# Portfolio optimization using GARCH-EVT-Copula: an application with FTSE MIB stocks

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# Summary

# Introduction

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The goal of this thesis is to replicate and extend the trading strategy proposed in *"Portfolio optimization based on GARCH-EVT-Copula forecasting models"* by Sahamkhadam et al. (2018)[1].

Key contributions and modifications:

- Use of a GJR-GARCH(1,1) model to account for volatility asymmetry (leverage effect).
- Application of skew-$t$ and skew-normal copulas to better capture tail dependence.
- Replacement of the original risk metric with Entropic Value at Risk (EVaR) for portfolio optimization.

[1] M. Sahamkhadam, A. Stephan, R. Ostermark. Portfolio optimization based on garch-evt-copula forecasting models. Int. Journal of Forecasting, 34:497–506, 2018.

# Methodology

# Model for returns

The logarithmic return at time $t$ is:

$$r_t = \log(p_t) - \log(p_{t-1})$$

where $p_t$ is the asset price at time $t$.

The return is modeled as:

$$\begin{aligned} r_t &= \mu_t + \epsilon_t \\ \epsilon_t &= \sigma_t z_t, \quad z_t \overset{\text{i.i.d.}}{\sim} (0, 1) \end{aligned}$$

where

- $\mu_t = \mathbb{E}[r_t \mid \mathcal{F}_{t-1}]$ is the conditional mean,
- $\epsilon_t$ is the zero-mean error term with $\text{Var}(\epsilon_t \mid \mathcal{F}_{t-1}) = \sigma_t^2$,
- $\mathcal{F}_{t-1}$ is the information set up to time $t-1$.

# ARMA Model

The conditional mean $\mu_t$ is modeled with an ARMA(1,1) model:

$$r_t = \phi_0 + \phi_1 r_{t-1} + \epsilon_t + \theta_1 \epsilon_{t-1}$$

where:

- $\phi_0$: intercept,
- $\phi_1$: autoregressive coefficient,
- $\theta_1$: moving average coefficient.

# GJR-GARCH Model

The conditional variance $\sigma_t^2$ is modeled with a GJR-GARCH(1,1) model:

$$\sigma_t^2 = \omega + \alpha\sigma_{t-1}^2 z_{t-1}^2 + \gamma\sigma_{t-1}^2 I(z_{t-1} < 0) z_{t-1}^2 + \beta\sigma_{t-1}^2$$

where:

- $\omega > 0$: constant term,
- $\alpha, \beta \geq 0$: ARCH and GARCH effects,
- $\gamma$: asymmetry coefficient (leverage effect),
- $I(z_{t-1} < 0)$: indicator function,
- $z_{t-1}$: standardized residuals.

# The `spd` R Package

The `spd` R package fits a semi-parametric model to data by combining:

- Non-parametric modelling of the bulk of the data (kernel density estimation)
- Parametric modelling of the tails (Peaks Over Threshold method)

Kernel density estimation is performed under the following assumptions:

- Gaussian kernel
- Bandwidth selected via the **oversmoothed bandwidth selector**:

$$\hat{\lambda}_{\mathrm{OS}} = \left( \frac{243\, R(K)}{25\, \mu_2(K)^2\, n} \right)^{1/5} s$$

# Copula model

A **copula** is a multivariate distribution function on the unit cube $[0,1]^d$ with uniform marginals.

### Definition

A $d$-dimensional copula is a function $C : [0,1]^d \to [0,1]$ that satisfies:

- $C(u_1, \ldots, u_d) = 0$ if any $u_i = 0$,
- $C(1, \ldots, u_i, \ldots, 1) = u_i$ for all $i$,
- $C$ is $d$-non-decreasing.

# Mean-EVaR Portfolio

The Entropic Value at Risk (EVaR) is a risk measure introduced by Ahmadi-Javid (2012):

$$\mathrm{EVaR}_\alpha = \inf_{z>0} \left\{ \frac{1}{z} \log \left( \frac{1}{1-\alpha} \, \mathbb{E} \left[ \exp(z\xi) \right] \right) \right\}$$

where $\xi = -\mathbf{w}'\mathbf{r}$ is the **portfolio loss**.

