

## STA2603 Distribution Theory II

- Always specify the domain of the function!
- Always specify the parameters when identifying the distribution!

Commonly used series expansions			
	$e^{tx}$	$= 1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \dots$	
$\log(1+x)$	$= \log_e(1+x)$	$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$	
$\log(1-x)$	$= \log_e(1-x)$	$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots$	

### Study Unit 2

- Frequency Function (discrete) AKA Mass function
- Density function (continuous).
- Distribution function is cumulative density function

### Discrete Random Variables

Distribution	Mass function $p_X(k)$	$M_X(t)$ mgf	Mean & Variance
Bernoulli $ber(p)$ one success	$\begin{cases} p^k (1-p)^{1-k} & k = 0; 1 \\ 0 & \text{elsewhere} \end{cases}$	$(1-p + pe^t)$	$\mu_X = p$ $\sigma_X^2 = p(1-p)$
sum of bernoulli Binomial $b(n; p)$ x successes in n trials	$\begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0; 1; \dots; n \\ 0 & \text{elsewhere} \end{cases}$	$(1-p + pe^t)^n$	$\mu_X = np$ $\sigma_X^2 = np(1-p)$
Geometric $geo(p)$ 1st success in x trials	$\begin{cases} (1-p)^{x-1} p, & x = 1; 2; 3; \dots \\ 0 & \text{elsewhere} \end{cases}$	$\frac{pe^t}{1 - (1-p)e^t}$	$\mu_X = \frac{1}{p}$ $\sigma_X^2 = \frac{1-p}{p^2}$
sum of geometric Negative Binomial $nb(r; p)$ r'th success in k trials	$\begin{cases} \binom{k-1}{r-1} p^r (1-p)^{k-r}, & k = 1; 2; \dots \\ 0 & \text{elsewhere} \end{cases}$	$\left[ \frac{pe^t}{1 - (1-p)e^t} \right]^r$	$\mu_X = \frac{r}{p}$ $\sigma_X^2 = \frac{r(1-p)}{p^2}$
limit of binomial Poisson $Po(\lambda)$ K rare events	$\begin{cases} \frac{\lambda^k}{k!} e^{-\lambda}, & k = 0; 1; \dots \\ 0 & \text{elsewhere} \end{cases}$	$e^{\lambda(e^t - 1)}$	$\mu_X = \lambda$ $\sigma_X^2 = \lambda$

$$\text{Hypergeometric : } p(x) = \begin{cases} \frac{\binom{r}{x} \binom{n-r}{m-x}}{\binom{n}{m}} & \text{if } x = 0, 1, 2, \dots, n \\ 0 & \text{otherwise} \end{cases}$$

Probability of x successes in a sample size  $m$  from sample space  $n$  and a subset  $r$

1.	$p_X(x_i) \geq 0$
2.	$\sum_{x=0}^{\infty} p_X(x) = 1$

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

$$\rightarrow P(X > x) = 1 - P(X \leq x)$$

→ To prove is distribution:

- 1) State probabilities are all non-negative.
- 2) Show probabilities all sum to 1.

$$\lim_{x \rightarrow a^+} F_X(x) = F_X(a) \text{ for all real values of } x$$

## Bernoulli Random Variables

$$\text{Indicator random variable } I_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \text{otherwise} \end{cases}$$

$$p_X(x) = \begin{cases} 1-p & \text{if } x=0 \\ p & \text{if } x=1 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1-p & \text{if } 0 < x < 1 \\ 1 & \text{if } x > 1 \end{cases}$$

## Binomial Distribution

- The sum of independent Bernoulli variables is a binomial random variable.
- To prove binomial probabilities sums to 1. Use finite binomial series expansion:
- 1) State binomial function sums to 1
  - 2) then given the binomial expansion of  $(a+b)^n = \sum_{x=0}^n \binom{n}{x} a^{n-x} b^x$
  - 3) let  $a = 1-p$  and  $b = p$
  - 4) Substitute above into  $(a+b)^n$
  - 5) therefore  $(1-p+p)^n = 1^n = 1$

