MATH1012 Mathematical Theory and Methods

Week 1

Role of lectures

Not for writing out all details in unit reader!

Instead, they are a *guided tour* of the mathematical landscape:

- Important definitions
- Key landmarks
- Common pitfalls
- Useful analogies

I will focus on explaining how to build the right *mental model* – the Big Picture – that can be fleshed out in practice classes and your own reading and practice.

Linear equations

Terminology: A *linear equation* is an equation involving only *numbers* and *variables* (to the first power).

- Linear $x + 2y + 3z + 42w = 1{,}208$ ✓
- Not linear $x + 2y^2 + 3z + 42w = 1{,}208$ ✗

  $xy + 2y + \cdots\cdots = 1{,}208$ ✗

Solutions

Terminology: A *solution* to a linear equation is a choice of numbers for the variables that *satisfies* the equation.

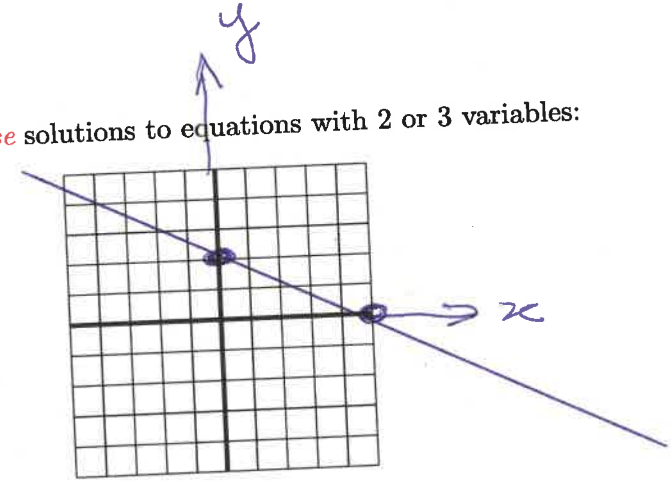
- $2x + 5y = 10$ ✓

- $x_1 + x_2 + x_3 = 1$ ✓

Geometry

$2x + 5y = 10$

We can *visualise* solutions to equations with 2 or 3 variables:



Systems of Linear Equations

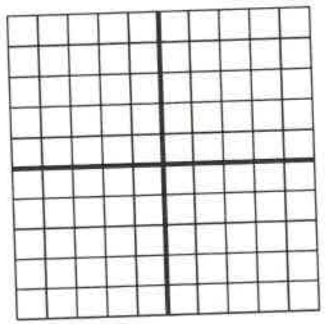
Terminology: A *system of linear equations* (SLE) is a collection of linear equations

$$\begin{array}{rcrcl} x & + & 2y & = & 4 \\ x & - & y & = & 1 \end{array}$$

A *solution* to an SLE must satisfy *all* the equations.

GEOMETRY 2

We can *visualise* solutions to SLEs with 2 or 3 variables:

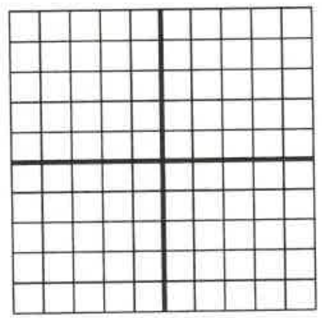


Counting solutions

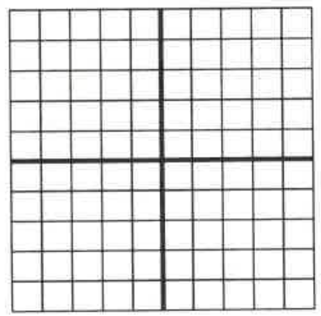
An SLE can have:

- No solutions at all (*inconsistent*)
- A *unique* solution (one choice)
- *Infinitely* many solutions

Example (no solution)



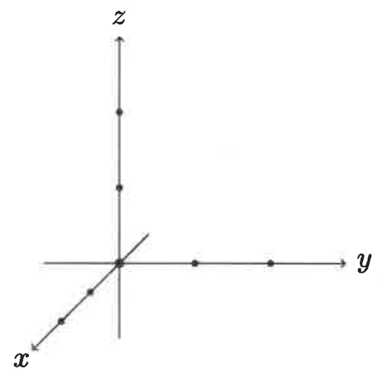
Example (infinitely many solutions)



GEOMETRY

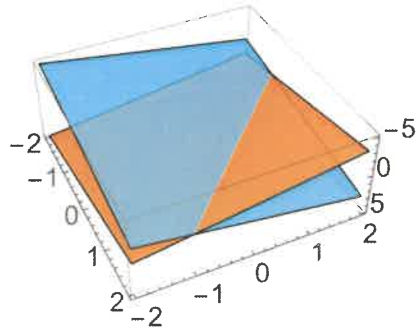
Two equations in three variables

$$\begin{aligned} x + y + z &= 1 \\ x + y - z &= 1 \end{aligned}$$



Pictured

$$\begin{aligned} x + y + z &= 1 \\ x + y - z &= 1 \end{aligned}$$



Geometry and Algebra

Two planes are either

- equal, or
- parallel, or
- meet in a line.

Which line?

$$\begin{aligned} x + y + z &= 1 \\ x + y - z &= 1 \end{aligned}$$

- (, ,) is on both planes
- (, ,) is on both planes

$$\begin{aligned} x + y + z &= 1 \\ x + y - z &= 1 \end{aligned}$$

Subtracting # 2nd equation from first:

$$0 + 0 + 2z = 0$$

dividing both sides by 2:

$$\boxed{\begin{aligned} z &= 0 \\ x + y &= 1 \end{aligned}}$$

$y = 1 - x$

$z = 0$

$S = \{ (x, 1-x, 0) : x \in \mathbb{R} \}$

Parametric

Express solutions *parametrically*

$$S = \{(\quad,\quad,\quad) \mid x \text{ is free}\}$$

or

$$S = \{(\quad,\quad,\quad) \mid x \in \mathbb{R}\}$$

$S = \{ (1-y, y, 0) : y \in \mathbb{R} \}$

Every coordinate is either *constant*, a *free variable* or a *linear combination*.

We can also take y as free variable.

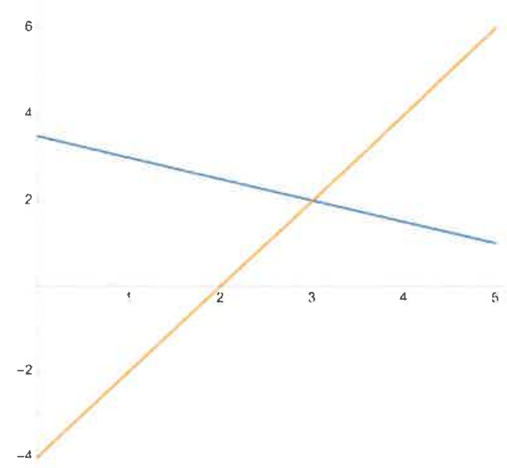
$$S = \{(\quad,\quad,\quad) \mid y \in \mathbb{R}\}$$

Purpose of EROs

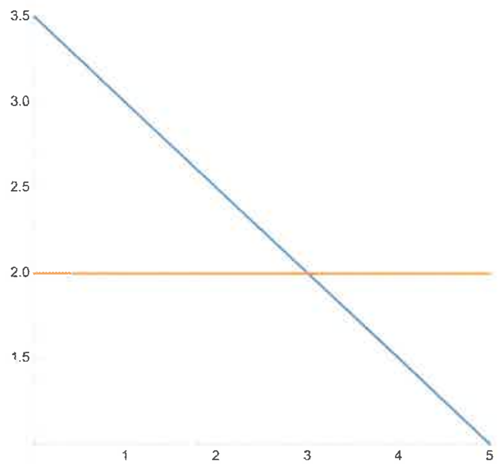
Theorem 1.7.
EROs don't change the *solution set* of an SLE!

(This seems plausible, but is not obvious)

Before ERO



After ERO



EXAMPLE

$$\begin{array}{rcrcl} x & + & 2y & = & 7 \\ 2x & - & y & = & 4 \end{array}$$

$2x + 4y = 14$

Do $(R2) \leftarrow (R2) - 2(R1)$ to get

$0 - 5y = -10$

$\div -5$

$y = 2$

$x + 4 = 7$

$x = 3$.

Elementary Row Operations

$$\begin{aligned} x + y &= 5 \\ 2x - 2y &= 7 \end{aligned}$$

- Swap two equations

- Multiply an equation by non-zero number

- Add a multiple of one equation to another

AUGMENTED MATRIX

Just a notation to get to the solution quicker.
Remember, each row corresponds to an equation.

$$\begin{array}{rcrcl} x & + & 2y & = & 7 \\ 2x & - & y & = & 4 \end{array}$$

Augmented Matrix

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 2 & -1 & 4 \end{array}\right] \quad R2 \leftarrow R2 - 2R1$$

Gaussian elimination

$$\begin{array}{ccc} x & y & = \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & -5 & -10 \end{array}\right]$$

row-echelon form

$x = 3$

$y = 2$ back-substitution

Row echelon form

- Zero-rows at bottom
- Find leading non-zeros
- Ensure they occur left-to-right

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 1 & 1 \end{array}\right]$$

BACK SUBSTITUTION

free variable

free variable

$$\begin{array}{cccc|c} x & y & z & w & \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

$x + y = 1/2$
$z = 1/2$
$w = 0$

Leading entries correspond to *basic* variables.
Other variables are *non-basic* or *free*.

basic

basic

Vital point

- All *non-basic* variables are free
- All *basic* variables are combos of constants / free vars

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

Gaussian elimination

Systematic way to use EROs to achieve row-echelon form.
At each stage, we consider a *pivot position* and associated *pivot entry*. First pivot position = top-left.

1. If pivot entry $= 0$ then swap pivot row with row below it so that new pivot entry $\neq 0$. If pivot entry and every entry below it are 0, move pivot position one column to the right.
2. If pivot entry $\neq 0$ then add a suitable multiple of the pivot row to every row below it so that every entry *below* the pivot entry becomes 0. Then move the pivot position one column to the right and one row down.

When the pivot position is moved off the matrix, then the process finishes and the matrix will be in row-echelon form.

Example: Gaussian elimination

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 3 & 2 \\ 2 & 1 & 2 & 0 & 0 \\ -1 & 3 & 4 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R2 \leftarrow R2 - 2R1 \\ R3 \leftarrow R3 + R1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 3 & 2 \\ 0 & 1 & 0 & -6 & -4 \\ 0 & 3 & 5 & 3 & 3 \end{array}\right] \quad R3 \leftarrow R3 - 3R2$$

$$\begin{array}{cccc} x & y & z & w \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 3 & 2 \\ 0 & 1 & 0 & -6 & -4 \\ 0 & 0 & 5 & 21 & 15 \end{array}\right]$$

basic basic basic free variable

$$\underline{x} = 2 - \left(3 - \tfrac{21}{5}w\right) - 3w$$
$$= \underline{-1 + \tfrac{6}{5}w}$$

Now ready for back-substitution

$$\begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 1 & 3 & 2 \\ 0 & 1 & 0 & -6 & -4 \\ 0 & 0 & 5 & 21 & 15 \end{array}\right] \end{array}$$

free

$$x + z + 3w = 2 \Rightarrow x = 2 - z - 3w$$
$$y - 6w = -4 \Rightarrow \underline{y = -4 + 6w}$$
$$5z + 21w = 15$$

$$\underline{z = 3 - \tfrac{21}{5}w}$$

Heads up!

Watch out for:

- Locating the pivot
- Zero columns

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

- Be systematic!
- Be smart.

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 1 & 1 & 1 \end{array}\right]$$

Counting solutions

Row-echelon form tells you how many solutions!

- *Inconsistent* SLE
  Row with one non-zero entry in last column
  $$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$$
- *Unique solution*
  All variables are basic
  $$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 5 & 2 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Counting solutions

Infinitely many

Row-echelon form tells you how many solutions!

- *~~Infinite~~ solutions*
  Some non-basic variables — one parameter for each non-basic variable

$$\begin{array}{ccccc} x & y & z & w & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 5 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right]$$

basic variables

free variable

$$\begin{array}{ccccc} x & y & z & w & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

basic variables

free variables

Historical parenthesis

The method of Gaussian elimination was known to *Chinese mathematicians* as early as 179 AD.

The method in Europe stems from the notes of *Isaac Newton*. In 1670, he wrote that all the algebra books known to him lacked a lesson for solving simultaneous equations, which Newton then supplied.

Carl Friedrich Gauss (1777–1855) in 1810 devised a notation for symmetric elimination that was adopted in the 19th century by professional hand computers. The algorithm that is taught in high school was named for Gauss only in the 1950s as a result of confusion over the history of the subject.

Source: Wikipedia

Summary of Gaussian elimination method

- Write the system of equations as an augmented matrix.
- Use Gaussian elimination (EROs) to get the augmented matrix into Row Echelon Form
- Rewrite the system with equations, and use back substitution from the last equation and working your way up.
- Write the solution set in parametric form.

More EROs

Back-substitution still some work, could be done in matrix form using more EROs.
Aim: basic variables have only one non-zero entry in the corresponding column and leading entries = 1.
Then it is much easier to write basic variables as combinations of constants and free variables.

$$\begin{array}{cccc} x & y & z & w \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -6/5 & -1 \\ 0 & 1 & 0 & -6 & -4 \\ 0 & 0 & 1 & 21/5 & 3 \end{array}\right]$$

Gauss-Jordan elimination

Systematic way to use EROs to achieve *reduced* row-echelon form.

At each stage, we consider a *pivot position* and associated *pivot entry*. First pivot position = top-left.

1. If pivot entry $= 0$ then swap pivot row with a row below it so that new pivot entry $\neq 0$. If pivot entry and every entry below it are 0, move pivot position one column to the right.
2. If pivot entry $\neq 0$ then divide row by entry so entry is now a 1. Then add a suitable multiple of the pivot row to *every other* row so that every other entry in that column is 0. Then move the pivot position one column to the right and one row down.

When the pivot position is moved off the matrix, then the process finishes and the matrix will be in reduced row-echelon form.

Reduced row echelon form

- row echelon form
- leading entries = 1
- leading entries = only non-zero entries in their column

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \quad \checkmark$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right] \quad \times$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \quad \times$$

all 3 in row-echelon form

only 1st in <u>reduced</u> row-echelon form

Writing solutions

From reduced row echelon form=very easy.

