Chapter 3

Quantization of Free Particle Fields

3.1 Introduction

The concept of associating particles with fields originated during the study of various physical phenomena involving electromagnetic radiation. For example, the observations and theoretical explanations of the black body radiation by Planck, the photoelectric effect by Einstein, and the scattering of a photon off an electron by Compton established that electromagnetic radiation can be described in terms of "discrete quanta of energy" called photon, identified as a massless particle of spin 1. Consequently, Maxwell's equations of classical electrodynamics, describing the time evolution of the electric ($\vec{E}(\vec{x},t)$) and magnetic ($\vec{B}(\vec{x},t)$) fields are interpreted to be the equations of motion of the photon, written in terms of the massless spin 1 electromagnetic field $A^{\mu}(\vec{x},t)$. Later, the quantization of the electromagnetic field $A^{\mu}(\vec{x},t)$ was formulated to explain the emission and absorption of radiation in terms of the creation and annihilation of photons during the interaction of the electromagnetic field with the physical systems. The concept of treating photons as quanta of the electromagnetic fields was successful in explaining the physical phenomena induced by the electromagnetic interactions; methods of field quantization were used leading to quantum electrodynamics (QED), the quantum field theory of electromagnetic interactions. The concept was later generalized by Fermi [23, 207] and Yukawa [208, 209] to formulate, respectively, the theory of weak and strong interactions in analogy with the theory of QED.

In order to describe QED, the quantum field theory of electromagnetic fields and their interaction with matter, in terms of the massless spin 1 fields $A^{\mu}(\vec{x},t)$ corresponding to photons, the equations of motion of $A^{\mu}(\vec{x},t)$ should be fully relativistic. This requires the reformulation of classical equations of motion for the fields to obtain the quantum equations of motion for the fields and find their solutions, in case of free fields as well as fields interacting with matter. This is generally done using perturbation theory for which a relativistically covariant perturbation theory is required.

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The path of transition from a classical description of fields to a quantum description of fields, requiring the quantization of fields, their equations of motion, propagation, and interaction with matter involves understanding many new concepts and mathematical methods. For this purpose, the Lagrangian formulation for describing the dynamics of particles and their interaction with the fields is found to be suitable. In this chapter, we attempt to explain this formulation in the case of free fields; we take up the case of interacting fields in the next chapter. The mathematical treatment in this chapter and the following chapter is based on the materials contained in Refs.[210] and [211].

3.2 Lagrangian Formulation for the Dynamics of Particles and Fields

3.2.1 Equation of motion for particles

The equation of motion of a particle in classical physics is described in terms of a set of variables called generalized coordinates $q^i (i = 1 - n)$ depending upon the degrees of freedom, for the motion of the particle. For example, for a particle moving in 3-dimensions, there will be three generalized coordinates, taken to be either the cartesian (x, y, z) or the spherical polar (r, θ, ϕ) coordinates. In most of the cases, the generalized coordinates are chosen to be position coordinates $x_i (i = 1 - n)$, which evolve with time t according to the predictions made by the equation of motion. Historically, these equations of motion are derived from Newton's laws of motion but can also be derived from the "principle of least action(S)", where the action S is derived in terms of the Lagrangian L as:

$$S = \int_{t_1}^{t_2} dt\, L = \int \mathcal{L}\, d\vec{x}\, dt, \tag{3.1}$$

where $L = \int d\vec{x}\, \mathcal{L}$. $\mathcal{L}$ is known as the Lagrangian density, which is function of the generalized coordinates x_i, generalized velocity $\dot{x}_i(t) = \left(\frac{dx_i(t)}{dt}\right)$, and time t, and therefore:

$$S = \int_{A(t_1)}^{B(t_2)} dt\, d\vec{x} \mathcal{L}(x_i(t), \dot{x}_i(t), t). \tag{3.2}$$

The principle of least action states that out of many possibilities of the motion of particles between A and B (Figure 3.1), the classical motion of the particles is such that the action S is extremum. Any variation in $x_i(t)$ like $x_i(t) \to x_i(t) + \delta x_i(t)$ vanishes at the end points A and B, where $\delta x_i(t_1) = \delta x_i(t_2) = 0$, as shown in Figure 3.1. For any path $x_i'(t)$ other than $x_i(t)$ such that $x_i'(t) = x_i(t) + \delta x_i(t)$, the change in action δS is given by:

$$\begin{aligned} \delta S &= \delta \int_A^B L(x_i(t), \dot{x}_i(t)) dt \\ &= \int_A^B \sum_i \left[\frac{\partial L}{\partial x_i} \delta x_i + \frac{\partial L}{\partial \dot{x}_i} \delta \dot{x}_i \right] dt \qquad \left(\because \delta \dot{x}_i(t) = \frac{d}{dt}(\delta x_i)\right) \end{aligned}$$

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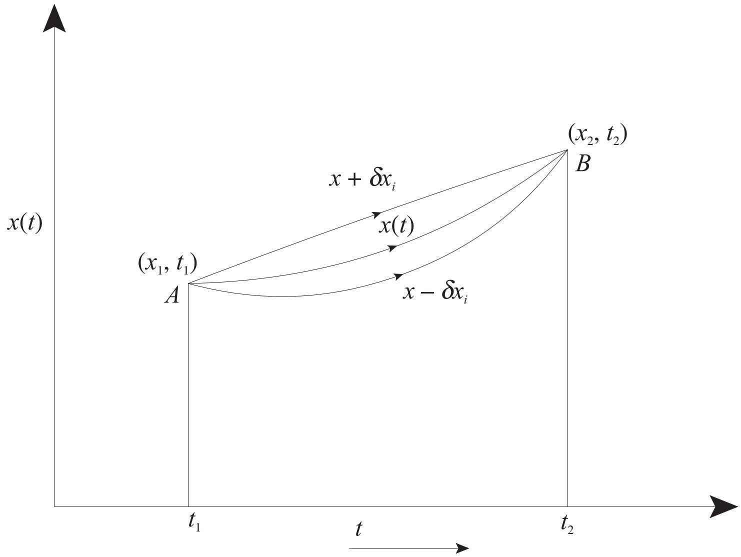

Figure 3.1 δ variation-extremum path.

$$= \int_A^B \sum_i \left\{ \frac{\partial L}{\partial x_i}\delta x_i + \frac{\partial L}{\partial \dot{x}_i}\frac{d}{dt}(\delta x_i) \right\} dt$$

$$= \int_A^B \sum_i \left\{ \frac{\partial L}{\partial x_i}\delta x_i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}(\delta x_i)\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right)(\delta x_i) \right\} dt$$

$$= \int_A^B \sum_i \left\{ \frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) \right\} \delta x_i \, dt$$

as $\delta x_i(t_1) = 0, \delta x_i(t_2) = 0,$ at the end points A and B.

Since the variations $\delta x_i(t)$ are arbitrary, the principle of least action, that is, $\delta S = 0$ leads to the well known Euler–Lagrange equation of motion for the particle, that is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0. \tag{3.3}$$

The momentum of the particle p_i corresponding to the motion along x_i is defined as:

$$p_i = \frac{\partial L}{\partial \dot{x}_i} \tag{3.4}$$

and the Hamiltonian (H) of the particle is defined as:

$$H = \sum_i p_i \dot{x}_i - L(x_i, \dot{x}_i, t) = \sum_i \frac{\partial L}{\partial \dot{x}_i}\dot{x}_i - L(x_i, \dot{x}_i, t). \tag{3.5}$$

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Let us consider the example of simple harmonic motion of a particle of mass m attached to a spring under a force $\vec{F}$, say along the x- direction, given by $F_x = -kx$, where k is the spring constant. Newton's equations of motion leads to the equations of motion for the particle being:

$$m\frac{d^2x}{dt^2} + kx = 0, \tag{3.6}$$

with the solution

$$x = x_0 \sin \omega t, \tag{3.7}$$

$$\text{where } \omega = \sqrt{\frac{k}{m}}.$$

This can be derived by considering a Lagrangian defined in terms of the position of the particle, that is, x(it has only one degree of freedom), written as:

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2. \tag{3.8}$$

The Euler–Lagrange equation of motion (Eq. (3.3)) leads to the equation:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad \text{and} \qquad p_x = m\dot{x}. \tag{3.9}$$

The Hamiltonian H is given by:

$$H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2 x^2 = T + V, \tag{3.10}$$

which is the total energy.

In classical physics, the equations of motion for a system derived by applying the Euler–Lagrange equation (Eq. (3.3)) are deterministic. This means that once initial conditions are given through the specification of position $\vec{x}$ and velocity $\vec{v} = \frac{\vec{p}}{m}$ at time $t = 0$, the dynamical behavior of the particle is completely determined.

In quantum physics, dynamical variables of classical physics, like the position x and the momentum p_x, are treated as operators, that is,

$$x \to \hat{x} \quad \text{and} \quad p_x \to \hat{p}_x = -i\frac{\partial}{\partial x}$$

or in general

$$\vec{x} \to \hat{\vec{x}}, \qquad \vec{p} \to \hat{\vec{p}} = -i\vec{\nabla},$$

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such that $\hat{x}$ and $\hat{p}_x$ do not commute, and the commutator of $\hat{x}$ and $\hat{p}_x$ is given by:

$$[\hat{x}, \hat{p}_x] = i, \tag{3.11}$$

implying that the physical observables, like the position x_i and the momentum p_i, are the eigenvalues of the operators $\hat{x}_i$ and $\hat{p}_i$, respectively, and cannot be determined simultaneously. In the presence of many degrees of freedom, the commutation relations take the form:

$$[\hat{x}_i, \hat{p}_j] = i\delta_{ij}, \tag{3.12}$$

$$[\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0. \tag{3.13}$$

This procedure of obtaining quantum equations of motion is also applied to the fields.

3.2.2 Quantization of a harmonic oscillator

We start with the Hamiltonian for the harmonic oscillator, that is,

$$H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2 x^2. \tag{3.14}$$

We first introduce the operators $\hat{q}$ and $\hat{p}$ as:

$$\hat{q} = \sqrt{\alpha}\hat{x} \quad \text{and} \quad \hat{p} = \frac{\hat{p}_x}{\sqrt{\alpha}}, \quad \text{where } \alpha = m\omega \tag{3.15}$$

which satisfy the commutation relations $[\hat{q}, \hat{p}] = i$.

Let us now introduce $\hat{a}^\dagger$ and $\hat{a}$ operators such that:

$$\begin{aligned} \hat{a} &= \tfrac{1}{\sqrt{2}}(\hat{q} + i\hat{p}), \\ \hat{a}^\dagger &= \tfrac{1}{\sqrt{2}}(\hat{q} - i\hat{p}), \end{aligned} \tag{3.16}$$

then it can be shown that:

$$\left[\hat{a}, \hat{a}^\dagger\right] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1, \tag{3.17}$$

$$\text{and } \hat{H} = \frac{\omega}{2}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}), \tag{3.18}$$

$$= \omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \omega\left(N + \frac{1}{2}\right), \qquad \text{where } N = \hat{a}^\dagger\hat{a}. \tag{3.19}$$

To see the physical interpretation of N, let us calculate the energy of a state ψ using Eq. (3.19) that is,

$$H\psi = \omega\left(N + \frac{1}{2}\right)\psi = E\psi \tag{3.20}$$

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and compare it with the result in conventional quantum mechanics where the energy of the harmonic oscillator is derived to be:

$$H\psi = \left(n + \frac{1}{2}\right)\omega\psi \quad \text{with } \; n = 0, 1, 2, 3... \tag{3.21}$$

Therefore, N can be identified as a number operator with integer eigenvalues 0,1,2,... such that for a state $|\psi\rangle$

$$N|\psi\rangle = n|\psi\rangle, \tag{3.22}$$

with the lowest value of $n = 0$. This can be also seen by considering the expectation value of N in any state ψ as:

$$\langle\psi|N|\psi\rangle = \langle\psi|\hat{a}^\dagger\hat{a}|\psi\rangle = \langle\hat{a}\psi|\hat{a}\psi\rangle \geq 0. \tag{3.23}$$

Therefore, the minimum value of $\langle\psi|N|\psi\rangle$ is zero corresponding to the eigenvalues of the lowest eigenstate, that is, the ground state of a harmonic oscillator. To see the physical interpretation of operators $\hat{a}$ and $\hat{a}^\dagger$, we note that they satisfy the following commutation relations

$$[N, \hat{a}^\dagger] = \hat{a}^\dagger \quad \text{and} \qquad [N, \hat{a}] = -\hat{a}. \tag{3.24}$$

Operating Eq. (3.24) on a state ψ, which satisfies Eq. (3.22), we obtain:

$$\begin{aligned} [N, \hat{a}^\dagger]|\psi\rangle &= \hat{a}^\dagger|\psi\rangle, \\ N\,\hat{a}^\dagger|\psi\rangle &= (n+1)\,\hat{a}^\dagger|\psi\rangle, \end{aligned} \tag{3.25}$$

$$\text{and similarly } \; N\,\hat{a}|\psi\rangle = (n-1)\,\hat{a}\,|\psi\rangle. \tag{3.26}$$

Equations (3.25) and (3.26) show that $\hat{a}$ and $\hat{a}^\dagger$ lower and raise the value of n by 1. By lowering the value of n by 1 in successive operations, we can reach the ground state $|\psi_0\rangle$ corresponding to the eigenvalues zero, such that:

$$\hat{a}|\psi_0\rangle = 0, \;\text{ that is, }\; |\psi_0\rangle = |0\rangle, \tag{3.27}$$

where $|0\rangle$ is the ground state and has energy

$$H|0\rangle = \omega(N + \frac{1}{2})|0\rangle = \frac{\omega}{2} \tag{3.28}$$

called the zero point energy of the harmonic oscillator. Since $\hat{a}$ and $\hat{a}^\dagger$ lower and raise the number by 1, respectively, we can write the action of $\hat{a}$ and $\hat{a}^\dagger$ on the normalized state $|n\rangle$, such that $\langle n|n\rangle = 1$. For $|n\rangle$ states, the following operator equations are obtained:

$$\hat{a}|n\rangle = \sqrt{n}|\,n-1\rangle \quad \text{and} \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle. \tag{3.29}$$

It is easy to see that a normalized state $|n\rangle$ can be generated from the ground state $|0\rangle$ by operating $\hat{a}^\dagger$, n number of terms to obtain:

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle, \quad n = 0, 1, 2, ..., \tag{3.30}$$

$$\text{with } \; H|n\rangle = \left(n + \frac{1}{2}\right)\omega|n\rangle. \tag{3.31}$$

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3.2.3 Equation of motion for fields

The motion of a classical particle with specified degrees of freedom as a function of time t is described by its position at a given point in space. On the other hand, the interpretation of the particle as a quanta of field $\phi(\vec{x},t)$ necessitates its description in terms of the equation of motion for the field $\phi(\vec{x},t)$ which is spread over a region of space and varies with time. Since the field $\phi(\vec{x},t)$ is a continuous function of space and time, this implies infinite degrees of freedom. Therefore, the equation of motion for fields are obtained by generalizing the principle of least action to a continuous system of infinite degrees of freedom. The Lagrangian for a particular field is written in terms of the fields and their derivatives using general principles of Lorentz invariance and other symmetries of the system, when in interaction. For a free field, the Lagrangian includes only the kinetic energy and mass terms and is constructed such that the equations of motion for various particles described in the last chapter are reproduced. These particles of spin 0, $\frac{1}{2}$, or 1 are now described by the fields carrying spin 0, $\frac{1}{2}$, or 1 and not by the single particle wave functions discussed in Chapter 2. These fields can then be quantized in terms of the creation and annihilation operators to describe the emission and absorption of particles corresponding to the fields.

