Chapter 4

Interacting Fields and Relativistic Perturbation Theory

4.1 Introduction

In the previous chapter, we have described free particles and fields in terms of Lagrangian, equations of motion, their solutions and quantization. However, in the physical world, particles and fields are visualized by their interactions with other particles and fields or among themselves. For example, the simple processes of Compton scattering, photoelectric effect, and Coulomb scattering involving photons and electrons, are well known. All the known elementary particles interact with each other through the four fundamental interactions, that is, electromagnetic, weak, strong, and gravitational, through the exchange of gauge fields. Since the particles themselves can be described in terms of fields, all the physical processes governed by the four fundamental interactions are examples of various types of fields in interaction with each other including self interaction, allowed by the general principles of physics. These interactions are described by an interaction Lagrangian $L_{\text{int}}(x)$, to be included along with the free Lagrangian $L_{\text{free}}(x)$, described in Chapter 2 for a quantum description of the evolution of physical systems. The interaction Lagrangians can be obtained by using the symmetry properties of the physical system defined by certain transformations called local gauge transformations and imposing the invariance of the free Lagrangian under these transformations. These will be discussed in some detail in Chapter 8, in the case of electromagnetic, weak, and strong interactions of scalar, vector, and spin $\frac{1}{2}$ particles.

In this chapter, we give some simple examples of interaction Lagrangians involving spin 0, spin $\frac{1}{2}$, and spin 1 particles and illustrate the general principles to write them. We use the example of electromagnetic interaction to demonstrate the general method of the relativistic perturbation theory to find out the solution of the equations of motion of fields in the presence of the interaction Lagrangian $L_{\text{int}}(x)$. It is assumed that the strength of the interaction Lagrangian can be quantified by a parameter which is small, so that perturbation theory can be applied. This

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is normally the situation in the case of electromagnetic and weak interactions; in a limited range of kinetic variables, it is also true in the case of strong interactions. The relativistic perturbation theory has been very useful in describing physical processes.

4.2 Simple Forms of Interaction Lagrangians of Fields

The form of the interaction Lagrangian of fields depends upon the type of field, that is, spin 0, spin $\frac{1}{2}$ and spin 1. However, we have seen in Chapter 2 that quadratic terms for various fields and their derivatives which form a scalar like ϕ^2, $\phi^*\phi$, $\bar{\psi}\psi$ or $W_\mu W^\mu$, give rise to the mass term, while the lowest order scalars formed from derivatives like $\partial_\mu\phi^*\partial^\mu\phi$, $\overline{\psi}\gamma^\mu\partial_\mu\psi$ or $(\partial_\mu A^\nu)(\partial^\mu A_\nu)$ give rise to the kinetic energy term in the Lagrangian. Therefore, any addition of such quadratic terms would redefine the mass or the kinetic energy term. The terms used to describe the interaction of fields should avoid such terms. Therefore, the interaction Lagrangian for one field (i.e., self interaction) is written in terms of fields having powers higher than the quadratic terms. On the other hand, the interaction Lagrangian for more than one type of fields can be written involving all the fields using the general principle of symmetries of interactions like Lorentz invariance, parity and T-invariance.

In the following sections, we give examples of simple forms of interaction Lagrangian for various interactions.

4.2.1 Electromagnetic interactions

These interactions involve the interaction of charged particles with the electromagnetic field. Since the electromagnetic field is described by a vector field $A^\mu(x)$, we should construct a vector current for particles of spin 0, spin $\frac{1}{2}$, or higher spin to construct a scalar quantity because electromagnetic interactions conserve parity. The simple vector currents which carry charge can be constructed for complex scalar fields ϕ or spin $\frac{1}{2}$ fields like ψ as:

$$j_\mu^\phi(x) = \phi^*(x)\partial_\mu\phi(x) - \phi\partial_\mu\phi^*(x), \tag{4.1}$$

$$j_\mu^\psi(x) = \bar{\psi}(x)\gamma_\mu\psi(x). \tag{4.2}$$

Hence, the simplest interaction Lagrangian conserving parity can be written as:

$$\mathcal{L}_{\text{int}} = g j_\mu^\phi(x) A^\mu(x) \qquad \text{for spin } 0, \tag{4.3}$$

$$\text{and} \quad \mathcal{L}_{\text{int}} = g j_\mu^\psi(x) A^\mu(x) \qquad \text{for spin } \frac{1}{2}, \tag{4.4}$$

where g is the strength of the interaction. In fact, such an interaction can be generated by the principle of minimal coupling or the local gauge invariance (see Chapter 7), which predicts $g = e$.

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4.2.2 Weak interactions

In the case of weak interactions, parity is violated; therefore, the interaction Lagrangian would have scalar and pseudoscalar terms simultaneously to give parity violating effects through their interference. Since weak interactions are four fermion interactions as proposed by Fermi (Chapter 1), we construct scalars and pseudoscalars from the bilinear convariants formed from fermion fields. We have seen in Chapter 2 that in the case of spinors, five bilinear covariants can be constructed which transform as scalar, vector, tensor, axial vector or pseudoscalar. Therefore, a scalar can be constructed using any of the five covariants. Consequently, the Lagrangian can be constructed as:

$$\mathcal{L}_{\text{int}}^{\text{spinors}}(x) = \sum_{i=S,V,T,A,P} g_i \bar{\psi}(x) O^i \psi(x) \bar{\psi}(x) O_i \psi(x), \tag{4.5}$$

where $O^i (i = S, V, T, A, P)$ are the operators involving Dirac matrices such that the bilinear covariants $\bar{\psi} O^i \psi$ transform as S, V, T, A, P. There would be additional terms in the Lagrangian transforming as pseudoscalar such that the Lagrangian is written as:

$$\mathcal{L}_{\text{int}}^{\text{weak}}(x) = \sum_i \bar{\psi}(x) O^i \psi(x) \bar{\psi}(x) (C_i + C_i' \gamma_5) \psi(x), \tag{4.6}$$

where C_i and C_i' are, respectively, the strength of the scalar and pseudoscalar couplings.

4.2.3 Strong interactions

The simplest interaction Lagrangian between nucleon fields $\psi(x)$ and scalar fields $\sigma(x)$ or pseudoscalar fields $\phi(x)$ was first given by Yukawa [208] and is written as:

$$\mathcal{L}_{\text{int}}^{\text{strong}}(x) = g_s \bar{\psi}(x) \psi(x) \sigma(x) \qquad \text{(for scalar fields)} \tag{4.7}$$

and

$$\mathcal{L}_{\text{int}}^{\text{strong}}(x) = g_{ps} \bar{\psi}(x) \gamma_5 \psi(x) \phi(x) \text{ or } g_{pv} \bar{\psi}(x) \gamma_\mu \gamma_5 \psi(x) \partial_\mu \phi(x) \quad \text{(for pseudoscalar fields)}, \tag{4.8}$$

where g_s is the strength of the scalar coupling of field $\sigma(x)$ and g_{ps} and g_{pv} are respectively the strengths of the pseudoscalar and pseudovector coupling of the pseudoscalar fields $\phi(x)$.

4.2.4 Self interaction of various fields

In the earlier sections, we have given examples of simple interaction Lagrangians in which two different fields interact. There may exist self interactions of spin 0, spin $\frac{1}{2}$, or spin 1 fields. In case of real scalar fields, the interaction Lagrangian could have terms which are higher than the quadratic term in fields $\phi(x)$, that is,

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$$\mathcal{L}_{\text{int}}^{\text{real scalar}}(x) = \sum_{n\geq 3} g_n \phi^n(x), \tag{4.9}$$

where g_n is the strength of the interaction in the nth term. It should be noted that the dimension of g_n in the case of real fields should be $(4-n)$ as $\int \mathcal{L} d\vec{x}$ has dimensions of energy. In the case of complex scalar fields, the interaction Lagrangian can be written as:

$$\mathcal{L}_{\text{int}}^{\text{complex scalar}}(x) = \sum_{n\geq 2} g_n (\phi^*(x)\phi(x))^n. \tag{4.10}$$

Similarly, the self interaction Lagrangian can be written for higher spin fields like the Yang–Mills field or gluon fields and their form is predicted by the local gauge field theories as will be discussed in Chapter 7.

4.3 Evolution of Physical Systems and the S-matrix

In quantum mechanics, the physical observables correspond to the expectation value of certain operators between the states describing the system and are generally dependent on time. If they are independent of time, they are conserved. Energy, momentum, and angular momentum are simple examples of such observables, which are the expectation values of the Hamiltonian, momentum, and angular momentum operators, respectively, in a given state $|\phi(t)\rangle$, in Hilbert space. We use a generic notation $\phi(t)$ to represent a state which could be specified later as $\chi(t)$, $\phi(t)$, or $\psi(t)$ corresponding to scalar, pseudoscalar, spin $\frac{1}{2}$, or any other particle of higher spin. Thus, the expectation value of any operator $\langle O(t)\rangle$ between these states is defined as:

$$\langle O(t)\rangle = \frac{\langle \phi(t)|O(t)|\phi(t)\rangle}{\langle \phi(t)|\phi(t)\rangle}. \tag{4.11}$$

In Eq. (4.11), the ket $|\phi(t)\rangle$, the bra $\langle\phi(t)|$, and the operator $O(t)$ are all time dependent. We can always perform unitary transformation to transfer the time dependence of the state $|\phi(t)\rangle$ to the operator $O(t)$ and vice versa, without changing the expectation value $\langle O(t)\rangle$. Therefore, there is some ambiguity in specifying the time dependence of $|\phi(t)\rangle$ and/or $O(t)$. This ambiguity gives rise to various ways of describing the evolution of the physical system, described by the state $|\phi(t)\rangle$ and operator $O(t)$, known as the Schrödinger picture (S), Heisenberg picture (H), and interaction picture (I). The interaction picture is most suitable for formulating the relativistic perturbation theory. In the following sections, we will introduce these pictures before describing the relativistic perturbation theory.

