MATH1012 Mathematical Theory and Methods

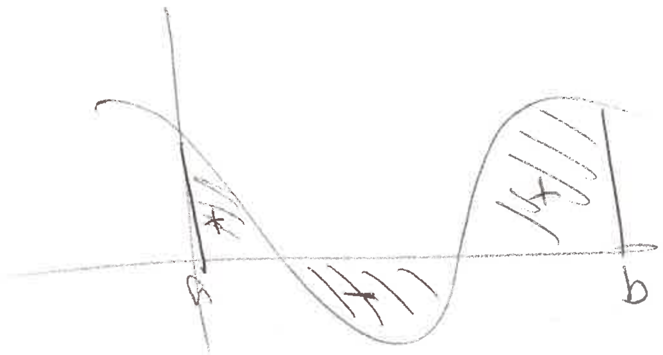
DES HILL

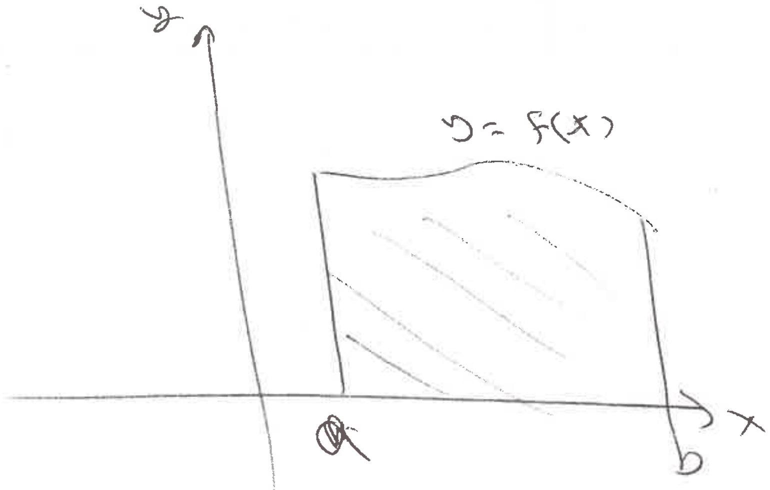
Week 6

Improper integrals

Recall from MATH1011:
For $[a, b]$ a finite interval and $f : [a, b] \longrightarrow \mathbb{R}$ a bounded function, we can define the *definite integral*

$$\int_a^b f(x)dx$$

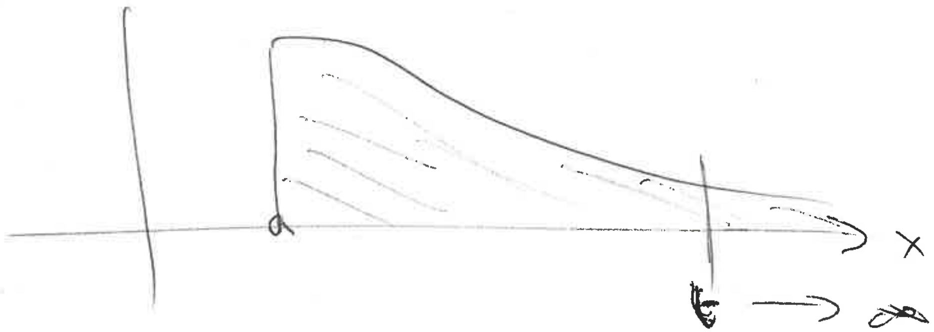




Some integrals are not of this form but nevertheless have a sensible interpretation.

- **Type I improper integral:**
  Integration over an infinite interval

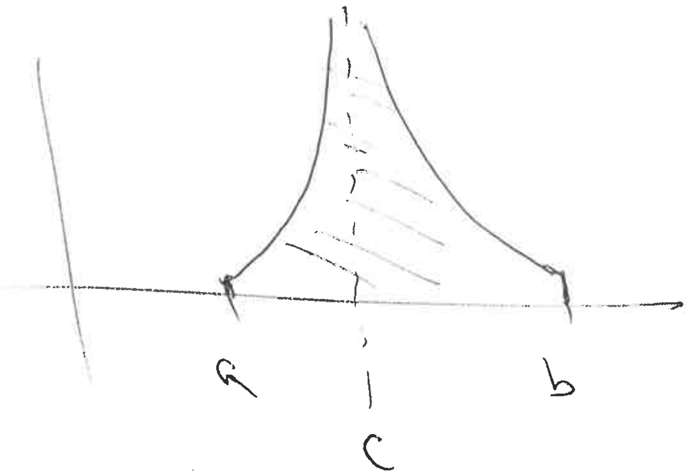
$$\int_a^\infty f(x)\,dx$$



- **Type II improper integral:**
  The interval is finite, but the function is unbounded

$$\int_a^b f(x)\,dx$$

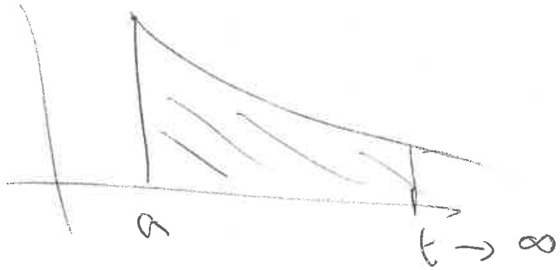
$f(c)$ undefined.



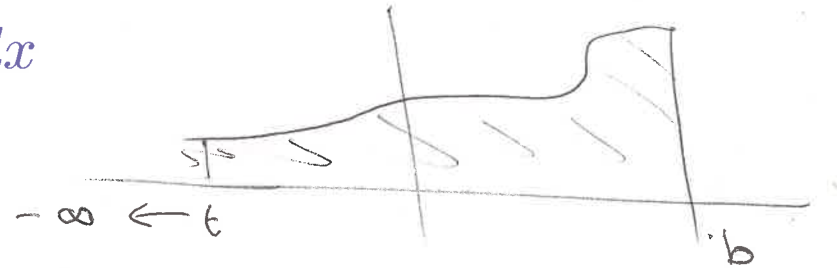
Type I: $[a, \infty)$ or $(-\infty, b]$ and $f(x)$ bounded

Define

$$\int_a^{\infty} f(x)dx = \lim_{t \to \infty} \int_a^t f(x)dx$$



$$\int_{-\infty}^{b} f(x)dx = \lim_{t \to -\infty} \int_t^b f(x)dx$$



The improper integral is defined as the *limiting value* of a suitably chosen *definite integral*.

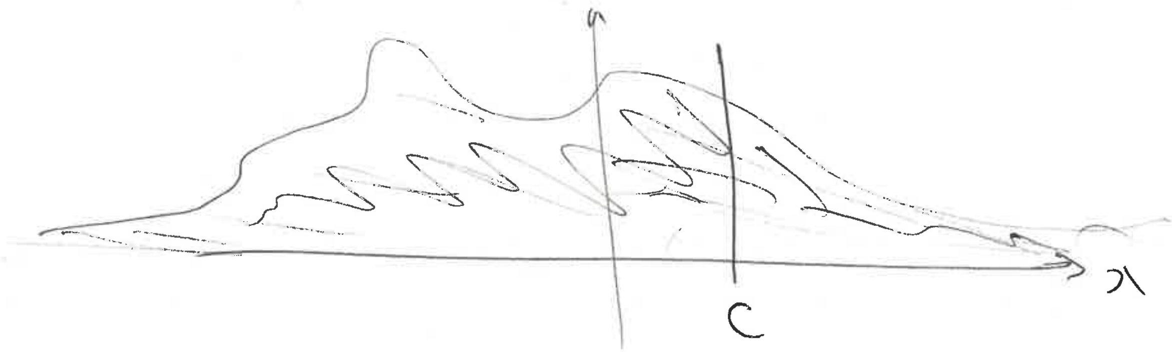
If the limit exists, we say the improper integral is *convergent*, and the limit still represents the "area" under the curve.

Otherwise, we say the improper integral is *divergent*.

Type I (doubly improper)

Define

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{C} f(x)dx + \int_{C}^{\infty} f(x)dx$$

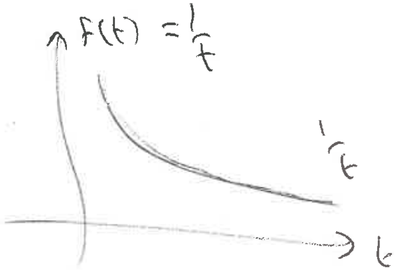


$\int_{-\infty}^{\infty} f(x)\,dx$ is convergent if and only if both improper integrals on the right-hand side are convergent.

The exact value of C is not important — choose for our own convenience.

Example

Is the following improper integral convergent or divergent?



$$\int_1^\infty \frac{1}{x^2}\,dx$$

We have

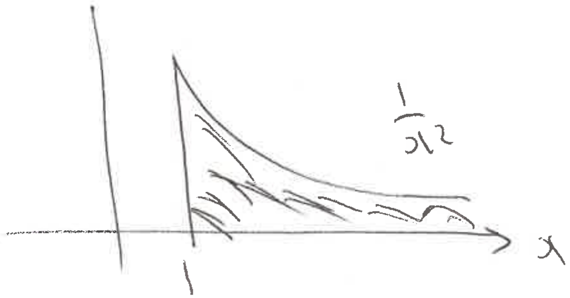
$$\int_1^t \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^t = 1 - \frac{1}{t}$$

and so

$$\lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left(1-\frac{1}{t}\right) = 1 - 0 = 1$$

hence

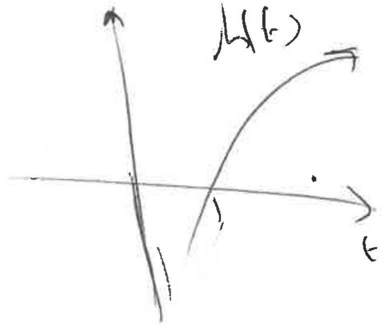
$$\int_1^\infty \frac{1}{x^2}\,dx = 1$$



Int. is convergent

ANOTHER EXAMPLE

Is the following improper integral convergent or divergent?



$$\int_1^\infty \frac{1}{x}dx$$

We have

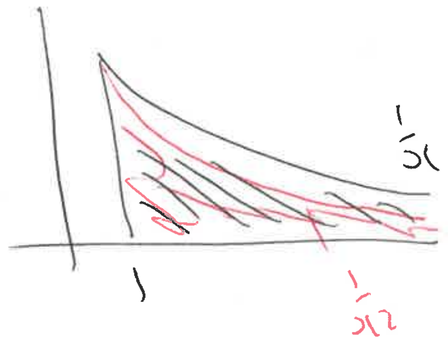
$$\int_1^t \frac{1}{x}dx = \Big[\ln x\Big]_1^t = \ln(t) - \ln(1) = \ln(t)$$

and so

$$\lim_{t\to\infty}\int_1^t \frac{1}{x}dx = \lim_{t\to\infty} \ln(t) \text{ is undefined}$$

hence

$$\int_1^\infty \frac{1}{x}dx \quad \text{diverges}$$



More general example

For which values of $p \neq 1$ is the following integral convergent?

$$\int_1^\infty x^{-p}\,dx = \int_1^\infty \frac{1}{x^p}\,dx \qquad \int x^n = \frac{x^{n+1}}{n+1} + c$$

We have

$$\int_1^t x^{-p}\,dx = \frac{1}{1-p}\left[x^{1-p}\right]_1^t$$
$$= \frac{1}{1-p}\left(t^{1-p} - 1\right)$$

So $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \frac{1}{1-p}\left(t^{1-p} - 1\right)$

diverges if $1-p > 0$ $\rightarrow p < 1$

converges if $1-p < 0 \Rightarrow p > 1$

$$\int_1^\infty \frac{1}{x^3}\,dx = \frac{1}{3-1} = \frac{1}{2}$$

So the integral converges only if $p > 1$ and has the value $\frac{1}{p-1}$

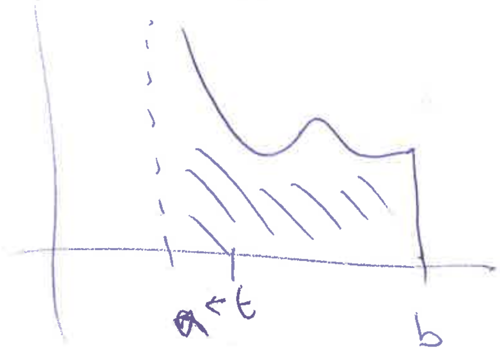
Type II

Integrals such as

$$\int_a^b f(x)dx$$

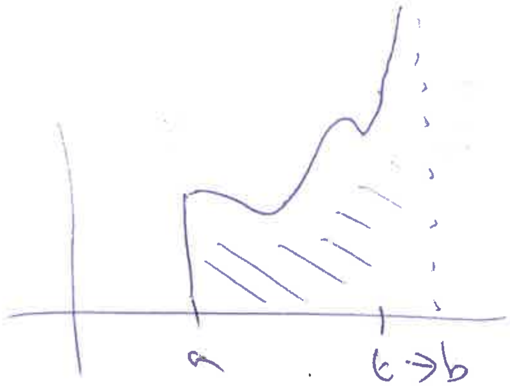
where f is not defined at an end point (vertical asymptote).

If $f(a)$ is not defined then define



$$\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$$

If $f(b)$ is not defined then define



$$\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$$

If the limit exists, we say the improper integral is *convergent*, otherwise *divergent*

Example

Is the following improper integral convergent or divergent?

$$\int_0^2 \frac{1}{\sqrt{2-x}}\,dx$$

Let $2-x = u \Rightarrow -1 = \frac{du}{dx}$

$\Rightarrow -dx = du$

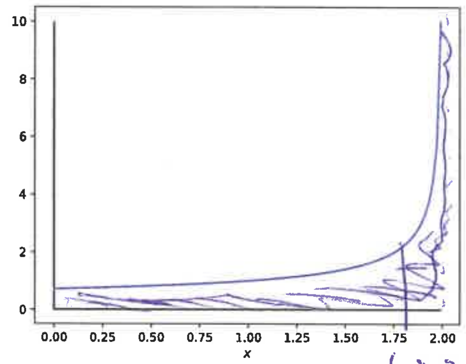
$$\int \frac{dx}{\sqrt{2-x}} = \int \frac{-du}{\sqrt{u}}$$

$$= \int -u^{-\frac{1}{2}}\,du$$

$$= -\frac{u^{\frac{1}{2}}}{\frac{1}{2}}$$

$$= -2\sqrt{u}$$

$$= -2\sqrt{2-x}$$



$t \to 2$

$$\int_0^2 \frac{1}{\sqrt{2-x}}\,dx = \lim_{t\to 2^-} \int_0^t \frac{1}{\sqrt{2-x}}\,dx$$

$$= \lim_{t\to 2^-} \left[-2\sqrt{2-x}\right]_0^t$$

$$= \lim_{t\to 2^-} -2\sqrt{2-t} + 2\sqrt{2}$$

$$= 2\sqrt{2}$$ Int. Converges to $2\sqrt{2} \approx 2.828$

Type II also

Integrals such as

$$\int_a^b f(x)dx$$

where $f(c)$ is not defined (vertical asymptote) for some c *inside* the interval (a, b).

$$= \lim_{t \to c^-} \int_a^t f(x)dx + \lim_{t \to c^+} \int_t^b f(x)dx$$

Define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

and the required integral is convergent if and only if both of the improper integrals on the right-hand side are convergent.

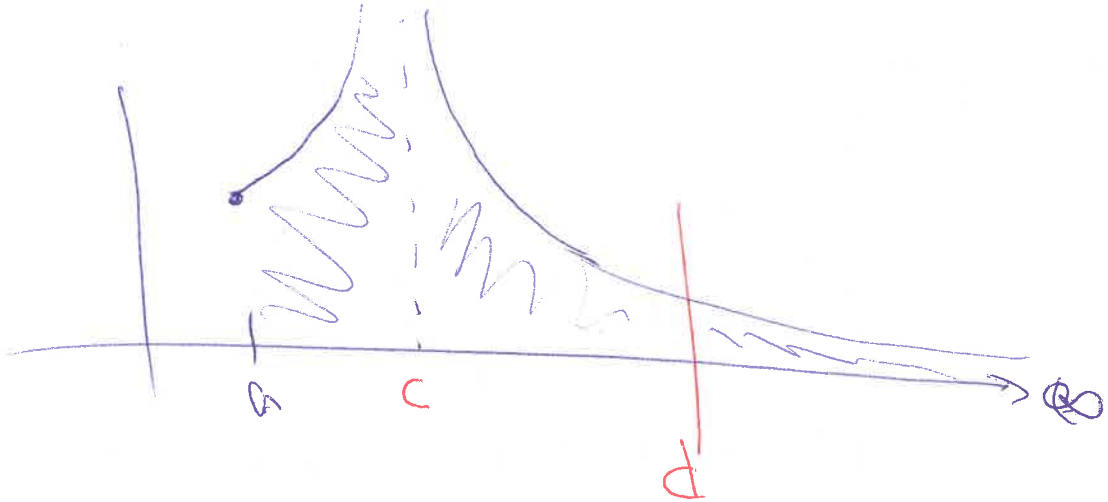
More complicated improper integrals

An integral could have a combination of different things: ∞, $-\infty$, asymptotes.

We must split the integral into a sum of improper integrals, each with only *one* problem in their domain.

The required integral is convergent if and only if *all* of the improper integrals are convergent.

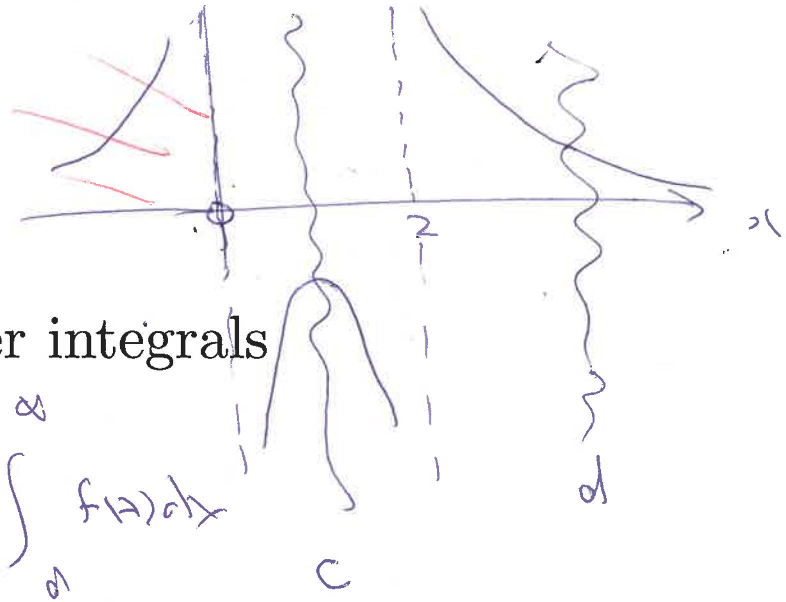
$$\int_a^\infty f(x)\,dx = \int_a^c f(x)\,dx + \int_c^d f(x)\,dx + \int_d^\infty f(x)\,dx$$



Example

Is the following improper integral convergent or divergent?

$$\int_0^\infty f(x)\,dx = \int_0^\infty \frac{2}{x(x-2)}\,dx$$



We split the integral into a sum of improper integrals

$$\int_0^c f(x)\,dx + \int_c^2 f(x)\,dx + \int_2^d f(x)\,dx + \int_d^\infty f(x)\,dx$$

We determine if each of these is convergent. Then

- If any one of them is divergent then the integral does not converge.
- If not, then the integral is convergent and its value is the sum of all of the limiting values.

Example continued

$$\int \frac{1}{x-c}\,dx = \ln|x-c|$$

$$\frac{2}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$$

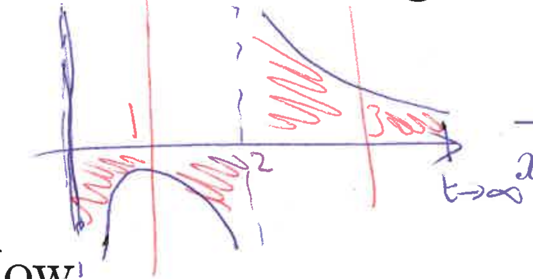
$$\frac{2}{x(x-2)} = \frac{A(x-2)+Bx}{x(x-2)}$$

$$\frac{2}{x(x-2)} = \frac{(A+B)x-2A}{x(x-2)}$$

$A+B=0 \Rightarrow B=-A=1$

$2=-2A \Rightarrow A=-1$

Partial fractions gives us



$$\frac{2}{x(x-2)} = \frac{1}{x-2} - \frac{1}{x}$$

Now

$$\begin{aligned}
\int_3^\infty \frac{2}{x(x-2)}\,dx &= \lim_{t\to\infty} \int_3^t \left(\frac{1}{x-2} - \frac{1}{x}\right) dx \\
&= \lim_{t\to\infty} \left[\ln|x-2| - \ln|x|\right]_3^t \\
&= \lim_{t\to\infty} \ln(t-2) - \ln t - \ln 1 + \ln 3 \\
&\to \quad \infty \quad \text{as} \quad t\to\infty
\end{aligned}$$

Do NOT cancel infinities

Because this improper integral diverges then the given integral

$$\int_0^\infty \frac{2}{x(x-2)}\,dx \qquad \text{diverges}$$

Sequences

Intuitively a sequence is an (infinite) string of numbers. E.g.,

$$1,\ 2,\ 4,\ 8,\ 16,\ \cdots$$

A *sequence*

$$(a_1,\ a_2,\ a_3,\ \cdots)$$

may be denoted

$$(a_n) \qquad \text{or} \qquad (a_n)_{n=1}^{\infty}$$

Sometimes the first element is a_0 and we denote it as

$$(a_n)_{n=0}^{\infty}$$

Sometimes the first element is a_m and we denote it as

$$(a_n)_{n=m}^{\infty}$$

Sequences form a vector space

$$(a_n) = (a_1, a_2, a_3, \cdots)$$

$$(b_n) = (b_1, b_2, b_3, \cdots)$$

- sum of sequences:

$$(a_n) + (b_n) = (a_1+b_1, a_2+b_2, a_3+b_3, \cdots)$$

- scalar multiplication:

$$k(a_n) = (ka_1, ka_2, ka_3, \cdots)$$

Examples

- $a_n = \frac{1}{n}$

$a_1 = 1$ $a_2 = \frac{1}{2}$ $a_3 = \frac{1}{3}$

- $a_n = (-1)^n n$

$a_1 = -1$ $a_2 = 2$ $a_3 = -3$

- $a_1 = 1$, $a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n > 2$

$a_1 = 1$ $a_2 = 1$ $a_3 = 2$ $a_4 = 3$ $a_5 = 5$ ...

$(1, 1, 2, 3, 5, 8, 13, 21, \ldots)$

Fibonacci

- $a_1 = 1$, $a_n = 1 + \frac{1}{a_{n-1}}$ for $n > 1$

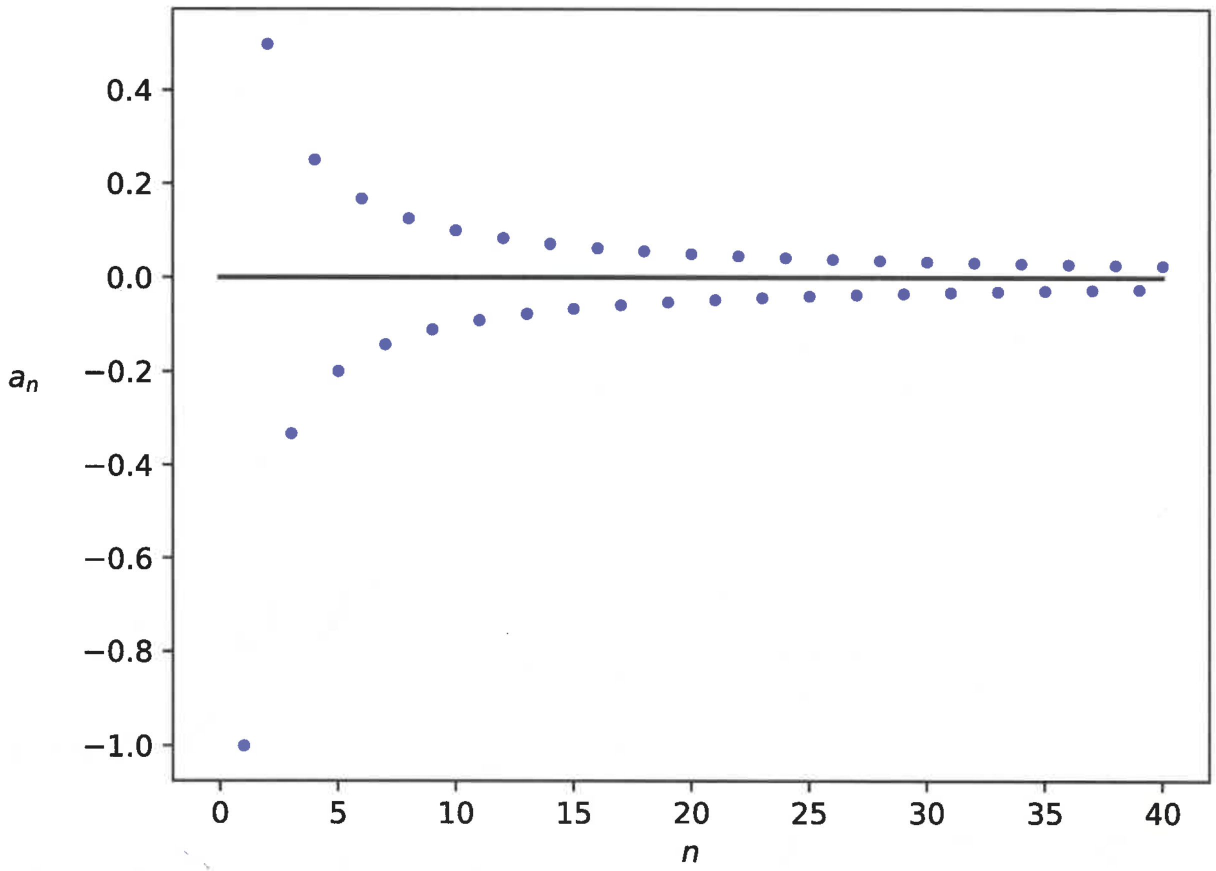
$(1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \ldots) = (1, 2, 1.5, 1.666, 1.6, \ldots)$

Visualisation

$\left(-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots\right)$

The sequence

$$(a_n)_{n=1}^{\infty} \qquad \text{where} \qquad a_n = \frac{(-1)^n}{n}$$



The sequences *converges* to 0

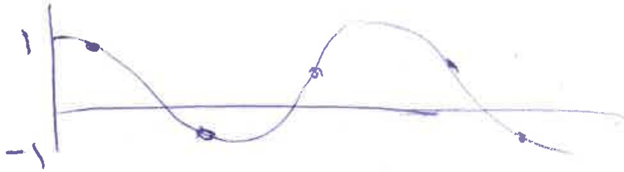
In the long run

What happens as $n \to \infty$? Intuitively,

Divergent

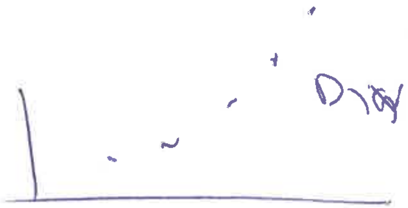
- $a_n = n$ $\quad (1, 2, 3, 4, \ldots)$

- $a_n = \cos n$



- $a_n = \frac{1}{n}$ $\quad (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$

- $a_1 = 1$, $a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n > 2$



- $a_1 = 1$, $a_n = 1 + \frac{1}{a_{n-1}}$ for $n > 1$

?=L

- $a_1 = 1$, $a_n = \sqrt{1 + a_{n-1}}$ for $n > 1$

$(1, \sqrt{2}, \sqrt{1+\sqrt{2}}, \sqrt{1+\sqrt{1+\sqrt{2}}}, \ldots)$ $\quad (1, 1.4, 1.53, 1.5, \ldots)$

Limit

We say that (a_n) is *convergent* and

$$\lim_{n\to\infty} a_n = L$$

if

> *"the sequence gets really close to L when n gets really big"*

and L is called the *limit* of the sequence (a_n).