In general, the action S in the case of fields is defined as:

$$S(\Omega) = \int L(\vec{x},t)dt = \int_\Omega d^4x\mathcal{L}(\phi_i(\vec{x},t),\phi_{i,\alpha}(\vec{x},t),t), \tag{3.32}$$

where $L(\vec{x},t)$ is the Lagrangian and $\mathcal{L}(\phi_i(\vec{x},t),\phi_{i,\alpha}(\vec{x},t))$ is the Lagrangian density for the field $\phi(\vec{x},t)$. $d^4x(=dx^0d\vec{x})$ is the four-dimensional element, $\phi_{i,\alpha} = \partial_\alpha\phi_i(x) = \frac{\partial\phi_i}{\partial x^\alpha}$, is the derivative of the field $\phi_i(x)$ and the integration is performed over a region of volume element Ω, in four-dimensional space–time. The principle of least action implies that under a variation of fields $\phi_i(x)$, the action S is extremum, that is, $\delta S = 0$. If we define the variation of fields $\phi_i(x)$ as $\delta\phi_i(x)$, that is,

$$\phi_i(x) \to \phi_i(x) + \delta\phi_i(x),$$

such that the variation at the surface $\mathcal{S}(\Omega)$ of volume Ω vanishes, that is, $\delta\phi_i(x) = 0$ on $\mathcal{S}(\Omega)$, then the principle of least action states that the change in the action δS is zero, that is,

$$\delta S = \int_\Omega \left(\frac{\partial\mathcal{L}}{\partial\phi_i(x)}\delta\phi_i(x) + \frac{\partial\mathcal{L}}{\partial\phi_{i,\alpha}(x)}\delta\phi_{i,\alpha}(x)\right)d^4x = 0 \tag{3.33}$$

as all the $\phi_i(x)$ and $\phi_{i,\alpha}(x)$ for $i = 1-n$ are linearly independent. The fields $\phi_i(x)$ can be taken as real or complex fields. In the case of complex fields, each $\phi_i(x)$ can be treated as two independent real fields, or equivalently, two independent fields $\phi(x)$ and $\phi^*(x)$. The principle of least action in Eq. (3.33) leads to the Euler–Lagrange equation of motion for the fields $\phi_i(x)$. Consider:

$$\delta S \quad = \quad \int_\Omega d^4x\left(\frac{\partial\mathcal{L}}{\partial\phi_i(x)}\delta\phi_i(x) + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi_i(x))}\partial_\alpha\delta\phi_i(x)\right)$$

$$= \int_\Omega d^4x \frac{\partial \mathcal{L}}{\partial \phi_i(x)} \delta\phi_i(x) - \int_\Omega d^4x\, \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_i(x))} \right) \delta\phi_i(x)$$

$$+ \frac{\partial \mathcal{L}}{\partial(\dot{\phi}_i(x))} \delta\phi_i(x) \bigg|_{t_i}^{t_f}. \tag{3.34}$$

At the extremum, $\delta\phi_i(\vec{x}, t_i) = \delta\phi_i(\vec{x}, t_f) = 0$, which leads to:

$$\delta S = \int_\Omega d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_i(x)} - \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_i(x))} \right) \right] \delta\phi_i(x), \tag{3.35}$$

and for an arbitrary $\delta\phi_i(x)$

$$\partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_i(x))} \right) - \frac{\partial \mathcal{L}}{\partial \phi_i(x)} = 0, \qquad \text{if } \delta S = 0. \tag{3.36}$$

In order to quantize the classical field, we apply the method of canonical quantization used in nonrelativistic quantum mechanics and define the canonical momentum density $\pi_i(\vec{x}, t)$ corresponding to the field $\phi_i(\vec{x}, t)$ as:

$$\pi_i(\vec{x}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i(\vec{x}, t)}. \tag{3.37}$$

With the definition of the conjugate momentum(Eq. (3.37)), the fields $\phi_i(\vec{x}, t)$ and the conjugate momenta $\pi_i(\vec{x}, t)$ become operators and field quantization is implemented through the commutation relations of $\hat{\phi}_i(\vec{x}, t)$ and $\hat{\pi}_j(\vec{x}, t)$ in analogy with the quantization in particle mechanics (Eqs. (3.12) and (3.13)), that is,

$$[\hat{\phi}_i(\vec{x}, t), \hat{\pi}_j(\vec{x}\,', t)] = i\delta(\vec{x} - \vec{x}\,')\delta_{ij}, \tag{3.38}$$

$$[\hat{\phi}_i(\vec{x}, t), \hat{\phi}_j(\vec{x}, t)] = [\hat{\pi}_i(\vec{x}, t), \hat{\pi}_j(\vec{x}, t)] = 0. \tag{3.39}$$

It should be noted that these are the commutation relations of fields at equal time, that is, $t = t'$ and are called equal time commutators (ETC). Thus, the transition from the classical field theory to the quantum field theory is made by introducing the commutation relations among the field variables $\phi_i(\vec{x}, t)$ and $\pi_j(\vec{x}, t)$. The method is known as second quantization, while the commutation relations given in Eqs. (3.12) and (3.13) for $\hat{x}$ and $\hat{p}_x$, etc. are known as first quantization. The quantization of the fields leads to the description of the field in terms of the particle which is acronymed as "the field quanta". These field quanta carry energy, momentum, and charge of the field.

3.2.4 Symmetries and conservation laws: Noether's theorem

Symmetries have played an important role in the development of physics and mathematics. They manifest through the invariance of the Lagrangian under certain transformations defining the symmetry. It has been shown that the invariance of the Lagrangian describing any system under the symmetry transformations implies the existence of conserved quantities. This is known as Noether's theorem [212] and the conserved quantities are called Noether's charges or Noether's currents, so named after its discoverer. Therefore, there is a conservation law associated with every continuous symmetry. For example, translational invariance leads to the conservation of linear momentum; rotational invariance leads to the conservation of angular momentum, etc.

Let us consider infinitesimal transformations in the case of internal symmetry where the dynamical variable x is unchanged as $\phi_i(x) \to \phi_i'(x) = e^{i\alpha}\phi_i(x)$, where α is independent of x, such that:

$$\phi_i(x) \to \phi_i'(x) \quad = \quad \phi_i(x) + \delta\phi_i(x). \tag{3.40}$$

The variation in the Lagrangian due to the variation in the field will be

$$\begin{aligned} \delta L &= \sum_i \left(\frac{\partial L}{\partial \phi_i(x)} \delta\phi_i(x) + \frac{\partial L}{\partial \phi_{i,\alpha}(x)} \delta\phi_{i,\alpha}(x) \right) \\ &= \sum_i \left(\partial_\alpha \left(\frac{\partial L}{\partial \phi_{i,\alpha}(x)} \right) \delta\phi_i(x) + \frac{\partial L}{\partial \phi_{i,\alpha}(x)} \delta\phi_{i,\alpha}(x) \right) \\ &= \sum_i \partial_\alpha \left(\frac{\partial L}{\partial \phi_{i,\alpha}(x)} \delta\phi_i(x) \right) \end{aligned} \tag{3.41}$$

If L is invariant under the transformation given in Eq. (3.40), that is, $\delta L = 0$, then the quantity in the parenthesis is known as the conserved current given by:

$$\partial_\alpha J_i^\alpha = 0, \tag{3.42}$$

where

$$J_i^\alpha = \frac{\partial L}{\partial(\partial_{i,\alpha}\phi(x))}\, \delta\phi_i(x). \tag{3.43}$$

The charge operator Q is defined as:

$$Q = \int d\vec{x}\, J^0, \tag{3.44}$$

and since the integral of a total charge vanishes, we may write

$$\frac{dQ}{dt} = \int d\vec{x}\partial_0 J^0 = -\int_V d\vec{x}\partial_i\, J^i = 0, -\int_S S_i \cdot J^i = 0 \tag{3.45}$$

which implies that Q is time independent. Thus,

$$[Q, H] = 0. \tag{3.46}$$

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Let us consider the case when the symmetry transformations involve the change in space coordinates as well as the fields given as:

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu \tag{3.47}$$

$$\text{and} \quad \phi_r(x) \to \phi'_r(x') = \phi_r(x) + \delta\phi_r(x). \tag{3.48}$$

Since

$$\delta\phi_r(x) = \phi'_r(x) - \phi_r(x), \tag{3.49}$$

we may define

$$\begin{aligned}\delta_T\phi_r(x) &= \phi'_r(x') - \phi_r(x) = \phi'_r(x') - \phi_r(x') + \phi_r(x') - \phi_r(x) \\ &= \delta\phi_r(x') + \frac{\partial\phi_r}{\partial x_\nu}\delta x_\nu.\end{aligned} \tag{3.50}$$

For the first order, the changes will be small such that:

$$\delta\phi_r(x') \approx \delta\phi_r(x) \tag{3.51}$$

$$\Rightarrow \delta_T\phi_r(x) = \delta\phi_r(x) + \frac{\partial\phi_r}{\partial x_\nu}\delta x_\nu. \tag{3.52}$$

Invariance of the Lagrangian is required, such that:

$$\mathcal{L}(\phi'_r(x'), \phi'_{r,\mu}(x')) - \mathcal{L}(\phi_r(x)\phi_{r,\mu}(x)) = 0. \tag{3.53}$$

With the help of Eq. (3.52), we can write:

$$\delta\mathcal{L} + \frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu = 0. \tag{3.54}$$

Moreover,

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_r}\delta\phi_r + \frac{\partial\mathcal{L}}{\partial\phi_{r,\mu}}\delta\phi_{r,\mu}. \tag{3.55}$$

Using the Euler–Lagrange equation:

$$\begin{aligned}\delta\mathcal{L} &= \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi_{r,\mu}}\delta\phi_r\right) \\ &= \frac{\partial}{\partial x^\mu}\left[\frac{\partial\mathcal{L}}{\partial\phi_{r,\mu}}\left(\delta_T\phi_r - \frac{\partial\phi_r}{\partial x_\nu}\delta x_\nu\right)\right].\end{aligned} \tag{3.56}$$

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Using Eq. (3.56) in Eq. (3.54), we have:

$$\begin{aligned}\frac{\partial}{\partial x^{\mu}}\left[\frac{\partial \mathcal{L}}{\partial \phi_{r,\mu}}\left(\delta_T\phi_r - \frac{\partial \phi_r}{\partial x_\nu}\delta x_\nu\right)\right] + \frac{\partial \mathcal{L}}{\partial x^{\mu}}\delta x^{\mu} &= 0 \\ \Rightarrow \frac{\partial}{\partial x^{\mu}}\left[\frac{\partial \mathcal{L}}{\partial \phi_{r,\mu}}\delta_T\phi_r\right] - \frac{\partial\, T^{\mu\nu}}{\partial x^{\mu}}\delta x_\nu &= 0, \end{aligned} \tag{3.57}$$

where

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \phi_{r,\mu}}\frac{\partial \phi_r}{\partial x_\nu} - \mathcal{L}g^{\mu\nu}. \tag{3.58}$$

$T^{\mu\nu}$ is known as the energy–momentum tensor. It can be shown that the conserved quantity J^{μ} is given by:

$$J^{\mu} = \frac{\partial \mathcal{L}}{\delta \phi_{r,\mu}}\delta_T\phi_r - T^{\mu\nu}\delta x_\nu. \tag{3.59}$$

Taking $\mu = \nu = 0$; $\mu = 0,\ \nu = i$

$$T^{00} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\dot{\phi} - \mathcal{L} = \mathcal{H}, T^{0i} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\,\frac{\partial \phi}{\partial x_i} \tag{3.60}$$

is interpreted as the energy-momentum density of the field. The total energy and the four momentum of the field are given by:

$$H = \int T^{00} d\vec{x}, \tag{3.61}$$

$$P^{\mu} = \int T^{0\mu} d\vec{x}. \tag{3.62}$$

The application of the conservation of J^{μ} using Eq. (3.42), leads to the conservation of momentum, energy, and angular momentum when applied to the symmetries corresponding to translational, rotational, and Lorentz transformations.

3.3 Quantization of Scalar Fields: Klein–Gordon Field

3.3.1 Real scalar field: Creation and annihilation operators

We consider the real scalar field $\phi(x)$, which corresponds to a particle of spin 0 and satisfies the Klein–Gordon equation given in Chapter 2. This equation for the field $\phi(x)$ can be derived from the Lagrangian density given by:

$$\mathcal{L}(x) = \frac{1}{2}(\phi_{,\alpha}(x)\phi^{,\alpha}(x) - m^2\phi^2(x)). \tag{3.63}$$

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Using the Euler–Lagrange equation, that is,

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathcal{L}}{\partial \phi_\alpha}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0, \tag{3.64}$$

we get

$$\begin{aligned} \partial_\mu \partial^\mu \phi(x) + m^2\phi(x) &= 0, \\ \Rightarrow (\Box + m^2)\phi(x) &= 0. \end{aligned} \tag{3.65}$$

The conjugate momentum $\pi(x)$ is obtained by the definition:

$$\pi(\vec{x}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}(\vec{x}, t). \tag{3.66}$$

The quantization of the real field ϕ using the canonical quantization procedure of nonrelativistic quantum mechanics is achieved by treating $\phi(x)$ and $\dot{\phi}(x)$ as operators satisfying the commutation relations:

$$[\phi(\vec{x}, t), \dot{\phi}(\vec{y}, t)] = i\delta^3(\vec{x} - \vec{y}), \tag{3.67}$$

$$[\phi(\vec{x}, t), \phi(\vec{y}, t)] = [\pi(\vec{x}, t), \pi(\vec{y}, t)] = [\dot{\phi}(\vec{x}, t), \dot{\phi}(\vec{y}, t)] = 0. \tag{3.68}$$

We have seen in Chapter 2 that the Klein–Gordon equation admits solutions like:

$$\phi(\vec{x}, t) = \frac{1}{\sqrt{2V\omega_{\vec{k}}}} e^{-ik\cdot x} \quad \text{and} \quad \phi^\dagger(\vec{x}, t) = \frac{1}{\sqrt{2V\omega_{\vec{k}}}} e^{ik\cdot x},$$

where

$$e^{\pm ik\cdot x} = e^{\pm ik_\mu x^\mu} = e^{\pm ik^\mu x_\mu} = e^{\pm i(k_0 t - \vec{k}\cdot\vec{x})}.$$

Here V is the normalization volume, $\omega_{\vec{k}} = \sqrt{|\vec{k}|^2 + m^2}$ and $k^\mu = (\omega_{\vec{k}}, \vec{k})$.

We expand $\phi(\vec{x}, t)$ in terms of the complete set of solutions of the Klein–Gordon equation and write:

$$\phi(x) = \phi^+(x) + \phi^-(x) = \sum_{\vec{k}} \frac{1}{\sqrt{2V\omega_{\vec{k}}}} \left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right), \tag{3.69}$$

$$\text{where } \phi^+(x) = \sum_{\vec{k}} \frac{1}{\sqrt{2V\omega_{\vec{k}}}} \left(a(\vec{k})e^{-ik\cdot x}\right) \text{ and} \tag{3.70}$$

$$\phi^-(x) = \sum_{\vec{k}} \frac{1}{\sqrt{2V\omega_{\vec{k}}}} \left(a^\dagger(\vec{k})e^{ik\cdot x}\right). \tag{3.71}$$

$a^\dagger(\vec{k})$ is the Hermitian conjugate of the operator $a(\vec{k})$. $\phi(x)$ is an operator in coordinate space,

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while $a(\vec{k})$ and $a^\dagger(\vec{k})$ are operators in momentum space. This form of $\phi(x)$ is necessitated because $\phi(x)$, being real, satisfies the relation $\phi(x) = \phi^\dagger(x)$.