4.3.1 Schrödinger, Heisenberg, and Interaction picture

We are familiar with the Schrödinger picture in which the equation of motion is given by:

$$i\frac{\partial \psi^S(t)}{\partial t} = H\psi^S(t), \tag{4.12}$$

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where we have put a superscript S over state $\psi(t)$ and operator H to refer to the Scrödinger picture. In this case, the time dependence is carried by the state $\psi^S(t)$; the Hamiltonian operator, H, is independent of time in most cases, leading to the conservation of energy. Equation (4.12) can be solved to get:

$$\psi^S(t) = e^{-iH(t-t_0)}\psi^S(t_0), \tag{4.13}$$

where $\psi^S(t_0)$ is the initial state at $t = t_0$, specified by the initial conditions. For simplicity of notation, let us set $t_0 = 0$. Then, Eq. (4.13) can be written as a transformation equation:

$$\psi^S(t) = U^S(t)\psi^S(0). \tag{4.14}$$

Here, H is a hermitian operator and $U^S(t) = e^{-iHt}$ is a unitary operator. In the Heisenberg picture, the time dependence is carried by the operator and the state is taken to be independent of time, defined at $t = t_0(= 0)$ by $\psi^H(0)$. Therefore,

$$\psi^H(t) = \psi^H(0)(= \psi^S(0))$$

and is related to $\psi^S(t)$ by a unitary transformation

$$\psi^H = U^{S^\dagger}(t)\psi^S(t). \qquad (\because \psi^S(0) = U^{S^\dagger}\psi^S(t)). \tag{4.15}$$

Therefore, in the Heisenberg picture, any operator O^S in the Schrödinger picture is given by:

$$O^H(t) = U^{S^\dagger}(t)O^S U^S(t), \tag{4.16}$$

making $\langle\psi^S(t)|O^S|\psi^S(t)\rangle = \langle\psi^H|O^H(t)|\psi^H\rangle$ invariant. Differentiating Eq. (4.16), we can derive the well-known Heisenberg equation of motion as:

$$\begin{aligned}
\frac{d}{dt}O^H(t) &= \frac{\partial}{\partial t}(U^{S^\dagger}(t)O^S U^S(t)) \\
&= \frac{\partial U^{S^\dagger}(t)}{\partial t}O^S U^S(t) + U^{S^\dagger}(t)O^S\frac{\partial U^S(t)}{\partial t} + U^{S^\dagger}(t)\frac{\partial O^S}{\partial t}U^S(t) \\
&= iU^{S^\dagger}(t)HO^S U^S(t) - iU^{S^\dagger}(t)O^S H U^S(t) \qquad \left(\because \frac{\partial O^S}{\partial t} = 0\right) \\
&= iHO^H(t) - iO^H(t)H \\
&= -i[O^H(t), H] \\
\Rightarrow \qquad i\frac{d}{dt}O^H(t) &= [O^H(t), H].
\end{aligned} \tag{4.17}$$

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These equations are well-suited to derive the equations of motion for free particles and find their solutions. In the case of Interaction picture, the Hamiltonian is given in terms of the free Hamiltonian (H_0) and an interaction Hamiltonian (H_I), that is,

$$H = H_0 + H_I, \tag{4.18}$$

$$\text{where} \quad H_0 = \int (\pi(x)\dot{\phi}(x) - \mathcal{L}_{\text{free}}) d\vec{x}, \tag{4.19}$$

$$\text{and} \quad H_I = -\int \mathcal{L}_{\text{int}} d\vec{x}, \tag{4.20}$$

where $\pi(x)$ is the momentum conjugate to $\phi(x)$.

In the presence of the interaction Lagrangian $\mathcal{L}_{\text{int}}(x)$, the equations of motion cannot be solved exactly as in the case of free fields. Therefore, the time dependence is defined by the state $\psi^I(t)$ in the interaction picture which is evolved from the Schrödinger picture $\psi^S(t)$ using H_0 instead of H, that is, it corresponds to the solution of the free Hamiltonian H_0, defined as:

$$\psi^I(t) = U_0^\dagger(t)\psi^S(t), \tag{4.21}$$

$$\text{where} \quad U_0(t) = e^{-iH_0 t} \tag{4.22}$$

$$\text{such that} \quad O^I(t) = U_0^\dagger(t) O^S U_0(t). \tag{4.23}$$

It should be noted that the free Hamiltonian is the same in all pictures, that is,

$$H_0^I = H_0^S = H_0. \tag{4.24}$$

Differentiating Eq. (4.23), we obtain the equation of motion in the interaction picture as:

$$i\frac{d}{dt}O^I(t) = [O^I(t), H_0]. \tag{4.25}$$

Putting Eq. (4.21) in Eq. (4.12), we obtain

$$i\frac{d}{dt}\psi^I(t) = H_I^I(t)\psi^I(t), \tag{4.26}$$

$$\text{where} \quad H_I^I(t) = e^{iH_0 t} H_I^S e^{-iH_0 t}. \tag{4.27}$$

Equation (4.26) determines the time development of $\psi^I(t)$ in the interaction picture, which could be solved to determine the evolution of the physical system in the presence of the interaction Hamiltonian calculated in this picture. It can be further shown that:

$$\psi^I(t) = U_0^\dagger(t)\psi^S(t) = U_0^\dagger(t)U^S(t)\psi^H(0) = U(t)\psi^H(0), \tag{4.28}$$

$$\text{with} \quad U(t) = U_0^\dagger(t)U^S(t) = e^{iH_0 t}e^{-iHt}, \tag{4.29}$$

$$\text{and} \quad \psi^I(t) = U(t)\psi^H(0)$$

$$= U(t)U^\dagger(t')\psi^I(t')$$

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$$= \quad U(t,t')\psi^I(t'), \tag{4.30}$$

$$\text{with } U(t,t') \quad = \quad U(t)U^\dagger(t') \tag{4.31}$$

connecting $\psi^I(t)$ at two time points t and t'.

4.3.2 S-Matrix and relativistic perturbation theory

Consider the equation of motion of $\psi^I(t)$ in the interaction picture

$$\frac{d}{dt}\psi^I(t) = -iH_I(t)\psi^I(t). \tag{4.32}$$

When $H_I(t) = 0$, that is, there is no interaction, $\psi^I(t)$ is a constant of motion. As soon as the interaction is switched on, it develops a time dependence and $\psi^I(t)$ evolves with time, depending upon the interaction Hamiltonian H_I. In a scattering process, the physical system is in an initial state $|i\rangle$, where the particles are not interacting; then, interaction (i.e., collision or scattering) takes place and depending upon the nature of the interaction, it leads to a set of final states of the system. After a considerable lapse of time, particles in the set of final state of the system are again noninteracting. We introduce the concept of asymptotic states to define the initial state at $t = -\infty$, that is, $\psi(-\infty) = |i\rangle$, where all the particle states are specified and the final states at $t = +\infty$, that is, $\psi(+\infty)$, which consists of particles in states to be specified depending upon the action of the interaction.

S-matrix is defined as an operator which changes $\psi(-\infty)$ to $\psi(\infty)$ as:

$$\psi^I(\infty) = S\psi^I(-\infty) = S|i\rangle \tag{4.33}$$

and describes the action of H_I on the system. The probability amplitude that a system is in state $|f\rangle$, after the collision is given by:

$$\langle f|\psi^I(\infty)\rangle \quad = \quad \langle f|S|i\rangle = S_{fi}, \tag{4.34}$$

which describes the probability amplitude for the state $|i\rangle$ to evolve into $|f\rangle$. Obviously, the normalization of $\psi^I(\infty)$, that is, $\langle\psi^I(\infty)|\psi^I(\infty)\rangle = 1$, ensures the conservation of probability, that is,

$$\sum_f |\langle f|S|i\rangle|^2 \quad = \quad \sum_f |S_{fi}|^2 = 1. \tag{4.35}$$

In order to calculate the S-matrix, we need to solve Eq. (4.32) to obtain $\psi^I(\infty)$. A formal solution of Eq. (4.32) is written as:

$$\psi^I(t) \quad = \quad -i\int H_I(t)\psi^I(t)dt, \tag{4.36}$$

$$\text{that is, } \psi^I(t) \quad = \quad |i\rangle + (-i)\int_{-\infty}^{t} dt_1 H_I(t_1)\psi^I(t_1), \tag{4.37}$$

using the initial condition of $\psi^I(-\infty) = |i\rangle$.

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Equation (4.37) can be solved iteratively, that is,

$$\psi^I(t) = |i\rangle + (-i)\int_{-\infty}^{t} dt_1 H_I(t_1)|i\rangle + (-i)^2 \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 H_I(t_1)H_I(t_2)\psi^I|t_2\rangle. \quad (4.38)$$

This would lead to an infinite series involving one additional multiplicative factor $H_I(t)$ in each order. Such a series will converge only if the interaction Hamiltonian H_I is weak and is specified by some parameter (say λ) which is less than 1. Using the expansion in Eq. (4.38), the S-matrix is defined as a perturbative series given by:

$$S = \sum_{n=0}^{\infty}(-i)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 ..... \int_{-\infty}^{t_{n-1}} dt_n \; H_I(t_1)H_I(t_2)....H_I(t_n) \quad (4.39)$$

$$= \sum_{n=0}^{\infty} S^{(n)} \quad (4.40)$$

and the matrix element is given by

$$S_{fi} = \langle f|S|i\rangle. \quad (4.41)$$

This is known as the S-matrix expansion in the relativistic perturbation theory.

