STA2603 Distribution Theory II

Study Unit 2

Frequency Function (discreet) = density function (continuous). Distribution functions is CDF.

Discreet Random Variables

Distribution	Mass function $p_X(k)$	$M_X(t)$ mgf	Mean & Variance
Bernoulli $ber(p)$	$\begin{cases} p^k(1-p)^{1-k} & k=0;1 \\ 0 & \text{elsewhere} \end{cases}$	$(1-p+pe^t)$	$\mu_X = p$ $\sigma_X^2 = p(1-p)$
Binomial $b(n;\ p)$	$\begin{cases} \binom{n}{x}p^x(1-p)^{n-x} & x=0;1;.;n \\ 0 & \text{elsewhere} \end{cases}$	$(1-p+pe^t)^n$	$\mu_X = np$ $\sigma_X^2 = np(1-p)$
Geometric $geo(\ p)$	$\begin{cases} (1-p)^{x-1}p, & x=1;2;3;\ldots \\ 0 & \text{elsewhere} \end{cases}$	$\dfrac{pe^t}{1-(1-p)e^t}$	$\mu_X = \dfrac{1}{p}$ $\sigma_X^2 = \dfrac{1-p}{p^2}$
Negative Binomial $nb(r;\ p)$	$\begin{cases} \binom{k-1}{r-1}p^r(1-p)^{k-r}, & k=1;2;\ldots \\ 0 & \text{elsewhere} \end{cases}$	$\left[\frac{pe^t}{(1-(1-p)e^t)}\right]^r$	$\mu_X = \dfrac{r}{p}$ $\sigma_X^2 = \dfrac{r(1-p)}{p^2}$
Poisson $Po(\lambda)$	$\begin{cases} \dfrac{\lambda^k}{k!}e^{-\lambda}, & k=0;1;\ldots \\ 0 & \text{elsewhere} \end{cases}$	$e^{\lambda(e^t-1)}$	$\mu_X = \lambda$ $\sigma_X^2 = \lambda$

$$\begin{array}{lll} 1. & p_X(x_i) & \geq & 0 \\ 2. & \sum_{x=0}^{\infty} p_X(x) & = & 1 \end{array}$$

Bernoulli Random Variables

Indicator random variable $I_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \text{otherwise} \end{cases}$

$$p_X(x) = \begin{cases} 1-p & \text{if } x=0 \\ p & \text{if } x=1 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & \text{if } x<0 \\ 1-p & \text{if } 0<x<1 \\ 1 & \text{if } x>1 \end{cases}$$

Binomial Distribution

The sum of independent Bernoulli variables is a binomial random variable.
To prove binomial probabilities sums to 1. Use finite binomial series expansion:

$$
\begin{aligned}
(a+b)^n &= a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{n-1} a^{n-(n-1)} b^{n-1} + \binom{n}{n} a^{n-n} b^n \\
&= (a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + b^n \\
&= \sum_{x=0}^{n} \binom{n}{x} a^{n-x} b^x
\end{aligned}
$$

and therefore it is a series expansion of $[(1-p)+p]^n$

Geometric Distribution

To prove that geometric distribution sums to 1, use Taylor series at x=0

$$
\begin{aligned}
(1-p)^{-1} &= 1 + \frac{(-1)}{1!}(-p) + \frac{(-1)(-1-1)}{2!}(-p)^2 + \frac{(-1)(-1-1)(-1-2)}{3!}(-p)^3 + .. \\
&= 1 + p + p^2 + p^3 + p^4 + \dots
\end{aligned}
$$

$$
\begin{aligned}
p\sum_{x=1}^{\infty}(1-p)^{x-1} &= p\sum_{r=0}^{\infty}(1-p)^r \\
&= p\left[1 + (1-p) + (1-p)^2 + (1-p)^3 + \dots\right] \\
&= p(1-(1-p))^{-1} \\
&= \frac{p}{p} \\
&= 1
\end{aligned}
$$

Negative Binomial Distribution

A negative binomial random variable can be expressed as the sum of r independent geometric variables.

Hypergeometric distribution

$$pX(x) = \begin{cases} \frac{\binom{r}{x}\binom{n-r}{m-x}}{\binom{n}{m}} & \text{if } x = 0, 1, 2, \ldots .n \\ 0 & \text{otherwise} \end{cases}$$

Poisson Distribution

To show it is a frequency function:

$$\begin{aligned} e^{\lambda} &= 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \ldots \\ &= \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} \end{aligned}$$

$$\begin{aligned} \sum_{x=0}^{\infty} \frac{\lambda^k}{x!} e^{-\lambda} &= e^{\lambda} e^{-\lambda} \\ &= 1. \end{aligned}$$

Poisson distribution can be used to approximate binomial distribution, if n is large and p is small.

