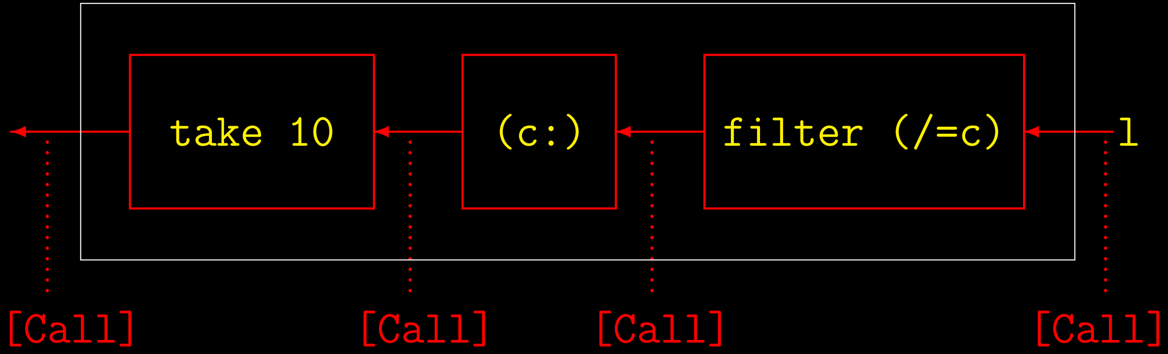
Cálculo de Programas

Aula T01

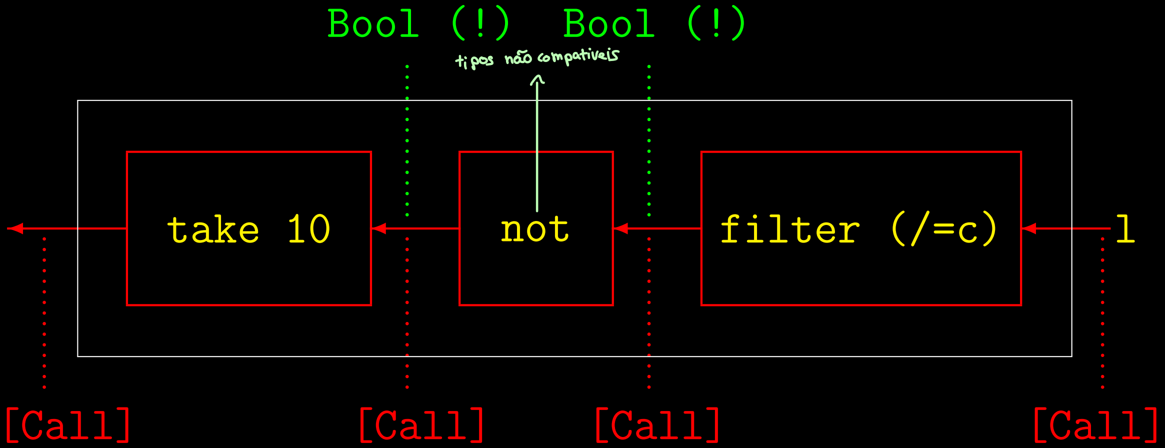
From a mobile phone manufacturer

```
store c :: [Call] -> [Call]
```



=> Os tipos se mantêm

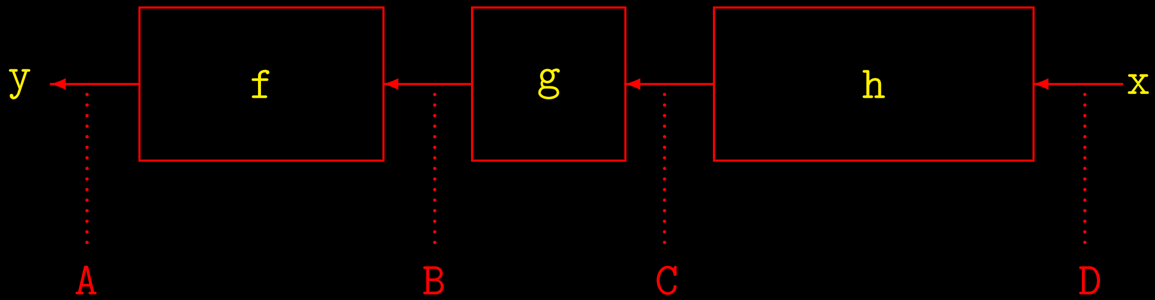
Uups!


