)\}$.

Conceptual understanding

EXERCISE 9. We saw in Prac 2 that $\{(x_1, x_2, \ldots, x_n) \mid x_1 + x_2 + \cdots + x_{n-1} = x_n\}$ is a subspace of $\mathbb{R}^n$. Find a basis for this subspace.

SOLUTION: Let $S := \{(x_1, x_2, \ldots, x_n) \mid x_1 + x_2 + \cdots + x_{n-1} = x_n\}$. An element in S can be written as $(x_1, x_2, \ldots, x_{n-1}, x_1 + x_2 + \cdots + x_{n-1}) = x_1(1, 0, \ldots, 0, 1) + x_2(0, 1, 0, \ldots, 0, 1) + \cdots + x_{n-1}(0, 0, \ldots, 0, 1, 1)$. So a spanning set is $\{\boldsymbol{e_i} + \boldsymbol{e_n} | i < n\}$. We now prove it is linearly independent. Suppose $\sum_{i=1}^{n-1} x_i(\boldsymbol{e_i} + \boldsymbol{e_n}) = \mathbf{0}$. It is equivalent to $(x_1, x_2, \ldots, x_{n-1}, x_1 + x_2 + \cdots + x_{n-1}) = (0, 0, \ldots, 0)$, so $x_1 = x_2 = \cdots = x_{n-1} = 0$. Thus it is a linearly independent set and hence a basis.

EXERCISE 10. Suppose that A is a matrix and that A' is a matrix in row-echelon form obtained from A by elementary row operations where

$$A' = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We are interested in what can be deduced about A if you are *only given* the matrix A'. (For example, we can definitely say that A has 3 rows and 5 columns). Answer each of the following questions about A if it is possible to answer just from knowing A' or explain why it is not possible to answer that particular question.

(1) Is the vector $(1, 1, 1, 1, 1)$ in the row space of A?
(2) Is the vector $(1, 0, 0)$ in the column space of A?
(3) What is the rank of A?
(4) What is the nullity of A?
(5) Find a basis for the row space of A
(6) Find a basis for the column space of A
(7) Find a basis for the null space of A

SOLUTION:

(1) The row space of A is equal to the row space of A' and so we can answer the first part by solving the SLE

$$(1, 1, 1, 1, 1) = \alpha(1, 2, 0, 0, 3) + \beta(0, 0, 1, 2, 1)$$

which is inconsistent. Thus $(1, 1, 1, 1, 1)$ is not in the row space of A' and hence not in the row space of A.
(2) We cannot tell from A' alone anything about the column space of A. The EROs do not alter the row space, but they definitely alter the column space.

(3) The rank of A is the dimension of its row space, which is equal to the dimension of the row space of A' which is 2.
(4) The nullity of A can be determined from the rank. As rank + nullity is equal to 5, the nullity is equal to 3.
(5) The row space of A has basis $\{(1, 2, 0, 0, 3), (0, 0, 1, 2, 1)\}$.
(6) We cannot determine a basis for the column space of A because the EROs alter it.
(7) EROs do not alter the null space of a matrix, and so we can determine the null space from the row echelon form. In particular, we see that x_1 and x_3 are basic variables, while x_2, x_4 and x_5 are free variables. From the second equation we can tell that $x_3 + 2x_4 + x_5 = 0$ and so $x_3 = -2x_4 - x_5$. From the first equation we see that $x_1 + 2x_2 + 3x_5 = 0$ and so $x_1 = -2x_2 - 3x_5$. Hence the solution set is

$$S = \{(-2x_2 - 3x_5, x_2, -2x_4 - x_5, x_4, x_5) : x_2, x_4, x_5 \in \mathbb{R}\}$$

and so the basis is

$$B = \{(-2, 1, 0, 0, 0), (0, 0, -2, 1, 0), (-3, 0, -1, 0, 1)\}$$

where these three vectors were obtained by extracting all the x_2 terms, then all the x_4 terms and finally all the x_5 terms.

Exercise 11. Suppose that A is an arbitrary 4×6 matrix. Explain why the columns of A are linearly dependent.

Solution: The rank of the matrix is at most 4 since the matrix has 4 rows. There are 6 column vectors, which span a subspace of dimension also at most 4, and so they are linearly dependent.

Exercise 12. Let A be an $n \times n$ matrix of rank n. Let S be a subset of the set of column vectors. Is S linearly independent?

Solution: Since the rank of A is equal to the number of columns, the n column vectors are linearly independent. Thus any subset of the column vectors is also linearly independent.

Exercise 13. Show that for same-sized square matrices the formula $(A-B)(A+B) = A^2 - B^2$ does not necessarily hold. (Find a 2×2 counter-example). Explain why this is so.