The conjugate momentum field $\dot{\phi}(\vec{x}, t)$ is then derived to be:

$$\dot{\phi}(\vec{x}, t) = \sum_{\vec{k}} \frac{1}{\sqrt{2V\omega_{\vec{k}}}}(-i\omega_{\vec{k}}) \left(a(\vec{k})e^{-ik\cdot x} - a^\dagger(\vec{k})e^{ik\cdot x} \right). \tag{3.72}$$

Using Eqs. (3.69) and (3.72), it can be shown that:

$$a(\vec{k}) = \frac{1}{\sqrt{2V\omega_{\vec{k}}}} \int d\vec{x}\, e^{ik\cdot x}(\omega_{\vec{k}}\phi(x) + i\dot{\phi}(x)), \tag{3.73}$$

$$a^\dagger(\vec{k}) = \frac{1}{\sqrt{2V\omega_{\vec{k}}}} \int d\vec{x}\, e^{-ik\cdot x}(\omega_{\vec{k}}\phi(x) - i\dot{\phi}(x)), \tag{3.74}$$

leading to the following commutation relations for $a(\vec{k})$ and $a^\dagger(\vec{k})$.

$$\begin{aligned}
\left[a(\vec{k}), a^\dagger(\vec{k}')\right] &= \iint \frac{d\vec{x}d\vec{x}'}{V\sqrt{2\omega_{\vec{k}}}\sqrt{2\omega_{\vec{k}'}}} e^{-ik\cdot x}e^{ik'\cdot x'}[\omega_{\vec{k}}\phi(x) + i\dot{\phi}(x), \omega_{\vec{k}}\phi(x') - i\dot{\phi}(x')] \\
&= \iint \frac{d\vec{x}d\vec{x}'}{V\sqrt{2\omega_{\vec{k}}}\sqrt{2\omega_{\vec{k}'}}} e^{-i(k\cdot x - k'\cdot x')}(-i\omega_{\vec{k}}[\phi(x), \dot{\phi}(x')] + i\omega_{\vec{k}'}[\dot{\phi}(x), \phi(x')]) \\
&= \frac{\omega_{\vec{k}} + \omega_{\vec{k}'}}{\sqrt{4\omega_{\vec{k}}\omega_{\vec{k}'}}} \int \frac{d\vec{x}}{V} e^{-i(k-k')\cdot x}
\end{aligned}$$

since $[\phi(x), \dot{\phi}(x')] = i\delta^3(x - x')$ the aforementioned expression results in,

$$\left[a(\vec{k}), a^\dagger(\vec{k}')\right] = \delta_{\vec{k}\vec{k}'} \quad , \tag{3.75}$$

$$\left[a(\vec{k}), a(\vec{k}')\right] = \left[a^\dagger(\vec{k}), a^\dagger(\vec{k}')\right] = 0. \tag{3.76}$$

Similarly, it can be shown that these relations for $a(\vec{k})$ and $a^\dagger(\vec{k})$ are precisely the relations derived earlier in this chapter in the case of the harmonic oscillator and can be interpreted, respectively, as the annihilation and creation operators. For example, the Hamiltonian can be derived as:

$$\begin{aligned}
H &= \int (\pi(x)\dot{\phi}(x) - \mathcal{L}(x))d\vec{x} \\
&= \int d\vec{x}\frac{1}{2}(\dot{\phi}^2(x) + (\nabla\phi(x))^2 + m^2\phi^2(x)) \\
&= \int d\vec{x}\frac{\omega_{\vec{k}}}{2}(a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k}))
\end{aligned} \tag{3.77}$$

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$$= \sum_{\vec{k}} \omega_{\vec{k}} \left(a^\dagger(\vec{k}) a(\vec{k}) + \frac{1}{2} \right)$$

$$= \sum_{\vec{k}} \omega_{\vec{k}} \left(N(\vec{k}) + \frac{1}{2} \right). \tag{3.78}$$

Similarly, the momentum operator $\vec{P}$ is given by:

$$\vec{P} = \sum_{\vec{k}} \vec{k} \left(N(\vec{k}) + \frac{1}{2} \right), \tag{3.79}$$

where $N(\vec{k}) = a^\dagger(\vec{k}) a(\vec{k})$ is the number operator as defined in the case of the harmonic oscillator. It can be shown that:

$$[H, a(\vec{k})] = -\omega_{\vec{k}} a(\vec{k}) \quad \text{and} \quad [H, a^\dagger(\vec{k})] = \omega_{\vec{k}} a^\dagger(\vec{k}). \tag{3.80}$$

The total number operator for all the oscillators in the system may be obtained using

$$N = \int d\vec{k}\, N(\vec{k}) = \int d\vec{k}\, a^\dagger(\vec{k}) a(\vec{k}). \tag{3.81}$$

However, a major difficulty is encountered in defining the energy of the ground state, which is a vacuum state in the case of field theory. The vacuum state $|0\rangle$ is defined as a state in which there is no particle, that is,

$$N(\vec{k})|0\rangle = 0 \quad \text{or} \quad a(\vec{k})|0\rangle = 0 \qquad \text{for all } \vec{k}.$$

Equivalently, in terms of the field operators:

$$\phi^+(x)|0\rangle = 0 \quad \text{and} \quad \langle 0|\phi^-(x) = 0 \ \text{ for all } x.$$

The energy of the vacuum state is then given by:

$$H|0\rangle = \left(\frac{1}{2} \sum_{\vec{k}} \omega_{\vec{k}} \right) |0\rangle \qquad \text{for all } \vec{k}, \tag{3.82}$$

leading to an infinite constant. However, energy of the physical states is measured with reference to the energy of the vacuum; only the energy differences are measured in physical processes. Therefore, we can reset the vacuum energy given by Eq. (3.82); energies are measured with reference to this point. Thus, the occurrence of this constant albeit being infinite is absorbed in the definition of vacuum state energy. This is achieved by considering the "normal ordering" of the creation(annihilation) operators $a^\dagger(\vec{k})(a(\vec{k}))$ or the field operators $\phi(x)(\phi^\dagger(x))$, which assures that a normal ordered product of the field operators gives zero when operated on a vacuum state. It should be noted that the factor of $\frac{1}{2}\sum_{\vec{k}} \omega_{\vec{k}}$ appears when we rewrite $a(\vec{k})a^\dagger(\vec{k})$ in terms of $a^\dagger(\vec{k})a(\vec{k})$ in Eq. (3.77) to bring $a(\vec{k})$, the annihilation operator to

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the right of the creation operator. As long as all the annihilation operators are kept to the right and physical measurements are made treating vacuum as reference point, we can ignore the problem of infinite constant. This is called the "normal ordering" of the product of operators. It is needed only when both the creation and annihilation operators appear together in the product and not in the case between two annihilation or two creation operators like $a^\dagger(\vec{k})a^\dagger(\vec{k})$ or $a(\vec{k})a(\vec{k})$, as they commute. For example:

$$\begin{aligned} N[a^\dagger(\vec{k}_1)a(\vec{k}_2)a(\vec{k}_3)a^\dagger(\vec{k}_4)] &= N[a^\dagger(\vec{k}_1)a^\dagger(\vec{k}_4)a(\vec{k}_2)a(\vec{k}_3)] \\ \text{or } N[\phi(x)\phi(y)] &= N\left[(\phi^+(x)+\phi^-(x))(\phi^+(y)+\phi^-(y))\right] \\ &= N\left[\phi^+(x)\phi^+(y)+\phi^+(x)\phi^-(y)+\phi^-(x)\phi^+(y)\right. \\ &\quad \left.+\phi^-(x)\phi^-(y)\right] \\ &= \phi^+(x)\phi^+(y)+\phi^-(y)\phi^+(x)+\phi^-(x)\phi^+(y) \\ &\quad +\phi^-(x)\phi^-(y). \end{aligned} \tag{3.83}$$

Thus, keeping the "normal ordering" in mind and the removal of the infinite constant appearing in the energy of the vacuum, the energy and momentum operators are given by:

$$\begin{aligned} H &= \textstyle\sum_{\vec{k}} \omega_{\vec{k}}\, a^\dagger(\vec{k})a(\vec{k}), \\ \vec{P} &= \textstyle\sum_{\vec{k}} \vec{k}\, a^\dagger(\vec{k})a(\vec{k}). \end{aligned} \tag{3.84}$$

The energy of a state $|\phi\rangle$ is given by:

$$\begin{aligned} \langle\phi|H|\phi\rangle &= \langle\phi|\sum_{\vec{k}}\omega_{\vec{k}}a^\dagger(\vec{k})a(\vec{k})|\phi\rangle \\ &= \sum_{\vec{k}}\omega_{\vec{k}}\langle\phi|a^\dagger(\vec{k})a(\vec{k})|\phi\rangle \\ &= \sum_{\vec{k}}\omega_{\vec{k}}|a(\vec{k})|\phi\rangle|^2. \end{aligned} \tag{3.85}$$

which is always positive. It also shows that the vacuum expectation value of any normal order product of the fields is always zero since $a(\vec{k})|0\rangle = 0$ or $\phi^+(x)|0\rangle = 0$.

3.3.2 Fock space

Fock space in quantum field theory is the space in which the states are defined by the occupation number of particles in that state. It is characterized by the eigenvalues of the number operator N defined as:

$$N = \sum_{\vec{k}} a^\dagger(\vec{k})a(\vec{k}), \tag{3.86}$$

which satisfies the commutation relations:

$$[N, a^\dagger(\vec{k})] = a^\dagger(\vec{k}), \qquad [N, a(\vec{k})] = -a(\vec{k}) \tag{3.87}$$

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and shows that $a(\vec{k})$ and $a^\dagger(\vec{k})$, respectively, annihilates and creates one particle in a state, in the Fock space. Now operating Eq. (3.87) on a state $|n(\vec{k})\rangle$ such that $N|n(\vec{k})\rangle = n|n(\vec{k})\rangle$ with n particles, we find:

$$[N, a^\dagger(\vec{k})]|n(\vec{k})\rangle = a^\dagger(\vec{k})|n(\vec{k})\rangle \quad \text{and} \quad [N, a(\vec{k})]|n(\vec{k})\rangle = -a|n(\vec{k})\rangle$$

$$\text{that is, } Na^\dagger(\vec{k})|n(\vec{k})\rangle = (n+1)a^\dagger(\vec{k})|n(\vec{k})\rangle \quad \text{and} \quad Na(\vec{k})|n(\vec{k})\rangle = (n-1)a(\vec{k})|n(\vec{k})\rangle. \tag{3.88}$$

This shows that $a^\dagger(\vec{k})$ creates one particle while $a(\vec{k})$ annihilates one particle. For example, the vacuum is a zero particle state. The one particle state, two particle states, and multi- particle states are created by the repeated operation of the creation operator $a^\dagger(\vec{k})$. For example, a single particle state $|\vec{k}\rangle$ is created as:

$$|\vec{k}\rangle = a^\dagger(\vec{k})|0\rangle \tag{3.89}$$

which satisfies $N|\vec{k}\rangle = |\vec{k}\rangle$, $H|\vec{k}\rangle = \omega_{\vec{k}}|\vec{k}\rangle$, $P|\vec{k}\rangle = \vec{k}|\vec{k}\rangle$, etc.

Similarly, a two particle state $|\vec{k}_1, \vec{k}_2\rangle$ is created by two operations of the creation operator, that is,

$$|\vec{k}_1, \vec{k}_2\rangle = a^\dagger(\vec{k}_1)a^\dagger(\vec{k}_2)|0\rangle. \tag{3.90}$$

For bosons, these states are even under particle interchange, that is, $|\vec{k}_1, \vec{k}_2\rangle = |\vec{k}_2, \vec{k}_1\rangle$, and satisfy

$$|\vec{k}_1, \vec{k}_2\rangle = |\vec{k}_2, \vec{k}_1\rangle = a^\dagger(\vec{k}_1)a^\dagger(\vec{k}_2)|0\rangle. \tag{3.91}$$

Moreover, they satisfy

$$\begin{aligned} H|\vec{k}_1, \vec{k}_2\rangle &= \left(\omega_{\vec{k}_1} + \omega_{\vec{k}_2}\right)|\vec{k}_1, \vec{k}_2\rangle, \\ \vec{P}|\vec{k}_1, \vec{k}_2\rangle &= \left(\vec{k}_1 + \vec{k}_2\right)|\vec{k}_1, \vec{k}_2\rangle. \end{aligned} \tag{3.92}$$

A zero particle state defines the vacuum, that is, $|0\rangle$, and satisfies

$$\begin{aligned} &\langle 0|0\rangle = 0, \\ &H|0\rangle = 0, \\ &\vec{P}|0\rangle = 0. \end{aligned} \tag{3.93}$$

A multi-particle state $|\vec{k}_1, \vec{k}_2, ..., \vec{k}_n\rangle$ is written as:

$$|\vec{k}_1, \vec{k}_2, ..., \vec{k}_n\rangle = a^\dagger(\vec{k}_1)a^\dagger(\vec{k}_2).....a^\dagger(\vec{k}_n)|0\rangle \tag{3.94}$$

which has to be properly normalized. It can be shown that a properly normalized state with n particles of momentum is written as:

$$|k(n)\rangle = \frac{1}{\sqrt{n!}}(a^\dagger(\vec{k}))^n|0\rangle. \tag{3.95}$$

3.4 Complex Scalar Field

3.4.1 Creation and annihilation operators

We now consider the case of complex scalar fields and apply all the results derived in the earlier section. The Lagrangian density for the complex scalar fields is written as:

$$\mathcal{L} = N(\phi^{\dagger}_{,\alpha}(x)\phi^{,\alpha}(x) - m^2\phi(x)^{\dagger}\phi(x)), \tag{3.96}$$

where N denotes the normal ordering of the fields $\phi(x)$ and $\phi^{\dagger}(x)$. These fields are treated as independent fields and are complex. The normal ordering is explicitly written here but is generally understood in the context of all the field operators when they appear in product form. The fields $\phi(x)(\phi^{\dagger}(x))$ can always be expressed as a sum (difference) of two real independent fields ($\phi_1(x)$ and $\phi_2(x)$) like:

$$\begin{aligned}\phi(x) &= \frac{\phi_1(x) + i\phi_2(x)}{\sqrt{2}}, && (3.97)\\ \phi^{\dagger}(x) &= \frac{\phi_1(x) - i\phi_2(x)}{\sqrt{2}}. && (3.98)\end{aligned}$$

The Lagrangian density can also be written in terms of the two real fields $\phi_1(x)$ and $\phi_2(x)$:

$$\mathcal{L} = \frac{N}{2}\sum_{i=1,2}(\phi_{i,\mu}(x)\phi^{i,\mu}(x) - m^2\phi_i^2). \tag{3.99}$$

We will, however, work with the complex fields $\phi(x)$ and $\phi^{\dagger}(x)$ in the following. Using the Euler–Lagrange equation, the Lagrangian density in Eq. (3.96) leads to the Klein–Gordon equation:

$$(\Box + m^2)\phi(x) = 0 \quad \text{or} \quad (\Box + m^2)\phi^{\dagger}(x) = 0. \tag{3.100}$$

The conjugate fields, that is, $\frac{\partial\mathcal{L}}{\partial\phi}$ and $\frac{\partial\mathcal{L}}{\partial\phi^{\dagger}}$ to ϕ and $\phi^{\dagger}$ are given by:

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(x)} = \dot{\phi}^{\dagger}(x) \quad \text{and} \quad \pi^{\dagger}(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}^{\dagger}(x)} = \dot{\phi}(x), \tag{3.101}$$

leading to the equal time commutation relations for the purpose of quantization being:

$$\begin{aligned}[\phi(\vec{x},t), \dot{\phi}^{\dagger}(\vec{y},t)] &= i\delta^3(\vec{x}-\vec{y}), && (3.102)\\ \left[\phi^{\dagger}(\vec{x},t), \dot{\phi}(\vec{y},t)\right] &= i\delta^3(\vec{x}-\vec{y}), && (3.103)\end{aligned}$$

$$\begin{aligned}[\phi(\vec{x},t), \phi(\vec{y},t)] &= [\dot{\phi}^{\dagger}(\vec{x},t), \dot{\phi}(\vec{y},t)] = [\phi^{\dagger}(\vec{x},t), \phi^{\dagger}(\vec{y},t)] = \\ [\dot{\phi}(\vec{x},t), \dot{\phi}(\vec{y},t)] &= \left[\phi(\vec{x},t), \phi^{\dagger}(\vec{y},t)\right] = [\dot{\phi}(\vec{x},t), \dot{\phi}^{\dagger}(\vec{y},t)] = 0. && (3.104)\end{aligned}$$

