Appendix A

Lorentz Transformation and Covariance of the Dirac Equation

A.1 Lorentz Transformations

Lorentz transformation (L.T.) is the rotation in Minkowski space. Minkowski space is the mathematical representation of the space time in Einstein's theory of relativity. Unlike the Euclidean space, Minkowski space treats space and time on a different footing and the space time interval between the two events would be the same in all frames of reference.

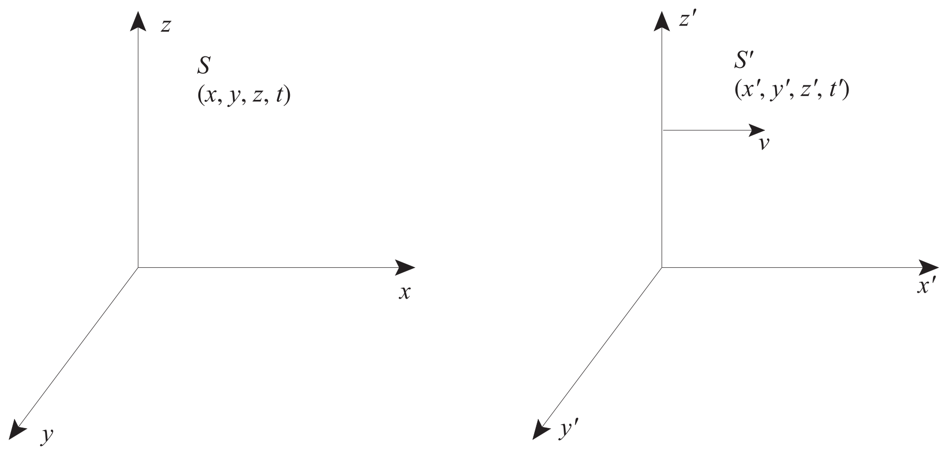

Figure A.1 Lorentz transformation.

If an event is observed in an inertial frame S at the coordinates (x, y, z, t) (Fig.A.1) and in another inertial frame S' at the coordinates (x', y', z', t') and the frame S' is moving with a constant velocity v with respect to S in the direction of x, then the measurements in S and S' are related by

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$$
\begin{aligned}
x' &= \gamma(x - vt), & x &= \gamma(x' + vt'), \\
y' &= y, & y &= y', \\
z' &= z, & z &= z', \\
t' &= \gamma(t - vx), & t &= \gamma(t' + vx'),
\end{aligned} \tag{A.1}
$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ and $c = 1$.

The L.H.S. of these equations describe the transformation of the coordinates from S to S' and the R.H.S. of the equations describe the transfer of coordinates from S' to S. Equation (A.1) may also be written as

$$
\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}. \tag{A.2}
$$

The invariant length element squared ds^2 is defined as

$$
\begin{aligned}
ds^2 &= c^2 dt^2 - dx^2 - dy^2 - dz^2, \\
&= c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2, \\
&= g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} dx'^\mu dx'^\nu, \\
&= dx^\mu dx_\mu = dx'^\mu dx'_\mu,
\end{aligned} \tag{A.3}
$$

where $x^\mu = \begin{pmatrix} t & x & y & z \end{pmatrix}$ is a contravariant vector and $x_\mu = \begin{pmatrix} t & -x & -y & -z \end{pmatrix} = g_{\mu\nu} x^\nu$ is a covariant vector. The metric tensor $g_{\mu\nu}$ is defined as

$$
g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.4}
$$

One may rewrite Eq. (A.2) as

$$
x'^\mu = \sum_{\nu=0}^{3} a^\mu_\nu x^\nu, \tag{A.5}
$$

where a^μ_ν is the Lorentz transformation matrix. Any set of quantities which have four components and transform like Eq. (A.5) under L.T. form a four vector, that is,

$$
A'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu A^\nu, \tag{A.6}
$$

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with the properties

$$\Lambda^{\mu}_{\nu}\Lambda^{\nu}_{\sigma} = \delta^{\mu}_{\sigma} \quad \text{and} \quad \det\left(\Lambda^{\mu}_{\nu}\right) = \pm 1.$$

For an infinitesimal L.T., Λ^{μ}_{ν} may also be written as $\Lambda^{\mu}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}_{\nu}$. The tensor ω^{μ}_{ν} generates the infinitesimal L.T. and it is considered to be a quantity that is smaller than unity. The major contributor is δ^{μ}_{ν}(unit matrix) and ω^{μ}_{ν} generates infinitesimal boost or infinitesimal rotation or both. This can be understood as demonstrated in the next section.

