Fundamentals of Elementary Algebra

Mark Bu

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Chapter 1

Exponents

1.1 Fundamentals of Exponents

Multiplication is a chain of addition, denoted by the form $x * n$. $2 * 5$, for example, symbolizes the addition of 5 different 2's.

Exponents, on the other hand, are a chain of self-multiplication. They are denoted in the form x^n where x is the base and n is the power. The base is multiplied by itself, and the number of multiplications are denoted by the power. We can also represent them as $\prod_{i=1}^{n} x$.

Exponents can be visualized as either parabolas or cubic curves. Given the function x^n, if n is an even integer and is greater than 0, the curve will be a parabola. If n is an odd integer and is greater than 1, the curve will be cubic.

Parabolas have outputs of positive values on both axes - this is because even values of n are guaranteed to return positive numbers, ex. with the function $\prod_{i=1}^{m}(-n) * (-n)$ that represents x^{2m}. The term $2m$ originates from the fact that if $n \geq 2$ and n is even, n can be represented as $2m$.

Cubic curves have outputs with signs that correlate to their respective inputs. This is because odd values of n that are negative are guaranteed to return negative numbers. Ex. $-n \prod_{i=1}^{m}(-n) * (-n)$ - we already covered that $\prod_{i=1}^{m}(-n) * (-n)$ returns even numbers, so $-n$ times an even number would always return a negative number as long as n is not a negative number itself (which in that case, $-n = n$ so there would be no negative sign in the first place unless the equation writer is illiterate).

1.2 Properties of Exponents

Exponent arithmetic follows rules that are seem slightly different from integer arithmetic.

Product Rule for Exponents: $x^m * x^n = x^{m+n}$.

How come $x^n = x^{m+n}$ and not $x^n = x^{m*n}$? This is because exponents are progressive. By this, I mean that exponents "scale" directly off of their past progress. For example, $2^4 = 2*2*2*2$ has to wait until $2*2 = 4$ is completed, then multiply that value by $4*2 = 8$, then multiply that by $8*2 = 16$, etc. Multiplying 2 with the chain $2*2*2*2$ will essentially be the same as "adding" an extra 2 into the visual chain, making it $2*2*2*2*2$ (note how a 2 is "added" into the chain). For another example, multiplying 2^2 by 2^4 will make a chain of 6 two's. However, suppose we assume the rule $x^n = x^{m*n}$ is true - in this case it would make a chain of 8 two's multiplied by each other. $(2*2*2*2)*(2*2) = 2*2*2*2*2*2$, NOT $2*2*2*2*2*2*2*2$. This is why exponent products are additive and not multiplicative.

Quotient Rule for Exponents: $x^m \div x^n = x^{m-n}$.

Understanding this rule is very simple - we have to first understand that $\frac{n^x}{n^x} = 1$ for all real values of n that are not 0. This is because of the reflexive property $n^x = n^x$, and something is inside itself 1 time. Applying this principle, in the example of $\frac{2*2*2*2*2}{2*2}$, we simply substitute all the equivalent two's into ones, making it $\frac{2*2*2*1*1}{1*1}$, which simplifies into $2*2*2 = 2^3$. This matches the quotient rule, as we just performed $2^5 \div 2^2$ and got 2^3 which is equivalent to 2^{5-2}.

Power of a Power Rule for Exponents: $(x^m)^n = x^{m*n}$.

Recall the "chain" principle from the product rule. In this case, we will be understanding the situation in which exponents can be multiplicative and not additive. If there is a chain $2*2*2*2 = 2^4$ powered by the integer 3, then by definition 2^4 is multiplied by itself 3 times. Note that the additive rule still applies, but in the same sense that regular multiplication is iterative addition - $2^4 * 2^4 = 2^8$, and it is multiplied one more time using the same additive rule, resulting in $2^8 * 2^4 = 2^{12}$, or 2^{4*3}. In chain form, it would be $2*2*2*2*2*2*2*2*2*2*2*2$.

Power of a Product Rule for Exponents: $(xy)^n = x^n y^n$.

