Chapter 2

Relativistic Particles and Neutrinos

2.1 Relativistic Notation

Neutrinos are neutral particles of spin $\frac{1}{2}$; they are completely relativistic in the massless limit. In order to describe neutrinos and their interactions, we need a relativistic theory of spin $\frac{1}{2}$ particles. The appropriate framework to describe the elementary particles in general and the neutrinos in particular, is relativistic quantum mechanics and quantum field theory. In this chapter and in the next two chapters, we present the essentials of these topics required to understand the physics of the weak interactions of neutrinos and other particles of spin 0, 1, and $\frac{1}{2}$.

We shall use natural units, in which $\hbar = c = 1$, such that all the physical quantities like mass, energy, momentum, length, time, force, etc. are expressed in terms of energy. In natural units:

$$\hbar c = 0.19732697 \text{ GeV fm} = 1 \qquad \text{and} \qquad 1 \text{ fm} = 5.06773 \text{ GeV}^{-1}.$$

The original physical quantities can be retrieved by multiplying the quantities expressed in energy units by appropriate powers of the factors $\hbar, c$, and $\hbar c$. For example, mass $m = E/c^2$, momentum $p = E/c$, length $l = \hbar c/E$, and time $t = \hbar/E$, etc.

2.1.1 Metric tensor

In the relativistic framework, space and time are treated on equal footing and the equations of motion for particles are described in terms of space–time coordinates treated as four- component vectors, in a four-dimensional space called Minkowski space, defined by x^μ, where $\mu = 0, 1, 2, 3$ and $x^\mu = (x^0 = t, x^1 = x, x^2 = y, x^3 = z)$ in any inertial frame, say S. In another inertial frame, say S', which is moving with a velocity $\beta (= \frac{v}{c})$ in the positive X direction, the space–time coordinates $x^{\mu'}(x^{0'} = t', x^{1'} = x', x^{2'} = y', x^{3'} = z')$ are related to x^μ through

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the Lorentz transformation given by:

$$x' = \frac{x - \beta t}{\sqrt{1-\beta^2}}, \quad y' = y, \quad z' = z, \quad \text{and} \quad t' = \frac{t - \beta x}{\sqrt{1-\beta^2}}, \tag{2.1}$$

such that

$$x'^2 = t'^2 - \vec{x}'^2 = t^2 - \vec{x}^2 = x^2 \text{ (constant)}, \tag{2.2}$$

remains invariant under Lorentz transformations. For this reason, the quantity $\sqrt{t^2 - |\vec{x}|^2}$ is called the length of the four-component vector x^μ in analogy with the length of an ordinary vector $\vec{x}$, that is, $|\vec{x}| = \sqrt{|\vec{x}|^2}$ which is invariant under rotation in three-dimensional Euclidean space. Therefore, the Lorentz transformations shown in Eq. (2.1) are equivalent to a rotation in a four-dimensional Minkowski space in which the quantity defined as $\sqrt{t^2 - |\vec{x}|^2}$, remains invariant, that is, it transforms as a scalar quantity under the Lorentz transformation. This is similar to a rotation in the three-dimensional Euclidean space in which the length of an ordinary vector $\vec{r}$, defined as $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$ remains invariant, that is, transforms as a scalar under rotation. The scalar product of the space–time four vector x^μ with itself is defined in terms of a quantity $g_{\mu\nu} (\mu, \nu = 0, 1, 2, 3)$ called metric tensor,

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g^{\mu\nu}, \tag{2.3}$$

such that the quantity $x^2 = x \cdot x$, defined as:

$$x \cdot x = \sum_{\mu,\nu} g_{\mu\nu} x^\mu x^\nu = t^2 - |\vec{x}|^2 = \sum g^{\mu\nu} x_\mu x_\nu,$$

transforms as a scalar and remains invariant under the Lorentz transformation. The metric tensor $g_{\mu\nu}$ allows us to define another vector x_μ as:

$$x_\mu = \sum_\nu g_{\mu\nu} x^\nu = (t, -\vec{x}), \tag{2.4}$$

such that the scalar product $x \cdot x$ can be represented as:

$$x \cdot x = \sum_\mu x^\mu x_\mu.$$

It is easy to see that $g_{\mu\nu}$ and $g^{\mu\nu}$ are symmetric tensors of second rank and satisfy the condition

$$\sum_\nu g_{\mu\nu} g^{\nu\rho} = \delta^\rho_\mu, \tag{2.5}$$

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where δ_μ^ρ is the Kronecker delta defined as:

$$\begin{aligned}\delta_\mu^\rho &= 1, \quad \mu = \rho \\ &= 0, \quad \mu \neq \rho\end{aligned} \tag{2.6}$$

because

$$x^\mu = \sum_{\nu,\rho} g^{\mu\nu} g_{\nu\rho} x^\rho.$$

This is possible only if Eq. (2.5) holds, such that:

$$\sum_{\mu,\rho} g^{\mu\nu} g_{\nu\rho} x^\rho = \sum_\rho \delta_\rho^\mu x^\rho = x^\mu. \tag{2.7}$$

2.1.2 Contravariant and covariant vectors

The four vector x^μ is defined as a contravariant vector with components $x^\mu = (x^0, x^1, x^2, x^3) = (t, \vec{x})$ represented by a superscript μ and the four vector x_μ is defined as a covariant vector $x_\mu = (x_0, -x^1, -x^2, -x^3) = (t, -\vec{x})$ represented by a subscript μ. The scalar product of a contravariant and covariant vectors can be defined with the help of the metric tensor. In fact, it is the metric tensor which defines the geometry of space and transforms as a symmetric tensor of rank 2. In general, a scalar product of any two vectors A^μ and B^μ is defined as:

$$A \cdot B = A^0 B^0 - \vec{A} \cdot \vec{B} = \sum_{\mu=0}^{3} A^\mu B_\mu = \sum_{\mu=0}^{3} A_\mu B^\mu.$$

We follow the Einstein's summation convention, where any repeated index implies a summation over that index and write:

$$A \cdot B = g_{\mu\nu} A^\mu B^\nu = g^{\mu\nu} A_\mu B_\nu = A_\mu B^\mu = A^\mu B_\mu.$$

Another example of the four vector is the energy–momentum four vector p^μ defined as:

$$p^\mu = (E, \vec{p}\,) \quad \text{and} \quad p_\mu = (E, -\vec{p}\,) \tag{2.8}$$

and the scalar product

$$p^2 = p^\mu p_\mu = E^2 - |\vec{p}|^2 = m^2, \tag{2.9}$$

such that $p^2 = m^2$, where m is the invariant mass of the particle, identified as the rest mass, that is, the energy of the particle at rest. For a particle moving with velocity $\vec{\beta}$, the energy and momentum are given by:

$$E = \gamma m \quad \text{and} \quad \vec{p} = \gamma \vec{\beta} m, \tag{2.10}$$

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where $\gamma = \frac{1}{\sqrt{1-|\vec{\beta}|^2}}$ such that

$$E^2 - |\vec{p}|^2 = \gamma^2(1 - |\vec{\beta}|^2)m^2 = m^2,$$

defines the relativistic relation between the energy E and momentum $\vec{p}$ of a real particle of mass m. It is straightforward to see that if the ordinary vector $\vec{x}$ is defined through the components ($\mu = 1, 2, 3$) of a contravariant vector x^μ, then the derivative operator $\vec{\nabla} = \frac{\partial}{\partial \vec{x}}$ is defined through the space covariant vector ∂_μ defined as:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \vec{x}}\right) = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right). \tag{2.11}$$

The fact that the 4-component derivative vector, that is, $\frac{\partial}{\partial x^\mu}$ transforms as a covariant vector can be seen as follows:

Consider any scalar function $\phi(x)$ for which a change $\delta\phi(x)$ in $\phi(x)$, due to a change δx^μ in x^μ, is defined through the equation:

$$\delta\phi(x) = \frac{\partial \phi}{\partial x^\mu}\delta x^\mu. \tag{2.12}$$

Since $\delta\phi$ is a scalar quantity and δx^μ transforms as a contravariant vector, the derivative operator $\frac{\partial}{\partial x^\mu} = \partial_\mu$ will transform as a covariant vector. Similarly,

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right), \tag{2.13}$$

such that

$$\begin{aligned} \partial_\mu A^\mu &= \frac{\partial A^0}{\partial t} + \vec{\nabla} \cdot \vec{A}, \\ \text{and } \partial_\mu \partial^\mu &= \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 = \Box, \end{aligned} \tag{2.14}$$

where $\Box$ is known as the D'Alembertian operator.

2.2 Wave Equation for a Relativistic Particle

2.2.1 Klein–Gordon equation for spin 0 particles

In nonrelativistic quantum mechanics, the Schrödinger equation is obtained by using the nonrelativistic energy–momentum relation for a particle given by:

$$E = \frac{|\vec{p}|^2}{2m}. \tag{2.15}$$

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Treating energy(E) and momentum($\vec{p}$) as operators in coordinate representation:

$$\hat{E} \rightarrow +i\frac{\partial}{\partial t}, \tag{2.16}$$

$$\hat{\vec{p}} \rightarrow -i\vec{\nabla}, \tag{2.17}$$

we obtain

$$i\frac{\partial \psi(\vec{x},t)}{\partial t} = \left(-\frac{1}{2m}\vec{\nabla}^2\right)\psi(\vec{x},t), \tag{2.18}$$

for wave function $\psi(\vec{x},t)$ which describes the motion of a free particle. This equation is linear in time derivative but quadratic in space derivative. Thus, space and time dependence of the wave function are not treated on equal footing. Therefore, the Schrödinger equation is not relativistically covariant and does not retain the same form in all the inertial frames under a Lorentz transformation. To describe the motion of a relativistic free particle, we start with the relativistic energy–momentum relation, that is,

$$E^2 = |\vec{p}|^2 + m^2 \quad \text{or} \quad E^2 - |\vec{p}|^2 = p^\mu p_\mu = m^2, \tag{2.19}$$

and use the coordinate representation of the 4-component energy–momentum operators p^μ and p_μ, that is,

$$\begin{aligned} \hat{p}^\mu &= i\partial^\mu = \left(i\tfrac{\partial}{\partial t}, -i\vec{\nabla}\right), \\ \hat{p}_\mu &= i\partial_\mu = \left(i\tfrac{\partial}{\partial t}, i\vec{\nabla}\right). \end{aligned} \tag{2.20}$$

If $\phi(\vec{x},t)$ is the wave function of a relativistic particle, using the operator form of p^μ and p_μ given by Eq. (2.20) in Eq. (2.19), we obtain:

$$\left((i\partial^\mu)(i\partial_\mu) - m^2\right)\phi(\vec{x},t) = 0,$$

$$(\Box + m^2)\phi(\vec{x},t) = 0, \tag{2.21}$$

where $\Box \equiv \partial_\mu\partial^\mu = -\vec{\nabla}^2 + \frac{\partial^2}{\partial t^2}$.

Equation (2.21) is the wave equation for a relativistic particle. Since the operator $(\Box + m^2)$ is a scalar operator, we assume $\phi(\vec{x},t)$ to be a scalar function of $(\vec{x},t)$, making the equation invariant under the Lorentz transformation. Moreover, being a scalar function, it describes only scalar particles, that is, particles of spin zero. This equation is known as the Klein–Gordon equation [192, 193].

We take the complex conjugate of Eq. (2.21) to obtain the relativistic wave equation for the complex conjugate wave function $\phi^*(\vec{x},t)$, that is,

$$\phi^*(\vec{x},t)(\Box + m^2) = 0. \tag{2.22}$$

The free particle Klein–Gordon equation given by Eq. (2.21) has a plane wave solution:

$$\phi(\vec{x}, t) = Ne^{-ip^{\mu}x_{\mu}} \equiv Ne^{-i(Et-\vec{p}\cdot\vec{x})}, \tag{2.23}$$

where N is an appropriate normalization constant. The momentum p^{μ} is the eigenvalue of the momentum operator $\hat{p}^{\mu}$ because

$$\hat{p}^{\mu}\phi(\vec{x}, t) = i\frac{\partial}{\partial x_{\mu}}\phi(\vec{x}, t) = iN\frac{\partial}{\partial x_{\mu}}e^{-ip^{\mu}x_{\mu}} = p^{\mu}\phi(\vec{x}, t),$$

which satisfies the equation:

$$\begin{aligned} p^2 - m^2 &= 0, \\ \text{i.e., } p_0^2 &= E^2 = |\vec{p}|^2 + m^2, \\ \text{leading to } p_0 &= \pm\sqrt{|\vec{p}|^2 + m^2} = \pm E_0. \end{aligned} \tag{2.24}$$

Therefore, the Klein–Gordon equation admits both positive and negative energy solutions. The presence of negative energy solutions presents a problem toward its physical interpretation. To see this, consider Eq. (2.21) and (2.22) for $\phi(\vec{x}, t)$ and $\phi^*(\vec{x}, t)$. We multiply Eq. (2.21) by $i\phi^*$ from the left and Eq. (2.22) by $i\phi$ from the right and subtract to get:

$$i(\phi^*\Box\phi - \phi\Box\phi^*) = 0$$

leading to

$$\partial_{\mu}(i\phi^*\partial^{\mu}\phi - i\phi\partial^{\mu}\phi^*) = 0,$$

that is,

$$\partial_{\mu}J^{\mu} = 0 \quad \text{with} \tag{2.25}$$

$$J^{\mu} = i(\phi^*\partial^{\mu}\phi - \phi\partial^{\mu}\phi^*), \tag{2.26}$$

which describes the continuity equation for the probability density $J^0(=\rho)$ and the current density $(\vec{J} = \vec{j})$ given by:

$$J^0 = i\left(\phi^*\frac{\partial}{\partial t}\phi - \phi\frac{\partial}{\partial t}\phi^*\right), \quad \text{and} \tag{2.27}$$

$$\vec{J} = -i\left(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*\right), \tag{2.28}$$

leading to the conservation of total probability P defined as $P = \int J^0 d\vec{x}$, that is,

$$\frac{dP}{dt} = \frac{\partial}{\partial t}\int J^0 d\vec{x} = -\int \vec{\nabla}\cdot\vec{J}d\vec{x} = 0, \quad \text{that is, } P \text{ is constant.} \tag{2.29}$$

However, in the case of the Klein–Gordon equation, we obtain the probability density ρ from the four vector J^μ as:

$$\rho = J^0 = 2p^0|N|^2\phi^*\phi = \pm 2E_0|N|^2 \ (\because \phi^*\phi = 1), \tag{2.30}$$

which gives negative probability for the negative energy solutions. The main cause of the appearance of the negative energy solution in the relativistic case is the quadratic relation between energy and momentum, that is, $E^2 = p^2 + m^2$, unlike the nonrelativistic case, where $E = m + \frac{|\vec{p}|^2}{2m}$, is always positive. It was for this reason that the Klein–Gordon equation was not used for sometime as a relativistic wave equation and attempts were made to linearize the relativistic wave equation, which led to the Dirac equation [28]. The Klein–Gordon equation was revived again as a genuine relativistic equation for spin zero particle, after Pauli and Weisskopf [194] suggested that ϕ should be interpreted as a field operator in quantum field theory and not as a wave function. The field $\phi(\vec{x}, t)$ and $\phi^*(\vec{x}, t)$, with appropriate normalization constant N, could be then used to describe the particles with positive and negative charges; the probability density ρ multiplied by the charge $(+e)$, that is,

$$\rho = ie\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right)$$

would describe the charge density which could be positive as well as negative.

Keeping this interpretation of probability density ρ in mind, we define the normalization constant N, such that the total number of particles in a volume V, given by $\int \rho dV$ remains unchanged. We, therefore, adopt a normalization in Eq. (2.23) for the Klein–Gordon wave function as:

$$N = \frac{1}{\sqrt{V}}.$$

With this value of N, the wave function is normalized such that there are $2E$ particles in volume V, that is, $\int \rho dV = 2E$. This is called the covariant normalization because with this normalization, the density of states, that is, the number of states per particle $\rho(E)$ is proportional to $\frac{d\vec{p}}{2E}$ which is relativistically covariant and occurs naturally in the 4-dimensional formulation of the fields in quantum field theory (this will be discussed in the next chapter).

Equivalently, if we take:

$$N = \frac{1}{\sqrt{2EV}},$$

then, there would be one particle in volume V, that is, $\int \rho dV = 1$.

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2.3 Dirac Equation for Spin $\frac{1}{2}$ Particles

In order to achieve linearization of the relativistic wave equation, Dirac [28] proposed a linear relation between the energy and momentum of the type

$$E = \vec{\alpha} \cdot \vec{p} + \beta m = \sum_{i=1}^{3} \alpha_i p_i + \beta m, \tag{2.31}$$

for a free particle, where $\alpha_i (i = 1, 2, 3)$ and β are to be chosen such that the relativistic relation between energy and momentum, that is,

$$E^2 = |\vec{p}|^2 + m^2 \quad \text{, that is, } E = \pm\sqrt{|\vec{p}|^2 + m^2} = \pm E_p \tag{2.32}$$

is reproduced. This requires that α_i and β must satisfy the following conditions, assuming Eqs. (2.31) and (2.32). A summation over the repeated indices is implied.

1.

$$\begin{aligned} (\alpha_i p_i + \beta m)(\alpha_j p_j + \beta m) = E^2 &= p_i p_i + m^2, \\ \text{that is, } \alpha_i \alpha_j p_i p_j + \alpha_i p_i \beta m + \beta m \alpha_j p_j + \beta^2 m^2 &= p_i p_i + m^2, \end{aligned}$$

since i and j are dummy indices, therefore, they can be interchanged and one can write:

$$\left(\frac{\alpha_i \alpha_j + \alpha_j \alpha_i}{2}\right) p_i p_j + m p_i \left(\frac{\alpha_i \beta + \beta \alpha_i}{2}\right) + m p_j \left(\frac{\alpha_j \beta + \beta \alpha_j}{2}\right) + \beta^2 m^2 = p_i p_j \delta_{ij} + m^2$$

leading to the conditions that:

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} \tag{2.33}$$

$$\{\alpha_i, \beta\} = 0 \tag{2.34}$$

$$\alpha_i{}^2 = \beta^2 = 1, \tag{2.35}$$

where $\{A, B\}$ is called the anticommutator of A and B operators defined as:

$$\{A, B\} = AB + BA. \tag{2.36}$$

Obviously, the quantities α_i and β are not numbers but operators satisfying an algebra given by Eqs. (2.33), (2.34) and (2.35). These operators are generally represented by matrices in n-dimensional space and are, therefore, $n \times n$ matrices with the dimension n yet to be decided.

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2. If there exist α_i and β with the properties discussed earlier, the Dirac equation can be represented as:

$$\begin{aligned} H\psi(\vec{x},t) &= i\frac{\partial\psi(\vec{x},t)}{\partial t} \\ \text{with } H &= \vec{\alpha}.\vec{p} + \beta m. \end{aligned} \tag{2.37}$$

   Since H is a Hermitian operator, $\alpha^\dagger = \alpha$ and $\beta^\dagger = \beta$, that is, the α_i and β matrices are also Hermitian.

3. Since ${\alpha_i}^2 = 1$ and $\beta^2 = 1$, the eigenvalues of α_i and β are ± 1.
4. The matrices α_i and β are traceless, which can be understood as follows:

   Eq. (2.33) implies that:

$$\begin{aligned} &\alpha_i\alpha_j + \alpha_j\alpha_i = 0 \text{ for } i \neq j \\ &\quad \text{that is, } \alpha_i\alpha_j = -\alpha_j a_i. \end{aligned} \tag{2.38}$$

   Multiplying by α_j on both sides and taking trace, we get

$$\begin{aligned} Tr(\alpha_i\alpha_j\alpha_j) &= -Tr(\alpha_j\alpha_i\alpha_j) = -Tr(\alpha_j\alpha_j\alpha_i) \\ \Rightarrow Tr(\alpha_i) &= -Tr(\alpha_i) \\ \Rightarrow Tr(\alpha_i) &= 0. \end{aligned} \tag{2.39}$$

   Similarly, using Eq. (2.34)

$$\alpha_i\beta = -\beta\alpha_i$$

   and multiplying by α_i on both sides and taking trace, we get

$$Tr(\beta) = 0. \tag{2.40}$$

5. Since all the four matrix operators $\alpha_i (i = 1, 2, 3)$ and β have eigenvalues ± 1 and are traceless, their dimensions must be even, that is, $n = 2, 4, 6...$. Since, there are only 3 traceless matrices in 2 dimensions, the minimum dimension of the matrices should be 4.
6. Since α_i and β are 4×4 matrices, the wave function ψ in Eq. (2.37) is a 4-component column vector.

