

assembles itself in statistical equilibrium into such a form that the binding energy per nucleon is maximized. This result of statistical mechanics (essentially a generalized Saha equation) economically characterizes the peak at iron, and the corresponding thermodynamic states are not unlike those anticipated in the terminal stages of stellar evolution.

It might at first seem that one would expect the synthesis of heavy elements to terminate with iron, at least if one envisions only equilibrium states of nuclear matter in which the total mass is converted to heavy nuclei. As an alternative, it was necessary to seek circumstances in which small traces of heavy nuclei could be processed to even heavier nuclei by nuclear reactions occurring as a *by-product* of the reactions among the dominant lighter nuclei. Capture of light charged nuclei (protons, for example) by heavy nuclei is inadequate because the coulomb barrier is much too large for barrier penetration at the temperatures characteristic of the burning of light nuclei. The solution appears to have been found by the demonstration that the very heavy nuclei can be formed efficiently by the capture of free neutrons liberated as a *by-product* of reactions between light charged particles. Such a strong correlation has been established between heavy nuclear abundances and their neutron-capture characteristics that the assumption must be considered as correct.

Thus by and large, one can say that the nuclear reactions in stars can produce the various nuclear species in abundances consistent with the anticipated reactions. The major features of the science are now so well demonstrated that few people doubt the existence of a correlation between nuclear abundances and nuclear properties. The major efforts are presently directed toward fitting the details of nuclear abundances to the various mechanisms and locating the sites of nucleosynthesis within the context of stellar evolution. It has become increasingly evident that the major problems lie not with synthesizing the elements in stars but with reinjecting them in a natural way into the interstellar medium. The question of the evolution of the chemical composition of the galaxy is inextricably entwined with the details of the rate of star formation throughout galactic history and the question of the composition of matter presently bound in stars. Such considerations illustrate the nature of the problems in stellar evolution and nucleosynthesis. The remainder of this book is devoted to the basic physical principles from which the science may be constructed. In closing this introduction we note that the spirit of the inquiry was expressed by the poet:

I believe a leaf of grass is no less
than the journeywork of the stars.

WALT WHITMAN "Leaves of Grass"

2 chapter

THERMODYNAMIC STATE OF THE STELLAR INTERIOR

The macroscopic properties of a star are intimately related to the microscopic phenomena occurring in the interior material. These phenomena and their rates depend upon the thermodynamic state of the material. One can calculate that in the interior environment the particles move very short distances compared to distances over which the temperature changes significantly before they collide with other particles. The rates of the fundamental atomic collision processes are, furthermore, very fast in comparison with rates of change of the local thermodynamic state. These facts enable one to assume a very important simplification in the description of the matter, *viz.*, local thermodynamic equilibrium. In the state of thermodynamic equilibrium, all properties of matter are calculable in terms of the chemical composition, the density, and the temperature. In his pioneering book "The Internal Constitution of the Stars," Sir Arthur Eddington has given the following vivid description:

The inside of a star is a hurly-burly of atoms, electrons, and aether waves. We have to call to aid the most recent discoveries of atomic physics to follow the intricacies of the dance. We started to explore the inside of a star; we soon find ourselves exploring the inside of an atom. Try to picture the tumult! Dishvelled atoms tear along at 50 miles a second with only a few tatters left of the elaborate cloaks of electrons torn from them in the scrummage. The lost electrons are speeding a hundred times faster to find new resting places. Look out!

there is nearly a collision as an electron approaches an atomic nucleus; but putting on speed it sweeps round it in a sharp curve. A thousand narrow shaves happen to the electron in 10^{-10} of a second; sometimes there is a side-slip at the curve, but the electron still goes on with increased or decreased energy. Then comes a worse slip than usual; the electron is fairly caught and attached to the atom, and its career of freedom is at an end. But only for an instant. Barely has the atom arranged the new scalp on its grille when a quantum of aether waves runs into it. With a great explosion the electron is off again for further adventures. Elsewhere two of the atoms are meeting full tilt and rebounding, with further disaster to their scanty remains of vesture.

As we watch the scene we ask ourselves, "Can this be the stately drama of stellar evolution?" . . .

This chaotic situation is reduced to tractable proportions by application of the principles of statistical mechanics. Because thermodynamic equilibrium is quickly achieved on the atomic (but not nuclear) scale, the rates of all atomic (i.e., electromagnetic, but not nuclear) reactions equal those of their inverse reactions. The hurly-burly of the individual electron is replaced by a steady macroscopic state whose properties are embodied in the principles of statistical mechanics. The functions of state are determined by the chemical composition, density, and temperature. Foremost among these is the pressure $P = P(\rho, T)$, commonly called the *equation of state*, from which the star derives its structural support against gravity. The burden of this chapter will be the discussion of the equation of state and related phenomena.

Near the surface of the stars, the equation of state of the gas is extremely complicated. The atomic constituents of the outer layers are in varying degrees of ionization. Application of the Saha ionization equation reveals that the hydrogen constituent becomes almost completely ionized by the time the temperature has risen to about 10^4 °K, whereas the helium is almost completely ionized by the time the temperature has risen to 10^5 °K, at which temperature the heavier elements have also lost a sizable number of their electrons to the continuum and are in relatively high stages of ionization. For temperatures higher than 10^5 °K, it becomes increasingly more accurate, insofar as the pressure is concerned, to talk of a completely ionized gas. Other important properties of the gas, such as its internal energy and its opacity to radiation, are strongly dependent upon the degree of ionization. For the common stellar composition, in which hydrogen and helium comprise more than 95 percent of the mass, the pressure at temperatures greater than 10^5 °K can be calculated to high accuracy by assuming complete ionization. Significantly, a large fraction of the mass of most stars does lie at temperatures higher than 10^5 °K. The bulk of the structure of most stars is determined, therefore, by an equation of state appropriate to completely ionized matter. From an analytical point of view, it is extremely 1 A. S. Eddington, "The Internal Constitution of the Stars," p. 19, Dover Publications, Inc., New York, 1959.

fortunate that this is so. A completely ionized gas behaves like a perfect gas to extremely high densities. Terrestrial matter reaches a density of only a few grams per cubic centimeter before it begins to resist compression, and the perfect-gas law begins to break down even before that density is reached. The rather large size of atoms and the interatomic forces between the electron clouds of the various atoms set a rather sudden limit to the density of un-ionized matter. The radii of nuclei, on the other hand, are only 10^{-5} of the radii of most atoms. A gas composed of nuclei and electrons, therefore, occupies only about 10^{-15} of the volume occupied by atoms. We may anticipate, therefore, that highly ionized matter can be compressed to extremely high densities before the perfect-gas law will break down as a result of the volume effect.

A *perfect gas* is defined as one in which there are no interactions between the particles of the gas. Although this criterion is never satisfied exactly in real gases, the approximation is physically sound if the average interaction energy between particles is much smaller than their thermal energies. This last condition may be satisfied by a weak interaction or by a sufficiently rarefied gas. In the ionized gas of a stellar interior the real interactions between particles are predominantly the coulomb interactions. It is fortunate that most physical circumstances in the stellar interior are such that the average coulomb energy of particles is much less than their characteristic kinetic energy, which is of the order kT for a nondegenerate gas. For this reason it will be adequate for most applications to use the equation of state of a perfect gas. We shall return later to the question of the real ionized gas and its applications.

2-1 MECHANICAL PRESSURE OF A PERFECT GAS

The microscopic source of pressure in a perfect gas is particle bombardment.¹ The reflection (or absorption) of these particles from a real (or imagined) surface in the gas results in a transfer of momentum to that surface. By Newton's second law ($F = dp/dt$), that momentum transfer exerts a force on the surface. The average force per unit area is called the *pressure*. It is the same mechanical quantity appearing in the statement that the quantity of work performed by the infinitesimal expansion of a contained gas is $dW = P dV$. In thermal equilibrium in stellar interiors, the angular distribution of particle momenta is isotropic; i.e., particles are moving with equal probabilities in all directions. When reflected from a surface, those moving normal to the surface will transfer larger amounts of momentum than those that glance off at grazing angles. Imagine that the surface in Fig. 2-1 is one of the surfaces of an evacuated can under particle bombardment. When particles are specularly reflected from that surface, the momentum transferred to the surface is $\Delta p_n = 2p \cos \theta$. Let $F(\theta, p) d\theta dp$ be the number of particles with momentum p in the range dp striking the surface per unit area per unit time from all directions inclined at angle θ to the normal in the range $d\theta$.

¹ In a nonperfect gas strong forces between the particles will represent an additional source or sink of energy for expansions and will therefore contribute to the pressure.

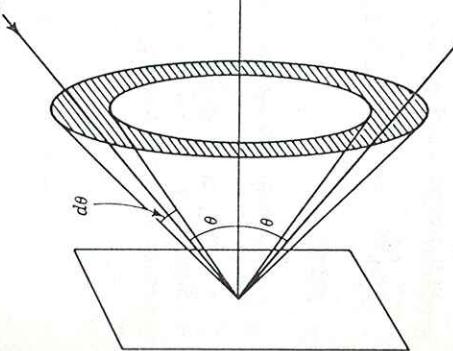


Fig. 2-1 A conical shell defining the set of directions having a spread $d\theta$ about the angle θ to the normal. The number of particles having $|p| = p$ in the range dp that strike a unit area in unit time within this conical shell of directions is designated $F(\theta, p) d\theta dp$.

The contribution to the pressure from those particles is

$$dP = 2p \cos \theta F(\theta, p) d\theta dp \quad (2-1)$$

so that the total pressure is

$$P = \int_{\theta=0}^{\pi/2} \int_{p=0}^{\infty} 2p \cos \theta F(\theta, p) d\theta dp \quad (2-2)$$

In thermodynamic equilibrium, the angular distribution of momenta is isotropic, whereas the distribution of the magnitudes of the momenta is given by statistical mechanics. The flux $F(\theta, p) d\theta dp$ may be calculated as the product of the number density of particles with momentum p in the range dp moving in the cone of directions inclined at angle θ in the range $d\theta$ times the volume of such particles capable of passing through the unit surface in unit time. That volume is the volume of the parallelepiped shown in Fig. 2-2 and is equal to the product of v_p , the velocity associated with momentum p , and $\cos \theta$, the cross-sectional area of the column. That is,

$$v(\theta, p) d\theta dp = v_p \cos \theta n(\theta, p) d\theta dp \quad (2-3)$$

where $n(\theta, p) d\theta dp$ is the number density of particles moving in the prescribed cone. For isotropic radiation, furthermore, the fraction of the number of particles moving in the cone of directions at angle θ in the range $d\theta$ is just

$$\frac{n(\theta, p) d\theta dp}{n(p) dp} = \frac{2\pi \sin \theta d\theta}{4\pi} \quad (2-4)$$

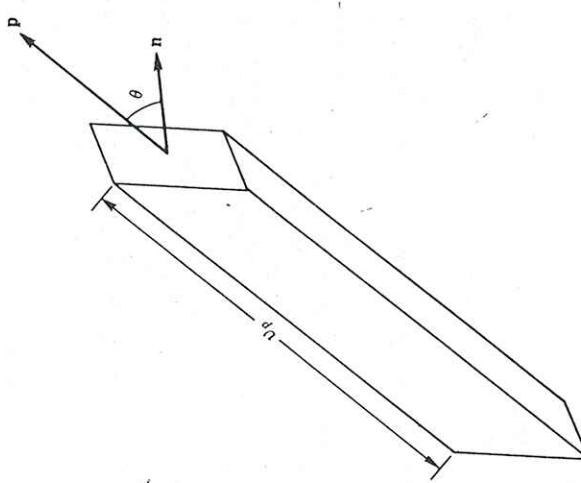


Fig. 2-2 The parallelepiped whose volume when multiplied by the density of particles about momentum p yields the number of particles per unit time of momentum p passing through the unit area n .

which is the fraction of the total spherical solid angle defined by the conical set of directions. The total number density of particles of momentum p in dp is $n(p) dp$. Evidently the gas pressure is

$$P = \int_0^{\pi/2} \int_0^{\infty} 2p \cos \theta v_p \cos \theta n(p) dp \frac{1}{2} \sin \theta d\theta \quad (2-5)$$

The explicit integration over angles is easily performed to yield

$$P = \frac{1}{3} \int_0^{\infty} p v_p n(p) dp \quad \text{perfect gas} \quad (2-6)$$

This pressure integral, valid for a perfect isotropic gas, must be evaluated for several sets of relevant astrophysical circumstances. The relationship of v_p to p depends upon relativistic considerations, whereas the distribution $n(p)$ depends upon the type of particles and the quantum statistics. The simplest perfect gas is the monatomic nondegenerate nonrelativistic one considered in the next subsection, which will be followed by a discussion of the degenerate electron gas and then a discussion of radiation pressure.

THE PERFECT MONATOMIC NONDEGENERATE GAS

In the most common case for which the gas density is small enough to be nondegenerate and for which the thermal velocities are nonrelativistic, the pressure

of a perfect gas is simply

$$P_\varrho = NkT \quad (2-7)$$

where N is the number of free particles per unit volume.

Problem 2-1: From Chap. 1, the momentum distribution of a nondegenerate nonrelativistic gas in thermal equilibrium is Maxwellian:

$$n(p) dp = \frac{N4\pi p^2 dp}{(2\pi mkT)^3} \exp - \frac{p^2}{2mkT}$$

For a nonrelativistic gas, derive Eq. (2-7) from the pressure integral. The contribution from the several constituents of the gas are additive (Dalton's law of partial pressures). Is Eq. (2-7) also correct for relativistic velocities?

Let the mean molecular weight of the perfect gas be designated by μ . Then the density is

$$\rho = N\mu M_u \quad (2-8)$$

where M_u is the mass of 1 amu. The number of particles per unit volume can then be expressed in terms of the density and the mean molecular weight as

$$N = \frac{\rho}{\mu M_u} = \frac{N_0 \rho}{\mu} \quad (2-9)$$

where $N_0 = 1/M_u$ is Avogadro's number and is equal to 6.0225×10^{23} mole⁻¹. Substitution into Eq. (2-7) gives the pressure of the gas in terms of the density and the mean molecular weight:

$$P_\varrho = \frac{N_0 k}{\mu} \rho T \quad (2-10)$$

The mean molecular weight rather clearly will depend upon the composition of the gas. It is common to let X_Z represent the fraction of the gas by weight of element Z ; that is, 1 g of gas contains X_Z g of the element of the atomic number Z . It follows that $\Sigma X_Z = 1$. Let us also suppose that each atom of element Z contributes n_Z free particles to the gas. For complete ionization, for instance, it will be true that $n_Z = Z + 1$, Z electrons plus the nucleus. Now let N_Z be the number density of atoms of element Z in the gas. From the definitions of all these quantities it is apparent that

$$N_Z = \frac{\rho_Z}{A_Z} N_0 = \rho \frac{X_Z}{A_Z} N_0 \quad (2-11)$$

Now the total number of free particles per cubic centimeter will be given by

$$N = \sum N_Z n_Z = \rho N_0 \sum \frac{X_Z n_Z}{A_Z} \quad (2-12)$$

where the sum is over all the elements Z . From a comparison of this last equa-

tion with Eq. (2-9), the mean molecular weight is given by

$$\frac{1}{\mu} = \sum \frac{X_Z n_Z}{A_Z} \quad (2-13)$$

It is conventional to use a slightly different terminology for the fraction by weight of the two most common elements in the stellar composition. In keeping with this convention, let X be the weight fraction of hydrogen, let Y be the weight fraction of helium, and let $1 - X - Y$ be the weight fraction¹ of all species heavier than helium. Then the mean molecular weight becomes

$$\mu = \left[\frac{X n_H}{1.008} + \frac{Y n_{He}}{4.004} + (1 - X - Y) \left\langle \frac{n_Z}{A_Z} \right\rangle \right]^{-1} \quad (2-14)$$

The quantity $\langle n_Z/A_Z \rangle$ is the average of n_Z/A_Z for $Z > 2$, each term being weighted proportional to X_Z .

Equation (2-14) can be further simplified for the case of complete ionization in the inner regions of stars. For complete ionization, the numbers of free particles contributed by the atoms of each element are $n_H = 2$, $n_{He} = 3$, and $n_Z = Z + 1$. When averaged over the species as they occur in nature, it is a convenient fact that the average atomic weight of element Z is approximately given by $A_Z = 2Z + 2$. The use of that approximation should be adequate in most cases where the fraction by weight of the species heavier than helium is small. With this approximation $\langle n_Z/A_Z \rangle$ in Eq. (2-14) becomes equal to $\frac{1}{2}$:

$$\mu \approx \frac{1}{2X + 3Y/4 + (1 - X - Y)/2} = \frac{2}{1 + 3X + 0.5Y} \quad (2-15)$$

It will also be convenient to have an auxiliary expression for the number density of electrons. Using exactly the same notation as above, we have

$$n_e = \sum N_Z (n_Z - 1) = \rho N_0 \sum \frac{X_Z}{A_Z} (n_Z - 1) \quad (2-16)$$

In the case of complete ionization $n_Z = Z + 1$, so that the number density of electrons becomes

$$n_e = \rho N_0 \sum \frac{X_Z Z}{A_Z} \quad \text{complete ionization} \quad (2-17)$$

Insertion of the composition-by-weight parameters given above for hydrogen and helium yields

$$n_e = \rho N_0 \left[X + \frac{2Y}{4} + (1 - X - Y) \left\langle \frac{Z}{A_Z} \right\rangle \right] \quad (2-18)$$

where $\langle Z/A_Z \rangle$ is the average for $Z > 2$, the average being taken with respect to X_Z . If the fraction by weight of elements heavier than He is small, it is often

¹ It is common to denote this last weight fraction by Z . To avoid confusion with the nuclear charge in the present discussion, we forego that notation for the moment. We shall use the symbol Z later, however, where the context will make its meaning clear.

adequate to assume $\langle Z/A_Z \rangle \approx \frac{1}{2}$, in which case

$$n_e \approx \frac{1}{2} \rho N_e (1 + X) \quad (2-19)$$

It is also common to use a quantity called the mean molecular weight per electron u_e , which is numerically equal to the average number of atomic mass units for each electron in the gas. From Eq. (2-17) it is evident that

$$\frac{1}{u_e} = \sum \frac{X_Z Z}{A_Z} \quad n_e = \frac{\rho N_0}{\mu_e} \quad (2-20)$$

If the ionization is complete, and if $\langle Z/A_Z \rangle \approx \frac{1}{2}$ for $Z > 2$,

$$u_e = \frac{2}{1 + X} \quad (2-21)$$

It is advisable for the reader to pause long enough to gain familiarity with the composition parameters and to mentally evaluate the errors in the various approximations.

Problem 2-2: To be sure of understanding the mean molecular weight of the completely ionized gas, calculate and interpret the values of μ under the following circumstances: (a) all hydrogen, that is, $X = 1$, $Y = 0$; (b) all helium, that is, $X = 0$, $Y = 1$; (c) all heavy elements, that is, $X = 0$, $Y = 0$. Which of these three values is exactly given by the approximate equation (2-15)?

Problem 2-3: Calculate the mean molecular weight per electron, u_e , for completely ionized conditions of all hydrogen ($X = 1$) and for all helium ($Y = 1$). Is Eq. (2-21) exact for $X = Y = 1.5$? Is it exact for $X = Z = 0.5$? What if the Z component is all C^{12} and O^{16} ?

Problem 2-4: Show that for conditions under which Eq. (2-15) is valid, the rate of change of the mean molecular weight with respect to the heavy-element content Z , always holding the hydrogen fraction constant, is equal to $\mu^2/4$; that is,

$$\left(\frac{\partial \mu}{\partial Z} \right)_X = \frac{\mu^2}{4}$$

In calculations of stellar structure, and particularly of the structure of evolving stars, a large variety of compositions will be encountered. The statement was made in Chap. 1 that the average composition of the surfaces of population I stars and of the interstellar medium is more or less uniform. It is appropriate, therefore, at this time to present a simplified table of the abundances of the elements (Table 2-1), which are the best that can be inferred for population I objects. Most of the entries are derived from abundances of elements in the solar system, because those are the ones for which the most extensive data exist. The most important exceptions are He and Ne, which are observed only in objects hotter than the sun. It is common to think of the chemical composition of the solar system as a standard, against which other compositions are to be compared. This procedure is no more than a matter of convenience, however, and it must be remembered that the composition of our solar system has no special cosmological significance.

Table 2-1 Relative abundances of most common species in population I†

Element	Atomic weight	Relative abundance	
		By number	By weight
H	1	1,000	1,000
He†	4	160	640
O	16	0.90	14
Ne†	20	0.50	10
C	12	0.40	4.8
N	14	0.11	1.5
Si	28	0.032	0.9
Mg	24	0.025	0.6
S	32	0.022	0.7
A	40	0.008	0.3
Fe	56	0.004	0.2
Na	23	0.002	0.05
Cl	36	0.002	0.07
Al	27	0.002	0.05
Ca	40	0.002	0.08
F	19	0.001	0.02
Ni	59	0.001	0.06
>Ni	>60	$\sim 10^{-4}$	~ 0.01

† L. H. Aller, "The Abundance of the Elements," Interscience Publishers, Inc., New York, 1961.

† Because the sun is a G2 star, its helium abundance is not well known. The value in this table comes from the hotter B stars in the solar neighborhood, which are much younger than the sun. There are some indications that in the sun $He/H \approx 0.10$ by number, which is about 60 percent of the amount of He found in B stars. A similar situation occurs for Ne, and it is more likely, but not certain, that in the sun $Ne/O \sim 0.1$.

logical significance. A simple calculation reveals that the abundance parameters corresponding to Table 2-1 are

$$X = 0.60 \quad Y = 0.38 \quad Z = 0.02$$

These composition parameters may be thought of as characteristic of the majority of population I stars. It must be reemphasized, however, that it is in the deviations of composition from uniformity that some of the most intriguing problems of stellar evolution and nucleosynthesis are to be found.

Problem 2-5: The center of a certain star contains 60 percent hydrogen by weight and 35 percent helium by weight. Evaluate numerically the equation of state. What is the pressure at the center of the star if the density there is 50 g/cm^3 and the temperature is $15 \times 10^6 \text{ K}$?

Of course, some error has been introduced by simplifying assumptions made in obtaining the equation of state. Atoms are never completely ionized, and it is

the Saha ionization equation that reveals the fraction of any given species that is ionized. In the relatively cool outer portions of a star, the number of free particles will depend upon the temperature. Elaborate techniques have been constructed for calculating a more realistic equation of state applicable to incomplete ionization. The reader who understands the ideas about it presented here, along with its restrictive assumptions, will have little trouble with a more sophisticated treatment of the equation of state.

Other than the lack of complete ionization in the cooler regions of the star, there are two extremely important physical circumstances that cause the equation of state for a perfect nondegenerate monatomic gas to be insufficient: (1) the pressure due to electromagnetic radiation in the interior of the star becomes comparable to the pressure due to particles, and (2) the electron gas becomes degenerate. We shall consider the second of these sets of circumstances, electron degeneracy, first.

ELECTRON DEGENERACY

Because electrons are particles with half-integral spin, the electron gas must obey Fermi-Dirac statistics. The density of electrons having momentum $|p| = p$ in the range dp is accordingly

$$n_e(p) dp = \frac{2}{h^3} 4\pi p^2 dp P(p) \quad (2-22)$$

where the occupation index for the Fermi gas is

$$P(p) = \left[\exp\left(\alpha + \frac{E}{kT}\right) + 1 \right]^{-1} \quad (2-23)$$

That $P(p)$ has a maximum value of unity is an expression of the Pauli exclusion principle, to which electrons must adhere. When $P(p)$ is unity, all the available electronic states of the gas are occupied. It follows that the maximum density of electrons in phase space is

$$[n_e(p)]_{\max} = \frac{2}{h^3} 4\pi p^2 \quad (2-24)$$

It is this restriction upon the number density of electrons in momentum space which creates *degeneracy pressure*. If one continually increases the electron density, the electrons are forced into high-lying momentum states because the lower states are occupied. These high-momentum electrons will make a large contribution to the pressure integral.

For any given temperature and electron density n_e , the value of the parameter α is determined from the demand that

$$n_e = \int_0^\infty n_e(p) dp = n_e(\alpha, T) \quad (2-25)$$

This relationship will be explored quantitatively at a later time, but for the present we note from Eq. (2-23) that if α is a large positive number, $P(p)$ will be

much less than unity for all energies. In this case the Fermi distribution reduces to the Maxwellian distribution. As the electron density is increased at constant temperature, the parameter α becomes smaller, going to large negative values at high density.

In the limit of large negative α ,

$$P(p) = \begin{cases} 1 & \text{for } \frac{E}{kT} < |\alpha| \\ 0 & \text{for } \frac{E}{kT} > |\alpha| \end{cases} \quad \text{complete degeneracy} \quad (2-26)$$

This transition occurs smoothly over a range of energy of several kT near the energy $E = |\alpha|kT$. If the energy $-akT$ is much larger than kT , the distribution function is nearly a step function. This limit is called *complete degeneracy*, and in this limit the quantity $|\alpha|kT = E_f$ is called the *Fermi energy*.

In the following discussion we shall calculate the pressure of a completely degenerate gas. The calculation will first be made for densities such that the energy E_f is nonrelativistic. It will then be repeated for densities high enough for E_f to correspond to relativistic electron velocities. Finally we shall calculate, in the nonrelativistic limit, the pressure of an electron gas for densities such that the distribution function is intermediate to the Maxwellian and the completely degenerate distributions.

Complete degeneracy In a completely degenerate gas, the density is high enough so that all the available electron states having energies less than some maximum energy are filled. Since the total number density of electrons is to be finite, the density of states can be filled only up to some limiting value of the electron momentum

$$n_e(p) dp = \begin{cases} \frac{2}{h^3} 4\pi p^2 dp & p < p_0 \\ 0 & p > p_0 \end{cases} \quad (2-27)$$

It is clear that complete degeneracy is the state of minimum kinetic energy, the ground state, so to speak, of a degenerate perfect electron gas. The total number density of electrons in a completely degenerate electron gas is related to the maximum momentum by

$$n_e = \int_0^{p_0} n_e(p) dp = \frac{8\pi}{3h^3} p_0^3 \quad (2-28)$$

Inversion of this last equation shows that the maximum momentum of a completely degenerate gas is determined by the electron density:

$$p_0 = \left(\frac{3h^3}{8\pi} n_e \right)^{\frac{1}{3}} \quad (2-29)$$

The energy associated with the momentum p_0 is the Fermi energy.

The pressure of a completely degenerate perfect electron gas can be computed from the integral of Eq. (2-6) by inserting Eq. (2-27) for $n_e(p)$. Because it is also necessary to insert the velocity of a particle of given momentum, it is common to distinguish between a nonrelativistic and a relativistic degenerate electron gas.

Nonrelativistic complete degeneracy If the energy associated with p_0 is much less than $m_e c^2$, or 0.51 Mev, then $v_p = p/m$ for all momenta in the degenerate distribution, and the pressure integral is elementary:

$$P_{e, nr} = \frac{8\pi}{15m\hbar^3} p_0^5 \quad (2-30)$$

where n signifies *nonrelativistic* electrons. Since the maximum momentum of the completely degenerate distribution is related to the electron density by Eq. (2-29), the electron pressure is determined by the electron density:

$$P_{e, nr} = \frac{\hbar^2}{20m} \left(\frac{3}{\pi} \right)^{\frac{1}{3}} n_e^{\frac{5}{3}} \quad (2-31)$$

The number density of the electrons may be written in terms of the mass density:

$$P_{e, nr} = \frac{\hbar^2}{20m} \left(\frac{3}{\pi} \right)^{\frac{1}{3}} N_0^{\frac{5}{3}} \left(\frac{\rho}{\mu_e} \right)^{\frac{5}{3}} \\ = 1.004 \times 10^{13} \left(\frac{\rho}{\mu_e} \right)^{\frac{5}{3}} \text{ dynes/cm}^2 \quad (2-32)$$

The value of μ_e is generally about 2 unless the gas contains considerable amounts of hydrogen. Inspection of this equation shows that the nonrelativistic-electron pressure varies as the $\frac{5}{3}$ power of the density. Since the pressure of a nondegenerate electron gas varies linearly with the density, it is clear that as the density increases, a point will be reached where the degenerate electron pressure becomes greater than the value that would be given by the formula for the pressure of a nondegenerate gas.

