PSYCHOMETRIKA—VOL. 45, NO. 3.
SEPTEMBER, 1980.
NOTES AND COMMENTS

# A NOTE ON IDENTIFICATION IN THE OBLIQUE AND ORTHOGONAL FACTOR ANALYSIS MODELS

JAMES ALGINA

UNVERSITY OF FLORIDA

Conditions for removing the indeterminancy due to rotation are given for both the oblique and orthogonal factor analysis models. The conditions indicate why published counterexamples to conditions discussed by Jöreskog are not identifiable.

Key Words: identification, orthogonal rotation, oblique rotation, factor analysis.

In the context of orthogonal factor analysis, Jöreskog [1969] gave a rule for specifying $r(r-1)/2$ of the elements in a $(p \times r)$ factor loading matrix in order to identify the matrix. Dunn [1973] and Jennrich [1978] have given counterexamples that show that specifications conforming to this rule need not identify the factor loading matrix. Dunn gave sufficient conditions for identification, based on the specification of $r(r-1)/2$ zeros. Jennrich gave sufficient conditions for the number of rotationally equivalent factor loading matrices, obtained by reflection of the rows of the initical matrix, to be $2^r$. His conditions are based on the specification of $r(r-1)/2$ nonzero elements. Jöreskog [1969] also gave a rule for restricting the diagonal elements of the factor dispersion matrix to unity and specifying $r(r-1)$ of the elements of a factor loading matrix in order to identify these two matrices. One purpose of this note is to give a counterexample to these conditions. A second purpose of this note is to report conditions for identification of the factor loading matrix in both the orthogonal and oblique factor analysis models. The conditions for the oblique model are based on Fisher's [1966] discussion of the identification problem in econometrics. The conditions for the orthogonal model are simple modification of those for the oblique model.

Suppose $S = AMA' + U^2$ is the oblique factor analysis model. It is well-known that all matrices $A^* = AT$ and $M^* = T^{-1}M(T^{-1})'$ satisfy the factor analysis model. The following is a counterexample to Jöreskog's [1969] conjecture with regard to identification in oblique factor analysis. Suppose

$$A = \begin{bmatrix} .8 & a_{12} \\ a_{21} & .8 \\ a_{31} & a_{32} \\ a_{41} & a_{42} \\ a_{51} & a_{52} \end{bmatrix}$$

where the $a_{ij}$ are not specified. Then, if it happens that $a_{21} = a_{12} = .4$ and the correlation between factor 1 and 2 is .6875, postmultiplying $A$ by

$$T = \begin{bmatrix} .5 & 1.0 \\ 1.0 & .5 \end{bmatrix}$$

The author would like to thank Gordon Bechtel and the reviewers for their comments and suggestions.
Requests for reprints should be sent to James Algina, 1413 Norman Hall, University of Florida, Gainesville, FL 32611.

0033-3123/80/0900-2823$00.50/0

yields

$$\begin{bmatrix} .8 & 1.0 \\ 1.0 & .8 \\ .5a_{31} + a_{32} & a_{31} + .5a_{32} \\ .5a_{41} + a_{42} & a_{41} + .5a_{42} \\ .5a_{51} + a_{52} & a_{51} + .5a_{52} \end{bmatrix}$$

with the interfactor correlation approximately equal to $-.14$. The fixed elements of $A$ and $M$ remain the same but the free elements change.

We now present the conditions for identification. The matrices $A$ $(p \times r)$ and $M$ $(r \times r)$ are said to be identified if and only if $T = I$. The $j^{\text{th}}$ column of $A$, say $\mathbf{a}_j$, is said to be identified if and only if the $j^{\text{th}}$ column of $T$, say $\mathbf{t}_j$, is $\mathbf{e}_j$, a column vector with one in row $j$ and zeros elsewhere. Let $C_j(m_j \times p)$ be a known matrix and $\mathbf{x}_j \neq \mathbf{0}$ a known vector such that

$$C_j \mathbf{a}_j = \mathbf{x}_j \tag{1}$$

expresses $m_j$ restrictions on $\mathbf{a}_j$.