Rotation of coordinates around the z-axis in 3-D

If we consider the rotation around the z-axis by an angle θ, then

$$\begin{aligned} x'_1 &= \cos\theta\, x_1 - \sin\theta\, x_2, \\ x'_2 &= \sin\theta\, x_1 + \cos\theta\, x_2, \\ x'_3 &= x_3. \end{aligned} \tag{A.7}$$

In the matrix form, Eq. (A.7) can be written as

$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

For an infinitesimal rotation, that is, $(\theta \to 0), \cos\theta \to 1, \sin\theta \to \theta$

$$\begin{aligned} \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} &= \begin{bmatrix} 1 & -\theta & 0 \\ \theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \\ &= \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\theta & 0 \\ \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \\ x'_i &= (\delta^i_j + \epsilon^i_j) x_j, \end{aligned} \tag{A.8}$$

where ϵ^i_j represents an infinitesimal rotation if $\theta = \epsilon$. Equation (A.7) can be rewritten as

$$\begin{aligned} x'_1 &= x_1 - \epsilon x_2, \\ x'_2 &= \epsilon x_1 + x_2, \\ x'_3 &= x_3. \end{aligned} \tag{A.9}$$

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The matrix representing the infinitesimal change is antisymmetric. Reconsidering Eq. (A.2),

$$\begin{aligned}
\begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} &= \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}, \\
&= \begin{bmatrix} \cosh\omega & -\sinh\omega & 0 & 0 \\ -\sinh\omega & \cosh\omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix},
\end{aligned}$$

where $\cosh\omega = \gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\sinh\omega = \gamma\beta = \frac{\beta}{\sqrt{1-\beta^2}}$. Now $x'^\mu = \Lambda^\mu_\nu x^\nu$ and Λ^μ_ν is given by

$$\Lambda^\mu_\nu = \begin{bmatrix} \cosh\omega & -\sinh\omega & 0 & 0 \\ -\sinh\omega & \cosh\omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{A.10}$$

For an infinitesimal boost or rotation, that is,

$$\Lambda^\mu_\nu = \begin{bmatrix} 1 & -\epsilon & 0 & 0 \\ -\epsilon & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \delta^\mu_\nu + \epsilon^\mu_\nu. \tag{A.11}$$

Therefore, Eq. (A.11) implies that

$$\begin{aligned}
\epsilon^{\mu\nu} &= g^{\nu\lambda}\epsilon^\mu_\lambda = \begin{bmatrix} 0 & +\epsilon & 0 & 0 \\ -\epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = -\epsilon^{\nu\mu} \\
x'^\mu &= \Lambda^\mu_\nu x^\nu = (\delta^\mu_\nu + \epsilon^\mu_\nu)x^\nu, \\
\Rightarrow \qquad g_{\mu\nu}x'^\mu x'^\nu &= g_{\mu\nu}x^\mu x^\nu, \\
\Rightarrow \qquad g_{\mu\nu}\Lambda^\mu_\rho \Lambda^\nu_\sigma x^\rho x^\sigma &= g_{\rho\sigma}x^\rho x^\sigma, \\
\Rightarrow \qquad \Lambda^\rho_\mu \Lambda^\mu_\sigma &= \delta^\rho_\sigma.
\end{aligned}$$

Now,

$$\begin{aligned}
x'_\mu x'^\mu &= \Lambda^\alpha_\mu x_\alpha \Lambda^\mu_\beta x^\beta, \\
&= (\delta^\alpha_\mu + \epsilon^\alpha_\mu)x_\alpha(\delta^\mu_\beta + \epsilon^\mu_\beta)x^\beta, \\
&= (\delta^\alpha_\mu\delta^\mu_\beta + \epsilon^\alpha_\mu\epsilon^\mu_\beta + \delta^\alpha_\mu\epsilon^\mu_\beta + \epsilon^\alpha_\mu\delta^\mu_\beta)x_\alpha x^\beta.
\end{aligned} \tag{A.13}$$

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The second term on the R.H.S. of Eq. (A.13) can be neglected as it is the product of two infinitesimal numbers, that is,

$$x'_\mu x'^\mu = x_\alpha x^\alpha + (\epsilon^\beta_\alpha + \epsilon^\alpha_\beta) x_\alpha x^\beta.$$

Since $x'_\mu x'^\mu$ should be an invariant quantity,

$$\begin{aligned} (\epsilon^\beta_\alpha + \epsilon^\alpha_\beta) &= 0, \\ \epsilon^\alpha_\beta &= -\epsilon^\beta_\alpha, \end{aligned} \tag{A.14}$$

that is, Eq. (A.14) shows antisymmetry. Therefore, the generator of an L.T. must be an antisymmetric matrix.