Solution: Lots of solutions possible: for instance take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then

$$(A - B)(A + B) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

while

$$A^2 - B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

We have $(A - B)(A + B) = A^2 + AB - BA - B^2$ by distributivity. Since matrices do not commute, we cannot eliminate the terms AB and $-BA$.

MATH1012 (Semester 1, 2024)

Practice Class 4: Inverses, determinants and linear transformations

Summary of what you have learnt

The inverse of an $n \times n$ matrix (if it exists) can be found using a super-augmented matrix and Gauss-Jordan elimination.
The determinant of an $n \times n$ matrix can be determined by expanding along any row or column. However it is far more efficient to use elementary row operations which only change the determinant in a controlled fashion.
Linear transformations are functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ preserving vector addition and scalar multiplication, that is:

(1) $f(\boldsymbol{u}+\boldsymbol{v}) = f(\boldsymbol{u}) + f(\boldsymbol{v})$ for all $\boldsymbol{u}, \boldsymbol{v}$ in $\mathbb{R}^n$;
(2) $f(\alpha \boldsymbol{v}) = \alpha f(\boldsymbol{v})$ for all $\boldsymbol{v}$ in $\mathbb{R}^n$ and all α in $\mathbb{R}$.

The *range* of a linear transformation f is the set $\{f(\boldsymbol{u}) \mid \boldsymbol{u} \in \mathbb{R}^n\}$, while the *kernel* of a linear transformation f is the set $\{\boldsymbol{u} \in \mathbb{R}^n \mid f(\boldsymbol{u}) = \boldsymbol{0}\}$.
The rank-nullity theorem for maps is

$$\dim(\text{range}(f)) + \dim(\text{Ker}(f)) = \dim(\text{domain}(f)) = n.$$

Foundational questions

Exercise 1. If possible, find the inverses of

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution: Form the matrix $[A \mid I]$ and then perform Gauss-Jordan elimination to obtain the matrix in reduced row-echelon form. If the left half of the matrix is the identity, then the right half is $[I \mid A^{-1}]$, otherwise the matrix is not invertible. The first two have inverses

$$A^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

but the third one is not invertible. Note that you can check your answer by multiplying the inverse with the original matrix and check you indeed get the identity matrix.Here are the details for the first one.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \; R_2 \leftarrow R_2 - R_1$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \end{array}\right] \; R_1 \leftarrow R_1 - 2R_2$$

Exercise 2.
If A, B, C are invertible, show that ABC is invertible by finding its inverse.

Solution: It is immediate that

$$(ABC)^{-1} = C^{-1}B^{1-}A^{-1}$$

because just multiplying ABC by $C^{-1}B^{-1}A^{-1}$ yields the identity.

Exercise 3. What are the determinants of the following matrices?

$$A_1 = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 1 & 3 & 2 \end{pmatrix} \qquad A_3 = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Solution: The determinant formula gives $\det A_1 = 1 \times (-1) - 2 \times 3 = -7$.
For the second perform the elementary row operation $R3 \leftarrow R3 - R1$, which does not change the determinant, to get

$$\det A_2 = \det \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = (1)(-1)(1) = -1$$

For A_3, we merely need multiply the diagonal entries to get -10.

Exercise 4. If A is an invertible matrix with determinant 5, then what are $\det(A^2)$, $\det(A^T)$ and $\det(A^{-1})$?

Solution: We know that $\det(AB) = \det(A)\det(B)$ and therefore

$$\det(A^2) = \det(A)^2 = 25.$$

The determinant of the transpose of a matrix is the same as the determinant of the matrix, so $\det(A^T) = 5$. This follows because expanding along a row of A will give exactly the same results as expanding along a column of A^T.
Finally we know that $AA^{-1} = I_n$ and so

$$\det(A)\det(A^{-1}) = \det(I_n) = 1$$

and hence $\det(A^{-1}) = 1/\det(A) = 1/5$.

Exercise 5. Suppose that A has two identical rows, what can you say about $\det(A)$? What about if it has two identical columns?

Solution: Say $R_i = R_j$. Then, performing the ERO $R_i \leftarrow R_i - R_j$ does not change the determinant, but has a row R_i of zeroes. Expanding along that row, we see that the determinant is 0. If A has two identical columns, then A^T has two identical rows, so $\det(A^T) = 0$. We know that $\det(A^T) = \det(A)$, so $\det(A) = 0$.

EXERCISE 6. Are the following functions linear transformations?