The most popular matrix representation used for α_i and β matrices is the Pauli–Dirac parametrization, in which, the three α_i matrices are written in terms of the three Pauli matrices σ_i as:

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \tag{2.41}$$

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where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.42}$$

and

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.43}$$

which satisfy the conditions given in Eqs. (2.33), (2.34) and (2.35). However, it should be emphasized that the Dirac equation is independent of any given parametrization of α_i and β matrices.

It is convenient to introduce a new set of matrices $\gamma^\mu (\mu = 0, 1, 2, 3)$ defined in terms of α_i and β as:

$$\gamma^0 = \beta \tag{2.44}$$

$$\gamma^i = \beta\alpha_i = \gamma^0\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{2.45}$$

which satisfy the relations:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{2.46}$$

$$\gamma^{0^2} = 1;\quad \gamma^{i^2} = -1 \tag{2.47}$$

$$\gamma^{0^\dagger} = \gamma^0;\quad \gamma^{i^\dagger} = -\gamma^i \tag{2.48}$$

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0. \tag{2.49}$$

Multiplying the Dirac equation, Eq. (2.37), by γ^0 from the left, we get:

$$\begin{aligned} i\gamma^0\frac{\partial\psi}{\partial t} &= (\gamma^0\alpha_i p^i + \gamma^0\beta m)\psi \\ \text{or } \left(i\gamma^0\frac{\partial}{\partial t} - \gamma^i p^i - m\right)\psi(\vec{x}, t) &= 0 \\ \left(i\gamma^0\frac{\partial}{\partial t} + i\gamma^i\partial_i - m\right)\psi(\vec{x}, t) &= 0 \quad (\because p^i = -i\frac{\partial}{\partial x^i} = -i\partial_i) \\ (i\gamma^\mu\partial_\mu - m)\,\psi(\vec{x}, t) &= 0 \\ (i\partial\!\!\!/ - m)\,\psi(\vec{x}, t) &= 0, \quad \text{where } \partial\!\!\!/ = \gamma^\mu\partial_\mu. \end{aligned} \tag{2.50}$$

The Hermitian conjugate of $\psi(\vec{x}, t)$ satisfies:

$$\psi^\dagger(\vec{x}, t)(-i\gamma^{\mu\dagger}\overleftarrow{\partial}_\mu - m) = 0, \tag{2.51}$$

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where $\overleftarrow{\partial}_\mu$ operates to the left on $\psi^\dagger(\vec{x},t)$. Multiplying Eq. (2.51) by γ^0 from the right, we have

$$\psi^\dagger(\vec{x},t)(-i\gamma^{\mu\dagger}\gamma^0\overleftarrow{\partial}_\mu - m\gamma^0) = 0.$$

Since ${\gamma^0}^2 = 1$, we can write this equation as:

$$\begin{aligned}\psi^\dagger(\vec{x},t)(-i\gamma^0\gamma^0\gamma^{\mu\dagger}\gamma^0\overleftarrow{\partial}_\mu - m\gamma^0) &= 0\\ \psi^\dagger(\vec{x},t)\gamma^0(-i\gamma^0\gamma^{\mu\dagger}\gamma^0\overleftarrow{\partial}_\mu - m) &= 0.\end{aligned}$$

Using $\psi^\dagger(\vec{x},t)\gamma^0 = \bar{\psi}(\vec{x},t)$ and $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$, we obtain:

$$\begin{aligned}\bar{\psi}(\vec{x},t)(-i\gamma^\mu\overleftarrow{\partial}_\mu - m) &= 0,\\ \Rightarrow \bar{\psi}(\vec{x},t)(i\overleftarrow{\not{\partial}} + m) &= 0.\end{aligned} \tag{2.52}$$

Thus, Eqs. (2.50) and (2.52) are the Dirac equations satisfied by $\psi(\vec{x},t)$ and $\bar{\psi}(\vec{x},t)$.

It can be shown that the following are true:

- The Dirac equation is covariant under the Lorentz transformation (discussed in Appendix A).
- There is a conserved current J^μ associated with the Dirac equation given by $J^\mu = \bar{\psi}\gamma^\mu\psi$. It can be shown that by multiplying Eq. (2.50) by $\bar{\psi}$ from the left and Eq. (2.52) by ψ from the right and then adding the modified Eq. (2.52) to the modified Eq. (2.50) we obtain:

$$\begin{aligned}i(\bar{\psi}\gamma^\mu\partial_\mu\psi + \bar{\psi}\gamma^\mu\overleftarrow{\partial}_\mu\psi) &= 0,\\ \text{i.e., } \partial_\mu\bar{\psi}\gamma^\mu\psi &= 0,\\ \text{or } \partial_\mu J^\mu = 0. &\end{aligned} \tag{2.53}$$

It may be noted that Eq. (2.53) is the covariant form of the continuity equation leading to the conservation of probability (Eq. (2.29)). Since $J^\mu = (J^0, J^i) = (\rho, \vec{J})$, we obtain the expressions for the probability density ρ and the probability current density $\vec{J}$ as:

$$\rho(\vec{x},t) = \bar{\psi}(\vec{x},t)\gamma^0\psi(\vec{x},t) = \psi^\dagger(\vec{x},t)\psi(\vec{x},t), \tag{2.54}$$

$$\vec{J}(\vec{x},t) = \bar{\psi}(\vec{x},t)\vec{\gamma}\psi(\vec{x},t) = \psi^\dagger(\vec{x},t)\vec{\alpha}\psi(\vec{x},t), \tag{2.55}$$

making the probability density $\rho(\vec{x},t)$ always positive, and thus avoiding the situation encountered in the Klein–Gordon equation. It should be kept in mind that $E^2 = |\vec{p}|^2 + m^2$ and the Dirac equation still admits the negative energy solutions as we will see in Section 2.3.2.

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2.3.1 Spin of a Dirac particle

The Hamiltonian operator of a Dirac particle is:

$$H = \vec{\alpha} \cdot \vec{p} + \beta m. \tag{2.56}$$

It is easy to see that it does not commute with the angular momentum operator $\vec{L} = \vec{r} \times \vec{p}$, that is,

$$[H, \vec{L}] = [\vec{\alpha} \cdot \vec{p} + \beta m, \vec{r} \times \vec{p}] \neq 0.$$

Therefore, $\vec{L}$ is not a constant of motion. To see this, let us consider, for simplicity, the operator L_3 and compute

$$\begin{aligned} [H, L_3] &= [\alpha_i p_i, (\vec{r} \times \vec{p})_3] \\ &= [\alpha_i p_i, x_1 p_2 - x_2 p_1]. \end{aligned} \tag{2.57}$$

Using the commutation relations for position and momentum operators in quantum mechanics

$$[p_i, x_j] = -i\delta_{ij}, \qquad [p_i, p_j] = 0,$$

and

$$[A, BC] = [A, B]C + B[A, C],$$

we obtain:

$$\begin{aligned} [H, L_3] &= \alpha_i [p_i, x_1] p_2 - \alpha_i [p_i, x_2] p_1 \\ &= -i(\alpha_1 p_2 - \alpha_2 p_1) \neq 0. \end{aligned} \tag{2.58}$$

In the relativistic case, we show that the constant of motion is not the $\vec{L}$ operator but an operator given by:

$$\vec{J} = \vec{L} + \frac{1}{2}\vec{\Sigma}, \text{ where } \Sigma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}, \quad i = 1 - 3 \tag{2.59}$$

is the 4-dimensional representation of Pauli matrices σ^i, that is, $\Sigma^i = \sigma^i \otimes I$, I being the unit matrix operator. To see this, define a tensor $\sigma^{\mu\nu}$ as:

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]. \tag{2.60}$$

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Writing the space components of $\sigma^{\mu\nu}$, we have:

$$\begin{aligned}
\sigma^{ij} &= \frac{i}{2}[\gamma^i, \gamma^j] \\
&= \frac{i}{2}\begin{pmatrix} -(\sigma^i\sigma^j - \sigma^j\sigma^i) & 0 \\ 0 & -(\sigma^i\sigma^j - \sigma^j\sigma^i) \end{pmatrix} \\
&= \epsilon_{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \quad (\because [\sigma^i, \sigma^j] = 2i\epsilon_{ijk}\sigma^k) \\
&= \epsilon_{ijk}\Sigma^k. \qquad (2.61)
\end{aligned}$$

For example,

$$\sigma^{12} = \Sigma^3 = \frac{i}{2}(\gamma^1\gamma^2 - \gamma^2\gamma^1).$$

Therefore,

$$\begin{aligned}
[H, \frac{1}{2}\Sigma_3] &= -\frac{i}{4}\left[\alpha_i p_i, (\alpha_1\alpha_2 - \alpha_2\alpha_1)\right] \\
&= -i(\alpha_2 p_1 - \alpha_1 p_2) = -[H, L_3]. \quad \text{(from Eq. (2.58))} \qquad (2.62)
\end{aligned}$$

Thus, we see that

$$[H, L_3 + \frac{1}{2}\Sigma_3] = 0. \qquad (2.63)$$

We also see that $(\Sigma_3)^2 = 1$, implying that Σ_3 has eigenvalues ± 1. Comparing the operator $\vec{J} = \vec{L} + \frac{1}{2}\vec{\Sigma}$ with the nonrelativistic total angular momentum $\vec{J} = \vec{L} + \vec{S} = \vec{L} + \frac{1}{2}\vec{\sigma}$, we conclude that $\vec{\Sigma}$ is indeed the 4-dimensional generalization of $\vec{\sigma}$, with $\frac{1}{2}\vec{\Sigma}$ being identified with the spin s of the Dirac particle. Eigenvalues of $\frac{1}{2}\Sigma_3 = \pm\frac{1}{2}$ imply that the particle has spin $\frac{1}{2}$. We, therefore, expect that ψ would have 2 components but it has 4 components as discussed in Section 2.3. The other two components are associated with the negative energy solutions as will be discussed in the next section. It is interesting to see that while Σ_3 does not commute with H, where Σ_3 is chosen to be the component along the '3' axis in the Euclidean space (Z-axis), the component of $\vec{\Sigma}$ along the momentum direction of the particle, that is, $\vec{\Sigma} \cdot \hat{p}$ commutes with the Hamiltonian and is a constant of motion, that is,

$$\left[H, \vec{\Sigma} \cdot \hat{p}\right] = 0. \qquad (2.64)$$

2.3.2 Plane wave solutions of the Dirac equation

As we have seen, the Dirac wave function $\psi(\vec{x}, t)$ satisfies the equation:

$$(i\gamma^\mu \partial_\mu - m)\psi(\vec{x}, t) = 0,$$

where $\psi(\vec{x}, t)$ is a 4-component column vector

$$\psi(\vec{x}, t) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \tag{2.65}$$

describing spin $\frac{1}{2}$ particles. The free particle solutions of Eq. (2.65) are plane waves given by:

$$\psi(\vec{x}, t) = N u(\vec{p}) e^{-ip^\mu x_\mu}; \quad p^\mu = (E, \vec{p}). \tag{2.66}$$

Then, $u(\vec{p})$ satisfies the equation

$$(\not{p} - m) u(\vec{p}) = 0. \tag{2.67}$$

To understand the physical meaning of Eq. (2.67), let us consider the particle at rest, so that a correspondence can be made with the nonrelativistic particle of spin $\frac{1}{2}$. For $\vec{p} = 0$, the equation becomes:

$$(E\gamma^0 - m) u(\vec{p}) \;\; = \;\; 0, \quad \text{with} \;\; u(\vec{p}) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}. \tag{2.68}$$

Using Eq. (2.68), one may write

$$\begin{pmatrix} E-m & 0 & 0 & 0 \\ 0 & E-m & 0 & 0 \\ 0 & 0 & -E-m & 0 \\ 0 & 0 & 0 & -E-m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \;\; = \;\; 0. \tag{2.69}$$

This has a solution given by the condition

$$\det(E\gamma^0 - mI) = 0,$$

$$\text{that is,} \quad (E-m)^2 (E+m)^2 = (E^2 - m^2)^2 = 0,$$

which has four eigenvalues, given by:

$$E_1 = E_2 = m \;\; \text{and} \;\; E_3 = E_4 = -m \tag{2.70}$$

with eigenvectors:

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \;\; \text{and} \;\; u_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{2.71}$$

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respectively. Thus, we have two positive energy states u_1 and u_2 which are degenerate and two negative energy states u_3 and u_4, which are also degenerate. The two positive energy states obviously correspond to the two spin states of a nonrelativistic spin $\frac{1}{2}$ particle. The Dirac equation, therefore, predicts, in addition, two negative energy states which are degenerate and correspond to spin states of spin $\frac{1}{2}$.

Now, consider the general case with $\vec{p} \neq 0$ and express the 4-component spinor $u(\vec{p})$ in terms of two 2-component spinors u_a and u_b and write

$$u(\vec{p}) = \begin{pmatrix} u_a(\vec{p}) \\ u_b(\vec{p}) \end{pmatrix} \text{ with } \; u_a(\vec{p}) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \text{ and } \; u_b(\vec{p}) = \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}. \tag{2.72}$$

The Dirac equation is then expressed as a matrix equation using the Pauli–Dirac representation for γ^μ and $p_i = -p^i$, $(i = 1-3)$, where p^i are ordinary vectors. Using Eq. (2.67), we obtain:

$$\begin{pmatrix} p_0 - m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -p_0 - m \end{pmatrix} \begin{pmatrix} u_a(\vec{p}) \\ u_b(\vec{p}) \end{pmatrix} = 0. \tag{2.73}$$

The equation has non-trivial solutions for

$$(p_0 - m)(p_0 + m) - (\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p}) = 0$$
$$\text{or } \; {p_0}^2 = |\vec{p}|^2 + m^2 \qquad (\because (\vec{\sigma}\cdot\vec{p}\,\vec{\sigma}\cdot\vec{p}) = \vec{p}\cdot\vec{p} + i\vec{\sigma}(\vec{p}\times\vec{p}) = |\vec{p}|^2).$$

Thus, the non-trivial plane wave solutions lead to two energy eigenvalues given by:

$$p_0 = \pm E = \pm\sqrt{|\vec{p}|^2 + m^2}, \tag{2.74}$$

and both would be doubly degenerate as shown in the case of solutions obtained for the limiting case of $\vec{p} = 0$ (Eq. (2.70)). The eigen functions $u_a(\vec{p})$ and $u_b(\vec{p})$ satisfy the relations for positive energy solutions, say $p_0 = E_+ = \sqrt{|\vec{p}|^2 + m^2}$, that is,

$$(E_+ - m)u_a(\vec{p}) - (\vec{\sigma}\cdot\vec{p})u_b(\vec{p}) = 0, \tag{2.75}$$

$$(\vec{\sigma}\cdot\vec{p})u_a(\vec{p}) - (E_+ + m)u_b(\vec{p}) = 0. \tag{2.76}$$

Equation (2.76) gives:

$$u_b(\vec{p}) = \frac{\vec{\sigma}\cdot\vec{p}}{E_+ + m} u_a(\vec{p}). \tag{2.77}$$

It should be noted that Eq. (2.75) also gives the same relation between $u_a(\vec{p})$ and $u_b(\vec{p})$, that is,

$$u_a(\vec{p}) = \frac{\vec{\sigma}\cdot\vec{p}}{E_+ - m} u_b(\vec{p}), \tag{2.78}$$

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Multiplying both the sides of Eq. (2.78) by $\vec{\sigma} \cdot \vec{p}$, we get:

$$(E_+ - m)(\vec{\sigma} \cdot \vec{p})\, u_a(\vec{p}) = (\vec{\sigma} \cdot \vec{p})\, (\vec{\sigma} \cdot \vec{p})\, u_b(\vec{p}).$$

Multiplying this by $(E_+ + m)$ on both the sides and using $(\vec{\sigma} \cdot \vec{p})^2 = |\vec{p}|^2$, we get:

$$\begin{aligned} u_b(\vec{p}) &= \frac{E_+^2 - m^2}{|\vec{p}|^2(E_+ + m)}(\vec{\sigma} \cdot \vec{p}) u_a(\vec{p}) \\ \text{or} \quad u_b(\vec{p}) &= \frac{\vec{\sigma} \cdot \vec{p}}{E_+ + m} u_a(\vec{p}). \end{aligned} \tag{2.79}$$

We see that the lower components $u_b(\vec{p})$ of $u(\vec{p})$ are given in terms of the upper components $u_a(\vec{p})$ through Eq. (2.79). The upper components are, therefore, independent and arbitrary and can be chosen such that they reproduce correctly the nonrelativistic limit of the solution, that is, $u_a = \chi_r (r = 1, 2)$ with $\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as shown in Eq. (2.71), with normalization $\chi_r^+ \chi_s = \delta_{rs}$.

The positive energy solutions are represented as:

$$u_r^+(\vec{p}) = \begin{pmatrix} 1 \\ \frac{\vec{\sigma} \cdot \vec{p}}{E_+ + M} \end{pmatrix} \chi_r \; = \begin{pmatrix} 1 \\ \frac{\vec{\sigma} \cdot \vec{p}}{E + M} \end{pmatrix} \chi_r. \tag{2.80}$$

Similarly, for negative energy solutions, we have:

$$\begin{aligned} p_0 = E_- = -E_+ &= -\sqrt{|\vec{p}|^2 + m^2}, \\ (E_- - m) u_a(\vec{p}) - (\vec{\sigma} \cdot \vec{p}) u_b(\vec{p}) &= 0, \\ \text{leading to} \quad u_a(\vec{p}) &= \frac{\vec{\sigma} \cdot \vec{p}}{E_- - m} u_b(\vec{p}). \end{aligned} \tag{2.81}$$

Now, choosing the independent solution for the negative energy state $u_b(\vec{p})$ as $\chi_r (r = 1, 2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we obtain:

$$\begin{aligned} u_r^-(\vec{p}) &= \frac{\vec{\sigma} \cdot \vec{p}}{E_- - m} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; \frac{\vec{\sigma} \cdot \vec{p}}{E_- - m} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{E_- - m} \\ 1 \end{pmatrix} \chi_r \qquad (r = 1, 2). \end{aligned} \tag{2.82}$$

$\because E_- = -\sqrt{|\vec{p}|^2 + m^2} = -E$, we can also write

$$u_r^-(\vec{p}) = \begin{pmatrix} -\frac{\vec{\sigma} \cdot \vec{p}}{E + m} \\ 1 \end{pmatrix} \chi_r \qquad (r = 1, 2). \tag{2.83}$$

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The two degenerate states $r = 1, 2$ corresponding to the positive and negative energy states refer to the spin up and spin down states of Dirac particles. We can explicitly write the Dirac spinors as

$$u^+_\uparrow(\vec{p}) = \begin{pmatrix} 1 \\ 0 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+M} \\ 0 \end{pmatrix};\ u^+_\downarrow(\vec{p}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+M} \end{pmatrix};\ u^-_\uparrow(\vec{p}) = \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{E+M} \\ 0 \\ 1 \\ 0 \end{pmatrix};\ u^-_\downarrow(\vec{p}) = \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{E+M} \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.84}$$

In literature, it is customary to define the positive energy spinors as $u_r(\vec{p})$ instead of $u^+_r(\vec{p})$ and the negative energy spinors as $v_r(\vec{p})$ instead of $u^-_r(\vec{p})$ to facilitate the description of antiparticles through the relation:

$$u_r(\vec{p}) = u^+_r(\vec{p}) \tag{2.85}$$

$$v_r(\vec{p}) = \epsilon^{rs} u^-_s(-\vec{p}), \tag{2.86}$$

$$\text{where } \epsilon_{11} = \epsilon_{22} = 0 \quad \text{and} \quad \epsilon_{12} = -\epsilon_{21} = 1$$

$$\Rightarrow v_1(\vec{p}) = u^-_2(-\vec{p}) \quad \text{and} \quad v_2(\vec{p}) = -u^-_1(-\vec{p}). \tag{2.87}$$

The positive energy spinor $u^+_r(\vec{p})(= u_r(\vec{p}))$ satisfies the relation:

$$(\not{p} - m)u^+_r(\vec{p}) = 0,$$

$$\Rightarrow (\not{p} - m)u_r(\vec{p}) = 0, \tag{2.88}$$

$$\text{and } \bar{u}_r(\vec{p})(\not{p} - m) = 0. \tag{2.89}$$

The negative energy spinor $u^-_r(\vec{p})$ satisfies the relation:

$$(\gamma^0 p_0 - \vec{\gamma}\cdot\vec{p} - m)u^-_r(\vec{p}) = 0,$$

$$\Rightarrow (\gamma^0 E_- - \vec{\gamma}\cdot\vec{p} - m)u^-_r(\vec{p}) = 0,$$

$$\text{or } (-\gamma^0 E + \vec{\gamma}\cdot\vec{p} - m)v_r(\vec{p}) = 0,$$

$$\Rightarrow (-\not{p} - m)v_r(\vec{p}) = 0,$$

$$\Rightarrow (\not{p} + m)v_r(\vec{p}) = 0, \tag{2.90}$$

$$\text{and } \bar{v}_r(\vec{p})(\not{p} + m) = 0. \tag{2.91}$$

Therefore, the solution for the Dirac wave function $\psi(\vec{x}, t)$ is written as:

$$\psi(\vec{x}, t) = N \begin{cases} u_r(\vec{p})e^{-ip\cdot x} & \text{for the positive energy states} \\ v_r(\vec{p})e^{ip\cdot x} & \text{for the negative energy states} \end{cases}, \tag{2.92}$$

where N is the normalization constant.