Otherwise we say that (a_n) is *divergent*

Rigorous definition of limit

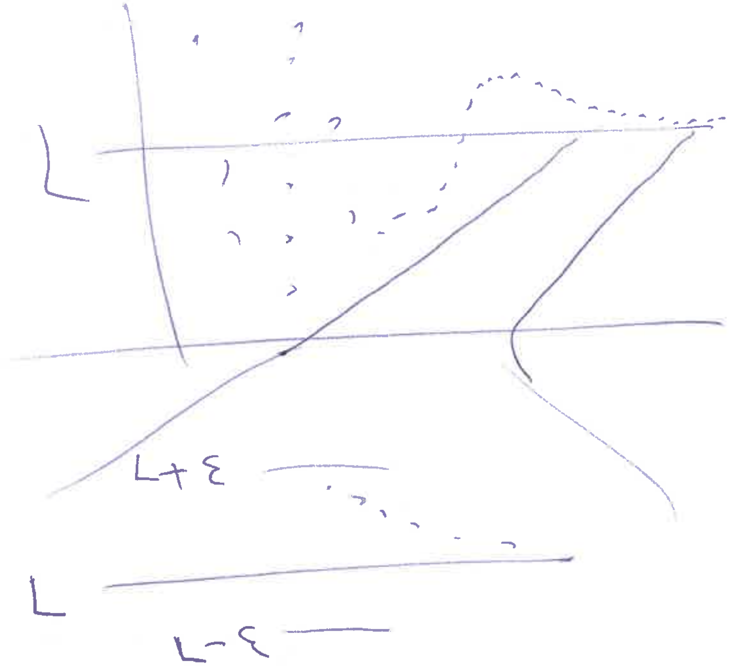
We say that

$$\lim_{n\to\infty} a_n = L$$

if for all $\varepsilon > 0$, there exists a number N (which depends on ε) such that

$$|a_n - L| < \varepsilon$$

for all $n > N$.



Diverging to infinity

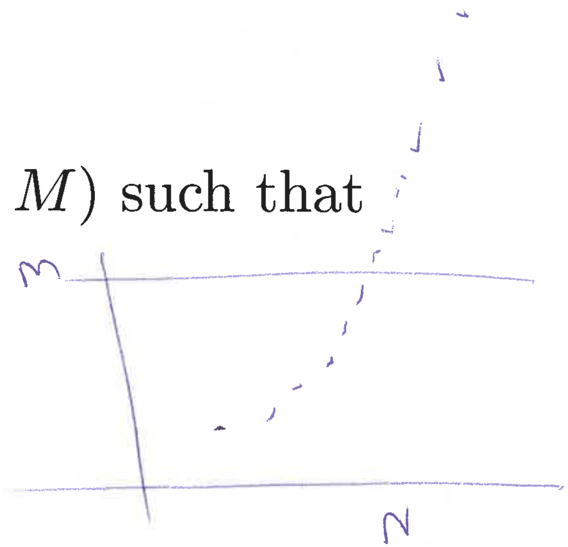
We say that

$$\lim_{n\to\infty} a_n = \infty$$

if for all M, there exists N (which depends on M) such that

$$a_n > M$$

for all $n > N$.



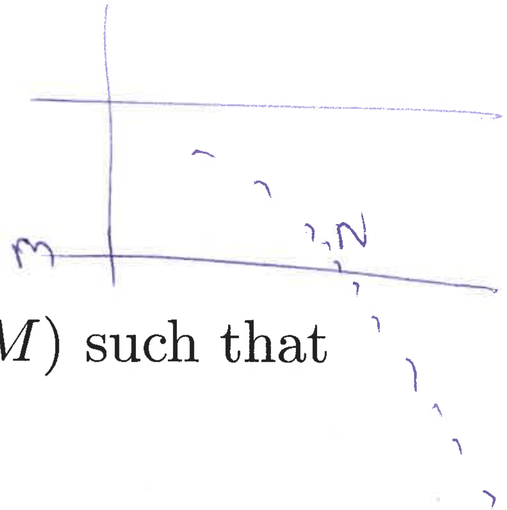
Similarly, we say that

$$\lim_{n\to\infty} a_n = -\infty$$

if for all M, there exists N (which depends on M) such that

$$a_n < M$$

for all $n > N$.

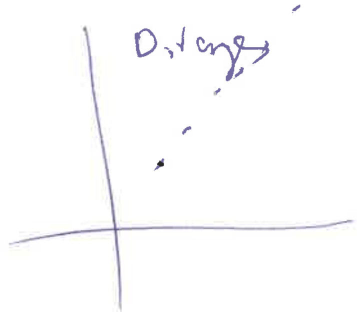


Example

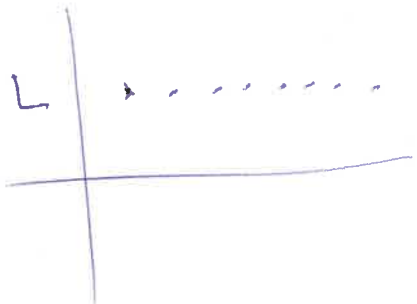
$$\lim_{n\to\infty} \frac{1}{n^p} = ?$$

$L = 0$ $\quad p > 0$ $\quad$ E.g $p = 1$ gives $a_n = \frac{1}{n}$

$L = \infty$ $\quad p < 0$ $\quad$ E.g $p = -1$ gives $a_n = \frac{1}{n^{-1}} = \frac{1}{1/n} = n$

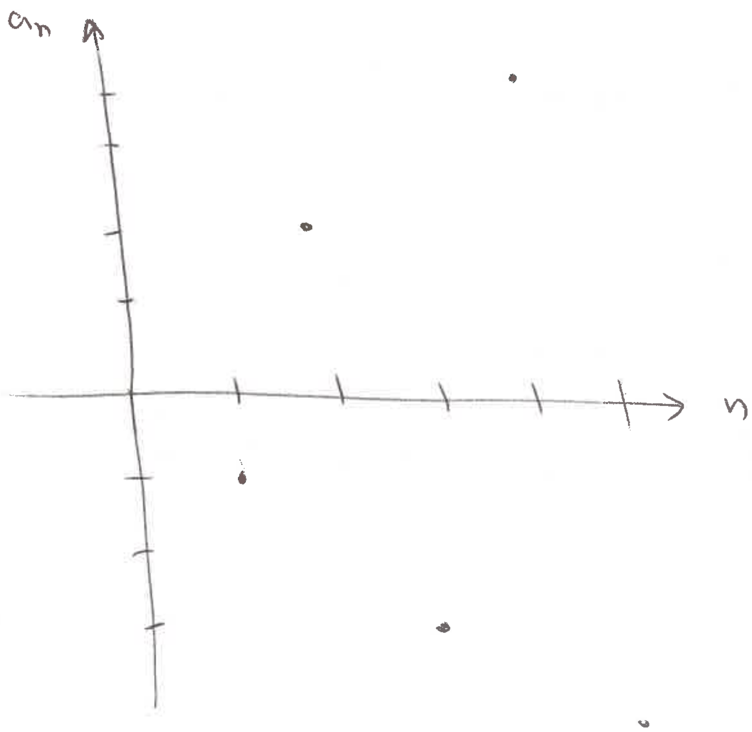


$L = 1$ $\quad p = 0$ $\quad$ $a_n = \frac{1}{n^0} = \frac{1}{1} = 1$



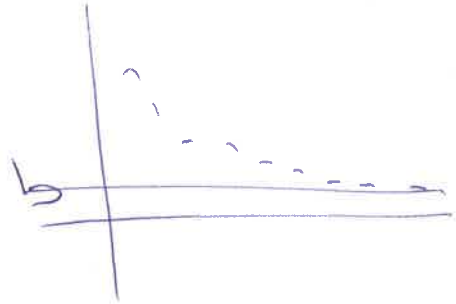
Another example

$$\lim_{n\to\infty} (-1)^n n = \text{ undefined }, \text{ the series diverges}$$



Limit laws for sequences

If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then



- $\lim_{n\to\infty}(a_n + b_n)$ $= \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

  $= a + b$

- $\lim_{n\to\infty}(a_n - b_n)$

  $= a - b$

- $\lim_{n\to\infty}(ka_n)$ $= ka$

- $\lim_{n\to\infty}(a_n b_n)$ $= ab$

- $\lim_{n\to\infty}(a_n/b_n)$ $= \frac{a}{b}$ $\quad b \neq 0$

Watch out: a, b have to be real numbers here, not ∞ or $-\infty$.

Example

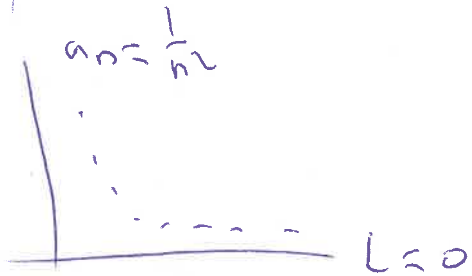
$$\lim_{n\to\infty} \frac{n^2+2n+4}{3n^2+1}$$

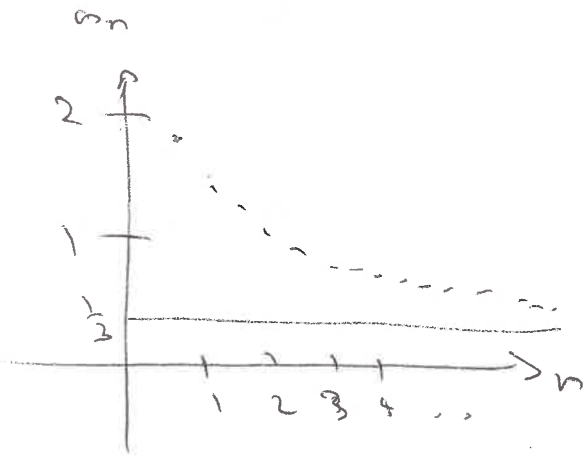
$$\lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{4}{n^2}}{3+\frac{1}{n^2}}$$

$$= \frac{\lim\limits_{n\to\infty}\left(1+2\left(\frac{1}{n}\right)+4\left(\frac{1}{n^2}\right)\right)}{\lim\limits_{n\to\infty}\left(3+\left(\frac{1}{n^2}\right)\right)}$$

$$= \frac{1+2(0)+4(0)}{3+0}$$

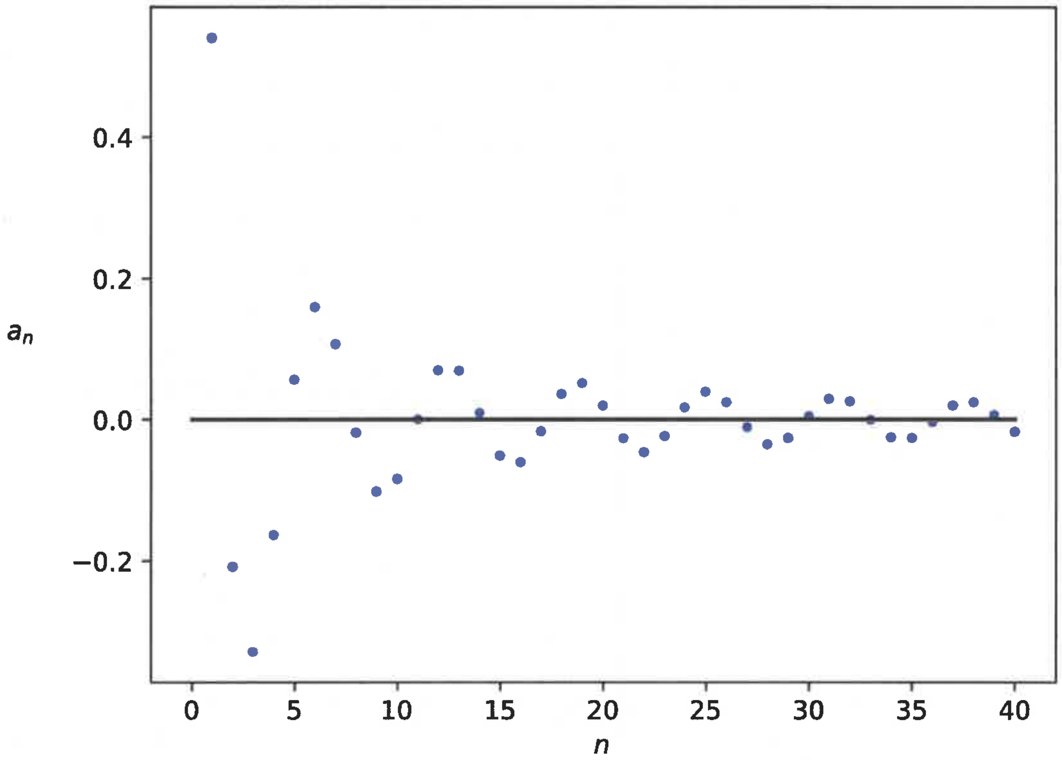
$$= \frac{1}{3}$$





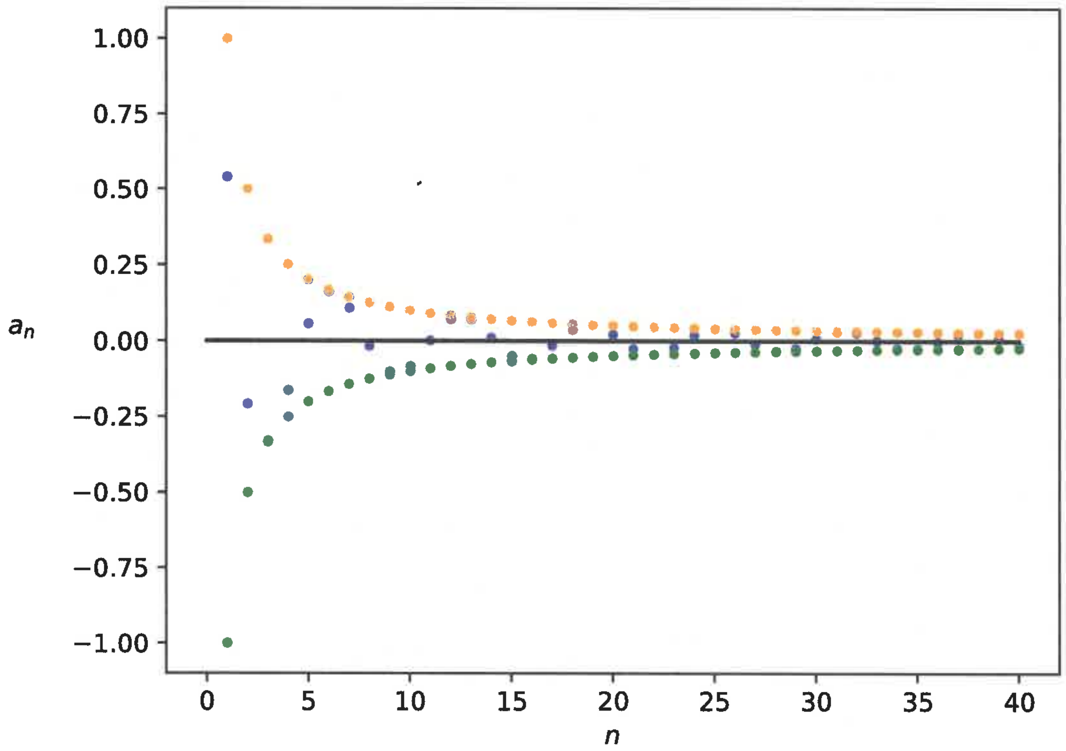
Squeezing

$$\lim_{n\to\infty} \frac{\cos n}{n} = ?$$



$$\lim_{n\to\infty} \frac{\cos n}{n} = ?$$

Notice that this sequence is 'squeezed' between $\left(\frac{1}{n}\right)$ and $\left(-\frac{1}{n}\right)$



The Squeeze Theorem

If

$$a_n \leq b_n \leq c_n$$

for all sufficiently large n and

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$$

then $\lim_{n\to\infty} b_n = L$

Previous example:

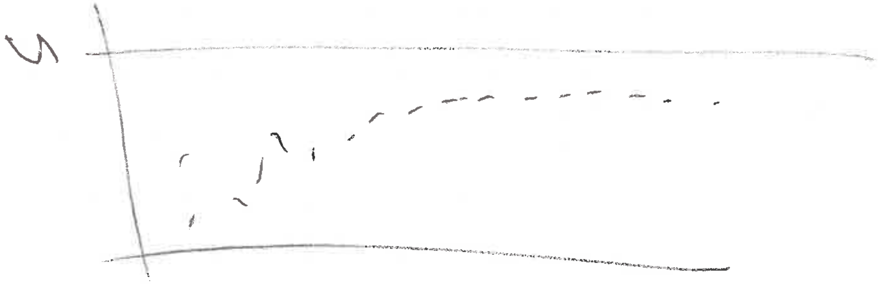
$$-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$$

and hence

$$\lim_{n\to\infty} \frac{\cos n}{n} = 0$$

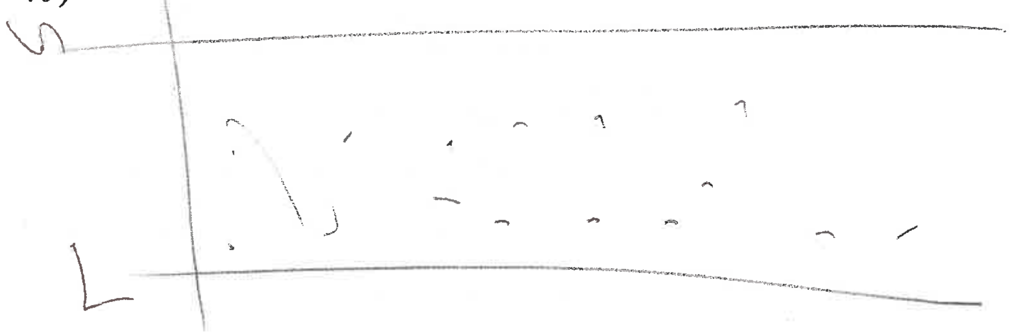
BOUNDED

A sequence (a_n) is *bounded above* if
There exists a number U such that $a_n \leq U$ for all n.



A sequence (a_n) is *bounded below* if
There exists a number L such that $a_n \geq L$ for all n.

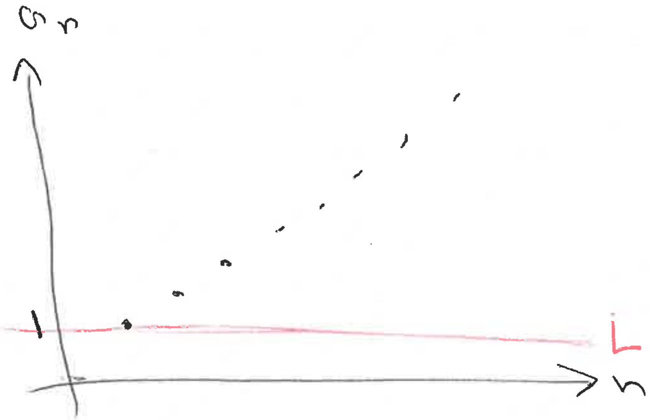
A sequence (a_n) is *bounded* if it is bounded above and below.



Examples

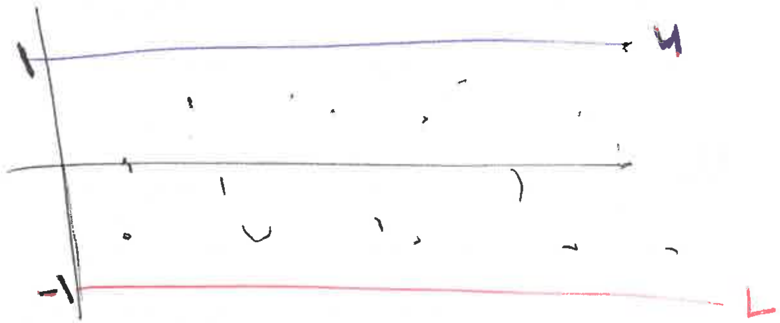
- $a_n = n$

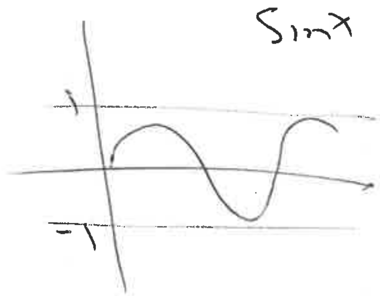
Bounded below



- $a_n = (-1)^n \sin n$

Bounded



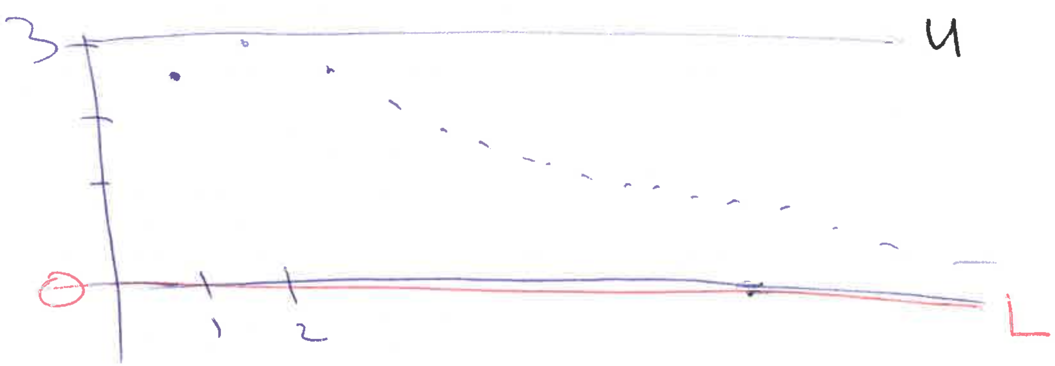


- $a_n = \dfrac{e^n}{n!}$ bounded

$e \simeq 2.718\ldots$

$n! = 1\times2\times3\times \cdots \times n$

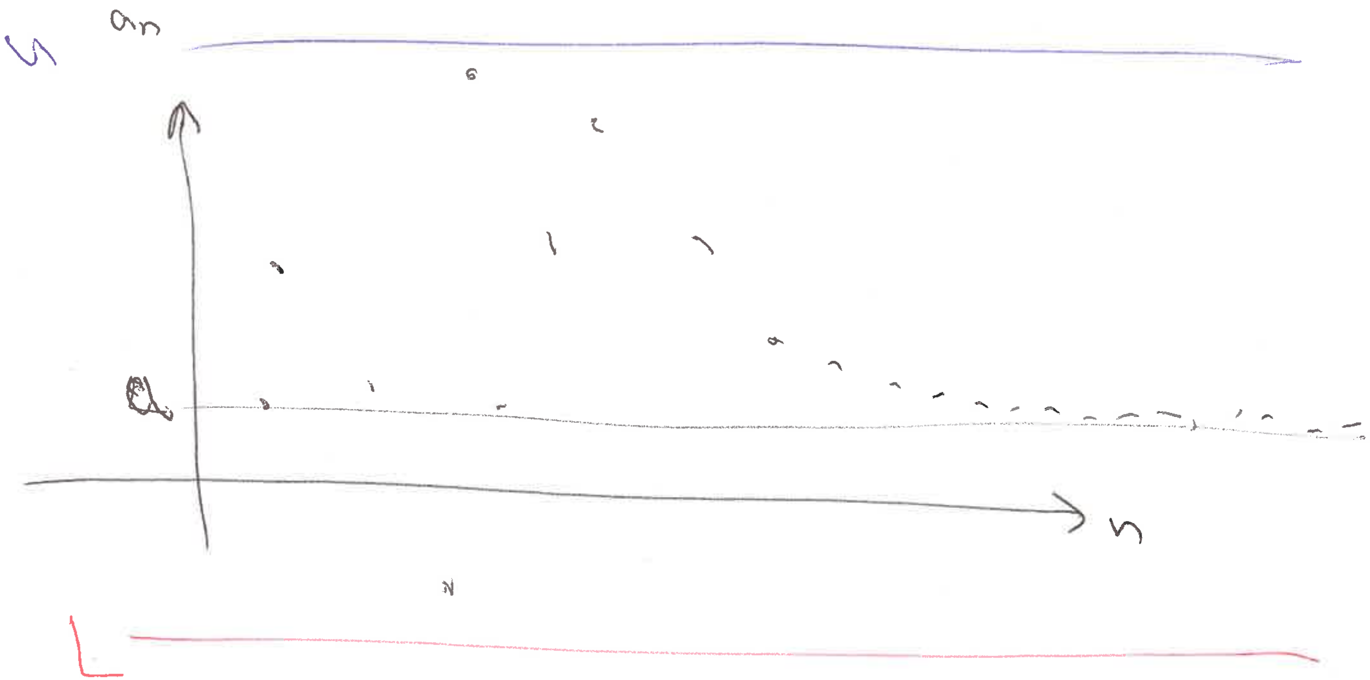
$4! = 1\times2\times3\times4 = 24$



Convergent implies bounded

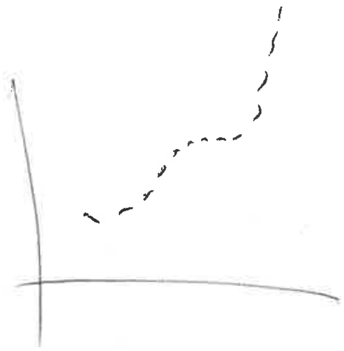
$\lim_{n\to\infty} a_n = a$

Any *convergent* sequence is bounded.



Monotone sequences

Monotone non-decreasing: $a_1 \leq a_2 \leq a_3 \leq a_4 \cdots$
(weakly increasing)



Monotone non-increasing: $a_1 \geq a_2 \geq a_3 \geq a_4 \cdots$
(weakly decreasing)

Example

$$a_n = \frac{1}{n}$$

Bounded?

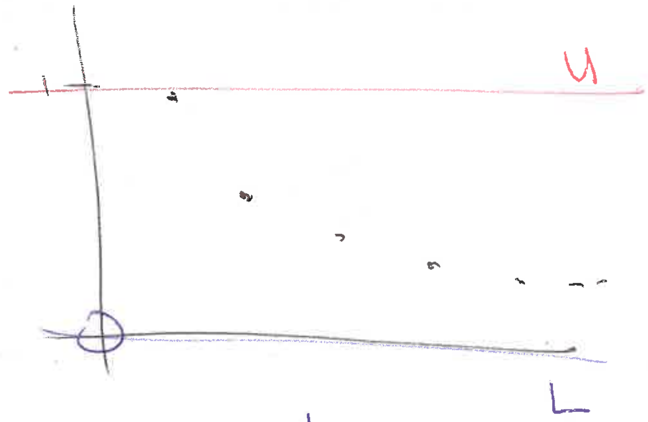
Yes

Monotone?

Non-increasing

Convergent?

