Notes on Basic Complex Morse Theory

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1 Some Fundamental Constructions

Notation. *In this note, we denote by* $\mathbb{P}^N := \mathbb{C}P^N$*; for every complex vector space* V*, we define* $\mathbb{P}(V)$ *to be its projectivization; we denote by* $\check{P}(V) := \mathbb{P}(V^*)$*. We denote by* $\mathcal{P}_{d,N}$ *the vector space of homogeneous complex polynomials of degree* d *in the variables* $z_0, \ldots, z_N$*, and by* $\mathbb{P}(d, N)$ *the projectivization of* $\mathcal{P}_{d,N}$*.*

Let X be a compact submanifold of $\mathbb{P}^N$ with dimension n. Each homogeneous polynomial $0 \neq P \in \mathcal{P}_{d,N}$ defines a hypersurface

$$Z_P := \{z \in \mathbb{P}^N \mid P(z) = 0\}.$$

We denote by $X_P := X \cap Z_P$. Note that Z_P and X_P only depends on the image of P in $\mathbb{P}(d, N)$.

We recall a concept.

Definition 1.1. A projective subspace of $\mathbb{P}(V)$ is $\mathbb{P}(W)$, where W is a subspace of V.

We use it to define a *linear system.*

Definition 1.2. For any projective subspace $U \subset \mathbb{P}(d, N)$, we define $(X_P)_{P\in U}$ a family of hypersurfaces of X. This is called a linear system. If U is a projective line ($\dim U = 1$), we say that $(X_P)_{P\in U}$ is a pencil.

Remark 1.1. *Generally speaking, one define linear system as follows. Let* L *be a holomorphic vector bundle on a complex manifold* X*, and suppose that* $s_0, \cdots, s_N$ *form basis of* $H^0(X, L)$*. Then we can define* $\phi_L : X \setminus \mathrm{Bs}(L) \to \mathbb{P}^N$*,* $x \mapsto (s_0(x); \cdots ; s_N(x))$*, where* $\mathrm{Bs}(L)$ *is the set of all points at* X *with* $s(x) = 0$ *for all section* s *of* L*. In this sense, we say that* $H^0(X, L)$ *is a linear system.*

We can see that the definition given above is a special case of this definition. More precisely, since X *is a projective subspace, we have that any linear bundle* L *over* X *is* $\mathcal{O}_X(d) := \mathcal{O}_{\mathbb{P}^N}(d)|_X$*, and its global section is nothing but some* $P|_X$*, where* $P \in \mathbb{P}(d, N)$*, thus we can say that* $U \subset \mathbb{P}(d, N)$ *gives a linear system (as a subspace of* $H^0(X, \mathcal{O}_X(d))$*).*

Additionally, we can see that the linear system defined above is ample, since X *is itself a submanifold of* $\mathbb{P}^N \subset \mathbb{P}(d, N)$*.*

As above, we can define the base locus of a family of sections of $L = \mathcal{O}(d)$, which is nothing but $P \in U \subset \mathbb{P}(d, N)$, of X.

Definition 1.3. The base locus of the linear system $(X_P)_{P\in U}$ is

$$B = B_U := \bigcap_{P\in U} X_P.$$

Any point $x \in X \setminus B$ defines a hyperplane

$$H_x := \{P \in U \mid P(x) = 0\} \subset U.$$

Note that H_x determines a point in $\check{U}$ (recall that $\check{U}$ is the space of annihilators of some hyperplane in U). Additionally, we have $U = \check{U}$ when $\dim U = 1$. Thus, a linear system defines a holomorphic map (which is just the map ϕ_L in the general definition)

$$f_U : X^* := X \setminus B \to \check{U}, x \mapsto H_x.$$

Definition 1.4. The modification of X determined by linear system $(X_P)_{P\in U}$ is the variety

$$\hat{X} = \hat{X}_U := \{(x, H) \in X \times \check{U} \mid P(x) = 0, \forall P \in H \subset U\}.$$

When $\dim U = 1$, the modification can be written as

$$\hat{X} = \{(x, P) \in X \times U \mid P(x) = 0\}.$$

We have a pair of holomorphic maps, say $\pi_X : \hat{X} \subset X \times \check{U} \to X, (x, H) \mapsto x$ and $\hat{f}_U : \hat{X} = X \times \check{U} \to \check{U}, (x, H) \mapsto H$. Then we have $\hat{f}_U = f_U \circ \pi_X$. The projection π_X induces a biholomorphic $\hat{X}^* := \pi_X^{-1}(X^*) \to X^*$. Thus, we have commutative diagram

$$\begin{tikzcd} \hat{X}^* \arrow[d, "\pi_X"] \arrow[dr, "\hat{f}_U"] & \\ X^* \arrow[r, "f_U"] & \check{U} \end{tikzcd}.$$

Remark 1.2. *When studying linear system defined by $U \subset \mathbb{P}(d, N)$, it suffices to consider the case $d = 1$, i.e., linear systems defined by hyperplanes. The reason is that there exists Veronese mapping (which is an embedding):*

$$V : \mathbb{P}^N \to \mathbb{P}(d, N),\ (z_0 : \cdots : z_N) \mapsto (z_0^{i_0} \cdots z_N^{i_N} : \cdots)_{(i_0, \cdots, i_N \mid \sum i_j = d)}.$$

Under this map, any hypersurface $Z_P \subset \mathbb{P}^N$ defined by $P \in \mathbb{P}(d, N)$ is mapped to some hyperplane in $\mathbb{P}(d, N) \cong \mathbb{P}^{\binom{N+d}{d}-1}$.

For example, when $N = d = 2$, we have a map $V : \mathbb{P}^2 \to \mathbb{P}^5$, $(x : y : z) \mapsto (x^2 : xy : xz : y^2 : yz : z^2)$, so the hypersurface Z_P defined by $P = a_0x^2 + a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2$ can be mapped to a hyperplane

$$\left\{(z_0 : \cdots : z_5) \mid \sum a_i z_i = 0\right\}.$$

Definition 1.5. A Lefschetz pencil on X is a pencil determined by one-dimensional subspace $U \subset P(d, N)$ satisfying

1. B is empty or smooth, complex codimension two submanifold of X;
2. $\hat{X}$ is a smooth manifold;
3. The holomorphic map $\hat{f} : \hat{X} \to U$ is a nonresonant Morse function, i.e., no two critical points correspond to a same critical value, and for any critical point x_0 of $\hat{f}$, there is a local coordinate (z_j) near x_0 and a local coordinate u near $\hat{f}(x_0)$, such that

$$u \circ \hat{f} = \sum_j z_j^2.$$

Definition 1.6. The projective tangent space T_xX of X at x is the smallest projective subspace of $\mathbb{P}^N$ containing all tangent directions to X at x.

We introduce the following theorem without proof.

Theorem 1.3. *Fix a compact complex submanifold $X \subset \mathbb{P}^N$. Then, for any generic projective line $U \subset P(d, N)$, the pencil $(X_P)_{P \in U}$ is Lefschetz.*

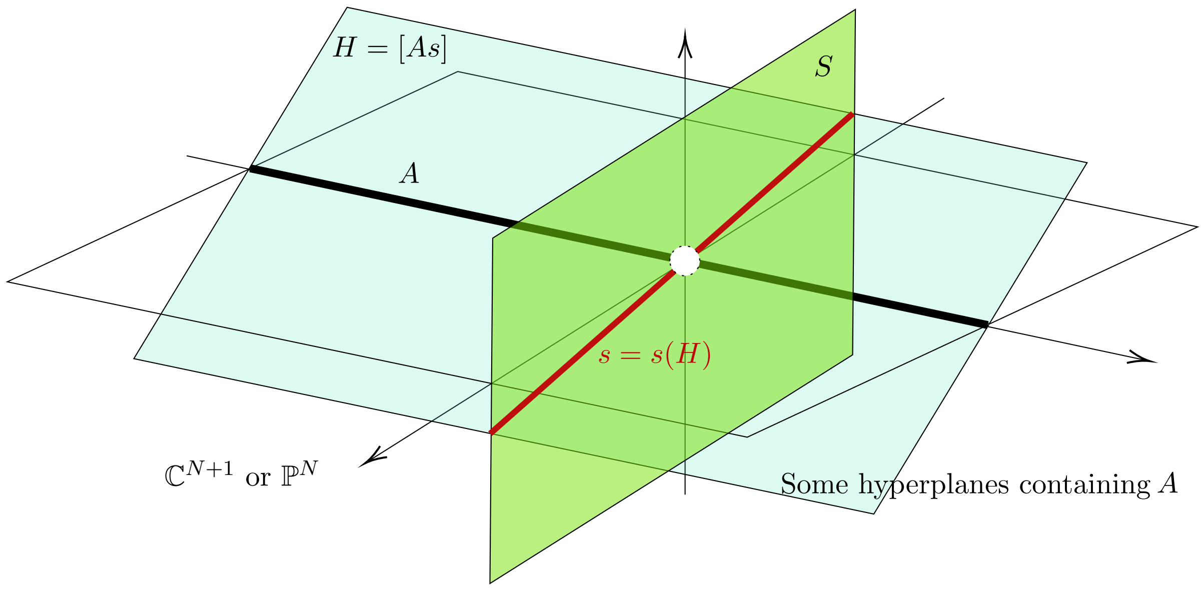
We now use our Remark 1.2 so that we can suppose that $d = 1$. In this case, the pencils can be given a more visual description. Let $X \subset \mathbb{P}^N$ be a compact complex submanifold, and let $A \subset \mathbb{P}^N$ be a projective subspace of codimension two, called an axis. The hyperplanes containing A form a one-dimensional projective space $U \subset \check{\mathbb{P}}^N \cong \mathbb{P}(1, N)$. This U can be identified with any line in $\mathbb{P}^N$ that not intersects A. Let S be such a line. Any hyperplane H containing A gives a point $s(H)$, so we get a map

$$U \to S,\ H \mapsto s(H).$$

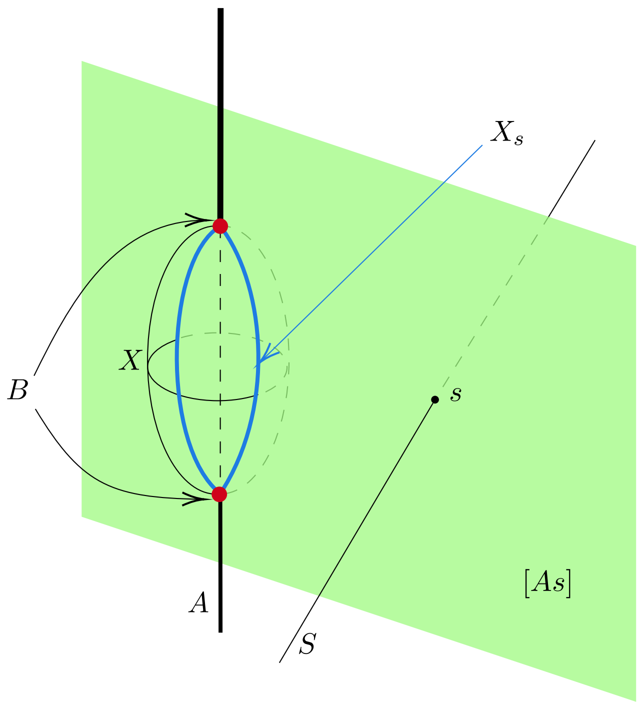
Conversely, any point $s \in S$ determines an unique hyperplane $[As]$ containing A and passing through s. The correspondence

$$S \to U,\ s \mapsto [As]$$

gives the inverse of the previous map. (See the figure below.)



The base locus of linear system $(X_s = [As] \cap X)_{s \in S}$ is $B = X \cap A$. All the hypersurfaces X_s pass through B. For generic A, B is a two-codimensional complex submanifold of X. (See the figure below.)



We have a natural map

$$f : X \setminus B \to S,\ x \mapsto S \cap [Ax].$$

We define the modified space as

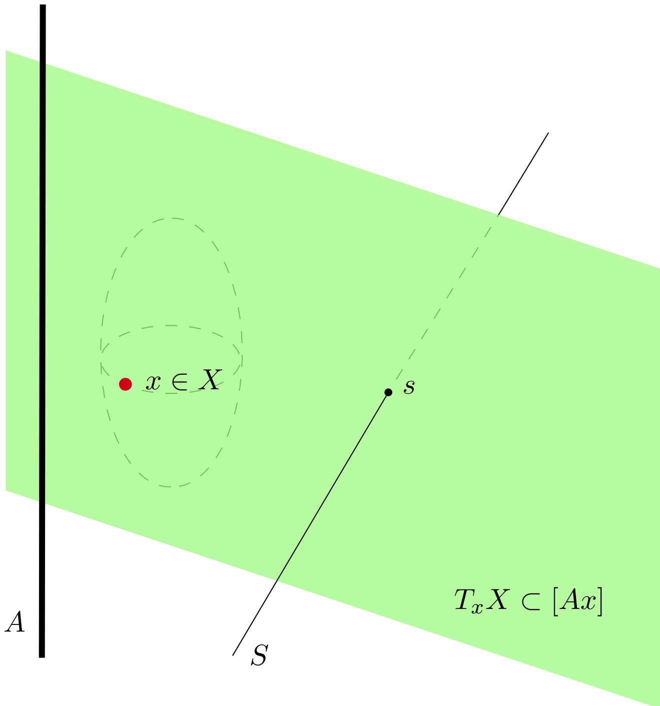
$$\hat{X} = \{(x, s) \in X \times S \mid x \in X_s\}$$

and similarly, we can define $\hat{f}(x, s) = s$ for $(x, s) \in \hat{X}$. We have $\hat{f} = f \circ \pi$ as above. We can define

$$\hat{B} := \pi^{-1}(B) = \{(b, s) \in B \times S \mid b \in [As]\} = B \times S,$$

and the natural projection $\pi_B : B \times S \to B$ is nothing but the projection π defined above. Set $\hat{X}_s = \hat{f}^{-1}(s)$, then π induces a homeomorphism $\hat{X}_s \to X_s$.

The critical points of $\hat{f}$ correspond to the hyperplanes containing A that contain the projective tangent plane to X.



The above figure gives us an intuitive understanding of the fact. The precise reason is the following.

Let $(x, s) \in \hat{X}$, and choose a local coordinate $x = (x_1, \cdots, x_n)$ of X and $s = t \in \mathbb{C}$ in $S \cong \mathbb{P}^1$. Then the hypersurface $H_s := [Ax]$ is defined by some one-degree homogeneous (in variable x) polynomial $F(x, t) = 0$. Then, locally, $\hat{X}$ is defined by $F = 0$, so we have $(v, w) \in T_{x,s}\hat{X}$ if and only if $dF_{(x,t)}(v, w) = 0$ locally at every (x, t) with $F(x, t) = 0$. Clearly we have $d\hat{f}(v, w) = w$ for all $(v, w) \in T_{(x,s)}\hat{X}$.

Now, suppose $T_xX \subset H_s = T_xH_s$, then for any $v = (v_1, \cdots, v_n) \in T_xX$, $v \in T_xH_s$. Note that H_s is defined by $F_s := F(-, s) = 0$, so we have $dF_s(v) = 0$, that is

$$\sum \frac{\partial F}{\partial x_i} v_i = 0.$$

Let $(v, w) \in T_{(x,s)}\hat{X}$ now, then since $dF(v, w) = 0$, we have

$$\frac{\partial F}{\partial t} w + \sum \frac{\partial F}{\partial x_i} v_i = 0,$$

so $\partial F/\partial t(s, x) \cdot w = 0$. Since H_s always changes as s changes, $\partial F/\partial t \neq 0$, so $w = 0$. This means $d\hat{f}(v, w) = 0$, so $d\hat{f} = 0$. Conversely, if $d\hat{f} = 0$, then for any $(v, w) \in T_{(s,x)}\hat{X}$, we have $w = 0$. Then $dF(v, w) = 0$ becomes $\sum \frac{\partial F}{\partial x_i} v_i = 0$, so $T_xX \subset T_xH_s$.

In general, let $S \cong U \cong \mathbb{P}^1$ be the space of parameter. The map $f : \hat{X} \to S$ is called the Lefschetz fibration associated with the Lefschetz pencil. If $B = \emptyset$, then $X = \hat{X}$ and the Lefschetz pencil is called a Lefschetz fibration.

2 Topological Preliminaries

For a fixed group G, there is a contravariant functor $\mathrm{Hom}(-, G) : \mathtt{Ab} \to \mathtt{Ab}$, sending homomorphism $\phi : A \to B$ to $\phi^\# : \mathrm{Hom}(-, B) \to \mathrm{Hom}(-, A)$, $f \mapsto f\phi$. In addition, $\mathrm{Hom}(-, G)$ is an additive functor, so $\phi^{\#} = 0$ if $\phi = 0$.

Lemma 2.1. *Let $(S_*(X), \partial)$ be a singular complex of topological space X. Then, for any abelian group G,*

$$0 \longrightarrow \mathrm{Hom}(S_0(X), G) \xrightarrow{\partial^{\#}} \mathrm{Hom}(S_1(X), G) \xrightarrow{\partial^{\#}} \mathrm{Hom}(S_2(X), G) \longrightarrow \cdots$$

is a complex, which is denoted by $\mathrm{Hom}(S_*(X), G)$.

Proof. Note that
$$\partial_n^{\#}\partial_{n-1}^{\#} = (\partial_{n-1}\partial_n)^{\#} = 0.$$
□

Definition 2.1. For $f \in \mathrm{Hom}(S_n(X), G)$, define
$$\delta^n f = \partial_{n+1}^{\#} f = f\partial_{n+1} \in \mathrm{Hom}(S_{n+1}(X), G).$$

Definition 2.2. Let G be an abelian group and let X be a space. If $n \geq 0$, then the group of (singular) n-cochains in X with coefficients in G is $\mathrm{Hom}(S_n(X), G)$. The group of n-cocycles is $\ker(\delta^n)$ and is denoted by $Z^n(X; G)$; the group of n-coboundaries is $\mathrm{im}(\delta^{n-1})$ and is denoted by $B^n(X; G)$. The n-th cohomology group of X with coefficients in G is
$$H^n(X; G) = \frac{Z^n(X; G)}{B^n(X; G)}.$$

Proposition 2.2. *For each fixed $n \geq 0$ and each abelian group G, cohomology is a contravariant functor $H^n(-; G) : \mathtt{Top} \to \mathtt{Ab}$.*

Proof. We have defined $H^n(X; G)$ for object X. Now, if $f : X \to Y$ is continuous, then we can define $f_{\#} : S_n(X) \to S_n(Y), \sigma \mapsto f\sigma$ to be the induced map on $\mathrm{Hom}(S_*(X), S_*(Y))$, and it satisfies $f_{\#}\partial = \partial f_{\#}$. Applying the contravariant functor $\mathrm{Hom}(-, G)$ to it gives $f^{\#}\delta = \delta f^{\#}$, where $f^{\#} : \mathrm{Hom}(S_n(Y), G) \to \mathrm{Hom}(S_n(X), G), h \mapsto hf_{\#}$. It is easy to see that $f^{\#}(Z^n(Y; G)) \subset (Z^n(X; G))$ and $f^{\#}(B^n(Y; G)) \subset (B^n(X; G))$, so it induces a homomorphism
$$f^* : H^n(Y; G) \to H^n(X; G), [h] \mapsto [f^{\#}(h)] = [hf_{\#}].$$
It is easy to show that $1^* = 1$ and $(fg)^* = g^* f^*$. □

Theorem 2.3 (Dimension Axiom)**.** *If $X = \{\mathrm{pt}\}$, then*
$$H^n(X; G) = \begin{cases} G & \text{if } n = 0 \\ 0 & \text{if } n \neq 0. \end{cases}$$

Proof. We see that $S_n(X) = \mathbb{Z}$ and that $\partial_n : S_n(X) \to S_{n-1}(X)$ is zero if n is odd and is an isomorphism if $n > 0$ is even. Now, for odd $n > 0$, we have $H^n(X; G) = \ker \partial_{n+1}^{\#} / \mathrm{im}\ \partial_n^{\#} = \{0\} / \mathrm{im}\ \partial_n^{\#} = 0$; for even $n > 0$, we have $H^n(X; G) = \ker \partial_{n+1}^{\#} / \mathrm{im}\ \partial_n^{\#} = \mathrm{Hom}(S_n(X), G)/\mathrm{Hom}(S_n(X), G) = 0$.