4.4 Dyson Expansion and Wick's Theorem

4.4.1 Dyson expansion

In order to calculate the expansion series in Eq. (4.40), Dyson [215] proposed a method which simplifies the calculation. To illustrate the Dyson expansion, let us consider the second term $S^{(2)}$ in the expansion integrated between t_0 and t, that is,

$$S^{(2)} = (-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \; H_I(t_1)H_I(t_2). \quad (4.42)$$

The limits of t_2 integration are from t_0 to t_1, that is, $t_2 \leq t_1$ and then t_1 is integrated between t_0 to t. Let us consider the (t_1, t_2) plane for integration as shown in Figure 4.1. The straight line shows the $t_1 = t_2$ line in the (t_1, t_2) plane. The integrand lying in the upper half shaded region ADB has $t_2 > t_1$ while the lower half region ABC, shaded by vertical lines, has $t_2 < t_1$. If $H_I(t_1)$ and $H_I(t_2)$ were commuting operators, then the integration over t_1 and t_2 in Eq. (4.42), can be interchanged giving equal results; the total result could be written as half of the integral over the region ADBC, that is,

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2) = \frac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \; H_I(t_1)\; H_I(t_2). \quad (4.43)$$

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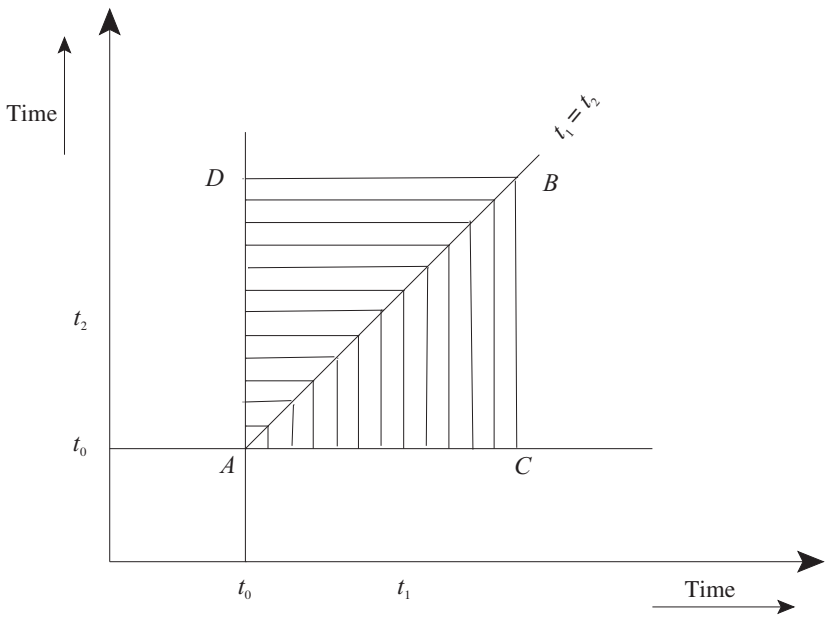

Figure 4.1 (t_1, t_2) plane for integration.

This can be done even in the case of non-commuting operators $H_I(t_1)H_I(t_2)$ by defining the time–ordered products of $H_I(t_1)H_I(t_2)$ such that: when $t_2 > t_1$ the integration is done along ADB but when $t_2 < t_1$ the integration is done along ACB i.e.

$$(-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \; H_I(t_1)H_I(t_2) = \frac{(-i)^2}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \; T[H_I(t_1)H_I(t_2)]. \tag{4.44}$$

Using this relation, Eq. (4.39) can be generalized to

$$S^{(n)} = \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 ... \int_{-\infty}^{\infty} dt_n \; T\left[H_I(t_1)H_I(t_2)...H_I(t_n)\right]. \tag{4.45}$$

Writing Eq. (4.45) in terms of the Hamiltonian density $\mathcal{H}(x)$, we obtain the following results for $S^{(n)}$:

$$S^{(n)} = \frac{(-i)^n}{n!} \int ... \int d^4x_1 \, d^4x_2 ... d^4x_n \; T\left[\mathcal{H}_I(x_1)\mathcal{H}_I(x_2)...\mathcal{H}_I(x_n)\right]. \tag{4.46}$$

The calculation of the matrix element $S_{fi} = \langle f|S|i\rangle = \sum_n \langle f|S_n|i\rangle$ can be performed in each order of perturbation expansion using the initial states $|i\rangle$ and $|f\rangle$, which are the eigenstates of the unperturbed free field Hamiltonian. These are assumed to be the states in Hilbert space which describe physical particles even though they are treated to be asymptotic states in the S-matrix theory considered here. This assumption needs further justification which can be found in standard textbooks on quantum field theory, for example, the text by Bjorken and Drell [211] and that of Mandl and Shaw [210].

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4.4.2 Wick's theorem

In any given order of the perturbative expansion S_n, the terms contain the time ordered-product of the various fields (see Chapter 3) defining $H_I(x)$ which itself has normal ordering. The time-ordering requires the fields (including creation and annihilation operators) at earlier times to be kept at the right, while normal-ordering requires that all annihilation operators are kept at the right of all creation operators. Hence, according to time-ordering, the creation operators at earlier times would be to the right of the annihilation operators at later time contrary to the normal-ordering, which requires all the creation operators to be at the left. We need a relation between the time-ordering and normal-ordering so that all the annihilation operators can be moved to the right in order to annihilate the vacuum and facilitate the calculations of S-matrix. This is provided by Wick's theorem [216].

We have seen in Chapter 3, that if $\phi_1(x)$ and $\phi_2(x)$ are the two fields, then:

$$\phi_1(x_1)\phi_2(x_2) = N[\phi_1(x_1)\phi_2(x_2)] + [\phi_1^+(x_1), \phi_2^-(x_2)] \quad \text{(for boson fields)} \tag{4.47}$$

$$= N[\phi_1(x_1)\phi_2(x_2)] + \{\phi_1^+(x_1), \phi_2^-(x_2)\} \quad \text{(for fermion fields).} \tag{4.48}$$

Taking the vacuum expectation value on both the sides, we get, for boson:

$$\langle 0|\phi_1(x_1)\phi_2(x_2)|0\rangle = \langle 0|N(\phi_1(x_1)\phi_2(x_2))|0\rangle + \langle 0|[\phi_1^+(x_1), \phi_2^-(x_2)]|0\rangle \tag{4.49}$$

and a similar equation for the fermions involving the vacuum expectation value of anticommutators $\langle 0|\{\phi_1(x_1), \phi_2(x_2)\}|0\rangle$ in place of commutators.

Since the vacuum expectation value of a normal product vanishes by definition, we get:

$$\langle 0|\phi_1(x_1)\phi_2(x_2)|0\rangle = \langle 0|[\phi_1^+(x_1), \phi_2^-(x_2)]|0\rangle \tag{4.50}$$

$$= [\phi_1^+(x_1), \phi_2^-(x_2)]. \tag{4.51}$$

Equation (4.51) follows because the commutator (or anticommutator in the case of fermions) is a c-number not an operator. Therefore, using Eqs. (4.49) and (4.51), we can write Eqs. (4.47) and (4.48) as:

$$\phi_1(x_1)\phi_2(x_2) = N(\phi_1(x_1)\phi_2(x_2)) + \langle 0|\phi_1(x_1)\phi_2(x_2)|0\rangle \tag{4.52}$$

with

$$N(\phi_1(x_1)\phi_2(x_2)) = +N(\phi_2(x_2)\phi_1(x_1)) \quad \text{(for bosons)} \tag{4.53}$$

$$= -N(\phi_2(x_2)\phi_1(x_1)) \quad \text{(for fermions),} \tag{4.54}$$

which requires all the creation operators to be at the left. Taking the time-ordering of Eq. (4.52), we get:

$$T(\phi_1(x_1)\phi_2(x_2)) = \theta(x_1^0 - x_2^0)N(\phi_1(x_1)\phi_2(x_2)) + \theta(x_2^0 - x_1^0)N(\phi_2(x_2)\phi_1(x_1)) + \langle 0|T(\phi_1(x_1)\phi_2(x_2))|0\rangle. \tag{4.55}$$

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For boson fields $\phi_1(x_1)\phi_2(x_2) = \phi_2(x_2)\phi_1(x_1)$ and $\theta(x_1^0 - x_2^0) + \theta(x_2^0 - x_1^0) = 1$, the equation can be written as:

$$T(\phi_1(x_1)\phi_2(x_2)) = N(\phi_1(x_1)\phi_2(x_2)) + \langle 0|T(\phi_1(x_1)\phi_2(x_2))|0\rangle. \tag{4.56}$$

This above equation is also valid in the case of fermions because a -1 appears due to the interchange of two fields $\phi_1(x_1)\phi_2(x_2)$ in anticommutators and another factor of -1 appears in the definition of the time-ordered product for the fermion fields.

A special notation is used for the vacuum expectation value of the time-ordered product of commutators as:

$$\underbrace{\phi_1(x_1)\phi_2}(x_2) = \langle 0|T(\phi_1(x_1)\phi_2(x_2))|0\rangle \tag{4.57}$$

called 'contraction' of $\phi_1(x_1)$ and $\phi_2(x_2)$. Using this notation, the time-ordered product of the two field operators is written as:

$$T(\phi_1(x_1)\phi_2(x_2)) = N(\phi_1(x_1)\phi_2(x_2)) + \underbrace{\phi_1(x_1)\phi_2}(x_2). \tag{4.58}$$

A nonzero value of the 'contraction' of the product of the two fields taken at $x_1^0 \neq x_2^0$ will appear only when a product state created by the operator field $\phi_2(x_2)$ is annihilated by the operator fields of $\phi_1(x_1)$; otherwise, it will vanish. The non-vanishing 'contraction' in Eq. (4.57) are Feynman propagators which have been discussed in Chapter 3. For example, in the case of scalar, spinor, and vector fields, they are written as:

$$\underbrace{\phi(x_1)\phi}(x_2) = i\Delta_F(x_1 - x_2). \tag{4.59}$$

$$\underbrace{\phi(x_1)\phi^\dagger}(x_2) = \phi^\dagger(x_2)\phi(x_1) = i\Delta_F(x_1 - x_2). \tag{4.60}$$

$$\underbrace{\psi_\alpha(x_1)\bar{\psi}_\beta}(x_2) = -\bar{\psi}_\beta(x_2)\psi_\alpha(x_1) = iS_{F\alpha\beta}(x_1 - x_2). \tag{4.61}$$

$$\underbrace{A^\mu(x_1)A^\nu}(x_2) = iD_F^{\mu\nu}(x_1 - x_2). \tag{4.62}$$

The definition of the time-ordered product is generalized to the product of n fields which is proved by following the method of induction. The generalization to n fields is written as:

$$T((\phi(x_1)...\phi(x_n))) = N(\phi(x_1)\phi(x_2)...\phi(x_n)) + \text{all possible contractions}. \tag{4.63}$$