Yes $\lim_{n\to\infty} \frac{1}{n} = 0$

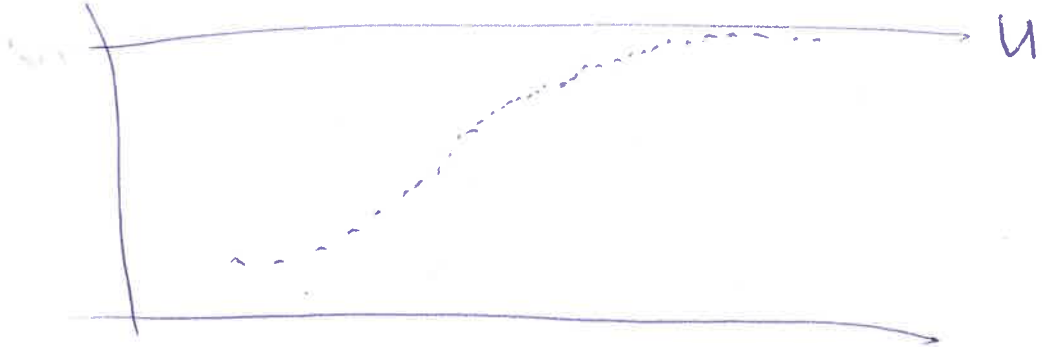


$$a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$$

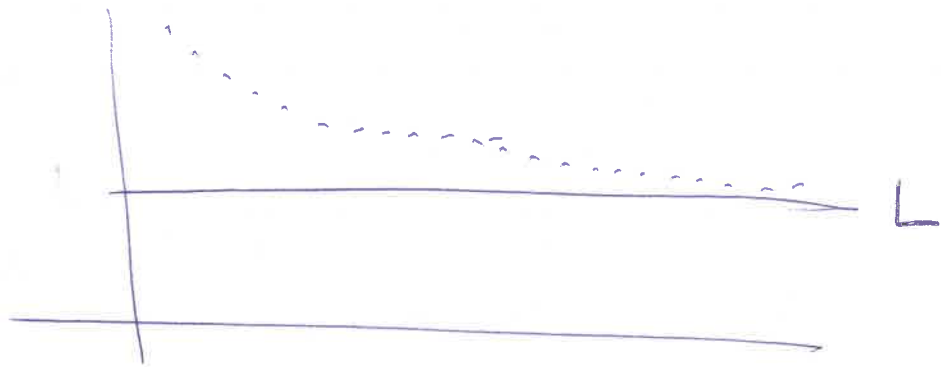
$$\Rightarrow a_{n+1} < a_n$$

Monotone sequence theorem

If (a_n) is monotone non-decreasing (for n sufficiently large) and bounded above, then (a_n) converges

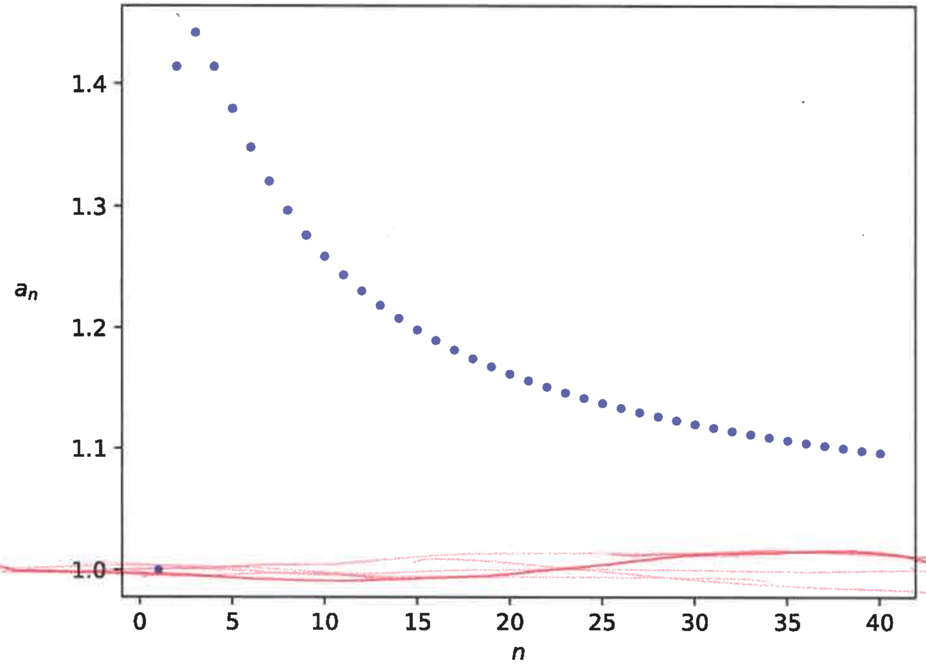


If (a_n) is monotone non-increasing (for n sufficiently large) and bounded below, then (a_n) converges



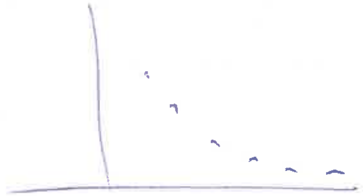
HARDER "STANDARD" LIMITS

$$\lim_{n \to \infty} n^{\frac{1}{n}} = 1$$



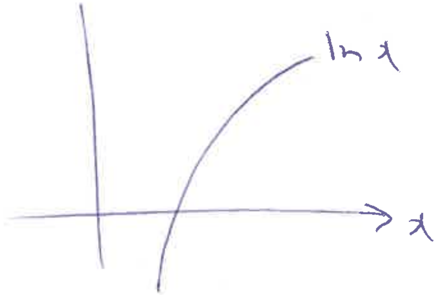
This sequence is monotone non-increasing for $n = 3, 4, 5, \cdots$ and is bounded below. Hence it converges.

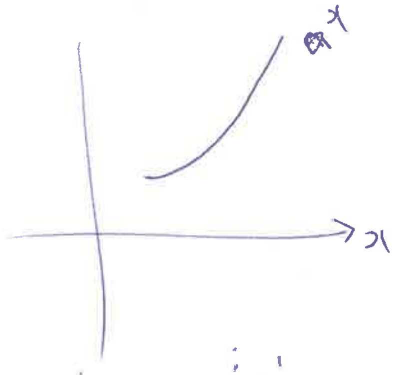
For every real number $\alpha > 0$,

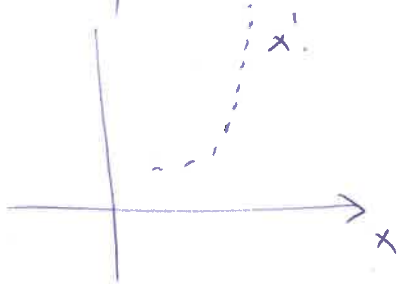


$$\boxed{\lim_{n\to\infty} \frac{\ln n}{n^\alpha} = 0}$$

The terms are monotone non-increasing for $n > e^{1/\alpha}$.
The log function grows *very slowly*.







For every real number α,

$$\boxed{\lim_{n\to\infty} \frac{\alpha^n}{n!} = 0}$$

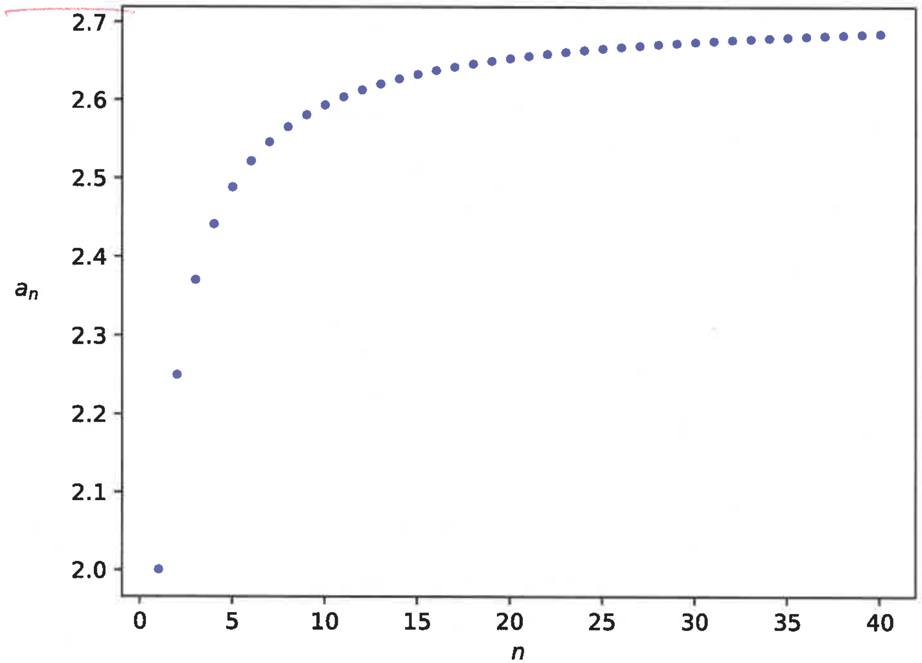
The terms are monotone non-increasing for $n \geq \alpha - 1$ if $\alpha > 0$.
Use squeezing for $\alpha < 0$.
The factorial function grows *very fast*.

For every real number c,

$$\boxed{\lim_{n\to\infty}\left(1+\frac{c}{n}\right)^n = e^a}$$

The sequence plot for $c = 1$ is

2.718... = e



The sequence is monotone non-decreasing.

If $c = 1$ then

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e}$$

Finding limits

We say that a sequence like

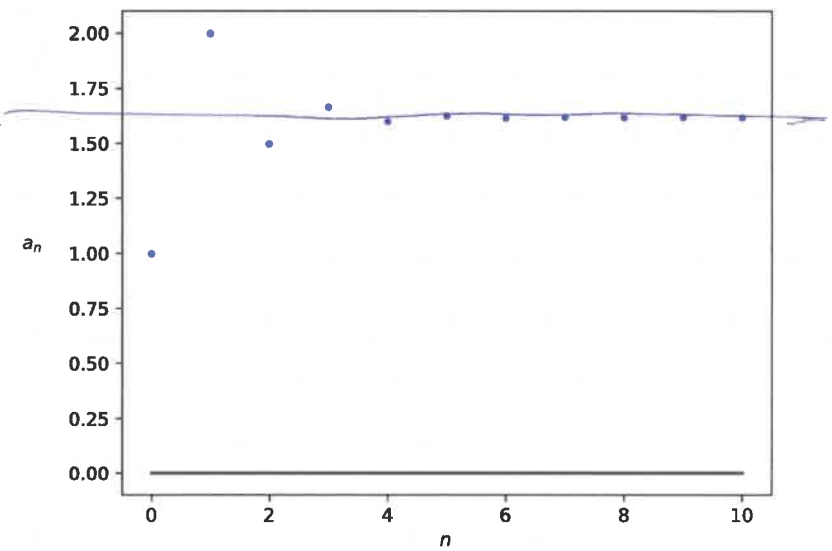
$$a_1 = 1, \quad a_n = 1 + \frac{1}{a_{n-1}} \quad \text{for} \quad n > 1$$

has been defined *iterativel*y, or *recursively*.

If we believe that such a sequence converges (e.g. graphically)

$$\phi = \frac{1+\sqrt{5}}{2} = a$$

Golden ratio



to a limit a, then can find the value of a?

Yes! We can, by replacing all sequence terms in the iteration by the limit a and then solving for a. We get

$$\lim_{n\to\infty} a_n = 1 + \frac{1}{\lim a_{n-1}} \Rightarrow a = 1 + \frac{1}{a} \Rightarrow a^2 = a + 1 \Rightarrow \underline{a^2 - a - 1 = 0}$$

We can use the *quadratic fomula*

$$\alpha x^2 + \beta x + \gamma = 0 \quad \Rightarrow \quad x = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

to determine a. We get

$$\alpha = 1, \quad \beta = -1, \quad \gamma = -1 \Rightarrow a = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

So $\boxed{a = \frac{1+\sqrt{5}}{2}}$ or $a = \frac{1-\sqrt{5}}{2}$

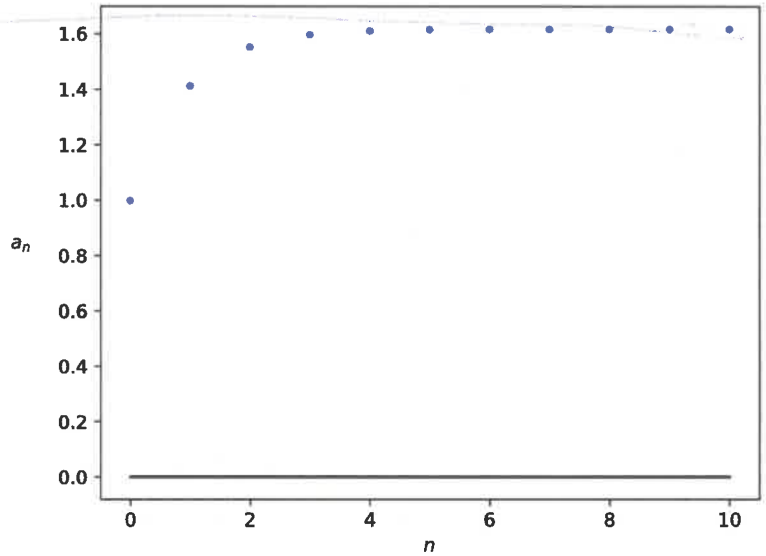
$\sqrt{5} = 2.236$ so $a = 1.618$ or ~~$a = -0.618$~~

a can't be negative

Another example

$$a_1 = 1, \quad a_n = \sqrt{1 + a_{n-1}} \quad \text{for} \quad n > 1$$

$1.618 = \frac{1+\sqrt{5}}{2} = a$



Replacing all sequence terms in the iteration by a gives

$$a = \sqrt{1+a} \Rightarrow a^2 = 1 + a \Rightarrow a^2 - a - 1 = 0$$

So $a = \frac{1+\sqrt{5}}{2}$ or $\cancel{a = \frac{1-\sqrt{5}}{2}}$

MATH1012 Mathematical Theory and Methods

Week 7

Series

An infinite series is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

Can we assign any *meaning* or *value* to such an expression?

Reminder: A *sequence* (a_n) is an infinite ordered list of values

$$a_1, a_2, a_3, \cdots$$

and that a sequence *converges* to a limit L if the *individual entries* get arbitrarily close to L as n tends to infinity.

Examples of series

Geometric series

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots$$

Harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots \qquad p \in \mathbb{R} \text{ is a constant}$$

Partial sums

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$

$$s_3 - s_2$$
$$= \cancel{a_1} + \cancel{a_2} + a_3 - \cancel{a_1} - \cancel{a_2}$$
$$= a_3$$

In general

$$s_N = \sum_{n=1}^{N} a_n$$

Note that

$$\boxed{a_n = s_n - s_{n-1}}$$

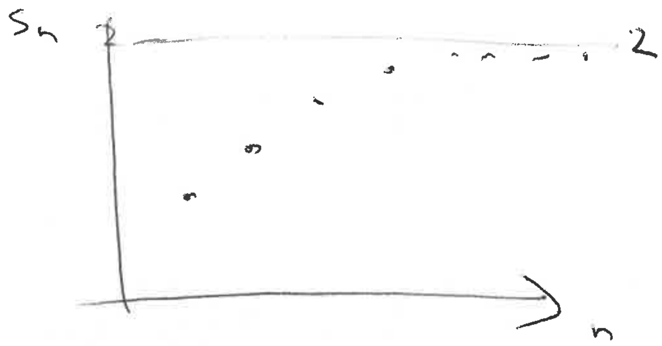
If the *sequence* of partial sums

$$s_1, s_2, s_3, \cdots$$

converges to L as $n \to \infty$, then we say

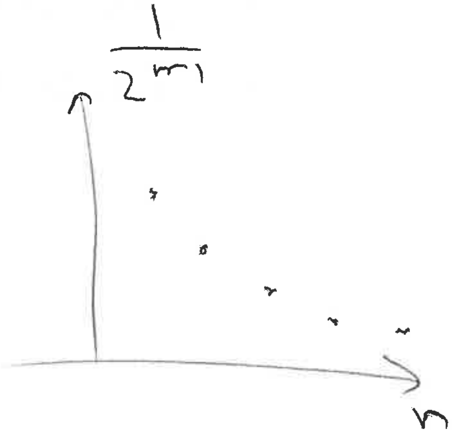
$$\sum_{n=1}^{\infty} a_n = L$$

Example



$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

The partial sums are



$$s_1 = 1 \qquad\qquad = 2 - 1$$

$$s_2 = 1 + \frac{1}{2} = \frac{3}{2} \qquad = 2 - \frac{1}{2}$$

$$s_3 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4} = 2 - \frac{1}{4}$$

We see that

$$s_n = 2 - \frac{1}{2^{n-1}}$$

and hence

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = 2 - \lim_{n\to\infty} \frac{1}{2^{n-1}}$$

$$= 2 - 0 = 2$$

Why study this and how do we do so

Most functions cannot be computed.
They are often *approximated* as the infinite sum of elementary functions of some type.
Understanding when these approximations converge is vital.

To study series we learn

1. A collection of “standard” sequences and series
2. Ways to compare series
3. A battery of tests to apply to unknown series

Pitfall warning

Mixing up *series* and *sequences* is a very common error.

- Sequence (a_n)
  The focus is on the *individual terms* as n gets large.

- Series $\sum_{n=1}^{\infty} a_n$
  The focus is on the *running total* obtained by summing.

A TEST FOR DIVERGENCE

For a series $\sum_{n=1}^{\infty} a_n$ to converges to some value L, that is,

$$L = \sum_{n=1}^{\infty} a_n$$

we must have

$$\lim_{n\to\infty} a_n = 0$$

To prove this, recall that $a_n = s_n - s_{n-1}$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = L - L = 0$$

Hence if

$$\lim_{n\to\infty} a_n \neq 0$$

then

$$\sum_{n=1}^{\infty} a_n$$

does not exist, we say it diverges

Example

Is the following *series* divergent?

$$\sum_{n=1}^{\infty} \frac{3n-5}{n+2}$$

a_n

$3 \neq 0 \Rightarrow$ diverges

$$a_n = \frac{3n-5}{n+2} \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n-5}{n+2}$$

$$= \lim_{n\to\infty} \frac{3-\frac{5}{n}}{1+\frac{2}{n}}$$

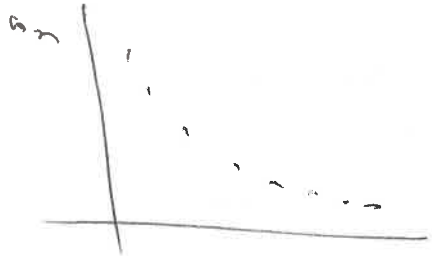
$$= \frac{\lim_{n\to\infty} 3-\frac{5}{n}}{\lim_{n\to\infty} 1+\frac{2}{n}} \Rightarrow \frac{3-0}{1+0} = 3$$

Is the converse true?

If

$$\lim_{n\to\infty} a_n = 0$$

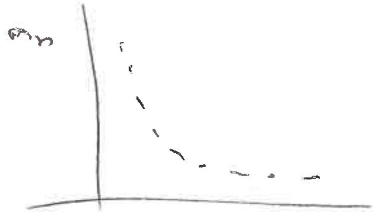
then what can be said?



Example: The harmonic series

$$\sum_{n=1}^{\infty} a_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \qquad \text{has} \qquad a_n = \frac{1}{n}$$

Certainly $a_n \to 0$ as $n \to \infty$, but what about the sum?



Example: p- series with $p = 2$

$$\sum_{n=1}^{\infty} a_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \qquad \text{has} \qquad a_n = \frac{1}{n^2}$$

Certainly $a_n \to 0$ as $n \to \infty$, but what about the sum?

THE HARMONIC SERIES

$$1 > \frac{1}{2}$$

$$\frac{1}{2}+\frac{1}{3} > \frac{1}{4}+\frac{1}{4} = \frac{1}{2}$$

$$\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7} > \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{1}{2}$$

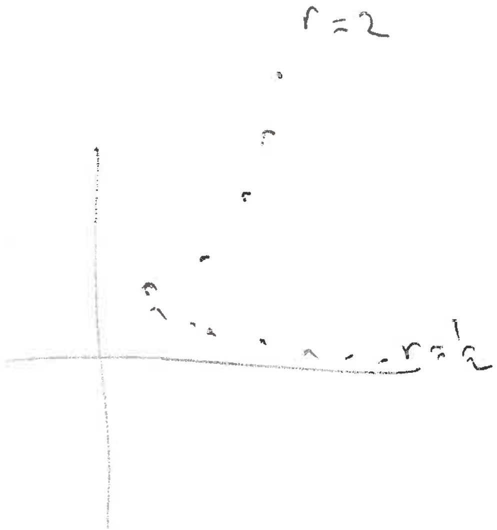
$$\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15} > \frac{1}{16}+\cdots+\frac{1}{16} = \frac{1}{2}$$

So $\displaystyle\sum_{n=1}^{\infty} a_n > \sum_{n=1}^{\infty} \frac{1}{2} = \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$ diverges

Hence

$$\sum_{n=1}^{\infty} a_n = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots \quad \text{diverges}$$

Geometric series



Given a real number $r \in \mathbb{R}$, the *geometric series* is

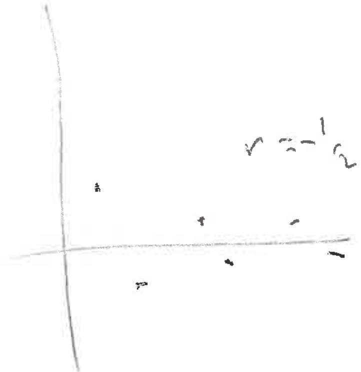
$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots$$

The test for divergence tells us that

$a_n = r^n \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} r^n$ converges to zero if $|r| < 1$

If $|r| < 1$ what is

$$s_N = 1 + r + r^2 + \cdots + r^N?$$



Geometric series

It's easy to verify that

$|r| < 1$

$$(1 + r + r^2 + \cdots + r^N)(1 - r) = 1 - r^{N+1}$$

Expand : $1 - \cancel{r} + \cancel{r} - \cancel{r^2} + \cancel{r^2} - \cancel{r^3} \ldots - r^{N+1}$ ✓

if and only if

Hence

$$s_N = 1 + r + r^2 + \cdots + r^N = \frac{1 - r^{N+1}}{1 - r} \quad \to \quad \frac{1}{1-r} \quad \text{iff} \quad |r| < 1$$

Example:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots$$

So $r = \frac{1}{2}$ and $|\frac{1}{2}| < 1$ ✓✓ so sum $= \dfrac{1}{1-\frac{1}{2}} = 2$

Series laws

If $S_a = \sum_{n=1}^{\infty} a_n$ and $S_b = \sum_{n=1}^{\infty} b_n$ are *convergent* series, then

- $S_{a\pm b} = S_a \pm S_b$
- $S_{ka} = kS_a, \quad k$ constant
- Note: $S_a S_b \neq S_{ab}$ and $S_a/S_b \neq S_{a/b}$

$$\sum_{n=1}^{\infty}(a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = S_a + S_b$$

Examples

$$\sum_{n=0}^{\infty} \frac{1}{2^n} + \frac{1}{3^n} = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n + \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n = \frac{1}{1-\frac{1}{2}} + \frac{1}{1-\frac{1}{3}} = 2 + \frac{3}{2} = \frac{7}{2}$$

$$\sum_{n=1}^{\infty} \frac{3}{4^n} = 3\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n = 3\left[\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n - 1\right] = 3\left[\frac{1}{1-\frac{1}{4}} - 1\right] = 3\left[\frac{4}{3}-1\right] = 3\cdot\frac{1}{3} = 1$$

Note: If $\sum_{n=1}^{\infty} a_n$ or $\sum_{n=1}^{\infty} b_n$ *diverges*, so does $\sum_{n=1}^{\infty}(a_n \pm b_n)$.
Note: If $\sum_{n=1}^{\infty} a_n$ *diverges*, so does $\sum_{n=1}^{\infty} ka_n$

Examples of series

Geometric series

$-1 < r < 1$

Converges to $\frac{1}{1-r}$

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots$$

$r < -1$ or $r > 1$

Diverges

Harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$ Diverges

p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots \qquad p \in \mathbb{R} \text{ is a constant}$$

$p = 1 \rightarrow$ Harmonic series $\rightarrow$ Diverges

Some more examples

$$\sum_{n=1}^{\infty} \frac{5}{n} = 5 \sum_{n=1}^{\infty} \frac{1}{n} = 5 \times (\text{Divergent series}) \quad \text{so diverges}$$

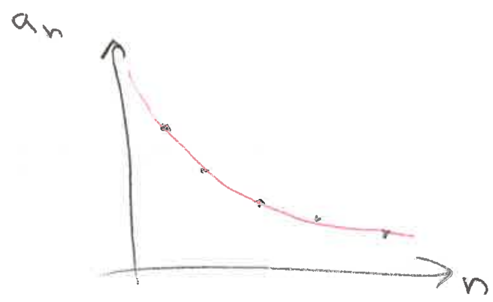
$$\sum_{n=0}^{\infty} \frac{2(3^n) - 3(2^n)}{4^n} = 2\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n - 3\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 2\left(\frac{1}{1-\frac{3}{4}}\right) - 3\left(\frac{1}{1-\frac{1}{2}}\right) = 2\times 4 - 3\times 2 = 2$$

$$\sum_{n=0}^{\infty} \frac{2^n + 4^n}{3^n} = \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n + \boxed{\sum_{n=0}^{\infty}\left(\frac{4}{3}\right)^n}$$

Diverges

$r = \frac{4}{3} > 1$

The integral test



Ingredients:

- A sequence $a_1, a_2, a_3, \ldots$ of *positive* values
- A *positive, decreasing continuous* function f such that

$$f(n) = a_n$$

Then

$$\sum_{n=1}^{\infty} a_n \qquad \text{and} \qquad \int_1^{\infty} f(x)dx$$

either *both* converge or *both diverge*.

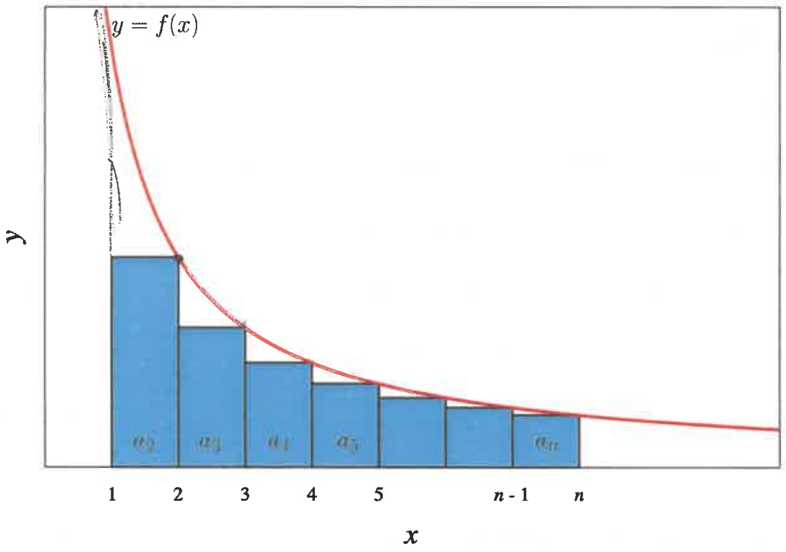
Note that the integral is *improper* so it may not exist (converge).

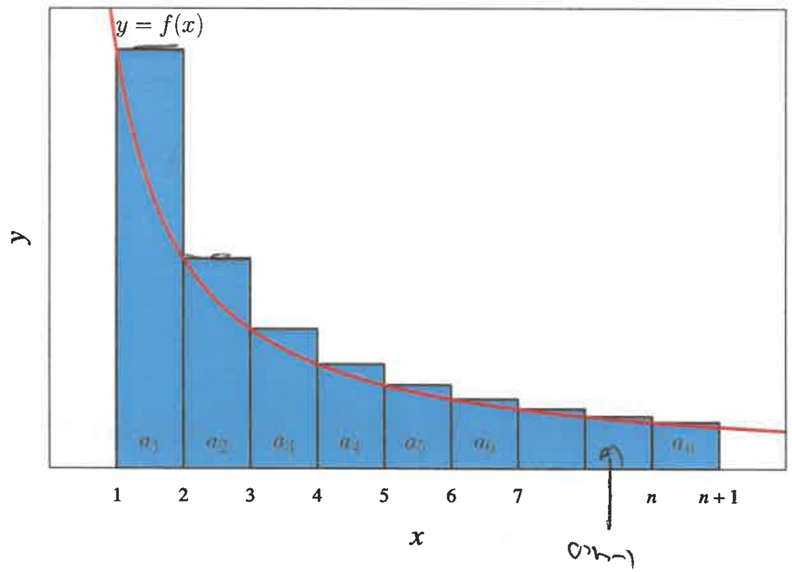
However, if the integral, and hence the sum, converge, the value of the integral tells us *nothing* about the value of the series.

The idea behind the proof

Considering Riemann sums, we get

$$a_2 + \ldots + a_n \;\leq\; \int_1^n f(x)dx \;\leq\; a_1 + a_2 + \ldots + a_{n-1}$$





$$s_n - a_1 = a_2 + \ldots + a_n \leq \int_1^n f(x)dx \leq a_1 + a_2 + \ldots + a_{n-1} = s_{n-1}$$

$s_n \leq a_1 + \int_1^n f(x)dx$

Note that the integral $\int_1^t f(x)dx$ is an *increasing* continuous function of t as $f(x)$ is *positive.*

- If the integral is unbounded then

  s_{n-1} unbounded so series diverges

- If the integral is bounded then

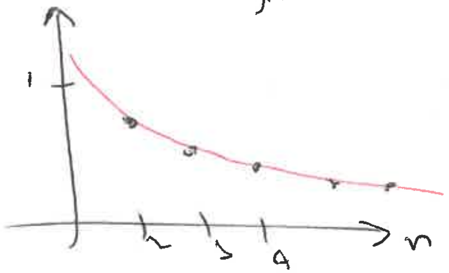
  s_n bounded so series converges

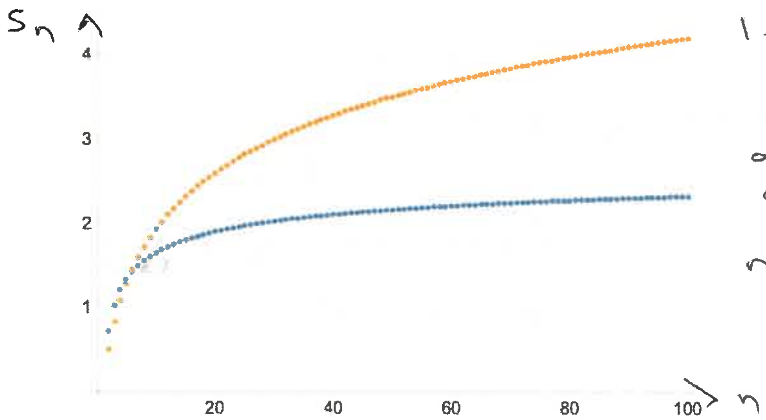
Example

Is the following series convergent?

