STA2603 Distribution Theory II

→ Always specify the domain of the function!
→ Always specify the parameters when identifying the distribution!

Commonly used series expansions

$$e^{tx} = 1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \dots\dots\dots$$

$$\log(1+x) = \log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\dots\dots$$

$$\log(1-x) = \log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots\dots\dots$$

Study Unit 2

→ Frequency Function (discreet)
→ Density function (continuous).
→ Distribution function is cumulative density function

Discreet Random Variables

Distribution	Mass function $p_X(k)$	$M_X(t)$ mgf	Mean & Variance
Bernoulli $ber(p)$ one success	$\begin{cases} p^k(1-p)^{1-k} & k=0;1 \\ 0 & \text{elsewhere} \end{cases}$	$(1-p+pe^t)$	$\mu_X = p$ $\sigma_X^2 = p(1-p)$
Binomial $b(n;\ p)$ x successes in n trials	$\begin{cases} \binom{n}{x}p^x(1-p)^{n-x} & x=0;1;.;n \\ 0 & \text{elsewhere} \end{cases}$	$(1-p+pe^t)^n$	$\mu_X = np$ $\sigma_X^2 = np(1-p)$
Geometric $geo(\ p)$ 1st success in x trials	$\begin{cases} (1-p)^{x-1}p, & x=1;2;3;\dots \\ 0 & \text{elsewhere} \end{cases}$	$\frac{pe^t}{1-(1-p)e^t}$	$\mu_X = \frac{1}{p}$ $\sigma_X^2 = \frac{1-p}{p^2}$
Negative Binomial $nb(r;\ p)$ r'th success in k trials	$\begin{cases} \binom{k-1}{r-1}p^r(1-p)^{k-r}, & k=1;2;\dots \\ 0 & \text{elsewhere} \end{cases}$	$\left[\frac{pe^t}{(1-(1-p)e^t)}\right]^r$	$\mu_X = \frac{r}{p}$ $\sigma_X^2 = \frac{r(1-p)}{p^2}$
Poisson $Po(\lambda)$ K rare events	$\begin{cases} \frac{\lambda^k}{k!}e^{-\lambda}, & k=0;1;\dots \\ 0 & \text{elsewhere} \end{cases}$	$e^{\lambda(e^t-1)}$	$\mu_X = \lambda$ $\sigma_X^2 = \lambda$

Hypergeometric : $p(x) = \begin{cases} \frac{\binom{r}{x}\binom{n-r}{m-x}}{\binom{n}{m}} & \text{if } x = 0,1,2,\dots .n \\ 0 & \text{otherwise} \end{cases}$

Probability of x successes in a sample size m from sample space n and a subset r

$$\begin{array}{lll} 1. & p_X(x_i) & \geq & 0 \\ 2. & \sum_{x=0}^{\infty} p_X(x) & = & 1 \end{array}$$

$\binom{n}{x} = \dfrac{n!}{x!\,(n-x)!}$

Bernoulli Random Variables

Indicator random variable $I_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \text{otherwise} \end{cases}$

$$p_X(x) = \begin{cases} 1-p & \text{if } x = 0 \\ p & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1-p & \text{if } 0 < x < 1 \\ 1 & \text{if } x > 1 \end{cases}$$

Binomial Distribution

The sum of independent Bernoulli variables is a binomial random variable.
To prove binomial probabilities sums to 1. Use finite binomial series expansion:

$$\begin{aligned} (a+b)^n &= a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n-1} a^{n-(n-1)} b^{n-1} + \binom{n}{n} a^{n-n} b^n \\ &= (a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + b^n \\ &= \sum_{x=0}^{n} \binom{n}{x} a^{n-x} b^x \end{aligned}$$

and therefore it is a series expansion of $[(1-p)+p]^n$

Geometric Distribution

To prove that geometric distribution sums to 1, use Taylor series at x=0

$$\begin{aligned} (1-p)^{-1} &= 1 + \frac{(-1)}{1!}(-p) + \frac{(-1)(-1-1)}{2!}(-p)^2 + \frac{(-1)(-1-1)(-1-2)}{3!}(-p)^3 + .. \\ &= 1 + p + p^2 + p^3 + p^4 + \ldots \end{aligned}$$

$$\begin{aligned} p\sum_{x=1}^{\infty}(1-p)^{x-1} &= p\sum_{r=0}^{\infty}(1-p)^r \\ &= p\left[1 + (1-p) + (1-p)^2 + (1-p)^3 + \ldots\right] \\ &= p\,(1-(1-p))^{-1} \\ &= \frac{p}{p} \\ &= 1 \end{aligned}$$

Negative Binomial Distribution

A negative binomial random variable can be expressed as the sum of r independent geometric variables.