As before:
Leading entries correspond to *basic* variables.
Other variables are *non-basic* or *free.*
All *basic* variables are combos of constants / free variables

Example continued

$x = 2-3t, \quad y = -2+3t, \quad z = 2-3t$

Example 1: Gauss-Jordan elimination

$$\begin{aligned} x + y + z + 3t &= 2 \\ 2x + 4y + 2z &= 0 \\ -x + 3y + 4z &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 2 \\ 2 & 4 & 2 & 0 & 0 \\ -1 & 3 & 4 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R2 \leftarrow R2 - 2R1 \\ R3 \leftarrow R3 + R1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 2 \\ 0 & 2 & 0 & -6 & -4 \\ 0 & 4 & 5 & 3 & 2 \end{array}\right] \quad R\cancel{2}2 \leftarrow \frac{1}{2} R2$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 2 \\ 0 & 1 & 0 & -3 & -2 \\ 0 & 4 & 5 & 3 & 2 \end{array}\right] \quad \begin{array}{l} R1 \leftarrow R1 - R2 \\ R\cancel{1}3 \leftarrow R3 - 4R2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 6 & 4 \\ 0 & 1 & 0 & -3 & -2 \\ 0 & 0 & 5 & 15 & 10 \end{array}\right] \quad R3 \leftarrow \frac{1}{5} R3$$

$$R1 - R3 \rightarrow R1 \quad \left[\begin{array}{cccc|c} 1 & 0 & 1 & 6 & 4 \\ 0 & 1 & 0 & -3 & -2 \\ 0 & 0 & 1 & 3 & 2 \end{array}\right]$$

basic   free

$$\begin{array}{c} \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 2 \\ 0 & 1 & 0 & -3 & -2 \\ 0 & 0 & 1 & 3 & 2 \end{array}\right] \end{array}$$

Example 1: Gauss-Jordan elimination continued

We now have the reduced row echelon form

$$\left[\begin{array}{cccr|r} 1 & 0 & 0 & 3 & 2 \\ 0 & 1 & 0 & -3 & -2 \\ 0 & 0 & 1 & 3 & 2 \end{array}\right]$$

so we can write the solution.

We can check by substituting in our original equations.

$$\begin{array}{rcrcrcrl} x & + & y & + & z & + & 3t & = 2 \\ 2x & + & 4y & + & 2z & & & = 0 \\ -x & + & 3y & + & 4z & & & = 0 \end{array}$$

Summary of Gauss-Jordan elimination method

- Write the system of equations as an augmented matrix.
- Use Gauss-Jordan elimination to get the augmented matrix into Reduced Row Echelon Form
- Write the solution set in parametric form (read directly from the matrix).

Example 2

$$
\begin{array}{rcrcrcrcr}
x_1 & + & x_2 & & & + & x_4 & = & 1 \\
x_1 & + & 3x_2 & - & x_3 & + & x_4 & = & 1 \\
 & & x_2 & - & x_3 & - & x_4 & = & -1
\end{array}
$$

$$
\left[\begin{array}{cccc|c}
1 & 1 & 0 & 1 & 1 \\
1 & 3 & -1 & 1 & 1 \\
0 & 1 & -1 & -1 & -1
\end{array}\right] \quad R2 \leftarrow R2 - R1
$$

$$
\left[\begin{array}{cccc|c}
1 & 1 & 0 & 1 & 1 \\
0 & 2 & -1 & 0 & 0 \\
0 & 1 & -1 & -1 & -1
\end{array}\right] \quad R2 \leftarrow \tfrac{1}{2} R2
$$

$$
\left[\begin{array}{cccc|c}
1 & 1 & 0 & 1 & 1 \\
0 & 1 & -1/2 & 0 & 0 \\
0 & 1 & -1 & -1 & -1
\end{array}\right] \quad \begin{array}{l} R1 \leftarrow R1 - R2 \\ R3 \leftarrow R3 - R2 \end{array}
$$

$$\begin{bmatrix} 1 & 0 & 1/2 & 1 & 1 \\ 0 & 1 & -1/2 & 0 & 0 \\ 0 & 0 & -1/2 & -1 & -1 \end{bmatrix} \quad R3 \leftarrow -2R3$$

$$\begin{bmatrix} 1 & 0 & 1/2 & 1 & 1 \\ 0 & 1 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix} \quad \begin{array}{l} R1 \leftarrow R1 - \frac{1}{2}R3 \\ R2 \leftarrow R2 + \frac{1}{2}R3 \end{array}$$

Example 2 continued

$$\begin{array}{c} \begin{array}{cccc|c} x & y & z & t & \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 \end{array}\right] \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \end{array} \end{array}$$

basic

free

$x = 0$

$y + t = 1 \Rightarrow y = 1 - t$

$z + 2t = 2 \Rightarrow z = 2 - 2t$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 1 & 3 & -1 & 1 & 1 \\ 0 & 1 & -1 & -1 & -1 \end{array}\right] \quad R2 \Leftarrow R2 - R1$$

EXAMPLE 2: ~~ALTERNATE~~ METHOD

Gaussian elimination

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & -1\cancel{1} \end{array}\right] \quad R3 \Leftarrow R3 - \frac{1}{2}R2$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & -1 \end{bmatrix}$$

Example 3

$$\begin{array}{ccccccccc} x_1 & & & + & x_3 & + & x_4 & = & 1 \\ x_1 & + & x_2 & - & x_3 & + & x_4 & = & 0 \\ x_1 & & & + & x_3 & + & x_4 & = & 3 \end{array}$$

Try one

Find all solutions to the SLE with augmented matrix

$$\left[\begin{array}{ccccc|c} 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

EXAMPLE 4: PARAMETRIC SYSTEM

SLE with a parameter k. Aim: solve in function of k.
For which value(s) of k has the system 0 solutions, 1 solutions, ∞ly many solutions.
Watch out you never divide nor multiply a row by 0.

$$\begin{array}{rcrcrcl} x & + & y & + & kz & = & 1 \\ kx & + & (k-1)y & + & z & = & k \\ x & + & y & + & k^2 z & = & k+1 \end{array}$$

inconsistent

$$\begin{bmatrix} 1 & 1 & k & 1 \\ k & k-1 & 1 & k \\ 1 & 1 & k^2 & k+1 \end{bmatrix} \quad \begin{array}{l} R2 \leftarrow R2 - kR1 \\ R3 \leftarrow R3 - R1 \end{array}$$

infinitely many solns

$$\left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & -1 & 1-k^2 & 0 \\ 0 & 0 & k^2-k & k \end{array}\right] \quad (k^2-k)z = k$$

basic

If $k=0$
or if $k=1$ than z is free.

Historical parenthesis 2

Gaussian elimination refers only to the procedure until the matrix is in echelon form. The term Gauss-Jordan elimination refers to the procedure which ends in reduced echelon form. The name is used because it is a variation of Gaussian elimination as described by *Wilhelm Jordan* in 1888. However, the method also appears in an article by *Clasen* published in the same year. Jordan and Clasen probably discovered Gauss-Jordan elimination independently.

Source: Wikipedia

Some key words

Linear equation, system of linear equations, constant, variable, solution, consistent, inconsistent, unique, line, plane, elementary row operation, matrix, augmented matrix, coefficient matrix, row-echelon form, Gaussian elimination, pivot, basic variable, non-basic variable, back-substitution, free parameter, reduced row-echelon form,Gauss-Jordan elimination.

MATH1012 Mathematical Theory and Methods

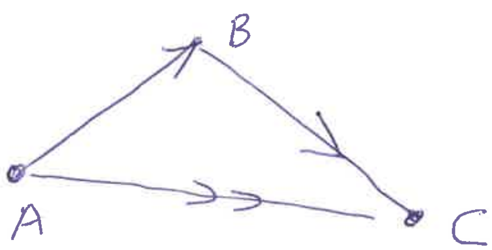
Week 2

CHAPTER 2

This chapter covers *vector spaces* and *subspaces*.

It contains the first material that some students find challenging.

In particular, the notion of *subspace* and demonstrating that a particular set is or is not a subspace seems to be a *"threshold concept"* that requires extra time and effort to master.



Vectors

A (real) *vector* is an ordered n-tuple of real numbers.

The set of *all vectors* of *arity* n is denoted $\mathbb{R}^n$.

Normally we view a vector simply as the coordinates of a point in n-dimensional space.

So $\mathbb{R}^2$ is the usual xy-*plane*, $\mathbb{R}^3$ is normal 3-space.

(x, y)

(x, y, z)

Vector addition

Vectors in the same *vector space* can be added *coordinate-wise*.

$$(1, 0, -1) + (2, 1, 3) = (3, 1, 2)$$
$$(2, -1, 0) + (6, 1, 2) = (8, 0, 2)$$

Vectors of different arity cannot be added

$$(4, 6) + (2, -1, 4) = \quad ??$$

SCALAR MULTIPLICATION

Vectors in a *real vector space* can be multiplied by *real numbers*.

$$10 \cdot (1, 2, -3) = (10, 20, -30)$$
$$-2 \cdot (0, 1, 3) = (0, -2, -6)$$

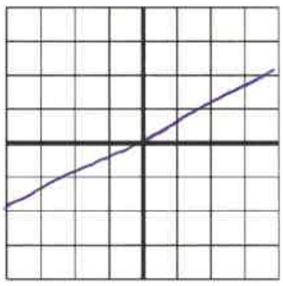
Remember: scalar $\times$ vector $=$ vector

Visualising sets of vectors

Frequently need to describe / analyse various *sets of vectors*, in particular the *solution sets* of SLEs.

- A unique solution: $S = \{(-1, 2, 3)\}$
- An infinite set of solutions $S = \{(1 - 2y, y) : y \in \mathbb{R}\}$

In 2d/3d we can *visualise* the solutions geometrically.



$(1, 2, 3, 7, 9, 10) \in \mathbb{R}^6$

Column vs row

Column vector $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 7 \\ 9 \\ 10 \end{bmatrix}$

Row vector

$\begin{bmatrix} 1 & 2 & 3 & 7 & 9 & 10 \end{bmatrix}$

Closure

Let S be a set of vectors.

- S is *closed under addition* if $x+y \in S$ whenever both $x, y \in S$.
- S is *closed under scalar multiplication* if $r \cdot x \in S$ whenever $r \in \mathbb{R}$ and $x \in S$.

THE WORD *"closed"*

A set S is *closed* under some *operation* if the result of the operation can never "escape from S" if the operation is applied to elements in S.

- $\mathbb{N} = \{1, 2, \ldots, \}$, is closed under addition

  Sum of *any two* natural numbers is a natural number.

- $\mathbb{N} = \{1, 2, \ldots, \}$ is not closed under subtraction

  Difference of two natural numbers *may* or *may not* be a natural number.

Closed under addition?

- $S = \{(1,2),(2,4),(3,6)\}$ ×
- $S = \{(x,2x) : x \in \mathbb{R}\}$ ✓ $(x, 2x) + (y, 2y) =$
  $(x+y, 2x+2y) =$
  $(x+y, 2(x+y))$
- $S = \{(x,y) : x \geqslant 0\}$ ✓

Showing closure

A question that frequently arises is that you are given S, and must decide whether or not it is closed under *vector addition* or *scalar multiplication*.

- If a set *is not* closed

- If a set *is* closed

Asymmetry

Proof and *disproof* are not symmetric:

- Proof

- Counterexample

Subspace

A set of vectors $S \subseteq \mathbb{R}^n$ is a *subspace* if

- $(0, 0, 0, \ldots, 0) \in S$
- S is closed under vector addition
- S is closed under scalar multiplication

Subspace vs Subset

- Subset

- Subspace

Proof or disproof

Given S, is it subspace or not?

- If you think "No"
- If you think "Yes"

EXAMPLES

- The line $x = y$ in $\mathbb{R}^2$ ✓ $S = \{(x, x) : x \in \mathbb{R}\}$

- The line $x + 1 = 2y$ in $\mathbb{R}^2$ ✗

EXAMPLES

- The circle $x^2 + y^2 = 1$ in $\mathbb{R}^2$

×

- The set $\{(0, 0, 0)\}$ in $\mathbb{R}^3$

$S = \{(0,0,0)\}$. ✓

IS THIS SET A SUBSPACE?

YES!

The solutions to the linear equation

$$x + 2y - 3z = 0$$

$$\begin{array}{cccc} & y & z & \\ \end{array}$$
$$\left[\begin{array}{cccc} 1 & 2 & -3 & 0 \end{array}\right]$$

basic

free

$$S = \{ (-2y + 3z, y, z) : y, z \in \mathbb{R} \}$$

$$(-2y + 3z, y, z) + (-2\tilde{y} + 3\tilde{z}, \tilde{y}, \tilde{z})$$

$$= (-2(y + \tilde{y}) + 3(z + \tilde{z}), y + \tilde{y}, z + \tilde{z})$$

(closed under vector +)

$$r \cdot (-2y + 3z, y, z) = (-2(ry) + 3(rz), ry, rz)$$

(closed under scalar multiplication)

$(0,0,0) \in S$ ✓

Is this set a subspace?

The solutions to the linear equation

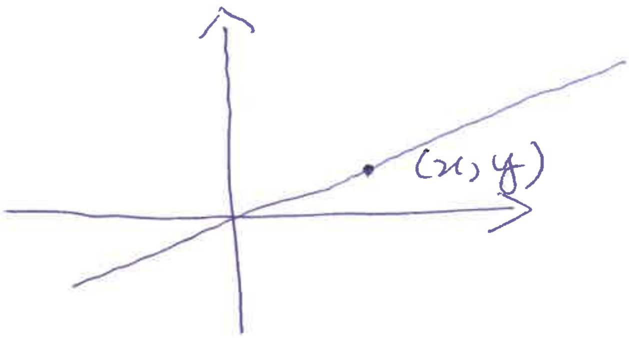
$$x + 2y - 3z = 10$$

Subspaces of $\mathbb{R}^2$

What are *all* the subspaces of $\mathbb{R}^2$?

$S = \{(0,0)\}$ ✓

$S = \{r\cdot(x,y) : r \in \mathbb{R}\}$ x, y fixed. $\neq$ both 0. ✓



$S = \mathbb{R}^2$ ✓

Subspaces of $\mathbb{R}^3$

What are *all* the subspaces of $\mathbb{R}^3$?

