Chapter 7

Gauge Field Theories and Fundamental Interactions

7.1 Introduction

Our present understanding of the physical phenomena in nature and the laws governing them is based on the assumption that quarks and leptons are the basic constituents of matter which interact with each other through strong (quarks only), weak, electromagnetic, and gravitational interactions. One of the major aims of scientists in the physics community has been to formulate a unified theory of all these fundamental interactions to describe the natural phenomenon. The first and earliest step in this direction was to unify electromagnetic and gravitational interactions, both of them being long range interactions, that is, proportional to $\frac{1}{r}$; many attempts were made to unify them in the early twentieth century. It was then believed that these were the only two fundamental interactions and the interactions could be described by field theories based on the principle of invariance under certain transformations called local gauge transformations because of their explicit dependence on space–time coordinates. In this type of field theories, the electromagnetic interaction between two charged particles is described by the exchange of a massless vector field $A_\mu(x)$ as proposed by Weyl [40], while the gravitational interaction between the two objects is described by the exchange of a tensor field $g_{\mu\nu}(x)$ as proposed by Weyl [322] and Einstein [323]. Later, after the discovery of the atomic nucleus and the experimental studies of the structure of nuclei and the phenomenon of nuclear radioactivity, two more fundamental interactions, viz., strong and weak interactions were revealed. The existence of strong interaction is responsible for binding neutrons and protons together and the weak interaction enables them to decay inside the nucleus. Both the interactions were found to be of short range. The need was felt to formulate a unified theory of all the four fundamental interactions, viz., the electromagnetic, strong, weak, and gravitational interactions. In analogy with the theory of electromagnetic interactions being mediated by a massless neutral vector field $A_\mu(x)$ called the photon, Yukawa [92] proposed that the newly

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discovered short range strong interactions are mediated by complex scalar field $U(x)$ (in fact, the scalar component of a four-vector field like ϕ in the case of electromagnetic interactions and $\bar{\Psi}_n \gamma_o \Psi_n = \Psi^\dagger \Psi$ in the case of weak interactions) corresponding to a "heavy particle" called meson with a mass 'm' appropriate to the short range 'r' of the strong interactions through the relation $m \propto \frac{1}{r}$. He further proposed that these meson fields $U(x)$ coupled strongly to the nucleons through strong interactions to provide nuclear binding, and coupled weakly to the leptons and nucleons through weak interactions describing the processes of β-decay, that is, $n \rightarrow p + e^- + \bar{\nu}_e$, and $p \rightarrow n + e^+ + \nu_e$ in nuclei. This idea led to the generalization of the concept of mediating fields which was used later to formulate the theoretical models of fundamental interactions. Such theoretical models of fundamental interactions were proposed earlier by Klein [324] and Kemmer [318] in the 1930s; in hindsight, these models can be considered as extensions of Weyl's theory of electromagnetism to describe strong and weak interactions.

In the theory of electromagnetism formulated by Weyl [40], the vector field $A_\mu(x)$ mediating the electromagnetic interaction was a neutral massless gauge field arising due to the requirement of invariance of the free electron Lagrangian under a local gauge transformation corresponding to U(1) symmetry. Kemmer [318] extended the formalism of the gauge field theory to describe nuclear interactions based on the SU(2) symmetry in isospin space which was an experimentally discovered symmetry in nuclear interactions. This model introduced a triplet of massless vector gauge fields corresponding to the three generators of SU(2). The two charge gauge fields of the triplet could describe the strong interactions between neutrons and protons or the weak interactions between them. In the case of weak interactions, this would imply the existence of neutral currents corresponding to the third component of the triplet for which there were no experimental evidence, even though their existence was speculated earlier by Wentzel [325] and Gamow and Teller [317].

Klein [192] also suggested the existence of a triplet of vector fields in the context of the Kaluza–Klein theory in the five dimensional space–time, deducing from the compactification of the fifth dimension. This triplet of vector fields is similar to the massless gauge fields corresponding to the SU(2) symmetry suggested by Kemmer. However, in this model, while the two charged fields of the triplet are used to describe the strong or weak charged interactions, the third neutral field was assigned to describe the electromagnetic interactions. Assigning charged and neutral fields to the same triplet implied that the strong (weak), and electromagnetic interactions had the same strength of coupling, a prediction not supported by experiments. Therefore, the study of the models proposed by Kemmer and Klein to explain fundamental interactions in analogy with the gauge field theoretical model of electromagnetic interactions given by Weyl [40] was not pursued any further for a long time.

It took more than 20 years before interest in the study of gauge field theoretical models of the fundamental interactions and their role in obtaining a unified theory of fundamental interactions was revived in the mid-1950s. The revival of interest was due to two major developments in our understanding of weak and strong interactions of elementary particles.

One was in the field of strong interactions, where an elegant extension of Weyl's theory to non-abelian gauge field theories was formulated to describe strong interactions [326, 327, 328]; the other was in the field of weak interactions, where a very successful phenomenological theory of the weak interaction was evolved from extensive experimental as well as theoretical studies of various weak interaction processes for almost 25 years after the Fermi's theory [329], leading to the $V - A$ theory of weak interactions. In the formulation of the gauge field theory of strong interactions[326, 327, 328], the strong interaction was assumed to be invariant under local gauge transformation corresponding to SU(2) symmetry in isospin space. The requirement of invariance under local SU(2) symmetry introduced the existence of a triplet of massless vector gauge fields called Yang-Mills fields, which were self interacting. While the masslessness and the self interaction of the gauge field assured that the theory could be renormalized, it was not appropriate to describe phenomenologically the strong or weak interactions, as the masslessness of the gauge fields implied long range. Introducing a mass term in the interaction or introducing more particles to generate mass through their interaction with the gauge fields would destroy the renormalizability of the theory. The attempts in this direction to formulate a gauge theory of strong and weak interactions were not successful.

The $V - A$ theory was very successful in describing the weak processes at low energies but was found inadequate at higher energies due to the violations of unitarity and the presence of divergences in the higher orders of perturbation theory. The theory was essentially not renormalizable. However, it was considered to be a low energy manifestation of an appropriate high energy theory of weak interactions presumably mediated by charged intermediate vector bosons (IVB) that are massive with a mass M corresponding to the range r of weak interactions; the bosons interact with the weak charged vector and axial vector $(V - A)$ currents of nucleons and leptons. The introduction of such naive IVB helped to partially solve the divergence problems present in the $V - A$ theory, but the theory still remained non-renormalizable(see Chapter 5). However, the possibility that the weak interactions are mediated by vector bosons which if identified with massless gauge fields corresponding to a local SU(2) symmetry in weak isospin space could lead to a renormalizable theory of weak and electromagnetic interactions. This was an attractive idea proposed by many during the 1950s [50, 51, 49], but it encountered many difficulties similar to those faced by the models proposed by Kemmer and Klein. In these models, weak interactions are mediated by the triplet of massless isovector vector fields, like the gauge fields of the Yang–Mills theory [326], which interact weakly with the vector, and axial vector currents of nucleons and leptons. While the charged components of the isovector vector fields interact with the charged weak currents, the third and the neutral component of the isovector fields is assumed to interact either with the electromagnetic current or a possible neutral weak current. Such models faced the following problems in achieving a satisfactory description of the weak interactions and its unification with the electromagnetic interactions.

a. The two charged gauge fields proposed to be mediating the weak interactions were predicted to be massless. The masslessness of the gauge fields rendered the theory renormalizable, but implied the range of weak interactions to be infinite which was phenomenologically incorrect.
b. If the third neutral component of the isovector vector gauge field is proposed to be weakly interacting like in the models of Kemmer [318] and Bludman [50], then the model predicts the existence of neutral currents, a result not supported by experiments at that time.

c. If the third neutral component of the isovector vector gauge field is identified with the electromagnetic field $A_\mu(x)$ putting the two charged weak fields and the electromagnetic field in the same triplet like in the models of Klein [324], Schwinger [49], and Leite Lopes [51], it leads to the following difficulties.

   i) It was hard to explain the simultaneous presence of different types of coupling of the three components of an isovector field to the nucleonic and leptonic weak currents. While the third neutral component of the isovector field would have a parity conserving coupling to the electromagnetic currents of charged leptons and nucleons, the charged components of the same isovector field would have parity conserving as well as parity violating couplings to the charged weak currents of nucleons and leptons. This is not allowed by the SU(2) symmetry if the three fields belong to the same triplet.

   ii) The same effective coupling strength for weak and electromagnetic interactions, that is, $e^2 = g^2/M_W^2$, where e and g are respectively, the strength of the coupling of the electromagnetic field (A_μ) and the weak fields ($W^\pm$) to the electromagnetic and weak currents implying a mass of vector field $M_W \sim \sqrt{\frac{g^2}{e^2}} \sim 30$ GeV. This is not compatible with the vector gauge fields being massless.

Thus, faced with the problem of massless vector bosons, a new idea of spontaneous symmetry breaking (SSB) inspired by the Bardeen–Cooper–Schrieffer (BCS) theory of superconductivity in solid state physics was brought to particle physics by Nambu [330] and Goldstone [162] to generate masses for the vector gauge fields in the theory. The initial application of such models of SSB to gauge field theories in the interaction Lagrangian for massless fermions and bosons with exact continuous symmetry like U(1) or SU(2) led to the appearance of massless spin 0 bosons known as the Nambu–Goldstone bosons. The theory succeeded in dynamically generating the mass of fermions and bosons, whereas experimentally there was no evidence for the existence of the massless and spinless, Nambu–Goldstone bosons, predicted by this type of theories. At first, it seemed to be a model-dependent result but soon, it was proved to be quite a general result in all the Lorentz invariant field theories with exact symmetry under the continuous symmetry transformation that are broken spontaneously [331, 332]. Therefore, the gauge field theories which are invariant under exact continuous local symmetry and are broken spontaneously by the Nambu–Goldstone mechanism had two undesirable features which prevented them from being a suitable theory of weak interactions: one was the appearance of the massless vector gauge bosons and the other was the appearance of the massless scalar Nambu–Goldstone bosons. Thus, the theories based on SSB were inapplicable to strong or weak interactions.

However, following the suggestion of Schwinger [333] that the invariance under the exact gauge symmetry need not always imply the existence of massless vector bosons if their coupling with the matter fields is strong, Anderson explicitly showed that [334] in a system with high density of electrons which are interacting via electromagnetic interactions, the massless photon acquires a mass through its interaction with the free electron gas. This was similar to the situation in superconductivity, where the photon acquires a mass responsible for explaining the Meissner effect through the spontaneous breaking of gauge symmetry of electromagnetic interactions. In both the cases of high density electrons and superconductivity, the physical systems were essentially non-relativistic. However, in a series of papers published independently by Englert and Brout [159], Higgs [158] and Guralnik [335, 336], Guralnik and Hagen [337], and Kibble [161], it was shown to be the case also in the Lorentz covariant relativistic field theories if the exact continuous symmetry is local and not global. In the case of a local gauge invariant field theory, which is spontaneously broken, the massless spin 0 field becomes the longitudinal component of the massless spin-1 gauge field, which appears due to the requirement of the local gauge invariance, making the field massive. This is known as the Higgs mechanism.

The Higgs mechanism applied to spontaneously broken local gauge field theories provided a way to generate masses for gauge fields. With an appropriate choice of the symmetry group to be SU(2) $\times$ U(1). Weinberg [157] and Salam [37] independently proposed a model for the unified theory of weak and electromagnetic interactions of leptons, which was later extended to quarks and hadrons using the GIM mechanism [64] for describing the weak interactions of hadrons. The model correctly reproduces all the aspects of the $V - A$ theory at low energies and predicts the existence of neutral currents which were observed in 1973.

Salam [37] and Weinberg [157] both speculated that spontaneously broken local gauge field theory was also renormalizable like the exact symmetric gauge field theories but gave no proof. It was t'Hooft and Veltman [338] and Lee and Zinn-Justin [339] who proved the renormalizability of spontaneously broken local gauge field theories using the formalism based on Feynman's path integrals. Thus, with the renormalizability of the model established firmly, the Weinberg–Salam–Glashow model provided a successful unified theory of weak and electromagnetic interactions.

In summary, the main concepts leading to a unified theory of electromagnetic and weak interactions are as follows:

- Gauge invariance of field theory.
- Spontaneously broken symmetries of gauge field theories.
- Higgs mechanism for the generation of mass of gauge fields.
- Choice of the gauge symmetry group and its fundamental representation.

In this chapter, we shall develop an understanding of the principle of gauge invariance and its consequences leading to the introduction of massless gauge fields as carriers of fundamental interactions and apply the concept of spontaneous symmetry breaking of local gauge field theories to generate the mass of gauge fields using Higgs mechanism.

7.2 Gauge Invariance in Field Theory

The idea of gauge invariance as a dynamical principle to describe electromagnetic interactions and its nomenclature was conceptualized by Weyl during his efforts to find a unified theory of electromagnetic and gravitational interactions [40]. Since then, efforts have been made to use this concept as the symmetry principle to formulate the theory of weak, strong, and electromagnetic interactions. In fact, the principle of gauge invariance is equivalent to the principle of phase invariance in quantum mechanics and field theory in the context of wave functions and fields; however, the terminology of gauge invariance is used even now to describe the physical concepts and underlying principles of phase invariance. As a necessary prerequisite to understand these efforts, we first describe the principle of gauge invariance in classical electrodynamics and then elaborate the role of gauge invariance as a dynamical principle in describing electromagnetic interactions in field theory.

In Chapter 2, we have already introduced the concept of gauge invariance in field theory, where we discussed the gauge invariance in classical electrodynamics and reviewed Maxwell's theory of electrodynamics. In this section, we will first obtain Maxwell's equations from a variational principle.

7.2.1 Maxwell's equations from a variational principle

It is instructive to note that Maxwell's equation for the electromagnetic field $A_\mu(x_\mu)$ can be derived from a suitable Lagrangian $\mathcal{L}(x)$ for the electromagnetic field in the presence of sources densities $J^\mu(x)$, that is,

$$\mathcal{L}(x) = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - J_\mu(x)A^\mu(x) \tag{7.1}$$

or from the Lagrangian of free electromagnetic field, $\mathcal{L}_{\text{free}}$ given by:

$$\begin{aligned}\mathcal{L}_{\text{free}}(x) &= -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)\\ &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)\\ &= -\frac{1}{2}\partial_\mu A_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu)\end{aligned} \tag{7.2}$$

$$\Rightarrow \qquad \frac{\partial \mathcal{L}_{\text{free}}}{\partial A_\nu} = 0, \qquad \text{and} \qquad \frac{\partial \mathcal{L}_{\text{free}}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}. \tag{7.3}$$

Putting this in the Euler Lagrange equation:

$$\frac{\partial \mathcal{L}_{\text{free}}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}_{\text{free}}}{\partial(\partial_\mu A_\nu)} = 0, \tag{7.4}$$

which results in

$$\partial_\mu F^{\mu\nu} = 0. \tag{7.5}$$

We may write the action using the Lagrangian in Eq. (7.1)

$$S = \int \mathcal{L} d^4x = \int \left[-\frac{1}{4}g_{\mu\lambda}g_{\nu\rho}F^{\lambda\rho}F^{\mu\nu} - J^\mu A_\mu\right] d^4x.$$

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Any variation in S leads to:

$$\delta S = \int \left[-\frac{1}{2} g_{\mu\lambda} g_{\nu\rho} F^{\lambda\rho} \delta F^{\mu\nu} - J^{\mu} \delta A_{\mu} \right] d^4x.$$

Using the property $F^{\lambda\rho} = -F^{\rho\lambda}$, we may write:

$$\delta S = \int \left[-F^{\lambda\rho} \partial_{\lambda} \delta A_{\rho} - J^{\mu} \delta A_{\mu} \right] d^4x.$$

Using the method of integration by parts in the first part of the right-hand side (R.H.S), we obtain:

$$\delta S = \int \left[\partial_{\lambda} F^{\lambda\rho} - J^{\rho} \right] \delta A_{\rho} d^4x$$

which vanishes ($\delta S = 0$) for arbitrary δA_{ρ}, such that:

$$\partial_{\mu} F^{\mu\nu} = J^{\nu} \tag{7.6}$$

which is nothing but the inhomogeneous Maxwell's equation in the medium.