Hypergeometric distribution

$$pX(x) = \begin{cases} \frac{\binom{r}{x}\binom{n-r}{m-x}}{\binom{n}{m}} & \text{if } x = 0,1,2,\dots .n \\ 0 & \text{otherwise} \end{cases}$$

Poisson Distribution

To show it is a frequency function:

$$\begin{aligned} e^{\lambda} &= 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \dots \\ &= \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} \end{aligned}$$

$$\begin{aligned} \sum_{x=0}^{\infty} \frac{\lambda^k}{x!} e^{-\lambda} &= e^{\lambda} e^{-\lambda} \\ &= 1. \end{aligned}$$

Poisson distribution can be used to approximate binomial distribution, if n is large and p is small.

Continuous Random Variables

1. $F_X(x)$ is a non-decreasing function
2. $\lim\limits_{x \to -\infty} F_X(x) = 0$
3. $\lim\limits_{x \to \infty} F_X(x) = 1$
4. $F_X(x)$ is everywhere continuous

$$\boxed{P(a < X < b) = \int_a^b f_X(x)dx}$$

1. $f_X(x) \geq 0$ and
2. $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$P(a < X < b) = P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b)$$

Distribution	Density function $f_X(x)$	$M_X(t)$ mgf	Mean & Variance
Uniform	$\begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$	$\frac{(e^{tb}-e^{ta})}{(b-a)t}$	$\mu_X = \frac{(a+b)}{2}$ $\sigma_X^2 = \frac{(b-a)^2}{12}$
Gamma $g(\alpha;\ \lambda)$	$\begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}e^{-\lambda x}, & x \ge 0 \\ 0 & \text{elsewhere} \end{cases}$	$\left(1-\frac{t}{\lambda}\right)^{-\alpha}$, $t<\lambda$	$\mu_X = \frac{\alpha}{\lambda}$ $\sigma_X^2 = \frac{\alpha}{\lambda^2}$
Exponential (special gamma with $\alpha = 1$)	$\begin{cases} \lambda e^{-\lambda x}, & x \ge 0 \\ 0 & \text{elsewhere} \end{cases}$	$\left(1-\frac{t}{\lambda}\right)^{-1}$	$\mu_X = \frac{1}{\lambda}$ $\sigma_X^2 = \frac{1}{\lambda^2}$
Chi-squared $\chi^2(r)$ (special gamma with $\alpha = \frac{n}{2};\ \lambda = \frac{1}{2}$)	$\begin{cases} \frac{1}{\Gamma\left(\frac{n}{2}\right)2^{\frac{n}{2}}} x^{\frac{n}{2}-1}e^{-\frac{x}{2}}, & x \ge 0 \\ 0 & \text{elsewhere} \end{cases}$	$(1-2t)^{-\frac{n}{2}}$	$\mu_X = n$ $\sigma_X^2 = 2n$
Normal $N(\mu;\ \sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$	$e^{\mu t+\frac{1}{2}\sigma^2 t^2}$	$\mu_X = \mu$ $\sigma_X^2 = \sigma^2$
Standard normal $N(0;\ 1)$	$\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}, \quad -\infty < x < \infty$	$e^{\frac{1}{2}t^2}$	$\mu_X = 0$ $\sigma_X^2 = 1$
Beta type 1	$\begin{cases} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$	Beyond the scope of this module	$\mu_X = \frac{a}{a+b}$ $\sigma_X^2 = \frac{ab}{(a+b)^2(a+b+1)}$