A.2 Covariance of Dirac Equation

Physical observables in all the inertial systems are the same. Therefore, it is important that the Dirac equation, upon which our physical interpretation is based, must be covariant under L.T. To prove that the Dirac equation is covariant under L.T., consider a spin $\frac{1}{2}$ particle (fermion) moving in space and an observer in the rest frame of reference (S), making measurements and determining that the properties of this fermion are described by the Dirac equation

$$(i\hbar\gamma^\mu \partial_\mu - m_0 c)\psi(x) = 0. \tag{A.15}$$

Covariance of Dirac equation means the following:

1. There must be an explicit rule to enable the observer in S' to calculate $\psi'(x')$, if $\psi(x)$ of the observer in S is given. Hence, $\psi'(x')$ of the S' frame describes a physical state of the fermion, as $\psi(x)$ describes the physical state in the S frame.

2. $\psi'(x')$ must be a solution of the Dirac equation in S', having the form

$$(i\hbar\gamma'^\mu \partial'_\mu - m_0 c)\psi'(x') = 0, \tag{A.16}$$

with γ'^μ satisfying the relations $\gamma'^\mu\gamma'^\nu + \gamma'^\nu\gamma'^\mu = 2g^{\mu\nu}$; $\gamma'^{0\dagger} = \gamma'^0$; $\gamma'^{0^2} = 1$ and $\gamma'^{i\dagger} = -\gamma'^i$, where $i = 1, 2, 3$. Rewriting Eq. (A.16), we obtain

$$i\gamma'^0 \frac{\partial \psi'(x')}{\partial t'} = \left(-i\gamma'^k \frac{\partial}{\partial x'^k} + m_0\right)\psi'(x'). \tag{A.17}$$

Multiplying by γ'^0 from the L.H.S., we obtain

$$i\frac{\partial \psi'(x')}{\partial t'} = \left(-i\gamma'^0\gamma'^k \frac{\partial}{\partial x'^k} + m_0\gamma'^0\right)\psi'(x').$$

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This is an equation similar to the Schrodinger equation. Therefore

$$i\frac{\partial\psi'(x')}{\partial t'} = \left(-i\gamma'^0\gamma'^k\frac{\partial}{\partial x'^k} + m_0\gamma'^0\right)\psi'(x') = \hat{H}'\psi', \tag{A.18}$$

where

$$\hat{H}' = -i\gamma'^0\gamma'^k\frac{\partial}{\partial x'^k} + m_0\gamma'^0. \tag{A.19}$$

This should be a Hermitian to get real eigenvalues, that is,

$$(\hat{H}')^\dagger = \hat{H}'. \tag{A.20}$$

It can be shown that all 4×4 γ'^μ matrices which satisfy all the properties obeyed by γ^μ are identical up to a unitary transformation, that is,

$$\gamma'^\mu = \hat{U}^\dagger\gamma^\mu\hat{U}, \qquad \hat{U}^\dagger = \hat{U}^{-1}. \tag{A.21}$$

Since the observables do not change under unitary transformation, for all purposes, we may replace γ'^μ by γ^μ. Now we construct the transformation between $\psi(x)$ and $\psi'(x')$. This transformation is required to be linear, since both the Dirac equation and the L.T. are linear in the space time coordinates. This requires that

$$\psi'(x') = \psi'(\hat{\Lambda}x) = \hat{S}(\hat{\Lambda})\psi(x) = \hat{S}(\hat{\Lambda})\psi(\hat{\Lambda}^{-1}x'), \tag{A.22}$$

where we have used $x = \hat{\Lambda}^{-1}x'$ as $x' = \hat{\Lambda}x$, since

$$x'^\mu = \Lambda^\mu_\nu x^\nu, \qquad \Lambda^\mu_\nu = \frac{\partial x'^\mu}{\partial x^\nu}. \tag{A.23}$$

Moreover, $\psi(x)$ may be written as

$$\psi(x) = \hat{S}^{-1}(\hat{\Lambda})\psi'(x') = \hat{S}^{-1}(\hat{\Lambda})\psi'(\hat{\Lambda}x) = \hat{S}(\hat{\Lambda}^{-1})\psi'(\hat{\Lambda}x). \tag{A.24}$$