It is also seen from Eqs. (7.1) and (7.2), that the presence of a mass term like $m^2 A_{\mu} A^{\mu}$ is not allowed in the Lagrangian since $m^2 A_{\mu} A^{\mu}$ transforms under the local gauge transformation as:

$$A^{\mu}(x) A_{\mu}(x) \to A^{'\mu}(x) A^{'}_{\mu}(x) = (A^{\mu}(x) - \partial^{\mu}\Lambda)(A_{\mu}(x) - \partial_{\mu}\Lambda) \neq A^{\mu}(x) A_{\mu}(x) \tag{7.7}$$

and violates the local gauge invariance. Thus, the invariance under local gauge invariance requires the electromagnetic field to be massless, a property inherent in Maxwell's equations of electrodynamics.

7.2.2 Gauge invariance in field theory and phase invariance

In field theory, the free particles of spin 0, 1/2, 1 are described by the scalar, spinor, and vector fields represented by ϕ, ψ, A^{μ}, which are solutions of the equations of motion for these fields derived from the given Lagrangian using Euler–Lagrange equations. For example, the Lagrangian for free scalar and spinor fields, ϕ and ψ, respectively, are written as:

$$\begin{aligned} \mathcal{L}^{\phi}_{\text{free}} &= \frac{1}{2}(\partial_{\mu}\phi(x))^{*}\partial^{\mu}\phi(x) - \frac{1}{2}m^2\phi^{*}(x)\phi(x), \qquad \phi = \phi_1 + i\phi_2, \\ \mathcal{L}^{\psi}_{\text{free}} &= \overline{\psi}(x)(i\gamma_{\mu}\partial^{\mu} - m)\psi(x). \end{aligned} \tag{7.8}$$

$\mathcal{L}^{A}_{\text{free}}$ for the electromagnetic field A is given in Eq. (7.2).

A phase transformation on the fields $\phi(x)$ or $\psi(x)$ is defined as the transformation in which

$$\phi(x) \to \phi'(x) = e^{i\alpha}\phi(x)\,; \qquad \phi^{*}(x) \to \phi'^{*}(x) = \phi^{*}(x)e^{-i\alpha} \tag{7.9}$$

$$\text{and} \quad \psi(x) \to \psi'(x) = e^{i\alpha}\psi(x); \qquad \psi^{*}(x) \to \psi'^{*}(x) = \psi^{*}(x)e^{-i\alpha}, \tag{7.10}$$

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where α is a parameter describing the phase transformations. If α is a constant, independent of the space–time coordinates $(\vec{x}, t)$, the transformation is called the global phase transformation (Figure 7.1), that is, the symmetry transformation is carried out by the same amount at each point in space and time. If the parameter depends on space and time, that is, $\alpha = \alpha(\vec{x}, t)$,

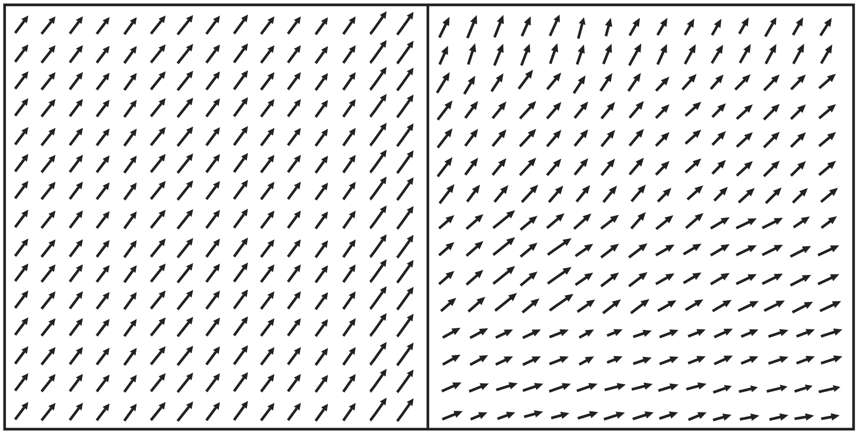

Figure 7.1 (Left) In a global transformation, α has a fixed value; therefore, $e^{i\alpha}$ has a constant value throughout space–time. (Right) In a local transformation, $e^{i\alpha(x)}$, where $x = x_\mu$ is a point in space–time, has a different value at each space–time point.

the transformations are called local phase transformations. For example, in case of rotation, the angles of rotation can change as one moves at different locations and times. It is clear that under the global phase transformation, the fields $\phi(\psi)$ and their derivatives $\partial^\mu\phi(\partial^\mu\psi)$ transform in the same way, that is,

$$\partial^\mu\phi(x) \to \partial^\mu\phi'(x) = e^{i\alpha}\partial^\mu\phi(x),$$

$$\partial^\mu\psi(x) \to \partial^\mu\psi'(x) = e^{i\alpha}\partial^\mu\psi(x), \tag{7.11}$$

with corresponding transformations for the complex conjugate of the field derivatives $\partial^\mu\phi^*(x)$ $[\partial^\mu\psi^*(x)]$. These transformations leave the Lagrangian for scalar (spinor fields) in Eq. (7.8) invariant under global phase transformations.

However, the situation is not so in the case of local phase transformations. Let us illustrate this for the case of spinor field ψ. Under the local phase transformation, prescribed by the parameter $\alpha(x)$, the field ψ transforms as:

$$\psi(x) \to \psi'(x) = e^{i\alpha(x)}\psi(x), \tag{7.12}$$

The field derivative transforms as:

$$\partial^\mu\psi(x) \to \partial^\mu\psi'(x) = e^{i\alpha(x)}[\partial^\mu\psi(x) + i(\partial^\mu\alpha(x))\psi(x)]. \tag{7.13}$$

The presence of the second term shows that the field derivative undergoes more changes in addition to the change in the phase of the initial field derivative, spoiling the invariance of the

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Lagrangian given in Eq. (7.8), which now acquires an extra term $\mathcal{L}_{\text{add}}$ given by:

$$\mathcal{L}_{\text{add}} = -\bar{\psi}(x)\gamma_\mu \partial^\mu \alpha(x)\psi(x). \tag{7.14}$$

The invariance of the Lagrangian under local phase transformations can be restored if the derivative operator is modified in such a way that the field $\psi(x)$ and its derivatives transform in the same way by acquiring the same phase as in the global gauge transformations. This is done by defining a covariant derivative D^μ as:

$$D^\mu = \partial^\mu - ieA^\mu(x), \tag{7.15}$$

where $-e$ is the charge on an electron and $A^\mu(x)$ is the vector field such that:

$$D^\mu \psi(x) \to (D^\mu \psi(x))' = e^{i\alpha(x)} D^\mu \psi(x)$$

requiring that $A^\mu(x)$ transforms as:

$$A^\mu(x) \to A^\mu(x') = A^\mu(x) + \frac{1}{e}\partial^\mu \alpha(x) \tag{7.16}$$

and making the term like $\bar{\psi}\gamma^\mu D_\mu \psi$ invariant under the local phase transformation. Thus, substituting the ordinary derivative ∂^μ everywhere in the Lagrangian by the covariant derivative D^μ, the Lagrangian for the scalar, spinor, or the vector fields can be made locally phase invariant. If the vector field $A^\mu(x)$ is identified with the electromagnetic field and e with the charge, then the transformation given in Eq. (7.16) as a requirement of the local phase invariance is same as the condition of the local gauge invariance in Maxwell's electrodynamics with $\Lambda(x) = -\frac{1}{e}\alpha(x)$(see Chapter 3). Therefore, the terms "local gauge invariance" and "local phase invariance" are equivalent in the context of quantum mechanics and field theory to describe the interactions of spin 0, $\frac{1}{2}$, and spin 1 particles. Earlier, transformations with the constant α were called gauge transformations of first kind and transformations with $\alpha(x)$ being space–time dependent were called gauge transformations of the second kind.

7.3 Local Gauge Symmetries and Fundamental Interactions

7.3.1 Introduction

Since the early days of twentieth century, symmetry principles have played an important role in formulating the theories of fundamental interactions like gravitation and electromagnetism. In general, symmetry principles are stated in terms of the invariance of the Lagrangian (describing a classical or quantum system of particles and fields) under certain transformations, which could be independent or dependent on the space–time coordinates, respectively. Accordingly, the corresponding symmetries are called global or local symmetries. The principle of invariance

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of classical systems under local transformations was used by Einstein and Weyl to formulate theories of gravitation and electromagnetism around the period when the special theory of relativity and quantum mechanics was proposed. It was Weyl who introduced the word "gauge" (*eich* in German) transformations to specify the way in which the transformations were to be implemented. He also formulated a field theory of electrodynamics based on the local gauge transformations in 1929 [40]. Einstein's theory of gravitation is also considered by many to be a gauge field theory based on the invariance under symmetry transformations that depend on space–time coordinates. Attempts to formulate the gauge field theories of nuclear interactions, like weak and strong interactions, started quite early in the 1930s; however, it took more than 30 years for these attempts to succeed. The aforementioned developments underscore the importance of symmetries in modern physics.

The invariance of the Lagrangian under gauge symmetry transformations leads, in general, to the Euler–Lagrange equations of motion and conserved current associated with the symmetry in terms of the fields describing the Lagrangian. In the case of local symmetry transformations, the invariance requirements necessitate the introduction of new vector fields in the Lagrangian known as gauge fields, which are massless and coupled to the conserved current. The nature and form of the coupling describe the interaction of the new gauge fields with the fields describing the Lagrangian.

In this way, the discovery of the principle of local gauge symmetry, which determines the form of the electromagnetic, weak and strong interactions, can be considered to be a major triumph of twentieth century physics, comparable to the special theory of relativity and quantum mechanics. While the special theory of relativity and quantum mechanics correctly describe the kinematics and dynamics of relativistic particles by defining equations of motion, the principle of local gauge symmetry provides the form and strength of the interactions (force) and it also specifies the interaction Lagrangian depending upon the underlying symmetry.

The theory of electromagnetic interaction of charged particles like electrons, which is mediated by photons, is formulated as a local gauge field theory based on abelian U(1) symmetry, with massless gauge fields as photons. It was Weyl who first demonstrated that the interaction Lagrangian for charged particles in an electromagnetic field can be derived from the free Lagrangian for the charged particles by imposing the invariance of free electron Lagrangian under the local gauge transformation corresponding to an abelian U(1) symmetry. The idea was applied by Yang and Mills [326], Shaw [328] and Utiyama [327] to extend the U(1) symmetry to the non-abelian local symmetries SU(2) in isospin space in order to formulate the theory of strong interactions and by many others to formulate the theory of weak interactions [37, 157, 220, 340, 341, 342]. The role of the local gauge symmetries in formulating a unified theory of fundamental interactions was indeed very important as evident from the early papers of Salam and Ward [341, 342] who stated in 1964 that: "It should be possible to generate the strong, weak and electromagnetic interaction terms with all their correct symmetry properties (as well as with clues regarding their relative strengths) by making local gauge transformations in the kinetic energy terms of the free Lagrangian for all particles."

In the following subsections, we attempt to demonstrate the use of the principle of local gauge symmetry in formulating the electromagnetic interaction Lagrangian of spin 0 and spin $\frac{1}{2}$ particles and extend it for deriving the interaction Lagrangian of non-abelian Yang–Mills fields and gluons toward formulating a theory of strong interactions based on SU(2) and SU(3) symmetry.

7.3.2 $U(1)$ gauge symmetry and the electromagnetic interactions of spin 0 particles

We start with the Lagrangian for the complex scalar field:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi(x))^* \partial^\mu \phi(x) - \frac{1}{2} m^2 \phi^*(x)\phi(x). \tag{7.17}$$

The Euler–Lagrange equation gives the equation of motion for the scalar field as:

$$\left.\begin{aligned} &(\partial_\mu \partial^\mu + m^2)\phi(x) = 0 \\ \text{and} \qquad &(\partial_\mu \partial^\mu + m^2)\phi^*(x) = 0. \end{aligned}\right\} \tag{7.18}$$

This Lagrangian is invariant under the global U(1) transformation:

$$\left.\begin{aligned} &\phi(x) \xrightarrow{U(1)} \phi'(x) = e^{i\alpha}\phi(x) \\ \text{and} \qquad &\phi^*(x) \xrightarrow{U(1)} \phi'^*(x) = e^{-i\alpha}\phi^*(x). \end{aligned}\right\}, \tag{7.19}$$

where α is a constant parameter. For an infinitesimal variation:

$$\left.\begin{aligned} \delta\phi = i\alpha\phi\,, \qquad & \delta\phi^* = -i\alpha\phi^*, \\ \delta(\partial_\mu\phi) = i\alpha\partial_\mu\phi\,, \qquad & \delta(\partial_\mu\phi^*) = -i\alpha\partial_\mu\phi^*. \end{aligned}\right\} \tag{7.20}$$

Now the invariance of $\mathcal{L}$ under U(1) transformation means $\delta\mathcal{L} = 0$, where

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) + \frac{\partial\mathcal{L}}{\partial\phi^*}\delta\phi^* + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\delta(\partial_\mu\phi^*). \tag{7.21}$$

Using the Euler–Lagrange equation $\frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)$, we may rewrite Eq. (7.21) as:

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\right)\delta\phi^* + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\delta(\partial_\mu\phi^*). \tag{7.22}$$

Using Eq. (7.20), Eq. (7.22) becomes:

$$\delta\mathcal{L} = i\alpha\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\phi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\phi^*\right] = 0. \tag{7.23}$$

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Using $\mathcal{L}$ from Eq. (7.17) in Eq. (7.23), we obtain:

$$\begin{aligned}
\Rightarrow \delta\mathcal{L} &= \frac{i\alpha}{2}\left[\partial_\mu(\partial^\mu\phi^*)\phi - \partial_\mu(\partial^\mu\phi)\phi^*\right] \\
&= \alpha\partial_\mu j^\mu, \\
\text{where } j^\mu &= -\frac{i}{2}\left[\phi^*(\partial^\mu\phi) - (\partial^\mu\phi^*)\phi\right].
\end{aligned} \tag{7.24}$$

Thus, if the Lagrangian is invariant under the global transformation defined by Eq. (7.19), then the current j^μ defined by Eq. (7.24) is conserved. Current conservation implies charge conservation.