Bool (!)
Bool (!)
tipos não compatíveis
take 10
not
filter (/=c)
l
[Call]
[Call]
[Call]
[Call]

Funções compostas

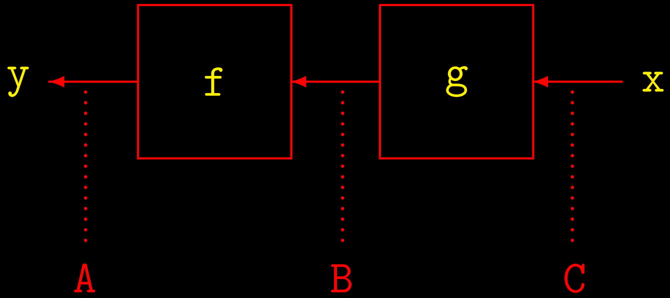
Em geral

$$y = f(g(h\ x))$$



Em geral

```
y = f(g x)
```



Simplificação

```
y = f(g x)
```

$$\mathtt{A} \xleftarrow{\mathtt{f}} \mathtt{B} \xleftarrow{\mathtt{g}} \mathtt{C}$$

Composição

$$y = f(g\ x)$$

$$A \xleftarrow{f} B \xleftarrow{g} C$$

$$A \xleftarrow{f\ \cdot\ g} C$$

$$y = (f \cdot g)\ x$$

Composição

$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

$$(a + b) + c = a + (b + c)$$

$$f \cdot g \cdot h$$

$$a + b + c$$

Composição

$$\mathit{store}\ c = \mathit{take}\ 10 \cdot \underbrace{(c:) \cdot \mathit{filter}\ (\neq c)}_{\mathit{store'}\ c}$$

isto é

$$\mathit{take}\ 10 \cdot ((c:) \cdot \mathit{filter}\ (\neq c))$$

igual a

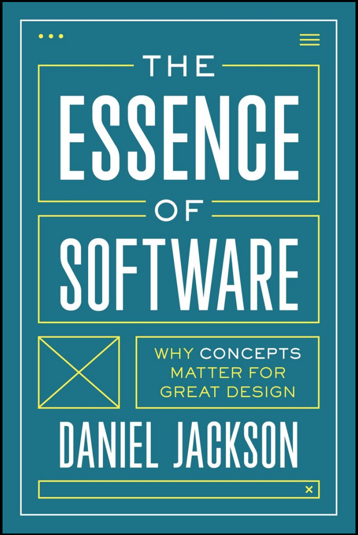
$$(\mathit{take}\ 10 \cdot (c:)) \cdot \mathit{filter}\ (\neq c)$$

Identidade

$$
\begin{tikzcd}
A \arrow[d, "f"'] & A \arrow[l, "id"'] \arrow[dl, "f"'] \arrow[d, "f"] \\
B & B \arrow[l, "id"]
\end{tikzcd}
$$

$$f \cdot id = f = id \cdot f$$

"(...) The best services revolve around **a small number of concepts** that are **well designed and easy** (...) **to understand and use**, and their innovations often involve simple but compelling new concepts."



$$C \xrightarrow{f} B \quad \text{e} \quad A \xrightarrow{g} C$$

Composição: $A \xrightarrow{f \cdot g} B$

$$B \xleftarrow{f} C \quad \text{e} \quad C \xleftarrow{g} A$$

$$
\begin{tikzcd}
B & & C \\
& A \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}
$$

$$(f\ a, g\ a)$$

Produto cartesiano

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

$$\frac{f\ a \in B \qquad g\ a \in C}{(f\ a, g\ a) \in B \times C}$$

"Split"

$$\begin{tikzcd}
B & B \times C & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\langle f, g \rangle \; a = (f \; a, g \; a)$$

Produto

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

$$\pi_1 : A \times B \to A$$
$$\pi_1(a, b) = a$$

$$\pi_2 : A \times B \to B$$
$$\pi_2(a, b) = b$$

Produto

$$\begin{tikzcd} B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\ & A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"'] \arrow[ur, "g"'] & \end{tikzcd}$$

$$\pi_1 \cdot \langle f, g \rangle = f$$

Produto

$$\langle f, g \rangle$$

f e g em paralelo

f “split” g

$$\langle f, g \rangle\ a = (f\ a, g\ a)$$

Produto

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "f"] & B \times C \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[u, "f \times g"] \arrow[ul] \arrow[ur] & C \arrow[u, "g"']
\end{tikzcd}$$

$$f \times g = \langle f \cdot \pi_1, g \cdot \pi_2 \rangle$$

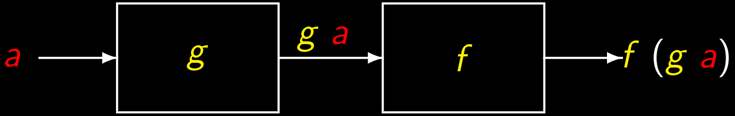
Recapitulando

$f \cdot g$	Composição sequencial
$\langle f, g \rangle$	Composição paralela (síncrona)
$f \times g$	Composição paralela (assíncrona)

