TÉCNICO LISBOA

Aprendizagem 2022/23

Homework IV

Deadline 7/11/2022 (Monday) 23:59 via Fenix as PDF

1)

E - Step

$\underset{\sim}{x_1}$

$$P(K \mid x_1) = \frac{P(x_1 \mid k)\, P(K)}{P(x_1)}$$

$k = 1$

$$P(x_1 \mid k=1) \sim N\left(\mu_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \Sigma_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$$

$$= \frac{1}{2\pi \sqrt{|\Sigma|}}\, e^{-\frac{1}{2}(x_1 - \mu_1)^T \Sigma_1^{-1} (x_1 - \mu_1)}$$

$|\Sigma_1| = 2 \times 2 - 1 \times 1$
$= 3$

$$\Sigma_1^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

$$x_1 - \mu_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2\pi\sqrt{3}}\, e^{-\frac{1}{2}\begin{bmatrix} -1 & 0 \end{bmatrix}\begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix}}$$

$$= \frac{1}{2\pi\sqrt{3}}\, e^{-1/3}$$

$$\simeq 0.0658$$

$$\text{Posterior}_1 = P(x_1 \mid k=1) \times \text{Prior}_1$$
$$= 0.1317 \times 0.5$$
$$= 0.033$$

$k = 2$

$$P(x_1 \mid k=2) \sim N\left(\mu_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Sigma_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)$$

$$= \frac{1}{2\pi \sqrt{|\Sigma_2|}} e^{-\frac{1}{2}(x_1 - \mu_2)^T \Sigma_2^{-1} (x_1 - \mu_2)}$$

$$|\Sigma_2| = 2 \times 2 = 4$$

$$\Sigma_2^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$x_1 - \mu_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{2\pi \sqrt{4}} e^{-\frac{1}{2}[1 \; 2]\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$$= \frac{1}{2\pi \sqrt{4}} e^{-5/4}$$

$$\simeq 0.0228$$

$$\text{Posterior}_2 = P(x_1 \mid k=2) \times \text{Prior}_2$$

$$= 0.0228 \times 0.5$$

$$= 0.0114$$

$$P(k=1 \mid x_1) = \frac{0.033}{0.033 + 0.0114}$$

$$= \boxed{0.7434}$$

$$P(k=2 \mid x_1) = \frac{0.0114}{0.033 + 0.0114}$$

$$= \boxed{0.2568}$$

$$\underset{\sim}{x}_2$$

$$P(k \mid x_2) = \frac{P(x_2 \mid k)\, P(k)}{P(x_2)}$$

$\underline{k=1}$

$$P(x_2 \mid k=1) \sim N\left(\mu_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \Sigma_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$$

$$= \frac{1}{2\pi \sqrt{|\Sigma|}} e^{-\frac{1}{2}(x_2 - \mu_1)^T \Sigma_1^{-1} (x_2 - \mu_1)}$$

$$|\Sigma|_1 = 2 \times 2 - 1 \times 1 = 3$$

$$\Sigma_1^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

$$x_2 - \mu_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

$$= \frac{1}{2\pi\sqrt{3}} e^{-\frac{1}{2}\begin{bmatrix} -3 & -1 \end{bmatrix}\begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \end{bmatrix}}$$

$$= \frac{1}{2\pi\sqrt{3}} e^{-7/3}$$

$$\simeq 0.00891$$

$$\text{Posterior}_1 = P(x \mid k=1) \times \text{Prior}_1$$

$$= 9.7716 \times 0.5$$

$$= 0.00446$$

$k=2$

$$P(x_2 \mid k=2) \sim N\left(\mu_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Sigma_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)$$

$$= \frac{1}{2\pi \sqrt{|\Sigma_2|}} e^{-\frac{1}{2}(x_2-\mu_2)^T \Sigma_2^{-1} (x_2-\mu_2)}$$

$$|\Sigma_2| = 2 \times 2 = 4$$

$$\Sigma_2^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$x_2 - \mu_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2\pi\sqrt{4}} e^{-\frac{1}{2}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

$$= \frac{1}{2\pi\sqrt{4}} e^{-1/2}$$

$$\simeq 0.0483$$

$$\text{Posterior}_2 = P(x \mid k=2) \times \text{Prior}_2$$
$$= 0.0483 \times 0.5$$
$$= 0.0241$$