Instead of the transformation given by Eq. (7.19), if we apply a local gauge transformation to the Lagrangian given by Eq. (7.17),

$$\begin{aligned}
\phi(x) \xrightarrow{U(1)} \phi'(x) &= e^{i\alpha(x)}\phi(x) \text{ and} \\
\phi^*(x) \xrightarrow{U(1)} \phi'^*(x) &= e^{-i\alpha(x)}\phi^*(x),
\end{aligned} \tag{7.25}$$

it leads to:

$$\begin{aligned}
\partial_\mu\phi(x) \xrightarrow{U(1)} \partial_\mu\phi'(x) &= e^{i\alpha(x)}\left[\partial_\mu\phi(x) + i(\partial_\mu\alpha(x))\phi(x)\right], \\
\partial_\mu\phi^*(x) \xrightarrow{U(1)} \partial_\mu\phi'^*(x) &= e^{-i\alpha(x)}\left[\partial_\mu\phi^*(x) - i(\partial_\mu\alpha(x))\phi^*(x)\right],
\end{aligned} \tag{7.26}$$

which destroys the invariance of the Lagrangian in Eq. (7.23) under the local gauge transformation. In order to preserve the invariance, a covariant derivative D_μ is defined as $D_\mu = \partial_\mu - ieA_\mu(x)$. The new vector field $A_\mu(x)$ is constrained to transform as:

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x), \tag{7.27}$$

as the covariant derivative field $D_\mu\phi(x)$ transforms as:

$$D_\mu\phi(x) \xrightarrow{U(1)} e^{i\alpha(x)}D_\mu\phi(x),$$

such that the Lagrangian remains invariant. The new Lagrangian written in terms of the covariant derivative $D_\mu\phi(x)$ is given by:

$$\mathcal{L} = \frac{1}{2}\left(D_\mu\phi(x)\right)^*\left(D^\mu\phi(x)\right) - \frac{1}{2}m^2\phi^*(x)\phi(x),$$

which, in terms of the ordinary derivative ∂_μ, may be written as:

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\left[(\partial_\mu + ieA_\mu(x))\phi^*(x)\right]\left[(\partial^\mu - ieA^\mu(x))\phi(x)\right] - \frac{1}{2}m^2\phi^*(x)\phi(x) \\
&= \frac{1}{2}\partial_\mu\phi^*(x)\partial^\mu\phi(x) + \frac{ie}{2}A_\mu(x)\phi^*(x)(\partial^\mu\phi(x)) - \frac{ie}{2}(\partial_\mu\phi(x))^*A^\mu(x)\phi(x) \\
&+ \frac{e^2}{2}A_\mu(x)A^\mu(x)\phi^*(x)\phi(x) - \frac{1}{2}m^2\phi^*(x)\phi(x)
\end{aligned}$$

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$$\begin{aligned}
&= \frac{1}{2}\partial_\mu \phi^*(x)\partial^\mu \phi(x) - \frac{1}{2}m^2\phi^*(x)\phi(x) + \frac{ie}{2}A_\mu(x)\left[\phi^*(x)(\partial^\mu\phi(x))\right. \\
&\qquad \left. -\phi(x)(\partial^\mu\phi^*(x))\right] \\
&+ \frac{e^2}{2}A_\mu(x)A^\mu(x)\phi^*(x)\phi(x). \qquad (7.28)
\end{aligned}$$

Comparing Eqs. (7.17) and (7.28), we observe that there are additional terms which are given by:

$$\begin{aligned}
\mathcal{L}_{\text{add}} &= \frac{ie}{2}A_\mu(x)\left[\phi^*(x)(\partial^\mu\phi(x)) - \phi(x)(\partial^\mu\phi(x))^*\right] + \frac{e^2}{2}A_\mu(x)A^\mu(x)\phi^*(x)\phi(x) \\
&= -A_\mu(x)j^\mu(x) + \frac{e^2}{2}A_\mu(x)A^\mu(x)\phi^*(x)\phi(x), \qquad (7.29)
\end{aligned}$$

where

$$j^\mu(x) = -\frac{ie}{2}\left[\phi^*(x)(\partial^\mu\phi(x)) - \phi(x)(\partial^\mu\phi^*(x))\right]. \qquad (7.30)$$

The two additional terms in the new Lagrangian, given by Eq. (7.29) give the interaction of the new field A_μ with the conserved current generated by ϕ and a quadratic interaction term of the field ϕ with the new field. If we identify the new vector field as an electromagnetic field, then the first term describes the interaction of the electromagnetic field with the electromagnetic current of the charged particle with the coupling strength e. To this, we add the kinetic energy term for the A_μ field in a gauge invariant way by introducing the electromagnetic field tensor $F^{\mu\nu}$. Therefore, the full Lagrangian for the scalar field consistent with the invariance under local U(1) gauge transformation becomes:

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\Big(\partial^\mu\phi^*(x)\partial_\mu\phi(x) - m^2\phi^*(x)\phi(x)\Big) - j_\mu(x)A^\mu(x) \\
&\quad + \frac{e^2}{2}A_\mu(x)A^\mu(x)\phi^*(x)\phi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \qquad (7.31)
\end{aligned}$$

It should be noted that the new field A_μ is a vector field. It need not necessarily represent a spin 1 field, as in the case of electrodynamics, but could be written as $A_\mu(x) = \partial_\mu\lambda(x)$ involving derivatives, where $\lambda(x)$ is a scalar field, satisfying the transformation property:

$$\lambda'(x) \to \lambda(x) + \frac{1}{e}\alpha(x). \qquad (7.32)$$

This will, however, involve additional momenta due to derivative coupling and would present problems in the renormalizability of the theory.

7.3.3 $U(1)$ gauge symmetry and the electromagnetic interactions of spin $\frac{1}{2}$ particles

Let us consider the Lagrangian density for a free spin $\frac{1}{2}$ Dirac particle:

$$\mathcal{L}_D = \bar{\psi}(x)(i\gamma_\mu\partial^\mu - m)\psi(x), \tag{7.33}$$

where $\psi(x)$ is a four-component complex spinor function and $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$. Consider the symmetry properties of Eq. (7.33) under a global phase transformation:

$$\left.\begin{aligned} \psi(x) \to \psi'(x) = &\quad e^{i\alpha}\psi(x), \\ \bar{\psi}(x) \to \bar{\psi}'(x) = &\quad e^{-i\alpha}\bar{\psi}(x), \end{aligned}\right\}, \tag{7.34}$$

where α is a real constant parameter. For infinitesimal transformations of $\psi(x)$:

$$\left.\begin{aligned} \psi(x) \to \psi'(x) = (1 + i\alpha)\psi(x) \Rightarrow \delta\psi(x) = \psi'(x) - \psi(x) = i\alpha\psi(x), \\ \bar{\psi}(x) \to \bar{\psi}'(x) = (1 - i\alpha)\bar{\psi}(x) \Rightarrow \delta\bar{\psi}(x) = \bar{\psi}'(x) - \bar{\psi}(x) = -i\alpha\bar{\psi}(x). \end{aligned}\right\} \tag{7.35}$$

The Lagrangian density remains invariant as:

$$\begin{aligned} \Rightarrow \delta\mathcal{L} &= i\delta\bar{\psi}(x)\gamma_\mu\partial^\mu\psi(x) + i\bar{\psi}(x)\gamma_\mu\partial^\mu(\delta\psi(x)) - m\delta\bar{\psi}(x)\psi(x) - m\bar{\psi}(x)\delta\psi(x) \\ &= i[-i\alpha\bar{\psi}(x)]\gamma_\mu\partial^\mu\psi(x) + i\bar{\psi}(x)\gamma_\mu\partial^\mu[i\alpha\psi(x)] \\ &\quad -m[-i\alpha\bar{\psi}(x)]\psi(x) - m\bar{\psi}(x)[i\alpha\psi(x)] \\ &= 0. \end{aligned} \tag{7.36}$$

This invariance of the Lagrangian density shows that $\psi(x)$ as well as $e^{i\alpha}\psi(x)$ have the same physical predictions, which in turn implies that α cannot be measured explicitly, that is, the phase of ψ remains arbitrary. Such transformations are known as global gauge transformations. α is, therefore, a parameter characterizing the transformation generated by a 1×1 unitary matrix which forms the group U(1). The group multiplication is commutative, that is, $U(\alpha_1)$ $U(\alpha_2) = U(\alpha_2)U(\alpha_1)$, and, therefore, the sets of unitary transformations form a unitary abelian group.

The invariance of the Lagrangian, shown in Eq. (7.36) also leads to:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta(\partial_\mu\psi) + \frac{\partial\mathcal{L}}{\partial\bar{\psi}}\delta\bar{\psi} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\delta(\partial_\mu\bar{\psi}) = 0. \tag{7.37}$$

Using infinitesimal transformations of $\psi(x)$ as discussed in Eq. (7.35), Eq. (7.37) results in:

$$\begin{aligned} \delta\mathcal{L} &= i\alpha\left[\frac{\partial\mathcal{L}}{\partial\psi}\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial_\mu\psi\right] - i\alpha\left[\frac{\partial\mathcal{L}}{\partial\bar{\psi}}\bar{\psi} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\partial_\mu\bar{\psi}\right] = 0 \\ &\Rightarrow i\alpha\left[\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right)\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial_\mu\psi\right] \\ &\qquad -i\alpha\left[\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right)\bar{\psi} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\partial_\mu\bar{\psi}\right] = 0 \end{aligned}$$

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$$\Rightarrow \quad i\alpha\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\psi\right] - i\alpha\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\bar{\psi}\right] = 0$$
$$\Rightarrow \quad i\alpha\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\psi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\bar{\psi}\right] = 0. \tag{7.38}$$

Using the definition of current j_μ as:

$$\begin{aligned} j^\mu &= i\alpha\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\psi - \bar{\psi}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right] \\ &= -\alpha\bar{\psi}\gamma_\mu\psi, \end{aligned} \tag{7.39}$$

and making use of Eq. (7.38), we get an equation for conserved current:

$$\partial_\mu j^\mu = 0. \tag{7.40}$$

Equation (7.40) also ensures that the charge $Q = \int j_0 d\vec{x}$ is conserved under U(1) global gauge transformation. Charge conservation is, therefore, a consequence of the invariance of the Dirac Lagrangian under global U(1) transformation. If the parameter α is made dependent on x, that is, $\alpha(x)$, then the transformation becomes local, as $\alpha(x)$ is different at each x and the group U(1) becomes the local U(1) group. If the parameter α is space–time dependent and the field $\psi(x)$ transforms as:

$$\psi(x) \to \psi'(x) = e^{i\alpha(x)}\psi(x)\text{ , and} \tag{7.41}$$
$$\bar{\psi}(x) \to \bar{\psi}'(x) = e^{-i\alpha(x)}\bar{\psi}(x)\text{ ,} \tag{7.42}$$
$$\partial_\mu\psi(x) \to \partial_\mu\psi'(x) = (\partial_\mu\psi(x))e^{i\alpha(x)} + i(\partial_\mu\alpha(x))\psi(x)e^{i\alpha(x)}, \tag{7.43}$$

then the mass term of the Lagrangian given in Eq. (7.33) will remain invariant as:

$$m\bar{\psi}(x)\psi(x) \to m\bar{\psi}'(x)\psi'(x) = m\bar{\psi}(x)e^{-i\alpha(x)}e^{i\alpha(x)}\psi(x) = m\bar{\psi}(x)\psi(x).$$

However, the kinetic energy term will not remain invariant as:

$$\begin{aligned} i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) &\to i\bar{\psi}'(x)\gamma^\mu\partial_\mu\psi'(x) \\ &= i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - \bar{\psi}(x)(\partial_\mu\alpha(x))\psi(x) \\ &\neq i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x). \end{aligned} \tag{7.44}$$

Therefore, unlike in the case of global gauge transformation, here ψ and $\partial_\mu\psi$ do not have the same transformation properties, making $\mathcal{L}' \neq \mathcal{L}$ under U(1). In order to impose invariance of $\mathcal{L}$ under local U(1) gauge transformation, a new field A_μ is introduced and the field transforms as:

$$A_\mu \to A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\alpha(x) \tag{7.45}$$

and the derivative is modified as $\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu$, such that ψ and $D_\mu\psi$ have the same transformation properties under U(1), that is,

$$\begin{aligned} \psi(x) \to \psi'(x) &= e^{i\alpha(x)}\psi(x) \text{ and} \\ D_\mu\psi(x) \to (D_\mu\psi(x))' &= e^{i\alpha(x)}D_\mu\psi(x) \end{aligned} \tag{7.46}$$

and similarly, for the conjugate field.

Therefore, under local U(1) gauge transformation, $\mathcal{L}'_{\mathcal{D}} = i\bar{\psi}(\gamma_\mu D^\mu - m)\psi$ is invariant and

$$\begin{aligned} \mathcal{L}_{\mathcal{D}} \to \mathcal{L}'_{\mathcal{D}} &= i\bar{\psi}'(x)\gamma^\mu D_\mu\psi'(x) - m\bar{\psi}'(x)\psi'(x) \\ &= i\bar{\psi}(x)e^{-i\alpha(x)}\gamma^\mu\left[\partial_\mu - ieA_\mu - i\partial_\mu\alpha(x)\right]e^{i\alpha(x)}\psi(x) \\ &\qquad -m\bar{\psi}(x)e^{-i\alpha(x)}e^{i\alpha(x)}\psi(x) \\ &= i\bar{\psi}(x)\gamma^\mu(\partial_\mu - ieA_\mu)\psi(x) - m\bar{\psi}(x)\psi(x) \\ &= \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) + e\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu \\ &= \mathcal{L}_{\mathcal{D}} + e\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu. \end{aligned} \tag{7.47}$$

Thus, the demand of local gauge invariance is fulfilled by introducing a covariant derivative D_μ. The price we have paid is the introduction of a new field $A_\mu(x)$ that also transforms as shown in Eq. (7.45) to make the Lagrangian locally invariant. Local gauge invariance has got great physical significance, for example, lepton number, charge, etc. are locally conserved.

Thus, the QED Lagrangian is obtained as:

$$\mathcal{L} = \bar{\psi}(x)i\gamma^\mu\partial_\mu\psi(x) - m\bar{\psi}(x)\psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu. \tag{7.48}$$

Equation (7.48) demonstrates that the principle of local gauge invariance generates the interaction Lagrangian for the interaction of the electromagnetic field A_μ known as the gauge field which couples to the Dirac particle (charge $-e$), with the current density $j^\mu(= -e\bar{\psi}(x)\gamma^\mu \psi(x)A_\mu)$. The gauge field A_μ is massless.

7.3.4 $SU(2)$ gauge symmetry and Yang–Mills fields

Suppose there are two non-interacting spin $\frac{1}{2}$ particles. The Lagrangian for the free particle is then given by:

$$\mathcal{L} = i\bar{\psi}_1(x)\gamma^\mu\partial_\mu\psi_1(x) - m_1\bar{\psi}_1(x)\psi_1(x) + i\bar{\psi}_2(x)\gamma^\mu\partial_\mu\psi_2(x) - m_2\bar{\psi}_2(x)\psi_2(x). \tag{7.49}$$

If the particles belong to a doublet representation under SU(2) group like the protons and neutrons which belong to a doublet representation of $SU(2)$ in isospin space, then

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}, \tag{7.50}$$

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and the Lagrangian $\mathcal{L}$ is written as:

$$\mathcal{L} = i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - M\bar{\psi}(x)\psi(x)\,, \tag{7.51}$$

where $\psi(x) = \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix}$ is a two-component column vector and $M = \begin{pmatrix}m_1 & 0\\ 0 & m_2\end{pmatrix}$.

In the limit of SU(2) symmetry, where $m_1 = m_2 = m$,

$$\mathcal{L} = i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - m\bar{\psi}(x)\psi(x). \tag{7.52}$$

If the global SU(2) transformation is applied to the Lagrangian given in Eq. (7.51), the transformation is given by:

$$\psi(x) \to \psi'(x) \quad = \quad U\psi(x) \qquad \text{and} \qquad \bar{\psi}(x) \to \bar{\psi}'(x) = \bar{\psi}(x)U^\dagger, \tag{7.53}$$

where U is a 2×2 unitary matrix and given by:

$$U = e^{\frac{i}{2}\sum\limits_{i=1}^{3}\tau_i.a_i}, \tag{7.54}$$

where τ_1, τ_2, τ_3 are the three components of Pauli matrices and a_1, a_2, a_3 are real numbers. $e^{\frac{i}{2}\vec{\tau}.\vec{a}}$ is a 2×2 matrix with determinant unity. Under such transformations:

$$\psi(x) \to \psi'(x) = U\psi(x) = e^{\frac{i}{2}\sum\limits_{i=1}^{3}\tau_i.a_i}\psi(x). \tag{7.55}$$

It may be verified that this Lagrangian remains invariant under the aforementioned transformation.

In 1954, Yang and Mills [326] extended the idea of global SU(2) gauge transformation to the local SU(2) gauge transformation. The transformation for an isodoublet Dirac field

$$\psi(x) = \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} \tag{7.56}$$

is

$$\psi(x) \xrightarrow{SU(2)} \psi'(x) = U\psi(x) = e^{i\sum\limits_{i=1}^{3}\alpha_i(x)T_i}\psi(x), \tag{7.57}$$

where

$$\alpha_i(x)T_i = \alpha_1(x)T_1 + \alpha_2(x)T_2 + \alpha_3(x)T_3, \tag{7.58}$$

α_{1-3}s are real parameters which depend on x_μ and $T_i = \frac{1}{2}\tau_i$, where $\tau_i's$ are the 2×2 isospin equivalent of Pauli matrices and

$$[T_i, T_j] = i\epsilon_{ijk}T_k. \tag{7.59}$$

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Now consider the Dirac Lagrangian:

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x), \tag{7.60}$$

which for an infinitesimal transformation,

$$\delta\psi(x) = i\alpha_i(x) T_i \psi(x) \text{ and } \delta\bar{\psi}(x) = -i\bar{\psi}(x)\alpha_i(x) T_i, \tag{7.61}$$

changes to say $\mathcal{L}'$. The change is given by:

$$\begin{aligned}\delta\mathcal{L} &= \left[\frac{\partial\mathcal{L}}{\partial\psi(x)}\delta\psi(x) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi(x))}\delta(\partial_\mu\psi(x))\right] \\ &\qquad + \left[\frac{\partial\mathcal{L}}{\partial\bar{\psi}(x)}\delta\bar{\psi}(x) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi}(x))}\delta(\partial_\mu\bar{\psi}(x))\right].\end{aligned} \tag{7.62}$$

Let us consider the change in Lagrangian density($\mathcal{L}$) resulting from a transformation $\delta\psi = i\alpha_i(x)T_i\psi$. Then:

$$\begin{aligned}\delta\mathcal{L} &= \left[\frac{\partial\mathcal{L}}{\partial\psi(x)}\delta\psi(x) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi(x))}\delta(\partial_\mu\psi(x))\right] \\ &= \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi(x))} i\alpha_i(x) T_i\psi(x)\right].\end{aligned} \tag{7.63}$$

Using the definition of current as:

$$j^\mu = i\alpha_i(x)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi(x))} T_i\psi(x) \tag{7.64}$$

and making use of the Dirac Lagrangian, Eq. (7.60), we obtain:

$$j^\mu = -\alpha_i(x)\bar{\psi}(x)\gamma^\mu \vec{T}\psi(x). \tag{7.65}$$

For constant α_is, the Lagrangian will remain invariant, that is, $\delta\mathcal{L} = 0$ if $\partial_\mu j^\mu = 0$.