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In analogy with the real field, we expand the fields $\phi(\vec{x},t)$ and $\phi^\dagger(\vec{x},t)$ in terms of the complete set of solutions, that is,

$$\phi(\vec{x},t) = \phi^+ + \phi^- = \sum_{\vec{k}} \frac{1}{\sqrt{2V\omega_{\vec{k}}}} \left[a(\vec{k})e^{-ik\cdot x} + b^\dagger(\vec{k})e^{ik\cdot x}\right], \tag{3.105}$$

$$\phi^\dagger(\vec{x},t) = (\phi^+)^\dagger + (\phi^-)^\dagger = \sum_{\vec{k}} \frac{1}{\sqrt{2V\omega_{\vec{k}}}} \left[b(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right]. \tag{3.106}$$

It should be noted that in this case, $\phi(\vec{x},t) \neq \phi^\dagger(\vec{x},t)$. The fields, $\phi^+(\vec{x},t)$, and $\phi^-(\vec{x},t)$ are the positive and negative energy parts of $\phi(x)$. Using Eqs. (3.104), (3.105) and (3.106), we can derive the following commutation relation for $a(\vec{k}), a^\dagger(\vec{k}), b(\vec{k}), b^\dagger(\vec{k})$ i.e.

$$\left[a(\vec{k}), a^\dagger(\vec{k}')\right] = \left[b(\vec{k}), b^\dagger(\vec{k}')\right] = \delta_{\vec{k}\,\vec{k}'}, \tag{3.107}$$

$$\begin{aligned}\left[a(\vec{k}), a(\vec{k}')\right] &= \left[b(\vec{k}), b(\vec{k}')\right] = \left[a^\dagger(\vec{k}), a^\dagger(\vec{k}')\right] \\ &= \left[b^\dagger(\vec{k}), b^\dagger(\vec{k}')\right] = \left[a(\vec{k}), b^\dagger(\vec{k}')\right] = 0.\end{aligned} \tag{3.108}$$

We have already explained the interpretation of $a(\vec{k})$ and $a^\dagger(\vec{k})$ as the annihilation and creation operators of field $\phi(x)$. In an analogous way, $b(\vec{k})$ and $b^\dagger(\vec{k})$ are interpreted as annihilation and creation of particles corresponding to field $\phi^\dagger(x)$. It is clear that if we continue with the Lagrangian density and the fields in terms of $\phi_1(x)$ and $\phi_2(x)$, then using Eq. (3.99) we would obtain:

$$\pi_1(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} = \dot{\phi}_1(x), \tag{3.109}$$

$$\pi_2(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_2} = \dot{\phi}_2(x). \tag{3.110}$$

From Eq. (3.97), we have:

$$\begin{aligned}\phi(\vec{x},t) &= \frac{1}{\sqrt{2}}(\phi_1(\vec{x},t) + i\phi_2(\vec{x},t)) \\ &= \frac{1}{\sqrt{2}} \sum_{\vec{k}} \frac{1}{\sqrt{2V\omega_{\vec{k}}}} \left[e^{-ik\cdot x}(a_1(\vec{k}) + ia_2(\vec{k})) + e^{ik\cdot x}(a_1^\dagger(\vec{k}) + ia_2^\dagger(\vec{k}))\right],\end{aligned} \tag{3.111}$$

where $a_1^\dagger(\vec{k})$ and $a_2^\dagger(\vec{k})$ will create quanta of fields $\phi_1(x)$ and $\phi_2(x)$, while $a_1(x)$ and $a_2(x)$ will annihilate quanta of $\phi_1(x)$ and $\phi_2(x)$. Similarly,

$$\phi^\dagger(\vec{x},t) = \frac{1}{\sqrt{2}} \sum_{\vec{k}} \frac{1}{2V\omega_{\vec{k}}} \left[e^{-ik\cdot x}(a_1(\vec{k}) - ia_2(\vec{k})) + e^{ik\cdot x}(a_1^\dagger(\vec{k}) - ia_2^\dagger(\vec{k}))\right]. \tag{3.112}$$

Equations (3.105) and (3.106) can be reproduced from Eqs. (3.111) and (3.112), using the following relations:

$$a(\vec{k}) = \frac{1}{\sqrt{2}}(a_1(\vec{k}) + ia_2(\vec{k})), \tag{3.113}$$

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$$
\begin{aligned}
b(\vec{k}) &= \frac{1}{\sqrt{2}}(a_1(\vec{k}) - ia_2(\vec{k})), && (3.114)\\
a^\dagger(\vec{k}) &= \frac{1}{\sqrt{2}}(a_1^\dagger(\vec{k}) - ia_2^\dagger(\vec{k})), && (3.115)\\
b^\dagger(\vec{k}) &= \frac{1}{\sqrt{2}}(a_1^\dagger(\vec{k}) + ia_2^\dagger(\vec{k})). && (3.116)
\end{aligned}
$$

However, there is a physical reason to consider the formulation of quantized fields of complex fields in term of $\phi(x)$ and $\phi^\dagger(x)$ instead of $\phi_1(x)$ and $\phi_2(x)$.

Using Eqs. (3.113)–(3.116) and Eqs. (3.58) and (3.62), the energy–momentum operator for the complex Klein–Gordon field can be derived to be:

$$P^\mu = (H, \vec{P}) = \frac{1}{2}\sum_{\vec{k}} k^\mu \Big(a^\dagger(\vec{k})a(\vec{k}) + a(\vec{k})a^\dagger(\vec{k}) + b^\dagger(\vec{k})b(\vec{k}) + b(\vec{k})b^\dagger(\vec{k}) \Big). \qquad (3.117)$$

Using the commutation relations in Eq. (3.107) and the concept of normal ordering discussed in Section 3.3.1, the operator P^μ is written as:

$$P^\mu = \sum_{\vec{k}} k^\mu \Big(a^\dagger(\vec{k})a(\vec{k}) + b^\dagger(\vec{k})b(\vec{k}) \Big). \qquad (3.118)$$

We can, therefore, define number operators for both types of particles created by the creation operators $a^\dagger(\vec{k})$ and $b^\dagger(\vec{k})$:

$$
\begin{aligned}
N_a &= \sum_{\vec{k}} a^\dagger(\vec{k})a(\vec{k}), && (3.119)\\
N_b &= \sum_{\vec{k}} b^\dagger(\vec{k})b(\vec{k}). && (3.120)
\end{aligned}
$$

The vacuum state $|0\rangle$ is defined as the state of minimum energy (redefined to zero energy) through the operation of $a(\vec{k})$ and $b(\vec{k})$ as:

$$a(\vec{k})|0\rangle = 0 \qquad \text{and} \qquad b(\vec{k})|0\rangle = 0 \qquad (3.121)$$

or equivalently $\phi^+(\vec{x})|0\rangle = 0$ and $(\phi^-)^\dagger(\vec{x})|0\rangle = 0$.

The one particle states in the Fock space are created by the action of $a^\dagger(\vec{k})|0\rangle$ and $b^\dagger(\vec{k})|0\rangle$ as:

$$|\vec{k}\rangle = a^\dagger(\vec{k})|0\rangle \quad \text{and} \quad |\vec{\bar{k}}\rangle = b^\dagger(\vec{k})|0\rangle. \qquad (3.122)$$

We already know the type of particles created by $a^\dagger(\vec{k})$. To understand the type of particles created by $b^\dagger(\vec{k})$, let us go to the next section.

3.4.2 Charge of the complex scalar field: Particles and antiparticles

The Lagrangian given in Eq. (3.96) is invariant under the global phase transformation

$$\phi(x) \to e^{-iq\theta}\phi(x), \qquad \phi^\dagger(x) \to \phi^\dagger(x)e^{iq\theta}, \tag{3.123}$$

where θ is independent of space–time and q is an arbitrary parameter. For infinitesimal θ, we have:

$$\delta\phi = -iq\theta\phi, \qquad \delta\phi^\dagger = iq\theta\phi^\dagger. \tag{3.124}$$

According to Noether's theorem, the conserved current J^μ associated with this global transformation is given by:

$$J^\mu = iq\big((\partial^\mu\phi)\phi^\dagger - (\partial^\mu\phi^\dagger)\phi\big), \tag{3.125}$$

such that $\partial_\mu J^\mu = 0$, corresponding to the conservation of the quantity $Q = \int d\vec{x}\, J^0(x)$, that is,

$$\begin{aligned} Q &= q\sum_{\vec{k}}\left[a^\dagger(\vec{k})a(\vec{k}) - b^\dagger(\vec{k})b(\vec{k})\right] \\ &= q\sum_{\vec{k}}\big(N_a(\vec{k}) - N_b(\vec{k})\big). \end{aligned} \tag{3.126}$$

Q is also known as the normal ordered charge operator. Q when operating on a vacuum state gives 0, that is,

$$Q|0\rangle = q\sum_{\vec{k}}\left[a^\dagger(\vec{k})a(\vec{k}) - b^\dagger(\vec{k})b(\vec{k})\right]|0\rangle = 0. \tag{3.127}$$

This can be seen as follows:
Q operating on the first of the two one particle states $|\vec{k}\rangle$ gives:

$$\begin{aligned} qN_a|\vec{k}\rangle &= q\sum_{\vec{k}'}\left[a^\dagger(\vec{k}')a(\vec{k}')\right]a^\dagger(\vec{k})|0\rangle \\ &= q\sum_{\vec{k}'}\left[a^\dagger(\vec{k}')a(\vec{k}')a^\dagger(\vec{k})\right]|0\rangle \\ &= q\sum_{\vec{k}'}a^\dagger(\vec{k}')\delta_{\vec{k}\vec{k}'}|0\rangle \\ &= q\, a^\dagger(\vec{k})|0\rangle \\ &= q\,|\vec{k}\rangle. \end{aligned} \tag{3.128}$$

$$\text{Similarly,} \qquad qN_b|\vec{k}\rangle = +q\,|\vec{k}\rangle, \tag{3.129}$$

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such that

$$q(N_a - N_b)|0\rangle = 0|0\rangle. \tag{3.130}$$

Using Eq. (3.118), the Hamiltonian H is defined as:

$$H = \sum_{\vec{k}} \omega_{\vec{k}} \left[a^\dagger(\vec{k})a(\vec{k}) + b^\dagger(\vec{k})b(\vec{k})\right], \tag{3.131}$$

and it can be shown that:

$$[H, Q] = 0 \tag{3.132}$$

so that the energy states could be simultaneously characterized by the eigenvalues of the charge operator Q. If the charge q is identified to be the charge of the particle created by $a^\dagger(\vec{k})$, then the particle created by $b^\dagger(\vec{k})$ has a charge $-q$. Since the energy E(or equivalent mass as the energy at rest) and charge are the only two quantum numbers associated with a charged scalar particle, they represent the particles and antiparticles of spin zero. The physical example of such particles are the charged pions π^+ and π^-. On the other hand, for a real particle field, the charge operator Q defined in Eq. (3.128) or Eq. (3.129) will be zero because $\phi(x) = \phi^\dagger(x)$ corresponding to $a = b$ and $a^\dagger = b^\dagger$.

If we had persisted with the description of the complex fields in terms of two real fields like $\phi_1(x)$ and $\phi_2(x)$, then both fields would separately correspond to neutral particles of zero charge and would not be adequate to describe the charged scalar particles. This is the reason why we describe the field quantization in this case in terms of $\phi(x)$ and $\phi^+(x)$ fields.

3.4.3 Covariant commutation relation

In this section, we will describe the concept of covariant commutation relations and its relation to the field propagators in the case of real scalar fields which would be generalized to the complex scalar fields. The commutation relations for the two fields $\phi(x_1)$ and $\phi(x_2)$ were defined for equal times in Eqs. (3.102)–(3.104) in context of the canonical quantization procedure of the fields. The covariant commutation relation (CCR) is defined for the two fields $\phi(x_1)$ and $\phi(x_2)$ at the two arbitrary space–time points x_1 and x_2 in terms of $\Delta(x_1 - x_2)$ as:

$$\Delta(x_1 - x_2) \quad = \quad -i[\phi(x_1), \phi(x_2)], \tag{3.133}$$

where

$$\begin{aligned} \phi(x_1) &= \phi^+(x_1) + \phi^-(x_1), \\ \phi(x_2) &= \phi^+(x_2) + \phi^-(x_2), \end{aligned}$$

and

$$\begin{aligned} [\phi(x_1), \phi(x_2)] &= [\phi^+(x_1) + \phi^-(x_1), \phi^-(x_2) + \phi^+(x_2)] \\ &= [\phi^+(x_1), \phi^-(x_2)] + [\phi^-(x_1), \phi^+(x_2)]. \end{aligned} \tag{3.134}$$

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Since $[\phi^+(x_1), \phi^+(x_2)] = [\phi^-(x_1), \phi^-(x_2)] = 0$, they contain only annihilation or only creation operators which commute. Consider the first term, that is,

$$\begin{aligned}[\phi^+(x_1), \phi^-(x_2)] &= \frac{1}{2V}\sum_{\vec{k}}\sum_{\vec{k}'}\frac{1}{\sqrt{\omega_{\vec{k}}\omega_{\vec{k}'}}}[a(\vec{k}), a^\dagger(\vec{k}')](e^{-ikx_1+ik'x_2}),\\ &= \frac{1}{2V}\sum_{\vec{k}}\frac{1}{\omega_{\vec{k}}}e^{-ik(x_1-x_2)}.\end{aligned} \tag{3.135}$$

Since $\vec{k} = \vec{k}'$, $\omega_{\vec{k}} = \sqrt{|\vec{k}|^2 + m^2} = \sqrt{|\vec{k}'|^2 + m^2} = \omega_{\vec{k}'}$. Taking the limit $V \to \infty$ and replacing the summation over $\vec{k}$ with an integral over $\vec{k}'$, we write:

$$[\phi^+(x_1), \phi^-(x_2)] = \frac{1}{2(2\pi)^3}\int \frac{d\vec{k}'}{\omega_{\vec{k}}}e^{-ik(x_1-x_2)}. \tag{3.136}$$

We introduce the function $\Delta^+(x)$ as:

$$\begin{aligned}\Delta^+(x) &= \frac{-i}{2(2\pi)^3}\int \frac{d\vec{k}}{\omega_{\vec{k}}}e^{-ik\cdot x}\\ \text{such that}\qquad [\phi^+(x_1), \phi^-(x_2)] &= i\Delta^+(x_1 - x_2).\end{aligned} \tag{3.137}$$

Similarly, we define the second term in the Eq. (3.134) in terms of $\Delta^-(x)$:

$$\Delta^-(x) = \frac{+i}{2(2\pi)^3}\int \frac{d\vec{k}}{\omega_{\vec{k}}}e^{+ik\cdot x} \tag{3.138}$$

$$\text{such that}\qquad [\phi^-(x_1), \phi^+(x_2)] = i\Delta^-(x_1 - x_2). \tag{3.139}$$

It is easy to see that:

$$[\phi^-(x_1), \phi^+(x_2)] = i\Delta^-(x_1 - x_2) = -i\Delta^+(x_2 - x_1). \tag{3.140}$$

Therefore,

$$\begin{aligned}\Delta(x_1 - x_2) &= \Delta^+(x_1 - x_2) + \Delta^-(x_1 - x_2)\\ &= -\frac{1}{(2\pi)^3}\int \frac{d\vec{k}}{\omega_{\vec{k}}}\sin(k(x_1 - x_2)).\end{aligned} \tag{3.141}$$

Thus, $\Delta(x_1 - x_2)$ satisfies the following properties:

(i) it is a real function of $(x_1 - x_2)$,

(ii) it is an odd function of $(x_1 - x_2)$,

(iii) $\Delta(x_1 - x_2)|_{(x_1)_0=(x_2)_0} = 0$,

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(iv) the equation $(\Box + m^2)\Delta(x_1 - x_2) = 0$ is satisfied.