2.3.3 Normalization of Dirac spinors

Let us now define the normalization of the Dirac wave function in a way similar to the wave function for the Klein–Gordon equation for uniformity and write

$$\psi_r(\vec{x},t) = \frac{1}{\sqrt{V}} \begin{cases} u_r(\vec{p})e^{-ip\cdot x} \\ v_r(\vec{p})e^{ip\cdot x} \end{cases} \tag{2.93}$$

$$\text{with} \quad u_r(\vec{p}) \;=\; N\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix}\chi_r \quad \text{and} \quad v_r(\vec{p}) = N\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix}\chi_r. \tag{2.94}$$

We use Eq. (2.83) and keep in mind that $u_r^-(-\vec{p})$ is written in terms of $v_r(\vec{p})$. Now, we determine the normalization of the spinors $u_r(\vec{p})$ and $v_r(\vec{p})$ with the normalization condition

$$\int \rho dV = 2E. \tag{2.95}$$

We use the expressions for ρ given in Eq. (2.54) and $u_r(\vec{p})$ and $v_r(\vec{p})$ given in Eq. (2.93) with the normalization for χ_r to obtain the condition for $u_r(\vec{p})$, that is

$$\begin{aligned} \int \rho dV \;&=\; \int \psi^\dagger \psi dV = u^\dagger(\vec{p})u(\vec{p}) = 2E, \\ \text{and} \;\; u_r^\dagger(\vec{p})u_s(\vec{p}) \;&=\; |N|^2\frac{2E}{E+m}\delta_{rs} = 2E\delta_{rs}, \end{aligned} \tag{2.96}$$

using $(\vec{\sigma}\cdot\vec{p})\;(\vec{\sigma}\cdot\vec{p}) = |\vec{p}|^2$ and $\chi_r^\dagger\chi_s = \delta_{rs}$, leading to $N = \sqrt{E+m}$. Similarly, we can obtain

$$v_r^\dagger(\vec{p})v_s(\vec{p}) \;=\; 2E\delta_{rs} \tag{2.97}$$

$$\text{and} \quad u_r^\dagger(\vec{p})v_s(-\vec{p}) \;=\; 0 \tag{2.98}$$

with N=$\sqrt{E+m}$.

If $\rho\, dV$ is normalized to 1, then

$$N = \sqrt{\frac{E+m}{2E}}.$$

So, for this normalization, we write

$$\psi_r(\vec{x},t) = \frac{1}{\sqrt{2EV}} \begin{cases} u_r(\vec{p})e^{-ip\cdot x} \\ v_r(\vec{p})e^{ip\cdot x}. \end{cases} \tag{2.99}$$

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We, therefore, write the components $u_r(\vec{p})$ and $v_r(\vec{p})$ of the Dirac wave function $\psi(\vec{x},t)$ as:

$$u_r(\vec{p}) = \sqrt{E+m}\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix}\chi_r e^{-ip\cdot x}, \tag{2.100}$$

$$v_r(\vec{p}) = \sqrt{E+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix}\chi_r e^{ip\cdot x}. \tag{2.101}$$

Using these normalizations for $u_r(\vec{p})$ and $v_r(\vec{p})$, we can derive the normalization condition:

$$\overline{u}_r(\vec{p})u_s(\vec{p}) = 2m\delta_{rs} \tag{2.102}$$

$$\bar{v}_r(\vec{p})v_s(\vec{p}) = -2m\delta_{rs} \tag{2.103}$$

$$\bar{u}_r(\vec{p})v_s(-\vec{p}) = 0, \tag{2.104}$$

where $\bar{u} = u^\dagger\gamma^0$. Multiplying Eq. (2.88) by $\overline{u}\gamma^\mu$ from the left and Eq. (2.89) by $\gamma^\mu u$ from the right and adding them together, we get

$$\begin{aligned}\overline{u}(\vec{p})\gamma_\mu(\not{p}-m)u(\vec{p}) + \overline{u}(\vec{p})(\not{p}-m)\gamma_\mu u(\vec{p}) &= \overline{u}(\vec{p})(\gamma_\mu\not{p}+\not{p}\gamma_\mu)u(\vec{p}) \\ &- 2m\overline{u}(\vec{p})\gamma_\mu u(\vec{p}) = 0,\end{aligned} \tag{2.105}$$

that is,

$$p^\mu\overline{u}(\vec{p})u(\vec{p}) = m\overline{u}(\vec{p})\gamma_\mu u(\vec{p}) \qquad (\because\ \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}).$$

Taking $\mu = 0$ component, we obtain:

$$\overline{u}_r(\vec{p})u_s(\vec{p}) = \frac{m}{E}u_r^\dagger(\vec{p})u_s(\vec{p}) = 2m\delta_{rs}. \tag{2.106}$$

Similarly, using Eqs. (2.90) and (2.91) and using $p^0 = -E$ for $v_r(\vec{p})$ solutions, we obtain:

$$\overline{v}_r(\vec{p})v_s(\vec{p}) = -2m\delta_{rs}. \tag{2.107}$$

Since $u(\vec{p})$ is a 4-component column matrix and $\overline{u}(\vec{p})$ is a 4-component row matrix, $u(\vec{p})\overline{u}(\vec{p})$ would be a 4×4 square matrix given by:

$$\begin{aligned}\sum_{r=1}^{2} u_r(\vec{p})\overline{u}_r(\vec{p}) &= \not{p}+m = p_0\gamma^0 - p_i\gamma^i + m, \\ &= \begin{pmatrix} E+m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E+M \end{pmatrix}.\end{aligned} \tag{2.108}$$

By operating Eq. (2.108), on the Dirac spinor $u_s(\vec{p})$, we get:

$$\sum_r u_r(\vec{p})\overline{u}_r(\vec{p})u_s(\vec{p}) = (\not{p}+m)u_s(\vec{p}),$$

$$\text{that is,}\quad \sum_r 2m\delta_{rs}u_r(\vec{p}) = (\not{p}+m)u_s(\vec{p}) = 2mu_s(\vec{p}), \tag{2.109}$$

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which is an identity. However, Eq. (2.108) can be proved by explicitly evaluating $\sum_r u_r(\vec{p})$ $\overline{u}_r(\vec{p})$ using $u_r(\vec{p})$ for $r = 1, 2$ from Eq. (2.100). Similarly, we can show that:

$$\sum_{r=1}^{2} v_r(\vec{p})\overline{v}_r(\vec{p}) = \not{p} - m, \tag{2.110}$$

such that

$$\sum_{r,s=1}^{2} (u_r(\vec{p})\bar{u}_s(\vec{p}) - v_r(\vec{p})\bar{v}_s(\vec{p})) = 2m\delta_{rs}. \tag{2.111}$$

The relations given in Eqs. (2.108) and (2.110) are called the completeness relations for the Dirac spinors for the positive and negative energy solutions.

2.4 Negative Energy States and Hole Theory

We see that the Dirac equation admits negative energy solutions in addition to the positive energy solutions with eigenvalues of $E = \pm E_0$, where $E_0 = \sqrt{|\vec{p}|^2 + m^2}$. These solutions are described by the Dirac spinor $u_r(\vec{p})$ and $v_r(\vec{p})$ corresponding to positive and negative energy solutions, respectively, with $r(= 1, 2)$ representing the two spin states of particles. Obviously, the negative energy states lie below the positive energy states separated by an energy gap of $2E_0$ as shown in Figure 2.1. We have also seen that the negative energy solutions do not

Figure 2.1 Energy spectrum of Dirac particles with energy gap $2E_0$.

present any problem in defining a positive definite probability density. It raises the question of defining the state of minimum energy, that is, vacuum state. Since $\vec{p}$ can take any value, the negative energy states have no lower bound in energy. However, in quantum mechanics, the lowest energy state, that is, ground state is considered to be the most stable state. In most of the cases, the higher states make transitions to the ground state due to interactions in physical systems. Therefore, all the positive energy states would eventually transit to negative energy states, leaving no states in the positive energy states. The situation seems quite unphysical. In this scenario, Dirac proposed that the ground state can be defined as a state in which all

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the negative energy states are filled. Thus, the ground state or the vacuum state in relativistic quantum mechanics is not a zero particle state but a many particle state. A given energy state can have only two spin $\frac{1}{2}$ particles in it, corresponding to the two spin states (up and down). If all the negative energy states are filled, then no other spin $\frac{1}{2}$ particles can occupy any state in the space of energy states, thus forbidding transition from positive energy states to negative energy states. This filled space of negative energy states is called the Dirac sea.

In order to make a transition to the energy states in the Dirac sea, a vacuum has to be created in the Dirac sea, that is, a negative energy state has to be excited to a positive energy state needing an energy which should be greater than $2m$, the minimum energy corresponding to $p = 0$. This will create a vacancy in the Dirac sea, called a hole, which is equivalent to the absence of a particle of negative energy and negative charge or equivalent to the presence of a particle of positive energy and positive charge. Therefore, such a transition creates an electron of positive energy state and a hole in the Dirac sea, creating an electron–hole pair. The 'hole in the Dirac sea' is interpreted as an antiparticle and the process is known as pair production, in which if sufficient energy $\geqslant 2m$ is available, then a pair of electron and its antiparticle can be created. Such an antiparticle was discovered by Anderson in 1932 and is called a positron. Since then, positron beams have been created and e^-e^+ scattering experiments have been performed. Positron beams have found applications in many areas of physics, chemistry, and biology.

2.5 Projection Operators

There are two energy projection operators, two spin and two helicity projection operators. They will be discussed in this section.

2.5.1 Energy projection operators

The four components of the Dirac wave functions correspond to two positive and two negative energy states $u_r(\vec{p})$ and $v_r(\vec{p})$ corresponding to particles and antiparticles. If we are interested only in $u_r(\vec{p})$ and $v_r(\vec{p})$ states, then we need to define projection operators, which can project out the positive and negative states from the Dirac wave functions. We define these operators as $\Lambda_+(p)$ and $\Lambda_-(p)$ such that:

$$\begin{aligned} \Lambda_+ u_r(\vec{p}) &= u_r(\vec{p}), & \Lambda_- v_r(\vec{p}) &= v_r(\vec{p}), \\ \Lambda_+ v_r(\vec{p}) &= 0, & \Lambda_- u_r(\vec{p}) &= 0. \end{aligned} \tag{2.112}$$

It is easy to see that:

$$\Lambda_+ = \frac{\not{p} + m}{2m} = \Sigma_r u_r(\vec{p})\bar{u}_r(\vec{p}) \quad \text{and} \quad \Lambda_- = \frac{-\not{p} + m}{2m} = -\Sigma_r v_r(\vec{p})\bar{v}_r(\vec{p}) \tag{2.113}$$

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The projection operators also satisfy the relations:

$$\Lambda_+^2 = \Lambda_+ \, ; \;\; \Lambda_-^2 = \Lambda_- . \tag{2.114}$$

$$\Lambda_- \Lambda_+ = \Lambda_+ \Lambda_- = 0. \tag{2.115}$$

$$\Lambda_+ + \Lambda_- = 1. \tag{2.116}$$

2.5.2 Spin projection operators

We have seen that the Dirac spinor has four components, two corresponding to the particles and two corresponding to the antiparticles. The two components of the particle/antiparticle spinors correspond to the spin up and spin down state of the spin $\frac{1}{2}$ particle and are written explicitly in Eq. (2.84). We can define the spin projection operator such that a particular spin state $u_r(r = 1)$ corresponding to spin up $|\uparrow\rangle$ or $u_r(r = 2)$ corresponding to spin down $|\downarrow\rangle$ state can be chosen for describing the particle.

The spin projection operator must commute with $\not{p}$ so that the four components have unique energy–momentum and spin eigenvalues. In the rest frame, the Dirac particle is described by mass m, momentum $p^\mu (= m, \vec{0})$, and unit vector $\hat{s}$ in the direction of the spin quantization axis and has only space component $\hat{s}(i = 1, 2, 3)$. In order to treat spin in a covariant way, we, therefore, define a four vector $n^\mu (= 0, \hat{s})$ in the rest system which satisfies, by construction, the conditions:

$$n^2 = n^\mu n_\mu = -1 \quad \text{and} \quad n \cdot p = n^\mu p_\mu = 0. \tag{2.117}$$

It can be shown that in an arbitrary frame

$$n^0 = \frac{\hat{s} \cdot \vec{p}}{m} \qquad \text{and} \qquad \hat{n} = \hat{s} + \frac{(\vec{p} \cdot \hat{s})\hat{p}}{m(E + m)}. \tag{2.118}$$

In the rest frame, it is straightforward to see that the action of the operator $\vec{\Sigma} \cdot \hat{s}$, where $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$ on the positive and negative energy solution $u_r(p)$ and $v_r(p)$ $(r = 1, 2)$ (Eqs. (2.84), (2.85) and (2.86)) gives:

$$\vec{\Sigma} \cdot \hat{s}\, u_1(\vec{p}, \vec{s}) = u_1(\vec{p}, \vec{s}), \qquad \vec{\Sigma} \cdot \hat{s}\, v_1(\vec{p}, \vec{s}) = -v_1(\vec{p}, \vec{s}),$$
$$\vec{\Sigma} \cdot \hat{s}\, u_2(\vec{p}, \vec{s}) = -u_2(\vec{p}, \vec{s}), \qquad \vec{\Sigma} \cdot \hat{s}\, v_2(\vec{p}, \vec{s}) = v_2(\vec{p}, \vec{s}). \tag{2.119}$$

This shows that the operator $\vec{\Sigma} \cdot \hat{s}$, defined in four-dimensions as $\begin{pmatrix} \vec{\Sigma} \cdot \hat{s} & 0 \\ 0 & \vec{\Sigma} \cdot \hat{s} \end{pmatrix}$, reproduces the correct eigenvalues for spin up and spin down wave functions for the positive energy solution but not for the negative energy solutions $v_r(\vec{p})(r = 1, 2)$. However, if we define an operator as

$$\begin{pmatrix} \vec{\Sigma} \cdot \hat{s} & 0 \\ 0 & -\vec{\Sigma} \cdot \hat{s} \end{pmatrix}, \tag{2.120}$$

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then the correct eigenvalues for all the four Dirac solutions corresponding to particles $u_r(\vec{p})$ and antiparticles $v_r(\vec{p})$ $(r = 1, 2)$ are reproduced. It is easy to see that the covariant generalization of this operator is $\gamma_5 \not{s}$, that is,

$$(\gamma_5 \not{s}) = \begin{pmatrix} \vec{\Sigma} \cdot \hat{s} & 0 \\ 0 & -\vec{\Sigma} \cdot \hat{s} \end{pmatrix}. \tag{2.121}$$

It can be verified that

$$[\gamma_5 \not{s}, \not{p}] = 0 \qquad \text{and} \qquad (\gamma_5 \not{s})^2 = 1, \tag{2.122}$$

where

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \gamma_5, \qquad \gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 \tag{2.123}$$

demonstrating that the eigenvalues of the covariant operator $\gamma_5 \not{s}$ are ± 1 and the individual spin states $u_r(\vec{p})$ and $v_r(\vec{p})$ are also the eigenstates of the energy–momentum operator p^μ. The spin projection operators $\Lambda_+(s)$ and $\Lambda_-(s)$ are, therefore, constructed as:

$$\Lambda_\pm(s) = \frac{1 \pm \gamma_5 \not{s}}{2} \tag{2.124}$$

in analogy with the energy projection operators $\Lambda_\pm(E)$ given in Eq. (2.113). Note here, that:

$$u(\vec{p})\bar{u}(\vec{p}) = (\not{p} + m)\frac{1 + \gamma_5 \not{s}}{2}, \tag{2.125}$$

$$v(\vec{p})\bar{v}(\vec{p}) = (\not{p} + m)\frac{1 - \gamma_5 \not{s}}{2}. \tag{2.126}$$

2.5.3 Helicity and helicity projection operators

We have seen in Section 2.3.1 that the spin $\vec{s}$ of the Dirac particle does not commute with the Hamiltonian and is, therefore, not a constant of motion. However, the component of spin along the direction of the momentum, that is, $\vec{s} \cdot \hat{p} = \frac{1}{2}\frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|}$ is a constant of motion because the operator $\frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|}$ commutes with the Hamiltonian, that is,

$$[\vec{\sigma} \cdot \hat{p}, H] = 0 \qquad \text{and} \qquad (\vec{\sigma} \cdot \hat{p})^2 = 1, \tag{2.127}$$

implying that the operator $(\vec{\sigma} \cdot \hat{p})$ can be used to specify the eigenstates of the the Dirac particles with eigenvalues ± 1. The representation $\vec{\sigma} \cdot \hat{p}$ is called the helicity operator and the Dirac particle states of both positive and negative energy have two states each with eigenvalues ± 1. These are called positive and negative helicity states. In literature, sometimes, the helicity operator is defined in terms of the spin operator $\vec{S}$, where $\vec{S} = \frac{1}{2}\vec{\sigma}$, as $\vec{S} \cdot \hat{p}$. In this case, the eigenvalues of the positive and negative helicity states are $\pm\frac{1}{2}$. In four-dimensional notation, the spin $\vec{\sigma}$ is $\Sigma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} = \gamma^5\gamma^0\gamma^i$, such that $\vec{\Sigma} \cdot \hat{p} = \gamma^5\gamma^0\vec{\gamma}.\hat{p}$. Moreover, the

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four-dimensional representation of the spin operator Σ_k can also be represented in terms of the space components of the antisymmetric tensor ($\sigma^{\mu\nu}$), as shown in Eq. (2.61). In view of the above discussions, the helicity projection operators $h_\pm$ are defined as:

$$h_\pm = \frac{1 \pm \vec{\Sigma} \cdot \vec{p}}{2}. \tag{2.128}$$

2.6 Massless Spin $\frac{1}{2}$ Particle and Weyl Equation

2.6.1 Equation of motion for massless particles

Let us consider the Dirac equation:

$$(\not{p} - m)u(\vec{p}) = 0,$$

where $u(\vec{p})$ is a 4-component spinor $u(\vec{p}) = \begin{pmatrix} u_a(\vec{p}) \\ u_b(\vec{p}) \end{pmatrix}$. Then, $u(\vec{p})$ satisfies Eqs. (2.75) and (2.76) that is,

$$p^0 u_a(\vec{p}) - (\vec{\sigma} \cdot \vec{p})u_b(\vec{p}) = m u_a(\vec{p}), \tag{2.129}$$

$$p^0 u_b(\vec{p}) - (\vec{\sigma} \cdot \vec{p})u_a(\vec{p}) = -m u_b(\vec{p}). \tag{2.130}$$

Adding and subtracting these equations, we obtain

$$(p^0 - \vec{\sigma} \cdot \vec{p})(u_a(\vec{p}) + u_b(\vec{p})) = m(u_a(\vec{p}) - u_b(\vec{p})), \tag{2.131}$$

$$(p^0 + \vec{\sigma} \cdot \vec{p})(u_a(\vec{p}) - u_b(\vec{p})) = m(u_a(\vec{p}) + u_b(\vec{p})). \tag{2.132}$$

Defining two linear combinations of $u_1(\vec{p})$ and $u_2(\vec{p})$ as:

$$u_L(\vec{p}) = \frac{u_a(\vec{p}) - u_b(\vec{p})}{2}, \tag{2.133}$$

$$u_R(\vec{p}) = \frac{u_a(\vec{p}) + u_b(\vec{p})}{2}, \tag{2.134}$$

which satisfy:

$$(p^0 - \vec{\sigma} \cdot \vec{p})u_R(\vec{p}) = m u_L(\vec{p}), \tag{2.135}$$

$$(p^0 + \vec{\sigma} \cdot \vec{p})u_L(\vec{p}) = m u_R(\vec{p}), \tag{2.136}$$

shows that the wave equation for $u_R(\vec{p})$ and $u_L(\vec{p})$ are coupled by the mass term. In the limit $m \to 0$, these equations get decoupled to give:

$$p^0 u_R(\vec{p}) = (\vec{\sigma} \cdot \vec{p})u_R(\vec{p}), \tag{2.137}$$

$$p^0 u_L(\vec{p}) = -(\vec{\sigma} \cdot \vec{p})u_L(\vec{p}). \tag{2.138}$$

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These are known as Weyl equations; they were proposed in 1937 for massless spin $\frac{1}{2}$ particles [40].