Now we consider the invariance under local gauge transformation, and define a covariant derivative D_μ as:

$$D_\mu = \partial_\mu - igW_\mu(x), \tag{7.66}$$

where

$$W_\mu(x) = W_\mu^i(x) T_i. \tag{7.67}$$

Here, $W_\mu^i s$ are the triplet $W_\mu^1, W_\mu^2, W_\mu^3$ of gauge fields, corresponding to each component of the isospin matrix T_i, such that:

$$\begin{aligned} & (D_\mu\psi(x))' &= U(D_\mu\psi(x)) \\ \Rightarrow \quad & (\partial_\mu - igW'_\mu)U\psi(x) &= U(\partial_\mu - igW_\mu)\psi(x)\end{aligned}$$

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$$\begin{aligned}
\Rightarrow \quad (\partial_\mu U)\psi(x) + U\partial_\mu\psi(x) - igW'_\mu U\psi(x) &= U\partial_\mu\psi(x) - igUW_\mu\psi(x) \\
\Rightarrow \quad (\partial_\mu U - igW'_\mu U)\psi(x) &= -igUW_\mu\psi(x) \\
\Rightarrow \quad \partial_\mu U - igW'_\mu U &= -igUW_\mu.
\end{aligned}$$

Multiplying the right-hand of the equation by U^{-1}, we get:

$$\begin{aligned}
(\partial_\mu U)U^{-1} - igW'_\mu UU^{-1} &= -igUW_\mu U^{-1} \\
\Rightarrow W'_\mu &= UW_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} \qquad (7.68) \\
\Rightarrow W'^i_\mu T_i &= UW^i_\mu T_i U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}.
\end{aligned}$$

For an infinitesimal transformation, that is, $U = 1 + i\alpha_i T_i$ and $U^{-1} = 1 - i\alpha_i T_i$, we may write:

$$\begin{aligned}
\Rightarrow W'^i_\mu T_i &= (1 + i\alpha_j T_j)W^i_\mu T_i(1 - i\alpha_j T_j) - \frac{i}{g}(i(\partial_\mu\alpha_j)T_j)(1 - i\alpha_j T_j) \\
&= W^i_\mu T_i - i\alpha_j W^i_\mu . i\epsilon_{ijk}T_k + \frac{1}{g}(\partial_\mu\alpha_j)T_j \\
&= \left(W^i_\mu - \epsilon_{ijk}\alpha_j W^k_\mu + \frac{1}{g}(\partial_\mu\alpha_i)\right)T_i \\
\Rightarrow W'^i_\mu &= W^i_\mu - \epsilon_{ijk}\alpha_j W^k_\mu + \frac{1}{g}(\partial_\mu\alpha_i) \\
\Rightarrow \vec{W}'_\mu &= \vec{W}_\mu - \vec{\alpha} \times \vec{W}_\mu + \frac{1}{g}(\partial_\mu\vec{\alpha}). \qquad (7.69)
\end{aligned}$$

We add three new fields W^1_μ, W^2_μ, W^3_μ and to make the field dynamic, the kinetic energy term $\frac{1}{4}G^i_{\mu\nu}G^{\mu\nu}_i$, where $G^i_{\mu\nu} = (\partial_\mu W^i_\nu - \partial_\nu W^i_\mu)$, is to be added to the Lagrangian. Defining a gauge invariant $G_{\mu\nu}$ as:

$$\begin{aligned}
G_{\mu\nu} &= D_\mu W_\nu - D_\nu W_\mu \\
&= \partial_\mu W_\nu - \partial_\nu W_\mu - ig[W_\mu, W_\nu] \\
\Rightarrow G^i_{\mu\nu}T^i &= (\partial_\mu W^i_\nu - \partial_\nu W^i_\mu)T^i - igW^i_\mu W^j_\nu[T_i, T_j] \\
&= (\partial_\mu W^i_\nu - \partial_\nu W^i_\mu)T^i - igW^i_\mu W^j_\nu i\epsilon^{ijk}T^k \\
&= (\partial_\mu W^i_\nu - \partial_\nu W^i_\mu)T^i + g\epsilon_{ijk}W^j_\mu W^k_\nu T^i.
\end{aligned}$$

For an arbitrary T^i,

$$G^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g\epsilon_{ijk}W^j_\mu W^k_\nu = -G^i_{\nu\mu}.$$

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The transformation of $G_{\mu\nu}$ under the SU(2) would give:

$$\begin{aligned}
G_{\mu\nu} \to G'_{\mu\nu} &= \partial_\mu W'_\nu - \partial_\nu W'_\mu \\
&= \partial_\mu \left[U W_\nu U^{-1} - \frac{i}{g}(\partial_\nu U)U^{-1} \right] - \partial_\nu \left[U W_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} \right] \\
&= \partial_\mu (U W_\nu U^{-1}) - \frac{i}{g}\partial_\mu \left[(\partial_\nu U)U^{-1} \right] - \partial_\nu \left[U W_\mu U^{-1} \right] \\
&\qquad + \frac{i}{g}\partial_\nu \left[(\partial_\mu U)U^{-1} \right] \\
&= \frac{i}{g}\left[(\partial_\mu U)(\partial_\nu U^{-1}) - (\partial_\nu U)(\partial_\mu U^{-1}) \right] + (\partial_\mu U) W_\nu U^{-1} \\
&\qquad + U W_\nu (\partial_\mu U^{-1}) \\
&- (\partial_\nu U) W_\mu U^{-1} - U W_\mu (\partial_\nu U^{-1}) + U(\partial_\mu W_\nu - \partial_\nu W_\mu)U^{-1}. \qquad (7.70)
\end{aligned}$$

Therefore, $G_{\mu\nu}$ is not invariant under gauge transformation. We redefine $G^i_{\mu\nu} = (\partial_\mu W^i_\nu - \partial_\nu W^i_\mu) - ig[W_\mu, W_\nu]$, where $[W_\mu, W_\nu]$ transforms as:

$$\begin{aligned}
[W_\mu, W_\nu] \to [W'_\mu, W'_\nu] &= (W'_\mu W'_\nu - W'_\nu W'_\mu) \\
&= \left(U W_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} \right)\left(U W_\nu U^{-1} - \frac{i}{g}(\partial_\nu U)U^{-1} \right) \\
&- \left(U W_\nu U^{-1} - \frac{i}{g}(\partial_\nu U)U^{-1} \right)\left(U W_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} \right) \\
&= U[W_\mu,\ W_\nu]U^{-1} + \frac{1}{g^2}(\partial_\mu U \partial_\nu U^{-1} - \partial_\nu U \partial_\mu U^{-1}) \\
&+ \frac{i}{g}\left[\partial_\mu U W_\nu U^{-1} + U W_\nu \partial_\mu U^{-1} - \partial_\nu U W_\mu U^{-1} \right. \\
&\qquad \left. - U W_\mu \partial_\nu U^{-1} \right]. \qquad (7.71)
\end{aligned}$$

Here, we have used the relation: $U^{-1}(\partial_\mu U) = -(\partial_\mu U^{-1})U$, inspired by $UU^{-1} = 1$.

When the transformations performed using Eqs. (7.70) and (7.71), are considered together, then the Lagrangian is found to be invariant under the local gauge transformation.

Thus, the complete Lagrangian invariant under SU(2) gauge transformation is given by:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + g\bar{\psi}\gamma^\mu \psi W_\mu - \frac{1}{4}G^i_{\mu\nu}G_i^{\mu\nu}, \qquad (7.72)$$

where the first term represents the free Lagrangian, the second term represents the interaction of the current with the gauge boson while the third term describes the self interaction of the gauge bosons. This is shown in Figure 7.2. Thus, it may be observed from Eqs. (7.60)–(7.71), that for the local gauge invariance of the Lagrangian under SU(2), the following aspects should be kept in mind:

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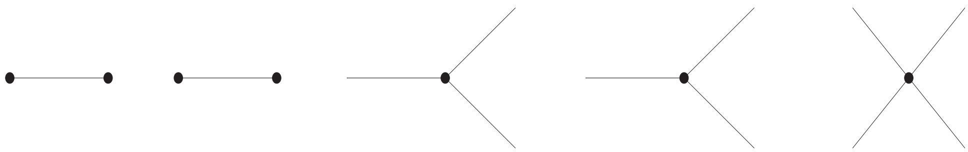

Figure 7.2

- ∂_μ should be replaced by a covariant derivative $D_\mu = \partial_\mu - igW_\mu$, where $W_\mu = W_\mu^i T^i$. W_μ^i stands for the three fields $W_\mu^1,\ W_\mu^2,\ W_\mu^3$.
- W_μ should transform as:
  $W'_\mu = UW_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}$
- The kinetic energy term $(-\frac{1}{4}G_{\mu\nu}^i G_i^{\mu\nu})$ is added to the field dynamics, where
  $G_{\mu\nu}^i = (\partial_\mu W_\nu^i - \partial_\nu W_\mu^i) - ig[W_\mu^i,\ W_\nu^i]$
  which transforms as :
  $G_{\mu\nu}^i \xrightarrow{SU(2)} G_{\mu\nu}^{i'} = UG_{\mu\nu}^i U^{-1}$.
- The Yang–Mills Lagrangian:
  $\mathcal{L}_{YM} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}G_{\mu\nu}^i G_i^{\mu\nu}$
  is invariant under the local SU(2) gauge transformation.
- The gauge fields $W_\mu^i(x)$ are massless.

7.3.5 $SU(3)_c$ gauge symmetry and QCD

The Lagrangian for free quark fields is defined as:

$$\mathcal{L} = \sum_{j=1}^{3} \bar{q}_j (i\gamma^\mu \partial_\mu - m_j) q_j, \tag{7.73}$$

where q_js are the color quark fields. It is assumed that the free Lagrangian is invariant under $SU(3)_C$ symmetry transformation in color space. The $SU(3)_C$ symmetry is generated by the 3×3 Gell–Mann matrices $\frac{1}{2}\lambda_i$, which are $n^2 - 1(3^2 - 1 = 8)$ in number; the matrices are also traceless and hermitian. The matrices λ_i are chosen in analogy with σ_i matrices of SU(2), extended to three dimensions and are given in Appendix B. These matrices λ_is, satisfy the following commutation and anticommutation relations:

$$\left[\frac{\lambda_i}{2}, \frac{\lambda_j}{2}\right] = if_{ijk}\left(\frac{\lambda_k}{2}\right), \tag{7.74}$$

$$\left\{\frac{\lambda_i}{2}, \frac{\lambda_j}{2}\right\} = \frac{1}{3}\delta_{ij} + d_{ijk}\left(\frac{\lambda_k}{2}\right), \tag{7.75}$$

where f_{ijk} are known as the structure constants.

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$f_{ijk}(d_{ijk})$ are anti-symmetric (symmetric) under the interchange of any pair of indices (i, j, k) and are listed in Appendix B.

Under the global SU(3) gauge transformation, the quark field $q(x)$ transforms as:

$$\left.\begin{aligned} q(x) \rightarrow \; q'(x) = Uq(x) = e^{i\alpha_i T_i} q(x) \\ \bar{q}(x) \rightarrow \bar{q}'(x) = U^{\dagger}\bar{q}(x) = e^{-i\alpha_i T_i}\bar{q}(x) \end{aligned}\right\}, \tag{7.76}$$

where U is an arbitrary 3×3 unitary matrix, $T_i = \frac{\lambda_i}{2}$, i=1–8 are the generators of the SU(3) group and α_i are the group parameters. For an infinitesimal transformation, we may write:

$$\left.\begin{aligned} q(x) \rightarrow q'(x) = (1 + i\alpha_i T_i)q(x), \\ \partial_\mu q(x) \rightarrow \partial_\mu q'(x) = (1 + i\alpha_i T_i)\partial_\mu q(x). \end{aligned}\right\} \tag{7.77}$$

The Lagrangian in Eq. (7.73) is invariant under the transformation defined in Eqs. (7.76) and (7.77).

Proceeding the same way as discussed in the case of U(1) and SU(2) gauge transformations, the required Lagrangian should also be invariant under local $SU(3)_C$ transformation, where the field transforms as:

$$q(x) \rightarrow \; q'(x) = Uq(x) = e^{i\alpha_i(x)T_i} q(x). \tag{7.78}$$

For an infinitesimal phase transformation:

$$\left.\begin{aligned} q(x) \rightarrow q'(x) = \left[1 + i\alpha_i(x)T_i\right] q(x) \\ \partial_\mu q(x) \rightarrow \partial_\mu q'(x) = \left[\partial_\mu + i(\partial_\mu \alpha_i(x))T_i + i\alpha_i(x)T_i\partial_\mu\right] q(x) \end{aligned}\right\}. \tag{7.79}$$

To fulfill the requirement of local gauge invariance, we must replace the ordinary derivative ∂_μ by the covariant derivative $D_\mu (= \partial_\mu + igT_i G^i_\mu)$ in Eq. (7.73), where G^i_μ are the eight gauge fields transforming as:

$$G^i_\mu(x) \rightarrow G'^i_\mu(x) = G^i_\mu(x) - \frac{1}{g}\partial_\mu \alpha_i(x), \tag{7.80}$$

following the discussion given in Section 7.3.4.

Introducing a kinetic energy term for the propagating field of gluons, the complete Lagrangian invariant under $SU(3)_c$ gauge transformation is given by:

$$\mathcal{L} \;=\; \sum_j \bar{q}_j (i\gamma^\mu \partial_\mu - m) q_j - g G^i_\mu \sum_j \bar{q}_j \, \gamma^\mu T_i q_j \; - \frac{1}{4} G^i_{\mu\nu} G^{i\mu\nu}. \tag{7.81}$$

For Eq. (7.81) to be locally gauge invariant under SU(3), the field strength can be defined as:

$$G^i_{\mu\nu} = \partial_\mu G^i_\nu - \partial_\nu G^i_\mu - g \, f_{ijk} G^j_\mu G^k_\nu. \tag{7.82}$$

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In Eq. (7.81), the first term is the free Lagrangian, the second term describes the interaction of the quark current with the gauge boson of the theory, that is, gluons G^i_μ, $[i = 1, 2, ..., 8]$(like the electromagnetic field in the case of U(1) symmetry). The last term involves three type of terms, that is,

$$\begin{aligned} G^i_{\mu\nu} G^{i\mu\nu} &= (\partial_\mu G^i_\nu - \partial_\nu G^i_\mu)(\partial^\mu G^\nu_i - \partial^\nu G^\mu_i) - g f_{ijk}(G^j_\mu G^k_\nu(\partial^\mu G^\nu_i - \partial^\nu G^\mu_i)) \\ &\quad - g f_{ilm}(G^\mu_l G^\nu_m(\partial_\mu G^i_\nu - \partial_\nu G^i_\mu)) + g^2 f_{ijk} f_{ilm} G^j_\mu G^k_\nu G^\mu_l G^\nu_m, \end{aligned} \tag{7.83}$$

which describes free gluons, and self interaction of gluons which is trilinear as well as quartic in the gluon field G^i_μ. Diagrammatically, this is shown in Figure 7.3. The Lagrangian given in

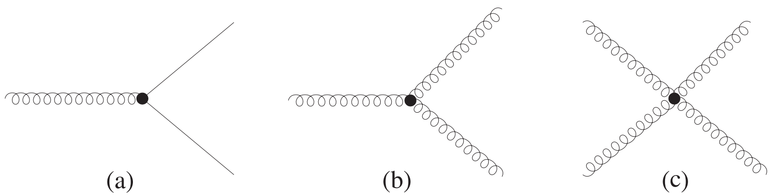

Figure 7.3 (a) Quark–gluon vertex, (b) triple gluon vertex, and (c) four gluon vertex.