Programação composicional

Recapitulando

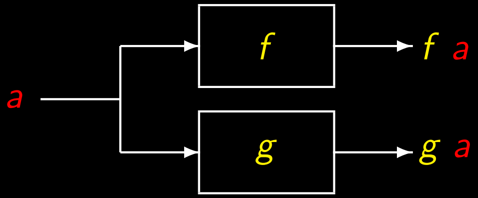
$$(f \cdot g)\ a = f\ (g\ a) \tag{2.6}$$



Composição de funções

Recapitulando

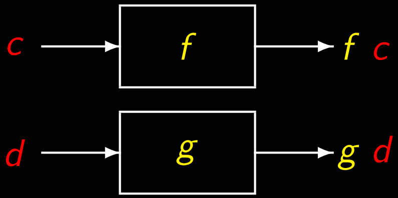
$$\langle f, g \rangle\ a = (f\ a, g\ a) \tag{2.20}$$



"Splits" de funções

Recapitulando

$$f \times g = \langle f \cdot \pi_1, g \cdot \pi_2 \rangle \tag{2.24}$$



Produtos de funções

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f , g \rangle"] \arrow[ur, "g"'] & \\
& D \arrow[u, "h"] &
\end{tikzcd}$$

$$
\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& D \arrow[ul, "f \cdot h"] \arrow[u, "\langle f \cdot h, g \cdot h \rangle"] \arrow[ur, "g \cdot h"'] &
\end{tikzcd}
$$

Fusão-$\times$

$$\langle f, g\rangle \cdot h = \langle f \cdot h, g \cdot h\rangle \qquad (2.26)$$

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
 & A \arrow[ul, "f"] \arrow[u, "\langle f, g\rangle"] \arrow[ur, "g"'] & \\
 & D \arrow[u, "h"] \arrow[uul, "f \cdot h"] \arrow[uur, "g \cdot h"'] &
\end{tikzcd}$$

$$
\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & & C \arrow[u, "k"'] \\
& A \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, "(h \times k) \cdot \langle f, g \rangle" description] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & & C \arrow[u, "k"'] \\
& A \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uul, "h \cdot f"'] \arrow[uur, "k \cdot g"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & & C \arrow[u, "k"'] \\
& A \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, "\langle h \cdot f, k \cdot g \rangle" description] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & & C \arrow[u, "k"'] \\
& A \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, "(h \times k) \cdot \langle f, g \rangle" description] &
\end{tikzcd}
$$

Absorção-$\times$

$$(h \times k) \cdot \langle f, g \rangle = \langle h \cdot f, k \cdot g \rangle \tag{2.27}$$

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
D \arrow[u, "h"] & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "h \times k"'] & E \arrow[u, "k"'] \\
& C \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$
\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "h \times k"] & C \arrow[u, "k"'] \\
& A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}
$$

Natural-π_1, natural-π_2

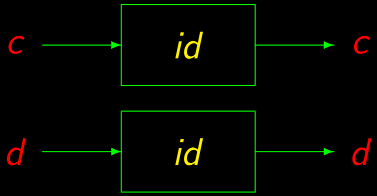
$$\pi_1 \cdot (h \times k) \;=\; h \cdot \pi_1 \tag{2.28}$$

$$\pi_2 \cdot (h \times k) = k \cdot \pi_2 \tag{2.29}$$

$$\begin{tikzcd}
D & D \times E \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & E \\
B \arrow[u, "h"] & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "h \times k"'] & C \arrow[u, "k"']
\end{tikzcd}$$

Functor-*id*-$\times$

$$id \times id = id \tag{2.31}$$



Produto de identidades é a identidade.

Functor-$\times$

$$(f \times h) \cdot (g \times k) = (f \cdot g) \times (h \cdot k) \tag{2.30}$$

Composição de produtos é o **produto** das composições.

Duas leis que faltam

Reflexão-$\times$

$$\langle \pi_1, \pi_2 \rangle = id \tag{2.32}$$

Eq-$\times$

$$\langle i, j \rangle = \langle f, g \rangle \;\Leftrightarrow\; \left\{ \begin{array}{l} i = f \\ j = g \end{array} \right. \tag{2.64}$$

$$
\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
& A \times B \arrow[ul, "\pi_1"] \arrow[u, "id"] \arrow[ur, "\pi_2"'] &
\end{tikzcd}
$$

E agora o mais importante...