Thus, the Dirac equation in frame S, that is,

$$\left(i\gamma^\mu\frac{\partial}{\partial x^\mu} - m_0\right)\psi(x) = 0,$$

can also be written as

$$\left(i\gamma^\mu\hat{S}^{-1}(\hat{\Lambda})\frac{\partial}{\partial x^\mu} - m_0 S^{-1}(\hat{\Lambda})\right)\psi'(x') = 0. \tag{A.25}$$

Multiplying L.H.S. of Eq. (A.25) by $\hat{S}(\hat{\Lambda})$ and using $\hat{S}(\hat{\Lambda})\hat{S}^{-1}(\hat{\Lambda}) = 1$, results in

$$\left(i\hat{S}(\hat{\Lambda})\gamma^\mu\hat{S}^{-1}(\hat{\Lambda})\frac{\partial}{\partial x^\mu} - m_0\right)\psi'(x') = 0, \tag{A.26}$$

and

$$\frac{\partial}{\partial x^{\mu}} = \frac{\partial x'^{\nu}}{\partial x^{\mu}}\frac{\partial}{\partial x'^{\nu}} = \Lambda^{\nu}_{\mu}\frac{\partial}{\partial x'^{\nu}}.$$

We may write Eq. (A.26) as

$$\left[i\left(\hat{S}(\hat{\Lambda})\gamma^{\mu}\hat{S}^{-1}(\hat{\Lambda})\Lambda^{\nu}_{\mu}\right)\frac{\partial}{\partial x'^{\nu}} - m_0\right]\psi'(x') = 0. \tag{A.27}$$

Now if the Dirac equation is to be invariant under L.T., then we must identify

$$\hat{S}(\hat{\Lambda})\gamma^{\mu}\hat{S}^{-1}(\hat{\Lambda})\Lambda^{\nu}_{\mu} = \gamma^{\nu}, \tag{A.28}$$

$$\hat{S}(\hat{\Lambda})\gamma^{\mu}\hat{S}^{-1}(\hat{\Lambda}) = \Lambda^{\mu}_{\nu}\gamma^{\nu}, \tag{A.29}$$

$$S^{-1}(\hat{\Lambda})\gamma^{\nu}\hat{S}(\hat{\Lambda}) = \Lambda^{\nu}_{\mu}\gamma^{\mu}. \tag{A.30}$$

There must exist a matrix connecting the two representations γ^{μ} and γ'^{μ}. It turns out that this matrix does exist and one of the possible forms could be

$$S(\Lambda) = e^{-\frac{i}{4}\sigma^{\mu\nu}\epsilon_{\mu\nu}},$$

where

$$\Lambda^{\nu}_{\mu} = \delta^{\nu}_{\mu} + \epsilon^{\nu}_{\mu}, \qquad \text{and} \qquad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}]. \tag{A.31}$$

Moreover,

$$\hat{S}^{-1} = \gamma^0\hat{S}^{\dagger}\gamma^0. \tag{A.32}$$

Another choice of S could be a 4×4 matrix given by

$$S = \begin{pmatrix} a_+ & 0 & 0 & a_- \\ 0 & a_+ & a_- & 0 \\ 0 & a_- & a_+ & 0 \\ a_- & 0 & 0 & a_+ \end{pmatrix} = \begin{pmatrix} a_+ I_{2\times 2} & a_-\sigma_1 \\ a_-\sigma_1 & a_+ I_{2\times 2} \end{pmatrix}, \tag{A.33}$$

where

$$a_+ = \sqrt{\frac{1}{2}(\gamma + 1)}, \qquad a_- = -\sqrt{\frac{1}{2}(\gamma - 1)},$$

$$I_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \text{and} \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

The components of the Dirac spinors do not transform as a four vector when one goes from one inertial system to another

$$\psi'(x') = \hat{S}(\hat{\Lambda})\psi(x).$$

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$\psi^\dagger\psi$ is not L.I. as

$$\begin{aligned}
\psi^\dagger\psi &= |\psi_1|^2+|\psi_2|^2+|\psi_3|^2+|\psi_4|^2\,, \\
(\psi^\dagger\psi)' &= (\psi'^\dagger)\psi', \\
&= (S\psi)^\dagger(S\psi)=\psi^\dagger S^\dagger S\psi, \\
&\neq \psi^\dagger\psi \qquad\quad \text{since}\,,\ \ S^\dagger S\neq 1.
\end{aligned} \tag{A.34}$$

$$\Rightarrow \qquad \begin{aligned}
S^\dagger &\neq S^{-1}, \\
\gamma^o S^\dagger\gamma^o &= S^{-1}, \\
\gamma^0 &= \begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix}.
\end{aligned} \tag{A.35}$$