(i) $\begin{array}{rccc} f_1: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (x,y) & \longmapsto & (x,-y) \end{array}$ (ii) $\begin{array}{rccc} f_2: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & |x+y| \end{array}$

(iii) $\begin{array}{rccc} f_3: & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & (x,y,z) & \longmapsto & (-y,x,z) \end{array}$ (iv) $\begin{array}{rccc} f_4: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & \begin{cases} \frac{x^2}{y} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases} \end{array}$

SOLUTION:

(i) Yes:

$$\begin{aligned} f_1((x,y)+(x',y')) &= f_1(x+x',y+y') = (x+x',-(y+y')) \\ &= (x,-y)+(x',-y') = f_1(x,y)+f_1(x',y') \end{aligned}$$

and

$$f_1(\alpha(x,y)) = f_1(\alpha x,\alpha y) = (\alpha x,-\alpha y) = \alpha(x,-y) = \alpha f_1(x,y).$$

(ii) No, it doesn't satisfy the second condition:

$$f_2((-1)(1,1)) = f_2(-1,-1) = 2 \qquad \text{while} \qquad (-1)f_2(1,1) = (-1)2 = -2.$$

You can also show it does not satisfy the first condition (but note that you only have to show one of them, not both).

(iii) Yes:

$$\begin{aligned} f_3((x,y,z)+(x',y',z')) &= f_1(x+x',y+y',z+z') = (-(y+y'),x+x',z+z') \\ &= (-y,x,z)+(-y',x',z') = f_3(x,y,z)+f_3(x',y',z') \end{aligned}$$

and

$$f_3(\alpha(x,y,z)) = f_3(\alpha x,\alpha y,\alpha z) = (-\alpha y,\alpha x,\alpha z) = \alpha(-y,x,z) = \alpha f_3(x,y,z).$$

(iv) No. f_4 doesn't preserve addition:

$$f_4((1,1)+(0,1)) = f_4(1,2) = 1/2 \qquad \text{while} \qquad f_4(1,1)+f_4(0,1) = 1+0 = 1.$$

Note that f_4 does preserve scalar multiplication!

EXERCISE 7. Let $f:\mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the linear transformation which maps $\boldsymbol{e_1}$ to $(1,1,2)$ and maps $\boldsymbol{e_2}$ to $(0,1,1)$. What is the image of $(1,2)$ under f? What is the formula for f (in other words, what is $f(x,y)$?

SOLUTION: We use the linearity of the function. The image of $(1,2)$ under f_5 is

$$f(1,2) = f(1,0)+f(0,2) = f(1,0)+2f(0,1) = (1,1,2)+2(0,1,1) = (1,3,4)$$

and the formula for f_5

$$f(x,y) = f(x,0)+f(0,y) = xf(1,0)+yf(0,1) = x(1,1,2)+y(0,1,1) = (x,x+y,2x+y).$$

EXERCISE 8. Find a basis for the range and a basis for the kernel of each of the following linear transformations (you do not need to show that they are linear transformations). These linear transformations have domain $\mathbb{R}^3$ and codomain $\mathbb{R}^3$.

$$f:(x,y,z) \to (x+y,y+z,x+z) \qquad\qquad g:(x,y,z) \to (x,x+z,z)$$

Solution: It is easiest to find a basis for the kernel first. This consists in solving a homogeneous system of linear equations. For f the system is $x+y=0, y+z=0, x+z=0$ which corresponds to the matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \quad \sim \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

so the kernel consists of the zero vector and has basis $\emptyset$ and dimension 0. Thus by the rank-nullity theorem, the range has dimension $\dim(domain) - \dim(kernel) = 3$. The range is spanned by $f(1,0,0)$, $f(0,1,0)$ and $f(0,0,1)$, so we know these three vectors will form a basis for the range: $\{(1,0,1),(1,1,0),(0,1,1)\}$.
For g: the system corresponds to

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \sim \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so the kernel consists of the vectors $(0,y,0)$ and has basis $\{(0,1,0)\}$ and dimension 1.
Thus by the rank-nullity theorem, the range has dimension $\dim(domain) - \dim(kernel) = 2$. The range is spanned by $g(1,0,0)$, $g(0,1,0)$ and $g(0,0,1)$ and we know two vectors will form a basis for the range and one will be redundant: out of $\{(1,1,0),(0,0,0),(0,1,1)\}$, it is obvious the second one is redundant so a basis for the range is $\{(1,1,0),(0,1,1)\}$.
Alternate method:
In each case, the range is spanned by $f(1,0,0)$, $f(0,1,0)$ and $f(0,0,1)$ and so to find a basis for the range, we must simply find a linearly independent subset of those vectors.
For f, the range is spanned by

$$\{(1,0,1),(1,1,0),(0,1,1)\}$$

which is linearly independent, since the system $\alpha_1(1,0,1)+\alpha_2(1,1,0)+\alpha_3(0,1,1) = (0,0,0)$, which is equivalent to $\alpha_1+\alpha_2 = 0, \alpha_2+\alpha_3 = 0, \alpha_1+\alpha_3 = 0$ which only has the zero solution. So a basis for the range is $\{(1,0,1),(1,1,0),(0,1,1)\}$. Since $\dim(range)+\dim(kernel) = 3$, the kernel has dimension 0, and so the empty set is the basis.
For g the range is spanned by

$$\{(1,1,0),(0,0,0),(0,1,1)\}$$

and clearly the second vector is dependent on the others, while the other two are linearly independent since they are not multiples of each other. Hence a basis for the range is

$$\{(1,1,0),(0,1,1)\}$$

The kernel is therefore 1-dimensional and it is clear that $(0,1,0)$ is in the kernel. Thus a basis for the kernel is $\{(0,1,0)\}$.