Recordar o **cancelamento**-$\times$:

$$
\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
 & A \arrow[ul, "f"] \arrow[u, "\langle f,g \rangle"'] \arrow[ur, "g"'] &
\end{tikzcd}
$$

$$\pi_1 \cdot \langle f, g \rangle = f \qquad \pi_2 \cdot \langle f, g \rangle = g$$

$$\begin{cases} \pi_1 \cdot \langle f, g \rangle = f \\ \pi_2 \cdot \langle f, g \rangle = g \end{cases}$$

$$k = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$$

Universal-$\times$

$$k = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$$

Universal-$\times$

Existência

$$k = \langle f, g \rangle \Rightarrow \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$$

“Existe uma solução — $k = \langle f, g \rangle$ — para as equações da direita”

Universal-$\times$

Unicidade

$$k = \langle f, g \rangle \;\Leftarrow\; \left\{ \begin{array}{l} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{array} \right.$$

"As equações da direita só têm uma solução: $k = \langle f, g \rangle$"

Equações!

$$\left\{\begin{array}{l} x = 2\,y \\ z = \frac{y}{3} \\ x + y + z = 10 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} x = 6 \\ z = 1 \\ y = 3 \end{array}\right.$$

Equações!

Problema

Resolver a equação

$$\langle f, g \rangle = id$$

em ordem a f *e a* g.

Equações!

Resolução

Em $k = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$ fazer $k = id$

$$id = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 = f \\ \pi_2 = g \end{cases}$$

Equações!

$$id = \langle f, g \rangle \;\Leftrightarrow\; \begin{cases} \pi_1 = f \\ \pi_2 = g \end{cases}$$

Substituindo:

$$id = \langle \pi_1, \pi_2 \rangle$$

Reflexão-$\times$

Problema

Resolver a equação

$$\langle h, k \rangle = \langle f, g \rangle$$

(1 equação, 4 incógnitas)

Resolução

$$\langle h, k \rangle = \langle f, g \rangle$$

$$\Leftrightarrow \qquad \{ \ \text{universal-}\times \ \}$$

$$\begin{cases} \pi_1 \cdot \langle h, k \rangle = f \\ \pi_2 \cdot \langle h, k \rangle = g \end{cases}$$

$$\Leftrightarrow \qquad \{ \ \text{cancelamento-}\times \ \}$$

$$\begin{cases} h = f \\ k = g \end{cases}$$

Eq-$\times$!

$\langle \pi_1, \pi_2 \rangle = id$

$\langle \pi_2, \pi_1 \rangle$?

$$\begin{tikzcd}
B & B \times A \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & A \\
& A \times B \arrow[ul, "\pi_2"] \arrow[u, "\langle \pi_2, \pi_1 \rangle"] \arrow[ur, "\pi_1"'] &
\end{tikzcd}$$

Problema

Resolver

$$\langle \pi_2, \pi_1 \rangle \cdot k = id$$

em ordem a k

$$\begin{tikzcd} A \times B \arrow[rr, bend left, "\langle \pi_2, \pi_1 \rangle"] & & B \times A \arrow[ll, bend left, "k"] \end{tikzcd}$$

Resolução

$$
\begin{array}{ll}
& \langle \pi_2, \pi_1 \rangle \cdot k = id \\
\Leftrightarrow & \{ \text{ fusão-}\times \text{ } \} \\
& \langle \pi_2 \cdot k, \pi_1 \cdot k \rangle = id \\
\Leftrightarrow & \{ \text{ universal-}\times \text{ } \} \\
& \left\{ \begin{array}{l} \pi_2 \cdot k = \pi_1 \\ \pi_1 \cdot k = \pi_2 \end{array} \right. \\
\Leftrightarrow & \{ \text{ trivial } \} \\
& \left\{ \begin{array}{l} \pi_1 \cdot k = \pi_2 \\ \pi_2 \cdot k = \pi_1 \end{array} \right. \\
\Leftrightarrow & \{ \text{ universal-}\times \text{ } \} \\
& k = \langle \pi_2, \pi_1 \rangle
\end{array}
$$

Swap

$$\begin{tikzcd} A \times B \arrow[rr, bend left, "\langle \pi_2, \pi_1 \rangle"] & & B \times A \arrow[ll, bend left, "\langle \pi_2, \pi_1 \rangle"] \end{tikzcd}$$

$$swap = \langle \pi_2, \pi_1 \rangle$$

$$swap \cdot swap = id$$

Até agora

$f \cdot g$ Composição sequencial

$\langle f, g \rangle$ Composição paralela

Associatividade

$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Associatividade?

$$\langle \langle f, g \rangle, h \rangle = \langle f, \langle g, h \rangle \rangle ?$$

Não! mas...