Now we look at $n = 0$. The end of singular complex is
$$0 \xleftarrow{0} S_1(X) \xleftarrow{0} S_2(X)\ ,$$
and applying $\mathrm{Hom}(-, G)$ gives
$$0 \xrightarrow{0} \mathrm{Hom}(S_1(X), G) \xrightarrow{0} \mathrm{Hom}(S_2(X), G)\ .$$
Thus $H^0(X; G) = \mathrm{Hom}(S_1(X), G) = \mathrm{Hom}(\mathbb{Z}, G) \cong G$. □

Theorem 2.4 (Homotopy Axiom)**.** *If $f, g : X \to Y$ are homotopic, then they induce the same homomorphisms $H^n(Y; G) \to H^n(X; G)$ for all $n \geq 0$.*

Corollary 2.5. *$H^n(X; G) \cong H^n(Y; G)$ if $X \approx Y$.*

Definition 2.3. A short exact sequence in `Ab`

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

splits if one of the following three equivalent condition holds:

1. there is some $s : B \to A$ such that $s \circ f = 1_A$;
2. there is some $r : C \to B$ such that $g \circ r = 1_C$;
3. $B \cong A \oplus C$.

Lemma 2.6 (Left exactness)**.** *Let G be an abelian group. If*

$$A' \xrightarrow{i} A \xrightarrow{p} A'' \longrightarrow 0$$

is exact, then

$$0 \longrightarrow \mathrm{Hom}(A'', G)) \xrightarrow{p^\#} \mathrm{Hom}(A, G) \xrightarrow{i^\#} \mathrm{Hom}(A', G)$$

is exact. Furthermore, if

$$0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{p} A'' \longrightarrow 0$$

is exact and splits, then

$$0 \longrightarrow \mathrm{Hom}(A'', G) \xrightarrow{p^\#} \mathrm{Hom}(A, G) \xrightarrow{i^\#} \mathrm{Hom}(A', G) \longrightarrow 0$$

is exact.

Applying this lemma to the splitting short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i = \cdot m} \mathbb{Z} \xrightarrow{p} \mathbb{Z}_m \longrightarrow 0$$

gives a short exact sequence

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}_m, G) \xrightarrow{p^\#} \mathrm{Hom}(\mathbb{Z}, G) \xrightarrow{i^\#} \mathrm{Hom}(\mathbb{Z}, G) \ .$$

Therefore, we have $\mathrm{Hom}(\mathbb{Z}, G) \cong \operatorname{im} p^\# = \ker i^\# = \{f : \mathbb{Z} \to G \mid f(m) = 0\} \cong G[m] := \{x \in G \mid mx = 0\}$. As a corollary, we have

$$\mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_{\gcd(m,n)}.$$

Thus, if we know two finitely generated abelian group A, B, then we can calculate $\mathrm{Hom}(A, B)$.

Applying this lemma to the split short exact sequence

$$0 \longrightarrow S_n(A) \longrightarrow S_n(X) \longrightarrow S_n(X)/S_n(A) \longrightarrow 0 \ ,$$

where $A \subset X$ is a subspace, gives a short exact sequence

$$0 \longleftarrow \mathrm{Hom}(S_n(A), G) \longleftarrow \mathrm{Hom}(S_n(X), G) \longleftarrow \mathrm{Hom}(S_n(X)/S_n(A), G) \longleftarrow 0 \ ,$$

and hence induce a short exact sequence of complexes

$$0 \longleftarrow \mathrm{Hom}(S_*(A), G) \longleftarrow \mathrm{Hom}(S_*(X), G) \longleftarrow \mathrm{Hom}(S_*(X)/S_*(A), G) \longleftarrow 0 \ .$$

Definition 2.4. Let $A \subset X$. Define the relative cohomology to be

$$H^n(X, A; G) := H_n(\mathrm{Hom}(S_*(X)/S_*(A), G)).$$

Recall that we can define $\bar{\partial}_{n+1} : S_{n+1}(X)/S_{n+1}(A) \to S_x(X)/S_x(A), [c] \mapsto [\partial_{n+1}c]$. Therefore,

$$H^n(X, A; G) = \ker \bar{\partial}^{\#}_{n+1} / \text{ im } \bar{\partial}^{\#}_{n+1}.$$

Due to the zig-zag lemma, we have the following theorem.

Theorem 2.7. *Let (X, A) be a topological pair and let G be an abelian group. Then we have a long exact sequence*

$$\cdots \longrightarrow H^n(X, A; G) \longrightarrow H^n(X; G) \longrightarrow H^n(A; G) \xrightarrow{d} H^{n+1}(X, A; G) \longrightarrow \cdots .$$

We still have excision axiom as in singular homolology.

Theorem 2.8 (Excision Axiom)**.** *Let $X_1, X_2 \subset X$ and $X = X_1^0 \cup X_2^0$. Then the inclusion $i : (X_1, X_1 \cap X_2) \to (X, X_2)$ induces isomorphisms for all $n \geq 0$,*

$$i^* : H^n(X, X_2; G) \cong H^n(X_1, X_1 \cap X_2; G).$$

Definition 2.5. Given an abelian group A, choose an short exact sequence

$$0 \longrightarrow R \xrightarrow{i} F \xrightarrow{p} A \longrightarrow 0 \ ,$$

where F is a free abelian group. Let B be another abelian group. By right exactness of tensor product, we get an exact sequence

$$R \otimes B \xrightarrow{i \otimes 1_B} F \otimes B \xrightarrow{p \otimes 1_B} A \otimes B \longrightarrow 0 \ ,$$

and we define $\mathrm{Tor}(A, B) = \ker(i \otimes 1_B)$. By left exactness of $\mathrm{Hom}(-, B)$, we get an exact sequence

$$0 \longleftarrow \mathrm{Hom}(R, B) \xleftarrow{i^{\#}} \mathrm{Hom}(F, B) \xleftarrow{p^{\#}} \mathrm{Hom}(A, B) \ ,$$

and we define $\mathrm{Ext}(A, B) = \mathrm{coker}\, i^{\#} = \mathrm{Hom}(R, B)/ \, \mathrm{im}\, i^{\#}$.

Proposition 2.9. *These are standard properties of the* $\mathrm{Ext}(-, -)$ *functor.*

1. *Given short exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

and abelian group G, we get an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(C, G) \arrow[r] & \mathrm{Hom}(B, G) \arrow[r] & \mathrm{Hom}(A, G) \arrow[dlll] \\
\mathrm{Ext}(C, G) \arrow[r] & \mathrm{Ext}(B, G) \arrow[r] & \mathrm{Ext}(A, G) \arrow[r] & 0
\end{tikzcd} .$$

2. *Given short exact sequence*

$$0 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 0$$

and abelian group A, we get an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(A, G) \arrow[r] & \mathrm{Hom}(A, H) \arrow[r] & \mathrm{Hom}(A, K) \arrow[dlll] \\
\mathrm{Ext}(A, G) \arrow[r] & \mathrm{Ext}(A, H) \arrow[r] & \mathrm{Ext}(A, K) \arrow[r] & 0
\end{tikzcd} .$$

3. *If F is free, then*
$$\operatorname{Ext}(F,G)=0.$$
4. *If D is divisible, i.e., for every $x \in G$ and every integer $n>0$, there is $y \in G$ such that $ny=x$, then*
$$\operatorname{Ext}(A,D)=0.$$
5. $$\operatorname{Ext}\left(\bigoplus A_i, G\right) \cong \prod \operatorname{Ext}(A_i,G).$$
6. $$\operatorname{Ext}\left(A, \prod G_i\right) \cong \prod \operatorname{Ext}(A,G_i).$$
7. $$\operatorname{Ext}(\mathbb{Z}_m,G) \cong G/mG.$$

Use there properties, one can compute $\operatorname{Ext}(A,B)$ for any finitely generated abelian groups A,B.

We recall the Universal Coefficient Theorem and Künneth Formula for homology.

Theorem 2.10 (Universal Coefficient Theorem for Homology)**.** *For a space X and abelian group G, there exists a split short exact sequence:*

$$0 \longrightarrow H_n(X)\otimes G \xrightarrow{\alpha} H_n(X;G) \longrightarrow \operatorname{Tor}(H_{n-1}(X),G) \longrightarrow 0\ ,$$

where $\alpha : [z]\otimes g \mapsto [z\otimes g]$. Thus,

$$H_n(X;G) \cong \operatorname{Tor}(H_{n-1}(X),G) \oplus H_n(X)\otimes G.$$

Theorem 2.11 (Künneth Formula for Homology)**.** *For spaces X,Y and abelian group G, we have a split exact sequence*

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(X)\otimes H_j(Y) \longrightarrow H_n(X\times Y) \longrightarrow \bigoplus_{p+q=n-1} \operatorname{Tor}(H_p(X),H_q(Y)) \longrightarrow 0\ ,$$

so

$$H_n(X\times Y) \cong \bigoplus_{i+j=n} H_i(X)\otimes H_j(Y) \oplus \bigoplus_{p+q=n-1} \operatorname{Tor}(H_p(X),H_q(Y)).$$

We have cohomology version of the Universal Coefficient Theorem and Kúnneth formula.

Theorem 2.12 (Dual Universal Coefficient Theorem)**.** *For a space X and abelian group G, there exists a split short exact sequence:*

$$0 \longrightarrow \operatorname{Ext}(H_{n-1}(X),G) \longrightarrow H^n(X;G) \xrightarrow{\beta} \operatorname{Hom}(H_n(X),G) \longrightarrow 0\ ,$$

where $\beta : H^n(X;G) = H^n(\operatorname{Hom}(S_*(X),G)) \ni [\phi] \mapsto (\phi' : z_n + B_n \mapsto \phi(z_n)) \in \operatorname{Hom}(H_n(X),G)$. Thus,

$$H^n(X;G) \cong \operatorname{Ext}(H_{n-1}(X),G) \oplus \operatorname{Hom}(H_n(X),G).$$

Definition 2.6. A chain complex C_* is of finite type if each of its terms C_n is finitely generated. A space A is of finite type it each of its homology groups $H_n(X)$ is finitely generated.

Notation. *We denote by $H^n(X) := H^n(X;\mathbb{Z})$.*

Theorem 2.13 (Universal Coefficient Theorem for Cohomology). *For a space X of finite type and abelian group G, there exists a split short exact sequence:*

$$0 \longrightarrow H^n(X) \otimes G \xrightarrow{\ \alpha\ } H^n(X;G) \longrightarrow \mathrm{Tor}(H^{n+1}(X),G) \longrightarrow 0 \ ,$$

where $\alpha : [z] \otimes g \mapsto [zg : \sigma \mapsto z(\sigma)g]$. Thus,

$$H^n(X;G) \cong H^n(X) \otimes G \oplus \mathrm{Tor}(H^{n+1}(X),G).$$

Theorem 2.14 (Künneth Formula for Homology). *For spaces X, Y of finite type and abelian group G, we have a split exact sequence*

$$0 \longrightarrow \bigoplus_{i+j=n} H^i(X) \otimes H^j(Y) \longrightarrow H^n(X \times Y) \longrightarrow \bigoplus_{p+q=n+1} \mathrm{Tor}(H^p(X), H^q(Y)) \longrightarrow 0 \ ,$$

so

$$H_n(X \times Y) \cong \bigoplus_{i+j=n} H^i(X) \otimes H^j(Y) \oplus \bigoplus_{p+q=n+1} \mathrm{Tor}(H^p(X), H^q(Y)).$$

Now we introduce the concept of the *cohomology ring* and the *cup product.*

Notation. *We denote by*

$$S^n(X,G) := \mathrm{Hom}(S_n(X), G),$$

and by

$$S^*(X;G) := \bigoplus_{n \geq 0} S^n(X,G).$$

Definition 2.7. For $\phi \in S^n(X,G)$, $c \in S_n(X)$, write

$$\langle \phi, c \rangle := \phi(c).$$

It is called the Kronecker paring. We use the same notation for $\phi \in H^n(X;G)$ and $c \in H_n(X)$.

We rewrite some properties we have mentioned above in the language of Kronecker paring. If $c \in S_{n+1}(X), \phi \in S^n(X,G)$, we have

$$\langle \delta^n(\phi), c \rangle = \langle \phi, \partial_{n+1} c \rangle.$$

If $f : X \to Y$ is continuous and $c \in S_n(X), \phi \in S^n(Y,G)$, then

$$\langle f^{\#}(\phi), c \rangle = \langle \phi, f_{\#} c \rangle.$$

In particular, if $c = \sigma$ is a simplex in X, then

$$\langle f^{\#}(\phi), \sigma \rangle = \langle \phi, f\sigma \rangle.$$

Since $S_n(X)$ is the free abelian group generated by all n-simplexes, $\phi \in S^n(X,G)$ is determined by $\langle \phi, \sigma \rangle$ as σ ranges over all continuous maps $\Delta^n \to X$.

Definition 2.8. If $0 \leq i \leq d$, define (affine) maps $\mu_i, \lambda_i : \Delta^i \to \Delta^d$ by

$$\mu_i : (t_0, \ldots, t_i) \mapsto (t_0, \ldots, t_i, 0, \ldots, 0)$$

and

$$\lambda_i : (t_0, \ldots, t_i) \mapsto (0, \ldots, 0, t_0, \ldots, t_i).$$

One calls μ_i a front face and λ_i a back face.

Let X be a space and R be a commutative ring. If $\phi \in S^n(X,R)$ and $\theta \in S^m(X,R)$, we define the cup product by

$$\phi \cup \theta \in S^{n+m}(X,R), \text{ such that } \langle \phi \cup \theta, \sigma \rangle = \langle \phi, \sigma\lambda_n \rangle \cdot \langle \theta, \sigma\mu_m \rangle$$

for any $(n+m)$-simplex σ in X. Here $\cdot$ is the multiplication in R.

Proposition 2.15. *This cup product makes $S^*(X, R)$ a graded ring.*

Proposition 2.16. *If $f : X \to Y$ is continuous, then*

$$f^{\#}(\phi \cup \theta) = f^{\#}(\phi) \cup f^{\#}(\theta).$$

Moreover, if $1_X \in S^0(X, R)$ and $1_Y \in S^0(Y, R)$ are constant 1 map, then $f^{\#}(1_Y) = 1_X$

Proof. Suppose $\phi \in S^n(Y, R)$ and $\theta \in S^m(Y, R)$, and suppose $\sigma \in S_{n+m}(X)$ is a simplex. Then

$$\begin{aligned}\langle f^{\#}(\phi \cup \theta), \sigma\rangle = \langle \phi \cup \theta, f_{\#}(\sigma)\rangle &= \langle \phi, f_{\#}(\sigma)\lambda_n\rangle\langle\theta, f_{\#}(\sigma)\mu_m\rangle \\ &= \langle \phi, f\sigma\lambda_n\rangle\langle\theta, f\sigma\mu_m\rangle \\ &= \langle f^{\#}(\phi), \sigma\lambda_n\rangle\langle f^{\#}(\theta), \sigma\mu_m\rangle \\ &= \langle f^{\#}(\phi) \cup f^{\#}(\theta), \sigma\rangle\end{aligned}$$

If $x \in X$, we have

$$\langle f^{\#}(1_Y), x\rangle = \langle 1_Y, f_{\#}(x)\rangle = 1.$$

□

Proposition 2.17. *If $\phi \in S^n(X, R)$ and $\theta \in S^m(X, R)$, then*

$$\delta(\phi \cup \theta) = \delta\phi \cup \theta + (-1)^n \phi \cup \delta\theta.$$

Definition 2.9. In $H^*(X; R)$, we define $[\phi] \cup [\theta] = [\phi \cup \theta]$. This makes $H^*(X; R)$ a graded ring.

Definition 2.10. Let $0 \le p \le q$. Define

$$\cap : H^p(X; R) \otimes H_q(X) \to H_{q-p}(X)$$

such that for all $v \in H^{q-p}(X; R)$, we have

$$\langle v \cup u, c\rangle = \langle v, u \cap c\rangle.$$

We introduce the Poincaré duality theorem

Theorem 2.18 (Poincaré)**.** *Let M be an m-dimensional oriented compact connected smooth manifold, then for all k, we have an isomorphism*

$$\mathscr{P}_M : H^k(M) \to H_{m-k}(M), u \mapsto u \cap [M],$$

where $[M]$ is a generator of $H_m(M) \cong \mathbb{Z}$.

Proposition 2.19 (Projection Formula)**.** *Let $f : X \to Y$ be continuous. Then we have*

$$f_*(f^*(u) \cap c) = u \cap f_*(c),$$

where $u \in H^k(Y)$ and $c \in H_n(X)$.

Proof. Pick any $v \in H^{n-k}(Y)$. Then

$$\begin{aligned}\langle v, u \cap f_*(c)\rangle &= \langle v \cup u, f_*(c)\rangle = \langle f^*(v \cup u), c\rangle \\ &= \langle f^*(v) \cup f^*(u), c\rangle = \langle f^*(v), f^*(u) \cap c\rangle = \langle v, f_*(f^*(u) \cap c)\rangle.\end{aligned}$$

□

Definition 2.11. Suppose X, Y are closed, connected and oriented smooth manifolds with dimension n, and $f : X \to Y$ is smooth. Pick $y \in Y$ being a regular value of f, which is dense due to Sard lemma, and define $d_f(y) = \sum_{x \in f^{-1}(y)} \operatorname{sgn}_x(f)$, where

$$\operatorname{sgn}_x(f) = \begin{cases} 1 & \text{if } f \text{ keeps the orientation near } x, \\ -1 & \text{otherwise.} \end{cases}$$

One can show that $d_f(y)$ is locally constant, so it is a fixed integer, denoted by $\deg f$.

Lemma 2.20. *We have*

$$f_*[X] = \deg f \cdot [Y].$$

Proof. Consider the inclusion $i : (Y, \emptyset) \to (Y, Y \setminus \{y\})$. It induces a homomorphism $i_* : H_n(Y) \to H_n(Y, Y \setminus \{y\})$. Let V be a chart neighborhood of y, then by excision, $H_n(Y, Y \setminus \{y\}) \cong H_n(V, V \setminus \{y\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}$. Thus, it has a generator. Let $\nu = i_*[Y]$, then ν is a generator.

For each $x_i \in f^{-1}(y)$, pick a small enough neighborhood U_i such that $f|_{U_i} : (U_i, U_i \setminus \{x_i\}) \to (Y, Y \setminus \{y\})$ is a homeomorphism and $\{U_i\}$ is a collection of pairwise disjoint open sets. By excision axiom,

$$H_n(X, X \setminus X \setminus f^{-1}(y)) \cong H_n \left(\bigcup U_i, \bigcup U_i \setminus \{x_i\} \right) \cong \bigoplus_i H_n(U_i, U_i \setminus \{x_i\}).$$

Suppose $\mathbb{Z} \cong H_n(U_i, U_i \setminus \{x_i\}) = \langle \mu_i \rangle$. Note that f induces locally an isomomorphism

$$f_* : H_n(U_i, U_i \setminus \{x_i\}) \cong H_n(Y, Y \setminus \{y\}),$$

and $f_*(\mu_i) = \operatorname{sgn}_{x_i}(f)\nu$.

Now, the inclusion $j : (X, \emptyset) \to (X, X \setminus f^{-1}(y))$ induces a homomorphism $j_* : H_n(X) \to H_n(X, X \setminus f^{-1}(y)) \cong \bigoplus H_n(U_i, U_i \setminus \{x_i\})$. The image of $[X]$ is $\sum \mu_i$. Therefore,

$$f_*[X] = f_* \sum \mu_i = \sum \operatorname{sgn}_{x_i}(f)\nu = \deg f \cdot i_*[Y] = \deg f \cdot [Y].$$

□

3 Topological Applications of Lefschetz Pencils

Suppose $X \subset \mathbb{P}^N$ is an n-dimensional algebraic manifold (the zero set of $N - n$ independent homogeneous polynomials such that is a complex submanifold of $\mathbb{P}^N$), and $S \subset \mathbb{P}(d, N)$ is a one-dimensional projective subspace defining a Lefschetz pencil $(X_s)_{s \in S}$ on X. As usual, denote by B the base locus

$$B = \bigcap_{s \in S} X_s,$$

which is a complex submanifold of X having dimension $n - 2$, and by $\hat{X}$ the modification

$$\hat{X} = \left\{ (x, s) \in X \times S \,\middle|\, x \in X_s \right\}.$$

We have an induced Lefschetz fibration $\hat{f} : \hat{X} \to S$ with fibers $\hat{X}_s := \hat{f}^{-1}(s)$, and a surjection $p : \hat{X} \to X$ that induces homeomorphisms $\hat{X}_s \cong X_s$. Observe that $\deg p = 1$. Set

$$\hat{B} := p^{-1}(B).$$

We have a tautological diffeomorphism

$$\hat{B} \cong B \times S, \ (x, s) \mapsto (x, s).$$

Since $S \cong S^2$ and $H_n(S^2) = \mathbb{Z}$ if $n = 0, 2$ and $H_n(S^2) = 0$ otherwise, we deduce from Künneth's formula that we have an isomorphism

$$H_q(\hat{B}) \cong H_q(B) \oplus H_{q-2}(B).$$

We have a natural injection

$$H_{q-2}(B) \to H_q(\hat{B}), c \mapsto c \times [S].$$

Using the inclusion $\hat{B} \to \hat{X}$, we have an injection

$$\kappa : H_{q-2}(B) \to H_q(\hat{X}).$$

Lemma 3.1. *The sequence*

$$0 \longrightarrow H_{q-2}(B) \xrightarrow{\kappa} H_q(\hat{X}) \xrightarrow{p_*} H_q(X) \longrightarrow 0$$

is exact and splits. In particular, $\hat{X}$ is connected if and only if X is connected and

$$\chi(\hat{X}) = \chi(X) + \chi(B).$$

Proof. The proof will be carried out in several steps.
[**Step 1.** p_* has a right inverse.]