Conceptual understanding

Exercise 9. Let A, B be two $n \times n$ matrices. Is it true that $|A+B| = |A| + |B|$?

Solution: We can easily find a counter-example, for instance:

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\qquad |A+B| = 0$ $\qquad |A|+|B| = 2$

Exercise 10. For which value(s) of k is the following matrix A invertible?

$$A = \begin{bmatrix} 1 & 0 & 0 & k \\ 0 & -k & 1 & -1 \\ 0 & -1 & k & k \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

Hint: compute the determinant, starting with some row operations; do not rush into using the determinant formula.

Solution: The easiest method is to calculate the determinant, and see for which k it is non zero. Thus we need the answer to be **factorised** so that we can easily tell for which k it is equal to 0.

$$\det(A) = \begin{vmatrix} 1 & 0 & 0 & k \\ 0 & -k & 1 & -1 \\ 0 & -1 & k & k \\ 1 & 0 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & k \\ 0 & 0 & 1-k^2 & -1-k^2 \\ 0 & -1 & k & k \\ 0 & 0 & 0 & 2-k \end{vmatrix} \begin{matrix} \\ R_2 \leftarrow R_2 - kR_3 \\ \\ R_4 \leftarrow R_4 - R_1 \end{matrix}$$

We now expand along the first column then the third row (because of all the zeroes):

$$\det(A) = \begin{vmatrix} 0 & 1-k^2 & -1-k^2 \\ -1 & k & k \\ 0 & 0 & 2-k \end{vmatrix} = (2-k) \begin{vmatrix} 0 & 1-k^2 \\ -1 & k \end{vmatrix}$$

$$= (2-k)(1-k^2) = (2-k)(1-k)(1+k).$$

So the matrix is always invertible except for $k \in \{2, 1, -1\}$.

Exercise 11. Show that if $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is linear and invertible, then the inverse transformation f^{-1} is also linear.

Solution: Since f is invertible, $f^{-1} : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is well-defined. We have to show it preserves vector addition and scalar multiplication.

Vector addition: Let $\boldsymbol{u}, \boldsymbol{v}$ in $\mathbb{R}^m$.

$$\begin{aligned} f^{-1}(\boldsymbol{u}) + f^{-1}(\boldsymbol{v}) &= f^{-1}(f(f^{-1}(\boldsymbol{u}) + f^{-1}(\boldsymbol{v}))) \\ &= f^{-1}(f(f^{-1}(\boldsymbol{u})) + f(f^{-1}(\boldsymbol{v}))) \text{ since } f \text{ is linear} \\ &= f^{-1}(\boldsymbol{u} + \boldsymbol{v}). \end{aligned}$$

Scalar multiplication: Let $\boldsymbol{v}$ in $\mathbb{R}^m$ and $\alpha \in \mathbb{R}$.

$$\begin{aligned} \alpha f^{-1}(\boldsymbol{u}) &= f^{-1}(f(\alpha f^{-1}(\boldsymbol{u}))) \\ &= f^{-1}(\alpha f(f^{-1}(\boldsymbol{u}))) \text{ since } f \text{ is linear} \\ &= f^{-1}(\alpha \boldsymbol{u}). \end{aligned}$$

MATH1012 (Semester 1, 2024)

Practice Class 5: Matrix of a linear transformations, change of basis, eigenvalues, eigenvectors, diagonalisation

Summary of what you have learnt

- Matrix multiplication, say $\boldsymbol{u} \to A\boldsymbol{u}$, is a linear transformation.
- Any linear transformation can be represented by a matrix
- The matrix depends on the choice of bases for the domain and codomain.
- Composition of linear transformations corresponds to matrix multiplication.
- Inverse transformation, if it exists, corresponds to inverse matrix.
- Transition matrix P_{CB} changes coordinates from B to C, that is, $\boldsymbol{v}_C = P_{CB}\boldsymbol{v}_B$.
- Transition matrices are invertible, $P_{BC} = P_{CB}^{-1}$.
- It can also be used to change basis for the matrix of a linear transformation:
$$[f]_{C'B'} = P_{C'C}[f]_{CB}P_{BB'}.$$
- Eigenvalues are solutions to the characteristic equation $\det(A - \lambda I) = 0$.
- Eigenspace E_λ is the solution of the system of linear equations $(A - \lambda I)\boldsymbol{v} = \mathbf{0}$.
- Sometimes we can find a matrix P such that $P^{-1}AP$ is a diagonal matrix.
- From the eigenvalues of A, we can deduce the eigenvalues of A^2, A^{-1}, etc.
- The trace of A is the sum of its eigenvalues.
- The determinant of A is the product of its eigenvalues.
- Algebraic and geometric multiplicity of eigenvalues.