$\langle\langle f, g\rangle, h\rangle$

$$\begin{tikzcd} A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\ & D \arrow[ul, "f"] \arrow[u, "{\langle f, g\rangle}"] \arrow[ur, "g"'] & \end{tikzcd}$$

$\langle\langle f, g\rangle, h\rangle$

$$\begin{tikzcd}
A \times B & (A \times B) \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& D \arrow[ul, "\langle f, g\rangle"] \arrow[u, "\langle\langle f, g\rangle, h\rangle" description] \arrow[ur, "h"'] &
\end{tikzcd}$$

$\langle f, \langle g, h \rangle \rangle$

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& D \arrow[ul, "g"] \arrow[u, "{\langle g, h \rangle}"] \arrow[ur, "h"'] &
\end{tikzcd}$$

$\langle f, \langle g, h \rangle \rangle$

$$\begin{tikzcd}
A & A \times (B \times C) \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \times C \\
& D \arrow[ul, "f"] \arrow[u, "{\langle f, \langle g, h \rangle \rangle}" description] \arrow[ur, "{\langle g, h \rangle}"'] &
\end{tikzcd}$$

$\langle\langle f, g\rangle, h\rangle$

$$\begin{tikzcd}
A \times B & (A \times B) \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& D \arrow[ul, "\langle f, g\rangle"] \arrow[u, "\langle\langle f, g\rangle, h\rangle"] \arrow[ur, "h"'] &
\end{tikzcd}$$

$$
\begin{tikzcd}
(A \times B) \times C \arrow[rr, bend left, "k"] & & A \times (B \times C) \\
& D \arrow[ul, "\langle\langle f, g\rangle, h\rangle"] \arrow[ur, "\langle f, \langle g, h\rangle\rangle"'] &
\end{tikzcd}
$$

$$k \cdot \langle\langle f, g\rangle, h\rangle = \langle f, \langle g, h\rangle\rangle$$

$$k \cdot \langle\langle f, g\rangle, h\rangle = \langle f, \langle g, h\rangle\rangle$$

$$k \cdot \underbrace{\langle\langle f, g\rangle, h\rangle}_{id?} = \langle f, \langle g, h\rangle\rangle$$

Resolver $\langle\langle f, g\rangle, h\rangle = id$

$$
\begin{aligned}
& \langle\langle f, g\rangle, h\rangle = id \\
\Leftrightarrow \quad & \{\ \text{universal-}\times\ \} \\
& \begin{cases} \pi_1 = \langle f, g\rangle \\ \pi_2 = h \end{cases} \\
\Leftrightarrow \quad & \{\ \text{universal-}\times\ \} \\
& \begin{cases} \pi_1 \cdot \pi_1 = f \\ \pi_2 \cdot \pi_1 = g \\ \pi_2 = h \end{cases}
\end{aligned}
$$

Substituir soluções

$$\begin{cases} \pi_1 \cdot \pi_1 = f \\ \pi_2 \cdot \pi_1 = g \\ \pi_2 = h \end{cases}$$

$$k \cdot \underbrace{\langle \langle f, g \rangle, h \rangle}_{id} = \langle f, \langle g, h \rangle \rangle$$

Substituir soluções

$$\begin{cases} \pi_1 \cdot \pi_1 = f \\ \pi_2 \cdot \pi_1 = g \\ \pi_2 = h \end{cases}$$

$$k = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, \pi_2 \rangle \rangle$$

Podemos melhorar...

$$k = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, \pi_2 \rangle \rangle$$

Podemos melhorar...

$$
\begin{array}{ll}
 & k = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, \pi_2 \rangle \rangle \\
\Leftrightarrow & \{ \ \pi_2 = id \cdot \pi_2 \ \} \\
 & k = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, id \cdot \pi_2 \rangle \rangle
\end{array}
$$

Podemos melhorar...