## Geometric Distribution

→ To prove that geometric distribution sums to 1, use Taylor series at  $x=0$

$$\begin{aligned} (1-p)^{-1} &= 1 + \frac{(-1)}{1!}(-p) + \frac{(-1)(-1-1)}{2!}(-p)^2 + \frac{(-1)(-1-1)(-1-2)}{3!}(-p)^3 + \dots \\ &= 1 + p + p^2 + p^3 + p^4 + \dots \\ p \sum_{x=1}^{\infty} (1-p)^{x-1} &= p \sum_{r=0}^{\infty} (1-p)^r \\ &= p [1 + (1-p) + (1-p)^2 + (1-p)^3 + \dots] \\ &= p (1 - (1-p))^{-1} \\ &= \frac{p}{p} \\ &= 1 \end{aligned}$$

→ To derive CDF

$$F(x) = \sum_{r=0}^x p(1-p)^r = \frac{p(1 - (1-p)^{x+1})}{1 - (1-p)} = 1 - (1-p)^{x+1}$$

## Negative Binomial Distribution

A negative binomial random variable can be expressed as the sum of  $r$  independent geometric variables.

$$p_X(x) = \begin{cases} \binom{x-1}{r-1} p^r (1-p)^{x-r} & \text{if } x = r, r+1, r+2, \dots; 0 \leq p \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

## Hypergeometric distribution

$$p_X(x) = \begin{cases} \frac{\binom{r}{x} \binom{n-r}{m-x}}{\binom{n}{m}} & \text{if } x = 0, 1, 2, \dots, n \\ 0 & \text{otherwise} \end{cases}$$

## Poisson Distribution

To show it is a frequency function:

$$\begin{aligned} e^\lambda &= 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \dots \\ &= \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} \end{aligned}$$

$$\begin{aligned} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} e^{-\lambda} &= e^\lambda e^{-\lambda} \\ &= 1. \end{aligned}$$

Poisson distribution can be used to approximate binomial distribution, if  $n$  is large and  $p$  is small.

## Continuous Random Variables

1.  $F_X(x)$  is a non-decreasing function
2.  $\lim_{x \rightarrow -\infty} F_X(x) = 0$
3.  $\lim_{x \rightarrow \infty} F_X(x) = 1$
4.  $F_X(x)$  is everywhere continuous

$$P(a < X < b) = \int_a^b f_X(x) dx$$

$$\begin{array}{l}
 1. \quad f_X(x) \geq 0 \quad \text{and} \\
 2. \quad \int_{-\infty}^{\infty} f_X(x) dx = 1
 \end{array}$$

$$P(a < X < b) = P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b)$$

Distribution	Density function $f_X(x)$	$M_X(t)$ mgf	Mean & Variance
Uniform	$\begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{elsewhere} \end{cases}$	$\frac{(e^{tb} - e^{ta})}{(b-a)t}$	$\begin{aligned} \mu_X &= \frac{(a+b)}{2} \\ \sigma_X^2 &= \frac{(b-a)^2}{12} \end{aligned}$
Gamma $g(\alpha; \lambda)$	$\begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$	$\left(1 - \frac{t}{\lambda}\right)^{-\alpha}, \quad t < \lambda$	$\begin{aligned} \mu_X &= \frac{\alpha}{\lambda} \\ \sigma_X^2 &= \frac{\alpha}{\lambda^2} \end{aligned}$
Exponential (special gamma with $\alpha = 1$ )	$\begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$	$\left(1 - \frac{t}{\lambda}\right)^{-1}$	$\begin{aligned} \mu_X &= \frac{1}{\lambda} \\ \sigma_X^2 &= \frac{1}{\lambda^2} \end{aligned}$
Chi-squared $\chi^2(r)$ (special gamma with $\alpha = \frac{n}{2}; \lambda = \frac{1}{2}$ )	$\begin{cases} \frac{1}{\Gamma(\frac{n}{2}) 2^{\frac{n}{2}}} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}, & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$	$(1 - 2t)^{-\frac{n}{2}}$	$\begin{aligned} \mu_X &= n \\ \sigma_X^2 &= 2n \end{aligned}$
Normal $N(\mu; \sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$	$e^{\mu t + \frac{1}{2}\sigma^2 t^2}$	$\begin{aligned} \mu_X &= \mu \\ \sigma_X^2 &= \sigma^2 \end{aligned}$
Standard normal $N(0; 1)$	$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad -\infty < x < \infty$	$e^{\frac{1}{2}t^2}$	$\begin{aligned} \mu_X &= 0 \\ \sigma_X^2 &= 1 \end{aligned}$
Beta type 1	$\begin{cases} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1}, & 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$	Beyond the scope of this module	$\begin{aligned} \mu_X &= \frac{a}{a+b} \\ \sigma_X^2 &= \frac{ab}{(a+b)^2(a+b+1)} \end{aligned}$