This $\Delta(x)$ defined in Eq. (3.141) can be written in covariant form as:

$$\begin{aligned}
\Delta(x) &= -\frac{i}{(2\pi)^3}\int d^4k\delta(k^2 - m^2)\epsilon(k_0)e^{-ik\cdot x}, \\
\text{where } \epsilon(k_0) &= +1 \quad \text{for } k_0 > 0 \\
&= -1 \quad \text{for } k_0 < 0
\end{aligned} \tag{3.142}$$

and $d^4k = dk_0\, d\vec{k}$. This can be verified by using the property of the δ function:

$$\begin{aligned}
\delta(k^2 - m^2) &= \delta(k_0^2 - (|\vec{k}|^2 + m^2)) = \delta(k_0^2 - \omega_{\vec{k}}^2) = \delta\left[(k_0 - \omega_{\vec{k}})(k_0 + \omega_{\vec{k}})\right] \\
&= \frac{1}{2\omega_{\vec{k}}}[\delta(k_0 - \omega_{\vec{k}}) + \delta(k_0 + \omega_{\vec{k}})].
\end{aligned} \tag{3.143}$$

(v) The equal time condition $(x_1)_0 - (x_2)_0 = 0$ implies that $(x_1 - x_2)^2 = ((x_1)_0 - (x_2)_0)^2 - (\vec{x_1} - \vec{x_2})^2 < 0$, that is, it corresponds to a space- like separation for which $[\phi(x_1), \phi(x_2)] = 0$. The invariance of $\Delta(x_1 - x_2)$, therefore, implies that it is zero for all space-like separation. This means that the physical observables depending on the fields at the two points separated by space- like separation do not interfere with each other. This is consistent with the special theory of relativity in which two events separated by space-like intervals cannot be connected by the Lorentz transformation. This is called the principle of microcausality.

(vi) A simple representation of the functions $\Delta^{\pm}(x_1 - x_2)$ is written in terms of the contour integral in the complex k_0 plane as:

$$\Delta^{\pm}(x_1 - x_2) = -\frac{1}{(2\pi)^4}\int_{C^{\pm}} \frac{d^4k e^{-ik\cdot(x_1 - x_2)}}{k^2 - m^2}, \tag{3.144}$$

where the contours $C^{\pm}$ are shown in Figure 3.2. The poles of the integrand in Eq. (3.144) are at $k_0 = \pm\omega_{\vec{k}}$ with $\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$ and the integration over $C^+(C^-)$ gives $\Delta^+(\Delta^-)$.

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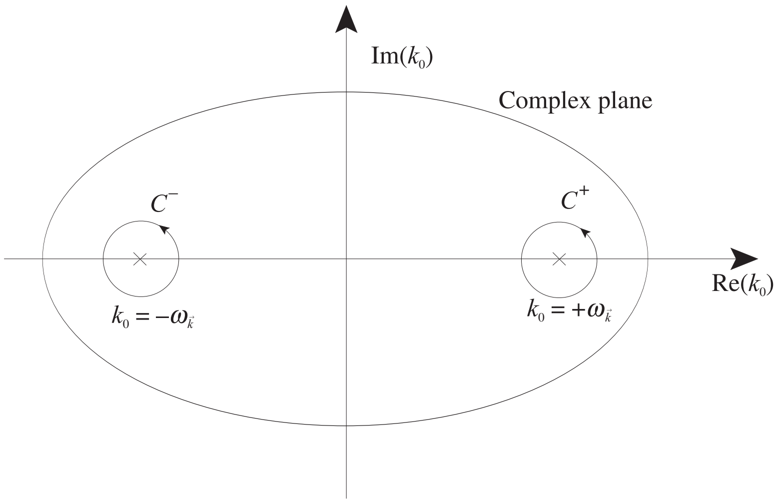

Figure 3.2 Contours for Eq. (3.144). C^+ is for Δ_F^+ and C^- is for Δ_F^-. The poles of the integral given in Eq. (3.144) are at $k_0 = \omega_{\vec{k}}$ for C^+ and $k_0 = -\omega_{\vec{k}}$ for C^-.

3.5 Time-ordered Product and Propagators for Scalar Fields

The covariant commutators $\Delta^{\pm}(x_1 - x_2)$ can be considered as the vacuum expectation values of the product of fields and can, therefore, be treated as the matrix element for creating and annihilating particles from vacuum as will be discussed in this section. Consider the product of two real scalar fields $\phi(x_1)\phi(x_2)$, that is,

$$\begin{aligned}
\phi(x_1)\phi(x_2) &= (\phi^+(x_1) + \phi^-(x_1))(\phi^+(x_2) + \phi^-(x_2)) && (3.145)\\
&= \phi^+(x_1)\phi^+(x_2) + \phi^+(x_1)\phi^-(x_2) + \phi^-(x_1)\phi^+(x_2) + \phi^-(x_1)\phi^-(x_2)\\
&= \phi^+(x_1)\phi^+(x_2) + \phi^+(x_1)\phi^-(x_2) + \phi^-(x_1)\phi^+(x_2)\\
&\quad +\phi^-(x_1)\phi^-(x_2) + \phi^-(x_2)\phi^+(x_1) - \phi^-(x_2)\phi^+(x_1)\\
&= \phi^+(x_1)\phi^+(x_2) + \phi^-(x_2)\phi^+(x_1) + \phi^-(x_1)\phi^-(x_2)\\
&\quad +\phi^-(x_1)\phi^+(x_2) + [\phi^+(x_1), \phi^-(x_2)] && (3.146)\\
&= N(\phi(x_1)\phi(x_2)) + [\phi^+(x_1), \phi^-(x_2)]. && (3.147)
\end{aligned}$$

Taking the vacuum expectation value on both sides of Eq. (3.147)

$$\langle 0|\phi(x_1)\phi(x_2)|0\rangle = \langle 0|N(\phi(x_1)\phi(x_2))|0\rangle + \langle 0|[\phi^+(x_1), \phi^-(x_2)]|0\rangle. \qquad (3.148)$$

Since the vacuum expectation value of a normal-ordered product vanishes, we get

$$\begin{aligned}
\langle 0|\phi(x_1)\phi(x_2)|0\rangle &= \langle 0|[\phi^+(x_1), \phi^-(x_2)]|0\rangle.\\
&= i\Delta^+(x_1 - x_2). && (3.149)
\end{aligned}$$

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A time-ordered product of two real scalar fields $\phi(x_1)$ and $\phi(x_2)$ is defined as:

$$\begin{aligned} T\left(\phi(x_1)\phi(x_2)\right) &= \phi(x_1)\phi(x_2) \quad \text{if } \; t_1 > t_2 \\ &= \phi(x_2)\phi(x_1) \quad \text{if } \; t_2 > t_1, \end{aligned} \tag{3.150}$$

That is, the field $\phi(x)$ at earlier time operates first, achieved by placing $\phi(x)$ towards the right of the field operating at a later time. The time-ordered product T can be expressed in a compact form using step function $\theta(t)$, where $\theta(t)$ is defined as:

$$\begin{aligned} \theta(t) &= 1 \quad \text{for } \; t > 0 \\ &= 0 \quad \text{for } \; t < 0. \end{aligned} \tag{3.151}$$

Then, T becomes

$$T(\phi(x_1)\phi(x_2)) = \theta(t_1 - t_2)\phi(x_1)\phi(x_2) + \theta(t_2 - t_1)\phi(x_2)\phi(x_1). \tag{3.152}$$

Another function called the Feynman Δ-function, $\Delta_F(x_1 - x_2)$, is defined as:

$$i\Delta_F(x_1 - x_2) = \langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle. \tag{3.153}$$

That is,

$$\Delta_F(x) = \theta(t)\Delta^+(x) - \theta(-t)\Delta^-(x), \tag{3.154}$$

which includes the vacuum expectation value of the products of the fields for both cases of time-ordering. Explicitly,

$$\Delta_F(x) = \pm\Delta^{\pm}(x) \quad \text{for} \quad t \gtrless 0. \tag{3.155}$$

These Δ-functions are physically interpreted to describe the propagation of particles between x_1 and x_2 depending upon $t_1 > t_2$ or $t_2 > t_1$. Let us consider $\Delta_F(x_1 - x_2)$ for $t_1 > t_2$. In this case:

$$\begin{aligned} \Delta_F(x_1 - x_2) &= -i\langle 0|\phi(x_1)\phi(x_2)|0\rangle \qquad t_1 > t_2 \\ &= -i\langle 0|\phi(x_2)\phi(x_1)|0\rangle \quad t_1 < t_2, \end{aligned} \tag{3.156}$$

in which the particle is created at x_2 and gets annihilated at x_1. In case $t_1 < t_2$, the particle is created at x_1 and gets annihilated at x_2. The physical processes are shown in Figure 3.3, where for a fixed t_1, the particle's creation and annihilation are depicted depending upon either $t_1 > t_2$ or $t_2 > t_1$.

In the case of Feynman Δ-function, $\Delta_F(x_1 - x_2)$, both time-ordered diagrams are included. In actual calculations, all values of x_1 and x_2 are integrated. Therefore, the Feynman Δ-function $\Delta_F(x_1 - x_2)$ is the appropriate description of the propagation of particles between its creation followed by its annihilation. This is therefore called the particle propagator. An integral

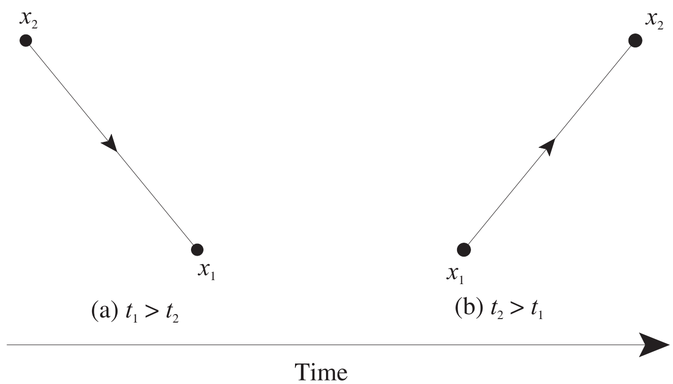

Figure 3.3 Propagation of bosons between x_1 and x_2 for (a) $t_1 > t_2$ and (b) $t_2 > t_1$.

representation of $\Delta_F(x_1 - x_2)$ which combines both $\Delta^+(x)$ and $\Delta^-(x)$ functions in Eq. (3.154) is given by:

$$\Delta_F(x) = \frac{1}{(2\pi)^4} \int_{C_F} \frac{d^4 k e^{-ik\cdot x}}{k^2 - m^2}, \tag{3.157}$$

where C_F is the contour specified to reproduce the results given in Eq. (3.154). This is achieved by performing the k_0 integration in the complex k_0 plane in Eq. (3.157) using Cauchy's integral formula. The denominator of the integrand is written as:

$$k^2 - m^2 = (k_0 - \omega_{\vec{k}})(k_0 - \omega_{\vec{k}}), \tag{3.158}$$

where $\omega_{\vec{k}} = \sqrt{|\vec{k}|^2 + m^2}$, so the integral gets contribution from the poles at $k_0 = \pm\omega_{\vec{k}}$ provided the integral vanishes at the boundaries of the contour C_F which is assured by

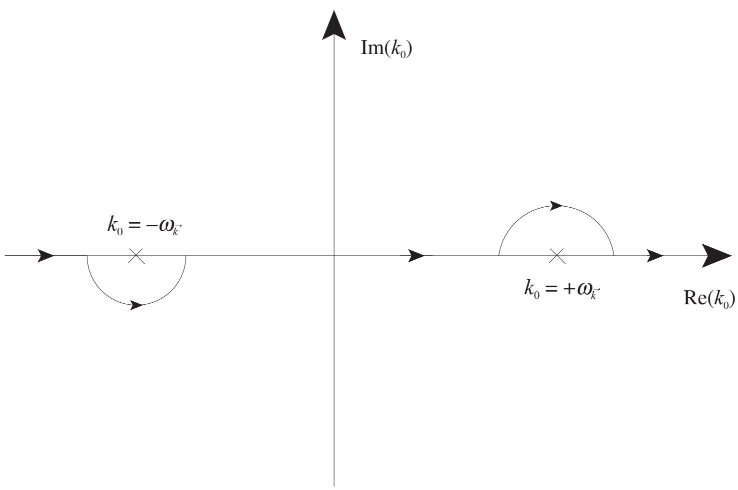

Figure 3.4 Contour for the boson propagator Δ_F.

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appropriately choosing the contour in the upper half or the lower half to perform the contour integration (as shown in Figure 3.4). We can obtain Eq. (3.144) for $\Delta^+(x)$ and $\Delta^-(x)$. In the case of $t_1 > t_2$, that is, $x^0 > 0$, $e^{-ik\cdot x} \to e^{-i(k_0 x^0 - \vec{k}\cdot\vec{x})}$, choosing the lower half in which $k_0 \to -i\infty$ along the boundary of the contour will make the integral vanish while for $x^0 < 0$, it will happen for the upper half in which $k_0 \to +i\infty$. Therefore, choosing the lower half including the pole at $k_0 = \omega_{\vec{k}}$ would give $\Delta^+(x)$ and choosing the upper half including the pole at $k_0 = -\omega_{\vec{k}}$ will give $\Delta^-(x)$. Instead of deforming the contour, the integrand can also be modified to displace the poles so that the pole at $k_0 = \pm\omega_{\vec{k}}$ is included in the lower (upper) half as we choose $x^0 > 0$ or $x^0 < 0$. This is done by writing:

$$k_0^2 - \omega_{\vec{k}}^2 \quad \longrightarrow \quad k_0^2 - (\omega_{\vec{k}} - i\eta)^2 \tag{3.159}$$

such that the poles are now at $k_0 = \pm(\omega_{\vec{k}} - i\eta)$ as shown in Figure 3.5. The integral representation of $\Delta_F(x)$ now becomes:

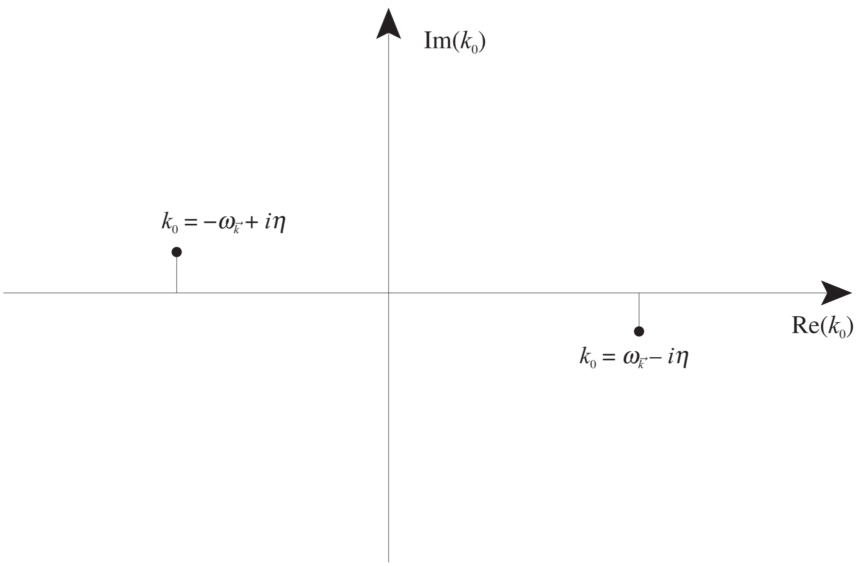

Figure 3.5 Position of displaced poles in the contour integration for Δ_F.

$$\begin{aligned}\Delta_F(x) &= \frac{1}{(2\pi)^4}\int \frac{d^4k\, e^{-ik\cdot x}}{k_0^2 - (\omega_{\vec{k}} - i\eta)^2} \\ &= \frac{1}{(2\pi)^4}\int \frac{d^4k\, e^{-ik\cdot x}}{k^2 - m^2 + i\epsilon}, \quad E = 2\omega_{\vec{k}}\eta.\end{aligned} \tag{3.160}$$

Equation (3.160) is the integral representation of the Feynman propagator for scalar particles.