We may thereby define an approximate boundary line in the ρT plane, dividing it into regions of nondegenerate and degenerate gas, respectively, by the condition that the pressures given by the nondegenerate-gas equation and the completely degenerate electron-gas equation be equal. That is, when¹

$$\frac{N_0 k}{\mu_e} \rho T = \frac{\hbar^2}{20m} \left(\frac{3}{\pi} \right)^{\frac{1}{3}} (N_0)^{\frac{5}{3}} \left(\frac{\rho}{\mu_e} \right)^{\frac{5}{3}} \quad (2-33)$$

Numerical evaluation of this equation shows that the completely degenerate electron pressure exceeds the nondegenerate electron pressure when

$$\frac{\rho}{u_e} > 2.4 \times 10^{-8} T^{\frac{1}{3}} \text{ g/cm}^3 \quad (2-34)$$

¹ It should perhaps be emphasized that Eq. (2-33) is never "true," since a gas cannot be simultaneously degenerate and nondegenerate. One might say that if ρ and T satisfy this equation, the state of the gas must be intermediate to nondegeneracy and complete degeneracy.

For densities greater than this value, the electron gas must be degenerate. Needless to say, the transition from nondegenerate to degenerate is not sudden and complete. The transition occurs smoothly for densities in the neighborhood of Eq. (2-34). The appropriate equation of state in the transition region will be discussed in the section on partial degeneracy.

It is instructive to apply Eq. (2-34) to two well-known astrophysical environments. At the center of the sun $\rho/\mu_e \approx 10^2$, and $T \approx 10^7$. For these values the inequality of Eq. (2-34) is strong in the opposite direction, so that one will anticipate using the nondegenerate electron pressure at the solar center. White-dwarf densities, on the other hand, are observationally known to be of order $\rho/\mu_e \approx 10^6$, whereas the interior temperatures are of order $T \approx 10^6$. For these values the inequality of Eq. (2-34) is strongly satisfied, and one must expect degeneracy pressure to dominate.

Relativistic complete degeneracy As the electron density is increased, the maximum momentum in a completely degenerate electron gas grows larger. Eventually a density is reached where the most energetic of the electrons in the degenerate distribution become relativistic. When that condition is reached, the substitution $v_p = p/m$ leading to Eq. (2-30) becomes incorrect. The velocity to be associated with the momentum p must be determined by relativistic kinematics.

Before calculating the pressure, let us estimate those densities for which it is necessary that some of the electrons be relativistic. For a relativistic particle, the total energy, which is the sum of the rest-mass energy plus the kinetic energy, forms a right triangle with the rest-mass energy and the momentum times the velocity of light, as illustrated in Fig. 2-3. The right-triangle relationship follows

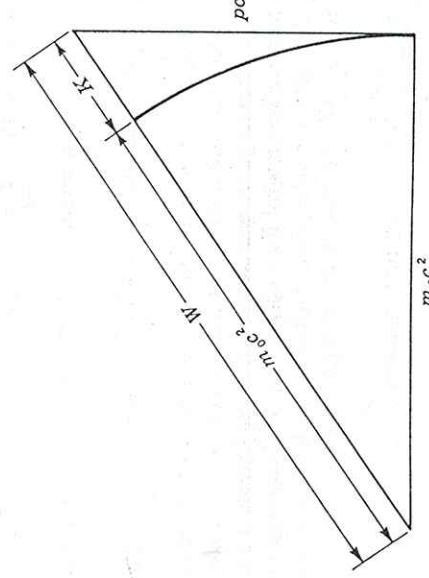


Fig. 2-3 The right triangle representing the relationship of the total energy of a particle to its momentum. The kinetic energy is the excess of the hypotenuse beyond the rest-mass energy.

from the relativistic expression for the total energy of a particle,

$$W^2 = p^2c^2 + m_0^2c^4 \quad (2-35)$$

where m_0 is the rest mass of the particle. The total energy is also given by the square of the velocity of light times the relativistic mass:

$$W = mc^2 = \frac{mc^2}{1 - (v/c)^2} \quad (2-36)$$

Equating W^2 in Eq. (2-35) to W^2 in Eq. (2-36) yields

$$pc = \frac{v}{c} W \quad (2-37)$$

What convenient order-of-magnitude criterion will ensure that particles are relativistic? One may say with adequate accuracy that particles become relativistic when v/c approaches unity and when the total energy W becomes appreciably greater than the rest-mass energy. As an order-of-magnitude criterion, it suffices to compute the density at which the electrons of maximum momentum have a total energy equal to, say, twice the rest-mass energy; from Eq. (2-37) the quantity pc will then be approximately $pc \sim 2mc^2$. On the other hand, the numerical value of pc is

$$\gamma_0 c = \hbar c \left(\frac{3}{8\pi} n_e \right)^{\frac{1}{3}} = 6.12 \times 10^{-11} n_e^{\frac{1}{3}} \quad \text{Mev} \quad (2-38)$$

in terms of the density and the mean molecular weight per electron, Eq. (2-38) may be expressed as

$$\gamma_0 c = 5.15 \times 10^{-3} \left(\frac{\rho}{\mu_e} \right)^{\frac{1}{3}} \quad \text{Mev} \quad (2-39)$$

This last equation reveals that $pc \approx 2mc^2 \approx 1$ Mev when

$$\frac{\rho}{\mu_e} = 7.3 \times 10^6 \text{ g/cm}^3 \quad \text{relativistic} \quad (2-40)$$

The natural conclusion is that as the density approaches this value, relativistic kinematics must be used in relating the velocity of an electron to its momentum. Densities this large are encountered in astrophysics, in white dwarfs, for instance. The pressure integral for a completely degenerate gas may be evaluated without difficulty for relativistic particles. Since the momentum of a relativistic particle is given by Eqs. (2-36) and (2-37) as

$$= \frac{m_0 v}{[1 - (v/c)^2]^{\frac{1}{2}}} \quad (2-41)$$

we can determine by inversion that

$$= \frac{p/m_0}{[1 + (p/m_0 c)^2]^{\frac{1}{2}}} \quad (2-42)$$

Insertion of this value for v_p in the pressure integral yields

$$p_e = \frac{8\pi}{3m\hbar^3} \int_0^{p_0} \frac{p^4 dp}{[1 + (p/mc)^2]^{\frac{1}{2}}} \quad (2-43)$$

In Eq. (2-43) and those which follow, the electron rest mass is designated simply by m . For evaluation of this integral it is convenient to define a new parameter θ such that

$$\sinh \theta = \frac{p}{mc} \quad dp = mc \cosh \theta d\theta \quad (2-37)$$

In terms of this new variable the pressure integral becomes

$$P_e = \frac{8\pi m^4 c^5}{3\hbar^3} \int_0^{\theta_0} \sinh^4 \theta d\theta \quad (2-44)$$

which may be integrated to give

$$P_e = \frac{8\pi m^4 c^5}{3\hbar^3} \left(\frac{\sinh^3 \theta_0 \cosh \theta_0}{4} - \frac{3 \sinh 2\theta_0}{16} + \frac{3\theta_0}{8} \right) \quad (2-45)$$

When written in terms of the Fermi momentum,

$$P_e = \frac{\pi m^4 c^5}{3\hbar^3} f(x) = 6.003 \times 10^{22} f(x) \quad \text{dynes/cm}^2 \quad (2-46)$$

where

$$x = \frac{p_0}{mc} = \frac{\hbar}{mc} \left(\frac{3}{8\pi} n_e \right)^{\frac{1}{3}} \quad (2-47)$$

$$f(x) = x(2x^2 - 3)(x^2 + 1)^{\frac{1}{3}} + 3 \sinh^{-1} x \quad (2-47)$$

The numerical value of the dimensionless parameter x is

$$x = 1.195 \times 10^{-10} n_e^{\frac{1}{3}} = 1.009 \times 10^{-2} \left(\frac{\rho}{\mu_e} \right)^{\frac{1}{3}} \quad (2-48)$$

Problem 2-6: The limit of small x , that is, $p_0 \ll mc$, must correspond to nonrelativistic particles. Show that

$$f(x) \approx \frac{8}{3} x^5 - \frac{4}{7} x^7 + \dots \quad x \rightarrow 0$$

and confirm that the pressure obtained from this limiting value of $f(x)$ reduces to the completely degenerate nonrelativistic electron pressure determined previously in Eq. (2-30).

Problem 2-7: The limit of large x must correspond to highly relativistic degeneracy. Show that

$$f(x) \approx 2x^4 - 2x^2 + \dots \quad x \rightarrow \infty$$

Show that the pressure obtained by inserting this limiting value of $f(x)$ into Eq. (2-46) is identical to that obtained by letting $y_p = c$ in the integral for the pressure given in Eq. (2-6). Does that make sense? Evidently the pressure is proportional to $\rho^{\frac{1}{3}}$ at very high density.

Table 2-2 Pressure of a complete degenerate gas†

x	$f(x)$	x	$f(x)$
0	0	2.0	26.7
0.1	1.60×10^{-6}	2.1	32.9
0.2	5.05×10^{-4}	2.2	40.1
0.3	3.77×10^{-3}	2.3	48.4
0.4	1.55×10^{-2}	2.4	58.0
0.5	4.61×10^{-2}	2.5	68.9
0.6	0.111	2.6	81.2
0.7	0.232	2.7	95.2
0.8	0.436	2.8	110.8
0.9	0.756	2.9	128.3
1.0	1.23	3.0	1.48×10^2
1.1	1.90	3.5	2.80×10^2
1.2	2.82	4.0	4.85×10^2
1.3	4.05	4.5	7.85×10^2
1.4	5.63	5.0	1.21×10^3
1.5	7.64	5.5	1.78×10^3
1.6	10.1	6.0	2.53×10^3
1.7	13.2	6.5	3.49×10^3
1.8	16.9	7.0	4.71×10^3
1.9	21.4	8.0	8.07×10^3

† S. Chandrasekhar, "An Introduction to the Study of Stellar Structure," p. 302; reprinted from the Dover Publications edition, Copyright 1939 by The University of Chicago, as reprinted by permission of The University of Chicago.

Table 2-3 Pressure of a completely degenerate gas is

$$\left(\frac{U}{V}\right)_{\text{kin}} = \frac{\pi m^4 c^5}{3h^3} g(x)$$

where $g(x) = 8x^3[(x^2 + 1)^{\frac{1}{2}} - 1] - f(x)$. Show also that $U \rightarrow \frac{3}{2}PV$ in the limit of small x and $U \rightarrow 3PV$ in the limit of large x .

Partial degeneracy The dividing line between degeneracy and nondegeneracy given in Eq. (2-34) defines only the region of the onset of degeneracy in the electron gas. That is, it indicates only the approximate condition under which electron degeneracy is becoming important in the equation of state. Actually, of course, there is a gradual transition from nondegeneracy toward complete degeneracy as the density rises. There is certainly no sharp transition between those extreme conditions. The electron occupation index gradually takes on the shape of a degenerate distribution with increase in density, as illustrated in Fig. 2-4.

The distribution of electron momenta is

$$n_e(p) dp = \frac{2}{h^3} \frac{4\pi p^2 dp}{\exp(\alpha + E/kT) + 1} \quad (2-49)$$

where α is a number that depends upon the electron density and the temperature. That is, α is fixed by the requirement that the total number of electrons equal the electron density n_e :

$$n_e = \int_0^\infty \frac{2}{h^3} \frac{4\pi p^2 dp}{\exp(\alpha + E/kT) + 1} = n_e(\alpha, T) \quad (2-50)$$

The integral for the pressure of the perfect electron gas becomes

$$P_e = \frac{8\pi}{3h^3} \int_0^\infty \frac{p^3 n_e dp}{\exp(\alpha + E/kT) + 1} \quad (2-51)$$

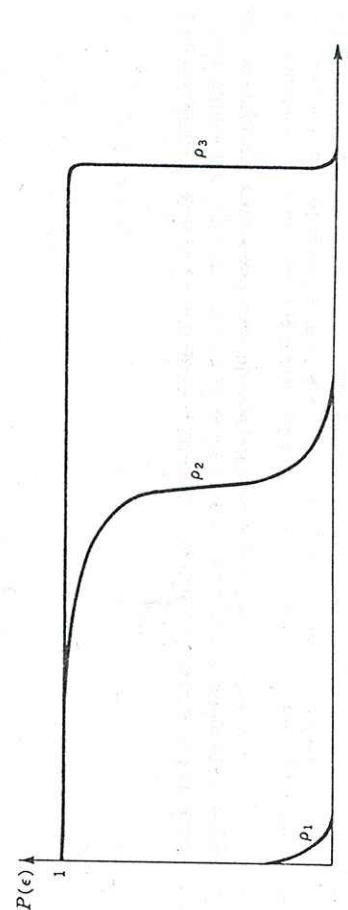


Table 2-2 lists some numerical values of $f(x)$. From this table and Eq. (2-46) the electron pressure can be evaluated for cases of semirelativistic complete degeneracy. The quantity x is to be evaluated from Eq. (2-48). This result is correct only for a completely degenerate gas. Approximate relativistic expressions for a partially degenerate gas can be obtained if desired.¹ However, densities must exceed 10^6 g/cm^3 for a degenerate gas to be relativistic [Eq. (2-40)], for which the degeneracy will be essentially complete unless $T > 10^9 \text{ }^\circ\text{K}$ [Eq. (2-34)]. Densities greater than 10^6 g/cm^3 at a temperature greater than $10^9 \text{ }^\circ\text{K}$ are probably found only in very late stages of stellar evolution. For all other classes of stars, degeneracy sets in at sufficiently low temperatures so that nonrelativistic kinematics should be adequate for the examination of partial degeneracy.

See, for instance, S. Chandrasekhar, "An Introduction to the Study of Stellar Structure," p. 392, Dover Publications, Inc., New York, 1957, or D. H. Menzel, P. L. Bhatnagar, and I. K. Sen, "Stellar Interiors," p. 35, John Wiley & Sons, Inc., New York, 1963.

Fig. 2-4 Schematic illustration of the occupation index of an electron gas for three different degrees of degeneracy. In this particular case $\rho_3 > \rho_2 \gg \rho_1$ and $T_2 > T_3$.

As stated at the end of the last section, for temperatures of less than 10^3 °K non-relativistic electron degeneracy sets in before relativistic degeneracy. Therefore, in considering the partially degenerate gas, we shall restrict ourselves to non-relativistic kinematics, keeping in mind that the results will be somewhat in error for extremely high temperatures ($T > 10^9$). That is, we shall once again let

$$v_p = p/m, \text{ whereupon} \quad (2-52)$$

$$P_e = \frac{8\pi}{3h^3m} \int_0^\infty \frac{p^4 dp}{\exp(\alpha + p^2/2mkT) + 1}$$

and

$$n_e = \frac{8\pi}{h^3} \int_0^\infty \frac{p^2 dp}{\exp(\alpha + p^2/2mkT) + 1} \quad (2-53)$$

With the aid of a dimensionless energy $u = p^2/2mkT$, these equations may be written in the form

$$P_e = \frac{8\pi kT}{3h^3} (2mkT)^{\frac{1}{2}} \int_0^\infty \frac{u^{\frac{5}{2}} du}{\exp(\alpha + u) + 1} \quad (2-54)$$

$$n_e = \frac{4\pi}{h^3} (2mkT)^{\frac{1}{2}} \int_0^\infty \frac{u^{\frac{3}{2}} du}{\exp(\alpha + u) + 1} \quad (2-55)$$

These two equations constitute a parametric representation of the equation of state, the parameter being the quantity α . The parametric representation is made more explicit by conventionally defining two new functions,¹

$$F_4(\alpha) = \int_0^\infty \frac{u^{\frac{5}{2}} du}{\exp(\alpha + u) + 1} \quad (2-56)$$

$$F_3(\alpha) = \int_0^\infty \frac{u^{\frac{3}{2}} du}{\exp(\alpha + u) + 1} \quad (2-57)$$

in which case the electron pressure and the electron density may be written as

$$P_e = \frac{8\pi kT}{3h^3} (2mkT)^{\frac{1}{2}} F_4(\alpha)$$

$$n_e = \frac{4\pi}{h^3} (2mkT)^{\frac{1}{2}} F_3(\alpha)$$

α	$\frac{2}{3}F_4$	F_3	α	$\frac{2}{3}F_3$	F_4
4.0	0.016 179	0.016 128	0.0	0.768 536	0.678 094
3.9	0.017 875	0.017 812	-0.1	0.839 082	0.733 403
3.8	0.019 748	0.019 670	-0.2	0.915 332	0.792 181
3.7	0.021 816	0.021 721	-0.3	0.997 637	0.854 521
3.6	0.024 099	0.023 984	-0.4	1.056 358	0.920 505
3.5	0.026 620	0.026 480	-0.5	1.181 862	0.990 209
3.4	0.029 494	0.029 233	-0.6	1.284 526	1.063 694
3.3	0.032 476	0.032 269	-0.7	1.384 729	1.141 015
3.2	0.035 868	0.035 615	-0.8	1.512 858	1.222 215
3.1	0.039 611	0.039 303	-0.9	1.639 302	1.307 327
3.0	0.043 741	0.043 366	-1.0	1.774 455	1.396 375
2.9	0.048 288	0.047 842	-1.1	1.918 709	1.489 372
2.8	0.053 324	0.052 770	-1.2	2.072 461	1.586 323
2.7	0.058 868	0.058 194	-1.3	2.236 106	1.657 226
2.6	0.064 981	0.064 161	-1.4	2.410 037	1.792 068
2.5	0.071 720	0.070 724	-1.5	2.594 650	1.900 833
2.4	0.079 148	0.077 938	-1.6	2.790 334	2.013 496
2.3	0.087 332	0.085 864	-1.7	2.997 478	2.130 027
2.2	0.096 347	0.094 566	-1.8	3.216 467	2.250 391
2.1	0.106 273	0.104 116	-1.9	3.447 683	2.374 548
2.0	0.117 200	0.114 588	-2.0	3.691 502	2.502 458
1.9	0.129 224	0.126 063	-2.1	3.948 288	2.634 072
1.8	0.142 449	0.138 627	-2.2	4.218 438	2.769 344
1.7	0.156 989	0.152 373	-2.3	4.502 287	2.908 224
1.6	0.172 967	0.167 397	-2.4	4.800 202	3.050 659
1.5	0.190 515	0.183 802	-2.5	5.112 536	3.196 598
1.4	0.209 777	0.201 696	-2.6	5.439 637	3.345 985
1.3	0.230 907	0.221 193	-2.7	5.781 847	3.498 775
1.2	0.254 073	0.242 410	-2.8	6.139 503	3.654 905
1.1	0.279 451	0.265 471	-2.9	6.512 937	3.814 326
1.0	0.307 232	0.290 501	-3.0	6.902 476	3.976 985
0.9	0.337 621	0.317 630	-3.1	7.308 441	4.142 831
0.8	0.370 833	0.346 989	-3.2	7.731 147	4.311 811
0.7	0.407 098	0.378 714	-3.3	8.170 906	4.453 876
0.6	0.446 659	0.412 937	-3.4	8.628 023	4.658 977
0.5	0.489 773	0.449 793	-3.5	9.102 801	4.837 066
0.4	0.538 710	0.489 414	-3.6	9.595 535	5.018 095
0.3	0.587 752	0.531 931	-3.7	10.106 516	5.202 092
0.2	0.643 197	0.577 470	-3.8	10.636 034	5.388 795
0.1	0.703 351	0.626 152	-3.9	11.184 369	5.578 378

The functions F_4 and F_3 have been tabulated for selected values of α in Table 2-3. Their values for other values of α may be interpolated in the range of α listed, and asymptotic values will soon be derived for extreme values of α .

In much of the literature the negative of α is used as the degeneracy parameter, in which case it is usually designated by η or Ψ ; or $\Psi = \eta = -\alpha$. Another common notation is $-\alpha kT = \mu$, which is called the *chemical potential*. Many people prefer to normalize the F 's in a different way, defining

$$U_n(\alpha) = \frac{1}{\Gamma(n+1)} F_n(\alpha)$$

Table 2-3 Table of Fermi-Dirac functions† (Continued)

α	$\frac{2}{3}F_4$	F_4	α	$\frac{2}{3}F_4$	F_4	α	$\frac{2}{3}F_4$	F_4	α	$\frac{2}{3}F_4$	F_4
-4.0	11.751 80	5.770 72	-8.0	52.901 73	15.380 48	-12.0	138.707 97	27.951 78	-14.0	201.709 50	35.142 97
-4.1	12.338 60	5.965 80	-8.1	54.453 85	15.662 24	-12.1	141.520 44	28.297 89	-14.1	205.242 49	35.517 00
-4.2	12.945 05	6.163 56	-8.2	56.034 24	15.945 80	-12.2	144.367 60	28.645 45	-14.2	208.812 95	35.892 38
-4.3	13.571 40	6.363 96	-8.3	57.643 07	16.231 14	-12.3	147.249 58	28.994 46	-14.3	212.421 01	36.269 08
-4.4	14.217 93	6.566 98	-8.4	59.280 52	16.518 26	-12.4	150.166 54	29.344 91	-14.4	216.066 81	36.647 12
-4.5	14.884 89	6.772 57	-8.5	60.946 78	16.807 14	-12.5	153.118 61	29.696 79	-14.5	219.750 48	37.026 49
-4.6	15.572 53	6.980 70	-8.6	62.642 01	17.097 76	-12.6	156.105 94	30.050 09	-14.6	223.472 15	37.407 18
-4.7	16.281 11	7.191 34	-8.7	64.386 39	17.390 13	-12.7	159.128 68	30.404 82	-14.7	227.231 96	37.789 18
-4.8	17.010 88	7.404 45	-8.8	66.120 09	17.684 23	-12.8	162.186 96	30.760 96	-14.8	231.030 03	38.172 50
-4.9	17.762 08	7.620 01	-8.9	67.903 29	17.980 04	-12.9	165.280 92	31.118 51	-14.9	234.866 50	38.557 12
-5.0	18.534 96	7.837 97	-9.0	69.716 16	18.277 56	-13.0	168.410 71	31.477 46	-15.0	238.741 50	38.943 04
-5.1	19.329 76	8.058 32	-9.1	71.558 86	18.576 77	-13.1	171.576 46	31.837 81	-15.1	242.655 15	39.330 27
-5.2	20.146 71	8.281 03	-9.2	73.431 57	18.877 68	-13.2	174.778 31	32.199 56	-15.2	246.607 59	39.187 79
-5.3	20.986 04	8.506 06	-9.3	75.334 45	19.180 26	-13.3	178.016 42	32.562 68	-15.3	250.598 95	40.108 59
-5.4	21.847 99	8.733 39	-9.4	77.267 68	19.484 51	-13.4	181.290 90	32.927 20	-15.4	254.629 36	40.499 69
-5.5	22.732 79	8.962 99	-9.5	79.231 41	19.790 41	-13.5	184.601 90	33.293 08	-15.5	258.698 93	40.892 06
-5.6	23.640 67	9.194 85	-9.6	81.225 82	20.097 96	-13.6	187.949 56	33.660 34	-15.6	262.807 81	41.285 71
-5.7	24.571 84	9.428 93	-9.7	83.251 06	20.407 15	-13.7	191.334 01	34.028 96	-15.7	266.956 12	41.680 64
-5.8	25.526 53	9.665 21	-9.8	85.307 30	20.717 97	-13.8	194.755 40	34.398 94	-15.8	271.148 38	42.076 83
-5.9	26.504 95	9.903 67	-9.9	87.394 71	21.030 42	-13.9	198.213 85	34.770 28	-15.9	275.371 53	42.474 29

† Taken from J. McDougall and E. C. Stoner, *Phil. Trans. Roy. Soc.*, **237**:67 (1938).

Problem 2-9: Show that in a perfect nonrelativistic electron gas

$$P_e = \frac{2}{3} \left(\frac{U}{V} \right)_{\text{kin}}$$

for any degree of degeneracy.

Problem 2-10: (a) Show that as $\alpha \rightarrow \infty$, $F_4/F_4 \rightarrow \frac{2}{3}$, for which case $P_e \rightarrow n_e kT$, the pressure of a Maxwellian electron gas. (b) Show that as $\alpha \rightarrow -\infty$, $F_4/F_4 \rightarrow \frac{3}{2}v_0$, for which case $P_e \rightarrow (8\pi/15m^3)p_0^5$, the pressure of a completely degenerate nonrelativistic electron gas.

From Eq. (2-57) it is apparent that

$$P_e = n_e kT \left(\frac{2}{3} \frac{F_4}{F_4} \right) \quad (2-58)$$

Thus, the factor $2F_4/3F_4$ measures the extent to which the electron pressure differs from that of a nondegenerate gas. This multiplication factor is plotted in Fig. 2-5 as a function of the parameter α . It can be seen that the gas pressure is essentially that of a nondegenerate gas for $\alpha > 2$.

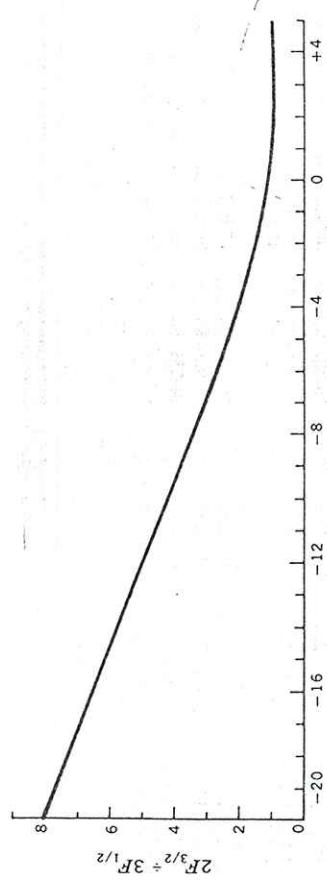


Fig. 2-5 The ratio $2F_4/3F_1$ as a function of the degeneracy parameter α . This ratio is equal to the ratio of the pressure of an electron gas to the pressure it would have if it were Maxwellian at the same density.

$$\frac{\rho N_0}{\mu_e} = \frac{4\pi}{h^3} (2mkT)^4 F_4(\alpha) \quad (2-59)$$

from which it follows that

$$\log \left(\frac{\rho}{\mu_e} T^{-4} \right) = \log F_4(\alpha) - 8.044 \quad (2-60)$$

This equation is plotted in Fig. 2-6, which relates $\log [(\rho/\mu_e)T^{-4}]$ to the degeneracy parameter α .

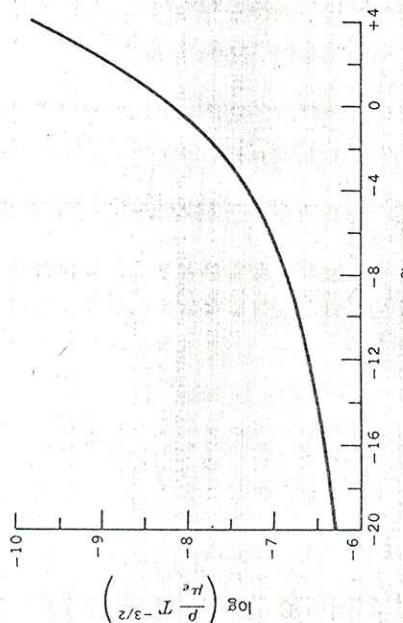


Fig. 2-6 The value of $(\rho/\mu_e)T^{-4}$ determines the degeneracy parameter α of an electron gas.

These equations describe the behavior of the equation of state of an electron gas in the partially degenerate region. For given ρ , T , Eq. (2-59) determines $F_4(\alpha)$, which in turn allows $F_1(\alpha)$ to be interpolated from Table 2-3. These calculations have used nonrelativistic kinematics because, in most stars, relativistic degeneracy is important only for such high densities that the degeneracy is essentially complete.

For many problems in nonrelativistic partial degeneracy, however, it is instructive to have appropriate expansions of the equation of state. Expansions that converge rapidly for weak degeneracy (nearly Maxwellian) and for strong degeneracy (nearly complete) are easily obtained.

Weak nonrelativistic degeneracy For notational ease, define $\Lambda = \exp(-\alpha)$. Then for $\alpha > 0$, which is seen from Fig. 2-5 to correspond to weak degeneracy, the number Λ is less than unity. Then $F_4(\Lambda)$ may be expanded:

$$\begin{aligned} F_4(\Lambda) &= \int_0^\infty \frac{u^4 du}{(1/\Lambda)e^u + 1} = \int_0^\infty \Lambda e^{-u} u^4 \frac{1}{1 + \Lambda e^{-u}} du \\ &= \Lambda \int_0^\infty e^{-u} u^4 [1 - \Lambda e^{-u} + (\Lambda e^{-u})^2 - (\Lambda e^{-u})^3 + \dots] du \quad (2-61) \end{aligned}$$

which may be integrated term by term to give

$$F_4(\Lambda) = -\frac{\sqrt{\pi}}{2} \sum_{n=1}^{\infty} \frac{(-1)^n \Lambda^n}{n^4}, \quad \Lambda < 1$$

or equivalently

$$F_4(\alpha) = -\frac{\sqrt{\pi}}{2} \sum_{n=1}^{\infty} \frac{(-1)^n e^{-n\alpha}}{n^4} \quad \alpha > 0 \quad (2-62)$$

Then Eq. (2-57) becomes

$$n_e = \frac{2(2\pi m k T)^4}{h^3} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{e^{-n\alpha}}{n^4} \quad \alpha > 0 \quad (2-63)$$

Problem 2-11: Show by the same technique used in obtaining Eq. (2-63) that

$$P_e = \frac{2kT(2\pi m k T)^4}{h^3} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{e^{-n\alpha}}{n^2} \quad \alpha > 0 \quad (2-64)$$

Problem 2-12: For large α , the series may be approximated by one term. Show that Eqs. (2-63) and (2-64) then reduce to the Maxwellian distribution.