$\{(0,0,0)\}$ ✓

$\{r\cdot(x,y,z) : r\in\mathbb{R}\}$ ✓ x,y,z fixed, not all 0

$\{r\cdot(x,y,z) + \tilde{r}\cdot(\tilde{x},\tilde{y},\tilde{z}) : r,\tilde{r}\in\mathbb{R}\}$

(x,y,z), $(\tilde{x},\tilde{y},\tilde{z})$ fixed

$\mathbb{R}^3$ ✓

Another example

Consider the set of vectors

$$S = \{(a, b, a + b) : a, b \in \mathbb{R}\}$$

Is this a subspace or not?

YES!!

Another example

A *linear combination* of $\boldsymbol{v}$ and $\boldsymbol{w}$ is any vector of the form

$$\alpha \boldsymbol{v} + \beta \boldsymbol{w} \quad = (\alpha, 0, \alpha) + (0, \beta, \beta)$$

where α, $\beta \in \mathbb{R}$. $\qquad = (\alpha, \beta, \alpha + \beta)$

If $\boldsymbol{v} = (1, 0, 1)$ and $\boldsymbol{w} = (0, 1, 1)$, then what is the set of *all linear combinations* of $\boldsymbol{v}$ and $\boldsymbol{w}$?

Span

Given $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots \boldsymbol{v_k}\}$, the *span* of T is

$$\text{span}(T) = \{\alpha_1 \boldsymbol{v_1} + \alpha_2 \boldsymbol{v_2} + \cdots + \alpha_k \boldsymbol{v_k} : \alpha_i \in \mathbb{R}\}$$

In other words, span(T) is exactly the set of all *linear combination* that can be formed from the vectors in T.

Spans are subspaces

nonempty (!)

The span of *anything* is a subspace!

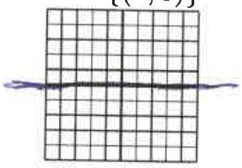
The span of T is the *smallest* subspace containing T.

Convention: $\mathrm{span}(\emptyset) = \{(0, 0, \dots 0)\}$,

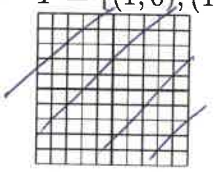
Visualising spans

What is span(T) for

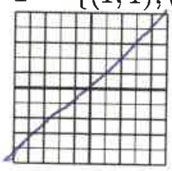
- $T = \{(1,0)\}$



- $T = \{(1,0),(1,1)\}$



- $T = \{(1,1),(2,2)\}$



$$(\alpha, 0) + (\beta, \beta)$$
$$= (\alpha + \beta, \beta)$$

Testing membership

$$\text{Is } (1,0,1) \text{ in } \operatorname{span}(\{\overbrace{(1,2,3),(2,3,4)}^{T}\})?$$

$$\alpha_1(1,2,3) + \alpha_2(2,3,4) =$$

$$(1,0,1) \stackrel{?}{=} (\alpha_1 + 2\alpha_2,\ 2\alpha_1 + 3\alpha_2,\ 3\alpha_1 + 4\alpha_2)$$

some α_1, α_2

$$\alpha_1 + 2\alpha_2 = 1$$
$$2\alpha_1 + 3\alpha_2 = 0$$
$$3\alpha_1 + 4\alpha_2 = 1$$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 3 & 0 \\ 3 & 4 & 1 \end{array}\right] \begin{array}{l} \\ R2 \leftarrow R2 - 2R1 \\ R_3 \leftarrow R_3 - 3R1 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & -2 & -2 \end{array}\right] \quad R3 \leftarrow R3 - 2R2$$

Testing membership

We get a system of linear equations!

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{array}\right] \leftarrow$$

system is inconsistent:

no solutions

NO: $(1,0,1)$ is not in

$\mathrm{span}(T)$.

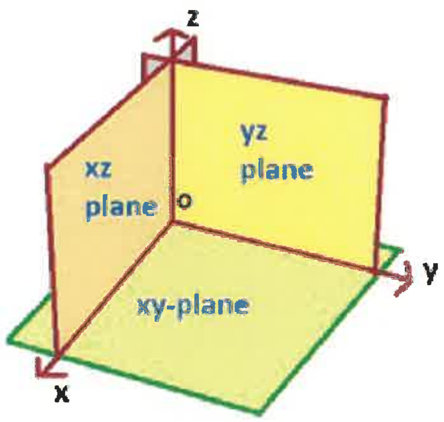
Two types of question

- Given a set T of vectors, find the *subspace* $\mathrm{span}(T)$

- Given a subspace S, find a *spanning set* of vectors for S

Spanning set → Subspace

If $T = \{(1,0,0),(0,1,0)\}$, then $\operatorname{span}(T)$ is the xy-plane.



Spanning set → Subspace

If $T = \{(1,0,0),(0,1,0)\}$, then what is $S = \operatorname{span}(T)$?

$$\downarrow$$

$$\alpha_1 \cdot (1,0,0) + \alpha_2 (0,1,0) =$$

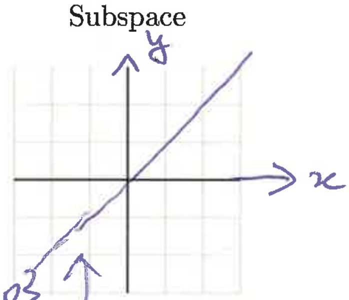
$$(\alpha_1, \alpha_2, 0)$$

Example 1

Spanning Set

$T\ \cancel{S} = \{(1,1)\}$

Subspace



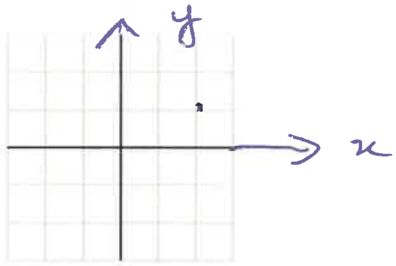
$\text{span}(T) = \{(\alpha,\alpha) : \alpha \in \mathbb{R}\}$

Example 2

Spanning Set

$T\ \cancel{S} = \{(1,1), (1,0)\}$

Subspace



$\operatorname{span}(T) =$

$\{ \alpha_1 \cdot (1,1) + \alpha_2 \cdot (1,0) : \alpha_1, \alpha_2 \in \mathbb{R} \}$

$\parallel$

$(\alpha_1+\alpha_2, \alpha_1)$

Subspace → Spanning Set

What is a *spanning set* for the plane $x = y$ in $\mathbb{R}^3$?

$$(x, x, z) =$$

$$x(1, 1, 0) + z(0, 0, 1)$$

$$T = \{ (1,1,0), (0,0,1) \}$$

Verifying a spanning set

Given a subspace S, how do we prove that T is a spanning set?

- $\text{span}(T) \subseteq S$

- $\text{span}(T) \supseteq S$

Proof

Show that $\operatorname{span}((1,1,0),(0,0,1)) = \{(x,x,z) : x, z \in \mathbb{R}\}$

FINDING A (SMALL) SPANNING SET FOR S

$\cancel{x}(0, 0, \ldots 0)$ $\emptyset$

Algorithm:

- Choose some vector $\boldsymbol{v_1} \in S$ and examine span$(\boldsymbol{v_1})$.
- If this equals S we are done.
- Otherwise choose some vector $\boldsymbol{v_2} \in S$ which is not in span$(\boldsymbol{v_1})$ and examine span$(\boldsymbol{v_1}, \boldsymbol{v_2})$
- etc

Other method: if S is given in terms of parameters, write a general vector of S as a linear combination where the *parameters are the coefficients*.

Find a spanning set

for $S = \{\underline{(x, 2x, z, -z)} : x, z \in \mathbb{R}\}$

$$x(1,2,0,0) + z(0,0,1,-1)$$

$$T = \{(1,2,0,0), (0,0,1,-1)\}$$

$\underline{0} = (0, 0, 0, \cdots 0) \in \mathbb{R}^n$

Linear Dependence

$\{\underline{0}\}$ is linearly dependent

Any other ~~A~~ A set of vectors is *linearly dependent* if one of them is a linear combination of the others.

- $T = \{(1,0,0), (0,1,1), (1,1,1)\}$

A set of vectors is *linearly independent* (or just *independent*) if it is not linearly dependent.

Dependent or independent?

- Set $\{\mathbf{0}\}$ DEPENDENT
- Set $\{\boldsymbol{v}\}$ with $\boldsymbol{v} \neq \mathbf{0}$ INDEPENDENT
- Set $\emptyset$ INDEPENDENT

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \quad R3 \leftarrow R3 - R2$$

Linear Independence

Is the following set of vectors independent or dependent? DEPENDENT.

- $T = \{(1,0,1),(0,2,1),(2,2,3)\}$

$$(1,0,1) \stackrel{?}{=} \beta_1(0,2,1) + \beta_2(2,2,3)$$

- $T = \{(1,0,0),(0,1,0),(0,0,1)\}$
- $T = \{(1,0),(2,0),(0,1)\}$

$$(2\beta_2,\ 2\beta_1+2\beta_2,\ \beta_1+3\beta_2)$$

$$\left[\begin{array}{cc|c} 0 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 3 & 1 \end{array}\right]$$

$$2\beta_2 = 1$$
$$2\beta_1 + 2\beta_2 = 0$$
$$\beta_1 + 3\beta_2 = 1$$

$$\begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix} \quad R3 \leftarrow R3 - \tfrac{1}{2}R1$$

Linear Independence

When is a set $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ of size 2 linearly dependent?

either $v_1 = \alpha_2 v_2$ some $\alpha_2 \in \mathbb{R}$

or $v_2 = \alpha_1 v_1$ some $\alpha_1 \in \mathbb{R}$

Linear Independence Test

In $\mathbb{R}^n$, a set $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}$ is *linearly independent* if and only if the homogeneous system of n linear equations

$$\alpha_1 \boldsymbol{v_1} + \alpha_2 \boldsymbol{v_2} + \cdots + \alpha_k \boldsymbol{v_k} = 0$$

in the unknowns $\alpha_1, \alpha_2, \ldots, \alpha_k$ has a *unique solution* $\alpha_1 = \alpha_2 = \ldots = \alpha_k = 0$.

(This saves testing each vector individually.)

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \qquad \begin{array}{l} \alpha_1 = 0 \\ \alpha_2 = 0 \\ \alpha_3 = 0 \end{array}$$

Linear Independence Test

Is $T = \{\underset{v_1}{(1,0,0)}, \underset{v_2}{(0,1,1)}, \underset{v_3}{(1,1,2)}\}$ independent or ~~dependent~~?

Solve

$$\alpha_1(1,0,0) + \alpha_2(0,1,1) + \alpha_3(1,1,2) = (0,0,0)$$
$$\| $$
$$(\alpha_1 + \alpha_3,\ \alpha_2 + \alpha_3,\ \alpha_2 + 2\alpha_3)$$

$$\alpha_1 + \qquad \alpha_3 = 0$$
$$0 + \alpha_2 + \alpha_3 = 0$$
$$\alpha_2 + 2\alpha_3 = 0$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \quad R_3 \leftarrow R_3 - R_2$$

Properties of dependence

- A *subset* of a linearly independent set is linearly independent
- A *superset* of a linearly dependent set is dependent

Subsets of independent sets

The set

$$\{(1,0,0),(0,1,1),(1,1,2)\}$$

is linearly *independent*.

So the set

$$\{(0,1,1),(1,1,2)\}$$

is also independent.

Supersets of dependent sets

The set

$$\{(1,1,1),(2,2,2)\}$$

is linearly dependent.

So the set

$$\{(1,1,1),(2,2,2),(1,0,0)\}$$

is also dependent.

Properties of dependence

- If $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}$ is a dependent set and $\boldsymbol{v_1}$ is a linear combination of the other vectors, then

$$\mathrm{span}(S) = \mathrm{span}(S \setminus \{\boldsymbol{v_1}\})$$

MENTAL MODEL

- Linearly dependent set
  - Set has some *redundancy*
  - Can remove dependent vector(s) without changing the span
- Linearly independent set
  - No redundancy
  - Every vector *needed*

So far

- Systems of linear equations
- Gaussian/Gauss-Jordan elimination
- Basic and non-basic variables
- Expressing set of solutions
- Vectors, vector addition, scalar multiplication
- Vector spaces $\mathbb{R}^2$, $\mathbb{R}^3$, $\ldots$, $\mathbb{R}^n$
- Closure under an operation
- Subspaces (contain $\mathbf{0}$, closed under $+$, closed under $\cdot$)
- Span of a set of vectors (baking a cake)
- Spanning set for a subspace (unbaking a cake)
- Linear independence

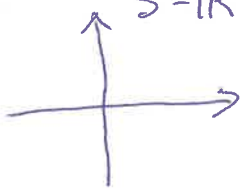
Basis

If S is a *subspace* then a *basis* for S is a set B of vectors such that

- $S = \text{span}(B)$
- B is linearly independent

Note that a basis is not unique.

$S = \mathbb{R}^2$



$B = \{(1,0), (0,1)\}$

B a basis of $\mathbb{R}^2$

$\tilde{B} = \{(1,0), (1,1)\}$

B another basis of $\mathbb{R}^2$,

Why bases?

A basis is the *best way* to specify a subspace.

- It contains *enough* information to exactly describe the subspace
- It does not have any *redundancy*

A very common question is "Give a basis for the following subspace ..."

Not unique

Give two different bases for the xy-plane in $\mathbb{R}^3$?

Standard Basis for $\mathbb{R}^n$

What is the simplest basis for $\mathbb{R}^3$?

$B = \{ (1,0,0), (0,1,0), (0,0,1) \}$

What is the simplest basis for $\mathbb{R}^n$?

$B = \{ (1,0,0,\cdots 0), (0,1,0,\cdots 0), \cdots\cdots$
$(0,0,\cdots 0,1) \}$

Called the *standard basis*.

($\uparrow$ n vectors in $\mathbb{R}^n$).

DIMENSION

Every basis for a subspace S has the *same number* of vectors.

This number is called the *dimension* of S, denoted $\dim(S)$.

$\dim(\mathbb{R}^n) = n$

Dimension of subspaces

In $\mathbb{R}^2$, what is

- the dimension of a *line* passing through 0? 1

- the dimension of the whole of $\mathbb{R}^2$? 2

- the dimension of $\{\mathbf{0}\}$? 0

Goldilocks

A basis for a subspace satisfies the “Goldilocks property”

- It is *just big enough* to be a spanning set
- It is *just small enough* to be an independent set

It is “just right”!

