MATH1012 (Semester 1, 2024)

Practice Class 11: Laplace transforms

Summary of what you have learned

- Definition of the Laplace transform $F(s) = \int_0^\infty e^{-st} f(t)dt$
- Table look-up and partial fractions
- Laplace transform of derivatives and integrals
- Solving differential equations with Laplace transforms
- Derivative of the Laplace transform.
- s-shift theorem
- Heaviside functions
- Heaviside shift theorem for Laplace transform

Foundational questions

Exercise 1. Find the Laplace transforms of

$(i)\quad f(t) = t^2 - 3t + 5 \qquad\qquad (ii)\quad g(t) = 3e^{-t} + \sin 6t$

Solution:

$$(i)\quad \mathcal{L}(f) = \mathcal{L}(t^2) - 3\mathcal{L}(t) + 5\mathcal{L}(1) = \frac{2}{s^3} - \frac{3}{s^2} + \frac{5}{s}$$

$$(ii)\quad \mathcal{L}(g) = 3\mathcal{L}(e^{-t}) + \mathcal{L}(\sin 6t) = \frac{3}{s+1} + \frac{6}{s^2+36}$$

Exercise 2. Find the inverse Laplace transforms of

$(i)\quad F(s) = \dfrac{s}{s^2+4} - \dfrac{2}{s-16} \qquad\qquad (ii)\quad G(s) = \dfrac{1}{s+3} + \dfrac{2}{s-3}$

Solution: Recall that $\mathcal{L}^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$ and $\mathcal{L}^{-1}\left(\frac{s}{s^2+\omega^2}\right) = \cos(\omega t)$. Then

$(i)\quad \mathcal{L}^{-1}(F) = \cos(2t) - 2e^{16t} \qquad\qquad (ii)\quad \mathcal{L}^{-1}(G) = e^{-3t} + 2e^{3t}$

Exercise 3. Use partial fractions to find the inverse Laplace transform of

$$F(s) = \frac{9+3s}{(s-1)(s+1)(s+2)}$$

Solution: Using the "cover the zero" method, we get

$$F(s) = \frac{2}{s-1} - \frac{3}{s+1} + \frac{1}{s+2}$$

By linearity and the formula $\mathcal{L}^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$, we deduce that

$$d(t) = 2e^{t} - 3e^{-t} + e^{-2t}$$

EXERCISE 4. Use partial fractions to find the inverse Laplace transforms of

$$(i) \quad F(s) = \frac{1}{s^4 - 16} \qquad\qquad (ii) \quad G(s) = \frac{s}{(s-1)(s^2-2s+2)}$$

SOLUTION: (i) The denominator factors as

$$s^4 - 16 = (s^2-4)(s^2+4) = (s-2)(s+2)(s^2+4).$$

Using the "cover the zero" method, we get

$$G(s) = \frac{1}{32}\frac{1}{s-2} - \frac{1}{32}\frac{1}{s+2} + \frac{As+B}{s^2+4}$$

Substituting values (for instance $s=0$ and $s=1$) gives that $B = -\frac{1}{8}$, $A = 0$ and hence

$$G(s) = \frac{1/32}{s-2} - \frac{1/32}{s+2} - \frac{1/8}{s^2+4} \qquad \to \qquad g(t) = \frac{1}{32}e^{2t} - \frac{1}{32}e^{-2t} - \frac{1}{16}\sin 2t$$

(ii) The polynomial $s^2 - 2s + 2 = (s-1)^2 + 1$ is irreducible (that is, it has no real zeros). The "cover the zero" method yields

$$F(s) = \frac{1}{s-1} + \frac{As+B}{s^2-2s+2}$$

Substituting values (for instance $s=0$ and $s=2$) gives $B = 2$,$A = -1$ and hence

$$F(s) = \frac{1}{s-1} + \frac{2-s}{(s-1)^2+1}$$

The second term is not exactly in the form of one of the rows in the table, so we need to decompose it further. We write it as

$$F(s) = \frac{1}{s-1} - \frac{s-1}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}$$

Hence

$$f(t) = e^t - e^t\cos t + e^t \sin t$$

EXERCISE 5. Without computing the integral, find the Laplace transform of

$$g(t) = \int_0^t (u^3 + \sin 2u)du$$

SOLUTION: We can use the theorem $\mathcal{L}(\int_0^t f(u)du) = \frac{F(s)}{s}$ with

$$f(t) = t^3 + \sin 2t \qquad \to \qquad F(s) = \frac{6}{s^4} + \frac{2}{s^2+4} \qquad \Rightarrow \qquad G(s) = \frac{F(s)}{s} = \frac{6}{s^5} + \frac{2}{s(s^2+4)}$$

EXERCISE 6. Use the formula $\mathcal{L}(tf(t)) = -F'(s)$ to find the Laplace transform of $g(t) = t\sin 3t$.