$$
\begin{array}{ll}
 & k = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, \pi_2 \rangle\rangle \\
\Leftrightarrow & \{\ \pi_2 = id \cdot \pi_2\ \} \\
 & k = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, id \cdot \pi_2 \rangle\rangle \\
\Leftrightarrow & \{\ f \times g = \langle f \cdot \pi_1, g \cdot \pi_2 \rangle\ \} \\
 & k = \langle \pi_1 \cdot \pi_1, \pi_2 \times id \rangle
\end{array}
$$

$$
\begin{tikzcd}
(A \times B) \times C \arrow[rr, bend left, "\langle \pi_1 \cdot \pi_1, \pi_2 \times id \rangle"] & & A \times (B \times C) \\
& D \arrow[ul, "{\langle \langle f, g \rangle, h \rangle}"] \arrow[ur, "{\langle f, \langle g, h \rangle \rangle}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
(A \times B) \times C \arrow[rr, bend left, "\langle \pi_1 \cdot \pi_1 , \pi_2 \times id \rangle"] & & A \times (B \times C) \arrow[ll, bend left, "j"] \\
& D \arrow[ul, "\langle \langle f , g \rangle , h \rangle"] \arrow[ur, "\langle f , \langle g , h \rangle \rangle"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
(A \times B) \times C \arrow[r, bend left, "\langle \pi_1 \cdot \pi_1, \pi_2 \times id \rangle"] & A \times (B \times C) \arrow[l, bend left, "\langle id \times \pi_1, \pi_2 \cdot \pi_2 \rangle"]
\end{tikzcd}
$$

$$\begin{tikzcd}(A \times B) \times C \arrow[rr, bend left, "assocr"] & & A \times (B \times C) \arrow[ll, bend left, "assocl"]\end{tikzcd}$$

$$assocr = \langle \pi_1 \cdot \pi_1, \pi_2 \times id \rangle$$
$$assocl = \langle id \times \pi_1, \pi_2 \cdot \pi_2 \rangle$$

Isomorfismo

$$
\begin{tikzcd}
(A \times B) \times C \arrow[r, bend left, "assocr"] \arrow[r, phantom, "\cong"] & A \times (B \times C) \arrow[l, bend left, "assocl"]
\end{tikzcd}
$$

$$assocr \cdot assocl = id$$

$$assocl \cdot assocr = id$$

Isomorfismo

$$\begin{tikzcd} A \times B \arrow[rr, bend left, "swap"] \arrow[rr, phantom, "\cong"] & & B \times A \arrow[ll, bend left, "swap"] \end{tikzcd}$$

$$swap \cdot swap = id$$

Isomorfismo

$$\begin{tikzcd} A \arrow[rr, bend left, "f"] \arrow[rr, phantom, "\cong"] & & B \arrow[ll, bend left, "g"] \end{tikzcd}$$

$$f \cdot g = id$$

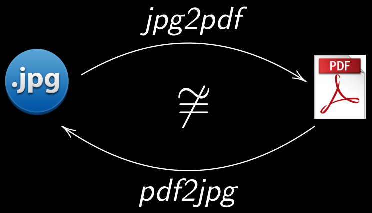
$$g \cdot f = id$$

Isomorfismo

$$\underbrace{\textit{iso}}_{\textit{a mesma}}\ (\iota\sigma o) + \underbrace{\textit{morfismo}}_{\textit{forma}}\ \ (\mu o\rho\phi\iota\sigma\mu o\zeta)$$

"Forma semelhante"

Problema prático!



$$jpg2pdf \cdot pdf2jpg \neq id$$

$$pdf2jpg \cdot jpg2pdf \neq id$$

Conversão de formatos

Necessidade

$$A \xrightarrow{f} B$$

Reutilizável

$$C \xrightarrow{r} D$$

Conversão de formatos

Necessidade Reutilizável

$$\begin{tikzcd} A \arrow[rr, "exporta"] \arrow[d, "f"'] \arrow[drr, phantom, "(\cong)"] & & C \arrow[d, "r"] \\ B & & D \arrow[ll, "importa"] \end{tikzcd}$$

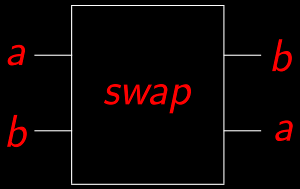
$$f = importa \cdot r \cdot exporta$$

A propósito de *swap*

$$\begin{tikzcd} A \times B \arrow[rr, bend left, "swap"] \arrow[rr, phantom, "\cong"] & & B \times A \arrow[ll, bend left, "swap"] \end{tikzcd}$$

Isomorfismos são computações **reversíveis**

A propósito de *swap*



swap é uma das unidades básicas da **programação quântica**

Problem

Retrieve the address of a civil servant, knowing that she/he can be identified either by a citizen card number (CC) or a fiscal number (NIF).

$$address : Iden \to Address$$

$$Iden = CC \cup NIF$$

Problem!