Handy facts about spanning sets

Let S be a k-dimensional subspace of $\mathbb{R}^n$. Then

- Any set of size $< k$ is not a spanning set for S

- Any spanning set for S has size $\geqslant k$ and contains a basis

- Any spanning set for S of size exactly k is a basis

Handy facts about linearly independent sets

Let S be a k-dimensional subspace of $\mathbb{R}^n$. Then

- Any subset of S of size $> k$ is linearly dependent

  If not then this subset T would be linearly independent + of size $> k$. Then $span(T) \subseteq S$ ~~...~~ .

- Any linearly independent set for S has size $\leq k$ and can be extended to a basis

- Any linearly independent set for S of size exactly k is a basis

$$\{(-b-3d-e,\ b,\ d,\ d,\ e) : b, d, e \in \mathbb{R}\}$$

$$\|$$

$$b\underline{(-1,0,1,0,0)} + d\underline{(-3,0,1,1,0)} + e\underline{(-1,0,0,0,1)}$$

EXAMPLE

Give a basis for the *solution space* of the SLE

$$\begin{aligned} a + b + 2c + d - e &= 0 \\ 2a + 2b + 5c + d - 2e &= 0 \end{aligned}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & -1 & 0 \\ 2 & 2 & 5 & 1 & -2 & 0 \end{array}\right] R2 \leftarrow R_2 - 2R1$$

$$\begin{array}{ccccc|c} a & b & c & d & e & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right]$$

basic free basic

$$\underline{c = d}$$

$$\begin{aligned} a &= -b - 2c - d - e \\ &= -b - 2(d) - d - e \\ &= \underline{-b - 3d - e} \end{aligned}$$

MATH1012 Mathematical Theory and Methods

Week 3

Example continued

UNIQUE LINEAR COMBINATION

ordered

Let $B = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}$ be a basis for a subspace S of $\mathbb{R}^m$. Then any vector $\boldsymbol{v}$ of S can be written in a *unique manner* as a linear combination

$$\boldsymbol{v} = \alpha_1 \boldsymbol{v_1} + \alpha_2 \boldsymbol{v_2} + \ldots + \alpha_k \boldsymbol{v_k}.$$

We need to show that

1. At *least* one linear combination is equal to $\boldsymbol{v}$. ✓
2. At *most* one linear combination is equal to $\boldsymbol{v}$.

if $\quad v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$

and $\quad v = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_k v_k$

$\underline{0} = (\alpha_1 - \beta_1) v_1 + (\alpha_2 - \beta_2) v_2 + \cdots + (\alpha_k - \beta_k) v_k$

linear $\Longrightarrow$ independence of B

$\alpha_1 - \beta_1 = 0 \Longrightarrow \alpha_1 = \beta_1$

$\alpha_2 - \beta_2 = 0 \Longrightarrow \alpha_2 = \beta_2$

$\cdots \qquad \cdots$

$\alpha_k - \beta_k = 0 \Longrightarrow \alpha_k = \beta_k$

PROOF

Coordinates

ordered

If $B = \{v_1, v_2, \ldots, v_k\}$ is ~~a~~ basis for a subspace S of $\mathbb{R}^m$, and

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k,$$

then the (uniquely determined) scalars in this linear combination are called the *coordinates* of the vector with respect to the basis B.

We write $(v)_B = (\alpha_1, \alpha_2, \ldots, \alpha_k)$.

Coordinates: example

Take $S = \mathbb{R}^3$.
What are the coordinates of $(1, 2, 3)$ with respect to

- $B = \{(1, 0, 1), (0, 1, 1), (0, 0, 1)\}$

$$(1,2,3) = 1\cdot(1,0,1) + 2\cdot(0,1,1) + 0\cdot(0,0,1)$$

$$(1,2,3)_B = (1,2,0)$$

- the standard basis

$$S = \{(1,0,0), (0,1,0), (0,0,1)\}$$

$$(1,2,3)_S = 1\cdot(1,0,0) + 2\cdot(0,1,0) + 3\cdot(0,0,1)$$
$$= (1,2,3)$$

Where are we?

Now we start Chapter 3 in the unit reader.

It covers matrices. We assume you are familiar with matrix addition and multiplication.

$$\begin{array}{c} \\ \end{array} \quad \begin{array}{cc} & j \\ B = & \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{2p} \\ b_{n1} & \cdots & b_{np} \end{bmatrix} \end{array}$$

$$A = \begin{array}{c} \\ i \\ \\ \end{array} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \boxed{a_{21}} & \boxed{a_{22}} & \cdots & \boxed{a_{2n}} \\ & \cdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad n \text{ columns}$$

m rows

Matrix Algebra

A is an $m \times n$ matrix

- ▶ Matrix Addition

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

- ▶ Matrix Multiplication

$$\underset{m\times n}{A} \; \underset{n\times p}{B} = \begin{array}{c} \\ i \\ \\ \end{array} \begin{bmatrix} & j\text{th} & \\ \cdots & \boxed{} & \\ & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

$m = n = p = 2$

$$\neq$$

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

usually $AB \neq BA$.

undefined unless $m = p$

Matrix Algebra

- Scalar Multiplication

$$10 \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 20 \\ 30 & 40 \end{bmatrix}$$

- Transpose

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Matrix Algebra

Lots and lots of properties (most of which are obvious) which are all listed in the unit reader.

What is $(AB)^T$?

$\Downarrow$

$B^T A^T$

Multiplication not commutative

Let A, B be square matrices. In general

$$AB \neq BA$$

Special square matrices

- Zero matrix O_n or just O ✓
- Identity matrix I_n or just I ✓
- Symmetric matrix

$$\begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix}$$

SLEs and matrices

The system with augmented matrix

$$\left[\begin{array}{c|c} \quad A \quad & \boldsymbol{b} \end{array}\right]$$

corresponds exactly to solving the matrix equation

$$A\boldsymbol{x} = \boldsymbol{b}$$

where $\boldsymbol{x}$ is a column vector of variables.

$$\begin{aligned} 2x + 3y + 4z &= 5 \\ 3x + 2y - z &= 2 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 5 \\ 3 & 2 & -1 & 2 \end{array}\right]$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

3×4 matrix

$m \times n$

Subspaces from matrices

Let A be an $m \times n$ matrix.

- The set of linear combinations of the *rows* of A is the *row space* of A. It is a subspace of $\mathbb{R}^n$.
- The set of linear combinations of the *columns* of A is the *column space* of A. It is a subspace of $\mathbb{R}^m$.

Subspaces from matrices

Let A be an $m \times n$ matrix.

- The *nullspace* of A is the set all vectors $x \in \mathbb{R}^n$ satisfying

$$Ax = 0$$

$$m \underset{n}{\begin{bmatrix} - & - & - \\ - & - & - \\ - & & - \end{bmatrix}} \underset{n \times 1}{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}} = \underset{m \times 1}{\begin{bmatrix} \cdot \\ \vdots \\ \cdot \end{bmatrix}} \overset{?}{=} 0_{m \times 1}$$

$$m\begin{bmatrix} & & \\ & & \\ & n & \end{bmatrix}\underbrace{\begin{bmatrix} \\ \\ \\ \end{bmatrix}}_{n\times 1} = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}}_{m\times 1}$$

$0_{n\times 1} \in$ in null-space.

$$\begin{bmatrix} A \end{bmatrix}\left(\begin{bmatrix} \cancel{x} \\ x \end{bmatrix} + \begin{bmatrix} y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

if x, y in nullspace

(closed under +)

NULLSPACE

$$\begin{bmatrix} A \end{bmatrix}\left(r\cdot\begin{bmatrix} x \end{bmatrix}\right) = r\cdot(A\cdot x) = \underline{0}$$

$\Rightarrow r\cdot x$ in null-space if x is,

The *nullspace* of the $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. ✓

Its dimension is called the *nullity* of A.

Sample Questions

✓✓ ✓ ?

- Is a given vector v in the null/row/column space of A

- Find a *basis* for the null/row/column space of A

- Find the *dimension* of the null/row/column space of A

Required Skills

By this stage, you *must* be able to quickly and accurately

- Solve a system of linear equations, determining the number of solutions
- Write down the solutions, both variable-by-variable, and as a set of vectors
- Find a basis for the solution space of a homogeneous SLE
- Determine the dimension of the solution space of a homogeneous SLE

Examples

We'll use the 3×4 matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 3 & 2 \\ 2 & 1 & 4 & 3 \end{bmatrix}$$

(This example has $m = 3$, $n = 4$)

$$\text{Nullspace} = \{(-x_3 - x_4,\ -2x_3 - x_4,\ x_3,\ x_4) : x_3, x_4 \in \mathbb{R}\}$$

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \hline \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} x_1 &= -x_3 - x_4 \\ x_2 &= -2x_3 - x_4 \end{aligned}$$

basic, base, free, free

etc. null-space has dimension 2 (nullity = 2)

Nullspace

Is $x = (0, -1, 1, -1) \in \text{nullsp}(A)$?

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 3 & 2 \\ 2 & 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

yes. ✓

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 3 & 2 \\ 2 & 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 3 & 2 & 0 \\ 2 & 1 & 4 & 3 & 0 \end{array}\right] \qquad \begin{aligned} R2 &\leftarrow R_2 - R_1 \\ R3 &\leftarrow R_3 - 2R_1 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{bmatrix} \qquad R3 \leftarrow R_3 - R2$$

~~#####~~ $= x_3(-1, -2, 1, 0) + x_4(-1, -1, 0, 1)$

Nullspace $= \text{span}\{(-1, -2, 1, 0), (-1, -1, 0, 1)\}$

linearly independent

(neither a multiple of the other)

~~###~~ is a basis

nullity $= 2$

Nullspace

Find a basis for the nullspace of

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 3 & 2 \\ 2 & 1 & 4 & 3 \end{bmatrix}$$

Homogeneous SLE

Consider a homogeneous SLE

$$Ax = 0 \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The *solution set* is just the nullspace of A.

The *dimension* of the solution space is the *number non-basic variables* in the row reduced form of A.

Working

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 3 & 2 & 0 \\ 2 & 1 & 4 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Row space

Is $x = (1, 1, 1, 1) \in \text{rowsp}(A)$?

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 3 & 2 \\ 2 & 1 & 4 & 3 \end{bmatrix}$$

$$(\alpha_1 + \alpha_2 + 2\alpha_3,\ \alpha_2 + \alpha_3,\ \alpha_1 + 3\alpha_2 + 4\alpha_3,\ \alpha_1 + 2\alpha_2 + 3\alpha_3)$$

$$= \alpha_1 (1, 0, 1, 1) + \alpha_2 (1, 1, 3, 2) + \alpha_3 (2, 1, 4, 3)$$

$$\stackrel{?}{=} (1, 1, 1, 1)$$

want to know if we can solve

$$\alpha_1 + \alpha_2 + 2\alpha_3 = 1$$

$$\alpha_2 + \alpha_3 = 1$$

$$\alpha_1 + 3\alpha_2 + 4\alpha_3 = 1$$

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 = 1$$

Working

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 3 & 4 & 1 \\ 1 & 2 & 3 & 1 \end{array}\right]$$

$R_3 \leftarrow R_3 - R_1$
$R_4 \leftarrow R_4 - R_1$

$$\downarrow$$

$$\left[\begin{array}{cccc} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$R_3 \leftarrow R_3 - 2R_2$
$R_4 \leftarrow R_4 - R_2$

$$\left(\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 \end{array}\right]\right) \checkmark$$

inconsistent system
(no $\alpha_1, \alpha_2, \alpha_3$ can satisfy these equations)

so $(1,1,1,1)$ is NOT in the row space of A.

Basis for rowspace

Several methods for finding basis of rowspace

- Take the rows of the matrix then delete redundant rows one by one. (? ✓)
- Perform row reduction (similar to Gaussian elimination method) ✓✓

Method 1

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 3 & 2 \\ 2 & 1 & 4 & 3 \end{bmatrix}$$

$R_1 + R_2 = R_3$

~~R_2 not a multiple of R_3~~

$\{R_1, R_2\}$ span r.s.

$\uparrow$

linearly independent

So $\{(1,0,1,1), (1,1,3,2)\}$ is a basis for the row space.

Method 2

ERO's do not change the row space
The non-zero rows of a matrix in row echelon form are independent, hence form a basis of the rowspace.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 3 & 2 \\ 2 & 1 & 4 & 3 \end{bmatrix} \quad \begin{array}{l} R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - 2R_1 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix} \quad R_3 \leftarrow R_3 - R_2$$

The dimension of the rowspace is the number of non-zero rows in the row reduced form of A, that is, the number of basic variables.

$$\begin{array}{c} \rightarrow \\ \rightarrow \\ \\ \end{array} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\{(1,0,1,1), (0,1,2,1)\}$ is a basis for the row space.

Difference

- The “delete-rows” method

  Finds a basis consisting of (some of) the *original rows*

- The “EROs” method

  Finds a basis, but almost always *not the original rows*

Column space

The *column space* of

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 3 & 2 \\ 2 & 1 & 4 & 3 \end{bmatrix}$$

is the *row space* of

$$A^T = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

Column space

To determine the column space of A: find row space of *transposed* matrix A^T

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix} \begin{array}{l} \\ \\ R_3 \leftarrow R_3 - R_1 \\ R_4 \leftarrow R_4 - R_1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{array}{l} \\ \\ R_3 \leftarrow R_3 - 2R_2 \\ R_4 \leftarrow R_4 - R_2 \end{array}$$

$$\begin{array}{l} \rightarrow \\ \rightarrow \\ \\ \\ \end{array} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\{(1,1,2), (0,1,1)\}$ is a basis for the row space of A^T, namely a basis for the column space of A.

Rank

- The *row rank* of A is the dimension of its row space
- The *column rank* of A is the dimension of its column space

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 3 & 2 \\ 2 & 1 & 4 & 3 \end{bmatrix}$$

Row rank

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix} \qquad 2$$

COLUMN RANK

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \quad 2$$

Key property

Key property: Row rank is *equal* to column rank

This is not at all obvious, nor is it very easy to show.

(It is proved in Theorem 3.12 in the unit reader.)