Since the contraction is done by considering the product of two fields in pairs, all possible contractions of n fields would consist of various terms, where the contraction is done with 1 pair of two fields, 2 pairs of two fields each, 3 pairs of 2 fields each, and so on, until all the fields are exhausted. In the case of n being even, all the fields will be contracted in pairs, while

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for the odd n, one field will be left out. For example, in the case of n=4, using $\phi(x_\alpha) = \phi_\alpha$,

$$\begin{aligned} T(\phi_1\phi_2\phi_3\phi_4) &= N(\phi_1\phi_2\phi_3\phi_4 + \underbrace{\phi_1\phi_2}\,\phi_3\phi_4 + \underbrace{\phi_1\phi_2\phi_3}\,\phi_4 \\ &\quad + \underbrace{\phi_1\phi_2\phi_3\phi_4} + \phi_1\,\underbrace{\phi_2\phi_3}\,\phi_4 + \phi_1\,\underbrace{\phi_2\phi_3\phi_4} \\ &\quad + \phi_1\phi_2\,\underbrace{\phi_3\phi_4} + \underbrace{\phi_1\phi_2}\,\underbrace{\phi_3\phi_4} + \overbrace{\phi_1\,\underbrace{\phi_2\phi_3}\,\phi_4} + \overbrace{\phi_1\,\underbrace{\phi_2\,\phi_3}\,\phi_4}). \end{aligned} \tag{4.64}$$

In the case of n fields, we, therefore write:

$$\begin{aligned} T(\phi_1\phi_2...\phi_n) &= N(\phi_1\phi_2...\phi_n) + N(\underbrace{\phi_1\phi_2}\,\phi_3...\phi_n) \\ &\quad + N(\underbrace{\phi_1\phi_2}\,\underbrace{\phi_3\phi_4}\,...\phi_n)... + (\phi_1\phi_2...\,\underbrace{\phi_{n-3}\phi_{n-2}}\,\underbrace{\phi_{n-1}\phi_n}) \\ &\quad + ... + N(\underbrace{\phi_1\phi_2}\,\underbrace{\phi_3\phi_4}\,...\,\underbrace{\phi_{n-1}\phi_n}) \quad \text{for even } n. \end{aligned} \tag{4.65}$$

In case we consider the fermion field $\psi(x)$, in place of $\phi(x)$, then the normal product of n fields is given as:

$$N(\psi_1, \psi_2, ..., \psi_n) = (-1)^P(\psi'_1, \psi'_2, ..., \psi'_n), \tag{4.66}$$

where $\psi'_\alpha(\alpha = 1...n)$ are the rearrangement of the fields $\psi_\alpha(\alpha = 1...n)$ and P is the number of interchanges of neighboring fermion field operators as they anticommute. There is no additional factor of -1 when boson fields are interchanged.

Equation (4.64) contains 0, 1, and 2 pairs of contractions to be continued until the last term, where all the fields (except one) are contracted in case n is even (odd).

When the interaction Hamiltonian density $H_I(x)$ contains a mixed time-ordered product like $\mathrm{T}(H_I(x_1)H_I(x_2)...)$, then

$$\begin{aligned} T[H_I(x_1)H_I(x_2)...H_I(x_n)] = T[&N(\phi_1(x_1)\phi_2(x_1)...)N(\phi_1(x_2)\phi_2(x_2)...)... \\ &N(\phi_1(x_n)\phi_2(x_n))]. \end{aligned} \tag{4.67}$$

It should be noted that the contraction terms from the right-hand side will contribute only when the two fields are evaluated at different times, that is, $t_1 \neq t_2$. Therefore, contraction of field products like $\phi_1(x_1)\phi_2(x_1)$ will not contribute because corresponding to the same group, they would have equal time commutators which would vanish [217].

This is known as Wick's theorem [216]. We see that the contracted terms give the Feynman propagators which describe the propagation of virtual particles. The non-contracted fields in a normal product contain a set of creation and annihilation operators. Therefore, when the S-matrix element like $\langle f|S|i\rangle$ is calculated, the action of the set of annihilation and creation operators in S acting on the initial state $|i\rangle$ must produce the particles present in the final state $|f\rangle$ for a non-vanishing matrix element. Such a matrix element may include propagation of virtual particles if the normal product contains contracted terms like the terms in Eqs. (4.64) and (4.65). Thus, using Wick's theorem for the expansion of the S-matrix, the transition matrix elements of $\langle f|S|i\rangle$ are calculated in any order of relativistic perturbation theory.

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4.5 S-matrix and Feynman Diagrams

The calculation of individual terms $S^{(n)}$ given in Eq. (4.46), in the expansion of S-matrix and its matrix element $S_{fi}^{(n)} = \langle f|S^{(n)}|i\rangle$ between the initial state $|i\rangle$ and the final state $|f\rangle$ is facilitated with the help of Feynman diagrams. This will be described in this section. For the purpose of illustration, let us consider the interaction Lagrangian $\mathcal{L}_{\mathcal{I}}(x)$ for the interaction of the electron with the electromagnetic field given in Eq. (4.4), that is,

$$\mathcal{L}_I^{em} = e\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu = e\bar{\psi}(x)\not{A}\psi(x) \tag{4.68}$$

such that

$$H_I^{em}(x) = -e\bar{\psi}(x)\not{A}(x)\psi(x), \tag{4.69}$$

where $\psi(x)$ and $A_\mu(x)$ are the fields for the electron and photon, respectively. In the case of quantized fields, the normal-ordering of fields in $H_I(x)$ is to be understood such that:

$$\begin{aligned} S^{(1)} &= +ie\int d^4x\, T(\bar{\psi}(x)\not{A}\psi(x)) \\ &= +ie\int d^4x\, N[(\bar{\psi}^+(x)+\bar{\psi}^-(x))(\not{A}^+(x)+\not{A}^-(x))(\psi^+(x)+\psi^-(x))]. \end{aligned} \tag{4.70}$$

There are eight terms in $S^{(1)}$ and they contribute to eight basic processes taking place at a point x in which four of them are associated with the absorption of photons (due to the $\phi^+(x)$ term) and the other four with the creation of photons. These four diagrams are associated with $\bar{\psi}^+(x)\psi^+(x), \bar{\psi}^+(x)\psi^-(x), \bar{\psi}^-(x)\psi^+(x)$, and $\bar{\psi}^-(x)\psi^-(x)$ and correspond to the annihilation of e^-e^+ pair, emission and absorption of e^+ and e^-, and creation of e^-e^+ pair as shown in Figure 4.2. However, all these processes are not possible for real physical electrons and photons due to the energy momentum conservation, so $S_{fi}^{(1)} = \langle f|S^{(1)}|i\rangle = 0$ for all $|i\rangle$

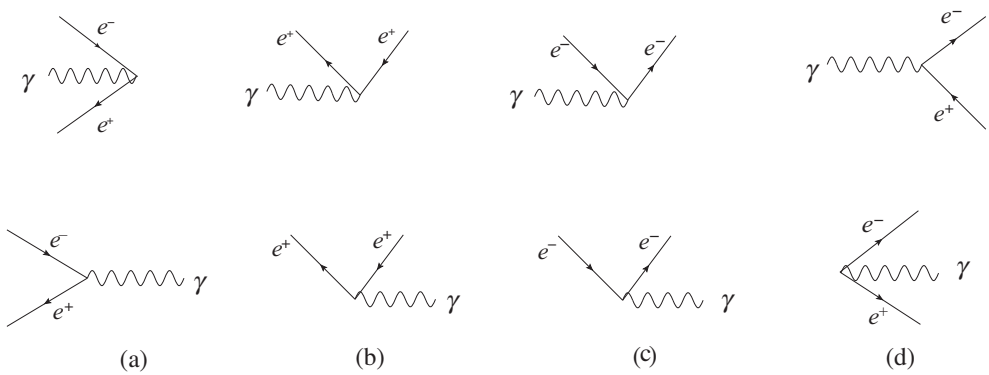

Figure 4.2 Feynman diagrams for eight basic processes in QED. (a) e^+e^- annihilation, (b) e^+ scattering, (c) e^- scattering, and (d) e^+e^- creation, accompanied by the absorption (upper row) and emission (lower row) of a photon.

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and $|f\rangle$, as shown in the next section. That is, all the processes possible from the mathematical structure of the S-matrix may not be physical. This statement is valid in each order of the S–matrix expansion, that is, $S^{(n)}$. These diagrams depiciting the physical as well as unphysical processes appearing in the S-matrix expansion are called Feynman diagrams. In order to obtain the real processes from $H_I(x)$ of Eq. (4.46), we go to the next order in perturbation theory in which we can also demonstrate the physical interpretation of the contracted terms in context of the real processes. Let us now consider the second order term $S^{(2)}$:

$$S^{(2)} = \sum_{i=0}^{m} S_i^{(2)}, \tag{4.71}$$

where m is the maximum number of contractions for $x_1^0 \neq x_2^0$, given by:

$$S_0^{(2)} = -\frac{e^2}{2!}\int d^4x_1 \int d^4x_2\, N\Big((\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2}\Big), \tag{4.72}$$

$$\begin{aligned} S_1^{(2)} &= -\frac{e^2}{2!}\int d^4x_1 \int d^4x_2\, N\Big((\bar{\psi}\not{A}\,\underbrace{\psi)_{x_1}(\bar{\psi}}\not{A}\psi)_{x_2} + (\underbrace{\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi}\big)_{x_2} \\ &\quad + (\bar{\psi}\,\underbrace{\not{A}\psi)_{x_1}(\bar{\psi}\not{A}}\,\psi)_{x_2}\Big), \end{aligned} \tag{4.73}$$

$$\begin{aligned} S_2^{(2)} &= -\frac{e^2}{2!}\int d^4x_1 \int d^4x_2\, N\Big((\underbrace{\bar{\psi}\,\not{A}\psi)_{x_1}(\bar{\psi}\not{A}}\,\psi)_{x_2} + (\underbrace{\bar{\psi}\,\not{A}\,\psi)_{x_1}(\bar{\psi}}\,\not{A}\,\psi)_{x_2} \\ &\quad + (\underbrace{\bar{\psi}\not{A}\,\psi)_{x_1}(\bar{\psi}}\,\not{A}\psi)_{x_2}\Big), \end{aligned} \tag{4.74}$$

$$S_3^{(2)} = -\frac{e^2}{2!}\int d^4x_1 \int d^4x_2\, N\Big(\underbrace{\bar{\psi}\,\not{A}\,\psi)_{x_1}(\bar{\psi}\,\not{A}\,\psi)_{x_2}}\Big). \tag{4.75}$$