Consider the Gysin morphism

$$p^! : H_q(X) \to H_q(\hat{X}), p^! := \mathscr{P}_{\hat{X}} p^* \mathscr{P}_X^{-1},$$

that is, the following diagram commutes:

$$\begin{tikzcd} H^{2n-q}(X) \arrow[r, "\mathscr{P}_X = \cdot\cap[X]"] \arrow[d, "p^*"'] & H_q(X) \arrow[d, "p^!"] \\ H^{2n-q}(\hat{X}) \arrow[r, "\mathscr{P}_{\hat{X}} = \cdot\cap[\hat{X}]"'] & H_q(\hat{X}) \end{tikzcd}.$$

We will show that $p_* \circ p^! = 1_{H_q(X)}$. Pick $c \in H_q(X)$ and let $u = \mathscr{P}_X^{-1}(c)$, i.e., $u \cap [X] = c$. Then

$$p^!(c) = \mathscr{P}_{\hat{X}} p^*(u) = p^*(u) \cap [\hat{X}],$$

so

$$p_* p^!(c) = p_*(p^*(u) \cap [\hat{X}]) = u \cap p_*([\hat{X}]) = \deg p \cdot u \cap [X] = u \cap [X].$$

[**Step 2.** Conclusion.]

We use the long exact sequence of $(\hat{X}, \hat{B})$ and (X, B) and the morphism between them induced by p_*. We have the following commutative diagram:

$$\begin{tikzcd} & & H_{q-2}(B) \arrow[d, hook, "\times[S]"] \arrow[dr, hook, "\kappa"] & & \\ H_{q+1}(\hat{X}) \arrow[r] \arrow[d, "p_*"] & H_{q+1}(\hat{X}, \hat{B}) \arrow[r, "d"] \arrow[d, "p'_*"] & H_q(B) \oplus H_{q-2}(B) \arrow[r, hook, "i_*"] \arrow[d, "\mathrm{pr}"] & H_q(\hat{X}) \arrow[r] \arrow[d, "p_*"] & H_q(\hat{X}, \hat{B}) \arrow[d, "p'_*"] \\ H_{q+1}(X) \arrow[r] & H_{q+1}(X, B) \arrow[r, "d"] & H_q(B) \arrow[r] & H_q(X) \arrow[r] & H_q(X, B) \end{tikzcd}$$

Since $p : \hat{X} \setminus \hat{B} \to X \setminus B$ is a homeomorphism, the induced morphism

$$p'_* : H_q(\hat{X}, \hat{B}) \to H_q(X, B)$$

is an isomorphism. Moreover, p_* is surjective since p is. Note that κ, being the composition of two injections, hence is injective. It remains to show that $\operatorname{im} \kappa = \ker p_*$, which is nothing but diagram chasing. □

Decompose the projective line S into two closed hemisphere

$$S = D_+ \cup D_-, \quad E = D_+ \cap D_- \cong S^1, \quad \hat{X}_\pm = \hat{f}^{-1}(D_\pm), \quad \hat{X}_E = \hat{f}^{-1}(E),$$

such that all the critical values of $\hat{f}$ is in the interior of D_+. Choose a generic point $\bullet \in E$. Denote by r the number of critical points (= the number of critical values) of the Morse function $\hat{f}$. We introduce a topological theorem now without proof.

Theorem 3.2 (Ehresmann Fibration Theorem)**.** *Suppose $\Phi : E \to B$ is a smooth map between smooth manifolds such that:*

1. *Φ is proper, i.e., $\Phi^{-1}(K)$ is compact for every compact $K \subseteq B$.*
2. *Φ is a submersion.*
3. *If $\partial E \neq \emptyset$, then the restriction $\partial\Phi := \Phi|_{\partial E}$ is also a submersion.*

Then $\Phi : (E, \partial E) \to B$ is a locally trivial, smooth fiber bundle, i.e., for any $b \in B$, there is a neighborhood U such that $\Phi^{-1}(U) \cong U \times F$ for some F.

Note that $\hat{f}|_{\hat{X}_-} : \hat{X}_- \to D_-$ has no critical point, so it is a submersion; since $\hat{X}$ is compact, $\hat{f}$ is proper. Thus, by Ehresmann Fibration Theorem, $\hat{f}|_{\hat{X}_-} : \hat{X}_- \to D_-$ is a locally trivial, smooth fiber bundle, i.e., there is a neighborhood $U \subset D_-$ of $\bullet$ such that $\hat{f}^{-1}(U) \cong U \times \hat{f}^{-1}(\bullet) = U \times \hat{X}_\bullet$; since D_- is contractible, it is trivial, so

$$\hat{X}_- \cong D_- \times \hat{X}_\bullet.$$

Similarly, we have

$$\hat{X}_E = \partial \hat{X}_\pm \cong \partial D_- \times \hat{X}_\bullet = E \times \hat{X}_\bullet.$$

Thus,

$$(\hat{X}_-, \hat{X}_E) \cong \hat{X}_\bullet \times (D_-, E).$$

In particular, $\hat{X}_\bullet$ is a deformation retraction of $\hat{X}_-$, so the inclusion $\hat{X}_\bullet \to \hat{X}_-$ induces an isomorphism

$$H_*(\hat{X}_\bullet) \cong H_*(\hat{X}_-).$$

Künneth formula gives

$$H_q(\hat{X}_\bullet \times (D_-, E)) \cong \bigoplus_{k+\ell=q} H_\ell(\hat{X}_\bullet) \otimes H_k(D_-, E) \oplus \bigoplus_{k+\ell=q-1} \mathrm{Tor}(H_\ell(\hat{X}_\bullet), H_k(D_-, E)).$$

Noting that

$$H_q(D_-, E) \cong \begin{cases} 0 & \text{if } q \neq 2, \\ \mathbb{Z} & \text{if } q = 2. \end{cases},$$

we have an isomorphism

$$H_{q-2}(\hat{X}_\bullet) \cong H_q(\hat{X}_\bullet \times (D_-, E)), c \mapsto c \times [D_-, E].$$

By excision axiom, we have an isomorphism

$$H_q(\hat{X}_-, \hat{X}_E) \cong H_q(\hat{X}, \hat{X}_+).$$

Thus, we get

$$H_{q-2}(\hat{X}_\bullet) \cong H_q(\hat{X}_\bullet \times (D_-, E)) \cong H_q(\hat{X}_-, \hat{X}_E) \cong H_q(\hat{X}, \hat{X}_+).$$

We use the long exact sequence of $(\hat{X}, \hat{X}_+, \hat{X}_\bullet)$,

$$\cdots \longrightarrow H_{q+1}(\hat{X}_+, \hat{X}_\bullet) \longrightarrow H_{q+1}(\hat{X}, \hat{X}_\bullet) \longrightarrow H_{q+1}(\hat{X}, \hat{X}_+) \xrightarrow{d} H_q(\hat{X}_+, \hat{X}_\bullet) \longrightarrow \cdots .$$

We introduce the following lemma, which will be proved later:

Lemma 3.3.
$$H_q(\hat{X}_+, \hat{X}_\bullet) = \begin{cases} 0 & \text{if } q \neq n = \dim_{\mathbb{C}} X, \\ \mathbb{Z}^r & \text{if } q = n, \end{cases}$$
where r is the number of critical points of $\hat{f}$.

Using this lemma, we deduce that if $q \neq n, n-1$,
$$H_{q+1}(\hat{X}, \hat{X}_\bullet) \cong H_{q+1}(\hat{X}, \hat{X}_+) \cong H_{q-1}(\hat{X}_\bullet).$$
For $q = n$, we get an exact sequence
$$0 \longrightarrow H_{n+1}(\hat{X}, \hat{X}_\bullet) \longrightarrow H_{n-1}(\hat{X}_\bullet) \longrightarrow H_n(\hat{X}_+, \hat{X}_\bullet) \longrightarrow H_n(\hat{X}, \hat{X}_\bullet) \longrightarrow H_{n-2}(\hat{X}_\bullet) \longrightarrow 0\ .$$

Proposition 3.4. *If X is connected and $n = \dim_{\mathbb{C}} X > 1$, then the fiber $\hat{X}_\bullet \cong X_\bullet$ is connected.*

Proof. Using the isomorphism established above, we have
$$H_0(\hat{X}, \hat{X}_\bullet) \cong H_{-2}(\hat{X}_\bullet) = 0, \quad H_1(\hat{X}, \hat{X}_\bullet) \cong H_{-1}(\hat{X}_\bullet) = 0.$$
Using the long exact sequence of $(\hat{X}, \hat{X}_\bullet)$
$$\begin{tikzcd} H_0(\hat{X}_\bullet) \arrow[r] & H_0(\hat{X}) \arrow[r] & H_0(\hat{X}, \hat{X}_\bullet) \\ H_1(\hat{X}_\bullet) \arrow[r] & H_1(\hat{X}) \arrow[r] & H_1(\hat{X}, \hat{X}_\bullet) \arrow[ull] \end{tikzcd},$$
we have $H_0(\hat{X}_\bullet) \cong H_0(\hat{X})$ and $H_1(\hat{X}_\bullet) \cong H_1(\hat{X})$. By Lemma 3.1, since X is connected, $\hat{X}$ is connected, so $H_0(\hat{X}_\bullet) \cong H_0(\hat{X}) \cong \mathbb{Z}$, so $\hat{X}_\bullet$ is connected.

□

Proposition 3.5.
$$\chi(\hat{X}) = 2\chi(\hat{X}_\bullet) + (-1)^n r, \quad \chi(X) = 2\chi(X_\bullet) - \chi(B) + (-1)^n r.$$

Proof. Due to Lemma 3.1, we deduce $\chi(\hat{X}) = \chi(X) + \chi(B)$. On the other hand, the long exact sequence of $(\hat{X}, \hat{X}_\bullet)$ gives $\chi(\hat{X}) - \chi(\hat{X}_\bullet) = \chi(\hat{X}, \hat{X}_\bullet)$. The isomorphism and long exact sequence deduced from Lemma 3.3 gives that
$$\chi(\hat{X}, \hat{X}_\bullet) = \chi(\hat{X}_\bullet) + (-1)^n r.$$
Therefore,
$$\chi(\hat{X}) = 2\chi(\hat{X}_\bullet) + (-1)^n r, \quad \chi(X) = 2\chi(X_\bullet) - \chi(B) + (-1)^n r.$$

□

Corollary 3.6 (Genus formula)**.** *For generic $P \in \mathcal{P}_{d,2}$, the projective plane curve*
$$C_P := \{(z_0 : z_1 : z_2) \in \mathbb{P}^2 \mid P(z_0, z_1, z_2) = 0\}$$
is a smooth Riemann surface of genus
$$g(C_P) = \frac{(d-1)(d-2)}{2}.$$

Proof. Fix a projective line $L \subset \mathbb{P}^2$ and a point $c \in \mathbb{P}^2 \setminus (C_P \cup \mathbb{L})$. We get a pencil of projective line $\{[c\ell] : \ell \in \mathbb{L}\}$ and a map $f = f_c : C_P \to \mathbb{L}, x \mapsto \mathbb{L} \cap [cx]$. In this case, the base locus

$$B = \bigcap_{\ell \in \mathbb{L}} (C_P \cap [c\ell]) = \emptyset,$$

so $X = \hat{X} = C_P$. Since every generic line intersects C_P in d times, we conclude that f, as a holomorphic map, has degree d. A point $x \in C_P$ is a critical point of f if and only if the line $[cx]$ is tangent to C_P.

For generic c the projection f_c defines a Lefschetz fibration. Modulo a linear change of coordinates, we can assume that all the critical points are situated in the region $z_0 \neq 0$ and c is the point at infinity $(0 : 1 : 0)$. In the affine plane $z_0 \neq 0$ with coordinates $x = z_1/z_0$, $y = z_2/z_0$, the point $c \in \mathbb{P}^2$ corresponds to the point at infinity on the lines parallel to the x-axis ($y = 0$). In this region, the curve C_P is described by the equation

$$F(x, y) = 0,$$

where $F(x, y) = P(1, x, y)$ is a degree d polynomial. Therefore, the critical points of the projection map are the points (x, y) on the curve $F(x, y) = 0$, where the tangent is parallel to the x-axis,

$$0 = \frac{dy}{dx} = -\frac{F_x}{F_y},$$

where F_x and F_y denote the partial derivatives of F with respect to x and y. Thus, the critical points are solutions of the system of polynomial equations:

$$\begin{cases} F(x, y) = 0, \\ F_x(x, y) = 0. \end{cases}$$

The first polynomial has degree d, while the second polynomial has degree $d-1$. For generic P, this system will have exactly $d(d-1)$ distinct solutions. The corresponding critical points will be nondegenerate. Using Proposition 3.5 with $X = \hat{X} = C_P$, $r = d(d-1)$, and $X_\bullet$ a finite set of cardinality d for generic point $\bullet \in \mathbb{L}$, we deduce

$$2 - 2g(C_P) = \chi(C_P) = 2d - d(d-1),$$

so that

$$g(C_P) = \frac{(d-1)(d-2)}{2}.$$

$\square$

Theorem 3.7 (Lefschetz Hyperplane Theorem)**.** *Suppose $X \subset \mathbb{P}^N$ is a smooth projective variety (closed complex submaifold) of dimension n. Then for any hyperplane $H \subset \mathbb{P}^N$ intersecting X transversally, the inclusion $X \cap H \hookrightarrow X$ induces isomorphisms*

$$H_q(X \cap H) \cong H_q(X)$$

if $q < \frac{1}{2}\dim_{\mathbb{R}}(X \cap H) = n - 1$ and an epimorphism if $q = n - 1$. Equivalently, (by using the long exact sequence of $(X, X \cap H)$,) this means that

$$H_q(X, X \cap H) = 0, \quad \forall q \leq n - 1.$$

Proof. Choose a codimension two projective subspace $A \subset \mathbb{P}^N$ such that the pencil of hyperplanes in $\mathbb{P}^N$ containing A defines a Lefschetz pencil on X. Then, the base locus $B = A \cap X$ is a smooth codimension two complex submanifold of X and the modification $\hat{X}$ is smooth as well.

There is a path H_t for $t \in [0,1]$ such that $H_0 = H$ and $H_1 = H_s$ a hyperplane in the pencil such that every H_t intersects X transversally. (It is due to Transversality Homotopy Theorem, i.e., for any smooth manifold Y, M, the mappings in $C^\infty(Y; M)$ that intersect $N \subset M$ transversally is open and dense in C^1-topology. In this case, the space of all mappings is $C^\infty([0,1]; \{\text{all hyperplanes in } \mathbb{P}^N\})$).

Since critical value of the Lefschetz fibration $\hat{f} : \hat{X} \to S$ is finite, where S denotes the projective line in $\check{\mathbb{P}}^N = \mathbb{P}(1, N)$ dual to A, we have that $\hat{f}$ is a proper and submersion map onto S_{reg}, which is connected, so by Ehresmann Fibration Theorem, every fibers of $\hat{f}$ over points in S_{reg} are diffeomorphic.

Now, by Ehresmann Fibration Theorem again, note that the map

$$\pi : \{(x,t) \in X \times [0,1] \mid x \in X \cap H_t\} \to [0,1], \quad (x,t) \mapsto t$$

is proper and submersive (by transversality), and note that $[0,1]$ is connected, so all fibers of π over $[0,1]$ are diffeomorphic. In particular, $X \cap H = X \cap H_0$ is diffeomorphic to $X \cap H_1 = X \cap H_s$. Therefore, $X \cap H$ is diffeomorphic to a generic divisor $X_\bullet$ of the Lefschetz pencil, or to a generic fiber $\hat{X}_\bullet$ of the associated Lefschetz fibration $\hat{f} : \hat{X} \to S$.

Using the long exact sequence of the pair $(X, X_\bullet)$ we see that it suffices to show that

$$H_q(X, X_\bullet) = 0, \quad \forall q \leq n-1.$$

We analyze the long exact sequence

$$\begin{tikzcd}
\cdots & & \\
H_{q-1}(\hat{X}_+ \cup \hat{B}, \hat{X}_\bullet \cup \hat{B}) \arrow[r] & H_{q-1}(\hat{X}, \hat{X}_\bullet \cup \hat{B}) \arrow[r] & H_{q-1}(\hat{X}, \hat{X}_+ \cup \hat{B}) \arrow[ull] \\
H_q(\hat{X}_+ \cup \hat{B}, \hat{X}_\bullet \cup \hat{B}) \arrow[r] & H_q(\hat{X}, \hat{X}_\bullet \cup \hat{B}) \arrow[r] & H_q(\hat{X}, \hat{X}_+ \cup \hat{B}) \arrow[ull] \\
& & \cdots \arrow[ull]
\end{tikzcd}$$

of the triple $(\hat{X}, \hat{X}_+ \cup \hat{B}, \hat{X}_\bullet \cup \hat{B})$. We have

$$H_q(\hat{X}, \hat{X}_+ \cup \hat{B}) \cong H_q(\hat{X}, \hat{X}_+ \cup B \times D_-) \overset{\text{excis.}}{\cong} H_q(\hat{X}_-, \hat{X}_E \cup B \times D_-).$$

Since we have

$$(\hat{X}_-, \hat{X}_E) \cong X_\bullet \times (D_-, E)$$

by Ehresmann fibration theorem, we get

$$H_q(\hat{X}_-, \hat{X}_E \cup B \times D_-) \cong H_q(X_\bullet \times D_-, X_\bullet \times E \cup B \times D_-) = H_q((X_\bullet, B) \times (D_-, E)),$$

where we define $(A, B) \times (C, D) := (A \times C, A \times D \cup B \times C)$. With this definition, we still have Künneth formula for product spaces. By Künneth formula, we have

$$H_q((\hat{X}_\bullet, B) \times (D_-, E)) \cong H_{q-2}(X_\bullet, B).$$

By excision axiom, we have an isomorphism

$$p_* : H_q(\hat{X}, \hat{X}_\bullet \cup \hat{B}) \cong H_q(X \setminus B, X_\bullet \setminus B) \cong H_q(X, X_\bullet).$$

Excise $B \times \operatorname{int} D_-$ from both of the terms of $(\hat{X}_+ \cup \hat{B}, \hat{X}_\bullet \cup \hat{B})$. The first term becomes

$$\hat{X}_+ \cup \hat{B} \setminus B \times \operatorname{int} D_- = \hat{X}_+;$$

The second term becomes

$$\hat{X}_\bullet \cup \hat{B} \setminus B \times \operatorname{int} D_- = \hat{X}_\bullet \cup (B \times D_+)\,.$$

Since $\hat{X}_\bullet \cap (B \times D_+) = B \times \{\bullet\} \approx B \times D_+$, we have that $\hat{X}_\bullet \cup (B \times D_+) \approx \hat{X}_\bullet$, so we have

$$H_q(\hat{X}_+ \cup \hat{B}, \hat{X}_\bullet \cup \hat{B}) \cong H_q(\hat{X}_+, \hat{X}_\bullet \cup (B \times D_+) \cong H_q(\hat{X}_+, \hat{X}_\bullet).$$

Thus, the long exact sequence of the triple $(\hat{X}, \hat{X}_+ \cup \hat{B}, \hat{X}_\bullet \cup \hat{B})$ becomes

$$\begin{tikzcd}
& \cdots & \\
H_q(\hat{X}_+, \hat{X}_\bullet) \arrow[r] & H_q(X, X_\bullet) \arrow[r] & H_{q-2}(X_\bullet, B) \arrow[ull] \\
H_{q+1}(\hat{X}_+, \hat{X}_\bullet) \arrow[r] & H_{q+1}(X, X_\bullet) \arrow[r] & H_{q-1}(X_\bullet, B) \arrow[ull] \\
& & \cdots \arrow[ull]
\end{tikzcd}$$