The nullity

The *nullity* of a matrix is the dimension of its nullspace.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

Rank-nullity theorem

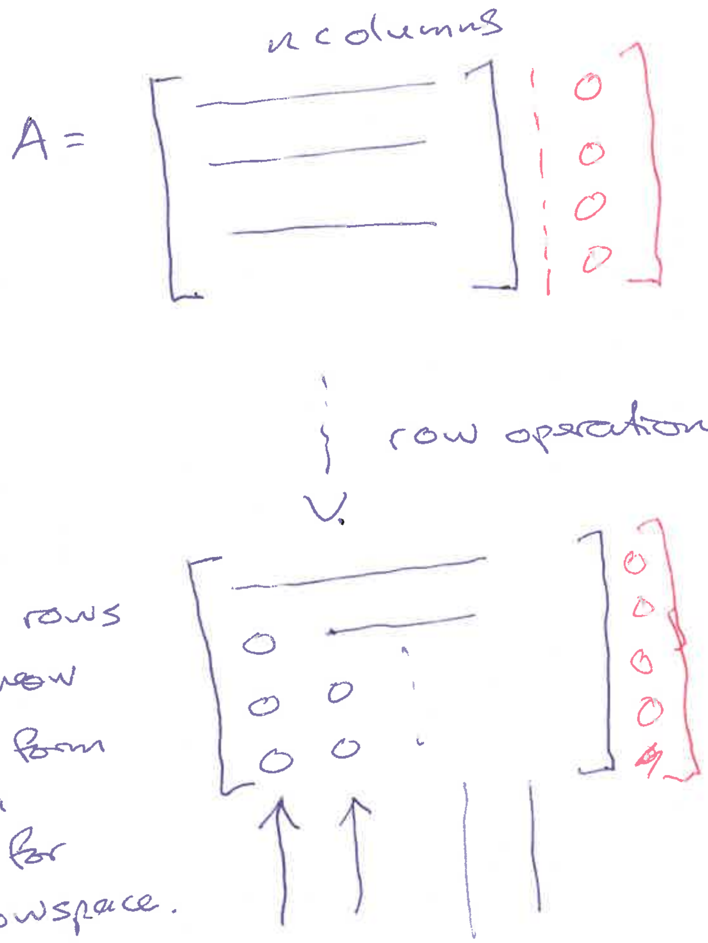
If A is an $m \times n$ matrix, then

$$\text{rank}(A) + \text{nullity}(A) = n$$

Theorem: "rank + nullity = number of columns"

$\underline{Ax = 0}$

WHY?



nonzero rows in the row echelon form give a basis for the rowspace.

basic

not basic

$\boxed{\text{free}}$

dimension of solution space

$= \boxed{\text{nullity of } A}$

number of nonzero rows

||

row rank A = rank A

$\cancel{\text{?}}$ $n = \text{rank}\, A + \text{nullity}\, A$

Back to SLEs

Consider the SLE $Ax = b$ and two solutions u_1 and u_2

SLE EXAMPLE

In week 1 we solved the following SLE

$$\begin{array}{rcrcrcrl} x & + & y & + & z & + & 3t & = 2 \\ 2x & + & 4y & + & 2z & & & = 0 \\ -x & + & 3y & + & 4z & & & = 0 \end{array}$$

and found its reduced row echelon form

$$\begin{array}{cccc|c} x & y & z & t & \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 2 \\ 0 & 1 & 0 & -3 & -2 \\ 0 & 0 & 1 & 3 & 2 \end{array}\right]$$

basic

free

$x = 2 - 3t$
$y = -2 + 3t$
$z = 2 - 3t$

$(2-3t, -2+3t, 2-3t, t)$

$\|$

$(2, -2, 2, 0) + t\underline{\underline{(-3, 3, -3, 1)}}$

Non-homogeneous SLE

Consider the SLE $A\boldsymbol{x} = \boldsymbol{b}$ and one solution $\boldsymbol{u_1}$.
Then every solution can be written as $\boldsymbol{u_1} + \boldsymbol{w}$ where $\boldsymbol{w}$ is a solution of the associated homogeneous system $A\boldsymbol{x} = \boldsymbol{0}$

We write $S = \boldsymbol{u_1} + \text{nullsp}(A)$. We like to express that solution using a basis for the null space.

Inverses

If A is an $n \times n$ matrix and

$$AB = I_n$$

then

$$BA = I_n$$

and we call B the *inverse* of A, and we say A is *invertible*.

(This is not obvious — it is proved in Thm 3.23)

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Finding the inverse

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{array}{cc} a & c \\ \multicolumn{2}{c}{\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 1 & 2 & 0 \end{array}\right]} \end{array} \quad R2 \leftarrow R2 - \tfrac{1}{2} R_1$$

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \quad \begin{array}{l} 2a = 1 - 3c = 4 \Rightarrow a = 2 \\ c = -1 \end{array}$$

$$\begin{array}{cc} b & d \\ \multicolumn{2}{c}{\left[\begin{array}{cc|c} 2 & 3 & 0 \\ 1 & 2 & 1 \end{array}\right]} \end{array} \quad R2 \leftarrow R2 - \tfrac{1}{2} R_1$$

$$\left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & \frac{1}{2} & 1 \end{array}\right] \quad \begin{array}{l} 2b = -3d = -6 \Rightarrow b = -3 \\ d = 2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

$R_1 \leftarrow R_1 + R_3$

$R2 \leftarrow R2 - R_3$

$$\underbrace{\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right.}_{I_3} \underbrace{\left|\begin{array}{ccc} 2 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{array}\right]}_{A^{-1}}$$

Find inverse

Solve $AB = I_n$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$R3 \leftarrow R3 - R1$

use EROs to get

I_3 here

then will have $B = A^{-1}$ here.

$$\left[\begin{array}{cccccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$R3 \leftarrow R3 - R2$

$$\left[\begin{array}{cccccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right]$$

$R3 \leftarrow -1 \times R3$

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

Method

To find the inverse of an $n \times n$ matrix A

- Form the "super-augmented" matrix

$$\left[\; A \;\middle|\; I_n \;\right]$$

- Perform *Gauss-Jordan* elimination to try to reach the *reduced* row-echelon form

$$\left[\; I_n \;\middle|\; X \;\right]$$

- If successful, then $X = A^{-1}$, and if not successful, A is not invertible.

Gauss-Jordan elimination

Find the *reduced row-echelon form* of the super-augmented matrix.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

Which matrices are invertible?

Only those for which the reduced row-echelon form is the identity.

In other words, those whose rank is equal to n.

> Invertible matrices have rank equal to n, and matrices with rank equal to n are invertible.

TFAE for an $n \times n$ matrix A

The **F**ollowing **A**re **E**quivalent:

- A is invertible
- A has full rank n,
- The rows of A are linearly independent
- The columns of A are linearly independent

More on inverses

both $n \times n$ matrices

If A, B are invertible, then

- AB is invertible
- A^2 is invertible
- A^T is invertible

$$(AB)(\underline{B^{-1}A^{-1}}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$$

$$(A^T)^{-1} = (A^{-1})^T$$

$$(A^n)^{-1} = (A^{-1})^n.$$

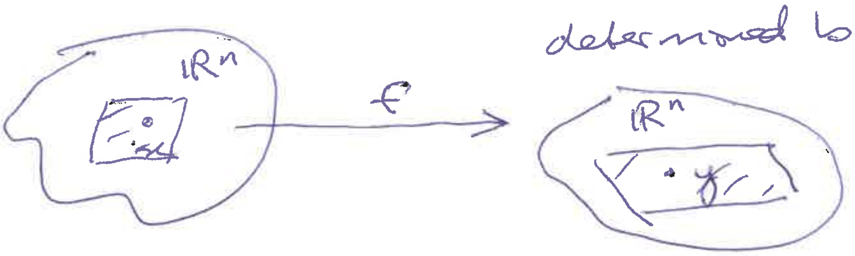
MATH1012 Mathematical Theory and Methods

Week 4

$$\underset{n\times n}{A}\ \underset{n\times 1}{[x]} = \underset{n\times 1}{[y]}$$

$$x \longmapsto y = f(x)$$

determined by A



Determinants

$$\mathrm{Vol}_n(f(\square)) = |\det A|\,\mathrm{Vol}_n(\square).$$

A *number* associated with any square matrix:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Alternative, but common, notation

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

INVERSES

Somehow related to inverses

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Three-by-three

What is

$$\det \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} ?$$

Defined recursively using *cofactors*

Technically

Let A be an $n \times n$ matrix.

- if $n = 1$, then $\det(A) = a_{11}$.
- if $n > 1$, then

$i \quad i \quad i$

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A[1, j]),$$

where $A[i, j]$ is the matrix obtained from A by deleting the ith row and jth column.

Expanding along the first row.

$i=1$: expand by first row

$$\det(A) = \sum_{j=1}^{n=3} (-1)^{1+j} a_{1j} \det(A[1,j])$$

$$(-1)^{1+1} a_{11} \det\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right) + (-1)^{1+2} a_{12} \det\left(\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\right)$$

$$+ (-1)^{1+3} a_{13} \det\left(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right) =$$

Three-by-three

We can expand along any row or column

$$\begin{array}{cc} & \begin{array}{ccc} & j & \end{array} \\ \det & \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \end{array}$$

$$= 1 \cdot (-2) \quad - \cdot 0 \left(\quad \right) + -1 \cdot -1$$

$$= -2 + 1 = \underline{\underline{-1}}.$$

2nd row:

$$= 0 \left(\quad \right) + 1 \det\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} - 1 \det\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \underline{\underline{-1}}$$

2nd column:

$$0 \det\begin{bmatrix} & \\ & \end{bmatrix} + 1 \det\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} - \det\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \underline{\underline{-1}}.$$

The pattern

Expanding requires using this pattern:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} + & - & + & - & + & - \\ - & + & - & + & - & + \\ + & - & + & - & + & - \\ - & + & - & + & - & + \\ + & - & + & - & + & - \\ - & + & - & + & - & + \end{bmatrix}$$

Pick the right row/column

$$\det \begin{bmatrix} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 2 \\ 1 & 2 & 1 & 5 & 7 \end{bmatrix}$$

expand about column 3 :

$$0 \cdot (\quad) + 0(\quad) + 0(\quad) + 0(1)$$

$$+ 1 \cdot \det \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix} = \text{(expand about column 2)}$$

$$\det \begin{bmatrix} 1 & -1 & 2 \\ 1 & -1 & 1 \\ 1 & 0 & 2 \end{bmatrix} = \text{(can subtract R2 from R1 without penalty)}$$

$$\det \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 2 \end{bmatrix} = -\det \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = 1.$$

First properties

- $\det(A^T) =$ det (A)
- if A has a row of zeroes then $\det(A) =$ 0

Upper triangular

If A is upper (or lower) triangular, then $\det(A) =$

expand top column 1

$$\det \begin{bmatrix} 1 & -2 & 6 & -1 & 2 \\ 0 & 3 & 5 & 2 & 1 \\ 0 & 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix} =$$

$$1 \cdot \det \begin{bmatrix} 3 & 5 & 2 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 7 \end{bmatrix} =$$

$$1 \cdot 3 \cdot \det \begin{bmatrix} 2 & -1 & 1 \\ 0 & 5 & 2 \\ 0 & 0 & 7 \end{bmatrix} =$$

$$1 \cdot 3 \cdot 2 \det \begin{bmatrix} 5 & 2 \\ 0 & 7 \end{bmatrix} =$$

$$1 \cdot 3 \cdot 2 \cdot 5 \det [7] = \underline{1 \cdot 3 \cdot 2 \cdot 5 \cdot 7} = 210$$

Expanding method doesn't scale

Computing the determinant of one $n \times n$ matrix requires computing the determinant of n matrices of order $n - 1$.

Computing the determinant of 100×100 matrix in this fashion would take longer than lifetime of the universe.

But coming to the rescue is our trusty friend — *row-reduction*.

First ingredient

A square matrix in row echelon form is upper triangular

Hence finding the determinant is *easy* for matrices in row-echelon form.

Second ingredient

EROs *do change* the determinant, but only in *predictable* ways

If A' is obtained from A by doing an ERO then

- $\det A' = -\det A$ if the ERO is Type 1: $R_i \longleftrightarrow R_j$

$$\det \begin{bmatrix} 0 & -1 & 6 \\ 1 & 3 & 5 \\ 6 & 8 & 2 \end{bmatrix} = -\det \begin{bmatrix} 1 & 3 & 5 \\ 0 & -1 & 6 \\ 6 & 8 & 2 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 6 & 8 & 2 \\ 1 & 3 & 5 \\ 0 & -1 & 6 \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 3 & 5 \\ 6 & 8 & 2 \\ 0 & -1 & 6 \end{bmatrix}$$

EROs

If A' is obtained from A by doing an ERO then

- $\det A' = \alpha \det A$ if the ERO is Type 2: $R_i \longleftarrow \alpha R_i$

$$\det \begin{bmatrix} 1 & 3 & 5 \\ 0 & -1 & 6 \\ 6 & 8 & 2 \end{bmatrix} =$$

- $\det A' = \det A$ if the ERO is Type 3: $R_i \longleftarrow R_i + \alpha R_j$

$$\det \begin{bmatrix} 1 & 3 & 5 \\ 0 & -1 & 6 \\ 3 & 4 & 1 \end{bmatrix} =$$

Technique

- Row reduce via sequence of EROs
- Keep track of total *cumulative change* in determinant
- Find determinant of row reduced matrix
- Figure out what original determinant is

Common Mistake (*Warning!*)

- Row-reduce A to A'
- Find $\det A'$
- Write this down as the answer

Example

$$\left(\det\right)\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 1 & 0 \end{bmatrix} \begin{matrix} R2 \leftarrow R2 - 2R1 \\ R3 \leftarrow R3 - 3R1 \end{matrix}$$

$$= \det \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & -5 & -3 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= -(1 \cdot -5 \cdot 1) = \underline{\underline{5}}.$$

Combining techniques

- Use EROs to get a column with only one non-zero element
- Expand along that column
- Repeat until you can compute the determinant by hand.