The term $S_0^{(2)}$ corresponds to the two sets of disconnected diagrams each, as shown in Figure 4.2 at the point x_1 and x_2, which correspond to an unphysical process. The first two terms in $S_1^{(2)}$ include the normal product of four fields $(\bar{\psi}\not{A})_{x_1}(\not{A}\psi)_{x_2}$ and $(\not{A}\psi)_{x_1}(\bar{\psi}\not{A})_{x_2}$ with the contraction of $\psi\,\underbrace{(x_1)\bar{\psi}}(x_2)$ and $\bar{\psi}\,\underbrace{(x_1)\psi}(x_2)$ fields, respectively, which correspond to the virtual propagation of electrons between x_1 and x_2 as discussed in Chapter 3. Using the anticommutation properties of ψ and $\bar{\psi}$ fields, it can be shown that:

$$N(\underbrace{(\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi}\big)_{x_2}) = N((\bar{\psi}\not{A}\,\underbrace{\psi)_{x_2}(\bar{\psi}}\,\not{A}\psi)_{x_1}), \tag{4.76}$$

leading to:

$$S_1^{(2)} = -e^2\int d^4x_1 d^4x_2\, N\left[(\bar{\psi}\not{A}\,\underbrace{\psi)_{x_1}(\bar{\psi}}\,\not{A}\psi)_{x_2} + \frac{1}{2}(\bar{\psi}\,\underbrace{\not{A}\psi)_{x_1}(\bar{\psi}\not{A}}\,\psi)_{x_2}\right] \tag{4.77}$$

$$= S_{1A}^{(2)} + S_{1B}^{(2)}, \tag{4.78}$$

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where

$$S_{1A}^{(2)} = -ie^2 \int d^4x_1 d^4x_2 \, N\Big[(\bar{\psi}\not{A})_{x_1}(\not{A}\psi)_{x_2} S_F(x_1 - x_2)\Big] \tag{4.79}$$

$$S_{1B}^{(2)} = -\frac{ie^2}{2} \int d^4x_1 d^4x_2 \, N\Big[(\bar{\psi}\psi)_{x_1}(\bar{\psi}\psi)_{x_2} D_F(x_1 - x_2)\Big], \tag{4.80}$$

Here, $S_F(x)$ and $D_F(x)$ are the Feynman propagators for the electron and photon, respectively. The term $S_{1A}^{(2)}$ contains two uncontracted field operators for electrons and two uncontracted operators for photons which create or absorb real particles at x_1 and x_2. They are represented by the external lines in the Feynman diagram and connected through the exchange of virtual electrons described by the Feynman propagator $S_F(x_1 - x_2)$.

There are many processes described by this term but real physical processes will correspond to those processes in which there are two particles in the initial state $|i\rangle$ and two particles in the final state $|f\rangle$ satisfying the energy–momentum conservation and the combination of creation and annihilation operators such that $\langle f|S|i\rangle$ is non-vanishing. The various real processes which are described by this term are, therefore,

$$(i)\ \gamma + e^- \to \gamma + e^- \quad (ii)\ \gamma + e^+ \to \gamma + e^+$$
$$(iii)\ \gamma + \gamma \to e^+ + e^- \quad (iv)\ e^+ + e^- \to \gamma + \gamma$$

known as the Compton scattering of (i) electrons and (ii) positrons; (iii) e^+e^- pair creation and (iv) e^+e^- annihilation.

The Feynman diagrams corresponding to these processes are shown in Figure 4.3. On the other hand, the term $S_{1B}^{(2)}$ has four uncontracted fermion operators which can describe the

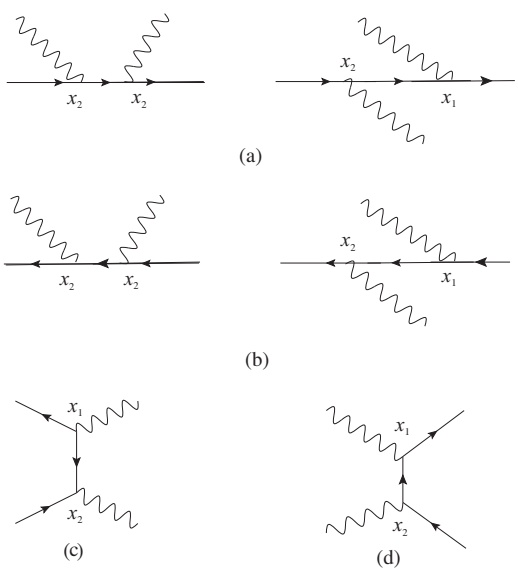

Figure 4.3 Feynman diagrams for (a) Compton scattering for e^-, (b) Compton scattering for e^+, (c) pair annihilation, and (d) pair creation.

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absorption and creation of electrons or positrons at point x_1 and x_2 connected by the photon propagator $D_F^{\mu\nu}(x)$; the processes would, therefore, consist of the absorption or creation of $e^-(e^+)$ as external lines and photons as propagators and can be described as follows.

$$(i)\ e^- + e^- \to e^- + e^-\ \ (ii)\ e^+ + e^+ \to e^+ + e^+\ \ (iii)\ e^- + e^+ \to e^- + e^+.$$

These processes are known as Møller scattering of (i) electron and (ii) positron; and (iii) Bhabha scattering of electron and positron. The Feynman diagrams corresponding to these processes are given in Figure 4.4. However, in these cases, there are additional constraints

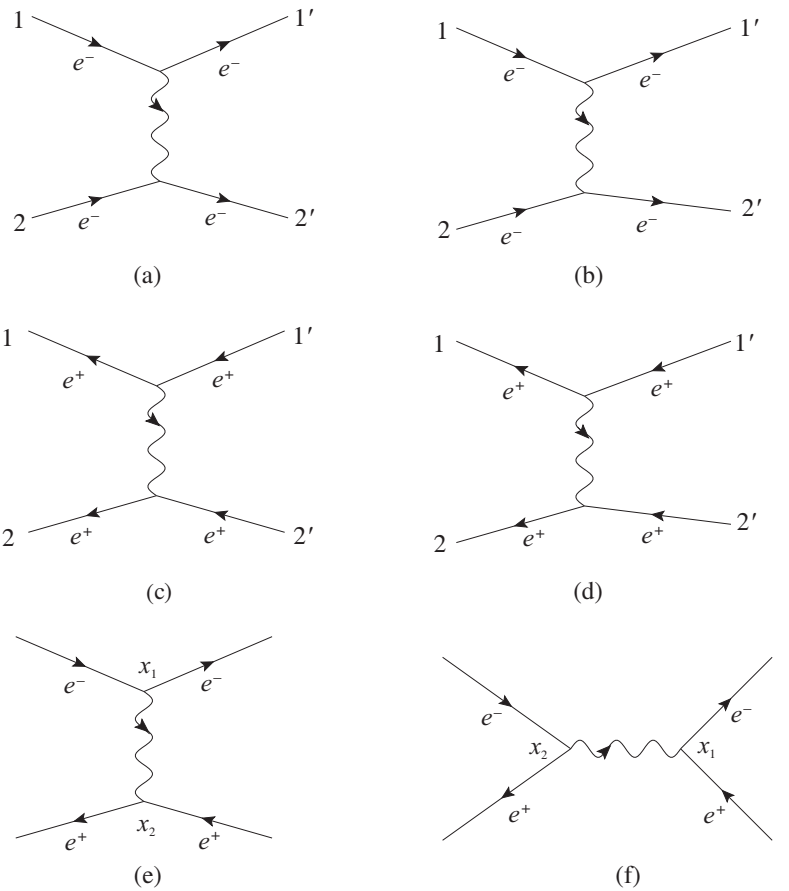

Figure 4.4 Feynman diagrams for (a, b) Møller scattering of $e^- - e^-$, (c, d) Møller scattering of $e^+ - e^+$, and (e, f) Bhabha scattering of $e^- - e^+$.

while evaluating the normal product due to two fermions being present in the initial and final states. The appropriate factors arising due to the antisymmetrization of two electrons or an electron and a positron state, or symmetrization of two photon states, in the initial and final states need to be taken into account. The details of these calculations are to be found in standard texts on quantum field theory (QFT) [210].

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The next term with two contraction $S_2^{(2)}$ can be written as:

$$S_2^{(2)} = S_{2A}^{(2)} + S_{2B}^{(2)}, \text{ where} \tag{4.81}$$

$$S_{2A}^{(2)} = -\frac{e^2}{2!}\int d^4x_1 d^4x_2 (N(\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2} + (\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2}) \tag{4.82}$$

$$S_{2B}^{(2)} = -\frac{e^2}{2!}\int d^4x_1 d^4x_2 N(\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2}). \tag{4.83}$$

The $S_{2A}^{(2)}$ term contains two uncontracted fermion fields $N(\bar{\psi}(x_1)\psi(x_2))$ and $N(\psi(x_1)\,\bar{\psi}(x_2))$ with two contractions corresponding to the electron and photon propagator $S_F(x_1 - x_2)$ and $D_F^{\mu\nu}(x_1 - x_2)$, respectively. The uncontracted fermion field will describe two physical processes according to the electrons (positrons) being presented in the initial and final states at x_1 and x_2 connected with electron and photon propagators between x_1 and x_2. The Feynman diagram corresponding to these processes are shown in Figure 4.5.

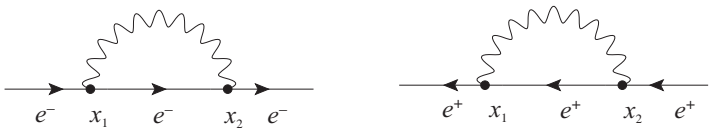

Figure 4.5 Electron (left) and positron (right) self energy.