$$P(k=1 \mid x_2) = \frac{0.00446}{0.0241 + 0.00446}$$

$$= 0.1562$$

$$P(k=2 \mid x_2) = \frac{0.0241}{0.0241 + 0.00446}$$

$$= 0.8438$$

$\underset{\sim}{x}_3$

$$P(K \mid x_3) = \frac{P(x_3 \mid K)\, P(K)}{P(x_3)}$$

$K=1$

$$P(x_3 \mid K=1) \sim N\left(\mu_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \Sigma_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$$

$$= \frac{1}{2\pi \sqrt{|\Sigma_1|}} e^{-\frac{1}{2}(x_3-\mu_1)^T \Sigma_1^{-1} (x_3-\mu_1)}$$

$$|\Sigma_1| = 2\times 2 - 1\times 1 = 3$$

$$\Sigma_1^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

$$x_3 - \mu_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$= \frac{1}{2\pi\sqrt{3}} e^{-\frac{1}{2}\begin{bmatrix} -1 & -2 \end{bmatrix}\begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}\begin{bmatrix} -1 \\ -2 \end{bmatrix}}$$

$$= \frac{1}{2\pi\sqrt{3}} e^{-1}$$

$$\simeq 0.0338$$

$$\text{Posterior}_1 = P(x \mid K=1) \times \text{Prior}_1$$
$$= 0.0338 \times 0.5$$
$$= 0.0169$$

$\underline{k=2}$

$$P(x_3 \mid k=2) \sim N\left(\mu_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Sigma_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)$$

$$= \frac{1}{2\pi \sqrt{|\Sigma_2|}} e^{-\frac{1}{2}(x_3-\mu_2)^T \Sigma_2^{-1} (x_3-\mu_2)}$$

$$|\Sigma_2| = 2 \times 2 = 4$$

$$\Sigma_2^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$x_3 - \mu_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2\pi \sqrt{4}} e^{-\frac{1}{2}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

$$= \frac{1}{2\pi \sqrt{4}} e^{-1/4}$$

$$\simeq 0.062$$

$$\text{Posterior}_2 = P(x \mid k=2) \times \text{Prior}_2$$

$$= 0.062 \times 0.5$$

$$= 0.031$$

$$P(k=1 \mid x_3) = \frac{0.0169}{0.031 + 0.0169}$$

$$= 0.3528$$

$$P(k=2 \mid x_3) = \frac{0.031}{0.031 + 0.0169}$$

$$= 0.6472$$

M - Step

$$\boxed{\mu_c = \frac{\sum_{i=1}^{N} P(c|x_i) \cdot x_i}{\sum_{i=1}^{N} p(c|x_i)}}$$

$$\boxed{\text{Prior}_c = p(c=k) = \frac{\sum P(c \mid x_i)}{\underbrace{\sum_{j=1}^{k} P(C=j|x_i)}_{=N}}}$$

$$\boxed{\Sigma_c^{ij} = \frac{\sum_{k=1}^{N} p(c|k) \cdot (x_k^i - \mu_c^i)(x_k^j - \mu_c^j)}{\sum_{k=1}^{N} p(c|x_k)}}$$

c = 1

$$\sum_{i=1}^{N} p(c|x_i) = 0.7434 + 0.1562 + 0.3538$$

$$= 1.2534$$

$$\mu_1 = \frac{0.7434 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0.1562 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 0.3538 \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{1.2534}$$

$$= \boxed{\begin{bmatrix} 0.7508 \\ 1.3108 \end{bmatrix}}$$

$$\Sigma_1^{(1,1)} = \frac{0.7434(1 - 0.7508)^2 + 0.1562(-1 - 0.7508)^2 + 0.3538(1 - 0.7538)^2}{1.2534}$$

$$= 0.4359$$

$$\Sigma^{(2,1)} = \Sigma^{(1,2)}$$

$$= \frac{0.7434(1 - 0.7508)(2 - 1.3108) + 0.1562(-1 - 0.7508)(1 - 1.3108) + 0.3538(1 - 0.7508)(0 - 1.3108)}{1.2534}$$

$$= 0.0775$$

$$\Sigma_1^{(2,2)} = \frac{0.7434(2-1.3108)^2 + 0.1562(1-1.3108)^2 + 0.3538(0-1.3108)^2}{1.2534}$$