Description	Density function $[f_X(x)]$
t-distribution $t_n$	$\frac{\Gamma[(n+1)/2]}{\Gamma(n/2)\sqrt{n\pi}} \left(1 + \frac{t^2}{n}\right)^{-\frac{1}{2}(n+1)}, \quad -\infty < t < \infty$
F-distribution $F_{m,n}$	$\frac{\Gamma[(m+n)/2]}{\Gamma(m/2)\Gamma(n/2)} \left(\frac{m}{n}\right)^{m/2} x^{m/2-1} \left(1 + \frac{m}{n}x\right)^{-(m+n)/2} \quad x \geq 0$
$k$ -th order statistic	$\frac{n!}{(k-1)!(n-k)!} [F_X(x)]^{k-1} [1 - F_X(x)]^{n-k} f_X(x), \quad -\infty < x < \infty$

→ The  $p$ th quantile:  $F(x_p) = p$  or  $P(X \leq x_p) = p$  and  $x_p = F^{-1}(p)$

→  $\ln(1) = 0$

$$\rightarrow \int_0^\infty e^{-y} dy = 1$$

## Exponential Density

Memoryless proof

$$\begin{aligned}
 P(T > t + s | T > s) &= \frac{P(T > t + s \text{ and } T > s)}{P(T > s)} \\
 &= \frac{P(T > t + s)}{P(T > s)} \\
 &= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} \\
 &= e^{-\lambda t}
 \end{aligned}$$

## Gamma Density

Gamma function

$$\Gamma(a) = \int_0^{\infty} x^{a-1} e^{-x} \quad \text{Properties } \Gamma(\alpha + 1) = \alpha \Gamma(\alpha) \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$c^n \Gamma(n) = \int_0^{\infty} x^{n-1} e^{-\frac{x}{c}}$$

$$\Gamma(2) = 1$$

If  $\alpha = 1$  then it becomes exponential density

If  $\lambda = 1$  then it is a one parameter gamma density

$$\begin{aligned}
 \int_{-\infty}^{\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda t} dt &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-\lambda t} dt \\
 &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-t/\frac{1}{\lambda}} dt \\
 &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \Gamma(\alpha) \left(\frac{1}{\lambda}\right)^\alpha \\
 &= 1.
 \end{aligned}$$

## Normal (Gaussian) distribution

→ Standard normal CDF =  $\Phi(X)$

→ Standard normal density =  $\phi(X)$

## Beta Density

$$\text{Beta function: } B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \quad \text{or } B(m, n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

To prove density function:

For  $0 < x < 1$  and  $m > 0, n > 0$

$$\begin{aligned}
 \int_0^1 \frac{1}{B(m, n)} x^{m-1} (1-x)^{n-1} dx &= \frac{1}{B(m, n)} B(m, n) \\
 &= 1.
 \end{aligned}$$

$$\begin{aligned}
 B(m+1; n+1) &= \frac{m \Gamma(m) n \Gamma(n)}{(m+n+1) \Gamma(m+n+1)} \\
 &= \frac{mn}{(m+n+1)(m+n)} \times \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} \\
 &= \frac{mn}{(m+n+1)(m+n)} B(m; n).
 \end{aligned}$$

Different characteristics of the beta function			
Characteristic	Comment	Proof / Examples	Ref no
Type 1 beta function $\int_0^1 x^{m-1} (1-x)^{n-1} dx$ $= B(m; n)$	Recognize and use beta integral in integration	$\int_0^1 x^5 (1-x)^6$ $= \int_0^1 x^{6-1} (1-x)^{7-1}$ $= B(6; 7)$	B1
Symmetry with respect to the parameters $m > 0; n > 0$	$B(m; n) = B(n; m)$		B2
Relationship between beta and gamma functions	$m > 0; n > 0$ Numerical value for any beta function	$B(m; n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$	B4
Changing the limits of the integral from $\int_0^1 \dots$ to $\int_0^\infty \dots$	$\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$ $= B(m; n)$ which is a type 2 beta function	$\int_0^\infty \frac{x^3}{(1+x)^5} dx$ $= \int_0^\infty \frac{x^{4-1}}{(1+x)^{4+1}} dx$ $= B(4; 1)$	

→ If  $m = n = 1$  then becomes standard uniform distribution.