We see that these equations are not invariant under parity transformation, which transforms $\vec{x} \to -\vec{x}$, $\vec{\sigma} \to \vec{\sigma}$, and $\vec{p} \to -\vec{p}$, such that $\vec{\sigma} \cdot \vec{p} \to -\vec{\sigma} \cdot \vec{p}$ and $p^0 \to p^0$. Since these equations violate parity, they were not used in particle physics until 1957, until parity violation was discovered in weak interactions.

The solutions of Eqs. (2.137) and (2.138) are obtained in a simple way. Multiplying both sides by p^0, we get:

$$p^{0^2} u_R(\vec{p}) = (\vec{\sigma} \cdot \vec{p}) p^0 u_R(\vec{p}) = (\vec{\sigma} \cdot \vec{p})(\vec{\sigma} \cdot \vec{p}) u_R(\vec{p}). \tag{2.139}$$

$$\text{Similarly,} \quad p^{0^2} u_L(\vec{p}) = (\vec{\sigma} \cdot \vec{p})(\vec{\sigma} \cdot \vec{p}) u_L(\vec{p}), \tag{2.140}$$

implying that both the wave functions $u_R(\vec{p})$ and $u_L(\vec{p})$ satisfy

$$\begin{aligned} (p^{0^2} - |\vec{p}|^2) u_{R(L)}(\vec{p}) &= 0, \\ \Rightarrow p^{0^2} = |\vec{p}|^2 \quad \text{that is} \quad p^0 &= \pm |\vec{p}|. \end{aligned} \tag{2.141}$$

Thus, for the positive energy solutions ($p^0 = +|\vec{p}|$), using Eqs. (2.137) and (2.138), we find:

$$\begin{aligned} p^0 u_R(\vec{p}) &= (\vec{\sigma} \cdot \vec{p}) u_R(\vec{p}), \\ p^0 u_L(\vec{p}) &= -(\vec{\sigma} \cdot \vec{p}) u_L(\vec{p}), \\ \text{or} \quad \frac{\vec{\sigma} \cdot \hat{p} |\vec{p}|}{p^0} u_R(\vec{p}) &= u_R(\vec{p}), \\ \text{Similarly,} \quad \frac{\vec{\sigma} \cdot \hat{p} |\vec{p}|}{p^0} u_L(\vec{p}) &= -u_L(\vec{p}), \end{aligned}$$

$$\Rightarrow (\vec{\sigma} \cdot \hat{p}) u_R(\vec{p}) = u_R(\vec{p}) \qquad \text{(Using (2.141))} \tag{2.142}$$

$$\text{and} \quad (\vec{\sigma} \cdot \hat{p}) u_L(\vec{p}) = -u_L(\vec{p}). \tag{2.143}$$

$\vec{\sigma} \cdot \hat{p}$ is called the chirality operator in the case of massless particles. In the four-dimensional representation, $\vec{\sigma} \cdot \hat{p}$ is represented by $\vec{\Sigma} \cdot \hat{p}$. We have already mentioned that $\vec{\Sigma} \cdot \hat{p}$ is the helicity operator which commutes with the Dirac Hamiltonian and is used to label the spin states as helicity states. The two different Weyl equations, Eqs. (2.142) and (2.143) describe the states with opposite helicity. The negative helicity state particle has its spin aligned in a direction opposite to its momentum. If we visualize the spin of a particle arising due to a circular motion, then the motion of a particle with negative helicity would correspond to the motion of a left-handed screw. Similarly, the motion of a particle with positive helicity would correspond to the motion of a right-handed-screw. Accordingly, the particles are labeled as left-handed with helicity $h = -1$ and right-handed with helicity $h = +1$. The term 'helicity' is also referred to as chirality and is the same in the case of massless particles.

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2.6.2 Equation of motion in Weyl representation

Weyl [40] showed that a massless fermion can be described by a two- component wave function, each component satisfying the fermion's equation of motion:

$$\psi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{2.144}$$

where ϕ_1 and ϕ_2 are the two component spinors. Weyl's choice of gamma matrices were:

$$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{2.145}$$

Using Eq. (2.144) we may write the Dirac equation as

$$\begin{aligned} \left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m \right) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} &= 0, \\ \Longrightarrow \left(i\gamma^o \frac{\partial}{\partial t} + i\vec{\gamma}.\vec{\nabla} - mI \right) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} &= 0, \end{aligned}$$

which results in

$$m\phi_1 = -i\frac{\partial \phi_2}{\partial t} + i\vec{\sigma} \cdot \vec{\nabla}\phi_2 \quad \text{and} \quad m\phi_2 = -i\frac{\partial \phi_1}{\partial t} - i\vec{\sigma} \cdot \vec{\nabla}\phi_1. \tag{2.146}$$

For a massless fermion($m = 0$), like the neutrinos:

$$i\frac{\partial \phi_2}{\partial t} = i\vec{\sigma} \cdot \vec{\nabla}\phi_2 \quad \text{and} \quad i\frac{\partial \phi_1}{\partial t} = -i\vec{\sigma} \cdot \vec{\nabla}\phi_1. \tag{2.147}$$

Notice that the upper and lower components are now decoupled. Using $E \to i\hbar\frac{\partial}{\partial t}$, $\vec{p} \to -i\hbar\vec{\nabla}$ and working in natural units ($\hbar = c = 1$), we get:

$$E\phi_2 = -\vec{\sigma}.\vec{p}\phi_2 \quad \text{and} \quad E\phi_1 = \vec{\sigma}.\vec{p}\phi_1. \tag{2.148}$$

Moreover, for a massless particle, the expectation value of energy and momentum are the same, that is, $\langle E \rangle = \langle |\vec{p}| \rangle$; this implies that for $\psi_R = \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix}$, $\langle \vec{\sigma}.\vec{p} \rangle = +\langle |\vec{p}| \rangle$ and for $\psi_L = \begin{pmatrix} 0 \\ \phi_2 \end{pmatrix}$, $\langle \vec{\sigma}.\vec{p} \rangle = -\langle |\vec{p}| \rangle$. In this expression, ψ_R represents a right-handed neutrino (+ve helicity) and ψ_L represents the left-handed neutrino (−ve helicity).

Defining the projection operators as $\frac{1}{2}(1 \pm \gamma^5)$, leads to:

$$\begin{aligned} \frac{1}{2}(1 + \gamma^5)\psi &= \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix} = \psi_R, \\ \frac{1}{2}(1 - \gamma^5)\psi &= \begin{pmatrix} 0 \\ \phi_2 \end{pmatrix} = \psi_L. \end{aligned}$$

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Thus, one is able to project out a right-handed neutrino spinor using $\frac{1}{2}(1+\gamma^5)$ and a left-handed neutrino spinor using $\frac{1}{2}(1-\gamma^5)$. The different experimental evidences agreed with the assumption that only ψ_L takes part in weak interactions.

2.6.3 Chirality and chirality projection operators

In order to elaborate the concept of chirality, let us consider the four-dimensional representation of the helicity operator

$$\vec{\Sigma}\cdot\vec{p} = \gamma^5\gamma^0\vec{\gamma}\cdot\vec{p}$$

and calculate the quantity

$$(\vec{\Sigma}\cdot\vec{p})u(\vec{p}) = (\gamma^5\gamma^0\vec{\gamma}\cdot\vec{p})u(\vec{p}).$$

Using the Dirac equation for the massless particle,

$$\not{p}u(\vec{p}) = 0 \quad \text{and} \quad p^0 = |\vec{p}|,$$

we get

$$\begin{aligned} (\gamma^5\gamma^0\vec{\gamma}\cdot\vec{p})u(\vec{p}) &= \gamma^5\gamma^0\gamma^0 p^0 u(\vec{p}), \\ \text{that is,}\quad (\vec{\Sigma}\cdot\vec{p})u(\vec{p}) &= \gamma^5|\vec{p}|u(\vec{p}), \\ \Rightarrow (\vec{\Sigma}\cdot\hat{p})u(\vec{p}) &= \gamma^5 u(\vec{p}), \end{aligned} \tag{2.149}$$

showing that γ^5 is the same as the helicity or chirality operator. This implies that:

$$\gamma^5 u_R(\vec{p}) = u_R(\vec{p}) \quad \text{and} \quad \gamma^5 u_L(\vec{p}) = -u_L(\vec{p}), \tag{2.150}$$

$$\Rightarrow \gamma^5(u_R(\vec{p}) + u_L(\vec{p})) = u_R(\vec{p}) - u_L(\vec{p}). \tag{2.151}$$

Writing $u(\vec{p}) = u_L(\vec{p}) + u_R(\vec{p})$, we obtain:

$$u_L(\vec{p}) = \frac{1-\gamma_5}{2}u(\vec{p}), \tag{2.152}$$

$$u_R(\vec{p}) = \frac{1+\gamma_5}{2}u(\vec{p}). \tag{2.153}$$

This has a simple interpretation. If $u(\vec{p})$ is a solution of the massless Weyl equation, that is, $\not{p}u(\vec{p}) = 0$, then $\gamma_5 u(\vec{p})$ is also a solution because $\gamma_5\not{p}u(\vec{p}) = -\not{p}\gamma_5 u(\vec{p}) = 0$. The orthogonal linear combinations of $u(\vec{p})$ and $\gamma_5 u(\vec{p})$, that is, $\frac{1+\gamma_5}{2}u(\vec{p})$ and $\frac{1-\gamma_5}{2}u(\vec{p})$, will also be a solution, which corresponds to definite chirality. We, therefore, define chirality projection operators Λ_L and Λ_R as:

$$\Lambda_L = \frac{1-\gamma_5}{2} \quad \text{and} \quad \Lambda_R = \frac{1+\gamma_5}{2}, \tag{2.154}$$

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which project out the left-handed(L) and right-handed(R) components of the massless particles. Such that:

$$\Lambda_L u_L(\vec{p}) = u_L(\vec{p}), \qquad \Lambda_R u_R(\vec{p}) = u_R(\vec{p}), \tag{2.155}$$

$$\Lambda_L u_R(\vec{p}) = 0, \qquad \Lambda_R u_L(\vec{p}) = 0, \tag{2.156}$$

$$\Lambda_L{}^2 = \Lambda_L, \qquad \Lambda_R{}^2 = \Lambda_R, \tag{2.157}$$

$$\Lambda_L \Lambda_R = \Lambda_R \Lambda_L = 0, \qquad \Lambda_L + \Lambda_R = 1. \tag{2.158}$$

2.7 Relativistic Spin 1 Particles

2.7.1 Massless spin 1 particles

Maxwell's equations for electromagnetic fields describe the wave equation for photons, that is, the massless spin 1 field. They are generally written in terms of the electric field ($\vec{E}(\vec{x},t)$) and magnetic field ($\vec{B}(\vec{x},t)$). In the covariant formulation of Maxwell's equations, they are written in terms of a 4-component vector field $A^\mu(\vec{x},t)(\mu = 0,1,2,3)$ (traditionally called potential). Since a real photon is transverse, it has only two nonzero components of its field $A^\mu(\vec{x},t)$ along the directions perpendicular to the direction of motion. Therefore, the 4-component field $A^\mu(\vec{x},t)$ is subjected to some constraints to eliminate the extra degrees of freedom. We first describe Maxwell's equations in a covariant form, that is, in terms of $A^\mu(\vec{x},t)$ and discuss the constraints on $A^\mu(\vec{x},t)$ to obtain a relativistic equation for massless spin 1 particle. Maxwell's equation for the fields $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ are written in natural units as:

$$\vec{\nabla}.\vec{B} = 0, \qquad \text{(Gauss's law in magnetism)} \tag{2.159}$$

$$\vec{\nabla}.\vec{E} = \rho, \qquad \text{(Gauss's law in electrostatics)} \tag{2.160}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \qquad \text{(Faraday's law)} \tag{2.161}$$

$$\vec{\nabla} \times \vec{B} = \vec{J} + \frac{\partial \vec{E}}{\partial t}, \qquad \text{(Ampere's law with Maxwell's modification),} \tag{2.162}$$

where Eqs. (2.159) and (2.161) are homogeneous differential equations and Eqs. (2.160) and (2.162) are inhomogeneous differential equations. Equation (2.159) implies that $\vec{B}$ is a divergenceless vector; therefore, it can be written as a curl of another vector, that is,

$$\vec{B} = \vec{\nabla} \times \vec{A}, \quad \text{where } \vec{A} \text{ is the vector potential.} \tag{2.163}$$

Using this definition of $\vec{B}$, Eq. (2.161) leads to

$$\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0. \tag{2.164}$$

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Since curl of a gradient is zero, we may write

$$\begin{aligned}\vec{E}+\frac{\partial\vec{A}}{\partial t} &= -\vec{\nabla}\phi,\\ \Rightarrow \qquad \vec{E} &= -\frac{\partial\vec{A}}{\partial t}-\vec{\nabla}\phi,\end{aligned} \tag{2.165}$$

where ϕ is the scalar potential. Using Eqs. (2.164) and (2.165) in Eqs. (2.160) and (2.162) lead to

$$\vec{\nabla}^2\phi+\frac{\partial}{\partial t}(\vec{\nabla}.\vec{A})=-\rho \tag{2.166}$$

$$\text{and}\quad\left(\frac{\partial^2\vec{A}}{\partial t^2}-\vec{\nabla}^2\vec{A}\right)+\vec{\nabla}\left(\vec{\nabla}.\vec{A}+\frac{\partial\phi}{\partial t}\right)=\vec{J}. \tag{2.167}$$

We have now obtained Maxwell's equations for $\vec{E}$ and $\vec{B}$ fields in terms of the vector and scalar potentials $\vec{A}\ (\vec{x},t)$ and $\phi\ (\vec{x},t)$ through the coupled equations. We can decouple them using the freedom due to arbitrariness inherent in defining the vector and scalar potentials $\vec{A}\ (\vec{x},t)$ and $\phi(\vec{x},t)$ through Eqs. (2.163) and (2.165). Note that the vector potential defined in Eq. (2.163), is not unique as we can always add a term expressed in terms of the gradient of a scalar like $\vec{\nabla}\Lambda$, with a vanishing curl, since $\vec{\nabla}\times\vec{\nabla}\Lambda=0$, that is

$$\vec{A}\to\vec{A}'=\vec{A}+\vec{\nabla}\Lambda, \tag{2.168}$$

without changing the magnetic field $\vec{B}$.

This change in $\vec{A}$ through Eq. (2.168) would change $\vec{E}$ defined through Eq. (2.165). However, if we make a simultaneous change in the scalar potential $\phi\ (\vec{x},t)$ such that:

$$\phi\to\phi'=\phi-\frac{\partial\Lambda}{\partial t}, \tag{2.169}$$

where Λ is also a function of x, then

$$\begin{aligned}\vec{E}\to\vec{E}' &= -\vec{\nabla}\phi'-\frac{\partial\vec{A}'}{\partial t}\\ &= -\vec{\nabla}\left(\phi-\frac{\partial\Lambda}{\partial t}\right)-\frac{\partial}{\partial t}(\vec{A}+\vec{\nabla}\Lambda)\\ &= -\vec{\nabla}\phi-\frac{\partial\vec{A}}{\partial t}=\vec{E}\end{aligned} \tag{2.170}$$

remains unchanged. Therefore, $\vec{E}$ and $\vec{B}$ remain unchanged during the simultaneous change of $\phi(\vec{x},t)$, and $\vec{A}(\vec{x},t)$ using a scalar function $\Lambda(\vec{x},t)$ through Eqs. (2.168) and (2.169) for scalar and vector potentials. These are called gauge transformations and the invariance of Maxwell's equations under these transformations is called gauge invariance in electrodynamics. It should

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be noted that these gauge transformations are dependent on the space–time coordinates and are essentially local transformations and not global transformations. Since $\Lambda(\vec{x},t)$ is arbitrary, we are free to choose $\Lambda(\vec{x},t)$. Historically, the two choices made, in describing the classical electrodynamics are the Lorenz gauge [195] and the Coulomb gauge. In the Lorenz gauge, $\Lambda(\vec{x},t)$ is chosen such that:

$$\vec{\nabla}.\vec{A} + \frac{\partial \phi}{\partial t} = 0, \tag{2.171}$$

which decouples Maxwell's equations for $\phi(\vec{x},t)$ and $\vec{A}(\vec{x},t)$ giving us

$$\vec{\nabla}^2\phi - \frac{\partial^2\phi}{\partial t^2} = -\rho \quad \text{and} \tag{2.172}$$

$$\vec{\nabla}^2\vec{A} - \frac{\partial^2\vec{A}}{\partial t^2} = -\vec{J}, \tag{2.173}$$

implying

$$\vec{\nabla}^2\Lambda - \frac{\partial^2\Lambda}{\partial t^2} = 0.$$

In the Coulomb gauge, we choose

$$\vec{\nabla}.\vec{A} = 0, \tag{2.174}$$

giving us

$$\vec{\nabla}^2\phi = -\rho \quad \text{and} \tag{2.175}$$

$$\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}^2\vec{A} + \vec{\nabla}\left(\frac{\partial\phi}{\partial t}\right) = \vec{J}. \tag{2.176}$$

In this gauge, the solution of the equation of motion for $\phi(\vec{x},t)$ gives

$$\phi(\vec{x},t) = \int \rho(x')\frac{e}{|\vec{x}-\vec{x}'|}d\vec{x}', \tag{2.177}$$

which is the reason for calling it the Coulomb gauge. Moreover, because of Eq. (2.174), the Coulomb gauge is also called the transverse or radiation gauge.

In the case of free fields, $\rho = 0$, $\vec{J} = 0$, such that Eq. (2.176) becomes:

$$\Box\vec{A} = 0, \tag{2.178}$$

leading to a solution of the type

$$\vec{A}(\vec{x},t) = \vec{\epsilon}(\vec{k})e^{-ik\cdot x}, \tag{2.179}$$

with $\vec{k} \cdot \vec{\epsilon} = 0$, that is, the photon field $\vec{A}(x)$ is transverse in the Coulomb gauge.