EXAMPLE

$$\det\begin{bmatrix} 1 & 0 & 1 & y \\ 0 & 0 & 1 & y \\ 0 & -1 & 0 & 1 \\ 1 & -y^2 & 0 & y \end{bmatrix} = \det\begin{bmatrix} 1 & 0 & 1 & y \\ 0 & 0 & 1 & y \\ 0 & -1 & 0 & 1 \\ 0 & -y^2 & -1 & 0 \end{bmatrix}$$

$$= 1 \times \det\begin{bmatrix} 0 & 1 & y \\ -1 & 0 & 1 \\ -y^2 & -1 & 0 \end{bmatrix} \quad R3 \leftarrow R3 - y^2 R2$$

$$= 1 \times \det\begin{bmatrix} 0 & 1 & y \\ -1 & 0 & 1 \\ 0 & 0 & -y^2 \end{bmatrix}$$

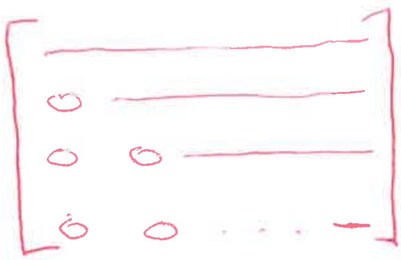
$$= 1 \times 1 \det\begin{bmatrix} 1 & y \\ 0 & -y^2 \end{bmatrix} = -y^2.$$

Properties of determinants

- $\det(\alpha A) = \alpha^n \det(A)$ for an $n \times n$ matrix A

- A matrix is invertible if and only if it has a non-zero determinant $\Longleftrightarrow$ product of diagonal elements in row-echelon form is nonzero

n



n

$\Longleftrightarrow$ row rank of $A = n$

PROPERTIES OF DETERMINANTS

Determinants are multiplicative:

$$\det(AB) = \det(A).\det(B) \quad \checkmark$$

Consequences:

- $\det(AB) = \det(BA)$ ✓
- $\det(A^k) =$ $(\det(A))^k$
  k positive integer
- $\det(A^{-1}) =$ $(\det(A))^{-1}$

provided A is invertible

Invertible matrix theorem

An $n \times n$ matrix A is invertible if and only if

- It has rank n
- It has nullity 0
- It has non-zero determinant
- Its rows are linearly independent
- Its columns are linearly independent
- Its row echelon form has n pivots (leading entries)
- Its reduced row echelon form is I_n
- The equation $Ax = 0$ has a unique solution
- Its rows are a basis for $\mathbb{R}^n$
- Its columns are a basis for $\mathbb{R}^n$

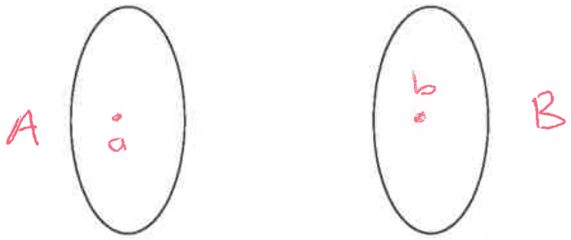
So far

- Systems of linear equations
- Vector spaces $\mathbb{R}^2, \mathbb{R}^3, \ldots, \mathbb{R}^n$
- Subspaces have bases and a dimension
- Matrix addition and multiplication
- row/column/null space
- Rank-nullity theorem
- Matrix inverses
- Determinants

Now we start Chapter 4: Linear transformations.

FUNCTION TERMINOLOGY

A *function* has a *domain* A and a *codomain* B



and maps each $a \in A$ to *exactly one* element in B.

$$a \longmapsto b$$

$$f : A \longrightarrow B$$

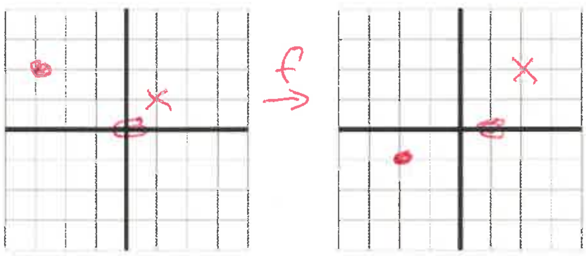
$$f(a) := b$$

Functions between vector spaces

Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f((x,y)) = (x+1, x+y)$$

$a = (-3, 2)$



$b = f(a) = (-2, -1)$

$a = (1,1)$ $\qquad$ $b = f(a) = (2,2)$

$f(0,0) = (1,0)$

$2f((0,0)) = f\big(\underline{(0,0) + (0,0)}\big) = f(\underline{0,0})$

$\neq f(0,0) + f(0,0) = (1,0) + (1,0)$

$= (2,0)$

Linear Transformations

A *linear transformation* is a function $f : \mathbb{R}^n \to \mathbb{R}^m$ such that

- $f(\boldsymbol{u} + \boldsymbol{v}) = f(\boldsymbol{u}) + f(\boldsymbol{v})$ for all vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$

- $f(\alpha \boldsymbol{u}) = \alpha f(\boldsymbol{u})$ for all vector $\boldsymbol{u} \in \mathbb{R}^n$ and scalar $\alpha \in \mathbb{R}$

Common Task

Given a function f, decide *whether or not* it is a linear transformation.

- If you think that the answer is "No"
  Present a *counterexample* — that is, two vectors v and w where $f(v+w) \neq f(v) + f(w)$.
- If you think that the answer is "Yes"
  Give a *symbolic proof* that works for arbitrary vectors.

$$f(\alpha(x,y)) \overset{?}{=} \alpha f(x,y) = \alpha(x+y, x+y)$$

✓

$$= f(\alpha x, \alpha y)$$

$$= (\alpha x + \alpha y, \alpha x + \alpha y)$$

also

EXAMPLE

Are the following functions linear transformations?

- $f : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x + y, x + y)$

YES

NO

- $g : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x^2, y^2)$

$$g(2\cdot(1,0)) \overset{?}{=} 2g(1,0) = 2(1,0) = (2,0)$$

$$= g(2,0) = (4,0)$$

- $h : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \mapsto (x, y + z, 0)$

yes

$$f((x,y)+(w,z)) \overset{?}{=} f(x,y) + f(w,z)$$

$$= f(x+w, y+z)$$

$$= (x+w+y+z, x+w+y+z)$$

$$f(x,y) + f(w,z) = (x+y, x+y) + (w+z, w+z)$$

$$= (x+y+w+z, x+y+w+z)$$

Property

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then

$$f(\mathbf{0}) = \mathbf{0}.$$

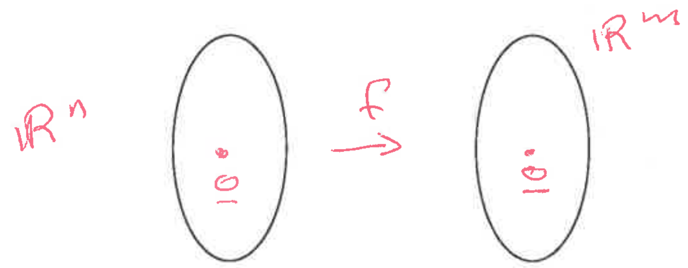
Proof: $f(\underline{0}) = f(0 \cdot \underline{0}) = 0 \cdot f(\underline{0})$

$= \underline{0}$

Kernel

is linear transformation

If $f : \mathbb{R}^n \to \mathbb{R}^m$, then the *kernel* of f is the vectors in the *domain* $\mathbb{R}^n$ that are mapped to $\mathbf{0}$.



kernel is a vector subspace of

Kernel

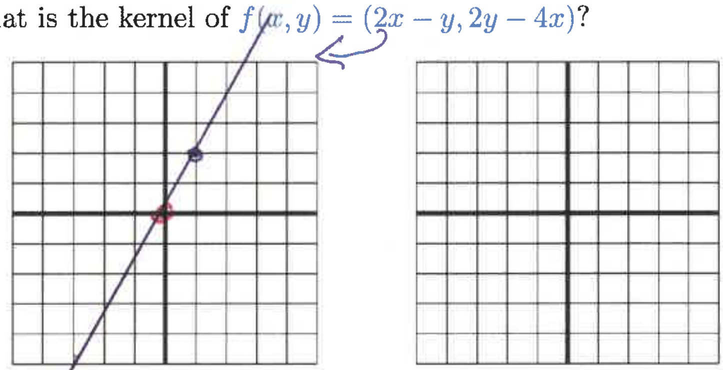
$f(x,y) = (0,0) \iff$

$= (2x-y,\ 2y-4x) \iff$ both $2x-y=0$ and $2y-4x=0$

What is the kernel of $f(x, y) = (2x - y, 2y - 4x)$?

$\iff (x,y) = (x, 2x)$

$\iff (x,y) = x(1,2)$



domain codomain

Exercise: what is the kernel of $g(x, y) = (x + y, x - y)$? $= (0,0) \iff$

$x+y=0$ and $x-y=0$ $\iff (x,y)=(0,0)$

$x=-y$ " $x=y$

$\mathrm{Ker}(g) = \{(0,0)\}$.

Subspace

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then the kernel of f is a subspace of $\mathbb{R}^n$.

Proof:

$\underline{0} \in \ker(f)$ ✓

$u, v \in \ker(f) \overset{?\,\checkmark}{\Longrightarrow} u+v \in \ker(f)$

$\Updownarrow \qquad\qquad \Updownarrow$

$f(u) = \underline{0}$ and $f(v) = \underline{0} \qquad f(u+v) = \underline{0}$

$\Updownarrow$

$f(u) + f(v)$

$u \in \ker(f) \overset{?\,\checkmark}{\Longrightarrow} \alpha \cdot u \in \ker(f)$

$\Updownarrow \qquad\qquad \Updownarrow$

$f(u) = \underline{0} \qquad f(\alpha \cdot u) = 0$

$\Updownarrow$

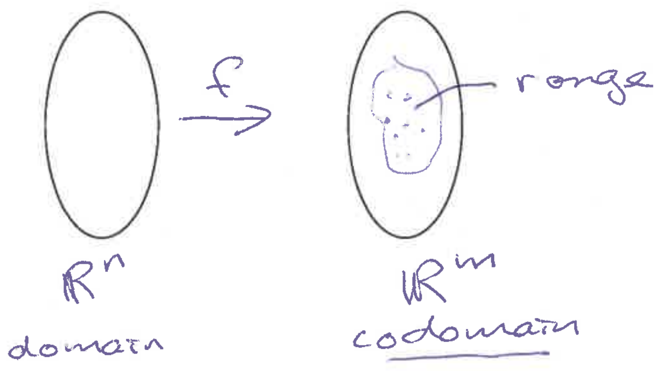
$\alpha f(u) = \underline{0}$

Range

If $f : \mathbb{R}^n \to \mathbb{R}^m$, then the *range* of f is the set of vectors

$$\underline{\{f(u) : u \in \mathbb{R}^n\}}.$$

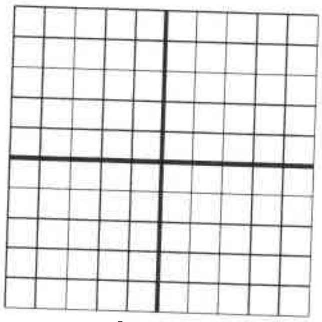
This is a set of vectors in the *codomain* of f, namely $\mathbb{R}^m$.



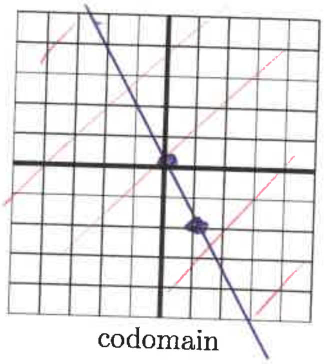
Range

$(z, -2z) = z(1, -2)$

What is the range of $f(x, y) = (2x - y, 2y - 4x)$?



domain



codomain

Exercise: what is the range of $g(x, y) = (x + y, x - y)$?

Subspace

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then the range of f is a subspace of $\mathbb{R}^m$
Proof:

- $f(\underline{0}) = \underline{0}$ so $\underline{0}$ in range(f)
- If w, z $\in$ range(f) then for some $u, v \in \mathbb{R}^n$

  ~~is~~ in

we have $w = f(u)$, $z = f(v)$ $\Rightarrow$

$f(\underline{u+v}) = f(u) + f(v) = \underline{w+z}$

in range of f

- $\alpha \in \mathbb{R}$, $w \in$ range(f) $\overset{?}{\Longrightarrow}$ $\alpha \cdot w \in$ range(f)

  $\Updownarrow$

  $w = f(u)$ for some $u \in \mathbb{R}^n$

  $\underline{\alpha \cdot w} = \alpha \cdot f(u) = f(\underline{\alpha \cdot u})$

  $\Rightarrow$ $\alpha \cdot w \in$ range(f).

DETERMINING LINEAR TRANSFORMATION

A linear transformation is *determined* by its action on a *basis*.

Suppose $f : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such that

$$f(1,0) = (1,1,1) \quad \text{and} \quad f(0,1) = (0,0,1)$$

$f(x,y) = f(\overline{x(1,0) + y(0,1)}) = x\,f(1,0) + y\,f(0,1)$

- $f(1,1) =$ $(1,1,2)$ $\quad = x(1,1,1) + y(0,0,1)$
- $f(2,4) =$ $(2,2,6)$ $\quad = (x, x, x+y)$
- $f(x,y) =$

Consequence

The range of the linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$ is the span of the images of any basis of $\mathbb{R}^n$. ✓

What is the range of $f(x, y) = (2x - y, 2y - 4x)$?

$n = 2$

$\{ (1,0), (0,1) \}$

$f(1,0) = (2,-4)$

$f(0,1) = (-1,2)$

$\text{range} = \{ x(2,-4) + y(-1,2) : x, y \in \mathbb{R} \}$

$= \{ z(2,-4) : z \in \mathbb{R} \}.$

Finding bases

The *range* and *kernel* of a linear transformation are *subspaces*.

So we will want to find a *basis* for each of these subspaces.

Fortunately, we already have the tools and techniques needed!

Basis for kernel

To find the kernel, we must solve $f(v) = 0$ and find basis for solution space.

Basis for range

If $f : \mathbb{R}^3 \to \mathbb{R}^m$ is linear then every vector in its *range* is a linear combination of

$$\{f(1,0,0), f(0,1,0), f(0,0,1)\}$$

Need to find *independent set* from this collection.

Example: find a basis for the range and kernel of f

Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(x, y, z) = (x + y, y + z)$

Kernel: $f(x,y,z) = (0,0)$

$x+y=0$

$y+z=0$

$(-y, y, -y) = y(-1,1,-1)$

$\{(-1,1,-1)\}$. ✓

range: $\{(1,0,0), (0,1,0), (0,0,1)\}$

basis for domain

f of these spans the range

$\{(1,0), (1,1), (0,1)\}$ spans

$\{(1,0), (0,1)\}$ basis.