$$CC = \mathbb{N}$$
$$NIF = \mathbb{N}$$

$$Iden = CC \cup NIF = \mathbb{N} \cup \mathbb{N} = \mathbb{N}$$

$$address : \mathbb{N} \rightarrow Address \text{ (!)}$$

In general

We need to fix

$$m : A \cup B \to C$$

starting from

$$A \cup B = \{ a \mid a \in A \} \cup \{ b \mid b \in B \}$$

Disjoint union

In need of something like

$$\{(1, a) \mid a \in A\} \cup \{(2, b) \mid b \in B\}$$

we define:

$$A + B = \{(1, a) \mid a \in A\} \cup \{(2, b) \mid b \in B\}$$

Disjoint union

Clearly,

$$A + B = \{\, i_1\ a \mid a \in A \,\} \cup \{\, i_2\ b \mid b \in B \,\}$$

upon further defining:

$$i_1\ a = (1, a)$$
$$i_2\ b = (2, b)$$

Disjoint union

Altogether:

$$m : A + B \to C$$
$$i_1 : A \to A + B$$
$$i_2 : B \to A + B$$

$$\begin{tikzcd}
A \arrow[r, "i_1"] \arrow[dr, "f"'] & A + B \arrow[d, "m"] & B \arrow[l, "i_2"'] \arrow[dl, "g"] \\
& C &
\end{tikzcd}$$

$$f = m \cdot i_1 \qquad g = m \cdot i_2$$

$$
\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
& C \arrow[ul, "f"] \arrow[u, "{\langle f, g \rangle}"] \arrow[ur, "g"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, "i_1"] \arrow[dr, "f"'] & A + B \arrow[d, "{[f, g]}"] & B \arrow[l, "i_2"'] \arrow[dl, "g"] \\
& C &
\end{tikzcd}
$$

+-Universal

$$k = [f, g] \Leftrightarrow \begin{cases} k \cdot i_1 = f \\ k \cdot i_2 = g \end{cases}$$

Compare with

$$k = \langle f, g \rangle \Leftrightarrow \begin{cases} \pi_1 \cdot k = f \\ \pi_2 \cdot k = g \end{cases}$$

“Alternating” functions

$$[f, g] : A + B \to C$$

$$[f, g]\ x \;=\; \begin{cases} x = i_1\ a \Rightarrow f\ a \\ x = i_2\ b \Rightarrow g\ b \end{cases}$$

$f + g$?

Sum of two functions

$$
\begin{tikzcd}
D \arrow[r, "i_1"] \arrow[d, "f"'] \arrow[dr] & D + E \arrow[d, "f + g"] & E \arrow[l, "i_2"'] \arrow[d, "g"] \arrow[dl] \\
B \arrow[r, "i_1"'] & B + C & C \arrow[l, "i_2"]
\end{tikzcd}
$$

$$f + g = [i_1 \cdot f, i_2 \cdot g]$$

Coproduct laws

"Just reverse the arrows", cf.

+-Absorption $$[h, k] \cdot (f + g) = [h \cdot f, k \cdot g] \quad (2.43)$$

+-Fusion $$f \cdot [h, k] = [f \cdot h, f \cdot k] \quad (2.42)$$

+-Reflexion $$[i_1, i_2] = id \quad (2.41)$$

Coproduct laws

+-Equality $$[h, k] = [f, g] \Leftrightarrow \begin{cases} h = f \\ k = g \end{cases} \qquad (2.66)$$

+-Functor $$(h + k) \cdot (f + g) = h \cdot f + k \cdot g \qquad (2.44)$$

+-Functor-*id* $$id + id = id \qquad (2.45)$$

and so on

Summary

$f \cdot g$	Sequential composition
$\langle f, g \rangle$	Parallel composition
$f \times g$	Product composition
$[f, g]$	Alternative composition
$f + g$	Coproduct (function sum)

Compositional programming

What about...

$$f \times (g + h) \stackrel{?}{=} f \times g + f \times h$$

...?

$$A \times (B + C) \stackrel{?}{\cong} A \times B + A \times C$$

Moreover, any "0" such that

$$A \times 0 = 0 \quad ? \; A + 0 = A \quad ??$$

Recall

$$\begin{cases} id : A \to A \\ id\ a = a \end{cases}$$

$$\begin{tikzcd} A \arrow[r, bend left, "id"] \arrow[r, phantom, "\cong"] & A \arrow[l, bend left, "id"] \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, phantom, "\cong"] & B \arrow[l, bend left, "g"] \end{tikzcd}$$

$$f \cdot g = id$$

$$g \cdot f = id$$

Recall

$$\begin{cases} swap : A \times B \to B \times A \\ swap\ (a, b) = (b, a) \end{cases}$$

$$\begin{tikzcd} A \times B \arrow[rr, bend left, "swap"] \arrow[rr, phantom, "\cong"] & & B \times A \arrow[ll, bend left, "swap"] \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[rr, bend left, "?"] \arrow[rr, phantom, "?"] & & A \times B \arrow[ll, bend left, "\pi_1"] \end{tikzcd}$$

$$\pi_1 \cdot \langle id, h \rangle = id$$

$$\langle id, h \rangle \cdot \pi_1 \neq id$$

Even worse than π_1...