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3.6 Quantization of Spin $\frac{1}{2}$ Fields

The Dirac equation for spin $\frac{1}{2}$ particles was discussed in Chapter 2. The equation for the Dirac field can be derived from the following Lagrangian:

$$\mathcal{L} = \overline{\psi}(x)\big(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\big)\psi(x), \tag{3.161}$$

to give:

$$\begin{aligned} \big(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\big)\psi(x) &= 0, & (3.162)\\ \text{and} \quad \overline{\psi}(x)\big(i\gamma^\mu \frac{\overleftarrow{\partial}}{\partial x^\mu} + m\big) &= 0. & (3.163)\end{aligned}$$

The conjugate fields are given by:

$$\begin{aligned} \pi(x) &= \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^\dagger, & (3.164)\\ \overline{\pi}(x) &= \frac{\partial \mathcal{L}}{\partial \dot{\overline{\psi}}} = 0. & (3.165)\end{aligned}$$

We have already seen from the solution of the Dirac equation that $\psi(x)$ is a 4-component column vector, so we denote it by $\psi_r(x)$ which can be represented by two 2-component column vectors $\psi_1(x)$ and $\psi_2(x)$ which are expressed in terms of 2-component spinors $u_r(p)$ and $v_r(p)(r = 1,2)$. In view of this, we rewrite the expression of the Dirac field in terms of the complete set of solutions as:

$$\psi(x) = \sum_{\vec{k}} \sum_{r=1,2} \frac{1}{\sqrt{2\omega_{\vec{k}}}} [a_r(\vec{k})u_r(\vec{k})e^{-ik\cdot x} + b_r^\dagger(\vec{k})v_r(\vec{k})e^{ik\cdot x}] \tag{3.166}$$

and

$$\psi^\dagger(x) = \sum_{\vec{k}} \sum_{r=1,2} \frac{1}{\sqrt{2\omega_{\vec{k}}}} [a_r^\dagger(\vec{k})u_r^\dagger(\vec{k})e^{-ik\cdot x} + b_r(\vec{k})v_r^\dagger(\vec{k})e^{ik\cdot x}], \tag{3.167}$$

where $\omega_{\vec{k}} = \sqrt{|\vec{k}|^2 + m^2}$.

The Hamiltonian density can be written as:

$$\begin{aligned} \mathcal{H}(x) &= \pi(x)\dot{\psi}(x) - \mathcal{L} = i\psi^\dagger(x)\dot{\psi}(x) - \mathcal{L}\\ &= i\psi^\dagger(x)\dot{\psi}(x) - i\psi^\dagger(x)\dot{\psi}(x) - i\psi^\dagger\gamma_0\gamma_i\frac{\partial}{\partial x^i}\psi + m\bar{\psi}\psi,\\ &= \psi^\dagger(x)(-i\gamma^0\vec{\gamma}\cdot\vec{\nabla} + \gamma^0 m)\psi(x)\\ &= \overline{\psi}(x)(-i\vec{\gamma}\cdot\vec{\nabla} + m)\psi(x), & (3.168)\end{aligned}$$

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leading to the Hamiltonian:

$$\begin{aligned} H &= \int d\vec{x}\,\mathcal{H} \\ &= \int d\vec{x}\,\overline{\psi}(-i\vec{\gamma}\cdot\vec{\nabla}+m)\psi(x) \end{aligned} \tag{3.169}$$

and the momentum operator

$$\vec{P} = -i\int d\vec{x}\overline{\psi}(x)\vec{\nabla}\psi(x), \tag{3.170}$$

Using $J^\mu = \overline{\psi}\gamma^\mu\psi$, we obtain:

$$Q = q\int d\vec{x}\, J^0 = q\int d\vec{x}\,\psi^\dagger(x)\psi(x). \tag{3.171}$$

Using Eqs. (3.166) and (3.167), respectively, for $\psi(x)$ and $\psi^\dagger(x)$, we find:

$$H = \sum_{\vec{k},r}\omega_{\vec{k}}[a_r^\dagger(\vec{k})a_r(\vec{k}) - b_r(\vec{k})b_r^\dagger(\vec{k})], \tag{3.172}$$

$$\vec{P} = \sum_{\vec{k},r}\vec{k}[a_r^\dagger(\vec{k})a_r(\vec{k}) - b_r(\vec{k})b_r^\dagger(\vec{k})]. \tag{3.173}$$

If we perform the quantization procedure using the creation and annihilation operators according to the commutation relation discussed in the case of complex scalar fields, given in Eqs. (3.107) and (3.108), we encounter the following difficulties.

i) Using the commutation relation for $b(\vec{k})$ operators, we obtain

$$b_r(\vec{k})b^\dagger(\vec{k}) = N_b(\vec{k}) + 1, \tag{3.174}$$

$$\text{where}\quad N_b(\vec{k}) = \sum_r b_r^\dagger(\vec{k})b_r(\vec{k}) \tag{3.175}$$

leading to

$$H = \sum_{\vec{k},r}\omega_{\vec{k}}[N_a(\vec{k}) - N_b(\vec{k}) - 1], \tag{3.176}$$

where $N_a(\vec{k}) = a_r^\dagger(\vec{k})a_r(\vec{k})$.

Therefore, $N_b(\vec{k})$ is the occupation number corresponding to the particle created by $b^\dagger(\vec{k})$. It can take any integer value making the contribution of the second term in H negative. This leads to energy eigenvalues which are unbounded from below. It will not allow the determination of a state of minimum energy, that is, the vacuum cannot be defined in this case.

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ii) The number operator $N_a(\vec{k})$ and $N_b(\vec{k})$ can take eigenvalues $0, 1, 2...$, that is, any integral number of particles occupying the states in Fock space. However, in the case of fermions, a state can have only one particle according to Pauli's exclusion principle. Therefore, these commutation rules are not appropriate for describing the quantization of fermions.

Instead, we postulate the following commutation (called anticommutation) rules for the creation and annihilation operators. We can then describe the quantization of "fermions" and apply for quantization of $a_r(\vec{k})$ and $b_r(\vec{k})$ operators as:

$$\{a_r(\vec{k}), a_s^\dagger(\vec{k}')\} = \{b_r(\vec{k}), b_s^\dagger(\vec{k}')\} = \delta_{rs}\delta_{\vec{k}\vec{k}'}, \tag{3.177}$$

$$\{a_r(\vec{k}), a_s(\vec{k})\} = \{a_r^\dagger(\vec{k}), a_s^\dagger(\vec{k})\} = \{b_r(\vec{k}), b_s(\vec{k})\} = \{b_r^\dagger(\vec{k}), b_s^\dagger(\vec{k})\} = 0. \tag{3.178}$$

In this case, we see that:

$$a_r(\vec{k})a_r(\vec{k}) = b_r(\vec{k})b_r(\vec{k}) = 0 \tag{3.179}$$

$$\begin{aligned}
\text{and} \quad N_a^2 &= \sum_{r,k}(a_r^\dagger(\vec{k})a_r(\vec{k})a_r^\dagger(\vec{k})a_r(\vec{k})) \\
&= \sum(a_r^\dagger(\vec{k})(1 - a_r^\dagger(\vec{k})a_r(\vec{k}))a_r(\vec{k})) \\
&= N_a - \sum a_r^\dagger(\vec{k})a_r^\dagger(\vec{k})a_r(\vec{k})a_r(\vec{k}) \\
&= N_a.
\end{aligned} \tag{3.180}$$

That is, $N_a(N_a - 1) = 0$ implying $N_a = 0, 1$ consistent with the exclusion principle. Similarly for N_b, we obtain $N_b(N_b - 1) = 0$. Therefore, using the anticommutation relation in Eq. (3.177), we may write:

$$b_r^\dagger(\vec{k})b_r(\vec{k}) - 1 = -b_r(\vec{k})b_r^\dagger(\vec{k}) \tag{3.181}$$

and the expressions for H and P given in Eqs. (3.172) and (3.173) as:

$$H = \sum_{k,r}\omega_{\vec{k}}(a_r^\dagger(\vec{k})a_r(\vec{k}) + b_r^\dagger(\vec{k})b_r(\vec{k}) - 1), \tag{3.182}$$

$$P = \sum_{k,r}\vec{k}\,(a_r^\dagger(\vec{k})a_r(\vec{k}) + b_r^\dagger(\vec{k})b_r(\vec{k}) - 1). \tag{3.183}$$

Therefore, using the normal-ordering for the operators and redefining the vacuum state as is done in the case of scalar fields, we obtain:

$$H = \sum_{k,r}\omega_{\vec{k}}[N_r(\vec{k}) + \bar{N}_r(\vec{k})], \tag{3.184}$$

$$\vec{P} = \sum\vec{k}[N_r(\vec{k}) + \bar{N}_r(\vec{k})]. \tag{3.185}$$

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Similarly, the charge operator Q is given by:

$$Q = q\sum_{k,r}[N_r(\vec{k}) - \bar{N}_r(\vec{k})], \tag{3.186}$$

for an electron $q = -e$. Equation (3.166) can be inverted to obtain $a_r(\vec{k})$ and $b_r^\dagger(\vec{k})$ in terms of ψ as:

$$a_r(\vec{k}) = \int d\vec{x}\frac{1}{\sqrt{2\omega_{\vec{k}}V}}e^{ik\cdot x}u_r^\dagger(\vec{k})\psi(\vec{x},t), \tag{3.187}$$

$$b_r^\dagger(\vec{k}) = \int d\vec{x}\frac{1}{\sqrt{2\omega_{\vec{k}}V}}e^{ik\cdot x}v_r^\dagger(\vec{k})\psi(\vec{x},t).. \tag{3.188}$$

These equations can be used to derive the commutation relations for the fields $\psi(\vec{x},t)$ and $\psi^\dagger(\vec{x},t)$. However, we can apply the rules of canonical quantization to postulate the anticommutation rules (in place of commutation rules) between the fields $\psi(\vec{x},t)$ and $\psi^\dagger(\vec{x},t)$ and their canonically conjugate fields $\pi_\psi(\vec{x},t)$ and $\pi_{\psi^\dagger}(\vec{x},t)$. Since $\pi_\psi(\vec{x},t) = i\psi^\dagger(\vec{x},t)$ and $\pi_{\psi^\dagger}(\vec{x},t) = 0$, we obtain:

$$\{\psi_\alpha(\vec{x},t), \psi_\beta(\vec{y},t)\} = 0 \tag{3.189}$$

$$\{\psi_\alpha(\vec{x},t), \psi_\beta^\dagger(\vec{y},t)\} = \delta(\vec{x}-\vec{y})\delta_{\alpha\beta} \tag{3.190}$$

$$\{\psi_\alpha^\dagger(\vec{x},t), \psi_\beta^\dagger(\vec{y},t)\} = 0. \tag{3.191}$$

These anticommutation rules can be obtained by using Eqs. (3.177) and (3.178). Conventionally, we can use Eqs. (3.166) and (3.167) for the anticommutation relations for the fields $\psi(\vec{x},t)$ and $\psi^\dagger(\vec{x},t)$ to obtain the anticommutation relations for the creation and annihilation operators as done in the case of scalar fields in Section 3.3.

With the help of the expansion of $\psi(\vec{x},t)$ and $\psi^\dagger(\vec{x},t)$ fields in terms of the creation and annihilation operators and their anticommutation relation, we can demonstrate the following:

i) These anticommutation rules for the Dirac field operators correctly reproduce the Dirac equation for $\psi(\vec{x},t)$ and $\psi^\dagger(\vec{x},t)$, that is,

$$i\dot{\psi}(\vec{x},t) = [\psi(\vec{x},t), H] \tag{3.192}$$

$$\text{leading to } (i\gamma^\mu\partial_\mu - m)\psi(\vec{x},t) = 0 \tag{3.193}$$

$$\text{and } \overline{\psi}(\vec{x},t)(i\gamma^\mu\partial_\mu + m) = 0. \tag{3.194}$$

ii) The vacuum state $|0\rangle$ is defined through the action of the annihilation operators:

$$a_r(\vec{k})|0\rangle = b_r(\vec{k})|0\rangle = 0 \quad \text{for each } r \text{ and } \vec{k} \tag{3.195}$$

$$N|0\rangle = 0. \tag{3.196}$$

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iii) The relation $[\psi(\vec{x},t),Q]=\psi(\vec{x},t)$ shows that if $Q|q\rangle=|q\rangle$, then

$$\langle q'|[\psi(\vec{x},t),Q]|q\rangle = \langle q'|\psi(\vec{x},t)|q\rangle, \tag{3.197}$$

$$\text{that is,}\quad (q-q')\langle q'|\psi(\vec{x},t)|q\rangle = \langle q'|\psi(\vec{x},t)|q\rangle, \tag{3.198}$$

implying that $q'=q-1$, that is, the action of the field $\psi(\vec{x},t)$ is to reduce charge by one unit.

iv) $[Q,H]=0$, implying that the charge is conserved.

v) The particles annihilated by $\psi^+(\vec{x},t)$ and the particles created by $\psi^-(\vec{x},t)$ have opposite charge with the same mass and spin. Therefore, they can be treated as particles and antiparticles.

vi) The operator $a^\dagger(\vec{k})$ and $b^\dagger(\vec{k})$ created particles and antiparticles in states characterized by spin s and momentum $\vec{k}$. A multiparticle state in the Fock space may therefore be written as:

$$|n\rangle = |n_1 n_2 ... a_n\rangle = (a^\dagger_{s_1}(\vec{k}_1))^{n_1}(a^\dagger_{s_2}(\vec{k}_2))^{n_2}...(b^\dagger_{r_1}(\vec{k}_1))^{n_1}(a^\dagger_{r_2}(\vec{k}_2))^{n_2}...|0\rangle, \tag{3.199}$$

where $n_1,n_2,...,n_n$ can be either 0 or 1 for each of the $(2s_i+1)$ or $(2r_i+1)$ spin states.

vii)

$$Q|\vec{k},s\rangle = |\vec{k},s\rangle, \tag{3.200}$$

$$Q|\bar{\vec{k}},\bar{s}\rangle = -|\bar{\vec{k}},\bar{s}\rangle, \tag{3.201}$$

$$\text{where}\quad |\vec{k},s\rangle = a^\dagger_s(\vec{k})|0\rangle \quad\text{and}\quad |\bar{\vec{k}},\bar{s}\rangle = b^\dagger_s(\vec{k})|0\rangle. \tag{3.202}$$

3.7 Covariant Anticommutators and Propagators for Spin $\frac{1}{2}$ Fields

In analogy with the Klein–Gordon fields, we will introduce the covariant anticommutators for Dirac fields using the expansion for $\psi(\vec{x},t)$ and $\psi^\dagger(\vec{x},t)$ to obtain:

$$\{\psi(\vec{x},t),\psi(\vec{y},t)\} = \{\bar{\psi}(\vec{x},t),\bar{\psi}(\vec{y},t)\}=0 \tag{3.203}$$

$$\begin{aligned}\{\psi(\vec{x},t),\bar{\psi}(\vec{y},t)\} &= \{\psi^+(\vec{x},t)+\psi^-(\vec{x},t),\bar{\psi}^+(\vec{y},t)+\bar{\psi}^-(\vec{y},t)\}\\ &= \{\psi^+(\vec{x},t),\bar{\psi}^-(\vec{y},t)\}+\{\psi^-(\vec{x},t),\bar{\psi}^+(\vec{y},t)\}.\end{aligned} \tag{3.204}$$

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Using the commutation relation for $a_r(\vec{k})$ and $b_r(\vec{k})$ or $a_r^\dagger(\vec{k})$ and $b_r^\dagger(\vec{k})$ given in Eqs. (3.177) and (3.178),

$$\{\psi_r^+(\vec{x},t), \bar{\psi}_s^-(\vec{y},t)\} = \frac{1}{2V}\sum_{r,\vec{k}}\sum_{s,\vec{k}'}\frac{1}{\sqrt{\omega_{\vec{k}}\omega_{\vec{k}'}}}\{a_r(\vec{k}), a_s^\dagger(\vec{k}')\}e^{-ik\cdot x+ik'y}u_r(\vec{k})\overline{u}_s(\vec{k}') \tag{3.205}$$

$$= \frac{1}{2V}\sum_{r,\vec{k}}\frac{1}{\omega_{\vec{k}}}e^{-ik(x-y)}u_r(\vec{k})\overline{u}_r(\vec{k}) \tag{3.206}$$

$$= \int \frac{d\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}}(\not{k}+m)e^{-ik\cdot(x-y)}, \tag{3.207}$$

where we also used $\sum_r u_r(\vec{k})\bar{u}_r(\vec{k}) = (\not{k}+m)$ from Chapter 2.

$$\{\psi_r^+(\vec{x},t), \overline{\psi}_s^-(\vec{x},t)\} = iS_{rs}^+(\vec{x}-\vec{y}), \tag{3.208}$$

where

$$S_{rs}^+ = -\frac{i}{(2\pi)^3}\int d\vec{k}\frac{(\not{k}+m)_{rs}}{2\omega_{\vec{k}}}e^{-ik\cdot x} \tag{3.209}$$

$$= \left(i\frac{\partial}{\partial x^\mu}+m\right)_{rs}\Delta^+(x-y). \tag{3.210}$$

We can now use these results for the integral representation for $\Delta^\pm(x)$ to obtain the integral representation for $S^\pm(x)$, that is,

$$S^\pm = -(i\not{\partial}+m)\int_{C^\pm}\frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot x}}{k^2-m^2} \tag{3.211}$$

$$= -\frac{1}{(2\pi)^4}\int_{C^\pm} d^4k\frac{1}{\not{k}-m}e^{-ik\cdot x}, \tag{3.212}$$

where $C^\pm$ are the contours in the complex (k_0) plane shown in Figure 3.2.