Using Lemma 3.3, if $q \neq n, n+1$, we have isomorphisms

$$H_q(X, X_\bullet) \cong H_{q-2}(X_\bullet, B)$$

and an exact sequence

$$\begin{aligned} 0 \longrightarrow H_{n+1}(X, X_\bullet) \longrightarrow H_{n-1}(X_\bullet, B) \longrightarrow H_n(\hat{X}_+, \hat{X}_\bullet) \longrightarrow \\ \longrightarrow H_n(X, X_\bullet) \longrightarrow H_{n-2}(X_\bullet, B) \longrightarrow 0 \end{aligned}\ . \tag{$\dagger$}$$

We now argue by induction on $n = \dim_{\mathbb{C}} X$. For $n = 1$, we have definitely $H_q(X, X_\bullet) = 0$ for all $q \leq 0$, obviously. For the inductive step, note that $X_\bullet$ is an $n-1$ dimensional complex submanifold of an $N-1$ subspace V of $\mathbb{P}^N$, and $B = A \cap X_\bullet$ is a transversal hyperplane section of $X_\bullet$, since $A \subset V$ is a hyperplane, and thus by induction hypothesis,

$$H_q(X_\bullet, B) = 0, \quad \forall q \leq n-2.$$

Thus,

$$H_q(X, X_\bullet) \cong H_{q-2}(X_\bullet, B) = 0 \quad \forall q \leq n-1.$$

□

Corollary 3.8. *If $X \subset \mathbb{P}^n$ is a hypersurface, then we can use Veronese embedding to embed $\mathbb{P}^n$ to $\mathbb{P}^N$ for some large N such that the image of X is a hyperplane. Then by Lefschetz hyperplane theorem, since $\dim_{\mathbb{C}} \mathbb{P}^n = n-1$, we have*

$$b_k(X) \cong b_k(\mathbb{P}^n), \quad \forall k \leq n-2.$$

Remark 3.9. *In the proof of Lefschetz hyperplane theorem, we note that*

$$H_q(X_\bullet, B) = 0, \quad \forall q \leq n-2.$$

Thus, the long exact sequence † becomes

$$0 \longrightarrow H_{n+1}(X, X_\bullet) \longrightarrow H_{n-1}(X_\bullet, B) \longrightarrow H_n(\hat{X}_+, \hat{X}_\bullet) \longrightarrow H_n(X, X_\bullet) \longrightarrow 0\ .$$

We now consider the long exact sequence of the pair $(\hat{X}_+, \hat{X}_\bullet)$

$$0 \longrightarrow H_n(\hat{X}_\bullet) \longrightarrow H_n(\hat{X}_+) \longrightarrow H_n(\hat{X}_+, \hat{X}_\bullet) \xrightarrow{\partial} H_{n-1}(\hat{X}_\bullet) \longrightarrow \cdots,$$

and we define the image of the connecting homomorphism ∂ to be the *module of vanishing circle*

$$\mathbb{V}(X_\bullet) := \partial\left(H_n(\hat{X}_+, \hat{X}_\bullet)\right) = \ker(H_{n-1}(\hat{X}_+) \to H_{n-1}(\hat{X}_+, \hat{X}_\bullet)) \subset H_{n-1}(\hat{X}_\bullet).$$

Using the corollary (the long exact sequence) of Lemma 3.3 and the long exact sequence of the pair $(X, X_\bullet)$, we get the following commutative diagram

$$\begin{tikzcd}
H_n(\hat{X}_+, \hat{X}_\bullet) \arrow[r, "\partial"] \arrow[d, two heads, "p_1"] & H_{n-1}(\hat{X}_\bullet) \arrow[r, "j_*"] \arrow[d, "p_2", "\cong"'] & H_{n-1}(\hat{X}_+) \arrow[r] \arrow[d, "p_3", "\cong"'] & 0 \\
H_n(X, X_\bullet) \arrow[r, "\widetilde{\partial}"] & H_{n-1}(X_\bullet) \arrow[r, "i_*"] & H_{n-1}(X) \arrow[r] & 0
\end{tikzcd}$$

Here, all p_i's are induced by the projection $p : \hat{X} \to X$. The morphism p_1 is onto because it appears in the long exact sequence †. The morphism p_2 must be an isomorphism because p induces a homeomorphism $\hat{X}_\bullet \cong X_\bullet$. Using the refined five lemma, we conclude that p_3 is an isomorphism. Thus,

$$\mathbb{V}(X_\bullet) = \ker j_* \cong \ker i_* = \operatorname{im} \widetilde{\partial},$$

and

$$\operatorname{rank} H_{n-1}(X_\bullet) = \operatorname{rank} \mathbb{V}(X_\bullet) + \operatorname{rank} H_{n-1}(X). \tag{*}$$

By the Dual Universal Coefficient Theorem and Lemma 3.3 (which guarantees that $H_{n-1}(\hat{X}_+, \hat{X}_\bullet)$ is free), we have

$$H^n(\hat{X}_+, \hat{X}_\bullet) \cong \operatorname{Hom}(H_n(\hat{X}_+, \hat{X}_\bullet), \mathbb{Z});$$

by the Dual Universal Coefficient Theorem and Lefschetz Hyperplane Theorem (which guarantees that $H_{n-1}(X, X_\bullet)$ is zero), we have

$$H^n(X, X_\bullet) \cong \operatorname{Hom}(H_n(X, X_\bullet), \mathbb{Z}).$$

Since $\operatorname{Hom}(-, \mathbb{Z})$ is a contravariant functor, we get the following commutative diagram

$$\begin{tikzcd}
H^n(\hat{X}_+, \hat{X}_\bullet) & H^{n-1}(\hat{X}_\bullet) \arrow[l, "\delta"'] & H^{n-1}(\hat{X}_+) \arrow[l] & 0 \arrow[l] \\
H^n(X, X_\bullet) \arrow[u, hook] & H^{n-1}(X_\bullet) \arrow[l, "\widetilde{\delta}"'] \arrow[u, "\cong"] & H^{n-1}(X) \arrow[l, "i^*"'] \arrow[u, "\cong"] & 0 \arrow[l]
\end{tikzcd}$$

We can define

$$\mathbb{I}(X_\bullet)^\vee := \ker \delta \subset H^{n-1}(\hat{X}_\bullet),$$

we will have

$$\ker \delta \cong \ker \widetilde{\delta} = \operatorname{im}\, i^*.$$

We define the *module of invariant cycles* to be the Poincaré dual of $\mathbb{I}(X_\bullet)^\vee$, (note that $\dim_{\mathbb{R}} X_\bullet = 2n-2$),

$$\mathbb{I}(X_\bullet) := \left\{u \cap [X_\bullet] \mid u \in \ker \widetilde{\delta} \cong \mathbb{I}(X_\bullet)^\vee\right\} \subset H_{(2n-2)-(n-1)}(X_\bullet) = H_{n-1}(X_\bullet).$$

Equivalently, recalling the Gysin morphism

$$i^! = \mathscr{P}_{X_\bullet} i^* \mathscr{P}_X^{-1} : H_{n+1}(X) \to H_{n-1}(X_\bullet),$$

we can define

$$\mathbb{I}(X_\bullet) := \operatorname{im} i^!.$$

From now on, we omit the tilde above δ and ∂ for simplicity.

Now, since i^* is injective, we conclude that $i^!$ is injective. Thus,

$$\operatorname{rank} \mathbb{I}(X_\bullet) = \operatorname{rank} H_{n+1}(X) = \operatorname{rank} H_{n-1}(X) = \operatorname{rank} \operatorname{im} i_*, \tag{**}$$

where the second equality is because Poincaré duality, and the third equality is because i_* is surjective. Also, we have

$$\operatorname{rank} H_{n-1}(X_\bullet) = \operatorname{rank} \operatorname{im} i_* + \operatorname{rank} \ker i_*. \tag{***}$$

Combining *, ** and ***, we deduce the following result.

Theorem 3.10 (Weak Lefschetz Theorem)**.** *For any projective manifold $X \subset \mathbb{P}^N$ with complex dimension n, and for a generic hyperplane H of $\mathbb{P}^N$, we have that the Gysin morphism $i^!$: $H_{n+1}(X) \to H_{n-1}(X \cap H)$ is injective, and that*

$$\operatorname{rank} H_{n-1}(X \cap H) = \operatorname{rank} \mathbb{I}(X \cap H) + \operatorname{rank} \mathbb{V}(X \cap H),$$

where

$$\mathbb{V}(X \cap H) = \ker(H_{n-1}(X \cap H) \to H_{n-1}(X)), \quad \mathbb{I}(X \cap H) = \operatorname{im} i^!.$$

Recall that

$$\begin{aligned}
\mathbb{I}(X_\bullet)^\vee = \ker \delta &= \{\omega \in H^{n-1}(\hat{X}_\bullet) \mid \omega\partial(u) = 0,\ \forall u \in H_n(\hat{X}_+, \hat{X}_\bullet)\} \\
&= \{\omega \in H^{n-1}(\hat{X}_\bullet) \mid \omega(v) = 0,\ \forall v \in \mathbb{V}(X_\bullet)\} \\
&= \{\omega \in H^{n-1}(\hat{X}_\bullet) \mid \langle \omega, v\rangle = 0,\ \forall v \in \mathbb{V}(X_\bullet)\}.
\end{aligned}$$

We now recall the definition of intersection product. For any $2n$-dimensional closed oriented manifold M, we have the Poincaré duality

$$\mathscr{P}_M : H^k(M) \cong H_{2n-k}(M), \omega \mapsto \omega \cap [M].$$

For $\alpha \in H_k(M)$ and $\beta \in H_{2n-k}(M)$, we define the intersection product of them to be

$$\alpha \cdot \beta := \langle \mathscr{P}_M^{-1}(\beta), \alpha\rangle.$$

This number is an integer, which is a counting of intersection of cycles α and β. If $k = n$, we say that the intersection product is the intersection form. Using this, we have

$$\begin{aligned}
\mathbb{I}(X_\bullet) &= \{\omega \cap [X_\bullet] \mid \omega \in H^{n-1}(X_\bullet) \text{ such that } \langle \omega, v\rangle = 0,\ \forall v \in \mathbb{V}(X_\bullet)\} \\
&= \{y \in H_{n-1}(X_\bullet) \mid y \cdot v = 0,\ \forall v \in \mathbb{V}(X_\bullet)\}.
\end{aligned}$$

Thus, we conclude

Proposition 3.11. *A middle dimensional cycle on $X_\bullet$ is invariant if and only if its intersection number with any vanishing cycle is zero.*

4 The Hard Lefschetz Theorem

In this section, unless specified otherwise, $H_*(X)$ denotes the homology with coefficients in $\mathbb{R}$. For every projective manifold $X \hookrightarrow \mathbb{P}^N$, we denote by X' its intersection with a generic hyperplane. Denote by $\omega \in H^2(X)$ the Poincaré dual of the hyperplane section X', i.e.,

$$\mathscr{P}_X(\omega) = \omega \cap [X] = [X'] \in H_{2n-2}(X).$$

If a cycle $c \in H_q(X)$ is represented by a smooth (real) oriented submanifold of dimension q, then its intersection with a generic hyperplane H is a $(q-2)$-cycle in $X \cap H = X'$. This intuitive operation $c \mapsto c \cap H$ is nothing but the Gysin morphism

$$i^! = \mathscr{P}_{X'} i^* \mathscr{P}_X^{-1} : H_q(X) \to H^{2n-q}(X) \xrightarrow{i^*} H^{2n-q}(X') \to H_{q-2}(X').$$

Note that, if we let $c \in H_q(X)$ and let $c = u \cap [X]$ be the Poincaré dual of $u \in H^{2n-q}(X)$, we have

$$i_* i^!(c) = i_* \mathscr{P}_{X'} i^*(u) = i_*(i^* u \cap [X']) = u \cap i_*[X'] = u \cap [X'],$$

and

$$\omega \cap c = \omega \cap (u \cap [X]) = (\omega \cup u) \cap [X] = (-1)^{2(2n-q)} u \cap (\omega \cap [X]) = u \cap [X'],$$

so we have a commutative diagram

$$\begin{tikzcd}
H_q(X) \arrow[r, "i^!"] \arrow[dr, "\omega\cap"'] & H_{q-2}(X') \arrow[d, "i_*"] \\
& H_{q-2}(X)
\end{tikzcd}$$

Proposition 4.1. *The following statements are equivalent.*

HL$_1$. $\mathbb{V}(X') \cap \mathbb{I}(X') = 0$.

HL$_2$. $\mathbb{V}(X') \oplus \mathbb{I}(X') = H_{n-1}(X')$.

HL$_3$. *The restriction of* $i_* : H_{n-1}(X') \to H_{n-1}(X)$ *to* $\mathbb{I}(X')$ *is an isomorphism.*

HL$_4$. *The map* $\omega\cap : H_{n+1}(X) \to H_{n-1}(X)$ *is an isomorphism.*

HL$_5$. *The restriction of intersection form* $H_{n-1}(X') \times H_{n-1}(X') \to \mathbb{Z}$ *to* $\mathbb{V}(X')$ *stays nondegenerate, i.e., for any nonzero* $\alpha \in H_{n-1}(X')$, *there must be some* β *such that* $\alpha \cdot \beta \neq 0$.

HL$_6$. *The restriction of intersection form* $H_{n-1}(X') \times H_{n-1}(X') \to \mathbb{Z}$ *to* $\mathbb{I}(X')$ *stays nondegenerate.*

Proof.

- **HL**$_1$ $\Leftrightarrow$ **HL**$_2$. By the weak Lefschetz theorem, we have

  $$\operatorname{rank} H_{n-1}(X') = \operatorname{rank} \mathbb{I}(X') + \operatorname{rank} \mathbb{V}(X'),$$

  so they are equivalent.
- **HL**$_2$ $\Rightarrow$ **HL**$_3$. Recall that

  $$\mathbb{V}(X') = \ker i_*,$$

  so by **HL**$_2$, we have that $i_*|_{\mathbb{I}(X')}$ is an isomorphism onto its image. On the other hand, the Lefschetz Hyperplane Theorem shows that the image of i_* is the whole $H_{n-1}(X)$.
- **HL**$_3$ $\Rightarrow$ **HL**$_4$. By the Weak Lefschetz Theorem, the Gysin morphism $i^! : H_{n+1} \to H_{n-1}(X')$ is injective with image $\mathbb{I}(X')$, and by **HL**$_3$, we have that $\omega\cap = i_* \circ i^!$ is an isomorphism.

- $\mathbf{HL}_4 \Rightarrow \mathbf{HL}_3$. If $i_* \circ i^!$ is an isomorphism, then we have that

$$i_* : \operatorname{im} i^! = \mathbb{I}(X') \to H_{n-1}(X)$$

is onto. Using **, we have

$$\dim \mathbb{I}(X') = \dim \operatorname{im} i_* = \dim H_{n-1}(X),$$

so i_* is injective.
- $\mathbf{HL}_2 \Rightarrow \mathbf{HL}_{5,6}$. This follows from Proposition 3.11. Combining with $\mathbf{HL}_2$, we conclude that $\mathbb{I}(X')$ is the orthogonal complement of $\mathbb{I}(X')$, with respect to the intersection form.
- $\mathbf{HL}_1 \Leftarrow \mathbf{HL}_{5,6}$. Suppose that we have a cycle

$$c \in \mathbb{I}(X') \cap \mathbb{V}(X'),$$

then for all $v \in \mathbb{V}(X')$, $z \in \mathbb{I}(X')$, we get

$$c \cdot v = c \cdot z = 0.$$

Therefore, $c = 0$.

□

Theorem 4.2 (Hard Lefschetz Theorem)**.** *The six equivalent statements above are true (for the homology with real coefficients).*

Its complete proof requires sophisticated analytical machinery (Hodge theory) and is beyond the scope of this note. For the sake of logical coherence, we placed corollaries of Hard Lefschetz theorem at the end of the note.

5 Vanishing Cycles and Local Monodromy

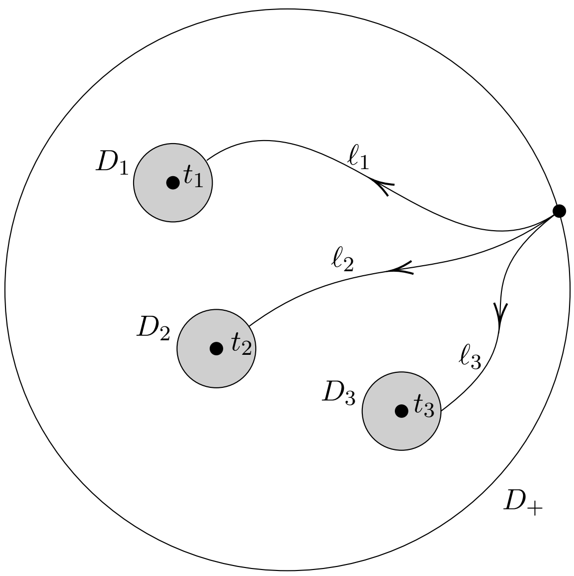
In this section, we finally give the promised proof of Lemma 3.3. Recall this lemma says that

$$H_q(\hat{X}_+, \hat{X}_\bullet) = \begin{cases} 0 & \text{if } q \neq n = \dim_{\mathbb{C}} X, \\ \mathbb{Z}^r & \text{if } q = n, \end{cases}$$

where r is the number of critical points of $\hat{f}$.

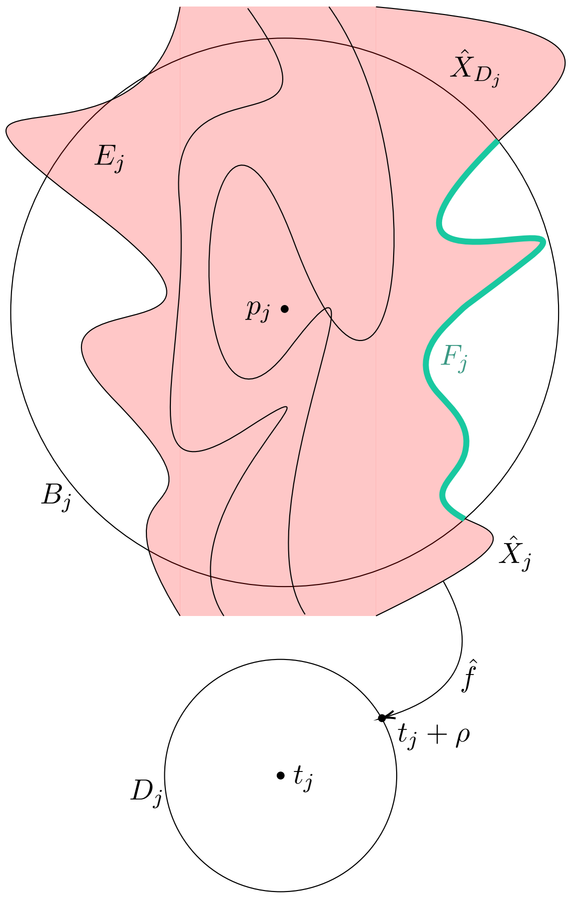
Recall we are given a Morse function $\hat{f} \colon \hat{X} \to \mathbb{P}^1$ and its critical values $t_1, \dots, t_r$ are all located in the upper closed hemisphere D_+. We denote the corresponding critical points by $p_1, \dots, p_r$, so that $\hat{f}(p_j) = t_j$ for all j.

We will identify D_+ with the unit closed disk at $0 \in \mathbb{C}$. Let $j = 1, \dots, r$.



Compare the figure above, we are doing the following to D_+:

1. Denote by D_j a closed disk of very small radius $\rho \in \mathbb{R}_+$ centered at $t_j \in D_+$ such that they are disjoint.
2. Connect $\bullet \in \partial D_+$ to $t_j + \rho \in \partial D_j$ by a smooth path ℓ_j such that the resulting paths $\ell_1, \dots, \ell_r$ are disjoint. Set $k_j := \ell_j \cup D_j$, let $\ell = \bigcup \ell_j$, and define $k = \bigcup k_j$.
3. Denote by B_j a small closed ball of radius R in $\hat{X}$ centered at p_j.