It turns out that in the case of spinors, we need minus signs for the third and fourth components, for Lorentz invariance and for this, we introduce the adjoint of the Dirac spinor

$$\overline{\psi}=\psi^\dagger\gamma^0=\begin{pmatrix}\psi_1^* & \psi_2^* & -\psi_3^* & -\psi_4^*\end{pmatrix},$$

where

$$\begin{aligned}
\overline{\psi}\psi &= |\psi_1|^2+|\psi_2|^2-|\psi_3|^2-|\psi_4|^2\,, \\
(\overline{\psi}\psi)' &= (\psi^\dagger\gamma^0\psi)'=\psi'^\dagger\gamma^0\psi', \\
&= (S\psi)^\dagger\gamma^0(S\psi)=\psi^\dagger S^\dagger\gamma^0 S\psi,
\end{aligned} \tag{A.36}$$

$$\begin{aligned}
&= \psi^\dagger\gamma^0\gamma^0 S^\dagger\gamma^0 S\psi, \\
&= \overline{\psi}S^{-1}S\psi, \\
&= \overline{\psi}\psi.
\end{aligned} \tag{A.37}$$

Therefore, $\overline{\psi}\psi$ is invariant under Lorentz transformation.

Now, we look for the four current density under Lorentz transformation

$$\begin{aligned}
j'^\mu(x') &= \overline{\psi}'(x')\gamma^\mu\psi'(x'), \\
&= \psi'^\dagger(x')\gamma^o\gamma^\mu\psi'(x'), \\
&= \left[\hat{S}(\hat{\Lambda})\psi(x)\right]^\dagger\gamma^o\gamma^\mu\left[\hat{S}(\hat{\Lambda})\psi(x)\right], \\
&= \psi^\dagger(x)\gamma^o\gamma^o\hat{S}^\dagger(\hat{\Lambda})\gamma^o\gamma^\mu\hat{S}(\hat{\Lambda})\psi(x), \\
&= \overline{\psi}(x)\hat{S}^{-1}(\hat{\Lambda})\gamma^\mu\hat{S}(\hat{\Lambda})\psi(x), \\
&= \overline{\psi}(x)\Lambda^\mu_\nu\gamma^\nu\psi(x), \\
&= \Lambda^\mu_\nu\overline{\psi}(x)\gamma^\nu\psi(x), \\
&= \Lambda^\mu_\nu j^\nu(x),
\end{aligned} \tag{A.38}$$

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which transforms as a four vector.

$$\begin{aligned}
\overline{\psi}'(x')\gamma^5\psi'(x') &= \psi'^{\dagger}(x')\gamma^o\gamma^5\psi'(x'), \\
&= \left[\hat{S}\psi(x)\right]^{\dagger}\gamma^o\gamma^5\left[\hat{S}\psi(x)\right], \\
&= \psi^{\dagger}(x)\gamma^o\gamma^o\hat{S}^{\dagger}\gamma^o\gamma^5\hat{S}\psi(x), \\
&= \overline{\psi}(x)\hat{S}^{-1}\gamma^5\hat{S}\psi(x), \\
&= \overline{\psi}(x)\hat{S}^{-1}\hat{S}\gamma^5\psi(x), \qquad (\gamma^5\hat{S} = \hat{S}\gamma^5) \\
&= \overline{\psi}(x)\gamma^5\psi(x), \qquad \text{(A.39)}
\end{aligned}$$

which is invariant under L.T.

Similarly, one can show that

1. $\overline{\psi}'(x')\gamma_5\gamma^\mu\psi'(x') = \det(\Lambda)\Lambda^\mu_\nu\overline{\psi}(x)\gamma_5\gamma^\mu\psi(x)$ transforms as a pseudovector.
2. $\overline{\psi}'(x')\hat{\sigma}^{\mu\nu}\psi'(x') = \Lambda^\mu_\alpha\Lambda^\nu_\beta\overline{\psi}(x)\hat{\sigma}^{\alpha\beta}\psi(x)$ is a tensor of rank two, with

$$\hat{\sigma}_{\mu\nu} = \frac{i}{2}[\gamma_\mu,\gamma_\nu] = \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$$