Problem 2-13: Suppose that α is large enough for only the first two terms of the series to be important. Show, then, that

$$\begin{aligned} P_e &\approx n_e k T \left[1 + \frac{n_e h^3}{2^4 (2\pi m k T)^4} + \dots \right] \\ &= n_e k T (1 + 10^{-16.435} n_e T^{-4} + \dots) \end{aligned}$$

strong nonrelativistic degeneracy The degeneracy becomes strong when α becomes a large negative number or, equivalently, when the parameter Λ becomes a large positive number. The expansion for large Λ employs a lemma due to Sommerfeld, which, as stated by Chandrasekhar,¹ is:

LEMMA: *If $\phi(u)$ is a sufficiently regular function which vanishes for $u = 0$, then we have the asymptotic formula*

$$\int_0^\infty \frac{du}{(1/\Lambda)e^u + 1} \frac{d\phi(u)}{du} = \phi(u_0) + 2 \left[c_2 \left(\frac{d^2\phi}{du^2} \right)_{u_0} + c_4 \left(\frac{d^4\phi}{du^4} \right)_{u_0} + \dots \right] \quad (2-65)$$

where $u_0 = \log \Lambda$ and c_2, c_4, \dots are numerical constants defined by

$$c_2 = 1 - \frac{1}{2^*} + \frac{1}{3^*} - \frac{1}{4^*} + \dots$$

he series for the constants c_p can be summed.² For instance,

$$= \frac{\pi^2}{12} \quad c_4 = \frac{7\pi^4}{720} \quad c_6 = \frac{31\pi^6}{30,240}$$

Problem 2-14: By applying Sommerfeld's lemma to the integrals F_4 and F_6 , show that

$$(\alpha) = \frac{2}{3}(-\alpha)^{\frac{1}{2}} \left(1 + \frac{\pi^2}{8\alpha^2} + \frac{7\pi^4}{640\alpha^4} + \dots \right) \quad (2-66)$$

$$(\alpha) = \frac{2}{3}(-\alpha)^{\frac{1}{2}} \left(1 + \frac{5\pi^2}{8\alpha^2} - \frac{7\pi^4}{384\alpha^4} + \dots \right)$$

a good expansion for $\alpha < -1$. These three-term expansions are accurate to three decimal places for $\alpha < -5.6$ and are quite useful for $\alpha < -3$.

Problem 2-15: Calculate $F_4(\alpha)$ and $\frac{2}{3}F_6(\alpha)$ for $\alpha = -3$ and compare the results with the values in Table 2-3.

$$= \frac{4\pi}{h^3} (2mkT)^{\frac{1}{2}} F_4(\alpha) \quad (2-67)$$

is evident from Eq. (2-66) that the physical meaning of α in the limit of strong degeneracy is

$$\alpha \approx \frac{1}{2mkT} \left(\frac{3h^3 n_e}{8\pi} \right)^{\frac{1}{2}} \quad (2-68)$$

¹ Chandrasekhar, "An Introduction to the Study of Stellar Structure," p. 389; reprinted in the Dover Publications edition, Copyright 1939 by The University of Chicago, as reprinted by permission of The University of Chicago.
² For instance, H. B. Dwight, "Tables of Integrals and Other Mathematical Data," eq. 47.2, 11, The Macmillan Company, New York, 1947.

which from Eq. (2-29) is

$$-\alpha \approx \frac{p_0^2}{2mkT} = \frac{E_f}{kT} \quad (2-69)$$

where E_f is the Fermi energy (the kinetic energy of an electron at the top of the Fermi sea). This result is the same one that was obtained from an inspection of the Fermi distribution function for large negative α . For incomplete degeneracy, however, the energies $|\alpha kT|$ and E_f have different definitions and physical meanings.

If the three-term expansion of $F_4(\alpha)$ is retained, Eq. (2-59) can be written as an approximate equation relating the value of α to the density and temperature:

$$(-\alpha)^{\frac{1}{2}} \left(1 + \frac{\pi^2}{8\alpha^2} + \frac{7\pi^4}{640\alpha^4} + \dots \right) = 1.66 \times 10^8 \frac{\rho}{\mu_e} T^{-\frac{1}{2}} \quad \text{for } \alpha < -3 \quad (2-70)$$

Problem 2-16: Show that the electron pressure is twice that of the Maxwellian electron-gas formula when $\rho/\mu_e = 5.0 \times 10^{-574}$. Compare this result with the approximate boundary of Eq. (2-34), which gave the density for which a completely degenerate gas formula yields the same pressure as the Maxwellian gas formula.

The properties of the equation of state of the perfect electron gas are shown graphically in Fig. 2-7, where the ρT plane is divided into various zones according to the extent of the electron degeneracy. The diagonal line represents the approximate boundary between nondegenerate and degenerate electron gas as given by Eq. (2-34). In the neighborhood of this boundary the equation of state is to be evaluated from the parametric equations (2-57), which apply to partial degeneracy. For densities as high as indicated by Eq. (2-40), an electron gas becomes relativistic. This boundary is shown by the vertical line in Fig. 2-7. In the neighborhood of this line, the pressure of a completely degenerate gas can be evaluated from Eq. (2-46). For very high temperatures ($T > 10^9$) not considered in this discussion, the electron gas can be both relativistic and only partially degenerate. This situation presents a slightly more difficult form of the equation of state. We shall not consider it here. Suffice it to say that the Fermi statistics yield the same expression for the pressure as Eq. (2-53), the difference being that relativistic kinematics are to be used.

Several additional comments concerning a degenerate electron gas are appropriate at this time. With regard to the mechanical pressure which is to support a star, it is clear that the calculations presented here account only for the pressure due to the electrons. The contribution from the particle pressure of the nuclei in the gas must be added. Since nuclei are never degenerate in common stars, the pressure due to them is simply that of a Maxwellian gas, whose equations have been developed previously. To calculate the partial pressure of this perfect nuclear gas one must, of course, use the appropriate value of the mean molecular weight. Since the electrons have in this case been accounted for independently, one must use only the mean molecular weight of the remaining ions.

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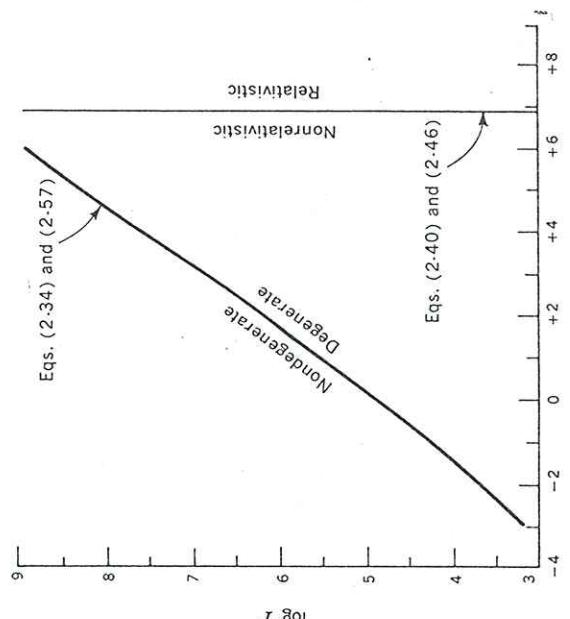


Fig. 2-7. Zones of the equation of state of an electron gas. The non-relativistic transition region between nondegeneracy and extreme degeneracy is located according to Eq. (2-34), and the pressure is given by Eq. (2-57) in this region. As ρ approaches 10^7 g/cm^3 , many of the electrons become relativistic, and the distribution becomes highly degenerate, in which case Eq. (2-46) adequately represents the pressure.

and nuclei. Let μ_i designate the mean molecular weight of the ions. The pressure due to particles is then the sum of the electron pressure and the nucleus pressure:

$$P_{\text{gas}} = P_e + \frac{N_0 k}{\mu_i} \rho T \quad (2-71)$$

In most practical cases where electron degeneracy does occur, the remaining nuclei are generally those of more advanced phases of stellar structure, consisting of helium nuclei, carbon nuclei, oxygen nuclei, or perhaps even heavier nuclei. In these circumstances the bulk of the pressure will be provided by the degenerate electron gas, the nuclei providing only a small additional term.

Problem 2-17: A gas composed of C^{12} and O^{16} has a density of $2.5 \times 10^5 \text{ g/cm}^3$ at 10^8 °K . Is this gas in the degenerate or nondegenerate region of the equation of state? Assuming the degeneracy is complete, is it completely nonrelativistic, partially relativistic, or extremely relativistic? Calculate the electron pressure from Table 2-2. Assuming that the degeneracy

is incomplete and nonrelativistic, calculate the electron pressure from Table 2-3. Why is the pressure calculated under assumptions of partial degeneracy greater than the pressure calculated for assumptions of complete degeneracy? Which numerical answer is more correct for the present problem? Why? What is the ratio of the electron pressure to the ion pressure?

Another interesting feature of the pressure of a completely degenerate gas is that it does not depend explicitly upon the temperature. Of course, at any finite temperature the electron gas is never completely degenerate, but in many cases the actual momentum distribution may be closely approximated by complete degeneracy. Whenever the energy associated with the momentum p_0 of the completely degenerate distribution greatly exceeds kT , the distribution of electron momenta will closely resemble that of complete degeneracy. It is in this case that the pressure is approximately independent of the temperature, being absolutely independent of the temperature for complete degeneracy.¹ This fact has the interesting consequence that a small rise in the temperature of an almost completely degenerate electron gas causes almost no change at all in the pressure. This last fact has far-reaching effects on stellar structure and on the evolution of stars. Those stages of stellar structure in which the electron gas is degenerate and is providing the main source of pressure for the gas must admit the possibility of abrupt rises in temperature with no corresponding increase in pressure. This situation actually occurs in certain stages of stellar evolution and leads to runaways in nuclear reaction rates (flash phenomena).

Problem 2-18: Show that the nonrelativistic electron pressure changes with temperature at constant volume according to

$$\begin{aligned} \left(\frac{\partial P_e}{\partial T} \right)_v &= \frac{8\pi k}{3h^3} (2mkT)^2 \left(\frac{5F_2}{2} - \frac{3}{2} F_1 \frac{dF_1/d\alpha}{dF_2/d\alpha} \right) \\ &= \frac{P_e}{T} \left(\frac{5}{2} - \frac{3}{2} \frac{F_1}{F_2} \frac{dF_1/d\alpha}{dF_2/d\alpha} \right) \end{aligned}$$

The quantity in parentheses in the second expression is unity for a nondegenerate gas and zero for a completely degenerate gas. Confirm this by evaluating it with the aid of the appropriate expansions.

Another important feature of the degenerate electron distributions is related to the transport of heat energy in the interiors of stars. The normal processes of energy transport in stellar interiors are altered somewhat when the electron gas becomes degenerate. The most important fact is that heat conductivity, which normally plays a secondary role to radiative transport and to convective transport, becomes important. In the case of nondegeneracy, the mean free path of charged particles is so small that heat conduction is extremely inefficient. When an electron gas is degenerate, however, the mean free path of electrons becomes

¹ Mathematically one shows that $\partial P_e / \partial T$ is very small by making an expansion of the parametric equation of state and evaluating for noncomplete degeneracy. The reader is referred to Chandrasekhar, *op. cit.*, chap. 10.

quite long. In order for an energetic electron to lose energy, it must fall into a lower-lying cell in momentum space as well as impart a new energy and momentum to the particle from which it scatters. The filling up of the available states in momentum space below a certain level hinders this process and renders energetic electrons quite free to move about in even a partially degenerate electron gas. This very good conductivity will tend to make partially degenerate electron gases isothermal.

White-dwarf stars are, to good approximation, supported by a completely degenerate electron gas. As those stars radiate their thermal energy, becoming increasingly cooler, the nearly degenerate momentum distribution becomes increasingly rectangular. Eventually the thermal energy is radiated away, the temperature falls toward zero, the light goes out, and the object remains an inert mass supported by a dense sea of completely degenerate electrons, or so the theory goes. This picture is in keeping with the observed properties of white dwarfs, which, from their observed masses and radii, are known to have densities as large as 10^6 g/cm^3 .

Pioneers in stellar structure encountered a subtle paradox in contemplating the above picture, however. Faithful application of the hitherto successful ionization equation seemed to imply that ions and electrons recombine at low temperatures. Since the density of un-ionized matter is at most a few grams per cubic centimeter, it would appear necessary that white dwarf expand as it cools. Yet it could be shown that the thermal energy is, at all stages, insufficient to do the necessary gravitational work. Eddington expressed the paradox as follows:

*I do not see how a star which has once got into this compressed condition is ever going to get out of it. So far as we know, the close packing of matter is only possible so long as the temperature is great enough to ionize the material. When the star cools down and regains the normal density ordinarily associated with solids, it must expand and do work against gravity. The star will need energy to cool. Sirius comes on solidifying will have to expand its radius at least tenfold, which means that 90 percent of its lost gravitational energy must be replaced. We can scarcely credit the star with sufficient foresight to retain more than 90 percent in reserve for the difficulty awaiting it. It would seem that the star will be in an awkward predicament when its supply of subatomic energy ultimately fails. Imagine a body continually losing heat but with insufficient energy to grow cold!*¹

The physical basis for the resolution of this problem is the thermodynamic peculiarity of a degenerate gas: the temperature no longer corresponds to kinetic energy. The electrons in a zero-temperature degenerate gas must still have large kinetic energy if the density is great. The classical ionization equation showed that at high densities atoms become ionized as kT approaches the order of magnitude of the electron binding energy, which is when the kinetic energy of the free-
Op. cit., p. 172.

electron gas approaches the kinetic energy of the bound electrons. The same approximate result applies in degenerate circumstances. Atoms are in an ionized state when the kinetic energy of the electron gas exceeds the kinetic energy of a bound electron.

The approximate truth of this statement can be seen from the following considerations. In a completely degenerate gas, all available electron states with momentum less than p_0 are occupied. The exclusion principle thus forbids the presence of bound electrons unless they are bound so tightly that their momentum exceeds p_0 , for otherwise there would be "too many" electrons in a momentum interval. Whereas a rigorous description of quantum statistics is considerably more complicated than this simple argument, the physical necessity of the result is evident.

The physical idea is also similar to that of the band structure of electronic states in solids. Ignore considerations of temperature completely for the moment. When the interatomic separations of atoms are large, the energy levels of electrons are just those associated with isolated atoms. Each energy level possesses a degeneracy equal to that of the atomic level times the total number of atoms. When the interatomic separation is decreased to the point where electronic levels of adjacent atoms overlap, however, a new feature is introduced by the exclusion principle. Since electrons are identical fermions, the mutual wave function of overlapping electrons must be antisymmetric in the electron coordinates. This antisymmetrization introduces a sharing of the indistinguishable electrons by all the atoms. In order that the electrons not be in exactly the same state, the many degenerate atomic energy levels of discrete energy regroup into a *continuous band* of energies for which each electron is shared by all atoms. The wave functions of those electrons in the band can be expressed by wave functions analogous to free electrons. This is what happens in a metal, for instance, for which the continuous band of *quasifree* electrons provides the source of electric conductivity. The same feature is carried to extremes at the densities of stellar interiors. Careful analysis shows that atoms are completely ionized by this mechanism for densities greater than about 10^3 g/cm^3 *independent of the temperature*. This physical effect has come to be called *pressure ionization*, and it resolves in a natural manner the paradox stated by Eddington.

This completes the introduction to the perfect electron gas. We have attempted to focus attention onto the physical principles rather than on the mathematical details. The serious student of stellar structure who has grasped these ideas may turn to more complete treatments for appropriate formulas applicable to the computation of physical problems.

THE PHOTON GAS

Particles are not the only source of mechanical pressure in a perfect gas. Pressure is also exerted by the radiation field in the interior of the star. By the radiation field we mean electromagnetic radiation, the omnipresent flux of photons inside a thermal enclosure. The pressure of the photon gas results from the fact

that each quantum of electromagnetic energy $h\nu$ carries with it a momentum equal to $h\nu/c$. If we imagine these photons being specularly reflected from a mirror, it is clear that momentum will be transferred to the mirror by the photons. If the environment is in thermodynamic equilibrium, the radiation flux is isotropic. In that case, the pressure integral for isotropic flux gives immediately the following interesting result for the radiation pressure P_r :

$$P_r = \frac{1}{3} \int_0^\infty \frac{h\nu}{c} cn(\nu) d\nu = \frac{1}{3} \int_0^\infty h\nu n(\nu) d\nu \\ = \frac{1}{3} u \quad (2-72)$$

where u is the energy density of photons, which was shown in Chap. 1 to be given by $u = a/T^4$.

It often happens in cases of physical interest that an enclosure is not strictly in thermodynamic equilibrium. In such objects as stars there exists a radiation field which is slightly anisotropic, resulting from the fact that there is a net excess of radiant energy flowing in one particular direction. In the case of a star, for instance, there is a net excess in the flux of electromagnetic radiation in the radial direction. At each point in the interior of such a star the situation corresponds nearly to one of thermodynamic equilibrium; i.e., the radiation field is nearly isotropic. It is convenient when considering such a slightly anisotropic radiation field to define the polar direction of a coordinate system as the direction of the net excess heat flow. In these cases of physical interest, azimuthal symmetry obtains about the direction of net flow. This symmetry corresponds to the assumption that there is no temperature gradient perpendicular to the direction of the net heat flow. In terms of this coordinate system we define a quantity called the *intensity of the radiation field* $I(\theta)$. Quantitatively, $I(\theta) d\Omega$ is the energy flux per square centimeter per second moving at a direction angle θ relative to the chosen axis inside a cone of directions defined by the solid angle $d\Omega$. Figure 2-8 shows the cross-sectional unit area inclined at angle θ to the chosen polar direction in the cone of directions corresponding to the solid angle $d\Omega$.

Let $u(\theta) d\Omega$ represent the *energy density* of radiation moving at angle θ in the set of directions $d\Omega$. It is clear from Fig. 2-8 that the flux $I(\theta) d\Omega$ passing through the unit area per second is given by the corresponding energy density $u(\theta) d\Omega$ times a unit column of length c , where c is the velocity of light:

$$I(\theta) d\Omega = cu(\theta) d\Omega \quad (2-73)$$

Since the integral of the directed energy density over the total solid angle of 4π is just the total energy density, we have

$$u = \int^{4\pi} u(\theta) d\Omega = \frac{1}{c} \int^{4\pi} I(\theta) d\Omega \quad (2-74)$$

Let H designate the net flux of energy transported per square centimeter per second in the polar direction. The flux per solid angle through a unit surface

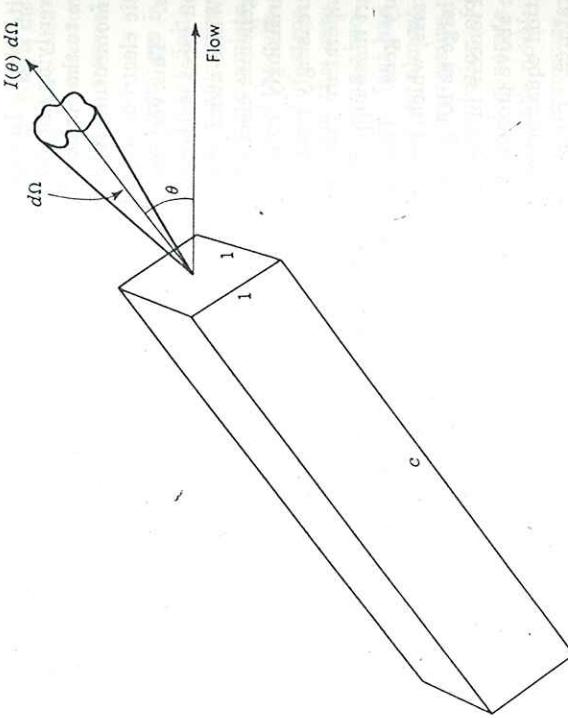


Fig. 2-8 The intensity $I(\theta)$ of the radiation field is defined such that $I(\theta) d\Omega$ is the energy flux moving in direction θ within the set of directions $d\Omega$.

normal to the polar direction is equal to $I(\theta) \cos \theta$, since the unit area has a projected area equal to $\cos \theta$ when viewed from the direction θ . It is apparent from Fig. 2-9 that the net flow of energy is given by

$$H = \int^{4\pi} I(\theta) \cos \theta d\Omega \quad (2-75a)$$

If, as is usually the case, the radiation field possesses azimuthal symmetry about the H axis,

$$H = 2\pi \int_0^\pi I(\theta) \cos \theta \sin \theta d\theta \quad (2-75b)$$

Evidently the heat flow vanishes for an isotropic radiation field [$I(\theta) = \text{const}$].

From the relationship between the energy $E = h\nu$ and the momentum $p = h\nu/c$ of a single photon, it follows that the flux of radiant energy $I(\theta)$ corresponds to a momentum flux $I(\theta)/c$. The resulting radiation pressure may be visualized as the compression force on a spring separating two imaginary unit areas between which the radiation field is excluded, as shown in Fig. 2-10. If the radiation field $I(\theta)$ is to remain unaltered by this imaginary mechanical system, a photon absorbed in plate I must be emitted in the same direction from plate II. The

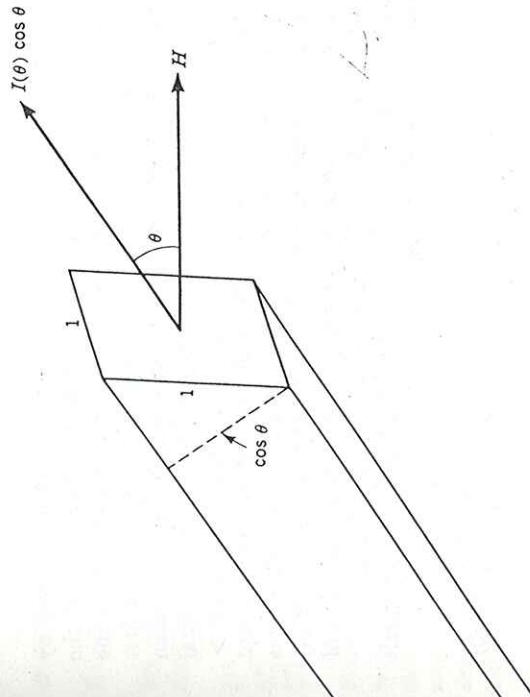


Fig. 2-9 The energy flux $I(\theta) d\Omega$ has associated with it an energy flow per unit area normal to the polar axis H equal to $I(\theta) d\Omega \cos \theta$.

resulting compressional force on the spring is, per unit area,

$$P_r = \int_{4\pi} I(\theta) \cos \theta \frac{d\Omega}{c} \cos \theta d\Omega = \frac{2\pi}{c} \int_0^\pi I(\theta) \cos^2 \theta \sin \theta d\theta \quad (2-76)$$

It is clear from these three simple calculations that the energy density u , the net flux of energy in the polar direction H , and the radiation pressure P_r are related to the three moments of the radiation field $I(\theta)$:

$$P_r = \frac{1}{c} \int I(\theta) d\Omega = \frac{2\pi}{c} \int_0^\pi I(\theta) \sin \theta d\theta \quad (2-77)$$

$$I = \int I(\theta) \cos \theta d\Omega = 2\pi \int_0^\pi I(\theta) \cos \theta \sin \theta d\theta$$

$$P_r = \frac{1}{c} \int I(\theta) \cos^2 \theta d\Omega = \frac{2\pi}{c} \int_0^\pi I(\theta) \cos^2 \theta \sin \theta d\theta \quad (2-78)$$

For cases appropriate to the interiors of stars, where near thermodynamic equilibrium obtains, the radiation field may be approximated by

$$I(\theta) = I_0 + I_1 \cos \theta + \dots$$

where I_0 represents the isotropic part of the radiation field and I_1 represents the

anisotropy in the radiation field corresponding to the net flux in the polar direction. Equation (2-78) represents the first two terms in the Fourier cosine expansion of the general radiation field. The integrals corresponding to the above discussion are easily evaluated for this radiation field, yielding

$$u = \frac{2\pi}{c} \int (I_0 + I_1 \cos \theta) \sin \theta d\theta = \frac{4\pi}{3} I_1 \quad (2-79)$$

$$H = 2\pi \int (I_0 + I_1 \cos \theta) \cos \theta \sin \theta d\theta = \frac{4\pi}{3} I_0$$

$$P_r = \frac{2\pi}{c} \int (I_0 + I_1 \cos \theta) \cos^2 \theta \sin \theta d\theta = \frac{4\pi}{3c} I_0 \quad (2-80)$$

In this approximation the energy density in the radiation field and the radiation pressure are independent of the anisotropic term in the radiation field. On the other hand, the net heat flux carried by radiation flow is dependent upon the existence of the anisotropic term. The relationship between the energy density u and the radiation pressure P_r is the same as that for an isotropic radiation field:

$$P_r = \frac{1}{3} u = \frac{1}{3} a T^4 \quad (2-81)$$

where the constant $a = 7.565 \times 10^{15}$ ergs cm^{-3} deg^{-4} . For a more general form of the radiation field involving higher powers of $\cos \theta$ in the expansion, this relationship between the energy density and the radiation pressure is not strictly

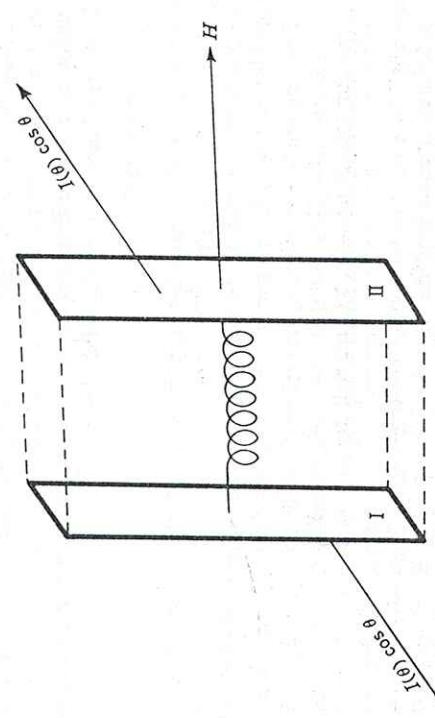


Fig. 2-10 Two imaginary plates separated by a spring. The radiation is excluded from the area between the plates, and each plate is required to emit the same radiation absorbed by its counterpart in order that the radiation field not be disturbed. The radiation pressure is the resulting compressional force on the spring.

correct, but in the cases of physical interest in the interiors of stars, the expression given by Eq. (2-78) is adequate. Judging from the relationships implied by Eq. (2-79), we may in fact rewrite Eq. (2-78) as

$$I(\theta) \approx \frac{c}{4\pi} u + \frac{3}{4\pi} H \cos \theta \quad (2-81)$$

We shall see shortly that the second term in Eq. (2-81) is numerically much smaller than the first in the interior of stars.

Problem 2-19: Suppose we have a radiation field in the form $I(\theta) = I_0 \exp(a \cos \theta)$. Note that the limit of small a corresponds to Eq. (2-78) with $a = I_1/I_0$. Calculate the relationship between u , H , and P_r . Does $P_r = u/3$? Do the expressions reduce to Eq. (2-79) to first order in a ?

We may consider one slight variation of Eq. (2-75a) at this point in order to clear up a relationship used in Chap. 1. Instead of calculating the net energy flux inside an enclosure, as in Eq. (2-75a), we may ask for the total flux which would emerge from a hole cut in the surface of the container. This calculation will correspond to the energy emitted per unit area from the surface of a blackbody. In this case the appropriate limits of the integral in Eq. (2-75b) are only from 0 to $\pi/2$, resulting in an emission

$$J = 2\pi \int_0^{\pi/2} (I_0 + I_1 \cos \theta) \cos \theta \sin \theta d\theta = \pi I_0 + \frac{2\pi}{3} I_1 \quad (2-82)$$

For an isotropic radiation field (or invoking the fact that in all real cases of interest the second term is very much less than the first) the radiation per square centimetre from the surface of a blackbody is related to the internal energy density inside the blackbody by the relationship

$$J = \pi I_0 = \frac{1}{4} \sigma u = \sigma T^4 \quad (2-83)$$

This is the source of the ratio between the two radiation constants introduced in Chap. 1; $\sigma = (c/4)a$ [see Eq. (1-29)].

We shall return to these three moments of the radiation field when we consider the question of the radiative transfer in stellar interiors. At that time the heat flow H will be related to the temperature gradient of the star. From Eq. (2-79) above, however, it is apparent that the energy density and the pressure in the enclosure are independent of a small anisotropy.