SOLUTION: Set $f(t) = \sin 3t$. Then we have

$$F(s) = \frac{3}{s^2+9} \qquad \Rightarrow \qquad G(s) = -F'(s) = \frac{6s}{(s^2+9)^2}$$

Exercise 7. Use the s-shift theorem $F(s-a) = \mathcal{L}(e^{at} f(t))$ to find the Laplace transforms of

$(i) \quad g(t) = (t^3 - 3t + 2)e^{-2t}$ $\qquad\qquad$ $(ii) \quad k(t) = e^{4t}(t - \cos t)$

Solution:

$$(i) \quad \text{Let} \quad f(t) = t^3 - 3t + 2 \qquad \to \qquad F(s) = \frac{6}{s^4} - \frac{3}{s^2} + \frac{2}{s}$$

$$\text{and hence} \quad G(s) = F(s+2) = \frac{6}{(s+2)^4} - \frac{3}{(s+2)^2} + \frac{2}{s+2}$$

$$(ii) \quad \text{Let} \quad f(t) = t - \cos t \qquad \to \qquad F(s) = \frac{1}{s^2} - \frac{s}{s^2+1}$$

$$\text{and hence} \quad K(s) = F(s-4) = \frac{1}{(s-4)^2} - \frac{s-4}{(s-4)^2+1}$$

Exercise 8. Use Laplace transforms to solve the initial value problem

$$y'' - 4y = 16 \sin 2t \qquad\qquad y(0) = 0 \qquad\qquad y'(0) = 0$$

Solution: The Laplace transform of the initial value problem is

$$(s^2 - 4)Y(s) = \frac{32}{s^2+4}$$

and hence the Laplace transform of its solution is

$$Y(s) = \frac{32}{(s^2-4)(s^2+4)} = \frac{32}{s^4 - 16}$$

Using the result of Exercise 4(i) we have

$$y(t) = e^{2t} - e^{-2t} - 2\sin 2t$$

Exercise 9. Use Laplace transforms to solve the initial value problem

$$y'' - 2y' + 2y = e^t \qquad\qquad y(0) = 0 \qquad\qquad y'(0) = 1$$

Solution: The Laplace transform of the initial value problem is

$$s^2 Y - 1 - 2sY + 2Y = \frac{1}{s-1}$$

We can write this as

$$(s^2 - 2s + 2)Y = \frac{1}{s-1} + 1 = \frac{s}{s-1}$$

and hence the Laplace transform of its solution is

$$Y(s) = \frac{s}{(s-1)(s^2 - 2s + 2)}$$

Using the result of Exercise 4(ii) we have

$$y(t) = e^t - e^t \cos t + e^t \sin t$$

Exercise 10. Use the Heaviside shift theorem to find the inverse Laplace transforms of

$$(i) \quad G(s) = \frac{2e^{-s}}{(s-5)^3} \qquad\qquad (ii) \quad H(s) = \frac{se^{-2s}}{s^2+9}$$

Solution: Recall that $\mathcal{L}^{-1}(e^{-as}F(s)) = f(t-a)H(t-a)$ for $a > 0$.
(i) Take

$$F(s) = \frac{2}{(s-5)^3} = G(s-5) \quad \text{where} \quad G(s) = \frac{2}{s^3} \quad \to \quad g(t) = t^2 \quad \text{so} \quad f(t) = t^2 e^{5t}$$

and hence $g(t) = f(t-1)H(t-1) = (t-1)^2 e^{5(t-1)} H(t-1) = \begin{cases} 0, & 0 \leqslant t < 1, \\ (t-1)^2 e^{5(t-1)}, & 1 \leqslant t. \end{cases}$

(ii) Take

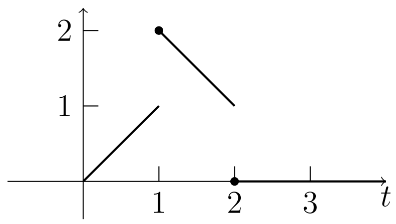
$$F(s) = \frac{s}{s^2+9} \quad \to \quad f(t) = \cos 3t \quad \Rightarrow \quad h(t) = f(t-2)H(t-2) = \cos(3t-6)H(t-2)$$

Exercise 11. Sketch the graph of

$$f(t) = \begin{cases} t, & 0 \leqslant t < 1, \\ 3-t, & 1 \leqslant t < 2, \\ 0, & 2 \leqslant t. \end{cases}$$

Write g using Heaviside functions and find its Laplace transform.