$$\begin{cases} \mathit{zero}\ x = 0 \\ \mathit{one}\ x = 1 \end{cases}$$

$$\mathit{one}\ 10 = 1$$

$$\mathit{one}\ \texttt{"string"} = 1$$

Even worse than π_1...

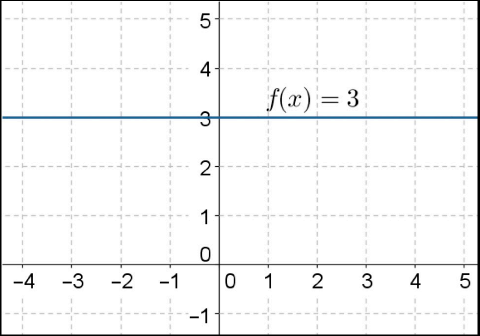
$$\begin{cases} \mathit{zero}\ x = 0 \\ \mathit{one}\ x = 1 \end{cases}$$

$$\mathit{one}\ 10 = 1$$

$$\mathit{one}\ \texttt{"string"} = 1$$

$$\mathit{zero}\ \texttt{"string"} = 0$$

$$f\ x = 3$$



Constant functions

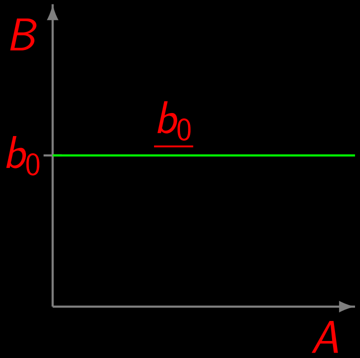
$3 \in \mathbb{N}_0$

$f\ x = 3$

$A \xrightarrow{f} \mathbb{N}_0$

$b_0 \in B$

$$\begin{cases} \underline{b_0} : A \to B \\ \underline{b_0}\ a = b_0 \end{cases}$$



(**Haskell**: `const` b_0)

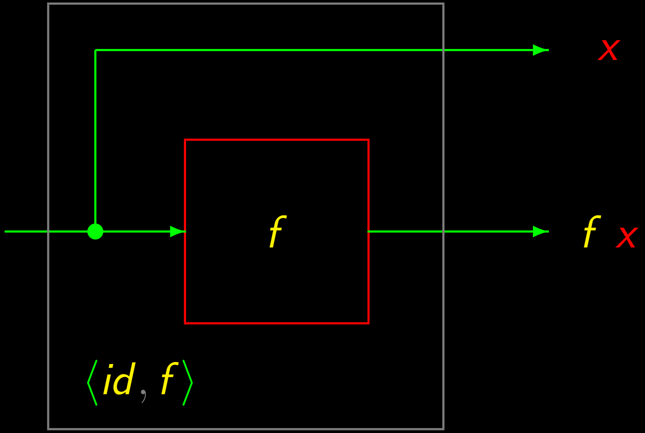
Properties

$$\underline{b} \cdot f = \underline{b}$$

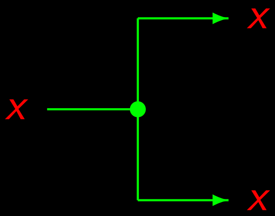
$$g \cdot \underline{b} = \underline{g\ b}$$

$$\begin{tikzcd}
 & C \arrow[dl, "\underline{b}"'] \arrow[d, "f"] \\
B \arrow[d, "g"'] & A \arrow[l, "\underline{b}"'] \arrow[dl, "\underline{g\ b}"] \\
D &
\end{tikzcd}$$

Data flow

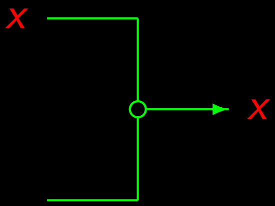


Data duplication



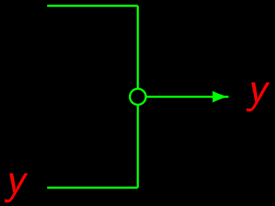
$$dup = \langle id, id \rangle$$

Data joins



$$join = [id, id]$$

Data joins



$$join = [id, id]$$