*Theorem 1:* Under the restrictions given in (1) a necessary and sufficient condition for $\mathbf{a}_j$ to be identified is that the associated rank of $C_j A$ is $r$ for $j = 1, \cdots, r$.

*Proof:* Let $\mathbf{a}_j^*$ be the $j^{\text{th}}$ column of $A^*$. Then $C_j \mathbf{a}_j^* = C_j A \mathbf{t}_j = \mathbf{x}_j$. This is a system of $m_j$ equations with $r$ unknowns. It may be seen that one solution is $\mathbf{t}_j = \mathbf{e}_j$. It is well-known that the solution to a system of equations is unique if and only if the rank of the coefficient matrix is equal to the number of unknowns [Searle, 1971, Theorem 4, p. 6] and this is the condition required by the theorem. □

The conditions state that in order to identify $A$ and $M$, $r$ linearly independent restrictions must be placed on each $\mathbf{a}_j$. Further $C_j$ cannot be a linear combination of the matrices expressing restrictions on the other $r - 1$ columns of $A$. Even if restrictions conforming to the above are adopted, the identification of $A$ can fail if it happens that one or more additional columns of $A$ satisfy the restrictions placed on $\mathbf{a}_j$. This seems unlikely to happen in practice. It should be noted that with restrictions of the kind under consideration, it is not necessary to restrict the diagonal elements of $M$ to be unity. Introducing restrictions on the diagonal elements of $M$ may reduce the fit of the model to the data. As indicated by the counterexample, placing $r - 1$ restrictions on each $\mathbf{a}_j$ and restricting the diagonal elements of $M$ need not identify $A$. However, the following condition can be established for identifying $A$ by placing $r(r - 1)$ restrictions on the columns of $A$ and restricting the diagonal elements of $M$ to unity. Let $m_j$ restrictions

$$C_j \mathbf{a}_j = \mathbf{0} \tag{2}$$

be placed on the $j^{\text{th}}$ column of $A$.

*Theorem 2:* Under the restrictions given by (2) a necessary and sufficient condition for $A$ to be identified is that the rank of $C_j A$ is $(r - 1)$ for $j = 1, \cdots, r$ and diag $M = I$.

*Proof:* Let $\mathbf{a}_j^*$ be the $j^{\text{th}}$ column of $A^*$. Then $C_j \mathbf{a}_j^* = C_j A \mathbf{t}_j = \mathbf{0}$ is a system of $m_j$ homogeneous equations with $r$ unknowns. One solution for $\mathbf{t}_j$ is a scalar multiple of $\mathbf{e}_j$. It is well-known that the dimension of the solution space is exactly $r - p(C_j A)$, where $p$ denotes rank. Therefore, if $p(C_j A) = r - 1$ the dimension of the solution space is exactly 1 and all solutions are a scalar multiple of $\mathbf{e}_j$. Therefore, $T$ must be a diagonal matrix. The condition diag $M = I$ serves to further restrict $T$ to be an identity matrix and therefore $A$ and $M$ are identified. □

A special case of the restrictions in (2) involves exclusion restrictions, that is specifying zeroes in each of the $r$ columns of $A$. Jöreskog [1979] referred to this case as case (iii) and proved that if diag $M = I$ and $A_j$, a submatrix of $A$ consisting of the rows of $A$ which have fixed zero elements in $\mathbf{a}_j$, has rank $(r - 1)$ for $j = 1, 2, \cdots, r$, then $A$ is identified. It should be clear that for exclusion restrictions, the rank conditions of Theorem 2 are equivalent to Jöreskog's [1979] rank conditions. A special case of the restrictions in (1) involves specifying nonzero values in each of the columns of $A$. This case is referred to as case (iv) by Jöreskog [1979]. Theorem 1 therefore provides conditions for identification of $A$ under case (iv).