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

$a_n = \frac{1}{n \ln n}$





$1 + \frac{1}{2} + \frac{1}{3} + \dots = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Comparison with harmonic series

$\frac{d}{dx}\left[\ln(\ln(x))\right] = \frac{1}{\ln(x)} \frac{d}{dx} \ln(x) = \frac{1}{\ln(x)} \cdot \frac{1}{x}$

$= \frac{1}{x \ln(x)}$

The corresponding integral is

$$\int_2^t \frac{1}{x \ln x} = [\ln(\ln(x))]_2^t = \ln(\ln(t)) - \ln(\ln(2))$$

$\ln(\ln(t))$ goes to ∞ hence series diverges

Another example: p-series

Is the following series convergent?

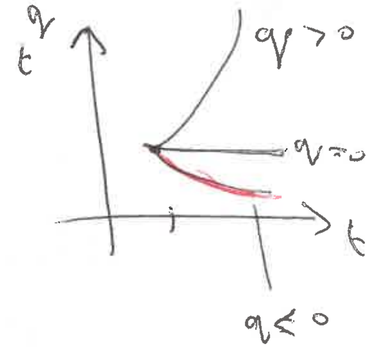
$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

Satisfies conditions for integral test

The corresponding integral is

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx = \lim_{t\to\infty} \left[\frac{x^{1-p}}{1-p} \right]_1^t$$

$$= \lim_{t\to\infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p}$$



$q = 1-p$ and $q<0 \Rightarrow 1-p<0$ we want $q<0$

$\Rightarrow p>1$ for convergence

and $p<1$ then divergence

Example continued

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Harmonic series diverges
($p=1$)

$$\sum_{n=1}^{\infty} \frac{1}{n^{1.0001}}$$

$p = 1.0001 > 1$ so converges

converges to $\pi^2/6$ $\longleftarrow$

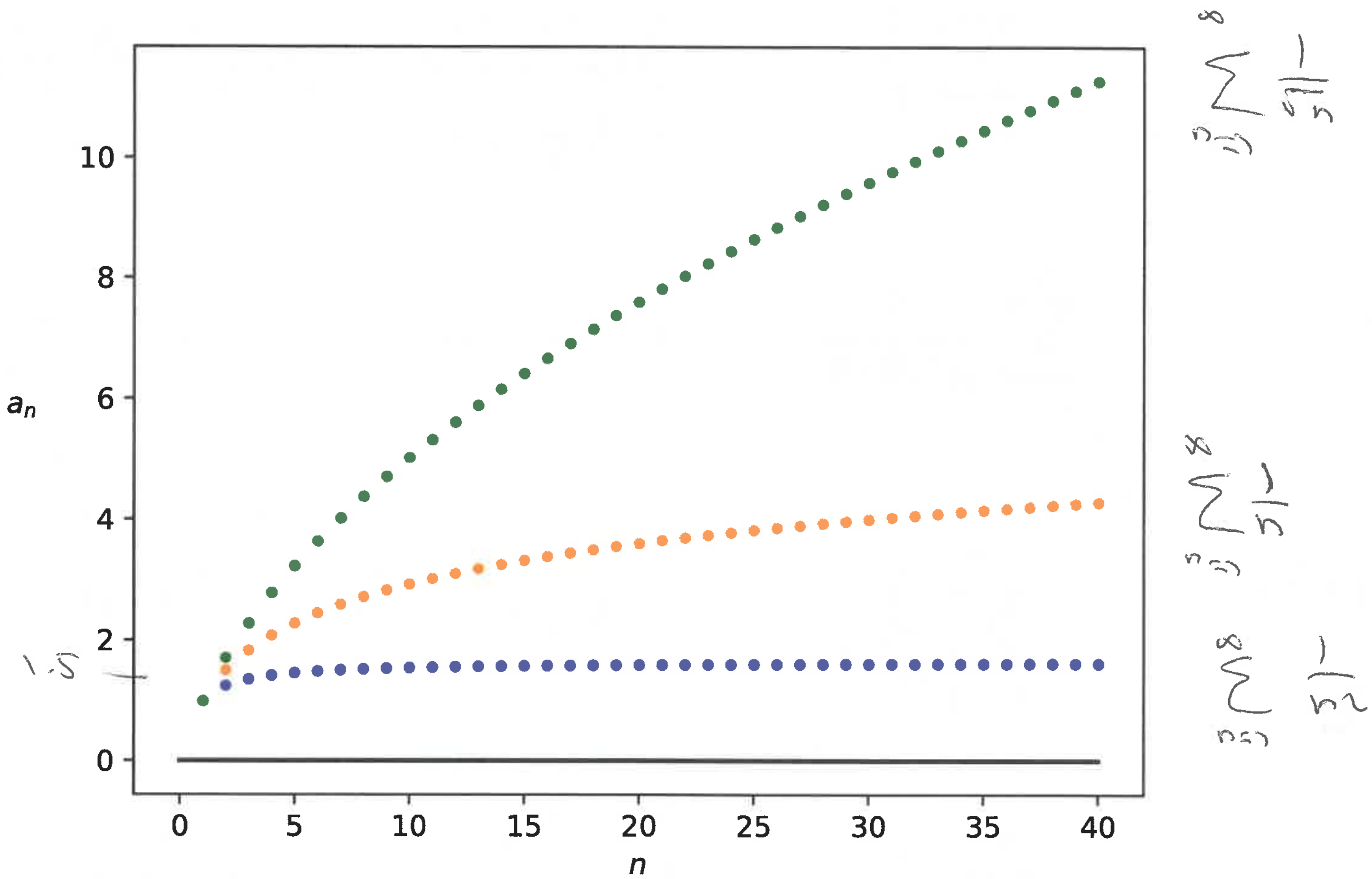
$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$p=2$ so converges

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

so $p = \frac{1}{2} < 1 \Rightarrow$ diverges

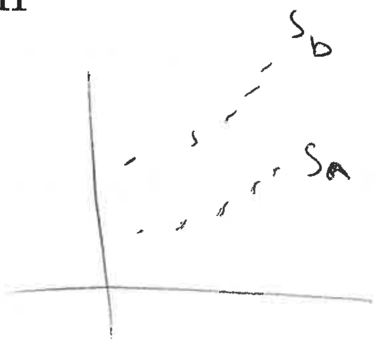
Graphically



The comparison test

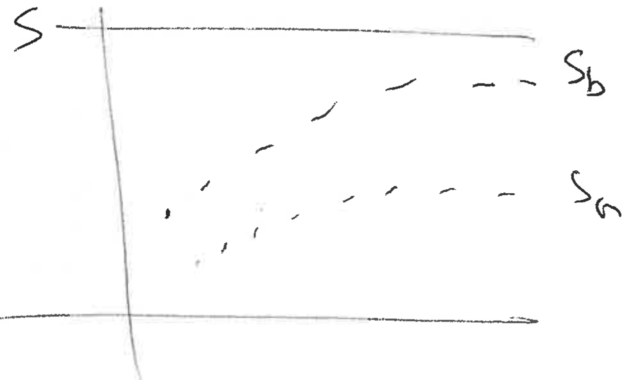
Suppose that $0 \leq a_n \leq b_n$ for all *sufficiently large* n. Then

- If $\sum_{n=1}^{\infty} a_n$ is divergent then $\sum_{n=1}^{\infty} b_n$ is divergent
- If $\sum_{n=1}^{\infty} b_n$ is convergent then $\sum_{n=1}^{\infty} a_n$ is convergent



In short,

- Larger than divergent is divergent
- Smaller than convergent is convergent

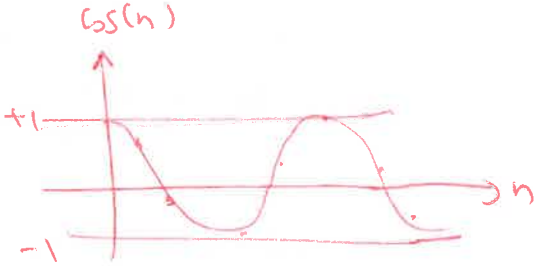


The comparison test applies to series with *non-negative terms*.

Some examples

Up to now we know convergence status of geometric series and p-series, so we try to compare with those.

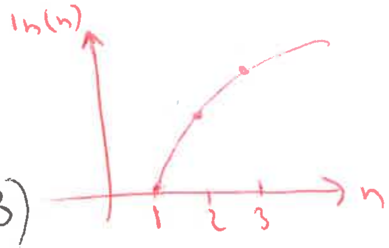
Are the following series convergent?



$$\sum_{n=1}^{\infty} \frac{|\cos(n)|}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$ which converges $(p=2)$

$$\sum_{n=1}^{\infty} \frac{\ln n}{n} > \sum_{n=1}^{\infty} \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^3}$$ which converges $(p=3)$



Limit comparison test

Series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, with $a_n \geq 0$ and $b_n > 0$. Let

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

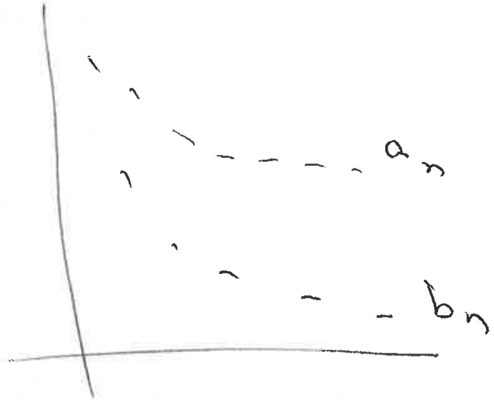
Either

- $c = 0$
  The a's are *much smaller* than the b's

  If $\sum b_n$ converges, then so does $\sum a_n$

- $0 < c < \infty$
  The a's and b's are (roughly) the same size

  If one series converges then so does the other



- $c = \infty$ (meaning a_n/b_n diverges to ∞)
  The a's are *much bigger* than the b's

  If $\sum a_n$ converges, then so does $\sum b_n$

EXAMPLE

$$\sum_{n=2}^{\infty} \frac{1}{n^3 - 1} \qquad a_n = \frac{1}{n^3 - 1}$$

Select a series to compare to:

$$\sum_{n=2}^{\infty} \frac{1}{n^3} \qquad b_n = \frac{1}{n^3}$$

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^3-1}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^3}{n^3-1}$$

$$= \lim_{n\to\infty} \frac{1}{1 - \frac{1}{n^3}} = \frac{1}{1-0} = 1$$

we know that $\sum_{n=2}^{\infty} b_n$ converges (p-series with $p=3$)

Hence $\sum_{n=2}^{\infty} a_n$ also converges

Another example

Given a series with possibly complicated terms, identify the *dominant* terms.

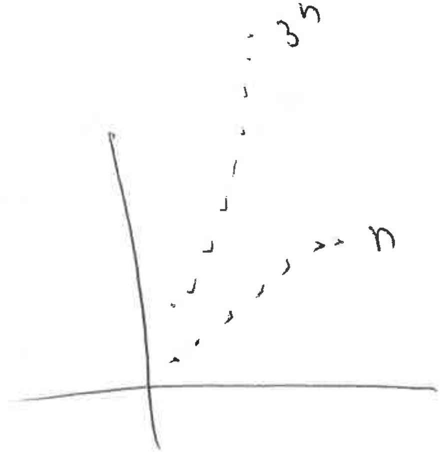
$$\sum_{n=1}^{\infty} \frac{1}{3^n - n}$$

$$a_n = \frac{1}{3^n - n}$$

Compare with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ $\qquad b_n = \frac{1}{3^n}$

Then $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{3^n - n}}{\frac{1}{3^n}} = \lim_{n\to\infty} \frac{3^n}{3^n - n}$

$$= \lim_{n\to\infty} \frac{1}{1 - \frac{n}{3^n}} = \frac{1}{1-0} = 1$$



And $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{1-\frac{1}{3}}$ $\qquad$ Geometric series

$$= 3/2$$

Yet another example

$$\sum_{n=2}^{\infty} \frac{3\sqrt{n}+5}{n^2-1}$$

$$a_n = \frac{3\sqrt{n}+5}{n^2-1}$$

Compare with $\displaystyle\sum_{n=2}^{\infty} \frac{3\sqrt{n}}{n^2}$ $\qquad b_n = \frac{3\sqrt{n}}{n^2}$

Then $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3\sqrt{n}+5}{n^2-1} \cdot \frac{n^2}{3\sqrt{n}}$

$$= \lim_{n\to\infty} \frac{3n^{\frac{5}{2}} + 5n^2}{3n^{5/2} - 3n^{\frac{1}{2}}} = \lim_{n\to\infty} \frac{3 + 5/n^{\frac{1}{2}}}{3 - 3/n^2}$$

$$= \frac{3-0}{3-0} = 1$$

$$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{3\sqrt{n}}{n^2} = 3\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$$

p - series with

$p = \frac{3}{2} > 1$

$\Rightarrow \sum_{n=2}^{\infty} b_n$ converges

Hence, by the limit comparison test,

$$\sum_{n=2}^{\infty} a_n = \sum_{n=}^{\infty} \frac{3\sqrt{n}+5}{n^2-1} \quad \text{converges}$$

Alternating series

Series where the terms alternate in sign. For example,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \quad = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$\parallel$

a_n

These are usually viewed as taking the *alternating sum*

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

$$a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$$

So $a_{n+1} < a_n$ ✓

of a positive sequence (a_n).

The above series has

$$a_n = \frac{1}{n}$$

The alternating series test

Alternating series behave well if (a_n) satisfies:

- $a_n \geq 0$,
- $\lim_{n\to\infty} a_n = 0$,
- $a_{n+1} \leq a_n$ (eventually)

then

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

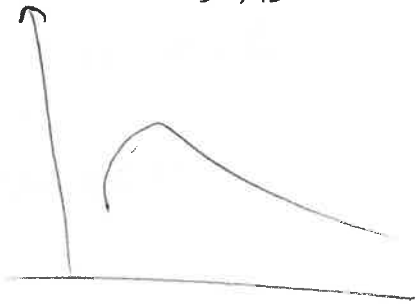
converges.

Example:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

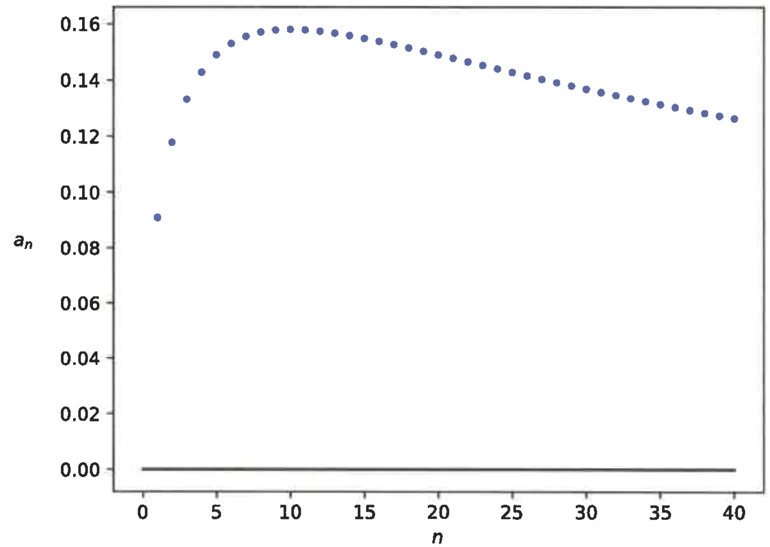
Example

$f(x) = \frac{\sqrt{x}}{x+10}$



a_n

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sqrt{n}}{n+10}$$



$a_{n+1} < a_n$

for $n \geq 10$

$\frac{\sqrt{n}}{n+10}$ is decreasing for $n \geq 10$

$$\lim_{n=\infty} \frac{\sqrt{n}}{n+10} = \lim_{n=\infty} \frac{1/\sqrt{n}}{1+\frac{10}{n}} \to \frac{0}{1+0} = 0$$

So by alternating series test, $\sum_{n=1}^{\infty} a_n$ converges

Absolute convergence

The series

$$\sum_{n=1}^{\infty} a_n \quad = \quad 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} \dots$$

is *absolutely convergent* if

$$\sum_{n=1}^{\infty} |a_n| \quad = \quad 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

is convergent.

Theorem: If the series

$$\sum_{n=1}^{\infty} a_n$$

is absolutely convergent then it is also convergent.

Conditional convergence

There are three possible situations:

- $\sum_{n=1}^{\infty} a_n$ diverges (so $\sum_{n=1}^{\infty} |a_n|$ also diverges).
- $\sum_{n=1}^{\infty} a_n$ converges
  - $\sum_{n=1}^{\infty} |a_n|$ converges. The series is *absolutely convergent*.
  - $\sum_{n=1}^{\infty} |a_n|$ diverges. The series is *conditionally convergent*.

Examples

The series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

is *conditionally convergent* because it converges but

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$ Harmonic Series diverges

The series

$-1 < r < -1$ Geometric $$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = \frac{1}{1+\frac{1}{2}}$$

is *absolutely convergent* because it converges and $= \frac{2}{3}$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$ also converges

$-1 < r < 1$

Geometric $= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$

MATH1012 Mathematical Theory and Methods

Week 8

The ratio test

Suppose (a_n) is a sequence such that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$$

- $L < 1$ means that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

  converges and $\sum_{n=1}^{\infty} |a_n|$ converges

- $L = 1$ gives no information
- $L > 1$ means that $\sum_{n=1}^{\infty} a_n$ is divergent

The proof is in the unit reader

EXAMPLE

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

$a_n = \frac{n^2}{2^n}$ so $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2}\right|$

So $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n^2+2n+1}{2n^2}\right|$

$= \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{1}{n^2}}{2} = \frac{1+0+0}{2} = \frac{1}{2}$

So $L = \frac{1}{2} < 1 \Rightarrow$ convergent

Example

Is the following series convergent?

$$\sum_{n=1}^{\infty}(-1)^n\frac{2^n}{n^2}$$

$a_n = \dfrac{(-1)^n 2^n}{n^2}$ so $\lim \left|\dfrac{a_{n+1}}{a_n}\right|$

$$= \lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)^2}\cdot\frac{n^2}{2^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2n^2}{n^2+2n+1}\right| = 2 > 1$$

So series diverges.

Example

The ratio test doesn't always work.

$$\sum_{n=1}^{\infty}(-1)^n \frac{n+1}{n^2}$$

$$a_n = \frac{(-1)^n (n+1)}{n^2} \Rightarrow \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \frac{n+2}{(n+1)^2} \cdot \frac{n^2}{n+1}$$

$$= \lim_{n\to\infty} \frac{n^3+2n^2}{n^3+\dots}$$

$$= \lim_{n\to\infty} \frac{1+2/n}{1+\dots} \rightarrow \frac{1}{1} = 1$$

Example

Is the following series convergent?

$$\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$$

We have

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{2^{n+1}+5}{3^{n+1}} \frac{3^n}{2^n+5} \\
&= \frac{1}{3} \frac{2^{n+1}+5}{2^n+5} \\
&= \frac{1}{3} \frac{2+\frac{5}{2^n}}{1+\frac{5}{2^n}}
\end{aligned}$$

So the limit as $n \to \infty$ is $L = \frac{2}{3}$, and hence series converges

What is the sum?

$$\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$$

Recall the geometric series

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \qquad -1 < r < 1$$

$$\sum_{n=0}^{\infty} \frac{2^n}{3^n} + \sum_{n=0}^{\infty} \frac{5}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n + 5 \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$$

$$= \frac{1}{1-\frac{2}{3}} + 5\left(\frac{1}{1-\frac{1}{3}}\right)$$

$$= 3 + \frac{15}{2} = \frac{21}{2}$$

Power series

A power series is a series of the form

$$\sum_{n=0}^{\infty} b_n (x-c)^n$$

where x is a variable, (b_n) is a sequence of *coefficients* and c is some fixed number, called the *centre* of the power series.

This is a function of x for all the values of x where the series is convergent.

$$f(x) = b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + \cdots$$

An important application is the calculation of function values by computers/calculators.

There are many important applications in science, engineering, economics, etc.

Using the ratio test

Let $a_n = b_n(x-c)$. Then

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{b_{n+1}(x-c)^{n+1}}{b_n(x-c)^n}\right| = \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| |x-c|$$

Let

$$R = \lim_{n\to\infty}\left|\frac{b_n}{b_{n+1}}\right| \qquad \Rightarrow \qquad L = \frac{|x-c|}{R}$$

There are three cases to consider

- $R = 0$

  $L = \infty$ unless $x = c$

- R does not exist (that is, the limit goes to infinity)

  $L = 0$ for any x

- R is a positive real number

Convergence

$$\sum_{n=0}^{\infty} b_n (x-c)^n$$

Number $R = \lim_{n\to\infty}\left|\frac{b_n}{b_{n+1}}\right|$ is called the *radius of convergence* of the power series.

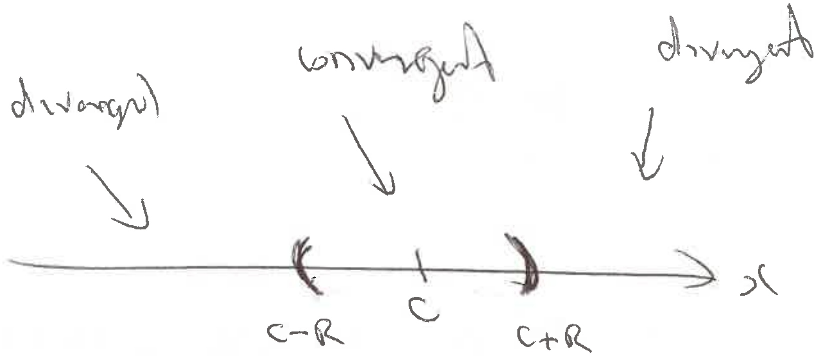
There are three possible behaviours:

- Absolutely convergent when $x = c$, divergent otherwise

  $R = 0$

- Absolutely convergent for all x

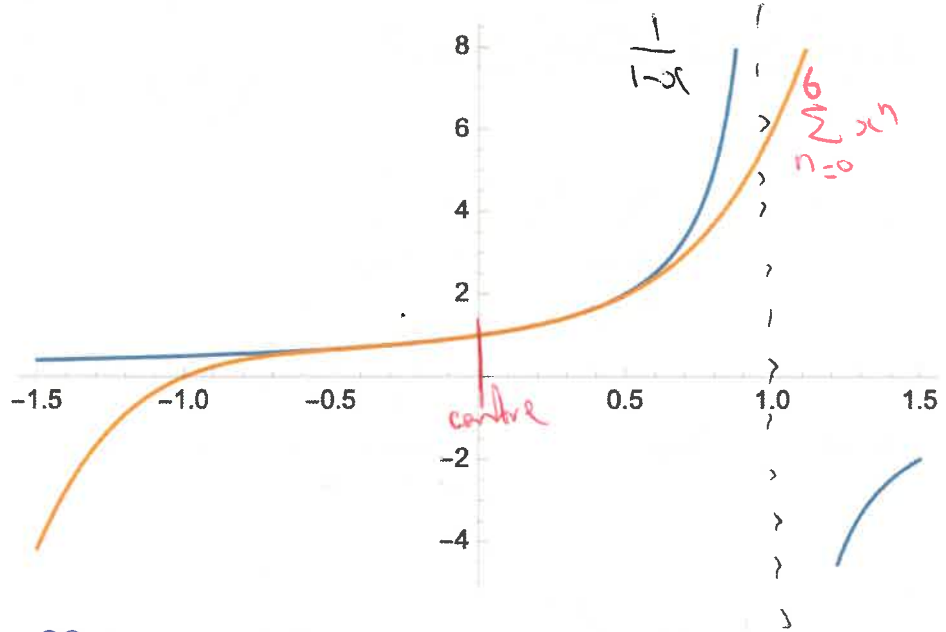
  $R = \infty$



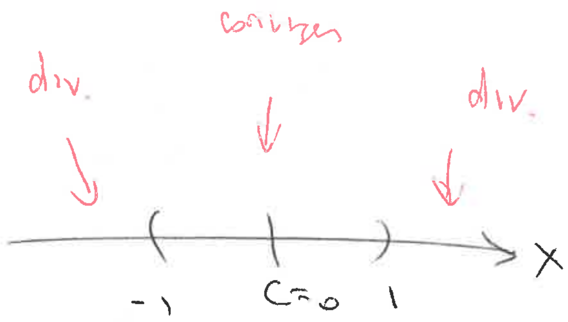
- If R is a positive real number
  - Absolutely convergent if $|x - c| < R$
  - Divergent if $|x - c| > R$
  - Has to be determined case by case if $|x - c| = R$

What's the point?

Where do polynomial approximations converge?



$$\sum_{n=0}^{\infty} b_n(x-c)^n = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{if} \qquad |x| < 1$$



Here we have $b_n = 1$ for all n, $c = 0$ and

$$R = \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left|\frac{1}{1}\right| = \lim_{n\to\infty} 1 = 1$$

More complicated example

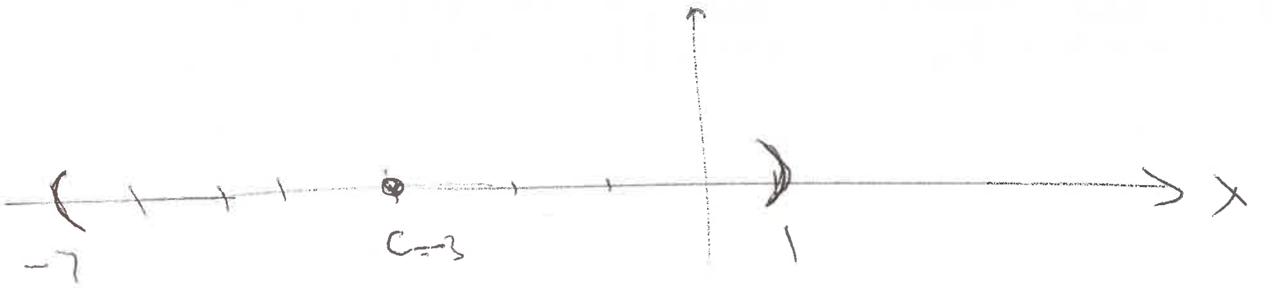
For which x is the following series absolutely convergent, conditionally convergent, divergent?

$$\sum_{n=0}^{\infty} b_n (x-c)^n = \sum_{n=0}^{\infty} \overbrace{\frac{(x+3)^n}{(n+1)4^n}}^{b_n}$$

Centre $c=-3$ and $b_n = \dfrac{1}{(n+1)4^n}$

So $\dfrac{b_n}{b_{n+1}} = \dfrac{(n+2)4^{n+1}}{(n+1)4^n} = \dfrac{4(n+2)}{n+1}$

$$R = \lim_{n\to\infty} \left|\frac{b_n}{b_{n+1}}\right| = \lim_{n\to\infty} \frac{4n+8}{n+1} = \lim_{n\to\infty} \frac{4+\frac{8}{n}}{1+\frac{1}{n}} = \frac{4+0}{1+0} = 4$$



what about $x=1$ and $x=-7$

So the power series:

- converges if $|x+3| < 4$
- diverges if $|x+3| > 4$
- What happens at the boundary?

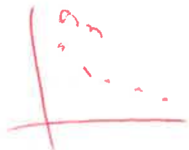
  - If $x+3=4 \ \rightarrow\ x=1$ then the sum is

Diverges

Harmonic series

$$\sum_{n=0}^{\infty}\frac{(x+3)^n}{(n+1)4^n}=\sum_{n=0}^{\infty}\frac{\cancel{4^n}}{(n+1)\cancel{4^n}}=\sum_{n=0}^{\infty}\frac{1}{n+1} = 1+\frac{1}{2}+\frac{1}{3}+\cdots$$

which diverges.