Solution: The graph of $g(t)$ is



In terms of Heaviside functions:

$$f(t) = t\,[H(t) - H(t-1)] + (3-t)\,[H(t-1) - H(t-2)]$$

Since $t \geqslant 0$, $H(t)$ is just the function 1. Thus

$$\begin{aligned} f(t) &= t + (3-2t)H(t-1) + (t-3)H(t-2) \\ &= t + [1 - 2(t-1)]H(t-1) + [(t-2) - 1]H(t-2) \\ &= t + g(t-1)H(t-1) + k(t-2)H(t-2), \end{aligned}$$

where

$$g(t) = 1 - 2t \;\to\; G(s) = \frac{1}{s} - \frac{2}{s^2} = \frac{s-2}{s^2} \text{ and } k(t) = t - 1 \;\to\; K(s) = \frac{1}{s^2} - \frac{1}{s} = \frac{1-s}{s^2}$$

Hence, using the Heaviside shift theorem:

$$F(s) = \frac{1}{s^2} + \frac{e^{-s}(s-2)}{s^2} + \frac{e^{-2s}(1-s)}{s^2}$$

Conceptual understanding

EXERCISE 12. A *boundary value problem* consists of a differential equation along with value data given at different points in the domain of the independent variable (the boundary points of its domain). Solve the boundary value problem

$$y'' + 7y' + 6y = 5e^{-2t} \qquad y(0) = 0 \qquad y'(1) = -3$$

Hint: Solve as in Exercises 8 and 9 but leaving $y'(0)$ as a constant C, and then determine C afterwards so that the solution satisfies $y'(1) = -3$.

SOLUTION: The Laplace transform of the initial value problem is

$$s^2Y(s) - C + 7sY(s) + 6Y(s) = \frac{5}{s+2} \qquad \Rightarrow \qquad (s+1)(s+6)Y(s) = \frac{5}{s+2} + C$$

Hence the Laplace transform of the solution is

$$Y(s) = \frac{5}{(s+1)(s+6)(s+2)} + \frac{C}{(s+1)(s+6)}$$

Employing partial fractions and using the "cover the zero" method, we can write this as

$$Y(s) = \frac{1}{s+1} + \frac{1/4}{s+6} - \frac{5/4}{s+2} + C\left(\frac{1/5}{s+1} - \frac{1/5}{s+6}\right)$$

Hence

$$y(t) = e^{-t} + \frac{e^{-6t}}{4} - \frac{5e^{-2t}}{4} + C\left(\frac{e^{-t}}{5} - \frac{e^{-6t}}{5}\right)$$

We now need to determine which C satisfies the initial condition $y'(1) = -3$. Firstly,

$$y'(t) = -e^{-t} - \frac{3e^{-6t}}{2} + \frac{5e^{-2t}}{2} + C\left(-\frac{e^{-t}}{5} + \frac{6e^{-6t}}{5}\right)$$

and setting $t = 1$ gives

$$-3 = -e^{-1} - \frac{3e^{-6}}{2} + \frac{5e^{-2}}{2} + C\left(-\frac{e^{-1}}{5} + \frac{6e^{-6}}{5}\right)$$

and hence

$$C = \frac{-3 + e^{-1} + \frac{3e^{-6}}{2} - \frac{5e^{-2}}{2}}{-\frac{e^{-1}}{5} + \frac{6e^{-6}}{5}}$$

EXERCISE 13. Find the inverse Laplace transform of

$$Y(s) = \frac{\pi}{2} - \arctan(s)$$

SOLUTION: Differentiate $Y(s)$ to get

$$Y'(s) = -\frac{1}{s^2+1}$$

The theorem $\mathcal{L}(-tf(t)) = F'(s)$ then gives

$$\mathcal{L}(-ty(t)) = -\frac{1}{s^2+1} \qquad \Rightarrow \qquad -ty(t) = -\sin t \qquad \Rightarrow \qquad y(t) = \frac{\sin t}{t}$$

Note: The $\frac{\pi}{2}$ is in $Y(s)$ to make it so that $Y(s) \to 0$ as $s \to \infty$, which must always be so.