Suppose that $S = AA' + U^2$ is the orthogonal factor analysis model. Again, it is well-known that all matrices $A^* = AT$, where $TT' = I$, satisfy the model. In order that $A$ be identified up to a column sign change, restrictions are again placed on the columns of $A$. Since $T$ is orthogonal, different numbers of restrictions will be placed on different columns of $A$. Let $E_j$ denote the first $(j - 1)$ rows of an $(r \times r)$ identity matrix, and $(A'C_j' : E_j')$ a partitioned matrix.

*Theorem 3:* Under the restrictions given by (1) a necessary and sufficient condition for identification of $A$ is that $p(C_1 A) = r$ and $p(A'C_j' : E_j') = r$ for $j = 2, \cdots, r$.

*Proof:* By the proof used for Theorem 1, $\mathbf{t}_1 = \mathbf{e}_1$. The restrictions on column 2 are $C_2 \mathbf{a}_2^* = C_2 A\mathbf{t}_2 = \mathbf{x}_2$ and $\mathbf{e}_1'\mathbf{t}_2 = 0$. These may be expressed as

$$\begin{bmatrix} C_2 A \\ \mathbf{e}_1' \end{bmatrix} [\mathbf{t}_2] = \begin{bmatrix} \mathbf{x}_2 \\ 0 \end{bmatrix}. \tag{3}$$

Again this is a system of equations for which $\mathbf{t}_2 = \mathbf{e}_2$ is a solution. If the rank condition is met, there is only one solution and this must be $\mathbf{e}_2$. The proof of necessity proceeds column by column. To show necessity for the first column, suppose there is $C_1 A$ with rank $< r$ such that the only solution to

$$\begin{bmatrix} C_1 A \\ T' \end{bmatrix} [\mathbf{t}_1] \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{e}_1 \end{bmatrix}$$

is $\mathbf{t}_1 = \mathbf{e}_1$. Since the orthogonality requirement $T'\mathbf{t}_1 = \mathbf{e}_1$ does not restrict $\mathbf{t}_1$ to be $\mathbf{e}_1$, this implies that there is exactly one solution to the equation $C_1 A\mathbf{t}_1 = \mathbf{x}_1$ which is a contradiction, provided that $p(C_1 A) < r$. Let $T_2'$ denote the last $r - 1$ rows of $T'$. Since the first row of $T'$ is $\mathbf{e}_1'$, $T = [\mathbf{e}_1 : T_2]$. To show necessity for the second column suppose there is a $(A'C_2' : \mathbf{e}_1)$ with rank less than $r$ such that the only solution to

$$\begin{bmatrix} C_2 A \\ \mathbf{e}_1' \\ T_2' \end{bmatrix} [\mathbf{t}_2] = \begin{bmatrix} \mathbf{x}_2 \\ \mathbf{e}_2 \end{bmatrix}$$

is $\mathbf{e}_2$. Again the orthogonality requirement $T_2'\mathbf{t}_2 = \mathbf{0}$ does not restrict $\mathbf{t}_2$ to be $\mathbf{e}_2$, and therefore the supposition implies that there is exactly one solution to (3) which is a contradiction provided that $p(A'C_2' : \mathbf{e}_1) < r$. Necessity for the other columns can be shown in a similar fashion. □

It might be noted that, in a fashion similar to the proof used for Theorem 2, it can be shown that if all $\mathbf{x}_j = \mathbf{0}$, the required ranks can be reduced by one in each column of $A$. Further if all such restrictions are exclusion restrictions, then Jöreskog's case (i) is obtained. This case was previously treated by Dunn. For this special case the required rank conditions are equivalent to Dunn's rank condition.

The above conditions make it easy to see why the counterexamples given by Dunn and Jennrich are not identifiable. Jennrich, treating Jöreskog's case (ii), specified $r(r-1)/2$ nonzero elements. However, the conditions require $r(r+1)/2$ specifications of nonzero elements. In Dunn's counterexample the rank conditions were violated. The conditions also show that Jennrich's conjecture that specification of $[r(r-1)/2]+1$ values in the appropriate positions will identify $A$ is wrong if the specified elements in $A$ are not all zero.

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*Manuscript received 1/30/79*
*First revision received 10/25/79*
*Final version received 2/25/80*