## Cauchy Density

$$f(x) = \frac{1}{\pi} \left( \frac{1}{1+x^2} \right) \quad -\infty < x < \infty$$

## Functions of random Variables

### Proposition A:

if  $X \sim N(\mu, \sigma^2)$  and  $Y = aX + b$ , then  $Y \sim N(a\mu + b, a^2\sigma^2)$

$$\begin{aligned}
 P(x_0 < X < x_1) &= F_X(x_1) - F_X(x_0) \\
 &= \Phi\left(\frac{x_1 - \mu}{\sigma}\right) - \Phi\left(\frac{x_0 - \mu}{\sigma}\right)
 \end{aligned}$$

→ 2 Methods

- 1) Cumulative distribution function  $F(x)$  method
  - find the **CDF** of the transformed variable
  - then differentiate to find the density function
  - and specify the region
- 2) Transformation method (Proposition B)

### PROPOSITION B

Let  $X$  be a continuous random variable with density  $f(x)$  and let  $Y = g(X)$  where  $g$  is a differentiable, strictly monotonic function on some interval  $I$ . Suppose that  $f(x) = 0$  if  $x$  is not in  $I$ . Then  $Y$  has the density function

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

for  $y$  such that  $y = g(x)$  for some  $x$ , and  $f_Y(y) = 0$  if  $y \neq g(x)$  for any  $x$  in  $I$ . Here  $g^{-1}$  is the inverse function of  $g$ ; that is,  $g^{-1}(y) = x$  if  $y = g(x)$ . ■



**PROPOSITION C**

Let  $Z = F(X)$ ; then  $Z$  has a uniform distribution on  $[0, 1]$ .

**Proof**

$$P(Z \leq z) = P(F(X) \leq z) = P(X \leq F^{-1}(z)) = F(F^{-1}(z)) = z$$

This is the uniform cdf. ■

**PROPOSITION D**

Let  $U$  be uniform on  $[0, 1]$ , and let  $X = F^{-1}(U)$ . Then the cdf of  $X$  is  $F$ .

**Proof**

$$P(X \leq x) = P(F^{-1}(U) \leq x) = P(U \leq F(x)) = F(x) \quad \blacksquare$$

Use  $D$  for generating pseudorandom numbers

$$[N(0, 1)]^2 \sim \chi_1^2$$

$$\text{Weibull density: } \frac{\beta}{\alpha^\beta} x^{\beta-1} e^{-\left(\frac{x}{\alpha}\right)^\beta}$$

**Study Unit 3****Discrete Random Variables**

$$p_{XY}(x, y) = P(X = x, Y = y)$$

$$1. \quad p_{X,Y}(x, y) \geq 0 \quad \text{for all } (x, y) \in R^2$$

$$2. \quad \sum_{x=0}^{n_1} \sum_{y=0}^{n_2} p_{X,Y}(x, y) = 1$$

**Continuous Random Variables**

Marginal density.

$$1. \quad f_{X,Y}(x, y) \geq 0 \quad \text{for all } (x, y) \in R^2$$

$$2. \quad \int \int_S f_{X,Y}(x, y) dy dx = 1$$

$$f_{X,Y}(x; y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x; y).$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x;v) dv \quad \text{for all } x \in R$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(u;y) du \quad \text{for all } y \in R$$

### Marginals sum to 1

Joint cumulative distribution.

$$\begin{aligned} F_{X,Y}(x;y) &= P(X < x; Y < y) \\ &= \int_{-\infty}^y \int_{-\infty}^x f_{X,Y}(u;v) dudv \end{aligned}$$

## Independent Random variables

Independent if  $F(x,y) = F_X(x)F_Y(y)$

Two discrete random variables will be independent if their joint mass function factors.

## Conditional Distributions

Discrete:

The law of total probability.  $p_X(x) = \sum_y p_{X|Y}(x|y)p_Y(y)$

Continuous:

The law of total probability.  $f_Y(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x)f_X(x) dx$

## Study Unit 4

## Expected Value of Random Variables

**Discrete**  $E(X) = \sum x p_X(x)$  Provided  $\sum |x| p_X(x) < \infty$

**Continuous**  $E(X) = \int_{-\infty}^{\infty} x f_X(x) dx$  Provided  $\int_{-\infty}^{\infty} |x| f_X(x) dx < \infty$

## Markov's Inequality

$$P(X \geq t) \leq \frac{E(X)}{t}$$

## Expectations of functions of random variables

→ Corollary A:  $E(XY) = E(X)E(Y)$

## Expectations of linear combinations of random variables

### THEOREM A

If  $X_1, \dots, X_n$  are jointly distributed random variables with expectations  $E(X_i)$  and  $Y$  is a linear function of the  $X_i$ ,  $Y = a + \sum_{i=1}^n b_i X_i$ , then

$$E(Y) = a + \sum_{i=1}^n b_i E(X_i)$$

→ Proof for  $n = 2$ :