A.3 Bilinear Covariants

The transition matrix element for any given physical process is a scalar quantity. The matrix element is written using currents involved in the leptonic and hadronic vertices. These currents are obtained using the Lagrangian which involves nature and strength of the interaction. Therefore, in order to construct these currents, one needs bilinear covariants such that the current takes the following form

$$j_\mu = \bar{\psi}(p')\ (\text{bilinear covariants})\ \psi(p).$$

Since $\bar{\psi}$ is a 1×4 matrix and ψ is a 4×1 matrix, in order to make j_μ a constant, these bilinear covariants must be some 4×4 matrices. These bilinear covariants are nothing but different combinations of γ matrices, which gives five types of bilinear covariants depending upon their behavior under Lorentz (described here earlier) and parity (Chapter 2) transformations. The quantities which are invariant under L.T. as well as under parity transformation (P.T.) are called scalars or pure scalars while the quantities which are invariant under L.T. but change sign under P.T. are called pseudoscalars. The quantities which transform like a four vector under L.T. and also change sign under P.T. are called vectors; the quantities which transform like a four vector under L.T. but do not change sign under P.T. are called pseudovectors or axial vectors. We also have quantities which transform like a tensor.

$$\begin{aligned}
\overline{\psi}(\vec{x},t)\ \sigma^{oi}\ \psi(\vec{x},t) &\xrightarrow{\hat{P}} -\overline{\psi}(-\vec{x},t)\ \sigma^{oi}\ \psi(-\vec{x},t). \\
\overline{\psi}(\vec{x},t)\ \sigma^{ij}\ \psi(\vec{x},t) &\xrightarrow{\hat{P}} \overline{\psi}(-\vec{x},t)\ \sigma^{ij}\ \psi(-\vec{x},t).
\end{aligned}$$

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In total, there are sixteen components; the tensor component is

$$\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu).$$

$$\begin{array}{rcll}
\overline{\psi}(x)\, 1\, \psi(x) & \longrightarrow & \text{Scalar (one component)} & \text{(A.40)}\\
\overline{\psi}(x)\, \gamma^5\, \psi(x) & \longrightarrow & \text{Pseudoscalar (one component)} & \text{(A.41)}\\
\overline{\psi}(x)\, \gamma^\mu\, \psi(x) & \longrightarrow & \text{Vector (four components)} & \text{(A.42)}\\
\overline{\psi}(x)\, \gamma^\mu\gamma^5\, \psi(x) & \longrightarrow & \text{Axial vector (four components)} & \text{(A.43)}\\
\overline{\psi}(x)\, \sigma^{\mu\nu}\, \psi(x) & \longrightarrow & \text{Antisymmetric tensor (six components).} & \text{(A.44)}
\end{array}$$

A.4 Nonrelativistic Reduction

Nonrelativistic reduction is applied to the process where kinetic energy of the particles is very small as compared to the rest of the mass ($E_k << M$). For example, in the β-decay processes

$$\begin{aligned}
n &\rightarrow p + e^- + \overline{\nu}_e,\\
X &\rightarrow Y + e^- + \overline{\nu}_e,
\end{aligned}$$

$E_n \approx M_n$ and $E_p \approx M_p$ or $E_X \approx M_X$ and $E_Y \approx M_Y$ and if four momenta for n or X is p and for proton or Y is p', then in the nonrelativistic limit

$$\begin{aligned}
E + M &\simeq 2E \text{ or } 2M,\\
E' + M &\simeq 2E' \text{ or } 2M,
\end{aligned}$$

$$\begin{aligned}
u(p) &= \sqrt{\frac{E+M}{2M}}\begin{pmatrix} \chi^s \\ \frac{\vec{\sigma}.\vec{p}}{2M}\chi^s \end{pmatrix},\\
\overline{u}(p') &= \sqrt{\frac{E'+M}{2M}}\begin{pmatrix} \chi^{s^\dagger} & -\frac{\vec{\sigma}.\vec{p}'}{2M}\chi^{s^\dagger} \end{pmatrix}.
\end{aligned}$$

1. Scalar interaction (I_4):

$$\begin{aligned}
\overline{u}(p')u(p) &= \sqrt{\frac{(E+M)(E'+M)}{2M.2M}}\begin{pmatrix} \chi^{s^\dagger} & -\frac{\vec{\sigma}.\vec{p}'}{2M}\chi^{s^\dagger} \end{pmatrix}\begin{pmatrix} \chi^s \\ \frac{\vec{\sigma}.\vec{p}}{2M}\chi^s \end{pmatrix},\\
&= \left(\chi^{s^\dagger}\chi^s - \frac{\vec{\sigma}\cdot\vec{p}'\vec{\sigma}\cdot\vec{p}}{4M_N^2}\chi^{s^\dagger}\chi^s\right),\\
&= \left(\chi^{s^\dagger}\chi^s - \frac{\vec{p}\cdot\vec{p}' + i\vec{\sigma}\cdot\vec{p}\times\vec{p}'}{4M_N^2}\chi^{s^\dagger}\chi^s\right).
\end{aligned}$$