Eq. (7.81) can be quantized and Feynman diagrams can be derived for doing calculations with gluon exchange. It should be noted that there are no mass terms like $G^i_\mu G^{i\mu}$ showing that all the gauge fields are massless. The Lagrangian describes the interaction of quarks with gluons and is the most reliable theory of strong interactions known as QCD.

7.3.6 Exact, broken, and spontaneously broken symmetries

i) **Exact symmetries:** In field theory, exact symmetry is characterized by two conditions. The first condition is that the interaction Lagrangian density $\mathcal{L}$ is invariant under symmetry transformations, that is, under a symmetry transformation U, the interaction Lagrangian of the system transforms as:

$$\mathcal{L} \xrightarrow{U} \mathcal{L}' = \mathcal{L}, \quad \text{such that} \qquad \delta\mathcal{L} = 0. \tag{7.84}$$

This leads to the Euler–Lagrange equations of motion which also satisfy the symmetry of the Lagrangian. The Euler–Lagrange equations of motion further lead to a current J^μ, which is conserved, that is $\partial_\mu J^\mu = 0$, leading to conservation of charge Q, defined as:

$$Q = \int J^0(\vec{x}, t) d\vec{x}. \tag{7.85}$$

The second condition is that the ground state of the physical system or the physical vacuum in case of field theory, is also invariant under the symmetry transformation and is, therefore, unique. These two conditions imply that there exist degenerate multiplets of states having the same energy as a requirement of the exact symmetry (as shown in the

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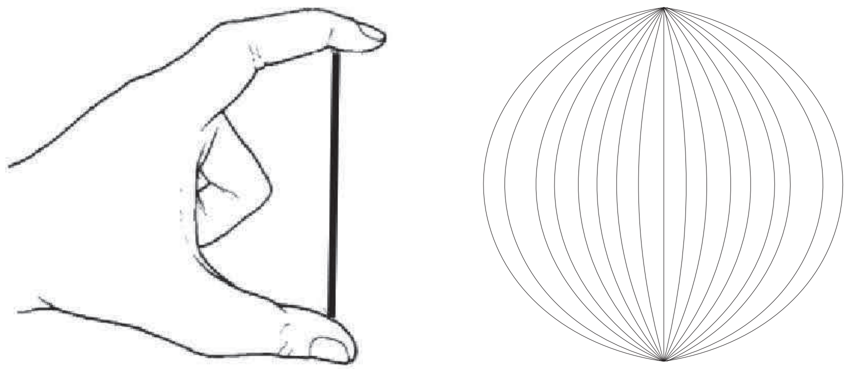

Figure 7.4 Unbroken symmetry (left): The plastic strip is in a rotationally invariant state. Spontaneously broken symmetry (right): If you squeeze the ends of a thin plastic strip together, it will bend in any direction, that is, there are infinite possibilities. As soon as it opts a particular direction, rotational invariance is lost.

left panel of Figure 7.4). Such degeneracy of states has been observed in many physical systems in the study of atoms, nuclei, and particles.

ii) **Broken or approximate symmetries:** However, many natural processes observed in the physical world do not exhibit exact symmetry, but only an approximate symmetry. These processes are described by separating the Lagrangian into two terms and writing it in the form $\mathcal{L} = \mathcal{L}_0 + \epsilon\mathcal{L}_1$, where $\mathcal{L}_0$ is invariant under the symmetry transformation and $\mathcal{L}_1$ is the part which breaks the symmetry explicitly; $\mathcal{L}_1$ is generally small with the smallness parameterized by a parameter ϵ. In many cases, it so happens that $\mathcal{L}_1$ corresponds to a lower symmetry as compared to the symmetry of $\mathcal{L}_0$; its transformation properties under the lower symmetry are used to lift the degeneracy of the multiplets of the states implied by the symmetry of $\mathcal{L}_0$. This form of treating approximate symmetries has been very useful in the study of atomic, nuclear, and particle spectroscopy. For example, in atomic or nuclear spectroscopy, the $\mathcal{L}_0$ Lagrangian is written as $\mathcal{L}_0 = \frac{p^2}{2M} - V(r)$, where $\vec{p}$ is the momentum operator $\vec{p}(= -i\vec{\nabla})$ of the electrons or nucleons and the potential $V(r)$ is either the Coulomb potential $\left(V(r) = -\frac{Ze^2}{|\vec{r}|}\right)$ in the case of atomic systems or a central potential of the Wood–Saxen type $\left(V(r) = \frac{V_0}{1+e^{\frac{r-R}{a}}}\right)$ in the case of nuclear systems. These potentials depend upon the radial distance $|\vec{r}|$ only, leaving $\mathcal{L}_0$ invariant under rotation; this leads to degenerate multiplets. In the presence of the magnetic field $\vec{B}$ or the spin orbit interaction, the symmetry breaking Lagrangian $\mathcal{L}_1$ is either $-\vec{\mu}.\vec{B}$ or $\vec{L}.\vec{S}$, where $\vec{\mu},\ \vec{L},\ \vec{S}$ are the magnetic moments, angular momenta, and the spin of the electron or nucleon, respectively. In the presence of $\mathcal{L}_1$, the rotational symmetry is reduced to only the rotational symmetry about the direction of the magnetic field $\vec{B}$ or the quantization axis of $\vec{L}$(or $\vec{S}$) which can be chosen to be along the Z-axis. The degeneracy

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of the states in the multiplets is, therefore, lifted and the energy of the states forming the multiplets depends upon the Z-direction of the quantum numbers μ_Z, L_Z, or S_Z.

Another example is the strong interaction among nucleons, which is invariant under the flavor symmetry SU(2) in absence of electromagnetic interactions. However, in the presence of electromagnetic interactions, the Lagrangian is written as:

$$\mathcal{L} = \mathcal{L}^0_{SU(2)} + \mathcal{L}_{\text{em}}, \tag{7.86}$$

where $\mathcal{L}_{\text{em}}$ depends upon the charge and breaks the SU(2) symmetry. The $\mathcal{L}^0_{SU(2)}$ part of the Lagrangian gives the degenerate mass of the nucleons (pions), while $\mathcal{L}_{\text{em}}$ breaks the SU(2) symmetry to give the $n - p$ ($\pi^+ - \pi^-$) mass difference. Similarly, in the case of strong interactions of elementary particles with unitary symmetry, the Lagrangian is written as:

$$\mathcal{L} = \mathcal{L}^0_{SU(3)} + \mathcal{L}_{\text{int}}. \tag{7.87}$$

In this case, the $\mathcal{L}^{SU(3)}$ part of the Lagrangian is invariant under SU(3) and gives the degenerate mass of the octet and decouplet states of SU(3). The $\mathcal{L}_{\text{int}}$ part of the Lagrangian breaks the SU(3) symmetry to give the splitting between the different isospin multiplets constituting the octet and decouplet and lifts the degeneracy.

iii) **Spontaneously broken symmetries:** There is another type of symmetry in which the Lagrangian is invariant under a symmetry transformation leading to the equations of motion, which are also invariant; however, the ground state of the physical system or the physical vacuum in the case of field theory is not invariant under the symmetry transformation leading to a vacuum state, which is not unique. There could be many states of lowest energy, that is, ground state, which are degenerate (as shown in the right panel of Figure 7.4). This means that under the symmetry transformation, while $\delta\mathcal{L} = 0$, leading to some local conservation laws as a consequence of properties of exact symmetry, the non-uniqueness of the vacuum destroys the mass (energy) degeneracy of the states in the multiplets, showing that the symmetry is no longer exact but is broken. This type of symmetry breaking is referred to as the dynamical or the spontaneous breaking of symmetry and the system is described to have a "hidden symmetry". It is not unusual to have a physical system in which the symmetry is broken by both mechanisms explicitly as well as spontaneously. A well-known example of such physical systems is a material exhibiting ferromagnetism near the Curie temperature T_c. In the case of $T > T_c$, if the system is placed in a magnetic field, then the spin dipoles are aligned in the direction of the magnetic field. The rotational symmetry is explicitly broken down to the rotational symmetry about the axis of rotation in the direction of the magnetic field $\vec{B}$. For $T > T_c$, in the paramagnetic phase, all the spin dipoles are randomly oriented (Figure 7.5). The system displays rotational symmetry and the ground state is rotationally invariant. For $T < T_c$, in the ferromagnetic phase, all the spin dipoles are aligned in a parallel direction (spontaneous magnetization), which is arbitrary; the rotational

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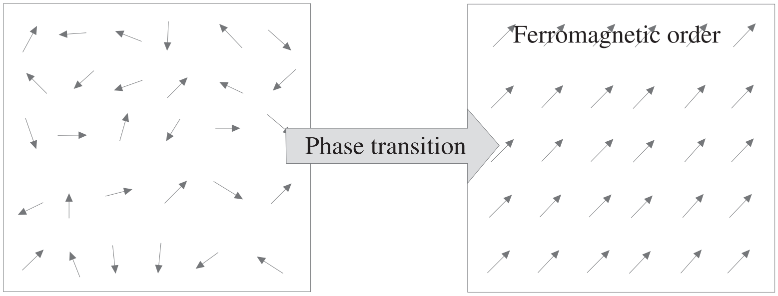

Figure 7.5 Phase transition in ferromagnetic materials. For $T > T_c$ (left), all the spin dipoles are randomly oriented. For $T < T_c$ (right), in the ferromagnetic phase, all the spin dipoles are aligned in a parallel direction (spontaneous magnetization).

symmetry is broken to a lower level to a symmetry around that arbitrary direction. This is the ground state corresponding to the lowest energy but it could correspond to any direction or orientation of spin and is, therefore, infinitely degenerate. This is an example of spontaneous breaking of symmetry(Figures 7.5 and 7.6). In the following sections,

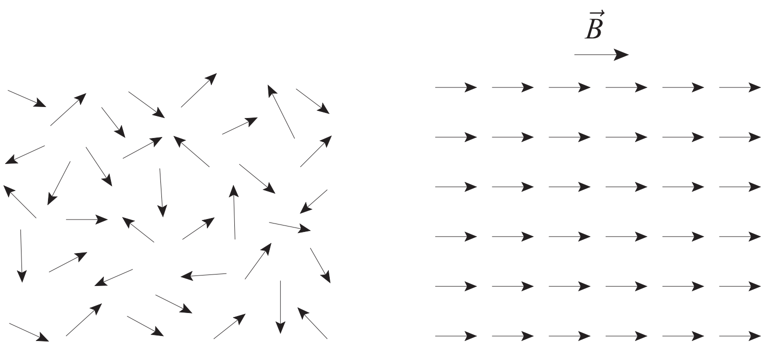

Figure 7.6 Paramagnetic material in the absence of external magnetic field (left); in the presence of external magnetic field (right), symmetry is lost.

we first describe some formal results regarding spontaneous symmetry breaking (SSB) in field theory and then discuss simple models to demonstrate them in some cases of discrete and continuous abelian and non-abelian field theories.

7.3.7 Spontaneous symmetry breaking: Some formal results

We have seen in earlier sections that the presence of an exact symmetry implies the invariance of the Lagrangian $\mathcal{L}$ of the system under some unitary transformation U, such that

$$\mathcal{L} \to \mathcal{L}' = U\mathcal{L}U^\dagger = \mathcal{L}, \tag{7.88}$$

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leading to the Euler–Lagrange equation of motion and local conservation laws. Some well-known examples are the conservation of current J^μ and the conservation of energy momentum tensor components $\theta^{\mu\nu}$ leading to (Chapter 3):

$$\partial_\mu J^\mu = 0 \qquad \text{and} \qquad \partial_\mu \theta^{\mu\nu} = 0, \tag{7.89}$$

associated with symmetry transformation, that is, leading to the conservation of charge and energy. To paraphrase these invariance principles and their consequences in terms of the Hamiltonian H, defined as:

$$H = \int \mathcal{H} d\vec{x}, \tag{7.90}$$

where $\mathcal{H}$ is the Hamiltonian density,

$$\mathcal{H} = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i} \dot{\phi}_i - \mathcal{L}(\phi_i, \dot{\phi}_i, t), \tag{7.91}$$

let us assume that a symmetry transformation defined by a unitary transformation U, leaves the Hamiltonian H invariant, that is,

$$H \to H' = UHU^\dagger = H. \tag{7.92}$$

Then,

$$[U, H] = 0. \tag{7.93}$$

If the operator Q, called the charge operator, is the generator of the infinitesimal unitary symmetry transformation $U(= e^{i\alpha Q})$, where α is a parameter, then Eq. (7.92) implies that:

$$(1 + i\alpha Q)H(1 - i\alpha Q) = H, \qquad \text{that is,} \qquad [Q, H] = 0. \tag{7.94}$$

Using Heisenberg's equation of motion, this leads to the equation of motion for Q, that is, $\frac{dQ}{dt} = [Q, H] = 0$ implying Q is a constant of motion.

This means that the quantity corresponding to the infinitesimal generator of the unitary transformation (U) is a constant of motion if U is an exact symmetry. The symmetry of H manifests in the degeneracies of the energy eigenstates corresponding to the irreducible representation of the unitary symmetry U. To demonstrate this, let us assume that $|A\rangle$ and $|B\rangle$ are the states corresponding to a given representation of U. Then they are connected by U as:

$$U|A\rangle = |B\rangle. \tag{7.95}$$

The energy E_B of the state $|B\rangle$ is defined as:

$$\begin{aligned} E_B &= \langle B|H|B\rangle \\ &= \langle AU^\dagger|H|UA\rangle \\ &= \langle A|H|A\rangle = E_A, \end{aligned} \tag{7.96}$$

leading to $E_B = E_A$; this shows that the states $|A\rangle$ and $|B\rangle$ are degenerate. Such degeneracies in the physical states have been observed experimentally in the atomic, nuclear, and hadron spectroscopy. However, the results in Eqs. (7.95) and (7.96) are not straightforward and depend upon the invariance of the ground state (the physical vacuum state in the case of field theory) implicitly implied in the aforementioned derivation. This can be seen by the simple argument: assume that $\phi^\dagger$ is the field operator which creates a single particle state $|A\rangle$ and $|B\rangle$ from the vacuum state $|0\rangle$, then:

$$|A\rangle = \phi_A^\dagger|0\rangle, \qquad |B\rangle = \phi_B^\dagger|0\rangle. \tag{7.97}$$

The operators $\phi_A^\dagger$ and $\phi_B^\dagger$ are related by the unitary transformation U as:

$$\phi_B^\dagger = U\phi_A^\dagger U^\dagger \qquad \Rightarrow \qquad \phi_B^\dagger U = U\phi_A^\dagger$$

giving us

$$\begin{aligned} \phi_B^\dagger U|0\rangle &= U\phi_A^\dagger|0\rangle = U|A\rangle = |B\rangle, \\ \phi_B^\dagger U|0\rangle &= |B\rangle = \phi_B^\dagger|0\rangle. \end{aligned} \tag{7.98}$$

Equation (7.98) is possible only if $U|0\rangle = |0\rangle$, that is, the physical vacuum state $|0\rangle$ is invariant under U. Since $U = 1 + i\alpha Q$, this leads to $Q|0\rangle = 0$, which implies that Q, the generator of the infinitesimal transformation, annihilates the physical vacuum state. In this case, the vacuum state $|0\rangle$ is unique because any symmetry operation U on $|0\rangle$ will again give $|0\rangle$.

However, there exists another possibility, where $U|0\rangle \neq |0\rangle \Rightarrow Q|0\rangle \neq 0$ [343]. In this case, if $U|0\rangle$ exists, then it is not unique and, therefore, it is degenerate. Moreover, it is infinitely degenerate in the case of field theory, where the field variable $\phi(x)$ can take any value depending upon x.