Foundational questions

Exercise 1. For each of the below linear maps, give its standard matrix along with a *geometric description* (that is, in terms of rotations, reflections, stretching etc).

$$f : \mathbb{R}^2 \to \mathbb{R}^2 : \ (x, y) \mapsto (x, -y) \qquad\qquad g : \mathbb{R}^3 \to \mathbb{R}^3 : \ (x, y, z) \mapsto (-y, x, z)$$

Solution: The standard matrices are

$$[f]_{SS} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad\qquad [g]_{SS} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

f fixes $\boldsymbol{e_1}$ and maps $\boldsymbol{e_2}$ to $-\boldsymbol{e_2}$: it is a reflection through the x-axis.
g fixes $\boldsymbol{e_3}$ and maps $\boldsymbol{e_1}$ to $\boldsymbol{e_2}$ and $\boldsymbol{e_2}$ to $-\boldsymbol{e_1}$: it is a rotation of angle 90 degrees around the z-axis (it helps to draw a picture).

Exercise 2. Are the above linear maps invertible?

Solution: Yes. In both cases the inverse matrices exist. After much calculation (or a bit of thinking about the geometry) we get

$$[f]_{SS} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad\qquad [g]_{SS} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

EXERCISE 3. Let $f : \mathbb{R}^2 \to \mathbb{R}^3 : (x, y) \mapsto (x, x + y, 2x + y)$.
(a) Find its standard matrix.
(b) Find its matrix with respect to the basis B of $\mathbb{R}^2$ and the basis C of $\mathbb{R}^3$, where

$$B = \{(1, 1), (1, -1)\} \qquad\qquad C = \{(1, 0, 1), (0, 1, 1), (1, 1, 0)\}$$

SOLUTION: Since $f : \mathbb{R}^2 \to \mathbb{R}^3$, both matrices are going to have size 3×2.
(a) f maps $\boldsymbol{e_1}$ to $(1, 1, 2)$ and $\boldsymbol{e_2}$ to $(0, 1, 1)$ so the standard matrix is:

$$A_{SS} = [f]_{SS} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$$

(b) We need to first compute the images of the basis B. We have

$$f(1, 1) = (1, 2, 3) \qquad\qquad f(1, -1) = (1, 0, 1).$$

We now need to express these vectors in coordinates with respect to C:

$$(1, 2, 3)_C = (1, 2, 0) \qquad\qquad (1, 0, 1)_C = (1, 0, 0).$$

These are easy to find in this case: if not we'd need to solve the systems

$$\alpha_1(1, 0, 1) + \alpha_2(0, 1, 1) + \alpha_3(1, 1, 0) = (1, 2, 3)$$
$$\alpha_1(1, 0, 1) + \alpha_2(0, 1, 1) + \alpha_3(1, 1, 0) = (1, 0, 1)$$

to find the coordinates. From above, we have

$$A_{CB} = [f]_{CB} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}$$

EXERCISE 4. (a) Find the change of basis matrix P_{CB} if B and C are the following $\mathbb{R}^2$ bases:

$$B = \{(11, 4), (9, 8)\} \qquad\qquad C = \{(1, -2), (5, 3)\}$$

SOLUTION: We need to write each element of B in terms of the elements of C. First we solve $(11, 4) = \alpha_1(1, -2) + \alpha_2(5, 3)$ and get solution $\alpha_1 = 1,\ \alpha_2 = 2$. Next we solve $(9, 8) = \beta_1(1, -2) + \beta_2(5, 3)$ and get solution $\beta_1 = -1,\ \beta_2 = 2$. Thus

$$P_{CB} = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}.$$

(b) If $(\boldsymbol{v})_B = (3, 1)$ then what is $(\boldsymbol{v})_C$?
If $(\boldsymbol{w})_C = (1, 5)$ then what is $(\boldsymbol{w})_B$?

SOLUTION:

$$(\boldsymbol{v})_C = P_{CB}\,(\boldsymbol{v})_B = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

and

$$(\boldsymbol{w})_B = P_{BC}\,(\boldsymbol{w})_C = P_{CB}^{-1}\,(\boldsymbol{w})_C = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(c) Find the change of basis matrix P_{SC} where S is the standard basis in $\mathbb{R}^2$.