This rule is the application of the commutative property of multiplication which states that $c(ab) = a(bc) = b(ac)$. Basically, multiplication order doesn't matter. $(2*3)^3$ can be represented in chain form as $(2*3)*(2*3)*(2*3)$. Rearranging the numbers, simply group together the same numbers to make $(2*2*2)*(3*3*3)$. This is equivalent to $2^3 * 3^3$ which satisfies the rule.

Power of a Quotient Rule for Exponents: $(\frac{x}{y})^n = \frac{x^n}{y^n}$.

Understanding this rule requires the understanding of fraction multiplication. For example, why is $\frac{2}{3} * \frac{3}{4} = \frac{6}{12} = \frac{1}{2}$? We can take a step back and look at multiplying and dividing fractions by integers. In integer-fraction multiplication, the numerators combine due to the fact that multiplication is iterative addition. $\frac{2}{3} * 5 = \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = \frac{10}{3} = \frac{2*5}{3}$. As for why the fraction's denominator is multiplied - looking at a simple case of fraction-integer division, $\frac{1}{3} \div 4$ is the same as $\frac{(\frac{1}{3})}{4}$. What goes into $\frac{1}{3}$ 4 times? $\frac{1}{12}$, because $\frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{4}{12} = \frac{1}{3}$. Since dividing by 4 can also be a multiplication of $\frac{1}{4}$, we have just shown that $\frac{1}{3} \div 4 = \frac{1}{3} * \frac{1}{4} = \frac{1}{12}$ which matches the denominator multiplication rule; Multiplying the denominator n times makes something n times smaller, because it means we are dividing it by n times more.

Now, combining both understandings together - using the commutative property of multiplication, since $\frac{3}{4}$ simply means that there are three different instances of $\frac{1}{4}$, $\frac{2}{3} * \frac{3}{4}$ can be rearranged as $\frac{2}{3} * 3 * \frac{1}{4}$, which in chain form is $(\frac{2}{3} * \frac{1}{4}) + (\frac{2}{3} * \frac{1}{4}) + (\frac{2}{3} * \frac{1}{4}) = \frac{2}{12} + \frac{2}{12} + \frac{2}{12} = \frac{6}{12} = \frac{2*3}{3*4}$, which satisfies the fraction multiplication rule stating that $\frac{2}{3} * \frac{3}{4} = \frac{2*3}{3*4}$. Returning back to the exponent rule, ex. $(\frac{2}{3})^4$ is essentially in chain form $\frac{2}{3} * \frac{2}{3} * \frac{2}{3} * \frac{2}{3}$.

Applying our intuition of fraction multiplication, it should be more obvious now that we would multiply the numerator 2 by itself 4 times and the denominator 3 by itself 4 times. This would be modeled as $\frac{2^4}{3^4}$ which fulfills the rule.

1.3 Zero, Negative, and Fractional Exponents

Exponents can have a power of fractional or negative exponents, and they can initially seem a bit trickier to understand. After all, how can something be multiplied by itself a total of -1 times?

There are many ways to prove this using algebra - and that's probably the best we are going to get. To visualize it intuitively without algebra, the best answer that I can give is "because I said so", and to "shut up and accept it". As immature as it might sound, it's addressing a similar question as something like "why is $1 + 1 = 2$?". 1 symbolizes "a single thing", so in a different universe maybe the symbol "2" symbolized the concept of 1 and maybe "3" symbolized the concept of two. In that case, in such a universe, the statement "$2 + 2 = 3$" would be true. In fact, maybe in a certain universe, our logic no longer works, and arithmetic works differently in its entirety.

The point is that mathematics is symbolic and most of its foundations are based upon axioms that people have decided to agree upon. Bertrand Russell and Alfred North Whitehead wrote the three-volume opus *Principia Mathemat-*

ica exploring and proving the concept of "1+1 = 2" and more similar topics in their entirety. That said, we can now move onto algebraically defining all three concepts of "weird" exponents. All of them involve the rules that were defined in section 1.2.

Rule of Zero Exponents: If $x \neq 0$, $x^0 = 1$.

Explaining TBD

Rule of Negative Exponents: $x^{-n} = \frac{1}{x^n}$.

Explaining TBD