This is demonstrated by the Fabri–Picasso theorem [343]. Consider the following vacuum expectation value of the operator $Q^2 \equiv QQ$,

$$\langle 0|QQ|0\rangle = \langle 0|\int d\vec{x} J^0(x)Q|0\rangle = \int d\vec{x}\langle 0|J^0(x)Q|0\rangle. \tag{7.99}$$

Since Q is derivable from the current operator $J^\mu(x)$ which is conserved under the continuous symmetry operation U (Eq. (7.40), Section 7.3.3), using the translational invariance, we can write:

$$J^\mu(x) = e^{-iP.x}J^\mu(0)e^{iP.x}, \tag{7.100}$$

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and

$$\langle 0|J^0(x)Q\rangle = \langle 0|e^{-iP.x}J^0(0)e^{iP.x}Q|0\rangle, \tag{7.101}$$

where P^μ is the momentum operator, the generator of the translation symmetry, and Q is the symmetry operator in internal space, such that

$$P^\mu|0\rangle = 0, \qquad [P^\mu, Q] = 0, \tag{7.102}$$

which gives

$$\langle 0|J^0(x)Q|0\rangle = \langle 0|J^0(0)Q|0\rangle \tag{7.103}$$

implying that $\langle 0|J^0(0)Q|0\rangle$ is independent of x and, therefore,

$$\langle 0|QQ|0\rangle = \int d\vec{x}\langle 0|j^0(x)Q|0\rangle = \int \langle 0|J^0(0)Q|0\rangle d\vec{x}. \tag{7.104}$$

This diverges because $\langle 0|j^0(0)Q|0\rangle$ is independent of x, and x extends to infinity in field theory. It implies that either $Q|0\rangle = 0$, making the norm finite or $Q|0\rangle$ has infinite norm. In the case of $Q|0\rangle = 0$, the physical vacuum is unique. This is because the two vacuum states $|A\rangle$ and $|B\rangle$ are related by the unitary transformation, that is, $|0\rangle_A$ and $|0\rangle_B$.

$$|0\rangle_B = e^{i\alpha Q}|0\rangle_A, \tag{7.105}$$

where α is a parameter and the infinitesimal generator of U.

$$|0\rangle_B = (1 + i\alpha Q + ...)|0\rangle_A, \tag{7.106}$$

implies that if $Q|0\rangle_A = 0$, then $|0\rangle_B = |0\rangle_A$.

In the other case, $Q|0\rangle$ has infinite norm. Therefore, either $Q|0\rangle$ does not exist, or if it exists, the states are infinitely degenerate leading to the infinite norm. In this case, the vacuum is not unique and the condition leading to the degeneracy of the eigenstate of the Hamiltonian is not satisfied; thus, destroying the degeneracy of the spectra of the states. This is the manifestation of a symmetry that is broken spontaneously. The realization is manifested through non-degenerate states, even though the Lagrangian and equations of motion are invariant(covariant) under symmetry transformations. Spontaneous breaking is also known as dynamical breaking of the symmetry. It happens in the case of ferromagnets for $T < T_C$, when all the spin dipoles are aligned in one direction which is arbitrary and, therefore, leads to infinite degeneracy.

7.3.8 Spontaneous symmetry breaking (SSB) and Goldstone's theorem

The phenomenon of spontaneous symmetry breaking in field theory was first studied by Nambu [330] and Goldstone [162] based on the analogy with the theory of superconductivity and applied to particle physics. Nambu's models were based on the BCS theory of superconductivity. In Nambu's model, the spontaneous breaking of chiral symmetry was achieved by quasi-particle-like excitations arising due to the interaction of fermion fields; an example of such an interaction is the creation of Cooper pairs of electrons in the BCS theory of superconductivity [344]. In Goldstone's model, the spontaneous breaking of gauge symmetry was achieved by introducing self interacting scalar fields and giving them non-zero vacuum expectation values leading to the scalar excitations of the fields, in analogy with the Ginzberg–Landau theory [345]. In both the approaches, while masses of fermions or vector bosons were generated, massless scalar or pseudoscalar bosons also appeared as a result of the SSB of an exact continuous symmetry; these massless particles were called Nambu–Goldstone bosons. In Nambu's model of the SSB of chiral symmetry, the appearance of massless pseudoscalar Nambu–Goldstone bosons may be interpreted as pions in the limit of $m_\pi \to 0$. However, in the case of other gauge symmetries, massless scalar bosons were generated. In the case of SSB of such gauge symmetries, the predictions of the existence of massless scalar Nambu–Goldstone bosons were not supported by any experimental evidence. The appearance of massless Nambu–Goldstone boson fields was thus an obstacle for these type of theories to be of any use in the phenomenology of weak interactions. At first, these were thought to be model-dependent results which could be improved by using a sophisticated field theoretic model but soon a general theorem by Goldstone, Salam, and Weinberg [331] proved the existence of massless boson fields to be an essential feature of any Lorentz invariant relativistic field theory. This put an end to such theories in which "the Goldstone bosons sit like a snake hiding in the grass ready to strike". The theorem states that "In a manifestly Lorentz invariant quantum field theory, if there is continuous symmetry under which the Lagrangian is invariant, then either the vacuum state is also invariant or there must exist spinless (scalar) particles of zero mass." This is the most quoted statement of the Goldstone theorem.

In Goldstone's approach, the spontaneous breaking of symmetry was achieved by introducing self interacting scalar fields and giving nonzero vacuum expectation values (VEV) of some scalar fields in the theory. This is possible if the physical vacuum is not invariant under symmetry transformations. Let us suppose that there exists a symmetry of the Lagrangian leading to a conserved current J_a^μ given by Noether's theorem such that:

$$\partial_\mu J_a^\mu(x) = 0, \qquad\qquad [\text{for some } a]$$

leading to a charge operator Q_a, which is defined as:

$$Q_a = \int J_a^0(x) d\vec{x}, \quad \text{such that} \tag{7.107}$$

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Q_a is conserved, reflecting the symmetry of the Lagrangian under which the physical vacuum is not invariant, that is,

$$\begin{aligned}[Q_a, H] &= 0, \text{ but} \\ Q_a|0\rangle &\neq 0. \end{aligned} \tag{7.108}$$

The fields ϕ_α in a given representation of the symmetry group will transform as:

$$[Q_a, \phi_\alpha(x)] = T^a_{\alpha\beta}\phi_\beta(x),$$

where $T^a_{\alpha\beta}$ is the matrix representation of Q_a. Taking the VEV on both sides gives:

$$T^a_{\alpha\beta}\langle 0|\phi_\beta(x)|0\rangle = \langle 0|[Q_a, \phi_a(x)]|0\rangle. \tag{7.109}$$

This shows that $\langle 0|\phi_\beta(x)|0\rangle \neq 0$ if $Q_a|0\rangle \neq 0$ and the symmetry is broken spontaneously. It means that if a component of a field in a given representation is assigned a non-zero VEV, then all the operators which do not commute with that field component break the symmetry spontaneously. For example, in the case of isotriplet of field $\phi_\alpha(x)(\alpha = 1, 2, 3)$, under SU(2) if $\langle 0|\phi_3(x)|0\rangle \neq 0$, then Q_1 and Q_2 generators do not annihilate the vacuum and break the SU(2) symmetry spontaneously.

To prove Goldstone's theorem [331], let us consider the Fourier transform of the VEV of the commutator $[J^\mu(x), \phi_\alpha(0)]$, that is,

$$M^\mu_\alpha(k) = \int d^4x e^{ik.x}\langle 0|[J^\mu(x), \phi_\alpha(0)]|0\rangle, \tag{7.110}$$

where k^μ is an arbitrary vector. For simplicity and definiteness, we drop the indices a as well as α, as it is valid for any a and α.

Let us saturate the commutator in Eq. (7.110) with a complete set of states $|n\rangle$, that is, $\sum_n |n\rangle\langle n| = 1$ satisfying the relation:

$$P^\mu|n\rangle = p^\mu_n|n\rangle,$$

where P^μ is the momentum operator and $|n\rangle$ is an eigenstate of momentum P^μ corresponding to the eigenvalue p^μ_n. The Goldstone theorem then states that one of the states $|n\rangle$ is necessarily a spin 0 massless state corresponding to $p^2_n = 0$ state. We write, after inserting complete set of $|n\rangle$ states,

$$M^\mu_\alpha(k) = \sum_n \int d^4x \left[\langle 0|J^\mu(x)|n\rangle\langle n|\phi_\alpha(0)|0\rangle - \langle 0|\phi_\alpha(0)|n\rangle\langle n|J^\mu(x)|0\rangle\right] e^{ik.x}. \tag{7.111}$$

Using translational invariance, we express $J^\mu(x)$ as:

$$J^\mu(x) = e^{-iP.x}J^\mu(0)e^{iP.x} \tag{7.112}$$

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and use it in Eq. (7.111) to give:

$$\begin{aligned} M_\alpha^\mu(k) &= \sum_n \int d^4x \left[e^{i(p_n+k).x} \langle 0|J^\mu(0)|n\rangle\langle n|\phi_\alpha(0)|0\rangle \right. \\ &\qquad \left. - e^{-i(p_n-k).x}\langle 0|\phi_\alpha(0)|n\rangle\langle n|J^\mu(0)|0\rangle \right] \\ &= \sum_n (2\pi)^4 \left[\delta^4(p_n+k)\langle 0|J^\mu(0)|n\rangle\langle n|\phi_\alpha(0)|0\rangle \right. \\ &\qquad \left. - \delta^4(p_n-k)\langle 0|\phi_\alpha(0)|n\rangle\langle n|J^\mu(0)|0\rangle \right]. \end{aligned}$$

In order to have spontaneous breaking of symmetry, $M_\alpha^\mu(k) \neq 0$ (see Eq. (7.101) and (7.111)), which implies that

1. $\langle n|\phi_\alpha(0)|0\rangle \neq 0$, that is, $|n\rangle$ is a single particle state created from the physical vacuum by the action of the field ϕ_α and is, therefore, a spinless boson due to $\phi_\alpha(0)$ being a scalar field.

2. The matrix element $\langle 0|J^\mu(0)|n\rangle$ should be proportional to a four vector as J^μ is a four vector implying that

$$\langle 0|J^\mu(0)|n\rangle = a(p_n^2)p_n^\mu, \tag{7.113}$$

with $a(p_n^2) \neq 0$ from the requirement of the Lorentz covariance. It should be noted that this is not the case in the non-relativistic theories [332], where additional terms could be present.

We also define

$$\langle n|J^\mu(0)|0\rangle = b(p_n^2)p_n^\mu, \tag{7.114}$$

where $b(p_n^2) \neq 0$, is another scalar which may be related to $a(p_n^2)$ depending upon the properties of $J^\mu(0)$. Using the properties of $\delta^4(p_n+k)$ and $\delta^4(p_n-k)$ and the requirement of the states $|n\rangle$ to be physical, that is, $p_n^0 = E_n > 0$, which leads to $\delta(E_n+k_0) = 0$ if $k_0 > 0$, and $\delta(E_n-k_0) = 0$ if $k_0 < 0$. Therefore, we will use the step function Θ in the expression of $M_\alpha^\mu(k)$ as:

$$M_\alpha^\mu(k) = (2\pi)^4[-a(k^2)k_\mu\langle n|\phi_\alpha(0)|0\rangle\Theta(-k_0) - b(k^2)k_\mu\langle 0|\phi_\alpha(0)|n\rangle\Theta(k_0)]. \tag{7.115}$$

Now we replace the step function Θ by a function $\epsilon(k_0)$as:

$$\epsilon(k_0) = \begin{cases} -1, & \text{if } \ k_0 < 0 \\ +1, & \text{if } \ k_0 \geq 0 \end{cases}. \tag{7.116}$$

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We write:

$$M^{\mu}_{\alpha}(k) = k^{\mu}[\rho_{1\alpha}(k^2)\epsilon(k_0) + \rho_{2\alpha}(k^2)], \tag{7.117}$$

where

$$\begin{aligned}
\rho_{1\alpha}(k^2) &= \frac{1}{2}(2\pi)^4[a(k^2)\langle n|\phi_\alpha(0)|0\rangle - b(k^2)\langle 0|\phi_\alpha(0)|n\rangle] \\
\rho_{2\alpha}(k^2) &= \frac{1}{2}(2\pi)^4[-a(k^2)\langle n|\phi_\alpha(0)|0\rangle - b(k^2)\langle 0|\phi_\alpha(0)|n\rangle].
\end{aligned} \tag{7.118}$$

3. The condition of the current being divergenceless, that is, $\partial_\mu J^\mu(x) = 0$ gives the corresponding condition on $M^\mu(k)$ as

$$\begin{aligned}
& \partial_\mu M^\mu_\alpha(k) = 0 \qquad \Rightarrow \quad k^2[\epsilon(k_0)\rho_{1\alpha}(k^2) + \rho_{2\alpha}(k^2)] = 0 \\
\Rightarrow \quad & k^2[-\rho_{1\alpha}(k^2) + \rho_{2\alpha}(k^2)] = 0 \qquad \text{for} \quad k_0 < 0 \\
\text{and} \quad & k^2[\rho_{1\alpha}(k^2) + \rho_{2\alpha}(k^2)] = 0 \qquad \text{for} \quad k_0 > 0
\end{aligned}$$

which implies that $\rho_{1\alpha}(k^2)$ and $\rho_{2\alpha}(k^2)$ can be written in terms of arbitrary parameters $c_{1\alpha}$ and $c_{2\alpha}$ as:

$$\rho_{1\alpha}(k^2) = c_{1\alpha}\delta(k^2) \qquad \text{and} \qquad \rho_{2\alpha}(k^2) = c_{2\alpha}\delta(k^2). \tag{7.119}$$

Therefore,

$$M^{\mu}_{\alpha}(k) = k^{\mu}[c_{1\alpha}\epsilon(k_0) + c_{2\alpha}]\delta(k^2). \tag{7.120}$$

If $c_{1\alpha}$ and $c_{2\alpha}$ are both zero, $M^\mu_\alpha(k) = 0$, implying that there is no spontaneous symmetry breaking. If either of them is non-zero, spontaneous breaking would be present. To evaluate $c_{1\alpha}$ and $c_{2\alpha}$, we evaluate $M^0_\alpha(\vec{k} = 0, k^0)$ in two ways.