  - If $x+3=-4$, that is, $x=-7$ the sum is

$$\sum_{n=0}^{\infty}\frac{(x+3)^n}{(n+1)4^n}=\sum_{n=0}^{\infty}\frac{(-4)^n}{(n+1)4^n}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} = \sum_{n=0}^{\infty}(-1)^n \overbrace{\frac{1}{n+1}}^{a_n}$$

Converges

which is conditionally convergent.

The *interval of convergence* is $[-7,1)$.

Summary

$$\sum b_n (x-c)^n$$

For a power series:

- Identify its centre (the value c)
- Find its radius of convergence R $= \lim_{n\to\infty} \left| \frac{b_n}{b_{n+1}} \right|$
- If $0 < R < \infty$ check $x = c - R$, $x = c + R$ separately

> When x lies in the interval of convergence, the power series behaves exactly like a giant polynomial.

Another example

For which x is the following series absolutely convergent, conditionally convergent, divergent?

$c = 1$ and $b_n = \frac{1}{n^2}$

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2}$$

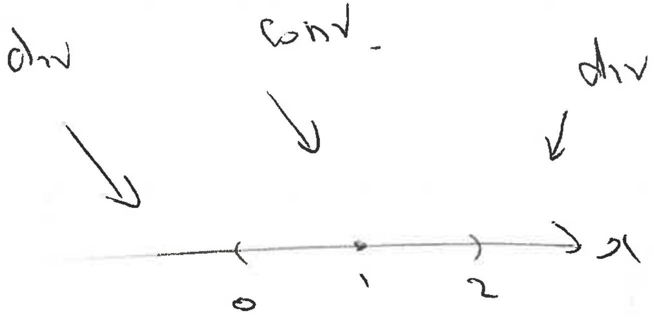
Radius of convergence $R = \lim_{n\to\infty} \left| \frac{b_n}{b_{n+1}} \right|$

$= \lim_{n\to\infty} \left| \frac{1}{n^2} \cdot \frac{(n+1)^2}{1} \right|$

$= \lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2} \right|$

$= \lim_{n\to\infty} \left| \frac{n^2+2n+1}{n^2} \right|$

$= \lim_{n\to\infty} \left| 1 + \frac{2}{n} + \frac{1}{n^2} \right| = 1$



So the power series:

- converges if $|x-1| < 1$, that is, $0 < x < 2$
- diverges if $|x-1| > 1$, that is, $x < 0$ or $x > 2$
- What happens at the boundary?

  - If $x = 0$ then the sum is

    $\sum (-1)^n a_n$ with $a_n = \frac{1}{n^2} \to 0$

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -1 + \frac{1}{4} - \frac{1}{9} + \cdots$$

    which converges absolutely.

  - If $x = 2$ the sum is

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

    p series with $p = 2 > 1$

    which converges absolutely.

The *interval of convergence* is $[0, 2]$.

14b

Differentiation

For a power series with radius of convergence $R > 0$:

- the series can be differentiated term by term
- the radius of convergence of the derived series is also R

$$f(x) = b_0 + b_1(x-c) + b_2(x-c)^2 + \ldots + b_n(x-c)^n + \cdots$$

$$f'(x) = b_1 + 2b_2(x-c) + 3b_3(x-c)^2 + \ldots + nb_n(x-c)^{n-1} + \cdots$$

In summation notation,

$$f(x) = \sum_{n=0}^{\infty} b_n(x-c)^n, \qquad f'(x) = \sum_{n=1}^{\infty} nb_n(x-c)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)b_n(x-c)^{n-2} \qquad \text{etc.}$$

Taylor series

Given the values of a function $f(x)$ and its derivatives at $x = c$, the power series

$$f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

is called the Taylor series of f about c (or centred at c).

The partial sum

$n! = 1 \times 2 \times 3 \times \cdots \times n$

$$T_{N,c}(x) = \sum_{n=0}^{N} \frac{f(x)^{(n)}}{n!}(x-c)^n$$

is called the *Taylor polynomial of degree N*.

The closer x is to c and the larger N is, the better the approximation of $f(x)$ by $T_{N,c}(x)$.

Example

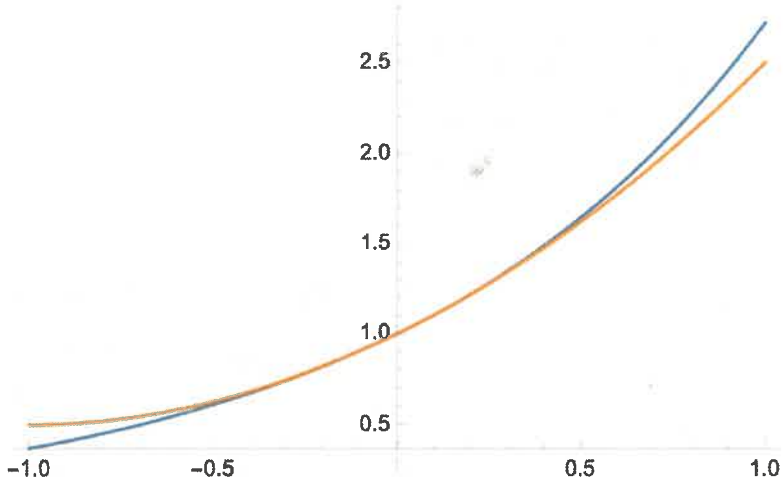
The Taylor series of $f(x) = e^x$ centred at $c = 0$. We have

$$f^{(n)}(x) = e^x \qquad \text{and hence} \qquad f^{(n)}(0) = 1$$

so the Taylor series of e^x centered at 0 is

$$T(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Here is a plot of $f(x) = e^x$ and $T_{2,0}(x) = 1 + x + \frac{x^2}{2}$



$f(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x$

$\vdots$

$f(0) = 1$

$f'(0) = 1$

$f''(0) = 1$

$\vdots$

Convergence

The series is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

For which values of x does this series converge?

$$R = \lim_{n\to\infty} \left| \frac{b_n}{b_{n+1}} \right| \quad \text{with} \quad b_n = \frac{1}{n!}$$

$$= \lim_{n\to\infty} \frac{1}{n!} \cdot \frac{(n+1)!}{1}$$

$$= \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} \frac{\cancel{1\cdot 2\cdot 3\cdot \ldots \cdot n}\cdot(n+1)}{\cancel{1\cdot 2\cdot 3 \cdot \ldots \cdot n}}$$

$$= \lim_{n\to\infty} (n+1) = \infty \quad \text{So} \quad \boxed{R = \infty}$$

The remainder

Suppose $T_{n,c}(x)$ is the Taylor polynomial of degree n centred at c for a function $f(x)$. Then

$$f(x) = T_{n,c}(x) + R_{n,c}(x)$$

where

$$R_{n,c}(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$

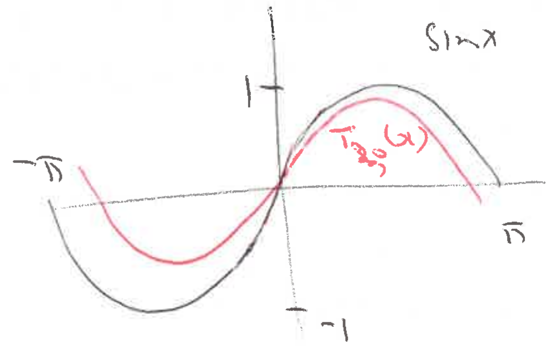
and z is somewhere between c and x.

The function $R_{n,c}(x)$ is the *'remainder'* or *'error term'* obtained by truncating the series at degree n.

We don't know the exact value of z, just that it is close to c. However this is often enough to find a good bound on the error, and hence deduce how many terms are needed to get a sufficiently accurate approximation.

Example

The Taylor series for $f(x) = \sin x$ about $c = 0$



$f'(x) = \cos x$ $\qquad$ $f'(0) = 1$

$f''(x) = -\sin x$ $\qquad$ $f''(0) = 0$

$f'''(x) = -\cos x$ $\qquad$ $f''(0) = -1$

$f^{iv}(x) = \sin x$ $\qquad$ $f^{iv}(x) = 0$

$\vdots$

Hence

odd numbers

$$T(x) = 1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1}$$

$n=0$ $\quad$ $n=1$ $\quad$ $n=2$ $\quad$ $n=3$

Its radius of convergence

$$b_n = \frac{(-1)^n}{(2n+1)!}$$

and hence

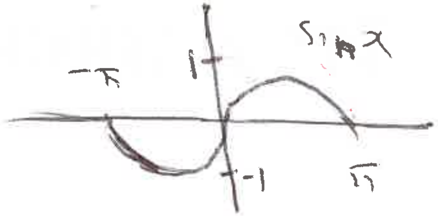
$$\left|\frac{b_n}{b_{n+1}}\right| = \left| \frac{(-1)^n}{(2n+1)!} \cdot \frac{(2(n+1)+1)!}{(-1)^{n+1}} \right|$$

$$= \left| - \frac{(2n+3)!}{(2n+1)!} \right|$$

$$= \frac{\cancel{1 \cdot 2 \cdot 3 \cdots (2n+1)}\,(2n+2)(2n+3)}{\cancel{1 \cdot 2 \cdot 3 \cdots (2n+1)}}$$

So $R = \lim_{n \to \infty} (2n+2)(2n+3) = \infty$

The size of the remainder on $|x| \le \pi$



We have

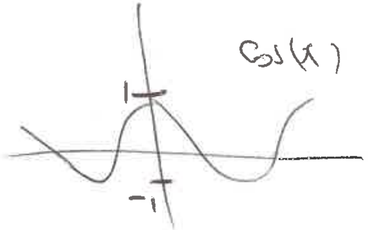
$$|R_{n,0}(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right|$$

$|\sin(x)| \le 1$

$|\cos(x)| \le 1$

The derivatives of $f(x) = \sin x$ are either $\pm \sin x$ or $\pm \cos x$, so

$$\left| f^{(n+1)}(z) \right| \le 1 \qquad \text{for all } \; n$$

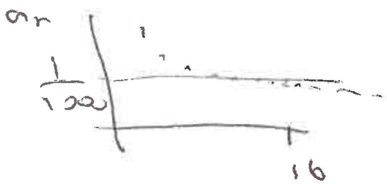


and hence

$$|R_{n,0}(x)| \; = \; \frac{1|x|^{n+1}}{(n+1)!} \le \frac{\pi^{n+1}}{(n+1)!} \qquad \text{for} \qquad |x| \le \pi$$

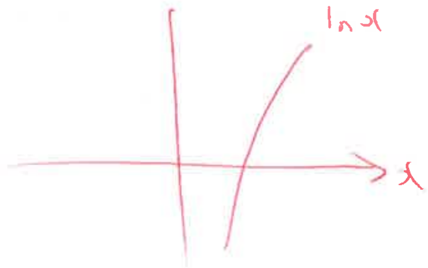
If we wish this to be smaller than say 0.001, then we must have

$$a_n \simeq \frac{\pi^{n+1}}{(n+1)!} < \frac{1}{1000} \quad \Rightarrow \quad 1000\, \pi^{n+1} < (n+1)! \quad \Rightarrow \quad n \ge 16$$



Example

$f(1) = 0$



The Taylor series for $f(x) = \ln x$ about $\underline{c = 1}$

$f'(x) = \frac{1}{x}$ $\quad$ $f'(1) = 1$

$f''(x) = -\frac{1}{x^2}$ $\quad$ $f''(1) = -1$

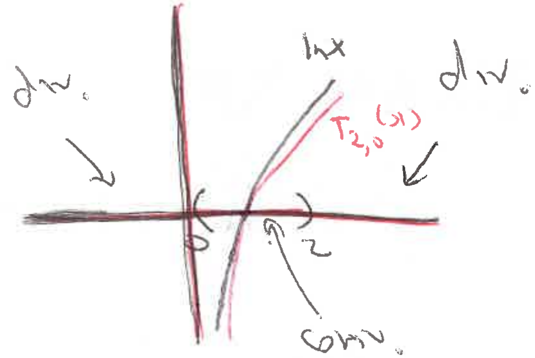
$f'''(x) = \frac{2}{x^3}$ $\quad$ $f'''(1) = 2 = 1.2$

$f^{iv}(x) = -\frac{6}{x^4}$ $\quad$ $f^{iv}(1) = -6 = 1.2.3$

Hence

$$T(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

Its radius of convergence



$$b_n = \frac{(-1)^{n+1}}{n}$$

So

$$\left|\frac{b_n}{b_{n+1}}\right| = \left|\frac{(-1)^{n+1}}{n} \cdot \frac{n+1}{(-1)^{n+2}}\right| = \left|\frac{n+1}{n}\right| \to 1 = R$$

Hence the Taylor series converges for

$$|x-1| < 1 \qquad \Longleftrightarrow \qquad 0 < x < 2$$

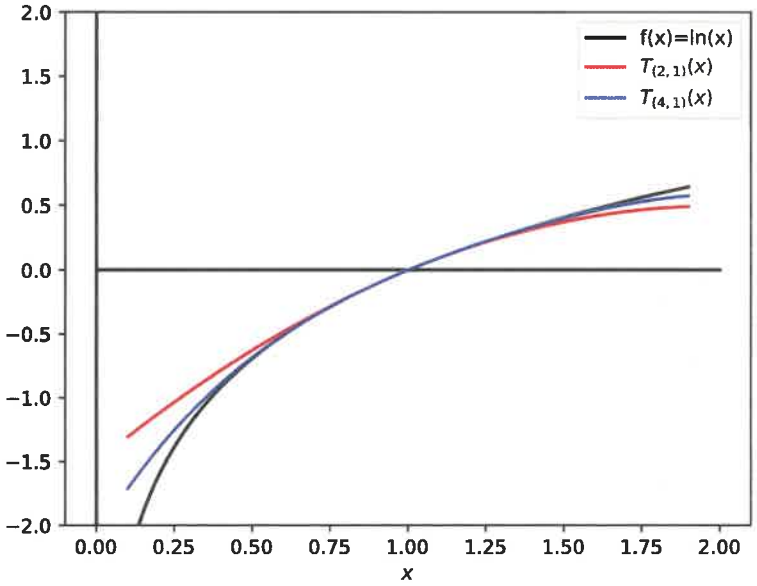
Note that at $x = 2$ the Taylor series is

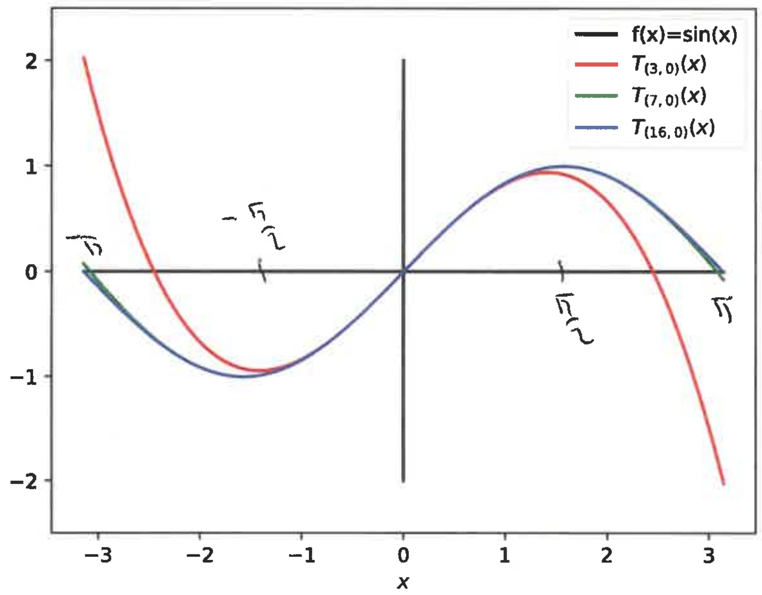
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \equiv -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \cdots$$

which is alternating and $\frac{1}{n} \to 0$ as $n \to \infty$ so converges. Hence

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2 \approx 0.69$$

Graphical comparisons





$$T_3 = x - \frac{x^3}{6}$$

MATH1012 Mathematical Theory and Methods

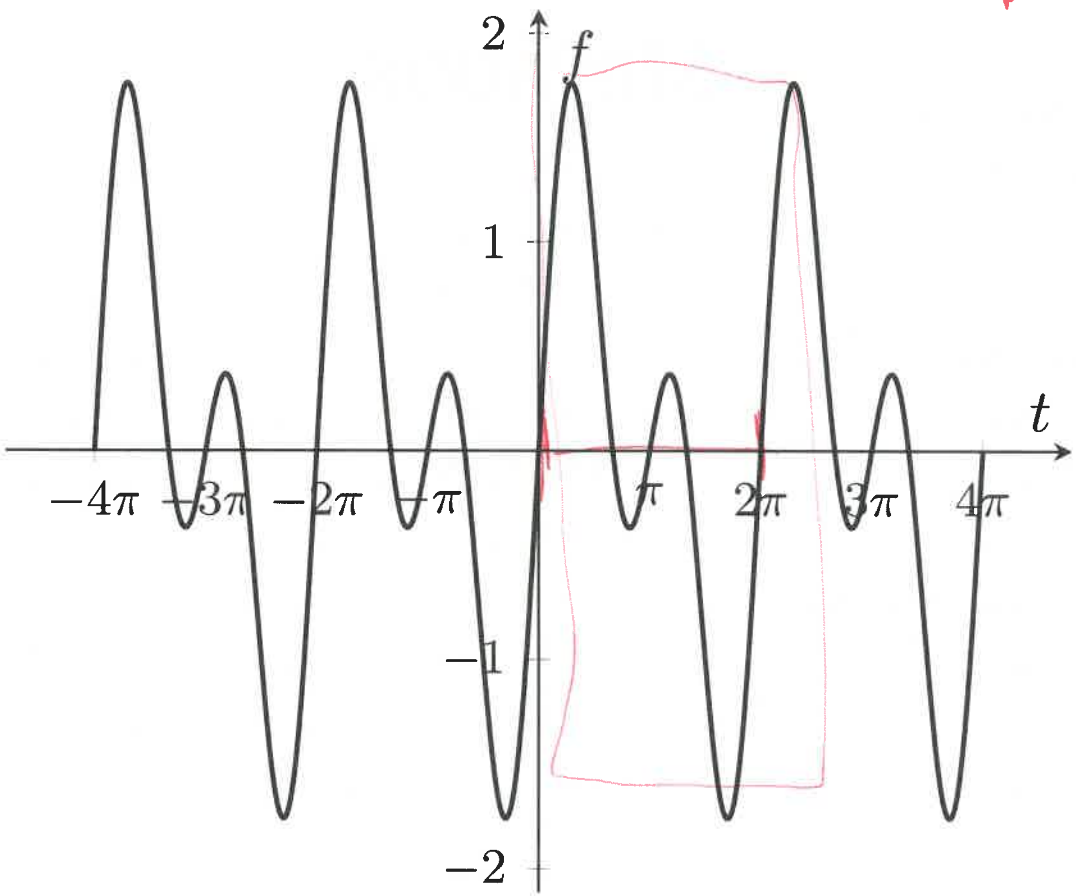
Week 9

Fourier Series

A function is *periodic* with period P if, for all t,

$$f(t+P) = f(t)$$

$P = 2\pi$

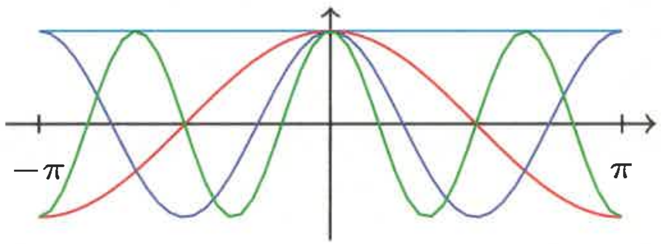


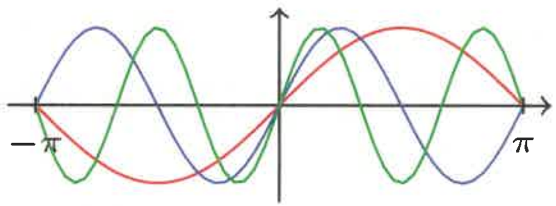
Fourier series

Many naturally occurring functions (waveforms) are periodic. For example, sound waves, ocean waves, heart rhythms etc.

Our aim is to express any given 2π-periodic function f as being a *linear combination* of:

$$1, \cos t, \cos 2t, \cos 3t, \ldots, \sin t, \sin 2t, \sin 3t, \ldots$$





In "waveform" language, we view the *signal* as being a combination of different *frequencies*.

The Fourier Series

The *Fourier series* expansion of $f(t)$ is a series in the form

$$\mathrm{FS}_f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

The numbers $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are the *Fourier coefficients*.

In the periodic case, a Fourier series is often more useful than a power series.

Start with a (measured, observed) periodic function f, and then

- Determine the Fourier coefficients
- Ensure that the Fourier series converges
- Use the Fourier series to analyse f

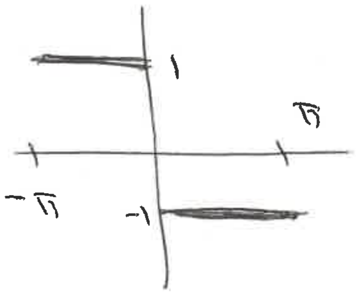
Example

This function is a Fourier series:

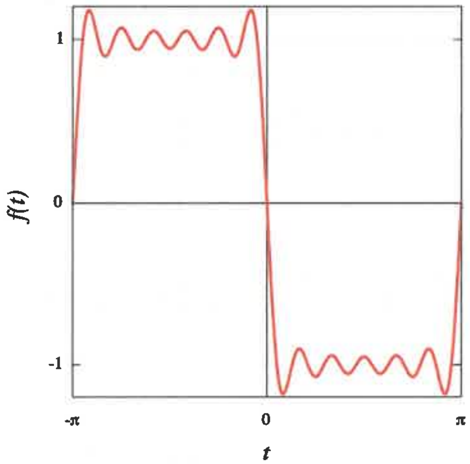
$b_n = 0$ for n even

$a_n = 0$ for all n

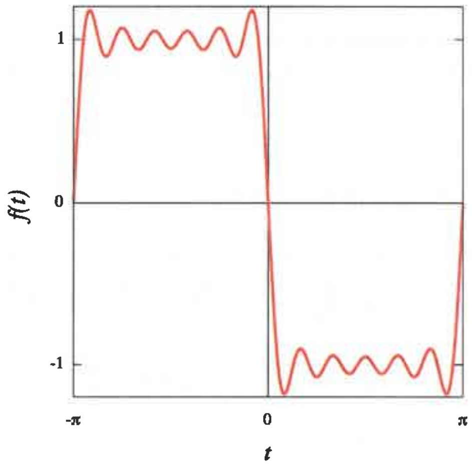
$a_0 = 0$



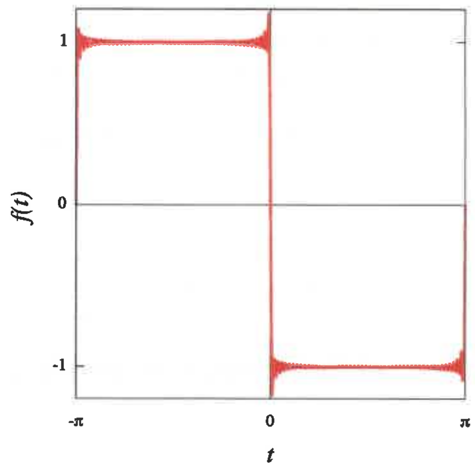
$$
\begin{aligned}
\mathrm{FS}_f(t) &= \sum_{\boxed{\substack{n=1\\ n\text{ odd}}}}^{\infty} \overset{b_n}{\frac{4}{n\pi}} \sin nt \\
&= \frac{4}{\pi} + \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\sin 5t + \cdots \\
&= \frac{4}{\pi}\left(\sin t + \tfrac{1}{3}\sin 3t + \tfrac{1}{5}\sin 5t + \cdots\right)
\end{aligned}
$$



$n = 11$



$n = 21$



$n = 101$

Our aim and how to achieve it

Given a 2π-periodic function $f(t)$ we wish to find coefficients

$$a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$$

so that

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt \qquad (*)$$

We can do this using the special properties, for *integer n*, $n \neq 0$

$$\int_{-\pi}^{\pi} \sin nt \, dt = 0 \qquad \int_{-\pi}^{\pi} \cos nt \, dt = 0$$

If we integrate $(*)$ from $-\pi$ to π and assume that the integral can be done term-by-term then

$$\boxed{\int_{-\pi}^{\pi} f(t)\, dt} = \int_{-\pi}^{\pi} \frac{a_0}{2}\, dt + 0 + 0 = \boxed{\pi a_0} \quad \Rightarrow \quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, dt$$

In more detail

$$\int_{-\pi}^{\pi} f(t)\,dt = \int_{-\pi}^{\pi} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt) \right) dt$$

$$= \int_{-\pi}^{\pi} \left(\frac{a_0}{2}\right) dt + \sum_{n=1}^{\infty} \underbrace{\int_{-\pi}^{\pi} a_n \cos(nt)\,dt}_{0} + \sum_{n=1}^{\infty} \underbrace{\int_{-\pi}^{\pi} b_n \sin(nt)\,dt}_{0}$$

$$= \pi a_0 + \sum_{n=1}^{\infty} 0 + \sum_{n=1}^{\infty} 0$$

$$= \pi a_0 \quad \Rightarrow \quad a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\,dt$$

The other coefficients

$\sin(A+B) = \sin A \cos B - \cos A \sin B$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

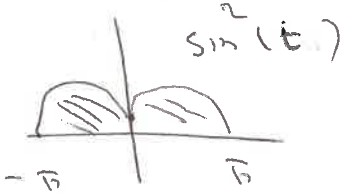
We can use the facts that for any *integers m, n* we have

$$\int_{-\pi}^{\pi} \sin(nt)\cos(mt)\,dt = 0$$

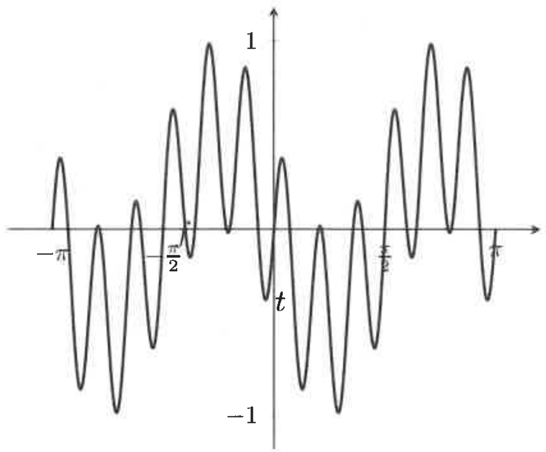
$n = m$

$\rightarrow \int_{-\pi}^{\pi} \sin^2(nt)\,dt$

$$\int_{-\pi}^{\pi} \sin(nt)\sin(mt)\,dt = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$



$$\int_{-\pi}^{\pi} \cos(nt)\cos(mt)\,dt = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$



Plot of $\sin(5t)\cos(7t)$

Finding the coefficients

$$
\begin{aligned}
f(t) = a_0& \\
&+ a_1 \cos(t) + a_2 \cos(2t) + a_3 \cos(3t) + \cdots \\
&+ b_1 \sin(t) + b_2 \sin(2t) + b_3 \sin(3t) + \cdots
\end{aligned}
$$

Evaluate

$$\int_{-\pi}^{\pi} f(t) \sin(2t)\, dt$$

$$
\begin{aligned}
= &\cancelto{0}{\int_{-\pi}^{\pi} a_0 \sin(2t)\, dt} + a_1 \cancelto{0}{\int_{-\pi}^{\pi} \cos(t)\sin(2t)\, dt} + a_2 \cancelto{0}{\int_{-\pi}^{\pi} \cos(2t)\sin(2t)\, dt} + \cdots \\
&+ b_1 \cancelto{0}{\int_{-\pi}^{\pi} \sin(t)\sin(2t)\, dt} + \boxed{b_2 \int_{-\pi}^{\pi} \sin(2t)\sin(2t)\, dt} = \pi b_2 \\
&+ b_3 \cancelto{0}{\int_{-\pi}^{\pi} \sin(3t)\sin(2t)\, dt} + \cdots
\end{aligned}
$$

Euler's formulas

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt)\, dt$$

An analogy

Compare Fourier series to our work on vector spaces

- The (infinite-dimensional) vector space of functions

  function space

- A basis for this vector space

  $\{ \frac{1}{2}, \cos t, \cos 2t, \cos 3t, \ldots, \sin t, \sin 2t, \sin 3t, \ldots \}$