These are called self energy diagrams for the electrons (positrons) and correspond to the modification of the properties of electrons (positron) converting a bare (mathematical) electron (positron) to the physical electron (positron). In actual calculations, the contribution of this diagram diverges but the divergent contribution is absorbed in the definition of physical properties of electrons through the concept of renormalization. It should be noted that in the higher order of perturbation theory, that is, $S^{(4)}, S^{(6)}$, etc, there will more number of such loop diagrams.

Similarly, the term $S_{2B}^{(2)}$ has two uncontracted electromagnetic fields $A^\mu(x_1), A^\mu(x_2)$ along with two fermion propagators $S_F(x_1 - x_2)$ and $S_F(x_2 - x_1)$. This describes a physical process in which there is a photon in the initial as well as in the final states connected by two fermion propagators for the electrons and the positrons corresponding to the Feynman diagram shown in Figure 4.6.

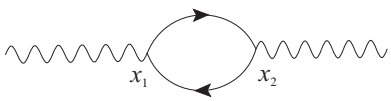

Figure 4.6 Photon self energy.

This diagram is called the photon self energy diagram in which an interacting photon can create a virtual e^-e^+ pair which recombines to give back the photon state. This can happen when a photon is passing through an external electromagnetic field or changing the distribution

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of virtual e^-e^+ pairs, like the polarization in dielectrics. These diagrams are, therefore, called vacuum polarization diagrams and give divergent contributions. Such divergences are removed through the renormalization procedure in QED [210].

Finally, the last term $S_3^{(2)}$ with three contractions does not have any free fields either in the initial or in the final state corresponding to three propagators between x_1 and x_2 as shown in Figure 4.7.

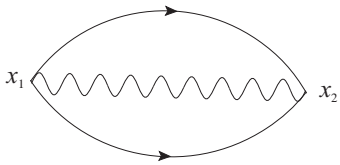

Figure 4.7 The vacuum diagram.

Such diagrams do not correspond to any physical process. Thus, the first term and the last term in $S^{(2)}$, which correspond to disconnected and fully connected Feynman diagrams, respectively, do not correspond to any physical process and can be ignored in the applications of the relativistic perturbation theory to physical processes.

4.6 Invariant Matrix Elements and Feynman Diagrams

4.6.1 Matrix elements in first order perturbation theory

We have seen in the last section that the various terms in the S-matrix expansion generate transition which may lead to physical processes involving the initial state $|i\rangle$ and the final state $|f\rangle$, subject to the conservation of energy and momentum. The uncontracted field operators appearing in the S-matrix expansion are combinations of fields such as $\psi(x)$, $\bar{\psi}(x)$, and $A^\mu(x)$, which are expressed in terms of creation and annihilation operators leading to the emission and absorption of real particles represented by the external lines in Feynman diagrams. For example, using the expression of $\psi(x)$, $\bar{\psi}(x)$, and $A^\mu(x)$, we can write

$$\bar{\psi}^{-}(x)|0\rangle = \sum_{r,\vec{p}} \sqrt{\frac{1}{2E_{\vec{p}}V}}\bar{u}_r(\vec{p})e^{ip\cdot x}|e_r^-(\vec{p})\rangle, \tag{4.84}$$

$$\psi^{-}(x)|0\rangle = \sum_{r,\vec{p}} \sqrt{\frac{1}{2E_{\vec{p}}V}}v_r(\vec{p})e^{ip\cdot x}|e_r^+(\vec{p})\rangle, \tag{4.85}$$

$$A^{\mu-}(x)|0\rangle = \sum_{r,\vec{k}} \sqrt{\frac{1}{2E_{\vec{k}}V}}\epsilon_r^\mu(\vec{k})e^{ik\cdot x}|\gamma(\vec{k})\rangle. \tag{4.86}$$

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Similarly, the action of $\psi(x)$, $\bar{\psi}(x)$, and $A^{\mu}(x)$ on the initial state $|i\rangle$, which consists of particle states of $e^{-}(\vec{p})$, $e^{+}(\vec{p})$, and $\gamma(\vec{k})$, will also lead to vacuum states, that is,

$$\psi^{+}(x)|e^{-}(\vec{p})\rangle = \sqrt{\frac{1}{2E_{\vec{p}}V}}u(\vec{p})e^{-ip\cdot x}|0\rangle, \tag{4.87}$$

$$\bar{\psi}^{+}|e^{+}(\vec{p})\rangle = \sqrt{\frac{1}{2E_{\vec{p}}V}}\bar{v}(\vec{p})e^{-ip\cdot x}|0\rangle, \tag{4.88}$$

and $$A^{\mu+}(x)|\gamma(\vec{k})\rangle = \sqrt{\frac{1}{2E_{\vec{k}}V}}\epsilon^{\mu}(\vec{k})e^{-ik\cdot x}|0\rangle \tag{4.89}$$

due to the action of the respective creation and annihilation operators.

Let us apply the first order relativistic perturbation theory to calculate a process such as $e^{-}(\vec{p}) \rightarrow e^{-}(\vec{p}\,') + \gamma(\vec{k}')$, which is one of the processes given in Figure 4.2 and described by the term $S^{(1)}$, given by:

$$S^{(1)} = ie\int d^4x N\Big(\bar{\psi}(x)\not{A}(x)\psi(x)\Big). \tag{4.90}$$

In this case, the initial state $|i\rangle$ and final state $|f\rangle$ are written as:

$$|i\rangle = |e^{-}(\vec{p})\rangle = a_e^{\dagger}(\vec{p})|0\rangle, \tag{4.91}$$

$$|f\rangle = |e^{-}(\vec{p}\,')\gamma(\vec{k}\,')\rangle = a_e^{\dagger}(\vec{p}\,')a_{\gamma}^{\dagger}(\vec{k}\,')|0\rangle, \tag{4.92}$$

where we have used a subscript $e(\gamma)$ on creation operators $a^{\dagger}$ to show that they create an electron(e) and a photon (γ). The S-matrix element $\langle f|S^{(1)}|i\rangle$ is calculated using Eq. (4.90) to give:

$$\langle f|S^{(1)}|i\rangle = ie\langle e^{-}(\vec{p}\,')\gamma(\vec{k}\,')|\int d^4x N\Big(\bar{\psi}(x)\not{A}(x)\psi(x)\Big)|e^{-}(\vec{p})\rangle. \tag{4.93}$$

Only the $\bar{\psi}^{-}\not{A}^{-}\psi^{+}(x)$ term in $N\Big(\bar{\psi}(x)\not{A}(x)\psi(x)\Big)$ will give a nonzero contribution to $\langle f|S^{(1)}|i\rangle$ for the process $e^{-} \rightarrow e^{-} + \gamma$.

$$\begin{aligned}\langle f|S^{(1)}|i\rangle &= ie\int d^4x\sqrt{\frac{1}{2E_{\vec{p}}V}}\sqrt{\frac{1}{2E_{\vec{p}\,'}V}}\sqrt{\frac{1}{2VE_{\vec{k}'}}}\bar{u}(\vec{p}\,')\not{\epsilon}u(\vec{p})e^{i(p'+k'-p)\cdot x}\\ &= (2\pi)^4\delta^4(p'+k'-p)\sqrt{\frac{1}{2E_{\vec{p}}V}}\sqrt{\frac{1}{2E_{\vec{p}\,'}V}}\sqrt{\frac{1}{2VE_{\vec{k}'}}}\mathcal{M}_{fi},\end{aligned} \tag{4.94}$$

where $\mathcal{M}_{fi} = ie\bar{u}(\vec{p}\,')\epsilon_{\mu}\gamma^{\mu}(\vec{k}')u(\vec{p})$. Equation (4.94) is the final result for $\langle f|S^{(1)}|i\rangle$. The matrix element $\mathcal{M}_{fi}$ is called the invariant matrix element or Feynman matrix element for the process $e^{-}(p) \rightarrow e^{-}(p') + \gamma(k')$. The factor $(2\pi)^4\delta^4(p' + k' - p)$ represents the conservation of momentum and energy. The Feynman diagram in momentum space

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corresponding to this process is given in Figure 4.8, where the lines labeled $e^-(p)$, $e^-(p')$ are the external lines describing the real electron in the initial and final states and the line $\gamma(k')$ is the external line describing the real photon. The vertex labeled $ie\gamma^\mu$ shows the strength and structure of the vertex corresponding to this transition. However, in this case, the energy momentum conservation condition, that is, $p' + k' = p$ is not compatible with the real particles satisfying $p^2 = p'^2 = m^2$, $k'^2 = 0$ and this process does not take place. Similarly, all the first order processes shown in Figures 4.2(a)–4.2(d) are not physical processes.

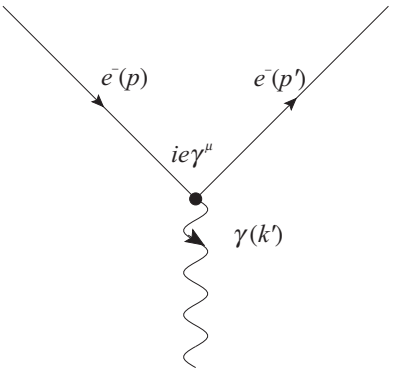

Figure 4.8 The process $e^- \to e^- + \gamma$.