EXERCISE 14. Consider bounded function $f(t)$ with Laplace transform $F(s)$. Show that

$$(i)\ \lim_{s\to\infty} F(s) = 0, \quad (ii)\ \lim_{s\to 0} F(s) = \int_0^\infty f(t)dt, \quad (iii)\ \lim_{s\to\infty} sF(s) = f(0), \quad (iv)\ \lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t)$$

You may assume that interchanging limits and integrals is valid.

SOLUTION:

$$(i) \quad \lim_{s\to\infty} F(s) = \lim_{s\to\infty} \int_0^\infty e^{-st} f(t)dt = \int_0^\infty \lim_{s\to\infty} e^{-st} f(t)dt = \int_0^\infty 0 f(t)dt = 0$$

$$(ii) \quad \lim_{s\to 0} F(s) = \lim_{s\to 0} \int_0^\infty e^{-st} f(t)dt = \int_0^\infty \lim_{s\to 0} e^{-st} f(t)dt = \int_0^\infty 1 f(t)dt = \int_0^\infty f(t)dt$$

$$(iii) \quad \mathcal{L}(f'(t)) = sF(s) - f(0) \quad \Rightarrow \quad \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} (\mathcal{L}(f'(t)) + f(0)) = 0 + f(0) = f(0)$$

$$(iv) \quad \lim_{s\to 0} sF(s) = \lim_{s\to 0} (\mathcal{L}(f'(t)) + f(0)) = \int_0^\infty f'(t)dt + f(0) = f(\infty) - f(0) + f(0) = f(\infty)$$

EXERCISE 15. Use mathematical induction to prove that $\mathcal{L}(t^n) = n!/s^{n+1}$ (for $s > 0$) for all non-negative integers n. (Recall that $0! = 1$)

SOLUTION: We have seen in lectures that $\mathcal{L}(1) = 1/s$, so the equality is true for $n = 0$. Assume true for some $n \geq 0$, so $\mathcal{L}(t^n) = n!/s^{n+1}$. We want to compute $\mathcal{L}(t^{n+1})$.

Method 1: Using integration by parts gives

$$\begin{aligned}
\mathcal{L}(t^{n+1}) &= \int_0^\infty e^{-st} t^{n+1} dt \\
&= -\frac{1}{s}[e^{-st} t^{n+1}]_0^\infty + \frac{n+1}{s} \int_0^\infty e^{-st} t^n dt \\
&= -\frac{1}{s}[0 - 0] + \frac{n+1}{s} \mathcal{L}(t^n) \\
&= \frac{n+1}{s} \cdot \frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}}
\end{aligned}$$

Method 2: Let $f(t) = t^{n+1}$ and use the theorem $\mathcal{L}(f'(t)) = s\mathcal{L}(f) - f(0)$. We have

$$\mathcal{L}(f'(t)) = \mathcal{L}((n+1)t^n) = (n+1)\mathcal{L}(t^n) = (n+1) \cdot n!/s^{n+1} = (n+1)!/s^{n+1}$$

and the theorem then says

$$(n+1)!/s^{n+1} = s\mathcal{L}(t^{n+1}) - 0 \qquad \Rightarrow \qquad \mathcal{L}(t^{n+1}) = \frac{(n+1)!}{s^{n+2}}$$

Method 3: Use the theorem $\mathcal{L}(tf(t)) = -F'(s)$ with $f(t) = t^n$. Now

$$F(s) = \mathcal{L}(f) = \mathcal{L}(t^n) = \frac{n!}{s^{n+1}} \qquad \Rightarrow \qquad \mathcal{L}(t^{n+1}) = \mathcal{L}(t \cdot t^n) = -F'(s) = \frac{(n+1)!}{s^{n+2}}$$

By all three methods the equality hold true for $n+1$. Hence, by mathematical induction, it is true for all integers $n \geq 0$. Note that

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}} \quad \Rightarrow \quad \mathcal{L}\left(\frac{t^n}{n!}\right) = \frac{1}{s^{n+1}} \quad \text{and } n \to n-1 \text{ gives} \quad \mathcal{L}\left(\frac{t^{n-1}}{(n-1)!}\right) = \frac{1}{s^n}$$