In the covariant notation, the gauge transformations in Eqs. (2.168) and (2.169) are written as:

$$A^{\mu} \to A^{\mu\,\prime} = A^{\mu} - \partial^{\mu}\Lambda \tag{2.180}$$

and the Lorenz condition in Eq. (2.171) is written as:

$$\partial^{\mu} A_{\mu} = 0. \tag{2.181}$$

2.7.2 Covariant form of Maxwell's equations and gauge invariance

Maxwell's equations of electrodynamics are also expressed in the covariant form using an antisymmetric field tensor $F^{\mu\nu}(\vec{x}, t)$ defined as:

$$F^{\mu\nu}(\vec{x}, t) = \frac{\partial A^{\mu}}{\partial x^{\nu}} - \frac{\partial A^{\nu}}{\partial x^{\mu}} = -F^{\nu\mu}(\vec{x}, t), \tag{2.182}$$

where $x^{\mu} = (t, \vec{x})$, $A^{\mu} = (\phi, \vec{A})$, and $A_{\mu} = g_{\mu\nu} A^{\nu}$. In the component form, the field tensor $F^{\mu\nu}(\vec{x}, t)$ is expressed in terms of electric and magnetic fields as:

$$F^{\mu\nu}(\vec{x}, t) \;=\; \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}, \tag{2.183}$$

such that, under gauge transformation:

$$\begin{aligned} F^{\mu\nu}(\vec{x}, t) \to F^{'\mu\nu}(\vec{x}, t) &= \partial^{\mu} A^{'\nu} - \partial^{\nu} A^{'\mu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} \\ \because \quad \delta F^{\mu\nu}(\vec{x}, t) &= F^{'\mu\nu}(\vec{x}, t) - F^{\mu\nu}(\vec{x}, t) = \partial^{\mu} \delta A^{\nu} - \partial^{\nu} \delta A^{\mu} \\ &= -\partial^{\mu}\partial^{\nu}\Lambda(x) + \partial^{\nu}\partial^{\mu}\Lambda(x) \\ &= 0, \end{aligned} \tag{2.184}$$

that is, the field tensor $F^{\mu\nu}(\vec{x}, t)$ remains invariant under the gauge transformation.

We define the four-component electromagnetic current as $J^{\mu} = (\rho, \vec{J})$ and can write the inhomogeneous Maxwell's equations (Eqs. (2.160) and (2.162)) as:

$$\partial_{\mu} F^{\mu\nu}(\vec{x}, t) = J^{\nu}(\vec{x}, t), \tag{2.185}$$

and the homogeneous Maxwell's equations (Eqs. (2.159) and (2.161)) as:

$$\partial^{\mu} F^{\nu\lambda}(\vec{x}, t) + \partial^{\nu} F^{\lambda\mu}(\vec{x}, t) + \partial^{\lambda} F^{\mu\nu}(\vec{x}, t) = 0. \tag{2.186}$$

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It is clear from the covariant formulation of Maxwell's equations, that:

a) Maxwell's equations are gauge invariant because under the gauge transformation

$$A^\mu(\vec{x},t) \longrightarrow A'^\mu(\vec{x},t) = A^\mu(\vec{x},t) - \partial^\mu \Lambda(\vec{x},t), \tag{2.187}$$

$F^{\mu\nu}(\vec{x},t)$ remains unchanged, that is,

$$F^{\mu\nu}(\vec{x},t) \longrightarrow F'^{\mu\nu}(\vec{x},t) = F^{\mu\nu}(\vec{x},t). \tag{2.188}$$

Maxwell's equations, Eqs. (2.185) and (2.186) remain unchanged.

b) The Lorentz covariant and gauge invariant field equations satisfied by A^μ are obtained by using Eqs. (2.183) and (2.185) leading to:

$$\Box A^\nu(\vec{x},t) - \partial^\nu(\partial_\mu A^\mu(\vec{x},t)) = J^\nu(\vec{x},t). \tag{2.189}$$

Therefore, using Eq. (2.185) it can be shown that the electromagnetic current is conserved,

$$\partial_\mu J^\mu(\vec{x},t) = \partial_\mu \partial_\nu F^{\nu\mu}(\vec{x},t) = 0. \tag{2.190}$$

This leads to the conservation of charge.

c) In the Lorenz gauge, that is,

$$\vec{\nabla}.\vec{A}(\vec{x},t) + \frac{\partial\phi(\vec{x},t)}{\partial t} = 0 \tag{2.191}$$

or

$$\partial_\mu A^\mu(\vec{x},t) = 0, \tag{2.192}$$

Eq. (2.189), for the free fields becomes

$$\Box A^\mu(\vec{x},t) = 0, \tag{2.193}$$

implying that the electromagnetic field $A^\mu(\vec{x},t)$ is massless.

This equation can be interpreted as a set of four wave equations for the massless scalar fields corresponding to $\mu = 0,1,2,3$ components of $A^\mu(\vec{x},t)$. However, a massive spin 1 particle has only three components of scalar fields, so that Eq. (2.193) has one additional component of scalar field which needs to be eliminated. Since, a scalar field can be constructed from $A^\mu(\vec{x},t)$, that is, $\partial_\mu A^\mu(\vec{x},t)$; therefore, the constraint $\partial_\mu A^\mu(\vec{x},t) = 0$ is imposed to eliminate one of the scalar fields. This is an alternative physical explanation of the gauge condition given in Eq. (2.192). However, the Lorenz gauge condition does not completely specify the electromagnetic field $A^\mu(\vec{x},t)$ corresponding to the real photons, which have only two components transverse

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to the direction of its propagation. This creates some difficulties in its quantization using covariant formulations (Chapter 3). Therefore, an appropriate basis is chosen to specify the electromagnetic field $A^\mu(\vec{x},t)$, such that $A^\mu(\vec{x},t)$ has only two transverse components and satisfies the condition 2.193.

2.7.3 Plane wave solution of photon

Equation (2.193) with the constraint given in Eq. (2.192) for the photon field admits a solution of the form

$$A^\mu(\vec{x},t) = \frac{1}{\sqrt{V}}\epsilon^\mu(\vec{k})e^{-ik\cdot x} \tag{2.194}$$

$$\text{with } k^2 = 0, \quad \text{that is,} \quad k^0 = \omega_k = |\vec{k}|$$

$$\text{and } k_\mu \cdot \epsilon^\mu(\vec{k}) = 0, \tag{2.195}$$

where $\epsilon^\mu(\vec{k})$ is the four-vector describing the spin or polarization states of the photon. Therefore, the four-field $A^\mu(\vec{x},t)$ described in terms of the polarization four-vector $\epsilon^\mu(\vec{k})$, can be expressed in terms of a set of four independent orthogonal polarization vectors ϵ_r^μ with ($\mu = 0,1,2,3$) components just like an ordinary vector $\vec{r}$, is expressed in terms of three independent ordinary vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ along the X,Y,Z axes. Expanding the field $A^\mu(x)$ in its Fourier components in momentum space, we write $A^\mu(\vec{x},t)$ with the normalization factor as

$$A^\mu(\vec{x},t) = \sum_{r,k}\frac{1}{\sqrt{2V\omega_k}}\epsilon_r^\mu(\vec{k})\left(a_r(\vec{k})e^{-ik\cdot x} + a_r^*(\vec{k})e^{ik\cdot x}\right). \tag{2.196}$$

A massless photon vector field of spin 1 described by $\vec{A}(\vec{x},t)$ is a transverse field with two polarization states which are perpendicular to the direction of propagation. In contrast, the vector field $\vec{A}(\vec{x},t)$, with mass $m \neq 0$, is described by three polarization states including the longitudinal component. In the Lorenz gauge description of $A^\mu(x)$, there are four polarization states ϵ_r^μ of with constraints in ϵ_r^μ, that is

$$\sum_r k.\epsilon_r(\vec{k}) = 0. \tag{2.197}$$

We take a photon moving in the z direction such that $k^\mu = (k^0,0,0,\vec{k})$ and define the polarization vector $\epsilon_r^\mu(\vec{k})$ as:

$$\epsilon_r^\mu(\vec{k}) = (0,\vec{\epsilon}_r(\vec{k})) \qquad r = 1,2,3, \tag{2.198}$$

where $\vec{\epsilon}_1(\vec{k})$ and $\vec{\epsilon}_2(\vec{k})$ are orthogonal to each other and also to $\vec{k}$, and

$$\epsilon_3^\mu(\vec{k}) = (0,\hat{k}), \text{ such that } \vec{\epsilon}_r(\vec{k})\cdot\vec{\epsilon}_s(\vec{k}) = \delta_{rs}. \quad \vec{k}\cdot\vec{\epsilon}_{r(s)} = 0; \; r,s = 1,2 \tag{2.199}$$

$\vec{\epsilon}_1(\vec{k})$ and $\vec{\epsilon}_2(\vec{k})$ are the transverse polarizations and $\vec{\epsilon}_3(\vec{k})$ is the longitudinal polarization of the photon. We also define a time-like vector, known as the scalar polarization $\epsilon_0^\mu(\vec{k})$ as:

$$\epsilon_0^\mu(\vec{k}) = n^\mu = (1,0,0,0), \tag{2.200}$$

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such that:

$$\epsilon_3^\mu(\vec{k}) = \frac{k^\mu - k \cdot n\, n^\mu}{\sqrt{(k \cdot n)^2 - k^2}}. \tag{2.201}$$

An explicit representation of $\epsilon_r^\mu(\vec{k})$ can be written as:

$$\begin{aligned} \epsilon_0^\mu(\vec{k}) &= (1,0,0,0), \quad \epsilon_1^\mu(\vec{k}) = (0,1,0,0), \\ \epsilon_2^\mu(\vec{k}) &= (0,0,1,0), \quad \epsilon_3^\mu(\vec{k}) = (0,0,0,1). \end{aligned} \tag{2.202}$$

We see that the orthonormalization conditions in covariant form are written as:

$$\epsilon_{r\mu}(\vec{k}) \cdot \epsilon_{r'}^\mu(\vec{k}) = -\zeta_r \delta_{rr'}, \tag{2.203}$$

$$\sum_r \zeta_r \epsilon_r^\mu(\vec{k}) \epsilon_r^\nu(\vec{k}) = -g^{\mu\nu}, \tag{2.204}$$

$$\text{where} \qquad \zeta_0 = -1 \quad \text{and} \quad \zeta_1 = \zeta_2 = \zeta_3 = +1. \tag{2.205}$$

2.7.4 Massive spin 1 particles

In the last section, we have developed the wave equation for the massless particle of spin 1, starting from Maxwell's equation. The general formalism of the wave equation for particle fields with spin $J \geq 1$ was first given by Dirac in 1936 [196], about 8 years after his paper on the relativistic equation for spin $\frac{1}{2}$ particles was published. The formulation of the wave equation for higher spin fields was also developed by Fierz and Pauli [197], Bargmann and Wigner [198], based on the principle that the wave equations for fields with $J \geq 1$ can be developed using the basic concepts of wave equations for spin $\frac{1}{2}$. In the specific case of spin 1 fields, it was applied by Duffin [199] and Kemmer [200] and Proca [201]. However, for the sake of simplicity and elucidation, we will rely on the analogy of the massless spin 1 photon with the equations of motion to extend it to $m \neq 0$ fields.

Massive spin 1 particles have three components of spin direction corresponding to the spin projections $S_z = \pm 1$ and 0. While a nonrelativistic spin 1 particle is described by a field $\vec{A}(\vec{x}, t)$ having three components corresponding to the spin directions, the relativistic formulation describes it in terms of a 4-vector field $A^\mu(\vec{x}, t)$ with four components subjected to subsidiary conditions to eliminate the fourth component. Then, all the components satisfy a wave equation similar to the Klein–Gordon equation for massive particles. The simplest way to visualize such an equation of motion for a free field is to replace the operator $\partial_\mu \partial^\mu$ in Eq. (2.189) with the operator $\partial_\mu \partial^\mu + M^2$, like the operator in the Klein–Gordon equation (Eq. (2.21)), and write the wave equation for the massive spin 1 field $W^\mu(x)$ as

$$(\Box + M^2) W^\mu(\vec{x}, t) - \partial^\mu \partial_\nu W^\nu(\vec{x}, t) = J^\mu(\vec{x}, t), \tag{2.206}$$

where $J^\mu(\vec{x}, t)$ represents a vector current coupled to $W^\mu(\vec{x}, t)$. This is called the Proca equation. In case of free fields, where $J^\mu(\vec{x}, t) = 0$, we get:

$$(\Box + M^2) W^\mu(\vec{x}, t) - \partial^\mu \partial \cdot W(\vec{x}, t) = 0. \tag{2.207}$$

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Taking divergence of Eq. (2.207), we get:

$$(\Box + M^2)\partial_\mu W^\mu(\vec{x},t) - \Box\partial_\nu W^\nu(\vec{x},t) = 0, \tag{2.208}$$

which gives the condition

$$M^2\partial_\mu W^\mu(\vec{x},t) = 0,$$

$$\text{However, } M \neq 0, \Rightarrow \partial_\mu W^\mu(\vec{x},t) = 0, \tag{2.209}$$

which is the condition to remove the fourth component as discussed in Section 2.7.1. Therefore, the equation for particle field $W^\mu(\vec{x},t)$ of spin 1, with mass M is written as

$$(\Box + M^2)W^\mu(\vec{x},t) \quad = \quad 0, \tag{2.210}$$

Equation (2.210) is the Proca equation for free fields. There is no further freedom due to gauge invariance because under a gauge transformation,

$$W^\mu(\vec{x},t) \to W^\mu(\vec{x},t) + \partial^\mu\chi(\vec{x},t). \tag{2.211}$$

Thus, Eq. (2.210) becomes

$$(\Box + M^2)W^\mu(\vec{x},t) + M^2\partial^\mu\chi(\vec{x},t) = 0 \quad \text{with} \quad \vec{\nabla}^2\chi = 0, \tag{2.212}$$

which is not the same as Eq. (2.210) due to the mass dependent term unless $\partial^\mu\chi = 0$ in Eq. (2.211) implying no gauge transformation. Therefore, the freedom to chose a gauge such that the degree of freedom of $W^\mu(\vec{x},t)$ is further reduced like the massless field $A^\mu(\vec{x},t)$, is not available. Consequently, there are three degrees of freedom for $W^\mu(\vec{x},t)$ fields. In case of the wave equations for the spin 1 field in the presence of external currents $J^\mu(x)$, the condition $\partial_\mu W^\mu(x) = 0$ is satisfied if the field is coupled to conserved currents, that is, $\partial_\mu J^\mu(x) = 0$. This can be shown after taking the divergence of the Proca equation in Eq. (2.206), such that it becomes

$$(\Box + M^2)\partial \cdot W(\vec{x},t) - \Box\partial \cdot W(\vec{x},t) = \partial_\mu J^\mu(\vec{x},t). \tag{2.213}$$

Therefore, in the case of a massive vector field $W^\mu(\vec{x},t)$, if it is coupled to a conserved current, that is, $\partial_\mu J^\mu = 0$, then $\partial_\mu W^\mu(\vec{x},t) = 0$. In analogy with the massless spin 1 field $A^\mu(\vec{x},t)$, the solution of Eq. (2.212) is written as

$$W^\mu(\vec{x},t) = \frac{1}{\sqrt{V}}\epsilon^\mu e^{-ik\cdot x}, \tag{2.214}$$

where ϵ^μ is the polarization 4-vector representing the spin states of the particle of spin 1, with $k^2 = M^2$ and $\epsilon \cdot k = 0$; the 4-vector can be expressed in terms of the four independent vectors ϵ_r^μ $(r = 0,1,2,3)$, as in the case of massless spin 1 particles, called polarization vectors. Because of the transversality condition $\varepsilon \cdot k = 0$, only three are independent and can be chosen to describe the polarization states of spin 1 particle, in terms of the transverse and longitudinal components of polarization. In the rest frame of the particle, the polarization states ϵ_r^μ $(r = 1, 2, 3)$ can be chosen to be along the X, Y, and Z directions, that is,

$$\epsilon_1^\mu = (0, 1, 0, 0); \quad \epsilon_2^\mu = (0, 0, 1, 0); \quad \epsilon_3^\mu = (0, 0, 0, 1), \tag{2.215}$$

such that

$$\epsilon_r^\mu \epsilon_{s\mu} = -\delta_{rs} \tag{2.216}$$

$$\text{and } P_{\mu\nu} = \sum_{\lambda=1}^{3} \epsilon_\mu^\lambda \epsilon_\nu^\lambda \tag{2.217}$$

has components as

$$P_{00} = 0,\ P_{11} = P_{22} = P_{33} = 1 \ \text{ and } P_{\mu\nu} = 0 \text{ for } \mu \neq \nu. \tag{2.218}$$

In a frame, where the particle is moving with momentum $\vec{k}$ in the Z direction, we get

$$\begin{aligned} k^\mu &= (k^0, 0, 0, \vec{k}), \\ \epsilon_1^\mu &= (0, 1, 0, 0), \quad \epsilon_2^\mu = (0, 0, 1, 0), \ \text{ and } \epsilon_3^\mu = \left(\frac{k}{m}, 0, 0, \frac{\omega}{m}\hat{k}\right). \end{aligned} \tag{2.219}$$

Therefore, $P_{\mu\nu}$ can be derived to be

$$P_{\mu\nu} = \frac{1}{m^2} \begin{pmatrix} |\vec{k}|^2 & 0 & 0 & -k^0|\vec{k}| \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -k^0|\vec{k}| & 0 & 0 & (k^0)^2 \end{pmatrix}, \tag{2.220}$$

either by actual substitution of the components of ϵ_r^μ given in Eq. (2.219) or by using a Lorenz transformation on P_{00}, using the transformation

$$P_{\mu\nu} = \Lambda_\mu^\rho \, \Lambda_\nu^\eta \, P_{\rho\eta}, \tag{2.221}$$

where $\Lambda_\mu^\rho(k)$ is a Lorentz transformation describing a boost along the Z-axis, described by the parameter $\gamma = \frac{\omega_k}{m}$ and $\gamma\beta = \frac{k}{m}$. It can be verified that the general form of $P_{\mu\nu}(k)$ will be given by

$$\sum_\lambda \epsilon_\mu^\lambda(k) \epsilon_\nu^\lambda(k) = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}. \tag{2.222}$$

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2.8 Wave Equation for Particle with Spin $\frac{3}{2}$

The relativistic wave equation for spin $\frac{3}{2}$ particles can be formulated using a 3-spinor description [196, 198] or a vector–spinor description [202] or a description based on a direct higher order representation of $\left(\frac{3}{2}\, 0\right) + \left(0\, \frac{3}{2}\right)$ under the Lorentz group [203, 204]. The equations of motion for the higher spin particles were also discussed by Bhabha [205] and Harish-Chandra [206].