Description	Density function $[f_X(x)]$
t-distribution t_n	$\frac{\Gamma[(n+1)/2]}{\Gamma(n/2)\sqrt{n\pi}}\left(1+\frac{t^2}{n}\right)^{-\frac{1}{2}(n+1)}, \quad -\infty < t < \infty$
F-distribution $F_{m,n}$	$\frac{\Gamma[(m+n)/2]}{\Gamma(m/2)\Gamma(n/2)}\left(\frac{m}{n}\right)^{m/2} x^{m/2-1}\left(1+\frac{m}{n}x\right)^{-(m+n)/2} \quad x \ge 0$
k-th order statistic	$\frac{n!}{(k-1)!(n-k)!}[F_X(x)]^{k-1}[1-F_X(x)]^{n-k}f_X(x), \quad -\infty < x < \infty$

The pth quantile: $F(x_p) = p$ or $P(X \le x_p) = p$ and $x_p = F^{-1}(p)$

Exponential Density

Gamma Density

Gamma function $\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}$ Properties $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$c^n\Gamma(n) = \int_0^\infty x^{n-1}e^{-\frac{x}{c}}$$

$\Gamma(2) = 1$

If $\alpha = 1$ then it becomes exponential density
If $\lambda = 1$ then it is a one parameter gamma density

$$\int_{-\infty}^{\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1}e^{-\lambda t}dt = \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-\lambda t}dt$$
$$= \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t/\frac{1}{\lambda}}dt$$
$$= \frac{\lambda^\alpha}{\Gamma(\alpha)}\Gamma(\alpha)\left(\frac{1}{\lambda}\right)^\alpha$$
$$= 1.$$

Normal (Gaussian) distribution

Beta Density

Beta function: $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ or $B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$

To prove density function:

For $0 < x < 1$ and $m > 0, n > 0$

$$\int_0^1 \frac{1}{B(m,n)} x^{m-1}(1-x)^{n-1}dx = \frac{1}{B(m,n)}B(m,n) = 1.$$

$$B(m+1;n+1) = \frac{m\Gamma(m)\,n\Gamma(n)}{(m+n+1)\Gamma(m+n+1)}$$
$$= \frac{mn}{(m+n+1)(m+n)} \times \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$
$$= \frac{mn}{(m+n+1)(m+n)}B(m;n).$$

Different characteristics of the beta function

Characteristic	Comment	Proof / Examples	Ref no
Type 1 beta function $\int_0^1 x^{m-1}(1-x)^{n-1}dx.$ $= B(m;n)$	Recognize and use beta integral in integration	$\int_0^1 x^5(1-x)^6$ $= \int_0^1 x^{6-1}(1-x)^{7-1}$ $= B(6;7)$	$B1$
Symmetry with respect to the parameters $m > 0;\ \ n > 0$	$B(m;n) = B(n;m)$		$B2$
Relationship between beta and gamma functions	$m > 0;\ \ n > 0$ *Numerical value* for any beta function	$B(m;n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$	$B4$
Changing the limits of the integral from $\int_0^1 \ldots$ to $\int_0^\infty \ldots$	$\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$ $= B(m;n)$ which is a type 2 beta function	$\int_0^\infty \frac{x^3}{(1+x)^5}dx$ $= \int_0^\infty \frac{x^{4-1}}{(1+x)^{4+1}}dx$ $= B(4;1)$	

Cauchy Density

$f(x) = \frac{1}{\pi}\left(\frac{1}{1+x^2}\right) \quad -\infty < x < \infty$

Functions of random Variables

Standard normal CDF $= \Phi(X)$ Standard normal density $= \phi(X)$

Proposition A:

if $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$, then $Y \sim N(a\mu + b, a^2\sigma^2)$

PROPOSITION B

Let X be a continuous random variable with density $f(x)$ and let $Y = g(X)$ where g is a differentiable, strictly monotonic function on some interval I. Suppose that $f(x) = 0$ if x is not in I. Then Y has the density function

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

for y such that $y = g(x)$ for some x, and $f_Y(y) = 0$ if $y \neq g(x)$ for any x in I. Here g^{-1} is the inverse function of g; that is, $g^{-1}(y) = x$ if $y = g(x)$. ■

PROPOSITION C

Let $Z = F(X)$; then Z has a uniform distribution on [0, 1].