The portfolio optimization problem is:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{maximize}} & \mathbf{w}'\boldsymbol{\mu} - \lambda \, \mathrm{EVaR}_\alpha(\mathbf{w}) \\ \text{subject to} & \mathbf{w} \in \mathcal{W} \end{array}$$

# Empirical Analysis

# Data Overview

Stocks included in the portfolio:

- **Intesa Sanpaolo (ISP.MI)** – Financial sector: Banking
- **UniCredit (UCG.MI)** – Financial sector: Banking
- **Eni (ENI.MI)** – Energy sector: Oil & Gas
- **Stellantis (STLAM.MI)** – Consumer Discretionary: Automotive
- **STMicroelectronics (STM.MI)** – IT sector: Semiconductors

Dataset details:

- Time period: `2019-01-01` to `2025-04-01`
- Frequency: Daily adjusted closing prices

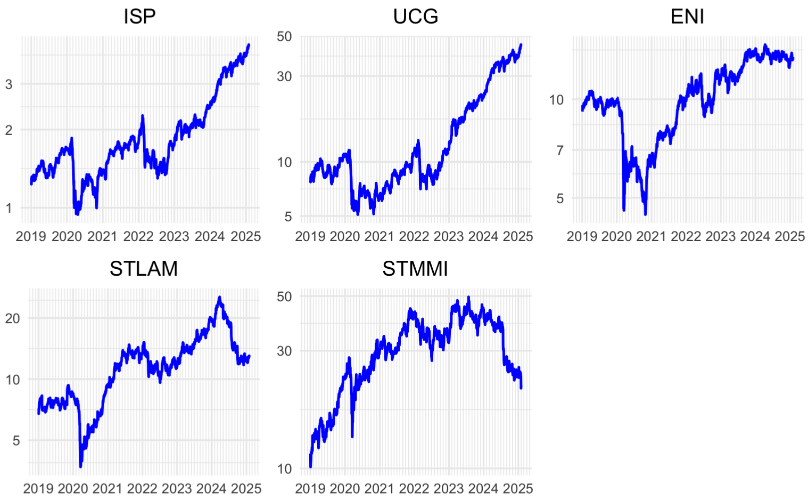


Figure 1: Base-10 logarithmic prices

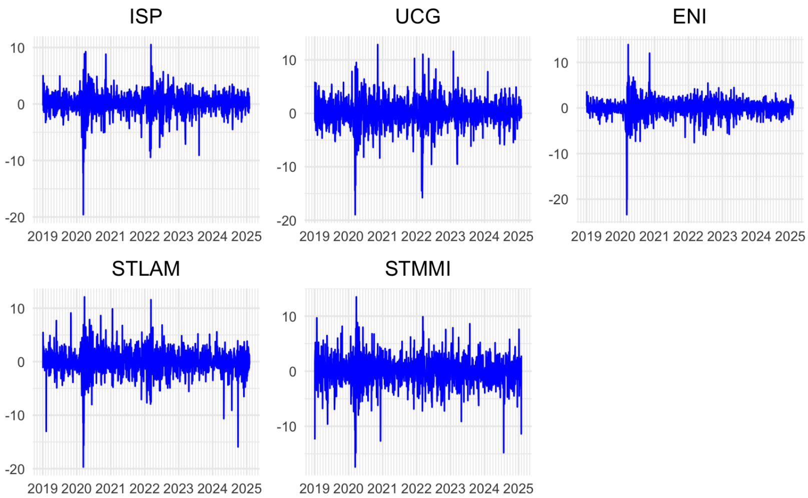


Figure 2: Log-returns

| Stock | Trimmed Mean | MAD | Jarque-Bera | Shapiro-Wilk |
|---|---|---|---|---|
| ISP | 0.1362 | 1.2975 | 0 | 0 |
| UCG | 0.1402 | 1.8629 | 0 | 0 |
| ENI | 0.0765 | 1.2573 | 0 | 0 |
| STLAM | 0.0955 | 1.5601 | 0 | 0 |
| STMMI | 0.0963 | 1.9952 | 0 | 0 |

Table 1: Descriptive statistics and normality test results

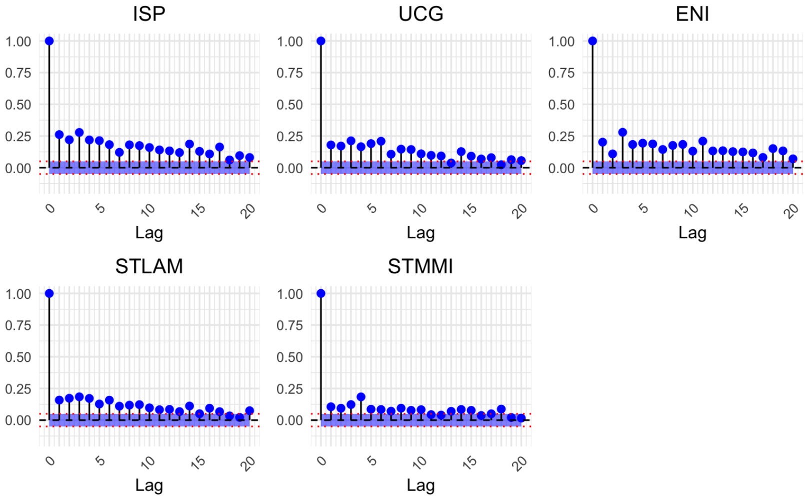


Figure 3: Autocorrelation function of absolute log-returns

# Algorithm for Portfolio Construction (1/2)

Test period: February and March 2025 (41 trading days).