Without going into details of these formulation, we use the vector–spinor description of spin $\frac{3}{2}$ given by Rarita and Schwinger [202] to discuss the relativistic wave equation of spin $\frac{3}{2}$ particles. In this theory, the fundamental quantity is $\psi^{\mu}_{\alpha}(\vec{x},t)$, which has the mixed transformation properties of a Lorentz vector denoted by μ and a Dirac spinor denoted by α. The proposed equation of motion for the free spin $\frac{3}{2}$ particle is

$$(i\gamma^{\nu}\partial_{\nu} - m)\,\psi^{\mu}_{\alpha}(\vec{x},t) = 0. \tag{2.223}$$

The main difficulty in this approach (as well as in other approaches based on multi-spinor description) is that $\psi^{\mu}_{\alpha}(\vec{x},t)$ includes the fields corresponding to lower spin particles, that is, spin $\frac{1}{2}$ in this case. For example, in the rest frame, the spin decomposition of spin $\frac{3}{2}$ fields in the vector–spinor formalism is given by

$$\left(\frac{1}{2} \oplus \frac{1}{2}\right) + \frac{1}{2} = (1 \oplus 0) \oplus \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}, \tag{2.224}$$

which has two spin $\frac{1}{2}$ components in addition to the physical spin $\frac{3}{2}$ particle. Therefore, the Rarita– Schwinger (RS) spinor $\psi^{\mu}_{\alpha}(\vec{x},t)$ is subjected to additional constraints on $\psi^{\mu}_{\alpha}(\vec{x},t)$ to eliminate the additional degrees of freedom arising due to the presence of spin $\frac{1}{2}$ particles. Since the Dirac spinor corresponding to spin $\frac{1}{2}$ is a Lorentz scalar ψ in the Minkowski space, we have to construct the Lorentz scalar from $\psi^{\mu}_{\alpha}(\vec{x},t)$ and constrain them to vanish as we have done in the case of spin 1 field, that is, $\partial_{\mu}A^{\mu}(\vec{x},t) = 0$. This is the only scalar quantity we can construct from the vector fields $A^{\mu}(x)$ or $W^{\mu}(x)$. In the present case, we can construct two scalars, that is, $\gamma_{\mu}\psi^{\mu}_{\alpha}(\vec{x},t)$ and $p_{\mu}\psi^{\mu}_{\alpha}(\vec{x},t)$; therefore, two subsidiary conditions are imposed on the RS spinor $\psi^{\alpha}_{\mu}(\vec{x},t)$, that is,

$$\gamma_{\mu}\psi^{\mu}_{\alpha}(\vec{x},t) = 0, \tag{2.225}$$

$$p_{\mu}\psi^{\mu}_{\alpha}(\vec{x},t) = 0. \tag{2.226}$$

However, the nature of the RS equation of motion is such that if Eq. (2.225) is assumed then Eq. (2.226) follows. In case of massless spin $\frac{3}{2}$ particles, the RS spinor $\psi^{\mu}_{\alpha}(\vec{x},t)$ satisfies a gauge condition

$$\psi^{\mu}_{\alpha}(\vec{x},t) \rightarrow \psi^{\mu}_{\alpha'}(\vec{x},t) = \psi^{\mu}_{\alpha}(\vec{x},t) + \partial^{\mu}\phi_{\alpha}(\vec{x},t), \tag{2.227}$$

where ϕ_{α} is the arbitrary function, which satisfies the relation:

$$\gamma^{\mu}\partial_{\mu}\phi_{\alpha}(\vec{x},t) = 0. \tag{2.228}$$

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Equation (2.226) implies that in the rest frame, $\psi^0(\vec{x},t) = 0$ and $\psi^\mu(\vec{x},t)$ is represented by the wave function of a vector particle $\vec{\psi}(\vec{x},t)$ in Lorentz space with 3-components ($\mu = 1,2,3$). Keeping in mind this simple interpretation of $\psi^\mu_\alpha(\vec{x},t)$ at rest, we can write a solution of Eq. (2.223) in the form

$$\psi^\mu_\alpha(\vec{x},t) = N u^\mu_\alpha(\vec{p}) e^{-ip\cdot x} \tag{2.229}$$

with $p^2 = m^2$, and an appropriate normalization constant N. The $u^\mu_\alpha(\vec{p})$ is then represented as (with α replaced by S_Δ for convenience)

$$\begin{aligned} u^\mu(\vec{p},S_\Delta) &= [\epsilon^\mu(\vec{p},\lambda)\otimes u(\vec{p},s)]^{\frac{3}{2}}_{S_\Delta} && (2.230)\\ &= \sum_{\lambda,s}\left(\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{3}{2} \\ \lambda & s & S_\Delta \end{array}\right)\epsilon^\mu(\vec{p},\lambda)\, u(\vec{p},s), && (2.231) \end{aligned}$$

where $u(\vec{p},s)$ and $\epsilon^\mu(\vec{p},\lambda)$ are spin $\frac{1}{2}$ and spin 1 polarization vectors, respectively. Evaluating the Clebsch–Gordan coefficients, for example,

$$u^\mu(\vec{p},3/2) = \begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 1 & \frac{1}{2} & \frac{3}{2} \end{pmatrix}\epsilon^\mu(\vec{p},1)\, u(\vec{p},1/2) = \epsilon^\mu(\vec{p},1)\, u(\vec{p},1/2), \tag{2.232}$$

$$u^\mu(\vec{p},-3/2) = \begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ -1 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}\epsilon^\mu(\vec{p},-1)u(\vec{p},-1/2) = \epsilon^\mu(\vec{p},-1)u(\vec{p},-1/2), \tag{2.233}$$

leads to the following explicit form of the spinors

$$\begin{aligned} u^\mu(\vec{p},\pm 3/2) &= \epsilon^\mu(\vec{p},\pm 1)\, u(\vec{p},\pm 1/2), && (2.234)\\ u^\mu(\vec{p},\pm 1/2) &= \sqrt{\frac{2}{3}}\,\epsilon^\mu(\vec{p},0)\, u(\vec{p},\pm 1/2) + \sqrt{\frac{1}{3}}\,\epsilon^\mu(\vec{p},\pm 1)\, u(\vec{p},\mp 1/2). && (2.235) \end{aligned}$$

The Rarita–Schwinger spinor for a spin $\frac{3}{2}$ particle is written down in the following form [202]

$$u^\mu(\vec{p},s) = \sqrt{\frac{E_\Delta + M_\Delta}{2M_\Delta}}\begin{pmatrix} \mathbf{I} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_\Delta + M_\Delta} \end{pmatrix} S^\mu_{\Delta N}\,\chi_s, \tag{2.236}$$

where χ_s is the four-components spin states for spin $\frac{3}{2}$ particle:

$$\chi_{+\frac{3}{2}} = \begin{pmatrix} 1\\0\\0\\0 \end{pmatrix},\ \chi_{+\frac{1}{2}} = \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}\ \chi_{-\frac{1}{2}} = \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}\ \chi_{-\frac{3}{2}} = \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}$$

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and $S^{\mu}_{\Delta N}$ is the four-components coupling matrices containing the Clebsch–Gordan coefficients for the coupling $1 \otimes \frac{1}{2} = \frac{3}{2}$:

$$S^0 = \frac{\vec{p}}{M_\Delta}\begin{pmatrix} 0 & \sqrt{2/3} & 0 & 0 \\ 0 & 0 & \sqrt{2/3} & 0 \end{pmatrix}, \quad S^1 = \begin{pmatrix} -\sqrt{1/2} & 0 & \sqrt{1/6} & 0 \\ 0 & -\sqrt{1/6} & 0 & \sqrt{1/2} \end{pmatrix},$$
$$S^2 = i\begin{pmatrix} \sqrt{1/2} & 0 & \sqrt{1/6} & 0 \\ 0 & \sqrt{1/6} & 0 & \sqrt{1/2} \end{pmatrix}, \quad S^3 = \frac{E_\Delta}{\vec{p}} S_0. \tag{2.237}$$

2.9 Discrete Symmetry: Parity, Time Reversal, and Charge Conjugation

The parity ($\hat{P}$), time reversal ($\hat{T}$) and charge conjugation ($\hat{C}$) operators, are the examples of the discrete symmetries and one can not describe them using infinitesimal transformations. The parity operator performs a reflection of the space coordinates about the origin ($\vec{r} \to -\vec{r}$) leading to a change in velocity ($\vec{v}$), momentum ($\vec{p}$) of the particle while the orbital angular momentum ($\vec{L} = \vec{r} \times \vec{p}$) and spin angular momentum ($\vec{S}$) do not change. Time reversal operator operates on the time coordinates of the system and changes velocity, momentum and angular momentum. Charge conjugation operator transforms a particle into an antiparticle in the same state such that the momentum, position etc. are unchanged, while charge, magnetic moment etc., change sign. Then we have composed symmetries like $\hat{C}\hat{P}$ and $\hat{C}\hat{P}\hat{T}$. $\hat{C}\hat{P}\hat{T}$ is conserved in all the four basic interactions and it leads to the mass, lifetime and the magnitude of the magnetic moment of particles and antiparticles to be the same. $\hat{C}$, $\hat{P}$, $\hat{T}$ and $\hat{C}\hat{P}$ are conserved in strong and electromagnetic interactions, while they are violated in the weak interaction processes. In the next subsection, we shall discuss $\hat{P}$, $\hat{T}$ and $\hat{C}$ in some detail.

2.9.1 Parity

The parity operator consists of inversion (Figure 2.2) of all three space components ($x, y, z \to -x, -y, -z$) such that

$$\hat{P}\psi(\vec{r}, t) = \eta\psi(-\vec{r}, t), \tag{2.238}$$

where $\psi(\vec{r}, t)$ is a scalar wave function and η is the phase factor, and the repetition of $\hat{P}$ operation gives back the same initial state i.e.

$$\hat{P}^2\psi(\vec{r}, t) = \eta\hat{P}\psi(-\vec{r}, t) = \eta^2\psi(\vec{r}, t) = \psi(\vec{r}, t), \tag{2.239}$$

implies that $\eta = \pm 1$, where $\eta = +1$ is known as the even parity state and $\eta = -1$ is known as the odd parity state.

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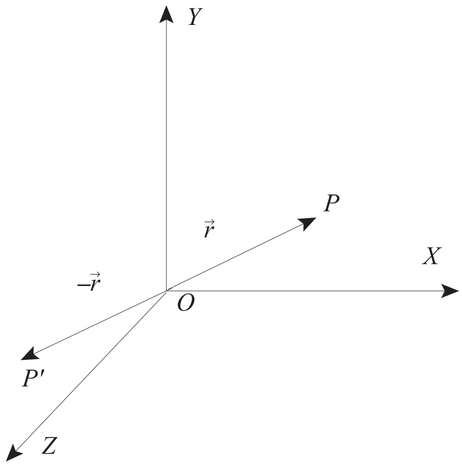

Figure 2.2 Mirror reflection resulting $\vec{r} \to -\vec{r}$ while $|\vec{r}|$ remains invariant.

Under the parity transformation the position vector $\vec{r} \xrightarrow{\hat{P}} -\vec{r}$ and velocity $\vec{v} \xrightarrow{\hat{P}} -\vec{v}$. The Maxwell's equations of motion are invariant under $\hat{P}$ as $\vec{E} \to -\vec{E}$, $\vec{B} \to \vec{B}$ and $\vec{\nabla} \to -\vec{\nabla}$ resulting:

$$\begin{aligned}
\vec{\nabla}.\vec{B}(\vec{r},t) &= 0 \\
\vec{\nabla}.\vec{E}(\vec{r},t) &= \frac{\rho}{\epsilon_0}; \\
\vec{\nabla} \times \vec{E}(\vec{r},t) &= -\frac{\partial \vec{B}(\vec{r},t)}{\partial t} \\
\vec{\nabla} \times \vec{B}(\vec{r},t) &= \frac{\partial \vec{E}(\vec{r},t)}{\partial t} + \vec{J}(\vec{r},t).
\end{aligned}$$

When we talk about parity of a particle or a system, there could be intrinsic parity and/or orbital parity. Here we shall discuss intrinsic parity. For example, consider a particle moving with momentum $\vec{p}$, its momentum eigenfunction is given by:

$$\psi_{\vec{p}}(\vec{r},t) = e^{-i(Et-\vec{p}.\vec{r})} \tag{2.240}$$

Under parity operation

$$\psi_{\vec{p}}(\vec{r},t) \xrightarrow{\hat{P}} \eta\psi_{\vec{p}}(-\vec{r},t) = \eta\psi_{-\vec{p}}(\vec{r},t), \tag{2.241}$$

such that if the particle is at rest ($\vec{p} = 0$), $\psi_{\vec{p}}(\vec{r},t)$ is an eigenstate of $\hat{P}$ with the eigenvalue η. The parity quantum number presented by such particle is known as the intrinsic parity which is nothing but the parity presented by the particle at rest.

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2.9.2 Dirac equation under parity transformation

The Dirac equation for a spin $\frac{1}{2}$ particle with mass m is expressed as

$$(i\gamma^\mu \partial_\mu - m)\psi(r) = 0, \tag{2.242}$$

where $\psi(r)$ is the Dirac spinor for a free particle given by

$$\psi(r) = N \begin{pmatrix} I \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+M} \end{pmatrix} \chi e^{-ip\cdot r}, \tag{2.243}$$

where χ is the upper component and $\frac{\vec{\sigma}\cdot\vec{p}}{E+M}\chi$ is the lower component. The upper and lower components have opposite parities under parity transformation, i.e.

$$u(\vec{r},t) \xrightarrow{\hat{P}} \eta_\psi u(-\vec{r},t), \tag{2.244}$$

$$v(\vec{r},t) \xrightarrow{\hat{P}} -\eta_\psi v(-\vec{r},t), \tag{2.245}$$

where $u(\vec{r},t)$ and $v(\vec{r},t)$ are the upper and lower components of the Dirac field.

Therefore, while defining the parity operation for a Dirac operator, it acts on $\vec{r}$ as well as on $\psi(\vec{r},t)$. Since $\psi(\vec{r},t)$ is a four component spinor i.e. ψ_β, represented by a column vector; P would be a 4×4 matrix operator such that the operation of P on $\psi(\vec{r},t)$ is written as

$$\psi(\vec{r},t) \xrightarrow{\hat{P}} \psi^P(-\vec{r},t) = \eta_\psi P_{\alpha\beta}\psi_\beta(-\vec{r},t). \tag{2.246}$$

We know that operating parity twice, we get the original spinor, thus

$$\eta_\psi = \pm 1 \qquad \text{and} \qquad P^2 = I.$$

One of the possible representation of $P_{\alpha\beta}$ is γ_0.

We have discussed in Appendix-A, how the Dirac equation is invariant under Lorentz transformation resulting the identity

$$\hat{S}(\hat{\Lambda})\gamma^\nu \hat{S}^{-1}(\hat{\Lambda}) = \Lambda^\nu_\mu \gamma^\mu, \tag{2.247}$$

where Λ^ν_μ is the transformation matrix, $S(\hat{\Lambda})$ is a 4×4 matrix and γ^μ is the Dirac γ-matrix.

For the parity operation, since $t,x,y,z \to t,-x,-y,-z$, resulting the transformation matrix as

$$\Lambda^\nu_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g^{\nu\mu}, \tag{2.248}$$

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and the parity operator satisfy the following constrain

$$\begin{aligned}
\Lambda^{\nu}_{\mu}\gamma^{\mu} &= \hat{P}\gamma^{\nu}\hat{P}^{-1} \qquad (2.249)\\
\Lambda^{\sigma}_{\nu}\Lambda^{\nu}_{\mu}\gamma^{\mu} &= \hat{P}\Lambda^{\sigma}_{\nu}\gamma^{\nu}\hat{P}^{-1}\\
\delta^{\sigma}_{\mu}\gamma^{\mu} &= \hat{P}\sum_{\nu=0}^{3} g^{\sigma\nu}\gamma^{\nu}\hat{P}^{-1}\\
\hat{P}^{-1}\gamma^{\sigma}\hat{P} &= g^{\sigma\sigma}\gamma^{\sigma}, \qquad (2.250)
\end{aligned}$$

where $g^{\sigma\sigma}$ shows that only diagonal elements contribute. One choice of $\hat{P}$ could be $e^{i\phi}\gamma^{o}$, where ϕ is some unobservable arbitrary phase, such that:

$$\begin{aligned}
\hat{P}^{-1} &= e^{-i\phi}\gamma^{0} \qquad (2.251)\\
\hat{P} &= e^{i\phi}\gamma^{0}\\
(\hat{P})^{4} &= 1 = (e^{i\phi})^{4}\\
\hat{P}^{-1} &= e^{-i\phi}\gamma^{0} = \hat{P}^{\dagger}, \qquad (2.252)
\end{aligned}$$

i.e. $\hat{P}$ is a unitary operator. Also

$$\begin{aligned}
\hat{P}^{-1} &= \gamma^{0}\hat{P}^{\dagger}\gamma^{0}, \qquad (2.253)\\
\psi'(\vec{x}',t) &= e^{i\phi}\gamma^{0}\psi(\vec{x},t). \qquad (2.254)
\end{aligned}$$

Scalar (1):

$$\begin{aligned}
\overline{\psi}'(x')\psi'(x') &= \psi'^{\dagger}(x')\gamma^{o}\psi'(x')\\
&= (P\psi(x))^{\dagger}\gamma^{0}(P\psi(x)); \qquad P=\gamma^{0}\\
&= \overline{\psi}(x)\psi(x) \qquad (2.255)
\end{aligned}$$

Pseudoscalar (γ_5):

$$\begin{aligned}
\overline{\psi}'(x')\gamma^{5}\psi'(x') &= \psi'^{\dagger}(x')\gamma^{o}\gamma^{5}\psi'(x')\\
&= -\overline{\psi}(x)\gamma^{5}\psi(x) \qquad (2.256)
\end{aligned}$$

Vector (γ^{μ}):

$$\begin{aligned}
\overline{\psi}'(x')\gamma^{\mu}\psi'(x') &= \psi'^{\dagger}(x')\gamma^{o}\gamma^{\mu}\psi'(x')\\
&= \overline{\psi}(x)\gamma_{\mu}\psi(x). \qquad (2.257)
\end{aligned}$$

Axial vector ($\gamma^{\mu}\gamma_5$):

$$\begin{aligned}
\bar{\psi}(\vec{r},t)\gamma^{\mu}\gamma_5\psi(\vec{r},t) &\xrightarrow{\hat{P}} \bar{\psi}^{P}(\vec{r},t)\gamma^{\mu}\gamma_5\psi^{P}(\vec{r},t)\\
&= |\eta_{\psi}|^{2}\bar{\psi}(-\vec{r},t)\gamma_0\gamma^{\mu}\gamma_5\gamma_0\psi(-\vec{r},t)\\
&= -\bar{\psi}(-\vec{r},t)\gamma_{\mu}\gamma_5\psi(-\vec{r},t). \qquad (2.258)
\end{aligned}$$

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Tensor ($\sigma^{\mu\nu}$): In the case of tensor interactions, we see the transformation under parity in the different components viz. σ^{0i} and σ^{ij}.

$$\begin{aligned}\bar{\psi}(\vec{r},t)\sigma^{0i}\psi(\vec{r},t) &\xrightarrow{\hat{P}} \bar{\psi}^P(\vec{r},t)\sigma^{0i}\psi^P(\vec{r},t)\\ &= |\eta_\psi|^2\bar{\psi}(-\vec{r},t)\gamma_0\,\frac{i}{2}\left(\gamma^0\gamma^i-\gamma^i\gamma^0\right)\gamma_0\psi(-\vec{r},t)\\ &= -\bar{\psi}(-\vec{r},t)\sigma^{0i}\psi(-\vec{r},t).\end{aligned} \tag{2.259}$$

Next we see the parity transformation on σ^{ij}:

$$\begin{aligned}\bar{\psi}(\vec{r},t)\sigma^{ij}\psi(\vec{r},t) &\xrightarrow{\hat{P}} \bar{\psi}^P(\vec{r},t)\sigma^{ij}\psi^P(\vec{r},t)\\ &= |\eta_\psi|^2\bar{\psi}(-\vec{r},t)\gamma_0\,\frac{i}{2}\left(\gamma^i\gamma^j-\gamma^j\gamma^i\right)\gamma_0\psi(-\vec{r},t)\\ &= \frac{i}{2}\bar{\psi}(-\vec{r},t)\left(\gamma^i\gamma^j-\gamma^j\gamma^i\right)\psi(-\vec{r},t)\\ &= \bar{\psi}(-\vec{r},t)\sigma^{ij}\psi(-\vec{r},t).\end{aligned} \tag{2.260}$$

Combining Eqs. (2.259) and (2.260), we find

$$\bar{\psi}(\vec{r},t)\sigma^{\mu\nu}\psi(\vec{r},t) \xrightarrow{\hat{P}} -\bar{\psi}(-\vec{r},t)\sigma_{\mu\nu}\psi(-\vec{r},t). \tag{2.261}$$

Therefore, under parity transformation, scalar and vector interactions are invariant, while pseudoscalar and axial vector change sign.