Problem 2-20: In a star like the sun, the entire luminosity originates from a region inside $r = 3 \times 10^{10}$ cm, at which point the temperature $T = 3 \times 10^6$ K. Calculate from L_\odot the energy flux H and show that the second term in Eq. (2-81) is very small compared to the first term.

The mechanical pressure of a perfect gas is to be computed as the sum of three terms:

$$P = P_{\text{ions}} + P_{\text{electrons}} + P_{\text{radiation}} \quad (2-84)$$

If the electron gas is nondegenerate, the sum of the first two terms is given by Eq. (2-10), where μ is the mean molecular weight of all free particles. If the electron gas is degenerate, the second term must be computed from one of the appropriate equations for degenerate electron pressure. In this case, the first term will be of the form of Eq. (2-10) except that μ will then represent the mean molecular weight of the ions only. Equation (2-84) will apply to all normal stellar interiors. It requires modification at extremely high temperatures ($T > 10^9$), when positron-electron pairs may be produced from energetic photons, and at high densities, where particle interactions may invalidate the perfect-gas approximation. If magnetic fields are present in low-density regions, Eq. (2-84) may require the addition of magnetic-field pressure. The difficult question of the coulomb interactions in the gas will be postponed to a later discussion.

Finally, we note that the pressure due to a perfect nondegenerate gas equals the pressure due to the radiation field when

$$\frac{N_0 k}{\mu} \rho T = \frac{1}{3} \sigma T^4 \quad (2-85)$$

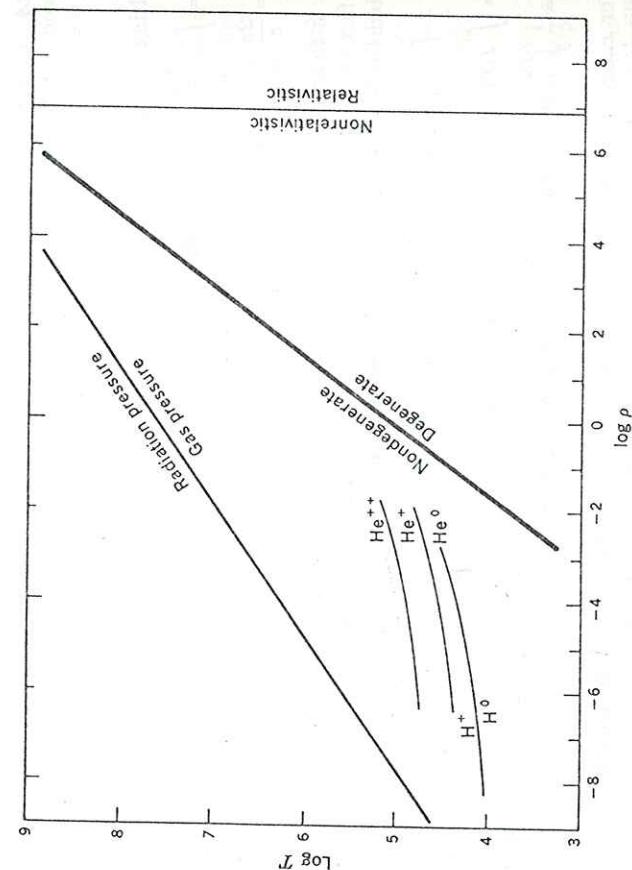


Fig. 2-11 Zones of the equation of state of a gas in thermodynamic equilibrium. Radiation pressure dominates the gas pressure in the upper left-hand corner. The remaining boundaries are similar to those in Fig. 2-7. Also included for comparison are the transition strips in a hydrogen-dominated gas between H^0 and H^+ , and between He^0 and He^+ .

or when

$$T = 3.20 \times 10^7 \left(\frac{\rho}{\mu}\right)^{\frac{1}{3}} \approx 3.6 \times 10^7 \rho^{\frac{1}{3}} \quad (2-86)$$

By this equation and the equations of the previous section the ρT plane may be roughly divided into three major regions in which:

- (1) The pressure is dominated by the photon gas.
- (2) The pressure is dominated by a nondegenerate gas.
- (3) The pressure is due to a degenerate electron gas.

Figure 2-11 shows this rough division into the various zones of the equation of state, calculated for a composition of nearly all hydrogen, as is characteristic of most stellar interiors. We have included for comparison the various ionization zones of hydrogen and helium calculated from the Saha equation for a composition predominantly of hydrogen. These lines correspond to those values of temperature and density for which the ionization is 50 percent accomplished.

2-2 QUASISTATIC CHANGES OF STATE

A star is not a static thing. It undergoes large expansions and contractions in the course of its evolution. Individual mass elements in convection zones rise and fall along the radius of the star. The very stability of the stellar structure is determined by its response to small perturbations. In this section we introduce the physical principles, mainly thermodynamic, of slow expansions.

According to the first law of thermodynamics, the internal energy of a gas may be changed by adding or withdrawing a quantity of heat energy or by doing work upon the gas by expansion or contraction. Specifically

$$dU = dQ + dW \quad (2-87)$$

If the process of change is performed infinitely slowly, so that one can think of the state of the system at any moment as being one of equilibrium, the process is referred to as *quasistatic*. Since quasistatic processes can be conducted in a reverse sense, they are generally referred to as *reversible processes*. An infinitesimal change of the volume of an enclosure containing a gas requires an infinitesimal amount of mechanical work on the gas given by

$$dW = -P dV \quad (2-88)$$

We shall explicitly introduce this equation into the first law of thermodynamics, whereupon it may be written

$$dQ = dU + P dV \quad (2-89)$$

It is from considerations of the equation of state that the proper expression to be

used for the pressure may be determined. The proper formula depends, as we have seen, upon the exact temperature and density of the gas in a star.

It is a fundamental proposition of thermodynamics that for quasistatic changes of state the heat increment may be written

$$dQ = T dS \quad (2-90)$$

where S is a function of state called the *entropy*. Because the entropy is a function of state, it may be computed for matter in thermodynamic equilibrium and depends only upon the equilibrium state of matter and not upon its past history. The change in entropy between two equilibrium states may be computed by the combination of Eqs. (2-90) and (2-89):

$$T dS = dU + P dV \quad (2-91)$$

wherein the change is evaluated along a sequence of equilibrium states reached by quasistatic changes. The internal energy is also a function of state, which means that for a given quantity of gas in equilibrium, U may be regarded as a function $U(V, T)$. For a quasistatic change of state of the gas, therefore,

$$dS = \frac{1}{T} \left[\left(\frac{\partial U}{\partial V} \right)_T + P \right] dV + \frac{1}{T} \left(\frac{\partial U}{\partial T} \right)_V dT \quad (2-92)$$

Because the entropy is a function $S(V, T)$ of the state of matter, dS may also be written

$$dS = \left(\frac{\partial S}{\partial V} \right)_T dV + \left(\frac{\partial S}{\partial T} \right)_V dT \quad (2-93)$$

Because the second partial derivatives are independent of the order in which they are taken, that is,

$$\frac{\partial}{\partial T} \frac{\partial S}{\partial V} = \frac{\partial}{\partial V} \frac{\partial S}{\partial T}$$

there exists an integrability condition for dS , viz.,

$$\frac{\partial}{\partial T} \left\{ \frac{1}{T} \left[\left(\frac{\partial U}{\partial V} \right)_T + P \right] \right\} = \frac{\partial}{\partial V} \left[\frac{1}{T} \left(\frac{\partial U}{\partial T} \right)_V \right] \quad (2-94)$$

The application of these simple ideas may lead to quite useful results. As an example of this fact, we now show thermodynamically that the energy per unit volume of an equilibrium photon gas is proportional to the fourth power of the temperature.

THE STEFAN-BOLTZMANN LAW

In the interior of an evacuated container in thermal equilibrium, the energy density of photons can be a function only of the temperature: $u = u(T)$. If the contemplated box has volume V , the internal energy in the box is then

$$U = V u(T) \quad (2-95)$$

whereupon the partial derivatives are

$$\left(\frac{\partial U}{\partial V}\right)_T = u(T) \left(\frac{\partial U}{\partial T}\right)_V = V \frac{du}{dT} \quad (2-96)$$

Independent of the photon energy density and spectrum, one knows that $P_r = u/3$ for an isotropic photon gas simply from the fact that photons are massless. (All massless particles move at the velocity of light and carry momentum $p = E/c$)

$$\frac{\partial}{\partial T} \left(\frac{1}{T} \frac{4}{3} u \right) = \frac{\partial}{\partial V} \left(\frac{1}{T} V \frac{du}{dT} \right) \quad (2-97)$$

If the differentiation is carried out, one obtains

$$\frac{du}{u} = 4 \frac{dT}{T} \quad (2-98)$$

which has the integral

$$u = a T^4 \quad (2-99)$$

This thermodynamic argument does not reveal the fact that the arbitrary constant of integration is the *Stefan-Boltzmann constant*, but it does show that the elementary thermodynamic properties of a photon gas are consistent with the quantum-statistical properties discussed in Chap. 1.

SPECIFIC HEATS AND ADIABATIC CHANGES OF A PERFECT NONDEGENERATE GAS

Because a nondegenerate gas represents the simplest case, and because many interesting concepts were historically introduced from considerations of such a gas, we shall first discuss the concepts of specific heats and adiabatic changes within that context.

The equation of state of the nondegenerate gas may be written in a thermodynamically more convenient form by introducing the *specific volume*, defined to be the volume of one gram of gas, in place of its reciprocal quantity, the density. Then Eq. (2-10) becomes

$$PV = \frac{R}{\mu} T \quad (2-100)$$

where R is the molar gas constant with a value $R = N_0 k = 8.314 \times 10^7$ ergs mole⁻¹ deg⁻¹. The ratio R/μ may be thought of as the gas constant for 1 g of gas, and its value depends upon the composition of the gas. Equation (2-100) combined with $U = U(T)$ defines the properties of a perfect nondegenerate gas. The first law of thermodynamics,

$$dQ = dU + P dV \quad (2-101)$$

has units of ergs per gram when V is defined to be the specific volume. For some other purposes one might want to discuss the thermodynamics in terms of 1 mole

of gas, in which case V would be the volume of 1 mole, the factor μ would not appear in Eq. (2-100), and Eq. (2-101) would then have units of ergs per mole. We shall normally choose to work with 1 g of gas, however.

In keeping with the assumption $U = U(T)$, Eq. (2-101) can be written

$$dQ = \frac{dU}{dT} dT + P dV \quad (2-102)$$

The specific heats of the gas are defined in the following way. Let α be a function of the physical variables. Then the rate of heat addition per unit rise in temperature, all the time keeping the function α constant, is called the specific heat at constant α , and is designated by c_α :

$$c_\alpha = \left(\frac{dQ}{dT} \right)_\alpha \quad (2-103)$$

This rate is to be determined from Eq. (2-102) in such a way that α remains constant. This demand is met in principle by changing the physical variables from (T, V) , as in Eq. (2-102), to the pair (T, α) . Then there exists an alternative differential expression

$$dQ = F_1 dT + F_2 d\alpha \quad (2-104)$$

where F_1 and F_2 represent two functions of the physical variables. When this is accomplished, the value $c_\alpha = F_1$ can be read off. For instance, the specific heat at constant volume can be read immediately from Eq. (2-102):

$$c_V = \left(\frac{dQ}{dT} \right)_V = \frac{dU}{dT} \quad (2-105)$$

In the special case where the gas particles possess no excited states, e.g., an ionized gas, the internal energy is simply the kinetic energy of translation, and c_V is a constant. To calculate the specific heat at constant pressure c_P , the first law must be changed to a form of Eq. (2-104) where α is to be equal to P . This may be easily done with the help of the equation of state, Eq. (2-100), whose differential is

$$P dV + V dP = \frac{R}{\mu} dT \quad (2-106)$$

This result allows Eq. (2-102) to be transformed to

$$dQ = \left(\frac{dU}{dT} + \frac{R}{\mu} \right) dT - V dP \quad (2-107)$$

which has the desired form. In conjunction with Eq. (2-105) it is evident that for a perfect nondegenerate gas

$$c_P = c_V + \frac{R}{\mu} \quad (2-108)$$

Problem 2-21: Show for a nonperfect gas [$U = U(Y, T)$] that

$$c_P - c_V = \left[\left(\frac{\partial U}{\partial V} \right)_T + P \right] \left(\frac{\partial Y}{\partial T} \right)_P$$

Demonstrate that the right-hand side reduces to R/μ for a perfect gas. This result will be useful in regions of partial ionization.

For many thermodynamic applications, the ratio of the specific heats, c_P/c_V , is an important quantity, hereafter designated by γ . From the point of view of the classical kinetic theory of gases, γ depends upon the number of degrees of freedom associated with the molecules of the gas. Quantum mechanics has explained the fact that γ also depends upon the temperature. When kT becomes smaller than rotational or vibrational quantum of energy, those degrees of freedom "freeze out."¹ Specifically,

$$\gamma = 1 + \frac{2}{f} \quad (2-109)$$

where f is the number of "unfrozen" degrees of freedom (translational, rotational, vibrational) of a molecule. In stellar interiors the gases are ionized particles possessing only the three translational degrees of freedom. Hence, for applications in the interiors of stars

$$\gamma = \gamma_{\text{perfect monatomic gas}} = \frac{5}{3} \quad (2-110)$$

Problem 2-22: For 1 mole of a perfect monatomic nondegenerate gas

$$U = \frac{3}{2}N\mu kT \quad \text{and} \quad PV = N\mu kT$$

Show that the amounts of heat required to raise the temperature by an amount ΔT at constant volume and constant pressure, respectively, are

$$\Delta Q_V = \frac{3}{2}N\mu k \Delta T \quad \Delta Q_P = \frac{5}{2}N\mu k \Delta T$$

Compare with Eq. (2-110).

Problem 2-23: Show that the ratio of the isothermal to isentropic compressibilities is always equal to the ratio of the specific heats c_P/c_V , just as it is for a perfect gas. In other words, show that

$$\frac{\chi_T}{\chi_S} = \frac{c_P}{c_V} \quad \chi_T \equiv -\frac{1}{V} \left(\frac{\partial V}{\partial P} \right)_T \quad \chi_S \equiv -\frac{1}{V} \left(\frac{\partial V}{\partial P} \right)_S$$

The notation constant S means constant entropy, or $dQ = 0$.

By employing the equation of state (2-100) and Eq. (2-105) the first law may be written for an ideal gas:

$$dQ = c_V dT + \frac{R}{\mu V} dV \quad (2-111)$$

¹ See, for instance, "F. K. Richtmyer, E. H. Kennard, and T. Lauritsen, 'Introduction to Modern Physics,' 5th ed., p. 405, McGraw-Hill Book Company, New York, 1955.

An adiabatic change is defined as a quasistatic change of state during which no heat is added, that is, $dQ = 0$. Setting dQ equal to zero and substituting $R/\mu = c_P - c_V$ from Eq. (2-108) yields

$$c_V \frac{dT}{T} + (c_P - c_V) \frac{dV}{V} = 0 \quad \text{ideal gas} \quad (2-112)$$

For an ideal gas c_P and c_V are constants, in which case Eq. (2-112) may be integrated:

$$TV^{\gamma-1} = \text{const} \quad (2-113)$$

Problem 2-24: With the aid of the equation of state, show that the track of the quasistatic adiabatic change in Eq. (2-113) has these three equivalent forms:

$$TV^{\gamma-1} = \text{const} \quad PV\gamma = \text{const} \quad P^{1-\gamma}T^\gamma = \text{const} \quad (2-114)$$

Problem 2-25: Show that the differential adiabatic change of the perfect nondegenerate gas is

$$\frac{dT}{T} + (\gamma - 1) \frac{dV}{V} = 0 \quad \frac{dP}{P} + \frac{\gamma}{1 - \gamma} \frac{dT}{T} = 0 \quad \frac{dP}{P} + \gamma \frac{dV}{V} = 0 \quad (2-115)$$

When the effects of radiation are considered in the next section, we shall see that the values of γ are not the same in each of the three equations (2-115), nor are they constant along the adiabatic track, nor are they equal to the ratio of specific heats.

QUASISTATIC CHANGES IN AN ENCLOSURE CONTAINING MATTER AND RADIATION

To good approximation the matter in a stellar interior is in thermodynamic equilibrium. Thus the particle gas at each interior point is accompanied by a photon gas characteristic of the local temperature. The presence of the photons introduces two important effects into quasistatic expansions, radiation pressure and ionization change.

Effects of radiation pressure From Eq. (2-86) it can be concluded that radiation pressure will be important only for temperatures high enough so that matter is essentially completely ionized. It will also be true in practically all common cases that radiation pressure will be relatively unimportant except for densities low enough for the electron gas to be nondegenerate. Thus with a small sacrifice in generality we can evaluate the effect of radiation pressure by considering adiabatic expansions of an ideal nondegenerate monatomic gas plus radiation pressure. Then the pressure becomes

$$P = P_0 + P_r = \frac{N\mu k}{\mu} \rho T + \frac{1}{3} \alpha T^4 \quad (2-116)$$

Since the particle internal energy is, for monatomic particles, just the kinetic

energy, the internal energy per gram is

$$U = \alpha T^4 V + \frac{N_0}{\mu} \left(\frac{3}{2} k T \right) \quad (2-117)$$

where V is the volume of 1 g of gas. Now for a quasistatic change, the first law of thermodynamics requires

$$dQ = \left(\frac{\partial U}{\partial T} \right)_V dT + \left(\frac{\partial U}{\partial V} \right)_T dV + P dV \quad (2-118)$$

and the partial derivatives are evaluated as

$$\left(\frac{\partial U}{\partial V} \right)_T = \alpha T^4 \quad \left(\frac{\partial U}{\partial T} \right)_V = 4\alpha T^3 V + \frac{3 N_0 k}{2 \mu} \quad (2-119)$$

Then Eq. (2-118) can be written at once as

$$dQ = \left(4\alpha T^3 V + \frac{3 N_0 k}{2 \mu} \right) dT + \left(\frac{3}{2} \alpha T^4 + \frac{N_0 k T}{\mu} \right) dV \quad (2-120)$$

The differential expression for an adiabatic change is obtained by setting $dQ = 0$. For many purposes it is useful to rearrange that equation into a form resembling that for the adiabatic changes of a particle gas. Following Chandrasekhar, it is useful to define the *adiabatic exponents* Γ_1 , Γ_2 , and Γ_3 by the equations

$$\frac{dP}{P} + \Gamma_1 \frac{dV}{V} = 0 \quad (2-121a)$$

$$\frac{dP}{P} + \frac{\Gamma_2}{1 - \Gamma_2} \frac{dT}{T} = 0 \quad (2-121b)$$

$$\frac{dT}{T} + (\Gamma_3 - 1) \frac{dV}{V} = 0 \quad (2-121c)$$

where the changes of state involved are adiabatic.

The definitions of the adiabatic exponents are made in this form to retain the analogy to the corresponding equations (2-115) for a perfect nondegenerate particle gas, for which all three adiabatic exponents are equal to γ . For a perfect gas with constant γ it is possible to integrate the equations immediately, of course, as in Eq. (2-114). The present equations are not immediately integrable because, as we shall see, the adiabatic exponents are functions of the thermodynamic state. Rather clearly, these definitions demand that $\Gamma_3 - 1 = (\Gamma_2 - 1)\Gamma_1/\Gamma_2$.

Now from Eq. (2-116) it follows that

$$\begin{aligned} dP &= \left(\frac{3}{2} \alpha T^4 + \frac{N_0 k T}{\mu} \right) \frac{dT}{T} - \frac{N_0 k T}{\mu} \frac{dV}{V} \\ &= (4P_r + P_o) \frac{dT}{T} - P_o \frac{dV}{V} \end{aligned} \quad (2-122)$$

The substitution of dP/P into Eq. (2-121a) yields

$$(4P_r + P_o) \frac{dT}{T} + [\Gamma_1(P_r + P_o) - P_o] \frac{dV}{V} = 0 \quad (2-123)$$

Equation (2-120), on the other hand, can be written for $dQ = 0$ as

$$(12P_r + \frac{3}{2} P_o) \frac{dT}{T} + (4P_r + P_o) \frac{dV}{V} = 0 \quad (2-124)$$

Comparison of Eqs. (2-123) and (2-124) defines Γ_1 in terms of the partial pressures

$$\frac{\Gamma_1(P_r + P_o) - P_o}{4P_r + P_o} = \frac{4P_r + P_o}{12P_r + \frac{3}{2} P_o} \quad (2-125)$$

It is conventional to designate by β the fraction of the total pressure contributed by the particle pressure

$$\beta P = P_o \quad (1 - \beta) P = P_r \quad (2-126)$$

Problem 2-26: By substituting Eq. (2-126) into (2-125), show that

$$\Gamma_1 = \frac{32 - 24\beta - 3\beta^2}{24 - 21\beta} \quad (2-127)$$

For a particle gas ($\beta = 1$), Γ_1 reduces to $\frac{5}{3}$, the value of γ for a monatomic gas.

For a photon gas ($\beta = 0$), Γ_1 reduces to $\frac{3}{2}$, the value of γ for a photon gas. The second adiabatic exponent is found by similar algebraic steps. Substituting dP/P from Eq. (2-122) into Eq. (2-121b) and comparing with Eq. (2-124) gives

$$\frac{12(1 - \beta) + \frac{3}{2}\beta}{4 - 3\beta + \Gamma_2/(1 - \Gamma_2)} = -\frac{4 - 3\beta}{\beta} \quad (2-128)$$

Problem 2-27: Solve for Γ_2 and show that it has the same limits as Γ_1 for $\beta \rightarrow 0, 1$.

$$\Gamma_2 = \frac{32 - 24\beta - 3\beta^2}{24 - 18\beta - 3\beta^2} \quad (2-129)$$

Problem 2-28: Show that

$$\Gamma_3 = \frac{32 - 27\beta}{24 - 21\beta} \quad (2-130)$$

Each adiabatic exponent decreases monotonically from a value of $\frac{5}{3}$ for $\beta = 1$ to a value of $\frac{3}{2}$ for $\beta = 0$. Thus the logarithmic derivatives in Eqs. (2-121a) to (2-121c) are seen to be related by coefficients $(\Gamma_1, \Gamma_2, \Gamma_3)$ whose values depend on the relative importance of gas pressure in the total pressure.

Latter discussions will indicate instabilities in stellar structure associated with values of Γ less than $\frac{4}{3}$. A careful analysis shows that stellar instability occurs if Γ_1 , when appropriately averaged over a star, becomes smaller than $\frac{4}{3}$. The present discussion shows that radiation pressure cannot (in the absence of positron-electron-pair production or ionization) reduce Γ below $\frac{4}{3}$. However, it is quite clear that stars dominated internally by radiation pressure will have adiabatic exponents close to $\frac{4}{3}$. Their binding energy is relatively small compared to stars for which particles provide the pressure.¹ The adiabatic exponents will also be relevant in the model of convection to be discussed later and to the theory of pulsation.

From the equations developed, it is possible to calculate the specific heats at constant volume and at constant pressure for an enclosure containing an ideal monatomic gas and radiation. From Eq. (2-120) we see that

$$C_V = \left(\frac{dQ}{dT} \right)_V = 4aT^3V + \frac{3}{2} \frac{N_0 k}{\mu} \left(1 + \frac{8aT^4/3}{N_0 k/\mu V} \right)$$

$$C_V = c_V \left(1 + \frac{8P_r}{P_a} \right) = c_V \left[1 + \frac{8(1 - \beta)}{\beta} \right] = c_V \frac{8 - 7\beta}{\beta} \quad (2-131)$$

where $c_V = 3N_0 k/2\mu$ is the specific heat of the particle gas alone.

Problem 2-29: Show that

$$C_P = c_V \frac{\frac{3}{2} - 8\beta - \beta^2}{\beta^2} \quad (2-132)$$

Problem 2-30: It was proved in Prob. 2-23 that $c_p/c_V = xT/xs$ for any gas. Evaluate these compressibilities for the present case of a monatomic gas plus radiation and prove thereby that

$$\frac{C_P}{C_V} = \frac{\Gamma_1}{\beta} \quad (2-133)$$

Check this result by Eqs. (2-131) and (2-132). What is the meaning of the infinity as $\beta \rightarrow 0$?

Equations (2-121) are not integrable as they stand. It is a simple matter, however, to rearrange variables until an integrable form of the adiabatic track is obtained. From the second law of thermodynamics, dQ/T must be the differential of the entropy S . From Eq. (2-120) we see that for an ideal monatomic nondegenerate gas

$$dS = \left(4aT^2V + \frac{3N_0 k}{2\mu T} \right) dT + \left(\frac{4}{3}aT^3 + \frac{N_0 k}{\mu V} \right) dV \quad (2-134)$$

Problem 2-31: Confirm that S satisfies the integrability condition

$$\frac{\partial^2 S}{\partial T \partial V} = \frac{\partial^2 S}{\partial V \partial T}$$

¹ See the discussion of the virial theorem later in this section for a simplified demonstration.

Show also that the elimination of the variable V in favor of a new variable $W = T^3V$ allows dS to be written as

$$dS = -\frac{3}{2} \frac{N_0 k}{\mu} \frac{dT}{T} + \frac{4a}{3} \frac{dW}{\mu} + \frac{N_0 k}{\mu} \frac{dW}{W} \quad (2-135)$$

Equation (2-135) can be integrated by inspection.

$$S = \text{const} + \frac{N_0 k}{\mu} \ln \frac{T^4}{\rho} + \frac{4a}{3} \frac{T^3}{\rho} \quad (2-136)$$

The second term is just the entropy per gram of an ideal nondegenerate monatomic gas, whereas the third term measures the entropy per gram of the photon component. The entropies are additive. Evidently the last term can be written in terms of the partial-pressure ratio, so that alternate forms are

$$S = \text{const} + \frac{N_0 k}{\mu} \left(\ln \frac{T^4}{\rho} + 4 \frac{P_r}{P_a} \right) \quad (2-137a)$$

and

$$S = \text{const} + \frac{N_0 k}{\mu} \left(\ln \frac{T^4}{\rho} + 4 \frac{1 - \beta}{\beta} \right) \quad (2-137b)$$

The increase in entropy of the final state f over that of the initial state i is

$$\Delta S = \frac{N_0 k}{\mu} \left\{ \ln \left[\left(\frac{T_f}{T_i} \right)^4 \frac{\rho_i}{\rho_f} \right] + 4 \left(\frac{1 - \beta_f}{\beta_f} - \frac{1 - \beta_i}{\beta_i} \right) \right\} \quad (2-138)$$

If a portion of a stellar interior is allowed to expand (or contract) reversibly without exchanging heat with its surroundings, then $\Delta S = 0$, and the change is an adiabatic one. Two examples of such changes might be (1) the gravitational contraction of a stellar core following the exhaustion of a nuclear fuel supply and (2) the expansion of a rising convective mass of gas. In no case is it obvious that such a change will be adiabatic. That no heat be exchanged demands that the expansion occur in times short enough so that only an insignificant fraction of the photon energy can diffuse into the surroundings during the expansion. For any physical expansion, therefore, it will be necessary to compute the heat exchange by radiative transfer during the expansion. If the entropy increase during the expansion,

$$\Delta S = \int_i^f \frac{dQ}{T} \quad$$

can be computed, Eq. (2-138) still provides one relationship between the initial and final states. Of course, these equations are valid only for temperatures high enough for ionization to be complete and for densities low enough for electrons to be nondegenerate. More elaborate formulas can be derived to apply to more general circumstances.

Problem 2-32: A gas composed of equal numbers of C_1 and O^{16} is initially at a density of 100 g/cm^3 and a temperature of 10^8 K . What is its density after an adiabatic compression to a temperature of 10^9 K ?