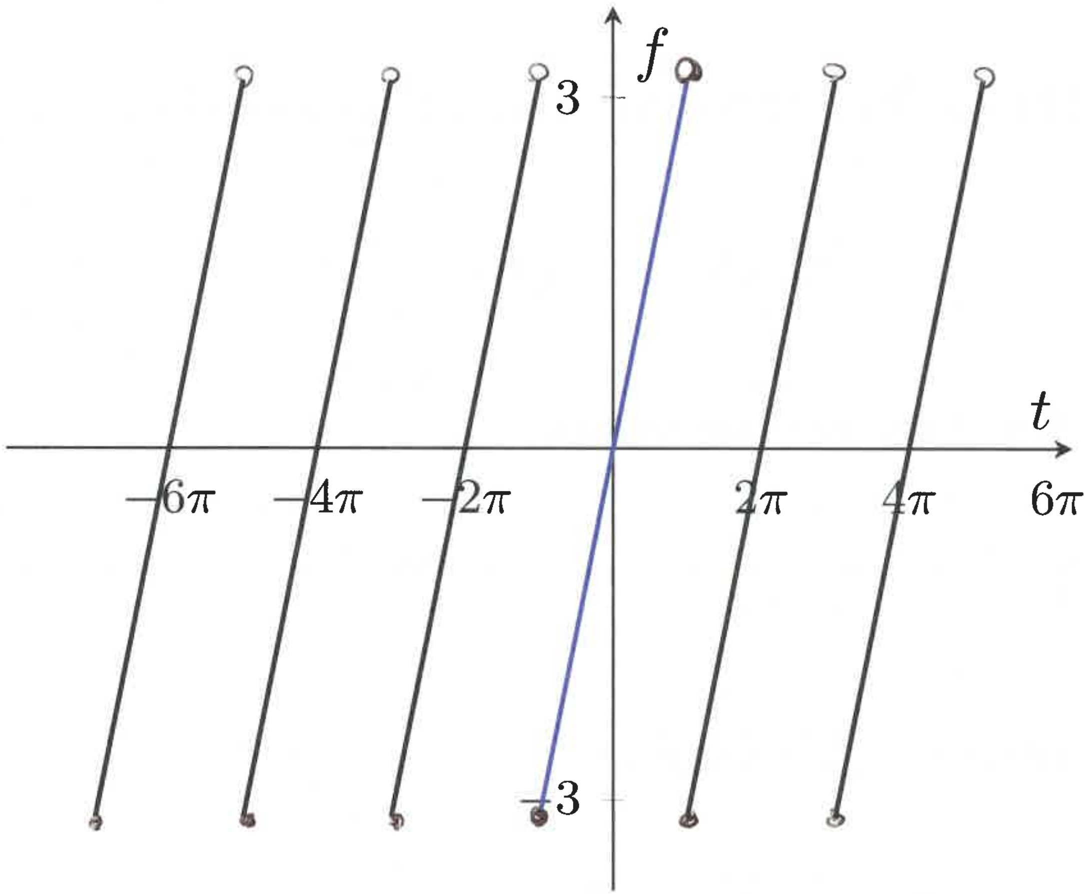
- Coordinates in this basis

  $a_0, a_1, a_2, a_3, \ldots, b_1, b_2, b_3 \ldots$

Example

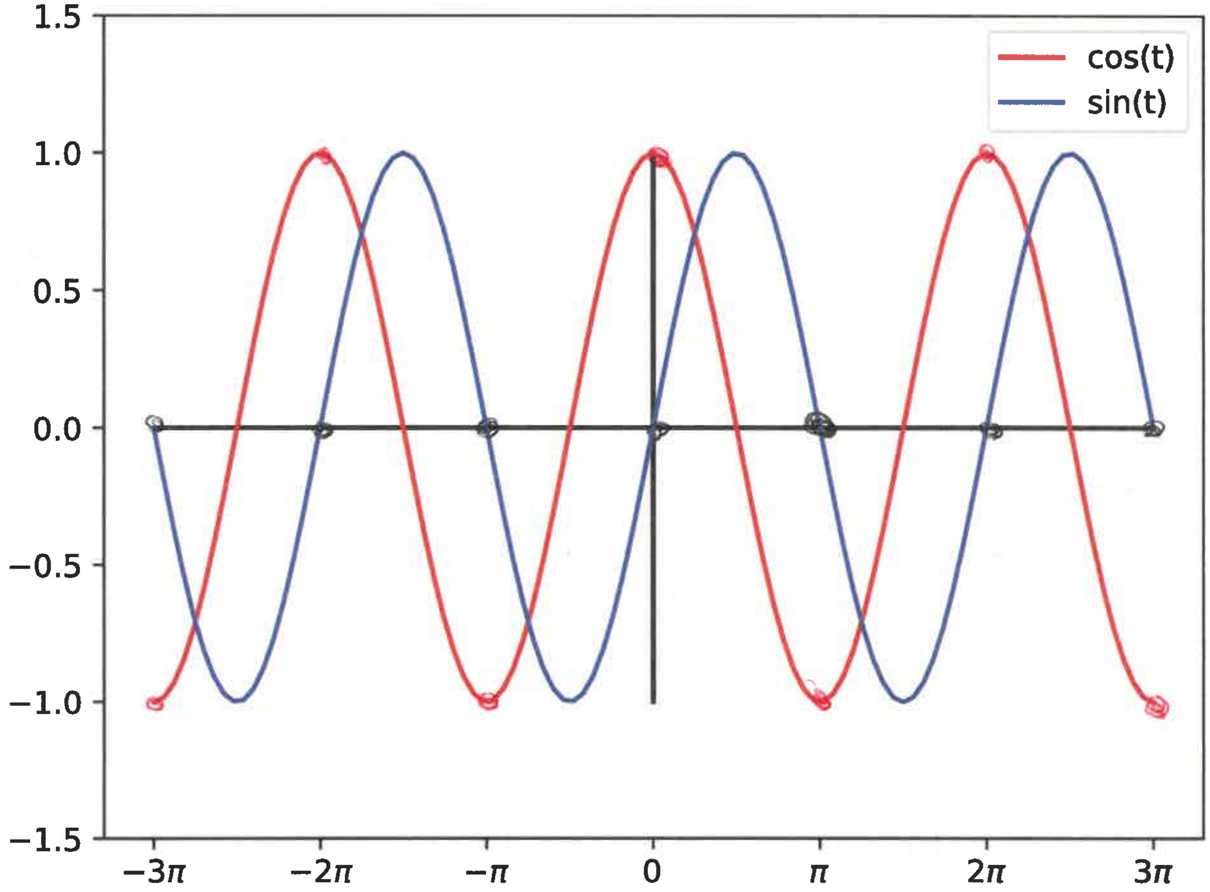
Consider the function $f(t) = t$ when $-\pi \leq t < \pi$, and its *periodic extension*

$P = 2\pi$



The graphs of sin and cos

These are



Note that

$$\sin(n\pi) = 0 \qquad n \text{ integer}$$
$$\cos(n\pi) = (-1)^n \qquad n \text{ integer}$$

12a

Integration by parts

Recall that

$$\int_a^b u'(t)v(t)dt = \Big[u(t)v(t)\Big]_a^b - \int_a^b u(t)v'(t)dt$$

Example

$$\int_{-\pi}^{\pi} t\sin(t)dt$$

$$u'(t) = \sin(t)$$
$$v(t) = t$$
$$u(t) = -\cos(t)$$
$$v'(t) = 1$$

$$= \Big[-t\cos(t)\Big]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} 1\,\cos(t)\,dt$$

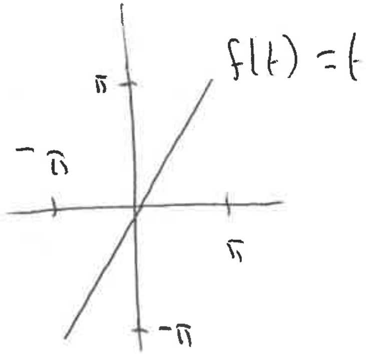
$$= \Big[-t\cos(t)\Big]_{-\pi}^{\pi} - \Big[\sin(t)\Big]_{-\pi}^{\pi}$$

$$= \Big[-\pi\underbrace{\cos(\pi)}_{(-1)} + \pi\underbrace{\cos(-\pi)}_{(-1)^{-1}}\Big] - \Big[\underbrace{\sin(\pi)}_{0} - \underbrace{\sin(-\pi)}_{0}\Big]$$

12b

The a's

We have



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t\,dt$$

$$= \frac{1}{\pi}\left[\frac{t^2}{2}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^2}{2}\right] = 0$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos(nt)\,dt$$

$$= \left[\tfrac{1}{n^2}\cos(nt) + \tfrac{t}{n}\sin(nt)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{n^2}\cos(n\pi) + \frac{\pi}{n}\cancel{\sin(n\pi)}^{\,0} - \frac{1}{n^2}\cos(-n\pi) - \frac{\pi}{n}\cancel{\sin(-n\pi)}^{\,0}$$

$$\frac{1}{n^2}(-1)^n - \frac{1}{n^2}(-1)^{-n}$$

$$= \frac{(-1)^n - (-1)^{-n}}{n^2}$$

$$= \frac{(-1)^n - (-1)^n}{n^2} = 0$$

THE b'S

We have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin(nt)\,dt$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2}\sin(nt) - \frac{t}{n}\cos(nt)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\pi}{n}\cos(n\pi) - \frac{\pi}{n}\cos(-n\pi)\right)$$

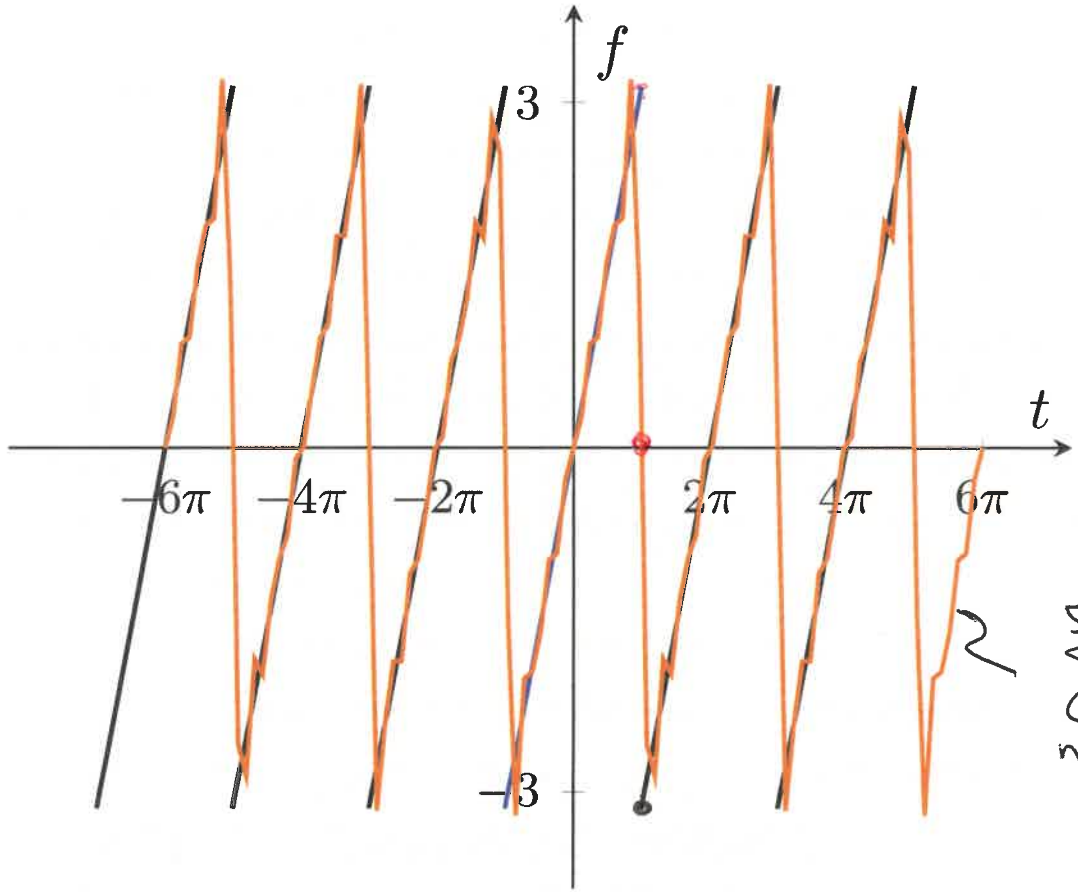
If n is even, then $\cos(n\pi) = \cos(-n\pi) = 1$
If n is odd, then $\cos(n\pi) = \cos(-n\pi) = -1$

$$b_n = \begin{cases} -\frac{2}{n}, & n \text{ even} \\ \frac{2}{n}, & n \text{ odd} \end{cases}$$

The Fourier series is

b_n

$$\mathrm{FS}_f(t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nt)$$



$f(t) = t, \quad -\pi \leq t < \pi$

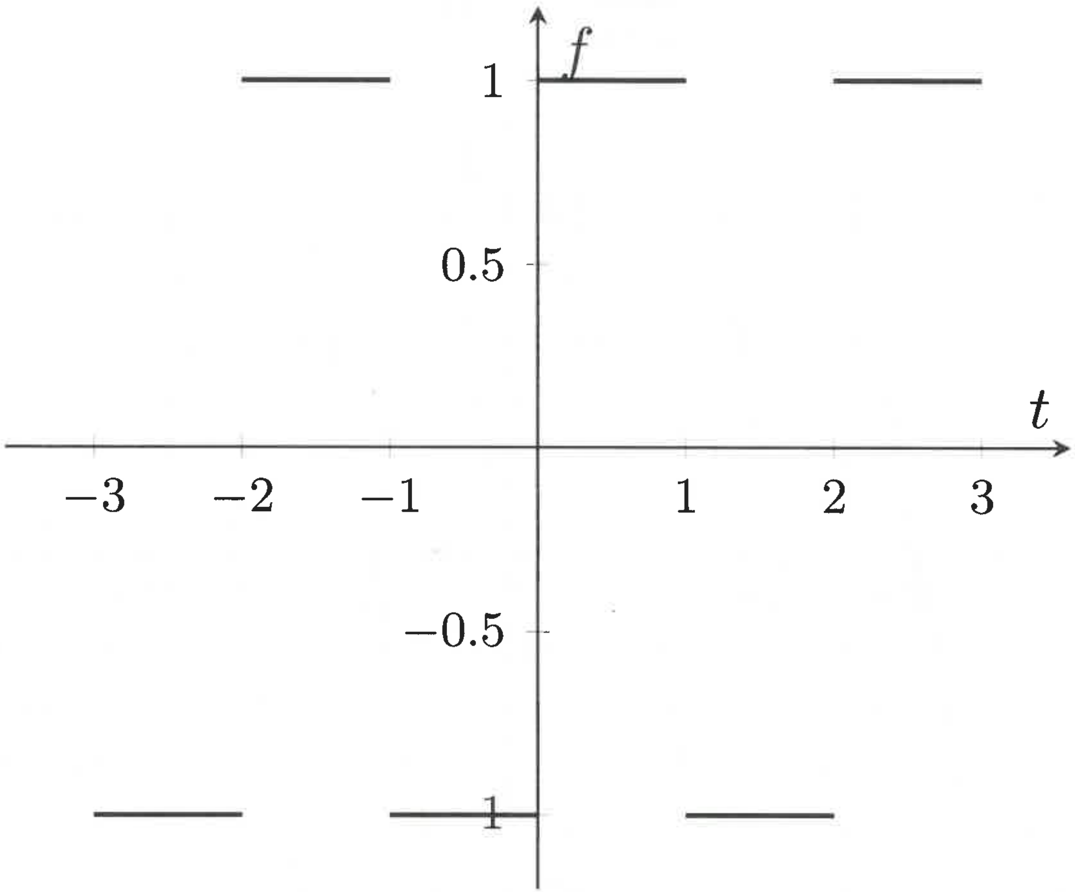
$f(t+2\pi) = f(t)$

$$\sum_{n=1}^{50} \frac{2(-1)^{n+1}}{n} \sin(nt)$$

What is $\mathrm{FS}_f(\pi)$? $\sin(n\pi) = 0 \Rightarrow \mathrm{FS}_f(\pi) = 0$

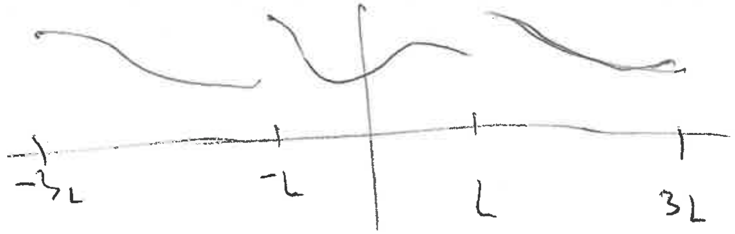
Varying the period

Functions to be approximated often have *other* periods.

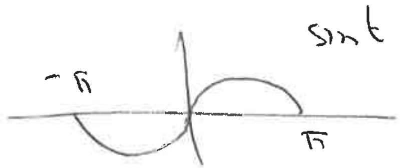


A function with period 2 = P

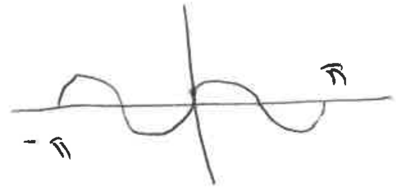
$2L$-PERIODIC



Normally assume one cycle of the function is defined on $[-L, L]$, and so it is $2L$-periodic.

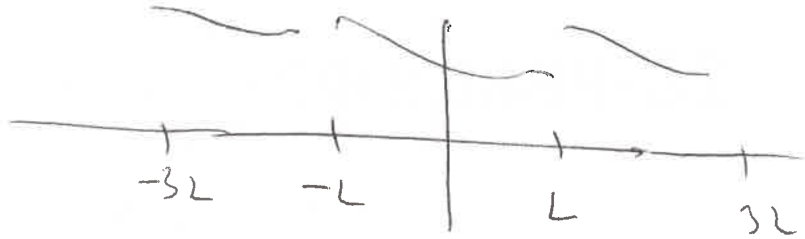


- $\sin t$ is 2π-periodic
- $\sin 2t$ is π-periodic
- $\sin 2\pi t$ is 1-periodic
- $\sin \pi t$ is 2-periodic
- $\sin \frac{\pi}{L} t$ is $2L$-periodic



$\sin(\alpha t)$ is $\frac{2\pi}{\alpha}$ periodic

Euler's formulas



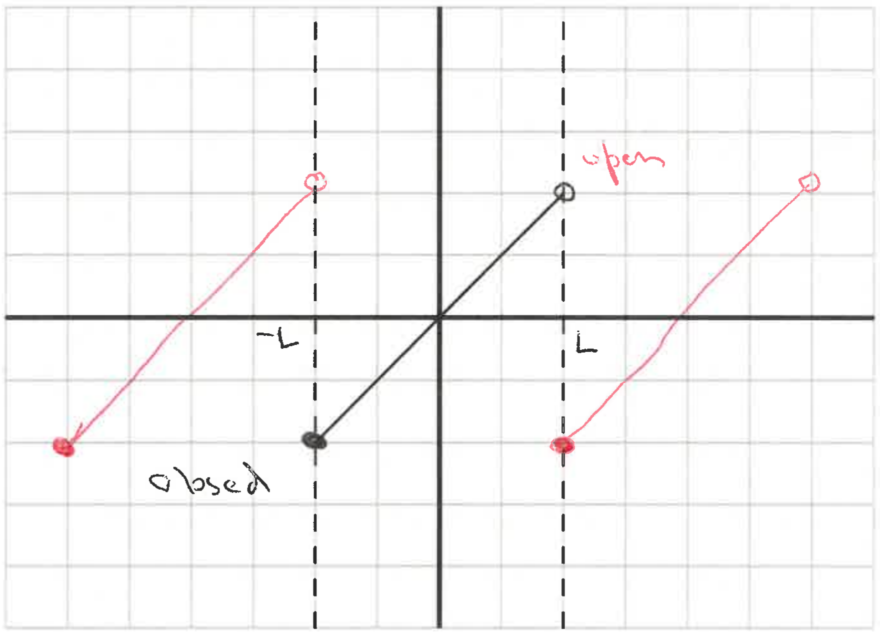
$$\mathrm{FS}_f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right)$$

where

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L}^{L} f(t)dt, \\
a_n &= \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt, \\
b_n &= \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt.
\end{aligned}$$

Periodic extension

A function defined only on $[-L, L)$ can be *extended* in a periodic fashion.



$f(t) = t$

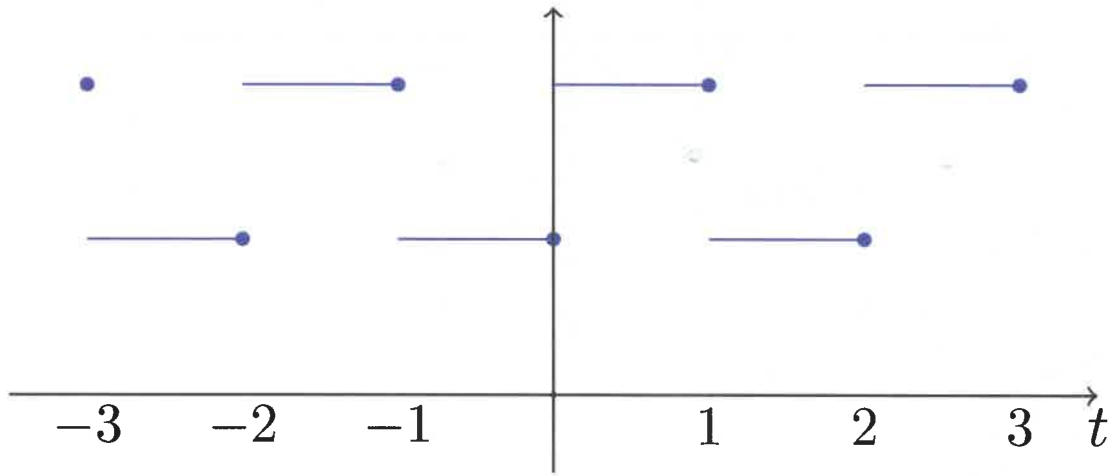
$-L \leq t < L$

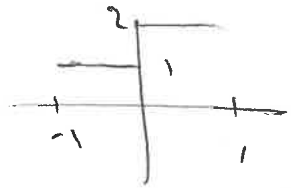
Example

Period $P = 2 = 2L \Rightarrow L = 1$

$$f(t) = \begin{cases} 1 & \text{if } -1 < t \leq 0 \\ 2 & \text{if } 0 < t \leq 1 \end{cases} \qquad f(t+2) = f(t)\,\forall t$$

Here the period is 2 so $L = 1$.





$$a_0 = \frac{1}{1}\int_{-1}^{1} f(t)dt = \int_{-1}^{0} f(t)dt + \int_{0}^{1} f(t)dt$$

$$= \int_{-1}^{1} 1\,dt + \int_{0}^{1} 2\,dt$$

$$= 1 + 2 \Rightarrow \boxed{a_0 = 3}$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(t)\cos\left(\frac{n\pi t}{1}\right) dt = \int_{-1}^{0} 1\cos(n\pi t)dt + \int_{0}^{1} 2\cos(n\pi t)dt$$

$$= \left[\frac{\sin(n\pi t)}{n\pi}\right]_{-1}^{0} + \left[2\frac{\sin(n\pi t)}{n\pi}\right]_{0}^{1}$$

$$= \frac{\cancelto{0}{\sin(0)} - \cancelto{0}{\sin(-n\pi)}}{n\pi} + \frac{\cancelto{0}{2\sin(n\pi)} - \cancelto{0}{2\sin(0)}}{n\pi}$$

$$\Rightarrow \boxed{a_n = 0 \text{ for all } n = 1, 2, 3, \ldots}$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(t)\sin\left(\frac{n\pi t}{1}\right)dt = \int_{-1}^{0}\sin(n\pi t)dt + \int_{0}^{1} 2\sin(n\pi t)dt$$

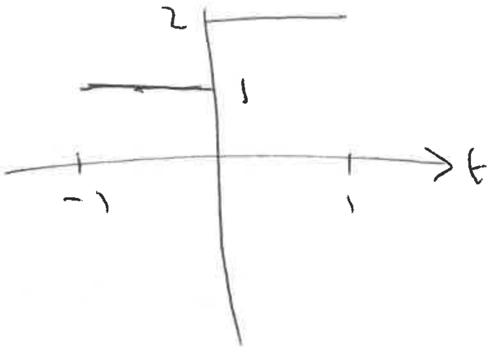
$$= \left[-\frac{\cos(n\pi t)}{n\pi}\right]_{-1}^{0} + \left[-2\frac{\cos(n\pi t)}{n\pi}\right]_{0}^{1}$$

$$= \frac{1}{n\pi}\left\{-\cos(0) + \cos(-n\pi) - 2\cos(n\pi) + 2\cos(0)\right\}$$

$$= \frac{1}{n\pi}\left\{-1 + (-1)^n - 2(-1)^n + 2\right\}$$

$$= \frac{1-(-1)^n}{n\pi} = \begin{cases} 0, & n \text{ even} \\ \frac{2}{n\pi}, & n \text{ odd} \end{cases}$$

$$
\begin{aligned}
\mathrm{FS}_f(t) &= \frac{3}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi} \sin(n\pi t) \\
&= \frac{3}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin(n\pi t) \\
&= \boxed{\frac{3}{2} + \frac{2}{\pi} \sum_{n=1,odd}^{\infty} \frac{1}{n} \sin(n\pi t)} \\
&= \frac{3}{2} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m-1} \sin((2m-1)\pi t)
\end{aligned}
$$



$\frac{a_0}{2}$ is the average value of the function

Convergence

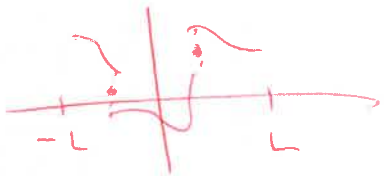
Let

$$S_N f(t) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{N} b_n \sin\left(\frac{n\pi t}{L}\right)$$

be the N^{th} partial sum of the Fourier series.

For a fixed value t, what is $\lim_{N\to\infty} S_N f(t)$?
Ideally, we'd like it to converge to $f(t)$ wherever "reasonable".

What is actually true?

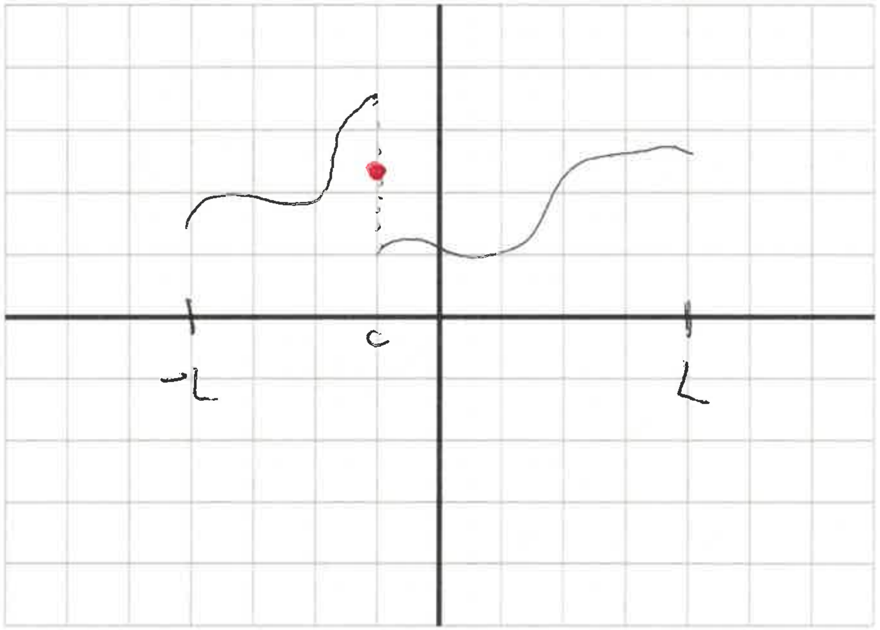


If f, f' are bounded and piecewise continuous on $[-L, L]$ then the Fourier series converges to $f(t)$ except at points of *discontinuity* of f.