4.6.2 Matrix elements in second order perturbation theory

Let us consider the term $S_1^{(2)}$ (Section 4.5), the second order of perturbation theory, to illustrate the use of Feynman propagators in momentum space for calculating the invariant matrix element $\mathcal{M}$. For definiteness, we take the physical process of Compton scattering, that is, $\gamma(k) + e^-(p) \to \gamma(k') + e^-(p')$. In this case,

$$|i\rangle = a_e^\dagger(\vec{p})a_\gamma^\dagger(\vec{k})|0\rangle\ ; \quad |f\rangle = a_e^\dagger(\vec{p}\,')a_\gamma^\dagger(\vec{k}')|0\rangle \tag{4.95}$$

and the relevant diagrams in $S_1^{(2)}$ will come from the $S_{1A}^{(2)}$ term from Eq. (4.79), where the initial photon is absorbed at x_1 and emitted at x_2 or absorbed at x_2 and emitted at x_1, corresponding to the two diagrams shown in Figure 4.3(a). These diagrams are reproduced in Figure 4.9 with appropriate labeling of the momentum of electrons and photons, to illustrate the difference between the two diagrams. The non-vanishing matrix element of $\langle f|S_{1A}^{(2)}|i\rangle$, corresponding to Figure 4.9, would be written as:

$$\begin{aligned}\langle f|S_{1A,a}^{(2)}|i\rangle \quad &= \quad -e^2 \int d^4x_1\, d^4x_2 \frac{1}{\sqrt{2VE_{\vec{p}}}} \frac{1}{\sqrt{2VE_{\vec{p}\,'}}} \frac{1}{\sqrt{2VE_{\vec{k}}}} \frac{1}{\sqrt{2VE_{\vec{k}'}}} \\ &\qquad \bar{u}(p')e^{ip'\cdot x_1} \not{\epsilon}(\vec{k}')e^{ik'\cdot x_1} \\ &\times \quad \frac{1}{(2\pi)^4} \int d^4q\; iS_F(q)e^{-iq\cdot(x_1-x_2)} \not{\epsilon}(\vec{k})e^{-ik\cdot x_2} u(\vec{p})e^{-ip\cdot x_2}. \end{aligned} \tag{4.96}$$

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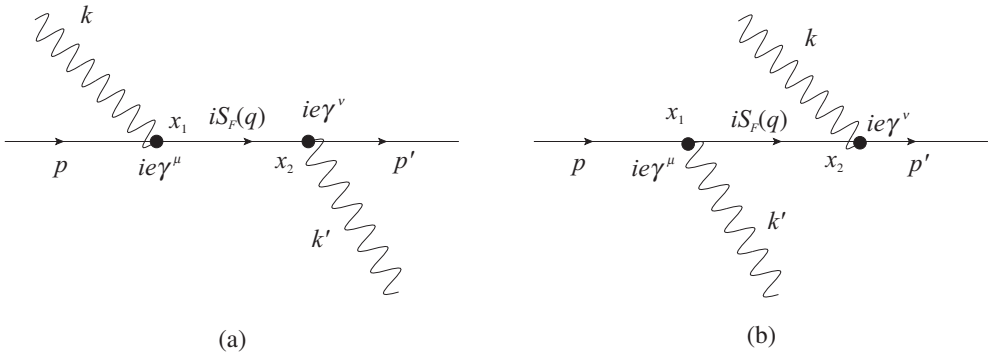

Figure 4.9 Compton scattering of electrons via (a) s-channel and (b) u-channel Feynman diagrams.

Performing the d^4x_1 and d^4x_2 integration,

$$\begin{aligned} = \quad & -ie^2\frac{(2\pi)^4(2\pi)^4}{(2\pi)^4}\int d^4q\delta^4(p'+k'-q)\delta^4(q-p-k)\bar{u}(\vec{p}\,')\not{\epsilon}(\vec{k}')S_F(q)\not{\epsilon}(\vec{k})u(\vec{p}) \\ & \times\frac{1}{\sqrt{2VE_{\vec{p}}}}\frac{1}{\sqrt{2VE_{\vec{p}\,'}}}\frac{1}{\sqrt{2VE_{\vec{k}}}}\frac{1}{\sqrt{2VE_{\vec{k}'}}}. \end{aligned} \tag{4.97}$$

Performing the d^4q integration, we get

$$\begin{aligned} \langle f|S^{(2)}_{1A,a}|i\rangle \quad &= \quad -ie^2(2\pi)^4\delta^4(p'+k'-p-k)\bar{u}(\vec{p}\,')\not{\epsilon}(\vec{k}')S_F(q=p+k)\not{\epsilon}(\vec{k})u(\vec{p}) \\ &= \quad (2\pi)^4\delta^4(p'+k'-p-k)\mathcal{M}^a_{fi} \\ &\quad \times\frac{1}{\sqrt{2VE_{\vec{p}}}}\frac{1}{\sqrt{2VE_{\vec{p}\,'}}}\frac{1}{\sqrt{2VE_{\vec{k}}}}\frac{1}{\sqrt{2VE_{\vec{k}'}}}, \end{aligned} \tag{4.98}$$

where

$$\mathcal{M}^a_{fi} = -ie^2\bar{u}(\vec{p}\,')\epsilon_\nu(\vec{k}')\gamma^\nu S_F(q=p+k)\epsilon_\mu(\vec{k})\gamma^\mu u(\vec{p}). \tag{4.99}$$

Similarly, for the second diagram, as shown in Figure 4.9, we have:

$$\begin{aligned} \langle f|S^{(2)}_{1A,b}|i\rangle &= (2\pi)^4\delta^4(p'+k'-p-k)\mathcal{M}^b_{fi} \\ &\times\frac{1}{\sqrt{2VE_{\vec{p}}}}\frac{1}{\sqrt{2VE_{\vec{p}\,'}}}\frac{1}{\sqrt{2VE_{\vec{k}}}}\frac{1}{\sqrt{2VE_{\vec{k}'}}}, \end{aligned} \tag{4.100}$$

where

$$\mathcal{M}^b_{fi} = -ie^2\bar{u}(\vec{p}\,')\epsilon_\nu(\vec{k})\gamma^\nu S_F(q=p-k')\epsilon_\mu\gamma^\mu(\vec{k}')u(\vec{p}) \tag{4.101}$$

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and

$$\mathcal{M}_{fi} = \mathcal{M}^a_{fi} + \mathcal{M}^b_{fi}.$$

Thus, we see how the propagator with a factor $iS_F(p) = \frac{i}{\not{p}-m+i\epsilon}$ in the momentum space, enters in the calculation of the invariant matrix element. The factors corresponding to the vertices and propagators are also shown in Figure 4.9.

4.6.3 Matrix elements for the closed loops

Some new feature appears in addition to the external lines and propagators when the invariant matrix element corresponding to the closed loops, which correspond to the second order terms in perturbation theory like $S_2^{(2)}$ in Eq. (4.74) are calculated. Consider a typical Feynman diagram corresponding to the self energy of the electron as shown in Figure 4.10, where the external and internal lines are properly labeled with corresponding momenta, in which

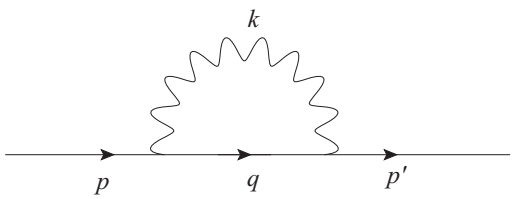

Figure 4.10 Electron self energy as an example of closed loop diagram.

an electron of momentum p emits a photon of momentum k, which is again absorbed by the electron such that $p = k + q = p'$. Thus, for a given momentum of photon k, the virtual electron has momentum $q = p - k$. Since, the photon momentum k can take any value, which fixes the electron momentum in virtual state to be $q = p - k$ and $k^2 \neq 0$, the contribution of all values of k and q should be included. This means that we should integrate over all values of k and q subject to the condition that $k + q = p$, that is, effectively integrating over only one of them, either k or q. This becomes evident if we perform the momentum integration in the S-matrix element corresponding to the term $S_{2A}^{(2)}$ in Eq. (4.82) with $|i\rangle = a_e^\dagger(\vec{p})|0\rangle$ and $|f\rangle = a_e^\dagger(\vec{p}\,')|0\rangle$. The non-vanishing term corresponding to $S_{2A}^{(2)}$ after integrating over d^4x_1 and d^4x_2 is given by:

$$\begin{aligned}\langle f|S_{2A}^{(2)}|i\rangle &= -e^2 \frac{1}{\sqrt{2VE_{\vec{p}}}} \frac{1}{\sqrt{2VE_{\vec{p}\,'}}} \int d^4q\, d^4k\, \delta^4(p' - k - q) \times \\ &\quad (2\pi)^4 \delta^4(k + q - p) i D_F^{\mu\nu}(k) \bar{u}(\vec{p}\,') \gamma_\mu i S_F(q) \gamma_\nu u(\vec{p}), \end{aligned} \tag{4.102}$$

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where $D_F^{\mu\nu}(k)$ is the photon propagator and $S_F(q)$ is the electron propagator. Performing the q integration, we obtain:

$$\langle f|S_{2A}^{(2)}|i\rangle = \frac{1}{\sqrt{2VE_{\vec{p}}}}\frac{1}{\sqrt{2VE_{\vec{p}'}}}(2\pi)^4\delta^4(p'-p)\mathcal{M}_{fi}, \tag{4.103}$$

$$\text{where } \mathcal{M}_{fi} = -\frac{e^2}{(2\pi)^4}\int d^4k\, i\, D_F^{\mu\nu}(k)\bar{u}(\vec{p})\gamma_\mu\, iS_F(p-k)\gamma_\nu u(\vec{p}). \tag{4.104}$$

Thus, the calculation of $\mathcal{M}_{fi}$ involves an integration over the loop momenta k, corresponding to the internal line of the photon. Such integration over the loop momenta also appears in the self energy of photon, as shown in Figure 4.6. It is this momentum integration over the loop in second order and higher orders, which gives rise to divergences, necessitating the need for the renormalization in QED.