The foregoing treatment was based on the assumption of a nondegenerate gas. Because the entropies are additive, however, the results can be generalized to a partially degenerate gas by separately computing the entropy for the Fermi electron gas. Using the fact that for any degree of degeneracy the internal energy per gram of a nonrelativistic gas is $U = \frac{3}{2}PV$, where V is the specific volume, the first law can be expressed as

$$dQ = dU + PdV = \frac{5}{2}PdV + \frac{3}{2}VdP \quad (2-139)$$

With the aid of Eq. (2-57) for the electron pressure, dQ becomes

$$dQ = \frac{5}{2} \left[\frac{8\pi kT}{3h^3} (2mkT)F_1(\alpha) \right] dV + \frac{3}{2}V \left[\frac{5}{2} \frac{8\pi k}{3h^3} (2mkT)F_1(\alpha) \right] dT + \frac{3}{2}V \left[\frac{8\pi kT}{3h^3} (2mkT)^{\frac{1}{2}} \frac{dF_1}{d\alpha} \right] d\alpha \quad (2-140)$$

Although this expression is in terms of three increments, dV , dT , and $d\alpha$, it will be realized that these increments are not independent. From the fact that

$$n_e = \frac{\rho N_0}{\mu_e} = \frac{N_0}{\mu_e} \frac{1}{V} = \frac{4\pi}{h^3} (2mkT)^{\frac{1}{2}} F_1(\alpha) \quad (2-141)$$

it follows that dV can be expressed in terms of dT and $d\alpha$. This operation yields

$$dV = -\frac{\mu_e V^2}{N_0} dn_e = -\frac{\mu_e V^2}{N_0} \left[\frac{4\pi}{h^3} (2mkT)^{\frac{1}{2}} \left(\frac{dF_1}{d\alpha} d\alpha + \frac{3}{2}F_1 \frac{dT}{T} \right) \right] \\ = -\frac{N_0}{\mu_e n_e} \left(\frac{dF_1/d\alpha}{F_1} d\alpha + \frac{3}{2} \frac{dT}{T} \right) \quad (2-142)$$

When this expression for dV is inserted into Eq. (2-140), the result for $dS = dQ/T$ is

$$dS = \frac{N_0 k}{\mu_e} \left(-\frac{5}{3} \frac{F_1}{F_1^2} \frac{dF_1}{d\alpha} + \frac{dF_1/d\alpha}{F_1} \right) d\alpha \quad (2-143)$$

It will be noticed that dS is a function of α times the increment $d\alpha$. It follows that the electron entropy is a function only of α .

Problem 2-33: It can be shown that $dF_1/d\alpha = -3F_1/2$. Confirm this result from the expansions appropriate for weak degeneracy, Eq. (2-62), and from the expansions appropriate for strong degeneracy, Eq. (2-66). Then show that the function

$$S(\alpha) = \frac{N_0 k}{\mu_e} \left(\frac{5}{2} \frac{F_1}{F_1^2} + \alpha \right) \quad (2-144)$$

reproduces dS .

From the additive properties of the entropy per gram, the generalized form of the entropy becomes

$$S = \text{const} + \frac{N_0 k}{\mu_e} \ln \frac{T^4}{\rho} + \frac{N_0 k}{\mu_e} \left[\frac{5}{3} \frac{F_1(\alpha)}{F_1(\alpha)} + \alpha \right] + \frac{4\alpha}{3} \frac{T^3}{\rho} \quad (2-145)$$

The three terms are the entropies of the ions, the electrons, and the radiation, respectively. The result is correct only if the electrons are nonrelativistic, however.

Problem 2-34: Can you derive an expression for the entropy of a partially degenerate relativistic electron gas?

Effects of ionization If the ionization of matter is incomplete, quasistatic changes will be accompanied by changes in the degree of ionization. From the Saha equation it follows that even a slight rise in the temperature of a partially ionized gas may considerably increase the ionization, which in turn may require a large amount of energy. A large energy requirement for a small temperature rise corresponds to a large value of the specific heat. It will therefore be expected that the thermodynamics of a partially ionized gas will differ considerably from that of a completely ionized (or completely neutral) gas. The differences occur because the number of free particles per gram is not constant and because energy is required to increase the number of free particles.

As the simplest of examples we shall compute c_V for a partially ionized gas of pure hydrogen. Now by definition $c_V = (dQ/dT)_V = (dU/dT)_V$, so that the first requirement is an expression for the internal energy of the partially ionized gas. Actually only the change in internal energy must be correctly represented. This end may be accomplished by noting that each free particle possesses translational energy equal to $3kT/2$, that an amount of energy approximately equal to χ_H is required for each hydrogen ionization, and that the internal energy of atoms represented by their population of excited states will change with temperature. For hydrogen the situation is simplified by the fact that the fractional population of excited states is very small, so that the internal energy per neutral atom can be neglected, and the average ionization potential can be taken to be that for the ground state. Then if N , H , and H^+ represent, respectively, the numbers per unit volume of free particles, of neutral hydrogens, and ionized hydrogens, the internal energy per gram is very nearly

$$U(T, V) = \frac{3}{2}NkTV + \chi_H H^+ V \quad (2-146)$$

where V is the specific volume. Then

$$dU = \left[\frac{3}{2}NkV + \frac{3}{2}kTV \left(\frac{\partial N}{\partial T} \right)_V + \chi_H V \left(\frac{\partial H^+}{\partial T} \right)_V \right] dT \\ + \left[\frac{3}{2}NkT + \frac{3}{2}kTV \left(\frac{\partial N}{\partial V} \right)_T + \chi_H H^+ + \chi_H V \left(\frac{\partial H^+}{\partial V} \right)_T \right] dV \quad (2-147)$$

and

$$C_V = \left(\frac{\partial U}{\partial T} \right)_V = \frac{3}{2} N k V + \frac{3}{2} k T V \left(\frac{\partial H^+}{\partial T} \right)_V + \chi_H V \left(\frac{\partial H^+}{\partial T} \right)_V \quad (2-148)$$

From the condition of charge neutrality in pure hydrogen,

$$n_e = H^+ \quad (2-149)$$

and the density of free particles is

$$N = H + H^+ + n_e = H + 2H^+ \quad (2-150)$$

Because the electron mass is negligible in comparison with the proton mass, the specific volume is related to the number densities by

$$H^+ + H = \frac{N_0}{V} \quad (2-151)$$

where N_0 is Avogadro's number. From these relationships between number densities it follows that

$$\left(\frac{\partial N}{\partial T} \right)_V = \left(\frac{\partial H^+}{\partial T} \right)_V = - \left(\frac{\partial H}{\partial T} \right)_V \quad (2-152)$$

Thus

$$\chi_V = \frac{3}{2} N k V \left[1 - \frac{2T}{3N} \left(\frac{3}{2} + \frac{\chi_H}{kT} \right) \left(\frac{\partial H}{\partial T} \right)_V \right] \quad (2-153)$$

Inasmuch as

$$3NkV = c_V^{(0)} \quad (2-154)$$

is the specific heat for a constant number of particles, the second term in Eq. (2-153) may be regarded as a correction term. The partial derivative in the second term must be evaluated from the Saha equation:

$$\frac{H^+ n_e}{H} = \frac{(H^+)^2}{H} = \frac{(2\pi m k T)^{\frac{3}{2}}}{h^3} \exp - \frac{\chi_H}{kT} = g(T) \quad (2-155)$$

By eliminating H^+ in favor of H with Eq. (2-151) we obtain

$$\frac{N_0/V - H}{H} = g(T) \quad (2-156)$$

so that

$$- \left[\frac{(N_0/V)^2}{H^2} + 1 \right] \left(\frac{\partial H}{\partial T} \right)_V = \left(\frac{\partial g}{\partial T} \right)_V \quad (2-157)$$

Problem 2-35: Show by further manipulation that

$$\left(\frac{\partial H}{\partial T} \right)_V = - \frac{1}{T} \left(\frac{3}{2} + \frac{\chi_H}{kT} \right) \frac{H^+ H}{H^+ + 2H} \quad (2-158)$$

Equation (2-153) now yields

$$c_V = c_V^{(0)} \left[1 + \frac{2}{3} \left(\frac{3}{2} + \frac{\chi_H}{kT} \right)^2 \frac{H^+ H}{(H + 2H^+)(2H^+ + H)} \right] \quad (2-159)$$

where $c_V^{(0)} = 3NkV/2$. Because the sum of H^+ and H is constant, the correction factor has a maximum at $H^+ = H$, the stage of 50 percent ionization.

Problem 2-36: It has already been pointed out that for most common densities hydrogen ionizes near 10^4 °K. Using that approximation for T , estimate the maximum enhancement factor for $c_V^{(0)}$.

Problem 2-37: Using the equation of state $PV = NkT$, show that

$$c_P = c_V^{(0)} \left[\frac{5}{3} + \frac{1}{3} \left(\frac{5}{2} + \frac{\chi_H}{kT} \right)^2 \frac{H^+ H}{(H^+ + H)^2} \right] \quad (2-160)$$

The specific heats for the pure hydrogen gas are shown in Fig. 2-12 as a function of the percent of hydrogen ionized. It can be seen that both c_V and c_P have maximum values about 30 times as great as their normal values. In examination of this figure it should be noted that $NV = N_0$ for neutral hydrogen and $NV = 2N_0$ for completely ionized hydrogen. Thus the specific heats per gram are twice as great at 100 percent ionization as for neutrality, and there is a slight asymmetry in the curves toward high ionization.

It is also evident that the ratio c_P/c_V drops below its normal value of $\frac{5}{3}$, leading one to wonder about the values of the adiabatic exponents in the regions of partial ionization. The first thing to notice is that the adiabatic exponents differ from each other in an ionization zone and they all differ from the ratio c_P/c_V . The adiabatic exponents are again defined by

$$\frac{dP}{P} + \Gamma_1 \frac{dV}{V} = 0 \quad (2-161)$$

$$\frac{dP}{P} + \frac{\Gamma_2}{1 - \Gamma_2} \frac{dT}{T} = 0 \quad (2-161)$$

$$\frac{dT}{T} + (\Gamma_3 - 1) \frac{dV}{V} = 0 \quad (2-161)$$

for adiabatic changes. Each function can easily be derived in terms of the specific heats and certain other partial derivatives. We first note the following relationship between the specific heats whenever the internal energy $U(V, T)$ is not independent of the volume, as is the case in regions of partial ionization. From the first law,

$$dQ = \left(\frac{\partial U}{\partial T} \right)_V dT + \left[\left(\frac{\partial U}{\partial V} \right)_T + P \right] dV \quad (2-162)$$

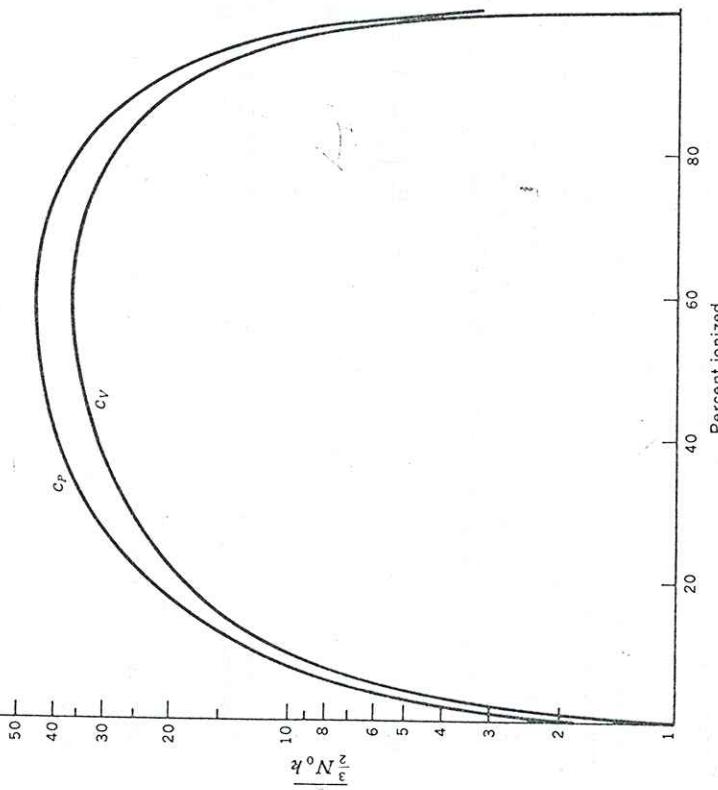


Fig. 2-12 The specific heats per gram c_P and c_V for a pure hydrogen gas as a function of its degree of ionization. In regions of partial ionization both specific heats are increased by large factors as a result of the large energy requirements for ionization changes. The specific heats at full ionization are twice as great as those of the neutral gas because the number of particles per gram is twice as great. Numerical values are given in Table 2-4.

and from the definitions of specific heats it follows that

$$\begin{aligned} c_P &= \left(\frac{\partial Q}{\partial T} \right)_P = \left(\frac{\partial U}{\partial T} \right)_V + \left[\left(\frac{\partial U}{\partial V} \right)_T + P \right] \left(\frac{\partial V}{\partial T} \right)_P \\ &= c_V + \left[\left(\frac{\partial U}{\partial V} \right)_T + P \right] \left(\frac{\partial V}{\partial T} \right)_P \end{aligned} \quad (2-163)$$

By rearrangement we have

$$\left(\frac{\partial U}{\partial V} \right)_T + P = (c_P - c_V) \left(\frac{\partial T}{\partial V} \right)_P \quad (2-164)$$

which yields in the first law

$$dQ = c_V dT + (c_P - c_V) \left(\frac{\partial T}{\partial V} \right)_P dV \quad (2-165)$$

For an adiabatic change

$$\frac{dT}{T} + \frac{(c_P - c_V)}{c_V T} \left(\frac{\partial T}{\partial V} \right)_P dV = 0 \quad \text{adiabatic} \quad (2-166)$$

Comparison with Eq. (2-161) yields

$$\Gamma_3 - 1 = \frac{V}{T} \frac{c_P - c_V}{c_V} \left(\frac{\partial T}{\partial V} \right)_P \quad (2-167)$$

To obtain Γ_1 it is necessary only to express dQ as a differential in dP and dV . To do so we express dT as

$$dT = \left(\frac{\partial T}{\partial P} \right)_V dP + \left(\frac{\partial T}{\partial V} \right)_P dV \quad (2-168)$$

so that

$$dQ = \left(\frac{\partial U}{\partial T} \right)_V \left(\frac{\partial T}{\partial P} \right)_V dP + \left[\left(\frac{\partial U}{\partial T} \right)_V + P + \left(\frac{\partial U}{\partial P} \right)_V \left(\frac{\partial T}{\partial V} \right)_P \right] dV \quad (2-169)$$

With the aid of Eq. (2-164) this expression reduces to

$$dQ = c_V \left(\frac{\partial T}{\partial P} \right)_V dP + c_P \left(\frac{\partial T}{\partial V} \right)_P dV \quad (2-170)$$

Setting $dQ = 0$ and dividing by $c_V (\partial T / \partial P)_V P$ yields

$$\frac{dP}{P} + \frac{c_P (\partial T / \partial V)_P}{P c_V (\partial T / \partial P)_V} dV = 0 \quad \text{adiabatic} \quad (2-171)$$

By use of the cyclic relation,

$$\left(\frac{\partial T}{\partial V} \right)_P = - \left(\frac{\partial T}{\partial P} \right)_V \left(\frac{\partial P}{\partial V} \right)_T \quad (2-172)$$

We obtain for an adiabatic change

$$\frac{dP}{P} - \frac{V}{P} \frac{c_P}{c_V} \left(\frac{\partial P}{\partial V} \right)_T \frac{dV}{V} = 0 \quad \text{adiabatic} \quad (2-173)$$

Comparison with Eq. (2-161) finally gives Γ_1 :

$$\Gamma_1 = - \frac{c_P}{c_V} \frac{V}{P} \left(\frac{\partial P}{\partial V} \right)_T \quad (2-174)$$

Problem 2-38: Show that

$$\frac{\Gamma_2}{1 - \Gamma_2} = - \frac{c_P T}{(c_P - c_V) (\partial T / \partial P)_V P} \quad (2-175)$$

Equations (2-167), (2-174), and (2-175) define the adiabatic exponents. Their evaluation in an ionization zone requires use of the formulas for c_P and c_V as well as evaluation of the partial derivatives appearing in the expressions. These quantities are relatively simple in the example of a pure hydrogen gas, as illustrated by the next problem.

Table 2-4 Properties of a hydrogen gas near $T = 10^4$ °K

$\frac{C_P}{\frac{3}{2}N^3k}$	$\frac{C_V}{\frac{3}{2}N^3k}$
1.66666	1.66666
1.66666	1.66666
1.2097	1.1593
1.1688	1.1160
1.1537	1.1000
1.1460	1.0920
1.1415	1.0872
1.1386	1.0842
1.1368	1.0822
1.1356	1.0810
1.1349	1.0803
1.1347	1.0801
1.1349	1.0803
1.1356	1.0810
1.1368	1.0822
1.1386	1.0842
1.1415	1.0872
1.1460	1.0920
1.1537	1.1000
1.1688	1.1160
1.2097	1.1593
1.66666	1.66666
1.66666	1.66666
1.0000	1.0000
5.8974	5.8974
10.5264	10.5264
14.8650	14.8650
18.891	18.891
22.5716	22.5716
28.1816	28.1816
32.5659	32.5659
36.4492	36.4492
39.7478	39.7478
42.3783	42.3783
44.6670	44.6670
45.3010	45.3010
45.7147	45.7147
45.4261	45.4261
44.5490	44.5490
42.5862	42.5862
39.4542	39.4542
35.0695	35.0695
28.3336	28.3336
23.9133	23.9133
22.3486	22.3486
13.5649	13.5649
2.0000	2.0000

$$= \frac{10(H^+ + H)^2 + 2(\frac{5}{2} + \chi_H/kT)^2 H^+ H}{(2) 176}$$

$$3\langle_{\frac{H}{2}}\rangle + 2\langle_{\frac{H}{2}}\rangle \langle_{\frac{H}{2}}\rangle + 2\langle_{\frac{H}{2}}\rangle + \langle_{\frac{H}{2}}\rangle + \langle_{\frac{H}{2}}\rangle$$

The values of the adiabatic exponents for the hydrogen gas at 10^4 °K are shown in Table 2-4 along with the values of c_v and c_P in units of $\frac{3}{2}Nok$. The striking feature is that all three adiabatic exponents fail to values near unity for partial ionization. Table 2-4 was computed by simply assuming $T = 10^4$ °K, but the values will not be much different if, for each density, one actually computes the temperature corresponding to the desired degree of ionization. These adiabatic expressions for γ_2 and γ_3 can be derived with the necessary manipulation.

The situation in a star will be further complicated by the mixture in composition, because each element will undergo stages of ionization at appropriate temperatures.¹ Since hydrogen will probably be the dominant element in the outer portions of the star, the adiabatic exponents will be similar to those of the table

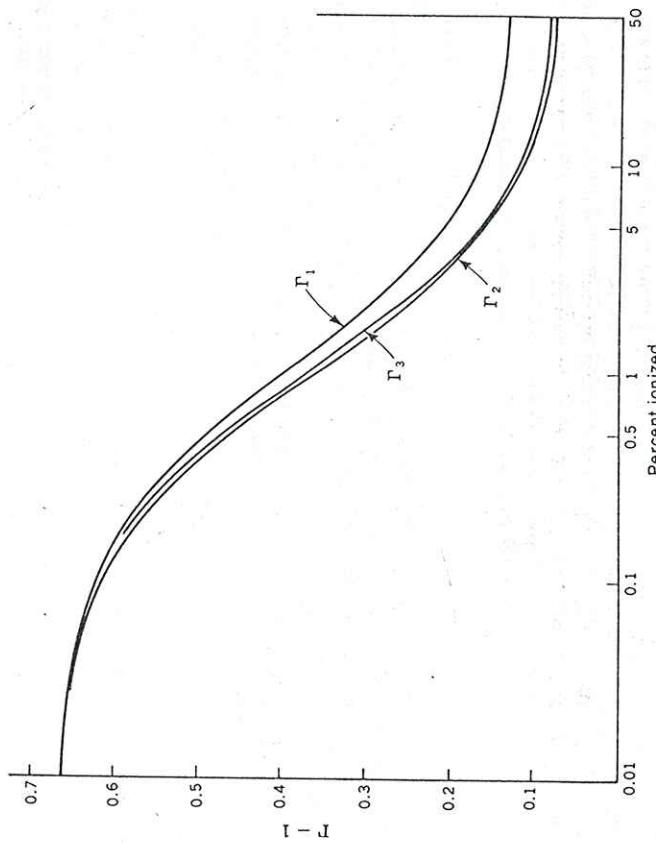


Fig. 2-13 The adiabatic exponents of a pure hydrogen gas as a function of its degree of ionization. Only the initial 50 percent is shown because the second 50 percent is its mirror image. The exponents change rapidly in the regions 0 to 1 or 99 to 100 percent ionization. Between 5 and 95 percent ionization, the values are considerably less than $\frac{4}{3}$ and therefore have a

initial ionization will modify that temperature gradient from the ones computed under the $\gamma = \frac{5}{3}$ assumption.

Radiation has another interesting effect upon the adiabatic exponents at high temperatures. In the presence of the coulomb field of a nucleus a photon may create a positron-electron pair if $h\nu > 2m_e c^2$.



This process is *thermodynamically similar to ionization* in that increased temperature is accompanied by an increase in the number of free particles at great expense to the energy of the photon gas. It is physically clear that the specific heats are increased thereby. Interestingly enough, it turns out that Γ_1 drops below $\frac{3}{4}$ for $n > 1 \times 10^9 \text{ K}$ if $\rho = 10^3 \text{ g/cm}^3$ and for $T > 5 \times 10^9 \text{ K}$ if $\rho = 10^6 \text{ g/cm}^3$. Adiabatic exponents are somewhat academic quantities in these conditions, however, because the associated annihilation of the electron pairs into neutrino pairs produces such a large heat loss that adiabatic changes are not really possible.¹

HYDROSTATIC EQUILIBRIUM AND

The pressure plays a primary role in the structure of stars because it provides resistance against gravitational collapse. Consider a static massive sphere held together by gravity. If a volume element of gas is to be held mechanically at a certain position in a star, neither being expelled outward by pressure nor falling to the center of gravitational attraction, then it will be necessary for the pressure and gravity forces on a volume element to sum to zero. This condition is called *hydrostatic equilibrium*. The balance in the case of a spherical gas cloud like the one illustrated in Fig. 2-14 leads to a simple differential equation for the pressure gradient. Consider the small cylindrical volume element with axis of length dr parallel to the radius vector at the point r and having a cross-sectional area equal to dA . Let dP be the pressure increment associated with dr . Then the radial force on this volume element due to the pressure differential is

$$\omega = P dA - (P + dP) dA = -dP dA \quad (2-177)$$

Since the pressure will actually decrease in the radial direction, the differential will be negative, and the pressure force will then be positive. If the volume element is not to be accelerated upward by F_P , it is necessary that it be exactly cancelled by the central gravitational force on the volume element. If dm is the mass of the volume element and $M(r)$ is the mass interior to the spherical surface radius r , the gravitational force is

$$= -G \frac{M(r) dm}{r^2}$$

or a thorough discussion see W. A. Fowler and F. Hoyle, *Astrophys. J. Suppl.*, **9**:201 (1964).

whoro

$$M(r) = \int_0^r \rho(r) 4\pi r^2 dr \quad (2-17q)$$

The gravitational constant $G = 6.670 \times 10^{-8}$ dyne $\text{cm}^2 \text{ g}^{-2}$. In writing these equations, we have clearly considered only the spherically symmetric problem. The important perturbation of rotation is discussed in Chap. 6. Because the mass of the small volume element is $\rho dA dr = dm$, we have

$$0 = F_P + F_G = - \frac{dP}{dr} - G \frac{\rho M}{r^2} \quad (2-180)$$

This force balance, together with the definition of Eq. (2-179), constitutes one of the most important conditions to be satisfied by static stellar structures. It applies equally well to the quiescent stages of contraction and expansion that normally occur during a stellar lifetime. These slow changes in stellar structures require times of the order of 10^6 years for their accomplishment, whereas any violations of Eq. (2-180) would cause sizable changes of structure to occur in a matter of hours. Only in highly dynamic situations is this condition violated, the most common examples being pulsating stars and exploding stars. If

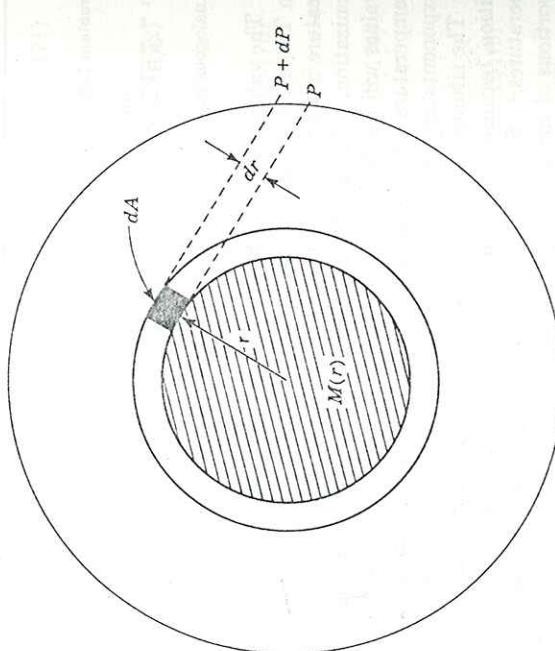


Fig. 2.14 In the hydrostatic equilibrium of a spherical body under self-gravitation, the weight of a small mass element is counterbalanced by the excess pressure on its lower surface.

very rapid adjustments of the structure are to be made, the inertial force resulting from the acceleration term must be added to the force balance. The result ; rather clearly

$$\frac{P}{r} = -\rho \frac{GM}{r^2} - \rho \frac{d^2r}{dr^2} \quad (2-181)$$

ince GM/r^2 is the local acceleration of gravity $g(r)$ at the interior point, it is clear that the last term will be negligible unless the mass acceleration d^2r/dr^2 is a significant fraction of g . Such a situation would be tantamount to free fall or explosion; in the vast majority of the less dramatic cases, Eq. (2-180) holds with high accuracy.

Even a casual examination of the condition of hydrostatic equilibrium reveals that very high central pressures will be demanded to support objects as massive as characteristic stars. In fact, we may estimate the order of magnitude of the sun's central pressure simply from the observed number $\mathfrak{M}_\odot = 2 \times 10^{33}$ g and $r_\odot = 7 \times 10^{10}$ cm. We can consider a point at half the solar radius and assume that (1) the pressure gradient is roughly the central pressure divided by the radius, (2) the density is roughly the average density of the sun, and (3) the interior mass is roughly one-half a solar mass. Making these insertions, we find that the central pressure is to order of magnitude given by

$$P_\odot \approx \frac{6}{4\pi} \frac{G\mathfrak{M}_\odot^2}{R_\odot^4} = 5 \times 10^{15} \text{ dynes/cm}^2 = 5 \times 10^9 \text{ atm} \quad (2-182)$$

Problem 2-40: Perform the steps leading to Eq. (2-182) and make the numerical substitutions.

The rough substitutions made for the properties of the median point of the sun may not appeal to some readers; however, the following inequality, due to Milne, places a lower limit on the central pressure of a star. Equations (2-180) and (2-179) may be combined to

$$\frac{P}{r} = -\frac{GM}{4\pi r^4} \frac{dM}{dr} \quad (2-183)$$

ince

$$\left(P + \frac{GM^2}{8\pi r^4} \right) = \frac{dP}{dr} + \frac{GM}{4\pi r^4} \frac{dM}{dr} - \frac{GM^2}{2\pi r^5} \quad (2-184)$$

he cancellation by Eq. (2-183) of the first two terms on the right-hand side shows that

$$\left(P + \frac{GM^2}{8\pi r^4} \right) < 0 \quad (2-185)$$

nce this function in parentheses decreases with increasing r , it must be greater ; the center, where its value is P_\odot , than it is at the radius of the star, where its

value is $G\mathfrak{M}^2/8\pi R^4$, where \mathfrak{M} is the total mass. Thus

$$P_\odot > \frac{G\mathfrak{M}^2}{8\pi R^4} = 4.4 \times 10^{14} \left(\frac{\mathfrak{M}}{\mathfrak{M}_\odot} \right)^2 \left(\frac{R_\odot}{R} \right)^4 \text{ dynes/cm}^2 \quad (2-186)$$

This lower limit is a full order of magnitude less than the more realistic estimate in Eq. (2-182), but the lower limit has the advantage of demonstrating rigorously that very high central pressures in terms of terrestrial standards are required. It is also physically interesting to note that the pressure estimated in Eq. (2-182) can be generated only if the temperature is around 10^7 °K for a gas with an average density near that of the sun. The point of this discussion is that *the elementary demand for hydrostatic equilibrium sets at once the order of magnitude of the physical variables in a stellar interior.*

At various stages in their lifetimes stars undergo expansions and contractions. For instance, the condition of hydrostatic equilibrium sets in long before a newly formed star has shrunk to its eventual main-sequence radius. It must undergo a rather extended period of slow contraction before settling into its static configuration. Or, as another example, when the necessary amount of hydrogen has been exhausted from a main-sequence core, it once again contracts gravitationally whereas the outer regions expand. It is of interest, therefore, to examine the thermodynamics of the simplest type of contraction, that of a *uniform contraction*. Chandrasekhar has stated the definition of such a process in the following words: "An expansion or contraction of a spherical distribution of matter is said to be uniform if the distance between any two points is altered in the same way as the radius of the configuration."¹¹ That is, if the subscripts f and i are associated with the final and initial configurations, whose *corresponding points* are related by

$$r_f = yr_i \quad dV_f = y^3 dV_i \quad (2-187)$$

then y is to be for a uniform contraction a constant scale factor that represents the ratio of the size of the final sphere to the size of the initial sphere. The final density is related to the initial density by

$$\rho_f = y^{-3} \rho_i \quad (2-188)$$

The condition of hydrostatic equilibrium is maintained throughout, so that

$$\frac{dP_f}{dr_f} = -\frac{GM_f}{r_f^2} \rho_f \quad \frac{dP_i}{dr_i} = -\frac{GM_i}{r_i^2} \rho_i \quad (2-189)$$

If these equations refer to *corresponding points*, as defined in Eq. (2-187), the interior masses will be equal: $M_f = M_i$. Then

$$\begin{aligned} dP_f &= -\frac{GM_f}{r_f^2} \rho_f dr_f = -\frac{GM_i}{(yr_i)^2} y^{-3} \rho_i y dr_i = y^{-4} \frac{GM_i}{r_i^2} \rho_i dr_i \\ &= y^{-4} dP_i \end{aligned} \quad (2-190)$$

¹¹ *Op. cit.*, p. 45.