- 1) Write out full expectation
- 2) Multiply out and separate integrals



- 3) First sums to 1
- 4) Second/third sum to expected values
- 5) State integral is convergent

## Variance and standard deviation

$$\text{Var}(X) = E([X - E(X)]^2)$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

<b>Discrete</b>	$\text{Var}(X) = \sum (x - \mu)^2 p_X(x)$
<b>Continuous</b>	$\text{Var}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$
<b>Standard deviation</b>	$SD = \sqrt{\text{Var}(X)}$

**Chebyshev's Inequality:** There is a high probability that  $X$  will deviate little from its mean if the variance is small.

Let  $X$  be a random variable with mean  $\mu$  and variance  $\sigma^2$ . Then, for any  $t > 0$ ,

$$P(|X - \mu| > t) \leq \frac{\sigma^2}{t^2}$$

Setting  $t = k\sigma$  then  $P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$  or what ever is asked.

## Covariance

$$\text{Cov}(X, Y) = E(X - \mu_X)(Y - \mu_Y)$$

$$= E(X, Y) - E(X)E(Y)$$

$$\text{Cov}(X, Y) = \int \int (x - \mu_X)(y - \mu_Y) f_{X,Y}(x, y) dx dy$$

If independent then:  $E(XY) = E(X)E(Y)$  and covariance is 0

### THEOREM A

Suppose that  $U = a + \sum_{i=1}^n b_i X_i$  and  $V = c + \sum_{j=1}^m d_j Y_j$ . Then

$$\text{Cov}(U, V) = \sum_{i=1}^n \sum_{j=1}^m b_i d_j \text{Cov}(X_i, Y_j)$$

### COROLLARY A

$$\text{Var}(a + \sum_{i=1}^n b_i X_i) = \sum_{i=1}^n \sum_{j=1}^n b_i b_j \text{Cov}(X_i, X_j).$$

### COROLLARY B

$$\text{Var}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \text{Var}(X_i), \text{ if the } X_i \text{ are independent.}$$

$$\text{Var}(X + Y) = \text{Cov}(X + Y, X + Y)$$

$$= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$$

$$E\left(\sum X_i\right) = \sum E(X_i)$$

$$\text{Var}\left(\sum X_i\right) = \sum \text{Var}(X_i) \text{ if } X_i \text{ are independent.}$$

## Correlation coefficient

**Revise Properly!!!!!!**

$$\rho = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

## Conditional Expectation and Prediction

$$E(Y|X = x) = \sum_y y p_{Y|X}(y|x)$$

$$E(Y|X = x) = \int y f_{Y|X}(y|x) dy$$

$$E[h(Y)|X = x] = \int h(y) f_{Y|X}(y|x) dy$$

### Law of total expectation

#### THEOREM A

$$E(Y) = E[E(Y|X)].$$

#### Proof

We will prove this for the discrete case. The continuous case is proved similarly. Using Theorem 4.1.1A we need to show that

$$E(Y) = \sum_x E(Y|X = x) p_X(x)$$

where

$$E(Y|X = x) = \sum_y y p_{Y|X}(y|x)$$

Interchanging the order of summation gives us

$$\sum_x E(Y|X = x) p_X(x) = \sum_y y \sum_x p_{Y|X}(y|x) p_X(x)$$

(It can be shown that this interchange can be made.) From the law of total probability, we have

$$p_Y(y) = \sum_x p_{Y|X}(y|x) p_X(x)$$

Therefore,

$$\sum_y y \sum_x p_{Y|X}(y|x) p_X(x) = \sum_y y p_Y(y) = E(Y) \quad \blacksquare$$

## Moment-generating functions

$$M_X(t) = E(e^{tX})$$

discrete :  $M_X(t) = \sum_x e^{tx} p_X(x)$

continuous :  $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx$

- Same mgf then same distribution.
- If the mgf can be determined it can be uniquely determines then probability distribution.
- The mgf provides an elegant way to compute the moments of a distribution.