However, $\vec{p} \approx \vec{p}\,'$ (in the nonrelativistic limit), as kinetic energy of the outgoing hadron is almost zero which implies negligible momentum transfer. Therefore,

$$\overline{u}(p')u(p) = \left(1 - \frac{|\vec{p}|^2}{4M^2}\right) = \left(1 - \frac{E^2 - M_N^2}{4M^2}\right) \simeq 1. \tag{A.45}$$

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2. Pseudoscalar interaction (γ_5):

$$\begin{aligned}
\overline{u}(p')\gamma^5 u(p) &= \begin{pmatrix}\chi^{s^\dagger} & -\frac{\vec{\sigma}.\vec{p}'}{2M}\chi^{s^\dagger}\end{pmatrix}\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\begin{pmatrix}\chi^s\\ \frac{\vec{\sigma}.\vec{p}}{2M}\chi^s\end{pmatrix},\\
&= \frac{\vec{\sigma}.\vec{p}}{2M}\chi^{s\dagger}\chi^s - \frac{\vec{\sigma}.\vec{p}}{2M}\chi^{s\dagger}\chi^s,\\
&= 0. \qquad\qquad (A.46)
\end{aligned}$$

Therefore, nonrelativistic reduction shows that the pseudoscalar term vanishes in β-decay, since the energy involved is very small.

3. Vector interaction (γ_μ): For the zeroth component, we substitute $\mu = 0$ in $\overline{u}(p)\gamma^\mu u(p)$,

$$\begin{aligned}
\overline{u}(p')\gamma^o u(p) &= \begin{pmatrix}\chi^{s^\dagger} & -\frac{\vec{\sigma}.\vec{p}'}{2M}\chi^{s^\dagger}\end{pmatrix}\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}\begin{pmatrix}\chi^s\\ \frac{\vec{\sigma}.\vec{p}}{2M}\chi^s\end{pmatrix},\\
&= \left(\chi^{s^\dagger}\chi^s + \frac{\vec{p}\cdot\vec{p}' + i\vec{\sigma}\cdot\vec{p}\times\vec{p}'}{4M_N^2}\chi^{s^\dagger}\chi^s\right), = 1,
\end{aligned}$$

because $\vec{p} \approx \vec{p}\,'$ and $T_N' = 0$. Substituting $\mu = i$ in $\overline{u}(p)\gamma^\mu u(p)$, we have

$$\begin{aligned}
\overline{u}(p')\gamma^i u(p) &= \begin{pmatrix}\chi^{s^\dagger} & -\frac{\vec{\sigma}.\vec{p}'}{2M}\chi^{s^\dagger}\end{pmatrix}\begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}\begin{pmatrix}\chi^s\\ \frac{\vec{\sigma}.\vec{p}}{2M}\chi^s\end{pmatrix},\\
&= \begin{pmatrix}\chi^{s^\dagger} & -\frac{\sigma_j p_j'}{2M}\chi^{s^\dagger}\end{pmatrix}\begin{pmatrix}\frac{\sigma_i\sigma_l p_l}{2M}\chi^s\\ -\sigma_i\chi^s\end{pmatrix},\\
&= \frac{\sigma_i\sigma_l p_l}{2M} + \frac{\sigma_j\sigma_i p_j'}{2M},\\
&= \frac{\delta_{il}p_l + i\epsilon_{ilm}\sigma_m p_l}{2M} + \frac{\delta_{ji}p_j' + i\epsilon_{jik}p_j'\sigma_k}{2M},
\end{aligned}$$

$$= \frac{p_i + p_i'}{2M} + \frac{i(\epsilon_{ilm}\sigma_m p_l - \epsilon_{ijk}p_j'\sigma_k)}{2M}.$$

4. Axial vector interaction ($\gamma_\mu\gamma_5$): The zeroth component for the axial vector interaction given by $\overline{u}(p')\gamma^\mu\gamma_5 u(p)$, can be obtained as

$$\begin{aligned}
\overline{u}(p')\gamma^o\gamma_5 u(p) &= \begin{pmatrix}\chi^{s^\dagger} - \frac{\vec{\sigma}.\vec{p}'}{2M}\chi^{s^\dagger}\end{pmatrix}\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\begin{pmatrix}\chi^s\\ \frac{\vec{\sigma}.\vec{p}}{2M}\chi^s\end{pmatrix},\\
&= \begin{pmatrix}\chi^{s^\dagger} - \frac{\vec{\sigma}.\vec{p}'}{2M}\chi^{s^\dagger}\end{pmatrix}\begin{pmatrix}\frac{\vec{\sigma}.\vec{p}}{2M}\chi^s\\ -\chi^s\end{pmatrix},
\end{aligned}$$