4.6.4 Feynman rules: A summary

We use the standard formula for the S-matrix element $\langle f|S|i\rangle$ in terms of the invariant matrix element as

$$\langle f|S|i\rangle = \delta_{fi} + (2\pi)^4\delta^4(P_f - P_i)\prod_{i=\text{initial particles}}\frac{1}{\sqrt{2VE_i}}\prod_{f=\text{final particles}}\frac{1}{\sqrt{2VE_f}}\mathcal{M}_{fi}, \tag{4.105}$$

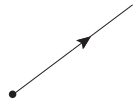
where P_f and P_i are the sum of 4-momenta of all the particles in final and initial state, respectively, and

$$\mathcal{M}_{fi} = \sum_n \mathcal{M}_{fi}^{(n)}, \tag{4.106}$$

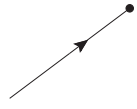
where $M_{fi}^{(n)}$ is the invariant matrix element calculated for all the connected diagrams in the nth order term in the S-matrix $S^{(n)}$. The calculation of $M_{fi}^{(n)}$ in momentum space is done using the following rules in QED:

(1) External Lines (i) Spin 0:

(a) Outgoing: 𝟙



(b) Incoming: 𝟙



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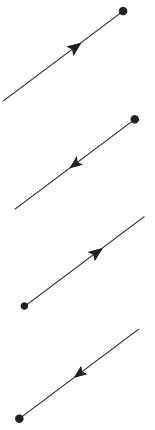
(ii) Spin $\frac{1}{2}$:

(a) Incoming particle: $u(\vec{p}, s)$

(b) Incoming antiparticle: $\bar{v}(\vec{p}, s)$

(c) Outgoing particle: $\bar{u}(\vec{p}, s)$

(d) Outgoing antiparticle: $v(\vec{p}, s)$

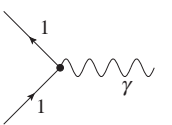


(iii) Spin 1:

(a) Incoming: $\epsilon_\mu(\vec{p}, \lambda)$

(b)Outgoing: $\epsilon^*_\mu(\vec{p}, \lambda)$

(2) Vertex factors: Electromagnetic interaction: $ie\gamma^\mu$



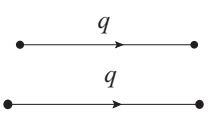
(3) Propagators:

(i) Spin 0: $\frac{i}{q^2 - m^2}$

(ii) Spin $\frac{1}{2}$: $\frac{i(\not{q} + m)}{q^2 - m^2}$

(iii) Spin 1:

(a) Massless: $\frac{-ig^{\mu\nu}}{q^2}$



Fermion propagator

Photon propagator

(4) For a closed Fermion loop, a factor of (-1) is taken.

(5) For a closed loop, integration is performed over the 4-momentum of the loop, that is, $\int \frac{d^4q}{(2\pi)^4}$. The factor $\frac{1}{(2\pi)^4}$ is inserted to satisfy the energy–momentum condition given consistently by a factor $(2\pi)^4\delta(P_f - P_i)$.

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(b) Massive: $\frac{-i\left(g^{\mu\nu}-\frac{q^\mu q^\nu}{M^2}\right)}{q^2-M^2}$

W, *Z* propagator

(6) For the vertex, a factor depending upon the strength and nature of interaction Lagrangian is added. In case of QED, the factor is $+ie\gamma^\mu$. It should be noted that an additional factor of (–1) is needed when we go from the Lagrangian to the Hamiltonian required to calculate $S^{(n)}$.

(7) Proper care of the factors of (–1) and (+1) should be taken when interchanging the fermion fields in the normal-ordered product of field operators.

(8) A definite order should be followed, that is, from right to left in writing the invariant matrix element $\mathcal{M}$, from the Feynman diagrams in momentum space.

4.7 Scattering Cross Sections and Particle Decay Rates

4.7.1 Scattering cross sections

The physical concept of cross sections in a scattering process is very useful in making quantitative studies of the elastic, inelastic, and deep inelastic scattering processes in the respective kinematical regions.

The name 'cross section' is derived from the study of collision processes in classical mechanics, in which a particle collides with a fixed spherical target with radius 'r', lying in the interaction region of total volume V and area A. The probability P that the incoming particle collides with the sphere of radius 'r' is given by:

$$P = \frac{\pi r^2}{A}, \tag{4.107}$$

such that, πr^2 is the quantity representing the cross section of area relevant for the collision process and is generally represented by σ. It can be given by:

$$\sigma = PA. \tag{4.108}$$

Thus, the cross section 'σ' is effectively an area in which the incident and target particle interact for the scattering to take place. This concept of cross section σ is generalized to any collision process in which a beam of particles scatter with a fixed target or two beams of particles from opposite directions collide with each other. Consider a beam of particles of density ρ, that is, number of particles per unit volume ($= \frac{n}{V}$) moving with velocity v. In time t, this beam, passing through an effective area of interaction A, will have $n = \rho vtA$ particles, that is,

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$$A = \frac{n}{\rho v t} = \frac{1}{\rho v t} \quad \text{(if } n \text{ is normalized to 1)} \tag{4.109}$$

such that:

$$\sigma = \frac{nP}{\rho v t} = \frac{P/t}{(v/V)} = \frac{P/t}{1/At}. \tag{4.110}$$

In Eq. (4.110), the numerator is the probability per unit time while the denominator is the flux of the incident particle. Therefore, the cross section is defined as:

$$\sigma = \frac{\text{Probability per unit time}}{\text{Flux of the incident particle}} = \frac{\text{Probability/time}}{\text{Incident flux}}. \tag{4.111}$$

Since the probability in the present case is $|\langle f|S|i\rangle|^2$, the cross section is given by:

$$\sigma = \frac{|\langle f|S|i\rangle|^2 \, V}{Tv}. \tag{4.112}$$

Using the expression for $\langle f|S|i\rangle$ given in Eq. (4.105), we obtain for $f \neq i$,

$$\sigma = \Big((2\pi)^4 \delta^4(P_f - P_i)\Big)^2 \frac{V}{Tv} \prod_{i=1,2} \prod_{f=1-n} \frac{1}{2E_f V} \frac{1}{2E_i V} |\mathcal{M}_{fi}|^2. \tag{4.113}$$

In Eq. (4.113), we obtain square of the $\delta^4(P_f - P_i)$ function, an operation which may not be normally well defined but can be evaluated in the limit of $V \to \infty$ and $T \to \infty$ using the integral representation of $\delta^4(P)$ function. For a rigorous proof, please see Ref.[218]. Here, we shall, demonstrate its evaluation in a heuristic way. For any function $f(P)$, the quantity

$$\delta^4(P) f(P) = \delta^4(P) f(P = 0), \tag{4.114}$$

assuming that the left-hand side always occurs under an integration over P. Therefore,

$$\delta^4(P)\delta^4(P) = \delta^4(P)\delta^4(P = 0). \tag{4.115}$$

Since the integral representation of $\delta^4(P)$ is given by:

$$\delta^4(P) \quad = \quad \frac{1}{(2\pi)^4} \int_{VT} e^{ip\cdot x} d^4x, \tag{4.116}$$

we can formally write,

$$(2\pi)^4 \delta^4(0) \; = \; VT \quad \text{(for } P \to 0\text{) leading to} \tag{4.117}$$

$$\sigma \; = \; (2\pi)^4 \delta^4(P_f - P_i) \frac{V^2}{v} \prod_i \left(\frac{1}{2E_i V}\right) \prod_f \left(\frac{1}{2E_f V}\right) |\mathcal{M}_{fi}|^2. \tag{4.118}$$

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The factor V^2 in Eq. (4.118) is compensated by $\frac{1}{V^2}$ occurring due to the normalization of two particle states in the initial state which are involved in the collision, making Eq. (4.118) independent of V, except for the powers of V, that is, V^{n_f}, occurring in the normalization of the particle states in the final state, where there are n_f particles which cancels with the V^{n_f} coming from the density of state of final particles as explained below.

In quantum mechanics, the momentum is not fixed, but always lies in between a range of $\vec{p}$ and $\vec{p} + d\vec{p}$. Therefore, we have to multiply Eq. (4.118), for the cross section, by the density of states $\rho(p)dp$, that is, the number of states lying between $\vec{p}$ and $\vec{p} + d\vec{p}$. This is given by the density of states in phase space

$$\rho(p)dp = \frac{Vd\vec{p}}{(2\pi)^3}$$

for each particle in the final state, in the standard case of plane–wave normalization of states. Therefore, the final expression for the cross section is given as:

$$\sigma \;=\; \frac{1}{4E_1E_2\,v}\int\prod_f \frac{d\vec{p}_f}{(2\pi)^3(2E_f)}(2\pi)^4\delta^4(P_f - P_i)|\mathcal{M}_{fi}|^2. \tag{4.119}$$

For a fixed target, v is the velocity of the incident particles; otherwise, it would be relative velocity, that is, $|v| = |\vec{v}_{rel}|$, for which

$$v_{rel} = \left|\frac{\vec{p}_1}{E_1} - \frac{\vec{p}_2}{E_2}\right|. \tag{4.120}$$

In such cases, $E_1E_2\,v = |\vec{p}_1E_2 - \vec{p}_2E_1| = \sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2}$.

Therefore, we get the covariant expression for the scattering cross section σ as:

$$\sigma = \frac{1}{4\sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2}}\int\prod_f \frac{d\vec{p}_f}{(2\pi)^3 2E_f}(2\pi)^4\delta^4(P_f - P_i)|\mathcal{M}_{fi}|^2. \tag{4.121}$$

4.7.2 Particle decay rates

In case of a particle of momentum $P^\mu(=E,\vec{P})$ decaying at rest or in flight in two or more particles, that is, $H(P) \to 1(p_1) + 2(p_2) + ...$, the S-matrix for the transition is given by:

$$\langle f|S|i\rangle = (2\pi)^4\delta^4(P - \sum_f p_f)\frac{1}{\sqrt{2E_iV}}\prod_{f=1}^n\sqrt{\frac{1}{2VE_f}}|\mathcal{M}_{fi}| \tag{4.122}$$

such that the probability for transition per unit time, that is, $\frac{|\langle f|S|i\rangle|^2}{T}$ is given by:

$$\frac{|\langle f|S|i\rangle|^2}{T} = (2\pi)^4\delta^4(P - \sum_f p_f)\cdot\frac{VT}{T}\frac{1}{2E_iV}\prod_{f=1}^n\frac{1}{2VE_f}|\mathcal{M}_{fi}|^2. \tag{4.123}$$

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Integrating over the density of final states $\rho(p)$, we obtain the decay rate Γ as:

$$\Gamma = \frac{1}{2E_i} \int \prod_f \frac{d\vec{p}_f}{(2\pi)^3 2E_f} (2\pi)^4 \delta^4 (P - \sum_f p_f) |\mathcal{M}_{fi}|^2. \tag{4.124}$$

The life time τ of the particles is inverse of the decay rate, that is, $\tau = \frac{1}{\Gamma}$.

It should be noted that in both cases of scattering cross section and particle decay rates in which more than one identical particles are produced in the final state, due consideration should be made of the (anti)symmetrization of states depending upon (fermions) bosons in calculating the density of state in phase space.

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