Proof

$$P(Z \leq z) = P(F(X) \leq z) = P(X \leq F^{-1}(z)) = F(F^{-1}(z)) = z$$

This is the uniform cdf. ■

PROPOSITION D

Let U be uniform on [0, 1], and let $X = F^{-1}(U)$. Then the cdf of X is F.

Proof

$$P(X \leq x) = P(F^{-1}(U) \leq x) = P(U \leq F(x)) = F(x)$$ ■

$[N(0,1)]^2 \sim \chi_1^2$

Weibull density: $\frac{\beta}{\alpha^\beta} x^{\beta-1} e^{-\left(\frac{x}{\alpha}\right)^\beta}$

Study Unit 3

Discreet Random Variables

$$\boxed{p_{XY}(x,y) = P(X = x, Y = y)}$$

1.	$p_{X,Y}(x,y)$	$\geq$	0	for all $(x,y) \in R^2$
2.	$\sum_{x=0}^{n_1} \sum_{y=0}^{n_2} p_{X,Y}(x,y)$	$=$	1	

Continuous Random Variables

Marginal density.

1.	$f_{X,Y}(x,y)$	≥ 0	for all $(x,y) \in R^2$
2.	$\int\int_S f_{X,Y}(x,y)\,dydx$	$= 1$	

$$f_{X,Y}(x;y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x;y).$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x;v)\,dv \qquad \text{for all } x \in R$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(u;y)\,du \qquad \text{for all } y \in R$$

Marginals sum to 1

Joint cumulative distribution.

$$\begin{aligned} F_{X,Y}(x;y) &= P(X < x;\ Y < y) \\ &= \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u;v)\,dudv \end{aligned}$$

Independent Random variables

Independent if $F(x,y) = F_X(x)F_Y(y)$

Two discreet random variables will be independent if th their joint mass function factors.

Conditional Distributions

Discreet:

The law of total probability. $p_X(x) = \sum_y p_{X|Y}(x|y)p_Y(y)$

Continuous:

The law of total probability. $f_Y(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x)f_X(x)\,dx$

Study Unit 4

Expected Value of Random Variables

Discrete	$E(X) = \sum x p_X(x)$	Provided $\sum_x \lvert x \rvert\, p_X(x) < \infty$
Continuous	$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\,dx$	Provided $\int_{-\infty}^{\infty} \lvert x \rvert\ f_X(x)\,dx < \infty$

Markov's Inequality

$P(X \geq t) \leq \frac{\mathrm{E}(X)}{t}$

Expectations of functions of random variables

→ Corollary A: $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$

Expectations of linear combinations of random variables

THEOREM A

If $X_1, \ldots, X_n$ are jointly distributed random variables with expectations $E(X_i)$ and Y is a linear function of the X_i, $Y = a + \sum_{i=1}^{n} b_i X_i$, then

$$E(Y) = a + \sum_{i=1}^{n} b_i E(X_i)$$

→ Proof for $n = 2$:
1) Write out full expectation
2) Multiply out and separate integrals
3) First sums to 1
4) Second/third sum to expected values
5) State integral is convergent

Variance and standard deviation

$Var(X) = \mathrm{E}\left([X - \mathrm{E}(X)]^2\right)$
$Var(X) = \mathrm{E}(X^2) - [E(X)]^2$

Discrete	$Var(X) = \sum_x (x-\mu)^2 \; p_X(x)$
Continuous	$Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 \; f(x)\, dx$
Standard deviation	$SD = \sqrt{Var(X)}$

Chebyshev's Inequality: There is a high probability that X will deviate little from its mean if the variance is small.

Let X be a random variable with mean μ and variance σ^2. Then, for any $t > 0$,

$$P(|X - \mu| > t) \leq \frac{\sigma^2}{t^2}$$

Setting $t = k\sigma$ then $P(|X - \mu| \geq k\sigma) \leq \dfrac{1}{k^2}$ or what ever is asked.