Compare the figure above, we are doing the following to $\hat{X}$ locally at p_j:

$$\hat{X}_{D_j} := \hat{f}^{-1}(D_j), \quad \hat{X}_j := \hat{f}^{-1}(t_j + \rho),$$
$$E_j := \hat{X}_{D_j} \cap B_j, \quad F_j := \hat{X}_j \cap B_j.$$

Proof of Lemma 3.3. The idea is:

1. Let $L = \hat{f}^{-1}(\ell)$ and $K = \hat{f}^{-1}(k)$. We will show that $\hat{X}_\bullet$ is a deformation retract of L, and K is a deformation retract of $\hat{X}_+$. Therefore, we get isomorphisms

$$H_*(\hat{X}_+, \hat{X}_\bullet) \cong H_*(\hat{X}_+, L) \cong H_*(K, L).$$

2. Use excision to get

$$\bigoplus_{1 \le j \le r} H_*(\hat{X}_{D_j}, \hat{X}_j) \cong H_*(K, L) \cong H_*(\hat{X}_+, \hat{X}_\bullet),$$

and prove that there is an isomorphism $H_*(\hat{X}_{D_j}, \hat{X}_j) \cong H_*(E_j, F_j)$.

3. Show that
$$H_q(E_j, F_j) \cong \begin{cases} 0 & \text{if } q \neq \dim_{\mathbb{C}} X = n, \\ \mathbb{Z} & \text{if } q = n. \end{cases}$$

The detail is as follows.

Observe that k is a strong deformation retract of D_+ and $\bullet$ deformation retract of ℓ. Using the Ehresmann fibration theorem , we deduce that we have fibrations
$$\hat{f} : L \to \ell, \quad \hat{f} : \hat{X}_+ \setminus \{p_1, \cdots, p_r\} \to D_+ \setminus \{t_1, \cdots, t_r\}.$$
Using the homotopy lifting property of fibrations, we obtain strong deformation retractions
$$L \to \hat{f}^{-1}(\bullet) = \hat{X}_\bullet, \quad \hat{X}_+ \setminus \{p_1, \cdots, p_r\} \to \hat{f}^{-1}\left(k \setminus \{t_1, \cdots, t_r\}\right) = K \setminus \{p_1, \cdots, p_r\}.$$
Therefore, we see that $\hat{X}_\bullet$ is a deformation retract of L, and K is a deformation retract of $\hat{X}_+$.

Excising $\bigcup \hat{f}^{-1}(\text{int } \ell_j)$ from the pair (K, L) gives an isomorphism
$$\bigoplus_{1 \leq j \leq r} H_*(\hat{X}_{D_j}, \hat{X}_j) \to H_*(K, L) \cong H_*(\hat{X}_+, \hat{X}_\bullet).$$
Define
$$Y_j := \hat{X}_{D_j} \setminus \operatorname{int} B_j, \quad Z_j := F_j \setminus \operatorname{int} B_j.$$
The map $\hat{f}$ induces a surjective submersion $Y_j \to D_j$, and by the Ehresmann fibration theorem, since D_j is contractible, we have
$$Y_j \cong D_j \times Z_j.$$
Thus, Z_j is a deformation retract of Y_j, and thus $\hat{X}_j = F_j \cup Z_j$ is a deformation retract of $F_j \cup Y_j$. Therefore, we get an isomorphism
$$H_*(\hat{X}_{D_j}, \hat{X}_j) \cong H_*(\hat{X}_{D_j}, F_j \cup Y_j) \overset{\text{excis. } Y_j}{\cong} H_*(E_j, F_j).$$

Now, it remains to show that
$$H_q(E_j, F_j) \cong \begin{cases} 0 & \text{if } q \neq \dim_{\mathbb{C}} X = n, \\ \mathbb{Z} & \text{if } q = n. \end{cases}$$
At this point we need to use the nondegeneracy of p_j. To simplify the presentation, in the sequel we will drop the subscript j.

Shrinking B if necessary, we can assume that there exist holomorphic coordinates (z_k) on B and centered at p, and u near and centered at $t = \hat{f}(p)$, such that
$$u = \sum z_i^2.$$
Since what we discuss above remains true if B is an arbitrary closed neighborhood of p, we can assume that now B is still a closed ball of radius R in $\hat{X}$ centered at p w.r.t the coordinate (z_i). Then
$$E = \left\{ \mathbf{z} = (z_i) : \sum |z_i|^2 \leq R^2, \left| \sum z_i^2 \right| \leq \rho \right\}, \quad F = F_\rho := \left\{ \mathbf{z} \in E : \sum z_i^2 = \rho \right\}.$$
Note that if $\mathbf{z} \in E$, we must have that $t\mathbf{z} \in E$ for all $t \in [0, 1]$, so E is contractible. Using the long exact sequence of the pair (E, F), we conclude that for $q \neq 0$, $H_q(E, F) \cong H_{q-1}(F)$ and $H_0(E, F) = 0$.

Set
$$z_j := x_j + iy_j, \quad \mathbf{x} := (x_1, \cdots, x_n), \quad \mathbf{y} := (y_1, \cdots, y_n).$$

Then the fiber F is

$$\left\{\mathbf{z} = \mathbf{x} + i\mathbf{y} : |\mathbf{x}|^2 - |\mathbf{y}|^2 = \rho,\ \mathbf{x}\cdot\mathbf{y} = 0, |\mathbf{x}|^2 + |\mathbf{y}|^2 \le R^2\ \right\}.$$

In particular, in F, we have

$$2|\mathbf{y}|^2 \le R^2 - \rho.$$

Now let

$$\mathbf{u} := \frac{\mathbf{x}}{\sqrt{\rho + |\mathbf{y}|^2}}, \quad \mathbf{v} := \frac{2\mathbf{y}}{R^2 - \rho}.$$

Thus F has description

$$|\mathbf{u}|^2 = 1,\ \mathbf{u}\cdot\mathbf{v} = 0,\ |\mathbf{v}|^2 \le 1.$$

This means

$$\{(\mathbf{u}, \mathbf{v})\} = \{(p, X) \in TS^{n-1} : |X| \le 1\},$$

which is homotopy equivalent to the sphere S^{n-1}, so

$$H_{q-1}(F) \cong \begin{cases} \mathbb{Z} & \text{if } q = 1, n, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$H_q(E, F) \cong \begin{cases} 0 & \text{if } q \ne \dim_{\mathbb{C}} X = n, \\ \mathbb{Z} & \text{if } q = n. \end{cases}$$

□

The proof gives a property of F.

Proposition 5.1. *$F = F_\rho$ is diffeomorphic to $\{(p, X) \in TS^{n-1} : |X| \le 1\}$, the disk bundle of the tangent bundle TS^{n-1}.*

We want to analyze in greater detail the picture of this proposition.

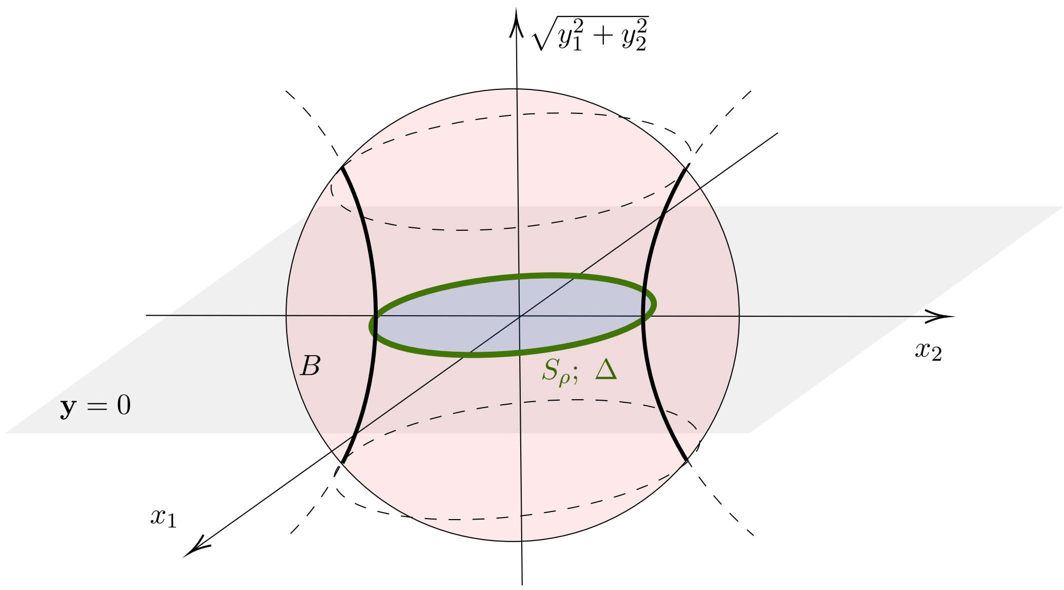
Let B be a small closed ball centered at $0 \in \mathbb{C}^n$. Consider the function $f : B \to \mathbb{C}$ defined by

$$f(z) = z_1^2 + \cdots + z_n^2.$$

We have observed that the regular fiber $F_\rho = f^{-1}(\rho)$ $(0 < \rho \ll 1)$ is diffeomorphic to a disk bundle over an $(n-1)$-sphere S. Here we let this sphere $S_\rho := S$ be with radius $\sqrt{\rho}$. This sphere is then defined as

$$S_\rho := \{\operatorname{Im}\mathbf{z} = 0\} \cap f^{-1}(\rho) = \{\mathbf{y} = 0,\ |\mathbf{x}|^2 = \rho.\}$$

As $\rho \to 0$ (i.e., when approaching the singular fiber $F_0 = f^{-1}(0)$), the radius of this sphere tends to zero. When $\rho = 0$, the fiber becomes the cone $z_1^2 + \cdots + z_n^2 = 0$. We refer to S_ρ as a *vanishing sphere*.



Example: $z_1^2 + z_2^2 = \rho$

The homology class in F_ρ determined by an orientation on this vanishing sphere generates $H_{n-1}(F) \cong \mathbb{Z}$ since this homology class is nontrivial. Such a homology class was called *vanishing cycle* by Lefschetz. We will denote by Δ a homology class obtained in this fashion, i.e., from a vanishing sphere and an orientation on it.

The inclusion $F_\rho \hookrightarrow \hat{X}_j \cong \hat{X}_\bullet$ induces an injective homomorphism

$$H_{n-1}(F_\rho) \to H_{n-1}(\hat{X}_\bullet).$$

By the proof of Lemma 3.3, we know that we have an injection

$$H_{n-1}(F_\rho) \xrightarrow[\cong]{\partial^{-1}} H_n(E, F_\rho) \cong H_n(\hat{X}_{D_j}, \hat{X}_j) \hookrightarrow H_n(\hat{X}_+, \hat{X}_\bullet).$$

We have the following commutative diagram:

$$\begin{tikzcd}
H_n(\hat{X}_+, \hat{X}_\bullet) \arrow[r, "\partial"] & H_{n-1}(\hat{X}_\bullet) \\
H_n(\hat{X}_{D_j}, \hat{X}_j) \arrow[u, hook] & \\
H_n(E, F_\rho) \arrow[u] \arrow[r, "\partial", "\cong"'] & H_{n-1}(F_\rho) = \langle \Delta \rangle \arrow[uu, hook]
\end{tikzcd}$$

Thus, Lefschetz's vanishing cycles coincide with what we previously named vanishing cycles.

Observe now that since $\partial : H_n(B, F) \to H_{n-1}(F)$ is an isomorphism, there exists a relative n-cycle $Z \in H_n(B, F)$ such that $\partial Z = \delta$. The relative cycle Z is known as the *thimble* associated with the vanishing cycle Δ. It is filled in by the family $\{S_\rho\}$ of shrinking spheres. It is represented by the blue disk in the figure above.

Denote by $D_\rho \subset \mathbb{C}$ the closed disk of radius ρ centered at the origin and by $B_R \subset \mathbb{C}^n$ the closed ball of radius R centered at the origin. Set

$$E_{R,\rho} := \{\mathbf{z} \in B_R \mid f(\mathbf{z}) \in D_\rho\}, \quad E^*_{R,\rho} := \{\mathbf{z} \in B_R \mid 0 < |f(\mathbf{z})| < \rho\},$$
$$\partial E_{R,\rho} := \{\mathbf{z} \in \partial B_R \mid f(\mathbf{z}) \in D_\rho\}.$$

We introduce the following technique result.

Lemma 5.2. *For any $\rho, R > 0$ such that $R^2 > \rho$, the maps*

$$f : E^*_{R,\rho} \to D_\rho \setminus \{0\} =: D^*_\rho, \quad f_\partial : \partial E_{R,\rho} \to D_\rho$$

are proper surjective submersions.

Proof. For each $w \in D_\rho \setminus \{0\}$, suppose $0 < |w| = s \le \rho$, and if we take $\mathbf{z} = (\sqrt{w}, 0, \cdots, 0)$, we will get

$$f(\mathbf{z}) = w, \quad 0 < |\mathbf{z}|^2 = s \le \rho < R^2 \Rightarrow \mathbf{z} \in E^*_{R,\rho}.$$

Thus, f is surjective.

To show f_∂ is surjective, take $w \in D_\rho$ and suppose $w = se^{i\theta}$ with $0 < s \le \rho$. We want to find $\mathbf{z}$ such that

$$\sum z_i^2 = w, \quad \sum |z_i|^2 = R^2.$$

Consider

$$\mathbf{z}(\phi) = \left(\frac{R}{\sqrt{2}}e^{i(\theta-\phi)/2}, \frac{R}{\sqrt{2}}e^{i(\theta+\phi)/2}, 0, \cdots, 0\right),$$

then $f(\mathbf{z}(\phi)) = kw$ for some real number k. Note that $|f(\mathbf{z}(\phi))|$ ranges from 0 to $R^2 > \rho > s$, so we can choose some ϕ such that $f(\mathbf{z}(\phi)) = w$.

The properness is easy, since both f and f_∂ are continuous, and B_R is compact.

Now we show the submersiveness. For $\mathbf{z} \in E_{R,\rho}$, we have

$$df_{\mathbf{z}}(\mathbf{v}) = 2z_i v_i.$$

Here we use Einstein's summation convention. For each $u \in T_{f(\mathbf{z})}\mathbb{C} = \mathbb{C}$, take $\mathbf{v} = u\bar{\mathbf{z}}/2|\mathbf{z}|$, then we have $df_{\mathbf{z}}(\mathbf{v}) = u$. Thus, f is a submersion. The tangent space of ∂B_R is described by $z_i\bar{v}_i + \bar{z}_i v_i = 0$. If $(df_\partial)_{\mathbf{z}}(\mathbf{v}) = 2z_i v_i = 0$ for all $\mathbf{v} \in T_{\mathbf{z}}\partial B_R$. Pick $\mathbf{v} = i\mathbf{z}$, then $\mathbf{v} \in T_{\mathbf{z}}\partial B_R$, but then $(df_\partial)_{\mathbf{z}}(\mathbf{v}) = iz_jz_j = 0$ means that $f(\mathbf{z}) = 0$, which violates the surjectiveness of f_∂. Therefore, f_∂ is submersive. □

By rescaling, we can assume $1 < \rho < 2 = R$. Set $B = B_R$, $D = D_\rho$, etc. According to the Ehresmann fibration theorem, we have two locally trivial fibrations:

- $F \hookrightarrow E^* \twoheadrightarrow D^*$ with standard fiber, the manifold with boundary, $F \cong f^{-1}(1) \cap B$;
- $\partial F \hookrightarrow \partial E \twoheadrightarrow D$ with standard fiber $\partial F \cong f^{-1}(1) \cap \partial B$. The bundle $\partial E \to D$ is a globally trivializable bundle because its base is contractible.

We can illustrate these things intuitively. Check the figure in the next page. The region bounded by the ball and lying between (but not including) the blue surface and the yellow surface is our E^*. The curves in the pattern of the blue and the red ones are ∂F's. The surfaces bounded by the ball and being in the pattern of the blue and the yellow ones are fiber F's. The boundary of disk D is the intersection of F with the x_1x_2-plane.

Here we can describe F as

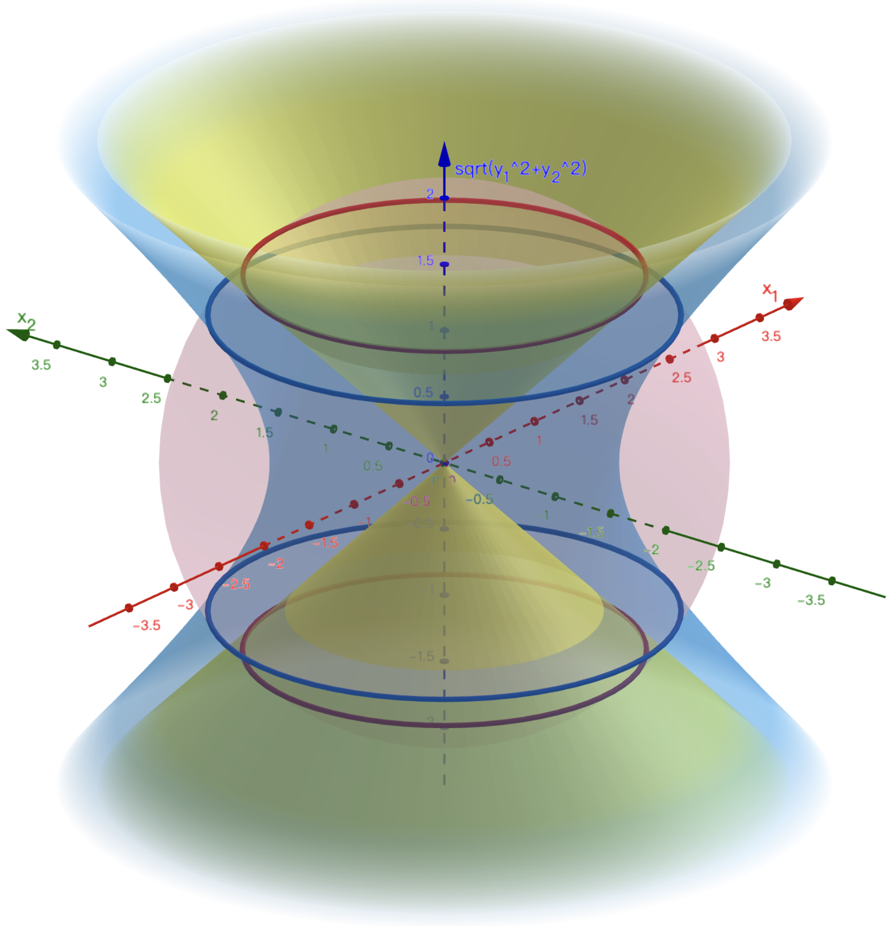
$$F = \{\mathbf{z} = \mathbf{x} + i\mathbf{y} : |\mathbf{x}|^2 + |\mathbf{y}|^2 \le 4, |\mathbf{x}|^2 = 1 + |\mathbf{y}|^2, \mathbf{x} \cdot \mathbf{y} = 0\}.$$

Denote by $\mathbb{M}$ the standard model for the fiber, incarnated as the unit disk bundle determined by the tangent bundle of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. It has description

$$\mathbb{M} = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^n \times \mathbb{R}^n : |\mathbf{u}| = 1, \mathbf{u} \cdot \mathbf{v} = 0, |\mathbf{v}| \le 1\}.$$

Its boundary is

$$\partial\mathbb{M} = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^n \times \mathbb{R}^n : |\mathbf{u}| = |\mathbf{v}| = 1, \mathbf{u} \cdot \mathbf{v} = 0\}.$$



We have a diffeomorphism between F and $\mathbb{M}$, as in the proof of Proposition 5.1,

$$\Phi : F \to \mathbb{M}, \quad \mathbf{z} = \mathbf{x} + i\mathbf{y} \mapsto (\mathbf{u}, \mathbf{v}) = \left(\frac{\mathbf{x}}{\sqrt{1 + |\mathbf{y}|^2}}, \sqrt{\frac{2}{3}}\mathbf{y} \right),$$

$$\Phi^{-1} : \mathbb{M} \to F, \quad (\mathbf{u}, \mathbf{v}) \mapsto \mathbf{z} = \mathbf{x} + i\mathbf{y} = \sqrt{1 + \frac{|\mathbf{v}|^2}{2/3}}\mathbf{u} + i\sqrt{\frac{3}{2}}\mathbf{v}.$$

This diffeomorphism Φ maps the vanishing sphere $\Sigma := \{\operatorname{Im} \mathbf{z} = 0\} \cap F$ to the sphere

$$\mathbb{S} = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^n \times \mathbb{R}^n : |\mathbf{u}| = 1, \mathbf{v} = 0\}.$$

We will say that $\mathbb{S}$ is the standard model for the vanishing cycle. The standard model for the thimble is the closed ball bounded by $\mathbb{S}$.