3.8 Time-ordered Products and Feynman Propagators

We define the time-ordered product or T-product for fermion fields $\psi(x)$ as:

$$T(\psi(x_1)\overline{\psi}(x_2)) = \psi(x_1)\overline{\psi}(x_2) \qquad (\text{for } t_1 > t_2) \tag{3.213}$$

$$= -\overline{\psi}(x_2)\psi(x_1) \qquad (\text{for } t_2 > t_1) \tag{3.214}$$

$$\text{or} \quad T(\psi(x),\overline{\psi}(x_2)) = \theta(t_1-t_2)\psi(x_1)\overline{\psi}(x_2) - \theta(t_2-t_1)\overline{\psi}(x_2)\psi(x_1). \tag{3.215}$$

This definition of time-ordered product for the fermion fields or T-product is different from the definition of T-product for boson fields in the sign of the second term. This also occurs in the definition of normal product.

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In the case of fermion fields:

$$\begin{aligned}
\psi(x_1)\bar{\psi}(x_2) &= (\psi^+(x_1)+\psi^-(x_1))(\bar{\psi}^+(x_2)+\bar{\psi}^-(x_2))\\
&= \psi^+(x_1)\bar{\psi}^+(x_2)+\psi^+(x_1)\bar{\psi}^-(x_2)+\psi^-(x_1)\bar{\psi}^+(x_2)+\psi^-(x_1)\bar{\psi}^-(x_2)\\
&= \bar{\psi}^-(x_2)\psi^+(x_1)-\bar{\psi}^-(x_2)\psi^+(x_1)+\psi^+(x_1)\bar{\psi}^+(x_2)\\
&\quad +\psi^+(x_1)\bar{\psi}^-(x_2)+\psi^-(x_1)\bar{\psi}^+(x_2)+\psi^-(x_1)\bar{\psi}^-(x_2)\\
&= N(\psi(x_1)\bar{\psi}(x_2))+\{\psi^+(x_1),\bar{\psi}^-(x_2)\}, \qquad (3.216)
\end{aligned}$$

where

$$N(\psi(x_1)\bar{\psi}(x_2)) = \psi^-(x_1)\bar{\psi}^+(x_2)+\psi^-(x_1)\bar{\psi}^-(x_2)-\bar{\psi}^-(x_2)\psi^+(x_1)+\psi^+(x_1)\bar{\psi}^+(x_2).$$

Taking the vacuum expectation value of the T-order product, we get:

$$\begin{aligned}
\langle 0|T\{\psi(x_1)\bar{\psi}(x_2)\}|0\rangle &= \langle 0|\theta(t_1-t_2)\psi(x_1)\bar{\psi}(x_2)\\
&\quad -\theta(t_1-t_2)\bar{\psi}(x_2)\psi(x_1)\\
&= \langle 0|\{\psi^+(x_1)\bar{\psi}^-(x_2)\}|0\rangle\\
&= iS^+(x_1-x_2) \quad \text{for } t_1>t_2 \qquad (3.217)\\
\text{and similarly, } \langle 0|T\{\bar{\psi}(x_2)\psi(x_1)\}|0\rangle &= \langle 0|\{\bar{\psi}^-(x_2),\psi^+(x_1)\}|0\rangle\\
&= iS^-(x_1-x_2) \quad \text{for } t_2>t_1. \qquad (3.218)
\end{aligned}$$

The Feynman propagator S_F is defined as:

$$\begin{aligned}
\langle 0|T\{\psi(x_1)\bar{\psi}(x_2)\}|0\rangle &= iS_F(x_1-x_2), \qquad (3.219)\\
\text{where} \quad S_F(x_1-x_2) &= \theta(t_1-t_2)S^+(x_1-x_2)-\theta(t_2-t_1)S^-(x_1-x_2)\\
&= \Big(i\gamma^\mu\frac{\partial}{\partial x^\mu}+m\Big)\Delta_F(x_1-x_2). \qquad (3.220)
\end{aligned}$$

Here, $\Delta_F(x_1-x_2)$ is given in Eq. (3.160).

We obtain the integral representation of $S_F(x)$ as:

$$S_F(x)=\frac{1}{(2\pi)^4}\int_{C_F}\frac{d^4k e^{-ik\cdot x}(\not{k}+m)}{k^2-m^2+i\epsilon}. \qquad (3.221)$$

Using the integral representation of $\Delta_F(x)$ and $S_F(x)$ given in Eqs. (3.160) and (3.220), respectively, the physical interpretation of $S^+(x_1-x_2)$ and $S^-(x_1-x_2)$ are given in Figure 3.6. In case of $t_1 > t_2$, the Dirac particle is created at x_2 and travels to x_1 , where it is annihilated, while for $t_2>t_1$, the Dirac antiparticle is created at x_1 (through $b^\dagger(\vec{k})$) and travels to x_2, where it is annihilated (through $b(\vec{k})$).

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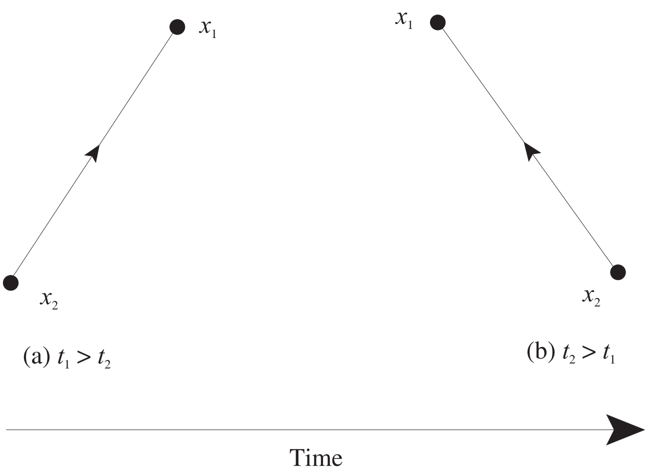

Figure 3.6 Propagation of a fermion between x_1 and x_2 for (a) $t_1 > t_2$ and (b) $t_2 > t_1$.

3.9 Quantization of Massless Electromagnetic Fields: Photons

The equation of motion for a massless spin 1 field $A^\mu(x)$ corresponding to photons has been discussed in Chapter 2, using Maxwell's equations of classical electrodynamics for electric field $\vec{E}(x)$ and magnetic field $\vec{B}(x)$. Defining an antisymmetric tensor of second rank $F^{\mu\nu}$ as:

$$F^{\mu\nu} = \partial^\nu A^\mu(x) - \partial^\mu A^\nu(x), \tag{3.222}$$

Maxwell's equations of motion can be written as:

$$\partial_\nu F^{\mu\nu}(x) = J^\mu(x) \tag{3.223}$$

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0, \tag{3.224}$$

or in terms of $A^\mu(x)$ as:

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu(x)) = J^\mu(x). \tag{3.225}$$

These equations of motion for $A^\mu(x)$ can be obtained by choosing a Lagrangian $\mathcal{L}(x)$ for the free field as:

$$\mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) \tag{3.226}$$

and applying the Euler–Lagrange equation. However, this Lagrangian is not suitable for the quantization procedure as the conjugate field $\pi^\mu(x)$ corresponding to $A^\mu(x)$ is given by:

$$\pi^\mu(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu(x)} = -F^{\mu 0}(x), \tag{3.227}$$

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implying that $\pi^0 = -F^{00}(x) = 0$, that is, one of the canonical fields is zero. This is not compatible with the canonical quantization conditions that need to be satisfied for all the four components of the $A^\mu(x)$ field and the conjugate fields $\Pi^\mu(x)$.

A Lagrangian which is suitable for the quantization and reproduces the equations of motion was first proposed by Fermi for the free field:

$$\mathcal{L} = -\frac{1}{2}(\partial_\nu A_\mu(x))(\partial^\nu A^\mu(x)). \tag{3.228}$$

Differentiating Eq. (3.228) with respect to $A^\mu_{,\alpha}$, we get:

$$\frac{\partial \mathcal{L}}{\partial A^\mu_{,\alpha}} = -\partial_\alpha A^\mu, \quad \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = -\dot{A}^\mu. \tag{3.229}$$

Using the Euler–Lagrange equation of motion, we obtain the equation of motion for the $A^\mu(x)$ field as:

$$\Box A^\mu(x) = 0, \tag{3.230}$$

and the conjugate field $\pi^\mu(x)$

$$\pi^\mu(x) = -\dot{A}^\mu, \tag{3.231}$$

which facilitates the process of canonical quantization.

It can be shown that the Lagrangian density given in Eqs. (3.226) and (3.228) lead to same action(S) in the Lorenz gauge giving the same equations of motion after applying the principle of least action.

The solutions of the equations of motion given in Eq. (3.230) can be written in analogy with the scalar field for each component of $A^\mu(x)$, that is,

$$\begin{aligned} A^\mu(x) &= \sum_{r,\vec{k}} \sqrt{\frac{1}{2V\omega_{\vec{k}}}} \left[\epsilon^\mu_r(\vec{k}) a_r(\vec{k}) e^{-ik\cdot x} + \epsilon^{*\mu}_r(\vec{k}) a^\dagger_r(\vec{k}) e^{ik\cdot x}\right], & (3.232) \\ &= A^+_\mu(x) + A^-_\mu(x), & (3.233) \end{aligned}$$

where k satisfies the relation $k^2 = k_0^2 - \vec{k}^2 = 0$, that is, $k_0 = |\vec{k}|$ and $r = 0, 1, 2, 3$ labels the four polarization states for the four-component vector field $A^\mu(x)$, represented by $\epsilon^\mu_r(\vec{k})$. As noted earlier in Chapter 2, all the four components are not independent but subject to the Lorenz condition; the choice of gauge can eliminate two of the independent components for real photons. The polarization vectors $\epsilon^\mu_r(r = 0 - 3)$ are chosen such that ϵ^μ_0 is time-like and $\epsilon^\mu_r(r = 1, 2, 3)$ are space-like. The operators $a_r(\vec{k})$ and $a^\dagger_r(\vec{k})$ are the annihilation and creation operators for the quanta of the $A^\mu(x)$ field, that is, the photon, in the state of polarization $r(= 0, 1, 2, 3)$ corresponding to the scalar $(r = 0)$, longitudinal $(r = 3)$ or transverse $(r = 1, 2)$ photons, assuming the Z-direction to be the direction of propagation. The normalization of the polarization vectors as discussed in Chapter 2 are given as:

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$$\epsilon_r(\vec{k})\epsilon_s(\vec{k}) = \epsilon_{r\mu}(\vec{k})\epsilon_s^\mu(\vec{k}) = -\rho_r\delta_{rs}, \tag{3.234}$$

where $\rho_0 = -1, \rho_1 = \rho_2 = \rho_3 = 1$, and the completeness relation is

$$\sum \rho_r \epsilon_r^\mu(\vec{k})\epsilon_s^\nu(\vec{k}) = -g^{\mu\nu}. \tag{3.235}$$

3.10 Commutation Relations and Quantization of $A^\mu(x)$

The canonical quantization of the electromagnetic field A^μ describing the photons is achieved by postulating the equal time commutation relations (ETCR) for $A^\mu(x)$ and the conjugate field $\pi^\mu(x)(= -\dot{A}^\mu(x))$ as:

$$\left[A^\mu(\vec{x},t), A^\nu(\vec{x}',t)\right] = \left[\dot{A}^\mu(\vec{x},t), \dot{A}^\nu(\vec{x}',t)\right] = 0, \tag{3.236}$$

$$\left[A^\mu(\vec{x},t), \dot{A}^\nu(\vec{x}',t)\right] = -ig^{\mu\nu}\delta(\vec{x}-\vec{x}'). \tag{3.237}$$

It can be seen that these equations are like the ETCR for the scalar Klein–Gordon field defining the quantization, except for the factor $-g^{\mu\nu}$ which takes care of the four independent components of $A^\mu(x)$. However, there is still some problem with the quantization, that is, the commutation relation in Eq. (3.237) does not satisfy the condition $\partial_\mu A^\mu = 0$. In order to implement this condition, we first quantize in the general case and impose this condition after quantization using the theory of Gupta [213] and Bleuler [214] as discussed later in Section 3.11.

The canonical commutation relations for the fields given in Eqs. (3.236) and (3.237) lead to the commutation relations for the creation and annihilation operators as:

$$\left[a_r(\vec{k}), a_s^\dagger(\vec{k}')\right] = \rho_r\delta_{rs}\delta_{\vec{k}\vec{k}'}, \tag{3.238}$$

$$\left[a_r(\vec{k}), a_s(\vec{k}')\right] = \left[a_r^\dagger(\vec{k}), a_s^\dagger(\vec{k}')\right] = 0. \tag{3.239}$$

These commutation relations are similar to the commutation relations for the scalar fields except in the case of $r = 0$, that is, the creation and annihilation operators corresponding to the scalar photons with polarization $\vec{\epsilon}_0$. This creates a problem as it allows the creation of scalar photon states with negative normalization, which can be seen as follows.

The vacuum state $|0\rangle$ is defined through the action of the annihilation operator as:

$$a_r(\vec{k})|0\rangle = 0 \quad \text{and} \quad \langle 0|a_r^\dagger(\vec{k}) = 0, \tag{3.240}$$

while the one photon state $|\vec{k}, r\rangle$ with momentum k^μ and polarization ϵ_r^μ is defined through the action of the creation operator $a_r^\dagger(\vec{k})$ as:

$$a_r^\dagger(\vec{k})|0\rangle = |\vec{k}, r\rangle\,, \quad \langle 0|a_r(\vec{k}) = \langle\vec{k}, r|$$

$$\text{and } \langle\vec{k}', 0|\vec{k}, 0\rangle = \langle 0|a_0(\vec{k}')a_0^\dagger(\vec{k})|0\rangle$$

$$\begin{aligned} &= \langle 0|[a_0(\vec{k}'), a_0^\dagger(\vec{k})]|0\rangle \\ &= \langle 0|0\rangle \rho_0\, \delta_{\vec{k}\vec{k}'} = -\delta_{\vec{k}\vec{k}'}. \end{aligned} \tag{3.241}$$

The Hamiltonian is calculated in terms of the creation and annihilation operators from the general expression of $\mathcal{H}$ as:

$$H = \int d\vec{x} \left[\pi^\mu(x)\dot{A}_\mu(x) - \mathcal{L}(x)\right], \tag{3.242}$$

with $\pi^\mu = -\dot{A}^\mu$ to give

$$\begin{aligned} H &= \sum_{r,\vec{k}} \rho_r \omega_{\vec{k}} a_r^\dagger(\vec{k}) a_r(\vec{k}) \\ &= \sum_{r,\vec{k}} \omega_{\vec{k}} N_r(\vec{k}), \end{aligned} \tag{3.243}$$

with $N_r(\vec{k}) = \rho_r a_r^\dagger(\vec{k}) a_r(\vec{k})$. We see that the number operator for scalar photons $r = 0$ is negative. This is related to the negative normalization of these photons. However, it does not create any problems in calculating the energy.