SOLUTION: It is easy to write each element of C as a linear combination of the standard basis vectors so we have

$$P_{SC} = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}.$$

(d) Let f be the linear map with standard matrix $[f]_{SS} = \begin{bmatrix} [0 & 5 \\ 6 & -7 \end{bmatrix}$. Find the matrix $[f]_{CC}$.

SOLUTION:

$$[f]_{CC} = P_{CS}[f]_{SS}P_{SC} = P_{SC}^{-1}[f]_{SS}P_{SC} = \frac{1}{13}\begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 5 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} 1 & 5 & -2 & 3 \end{bmatrix} = \begin{bmatrix} -10 & 0 \\ 0 & 3 \end{bmatrix}.$$

EXERCISE 5. (a) Find the eigenvalues and a basis for the corresponding eigenspaces of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

SOLUTION:

$$(A - \lambda I) = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 1 & 2 & -\lambda \end{bmatrix} = -\lambda(1-\lambda)^2 \qquad \Rightarrow \qquad \lambda = 0, 1.$$

In fact the matrix is lower triangular so can just read the eigenvalues off the diagonal. The eigenspace corresponding to $\lambda = 0$ is the solution of $A\boldsymbol{v} = \mathbf{0}$. EROs give

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \boldsymbol{v} = \{(0,0,z) \mid z \in \mathbb{R}\} \longrightarrow E_0 = \operatorname{span}\{(0,0,1)\}$$

The eigenspace corresponding to $\lambda = 0$ is the solution of $(A - I)\boldsymbol{v} = \mathbf{0}$. That is,

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & -1 & 0 \end{bmatrix} \longrightarrow \boldsymbol{v} = \{(x, y, x{+}2y) \mid x, y \in \mathbb{R}\} \longrightarrow E_1 = \operatorname{span}\{(1,0,1),(0,1,2)\}$$

Happily, we didn't need to do any EROs for this eigenvalue.

(b) Determine the algebraic and geometric multiplicities of each eigenvalue.

SOLUTION: From the characteristic polynomial, $\lambda = 0$ has algebraic multiplicity 1, and $\lambda = 1$ has algebraic multiplicity 2. From the dimension of the eigenspaces, we see that the geometric multiplicities are the same values. Thus this matrix is diagonisable.

(c) If possible, find a basis B consisting entirely of eigenvectors of A.

SOLUTION: The basis elements of the eigenspaces provide the basis elements of B:

$$B = \{(0,0,1),(1,0,1),(0,1,2)\}.$$

(d) Let P be the transition matrix P_{SB}. Determine $P^{-1}AP$ without computing the product.

SOLUTION: P is the matrix whose columns are eigenvectors of A, and $P^{-1}AP = D$, the diagonal matrix of corresponding eigenvalues. Hence

$$P = P_{SB} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If you chose a different order for basis B, you might get a different diagonal matrix. We can think of A as the matrix of a linear map with respect to the standard basis:

$$P^{-1}AP = P_{SB}^{-1} A_{SS} P_{SB} = P_{BS} A_{SS} P_{SB} = A_{BB}.$$

(e) Write down the eigenvalues of A^T, A^{42} and $A + 3I$.

SOLUTION:

$$\begin{array}{ll} A^T & \text{has eigenvalues} \quad 0 \quad \text{and} \quad 1. \\ A^{42} & \text{has eigenvalues} \quad 0^{42} \quad \text{and} \quad 1^{42}, \quad \text{that is,} \quad 0 \quad \text{and} \quad 1. \\ A + 3I & \text{has eigenvalues} \quad 0 + 3 \quad \text{and} \quad 1 + 3, \quad \text{that is,} \quad 3 \quad \text{and} \quad 4. \end{array}$$

EXERCISE 6. The matrix

$$A = \begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$$

has just the one real eigenvalue $\lambda = 1$. Determine the dimension of its corresponding eigenspace.

SOLUTION: We need to solve the homogeneous system

$$(A - \lambda I)\boldsymbol{v} = \mathbf{0} \qquad \Rightarrow \qquad (A - I)\boldsymbol{v} = \mathbf{0} \qquad \text{where} \qquad A - I = \begin{bmatrix} -4 & -7 & -5 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

After applying EROs to the augmented matrix we get the reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solution of this system, eigenspace E_1 has basis $\{(-3, 1, 1)\}$, and so has dimension 1.

EXERCISE 7. The following matrix is symmetric, and hence diagonalisable.

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}.$$

It can be shown that its characteristic equation is $-\lambda(-3 - \lambda)^2 = 0$.

(a) Find a basis for each corresponding eigenspace.