Fix a trivialization $\partial E \cong \partial F \times D$, and a Riemannian metric h compatible with the almost complex structure on tangent spaces on ∂F. Equip ∂E with the metric $g_\partial = h \oplus h_0$, where h_0 is the Euclidean metric on D. By the tubular neighborhood theorem, there is a neighborhood U of ∂E in E diffeomorphic to $\partial E \times [0, \varepsilon)$, and we define a metric g_U on U by $g_U = g_\partial \oplus dt^2$, that is, for $v = u + k \cdot \partial/\partial t$,

$$\|v\|_{g_U}^2 = \|u\|_{g_\partial}^2 + k^2.$$

Now picking a partition of unity $\{\rho\}$ of open covering $\{U, E \setminus \partial E\}$ of E, we define

$$g = \rho_U g_U + \rho_{E \setminus \partial E} g^\circ,$$

where g° is any metric of the interior of E. Thus, we extend the metric g_∂ on ∂E to the whole E. Define H to be the normal bundle of the fibers of f using the metric, i.e., for each $z \in E^*$, we define

$$H_z = \{v \in T_zE^* : \, g_z(v, w) = 0, \, \forall w \in \ker(df_z) = T_zF\}.$$

Note that

$$T_zE^* \cong H_z \oplus T_zF, \quad T_zE^* \cong T_zF \oplus T_{f(z)}D^*,$$

so the differential df_z induces an isomorphism

$$H_z \cong T_{f(z)}D^*, \, \forall z \in E^*.$$

Suppose $\gamma : [0, 1] \to D^*$ is a smooth path beginning and ending at arbitrary point $\bullet$. We obtain for each $p \in F = f^{-1}(\bullet) \cap B$ a smooth path $\widetilde{\gamma}_p$ that is tangent to the horizontal subbundle H , and it is a lift of γ starting at p, i.e., $f \circ \widetilde{\gamma}_p = \gamma$. We can hence define a map

$$h_\gamma : F = f^{-1}(\bullet) \to F, \, p \mapsto \widetilde{\gamma}_p(1).$$

Lemma 5.3. $h_\gamma \in \mathrm{Diff}(F)$ *satisfies*

$$h_\gamma|_{\partial F} = \mathrm{id}.$$

Proof. In local coordinate, $\widetilde{\gamma}_p$ is the solution of the following equation

$$\left.\frac{d\widetilde{\gamma}_p}{dt}\right|_t \in H_{\widetilde{\gamma}_p(t)}, \; f \circ \widetilde{\gamma}_p = \gamma, \; \widetilde{\gamma}_p(0) = p.$$

Since we have the isomorphism

$$df_z : H_z \cong T_{f(z)}D^*,$$

the equation is then an ODE

$$\left.\frac{d\widetilde{\gamma}_p}{dt}\right|_t = (df_z)^{-1}\left.\frac{d\gamma}{dt}\right|_t, \; \widetilde{\gamma}_p(0) = p.$$

The standard results on the smooth dependence of solutions of ODEs on initial data show that h_γ is a smooth map. Noting that h_γ has a smooth inverse $h_{-\gamma}$, we proved that $h_\gamma : F \to F$ is a diffeomorphism. In $\partial E \cong \partial F \times D$ (where we regard them as the same thing in the sequel), the subbundle H is the subbundle of $T\partial F \oplus TD$ that is $g = g_\partial$-orthogonal to fibers of f. We have for each $(x, c) \in \partial E$, $f(x, c) = c$. Thus, $f^{-1}(c) \cap \partial E = \partial F \times \{c\}$, so it has tangent space $T_x\partial F \oplus 0$ at (x, c). By the definition of H, we have that

$$H_{(x,c)} = \{(v, w) \in T_x\partial F \oplus T_cD : \, h(v, u) + h_0(w, 0) = 0, \, \forall u \in T_x\partial F\} = 0 \oplus T_cD.$$

Therefore, the lift $\widetilde{\gamma}_p$ of γ must be in ∂E if $p \in \partial E$, so $\widetilde{\gamma}_p(t) = (p, \gamma(t))$. In conclusion, $\widetilde{\gamma}_p(1) = (p, \bullet) = \widetilde{\gamma}_p(0) = p$, that is,

$$h_\gamma|_{\partial F} = \mathrm{id}.$$

□

The map h_γ is not canonical, because it depends on several choices: the choice of trivialization $\partial E \cong \partial F \times D$, the choice of metric h on ∂F , and the choice of the extension g of g_∂.

Definition 5.1. *Two diffeormorphisms $G_0, G_1 \in \mathrm{Diff}(F)$ such that $G_i|_{\partial F} = \mathrm{id}$ are called isotopic if there is a smooth homotopy $G : [0, 1] \times F \to F$ connecting them such that for each $t \in [0, 1]$, $G_t(-) := G(t, -)$ is a diffeomorphism satisfying $G_t|_{\partial F} = \mathrm{id}$.*

Lemma 5.4. *The isotopy class of h_γ depends only on the image of γ in $\pi_1(D^*, \bullet)$.*

a less important proof. We need to prove that the isotopy class of h_γ is independent of the various choices listed above.

First, we consider two different trivializations $\Phi_0, \Phi_1 : \partial E \to \partial F \times D$. Then we get a diffeomorphism $\Phi_0\Phi_1^{-1} : \partial F \times D \to \partial F \times D$, $(p, v) \mapsto (p, \phi_p(v))$ with ϕ_p depending smoothly on p and $\phi_p \in \mathrm{Diff}(D)$. Under the trivialization Φ_1, the diffeomorphism ϕ_p gives a new metric h_1 on D by pulling back, say

$$(h_1)_v(x, y) = (h_0)_{\phi_p(v)}(d\phi_p(x), d\phi_p(y)).$$

It is the metric induced by trivialization Φ_2. We now show that there is a one-parameter family h_t of metric, depending smoothly on the parameter t, connecting h_0 and h_1. Indeed, let

$$h_t = (1 - t)h_0 + th_1,$$

then h_t is a metric by easy verification. Now we define $g_{\partial,t} = h \oplus h_t$, and hence we can define the final g_t on E. It is a one-parameter family of metric depending smoothly on the parameter t, which connects the metrics induces by two trivializations. By the smooth dependence of initial value in ODE, we can thus define a one-parameter family of diffeomorphisms $h_{\gamma,t}$ depending smoothly on the parameter t. By the proof of Lemma 5.3, $h_{\gamma,t}|_{\partial F} = 0$, so the map $G_t := h_{\gamma,t}$ is an isotropy.

Second, we consider two different metric h^0, h^1 on ∂F. Let $h^t = (1 - t)h^0 + th^1$. We have verified that h^t is a metric. Now, suppose J is the almost complex structure on $T_p\partial F$, then

$$h^t(Jx, Jy) = (1 - t)h^0(Jx, Jy) + th^1(Jx, Jy) = (1 - t)h^0(x, y) + th^1(x, y) = h^t(x, y).$$

Thus, $\{h^t\}$ is a one-parameter family of metric, depending smoothly on the parameter t, connecting h^0 and h^1. The remaining process is just the same as the first part.

Third, consider the different choices of the extension g. The idea is the same as above. Consider two metrics g_0 and g_1 on E such that $g_0|_{\partial E} = g_1|_{\partial F}$, then

$$g_t = (1 - t)g_0 + tg_1$$

is a one-parameter family of metric, depending smoothly on the parameter t.

Now we show that isotopy class of h_γ is independent of the choice of representation of $[\gamma] \in \pi_1(D^*, \bullet)$. Choose two paths $\gamma_0, \gamma_1 \in [\gamma]$. Then there is a smooth homotopy $H : [0, 1]^2 \to D^*$, such that $H(i, t) = \gamma_i(t)$ and $H(s, 0) = H(s, 1) = \bullet$. Let $\gamma_s(t) = H(s, t)$. Then the induced family $\{\widetilde{\gamma}_{p,s}\}$ depends smoothly on s and p, so we can define

$$G(s, t) = \widetilde{\gamma}_{p,s}(1).$$

As proved in Lemma 5.3, G is a isotropy. □

Now, the induced map

$$[h_\gamma] : H_*(F) \to H_*(F)$$

is called the *(homological) monodromy* along the loop γ. The corresponding

$$[h] : \pi_1(D, \bullet) \to \mathrm{Aut}(H_*(F)),\ [\gamma] \mapsto [h_\gamma]$$

is a group homomorphism called the *local (homological) monodromy.* Since $h_\gamma|_{\partial F} = \mathrm{id}$, we obtain another homomorphism

$$[h]^{rel} : \pi_1(D, \bullet) \to \mathrm{Aut}(H_*(F, \partial F)),$$

which we will call *local relative monodromy.*

Observe that if z is a singular n-chain in F such that $\partial z \in \partial F$ (hence z defines an element $[z] \in H_n(F, \partial F)$), then for every $\gamma \in \pi_1(D^*, \bullet)$ we have $\partial z = \partial h_\gamma z$, so

$$\partial(z - h_\gamma z) = 0,$$

so that the singular chain $z - h_\gamma z$ is a cycle in F. Thus, we can define

$$\mathbf{var}_\gamma(z) = [h_\gamma]^{rel} z - z \in H_{n-1}(F)$$

for $z \in H_{n-1}(F, \partial F)$, $\gamma \in \pi_1(D^*, \bullet)$. This gives a map

$$\mathbf{var} : \pi_1(D^*, \bullet) \to \mathrm{Hom}(H_{n-1}(F, \partial F), H_{n-1}(F)).$$

It is called the *variation map.*

The local Picard–Lefschetz formula will provide an explicit description of this variation map. To formulate it, we need to make a topological digression.

An orientation $\mathbf{or} = \mathbf{or}_F$ on F defines a nondegenerate intersection pairing

$$*_{\mathbf{or}} : H_{n-1}(F) \times H_{n-1}(F) \to \mathbb{Z}, \quad \text{or} \quad H_{n-1}(F, \partial F) \times H_{n-1}(F) \to \mathbb{Z}$$

formally defined by

$$c_1 *_{\mathbf{or}} c_2 := \langle \mathscr{P}_{\mathbf{or}}^{-1} i_* c_1, c_2 \rangle, \quad \text{or } \langle \mathscr{P}_{\mathbf{or}}^{-1} c_1, c_2 \rangle \text{ resp.},$$

where $i_* : H_{n-1}(F) \to H_{n-1}(F, \partial F)$ is the morphism induced by inclusion $F \hookrightarrow (F, \partial F)$ and $\mathscr{P}_{\mathbf{or}} : H^{n-1}(F) \cong H_{(2n-2)-(n-1)}(F, \partial F)$, $u \mapsto u \cap [F, \partial F]$ is the Poincaré-Lefschetz duality (the generalization of Poincaré duality in the case of manifold with boundary). Here $[F, \partial F]$ is the generator of $H_{2n-2}(F, \partial F) \cong \mathbb{Z}$ given by $\mathbf{or}_F$.

Note that $H^k(F) \cong H^k(S^{n-1})$, and $H_{n-1}(F, \partial F) \cong H^{n-1}(F) \cong \mathbb{Z}$. Since F is the unit disk bundle in the tangent bundle $T\Sigma$, a generator of $H_{n-1}(F, \partial F)$ can be represented by a disk ∇ in this disk bundle. The generator is fixed by a choice of orientation on ∇. Thus $\mathbf{var}_\gamma$ is completely understood once we understand its action on ∇. See the figure in the next page.

The group $H_{n-1}(F) \cong \mathbb{Z}$ also has one generator, but with two choices after $\mathbf{or}_\nabla$ is fixed. The generator can be represented by an embedded $(n-1)$-sphere Σ equipped with one of the two possible orientations. Since $\mathbf{var}_\gamma([\nabla]) \in H_{n-1}(F)$, there must some integer $\nu_\gamma(\nabla)$ such that

$$\mathbf{var}_\gamma([\nabla]) = \nu_\gamma(\nabla)[\Sigma].$$

The integer $\nu_\gamma(\nabla) \in \mathbb{Z}$ is determined by the *Picard-Lefschetz number*

$$m_\gamma(\mathbf{or}_F) := [\nabla] *_{\mathbf{or}_F} \mathbf{var}_\gamma(\nabla).$$

By the definition, we get

$$m_\gamma(\mathbf{or}_F) = [\nabla] *_{\mathbf{or}_F} \mathbf{var}_\gamma(\nabla) = \nu_\gamma(\nabla)[\nabla] *_{\mathbf{or}_F} [\Sigma].$$

Note that $[\nabla] * [\Sigma] = \pm 1$ where the sign can be determined by orientations. Hence,

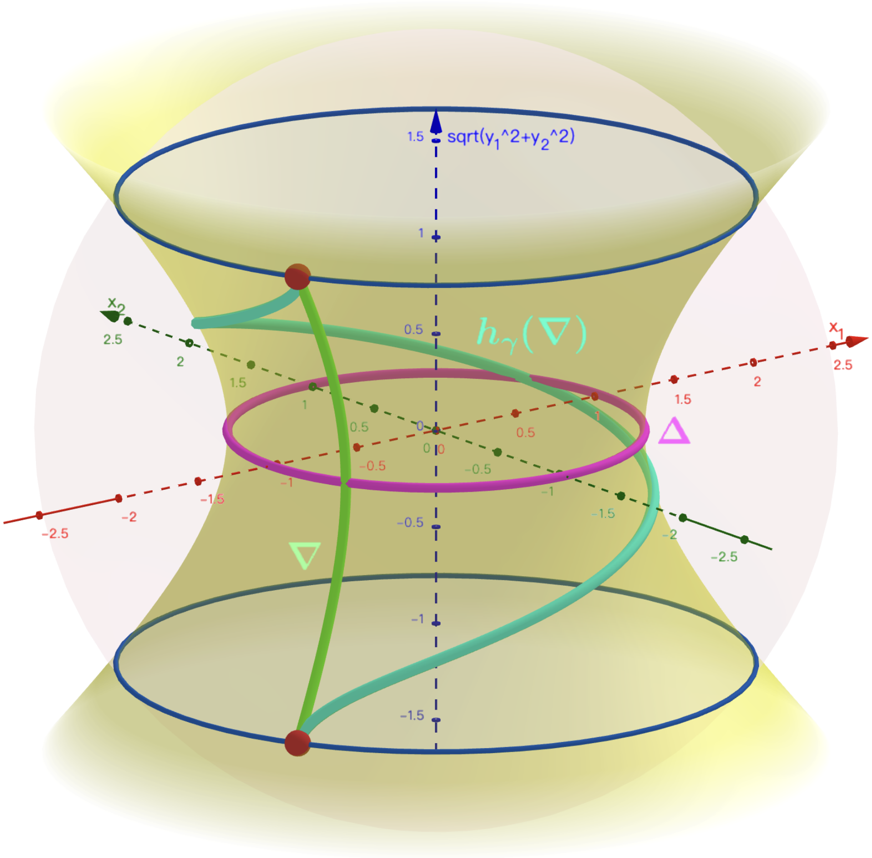
$$\nu_\gamma(\nabla) = m_\gamma(\mathbf{or}_F)[\nabla] *_{\mathbf{or}_F} [\Sigma],$$

so

$$\mathbf{var}_\gamma(\nabla) = m_\gamma(\mathbf{or}_F)([\nabla] *_{\mathbf{or}_F} [\Sigma])[\Sigma] = ([\nabla] *_{\mathbf{or}_F} [\Sigma])([\nabla] *_{\mathbf{or}_F} \mathbf{var}_\gamma(\nabla))[\Sigma].$$

The integer m_γ depends on choices of orientations on $\mathbf{or}_F$, $\mathbf{or}_\nabla$, and $\mathbf{or}_\Sigma$, but ν_γ depends only on the orientations on ∇ and Σ. Let us explain how to fix such orientations.

The diffeomorphism Φ maps the vanishing sphere $\Sigma \subset F$ to $\mathbb{S} = \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^n : \mathbf{v} = 0, |\mathbf{u}| = 1\}$. This is oriented as the boundary of the unit disk via the outer-normal-first



convention (i.e., if $du_1 \wedge \cdots \wedge du_n$ expresses the orientation of the unit disk $\{|\mathbf{u}| \leq 1\}$, and w is a outward-pointing vector field, then $\iota_w(u_1 \wedge \cdots \wedge du_n)$ is the orientation of $\mathbb{S}$ locally.) We denote by $\Delta \in H_{n-1}(F)$ the cycle determined by Σ with this orientation.

Now let

$$\mathbf{u}_\pm = (\pm 1, 0, \cdots, 0), \quad P_\pm = (\mathbf{u}_\pm, \mathbf{0}) \in \mathbb{S} \subset \mathbb{M}.$$

The orientation on $\mathbb{M}$ is determined by the fiber-first convention

$$\mathbf{or}_{\text{total space}} = \mathbf{or}_{\text{fiber}} \wedge \mathbf{or}_{\text{base}}.$$

Observe that since $\mathbb{M}$ is one part the tangent bundle of $\mathbb{S}$, an orientation on $\mathbb{S}$ determines tautologically an orientation in each fiber of $\mathbb{M}$. Thus the orientation on $\mathbb{S}$ as boundary of an Euclidean ball determines via the above formula an orientation on $\mathbb{M}$. We will refer to this orientation as the *bundle orientation*.

Near $P_+ \in \mathbb{M}$ (i.e., fibers with base points near $P_+ \in \mathbb{S}$), we can use as local coordinates the pair

$$(\boldsymbol{\xi}, \boldsymbol{\eta}) = (u_2, \cdots, u_n, v_2, \cdots, v_n).$$

The orientation at P_+ of $\mathbb{S}$ is

$$d\boldsymbol{\xi} := du_2 \wedge \cdots \wedge du_n,$$

Via Φ, we see that the orientation of Σ at $\Phi^{-1}(P_+)$ is $x_2 \wedge \cdots \wedge x_n$. The bundle orientation of $\mathbb{M}$ is described in these coordinates near P_+ by the form

$$d\boldsymbol{\eta} \wedge d\boldsymbol{\xi} = dv_2 \wedge \cdots \wedge dv_n \wedge du_2 \wedge \cdots \wedge du_n,$$

so, via Φ, the orientation of F near $\Phi^{-1}(P_+)$ is

$$\mathbf{or}_{\text{bundle}} := dy_2 \wedge \cdots \wedge dy_n \wedge dx_2 \wedge \cdots \wedge dx_n.$$

Using the identification Φ between F and $\mathbb{M}$ we deduce that we can represent ∇ as the fiber $\mathbb{T}_+$ of $M \to S$ over the north pole P_+ with orientation $d\boldsymbol{\eta}$. Thus, we can denote by $\nabla \in H_{n-1}(F, \partial F)$ the cycle determined by $\Phi^{-1}(\mathbb{T}_+)$ with the above orientation, i.e., $dy_2 \wedge \cdots \wedge dy_n$.

On the other hand, instead of the bundle orientation, F has a natural orientation as a complex manifold. We will refer to it as the *complex orientation*. The collection $(z_2, \cdots, z_n)$ defines holomorphic local coordinates on F near $\Phi^{-1}(P_+)$, so that

$$\mathbf{or}_{\text{complex}} = dx_2 \wedge dy_2 \wedge \cdots \wedge dx_n \wedge dy_n.$$

We see that

$$\mathbf{or}_{\text{complex}} = (-1)^{n(n-1)/2} \mathbf{or}_{\text{bundle}}.$$

We denote by $\circ$ (respectively $*$) the intersection number of

$$H_{n-1}(F, \partial F) \times H_{n-1}(F) \to \mathbb{Z}, \quad \text{or} \quad H_{n-1}(F) \times H_{n-1}(F) \to \mathbb{Z}$$

with respect to the bundle orientation (respectively the complex orientation). Via Φ, we see

$$\nabla \circ \Delta = 1,$$

so

$$\nabla * \Delta = (-1)^{n(n-1)/2} \nabla \circ \Delta = (-1)^{n(n-1)/2}.$$

Also, thanks to Poincaré-Hopf theorem,

$$\Delta \circ \Delta = (-1)^{n(n-1)/2} \Delta * \Delta = e(TS^{n-1})[S^{n-1}] = \chi(S^{n-1}) = \begin{cases} 0 & \text{if } n \text{ is even}, \\ 2 & \text{if } n \text{ is odd}. \end{cases}$$

The loop $\gamma_1 : [0,1] \to \mathbb{D}^*$, $\gamma_1(t) e^{2\pi i t}$ generates the fundamental group of $\mathbb{D}^*$, and thus the variation map is completely understood once we understand the morphism of $\mathbb{Z}$-modules

$$\mathbf{var}_1 = \mathbf{var}_{\gamma_1} : H_{n-1}(F, \partial F) \to H_{n-1}(F).$$

Since $H_{n-1}(F, \partial F) \cong \mathbb{Z}$ is free and by the dual universal coefficient theorem we have $H^{n-1}(F) \cong \operatorname{Hom}(H_{n-1}(F, \partial F), Z)$. Once an orientation or_F on F is chosen, we have a Poincaré duality isomorphism

$$H_{n-1}(F) \cong \operatorname{Hom}\left(H_{n-1}(F, \partial F), \mathbb{Z}\right),$$

and the morphism $\mathbf{var}_1$ is completely determined by the Picard–Lefschetz number

$$m_1(\mathbf{or}_F) := m_{\gamma_1}(\mathbf{or}_F) = \nabla *_{\mathbf{or}_F} \mathbf{var}_1(\nabla).$$

We have the following fundamental result.