2.9.3 Charge conjugation

Charge conjugation ($\hat{C}$) turns a particle into an antiparticle

$$\hat{C}\psi(\text{particle}) \longrightarrow \psi(\text{antiparticle}) = \eta_C\psi^C(r), \tag{2.262}$$

in the same state leaving position, momentum and angular momentum unchanged. If a set of particles is obeying certain physical law, and the set of corresponding antiparticles, if also obey the same law then the law is said to be invariant under the charge conjugation.

Its effect on electric field $\vec{E}$ and magnetic field $\vec{B}$ is the following:

$$\begin{aligned}\vec{E} &= \frac{1}{4\pi\epsilon_0}\Big(\frac{q}{|\vec{r}|^3}\Big)\vec{r} \xrightarrow{\hat{C}} -\vec{E},\\ \vec{B} &= \frac{\mu_0}{4\pi}\Big(\frac{Id\vec{l}\times\vec{r}}{|\vec{r}|^3}\Big) \xrightarrow{\hat{C}} -\vec{B},\\ \vec{\nabla}.\vec{E} &= 4\pi\rho \xrightarrow{\hat{C}} -\vec{\nabla}.\vec{E} \quad\Rightarrow\quad \rho \xrightarrow{\hat{C}} -\rho,\\ \vec{\nabla}\times\vec{B} &= \frac{\partial\vec{E}}{\partial t}+\vec{J} \quad\Rightarrow\ \vec{J} \xrightarrow{\hat{C}} -\vec{J}.\end{aligned}$$

Writing $\vec{E}$ and $\vec{B}$ in terms of potential A_μ and four current density J_μ

$$\begin{aligned}
\vec{E} &= -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \\
\vec{B} &= -\vec{\nabla} \times \vec{A}, \quad \text{we may see that under } \hat{C} \\
A_\mu &\xrightarrow{\hat{C}} -A_\mu \\
\text{and} \qquad J_\mu &\xrightarrow{\hat{C}} -J_\mu
\end{aligned}$$

Strong and electromagnetic interactions are invariant under $\hat{C}$ operation i.e.

$$\begin{aligned}
\left[\hat{C}, \mathcal{H}_{\text{em}}\right] &= 0, \\
\left[\hat{C}, \mathcal{H}_{\text{strong}}\right] &= 0,
\end{aligned}$$

while it gets violated in the case of weak interactions i.e.

$$\left[\hat{C}, \mathcal{H}_{\text{weak}}\right] \neq 0,$$

where $\mathcal{H}_{\text{em}}$, $\mathcal{H}_{\text{strong}}$ and $\mathcal{H}_{\text{weak}}$, respectively, represent the Hamiltonian of the electromagnetic, strong and weak interactions.

Dirac field

Consider a system of Dirac fermions interacting with an electromagnetic field

$$\left[i\gamma^\mu(\partial_\mu + ieA_\mu) - m\right]\psi(x) = 0. \tag{2.263}$$

Adjoint equation is

$$\bar{\psi}(x)\left[i\gamma^\mu(\partial_\mu - ieA_\mu) + m\right] = 0. \tag{2.264}$$

Taking the transpose of the adjoint equation

$$\left[i(\gamma^\mu)^T(\partial_\mu - ieA_\mu) + m\right]\bar{\psi}(x)^T = 0. \tag{2.265}$$

Recall positron is an antielectron which when interacts with an electromagnetic field would satisfy

$$\left[i\gamma^\mu(\partial_\mu - ieA_\mu) - m\right]\psi^C(x) = 0, \tag{2.266}$$

where $\psi^C(x)$ is the charge conjugation of $\psi(x)$.

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How $\psi^C(x)$ is related to $\psi(x)$? Notice that Eqs. (2.265) and (2.266) are very similar. Multiply in Eq. (2.265) by C from the left, that is

$$C\left[i(\gamma^\mu)^T(\partial_\mu - ieA_\mu) + m\right]\bar{\psi}(x)^T = 0$$

$$\left[iC(\gamma^\mu)^T C^{-1}C(\partial_\mu - ieA_\mu) + CC^{-1}Cm\right]\bar{\psi}(x)^T = 0$$

$$\left[iC(\gamma^\mu)^T C^{-1}C(\partial_\mu - ieA_\mu) + Cm\right]\bar{\psi}(x)^T = 0$$

$$\left[i\gamma^\mu(\partial_\mu - ieA_\mu) - m\right]C\bar{\psi}(x)^T = 0,$$

where $C(\gamma^\mu)^T C^{-1} = -\gamma^\mu$ implies that $C^{-1}\gamma^\mu C = -(\gamma^\mu)^T$

$$\psi^C(x) = \eta_\psi C\bar{\psi}^T(x). \tag{2.267}$$

In $C^{-1}\gamma^\mu C = -(\gamma^\mu)^T$ we multiply by C from the left

$$\begin{aligned} \gamma^\mu C &= -C(\gamma^\mu)^T \\ \gamma^\mu C &= +C^T(\gamma^\mu)^T \\ &= +(\gamma^\mu C)^T \\ \Rightarrow \qquad C &= -C^{-1} = -C^\dagger = -C^T \end{aligned}$$

Unlike the parity transformation where we determine $P = \gamma_0$, in the case of charge conjugation, the matrix C depends on the structure of the Dirac matrices. In the Pauli-Dirac representation, C is defined as

$$C = i\gamma^2\gamma^0. \tag{2.268}$$

Using the expression of C from Eq. (2.268) in Eq. (2.267), we find

$$\psi^C(r) \;=\; i\eta_C\gamma^2\gamma^0\bar{\psi}^T(r). \tag{2.269}$$

Similarly, for the adjoint Dirac spinor, we find

$$\begin{aligned} \bar{\psi}(r) &\xrightarrow{\hat{C}} \bar{\psi}^C(r) = \psi^{C^\dagger}(x)\gamma_0 \\ &= -\psi^T(r)C^{-1}, \\ &= i\eta_C^*\psi^T(r)\gamma^2\gamma^0. \end{aligned} \tag{2.270}$$

If F is a 4×4 matrix, then the bilinear covariants (Appendix-A) of the form $\bar{\psi}_b F\psi_a$ transform as (using Eqs. (2.269) and (2.270))

$$\bar{\psi}_b F\psi_a \xrightarrow{\hat{C}} \bar{\psi}_a\left(\gamma^2\gamma^0 F^T\gamma^2\gamma^0\right)\psi_b = \bar{\psi}_a\left(\gamma_2\gamma_0 F^T\gamma_2\gamma_0\right)\psi_b \tag{2.271}$$

under charge conjugation operation.

In the following we show the transformation of different bilinear covariants under C:

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(i) Scalar: I

$$\bar{\psi}_b \psi_a = \bar{\psi}_a \left(\gamma_2 \gamma_0 \gamma_2 \gamma_0\right) \psi_b = \bar{\psi}_a \psi_b. \tag{2.272}$$

(ii) Pseudoscalar: γ_5

$$\begin{aligned}\bar{\psi}_b \gamma_5 \psi_a &= \bar{\psi}_a \left(\gamma_2 \gamma_0 \gamma_5^T \gamma_2 \gamma_0\right) \psi_b \\ &= \bar{\psi}_a \left(\gamma_2 \gamma_0 \gamma_5 \gamma_2 \gamma_0\right) \psi_b \\ &= \bar{\psi}_a \gamma_5 \psi_b. \end{aligned} \tag{2.273}$$

(iii) Vector: γ^μ.
Since the transpose of γ^μ ($\mu = 0 - 3$) behave differently, therefore, we perform the transformation for the components $\mu = 0, 1, 2, 3$, individually. Starting with $\mu = 0$

$$\begin{aligned}\bar{\psi}_b \gamma_0 \psi_a &= \bar{\psi}_a \left(\gamma_2 \gamma_0 \gamma_0^T \gamma_2 \gamma_0\right) \psi_b \\ &= -\bar{\psi}_a \gamma_0 \psi_b. \end{aligned} \tag{2.274}$$

For $\mu = 1$ and 3, $(\gamma^\mu)^T$ behaves the same way, thus,

$$\begin{aligned}\bar{\psi}_b \gamma_1 \psi_a &= \bar{\psi}_a \left(\gamma_2 \gamma_0 \gamma_1^T \gamma_2 \gamma_0\right) \psi_b \\ &= -\bar{\psi}_a \left(\gamma_2 \gamma_0 \gamma_1 \gamma_2 \gamma_0\right) \psi_b \\ &= \bar{\psi}_a \gamma_0 \gamma_1 \gamma_0 \psi_b \\ &= -\bar{\psi}_a \gamma_1 \psi_b. \end{aligned} \tag{2.275}$$

For $\mu = 2$, we find

$$\begin{aligned}\bar{\psi}_b \gamma_2 \psi_a &= \bar{\psi}_a \left(\gamma_2 \gamma_0 \gamma_2^T \gamma_2 \gamma_0\right) \psi_b \\ &= \bar{\psi}_a \left(\gamma_2 \gamma_0 \gamma_2 \gamma_2 \gamma_0\right) \psi_b \\ &= -\bar{\psi}_a \gamma_2 \psi_b. \end{aligned} \tag{2.276}$$

From Eqs. (2.274), (2.275) and (2.276), we conclude

$$\bar{\psi}_b \gamma^\mu \psi_a = -\bar{\psi}_a \gamma^\mu \psi_b. \tag{2.277}$$

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(iv) Pseudoscalar: $\gamma^\mu\gamma_5$

$$\begin{aligned}\bar{\psi}_b\gamma^\mu\gamma_5\psi_a &= \bar{\psi}_a\left(\gamma_2\gamma_0(\gamma^\mu\gamma_5)^T\gamma_2\gamma_0\right)\psi_b \\ &= -\bar{\psi}_a\left(\gamma_2\gamma_0\gamma_5\gamma^\mu\gamma_2\gamma_0\right)\psi_b \\ &= \bar{\psi}_a\gamma^\mu\gamma_5\psi_b. \end{aligned} \tag{2.278}$$

(v) Tensor: $\sigma^{\mu\nu}$

$$\begin{aligned}\bar{\psi}_b\sigma^{\mu\nu}\psi_a &= \bar{\psi}_a\left(\gamma_2\gamma_0(\sigma^{\mu\nu})^T\gamma_2\gamma_0\right)\psi_b \\ &= \frac{i}{2}\bar{\psi}_a\left(\gamma_2\gamma_0(\gamma^{\nu T}\gamma^{\mu T}-\gamma^{\mu T}\gamma^{\nu T})\gamma_2\gamma_0\right)\psi_b \\ &= -\frac{i}{2}\bar{\psi}_a\left(\gamma_2\gamma^\nu\gamma^\mu\gamma_2-\gamma_2\gamma^\mu\gamma^\nu\gamma_2\right)\psi_b \\ &= -\bar{\psi}_a\sigma^{\mu\nu}\psi_b. \end{aligned} \tag{2.279}$$

2.9.4 Time reversal

If a physical law which is invariant when a time t changes by $-t$, we say that the law is invariant under the time reversal operation, such that

$$\begin{aligned}\vec{r} &\xrightarrow{\hat{T}} \vec{r} \\ t &\xrightarrow{\hat{T}} -t \\ \vec{p} &\xrightarrow{\hat{T}} -\vec{p} \\ \vec{L} &\xrightarrow{\hat{T}} -\vec{L} \\ \vec{J} &\xrightarrow{\hat{T}} -\vec{J}.\end{aligned}$$

Suppose we make a movie of a particle falling under the influence of gravity (Fig. 2.3), A is the initial point and B is the final point, and the motion is described by:

$$m\frac{d^2z}{dt^2} = -mg. \tag{2.280}$$

Now changing t to $-t$ is equivalent to turning the movie backwards in time i.e. now B is the initial point and A is the final point and the equation of motion is still described by Eq. (2.280). We say that the equation of motion is invariant under time reversal operation.

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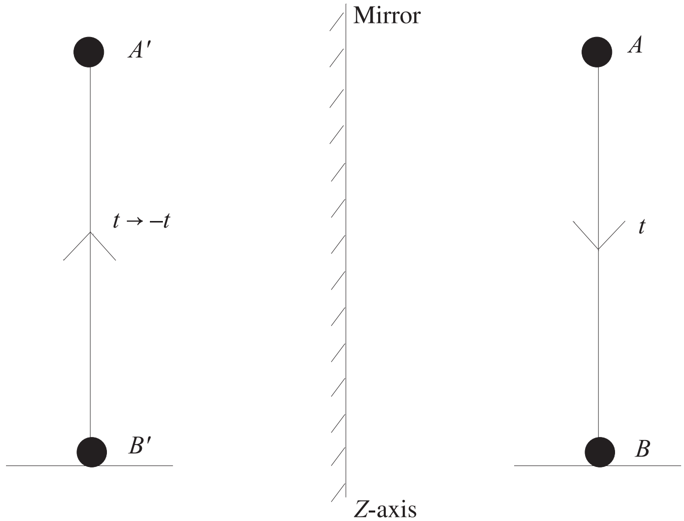

Figure 2.3 Time reversal $t \to -t$.

Under the time reversal operation,

electric field	$\vec{E}(\vec{x},t) = \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{\lvert\vec{r}\rvert^3}\vec{r} \xrightarrow{\hat{T}} \vec{E}(\vec{x},t)$
magnetic field	$\vec{B}(\vec{x},t) = \dfrac{\mu_0}{4\pi}\dfrac{Id\vec{l}\times\vec{r}}{\lvert\vec{r}\rvert^3} \xrightarrow{\hat{T}} -\vec{B}(\vec{x},-t)$
charge density	$\rho(\vec{x},t) \xrightarrow{\hat{T}} \rho^T(\vec{x},t) = \rho(\vec{x},-t)$
current density	$\vec{J}(\vec{x},t) \xrightarrow{\hat{T}} \vec{J}^T(\vec{x},t) = -\vec{J}(\vec{x},-t).$

Effect of time reversal operation on the Maxwell's equations

$$
\begin{aligned}
\vec{\nabla}.\vec{B}(\vec{r},t) &= 0 \\
\vec{\nabla}.\vec{E}(\vec{r},t) &= \frac{\rho}{\epsilon_0} \\
\vec{\nabla}\times\vec{E}(\vec{r},t) &= -\frac{\partial\vec{B}(\vec{r},t)}{\partial t} \\
\vec{\nabla}\times\vec{B}(\vec{r},t) &= \frac{\partial\vec{E}(\vec{r},t)}{\partial t} + \vec{J}(\vec{r},t) \\
\vec{E}(\vec{r},t) &= -\vec{\nabla}\phi(\vec{r},t) - \frac{\partial\vec{A}(\vec{r},t)}{\partial t} \\
\vec{B}(\vec{r},t) &= -\vec{\nabla}\times\vec{A}(\vec{r},t)
\end{aligned}
\tag{2.281}
$$

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Dirac fields

Under time reversal invariance, the Dirac spinor transforms as

$$\psi(\vec{r},t) \xrightarrow{\hat{\mathcal{T}}} \psi^{\mathcal{T}}(\vec{r},t) = \eta_T \mathcal{T} \psi^*(\vec{r},-t). \tag{2.282}$$

To relate the time reversed and unreversed spinors, we start with the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(r) = 0. \tag{2.283}$$

Taking the complex conjugate of the above equation

$$\begin{aligned} (-i\gamma^{\mu *}\partial_\mu - m)\psi^*(r) &= 0 \\ \Rightarrow \qquad \left(-i\gamma^{0*}\frac{\partial}{\partial t} - i\gamma^{i*}\frac{\partial}{\partial x^i} - m\right)\psi^*(\vec{r},t) &= 0. \end{aligned} \tag{2.284}$$

Applying $\hat{\mathcal{T}}$ on the above equation, we find

$$\left(-i\gamma^{0*}\frac{\partial}{\partial(-t)} - i\gamma^{i*}\frac{\partial}{\partial x^i} - m\right)\psi^*(\vec{r},-t) = 0. \tag{2.285}$$

If $\psi^{\mathcal{T}}(\vec{r},t)$ is the solution of the Dirac equation then

$$\begin{aligned} (i\gamma^\mu \partial_\mu - m)\psi^{\mathcal{T}}(r) = 0 \\ \left(i\gamma^0\frac{\partial}{\partial t} + i\gamma^i\frac{\partial}{\partial x^i} - m\right)\mathcal{T}\psi^*(\vec{r},-t) = 0. \end{aligned} \tag{2.286}$$

Multiplying $\mathcal{T}$ from the left in Eq. (2.285),

$$\left(i\mathcal{T}\gamma^{0*}\frac{\partial}{\partial t} - i\mathcal{T}\gamma^{i*}\frac{\partial}{\partial x^i} - \mathcal{T}m\right)\psi^*(\vec{r},-t) = 0. \tag{2.287}$$

Comparing Eqs. (2.286) and (2.287)

$$\mathcal{T}\gamma^{0*}\mathcal{T}^{-1} = \gamma^0 \qquad \text{and} \qquad \mathcal{T}\gamma^{i*}\mathcal{T}^{-1} = -\gamma^i. \tag{2.288}$$

Similarly, we find

$$\mathcal{T}^{-1}\gamma^0\mathcal{T} = \gamma^{0^*} = (\gamma^{0\dagger})^T = \gamma^{0^T}, \tag{2.289}$$

$$\mathcal{T}^{-1}\gamma^i\mathcal{T} = -\gamma^{i^*} = -(\gamma^{i\dagger})^T = \gamma^{i^T}. \tag{2.290}$$

If Eq. (2.288) holds, then the time reversed wave function $\psi^{\mathcal{T}}(r)$ would satisfy the Dirac equation.

Now, we see the transformation of different γ matrices under $\mathcal{T}$:

(i) Vector: γ^μ

$$\mathcal{T}^{-1}\gamma^\mu\mathcal{T} = (\gamma^\mu)^T. \tag{2.291}$$

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(ii) Pseudoscalar: γ^5

$$\begin{aligned}\mathcal{T}^{-1}\gamma^5\mathcal{T} &= -i\mathcal{T}^{-1}(\gamma^0\gamma^1\gamma^2\gamma^3)\mathcal{T} \\ &= -i(\mathcal{T}^{-1}\gamma^0\mathcal{T})(\mathcal{T}^{-1}\gamma^1\mathcal{T})(\mathcal{T}^{-1}\gamma^2\mathcal{T})(\mathcal{T}^{-1}\gamma^3)\mathcal{T}) \\ &= -i\gamma^{0^T}\gamma^{1^T}\gamma^{2^T}\gamma^{3^T} = (\gamma^5)^T. \end{aligned} \tag{2.292}$$

(iii) Axial vector: $\gamma^\mu\gamma^5$

$$\begin{aligned}\mathcal{T}^{-1}\gamma^\mu\gamma^5\mathcal{T} &= \mathcal{T}^{-1}\gamma^\mu\mathcal{T}\mathcal{T}^{-1}\gamma^5\mathcal{T} \\ &= \gamma^{\mu^T}\gamma^{5^T} = -(\gamma^\mu\gamma^5)^T. \end{aligned} \tag{2.293}$$

(iv) Tensor: $\sigma_{\mu\nu}$

$$\begin{aligned}\mathcal{T}^{-1}\sigma^{\mu\nu}\mathcal{T} &= \frac{i}{2}\mathcal{T}^{-1}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\mathcal{T} \\ &= \frac{i}{2}(\mathcal{T}^{-1}\gamma^\mu\mathcal{T}\mathcal{T}^{-1}\gamma^\nu\mathcal{T} - \mathcal{T}^{-1}\gamma^\nu\mathcal{T}\mathcal{T}^{-1}\gamma^\mu\mathcal{T}) \\ &= \frac{i}{2}(\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu)^T = -(\sigma^{\mu\nu})^T. \end{aligned}$$

Using Eq. (2.288) the matrix $\mathcal{T}$ is constructed. One choice of $\mathcal{T}$ is

$$\mathcal{T} = i\gamma^1\gamma^3. \tag{2.294}$$

The matrix $\mathcal{T}$ satisfies the relation

$$\mathcal{T} = -\mathcal{T}^\dagger = -\mathcal{T}^{-1} = \mathcal{T}^T, \tag{2.295}$$

which shows that the matrix $\mathcal{T}$ is antisymmetric.