Solution: The eigenvalues are $\lambda = 0$ and $\lambda = -3$. Finding E_0 corresponds to solving the homogeneous system with matrix A. That is,

$$\begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow E_0 = \text{span}\{(1,1,1)\}.$$

Finding E_{-3} corresponds to solving the homogeneous system with matrix $A + 3I$:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow E_{-3} = \text{span}\{(-1,1,0),(-1,0,1)\}.$$

(b) A computer package determined that its eigenvalues were $\lambda = 0, -3, -3$ and that

$$E_0 = \text{span}\{(1,1,1)\} \qquad \text{and} \qquad E_{-3} = \text{span}\{(-5,1,4),(1,-3,2)\}.$$

Is this answer consistent with your results from part (a)?

Solution: Yes. E_{-3} stays the same, we just have two different bases for it. Indeed

$$(-5,1,4) = (-1,1,0) + 4(-1,0,1) \in \text{span}\{(-1,1,0),(-1,0,1)\}$$

and

$$(1,-3,2) = -3(-1,1,0) + 2(-1,0,1) \in \text{span}\{(-1,1,0),(-1,0,1)\},$$

so

$$\text{span}\{(-5,1,4),(1,-3,2)\} \subseteq \text{span}\{(-1,1,0),(-1,0,1)\}.$$

Since both spans have the same dimension, they must be equal.

Conceptual understanding

Exercise 8. (a) Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 1 \\ -5 & 4 \end{bmatrix}$$

Note: This is harder than it seems — the answers involve complex numbers

Solution: The characteristic equation is

$$\lambda^2 - 6\lambda + 13 = 0$$

which has roots $\lambda = 3 \pm 2i$ which are the two complex eigenvalues. An eigenvector for $\lambda_1 = 3 + 2i$ is $(1, 1+2i)$, and an eigenvector for $\lambda_2 = 3 - 2i$ is $(1, 1-2i)$.

(b) Find $\det(A)$ and $\text{trace}(A)$ (the sum of the diagonal elements). Also, find the product and sum of the eigenvalues of A.

Solution:

$$\det(A) = (2)(4) - (1)(-5) = 8 + 5 = 13, \qquad \lambda_1\lambda_2 = (3+2i)(3-2i) = 9 + 4 = 13$$

and

$$\text{trace}(A) = 2 + 4 = 6, \qquad \lambda_1 + \lambda_2 = (3+2i) + (3-2i) = 6.$$

EXERCISE 9. Determine the standard matrices of each of the following linear transformations, and compute their determinants:

(1) an anticlockwise rotation of angle θ around the origin.
(2) (Harder) a reflection in a line through the origin forming an angle $\frac{\theta}{2}$ with the x-axis.

SOLUTION:

(1)

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{which has determinant} \quad \cos(\theta)^2 + \sin(\theta)^2 = 1.$$

(2)

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \quad \text{which has determinant} \quad -\cos(\theta)^2 - \sin(\theta)^2 = -1.$$

EXERCISE 10. Use the matrices found in the previous exercise to determine:

(1) The composition of a rotation of angle θ_1 and a rotation of angle θ_2, around the origin.
(2) The composition of a reflection with itself.

Hint: You may need the trig identities:

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

SOLUTION:

(1)

$$\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 \end{bmatrix}$$
$$= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}.$$

So the composition is a rotation of angle $\theta_1 + \theta_2$.

(2)

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So composing a reflection by itself yields the identity.

EXERCISE 11. Show that if λ is an eigenvalue of A then $\lambda + k$ is an eigenvalue of $A + kI$.

SOLUTION: By hypothesis there exists $\boldsymbol{v} \neq \mathbf{0}$ such that $A\boldsymbol{v} = \lambda\boldsymbol{v}$. Then

$$(A+kI)\boldsymbol{v} = A\boldsymbol{v} + kI\boldsymbol{v} = \lambda\boldsymbol{v} + k\boldsymbol{v} = (\lambda+k)\boldsymbol{v}.$$

Thus $\lambda + k$ is an eigenvalue of $A + kI$.

EXERCISE 12. Suppose that $\lambda = 0$ is an eigenvalue of B. Show that B is not invertible, and show that the eigenspace E_0 is the nullspace of B.

SOLUTION: Since 0 is an eigenvalue of B, we have that $\det(B - 0I) = \det(B) = 0$. Therefore B is not invertible. Moreover E_0 is the solution of the system $(B - 0I)\boldsymbol{v} = \mathbf{0}$, that is, it is exactly the nullspace of B.

EXERCISE 13. Let B be a basis of $\mathbb{R}^n$, S the standard basis of $\mathbb{R}^n$ and f a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let $[f]_{BB}$, $[f]_{SS}$ be the matrices of f with respect to the respective bases.

(a) Show that $[f]_{SS}$ and $[f]_{BB}$ have the same characteristic polynomial.