Theorem 5.5 (Local Picard–Lefschetz formula)**.**

$$\begin{aligned}
m_1(\mathbf{or}_{\text{bundle}}) &= \nabla \circ \mathbf{var}_1(\nabla) = (-1)^n, \\
m_1(\mathbf{or}_{\text{complex}}) &= \nabla * \mathbf{var}_1(\nabla) = (-1)^{n(n+1)/2}, \\
\mathbf{var}_1(\nabla) &= m_1(\mathbf{or}_F)(\nabla *_{\mathbf{or}_F} \Delta)\Delta \\
&= ((-1)^n \cdot 1)\nabla = ((-1)^{n(n+1)/2} \cdot (-1)^{n(n-1)/2})\Delta \\
&= (-1)^n \Delta, \\
\mathbf{var}_1(z) &= (-1)^n (z \circ \nabla)\nabla = (-1)^{n(n+1)/2}(z * \nabla)\nabla, \ \forall z \in H_{n-1}(F, \partial F).
\end{aligned}$$

6 Proof of the Picard–Lefschetz Formula

For each $w \in D = D_\rho$, set

$$F_w = f^{-1}(w) \cap B.$$

In section 5, we have defined that $F = F_1$. Note that

$$\partial F_{a+bi} = \{\mathbf{x} + i\mathbf{y} \, : \, |\mathbf{x}|^2 - |\mathbf{y}|^2 = a, \, 2\mathbf{x} \cdot \mathbf{y} = b, \, |\mathbf{x}|^2 + |\mathbf{y}|^2 = 4\}.$$

For every $w = a + bi \in D$, we define

$$\Gamma_w : \partial F_w \to \partial \mathbb{M},$$
$$\mathbf{x} + i\mathbf{y} \mapsto \begin{cases} \mathbf{u} = c_1(w)\mathbf{x}, \\ \mathbf{v} = c_3(w)(\mathbf{y} + c_2(w)\mathbf{x}). \end{cases}$$

where

$$c_1(w) = \left(\frac{2}{4+a}\right)^{1/2}, \quad c_2(w) = -\frac{b}{4+a}, \quad c_3(w) = \left(\frac{8+2a}{16-a^2-b^2}\right)^{1/2}.$$

Observe that $\Gamma_1 = \Phi|_{\partial F}$, the diffeomorphism between ∂F and $\partial \mathbb{M}$ constructed in section 5. The family $(\Gamma_w)_{|w|<\rho}$ defines a trivialization

$$\Gamma : \partial E \to \partial \mathbb{M} \times D, \; \mathbf{z} \mapsto (\Gamma_{f(\mathbf{z})}(\mathbf{z}), \, f(\mathbf{z})).$$

The manifold $E_{|w|=1} = f^{-1}(|w| = 1)$ is a smooth compact manifold with boundary

$$\partial E_{|w|=1} = f^{-1}(\{|w| = 1\}) \cap \partial B.$$

The boundary $\partial E_{|w|=1}$ is the restriction to the unit circle $\{|w| = 1\}$ of the trivial fibration $\partial E \to D$. The trivialization Γ above induces a trivialization $\Gamma : \partial E_{|w|=1} \cong \partial \mathbb{M} \times \{|w| = 1\}$.

Fix a vector field V on $E_{|w|=1}$ such that

$$f_*(V) = 2\pi \frac{\partial}{\partial \theta}, \quad \Gamma_*(V|_{\partial E_{|w|=1}}) = 2\pi \frac{\partial}{\partial \theta}.$$

Denote by $\mu_t \in \mathrm{Diff}(E_{|w|=1})$ the flow of V. For each $\mathbf{z} \in F$, we have

$$f(\mathbf{z}) = 1, \quad \frac{d}{dt}\mu_t(\mathbf{z}) = V_{\mu_t(\mathbf{z})},$$

so

$$\frac{d}{dt} f(\mu_t(\mathbf{z})) = 2\pi \frac{\partial}{\partial \theta}.$$

Thus,

$$f(\mu_t(\mathbf{z})) = f(\mathbf{z})e^{2\pi i t} = e^{2\pi i t}.$$

The inverse μ_{-t} maps $F_{e^{2\pi i t}}$ to F, so μ_t defines a diffeomorphism $\mu_t : F \to F_{e^{2\pi i t}}$. For the same reason, this diffeomorphism is compatible with the trivialization Γ, i.e., the following diagram commutes:

$$\begin{tikzcd} \partial F \arrow[r, "\Gamma_1 = \Phi"] \arrow[d, "\mu_t"'] & \partial \mathbb{M} \\ \partial F_{e^{2\pi i t}} \arrow[ur, "\Gamma_{e^{2\pi i t}}"'] & \end{tikzcd}$$

Consider also the flow $\Omega_t(\mathbf{z}) := e^{\pi i t}\mathbf{z}$ on $E_{|w|=1}$. We have

$$f \circ \Omega_t(\mathbf{z}) = e^{2\pi i t} f(\mathbf{z}),$$

so

$$\Omega_t(F) = F_{e^{2\pi it}}.$$

However, since $\Omega_1|_{\partial F} = -1 \neq 1$, this flow is not compatible with the trivialization Γ of ∂E.

We now pick $T_\pm \subset F$ representing ∇. More precisely, we define T_+ so that $\mathbb{T}_+ := \Phi(T_+)$ is the fiber of $\mathbb{M} \to \mathbb{S}$ over the north pole P_+. We have shown in section 5 that $\mathbb{T}_+$ (and T_+, resp.) is oriented by

$$dv_2 \wedge \cdots \wedge dv_n, \quad (dy_2 \wedge \cdots \wedge dy_n \text{ resp.}).$$

Define T_- so that $\mathbb{T}_- := \Phi(T_-)$ is the fiber of $\mathbb{M} \to \mathbb{S}$ over the north pole P_-. The orientation of $\mathbb{S}$ at P_- is given by $-du_2 \wedge \cdots \wedge du_n$, so the orientation of $\mathbb{T}_-$ is given by $-dv_2 \wedge \cdots \wedge dv_n$. Inside F the chain $\nabla = T_-$ is described by

$$\mathbf{x} = (1 + |\mathbf{y}|^2/\alpha^2)^{1/2}\mathbf{u}_- \quad \Leftrightarrow \quad x_1 < 0,\, x_2 = \cdots = x_n = 0,$$

where $\alpha = \sqrt{2/3}$, and it is oriented by $-dy_2 \wedge \cdots \wedge dy_n$.

Note that $\Omega_1 = -1$, so that taking into account the orientations, we have

$$\Omega_1(T_+) = (-1)^n T_- = (-1)^n \nabla.$$

For simplicity, set

$$m = m_1(\mathbf{or}_{\text{bundle}}) = \nabla \circ \mathbf{var}_1(\nabla).$$

Now, observe that for each $\mathbf{z} \in F$,

$$f \circ \mu_t(\mathbf{z}) = e^{2\pi it}$$

and

$$f_* \frac{d\mu_t(\mathbf{z})}{dt} = 2\pi \frac{\partial}{\partial\theta} \in T_{f(\mu_t(\mathbf{z}))} D^*,$$

so

$$\mathbf{var}_1(\nabla) = \mu_1(T_+) - T_+.$$

Therefore, we get

$$m = (-1)^n \Omega_1(T_+) \circ (\mu_1(T_+) - T_+).$$

Note that T_+ and T_- are disjoint, we have

$$\boxed{m = (-1)^n \Omega_1(T_+) \circ \mu_1(T_+).}$$

The compatibility of μ_t and Γ implies that

$$\Gamma_{e^{2\pi it}} \mu_t(\partial T_+) = \Phi(\partial T_+) = \partial \mathbb{T}_+ = \{(\mathbf{u}_+, \mathbf{v}) \in \mathbb{M} :\, |\mathbf{v}| = 1\}.$$

On the other hand,

$$\Gamma_{e^{2\pi it}} \Omega_t(\partial T_+) = \left\{ \Gamma_{e^{2\pi it}} \Omega_t \left(\sqrt{\frac{1+\alpha^2}{\alpha^2}} \mathbf{u}_+, \alpha^{-1}\mathbf{v} \right) : |\mathbf{v}| = 1 \right\}.$$

By the definition of Γ and Ω, we have that

$$\Gamma_{e^{2\pi it}} \Omega_t(\partial T_+) \cap \Gamma_{e^{2\pi it}} \mu_t(\partial T_+) = \Gamma_{e^{2\pi it}} \Omega_t(\partial T_+) \cap \partial \mathbb{T}_+ = \emptyset.$$

Therefore, the deformations

$$\Omega_1(T_+) \to \Omega_{1-s(1-t)}(T_+), \quad \mu_1(T_+) \to \mu_{1-s(1-t)}(T_+)$$

do not change the intersection numbers, so

$$\boxed{\Omega_1(T_+) \circ \mu_1(T_+) = \Omega_t(T_+) \circ \mu_t(T_+),\ \forall t \in (0, 1].}$$

Thus, it remains to show that

$$\Omega_t(T_+) \circ \mu_t(T_+) = 1.$$

We will show it holds for sufficiently small $t > 0$. Set

$$A_t = \Omega_t(T_+), \quad B_t = \mu_t(T_+).$$

For sufficiently small $\varepsilon > 0$, set $C_\varepsilon := \{e^{2\pi it} : t \in [0, \varepsilon]\}$. Since C_ε is contractible, we can extend the trivialization $\Gamma : \partial E|_{C_\varepsilon} \to \partial\mathbb{M} \times C_\varepsilon$ to

$$\widetilde{\Gamma} : E|_{C_\varepsilon} \to \mathbb{M} \times C_\varepsilon,$$

such that $\widetilde{\Gamma}_1 := \widetilde{\Gamma}|_F = \Phi$.

For $t \in [0, \varepsilon]$, we use the following commutative diagram to define $\omega_t, h_t \in \mathrm{Diff}(\mathbb{M})$:

$$\begin{tikzcd} F \arrow[r, "\widetilde{\Gamma}_1=\Phi"] \arrow[d, "\Omega_t"'] & \mathbb{M} \arrow[d, "\omega_t"] \\ F_{e^{2\pi it}} \arrow[r, "\widetilde{\Gamma}_{e^{2\pi it}}"'] & \mathbb{M} \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "\widetilde{\Gamma}_1=\Phi"] \arrow[d, "\mu_t"'] & \mathbb{M} \arrow[d, "h_t"] \\ F_{e^{2\pi it}} \arrow[r, "\widetilde{\Gamma}_{e^{2\pi it}}"'] & \mathbb{M} \end{tikzcd}$$

Set

$$\mathbb{A}_t := \widetilde{\Gamma}_{e^{2\pi it}}(A_t) = \omega_t(\mathbb{T}_+), \quad \mathbb{B}_t := \widetilde{\Gamma}_{e^{2\pi it}}(B_t) = h_t(\mathbb{T}_+).$$

Clearly,

$$A_t \circ B_t = \mathbb{A}_t \circ \mathbb{B}_t.$$

Note that $h_t|_{\partial\mathbb{M}} = \mathrm{id}$, so $\mathbb{B}_t$ is homotopic to $\mathbb{T}_+$ via homotopies that are trivial along the boundary. Such homotopies do not alter the intersection number, and we have

$$\boxed{\mathbb{A}_t \circ \mathbb{B}_t = \mathbb{A}_t \circ \mathbb{T}_+.}$$

Along $\partial\mathbb{M}$ we have

$$\omega_t|_{\partial\mathbb{M}} = \Psi_t := \Gamma_{e^{2\pi it}} \circ \Omega_t \circ \Gamma_1^{-1}.$$

For $0 < \delta < 1/2$, then for sufficiently small t,

$$\mathbb{A}_t \subset U_\delta := \{(\boldsymbol{\xi}, \boldsymbol{\eta}) : |\boldsymbol{\xi}| < \delta, |\boldsymbol{\eta}| < 1\},$$

a neighborhood of $\mathbb{T}_+$, where as in section 5 we set $\boldsymbol{\xi} = (u_2, \cdots, u_n)$ and $\boldsymbol{\eta} = (v_2, \cdots, v_n)$. More precisely, if $P = (\mathbf{u}, \mathbf{v})$ is a point of $\mathbb{M}$ such that $\mathbf{u}$ is near $\mathbf{u}_+$, then its local coordinate is $(\boldsymbol{\xi}, \boldsymbol{\eta}) = \mathbf{pr}(\mathbf{u}, \mathbf{v})$, where

$$\mathbf{pr} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \ (\mathbf{u}, \mathbf{v}) \mapsto (u_2, \cdots, u_n, v_2, \cdots, v_n).$$

We write now $\omega_t|_{\partial\mathbb{M}}$ in $(\boldsymbol{\xi}, \boldsymbol{\eta})$ coordinate:

$$\omega_t|_{\partial\mathbb{M}}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \mathbf{pr} \circ \Psi_t(\mathbf{u}(\boldsymbol{\xi}, \boldsymbol{\eta}), \mathbf{v}(\boldsymbol{\xi}, \boldsymbol{\eta})) = \mathbf{pr} \circ \Gamma_{e^{2\pi it}} \circ \Omega_t \circ \Gamma_1^{-1}(\mathbf{u}(\boldsymbol{\xi}, \boldsymbol{\eta}), \mathbf{v}(\boldsymbol{\xi}, \boldsymbol{\eta})).$$

Thus, we since each term is linear, we can present $\omega_t|_{\partial\mathbb{M}}$ as a matrix $L_t \in M_{2n-2}(\mathbb{R})$, say

$$L_t \begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{pmatrix} = C(t)R(t)C(0)^{-1} \begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{pmatrix}.$$

Here

$$C(t) = \begin{pmatrix} c_1(e^{2\pi it}) & 0 \\ c_2(e^{2\pi it})c_3(e^{2\pi it}) & c_3(e^{2\pi it}) \end{pmatrix}, \quad R(t) = \begin{pmatrix} \cos \pi t & -\sin \pi t \\ \sin \pi t & \cos \pi t \end{pmatrix}.$$

Now, we have two submanifolds, $\mathbb{A}_t = \omega_t(\mathbb{T}_+)$ and $L_t(\mathbb{T}_+)$. They are the same on boundary, and they are both in a neighborhood U_δ for sufficiently small t. Thus, note that in $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$, U_δ is convex, so we can construct a homotopy $H : [0,1] \times \mathbb{T}_+ \to U_\delta$ with

$$H(s,x) = (1-s)\omega_t(x) + sL_t(x).$$

It connects $\mathbb{A}_t$ and L_t with boundary unchanged, so if t is small enough, we have

$$\mathbb{A}_t \circ \mathbb{T}_+ = L_t(T_+) \circ \mathbb{T}_+$$

For small enough t, we have

$$L_t = L_0 + tL_0' + O(t^2),$$

and we can now deform $L_t(\mathbb{T}_+)$ to $\Sigma_t := (L_0 + tL_0')(\mathbb{T}_+)$ such that during the deformation the boundary of the deforming relative cycle does not intersect the boundary of $\mathbb{T}_+$. Such deformation again does not alter the intersection number.

Now we calculate Σ_t. $L_0 = 1$ and

$$L_0' = C'(0)C(0)^{-1} + C(0)JC(0)^{-1}, \quad J = \pi \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}.$$

A simple calculation gives that

$$\begin{aligned} c_1(1) &= \sqrt{\frac{2}{5}} > 0, & c_2(1) &= 0, & c_3(1) &= \sqrt{\frac{2}{3}} > 0 \\ \frac{d}{dt}\Big|_0 c_1(e^{2\pi it}) &= 0, & \frac{d}{dt}\Big|_0 c_2(e^{2\pi it}) &= -\frac{2\pi}{25}, & \frac{d}{dt}\Big|_0 c_3(e^{2\pi it}) &= 0 \end{aligned}$$

Thus,

$$\begin{aligned} C'(0)C(0)^{-1} &= -\frac{2\pi}{25}\begin{pmatrix} 0 & 0 \\ c_3(1)/c_1(1) & 0 \end{pmatrix}, \\ C(0)JC(0)^{-1} &= \pi \begin{pmatrix} & -c_1(1)/c_3(1) \\ c_3(1)/c_1(1) & \end{pmatrix}. \end{aligned}$$

The upshot is that

$$L_0' = \begin{pmatrix} & -a \\ b & \end{pmatrix}, \quad a, b > 0.$$

Therefore, Σ_t is the portion inside U_δ of the $n-1$-subspace

$$\boldsymbol{\eta} \mapsto (L_0 + tL_0')\begin{pmatrix} 0 \\ \boldsymbol{\eta} \end{pmatrix} = \begin{pmatrix} -ta\boldsymbol{\eta} \\ \boldsymbol{\eta} \end{pmatrix}.$$

Observe that Σ_t intersects the $n-1$-subspace $\mathbb{T}_+$ given by $\boldsymbol{\xi} = 0$ at the origin, so

$$\Sigma_t \circ \mathbb{T}_+ = \pm 1.$$

To determine the sign, we observe that the orientation of Σ_t is given by

$$(-tadu_2 + dv_2) \wedge \cdots \wedge (-tadu_n + dv_n).$$

Note that

$$\begin{aligned} &(-tadu_2 + dv_2) \wedge \cdots \wedge (-tadu_n + dv_n) \wedge dv_2 \wedge \cdots \wedge dv_n \\ =& -(ta)^{n-1} du_2 \wedge \cdots \wedge du_n \wedge dv_2 \wedge \cdots \wedge dv_n \\ =&(-1)^{(n-1)+(n-1)^2}(ta)^{n-1} dv_2 \wedge \cdots \wedge dv_n \wedge du_2 \wedge \cdots \wedge du_n \\ =&(ta)^{n-1} dv_2 \wedge \cdots \wedge dv_n \wedge du_2 \wedge \cdots \wedge du_n. \end{aligned}$$

Thus,

$$\boxed{\Sigma_t \circ \mathbb{T}_+ = 1,}$$

so we get $m = (-1)^n$. This completes the proof of the local Picard–Lefschetz formula.

7 Global Picard–Lefschetz Formulæ

We recall what we have established. Consider a Lefschetz pencil (X_s) on $X \hookrightarrow \mathbb{P}^N$ with associated Lefschetz fibration $\hat{f} \colon \hat{X} \to S \cong \mathbb{P}^1$ such that all its critical values $t_1, \cdots, t_r$ are situated in the upper hemisphere $D_+ \subset S$. We denote its critical points by $p_1, \cdots, p_r$, so that

$$\hat{f}(p_j) = t_j, \quad \forall j.$$

We will identify D_+ with the closed unit disk centered at $0 \in \mathbb{C}$. Assume $|t_j| < 1$ for $j = 1, \cdots, r$. Fix a base point $\bullet \in \partial D_+$. For $j = 1, \cdots, r$ we define:

1. D_j is a closed disk of small radius ρ centered at $t_j \in D_+$. If $\rho \ll 1$, these disks are pairwise disjoint.
2. $\ell_j : [0, 1] \to D_+$ is a smooth embedding connecting $\bullet \in \partial D_+$ to $t_j + \rho$, such that the paths $\ell_1, \cdots, \ell_r$ are pairwise disjoint. Set $k_j := \ell_j \cup D_j$, $\ell = \bigcup \ell_j$, and $k = \bigcup k_j$.
3. B_j is a small ball in $\hat{X}$ centered at p_j.