### Calculating moments

→ First principles

→ Discrete:  $E(X^r) = \sum x^r p_X(x)$

→ Continuous:  $E(X^r) = \int x^r f_X(x) dx$

→ Using moment-generating function

→  $M^{(r)}(0) = E(X^r)$

→ Using Taylor series expansion

$$\begin{aligned}
 e^{tx} &= 1 + (tx) + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \dots \\
 &= 1 + (x) \cdot \frac{t}{1!} + (x)^2 \cdot \frac{t^2}{2!} + (x)^3 \cdot \frac{t^3}{3!} + \dots
 \end{aligned}$$

$$E(e^{tX}) = M_X(t) = 1 + E(X) \frac{t}{1!} + E(X^2) \frac{t^2}{2!} + E(X^3) \frac{t^3}{3!} + \dots$$

## Properties of moment-generating functions

If  $X$  has the mgf  $M_X(t)$  and  $Y = a + bX$ , then  $Y$  has the mgf  $M_Y(t) = e^{at} M_X(bt)$ .

If  $X$  and  $Y$  are independent random variables with mgf's  $M_X$  and  $M_Y$  and  $Z = X + Y$ , then  $M_Z(t) = M_X(t)M_Y(t)$  on the common interval where both mgf's exist.

→ The joint moment-generating function of two random variables  $X$  and  $Y$

$$M_{XY}(s, t) = E(e^{tX+sY}) = \sum_x \sum_y e^{tx+sy} p_{X,Y}(x, y) \text{ if } (X, Y) \text{ is discrete}$$

$$M_{XY}(s, t) = E(e^{tX+sY}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{tx+sy} f_{X,Y}(x, y) dx dy \text{ if } (X, Y) \text{ is continuous}$$

→ If  $X$  and  $Y$  are independent random variables then

$$M_{XY}(s, t) = M_X(s) M_Y(t)$$

## Approximate problems

Go over examples in rice again

→ Propagation of error or  $\delta$  method

→ The first order approximation of  $\mu_X$  is

$$Y = g(X) \approx g(\mu_X) + (X - \mu_X) g'(\mu_X)$$

→ and the mean and the variance of  $Y$  are

$$\mu_Y \approx g(\mu_X) \quad \text{and} \quad \sigma_Y^2 \approx \sigma_X^2 [g'(\mu_X)]^2.$$

→ The second order approximation of  $\mu_X$  is

$$Y = g(X) \approx g(\mu_X) + (X - \mu_X) g'(\mu_X) + \frac{1}{2} (X - \mu_X)^2 g''(\mu_X)$$

↑ Expected Value = 0

→ and the new improved mean of  $Y$  is

$$E(Y) \approx g(\mu_X) + \frac{1}{2} \sigma_X^2 g''(\mu_X).$$

## Study Unit 5

## Law of large numbers

### THEOREM A *Law of Large Numbers*

Let  $X_1, X_2, \dots, X_i \dots$  be a sequence of independent random variables with  $E(X_i) = \mu$  and  $\text{Var}(X_i) = \sigma^2$ . Let  $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$ . Then, for any  $\varepsilon > 0$ ,

$$P(|\bar{X}_n - \mu| > \varepsilon) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

#### Proof

We first find  $E(\bar{X}_n)$  and  $\text{Var}(\bar{X}_n)$ :

$$E(\bar{X}_n) = \frac{1}{n} \sum_{i=1}^n E(X_i) = \mu$$

Since the  $X_i$  are independent,

$$\text{Var}(\bar{X}_n) = \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i) = \frac{\sigma^2}{n}$$

The desired result now follows immediately from Chebyshev's inequality, which states that

$$P(|\bar{X}_n - \mu| > \varepsilon) \leq \frac{\text{Var}(\bar{X}_n)}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2} \rightarrow 0, \quad \text{as } n \rightarrow \infty \quad \blacksquare$$

## Central limit Theorem

→ Remember approximation  $\approx$  and not  $=$  when doing calculations because only equal in the limit.

Let  $X_1, X_2, \dots$  be a sequence of independent random variables having mean 0 and variance  $\sigma^2$  and the common distribution function  $F$  and moment-generating function  $M$  defined in a neighborhood of zero. Let

$$S_n = \sum_{i=1}^n X_i \quad \text{no need to prove}$$

Then

$$\lim_{n \rightarrow \infty} P\left(\frac{S_n}{\sigma\sqrt{n}} \leq x\right) = \Phi(x), \quad -\infty < x < \infty$$

## Study Unit 6

### $\chi^2$ , t and F distributions

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