First, using Eq. (7.110), we write:

$$\begin{aligned}
M^0_\alpha(\vec{k}=0,k^0) &= \int d^4x \langle 0|[J^0(x),\phi_\alpha(0)]|0\rangle e^{ik_0x_0} \\
&= \int d\vec{x}\, 2\pi\delta(k_0) < 0|[J^0(x),\phi_\alpha(0)]|0 >, \text{where} \\
&\qquad\qquad \int dx_0 e^{ik_0x_0} = 2\pi\delta(k_0) \\
&= 2\pi\delta(k_0)\langle 0|[Q(t),\phi_\alpha(0)]|0\rangle.
\end{aligned} \tag{7.121}$$

Since the equations of motion respect the symmetry of the Lagrangian, the operator $Q(t)$ is independent of time (Eq. (7.107)). Therefore,

$$M^0_\alpha(\vec{k}=0,k^0) = 2\pi\delta(k_0)\eta, \qquad \text{where} \quad \eta = \langle 0|[Q,\phi_\alpha(0)]|0\rangle \tag{7.122}$$

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such that

$$\int M_\alpha^0(\vec{k}=0,k^0)dk^0 = \int_{-\infty}^{+\infty} 2\pi\delta(k_0)\eta dk^0 = 2\pi\eta. \tag{7.123}$$

This integral can also be evaluated using the covariant form of $M_\alpha^\mu(k)$ in Eq. (7.120), that is,

$$\begin{aligned}
\int M_\alpha^0(\vec{k}=0,k^0)dk^0 &= \int_{-\infty}^{+\infty} dk_0[c_{1\alpha}\epsilon(k_0)+c_{2\alpha}]k_0\delta(k_0^2-\vec{k}^2) \\
&= \int_{-\infty}^{+\infty} dk_0[c_{1\alpha}\epsilon(k_0)+c_{2\alpha}]k_0 \\
&\qquad \left(\frac{\delta(k_0-|\vec{k}|)+\delta(k_0+|\vec{k}|)}{2|\vec{k}|}\right) \\
&= \int_{-\infty}^{0} dk_0[-c_{1\alpha}k_0+c_{2\alpha}k_0]\left(\frac{\delta(k_0+|\vec{k}|)}{2|\vec{k}|}\right) \\
&\quad + \int_{0}^{+\infty} dk_0[c_{1\alpha}k_0+c_{2\alpha}k_0]\left(\frac{\delta(k_0-|\vec{k}|)}{2|\vec{k}|}\right) \\
&= \frac{|\vec{k}|}{2|\vec{k}|}(c_{1\alpha}-c_{2\alpha})+\frac{|\vec{k}|}{2|\vec{k}|}(c_{1\alpha}+c_{2\alpha}) \\
&= c_{1\alpha}.
\end{aligned} \tag{7.124}$$

Comparing the two evaluations, Eqs. (7.123) and (7.124) we get: $c_{1\alpha} = 2\pi\eta$. Therefore, if $\eta \neq 0$, then $M_\alpha^\mu(k) \neq 0$. Using Eqs. (7.113), and (7.114), we can write:

$$\begin{aligned}
M_\alpha^\mu(k) &= (2\pi)^4 \sum_n \left[\delta(k+p_n)a(p_n^2)\langle n|\phi_\alpha(0)|0\rangle - \delta(k-p_n)b(p_n^2)\langle 0|\phi_\alpha(0)|n\rangle\right] p_n^\mu \\
&= 2\pi\eta k^\mu\delta(k^2)\epsilon(k).
\end{aligned} \tag{7.125}$$

If $\eta \neq 0$, only $p_n = \pm k$ states in Eq. (7.125) contribute. both have $p_n^2 = k^2 = 0$, that is, massless bosons, in order that $M_\alpha^\mu(k) \neq 0$. These are called Goldstone bosons. Therefore, there is at least one Goldstone boson corresponding to the spontaneous breaking of symmetry. There could be more than one. This is the Goldstone theorem.

7.3.9 Spontaneously broken discrete symmetry: A simple model

Let us take the Lagrangian for a self interacting real scalar field ϕ as:

$$\mathcal{L} = T - V = \frac{1}{2}(\partial_\mu\phi(x))(\partial^\mu\phi(x)) - \frac{1}{2}\mu^2\phi^2(x) - \frac{1}{4}\xi\phi^4(x). \tag{7.126}$$

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ϕ^4 represents self interaction of the field with coupling strength ξ ($\xi>0$); the potential is symmetric about ϕ, that is, $V(\phi) = V(-\phi)$. The lowest energy corresponding to the state of minimum energy pertaining to the minimum of the Hamiltonian

$$H = T + V = -\frac{1}{2}\dot{\phi}^2 + (\nabla\phi)^2 + V(\phi), \qquad \text{where } \dot{\phi} = \frac{\partial\phi}{\partial t} \tag{7.127}$$

is obtained by equating

$$\frac{\partial V}{\partial \phi} = \mu^2\phi + \xi\phi^3 = (\mu^2 + \xi\phi^2)\phi = 0. \tag{7.128}$$

Notice from Eq. (7.128), that for the minimum, there are two possibilities that is: either $\phi = 0$ or $\mu^2 + \xi\phi^2 = 0$

1. if the ground state is at $\phi = 0$ and $\mu^2>0$, then the potential has a unique minima, which corresponds to the vacuum state(Figure 7.7).

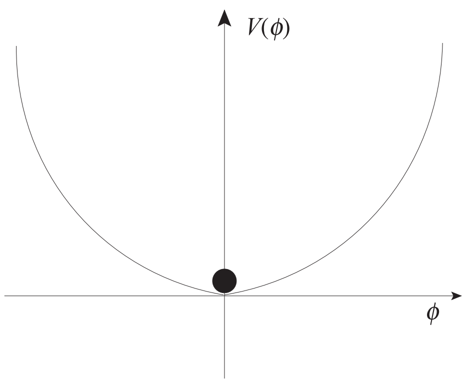

Figure 7.7 The potential $V(\phi) = \frac{1}{2}\mu^2\phi^2(x) + \frac{1}{4}\xi\phi^4(x)$ for $\mu^2 > 0$.

2. if ϕ satisfies the equation $\mu^2 + \xi\phi^2 = 0$,

$$\Rightarrow \qquad \phi = \pm\sqrt{\frac{-\mu^2}{\xi}} = \pm\lambda \text{ (say)}, \tag{7.129}$$

which corresponds to two degenerate lowest energy states as shown in Figure 7.8(a) for $\mu^2 < 0$. Therefore, the real minima is not $\phi = 0$ but $\phi = +\lambda$ or $\phi = -\lambda$, and $\phi = 0$ is an unstable point. Choosing any one of them (either Figure 7.8(b) or Figure 7.8(c)) will break the symmetry.

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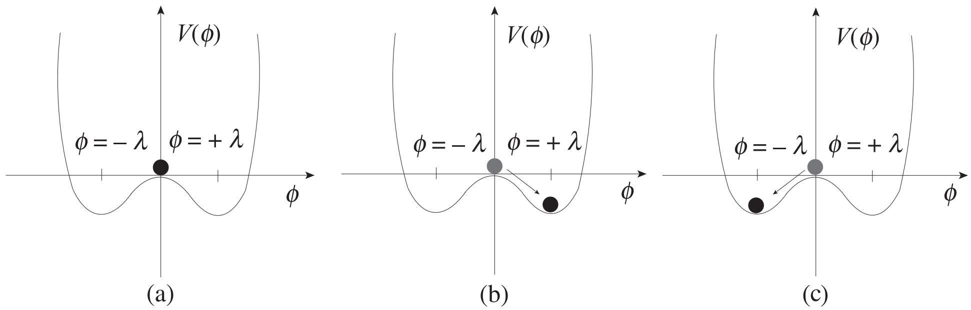

Figure 7.8 The potential $V(\phi) = \frac{1}{2}\mu^2\phi^2(x) + \frac{1}{4}\xi\phi^4(x)$ for $\mu^2 < 0$. There is an equal probability of the ball going to the left as to the right; both are the ground states for the system. The moment the ball takes a position, the left–right symmetry is broken as the system has preferred a particular place. This is an example of discrete symmetry, with just two ground states.

Let us take the case when $\phi = +\lambda$, and rescale the field $\phi(x) = \lambda + \eta(x)$, where $\eta(x)$ represents the fluctuations of ϕ around $\phi_0 = \lambda$ in terms of the new field $\eta(x)$. The Lagrangian given in Eq. (7.126) is now written as:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu\eta(x))(\partial^\mu\eta(x)) - \frac{1}{2}\mu^2(\eta^2(x) + 2\eta(x)\lambda + \lambda^2) \\ &\quad -\frac{1}{4}\xi\left(\eta^4(x) + 4\eta^3(x)\lambda + 6\eta^2(x)\lambda^2 + 4\eta(x)\lambda^3 + \lambda^4\right). \end{aligned} \tag{7.130}$$

Using $\mu^2 = -\xi\lambda^2$ from Eq. (7.129), and dropping constant terms like $\xi\lambda^4$ as they would not contribute to the field equations for the system, the Lagrangian is obtained as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\eta(x))(\partial^\mu\eta(x)) - \xi\lambda^2\eta^2(x) - \xi\lambda\eta^3(x) - \frac{1}{4}\xi\eta^4(x). \tag{7.131}$$

Recall the expression for the Lagrangian density for a free real scalar field that gives the Klein–Gordon equation with mass m, that is,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi(x))(\partial^\mu\phi(x)) - \frac{1}{2}m^2\phi^2(x). \tag{7.132}$$

When we compare it with Eq. (7.131), we obtain

$$\xi = \frac{m^2}{2\lambda^2} \quad \text{or equivalently} \qquad m = \sqrt{2\lambda^2\xi} = \sqrt{-2\mu^2}, \quad \text{using Eq. (7.129)}$$

The physical significance of the different terms of the Lagrangian density given in Eq. (7.131) is as follows.

1. The first term represents the kinetic energy of the scalar field $\eta(x)$.
2. The second term represents the mass term which has been identified by expanding the powers of the new field $\eta(x)$, with $m_\eta = \sqrt{-2\mu^2}$.

3. The third and the fourth terms represent the self interaction of the field $\eta(x)$ with three legs and four legs respectively as shown in Figure 7.9
4. The Lagrangian density defined in Eq. (7.131) is not symmetric in $\eta(x)$, although it was symmetric in Eq. (7.126) for $\phi(x) \to -\phi(x)$.

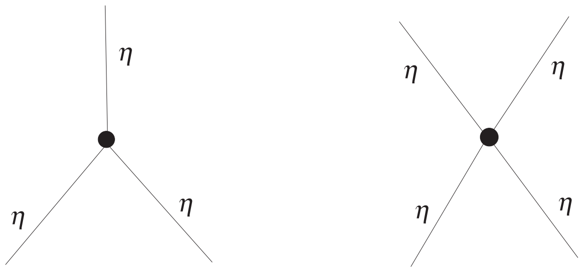

Figure 7.9 Self interaction of the $\eta(x)$ field with three legs and four legs.

The symmetry is, therefore, broken in the revised Lagrangian. This is an example of the spontaneous breaking of the discrete symmetry when one of the minima is chosen to be the physical ground state which generates mass of the $\eta(x)$ field.

7.3.10 Spontaneously broken continuous global symmetry under $U(1)$ gauge transformation and Goldstone boson

Let us consider a more realistic situation, a complex scalar field ϕ (for spin 0 particle) such that $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, where ϕ_1 and ϕ_2 are the two real scalar fields and writing the Lagrangian density as:

$$\begin{aligned} \mathcal{L} &= (\partial_\mu \phi(x))^* (\partial^\mu \phi(x)) - \mu^2 \phi^*(x)\phi(x) - \xi(\phi^*(x)\phi(x))^2 \qquad (7.133) \\ &= \frac{1}{2}(\partial_\mu \phi_1(x))(\partial^\mu \phi_1(x)) + \frac{1}{2}(\partial_\mu \phi_2(x))(\partial^\mu \phi_2(x)) - \frac{\mu^2}{2}(\phi_1^2(x) + \phi_2^2(x)) \\ &\quad -\frac{\xi}{4}\left(\phi_1^4(x) + \phi_2^4(x) + 2\phi_1^2(x)\phi_2^2(x)\right), \qquad (7.134) \end{aligned}$$

where the first two terms are the kinetic energy terms and the last two terms are the potential energy terms. The symmetry of the Lagrangian may be described by rotation in (ϕ_1, ϕ_2) space such that for any rotation angle θ:

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \qquad (7.135)$$

which gives $\phi_1'^2 + \phi_2'^2 = \phi_1^2 + \phi_2^2$ i.e. under the group SO(2) of rotation in the plane, where the Lagrangian is invariant.

If $\mu^2 > 0$, then we have the vacuum state defined by:

$$\langle\phi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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and for small fluctuations of the field i.e. neglecting quartic terms of ϕ_1 and ϕ_2, the Lagrangian density given in Eq. 7.134, simply becomes the Lagrangian for a two particle non-interacting system (with the same mass μ) i.e.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1(x))^2 + \frac{1}{2}(\partial_\mu \phi_2(x))^2 - \frac{\mu^2}{2}(\phi_1^2(x) + \phi_2^2(x)). \tag{7.136}$$

For $\mu^2<0$, the minimum of the potential lie on the circle of radius $\frac{\mu^2}{\xi}$ as shown in Figure 7.10, we rewrite the Lagrangian in Eq. (7.134), with a reversed sign ($\mu^2 \to -\mu^2$) for the mass term, that is

$$\begin{aligned}\mathcal{L} \;=\; & \frac{1}{2}(\partial_\mu \phi_1(x))(\partial^\mu \phi_1(x)) + \frac{1}{2}(\partial_\mu \phi_2(x))(\partial^\mu \phi_2(x)) + \frac{\mu^2}{2}(\phi_1^2(x) + \phi_2^2(x)) \\ & -\frac{\xi}{4}\left(\phi_1^4(x) + \phi_2^4(x) + 2\phi_1^2(x)\phi_2^2(x)\right).\end{aligned} \tag{7.137}$$

It should be noted that the mass term appears in the Lagrangian with a 'wrong' sign, which has been done to ensure that $\mu^2<0$ and we get a real value of vacuum expectation, that is:

$$\Rightarrow (\phi_1)^2_{\text{min}} + (\phi_2)^2_{\text{min}} = \frac{\mu^2}{\xi} \tag{7.138}$$

and for a particular ground state corresponding to the vacuum, the choice can be:

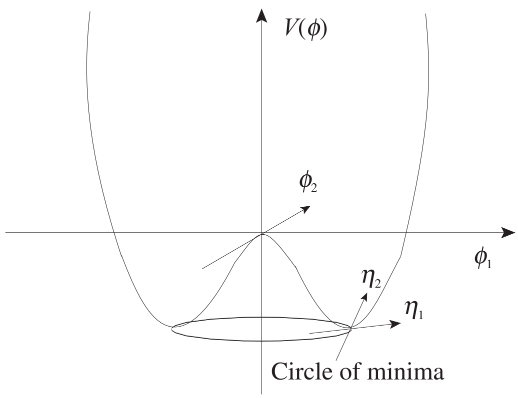

Figure 7.10 The potential $V(\phi)$ for $\mu^2 < 0$ and $\xi > 0$.

$$\begin{aligned}(\phi_1)_{\text{min}} &= \frac{\mu}{\sqrt{\xi}} = \lambda_1(say), \qquad (\phi_2)_{\text{min}} = 0 = \lambda_2(say) \\ \text{i.e.} \qquad \langle\phi\rangle_0 &= \frac{1}{\sqrt{2}}\begin{pmatrix}\lambda_1 \\ 0\end{pmatrix}\end{aligned} \tag{7.139}$$

which has been shown in Figure 7.10.

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This time for the fluctuation about the local minima we need two fields say $\eta_1(x)$ and $\eta_2(x)$ such that:

$$\phi = \frac{1}{\sqrt{2}}\left(\frac{\mu}{\sqrt{\xi}} + \eta_1(x) + i\eta_2(x)\right)$$

which results:

$$\begin{aligned}
\phi_1(x) &= \frac{\mu}{\sqrt{\xi}} + \eta_1(x), \\
\phi_2(x) &= \eta_2(x) \\
\phi_1^2(x) + \phi_2^2(x) &= \eta_1^2(x) + 2\frac{\mu}{\sqrt{\xi}}\eta_1(x) + \frac{\mu^2}{\xi} + \eta_2^2(x).
\end{aligned} \tag{7.140}$$

Using the above expression in Eq. 7.137, we get:

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}(\partial_\mu\eta_1(x))(\partial^\mu\eta_1(x)) + \frac{1}{2}(\partial_\mu\eta_2(x))(\partial^\mu\eta_2(x)) + \frac{\mu^2}{2}\left(\frac{\mu^2}{\xi} + \eta_1^2 + 2\frac{\mu}{\sqrt{\xi}}\eta_1 + \eta_2^2\right) \\
&- \frac{\xi}{4}\left(\eta_1^4 + \frac{\mu^4}{\xi^2} + 6\frac{\mu^2}{\xi}\eta_1^2 + 4\frac{\mu^3}{\xi\sqrt{\xi}}\eta_1 + 4\frac{\mu}{\sqrt{\xi}}\eta_1^3 + \eta_2^4\right) \\
&- \frac{1}{2}\xi\left(\frac{\mu^2}{\xi}\eta_2^2 + \eta_1^2\eta_2^2 + 2\frac{\mu}{\sqrt{\xi}}\eta_1\eta_2^2\right).
\end{aligned} \tag{7.141}$$

Looking for the mass terms in the Lagrangian, and writing only the kinetic energy term and the term proportional to η_1^2 or η_2^2 we find the Lagrangian as:

$$\mathcal{L} = \left[\frac{1}{2}(\partial_\mu\eta_1(x))(\partial^\mu\eta_1(x)) - \mu^2\eta_1^2\right] + \left[\frac{1}{2}(\partial_\mu\eta_2(x))(\partial^\mu\eta_2(x))\right]. \tag{7.142}$$

It may be noticed that the first term of the Lagrangian is the Lagrangian for the Klein-Gordon field for free particle of (mass m = $\sqrt{2}\mu$) and the second term is the Lagrangian for a field which is massless (i.e. m=0). Also the terms of the Lagrangian given in Eq. (7.141) define different interactions of the field which are shown using the Feynman prescription in Figure 7.11. It may

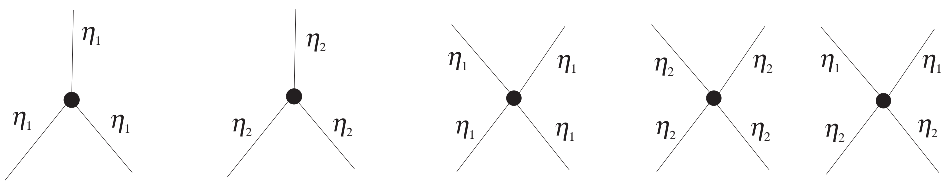

Figure 7.11 Different interaction terms for η_1 and η_2 fields.

be noticed that the Lagrangian which was symmetric in Eq. (7.133), in its new form given in Eq. (7.141) does not look symmetric and that means symmetry has been broken by making a choice of a particular vacuum state. One of the fields, defined by η_2 is massless and is a consequence of Goldstone's theorem discussed in Section 7.3.8, which says that spontaneous breaking of a continuous global symmetry is always accompanied by at least one massless scalar boson known as the Goldstone boson.