Continuous Random Variables

1. $F_X(x)$ is a non-decreasing function
2. $\lim_{x \to -\infty} F_X(x) = 0$
3. $\lim_{x \to \infty} F_X(x) = 1$
4. $F_X(x)$ is everywhere continuous

$$\boxed{P(a < X < b) = \int_a^b f_X(x)dx}$$

1. $f_X(x) \geq 0$ and
2. $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$P(a < X < b) = P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b)$$

The pth quantile: $F(x_p) = p$ or $P(X \leq x_p) = p$ and $x_p = F^{-1}(p)$

Exponential Density
Gamma Density

Gamma function $\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}$ Properties $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ $c^n\Gamma(n) = \int_0^\infty x^{n-1}e^{-\frac{x}{c}}$

$\Gamma(2) = 1$

If $\alpha = 1$ then it becomes exponential density
If $\lambda = 1$ then it is a one parameter gamma density

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1}e^{-\lambda t}dt &= \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-\lambda t}dt \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t/\frac{1}{\lambda}}dt \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)}\Gamma(\alpha)\left(\frac{1}{\lambda}\right)^\alpha \\ &= 1.\end{aligned}$$

Normal (Gaussian) distribution
Beta Density

Beta function: $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ or $B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$

To prove density function:

For $0 < x < 1$ and $m > 0, n > 0$

$$\begin{aligned}\int_0^1 \frac{1}{B(m,n)} x^{m-1}(1-x)^{n-1}dx &= \frac{1}{B(m,n)}B(m,n) \\ &= 1.\end{aligned}$$

$$\begin{aligned}B(m+1;n+1) &= \frac{m\Gamma(m)\,n\Gamma(n)}{(m+n+1)\Gamma(m+n+1)} \\ &= \frac{mn}{(m+n+1)(m+n)} \times \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \\ &= \frac{mn}{(m+n+1)(m+n)}B(m;n).\end{aligned}$$

Different characteristics of the beta function

Characteristic	Comment	Proof / Examples	Ref no
Type 1 beta function $\int_0^1 x^{m-1}(1-x)^{n-1}dx. = B(m;n)$	Recognize and use beta integral in integration	$\int_0^1 x^5(1-x)^6 = \int_0^1 x^{6-1}(1-x)^{7-1} = B(6;7)$	$B1$
Symmetry with respect to the parameters $m > 0;\ n > 0$	$B(m;n) = B(n;m)$		$B2$
Relationship between beta and gamma functions	$m > 0;\ n > 0$ *Numerical value* for any beta function	$B(m;n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$	$B4$
Changing the limits of the integral from $\int_0^1 \ldots$ to $\int_0^\infty \ldots$	$\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx = B(m;n)$ which is a type 2 beta function	$\int_0^\infty \frac{x^3}{(1+x)^5}dx = \int_0^\infty \frac{x^{4-1}}{(1+x)^{4+1}}dx = B(4;1)$	

Functions of random Variables

Standard normal CDF $= \Phi(X)$ Standard normal density $= \phi(X)$
Proposition A: if $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$, then $Y \sim N(a\mu + b, a^2\sigma^2)$

PROPOSITION B

Let X be a continuous random variable with density $f(x)$ and let $Y = g(X)$ where g is a differentiable, strictly monotonic function on some interval I. Suppose that $f(x) = 0$ if x is not in I. Then Y has the density function

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

for y such that $y = g(x)$ for some x, and $f_Y(y) = 0$ if $y \neq g(x)$ for any x in I. Here g^{-1} is the inverse function of g; that is, $g^{-1}(y) = x$ if $y = g(x)$. ■

PROPOSITION C

Let $Z = F(X)$; then Z has a uniform distribution on [0, 1].

Proof

$$P(Z \leq z) = P(F(X) \leq z) = P(X \leq F^{-1}(z)) = F(F^{-1}(z)) = z$$

This is the uniform cdf. ■

PROPOSITION D

Let U be uniform on [0, 1], and let $X = F^{-1}(U)$. Then the cdf of X is F.

Proof

$$P(X \leq x) = P(F^{-1}(U) \leq x) = P(U \leq F(x)) = F(x)$$ ■

$[N(0,1)]^2 \sim \chi_1^2$

Weibull density: $\frac{\beta}{\alpha^\beta} x^{\beta-1} e^{-\left(\frac{x}{\alpha}\right)^\beta}$

Study Unit 3: Joint Distributions

Discreet Random Variables

$$\boxed{p_{XY}(x,y) = P(X = x, Y = y)}$$

1.	$p_{X,Y}(x,y)$	$\geq$	0	for all $(x,y) \in R^2$
2.	$\sum_{x=0}^{n_1} \sum_{y=0}^{n_2} p_{X,Y}(x,y)$	$=$	1	

Continuous Random Variables

Marginal density.

1.	$f_{X,Y}(x,y)$	≥ 0	for all $(x,y) \in R^2$
2.	$\int\int_S f_{X,Y}(x,y)\,dydx$	$=1$	

$$\boxed{f_{X,Y}(x;y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x;y).}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x;v)\,dv \qquad \text{for all } x \in R$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(u;y)\,du \qquad \text{for all } y \in R$$

Marginals sum to 1

Joint cumulative distribution.

$$\begin{aligned} F_{X,Y}(x;y) &= P(X < x;\ Y < y) \\ &= \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u;v)\,dudv \end{aligned}$$

Always specify the domain of the function!
Always specify the parameters when identifying the distribution!

Independent Random variables

Independent if $F(x,y) = F_X(x)F_Y(y)$

Two discreet random variables will be independent if th their joint mass function factors.

Conditional Distributions

Discreet:

The law of total probability. $p_X(x) = \sum_y p_{X|Y}(x|y)p_Y(y)$

Continuous:

The law of total probability. $f_Y(y) = * \int_{-\infty}^{\infty} f_{Y|X}(y|x)f_X(x)\,dx$