The energy of a single particle state $|\vec{k}, r\rangle$ is, therefore, given by E_{kr} and is obtained from the eigenvalue equation

$$H|\vec{k}, r\rangle = \omega_{\vec{k}}|\vec{k}, r\rangle = \omega_{\vec{k}} a_r^\dagger(\vec{k})|0\rangle, \tag{3.244}$$

since,

$$\begin{aligned} H|\vec{k}, r\rangle &= \sum_{\vec{k}', r'} \omega'_{\vec{k}} \rho'_r a_{r'}^\dagger(\vec{k}') a_{r'}(\vec{k}') a_r^\dagger(\vec{k})|0\rangle \\ &= \sum_{\vec{k}', r'} \omega'_{\vec{k}} \rho'_r a_{r'}^\dagger(\vec{k}')[a_{r'}(\vec{k}'), a_r^\dagger(\vec{k})]|0\rangle \\ &= \sum_{\vec{k}', r'} \omega'_{\vec{k}} \rho'_r a_{r'}^\dagger(\vec{k}') \rho_r \delta_{rr'} \delta_{kk'}|0\rangle \\ &= \rho_r^2 \omega_{\vec{k}} a_r^\dagger(\vec{k})|0\rangle = \omega_{\vec{k}}|\vec{k}, r\rangle. \end{aligned} \tag{3.245}$$

Therefore, a number operator $N_r(\vec{k})$ can still be defined as:

$$N_r(\vec{k}) = \rho_r a_r^\dagger(\vec{k}) a_r(\vec{k}), \tag{3.246}$$

ignoring the problem of the negative normalization of the scalar photon at present, which is solved by choosing the appropriate gauge through the mechanism of gauge fixing.

3.11 Lorenz Condition and Gupta–Bleuler Formalism

The covariant formalism of the quantization of massless photons presented in the previous section in analogy with the covariant formalism for the quantization of the scalar field is beset with the following problems:

i) It can be seen that the Lorenz conditions which are needed in this formalism to reproduce Maxwell's equations are not implemented in the commutation relations postulated in Eqs. (3.236) and (3.237), that is,

$$[\partial_\mu A^\mu(\vec{x},t), \dot{A}^\nu(\vec{x}\,',t)] = -i\partial_\mu g^{\mu\nu}\delta(\vec{x}-\vec{x}\,') \neq 0. \tag{3.247}$$

ii) The commutation relations for the annihilation and creation operators corresponding to the $r = 0$ component of polarization vectors, that is, ϵ_0^μ, are not consistent with the canonical commutation relations for the scalar fields.

Both the aforementioned problems are manifestation of the non-implementation of the Lorenz condition. Therefore, the covariant theory is not equivalent to the massless photon corresponding to Maxwell's theory. The problem was resolved by Gupta [213] and Bleuler [214] independently, by imposing a weaker condition, that is,

$$\partial_\mu A^{\mu+}|\psi\rangle = 0\,, \quad \langle\psi|\partial_\mu A^{\mu-} = 0, \tag{3.248}$$

involving absorption operators only, implying that at the level of matrix element:

$$\langle\psi|\partial_\mu A^\mu|\psi\rangle = \langle\psi|\partial_\mu A^{\mu+} + \partial_\mu A^{\mu-}|\psi\rangle = 0. \tag{3.249}$$

The Lorenz condition [195] is realized weakly at the expectation values level and not at the operator identity level. Since the physical observables are theoretically obtained through the calculation of expectation values of the operators, this implementation of the Lorenz condition should be considered satisfactory.

The equation implies that:

$$\langle\psi|\partial_\mu A^{\mu+}|\psi\rangle = -i\langle\psi|\sum_{\vec{k}}\sum_{r=0}^{3}\frac{1}{\sqrt{2V\omega_{\vec{k}}}}k_\mu\epsilon_r^\mu(\vec{k})a_r(\vec{k})e^{-ik\cdot x}|\psi\rangle = 0. \tag{3.250}$$

Let us assume that the photon is moving along the z-axis, that is, $\vec{k} = |\vec{k}|\hat{z}$ such that:

$$k^\mu = (k_0, 0, 0, |\vec{k}|) \text{ and } k_\mu = (k_0, 0, 0, -|\vec{k}|). \tag{3.251}$$

Since $k_0 = |\vec{k}| = \omega_{\vec{k}}$, with the choice of ϵ_r^μ for $r = 0, 1, 2, 3$ given in Eq. (2.202), we obtain:

$$k_\mu \sum_{r=0}^{3} a_r\epsilon_r^\mu = \omega_{\vec{k}}\left(a_0(\vec{k}) - a_3(\vec{k})\right), \tag{3.252}$$

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which gives

$$\langle\psi|\partial_\mu A^{\mu+}|\psi\rangle = -i\langle\psi|\sum_{\vec{k}}\frac{1}{\sqrt{2V\omega_{\vec{k}}}}\omega_{\vec{k}}(a_0(\vec{k})-a_3(\vec{k}))e^{-ik\cdot x}|\psi\rangle$$

$$= 0, \tag{3.253}$$

$$\text{that is, } a_0(\vec{k})|\psi\rangle - a_3(\vec{k})|\psi\rangle = 0 \quad \text{for all } \vec{k} \tag{3.254}$$

$$\text{and } \langle\psi|a_0^\dagger(\vec{k}) = \langle\psi|a_3^\dagger(\vec{k}). \tag{3.255}$$

This implies that the contribution of the scalar photons and longitudinal photons cancel each other. Since the physical operators in Fock space will involve summation over all the photon polarizations in a covariant formulation, only transverse photons will contribute. The theory then becomes equivalent to Maxwell's theory. This can be seen in case of the Hamiltonian operator:

$$\langle\psi|H|\psi\rangle = \langle\psi|\sum_{\vec{k}}\sum_{r=0}^{3}\rho_r\omega_{\vec{k}}a_r^\dagger(\vec{k})a_r(\vec{k})|\psi\rangle$$

$$= \langle\psi|\omega_{\vec{k}}(a_1^\dagger(\vec{k})a_1(\vec{k})+a_2^\dagger(\vec{k})a_2(\vec{k})+a_3^\dagger(\vec{k})a_3(\vec{k})-a_0^\dagger(\vec{k})a_0(\vec{k}))|\psi\rangle$$

$$= \langle\psi|\sum_{\vec{k}}\sum_{r=1}^{2}\omega_{\vec{k}}a_r^\dagger(\vec{k})a_r(\vec{k})|\psi\rangle. \tag{3.256}$$

Therefore, in the case of free fields of real photons, only transverse components will contribute. It is important to realize that the Lorenz condition eliminates both the additional photon degrees of freedom arising due to the scalar and longitudinal components included due to covariant formulation. However, in the case of interacting fields where virtual photons can be produced in the intermediate states, the situation is not so simple. In such cases, the contribution of scalar and longitudinal photons cannot be ignored. In any physical process involving real photons, the initial and final states will be transverse photons which are described by free fields, being asymptotic states, while the intermediate states will be virtual states, needing a covariant description of photons in terms of all the polarization components included in the photon propagator.

3.12 Time-ordered Product and Propagators for Spin 1 Fields

In Section 3.3, we defined the covariant commutators and their vacuum expectation values and related them to the propagators $\Delta^{\pm}(x)$ and $\Delta(x)$ for the scalar field $\phi(x)$. We also defined the T-order product of scalar fields and their expectation values and related them to the propagators $\Delta^{\pm}(x)$ and $\Delta_F(x)$. We can extend the formalism to apply them directly in case of the four-component vector field $A^\mu(x)$ in the following way:

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We define the covariant commutator of the fields $A^\mu(x)$ in analogy with Eq. (3.133) to write:

$$[A^\mu(x_1), A^\nu(x_2)] = iD^{\mu\nu}(x_1 - x_2), \tag{3.257}$$

where

$$D^{\mu\nu}(x_1 - x_2) = \lim_{m\to 0}(-g^{\mu\nu}\Delta(x_1 - x_2)). \tag{3.258}$$

Here $\Delta(x_1 - x_2)$ is the invariant Δ-function given as:

$$\Delta(x_1 - x_2) = -\frac{1}{(2\pi)^3}\int \frac{d\vec{k}}{\omega_{\vec{k}}} \sin[k(x_1 - x_2)]. \tag{3.259}$$

Similarly, the vacuum expectation value of the time-ordered product(T-product) is defined to give the Feynman propagator $D_F^{\mu\nu}(x)$ as:

$$\langle 0| T\left[A^\mu(x_1), A^\nu(x_2)\right]|0\rangle = iD_F^{\mu\nu}(x_1 - x_2), \tag{3.260}$$

where

$$\begin{aligned} D_F^{\mu\nu}(x_1 - x_2) &= -\lim_{m\to 0}(g^{\mu\nu}\Delta_F(x_1 - x_2)) \\ &= -\frac{g^{\mu\nu}}{(2\pi)^4}\int_{CF}\frac{d^4k\, e^{-ik\cdot(x_1-x_2)}}{k^2 + i\epsilon} \qquad (3.261) \\ &= \frac{1}{(2\pi)^4}\int_{CF} d^4k\, D_F^{\mu\nu}(\vec{k})\, e^{-ik\cdot(x_1-x_2)}. \qquad (3.262) \end{aligned}$$

Here

$$D_F^{\mu\nu}(\vec{k}) = \frac{-g^{\mu\nu}}{k^2 + i\epsilon}, \tag{3.263}$$

is the propagator in momentum space. In terms of the explicit polarization states $\epsilon_r(\vec{k})$, $D_F^{\mu\nu}(\vec{k})$ can be written using all the polarization states, that is,

$$\begin{aligned} D_F^{\mu\nu}(\vec{k}) &= \frac{1}{k^2 + i\epsilon}\left[\sum_{k,r}\rho_r\epsilon_r^\mu(\vec{k})\epsilon_r^\nu(\vec{k})\right] \\ &= \frac{1}{k^2 + i\epsilon}\left[\sum_{k,r=1}^{2}\epsilon_r^\mu(\vec{k})\epsilon_r^\nu(\vec{k}) + \frac{(k^\mu - (k.n)n^\mu)(k^\nu - (k.n)n^\nu)}{(k.n)^2 - k^2} - n^\mu n^\nu\right]. \end{aligned} \tag{3.264}$$

Using the explicit values of the polarization vectors,

$$\epsilon_0^\mu(\vec{k}) = n^\mu = (1,0,0,0) \qquad \text{and} \qquad \epsilon_3^\mu(\vec{k}) = \left[\frac{k^\mu - (k.n)n^\mu}{\sqrt{(k.n)^2 - k^2}}\right]. \tag{3.265}$$

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The three terms in Eq. (3.264) correspond to the propagation of the transverse photons, longitudinal photons, and the scalar photons. However we express $D_F^{\mu\nu}(\vec{k})$ as:

$$D_F^{\mu\nu}(\vec{k}) = D_{F,T}^{\mu\nu}(\vec{k}) + D_{F,C}^{\mu\nu}(\vec{k}) + D_{F,R}^{\mu\nu}(\vec{k}), \tag{3.266}$$

where

$$D_{F,T}^{\mu\nu}(\vec{k}) = \frac{1}{k^2 + i\epsilon}\left[\sum_{\vec{k},r=1}^{2} \epsilon_r^{\mu}(\vec{k})\epsilon_r^{\nu}(\vec{k})\right] \tag{3.267}$$

$$D_{F,C}^{\mu\nu}(\vec{k}) = \frac{n^\mu n^\nu}{(k.n)^2 - k^2} \tag{3.268}$$

$$D_{F,R}^{\mu\nu}(\vec{k}) = \frac{1}{k^2 + i\epsilon}\left[\frac{k^\mu k^\nu - (k.n)\left(k^\mu n^\nu + k^\nu n^\mu\right)}{(k.n)^2 - k^2}\right]. \tag{3.269}$$

To understand the physical interpretation of the longitudinal and scalar terms, let us consider the term proportional to $n^\mu n^\nu$, that is,

$$D_{F,C}^{\mu\nu}(\vec{k}) = \frac{n^\mu n^\nu}{(k.n)^2 - k^2}$$

and the rest, $D_{F,R}^{\mu\nu}(\vec{k})$ as:

$$D_{F,R}^{\mu\nu}(\vec{k}) = \frac{1}{k^2 + i\epsilon}\frac{k^\mu k^\nu - (k.n)(k^\mu n^\nu + k^\nu n^\mu)}{(k.n)^2 - k^2}. \tag{3.270}$$

The physical interpretation of $D_{F,C}^{\mu\nu}(\vec{k})$ can be seen by calculating it in coordinate space in which

$$D_{F,C}^{\mu\nu}(x) = \frac{1}{(2\pi)^4}\int \frac{n^\mu n^\nu}{(k.n)^2 - k^2} e^{-ik\cdot x} d^4x. \tag{3.271}$$

Using n^μ from Eq. (3.265), we obtain the only non-vanishing component of $D_{F,C}^{\mu\nu}(x)$ as:

$$\begin{aligned} D_{F,C}^{00}(x) &= \frac{1}{(2\pi)^3}\int \frac{d\vec{k}e^{-i\vec{k}\cdot\vec{x}}}{|\vec{k}|^2}\delta(x_0) \\ &= \frac{1}{4\pi|\vec{x}|}\delta(x_0) = \phi(x), \end{aligned} \tag{3.272}$$

that is, instantaneous Coulomb potential between the two static charges. On the other hand, the contribution of the $D_{F,R}^{\mu\nu}(\vec{k})$ term to any physical process involving real photons specified by the polarization vector $\epsilon^\mu(\vec{k})$ and the external electromagnetic current j^μ of a charged particle described by:

$$j^\mu(x) = \frac{1}{(2\pi)^4}\int d^4k\, j^\mu(\vec{k})e^{-ikx}, \tag{3.273}$$

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which is conserved, vanishes. This is because any matrix element corresponding to a physical process involving electromagnetic interaction, shown in Figure 3.7 between the two fermion currents $j^\mu(x_1)$ and $j^\mu(x_2)$ through the photon propagator gets contribution from $D^{\mu\nu}_{F,R}$ as:

$$\int d^4x_1\, d^4x_2\, j_\mu(x_1)\, D^{\mu\nu}_{F,R}(x_1 - x_2)\, j_\nu(x_2). \tag{3.274}$$

Since the currents $j^\mu(x_1)$ and $j^\mu(x_2)$ are conserved, that is, $k \cdot j^{(1)}_\mu = k \cdot j^{(2)}_\mu = 0$, the contribution of $D^{\mu\nu}_{F,R}$ to the physical process vanishes. Thus, the covariant description of the photon field is equivalent to the description of photon fields in Coulomb gauge where it is described by $A^\mu(= \phi, \vec{A})$ as discussed in Chapter 2.

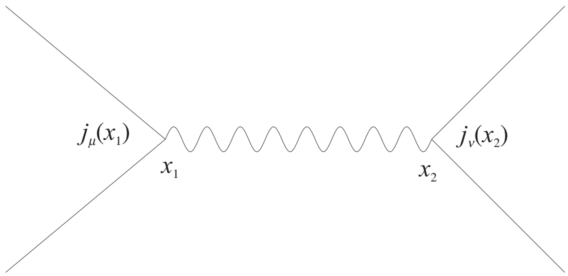

Figure 3.7 Interaction of two electromagnetic currents through photon exchange.

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