Piecewise continuous

Only finitely many points of discontinuity and at each one, the Fourier series converges to

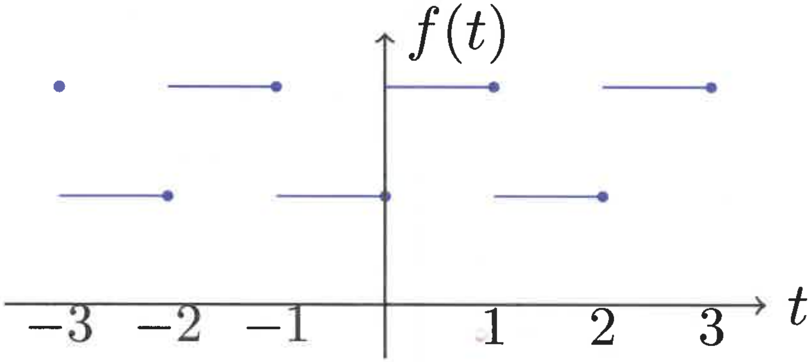
$$\frac{\lim_{t \to c^-} f(t) + \lim_{t \to c^+} f(t)}{2}$$



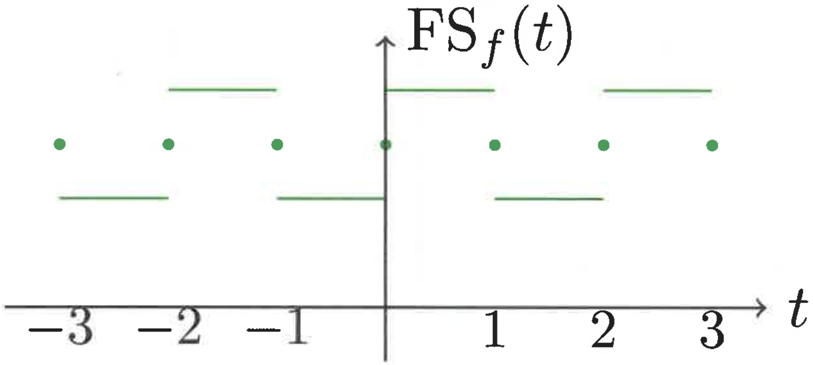
Example

The graph of the Fourier series of the function

$$f(t) = \begin{cases} 1 & \text{if } -1 < t \leq 0 \\ 2 & \text{if } 0 < t \leq 1 \end{cases} \qquad f(t+2) = f(t), \ \forall t$$



is

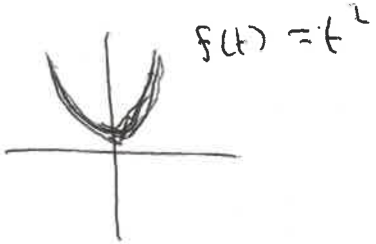


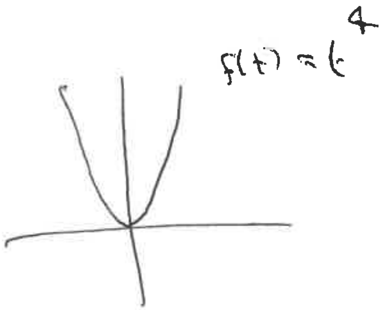
Note: We don't need to compute the Fourier series coefficients.

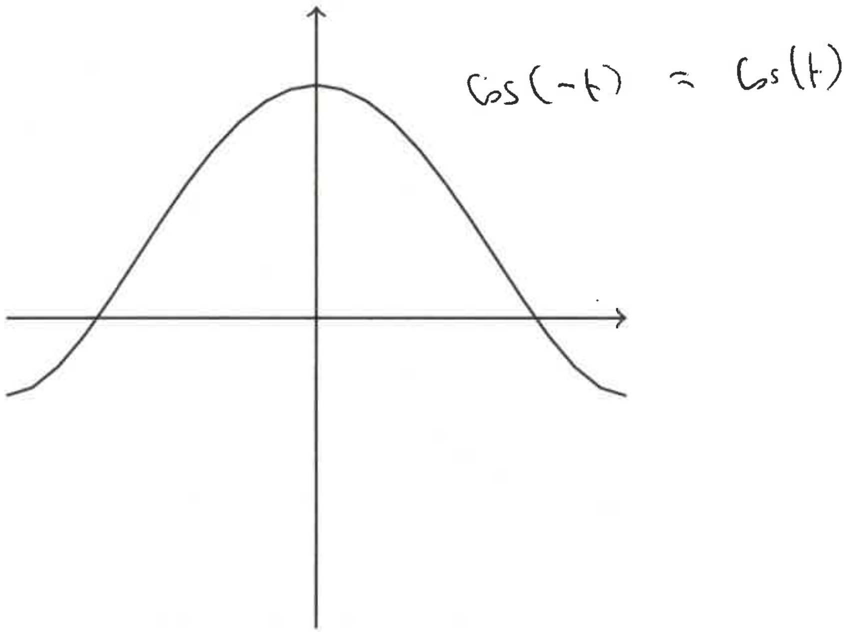
EVEN FUNCTIONS

A function $f(t)$ is *even* if

$$f(-t) = f(t)$$





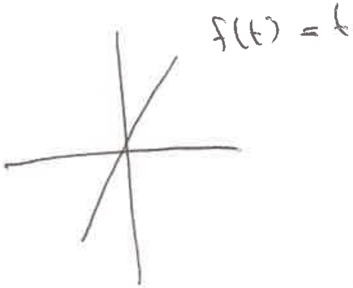


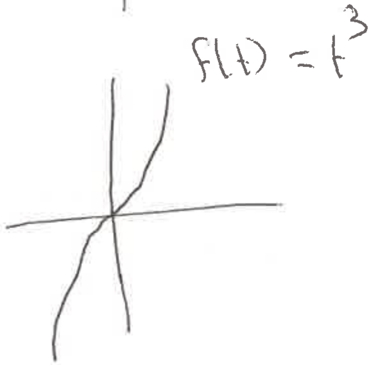
Its graph has *vertical reflection symmetry*.

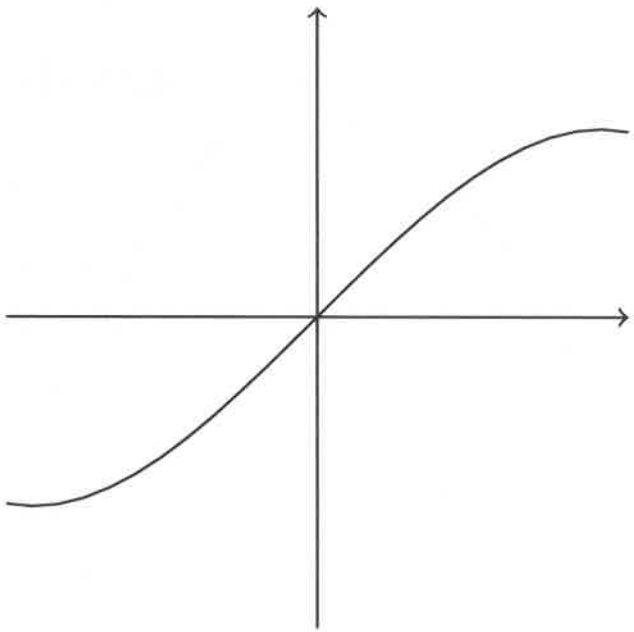
Odd functions

A function $f(t)$ is *odd* if

$$f(-t) = -f(t)$$







$\sin(-t) = -\sin(t)$

Its graph has 180^o-*rotational symmetry*

Evenness and Oddness is refered to as parity

PROPERTIES OF EVEN AND ODD FUNCTIONS

- The sum of two even functions is even.

$$(even) + (even) = (even)$$

$$t^2 + t^4 = t^2 + t^4$$

- The sum of two odd functions is odd.

$$(odd) + (odd) = (odd)$$

$$t + t^3 = t + t^3$$

- The product of two even functions is even.

$$(even) \cdot (even) = (even) \qquad t^2 t^4 = t^6$$

- The product of two odd functions is even.

$$(odd) \cdot (odd) = (even) \qquad t^3 t^5 = t^8$$

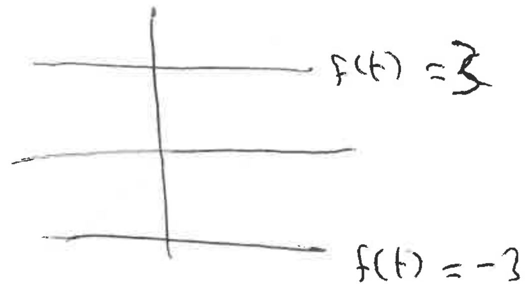
- The product of an even and an odd function is odd.

$$(even) \cdot (odd) = (odd) \qquad t^4 t^3 = t^7$$

Some more properties

Evenness/oddness is referred to as *parity*.

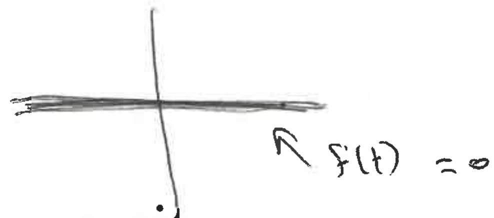
The constant function is even:



$$f(t) = c, \qquad c \neq 0$$

What about

$$f(t) = 0?$$



Scaling: Multiplying by a constant preserves parity:

$f(t) = 2\cos t$ is even $\qquad\qquad f(t) = 4t^3$ is odd

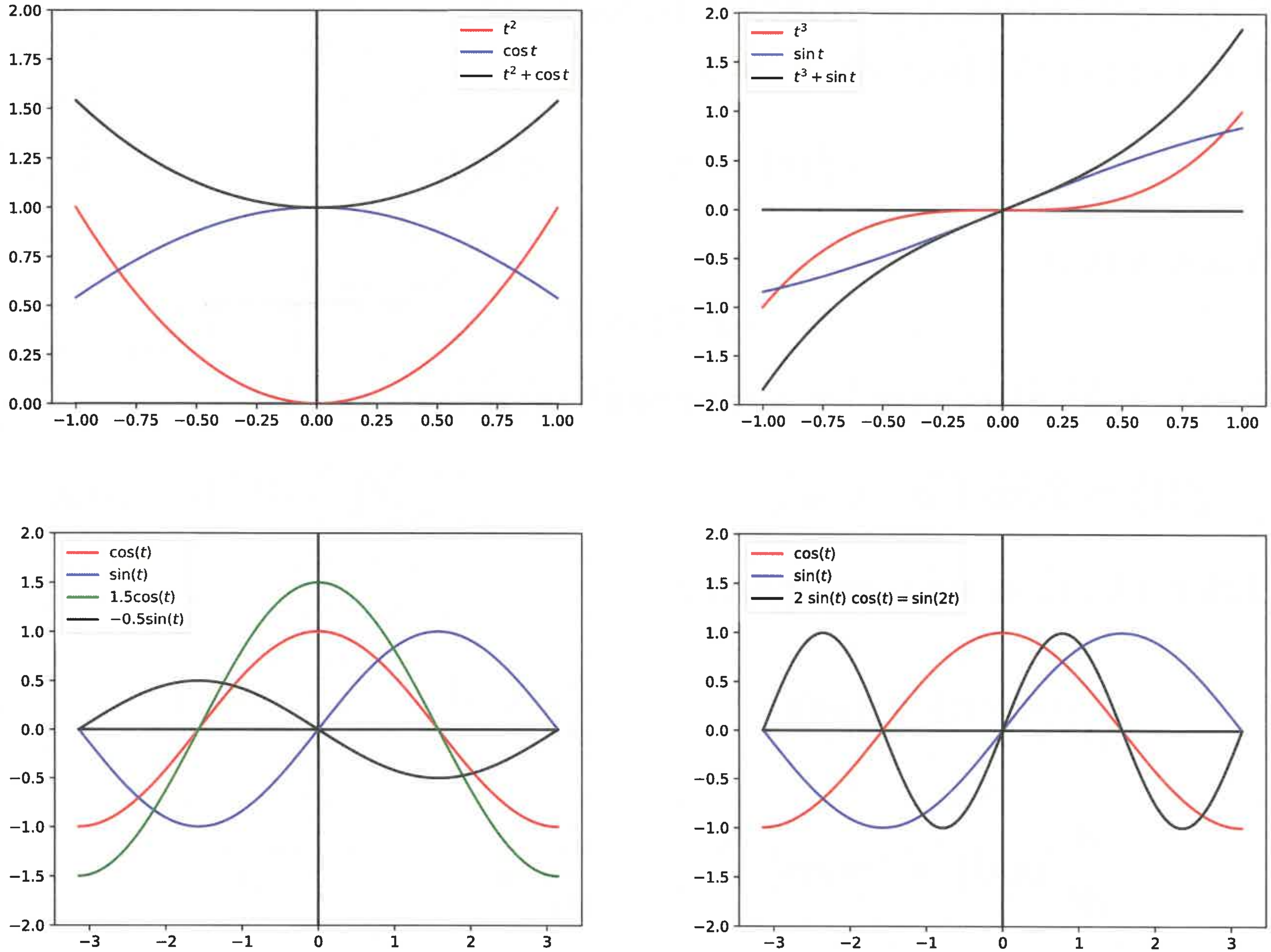
Differentiation switches parity:

$$\frac{d}{dt}(even) = (odd) \qquad \frac{d}{dt}\cos t = -1\sin t$$

$$\frac{d}{dt}(odd) = (even) \qquad \frac{d}{dt}t^3 = 3t^2$$

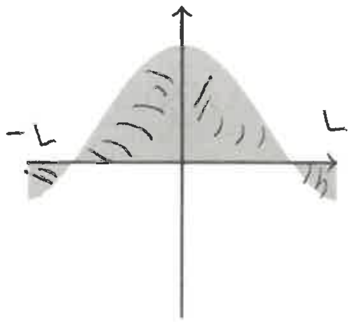
EXAMPLES

$\left.\begin{array}{l}\sin(ct) \text{ is odd} \\ \cos(ct) \text{ is even}\end{array}\right\}$ for any $c \neq 0$



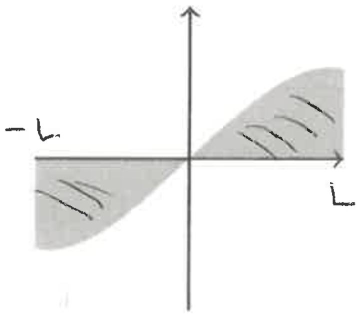
Integration of an even function over a symmetric interval:

$$\int_{-L}^{L} \underset{\text{even}}{f(t)}dt = 2\int_{0}^{L} f(t)\,dt$$



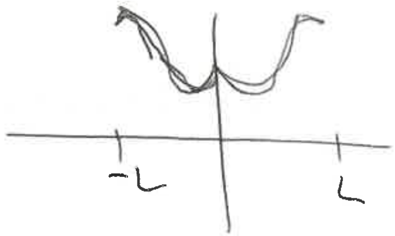
Integration of an odd function over a symmetric interval:

$$\int_{-L}^{L} \underset{\text{odd}}{f(t)}dt = 0$$



Fourier cosine series for even functions

If $f(t)$ is an *even* function with period $2L$, then



$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(t)\sin\left(\frac{n\pi t}{L}\right)}_{odd\ function} dt = 0$$

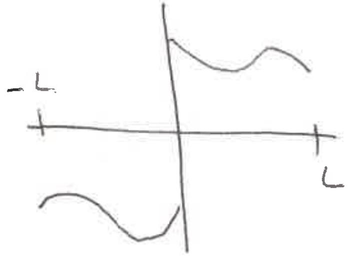
and

$$a_0 = \frac{1}{L}\int_{-L}^{L} \underbrace{f(t)}_{even\ function} dt = \frac{2}{L}\int_{0}^{L} f(t)\,dt$$

and

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(t)\cos\left(\frac{n\pi t}{L}\right)}_{even\ function} dt = \frac{2}{L}\int_{0}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right) dt$$

Fourier sine series for odd functions



If $f(t)$ is an *odd* function with period $2L$, then

$$a_0 = \frac{1}{L}\int_{-L}^{L} \underbrace{f(t)}_{odd\ function} \, dt = 0$$

and

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(t)\cos\left(\frac{n\pi t}{L}\right)}_{odd\ function} dt = 0$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(t)\sin\left(\frac{n\pi t}{L}\right)}_{even\ function} dt = \frac{2}{L}\int_{0}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right) dt$$

Even functions and odd functions

Even functions have Fourier series which only have constant and cos terms. They're sometimes just called *Fourier cosine series*

$$\mathrm{FS}_{f\ \mathrm{even}}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right)$$

$b_n = 0$

where

$$a_0 = \frac{2}{L}\int_0^L f(t)dt \qquad a_n = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

Odd functions have Fourier series which only have sin terms. They're sometimes just called *Fourier sine series*

$$\mathrm{FS}_{f_{\mathrm{odd}}}(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right)$$

$a_0 = 0$

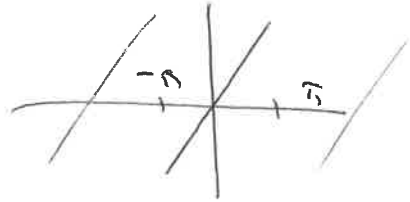
$a_n = 0$

where

$$b_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

Our previous examples

$$f(t) = t \qquad \text{when} \qquad -\pi \le t < \pi$$



$a_0 = 0$

$a_n = 0$

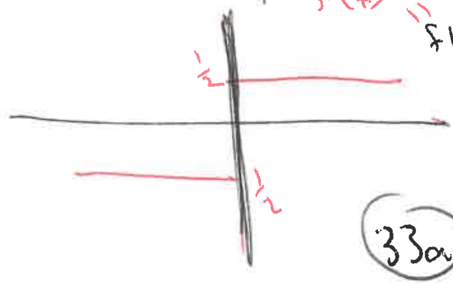
and $\quad b_n = \ldots$

$$f(t) = \begin{cases} 1 & \text{if } -1 < t \le 0 \\ 2 & \text{if } 0 < t \le 1 \end{cases}$$

$\frac{a_0}{2}$ = average value $= \frac{3}{2}$

$\hat{f}(t) = f(t) - \frac{a_0}{2}$

$\hat{f}(t)$ is odd $\Rightarrow a_n = 0$



$$FS_f(t) = \frac{3}{2} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right)$$

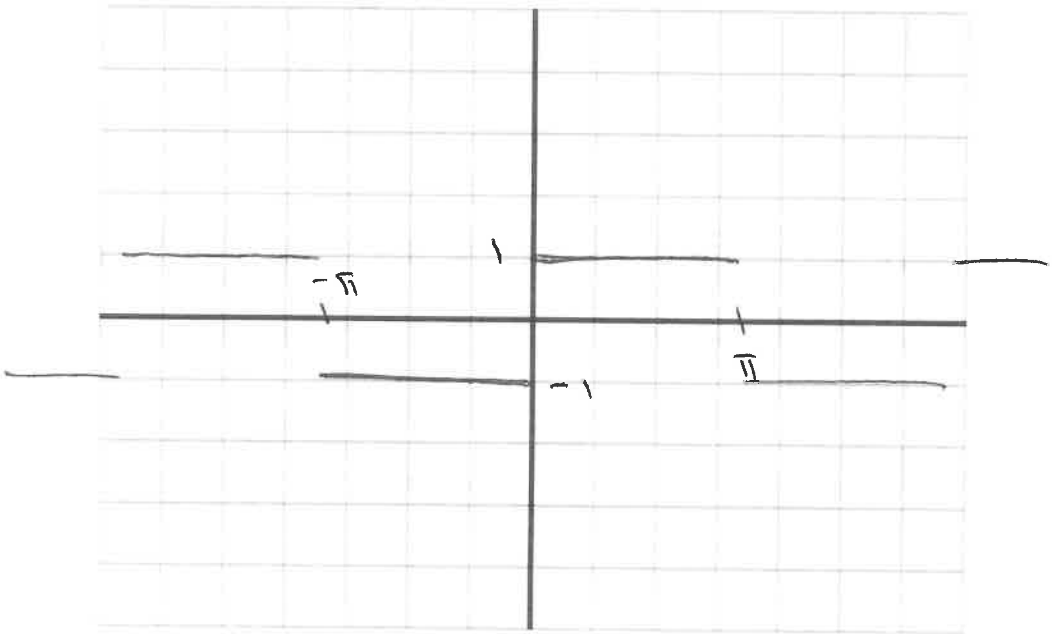
and $\quad b_n = \ldots$

33a

EXAMPLE

Consider the periodic extension of

$$f(t) = \begin{cases} -1, & -\pi \leq t \leq 0 \\ 1, & 0 < t < \pi \end{cases}$$



The periodic extension is an odd function and so it will have a Fourier sine series. We know that $a_0 = 0$ and $a_n = 0$. We need

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin(nt)\,dt$$

$$= \frac{2}{\pi}\int_0^{\pi} 1 \sin(nt)\,dt$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin(nt)\,dt$$

$$= \frac{2}{\pi}\left[-\frac{\cos(nt)}{n}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{1-(-1)^n}{n}\right]$$

$$= \begin{cases} 0, & n \text{ even} \\ \frac{4}{n\pi}, & n \text{ odd} \end{cases}$$

and hence

$$\mathrm{FS}_f(t) = \frac{4}{\pi}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin nt}{n}$$

Half-range expansions

Let $f(t)$ be defined on $[0, L]$. An extension of $f(t)$ to $[-L, L]$ is called a *half-range expansion.*

Even half-range expansion

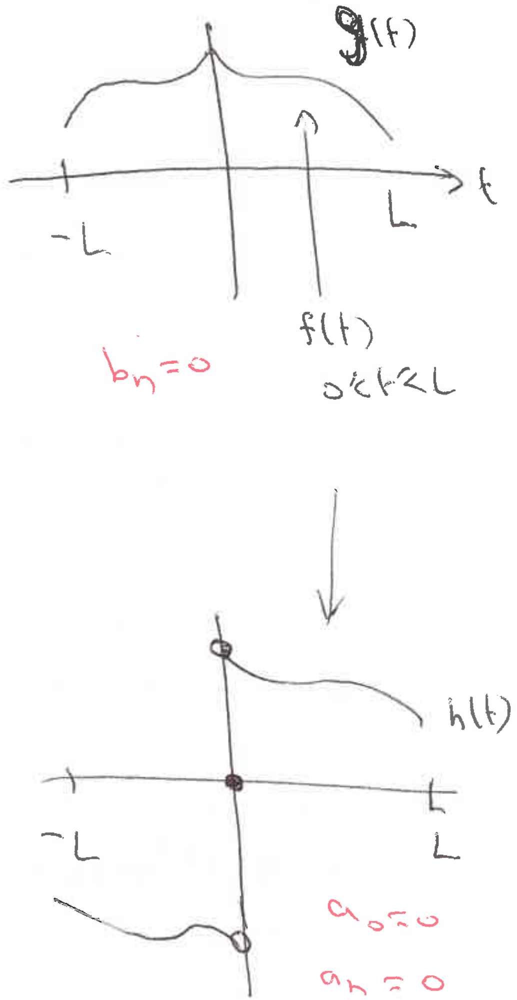
$$g(t) = \begin{cases} f(t), & \text{if } 0 \leq t \leq L \\ f(-t), & \text{if } -L \leq t \leq 0 \end{cases}$$

- $g(t)$ is an even function
- It has a Fourier cosine series

Odd half-range expansion

$$h(t) = \begin{cases} f(t), & \text{if } 0 < t \leq L \\ 0 & \text{if } t = 0 \\ -f(-t) & \text{if } -L \leq t < 0 \end{cases}$$

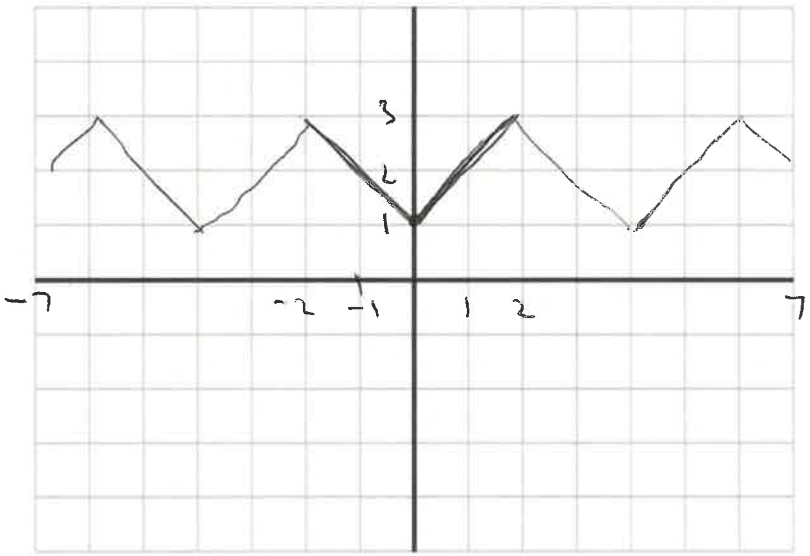
- $h(t)$ is an odd function
- It has a Fourier sine series

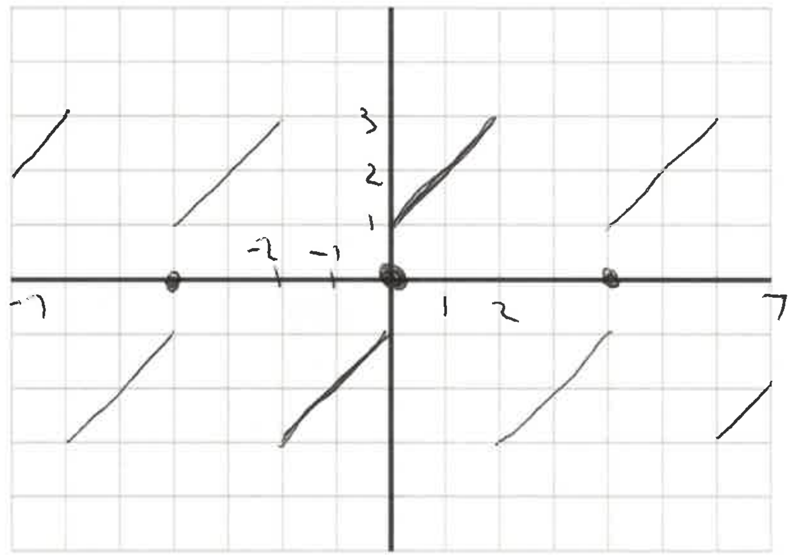


Example

The even and odd half-range expansion of $f(t) = 1+t$ on $[0,2]$ are

$b_n = 0$





$a_0 = 0$

$a_n = 0$

$$g(t) = \begin{cases} 1+t, & 0 \le t \le 2 \\ 1-t, & -2 \le t \le 0 \end{cases}$$

$f(-t)$

$$h(t) = \begin{cases} 1+t, & 0 < t < 2 \\ 0 & t = 0 \\ -1+t, & -2 < t < 0 \end{cases}$$

$-f(-t) = -(1-t)$

MATH1012 Mathematical Theory and Methods

Week 10

Differential equations (DE)

These are equations which involve derivatives. For example,

$$\frac{dy}{dx} = \frac{y}{x}$$ for $y = f_n(x)$

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \sin(3x)$$ for $y = f_n(x)$

$$\frac{d^3f}{dt^3} = f^2$$ for $f = f_n(t)$

$$\left(\frac{dh}{dx}\right)^2 + h^2 = x^2$$ $h = f_n(x)$

Our job is no find all of the functions which satisfy a given differential equation, that is, *solve the differential equation.*

An application

A naïve model of population growth is the differential equation

$$\frac{dP}{dt} = rP$$

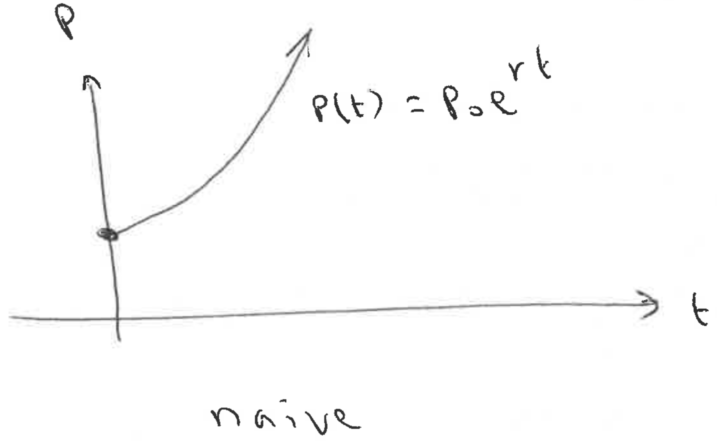
for P = fn of t

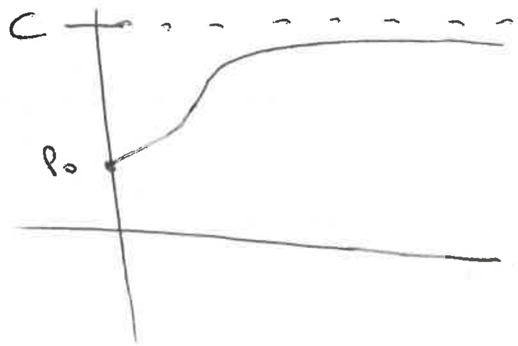
A more realistic model is

$$\frac{dP}{dt} = rP(C - P)$$

where $P(t)$ is the population size at time t, r is the growth rate of the species and C is the carrying capacity of the environment. We'll need to solve subject to an initial population size

$$P(0) = P_0$$





Classification

The order of a differential equation is the order of the highest derivative derivative which appears in it

$$\frac{dy}{dx} = \frac{y}{x} \qquad \text{is} \quad \text{first} \quad \text{order}$$

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \sin(3x) \qquad \text{is} \quad \text{second} \quad \text{order}$$

$$\frac{d^3f}{dt^3} = f^2 \qquad \text{is} \quad \text{third} \quad \text{order}$$

$$\left(\frac{dh}{dx}\right)^2 + h^2 = x^2 \qquad \text{is} \quad \text{first} \quad \text{order}$$

Separation of variables

A first order differential equation is said to be separable if it can be written in the form

$$\frac{dy}{dx} = F(x)G(y)$$

We can solve such differential equations by a process called separation of variables.

