Heat Transmission Formula

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Problem

I created a custom Unreal component called `UTemperatureComponent`. The component is just a `UCapsuleComponent` with a `double` attribute, representing the temperature of the `AActor` it is attached to; the capsule is needed to determine, through an apposite object channel I created, when two `AActor`s with the component are close enough to exchange heat.

I now need to find a formula that determines how the temperatures of the two objects vary over time when they exchange heat, so that I can update the values frame-by-frame through the `TickComponent` method.

I am going to assume that bodies with `UTemperatureComponent`s exchange heat via conduction. This is just a raw approximation of reality: conduction requires physical contact, while I am assuming that even distant bodies whose temperature capsules intersect can exchange heat in this way. However my plan is to:

- Give every `AActor` a very small temperature capsule, that barely goes beyond its mesh. This way, different `AActor`s can exchange heat only if they're very close to each other, simulating physical contact

- Have a scatterer in the map that spawns a cubic lattice of point-like `AActor`s with the `UTemperatureComponent`; these `AActor`s are set to be invisible and ignore any physical collision, but each of them has the temperature capsule set so that it intersects the neighbour ones in the lattice.

I believe that the two points above, the second one in particular, allow me to reproduce reality, even with the conduction approximation. The lattice simulates a fluid in which all "real" `AActor`s, i.e. actors that are visible to the player and have physical collisions, move around. If an `AActor` like, for example, a very hot fireball, is spawned into the map, the fireball first warms all the point-like `AActor`s of the lattice that are immediately nearby and intersect the fireball's temperature capsule. The next frame, the points of the lattice that do not cross the fireball temperature capsule directly but do cross the capsules of the points that were warmed before also get warm, while the first points get even warmer. This cycle repeats, causing the heat emitted by the fireball to propagate through all the map while the fireball itself gets colder.

Procedure

Given a cold body at temperature $T_{i,C}$ and a hot body at temperature $T_{i,H}$, suppose they can exchange heat Q only via a conducting rod of lenght L, section A, and thermal conductivity k. Then, the *heat flux* between the two bodies is given by (see Ref. [1] for details):

$$\frac{\mathrm{d}q}{\mathrm{d}t} = kA\frac{T_{i,H} - T_{i,C}}{L} \tag{1}$$

The heat flux causes an infinitesimal temperature raise in the cold body (see Ref. [2])

$$\mathrm{d}q = C_C \mathrm{d}T \,, \tag{2}$$

where $C_C = c_C m_C$ is the *heat capacity* of the cold body, which depends on the *specific heat capacity* c_C and the body *mass* m_C. Substituting (2) into (1), we get the equation for the variation of temperature:

$$\mathrm{d}T = \frac{kA}{C_C L} \left(T_{i,H} - T_{i,C} \right) \mathrm{d}t \tag{3}$$

This equation only holds for infinitesimal time intervals $\mathrm{d}t$, since $T_{i,H}$ and $T_{i,C}$ vary with every infinitesimal temperature exchange $\mathrm{d}T$. For our purposes, it is ok: the parameter `DeltaTime` of the `TickComponent` method is generally quite small. The hot body undergoes a similar process, the only difference being that the $\mathrm{d}T$ is actually negative. The final temperatures after one tick are:

$$T_{f,C} = T_{i,C} + \frac{kA}{C_C L} \left(T_{i,H} - T_{i,C} \right) \mathrm{d}t \tag{4}$$

$$T_{f,H} = T_{i,H} - \frac{kA}{C_H L} \left(T_{i,H} - T_{i,C} \right) \mathrm{d}t \tag{5}$$

Tick after tick, the difference $T_{i,H} - T_{i,C}$ tends to zero, causing the temperature updates to be less and less relevant: an infinite amount of time must pass before the two bodies reach perfect thermal equilibrium.

Test

The validity of the formula has been verified by creating two `AActor`s with overlapping `UTemperatureComponent`s, one at $T_{i,H} = 298\,\mathrm{K}$ and the other at $T_{i,C} = 293\,\mathrm{K}$. They were put in the level at a distance $L = 10\,\mathrm{cm}$ and I set $C_H = C_C = 4196\,\mathrm{J/K}$. I used $kA = 1\,\mathrm{J} \cdot \mathrm{m/s} \cdot \mathrm{K}$ for simplicity, I haven't decided how to deal with these quantities yet (they'll probablly end up being "magic numbers" I'll adjust according to gameplay necessities). The equilibrium temperature of the two bodies is (see Ref. [1]):

$$T_{eq} = \frac{C_C T_{i,C} + C_H T_{i,H}}{C_C + C_H} \tag{6}$$

Since $C_H = C_C$ in our case, this implies $T_{eq} = (T_{i,H} + T_{i,C})/2 = 295.5\,\mathrm{K}$. After approximately $t \sim 35\,\mathrm{min}$ the results were $T_{i,C} = 295.499941\,\mathrm{K}$ and $T_{i,H} = 295.500059\,\mathrm{K}$, proving that the formula works.

Warning: do not use (3) to find out how much time it is needed to get to T_{eq} because it won't work. You'll get $t \sim 3\,\mathrm{min}$, which is way shorter than the experimental one. Why? Once again: (3) only works with infinitesimal times because it doesn't account that the initial temperatures $T_{i,H}$ and $T_{i,C}$ also depend on time.

References

[1] https://phas.ubc.ca/~kiefl/ch15_part2.pdf

[2] https://physics.stackexchange.com/questions/248202/calculating-the-time-for-two-bodies-to-reach-thermodynamic-equilibrium