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7.3.11 Spontaneously broken continuous local $U(1)$ symmetry and Higgs mechanism

Let us start by considering locally gauge invariant Lagrangian that gives rise to spontaneously broken symmetries

$$\mathcal{L} = (D_\mu\phi(x))^*(D^\mu\phi(x)) - \mu^2\phi^*(x)\phi(x) - \xi(\phi^*(x)\phi(x))^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{7.143}$$

where $\phi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x))$. The covariant derivative $D_\mu = \partial_\mu - ieA_\mu$ is required for the local invariance of the Lagrangian and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

For a local gauge transformation under $U(1)$

$$\phi(x) \to \phi'(x) = e^{i\alpha(x)}\phi(x) \qquad \text{and} \qquad \phi'^*(x) = e^{-i\alpha(x)}\phi^*(x)$$

and

$$A^\mu \to A^{\mu\prime} = A^\mu + \frac{1}{e}\partial^\mu\alpha(x).$$

These transformations lead to:

$$\begin{aligned}\mathcal{L} = {} & ((\partial_\mu - ieA_\mu)\phi(x))^*((\partial^\mu - ieA^\mu)\phi(x)) - \mu^2\phi^*(x)\phi(x) \\ & -\xi(\phi^*(x)\phi(x))^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.\end{aligned} \tag{7.144}$$

For $\mu^2 > 0$ the potential has a minimum identified at $\phi = 0$ and the symmetry of the Lagrangian is preserved. Whereas for $\mu^2 < 0$, we follow the same analogy as that has been used in the previous section and we expand $\phi_1(x)$ and $\phi_2(x)$ about their minimum values:

$$\phi_1(x) = \eta_1(x) + \frac{\mu}{\sqrt{\xi}}, \qquad \phi_2(x) = \eta_2(x), \tag{7.145}$$

the Lagrangian density becomes:

$$\begin{aligned}\mathcal{L} = {} & \left[\frac{1}{2}(\partial_\mu\eta_1(x))(\partial^\mu\eta_1(x)) - \mu^2\eta_1^2(x)\right] + \left[\frac{1}{2}(\partial_\mu\eta_2(x))(\partial^\mu\eta_2(x))\right] \\ & + \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\frac{e^2\mu^2}{\xi}A_\mu A^\mu\right] \\ & - \Bigg(\left[e\left(\eta_1(x)\partial_\mu\eta_2(x) - \eta_2(x)\partial_\mu\eta_1(x)\right)\right]A^\mu + \frac{\mu}{\sqrt{\xi}}e^2\eta_1(x)A_\mu A^\mu \\ & + \frac{e^2}{2}(\eta_1^2(x) + \eta_2^2(x))A_\mu A^\mu - \mu\sqrt{\xi}(\eta_1^3(x) + \eta_1\eta_2^2(x)) \\ & - \frac{1}{4}\xi(\eta_1^2(x) + \eta_2^2(x))^2\Bigg) - \frac{\mu e}{\sqrt{\xi}}(\partial_\mu\eta_2(x))A^\mu + \left(\frac{\mu^2}{2\sqrt{\xi}}\right)^2.\end{aligned} \tag{7.146}$$

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It may be noticed that the first two terms of the Lagrangian are the same as Eq. 7.142, a scalar field $\eta_1(x)$ with mass μ, another field $\eta_2(x)$ which is massless corresponding to the Goldstone boson. Interestingly now there is an additional mass term appearing with the gauge field A^μ, which is the result of spontaneous symmetry breaking. When this is compared with the Lagrangian for a massive vector field with mass M

$$\mathcal{L}_{Proca} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 A_\mu A^\mu,$$

i.e. the Lagrangian that leads to Proca equation

$$\partial_\nu(\partial^\nu A^\mu - \partial^\mu A^\nu) + M^2 A^\mu = 0,$$

gives $M = \frac{\mu e}{\sqrt{\xi}}$. Other terms of the Lagrangian in Eq. 7.146 lead to various couplings of the

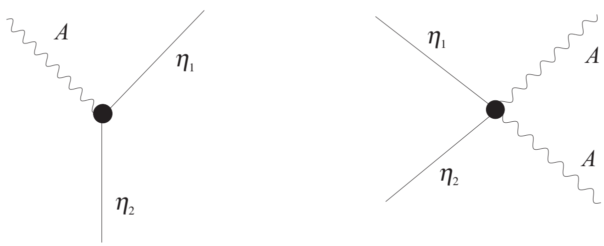

Figure 7.12 New interaction.

fields like shown in Figure 7.11, as well as some new ones due to the presence of vector field A^μ for example as shown in Figure 7.12. The Lagrangian density given in Eq. (7.146) describes

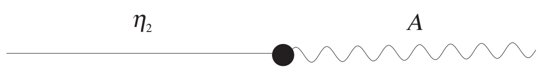

Figure 7.13 Interaction of massless field η_2 with A^μ.

the interaction of a massive vector field $A^\mu(x)$ and two scalar fields, one massive ($\eta_1(x)$) and one massless ($\eta_2(x)$) field (which is the Goldstone field), as shown in Figure 7.13. This means that the transformed Lagrangian has now fields with five degrees of freedom in contrast with the original Lagrangian which had fields with four degrees of freedom i.e. two for massless gauge field $A^\mu(x)$ and two for scalar fields $\phi_1(x)$ and $\phi_2(x)$. Therefore, the transformation has created a spurious degree of freedom for the fields which is not physical. We, therefore make a transformation using the gauge freedom such that this spurious degree of freedom disappears. We would see that this also removes the mixing of the gauge field A^μ with the Goldstone field

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which appears through the term $\frac{\mu e}{\sqrt{\xi}}(\partial_\mu \eta_2(x))A^\mu$ in Eq. (7.146). For this let us consider:

$$\begin{aligned}\phi &= \frac{1}{\sqrt{2}}\left(\eta_1(x) + \frac{\mu}{\sqrt{\xi}} + i\eta_2(x)\right) \\ &\simeq \frac{1}{\sqrt{2}}\left(\eta_1(x) + \frac{\mu}{\sqrt{\xi}}\right) e^{i\eta_2(x)\sqrt{\xi}/\mu} \end{aligned} \tag{7.147}$$

to the lowest order in $\eta_2(x)$. If we substitute a new transformed set of real fields as in Eq. (7.147) (for convenience still calling them $\eta_1(x)$ and $\eta_2(x)$) and make the following transformation on the gauge field A^μ:

$$A^\mu \to A^\mu + \frac{\sqrt{\xi}}{e\mu}\partial^\mu \eta_2(x) \tag{7.148}$$

in our original Lagrangian given in Eq.7.144, the expression for the Lagrangian density is obtained as:

$$\begin{aligned}\mathcal{L} &= \left[\frac{1}{2}(\partial_\mu \eta_1(x))(\partial^\mu \eta_1(x)) - \mu^2\eta_1^2(x)\right] + \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\frac{e^2\mu^2}{\xi}A_\mu A^\mu\right] \\ &+ \left(\frac{\mu}{\sqrt{\xi}}e^2\eta_1(x)A_\mu A^\mu + \frac{e^2}{2}\eta_1^2(x)A_\mu A^\mu - \mu\sqrt{\xi}\eta_1^3(x) - \frac{1}{4}\xi\eta_1^4(x)\right) + \left(\frac{\mu^2}{2\sqrt{\xi}}\right)^2. \end{aligned} \tag{7.149}$$

From Eq. 7.149, if we look at the mass terms in the Lagrangian it may be observed that there are two terms, one associated with single scalar field $\eta_1(x)$ (Higgs) and another term associated with the massive vector field A^μ, while the field associated with the Goldstone boson has disappeared.

7.3.12 Spontaneously broken local $SU(2)$ gauge symmetry and Higgs mechanism

We consider $SU(2)$ doublet of complex scalar field ϕ, consists of ϕ_α and ϕ_β with each component $\phi_1 + i\phi_2$ and $\phi_3 + i\phi_4$ respectively, such that:

$$\phi = \begin{pmatrix}\phi_\alpha \\ \phi_\beta\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\phi_2 \\ \phi_3 + i\phi_4\end{pmatrix}; \phi_{1-4} \text{ are real scalar fields, and} \tag{7.150}$$

$$\phi^\dagger\phi = \frac{1}{2}\left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\right). \tag{7.151}$$

Let us write the Lagrangian density in terms of complex scalar field:

$$\mathcal{L} = (\partial_\mu \phi(x))^\dagger(\partial^\mu \phi(x)) - \mu^2\phi^\dagger(x)\phi(x) - \lambda(\phi^\dagger(x)\phi(x))^2, \tag{7.152}$$

where μ is the mass of the particle associated with complex scalar field and λ is a real parameter. The first term in Eq. 7.152 corresponds to the kinetic energy term and the second term, for $\mu^2 > 0$ corresponds to the mass term and the third term represents ϕ^4 self interaction. If λ is equal to zero, then the above Lagrangian is nothing but the Lagrangian for the Klein-Gordon field with $\mu = m$.

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The above Lagrangian is invariant under $SU(2)$ global gauge transformation of the field

$$\phi(x) \to \phi'(x) = e^{i\alpha_i T_i}\phi(x). \tag{7.153}$$

In the above equation α is a parameter independent of space and time and $T_i = \frac{\tau_i}{2}$, where τ_i are the components of the Pauli matrices.

We are interested in the invariance under local $SU(2)$ gauge transformation:

$$\phi(x) \to \phi'(x) = e^{i\alpha_i(x) T_i}\phi(x). \tag{7.154}$$

Notice α_i is now a function of x, showing the space-time dependence. To ensure the invariance, we follow the algebra described in Section 7.3.4 and replace ∂_μ by the covariant derivative D_μ as:

$$D_\mu = \partial_\mu + igT_i W^i_\mu, \qquad \text{where } \; i = 1-3$$

and W^i_μ are the three gauge fields. For an infinitesimal gauge transformation under $SU(2)$

$$\begin{aligned} \phi(x) \to \phi'(x) &= (1 + i\alpha_i(x)T_i)\phi(x) \\ \Rightarrow \qquad \delta\phi(x) &= i\alpha_i(x)T_i\phi(x). \end{aligned}$$

Following the prescription of Section 7.3.4 the three gauge fields transform as:

$$W^i_\mu \to W^{i\prime}_\mu = W^i_\mu - \epsilon_{ijk}\alpha_j W^k_\mu - \frac{1}{g}(\partial_\mu \alpha_i(x)). \tag{7.155}$$

The gauge invariant Lagrangian is then given by:

$$\mathcal{L} = (\partial_\mu\phi + igT_i W^i_\mu\phi)^\dagger(\partial^\mu\phi + igT_i W^{\mu i}\phi) - V(\phi) - \frac{1}{4}G^i_{\mu\nu}G^{\mu\nu}_i, \text{ where} \tag{7.156}$$

$$\begin{aligned} G^i_{\mu\nu} &= \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g(W_\mu \times W_\nu)^i \\ V(\phi) &= \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2. \end{aligned} \tag{7.157}$$

For $\mu^2 > 0$, the Lagrangian in Eq. 7.156 describes a system of four scalar particles, each of mass μ, interacting with the three massless gauge bosons W^i_μ. For $\mu^2 < 0$, the potential $V(\phi)$ of Eq. 7.157 has a minimum at a finite value of $|\phi|$ i.e.

$$\mu^2 + 2\lambda\phi^\dagger\phi = 0 \tag{7.158}$$

$$\Rightarrow \qquad \phi^\dagger\phi = \frac{1}{2}(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2) = -\frac{\mu^2}{2\lambda} = \zeta^2 > 0. \tag{7.159}$$

Various choices of ϕ satisfying Eq. 7.159 are possible i.e. $\phi(x)$ may be expanded about some local minima and that choice is not unique. For example, one choice could be:

$$\phi_1 = \phi_2 = \phi_4 = 0 \quad \text{and} \quad \phi_3^2 = -\frac{\mu^2}{\lambda} = v^2. \tag{7.160}$$

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Applying the above constrain is equivalent to applying spontaneous symmetry breaking of the $SU(2)$ symmetry.

By expanding $\phi(x)$, about this vacuum

$$\phi(0) = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \tag{7.161}$$

and due to gauge invariance, the field $\phi(x)$ may be expanded as:

$$\phi(x) = \begin{pmatrix} 0 \\ \frac{v+h(x)}{\sqrt{2}} \end{pmatrix}, \quad \text{where } h(x) \text{ is Higgs field} \tag{7.162}$$

and use it in Eq. 7.156. Thus, it appears that we had started with four scalar fields, and are left finally with the Higgs field $h(x)$. This can be understood as follows:

Suppose we parameterize the fluctuations from the vacuum $\phi(0)$ in terms of four real fields viz. $\theta_1(x)$, $\theta_2(x)$, $\theta_3(x)$ and $h(x)$ using:

$$\begin{aligned}
\phi(x) &= e^{i\vec{\tau}.\vec{\theta}(x)/v} \begin{pmatrix} 0 \\ \frac{v+h(x)}{\sqrt{2}} \end{pmatrix}, \text{ which for small perturbations} \\
&= \left(1 + \frac{i}{v}\left[\tau_1\theta_1 + \tau_2\theta_2 + \tau_3\theta_3\right]\right) \begin{pmatrix} 0 \\ \frac{v+h(x)}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\theta_2 + i\theta_1}{\sqrt{2}} \\ \frac{v+h(x)-i\theta_3}{\sqrt{2}} \end{pmatrix}.
\end{aligned}$$

It may be noticed that the four fields are independent and are completely able to parameterize the deviations from the vacuum $\phi(0)$. Since the Lagrangian is locally invariant under $SU(2)$, therefore, the three fields defined by $\theta(x)$ corresponding to the massless Goldstone bosons can be gauged. The masses generated from the gauge bosons W^i_μ can be determined by substituting $\phi(0)$ from Eq. 7.161 into the Lagrangian, and the relevant term in the Lagrangian of our interest is

$$\left(igT_iW^i_\mu\phi(x)\right)^\dagger \left(igT_iW^{i\mu}\phi(x)\right) = \frac{g^2v^2}{8}\left\{(W^1_\mu)^2 + (W^2_\mu)^2 + (W^3_\mu)^2\right\}. \tag{7.163}$$

Comparing it with mass term associated with a boson field say $\frac{1}{2}M^2A^2_\mu$, we conclude

$$M = \frac{1}{2}gv. \tag{7.164}$$

Thus the Lagrangian describes the three massive gauge fields and one massive scalar field h(x). The gauge fields have eaten up the Goldstone bosons and in turn have become massive. Therefore, with the help of Higgs mechanism, we get rid of massless particles. However, this is not the end of the story, as we need the theory to be renormalizable, the proof of which was provided by 't 'Hooft as discussed in the next chapter, where we shall also discuss Glashow Weinberg-Salam theory of electroweak interaction.

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