Matrix multiplication

Let A be an arbitrary $m \times n$ matrix and define $f : \mathbb{R}^n \to \mathbb{R}^m$ by

$$f(\boldsymbol{v}) = A\boldsymbol{v}$$

where $\boldsymbol{v}$ is an $n \times 1$ column vector.

Then f is linear.

Matrix of transformation

- Linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$
- $B = \{v_1, v_2, \dots, v_n\}$ basis for $\mathbb{R}^n$
- $C = \{w_1, w_2, \dots, w_m\}$ basis for $\mathbb{R}^m$

B

C

v_j

$\underbrace{f(v_j)_C}$

The matrix of f with respect to B and C is

$A_{CB} =$
$[f]_{CB}$
${}_C T_B$

$f(v_j) = b_1 w_1 + b_2 w_2 + \cdots b_m w_m$

$$= \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{array}{cc} & j \\ m \text{ rows} & \begin{bmatrix} & \vdots & \\ & \sim & \\ & \vdots & \end{bmatrix} \\ & n \text{ columns} \end{array}$$

Use of matrix

If A_{CB} is the matrix of linear transformation f, then

$$A_{CB}(\boldsymbol{v})_B = (f(\boldsymbol{v}))_C.$$

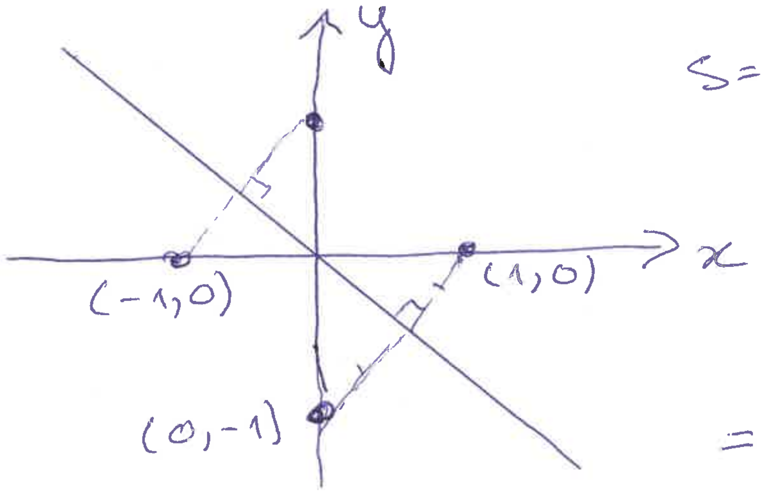
If we take the standard bases for B and C:

$$A_{SS}(\boldsymbol{v})_{\cancel{S}} = (f(\boldsymbol{v}))_{\cancel{S}}.$$

Example

f

Find the matrix of the reflection in $\mathbb{R}^2$ through the line $x + y = 0$ w.r.t the standard basis S (for domain and codomain)



$S = \{(1,0), (0,1)\}$

$f(1,0) \overset{S}{=} (0,-1)$
$= 0 \cdot (1,0) - 1 \cdot (0,1)$

$f(0,1) = (-1,0)$
$= -1 \cdot (1,0) + 0 \cdot (0,1)$

$$\begin{array}{cc} & \begin{array}{cc} j=1 & j=2 \end{array} \\ A_{SS} = & \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \end{array}$$

Use of matrix

If A_{CB} is the matrix of linear transformation f, then

$$A_{CB}(v)_B = (f(v))_C.$$

If we take the standard bases for B and C:

$$A_{SS}(v)_{\cancel{S}} = (f(v))_{\cancel{S}}.$$

Example

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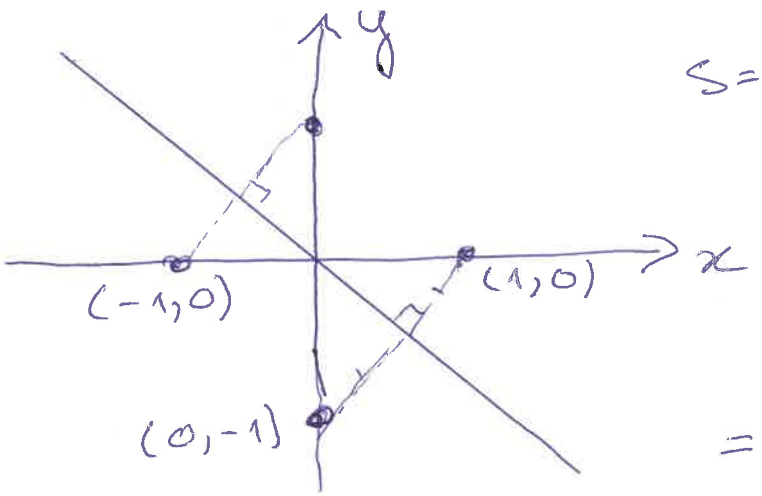

$S = \{(1,0), (0,1)\}$

$f(1,0) \overset{?}{=} (0,-1)$

$= 0 \cdot (1,0) - 1 \cdot (0,1)$

$f(0,1) = (-1,0)$

$= -1 \cdot (1,0) + 0 \cdot (0,1)$

$$A_{SS} = \begin{array}{cc} j=1 & j=2 \\ \end{array}$$
$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Example

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(x, y) = (x + y, x - y)$$

and let $B = \{(1,1),(2,1)\}$ and $C = \{(1,1),(1,0)\}$.

$$\Big\downarrow f$$

$$f(1,1) = (2,0) = 0 \cdot (1,1) + 2 \cdot (1,0)$$

$$[f]_{CB} = \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}$$

$$f(2,1) = (3,1) = 1 \cdot (1,1) + 2(1,0)$$

Example

Find the matrix of g w.r.t bases C, D where

$$C = \{(1,1),(1,0)\}$$
$$D = \{(1,0),(0,1)\}$$

and $g : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$g(x,y) = (x+2y, 2x+y)$$

$$g(1,1) = (3,3) = 3\cdot(1,0) + 3(0,1)$$
$$g(1,0) = (1,2) = 1\cdot(1,0) + 2\cdot(0,1)$$

$$[g]_{DC} = \begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix}$$

EXERCISE

Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(x, y, z) = (x + y, y + z)$

$B = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\} \quad C = \{(1, 1), (1, -1)\}$

$$f(1,1,1) = (2,2) = 2\cdot(1,1) + 0\cdot(1,-1)$$

$$f(1,1,0) = (2,1) = a\cdot(1,1) + b(1,-1)$$

$$= (a+b, a-b)$$

$$a+b=2 \qquad a-b=1$$

$$a = \frac{3}{2} \qquad b = \frac{1}{2}$$

$$f(1,0,0) = (1,0) = \frac{1}{2}\cdot(1,1) + \frac{1}{2}(1,-1)$$

$$[f]_{CB} = \begin{bmatrix} 2 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Example

f

Find the matrix of the identity transformation in $\mathbb{R}^3$ w.r.t any basis B (for domain and codomain)

$$B = \{v_1, v_2, v_3\}$$

$$f(v_1) = v_1 = 1 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3$$

$$f(v_2) = v_2 = 0 \cdot v_1 + 1 \cdot v_2 + 0 \cdot v_3$$

$$f(v_3) = v_3 = 0 \cdot v_1 + 0 \cdot v_2 + 1 \cdot v_3$$

$$[f]_{BB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Back to kernel and range

Recall

$$f(\boldsymbol{v}) = A(\boldsymbol{v}) \text{ taking } A = [f]_{\underset{\sim}{S}\underset{\sim}{S}}.$$

Kernel(f)= $\{ v : A v = \underset{\sim}{0} \}$ = nullspace of A

Range(f)= $\{$ columns of A $\}$ = columnspace of A

ANOTHER RANK-NULLITY THEOREM

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then

$$\dim(\text{range}(f)) + \dim(\ker(f)) = n$$

nullspace of A

nullity of A

dimension of column space of A

=

dimension of row space of A

=

rank A

= n

Composition of functions

If $f : \mathbb{R}^\ell \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^n$ then

$$g \circ f : \mathbb{R}^\ell \to \mathbb{R}^n$$

is defined by

$$(g \circ f)(v) = g(f(v))$$

Then $g \circ f$ is *linear*.

Matrix of composition

Earlier, we used $f(x,y) = (x+y, x-y)$ and $g(x,y) = (x+2y, y+2x)$ with bases

$$B = \{(1,1),(2,1)\}$$
$$C = \{(1,1),(1,0)\}$$
$$D = \{(1,0),(0,1)\}$$

and found

$$[f]_{CB} = \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix} \quad [g]_{DC} = \begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix}$$

What is the matrix of $g \circ f$ with respect to B and D?

$$[g\circ f]_{DB} = \begin{bmatrix} 2 & 5 \\ 4 & 7 \end{bmatrix}$$

$$(1,1) \xmapsto{f} (2,0) \xmapsto{g} (2,4) = 2\cdot(1,0)+4(0,1)$$
$$(2,1) \xmapsto{f} (3,1) \xmapsto{g} (5,7) = 5\cdot(1,0)+7\cdot(0,1)$$

Working

$$(1,1) \xrightarrow{f} \qquad \xrightarrow{g}$$

$$(2,1) \xrightarrow{f} \qquad \xrightarrow{g}$$

$$\begin{aligned} &= \quad (1,0) + \quad (0,1) \\ &= \quad (1,0) + \quad (0,1) \end{aligned}$$

So the matrix is

$$[g \circ f]_{DB} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Matrix of composition

The matrix of a *composition* of two linear transformations is the matrix *product* of the matrices.

$$\begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 4 & 7 \end{bmatrix}$$

INVERSE FUNCTION

Let $f : A \longrightarrow B$ be a ~~function~~. (linear transformation)

We say that f is *invertible* if there exists a function $g : B \longrightarrow A$ such that $g(f(x)) = x$. ✓ and $f(g(y)) = y$. (✓?)

If this function exists it is denoted f^{-1}.

For example, suppose $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is given by

$$f(x, y) = (\underbrace{x + y}_{w}, \underbrace{x - y}_{z})$$

$$x = \tfrac{1}{2}(w+z) \qquad y = \tfrac{1}{2}(w-z)$$

$$g(w,z) = \left(\tfrac{1}{2}(w+z), \tfrac{1}{2}(w-z)\right)$$

Inverse of a linear transformation

$f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ a linear transformation is invertible ✓

- only possible if $m = n$

- f^{-1} also a linear transformation ✓

- the standard matrix of f^{-1} is

$$[1]_{SS} = [f^{-1}]_{SS} \cdot [f]_{SS}$$

$$[1]_{SS} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$$[f^{-1}]_{SS} = [f]_{SS}^{-1}$$

matrix inverse

MATH1012 Mathematical Theory and Methods

Week 5

So far

- Systems of linear equations
- Vector spaces $\mathbb{R}^2$, $\mathbb{R}^3$, $\ldots$, $\mathbb{R}^n$
- Subspaces have bases and a dimension
- Matrix addition and multiplication
- row/column/null space
- Rank-nullity theorem
- Matrix inverses
- Determinants
- Linear transformations
- Range, Kernel
- Composition of transformations

Now we start Chapter 5.

Change of basis

Let B and C be two bases for the same m-dimensional subspace V of $\mathbb{R}^n$ (often $n = m$ and $V = \mathbb{R}^n$).

What is the link between the coordinates with respect to each basis?

$$\underline{\underline{(\boldsymbol{v})_B}} \longleftrightarrow \underline{\underline{(\boldsymbol{v})_C}}$$

We want *transition matrix* P_{CB} such that $(\boldsymbol{v})_C = P_{CB}(\boldsymbol{v})_B$

Change of basis

Recall from Chapter 4 that

$$[f]_{CB}(\boldsymbol{v})_B = (f(\boldsymbol{v}))_C$$

So take f to be the *identity mapping*, and then the effect of multiplying by $[f]_{CB}$ is to translate a B-coordinate vector to a C-coordinate vector.

Transition matrix

The i-th column P_{CB} should contain the C-coordinates of the i-th vector in B.

Say $B = \{\boldsymbol{u_1}, \dots, \boldsymbol{u_m}\}$.

$$P_{CB} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Example: Transition matrix

Suppose that

$$B = \{(1,1,0),(1,2,3),(0,0,5)\}$$
$$S = \{(1,0,0),(0,1,0),(0,0,1)\}$$

$$P_{SB} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 3 & 5 \end{bmatrix}$$

Example: Transition matrix

Suppose that

$$B = \{(1,1,0),(1,0,0),(0,0,-1)\}$$
$$C = \{(0,0,1),(1,0,1),(0,1,0)\}$$

Then

$$P_{CB} = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$(1,1,0) = -1\cdot(0,0,1) + 1\cdot(1,0,1) + 1\cdot(0,1,0)$$

$$(1,0,0) = -1\cdot(0,0,1) + 1\cdot(1,0,1) + 0\cdot(0,1,0)$$

$$(0,0,-1) = -1\cdot(0,0,1) + 0\cdot(1,0,1) + 0\cdot(0,1,0)$$

What is P_{BC}?

✓

$P_{CB}(\boldsymbol{v})_B = (\boldsymbol{v})_C$ and $P_{BC}(\boldsymbol{v})_C = (\boldsymbol{v})_B$ and so

$$P_{CB}P_{BC}(\boldsymbol{v})_C = (\boldsymbol{v})_C$$

So P_{BC} is the *inverse* of P_{CB}.

$P_{BC} = (P_{CB})^{-1}$ ✓

Another basis

You may be given the *standard matrix* of a linear transformation f and then asked to find the matrix with respect to bases B and C.

We claim that

$$[f]_{CB} = P_{CS}[f]_{SS}P_{SB}$$

G E

$$\begin{aligned} P_{CS}[f]_{SS}P_{SB}(\boldsymbol{v})_B &= P_{CS}[f]_{SS}(\boldsymbol{v})_S \\ &= P_{CS}(f(\boldsymbol{v}))_{SS} \\ &= (f(\boldsymbol{v}))_C \end{aligned}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} \end{array}\right] \quad R1 \leftarrow R_1 - 2R2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} \end{array}\right]$$

$$P_{BS} = (P_{SB})^{-1} = \frac{1}{3}\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}.$$

A better basis?