Since Eq. (2-190) applies to the pressure difference between corresponding points, it applies equally well to the pressures themselves, so that

$$P_f = y^{-4} P_i \quad (2-191)$$

Problem 2-41: Show that a uniform contraction of a perfect nondegenerate gas results in the following ratios for the physical variables:

$$\frac{P_f}{P_i} = \left(\frac{R_i}{R_f} \right)^4 \quad \frac{\rho_f}{\rho_i} = \left(\frac{R_i}{R_f} \right)^3 \quad \frac{T_f}{T_i} = \frac{R_i}{R_f} \quad (2-192)$$

To the extent that contractions of real stars resemble uniform contractions of a perfect gas, the pressure rises much more rapidly than the temperature.

THE VIRIAL THEOREM

The virial theorem is a statistical statement about mutually interacting particles. Consider a general system of mass points m_i with positions \mathbf{r}_i under influence of the force \mathbf{F}_i . Then the equations of motion are

$$\frac{d}{dt} \mathbf{p}_i = \mathbf{F}_i \quad (2-193)$$

Consider the quantity $\sum_i \mathbf{p}_i \cdot \mathbf{r}_i$, where the sum is over all the particles of the system. The time derivative of the sum is

$$\frac{d}{dt} \sum_i \mathbf{p}_i \cdot \mathbf{r}_i = \sum_i \frac{d\mathbf{p}_i}{dt} \cdot \mathbf{r}_i + \sum_i \mathbf{p}_i \cdot \frac{d\mathbf{r}_i}{dt} \quad (2-194)$$

The second term in a nonrelativistic case is just $\sum m_i v_i^2 = 2K$, where K is the total kinetic energy of the assembly of particles. The left-hand side may be rewritten as

$$\frac{d}{dt} \sum_i m_i \frac{d\mathbf{r}_i}{dt} \cdot \mathbf{r}_i = \frac{d}{dt} \sum_i \frac{1}{2} \frac{d}{dt} (m_i v_i^2) = \frac{1}{2} \frac{d^2 I}{dt^2} \quad (2-195)$$

where $I = \sum_i m_i v_i^2$ is the *spherical moment of inertia*. By taking these features into account, Eq. (2-194) may be written as

$$\frac{1}{2} \frac{d^2 I}{dt^2} = 2K + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \quad (2-196)$$

The sum in Eq. (2-196) is called the *virial of Clausius*. It includes only the long-range field forces and the external forces, for all *collisions at a point* contribute two terms whose sum is zero. For static configurations, furthermore, the moment of inertia is constant, giving

$$K = -\frac{1}{2} \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \quad \text{static} \quad (2-197)$$

As a first application of this important theorem, consider 1 g of gas contained

under pressure in a box of volume V . The forces in the virial fall into two categories, the external pressure force at the surface of the container and the interparticle forces. For the external pressure force we have

$$\sum_{\text{pressure}} \mathbf{F}_i \cdot \mathbf{r}_i = \int_{\text{surface}} (-P) dS \cdot \mathbf{r} = -P \int_S \mathbf{r} \cdot \mathbf{n} dS \quad (2-198)$$

where \mathbf{n} is a unit vector normal to the surface. By Gauss' theorem this integral becomes

$$-P \int_V \nabla \cdot \mathbf{r} dV = -3PV \quad (2-199)$$

The interparticle forces on the other hand may be thought of as occurring in pairs:

$$\sum_i \mathbf{F}_i \cdot \mathbf{r}_i = \sum_{\text{pairs}} \mathbf{F}_{ij} \cdot \mathbf{r}_i + \mathbf{F}_{ji} \cdot \mathbf{r}_j \quad (2-200)$$

where $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ is defined as the force on particle i due to particle j . Reassembling the virial equation, we have

$$K = \frac{3}{2}PV - \frac{1}{2} \sum_{\text{pairs}} \mathbf{F}_{ij} \cdot (\mathbf{r}_i - \mathbf{r}_j) \quad (2-201)$$

Consider first the case of the perfect gas, where the interparticle forces are zero by definition. The kinetic energy K is in the nondegenerate case the product of $\frac{3}{2}kT$ times the number of particles. Thus with N representing the number of free particles per unit volume, we have

$$P = NkT \quad (2-202)$$

in keeping with Eq. (2-7).

In the case of a star, the situation is different. The gas experiences two forces. The virial contains electric forces from coulomb collisions between the charged particles of the ionized gas as well as gravitational forces. Although electric forces are intrinsically much stronger than gravitational forces, their contribution to the virial is generally quite small. The strong electric forces maintain charge neutrality in the ionized gas, so that there is no net electric force from the bulk of the star. Only in close scattering events do the particles experience unbalanced electric forces, and in those cases the contribution to the virial consists of two nearly equal and opposite terms. There will exist a weak radial electric field that prohibits the positive nuclei from falling to the center and leaving the electrons behind, but this field exerts equal and opposite forces on the electrons and nuclei within any small volume element dV and hence does not contribute to the virial. So let us temporarily ignore the coulomb force, a subject to which we shall return for closer analysis in Sec. 2-3. For a perfect-gas star, the only forces on the particles are gravitational, and explicitly

$$\mathbf{F}_{ij} = -\frac{Gm_i m_j}{(r_{ij})^3} (\mathbf{r}_i - \mathbf{r}_j) \quad (2-203)$$

The static virial theorem becomes in this case

$$K = -\frac{1}{2} \sum_{\text{pairs}} \mathbf{F}_{ij} \cdot (\mathbf{r}_i - \mathbf{r}_j) = \frac{1}{2} \sum_{\text{pairs}} \frac{Gm_i m_j}{r_{ij}} \quad (2-204)$$

Each term in the sum equals the negative of the potential energy due to the interaction of m_i with m_j . When summed over all pairs,

$$K = -\frac{\Omega}{2} \quad (2-205)$$

where Ω is the total potential energy of the star. This well-known result of the inverse-square force is itself often referred to as the *virial theorem*. Because the kinetic energy of nondegenerate particles in a star is measured by the average temperature, Eq. (2-205) has the very important consequence that the temperature of a stellar interior rises as the star contracts gravitationally.

Problem 2-42: Consider the Bohr model of the hydrogen atom with an electron circling a proton. Show that Eq. (2-205) is satisfied.

Notice that K was defined to be the translational kinetic energy of particles and does not include the energy of internal degrees of freedom (rotational, vibrational, or excitation energy) or the kinetic energy of trapped photons. For the sake of illustrating an important principle in a simple way, consider a perfect gas characterized by a uniform value of the adiabatic exponent γ . It follows that or adiabatic expansions

$$\Omega U = -P dV \quad (2-206)$$

$$\frac{\Omega T}{T} + (\gamma - 1) \frac{dV}{V} = 0 \quad (2-206)$$

the second condition coming from Eq. (2-115). By elimination of dV we have

$$\begin{aligned} \Omega U &= \frac{PV}{T} \frac{dT}{\gamma - 1} \\ &= \frac{N_0 k}{\mu} \frac{dT}{\gamma - 1} = \frac{2}{3} \frac{dK}{\gamma - 1} \end{aligned} \quad (2-207)$$

The assumption of uniform γ allows Eq. (2-207) to be integrated over the entire star:

$$\Omega = \frac{3}{2}(\gamma - 1)U \quad (2-208)$$

Let us define an energy E as the sum of the gravitational energy Ω and the internal energy U . Then by the virial theorem

$$\begin{aligned} \gamma &= U + \Omega = U - 2K \\ &= -(3\gamma - 4)U = \frac{3\gamma - 4}{3(\gamma - 1)}\Omega \end{aligned} \quad (2-209)$$

The energy E contains all the particle energy but does not include photons. Some straightforward implications of Eq. (2-209) follow.

(1) For the perfect monatomic gas $\gamma = \frac{5}{3}$ and $E = -U = \Omega/2$. The total energy (excluding that in photons) is negative and equal to half of the gravitational binding energy. But the total amount of energy must be conserved. Thus, as a star shrinks, one half of the binding energy appears as thermal motion whereas the other half must go into the production of radiation, most of which is usually lost into space. This reasoning lay behind the discussion in Chap. 1 of the inadequacy of gravity as a source of energy for the full lifetimes of stars. This result has great significance for stellar evolution, however. Static stellar structures are those in which the luminosity is replenished by energy-liberating nuclear reactions in the interior, but whenever a given nuclear fuel is exhausted in the interior, a gravitational contraction ensues to replace the energy radiated from the star. This contraction increases the magnitude of Ω , one-half of which appears as additional kinetic energy; i.e., the *temperature rises*. This rise continues until the temperature becomes sufficient to ignite the next source of nuclear fuel, which usually halts the contraction. Thus it is that gravitation provides the driving source which compels the stars to evolve through a sequence of nuclear burning phases.

(2) The energy E changes from negative to positive as γ decreases through a value $\gamma = \frac{4}{3}$. If the sum of internal and gravitational energies is positive, the mass can fly apart. This result would be contradictory to the initial assumption of the virial theorem that the mass is static $\left(\frac{1}{2} \frac{d^2 I}{dt^2} = 0\right)$, and hence that assumption must itself be incorrect for $\gamma \leq \frac{4}{3}$. Thus dynamic radial instabilities occur when γ reaches as low as $\frac{4}{3}$. In real stars, the constant γ must be generalized to include effects of radiation pressure and ionization. When that generalized γ has an average value over the star that is less than $\frac{4}{3}$, radial instabilities ensue.¹ Since $\gamma = \frac{4}{3}$ for a photon gas, we note that a star internally dominated by radiation pressure will have a relatively "loose" structure. Effects of ionization can cause γ to drop well below $\frac{4}{3}$ in the regions of ionization.

Problem 2-43: The gravitational binding of the sun is of the order

$$\Omega \approx -\frac{GM^2}{R} = -4 \times 10^{38} \text{ ergs}$$

Since the sun is predominately a perfect gas, the thermal kinetic energy is about $K \approx 2 \times 10^{38}$ ergs by the virial theorem. If the internal temperature of the sun were $10^7 (1 - r/R)^2$ K,

¹ An introduction to stellar stability is contained in the discussion of pulsation in Chap. 6. The principles of stability are complicated in detail, however, and a full discussion cannot be included in this book. Fortunately there are monographs that expertly summarize the current state of knowledge. See particularly P. Ledoux, *Stellar Stability*, in L. H. Aller and D. B. McLaughlin (eds.), "Stellar Structure," The University of Chicago Press, Chicago, 1965, and other papers cited in that review.

what is the total trapped energy in the radiation field? Apparently most of the radiant energy escaped during the contraction.

The virial theorem can be approached in a different way that employs only macroscopic quantities. The assumption that the star is static, which is necessary to draw any useful information from the virial theorem, is expressible by the condition of hydrostatic equilibrium:

$$\frac{dP}{dr} = -\rho \frac{GM(r)}{r^2} \quad (2-210)$$

Multiply both sides of this equation by

$$V(r) dr = \frac{4}{3}\pi r^3 dr \quad (2-211)$$

Then one obtains

$$V(r) dP = -\frac{1}{3}4\pi r^3 \rho dr \frac{GM(r)}{r} \quad (2-212)$$

When this equation is integrated over the star, the left-hand side becomes

$$\int V dP = PV \Big|_0^R - \int P dV$$

In an idealized star, the pressure goes to zero near the surface $r = R$, whereas the volume vanishes near the center. The right-hand side of Eq. (2-212), on the other hand, is just one-third the gravitational potential energy. Thus we have

$$-3 \int P dV = \Omega \quad (2-213)$$

The result is somewhat more general than the previous treatment because no assumption has been made about the nature of the interior particles. However the earlier results are easily recovered as special cases. For the perfect gas

$$P = \begin{cases} \frac{2}{3} \left(\frac{K}{V} \right) & \text{nonrelativistic} \\ \frac{1}{3} \left(\frac{K}{V} \right) & \text{relativistic} \end{cases} \quad (2-214)$$

which gives immediately the earlier result

$$\Omega = \begin{cases} -2K & \text{nonrelativistic} \\ -K & \text{relativistic} \end{cases} \quad (2-215)$$

whereas the total energy of a monatomic gas

$$E = \begin{cases} \Omega + K = \frac{\Omega}{2} & \text{nonrelativistic} \\ 0 & \text{relativistic} \end{cases} \quad (2-216)$$

As a final application of the virial theorem we shall consider what effect the coulomb interactions of the ionized gas are likely to have on the value of the pressure. To do so, return to Eq. (2-201) describing the nonrelativistic gas in a container. In this case we ignore the gravitational interaction in the sum over pairs. Because the coulomb force between pairs is an inverse-square force, the sum is easily related to the coulomb potential energy.

Problem 2-44: Show that for the ionized nonrelativistic gas in a box

$$K = \frac{3}{2}PV - \frac{1}{2}U_e \quad (2-217)$$

where U_e is the coulomb potential energy.

From Eq. (2-217) we see that the pressure on the walls is

$$P = \frac{2}{3} \frac{K}{V} + \frac{1}{3} \frac{U_e}{V} \quad (2-218)$$

In the nondegenerate case

$$P = NkT + \frac{1}{3} \frac{U_e}{V} \quad (2-219)$$

where N is the number of free particles per unit volume. The interesting point is that there is an additional *coulomb pressure* equal to one-third of the coulomb energy density. The reason that this term has been neglected in the discussion of the virial theorem is that it was argued to be small. If the positive charges are, on the average, equidistant from both positive and negative charges, the net coulomb energy is zero. The gravitational energy dominates in the star because all the individual interactions occur with the same sign, whereas the individually much greater coulomb interactions cancel out because of charge neutrality. The coulomb interactions do play a significant role in some phenomena, however, because the plasma polarizes; i.e., the electrons cluster near the ions to some extent. A discussion of this phenomenon follows.

2-3 THE IONIZED REAL GAS

Gases having interactions between constituent particles are called *real gases*. The name reflects the fact that nature always provides some interaction, and for the dynamics of the gas the relevant question is the relative magnitude of the average interaction and the average kinetic energy. In an ionized gas the strongest forces are the coulomb forces between the charges. Much less important are the interactions between magnetic moments and the nuclear-force interactions. Although the latter is quite strong, the particles are seldom within the range of the nuclear force except at nuclear density ($\rho > 10^{14} \text{ g/cm}^3$). For the stellar interior, the important problem is to ascertain which properties of the real gas may introduce observable consequences into astrophysics. The first of these is that the particle pressure deviates from that of an ideal gas, the

size of the effect depending upon the coulomb energy density. A second important consequence is that the energetics of atomic and nuclear reactions is modified, the most important application being to the ionization equilibrium. And third, the transport phenomena in the ionized gas are influences by collective phenomena of the plasma.

In the presence of forces, the internal energy of a monatomic gas must include the potential energy of the interactions

$$U_{ad} = \sum \frac{p^2}{2m} + \Phi \quad (2-220)$$

where Φ is the potential energy. Because Φ depends upon the average interparticle distance, it is a density-dependent quantity. The pressure is given by the change in internal energy associated with adiabatic compression,

$$U_{ad} = -P dV \quad (2-221)$$

if the internal energy is density-dependent because of the interactions, a corresponding pressure results.

All the important effects to be discussed occur because the charge density is made nonuniform by the coulomb interactions. To have a concise notation for the composition of the ionized gas, let \bar{n}_z represent the average number density of each species of charge Z , with $Z = -1$ corresponding to the electron density, because the gas is macroscopically neutral,

$$Z\bar{n}_z = 0 \quad (2-222)$$

To calculate the electrostatic energy of the gas one must know the electrostatic potential ϕ_z at each charge Z due to all the other particles of the gas. Then the coulomb energy per unit volume is

$$U = \frac{1}{2} \sum_z eZ\bar{n}_z \phi_z \quad (2-223)$$

In the material to follow we shall estimate the important effects for two limiting cases of physical interest, the nearly perfect gas and the zero-temperature gas.

E NEARLY PERFECT GAS AT LOW DENSITY

The gas may be called *nearly perfect* if the coulomb energy between particles is much smaller than their thermal energy. If the ions have charge Z , the coulomb energy between neighbors is of order $(Ze)^2/r$, where $r \approx \bar{n}_z^{-1}$ is the average separation of the ions. For this coulomb energy to be small compared to kT requires that

$$\ll \left(\frac{kT}{Ze^2} \right)^3 \quad (2-224)$$

Fig. 2-15 The electrostatic potential V_i as a function of the distance from the ion of charge Z_i . This potential drops more rapidly than the coulomb potential Ze/r because of electron clustering near the ion.

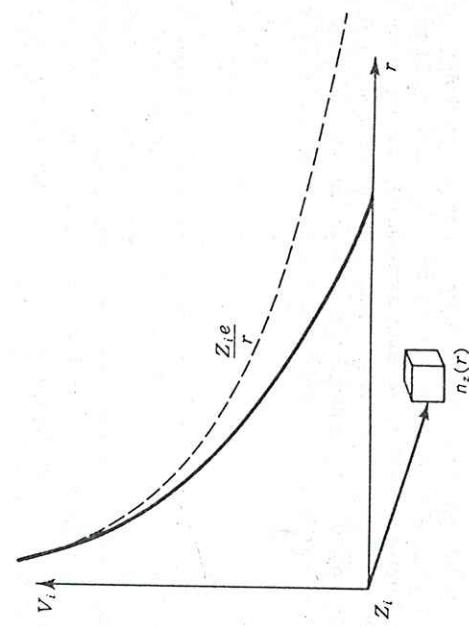


Fig. 2-15 The electrostatic potential V_i as a function of the distance from the ion of charge Z_i . This potential drops more rapidly than the coulomb potential Ze/r because of electron clustering near the ion.

Consistent with the assumption that the coulomb energy is small compared to kT is the assumption that each density may be expanded:

$$n_Z = \bar{n}_Z \exp - \frac{ZeV_i}{kT} \approx \bar{n}_Z \left(1 - \frac{ZeV_i}{kT} + \dots \right) \quad (2-227)$$

Substitution of this approximation in Poisson's equation gives

$$\nabla^2 V_i \approx -4\pi e \sum Z \bar{n}_Z \left(1 - \frac{ZeV_i}{kT} \right) = + \frac{4\pi e^2}{kT} \sum Z^2 \bar{n}_Z V_i \quad (2-228)$$

the last step resulting from the charge-neutrality condition.

The equation for the potential in the ion sphere may now be written:

$$\nabla^2 V_i - \kappa^2 V_i = 0 \quad (2-229)$$

where κ is an inverse length defined by

$$\kappa^2 = \frac{4\pi e^2}{kT} \sum Z^2 \bar{n}_Z \quad (2-230)$$

Problem 2-45: Show that the spherically symmetric solution of Eq. (2-229) is

$$V_i = \text{const } e^{-\kappa r} / r \quad (2-231)$$

From the result of the preceding problem and the fact that $V_i \rightarrow eZ_i/r$ as $r \rightarrow 0$, the potential in the ion sphere must be

$$V_i = \frac{eZ_i e^{-\kappa r}}{r} \quad (2-232)$$

Thus the field drops off rapidly for distances $r > 1/\kappa$. The length $1/\kappa = R_D$ is called the *Debye-Hückel radius* and is a measure of the size of the ion cloud. The condition of weak interaction energy (compared to kT) can easily be seen to be equivalent to the assumption that $1/\kappa$ is much greater than the average distance between neighboring particles.

The potential at Z_i due to all other charges in its ion cloud may be obtained by expanding Eq. (2-232) very near to Z_i :

$$V_i(r) = \frac{eZ_i}{r} - eZ_i \kappa + \dots \quad (2-233)$$

where the higher-order terms in the expansion vanish as $r \rightarrow 0$. The first term is the coulomb field of the ion itself, whereas the second must be the potential at the ion due to the other charges. It is perhaps instructive to notice that the potential at the ion (but not elsewhere) is the same as if the charge Z_i were surrounded by a charge $-Z_i$ spread over a spherical shell at the Debye-Hückel radius. The potential at any charge Z due to all other charges is then

$$\phi_Z \approx -eZ\kappa = -\frac{eZ}{R_D} \quad (2-234)$$

where

$$R_D = \left(\frac{kT}{4\pi e^2 \sum Z^2 \bar{n}_Z} \right)^{\frac{1}{2}} \quad (2-235)$$

Problem 2-46: In the sum $\sum Z^2 \bar{n}_Z$ the electrons are represented by $Z = -1$. Show with aid of the charge-neutrality condition that the sum may be expressed as a sum over positive ions only as

$$\sum Z^2 \bar{n}_Z = \sum_{+Z} (Z^2 + Z) \bar{n}_Z = \sum_{+Z} (Z^2 + Z) \frac{\rho X_Z}{A_Z} N_0 \quad (2-236)$$

It will frequently be convenient to define a quantity dependent only upon the composition by

$$\xi = \sum_{+Z} (Z^2 + Z) \frac{X_Z}{A_Z} \quad (2-237)$$

whereupon

$$R_D = \left(\frac{kT}{4\pi e^2 \rho N_0 \xi} \right)^{\frac{1}{2}} \quad (2-238)$$

The coulomb potential energy per unit volume is immediately obtainable by inserting Eq. (2-234) into Eq. (2-223):

$$\left(\frac{U}{V} \right)_c = -e^3 \left(\frac{\pi}{kT} \right)^{\frac{1}{2}} \left(\sum \bar{n}_Z Z^2 \right)^{\frac{1}{2}} = -e^3 \left(\frac{\pi}{kT} \right)^{\frac{1}{2}} (\rho N_0 \xi)^{\frac{1}{2}} \quad (2-239)$$

With the aid of these results it becomes possible to discuss two rather complex features of the nearly perfect ionized gas, the pressure due to the coulomb energy and the effective ionization potential. Although the phenomena themselves are general, the correctness of the subsequent results is limited to the temperature-density domain for which the Debye-Hückel treatment is valid. Since the kinetic energy per unit volume is just $\frac{3}{2}kT$ times the number of particles per unit volume, the Debye-Hückel treatment is valid only if $(\rho N_0 / \mu) \frac{3}{2}kT$ is much greater than Eq. (2-239).

Coulomb pressure In analyzing thermodynamic changes of state one considers a specific quantity of gas. If that quantity is chosen to be 1 g, its internal energy U is the internal energy density times the volume of 1 g ($V = 1/\rho$). The coulomb energy per gram is then

$$U_c = -e^3 \left(\frac{\pi}{kT V} \right)^{\frac{1}{2}} (N_0 \xi)^{\frac{1}{2}} = \frac{B}{(V T)^{\frac{1}{2}}} \quad (2-240)$$

By so defining a constant B the dependence $U_c \propto (V T)^{-\frac{1}{2}} = (\rho / T)^{\frac{1}{2}}$ is simple and explicit. Furthermore, let the number of particles in 1 g of gas be represented simply by N . Then the internal energy and pressure can be simply

expressed as the corresponding quantities for the ideal gas plus the corrections necessitated by the coulomb imperfection:

$$U = \frac{3}{2} N k T + U_e, \quad P = \frac{N}{V} k T + P_e. \quad (2-241)$$

The differential of the entropy is

$$dS = \frac{1}{T} \left[\left(\frac{\partial U}{\partial V} \right)_T + P \right] dV + \frac{1}{T} \left(\frac{\partial U}{\partial T} \right)_V dT \quad (2-242)$$

where the partial derivatives are

$$\begin{aligned} \left(\frac{\partial U}{\partial V} \right)_T &= \left(\frac{\partial U_e}{\partial V} \right)_T = -\frac{1}{2} \frac{B}{V^4 T^4} \\ \left(\frac{\partial U}{\partial T} \right)_V &= \frac{3}{2} N k - \frac{1}{2} \frac{B}{V^4 T^4} \end{aligned} \quad (2-243)$$

Insertion of these values gives

$$dS = \frac{1}{T} \left(-\frac{1}{2} \frac{B}{V^4 T^4} + \frac{N}{V} k T + P_e \right) dV + \frac{1}{T} \left(\frac{3}{2} N k - \frac{1}{2} \frac{B}{V^4 T^4} \right) dT \quad (2-244)$$

Problem 2-47: Show that the integrability condition on dS demands that

$$\frac{\partial}{\partial T} \frac{P_e}{T} = -\frac{1}{2} \frac{B}{V^4 T^4} \quad (2-245)$$

The integral of Eq. (2-245) is

$$\frac{P_e}{T} = \frac{1}{3} \frac{B}{V^4 T^4} + f(V) \quad (2-246)$$

where $f(V)$ is some arbitrary function of V . But since the coulomb pressure must vanish as $T \rightarrow \infty$, the function $f(V) = 0$. By reincorporating the definition of B it follows that

$$P_e = \frac{1}{3} \left(\frac{U}{V} \right)_e = -\frac{e^3}{3} \left(\frac{\pi}{k T} \right)^4 (\rho N_0 \xi)^4 \quad (2-247)$$

The coulomb pressure is negative, corresponding to the fact that compression of the charges performs work rather than requires work. This calculation has confirmed Eq. (2-219), which was derived from the virial theorem.

The total pressure is then

$$P = \frac{N k}{\mu} \rho T - \frac{e^3}{3} \left(\frac{\pi}{k T} \right)^4 (\rho N_0 \xi)^4 \quad (2-248)$$

This result can be valid only if the second term is a small correction to the perfect-gas pressure, as can easily be seen by the following consideration. The coulomb pressure is one-third of the coulomb-energy density, whereas the perfect

gas pressure is two-thirds of the kinetic-energy density; but the Debye-Hückel treatment is valid only if the coulomb-energy density is much less than the kinetic-energy density. Quantitatively the criterion demands

$$\begin{aligned} U_e &\ll U_{\text{kin}} \\ e^3 \left(\frac{\pi}{k T} \right)^4 (\rho N_0 \xi)^4 &\ll \frac{3}{2} \frac{N_0 k}{\mu} \rho T \\ T &\gg 10^6 \rho^{\frac{1}{3}} \end{aligned} \quad (2-249)$$

This condition will not be satisfied in major portions of many stars, so that Eq. (2-247) for the coulomb pressure should be used with caution.

Problem 2-48: Compute the coulomb pressure and the perfect-gas pressure for a helium gas at $T = 10^6$ °K and $\rho = 10^{-2}$ g/cm³.
Ans: $P_e = -4.8 \times 10^8$ dynes/cm², $P_g = 6.3 \times 10^{11}$ dynes/cm².

Depression of the continuum and effective ionization potential Interactions between gas particles alter the energetics of the gas. The ionization of an ion in a perfect gas is no different from the ionization in a vacuum, but in a real ionized gas the potential energy of both the bound and free electrons is altered by the coulomb energy. In the present section these effects will be discussed within the framework of the Debye-Hückel model.

From Eq. (2-233) the average potential around each electron is

$$V_e(r) = -\frac{e}{r} + e_k - \dots \quad (2-250)$$

so that the potential energy of each free electron due to interactions with other charges is

$$(PE)_e = -e^2 e_k = -\frac{e^2}{R_D} \quad (2-251)$$

This negative energy reflects the fact that the free electron is actually bound to the plasma as a whole, even though it is free to navigate through the plasma. The ionization of an atom requires sufficient energy to liberate the electron to the continuum of states with zero kinetic energy. But the energy of a zero-kinetic-energy electron is $E = -e^2/R_D$, and one commonly says that the continuum of states (as opposed to discrete bound states) has been depressed by an amount $E_0 = e^2/R_D$, and the energy required to ionize a bound electron is accordingly reduced.