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$$
\begin{aligned}
&= \frac{\vec{\sigma}\cdot(\vec{p}+\vec{p}')}{2E}, \\
&= \frac{\vec{\sigma}\cdot\vec{P}}{2M} \quad (\text{for } E = E' = M \text{ and } \vec{P} = \vec{p}+\vec{p}'.)
\end{aligned} \tag{A.47}
$$

Similarly, for the ith component, we find

$$
\begin{aligned}
\bar{u}(p')\gamma_i\gamma_5 u(p) &= \left(\chi^{s^\dagger} - \tfrac{\vec{\sigma}.\vec{p}'}{2M}\chi^{s^\dagger}\right)\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} \chi^s \\ \frac{\vec{\sigma}.\vec{p}}{2M}\chi^s \end{pmatrix}, \\
&= \left(\chi^{s^\dagger} - \tfrac{\vec{\sigma}.\vec{p}'}{2M}\chi^{s^\dagger}\right)\begin{pmatrix} \sigma_i\chi^s \\ -\sigma_i\frac{\vec{\sigma}.\vec{p}}{2M}\chi^s \end{pmatrix}, \\
&= \left(\vec{\sigma} + \frac{\vec{\sigma}(\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p}')}{4E^2}\right) \qquad (\text{for } E = E' = M).
\end{aligned} \tag{A.48}
$$

5. Tensor interaction ($\sigma_{\mu\nu}q^\nu$): We expand $\sigma_{\mu\nu}q^\nu$ as

$$
\sigma_{\mu\nu}\, q^\nu = \sigma_{\mu 0}\, q^0 - \sigma_{\mu i}\, q^i.
$$

Since $q^0 = E - E' = 0$ for $E = E'$, only $\nu = i$ will contribute, that is,

$$
\begin{aligned}
\sigma_{\mu\nu}\, q^\nu &= -\sigma_{\mu i}\, q^i = -\left[\sigma_{0i}\, q^i,\ \sigma_{ji}\, q^i\right] \\
&= -i\left[\begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{q} \\ \vec{\sigma}\cdot\vec{q} & 0 \end{pmatrix}, \begin{pmatrix} \vec{\sigma}\times\vec{q} & 0 \\ 0 & \vec{\sigma}\times\vec{q} \end{pmatrix}\right].
\end{aligned} \tag{A.49}
$$

Hence,

$$
i\,\bar{u}(p')\,\sigma_{\mu\nu}\, q^\nu u(p) = -i\left[\bar{u}(p')\,\sigma_{0i}\, q^i\, u(p),\ \bar{u}(p')\,\sigma_{ji}\, q^i\, u(p)\right]. \tag{A.50}
$$

We have

$$
\begin{aligned}
-i\,\bar{u}(p')\,\sigma_{0i}\, q^i u(p) &= -\left[\left(\chi^{s^\dagger} - \tfrac{\vec{\sigma}.\vec{p}'}{2M}\chi^{s^\dagger}\right)\begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{q} \\ \vec{\sigma}\cdot\vec{q} & 0 \end{pmatrix}\begin{pmatrix} \chi^s \\ \frac{\vec{\sigma}.\vec{p}}{2M}\chi^s \end{pmatrix}\right] \\
&= -\left[\frac{(\vec{\sigma}\cdot\vec{q})(\vec{\sigma}\cdot\vec{q})}{2M}\right]
\end{aligned} \tag{A.51}
$$

and

$$
\begin{aligned}
-i\,\bar{u}(p')\,\sigma_{ji}\, q^j u(p) &= -\left[\left(\chi^{s^\dagger} - \tfrac{\vec{\sigma}.\vec{p}'}{2M}\chi^{s^\dagger}\right)\begin{pmatrix} \vec{\sigma}\times\vec{q} & 0 \\ 0 & \vec{\sigma}\times\vec{q} \end{pmatrix}\begin{pmatrix} \chi^s \\ \frac{\vec{\sigma}.\vec{p}}{2M}\chi^s \end{pmatrix}\right] \\
&= i\left[\left(\vec{\sigma}\times\vec{q} - \frac{(\vec{\sigma}\times\vec{q})(\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p}')}{4M^2}\right)\right].
\end{aligned} \tag{A.52}
$$

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