Denote by γ_j the loop in $S^* := S \setminus \{t_1, \cdots, t_r\}$ based at $\bullet$, obtained by traveling along ℓ_j from $\bullet$ to $t_j + \rho$, then traversing ∂D_j counterclockwise, and then returning to $\bullet$ along ℓ_j. The loops γ_j generate the fundamental group:

$$\pi_1(S^*, \bullet), \quad S^* := S \setminus \{t_1, \cdots, t_r\}.$$

Set

$$\hat{X}_S^* := \hat{f}^{-1}(S^*).$$

We have a fibration

$$\hat{f} : \hat{X}_S^* \to S^*,$$

and there exists a monodromy action

$$\mu \colon \pi_1(S^*, \bullet) \to \operatorname{Aut}(H_*(\hat{X}_\bullet, \mathbb{Z})),$$

called the *monodromy of the Lefschetz fibration*. Since $\hat{X}_\bullet$ is canonically diffeomorhpic to $X_\bullet$, we will write $X_\bullet$ instead of $\hat{X}_\bullet$.

From the proof of the local Picard–Lefschetz formula we deduce that for each critical point p_j of $\hat{f}$, there exists an oriented $(n-1)$-sphere Σ_j embedded in the fiber $X_{t_j+\rho}$ which bounds a *thimble*, i.e., an oriented embedded n-disk $Z_j \subset \hat{X}_+$. This disk is spanned by the family of vanishing spheres in the fibers over the radial path from $t_j + \rho$ to t_j.

We denote by $\Delta_j \in H_{n-1}(X_{t_j+\rho}, \mathbb{Z})$ the homology class determined by the vanishing sphere Σ_j in the fiber over $t_j + \rho$. In fact, we have shown in section 5 that

$$\Delta_j * \Delta_j = (-1)^{n(n-1)/2} \left(1 + (-1)^{n-1}\right) = \begin{cases} 0 & \text{if } n \text{ is even}, \\ -2 & \text{if } n \equiv -1 \mod 4, \\ 2 & \text{if } n \equiv 1 \mod 4. \end{cases}$$

The above intersection pairing is the one determined by the complex orientation of $X_{t_j+\rho}$.

Note that for each j, the path $\ell_j : [0, 1] \to S^*$ pulls back X_{S^*} to $\ell^*(X_{S^*})$, and it becomes a fiber bundle over $[0, 1]$, which is contractible, so $\ell^*(X_{S^*}) \cong [0, 1] \times X_\circ$, where $\circ$ is any point on ℓ_j. Thus we have $X_j := X_{t_j+\rho}$ is diffeomorphic to $X_\bullet$, so we have a canonical isomorphism

$$H_*(X_j, \mathbb{Z}) \cong H_*(X_\bullet, \mathbb{Z}).$$

For this reason we will freely identify $H_*(X_\bullet, \mathbb{Z})$ with any $H_*(X_j, \mathbb{Z})$.

Using the local Picard–Lefschetz formula, we obtain the following important result.

Theorem 7.1 (Global Picard–Lefschetz formula)**.** *If $z \in H_{n-1}(X_\bullet, \mathbb{Z})$, then*

$$\mathbf{var}_{\gamma_j}(z) := \mu_{\gamma_j}(z) - z = (-1)^{n(n-1)/2}(z * \Delta_j)\Delta_j$$

Proof. We only state the proof for the homology with real coefficients. We think of cohomology $H^*(X_j)$ as the de Rham cohomology. Then by Poincaré duality and the universal coefficient theorem, we have

$$H^*(X_j) \cong (H_c^{2n-2-*}(X_j))^* = (H^{2n-2-*}(X_j))^* \cong H_{2n-2-*}(X_j).$$

We use the convention as follows. For submanifold $S \subset X_\bullet$, its Poincaré dual $\eta_S \in H^*(X_\bullet)$ satisfies $\int_S \omega = \int_{X_\bullet} \omega \wedge \eta_S$, for each $\omega \in H^{\dim S}(X_\bullet)$. This implies that we have $S_1 * S_2 = \int_{X_\bullet} \eta_{S_1} \wedge \eta_{S_2}$.

Represent the Poincaré dual of z by $\zeta \in H^{n-1}(X_\bullet)$ and the Poincaré dual of Δ_j by $\delta_j \in H^{n-1}(X_\bullet)$. Then, by localization principle, we can assume that δ_j is supported in an arbitarily small neighborhood of Δ_j in X_j. Thus, we assume $\operatorname{supp}\delta_j \subset U_j$, where U_j is a open tubular neighborhood of Δ_j that is diffeomorphic to the open disk bundle of $T\Sigma_j$.

We represent the monodromy μ_{γ_j} by a diffeomorphism h_j of X_j which is trivial outside a compact subset of U_j. In particular, h_j preserves the orientation. We claim that the Poincaré dual of $\mu_{\gamma_j}(z)$ is $(h_j^{-1})^*\zeta$. For simplicity, we only prove the case when z is represented by an oriented submanifold Z. The complete proof and the rigorous definition of the Poincaré duality map will be discussed in Remark 7.2. In the case we will discuss now, $\mu_{\gamma_j}(z)$ is represented by $h_j(Z)$, so for each $\omega \in H^{n-1}(X_\bullet)$, we have

$$\begin{aligned} \int_{h_j(Z)} \omega &= \int_Z h_j^*\omega = \int_{X_j} h_j^*\omega \wedge \zeta \\ &= \int_{X_j} h_j^*(\omega \wedge (h_j^{-1})^*\zeta) = \int_{X_j} \omega \wedge (h_j^{-1})^*\zeta. \end{aligned}$$

Note that outside U_j we will have $(h_j^{-1})^*\zeta = \zeta$, so $(h_j^{-1})^*\zeta - \zeta \in H_c^{n-1}(U_j)$.

On the other hand, by the Thom isomorphism theorem, we have

$$H_{cv}^*(U_j) \cong H^{*-(n-1)}(\Sigma_j).$$

Since the base $\Sigma_j \cong S^{n-1}$ is compact, $H_{cv}^*(U_j) = H_c^*(U_j)$. Thus,

$$H_c^{n-1}(U_j) \cong H^0(\Sigma_j) \cong \mathbb{R}.$$

Note that we have a nonzero $\delta \in H_c^{n-1}(U_j)$, so $H_c^{n-1}(U_j) = \langle \delta \rangle$. Therefore, there is some $c \in \mathbb{R}$ such that

$$(h_j^{-1})^*\zeta - \zeta = c\delta.$$

We then have

$$\int_{\nabla_j} \delta_j = \Delta_j * \nabla_j = (-1)^{n-1}\nabla_j * \Delta_j,$$

so

$$c(-1)^{n-1}\nabla_j * \Delta_j = \int_{\nabla_j} c\delta_j = \int_{\nabla_j} (h_j^{-1})^*\zeta - \zeta = \int_{h_j^{-1}(\nabla_j) - \nabla_j} \zeta.$$

We have shown in section 4 that

$$\nabla_j * \Delta_j = (-1)^{n(n-1)/2}.$$

The singular chain $h_j^{-1}(\nabla_j) - \nabla_j = h_j^{-1}(\nabla_j - h_j(\nabla_j)) = h_j^{-1}(-\mathbf{var}_{\gamma_j}(\nabla_j))$ is homologous in X_j to $-(-1)^n\Sigma_j$ by the local Picard-Lefschetz formula. Thus

$$c = (-1)^{n-1}(-1)^{n(n-1)/2} \int_{-(-1)^n\Sigma_j} \zeta = (-1)^{n(n-1)/2} z * \Delta_j$$

Thus,

$$(h_j^{-1})^*\zeta - \zeta = (-1)^{n(n-1)/2}(z * \Delta_j)\delta_j.$$

Via Poincaré dual, we have

$$\mu_{\gamma_j}(z) - z = (-1)^{n(n-1)/2}(z * \Delta_j)\Delta_j.$$

□

Remark 7.2. *We begin with the rigorous definition of the Poincaré dual. For a compact, connected, orientable, smooth n-dimensional manifold M, we denote by $H^*(M)$ its de Rham cohomology, and fix an orientation* **or** *on it. We set $H_*(M) := (H^*(M))^*$. Using the Poincaré duality and the universal coefficient theorem, it can be identified with $H_*(M, \mathbb{R})$. We define the Kronecker pairing*

$$H^k(M) \otimes H_k(M) \to \mathbb{R}, \quad (\alpha, z) \mapsto \langle \alpha, z \rangle := z(\alpha).$$

The orientation **or** *determines an element $[M] \in H_n(M)$ via*

$$\langle \alpha, [M] \rangle = \int_M \alpha.$$

Then we have a natural isomorphism

$$\mathscr{P} : H^{n-k}(M) \to H_k(M), \quad \text{such that } \forall \beta \in H^k(M),\ \langle \beta, \mathscr{P}(\alpha) \rangle = \langle \beta \wedge \alpha, [M] \rangle.$$

For any smooth map $f : M \to M$, we define

$$f_* : H_*(M) \to H_*(M), \quad f_* = \mathscr{P} \circ f^* \circ \mathscr{P}^{-1}.$$

It is nothing but the push-forward when we think of $H_*(M)$ as $H_*(M, \mathbb{R})$.

Now we show that if $f \in \mathrm{Diff}(M)$, then for each $\alpha \in H^{n-k}(M)$,

$$f_*\mathscr{P}(\alpha) = (\deg f)\mathscr{P}((f^{-1})^*\alpha).$$

Indeed, it will be easy if we use singular (co)homology:

$$f_*\mathscr{P}(\alpha) = f_*(\alpha \cap [M]) = (f^{-1})^*\alpha \cap f_*[M] = (\deg f)(f^{-1})^*\alpha \cap [M] = (\deg f)\mathscr{P}((f^{-1})^*\alpha).$$

Definition 7.1. *The monodromy group of the Lefschetz pencil $(X_s)_{s\in S}$ of X is the subgroup $\mathfrak{G} \leq \mathrm{Aut}(H_{n-1}(X_\bullet, \mathbb{Z}))$ generated by monodromies μ_{γ_j}.*

Proposition 7.3. *The module of invariant cycles $\mathbb{I}(X_\bullet)$ consists of the cycles invariant under the action of the monodromy group $\mathfrak{G}$.*

Proof. We have shown that the invariant module

$$\mathbb{I}(X_\bullet) = \{y \in H_{n-1}(X_\bullet) \mid y * v = 0, \forall v \in \mathbb{V}(X_\bullet)\},$$

and the vanishing module $\mathbb{V}(X_\bullet)$ is generated by the vanishing cycles Δ_j. Thus,

$$\mathbb{I}(X_\bullet) = \{y \in H_{n-1}(X_\bullet) \mid y * \Delta_j = 0, \forall j\}.$$

Using the global Picard-Lefschetz formula, we get

$$y * \Delta_j = 0 \quad \Leftrightarrow \quad \mu_{\gamma_j}(y) = y,$$

so

$$\mathbb{I}(X_\bullet) = \{y \in H_{n-1}(X_\bullet) \mid \mu_{\gamma_j}(y) = y, \forall j\}.$$

□

8 Corollaries of Hard Lefschetz Theorem

We first recall the Hard Lefschetz Theorem.

Theorem 8.1. *The following six equivalent statements are all true for the homology with real coefficients.*

HL$_1$. $\mathbb{V}(X') \cap \mathbb{I}(X') = 0$.
HL$_2$. $\mathbb{V}(X') \oplus \mathbb{I}(X') = H_{n-1}(X')$.
HL$_3$. *The restriction of* $i_* : H_{n-1}(X') \to H_{n-1}(X)$ *to* $\mathbb{I}(X')$ *is an isomorphism.*
HL$_4$. *The map* $\omega\cap : H_{n+1}(X) \to H_{n-1}(X)$ *is an isomorphism.*
HL$_5$. *The restriction of intersection form* $H_{n-1}(X') \times H_{n-1}(X') \to \mathbb{Z}$ *to* $\mathbb{V}(X')$ *stays nondegenerate, i.e., for any nonzero* $\alpha \in H_{n-1}(X')$*, there must be some* β *such that* $\alpha \cdot \beta \neq 0$.
HL$_6$. *The restriction of intersection form* $H_{n-1}(X') \times H_{n-1}(X') \to \mathbb{Z}$ *to* $\mathbb{I}(X')$ *stays nondegenerate.*

Define inductively

$$X^{(0)} = X, \quad X^{(q+1)} := (X^{(q)})', \quad q \geq 0.$$

Then we have a decreasing filtration

$$X = X^{(0)} \supset X^{(1)} \supset \cdots \supset X^{(n)} \supset \emptyset,$$

where $\dim_{\mathbb{C}} X^{(q)} = n - q$. Denote by

$$\mathbb{I}_q(X) := \operatorname{im}\left(i^! : H_{n-q+2}(X^{(q-1)}) \to H_{n-q}(X^{(q)})\right) \subset H_{n-q}(X^{(q)})$$

the invariant module. Its Poincaré dual in $X^{(q)}$ is

$$\mathbb{I}_q(X)^\vee = \mathscr{P}^{-1}_{X^{(q)}}(\mathbb{I}_q(X)) = \operatorname{im}\left(i^* : H^{n-q}(X^{(q-1)}) \to H^{(n-q)}(X^{(q)})\right).$$

The Lefschetz hyperplane theorem implies that the morphisms

$$i_* : H_k(X^{(q)}) \to H_k(X^{(j)}), \quad q \geq j$$

are isomorphisms for each $k < \dim_{\mathbb{C}} X^{(q)} = n - q$. By the universal coefficient theorem, we have isomorphisms

$$i^* : H^k(X^{(j)}) \to H^k(X^{(q)}), \quad q \geq j$$

for each $k < n - q$. By **HL**$_3$ and $H_k(X^{(q)}) \cong H_k(X^{(j)})$, we conclude that

$$i_* : \mathbb{I}_q(X) \to H_{n-q}(X^{(q-1)}) \cong H_{n-q}(X)$$

are isomorphisms. Using the isomorphisms $i^* : H^k(X^{(j)}) \to H^k(X^{(q)})$ deduced from the Lefschetz Hyperplane theorem, we get isomorphisms

$$H^{n-q}(X) \xrightarrow{i^*} H^{n-q}(X') \xrightarrow{i^*} \cdots \xrightarrow{i^*} H^{n-q}(X^{(q-1)})$$

Thus,

$$\mathbb{I}_q(X)^\vee = \operatorname{im}\left(i^* : H^{n-q}(X^{(q-1)}) \to H^{(n-q)}(X^{(q)})\right) = \operatorname{im}\left(i^* : H^{n-q}(X) \to H^{(n-q)}(X^{(q)})\right).$$

Using Poincaré duality, we conclude that the morphisms

$$i^! : H_{n+q}(X) \to \mathbb{I}_q(X)$$

are isomorphisms. By $\mathbf{HL}_6$, the restriction of the intersection form of $H_{n-q}(X^{(q)})$ to $\mathbb{I}_q(X)$ is nondegenerate. The isomorphisms i_* and $i^!$ carries the intersection form to $H_{n+q}(X) \cong H_{n-q}(X)$. When $n-q$ is odd, then the intersection form is skew-symmetric, and thus the nondegeneracy assumption implies

$$\dim H_{n-q}(X) = \dim H_{n+q}(X) \in 2\mathbb{Z}.$$

We have thus proved the following result.

Corollary 8.2. *The odd dimensional Betti numbers $b_{2k+1}(X)$ are even.*

Note that if we write $X^{(q)} = X^{(q-1)} \cap H_q$, where H_q is a generic hyperplane, then

$$\omega^q = (\mathscr{P}_X^{-1}([X']))^q = \bigcup_{i=1}^{q} \mathscr{P}_X^{-1}([X \cap H_i]) = \mathscr{P}_X^{-1}\left(\left[X \cap \bigcap_{i=1}^{q} H_i\right]\right) = \mathscr{P}_X^{-1}([X^{(q)}]).$$

Therefore, we have a decomposition

$$\begin{tikzcd} H_k(X) \arrow[r, "i^!"] \arrow[dr, "\omega^q \cap"'] & H_{k-2q}(X^{(q)}) \arrow[d, "i_*"] \\ & H_{k-2q}(X) \end{tikzcd}.$$

We then obtain a generalization of $\mathbf{HL}_4$ since we have concluded that both $i_* : \mathbb{I}_q(X) \to H_{n-q}(X)$ and $i^! : H_{n+q}(X) \to \mathbb{I}_q(X)$ are isomorphisms.

Corollary 8.3. *The map $\omega^q \cap : H_{n+q}(X) \to H_{n-q}(X)$ is an isomorphism.*

Clearly, this corollary is an equivalent statement of the Hard Lefschetz Theorem. We can further generalize it to Lefschetz Decomposition Theorem as follows. We first need to introduce some concepts.

Definition 8.1. An element $c \in H_{n+q}(X)$, $0 \le q \le n$, is called primitive if $\omega^{q+1} \cap c = 0 \in H_{n-2}(X)$; an element $z \in H_{n-q-2}(X)$, $0 \le q \le n$, is called effective if $\omega \cap z = 0 \in H_{n-q-2}(X)$. We will denote by $P_{n+q}(X) \subset H_{n+q}(X)$ the subspace of all primitive elements, and by $E_{n-q} \subset H_{n-q}(X)$ the subspace of all effective elements.

Observe that

$$c \in P_{n+q}(X) \quad \Longleftrightarrow \quad \omega^q \cap c \in E_{n-q}(X).$$

Theorem 8.4 (Lefschetz decomposition)**.** *Every element $c \in H_{n+q}(X)$ decomposes uniquely as*

$$c = \sum_{i \ge 0} \omega^i \cap c_i,$$

where $c_i \in P_{n+q+2i}(X)$; every element $z \in H_{n-q}(X)$ decomposes uniquely as

$$z = \sum_{i \ge 0} \omega^{q+i} \cap z_i,$$

where $z_i \in P_{n+q+2i}(X)$.

Proof. Note that

$$\sum_{i \ge 0} \omega^{q+i} \cap z_i = \omega^q \cap \left(\sum_{i \ge 0} \omega^i \cap z_i\right),$$

so it suffices to show the first decomposition holds for $c \in H_{n+q}(X)$.

First consider the case when $q = n, n-1$. A dimensional consideration gives that $P_{n+q}(X) = H_{n+q}(X)$. Thus, the decomposition is trivially true for $q \geq n-1$. We then use the descending induction on q, so it remains to show that for each $c \in H_{n+q}(X)$, we can write it uniquely as

$$c = c_0 + \omega \cap c_1, \quad c_0 \in P_{n+q}(X),\ c_q \in H_{n+q+2}(X).$$

Since $\omega^{q+2} \cap : H_{n+q+2}(X) \to H_{n-q-2}(X)$ is an isomorphism, there is a unique $z \in H_{n+q+2}(X)$ such that

$$\omega^{q+2} \cap z = \omega^{q+1} \cap c.$$

Define

$$c_0 := c - \omega \cap z,$$

then

$$\omega^{q+1} \cap c_0 = \omega^{q+1} \cap c - \omega^{q+2} \cap z = 0 \quad \Longrightarrow \quad c_0 \in P_{n+q}(X).$$

Therefore, the decomposition

$$c = c_0 + \omega \cap z$$

exists. Now we verify the uniqueness. Suppose there is another decomposition

$$c = c_0' + \omega \cap c,$$

then

$$\delta + \omega \cap \varepsilon = 0,$$

where $\delta = c_0 - c_0' \in P_{n+q}(X)$ and $\varepsilon = z - c \in H_{n+q+2}(X)$. Then,

$$0 = \omega^{q+1} \cap (\delta + \omega \cap \varepsilon) = \omega^{q+2} \cap \varepsilon \implies \varepsilon = 0 \implies \delta = 0.$$

□

The Lefschetz decomposition shows that the homology of X is completely determined by its primitive part. Moreover, the above proof shows that

$$0 \leq \dim P_{n+q}(X) = \dim H_{n+q}(X) - \dim(\omega \cap H_{n+q+2}(X)) = b_{n+q} - b_{n+q+2} = b_{n-q} - b_{n-q-2}.$$

Here we use the fact that $\omega\cap : H_k(X) \to H_{k-2}(X)$ is injective if $k > n$. It is because if $\omega \cap a = 0$, then

$$\omega^{k-n} \cap a = \omega^{k-n-1} \cap (\omega \cap a) = 0 \implies \omega \cap a = 0.$$

This implies the unimodality of the Betti numbers of a closed projective manifold,

$$\begin{aligned} 1 = b_0 \leq b_2 \leq \cdots \leq b_{2\lfloor n/2 \rfloor}, &\quad b_{2\lfloor n/2 \rfloor} \geq \cdots \geq b_{2n} \\ b_1 \leq b_3 \leq \cdots \leq b_{2\lfloor (n-1)/2 \rfloor + 1}, &\quad b_{2\lfloor (n-1)/2 \rfloor + 1} \geq \cdots \geq b_{2n-1}. \end{aligned}$$

These inequalities introduce additional topological restrictions on algebraic manifolds.