The energy required for ionization is further reduced by the fact that the energy of the bound electron state is also changed and in the direction such that the binding energy is reduced. This effect comes about because the electron moves in a shielded potential rather than a pure $1/r$ potential. Consider a single bound electron. Since it is sensible to assume that the radius of the ground-state

orbital is much less than R_D , the electron bound to charge Z moves in a potential that is approximated by

$$V(r) \approx \frac{Ze}{r} - \frac{(Z-1)e}{R_D} \quad (2-252)$$

The first term is just the $1/r$ potential from the charge Z that provides the normal binding energy for the atom, whereas the second term is the potential due to the Debye sphere surrounding the ion of charge $Z-1$. The potential energy of the

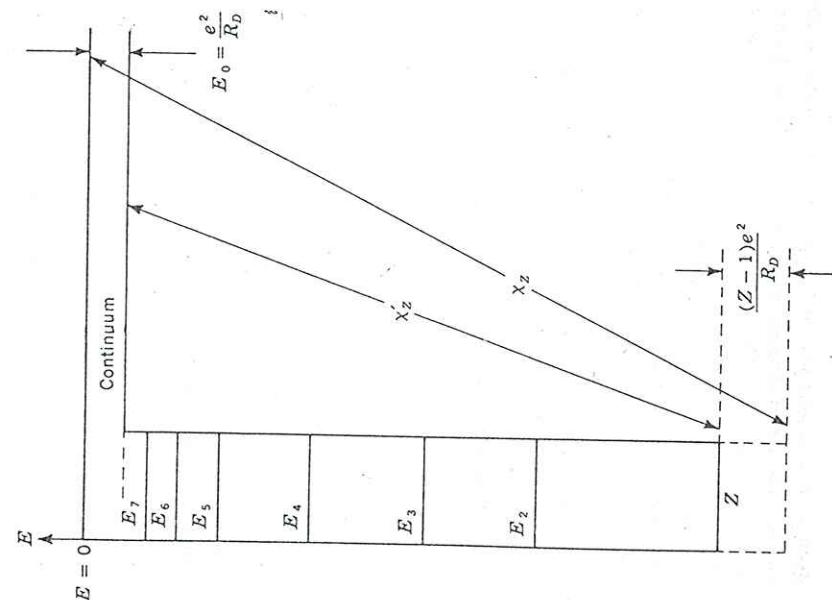


Fig. 2-16 The effective ionization potential x'_z . The laboratory ionization potential is reduced by coulomb interactions in the dense plasma. The energy of continuum electrons is lowered by e^2/R_D , whereas the energy of a tightly bound electron with Bohr radius $a \ll R_D$ is raised by $(Z-1)e^2/R_D$. Thus the ionization potential is effectively reduced by the amount Ze^2/R_D .

bound electron is

$$(PE)_e \approx -\frac{Ze^2}{a} + \frac{(Z-1)e^2}{R_D} \quad (2-253)$$

where a is the radius of the orbital. (For a hydrogenlike ion $a = a_0/Z$.) From Eq. (2-253) it can be seen that, to first approximation, the ground-state energy is elevated by an amount $(Z-1)e^2/R_D$. Figure 2-16 shows the energy-level diagram of an ion, where x_z is the laboratory ionization potential and x'_z is the effective ionization potential in the plasma. In the calculation of the ionization equilibrium, therefore, the laboratory ionization potential must be replaced by an *effective ionization potential* whose value is

$$x'_z = x_z - \frac{Ze^2}{R_D} \quad (2-254)$$

The excited bound states are also shifted upward in energy, so that some of the states that are bound in the laboratory move up into the continuum of states in a dense plasma. This phenomenon resolves in a natural way the paradox mentioned in Chap. 1, *viz.*, that the partition functions appear to be infinite. In the crudest approximation, one would simply regard all those states bound by less than Ze^2/R_D in the laboratory as being unbound in the star; and even though Ze^2/R_D might be a much smaller energy than x_z , that energy range is the one which contains the infinite number of bound states, so that its elimination not only makes the partition functions finite but often means that they can be estimated by the sum of only a few terms. In a more exact calculation, the excited-state energies will be increased by an amount different from that of the ground state because there are smaller energy shifts because of the differences in radii of the excited states. The present simplified discussion should be regarded only as an illustration of the dominant terms of a problem that is extremely complicated in its details.¹ Correction of the ionization potential is an important part of the calculation of the radiative opacity of the gas.

Problem 2-49: Estimate the partition function for a five-times-ionized carbon atom (hydrogen-like) if it is embedded in a hydrogen gas at $T = 5 \times 10^5$ °K and $\rho = 5 \times 10^{-3}$ g/cm³. In the simple approximation of the text, how much is the binding energy of that electron reduced from its normal value of 36 Ry?

Ans: $G_c = 2.6$, $\Delta x \approx -1.0$ Ry.

THE ZERO-TEMPERATURE IONIZED GAS

As the density of the gas increases, the coulomb energy increases, and one says that the gas becomes less perfect. At densities high enough for the electron gas to be degenerate, however, that rule of thumb becomes incorrect. A degenerate gas becomes more perfect as the density rises. This peculiarity comes about

¹J. C. Stewart and K. D. Pyatt, *Astrophys. J.*, 144:1203 (1966), and C. A. Rouse, *Phys. Rev.*, 159:41 (1967).

because the relevant energy for the gas becomes the Fermi energy E_f rather than the thermal energy kT . The perfect-gas condition $E_e \ll kT$ must be replaced by $E_e \ll E_f$, but because the Fermi energy increases as a higher power of the density than the coulomb energy, a nearly perfect degenerate gas is more nearly perfect at higher density. Quantitatively the argument goes as follows. If $\langle r \rangle$ is an appropriate average separation of electrons from the ions of charge Z , then the condition that the gas be perfect becomes

$$\frac{Ze^2}{\langle r \rangle} \ll E_f \quad (2-255)$$

or

$$Ze^2 n_e^{\frac{1}{3}} \ll \frac{\hbar^2}{2m} (3\pi^2 n_e)^{\frac{1}{3}} \quad (2-256)$$

This equation clearly becomes true for sufficiently large values of the electron density; in fact, for $n_e > 10^2 Z^3$, a condition that seems always satisfied in astrophysical degenerate electron gases. The consequence of this fact is that the electron gas is nearly uniform; i.e., the electronic charge density is, in first approximation, constant.

As was discussed earlier, as long as $E_f \gg kT$, the state of the electron gas is almost independent of T . The same cannot be said of the ions, because their energy is predominantly thermal. For arbitrary temperature, the configuration of the gas may be quite complicated, but it is not too difficult to see what must happen as the temperature is reduced to very low values. Assume for this discussion that the density is great enough so that the matter must remain *pressurized* even at zero temperature. As $T \rightarrow 0$ the thermal energy of the ions vanishes, so that they will assume a configuration that minimizes the potential energy. This stable configuration is such that the ions maximize the inter-ion separation and is more like a lattice than a gas. For small displacements of an ion from its equilibrium point in the lattice, there will be a restoring force resulting from an increase in the coulomb potential energy. The result is that each ion situated in a harmonic-oscillator potential. Now although a classical oscillator can have zero energy, a quantum oscillator cannot. It must oscillate with a zero-point energy equal to $\frac{3}{2}\hbar\omega$, where ω is the frequency of the oscillator ($\omega = \sqrt{k/m}$, here k is the force constant of the equivalent oscillator). Salpeter¹ and others have discussed this phenomenon and showed that the zero-point energy will be less than the coulomb energy holding the ions in the lattice structure and hence the lattice structure is correct at zero temperature. Although most interesting astrophysical environments are not strictly at zero temperature, it may turn out high density that the oscillator energy dominates kT , in which case a zero-temperature approximation is sensible. Because the major coulomb effects are easy to estimate in this limit, we now turn our attention to those effects in the zero-temperature ionized gas.

¹ E. Salpeter, *Astrophys. J.*, 134:669 (1961).

Draw an imaginary sphere of radius R_z around each ionized nucleus of charge Z such that the sphere contains Z free electrons which neutralize the entire sphere. That is, the gas is imagined as being divided into neutral spheres about each nucleus Z which contain the Z electrons closest to that nucleus. Assume, furthermore, that the Z electrons in each ion sphere are spread uniformly over the volume of the sphere and that the average electron density is unperturbed by the nuclear charges. (These assumptions are incorrect in detail, but as a simple approximation they allow an easy numerical estimate of the physical effects being considered. The nuclear charge will actually polarize the electron density to a degree dependent upon density and temperature.)

Since the ion spheres are spherically symmetric and neutral, the electrostatic potential energy of electrons in a given ion sphere can be calculated from the geometry of that sphere alone. And the first point to notice is that the free electrons do possess an average potential energy which is negative. Consider the total electrostatic energy of one sphere. It may be calculated as the sum of the potential energies from electron-electron interactions and of electron-nucleus interactions.

(1) *The electron-electron interaction:* To assemble the uniform sphere with total charge Z_e requires electrostatic energy equal to

$$U_{ee} = + \int_0^{R_z} \frac{q_r dq}{r} \quad (2-257)$$

as illustrated in Fig. 2-17, where

$$q_r = \left(\frac{r}{R_z}\right)^3 Z_e \quad dq = \frac{4\pi r^2 dr}{\frac{4}{3}\pi R_z^3} Z_e \quad (2-258)$$

The integration gives

$$U_{ee} = + (Ze)^2 \frac{3}{R_z^6} \int_0^{R_z} r^4 dr = \frac{3}{5} \frac{(Ze)^2}{R_z^5} \quad (2-259)$$

(2) *The electron-nucleus interaction:* To assemble the uniform sphere of charge Z_e about the nucleus Z requires electrostatic energy

$$U_{ez} = - Ze \int_0^{R_z} \frac{dq}{r} = - \frac{3}{2} \frac{(Ze)^2}{R_z^5} \quad (2-260)$$

Thus the total electrostatic energy of the ion sphere is

$$U = U_{ee} + U_{ez} = \left(\frac{3}{5} - \frac{3}{2}\right) \frac{(Ze)^2}{R_z^5} = - \frac{9}{10} \frac{(Ze)^2}{R_z^5} \quad (2-261)$$

Since this electrostatic energy is shared by Z electrons, an electron in a given ion

orbital is much less than R_D , the electron bound to charge Z moves in a potential that is approximated by

$$V(r) \approx \frac{Ze}{r} - \frac{(Z-1)e}{R_D} \quad (2-252)$$

The first term is just the $1/r$ potential from the charge Z that provides the normal binding energy for the atom, whereas the second term is the potential due to the Debye sphere surrounding the ion of charge $Z-1$. The potential energy of the

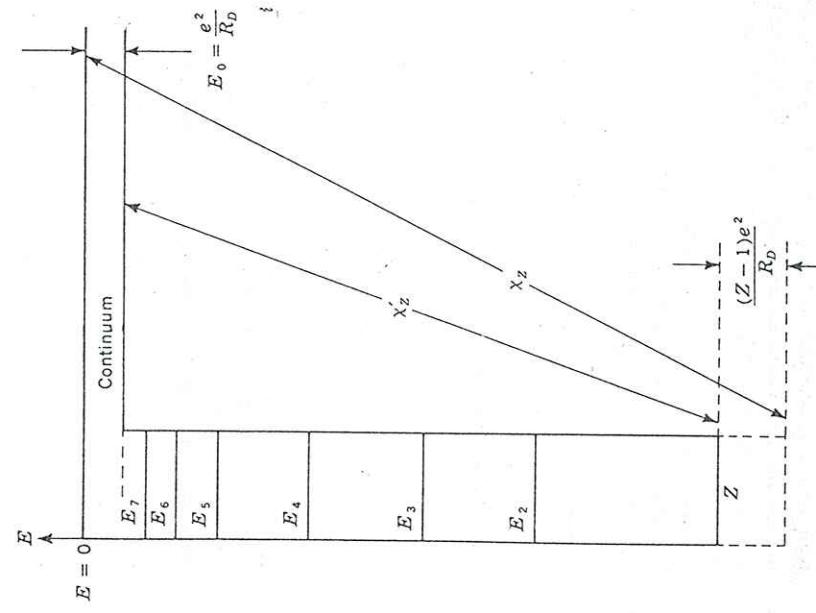


Fig. 2-16 The effective ionization potential x'_z . The laboratory ionization potential is reduced by coulomb interactions in the dense plasma. The energy of continuum electrons is lowered by e^2/R_D , whereas the energy of a tightly bound electron with Bohr radius $a \ll R_D$ is raised by $(Z-1)e^2/R_D$. Thus the ionization potential is effectively reduced by the amount Ze^2/R_D .

bound electron is

$$(PE)_e \approx -\frac{Ze^2}{a} + \frac{(Z-1)e^2}{R_D} \quad (2-253)$$

where a is the radius of the orbital. (For a hydrogenlike ion $a = a_0/Z$.)

From Eq. (2-253) it can be seen that, to first approximation, the ground-state energy is elevated by an amount $(Z-1)e^2/R_D$. Figure 2-16 shows the energy-level diagram of an ion, where x_z is the laboratory ionization potential and x'_z is the effective ionization potential in the plasma. In the calculation of the ionization equilibrium, therefore, the laboratory ionization potential must be replaced by an *effective ionization potential* whose value is

$$x'_z = x_z - \frac{Ze^2}{R_D} \quad (2-254)$$

The excited bound states are also shifted upward in energy, so that some of the states that are bound in the laboratory move up into the continuum of states in a dense plasma. This phenomenon resolves in a natural way the paradox mentioned in Chap. 1, *viz.*, that the partition functions appear to be infinite. In the crudest approximation, one would simply regard all those states bound by less than Ze^2/R_D in the laboratory as being unbound in the star; and even though Ze^2/R_D might be a much smaller energy than x_z , that energy range is the one which contains the infinite number of bound states, so that its elimination not only makes the partition functions finite but often means that they can be estimated by the sum of only a few terms. In a more exact calculation, the excited-state energies will be increased by an amount different from that of the ground state because there are smaller energy shifts because of the differences in radii of the excited states. The present simplified discussion should be regarded only as an illustration of the dominant terms of a problem that is extremely complicated in its details.¹ Correction of the ionization potential is an important part of the calculation of the radiative opacity of the gas.

Problem 2-49: Estimate the partition function for a five-times-ionized carbon atom (hydrogen-like) if it is embedded in a hydrogen gas at $T = 5 \times 10^6^\circ\text{K}$ and $\rho = 5 \times 10^{-3} \text{ g/cm}^3$. In the simple approximation of the text, how much is the binding energy of that electron reduced from its normal value of 36 Ry?

Ans: $G_e = 2.6$, $\Delta x \approx -1.0 \text{ Ry}$.

THE ZERO-TEMPERATURE IONIZED GAS

As the density of the gas increases, the coulomb energy increases, and one says that the gas becomes less perfect. At densities high enough for the electron gas to be degenerate, however, that rule of thumb becomes incorrect. A degenerate gas becomes more perfect as the density rises. This peculiarity comes about

¹J. C. Stewart and K. D. Pyatt, *Astrophys. J.*, 144:1203 (1966), and C. A. Rouse, *Phys. Rev.*, 159:41 (1967).

because the relevant energy for the gas becomes the Fermi energy E_f rather than the thermal energy kT . The perfect-gas condition $E_e \ll kT$ must be replaced by $E_e \ll E_f$, but because the Fermi energy increases as a higher power of the density than the coulomb energy, a nearly perfect degenerate gas is more nearly perfect at higher density. Quantitatively the argument goes as follows. If $\langle r \rangle$ is an appropriate average separation of electrons from the ions of charge Z , then the condition that the gas be perfect becomes

$$\frac{Ze^2}{\langle r \rangle} \ll E_f \quad (2-255)$$

This equation clearly becomes true for sufficiently large values of the electron density, in fact, for $n_e > 10^{32} Z^3$, a condition that seems always satisfied in astrophysical degenerate electron gases. The consequence of this fact is that the electron gas is nearly uniform; i.e., the electronic charge density is, in first approximation, constant.

As was discussed earlier, as long as $E_f \gg kT$, the state of the electron gas is almost independent of T . The same cannot be said of the ions, because their energy is predominantly thermal. For arbitrary temperature, the configuration of the gas may be quite complicated, but it is not too difficult to see what must happen as the temperature is reduced to very low values. Assume for this discussion that the density is great enough so that the matter must remain *pressurized* even at zero temperature. As $T \rightarrow 0$ the thermal energy of the ions vanishes, so that they will assume a configuration that minimizes the potential energy. This stable configuration is such that the ions maximize the inter-ion separation and is more like a lattice than a gas. For small displacements of an ion from its equilibrium point in the lattice, there will be a restoring force resulting from an increase in the coulomb potential energy. The result is that each ion is situated in a harmonic-oscillator potential. Now although a classical oscillator can have zero energy, a quantum oscillator cannot. It must oscillate with a zero-point energy equal to $\frac{3}{2}\hbar\omega$, where ω is the frequency of the oscillator ($\omega = \sqrt{k/m}$, here k is the force constant of the equivalent oscillator). Salpeter¹ and others have discussed this phenomenon and showed that the zero-point energy will be less than the coulomb energy holding the ions in the lattice structure and hence the lattice structure is correct at zero temperature. Although most interesting astrophysical environments are not strictly at zero temperature, it may turn out at high density that the oscillator energy dominates kT , in which case a zero-temperature approximation is sensible. Because the major coulomb effects are easy to estimate in this limit, we now turn our attention to those effects in the zero-temperature ionized gas.

¹ E. Salpeter, *Astrophys. J.*, 134:669 (1961).

Draw an imaginary sphere of radius R_z around each ionized nucleus of charge Z such that the sphere contains Z free electrons which neutralize the entire sphere. That is, the gas is imagined as being divided into neutral spheres about each nucleus Z which contain the Z electrons closest to that nucleus. Assume, furthermore, that the Z electrons in each ion sphere are spread uniformly over the volume of the sphere and that the average electron density is unperturbed by the nuclear charges. (These assumptions are incorrect in detail, but as a simple approximation they allow an easy numerical estimate of the physical effects being considered. The nuclear charge will actually polarize the electron density to a degree dependent upon density and temperature.)

Since the ion spheres are spherically symmetric and neutral, the electrostatic potential energy of electrons in a given ion sphere can be calculated from the geometry of that sphere alone. And the first point to notice is that the free electrons do possess an average potential energy which is negative. Consider the total electrostatic energy of one sphere. It may be calculated as the sum of the potential energies from electron-electron interactions and of electron-nucleus interactions.

(1) *The electron-electron interaction:* To assemble the uniform sphere with total charge Z_e requires electrostatic energy equal to

$$U_{ee} = + \int_0^{R_z} \frac{q_r dq}{r} \quad (2-257)$$

as illustrated in Fig. 2-17, where

$$q_r = \left(\frac{r}{R_z} \right)^3 Z_e \quad dq = \frac{4\pi r^2 dr}{\frac{4}{3}\pi R_z^3} Z_e \quad (2-258)$$

The integration gives

$$U_{ee} = + (Z_e)^2 \frac{3}{R_z^6} \int_0^{R_z} r^4 dr = \frac{3}{5} \frac{(Z_e)^2}{R_z^2} \quad (2-259)$$

(2) *The electron-nucleus interaction:* To assemble the uniform sphere of charge Z_e about the nucleus Z requires electrostatic energy

$$U_{ez} = - Z_e \int_0^{R_z} \frac{dq}{r} = - \frac{3}{2} \frac{(Z_e)^2}{R_z} \quad (2-260)$$

Thus the total electrostatic energy of the ion sphere is

$$U = U_{ee} + U_{ez} = \left(\frac{3}{5} - \frac{3}{2} \right) \frac{(Z_e)^2}{R_z} = - \frac{9}{10} \frac{(Z_e)^2}{R_z} \quad (2-261)$$

Since this electrostatic energy is shared by Z electrons, an electron in a given ion

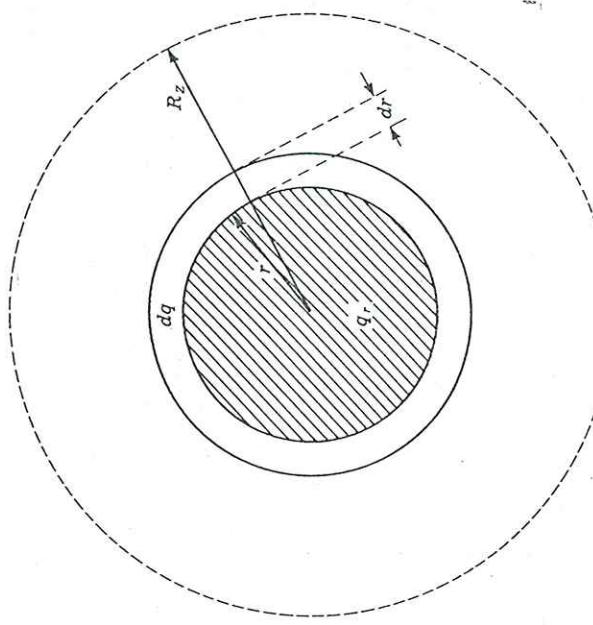


Fig. 2.17 The ion sphere in a degenerate zero-temperature plasma. The electron gas is uniform in these circumstances, and each ion Z is surrounded by a sphere of radius R_z which contains Z free electrons. In this zero-temperature limit other ions are excluded from the sphere so that $\frac{4}{3}\pi R_z^3 n_e = Z$.

sphere has an average potential energy equal to

$$\langle -eV \rangle_z = -\frac{9}{10} \frac{Ze^2}{R_z} \quad (2-262)$$

This negative energy physically reflects the fact that the electrons are actually bound to the ionized gas. This is again the idea behind the depression of the continuum, *viz.*, that continuum electrons moving with kinetic energy $p^2/2m$ have total energy equal to $p^2/2m + E_0$, where E_0 is the electron potential energy averaged over all the ion spheres of the system.

The magnitude of the continuum depression can be calculated with this model. The number of ion spheres per unit volume of type Z is

$$n_Z = \frac{\rho X_Z}{A_Z} N_0 \quad (2-263)$$

and each one contains Z electrons with potential energy $\langle -eV \rangle_z$, so that the

potential energy per unit volume from ion spheres of charge Z is

$$\langle -eV \rangle_z Z n_Z = \langle -eV \rangle_z \frac{ZX_Z}{A_Z} \rho N_0 \quad (2-264)$$

and the total electron potential energy per unit volume is just the sum over all ion spheres:

$$\left(\frac{U}{V}\right)_e = \rho N_0 \sum_Z \langle -eV \rangle_z \frac{ZX_Z}{A_Z} \quad (2-265)$$

Since the total number of electrons per unit volume is $n_e = \rho N_0 / \mu_e$, the average potential energy per electron is

$$E_0 = \frac{\langle U/V \rangle_e}{n_e} = \mu_e \sum_Z \langle -eV \rangle_z \frac{ZX_Z}{A_Z} \quad (2-266)$$

To evaluate E_0 numerically requires evaluation of $\langle -eV \rangle_z$ for each ion sphere. It is evident that for this purpose the radii of the ion spheres are required. Since the electrons are assumed to be uniformly distributed, the volume of each ion sphere must be such that

$$n_e \left(\frac{4}{3}\pi R_z^3\right) = Z \quad (2-267)$$

or

$$R_z = \left(\frac{3}{4\pi n_e}\right)^{\frac{1}{3}} \quad (2-268)$$

Substituting into Eq. (2-262) give

$$E_0 = -\mu_e \sum_Z \frac{9}{10} \left(\frac{3}{4\pi n_e}\right)^{\frac{1}{3}} \frac{ZX_Z}{A_Z} = -\frac{9}{10} \left(\frac{4\pi}{3}\right)^{\frac{1}{3}} \mu_e n_e^{\frac{1}{3}} e^2 \sum_Z \frac{ZX_Z}{A_Z} \quad (2-269)$$

or

$$E_0 = -\frac{9}{10} \left(\frac{4\pi}{3} \mu_e^{\frac{1}{3}} \rho N_0\right)^{\frac{1}{3}} e^2 \left(X + 2^{-\frac{1}{3}} Y + \sum_{Z>2} \frac{Z^{\frac{1}{3}} X_Z}{A_Z}\right) \quad (2-270)$$

The most useful way to evaluate this energy is to multiply numerator and denominator by the Bohr radius $a_0 = 0.528 \times 10^{-8}$ cm and to note that $e^2/a_0 = 2\mathcal{R}$, where \mathcal{R} is the Rydberg constant and numerically equals the binding energy of the hydrogen ground state. Then the continuum depression in Rydbergs is

$$E_0 = -1.30(\mu_e^{\frac{1}{3}} \rho)^{\frac{1}{3}} \left(X + 0.79Y + \sum_{Z>2} \frac{Z^{\frac{1}{3}} X_Z}{A_Z}\right) \text{ Ry} \quad (2-271)$$

It is evident that, in this approximation, the continuum depression depends upon the density as $\rho^{\frac{1}{3}}$ and upon the composition. (In a more complete treatment, E_0 also depends upon the temperature, a dependence that results from the spatial

nonuniformity of the electron density and the breakup of the ion lattice.) The continuum depression is also seen to increase with the average Z of the ions, being smallest for essentially pure hydrogen. For population I composition ($X = 0.60$, $Y = 0.38$, $Z = 0.02$)

$$E_0 \approx -1.44\rho^4 Q \quad (2-272)$$

All free electrons in this approximation are considered as having total energy $E = p^2/2m + E_0$. This fact makes a slight change in the previous expression for the number density of electrons. Now we have

$$n_e(p) dp = \frac{2}{h^3} \exp(\alpha + E/kT) \frac{4\pi p^2 dp}{(E_0 + E_0/kT + p^2/2mkT) + 1} \quad (2-273)$$

Thus

$$n_e = \int_0^\infty n_e(p) dp = \frac{4\pi}{h^3} (2mkT)^{1/2} F_1 \left(\alpha + \frac{E_0}{kT} \right) \quad (2-274)$$

Exactly the same thing occurs in the pressure integral for a perfect gas, so that the equation of state is represented, to first approximation, by the same parametric representation as before, where α is replaced by $\alpha + E_0/kT$. The coulomb energy does contribute a correction to the pressure which must be computed separately, however.

The fact that the coulomb potential energy is negative and increases with density means that it must reduce the pressure from the value appropriate to a non-interacting gas. Although a compression must do work against the free-particle momenta, it does negative work against the coulomb field.

Problem 2-50: Show from the first law of thermodynamics that the coulomb pressure from the homogeneous-ion-sphere model is

$$P_c = -\frac{3}{10} \left(\frac{4\pi}{3} \right)^{1/3} \frac{(\rho N_0)^{4/3}}{\mu_e^{1/3}} e^2 \left(X + 2^{-1}Y + \sum_{Z>2} \frac{Z^3 X_Z}{A_Z} \right) \quad (2-275)$$

It will be noticed from Eq. (2-275) that

$$P_c = \frac{1}{3} \left(\frac{U}{V} \right)_e \quad (2-276)$$

where $(U/V)_e$ is the energy density in the coulomb field. It is remarkable that the relationship of the pressure to the coulomb-energy density is the same as that obtained for the photon gas and for the Debye-Hückel gas, in both cases confirming the result of the virial theorem. Of course the calculation is inexact. In the present case, for example, the next largest correction (much smaller than this one) comes from distortion of the electron density. This calculation proceeds

from a Thomas-Fermi model of ion sphere.¹ The coulomb corrections to the pressure are not always negligible and should, for example, be included in models of white dwarfs.²

Problem 2-51: Show that the numerical value of the coulomb pressure is

$$P_c = -5.68 \times 10^{15} \rho^{-1/3} \left(X + 0.79Y + \sum_{Z>2} \frac{Z^3 X_Z}{A_Z} \right) \text{ dyne/cm}^2 \quad (2-277)$$

Problem 2-52: Calculate in this approximation the density for which completely (pressure) ionized iron has zero pressure.

$$\text{Ans: } \rho = 260 \text{ g/cm}^3.$$

Just as in the Debye-Hückel case, the coulomb interactions reduce the ionization potential of bound electrons. Again to simplify the matter, assume that the only bound electrons are in orbitals having a Bohr radius much smaller than the ion sphere:

$$a_{Zn} = \frac{n^2}{Z} a_0 < R_Z \quad (2-278)$$

Then to the normal energy

$$E_{Zn} = -\frac{1}{2} \frac{Ze^2}{a_{Zn}} \quad (2-279)$$

must be added the interaction of an electron at the center of the ion sphere to the other electrons of the sphere:

$$E'_{Zn} = E_{Zn} + \frac{3}{2} \frac{Ze^2}{R_Z} \quad (2-280)$$

Problem 2-53: Show that in Rydbergs

$$E'_{Zn} = -\frac{Z^2}{n^2} + 2.16 \left(\frac{\rho Z^2}{\mu_e} \right)^{1/3} \text{ Ry} \quad (2-281)$$

Equation (2-281) is only a rough approximation because it has assumed (1) a homogeneous electron density, (2) that the nuclear charge Z is unshielded by free electrons, and (3) that the bound-electron wave function is unperturbed by the free electrons. A more satisfactory treatment would actually involve the solution for the bound-electron wave function in a shielded rather than a pure coulomb potential. If, however, the bound-electron radius is much less than the ion-sphere radius, this expression will give a reasonable approximation to the energy of the bound electron.

¹ The reader is referred to Salpeter, *op. cit.*, for this treatment.

² For a thorough discussion of coulomb effects on white dwarfs see H. Van Horn, *Astrophys. J.*, 161:227 (1968).