SOLUTION:

$$[f]_{BB} = P_{SB}^{-1}[f]_{SS}P_{SB} = P^{-1}AP \quad \text{where} \quad A = [f]_{SS} \quad \text{and} \quad P = P_{SB}.$$

Therefore

$$\begin{aligned}
\det([f]_{BB} - \lambda I) &= \det(P^{-1}AP - \lambda I) \\
&= \det(P^{-1}AP - \lambda P^{-1}P) \\
&= \det(P^{-1}(AP - \lambda P)) \\
&= \det(P^{-1}(A - \lambda I)P) \\
&= \det(P^{-1})\det(A - \lambda I)\det(P) \\
&= \det(P)^{-1}\det(P)\det(A - \lambda I) = \det(A - \lambda I)
\end{aligned}$$

and we have used here the fact that the determinant is multiplicative:

$$\det(AB) = \det(A)\det(B).$$

(b) Show that $\det([f]_{BB}) = \det([f]_{SS})$.

SOLUTION: By the change of basis formula:

$$[f]_{BB} = P_{BS}[f]_{SS}P_{SB} = P_{SB}^{-1}[f]_{SS}P_{SB}$$

and hence

$$\begin{aligned}
\det([f]_{BB}) &= \det(P_{SB}^{-1}[f]_{SS}P_{SB}) = \det(P_{SB}^{-1})\det([f]_{SS})\det(P_{SB}) \\
&= \det([f]_{SS})\det(P_{SB})\det(P_{SB})^{-1} = \det([f]_{SS}).
\end{aligned}$$

Alternatively, take $\lambda = 0$ in part (a).

EXERCISE 14. (a) If $[f]_{SS}$ is the stabdard matrix of a reflection about a line through the origin in $\mathbb{R}^2$, can you say what its eigenvalues are? Is there a basis B such that $[f]_{BB}$ is diagonal?

SOLUTION: The reflection is about a line L through the origin, and all the vectors on that line are fixed, so are eigenvectors with eigenvalue 1. The line L' through 0 perpendicular to L has all its vectors sent to their negative, so are eigenvectors with eigenvalue -1. Hence in a basis B consisting of $\boldsymbol{v_1} \in L$ and $\boldsymbol{v_2} \in L'$, we have

$$[f]_{BB} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(b) Let $[g]_{SS}$ be the standard matrix of an anticlockwise rotation of angle $\theta \neq \pi$ about the origin in $\mathbb{R}^2$. Can you say what its eigenvalues are (only real eigenvalues, not complex)? Is there a basis B of $\mathbb{R}^2$ such that $[g]_{BB}$ is diagonal?

SOLUTION: We easily see that no vector is transformed into a multiple of itself, so there are no real eigenvalues. The basis B does not exist.

EXERCISE 15. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 8 \end{bmatrix}$$

which has characteristic polynomial $-\lambda^3 + 13\lambda^2 + 7\lambda - 2$.

(a) What is $-A^3 + 13A^2 + 7A - 2I$?

SOLUTION: It is the zero matrix. The Cayley-Hamilton theorem says that a matrix is always a zero of its own characteristic polynomial.

(b) Use the Cayley-Hamilton theorem to express both A^3 and A^{-1} as a linear combination of A^2, A and I.

SOLUTION: We directly get $A^3 = 13A^2 + 7A - 2I$ and if we multiply the above expression by A^{-1} we get

$$-A^2 + 13A + 7I - 2A^{-1} = 0 \qquad \Rightarrow \qquad A^{-1} = -\frac{1}{2}A^2 + \frac{13}{2}A + \frac{7}{2}I.$$

EXERCISE 16. Let A be an $n \times n$ matrix with characteristic polynomial $p(\lambda)$ and n eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ (with repetitions if algebraic multiplicity not equal to 1).

(a) Show that $\det(A) = p(0)$.

(b) Deduce that $\det(A) = \lambda_1.\lambda_2 \ldots \lambda_n$.

SOLUTION:

(a) The characteristic polynomial is $p(\lambda) = \det(A - \lambda I)$. If we take $\lambda = 0$ we get the result.

(b) In addition,

$$p(\lambda) = (\lambda_1 - \lambda).(\lambda_2 - \lambda) \ldots (\lambda_n - \lambda) \qquad \text{and} \qquad \lambda = 0 \quad \Rightarrow \quad p(0) = \lambda_1.\lambda_2 \ldots \lambda_n.$$

EXERCISE 17. The Cayley-Hamilton theorem states that a matrix satisfies its own characteristic equation. A matrix which satisfies a polynomial matrix equation of lower order than that of itself is called *derogatory*. Show that the following matrix is derogatory in that

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} \qquad \text{satisfies} \qquad A^2 - 3A + 2I = 0$$

SOLUTION: By direct calculation.