If we think of the derivative $\frac{dy}{dx}$ as a fraction (just like we do with the chain rule) then we can write the DE in the form

$$\frac{1}{G(y)}dy = F(x)dx$$

and then each side can be integrated, so that

$$\int \frac{1}{G(y)}dy = \int F(x)dx \qquad \text{or} \qquad \int \frac{dy}{G(y)} = \int F(x)dx$$

from which we may hope to find y as a function of x

The population growth models can be solved this way.
For the naïve model we get

$$\frac{dP}{dt} = rP \quad \Rightarrow \quad \frac{dP}{P} = r\,dt \quad \Rightarrow \quad \int \frac{dP}{P} = \int r\,dt$$

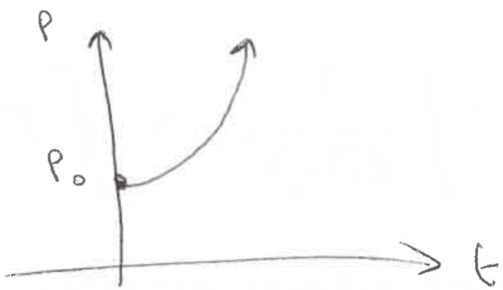
Integrate to get $\qquad \ln|P| = rt + C$

$$\left(\ln|P| + c_1 = rt + c_2 \quad \Rightarrow \quad \ln|P| = rt + \overbrace{c_2 - c_1}^{C} \right)$$

Hence $\qquad P(t) = e^{rt+C} = e^{rt} e^{C}$

Now $\qquad P(0) = P_0 \Rightarrow P_0 = e^{C} \Rightarrow e^{C} = P_0$

So $\qquad \boxed{P(t) = P_0 e^{rt}}$



Example

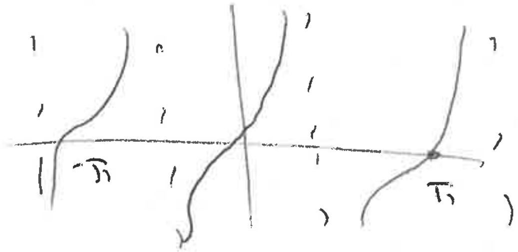
Solve $\frac{dy}{dx} = 2x(1+y^2)$ subject to $y(0) = 0$. We have

$$\frac{dy}{dx} = 2x(1+y^2) \Rightarrow \int \frac{dy}{1+y^2} = \int 2x\,dx$$

Integrate to get Arctan $y = x^2 + C$

$\tan^{-1} y$

Solve for y to get $y(x) = \tan(x^2 + C)$



Now $y(0) = 0$ requires $0 = \tan C$

So $\underline{\underline{C = 0}}$ ~~or π or 2π, ...~~

So $y(x) = \tan(x^2)$

First order linear differential equations

These are differential equations which can be written in the form

$$\frac{dy}{dx} + f(x)y = g(x)$$

where $f(x)$ and $g(x)$ are given functions of x.

We can solve such differential equations by multiplying both sides by an *integrating factor*

$$I(x) = e^{\int f(x)dx}$$

We get

$$I(x)\frac{dy}{dx} + I(x)f(x)y = I(x)g(x)$$

We chose this special form for the integrating factor because

$$\frac{dI}{dx} = \frac{d}{dx} e^{\int f(x)dx} = e^{\int f(x)dx} \cdot \frac{d}{dx}\left[\int f(x)\,dx\right] = I(x)\,f(x)$$

$$\frac{d}{dx} e^{h(x)} = e^{h(x)} \cdot h'(x)$$

Chain Rule

Hence we can write the multiplied DE in the form

$$I(x)\frac{dy}{dx} + \frac{dI}{dx}y = I(x)g(x)$$

and observe that the left-hand side is a product derivative:

$$\boxed{\frac{d}{dx}\{I(x)y\} = I(x)g(x)}$$

Integrate both sides to get

$$I(x)\,y = \int I(x)\,g(x)\,dx + C$$

Then

$$y(x) = \frac{1}{I(x)}\left\{\int I(x)\,g(x)\,dx + C\right\}$$

Example

The differential equation

$\frac{dy}{dx} + f(x)y = g(x)$

$$\frac{dy}{dx} + \frac{2}{x}y = x^4$$

is linear so we can solve it via an integrating factor. We have

and first order

$f(x) = \frac{2}{x}$ so $I(x) = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = e^{\ln(x^2)} = x^2$

Multiply throughout by $I(x) = x^2$ to get

$x^2\frac{dy}{dx} + 2xy = x^6 \Rightarrow \frac{d}{dx}(x^2 y) = x^6$

Integrate to get $x^2 y = \frac{x^7}{7} \boxed{+ C}$

So $\boxed{y(x) = \frac{x^5}{7} + \frac{C}{x^2}}$

We can ignore an integration constant when calculating the integrating factor, but we *must not* omit an integration constant when integrating the multiplied differential equation

$$\frac{dy}{dx} + \frac{2}{x} y = x^4$$

$$I(x) = e^{\int \frac{2}{x} dx} = e^{2\ln|x| + c} = e^{2\ln|x|} e^{c}$$

$$= A e^{2\ln|x|} \quad \text{where} \quad A = e^{c}$$

So $I(x) = A e^{\ln(x^2)} = A x^2$

Multiply throughout to get

$$\cancel{A} x^2 \frac{dy}{dx} + 2\cancel{A} x y = \cancel{A} x^6$$

Second order linear differential equations

These are differential equations which can be written in the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = g(x)$$

where $p(x)$, $q(x)$ and $g(x)$ are given functions of x.

If $g(x) = 0$ it is said to be homogeneous, and if $g(x) \neq 0$ it is said to be inhomogeneous.

non-homogeneous

Theorem

(Principle of Superposition) If $y_1(x)$ and $y_2(x)$ are solutions of a second order linear homogeneous differential equation, then so is $c_1y_1(x) + c_2y_2(x)$ for any constants c_1, c_2.

The proof can be found in the Unit Reader and is not difficult.

Functions $y_1(x)$ and $y_2(x)$ are said to be linearly independent on an interval I if

$$c_1 y_1(x) + c_2 y_2(x) = 0 \qquad \text{only if} \qquad c_1, c_2 = 0 \qquad \text{for all } x$$

Examples

(1) $\underbrace{y_1(x) = x^2 \quad \text{and} \quad y_2(x) = x^3}_{\text{lin ind.}} \rightarrow c_1 x^2 + c_2 x^3 = 0 \Rightarrow c_1 = 0, \; c_2 = 0$

(2) $y_1(x) = e^x$ and $y_2(x) = 2e^x$

$\rightarrow \underbrace{c_1 e^x + c_2 2e^x}_{\text{lin. dep.}} = (c_1 + 2c_2)e^x = 0$ if $c_1 = -2c_2$

THEOREM

Let y_1, y_2 be any two linearly independent solutions of a second order linear homogeneous DE. Then the general solution of the DE has the form $y(x) = c_1 y_1(x) + c_2 y_2(x)$ where the constants $c_1, c_2 \in \mathbb{R}$ are arbitrary.

Linear Algebra

Is at the heart of solving linear differential equations.
The n^{th} order homogeneous differential equation

$$\frac{d^n y}{dx^n} + p_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_1(x)\frac{dy}{dx} + p_0(x)y = 0$$

will have general solution

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

The solution set of the differential equation is a subspace of the (infinite dimensional) vector space of functions

- A basis for this subspace is

  $\{ y_1(x), y_2(x), \ldots, y_n(x) \}$

- Coordinates in this basis are

  $c_1, c_2, \ldots, c_n$

Auxiliary conditions fix a point in this subspace

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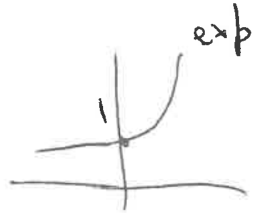
Constant coefficients

If the coefficients (the factors multiplying the derivatives) are constant we can get solutions. The differential equation

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$



where p and q are constants, will have solutions of the form $y(x) = e^{mx}$ where m is a constant. Substitution gives

Cancel

$$m^2 e^{mx} + pm e^{mx} + q e^{mx} = 0 \Rightarrow (m^2 + pm + q)\cancel{e^{mx}} = 0$$

and hence we're required to solve the *characteristic equation* (also called the *auxiliary equation*)

$$m^2 + pm + q = 0$$

Not the same as the characteristic equation for determining eigenvalues of a matrix, although the two are indeed related.

The characteristic equation is a quadratic equation, and hence we can find its solutions by using the formula $p^2 - 4q$ discriminant

$$\boxed{\begin{aligned} &ax^2 + bx + c = 0 \\ &\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}}$$

$$m = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

$$m^2 + pm + q = 0$$

Examples

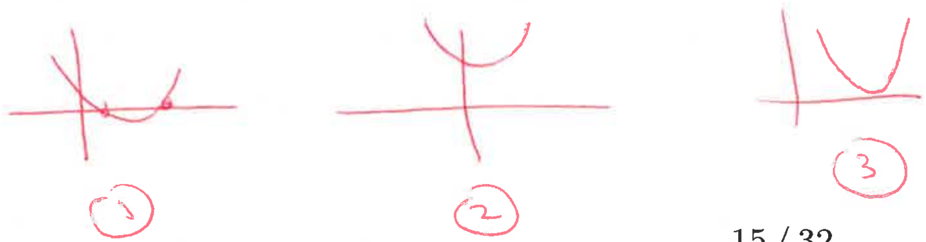
(1) $m^2 + 3m + 2 = 0 \Rightarrow m = \dfrac{-3 \pm \sqrt{9-8}}{2} = \dfrac{-3 \pm 1}{2} = -2, -1$

(2) $m^2 - 2m + 2 = 0 \Rightarrow m = \dfrac{2 \pm \sqrt{4-8}}{2} = \dfrac{2 \pm \sqrt{-4}}{2} = \dfrac{2 \pm i\sqrt{4}}{2}$

so $m = 1 \pm i$ where $i = \sqrt{-1}$

(3) $m^2 - 4m + 4 = 0 \Rightarrow m = \dfrac{4 \pm \sqrt{16-16}}{2} = \dfrac{4 \pm 0}{2} = 2, 2$

We have three cases to consider.



Case 1: $p^2 - 4q > 0$

The characteristic equation has two distinct real roots m_1, m_2, yielding solutions $y_1(x) = e^{m_1 x}$ and $y_2(x) = e^{m_2 x}$.

Hence the general solution of the differential equation is

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

where c_1, c_2 are constants.

Example

Initial conditions

Solve

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0 \qquad \text{subject to} \quad y(0) = 1 \text{ and } \frac{dy}{dx}(0) = 3$$

$y = e^{mx} \rightarrow (m^2 + 3m + 2)\cancel{e^{mx}} = 0 \rightarrow m = -1, -2$

So general soln is

$$y(x) = c_1 e^{-x} + c_2 e^{-2x}$$

Initial conditions give

$y(0) = 1 \Rightarrow c_1 + c_2 = 1 \longrightarrow c_2 = 1 - c_1$

Then $y'(x) = -c_1 e^{-x} - 2c_2 e^{-2x}$

and $y'(0) = 3 \Rightarrow -c_1 - 2c_2 = 3$

Hence $-c_1 - 2(1 - c_1) = 3 \Rightarrow c_1 = 5 \rightarrow c_2 = -4$

Hence $\boxed{y(x) = 5e^{-x} - 4e^{-2x}}$

$y(x) = A e^{(a+ib)x} + B e^{(a-ib)x}$

and use $e^{i\theta} = \cos\theta + i\sin\theta$ Eulers formula

Case 2: $p^2 - 4q < 0$

The characteristic equation will have *complex conjugate roots*, which we can write as $m = a \pm ib$ where $i^2 = -1$. These yield solutions $y_1(x) = e^{ax}\cos(bx)$ and $y_2(x) = e^{ax}\sin(bx)$.

Hence the general solution of the differential equation is

$$y(x) = c_1 e^{ax}\cos(bx) + c_2 e^{ax}\sin(bx)$$

where c_1, c_2 are constants.

Example

Solve

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0 \qquad \text{subject to} \quad y(0) = 0 \text{ and } \frac{dy}{dx}(0) = 1$$

$y = e^{mx} \rightarrow (m^2 - 2m + 2)\cancel{e^{mx}} = 0 \Rightarrow m = 1 \pm i = \underbrace{1}_{a} \pm \underbrace{1}_{b} i$

General soln is

$$y(x) = c_1 e^x \cos(x) + c_2 e^x \sin(x)$$

Initial conditions

$y(0) = 0 \Rightarrow 0 = c_1 . 1 . 1 + c_2 . 1 . 0 \Rightarrow c_1 = 0$

Hence $y(x) = c_2 e^x \sin(x) \rightarrow y'(x) = c_2 e^x \sin(x) + c_2 e^x \cos(x)$

Then $y'(0) = 1 \Rightarrow 1 = c_2$

So

$$\boxed{y(x) = e^x \sin(x)}$$

" Method of Reduction of Order "

Case 3: $p^2 - 4q = 0$

The characteristic equation has a single (repeated) root $m = -\frac{p}{2}$, and the general solution of the differential equation is

$$y(x) = c_1 e^{mx} + c_2 x e^{mx}$$

where c_1, c_2 are constants.

Example

Solve

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0 \qquad \text{subject to} \quad y(0) = 1 \text{ and } \frac{dy}{dx}(0) = 0$$

$y = e^{mx} \rightarrow (m^2 - 4m + 4)\cancel{e^{mx}} = 0 \Rightarrow m = 2$

General solution is

$$y(x) = c_1 e^{2x} + c_2 x e^{2x}$$

Initial conditions :

$$y(0) = 1 \Rightarrow 1 = c_1 + 0 \Rightarrow c_1 = 1$$

$$y'(x) = 2c_1 e^{2x} + c_2 e^{2x} + 2c_2 x e^{2x}$$

So $y'(0) = 0 \Rightarrow 0 = 2c_1 + c_2 + 0 \Rightarrow c_2 = -2c_1 = -2$

Hence $\boxed{y(x) = e^{2x} - 2xe^{2x}}$

Initial and boundary value problems

A second order differential equation is usually accompanied by two extra conditions on the unknown function

If those conditions are given at the same point then we call the problem an initial value problem. For example,

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0 \qquad \text{subject to} \quad y(0) = 1 \text{ and } \frac{dy}{dx}(0) = 3$$

If those conditions are given at different points then we call the problem a boundary value problem. For example,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0 \qquad \text{subject to} \quad y(0) = 0 \text{ and } y(1) = 2$$

Collectively, initial and boundary conditions are referred to as auxiliary conditions

Nonhomogeneous, constant coefficient

To solve the second order linear DE with constant coefficients

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = g(x)$$

where the right hand side $g(x)$ is nonzero, we do the following:

- First find the solution $\boxed{y_c(x)}$ of the associated homogeneous differential equation

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$

  This is called the complementary function
- Then find any particular solution $y_p(x)$ of the original nonhomogeneous differential equation
- The general solution of the nonhomogeneous differential equation is

$$y(x) = y_c(x) + y_p(x)$$

Method of undetermined coefficients

There is an easy way to find $y_p(x)$ if $g(x)$ has one of a number of particular forms. In such cases, a trial form for $y_p(x)$ is given in the following table:

$g(x)$	Trial form of $y_p(x)$
$a_n(x) = a_n x^n + \cdots a_1 x + a_0$	$A_n(x) = A_n x^n + \cdots A_1 x + A_0$
$ae^{\alpha x}$	$Ae^{\alpha x}$
$a\cos\beta x$ or $b\sin\beta x$	$A\cos\beta x + B\sin\beta x$
$a_n(x)e^{\alpha x}$	$A_n(x)e^{\alpha x}$
$ae^{\alpha x}\cos\beta x$ or $be^{\alpha x}\sin\beta x$	$Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x$
$a_n(x)e^{\alpha x}\cos\beta x$ or $b_n(x)e^{\alpha x}\sin\beta x$	$e^{\alpha x}(A_n(x)\cos\beta x + B_n(x)\sin\beta x)$

Example

Find the general solution of

Homog: $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 1 + 2x^2$$

$y = e^{mx} \Rightarrow m^2 - 3m + 2 = 0$

$\Rightarrow m = 1, 2$

The complementary function $y_c(x)$ is almost the solution found in the Case 1 example, namely

Complementary function

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0 \qquad \Rightarrow \qquad y_c = c_1e^x + c_2e^{2x}$$

To find a particular solution $y_p(x)$ we observe that $g(x)$ is a polynomial of degree $n = 2$, so we should try

$$y_p(x) = A_2x^2 + A_1x + A_0$$

where the coefficients A_2, A_1, A_0 are to be determined.

$y_p'(x) = 2A_2x + A_1$ and $y_p''(x) = 2A_2$

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 1 + 2x^2$$

$$y_p(x) = A_2x^2 + A_1x + A_0$$

$y_p'(x) = 2A_2x + A_1$

$y_p''(x) = 2A_2$

Substitution gives

$$2A_2 - 3(2A_2x + A_1) + 2(A_2x^2 + A_1x + A_0) = 1 + 2x^2$$

$$\Rightarrow \boxed{2A_2}\,x^2 + \underbrace{\boxed{(2A_1 - 6A_2)}}_{=0}\,x + (2A_0 - 3A_1 + 2A_2) = 1 + \boxed{2}x^2$$

hence $2A_2 = 2 \Rightarrow A_2 = 1$

then $2A_1 - 6A_2 = 0 \Rightarrow A_1 = 3A_2 \Rightarrow A_1 = 3$

and $2A_0 - 3A_1 + 2A_2 = 1 \Rightarrow 2A_0 = 1 + 3A_1 - 2A_2$

$\Rightarrow 2A_0 = 8 \Rightarrow A_0 = 4$

Hence the general solution of the nonhomogeneous equation is

$$y(x) = y_c(x) + y_p(x) = \underbrace{c_1 e^x + c_2 e^{2x}}_{y_c} + \underbrace{4 + 3x + x^2}_{y_p}$$

where c_1, c_2 are arbitrary constants.

If we had been asked for the solution which satisfies the initial conditions $y(0) = 1$ and $\frac{dy}{dx}(0) = 2$, we would now (not earlier) determine the required values of c_1 and c_2. Firstly, we need

$$y'(x) = c_1 e^x + 2c_2 e^{2x} + 3 + 2x$$

where $y'(x)$ (or simply y') is another way of writing $\frac{dy}{dx}$.
Hence we need to solve

$$\begin{array}{rrcll} y(0)=1 \Rightarrow & c_1 + c_2 + 4 & = & 1 & (1) \\ y'(0)=2 \Rightarrow & c_1 + 2c_2 + 3 & = & 2 & (2) \end{array}$$

$(2) - (1)$ gives

$c_2 - 1 = 1$

$\Rightarrow c_2 = 2 \Rightarrow c_1 = -5$

giving $c_2 = 2$ and $c_1 = -5$, and so the required solution is

$$y(x) = -5e^x + 2e^{2x} + 4 + 3x + x^2$$

Example

Find the general solution of

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^{3x}$$

The complementary function $y_c(x)$ is the same as before:

$$y_c(x) = c_1e^x + c_2e^{2x}$$

For a particular solution try $y_p(x) = Ae^{3x}$. Substitution gives

$$\cancel{9Ae^{3x}} - 3\left(\cancel{3Ae^{3x}}\right) + 2Ae^{3x} = 2e^{3x}$$

$$\Rightarrow \quad 2A\cancel{e^{3x}} = 2\cancel{e^{3x}} \quad \Rightarrow \quad 2A = 2 \quad \Rightarrow \quad A = 1$$

So $y_p(x) = e^{3x}$ and the general solution is

$$y(x) = y_c(x) + y_p(x) = \underbrace{c_1e^x + c_2e^{2x}}_{y_c} + \underbrace{e^{3x}}_{y_p}$$

When $g(x)$ is the sum of different expressions in the left column of the table then $y_p(x)$ can be taken as the sum of the corresponding particular solutions.

Example

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^{-x} + 5\sin x$$

For the particular solution we try

$$y_p(x) = Ce^{-x} + A\cos x + B\sin x$$

$y_p'(x) = -Ce^{-x} - A\sin x + B\cos x$ and $y'' = Ce^{-x} - A\cos x - B\sin x$

We cannot automatically set $A = 0$, even though $\cos x$ does not appear on the right hand side of the differential equation.

Substituting y_p in the ODE with $\sin x$ on the right hand side,

$y'' - 3y' + 2y = g(x)$

$$(Ce^{-x} - A\cos x - B\sin x) - 3(-Ce^{-x} - A\sin x + B\cos x)$$

$$+2(Ce^{-x} + A\cos x + B\sin x) = 2e^{-x} + 5\sin x$$

We can collect together terms on the left-hand side:

$$(-A - 3B + 2A)\cos x + (-B + 3A + 2B)\sin x$$

$$(C + 3C + 2C)e^{-x} = 2e^{-x} + 5\sin x$$

The red terms tell us that $6C = 2$ and hence $C = \frac{1}{3}$
Equating the coefficients of $\cos x$ and $\sin x$ tells us that

$$\begin{aligned} A - 3B &= 0 \quad (1) \\ 3A + B &= 5 \quad (2) \end{aligned}$$

$3 \times (1) = (3)$

$$\begin{aligned} 3A - 9B &= 0 \\ 3A + B &= 5 \quad (2) \end{aligned}$$

$(2) - (3)$

$10B = 5$

$\Rightarrow B = \frac{1}{2}$

$\Rightarrow A = 3B = \frac{3}{2}$

Solving these gives $B = \frac{1}{2}$ and $A = \frac{3}{2}$ and hence

$$y_p = \frac{1}{3}e^{-x} + \frac{3}{2}\cos x + \frac{1}{2}\sin x$$

General sol^n

$y(x) = c_1 e^{x} + c_2 e^{2x} + \frac{1}{3}e^{-x} + \frac{3}{2}\cos x + \frac{1}{2}\sin x$

ONE THING TO WATCH OUT FOR

If $g(x)$ has a term which appears in the complementary function then the suggested guess won't work. We should multiply our original guess by x and try again.

Example: Find the general solution of

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^x$$

We found before that the complementary function is

$$y_c(x) = c_1 e^x + c_2 e^{2x}$$

Trying $y_p(x) = Ae^x$ fails

Since e^x appears in this solution (when $c_1 = 1$ and $c_2 = 0$), so we must multiply the suggested guess Ae^x by x to get

From "Reduction of Order"

$$y_p(x) = Axe^x$$

We can get, from the Product Rule

$$y_p'(x) = A(e^x + xe^x) \qquad \text{and} \qquad y_p''(x) = A(2e^x + xe^x)$$

Substitution into the nonhomogeneous equation gives

$$A(2e^x + \cancel{xe^x} - 3e^x - \cancel{3xe^x} + \cancel{2xe^x}) = e^x$$

and hence

$$-Ae^x = e^x \quad \Rightarrow \quad A = -1 \quad \Rightarrow \quad y_p(x) = -xe^x$$

so the general solution is

$$y(x) = y_c(x) + y_p(x) = \underbrace{c_1 e^x + c_2 e^{2x}}_{y_c} - \underbrace{xe^x}_{y_p}$$

Initial conditions:

$y(0) = 1 \Rightarrow c_1 + c_2 = 1$

$y'(0) = 0 \Rightarrow c_1 + 2c_2 - 1 = 0$

$\Rightarrow c_1 + 2c_2 = 1 \rightarrow \boxed{c_2 = 0}$ and $\boxed{c_1 = 1}$

$y'(x) = c_1 e^x + 2c_2 e^{2x} - e^x - xe^x$

What about other sorts of right-hand sides?

What if our differential equation was

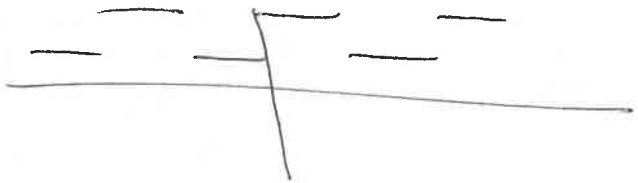
$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = g(x)$$

and $g(x)$ wasn't one of the method-of-undetermined-coefficients types? For example,

$$g(x) = \tan(x) \qquad g(x) = \ln(x) \qquad g(x) = x^p$$

For such, we can use the method of Variation of Parameters. It's in the Unit Reader but is not examinble.

If $g(x)$ is periodic, for example

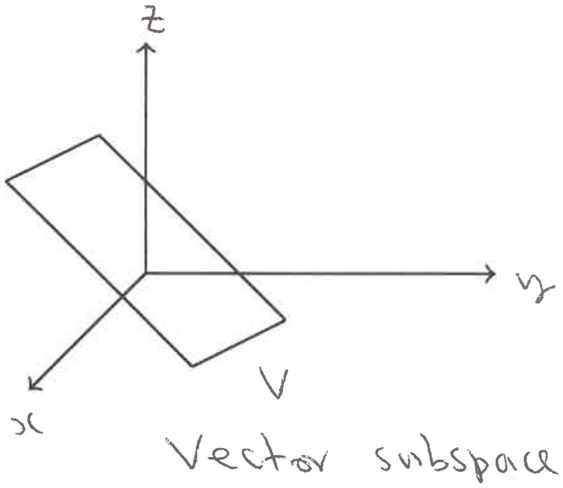


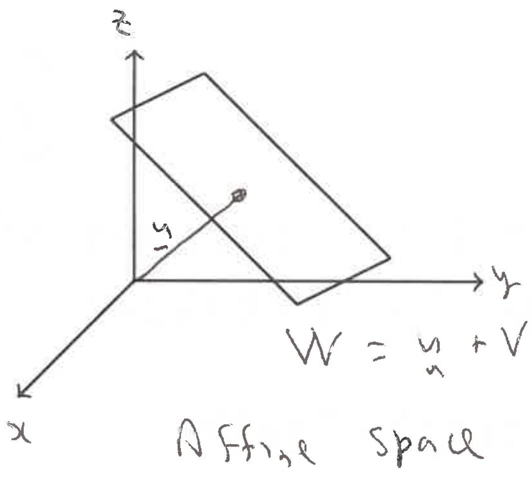
$$g(t) = \begin{cases} 1 & \text{if } -1 < t \leq 0 \\ 2 & \text{if } 0 < t \leq 1 \end{cases} \qquad g(t+2) = g(t)$$

then we can use Fourier series. That solution method is not covered in First Year.

Affine spaces

These are 'translated' vector spaces. Geometrically:





The n^{th} order inhomogeneous differential equation

$$\frac{d^n y}{dx^n} + p_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_1(x)\frac{dy}{dx} + p_0(x)y = g(x)$$

will have general solution

$$y(x) = \underbrace{c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)}_{y_c} + y_p(x)$$

which is an affine function space