Now the *effective ionization potential* is the difference in energy between the lowest energy (zero kinetic energy) of the continuum electron states and the energy (negative) of the bound electron; that is,

$$\chi_{\text{eff}} = E_0 - E'_{2n} = E_0 + \frac{1}{2} \frac{Ze^2}{a_2} - \frac{3}{2} \frac{Ze^2}{R_2}$$

$$= \frac{Z^2}{n^2} - 2.16 \left(\frac{\rho Z^2}{\mu_e} \right)^{1/4} - 1.30(\mu_e^2 \rho)^{1/4} \left(X + 0.79Y + \sum_{Z>2} \frac{Z^3 X^2}{A_Z} \right) \text{ Ry} \quad (2-282)$$

The first term is the normal low-density ionization potential; $\chi = Z^2/n^2$ Ry. The remaining terms represent the reduction in ionization brought about by the coulomb interactions. These interactions reduce χ by a constant amount, the continuum depression E'_0 , plus another term which varies from state to state. This reduction will rapidly cut off the partition functions of ions by restricting their allowable states to the more tightly bound ones. When the energy of a bound electron is less negative than E'_0 , the electrons are to be considered as free. We also note here another view of the phenomenon of pressure ionization, because there exists, for each Z , a sufficiently high density that $\chi_{\text{eff}} < 0$.

Problem 2-54: Estimate the density at which iron must be completely pressure-ionized.
Ans: $\rho \approx 2 \times 10^4$.

A summary is in order at this point. The coulomb interactions render the ionized gas imperfect. The two most important effects are a negative coulomb pressure and a reduction of the ionization potentials. These effects have been quantitatively investigated for two extreme environments: (1) at sufficiently low density and high temperature, kT dominates coulomb energies, and the non-degenerate gas may be treated approximately by the Debye-Hückel method; (2) at high density and low temperature, the electron gas is degenerate and near perfect, but the ions are forced into a lattice because their coulomb energies exceed kT . The treatments given were only the simplest approximations possible in a very difficult subject, so that the formulas derived should be regarded with caution inasmuch as the assumptions made may not be satisfied in most astrophysical environments.

As a final note, the reader may be interested to know that the same type of problem occurs in the equation of state of nuclear matter. The properties of neutron stars, if they exist, depend upon the corrections to the independent-particle-model equation of state necessitated by interactions in the gas at nuclear density. In this case, of course, the perturbing interactions stem from the nuclear force rather than the coulomb force.¹

Another application of the coulomb interactions will be postponed until Chap. 5, where it will be seen that the energetics of nuclear reactions is shifted by a J. N. Bahcall and R. A. Wolf, *Phys. Rev.*, 140:1445 (1965).

2.4 POLYTROPES

The static configuration of a gaseous sphere held together by self-gravitation must satisfy the condition of hydrostatic equilibrium. The essence of the equation of hydrostatic equilibrium is that the pressure at each point in a stellar interior is sufficient to just balance the weight of the overlying layers of the star. Furthermore, the pressure itself is determined by the equation of state applicable to the local conditions in the stellar interior. These considerations do not in themselves determine the structure of a star. Any specified pressure that may be required to support the overlying layers is obtainable from an unlimited number of combinations of density and temperature at that point. What are clearly needed are more conditions on the density and temperature in a stellar interior that relate to other physical processes that go on there.

It was about the beginning of the twentieth century when several notable physicists, *viz.*, Lane, Ritter, Kelvin, Emden, and Fowler, considered the question of what limitations could be placed on the structure of a star just from the condition of hydrostatic equilibrium alone. They quickly concluded that some other condition relating the physical variables in the stellar interior is necessary in order to be able to specify the structure. The necessary relationships are to be found in the production and transport of thermal energy, subjects to be discussed in subsequent chapters. One explicit auxiliary condition that has been found to correspond to certain idealized physical situations, however, is a pressure expressible in terms of some power of the density only. For historical reasons¹ the assumed pressure-density relationship is written as

$$P = K \rho^{(n+1)/n} \quad (2-283)$$

where the number n is called the polytropic index. Gaseous spheres in hydrostatic equilibrium in which the pressure and density are related by Eq. (2-283) at each point along the radius are called *polytropes*. The constant K depends upon the nature of the polytrope. It was shown by Lane and Emden that if a polytropic pressure-density relation is assumed, the properties of the structure can be computed.

Since, of course, any explicit relationship between the pressure and the density would make possible the solution for the structure of a self-gravitating gaseous sphere in hydrostatic equilibrium, one might ask why a relationship of the form¹

¹ The nomenclature is patterned after quasistatic changes of state of an ideal gas for which a generalized specific heat is held constant. It was found by early workers in kinetic theory that for such changes of state, called *polytropic changes* by R. Emden in his classical treatise "Gaskugeln," B. G. Teubner, Leipzig, 1907, the variables change along a track $P = K \rho^{\gamma'}$, where γ' is determined by C_P , C_V , and C , the specific heat characterizing the process. These matters of historical interest are elegantly summarized in Chandrasekhar, *op. cit.*, Chaps. 2 and 4.

of Eq. (2-283) should be chosen for study. The reason lies in the fact that some idealized physical circumstances for a star would lead naturally to equations of that form. To clarify this point, we shall consider examples of stars that can be represented by polytropes.

As a first example, we may follow Kelvin in considering a star that is "boiling" in a state that he described as *adiabatic convective equilibrium*. If the whole interior of a star is completely convective, mass elements are both rising and falling in the interior of a star. A star is said to be in convective equilibrium if any mass element after rising and falling from its initial temperature and density to a new temperature and density finds itself at the same temperature and density as the surroundings. The convective equilibrium is adiabatic if the convective cells move without heat exchange. It will be evident that if some mechanism continuously stirs and mixes the entire interior of a star, it must soon come to a condition of convective equilibrium, for any differences in temperature and density of the surroundings in a star from those of an element that has risen from some lower portion of a star will quickly be eliminated. If radiation pressure is an unimportant determinant in the structure, adiabatic changes are of the form

$$P = K\rho^{\frac{5}{3}} \quad (2-284)$$

where $\gamma = \frac{5}{3}$ for an ideal monatomic gas. If such a rising or falling element is, furthermore, at the same conditions of temperature, density, and pressure as the surrounding matter at all times, it follows that the run of pressure and density in the star is such that

$$P(r) = K\rho(r)^{\frac{5}{3}} \quad (2-285)$$

It seems, therefore, that a star in convective equilibrium in which radiation pressure is not important is a polytrope of exponent $\gamma = \frac{5}{3}$, which is also a polytrope of index $n = 1.5$. In such a way Kelvin was led quite naturally to at least one physical possibility that would correspond to the structure of a polytrope.

As a second example, consider a star in which radiation pressure is not unimportant. We have defined the quantity β such that

$$P_o = \frac{N_0 k}{\mu} \rho T = \beta P \quad P_r = \frac{1}{3} \alpha T^4 = (1 - \beta)P \quad (2-286)$$

for a nondegenerate gas. Equating the values of the pressure from these two equations gives immediately at each point

$$T = \left(\frac{N_0 k}{\mu} \frac{3}{a} \frac{1 - \beta}{\beta} \right)^{\frac{1}{4}} \rho^{\frac{1}{3}} \quad (2-287)$$

Since we also have

$$P = \frac{N_0 k}{\mu} \frac{\rho T}{\beta} = K \lambda^{(n+1)/n} \phi^{n+1} \quad (2-288)$$

we see that

$$P = \left[\left(\frac{N_0 k}{\mu} \right)^4 \frac{3}{a} \frac{1 - \beta}{\beta^4} \right]^{\frac{1}{n}} \rho^{\frac{1}{n}} \quad (2-289)$$

This equation is true at each point in the interior of the star we are considering. The ratio of gas pressure to total pressure β does, in general, depend upon the distance from the center of the star. If, however, one had a special configuration in which the quantity β was a constant, i.e., such that the gas pressure was a constant fraction of the total pressure throughout the star, then the expression in brackets in Eq. (2-289) reduces to a constant, and one has an equation of the form

$$P = K\rho^{\frac{1}{n}} \quad (2-290)$$

This model star would correspond to a polytrope of polytropic exponent $\frac{1}{n}$ or index 3. We can see later that this particular polytrope corresponds more closely to stars in radiative equilibrium, i.e., stars for which the energy is transported by radiative transfer rather than by convection. It will, in fact, be shown that the polytrope of index 3 corresponds to that star in radiative equilibrium such that at each distance r from the center of the star, the product of the energy liberated per unit mass from all the material interior to r times the opacity of the gas at the point r is a constant. The properties of a nondegenerate polytrope of index 3 have also been highly developed in the analytical study of gaseous configurations, especially by Eddington. This model star is frequently called the *standard model*. We shall use the standard model often in an attempt to get a first approximation to the runs of temperatures and densities in the interiors of stars.

A third example may be provided by stars supported by the pressure of a completely degenerate electron gas (white dwarfs). That pressure has been shown to vary as ρ^3 or ρ^5 , according to whether the electron velocities are non-relativistic or relativistic. The corresponding polytropes can provide useful insights into their structure.

These examples give some indication of the physical reasons that lie behind considering the structure of gaseous spheres in hydrostatic equilibrium for which the pressure and density are related by an equation of the form of Eq. (2-283). Motivated by the fact that the density is proportional to T^n in a nondegenerate polytrope of index n , a convenient definition is

$$\rho \equiv \lambda \phi^n \quad (2-291)$$

where λ is a scaling parameter whose value depends upon the definition of the quantity ϕ . This representation for the run of density throughout the star will be convenient for the study of polytropes, where we shall identify the parameter λ with the central density of the star, thereby normalizing the function ϕ to unity at the center. For this representation, the pressure is

$$P = K \rho^{(n+1)/n} = K \lambda^{(n+1)/n} \phi^{n+1} \quad (2-292)$$

The solution for the structure of a polytrope, then, depends upon the coupling of Eq. (2-283) to the condition of hydrostatic equilibrium

$$\frac{dP}{dr} = -\rho \left(\frac{GM_r}{r^2} \right) \frac{dM_r}{dr} = 4\pi r^2 \rho \quad (2-293)$$

from which it follows that

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\frac{r^2 dP}{\rho} \right) = \frac{1}{r^2} \frac{d}{dr} (-GM_r) = -4\pi G \rho \quad (2-294)$$

Substitution of the values of pressure and density for a polytrope of index n into this last equation yields

$$(n+1)K\lambda^{1/n} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) = -4\pi G \rho \quad (2-295)$$

This equation can be made more attractive by defining a unit of length

$$a \equiv \left[\frac{(n+1)K\lambda^{(1-n)/n}}{4\pi G} \right]^{\frac{1}{n+1}} \quad (2-296)$$

and by defining a dimensionless distance variable $\xi = r/a$, whereupon Eq. (2-295) reduces to

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\phi}{d\xi} \right) = -\phi^n \quad (2-297)$$

This equation is called the *Lane-Emden equation* for the structure of a polytrope for index n . The solution for ϕ as a function of ξ completely determines the structure of the polytrope except for the choice of the central density. By setting λ equal to the central density, it is easy to see that the temperaturelike variable ϕ must obey certain boundary conditions at the center of the star, i.e., at $\xi = 0$; viz.,

$$\phi = 1 \quad \frac{d\phi}{d\xi} = 0 \quad \text{at } \xi = 0 \quad (2-298)$$

Problem 2-55: Show that $(d\phi/d\xi)_0 = 0$. Hint: Expand the equation of hydrostatic equilibrium near the center.

The solution ϕ which satisfies the Lane-Emden equation of index n under these boundary conditions is called the *Lane-Emden function of index n* .

Explicit solutions of the Lane-Emden equation for general values of n apparently do not exist. For values of n other than $n = 0, 1$, and 5 , numerical techniques must be employed for the determination of the Lane-Emden function. We first note that if $\phi(\xi)$ is a solution of the equation, then $\phi(-\xi)$ is also a solution. This observation implies that if ϕ is expressed as a power series in ξ , only even powers of ξ appear; that is,

$$\phi(\xi) = C_0 + C_2 \xi^2 + C_4 \xi^4 + \dots \quad (2-299)$$

Problem 2-56: By substituting Eq. (2-299) into the Lane-Emden equation, show that the first three terms of the series for ϕ are

$$\phi = 1 - \frac{1}{6} \xi^2 + \frac{n}{120} \xi^4 - \dots \quad (2-300)$$

By taking a sufficient number of terms in this alternating series for ϕ , one can calculate the solution to any desired accuracy for values of $\xi < 1$. For values of $\xi > 1$, this solution can be continued from the differential equation by standard numerical methods. The solutions are found to decrease monotonically from the center and for values of n less than 5 have a zero for some finite value of ξ , say, $\xi = \xi_1$. At $\xi = \xi_1$ it is clear that ϕ 's being equal to zero makes the pressure vanish, and the configuration may be said to have a physical boundary at that point. Table 2-5 lists the value of ξ_1 and the derivative of ϕ at $\xi = \xi_1$ for the various Lane-Emden functions of index n . We shall shortly see that the value of ξ_1 and the slope of ϕ at $\xi = \xi_1$ are important for determining large-scale properties of the various gaseous configurations.

It is evident that the structure of each polytrope is specified in terms of the dimensionless length ξ . An inspection of the length a used in forming this dimensionless length shows that its value is determined by two numbers; the first is the constant K , which occurs in Eq. (2-283) relating the pressure to the density, and the second is the parameter λ , which we have taken to be the central density for the solution of this problem. It is apparent, therefore, that each Lane-Emden function ϕ_n represents, for a specified value of the constant K , a one-parameter family of solutions, the parameter being the central density λ . As an example, we may turn to the polytropes that were considered in introducing this whole discussion. In the case of the completely convective polytrope of index 1.5, it is easy to see that the value of K is determined by the particular adiabat of the gas. In adiabatic convective equilibrium the entropy per gram of material is constant, and it specifies the value of K .

Problem 2-57: Ignore the entropy of the radiation and show from Eq. (2-136) that the entropy per gram can be written

$$S = \frac{3}{2} \frac{Nak}{\mu} \ln \frac{P}{\rho^4} + \text{const} \quad (2-301)$$

that is, $S = S(K)$.

In the case of the standard model, or polytrope of index 3, we see from Eq. (2-289) that the value of the constant K is given by

$$K = \left[\left(\frac{Nak}{\mu} \right)^4 \frac{3}{a} \frac{1 - \beta^4}{\beta^4} \right]^{\frac{1}{n+1}} \quad (2-302)$$

where β represents the ratio of the gas pressure to the total pressure and is a constant throughout the standard model. The selection of a value of K in either

for these two instances still allows a complete run of corresponding solutions as determined by the central density λ . It is quite clear that considerable leeway still exists for the actual structure of the various polytropes being considered. To understand the way in which these various factors come into play, we need to consider several more large-scale properties of the configuration that can be derived from the material presented so far.

1) *Radius:* The radius of the configuration is by definition determined by the first zero of the Lane-Emden function of order n . Thus

$$a\xi_1 = \left[\frac{(n+1)K}{4\pi G} \right]^{\frac{1}{n-m}} \lambda^{(n-m)/2n} \quad (2-303)$$

2) *Mass:* The mass M interior to the normalized radius ξ is given by

$$I(\xi) = \int_0^{a\xi} 4\pi r^2 \rho dr = 4\pi a^3 \int_0^{\xi} \lambda \phi^n \xi^2 d\xi \quad (2-304)$$

by using the Lane-Emden equation itself, the integral in Eq. (2-304) can be transformed to

$$\begin{aligned} I(\xi) &= -4\pi a^3 \lambda \int_0^{\xi} \frac{d}{d\xi} \xi^2 \frac{d\phi}{d\xi} d\xi \\ &= -4\pi a^3 \lambda \xi^2 \frac{d\phi}{d\xi} \end{aligned} \quad (2-305)$$

Substituting for the unit of length a and evaluating the above expression at $\xi = \xi_1$

$$\begin{aligned} &\frac{\rho_e}{\bar{\rho}} = -\xi_1^2 \left(\frac{d\phi}{d\xi} \right)_{\xi=\xi_1} \\ &= -4\pi a^3 \lambda \xi_1^2 \frac{d\phi}{d\xi} \end{aligned} \quad (2-306)$$

gives the total mass of the star:

$$\Sigma = -4\pi \left[\frac{(n+1)K}{4\pi G} \right]^{\frac{1}{n-m}} \lambda^{(3-n)/2n} \left(\xi_1^2 \frac{d\phi}{d\xi} \right)_{\xi_1} \quad (2-306)$$

We note that in the case $n = 3$, the mass depends only upon K and is independent of λ . The product of ξ_1^2 times the slope of ϕ evaluated at the first zero, ξ_1 , is one of the quantities listed in Table 2-5.

Problem 2-58: Show that the mass of the standard-model polytrope of index 3 is given numerically by

$$\Sigma = 18.0 \frac{\sqrt{1-\beta}}{\mu^2 \rho^2} \Sigma_{\odot} \quad (2-307)$$

For a given composition μ , the mass determines the value of β .

Problem 2-59: Imagine that a white dwarf is a body supported by the pressure of completely degenerate electrons. As the mass of the structure is increased, the central density becomes so high that the degeneracy becomes relativistic at the center, such that (confirm this)

$$P_e \rightarrow 1.244 \times 10^{15} \left(\frac{\rho}{\mu_e} \right)^3 \text{ dynes/cm}^2 \quad (2-308)$$

and falls off to nonrelativistic degeneracy in the outer portions of the star. As the mass is continually increased, the star shrinks to ever higher densities and ever smaller radius, until the electrons become highly relativistic everywhere. Then Eq. (2-308) is applicable throughout the star. Show that at this point the mass is

$$\Sigma = \frac{5.80}{\mu_e^2} \Sigma_{\odot} \quad (2-309)$$

This mass is called the *Chandrasekhar limit*, since Chandrasekhar showed that this was the maximum mass that could be supported by electron degeneracy. (Other physical effects such as rotation and inverse beta decay have been ignored.) It seems clear that this value must represent a limiting mass because the electrons can be relativistic throughout only if the mass is sufficient to squeeze the volume to a point. Inasmuch as white dwarfs are observed to have nonzero radii, their masses must be less than Eq. (2-309). The question of what happens if the mass exceeds this limit is a difficult one and will not be considered here.

(2) Ratio of mean density to central density: The mean density of the configuration is given by the total mass of the configuration divided by the volume of the configuration, whereas the central density is equal to λ . Thus, the ratio of the mean density to the central density can be determined from Eqs. (2-303) and (2-306) to be

$$\frac{\bar{\rho}}{\rho_e} = -\frac{3}{\xi_1} \left(\frac{d\phi}{d\xi} \right)_{\xi_1} \quad (2-310)$$

It is evident that the ratio of mean density to central density depends only upon the index of the polytrope. In fact, we may take this to be the main feature of the polytropic index; viz., the extent to which the matter is concentrated toward

Table 2-5 Constants of the Lane-Emden functions[†]

ξ	$-\xi_1^2 \left(\frac{d\phi}{d\xi} \right)_{\xi=\xi_1}$	$\frac{\rho_e}{\bar{\rho}}$
2.4494	4.8988	1.0000
5	2.7528	3.7871
0	3.14159	3.14159
5	3.65375	2.71406
0	4.35287	2.41105
5	5.35528	2.18720
0	6.89635	2.01824
25	8.01894	1.94980
5	9.53581	1.89056
0	14.97155	1.79723
5	31.83646	1.73780
9	169.47	1.7355
0	∞	1.73205

[†] Chandrasekhar, "An Introduction to the Study of Stellar Structure," p. 96; reprinted from the Dover Publications edition, copyright 1939 by The University of Chicago, as reprinted by permission of The University of Chicago.

the center of the star. Although we have not shown it here, the ratio of the central density to the mean density varies between the limits of unity (a star of uniform density) for a polytrope of index zero to value of infinity (a star infinitely concentrated toward the center) for a polytrope of index 5, passing through all intermediate values as n increases from zero to 5.

Problem 2-50: Show that the central density in the standard model exceeds the mean density by the factor $\rho_c/\rho = 54.2$.

(4) *The central pressure:* Since ϕ is normalized to unity at $\xi = 0$ by the interpretation of λ , the central pressure may be written, from Eq. (2-292), as

$$P_c = K\lambda^{(n+1)/n} \quad (2-311)$$

To express the central pressure in terms of macroscopic properties, we note that Eq. (2-303) can be written in the form

$$R = \left[\frac{(n+1)}{4\pi G} \xi_1^2 \right]^{\frac{1}{n}} [K\lambda^{(1-n)/n}]^{\frac{1}{n}} \quad (2-312)$$

from which

$$K\lambda^{(1-n)/n} = \frac{4\pi R^2 G}{(n+1)\xi_1^2} \quad (2-313)$$

Hence, the central pressure is given by

$$\begin{aligned} P_c &= (K\lambda^{(1-n)/n})^2 = K\lambda^{(1-n)/n} \rho_c^2 \\ &= \frac{4\pi R^2 G}{(n+1)\xi_1^2} \left[\frac{\xi_1}{3} \frac{1}{(d\phi/d\xi_1)} \right]^2 \rho^2 \\ &= \frac{1}{4\pi(n+1)(d\phi/d\xi_1)} \frac{G^2 \pi^2}{R^4} \end{aligned} \quad (2-314)$$

Problem 2-61: Show that for the standard model the central pressure is given numerically by

$$P_c = 1.24 \times 10^{17} \left(\frac{\pi}{3\pi_\odot} \right)^2 \left(\frac{R_\odot}{R} \right)^4 \text{ dynes/cm}^2 \quad (2-315)$$

This result gives a larger and more realistic estimate of the central pressure of the sun than the earlier rough arguments because it allows for the central condensation.

(5) *The central temperature:* The central temperature may be computed from the central pressure and the central density by use of the appropriate equation of state. For the case of the ideal ionized nondegenerate gas the relevant relationships are

$$P_\theta = \frac{N_0 k}{\mu} \rho_c T_c = \beta_c P_c \quad (2-316)$$

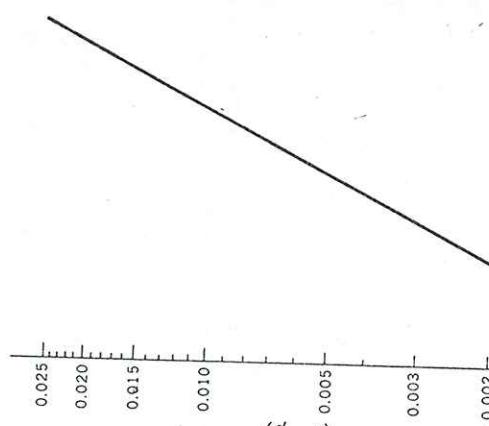


Fig. 2-18 The value of $1 - \beta$ for the standard model of low-mass stars.
(2-318)

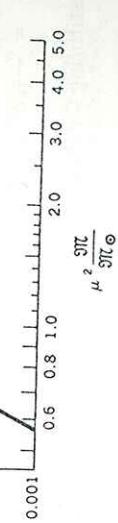
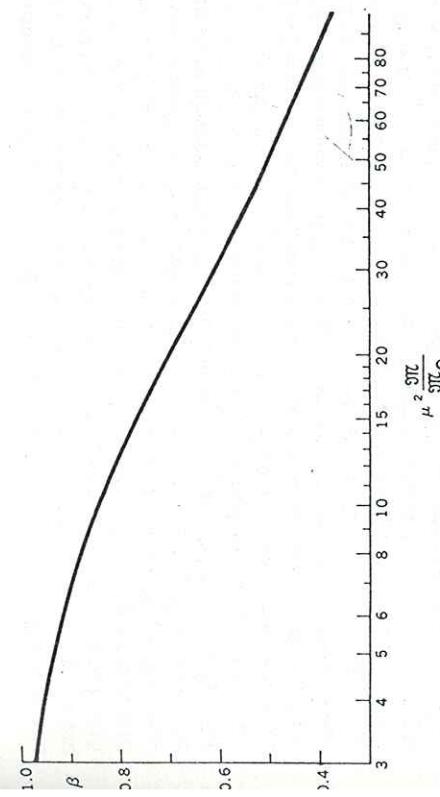


Fig. 2-19 The value of $1 - \beta$ for the standard model of low-mass stars.
(2-319)

Problem 2-62: Show that for the standard model the central temperature is

$$T_c = 4.6 \times 10^6 \mu \beta \left(\frac{\pi}{3\pi_\odot} \right)^{\frac{1}{n}} \rho_c^{\frac{1}{n}} \quad (2-318)$$

We may note that the equation above for the central temperature of the standard model contains quantities that are not independent of each other. In Prob. 2-58 it was demonstrated that the mass of the standard model is related to the ratio of gas pressure to total pressure; therefore, for a fixed value of μ it is apparent that β is a function of the mass (or vice versa), although the solution cannot be written explicitly. Figures 2-18 and 2-19 show graphically the dependence of β upon the quantity $\mu^2(\pi/\pi_\odot)$. These figures indicate the growing importance of radiation pressure with increasing mass. Since the mean molecular

Fig. 2-19 The value of $1 - \beta$ for the standard model of high-mass stars.

weight μ lies between $\frac{1}{2}$ and 2, the quantity $\mu^2(\mathfrak{M}/\mathfrak{M}_\odot)$ is not greatly different from the mass expressed in solar masses. At any rate, this additional constraint must be taken into account when using Eq. (2-318) for the central temperature of the standard model.

We may in fact use Eq. (2-318) at this point to make an estimate of the central temperatures of main-sequence stars. Main-sequence stars certainly are not polytropes of index 3, but we may nonetheless expect to see the trend of central temperatures by representing all main-sequence stars by the standard model. It is clear that the central density is an unknown in Eq. (2-318); however, an earlier problem demonstrated that $\rho_e = 54.2\bar{\rho}$ for the standard model, so that Eq. (2-318) may be altered to read

$$T_e = 17.4 \times 10^6 \mu \beta \left(\frac{\mathfrak{M}}{\mathfrak{M}_\odot} \right)^{\frac{1}{2}} \quad (2-319)$$

Problem 2-63: A certain type O star has $\mathfrak{M} = 30\mathfrak{M}_\odot$, $R = 6.6R_\odot$, $X = 0.70$, and $Y = 0.30$. Estimate the importance of radiation pressure and the central temperature by approximating the star by the standard model. A much better calculation on electronic computers yields $\beta_e = 0.77$, $T_e = 3.7 \times 10^7$.

The mean density $\bar{\rho}$ is just the mass divided by the volume, so that we can obtain these properties of main-sequence stars from Table 1-1. In fact, when the properties of Table 1-1 are coupled with the mass-luminosity relationship and with the fact that $\bar{\rho}_0 = 1.4 \text{ g/cm}^3$, it appears that an approximation to the

average density of main-sequence stars is

$$\bar{\rho} \approx \frac{1.4}{\mathfrak{M}/\mathfrak{M}_\odot} \quad \text{g/cm}^3 \quad (2-320)$$

Problem 2-64: Using the data of Table 1-1, show that a main-sequence star of $63\mathfrak{M}_\odot$ has an average density approximately equal to 0.28 g/cm^3 . What is the percentage difference between this value and that inferred from Eq. (2-320)? How much of a percentage error in the central temperature would be introduced by that percentage error in the average density?

It is of some interest that the density of main-sequence stars decreases with increasing mass. This fact is a consequence of the virial theorem, which demands higher temperatures for higher values of the potential energy of self-gravitation. These higher temperatures become sufficient to support the star in a more extended configuration of lower density.

It is also clear from Fig. 2-18 that $\beta \approx 1$ for stars of main-sequence mass. The mean molecular weight has a value near 0.7 for the centers of stars that have partially depleted their hydrogen (say, $X \approx 0.5$, $Y \approx 0.5$). By combining these approximations, Eqs. (2-319) and (2-320) yield

$$T_e \approx 14 \times 10^6 \left(\frac{\mathfrak{M}}{\mathfrak{M}_\odot} \right)^{\frac{1}{2}} \quad (2-321)$$

as anticipated central temperatures for main-sequence stars. When the question of thermonuclear reactions is considered, Eq. (2-321) will prove a helpful guide.

Many other interesting physical quantities can be calculated for these model stars called polytropes. An extensive discussion of the mathematical considerations related to this well-developed subject will be found in the monograph on stellar structure by Chandrasekhar. Furthermore, we have considered only uniform polytropes of homogeneous composition, whereas it is possible to divide stars into polytropic shells or mixed polytropes. A great deal of intuitive appreciation for the complexities of the physics of stellar structure may be obtained by an extensive analysis of the structure of polytropes. We have employed only the simplest features in this section as an introduction to the subject. Modern research has shown that the usefulness of polytropes is for the most part limited to this introductory acquaintance. Accurate and detailed models of the structure of real stars may be obtained only from detailed computer calculation. The additional physics needed to make these detailed calculations will constitute the burden of the following chapter.