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (y - 3x, 2x - 2y)$

$$[f]_{SS} = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}$$

If $B = \{(-1, 1), (1, 2)\}$ then what is $[f]_{BB}$?

$$[f]_{BB} = P_{BS}[f]_{SS}P_{SB}$$

$$P_{BS} = (P_{SB})^{-1} \qquad \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} -1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{bmatrix} \quad R2 \leftarrow R_2 + R1$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 1 \end{bmatrix} \quad R2 \leftarrow \frac{1}{3} R2$$

Transition matrices

$$(-1,1) = -1(1,0) + 1(0,1)$$
$$(1,2) = 1(1,0) + 2(0,1)$$

so

$$P_{SB} = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$$

Then

$$P_{BS} = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} =$$

$$\frac{1}{3}\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ -4 & -2 \end{bmatrix} =$$

$$\frac{1}{3}\begin{bmatrix} -12 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix},$$

AND $[f]_{BB}$?

$$[f]_{BB} = P_{BS}[f]_{SS}P_{SB}$$

$$\underline{\underline{[f]_{BB}}} = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix} \longleftarrow$$

Why is this useful?

In linear algebra applications, matrices often represent a *transition* from one state to another.

$$v_0, \quad v_1 = Av_0, \quad v_2 = Av_1, \quad v_3 = Av_2 \ldots$$

What is the *long term behaviour* of the system?

So far

- Systems of linear equations
- Vector spaces $\mathbb{R}^2$, $\mathbb{R}^3$, $\ldots$, $\mathbb{R}^n$
- Subspaces have bases and a dimension
- Matrix addition and multiplication
- row/column/null space
- Rank-nullity theorem
- Matrix inverses
- Determinants
- Linear transformations
- Range, Kernel
- Composition of transformations
- Transition matrices for coordinates and linear transformations

Now we start Chapter 6.

EIGENVECTORS

If A is an $n \times n$ matrix, then a *non-zero vector* $\boldsymbol{v} \in \mathbb{R}^n$ is called an *eigenvector* of A with *eigenvalue* $\lambda \in \mathbb{R}$ if

$$A\boldsymbol{v} = \lambda \boldsymbol{v}$$

Example

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue 3

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ " ... ... -1.

Finding eigenvalues?

Is 2 an eigenvalue of A?

eigenvector $\stackrel{?}{\neq} \underline{0}$. NO!

Solve

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{aligned} x + 2y &\stackrel{?}{=} 2x \\ 2x + y &\stackrel{?}{=} 2y \end{aligned} \iff \begin{aligned} -x + 2y &= 0 \\ 2x - y &= 0 \end{aligned}$$

only solutions are $x = y = 0$.

More systematic

(A is n x n)

If

$$Av = \lambda v$$

then

$$\underline{(A - \lambda I)v = 0}$$

When does this system have non-zero solutions?

$\iff$ v in nullspace of $A - \lambda I$

$\ne$

$\underline{0}$

$$\iff \underline{\det(A - \lambda I) = 0}$$

Determinant condition

The matrix

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$$

has determinant

$$(1-\lambda)^2 - 4 = 1 - 2\lambda + \lambda^2 - 4$$

When is this zero?

$$= \lambda^2 - 2\lambda - 3$$

$$= (\lambda - 3)(\lambda + 1)$$

$$= 0 \iff$$

either $\lambda = 3$ or $\lambda = -1$.

The eigenvalue 3

Solve

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$$

that is the homogeneous system with matrix

$$\begin{aligned} x + 2y &= 3x \\ 2x + y &= 3y \end{aligned} \iff$$

$$\begin{aligned} -2x + 2y &= 0 \\ 2x - 2y &= 0 \end{aligned} \iff x = y$$

solutions $\begin{bmatrix} x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The eigenvalue -1

Solve

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -1 \begin{bmatrix} x \\ y \end{bmatrix}$$

that is the homogeneous system with matrix

$$A\begin{bmatrix}1\\1\end{bmatrix} = 3\begin{bmatrix}1\\1\end{bmatrix}$$

$$A\begin{bmatrix}-1\\1\end{bmatrix} = -1\begin{bmatrix}-1\\1\end{bmatrix}$$

High powers

Suppose that A is a square matrix such that $Av = \lambda v$. Then

$$A^2v = AAv = A\,3\begin{bmatrix}1\\1\end{bmatrix} = 3A\begin{bmatrix}1\\1\end{bmatrix} = 3^2\begin{bmatrix}1\\1\end{bmatrix}$$

and

$$A^3v = 3^3\begin{bmatrix}1\\1\end{bmatrix}\ldots\ldots$$

and so on.

$$A^{500}v = 3^{500}\begin{bmatrix}1\\1\end{bmatrix} \quad (k=500)$$

An eigenvector v of A with eigenvalue λ is also an eigenvector A^k with eigenvalue λ^k.

Why do we care?

What is the value of

$$A^{500}\begin{bmatrix}1\\3\end{bmatrix}? \quad ?$$

Characteristic polynomial

If A is an $n \times n$ matrix, then

$$\det(A - \lambda I)$$

is a *polynomial* of degree n in λ, called the *characteristic polynomial* of A.

The equation $\det(A - \lambda I) = 0$ is the *characteristic equation*.

Its *solutions* are the *eigenvalues* of A.

Subspaces

For *any number* $\lambda \in \mathbb{R}$, the solutions to

$$Av = \lambda v$$

form a *subspace*.

Eigenspaces

When λ is an *eigenvalue*, the subspace has dimension at least 1 and is called an *eigenspace*, and denoted E_λ.

Standard task

Here is an $n \times n$ matrix A, what are its eigenspaces? Describe them by giving a basis for each.

- Calculate $\det(A - \lambda I)$ solve $\det(A - \lambda I) = 0$
- Find the roots $\lambda_1, \lambda_2, \ldots, \lambda_k$ $(k \leq n)$
- Solve $(A - \lambda_1 I)\boldsymbol{v} = \mathbf{0}$ and find a basis for E_{λ_1}
- Solve $(A - \lambda_2 I)\boldsymbol{v} = \mathbf{0}$ and find a basis for E_{λ_2}
- ...
- Solve $(A - \lambda_k I)\boldsymbol{v} = \mathbf{0}$ and find a basis for for E_{λ_k}

Example

The matrix

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

has characteristic polynomial $\lambda^2 - 3\lambda - 10$. Find a basis for each of its eigenspaces.

$$0 = \det\left(\begin{bmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{bmatrix} \right) =$$

$$(1-\lambda)(2-\lambda) - 12 = 2 - 3\lambda + \lambda^2 - 12$$

$$= \lambda^2 - 3\lambda - 10.$$

$$= (\lambda - 5)(\lambda + 2)$$

$$\Leftrightarrow \lambda = 5 \text{ or } \lambda = -2. \qquad \text{For } E_5:$$

$$\lambda = 5$$

$$(1-\lambda)x + 3y = 0$$

$$4x + (2-\lambda)y = 0$$

$$-4x + 3y = 0$$

$$4x - 3y = 0$$

$$v = \begin{bmatrix} x \\ \frac{4}{3}x \end{bmatrix} = x \begin{bmatrix} 1 \\ 4/3 \end{bmatrix}$$

Example 1

Consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Give a *geometric description* of this linear transformation.

What are its eigenvalues and eigenspaces?

$$\det \left[A - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right]$$

$$\det \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = 0 \quad \text{char. eqn.}$$

$$\| $$

$$\lambda^2 + 1$$

$\lambda^2 = -1$ no real solns.

$(\lambda = \pm i.)$

Eigenvalues

The equation $\det(A - \lambda I) = 0$ is a *polynomial equation* of degree n and so it has exactly n solutions

$$\lambda_1, \lambda_2, \ldots, \lambda_n$$

(Eigenvalues might be *repeated* and might be *complex numbers*.)

$$\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ A_{21} & A_{22} \cdots & A_{2n} \\ \cdots & \cdots & \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}$$

Properties of eigenvalues

- The *sum of the eigenvalues* is equal to the *trace* of the matrix

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = A_{11} + A_{22} + \cdots + A_{nn}$$

- The *product of the eigenvalues* is equal to the *determinant* of the matrix

$$\lambda_1 \times \lambda_2 \times \cdots \times \lambda_n = \det A$$

- The matrix A is invertible if and only if 0 is not an eigenvalue of A.

Properties of eigenvalues II

- the eigenvalues of A^T are $\lambda_1, \lambda_2, \ldots, \lambda_n$.
- the eigenvalues of kA (where $k \in \mathbb{R}$) are $k\lambda_1, k\lambda_2, \ldots, k\lambda_n$.
- the eigenvalues of A^k (where $k \in \mathbb{Z}$) are $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$.
- Cayley-Hamilton Theorem: A satisfies its own characteristic equation.

$$\det(A - \lambda I) = c_0\lambda^n + c_1\lambda^{n-1} + \cdots + c_n \quad \checkmark$$

$$c_0A^n + c_1A^{n-1} + \cdots c_{n-1}A + c_nI = \underline{0}$$

Theorem.

Example 2

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

which has characteristic polynomial

$$\det \begin{bmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{bmatrix} = (\lambda - 1)^2 = 0 \quad \text{char. eqn.}$$

What are its eigenvalues and eigenspaces?

$\underline{\lambda = 1}$ (repeated root $\underline{\text{multiplicity } 2}$)

E_1

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

free

v

$$v = \begin{bmatrix} x \\ 0 \end{bmatrix} = x \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

E_1 has basis $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

geometric multiplicity $\dim E_1 = 1$ ~~algebraic multiplicity~~

algebraic multiplicity is 2

Multiplicity of eigenvalue

The *algebraic multiplicity* of a particular eigenvalue k is m if

$$\det(A - \lambda I) = \cdots (k - \lambda)^m \cdots$$

The *geometric multiplicity* of a particular eigenvalue k is the *dimension* of the eigenspace E_k.

Back to Example 2

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Algebraic multiplicity= 2

Geometric multiplicity= 1

ALGEBRAIC VS GEOMETRIC

For any eigenvalue:

$$1 \leqslant \text{ geometric multiplicity} \leqslant \text{ algebraic multiplicity } \leq n$$

- The *algebra* tells us the *maximum possible* multiplicity
- The *geometry* tells us the *actual* multiplicity

Diagonalisable

A linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ is *diagonalisable* if there is a basis B for $\mathbb{R}^n$ such that $[f]_{BB}$ is a *diagonal matrix*.

A matrix A is called *diagonalisable* if the corresponding linear transformation is diagonalisable.

This happens exactly if we can find a basis for $\mathbb{R}^n$ consisting of eigenvectors for A.

Example

What are the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}?$$

Find characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix}$$
$$= \lambda(\lambda - 1)(\lambda - 2)$$

?

$$(1-\lambda)\det\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)\left((1-\lambda)^2 - 1\right)$$

$$= (1-\lambda)(\cancel{1} + \lambda^2 - 2\lambda \cancel{-1})$$

$$= \lambda(1-\lambda)(\lambda - 2)$$

Working

Solve three systems of linear equations:

- $Ax = 0x$ has solutions $S = \{t(0,1,-1) : t \in \mathbb{R}\}$
- $Ax = 1x$ has solutions $S = \{t(1,0,0) : t \in \mathbb{R}\}$
- $Ax = 2x$ has solutions $S = \{t(2,1,1) : t \in \mathbb{R}\}$

$n = 3$

So $B = \{(0,1,-1),(1,0,0),(2,1,1)\}$ is a basis for $\mathbb{R}^n$ of eigenvectors.

$$A_{BB} = P_{BS} \, A_{SS} \, \underline{\underline{P_{SB}}}$$

$$P_{BS} = \left(\underline{\underline{P_{SB}}}\right)^{-1}$$

$$P_{SB} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

DIAGONALISABILITY TEST

A matrix A is diagonalisable if and only if

- It has all *real* eigenvalues (no complex)
- Every eigenvalue has the *maximum possible* geometric multiplicity

In other words, each eigenvalue has geometric multiplicity equal to its algebraic multiplicity.

Special case: if the matrix has n *distinct* eigenvalues, then it is diagonalisable.

Working: Change of basis

The matrix

$$P = P_{SB} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

and

$$A_{BB} = P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Summary

To diagonalise a matrix A

- Find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ and their multiplicities

  If some λ has geometric multiplicity *too low* then *fail*.
- Find a *basis* for each eigenspace $E_{\lambda_1}, E_{\lambda_2}, \ldots, E_{\lambda_k}$.
- Put the *basis vectors* into the *columns* of P

  Note that P is a *square* matrix.

Then $\underline{P^{-1}AP = D}$ where D is the diagonal matrix of eigenvalues.

$$\Updownarrow$$

$$AP = PD$$

$$\Updownarrow$$

$$\underline{A = PDP^{-1}}$$

High powers

What is A^{5}, if A is diagonalisable?

$$\begin{aligned}
A^5 &= (PDP^{-1})(PDP^{-1})(PDP^{-1})(PDP^{-1})(PDP^{-1}) \\
&= PD(P^{-1}P)D(P^{-1}P)D(P^{-1}P)D(P^{-1}P)DP^{-1} \\
&= PDIDIDIDIDP^{-1} \\
&= PD^5P^{-1}
\end{aligned}$$

$$A^{500} = P \begin{bmatrix} \lambda_1^{500} & 0 & 0 \\ 0 & \lambda_2^{500} & 0 \\ 0 & 0 & \lambda_3^{500} \end{bmatrix} P^{-1}$$

If

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \qquad D^5 = \begin{bmatrix} \lambda_1^5 & 0 & 0 \\ 0 & \lambda_2^5 & 0 \\ 0 & 0 & \lambda_3^5 \end{bmatrix}$$

Special Case

A matrix is *symmetric* if $A = A^T$.

If A is a symmetric matrix, then

- Each eigenvalue of A is *real*
- Each eigenvalue has geometric multiplicity equal to algebraic multiplicity
- Eigenvectors from distinct eigenspaces are *orthogonal*

In other words, symmetric matrices are *diagonalisable*.

Applications

There are lots of applications of eigenvalues/eigenspaces to

- engineering (stress tensors)
- physics (mechanics)
- study of graphs (Google Page Rank)
- data analysis
- computer graphics (image compression)

This concludes our study of Linear Algebra.