Steps to compute portfolio weights $\mathbf{w}_t$:

1. Fit a dynamic ARMA-GJR-GARCH(1,1) model with skew-$t$ innovations to each return series $r_t$, extract the standardized residuals $z_t$ and store them in the matrix $Z \in \mathbb{R}^{L \times d}$.
2. Fit a semiparametric model on the residuals $z_t$ using the `spd` package.
3. Estimate the parameters of a multivariate skew-$t$ or skew-normal distribution using the data in matrix $Z$.

# Algorithm for Portfolio Construction (2/2)

4. Simulate new pseudo-observations $u_i^{new}$ from the copula corresponding to the fitted distribution and save them in a tensor of dimensions $L \times d \times M$.
5. Apply the quantile function from Step 2 to $u_i^{new}$ in the tensor to obtain new standardized residuals $z_t^{new}$.
6. Compute the one-step-ahead forecast of the standardized residuals $\hat{z}_{t+1}$ as the median across the simulation dimension (last slice) of the tensor.
7. Predict the returns $\hat{\mathbf{r}}_{t+1}$, then solve the portfolio optimization problem to obtain the portfolio weights $\mathbf{w}_t$ by maximizing the objective function.

# Point 3: Multivariate Distributions

- **Multivariate Skew-Normal:**

$$\boldsymbol{Y} \sim \mathrm{SN}_d(\boldsymbol{\xi}, \Omega, \boldsymbol{\alpha})$$

$$f(\boldsymbol{y}) = 2\,\phi_d(\boldsymbol{y}-\boldsymbol{\xi};\Omega)\,\Phi\left(\boldsymbol{\alpha}^\top \omega^{-1}(\boldsymbol{y}-\boldsymbol{\xi})\right) \qquad \boldsymbol{y} \in \mathbb{R}^d$$

- **Multivariate Skew-$t$:**

$$\boldsymbol{Y} \sim \mathrm{ST}_d(\boldsymbol{\xi}, \Omega, \boldsymbol{\alpha}, \nu)$$

$$f(\boldsymbol{y}) = |\omega|^{-1} \cdot f_{\boldsymbol{Z}}(\omega^{-1}(\boldsymbol{y}-\boldsymbol{\xi})) \qquad \boldsymbol{y} \in \mathbb{R}^d$$

**Important:** The skewness parameter $\boldsymbol{\alpha}$ should be different from zero to capture asymmetry.

# Point 7: Optimization Problem

$$
\begin{aligned}
\underset{\mathbf{w},\, t>0}{\text{maximize}} \quad & \mathbf{w}'\hat{\boldsymbol{r}}_{t+1} \\
& - \lambda \left( t \log \left( \sum_{i=1}^{T} \exp \left\{ -\frac{1}{t} \mathbf{w}' \mathbf{r}_i \right\} \right) - t \log \left[ (1-\alpha) T \right] \right) \\
\text{subject to} \quad & \sum_{i=1}^{d} w_i = 1 \\
& 0.1 \leq w_i \leq 0.25, \quad \forall i = 1, \ldots, d
\end{aligned}
$$

- The first constraint enforces **full investment**
- The second constraint imposes **holding constraints**

# Final Results: Portfolios Overview

We compare the performance of three portfolios:

- **Equal Weight Portfolio (EWP)**
- **Portfolio 1** – constructed using skew-$t$ copula
- **Portfolio 2** – constructed using skew-normal copula

The out-of-sample backtest period is from `2025-02-03` to `2025-03-31`. Portfolios are rebalanced every 5 trading days. The portfolio return at time $t$ is defined as:

$$R_t^{\text{portf}} \equiv \mathbf{w}_{t-1}' \cdot \mathbf{r}_t,$$

The cumulative return is calculated as:

$$R_{\text{cum}} = \prod_{t=1}^{T} (1 + R_t^{\text{portf}}) - 1$$

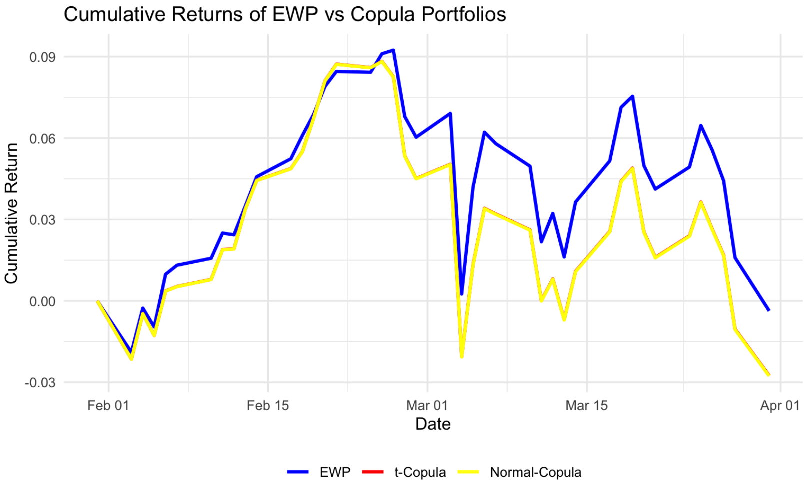


Figure 4: Cumulative returns over the out-of-sample period

# Final Results: Cumulative Returns

| **Portfolio** | **EWP** | **Portfolio 1** | **Portfolio 2** |
|---|---|---|---|
| Cumulative Return (%) | $-0.3687$ | $-2.7546$ | $-2.7871$ |

Table 2: Final cumulative returns (%)

The Equal Weight Portfolio (EWP) achieves a higher cumulative return than both Portfolio 1 and Portfolio 2 over the evaluation period.

# Copula Portfolios Weights Comparison

| Skew-Normal | | | | | Skew-t | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **ISP** | **UCG** | **ENI** | **STLAM** | **STMMI** | **ISP** | **UCG** | **ENI** | **STLAM** | **STMMI** |
| 0.1643503 | 0.1 | 0.25 | 0.2356497 | 0.25 | 0.1654827 | 0.1 | 0.25 | 0.2345173 | 0.25 |
| 0.1650007 | 0.1 | 0.25 | 0.2349993 | 0.25 | 0.1654942 | 0.1 | 0.25 | 0.2345058 | 0.25 |
| 0.1644812 | 0.1 | 0.25 | 0.2355188 | 0.25 | 0.1652240 | 0.1 | 0.25 | 0.2347760 | 0.25 |
| 0.1636792 | 0.1 | 0.25 | 0.2363208 | 0.25 | 0.1651341 | 0.1 | 0.25 | 0.2348659 | 0.25 |
| 0.1604227 | 0.1 | 0.25 | 0.2395773 | 0.25 | 0.1601833 | 0.1 | 0.25 | 0.2398167 | 0.25 |
| 0.2153573 | 0.1 | 0.25 | 0.1846427 | 0.25 | 0.2158709 | 0.1 | 0.25 | 0.1841291 | 0.25 |
| 0.2156433 | 0.1 | 0.25 | 0.1843567 | 0.25 | 0.2160127 | 0.1 | 0.25 | 0.1839873 | 0.25 |
| 0.2157493 | 0.1 | 0.25 | 0.1842507 | 0.25 | 0.2170222 | 0.1 | 0.25 | 0.1829778 | 0.25 |
| 0.2210336 | 0.1 | 0.25 | 0.1789664 | 0.25 | 0.2227923 | 0.1 | 0.25 | 0.1772077 | 0.25 |

Table 3: Portfolio weights across rebalancing periods

The weights are nearly identical across all periods, suggesting that the skew-$t$ copula offers no additional advantage in this case.

# Conclusions and improvements

# Asset Selection

**Problems:**

- Based on personal trading experience.
- Lack of generality and statistical rigor.

**Possible Improvements:**

- Diversify across sectors, regions, and asset classes.
- Apply clustering techniques on raw financial data (e.g., from Yahoo Finance).
- Use ETF portfolios that track major indices (e.g., S&P 500, MSCI World).

# Statistical Modelling

**Limitations:**

- Only one innovation distribution considered.
- Only a single GARCH-type model (GJR-GARCH) applied.

**Possible Improvements / Alternatives:**

- Sequential model selection using information criteria (e.g., AIC, BIC).
- Multivariate frameworks: VECM + DCC-GARCH.
- Stochastic Volatility models for more flexible volatility dynamics.

# Portfolio Construction

**Limitations:**

- Based on a single realization of historical data.
- Noisy parameter estimates due to limited sample.
- Risk of "noise maximization" in optimization.

**Possible Improvement:**

- Apply time-series bootstrap methods to generate alternative scenarios and compute more robust weights by repeating the backtest over replications.

Thanks for your attention!