Covariance

$$\begin{aligned} Cov(X,Y) &= \mathrm{E}(X-\mu_X)(Y-\mu_Y) \\ &= \mathrm{E}(X,Y) - \mathrm{E}(X)\mathrm{E}(Y) \end{aligned}$$

$$\boxed{Cov(X,Y) = \int\int (x-\mu_X)(y-\mu_Y)\, f_{X,Y}(x,y)\, dxdy}$$

If independent then: $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$ and covariance is 0

THEOREM A

Suppose that $U = a + \sum_{i=1}^{n} b_i X_i$ and $V = c + \sum_{j=1}^{m} d_j Y_j$. Then

$$\mathrm{Cov}(U, V) = \sum_{i=1}^{n} \sum_{j=1}^{m} b_i d_j \mathrm{Cov}(X_i, Y_j)$$

COROLLARY A

$\mathrm{Var}(a + \sum_{i=1}^{n} b_i X_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_i b_j \mathrm{Cov}(X_i, X_j)$.

COROLLARY B

$\mathrm{Var}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \mathrm{Var}(X_i)$, if the X_i are independent.

$$\begin{aligned} \mathrm{Var}(X+Y) &= \mathrm{Cov}(X+Y, X+Y) \\ &= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y) \end{aligned}$$

$\mathrm{E}\left(\sum X_i\right) = \sum \mathrm{E}(X_i)$

$\mathrm{Var}\left(\sum X_i\right) = \sum \mathrm{Var}(X_i)$ if X_i are independent.

Correlation coefficient

Revise Properly!!!!!!

$$\rho = \frac{\mathrm{Cov}(X,\ Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$$

Conditional Expectation and Prediction

$$E(Y|X=x) = \sum_y y p_{Y|X}(y|x)$$

$$E(Y|X=x) = \int y f_{Y|X}(y|x)\,dy$$

$$E[h(Y)|X=x] = \int h(y) f_{Y|X}(y|x)\,dy$$

Law of total expectation

THEOREM **A**

$E(Y) = E[E(Y|X)]$.

Proof

We will prove this for the discrete case. The continuous case is proved similarly. Using Theorem 4.1.1A we need to show that

$$E(Y) = \sum_x E(Y|X=x) p_X(x)$$

where

$$E(Y|X=x) = \sum_y y p_{Y|X}(y|x)$$

Interchanging the order of summation gives us

$$\sum_x E(Y|X=x) p_X(x) = \sum_y y \sum_x p_{Y|X}(y|x) p_X(x)$$

(It can be shown that this interchange can be made.) From the law of total probability, we have

$$p_Y(y) = \sum_x p_{Y|X}(y|x) p_X(x)$$

Therefore,

$$\sum_y y \sum_x p_{Y|X}(y|x) p_X(x) = \sum_y y p_Y(y) = E(Y) \qquad \blacksquare$$

Moment-generating functions

$$\boxed{M_X(t) = E\left(e^{tX}\right)}$$

discrete : $M_X(t) = \sum_x e^{tx} p_X(x)$

continuous : $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$

→ Same mgf then same distribution.
→ If the mgf can be determined it can be uniquely determines then probability distribution.
→ The mgf provides an elegant way to compute the moments of a distribution.

Calculating moments

→ First principles

→ Discreet: $\mathrm{E}(X^r) = \sum x^r p_X(x)$

→ Continuous: $\mathrm{E}(X^r) = \int x^r f_X(x)$

→ Using moment-generating function

→ $M^{(r)}(0) = E(X^r)$

→ Using Taylor series expansion

$$\begin{aligned} e^{tx} &= 1 + (tx) + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots \\ &= 1 + (x) \cdot \frac{t}{1!} + (x)^2 \cdot \frac{t^2}{2!} + (x)^3 \cdot \frac{t^3}{13!} + \ldots \end{aligned}$$

$$\boxed{E\left(e^{tX}\right) = M_X(t) = 1 + E(X)\frac{t}{1!} + E\left(X^2\right)\frac{t^2}{2!} + E\left(X^3\right)\frac{t^3}{3!} + \cdots}$$

Properties of moment-generating functions

If X has the mgf $M_X(t)$ and $Y = a + bX$, then Y has the mgf $M_Y(t) = e^{at} M_X(bt)$.

If X and Y are independent random variables with mgf's M_X and M_Y and $Z = X + Y$, then $M_Z(t) = M_X(t)M_Y(t)$ on the common interval where both mgf's exist.

Study Unit 5

Study Unit 6