Using Eq. (2.294) in Eq. (2.282), we find

$$\psi^{\mathcal{T}}(\vec{r},t) = \eta_T\mathcal{T}\psi^*(\vec{r},-t) = i\eta_T\gamma^1\gamma^3\psi^*(\vec{r},-t) \tag{2.296}$$

$$= i\eta_T\gamma^0\gamma^1\gamma^3\bar{\psi}^T(\vec{r},-t). \tag{2.297}$$

Similarly, the adjoint Dirac spinor is expressed, under time reversal, as

$$\begin{aligned}\bar{\psi}^{\mathcal{T}}(\vec{r},t) &= \psi^{\mathcal{T}^\dagger}\gamma^0 \\ &= -i\eta_T^*\psi^T(\vec{r},-t)\gamma^3\gamma^1\gamma^0. \end{aligned} \tag{2.298}$$

The bilinear term $\bar{\psi}_a F\psi_b$ transforms as

$$\bar{\psi}_a(\vec{r},t)F\psi_b(\vec{r},t) \xrightarrow{\hat{\mathcal{T}}} \bar{\psi}_b(\vec{r},-t)\gamma^0\gamma^1\gamma^3F^T\gamma^3\gamma^1\gamma^0\psi_a(\vec{r},-t), \tag{2.299}$$

where F is a 4×4 matrix.

Now, we see the transformation of different bilinear covariants under time reversal:

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Table 2.1 Transformations of Dirac spinor and adjoint spinors under various symmetry operations.

Symmetry transformation	Dirac spinor $\psi(\vec{r},t)$	Adjoint Dirac spinor $\bar{\psi}(\vec{r},t)$
$\hat{P}$	$\eta_P\gamma^0\psi(-\vec{r},t)$	$\eta_P\bar{\psi}(-\vec{r},t)\gamma^0$
$\hat{C}$	$-i\eta_C\gamma^2\psi^*(\vec{r},t)$	$i\eta_C^*\psi^T(\vec{r},t)\gamma^2\gamma^0$
$\hat{\mathcal{T}}$	$i\eta_T\gamma^0\gamma^1\gamma^3\bar{\psi}^T(\vec{r},-t)$	$-i\eta_T^*\psi^T(\vec{r},-t)\gamma^3\gamma^1\gamma^0$
$\hat{C}\hat{P}$	$-i\eta_P\eta_C\gamma^0\gamma^2\psi^*(-\vec{r},t)$	$i\eta_P\eta_C^*\bar{\psi}^T(-\vec{r},t)\gamma^2$
$\hat{C}\hat{P}\hat{\mathcal{T}}$	$i\eta_{CPT}\gamma^5\psi(\vec{r},t)$	$-i\eta_{CPT}^*\psi^T(\vec{r},t)\gamma^5\gamma^0$

(i) Scalar: I

$$\begin{aligned}\bar{\psi}_a(\vec{r},t)\psi(\vec{r},t)_b &\xrightarrow{\hat{\mathcal{T}}} \bar{\psi}_b(\vec{r},-t)\gamma^0\gamma^1\gamma^3\gamma^3\gamma^1\gamma^0\psi_a(\vec{r},-t)\\ &= \bar{\psi}_b(\vec{r},-t)\psi_a(\vec{r},-t).\end{aligned} \tag{2.300}$$

(ii) Pseudoscalar: γ_5

$$\begin{aligned}\bar{\psi}_a(\vec{r},t)\gamma^5\psi_b(\vec{r},t) &\xrightarrow{\hat{\mathcal{T}}} \bar{\psi}_b(\vec{r},-t)\gamma^0\gamma^1\gamma^3\gamma^{5^T}\gamma^3\gamma^1\gamma^0\psi_a(\vec{r},-t)\\ &= \bar{\psi}_b(\vec{r},-t)\gamma^0\gamma^1\gamma^3\gamma^5\gamma^3\gamma^1\gamma^0\psi_a(\vec{r},-t)\\ &= \bar{\psi}_b(\vec{r},-t)\gamma^0\gamma^5\gamma^0\psi_a(\vec{r},-t)\\ &= -\bar{\psi}_b(\vec{r},-t)\gamma^5\psi_a(\vec{r},-t).\end{aligned} \tag{2.301}$$

(iii) Vector: γ^μ
For γ^μ, we look for the different components. For $\mu = 0$:

$$\begin{aligned}\bar{\psi}_a(\vec{r},t)\gamma^0\psi_b(\vec{r},t) &\xrightarrow{\hat{\mathcal{T}}} \bar{\psi}_b(\vec{r},-t)\gamma^0\gamma^1\gamma^3\gamma^{0^T}\gamma^3\gamma^1\gamma^0\psi_a(\vec{r},-t)\\ &= \bar{\psi}_b(\vec{r},-t)\gamma^0\psi_a(\vec{r},-t).\end{aligned} \tag{2.302}$$

For $\mu = 1$:

$$\begin{aligned}\bar{\psi}_a(\vec{r},t)\gamma^1\psi_b(\vec{r},t) &\xrightarrow{\hat{\mathcal{T}}} \bar{\psi}_b(\vec{r},-t)\gamma^0\gamma^1\gamma^3(\gamma^1)^T\gamma^3\gamma^1\gamma^0\psi_a(\vec{r},-t)\\ &= -\bar{\psi}_b(\vec{r},-t)\gamma^1\psi_a(\vec{r},-t).\end{aligned} \tag{2.303}$$

For $\mu = 2$:

$$\begin{aligned}\bar{\psi}_a(\vec{r},t)\gamma^2\psi_b(\vec{r},t) &\xrightarrow{\hat{\mathcal{T}}} \bar{\psi}_b(\vec{r},-t)\gamma^0\gamma^1\gamma^3(\gamma^2)^T\gamma^3\gamma^1\gamma^0\psi_a(\vec{r},-t)\\ &= -\bar{\psi}_b(\vec{r},-t)\gamma^2\psi_a(\vec{r},-t).\end{aligned} \tag{2.304}$$

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Table 2.2 Transformations of Dirac bilinear covariants under various symmetry operations.

Bilinear covariants → Symmetry operations ↓	$\bar{\psi}_a\psi_b$	$\bar{\psi}_a\gamma^5\psi_b$	$\bar{\psi}_a\gamma^\mu\psi_b$	$\bar{\psi}_a\gamma^\mu\gamma^5\psi_b$	$\bar{\psi}_a\sigma^{\mu\nu}\psi_b$
$\hat{P}$	$\bar{\psi}_a\psi_b$	$-\bar{\psi}_a\gamma^5\psi_b$	$\bar{\psi}_a\gamma_\mu\psi_b$	$-\bar{\psi}_a\gamma_\mu\gamma^5\psi_b$	$-\bar{\psi}_a\sigma_{\mu\nu}\psi_b$
$\hat{\mathcal{T}}$	$\bar{\psi}_b\psi_a$	$-\bar{\psi}_b\gamma_5\psi_a$	$\bar{\psi}_b\gamma_\mu\psi_a$	$\bar{\psi}_b\gamma_\mu\gamma^5\psi_a$	$\bar{\psi}_b\sigma_{\mu\nu}\psi_a$
$\hat{C}$	$\bar{\psi}_b\psi_a$	$\bar{\psi}_b\gamma^5\psi_a$	$-\bar{\psi}_b\gamma^\mu\psi_a$	$\bar{\psi}_b\gamma^\mu\gamma^5\psi_a$	$-\bar{\psi}_b\sigma^{\mu\nu}\psi_a$
$\hat{C}\hat{P}$	$\bar{\psi}_b\psi_a$	$-\bar{\psi}_b\gamma^5\psi_a$	$-\bar{\psi}_b\gamma_\mu\psi_a$	$-\bar{\psi}_b\gamma_\mu\gamma^5\psi_a$	$\bar{\psi}_b\sigma^{\mu\nu}\psi_a$
$\hat{C}\hat{P}\hat{\mathcal{T}}$	$\bar{\psi}_a\psi_b$	$\bar{\psi}_a\gamma^5\psi_b$	$-\bar{\psi}_a\gamma^\mu\psi_b$	$-\bar{\psi}_a\gamma^\mu\gamma^5\psi_b$	$\bar{\psi}_a\sigma^{\mu\nu}\psi_b$
$\hat{G}$	$-\bar{\psi}_b\psi_a$	$-\bar{\psi}_b\gamma^5\psi_a$	$\bar{\psi}_b\gamma^\mu\psi_a$	$-\bar{\psi}_a\gamma^\mu\gamma^5\psi_b$	$\bar{\psi}_b\sigma^{\mu\nu}\psi_a$

For $\mu = 3$:

$$\begin{aligned}
\bar{\psi}_a(\vec{r},t)\gamma^3\psi_b(\vec{r},t) &\xrightarrow{\hat{\mathcal{T}}} \bar{\psi}_b(\vec{r},-t)\gamma^0\gamma^1\gamma^3(\gamma^3)^T\gamma^3\gamma^1\gamma^0\psi_a(\vec{r},-t)\\
&= \bar{\psi}_b(\vec{r},-t)\gamma^0\gamma^3\gamma^0\psi_a(\vec{r},-t)\\
&= -\bar{\psi}_b(\vec{r},-t)\gamma^3\psi_a(\vec{r},-t). \qquad (2.305)
\end{aligned}$$

Thus, we find that the zeroth component does not change sign under $\hat{T}$ operation while the ith component changes sign and we may write

$$\bar{\psi}_a(\vec{r},t)\gamma^\mu\psi_b(\vec{r},t) \xrightarrow{\hat{\mathcal{T}}} \bar{\psi}_b(\vec{r},-t)\gamma_\mu\psi_a(\vec{r},-t). \qquad (2.306)$$

(iv) Axial vector: $\gamma^\mu\gamma_5$

$$\begin{aligned}
\bar{\psi}_a(\vec{r},t)\gamma^\mu\gamma_5\psi_b(\vec{r},t) &\xrightarrow{\hat{\mathcal{T}}} \bar{\psi}_b(\vec{r},-t)\gamma^0\gamma^1\gamma^3(\gamma^\mu\gamma_5)^T\gamma^3\gamma^1\gamma^0\psi_a(\vec{r},-t)\\
&= \bar{\psi}_b(\vec{r},-t)\gamma^0\gamma^1\gamma^3{\gamma_5}^T{\gamma^\mu}^T\gamma^3\gamma^1\gamma^0\psi_a(\vec{r},-t). \qquad (2.307)
\end{aligned}$$

For $\mu = 0$, we find

$$\bar{\psi}_a(\vec{r},t)\gamma^0\gamma^5\psi_b(\vec{r},t) \xrightarrow{\hat{\mathcal{T}}} \bar{\psi}_b(\vec{r},-t)\gamma^0\gamma^5\psi_a(\vec{r},-t), \qquad (2.308)$$

following the same analogy as performed in Eq. (2.302). Similarly for $\mu = i$,

$$\bar{\psi}_a(\vec{r},t)\gamma^i\gamma^5\psi_b(\vec{r},t) \xrightarrow{\hat{\mathcal{T}}} -\bar{\psi}_b(\vec{r},-t)\gamma^i\gamma^5\psi_a(\vec{r},-t). \qquad (2.309)$$

Thus

$$\bar{\psi}_a(\vec{r},t)\gamma^\mu\gamma^5\psi_b(\vec{r},t) \xrightarrow{\hat{\mathcal{T}}} \bar{\psi}_b(\vec{r},-t)\gamma_\mu\gamma^5\psi_a(\vec{r},-t). \qquad (2.310)$$

(v) Tensor: $\sigma^{\mu\nu}$

$$\begin{aligned}\bar{\psi}_a(\vec{r},t)\sigma^{\mu\nu}\psi_b(\vec{r},t) &\xrightarrow{\hat{\mathcal{T}}} \bar{\psi}_b(\vec{r},-t)\gamma^0\gamma^1\gamma^3(\sigma^{\mu\nu})^T\gamma^3\gamma^1\gamma^0\psi_a(\vec{r},-t)\\ &= \frac{i}{2}\bar{\psi}_b(\vec{r},-t)\gamma^0\gamma^1\gamma^3\left(\gamma^{\nu T}\gamma^{\mu T}-\gamma^{\mu T}\gamma^{\nu T}\right)\\ &\quad\gamma^3\gamma^1\gamma^0\psi_a(\vec{r},-t).\end{aligned} \tag{2.311}$$

For $\sigma^{\mu\nu}$, we perform time reversal on the components σ^{0i} and σ^{ij}. For σ^{0i}, Eq. (2.311) becomes

$$\bar{\psi}_a(\vec{r},t)\sigma^{0i}\psi_b(\vec{r},t) \xrightarrow{\hat{\mathcal{T}}} \frac{i}{2}\bar{\psi}_b(\vec{r},-t)\gamma^0\gamma^1\gamma^3\left({\gamma^i}^T{\gamma^0}^T-{\gamma^0}^T{\gamma^i}^T\right)\gamma^3\gamma^1\gamma^0\psi_a(\vec{r},-t). \tag{2.312}$$

For $i=1$, we obtain

$$\begin{aligned}\bar{\psi}_a(\vec{r},t)\sigma^{01}\psi_b(\vec{r},t) &\xrightarrow{\hat{\mathcal{T}}} -\frac{i}{2}\bar{\psi}_b(\vec{r},-t)\left[\gamma^0\gamma^1\gamma^3\gamma^1\gamma^0\gamma^3\gamma^1\gamma^0\right.\\ &\quad\left.-\gamma^0\gamma^1\gamma^3\gamma^0\gamma^1\gamma^3\gamma^1\gamma^0\right]\psi_a(\vec{r},-t)\\ &= -\frac{i}{2}\bar{\psi}_b(\vec{r},-t)\left[\gamma^1\gamma^0-\gamma^0\gamma^1\right]\psi_a(\vec{r},-t)\\ &= \bar{\psi}_b(\vec{r},-t)\sigma^{01}\psi_a(\vec{r},-t).\end{aligned} \tag{2.313}$$

Similarly, one can do for $i=2$ and 3. Thus

$$\bar{\psi}_a(\vec{r},t)\sigma^{0i}\psi_b(\vec{r},t) \xrightarrow{\hat{\mathcal{T}}} \bar{\psi}_b(\vec{r},-t)\sigma^{0i}\psi_a(\vec{r},-t). \tag{2.314}$$

For $\mu=i$ and $\nu=j$, we get

$$\bar{\psi}_a(\vec{r},t)\sigma^{ij}\psi_b(\vec{r},t) \xrightarrow{\hat{\mathcal{T}}} \frac{i}{2}\bar{\psi}_b(\vec{r},-t)\gamma^0\gamma^1\gamma^3\left({\gamma^j}^T{\gamma^i}^T-{\gamma^i}^T{\gamma^j}^T\right)\gamma^3\gamma^1\gamma^0\psi_a(\vec{r},-t). \tag{2.315}$$

For $i=1$ and $j=2$, we obtain

$$\begin{aligned}\bar{\psi}_a(\vec{r},t)\sigma^{12}\psi_b(\vec{r},t) &\xrightarrow{\hat{\mathcal{T}}} -\frac{i}{2}\bar{\psi}_b(\vec{r},-t)\left[\gamma^0\gamma^1\gamma^3\gamma^2\gamma^1\gamma^3\gamma^1\gamma^0\right.\\ &\quad\left.-\gamma^0\gamma^1\gamma^3\gamma^1\gamma^2\gamma^3\gamma^1\gamma^0\right]\psi_a(\vec{r},-t)\\ &= -\bar{\psi}_b(\vec{r},-t)\sigma^{12}\psi_a(\vec{r},-t).\end{aligned} \tag{2.316}$$

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For $i = 1$ and $j = 3$, we get

$$\begin{aligned}\bar{\psi}_a(\vec{r},t)\sigma^{13}\psi_b(\vec{r},t) &\xrightarrow{\hat{\mathcal{T}}} \frac{i}{2}\bar{\psi}_b(\vec{r},-t)\left[\gamma^0\gamma^1\gamma^3\gamma^3\gamma^1\gamma^3\gamma^1\gamma^0\right.\\ &\quad \left. -\gamma^0\gamma^1\gamma^3\gamma^1\gamma^3\gamma^3\gamma^1\gamma^0\right]\psi_a(\vec{r},-t)\\ &= -\bar{\psi}_b(\vec{r},-t)\sigma^{13}\psi_a(\vec{r},-t). \end{aligned} \tag{2.317}$$

Similarly, for $i = 2$ and $j = 3$, we get

$$\bar{\psi}_a(\vec{r},t)\sigma^{23}\psi_b(\vec{r},t) \xrightarrow{\hat{\mathcal{T}}} -\bar{\psi}_b(\vec{r},-t)\sigma^{23}\psi_a(\vec{r},-t). \tag{2.318}$$

From Eqs. (2.316), (2.317) and (2.318), it is established that

$$\bar{\psi}_a(\vec{r},t)\sigma^{ij}\psi_b(\vec{r},t) \xrightarrow{\hat{\mathcal{T}}} -\bar{\psi}_b(\vec{r},-t)\sigma^{ij}\psi_a(\vec{r},-t). \tag{2.319}$$

Using Eqs. (2.314) and (2.319), we get

$$\bar{\psi}_a(\vec{r},t)\sigma^{\mu\nu}\psi_b(\vec{r},t) \xrightarrow{\hat{\mathcal{T}}} \bar{\psi}_b(\vec{r},-t)\sigma_{\mu\nu}\psi_a(\vec{r},-t). \tag{2.320}$$

(vi) $\sigma^{\mu\nu}\gamma_5$

$$\begin{aligned}\bar{\psi}_a(\vec{r},t)\sigma^{\mu\nu}\gamma_5\psi_b(\vec{r},t) &\xrightarrow{\hat{\mathcal{T}}} \bar{\psi}_b(\vec{r},-t)\gamma^0\gamma^1\gamma^3(\sigma^{\mu\nu}\gamma_5)^T\gamma^3\gamma^1\gamma^0\psi_a(\vec{r},-t)\\ &= \frac{i}{2}\bar{\psi}_b(\vec{r},-t)\gamma^0\gamma^1\gamma^3\gamma_5\left(\gamma^{\nu T}\gamma^{\mu T}-\gamma^{\mu T}\gamma^{\nu T}\right)\\ &\quad \gamma^3\gamma^1\gamma^0\psi_a(\vec{r},-t). \end{aligned} \tag{2.321}$$

Using the properties of γ matrices given in Appendix-C, the components $\sigma^{0i}\gamma_5$ and $\sigma^{ij}\gamma_5$ transforms under time reversal transformation as

$$\bar{\psi}_a(\vec{r},t)\sigma^{0i}\gamma_5\psi_b(\vec{r},t) \xrightarrow{\hat{\mathcal{T}}} -\bar{\psi}_b(\vec{r},-t)\sigma^{0i}\gamma_5\psi_a(\vec{r},-t), \tag{2.322}$$

$$\bar{\psi}_a(\vec{r},t)\sigma^{ij}\gamma_5\psi_b(\vec{r},t) \xrightarrow{\hat{\mathcal{T}}} \bar{\psi}_b(\vec{r},-t)\sigma^{ij}\gamma_5\psi_a(\vec{r},-t). \tag{2.323}$$

Combining the two equations, we get the transformation of $\sigma^{\mu\nu}\gamma_5$ under $\hat{\mathcal{T}}$, as

$$\bar{\psi}_a(\vec{r},t)\sigma^{\mu\nu}\gamma_5\psi_b(\vec{r},t) \xrightarrow{\hat{\mathcal{T}}} -\bar{\psi}_b(\vec{r},-t)\sigma_{\mu\nu}\gamma_5\psi_a(\vec{r},-t). \tag{2.324}$$

In Tables 2.1 and 2.2, respectively, the transformation of Dirac spinor as well as adjoint spinor and Dirac bilinear covariants under $\hat{C}$, $\hat{P}$, $\hat{T}$, $\hat{C}\hat{P}$, $\hat{C}\hat{P}\hat{T}$ and $\hat{G}$ are shown.