$$= 0.7788$$

Therefore, $\Sigma_1 = \begin{bmatrix} 0.4359 & 0.0775 \\ 0.0775 & 0.7788 \end{bmatrix}$

$$\pi_1 = \frac{1.2534}{3}$$

$$= 0.4178$$

$\underset{\sim\sim\sim}{c=2}$

$$\sum_{n=1}^{N} p(c \mid x_n) = 0.2568 + 0.8438 + 0.6472$$

$$= 1.7478$$

$$\mu_2 = \frac{0.2568\begin{bmatrix}1\\2\end{bmatrix} + 0.8438\begin{bmatrix}-1\\1\end{bmatrix} + 0.6472\begin{bmatrix}1\\0\end{bmatrix}}{1.7478}$$

$$= \begin{bmatrix}0.0344\\0.7766\end{bmatrix}$$

$$\Sigma_2^{(1,1)} = \frac{0.2568(1-0.0344)^2 + 0.8438(-1-0.0344)^2 + 0.6472(1-0.0344)^2}{1.7478}$$

$$= 0.9988$$

$$\Sigma^{(2,1)} = \Sigma^{(1,2)}$$

$$= \frac{0.2568(1-0.0344)(2-0.7766) + 0.8438(-1-0.0344)(1-0.7766) + 0.6472(1-0.0344)(0-0.7766)}{1.7478}$$

$$= -0.2157$$

$$\Sigma^{(2,2)} = \frac{0.2568(2-0.7766)^2 + 0.8438(1-0.7766)^2 + 0.6472(0-0.7766)^2}{1.7478}$$

$$= 0.6517$$

Therefore, $\Sigma_2 = \begin{bmatrix} 0.9988 & -0.2157 \\ -0.2157 & 0.4673 \end{bmatrix}$

$$\tilde{\pi}_2 = \frac{1.7478}{3}$$

$$= 0.5826$$

To summarize, the new parameters are:

$$\mu_1 = \begin{bmatrix} 0.7508 \\ 1.3108 \end{bmatrix} \qquad \mu_2 = \begin{bmatrix} 0.0344 \\ 0.7766 \end{bmatrix}$$

$$\Sigma_1 = \begin{bmatrix} 0.4359 & 0.0775 \\ 0.0775 & 0.7788 \end{bmatrix} \qquad \Sigma_2 = \begin{bmatrix} 0.9988 & -0.2157 \\ -0.2157 & 0.4673 \end{bmatrix}$$

$$\tilde{\pi}_1 = 0.4178 \qquad \tilde{\pi}_2 = 0.5826 \quad //$$

2.a) Since we are considering a hard assignment under MAP assumption, we must assign each observation to the highest posterior probability:

$$P(k=1|x_1) = \frac{0.033}{0.033+0.0114} = 0.7434 \qquad P(k=1|x_2) = \frac{0.00446}{0.0241+0.00446} = 0.1562 \qquad P(k=1|x_3) = \frac{0.0169}{0.031+0.0169} = 0.3528$$

$$P(k=2|x_1) = \frac{0.0114}{0.033+0.0114} = 0.2568 \qquad P(k=2|x_2) = \frac{0.0241}{0.0241+0.00446} = 0.8438 \qquad P(k=2|x_3) = \frac{0.031}{0.031+0.0169} = 0.6472$$

Therefore, x_1 is assigned to cluster 1 and x_2 and x_3 are assigned to cluster 2. //

2 . b) Silhouette of c_2 (largest cluster) using Euclidean distance:

$$s(c_2) = \frac{s(x_2) + s(x_3)}{2}$$

$$a(x_2) = d(x_2, x_3) = \sqrt{(-1-1)^2 + (1-0)^2} = \sqrt{5}$$

$$b(x_2) = d(x_2, x_1) = \sqrt{(-1-1)^2 + (1-2)^2} = \sqrt{5}$$

$$a(x_2) \leq b(x_2)$$

$$\Downarrow$$

$$s(x_2) = 1 - \frac{a(x_2)}{b(x_2)} = 1 - 1 = 0$$

$$a(x_3) = d(x_3, x_2) = a(x_2) = \sqrt{5}$$

$$b(x_3) = d(x_3, x_1) = \sqrt{(1-1)^2 + (0-2)^2} = 2$$

$$a(x_3) > b(x_3)$$

$$\Downarrow$$

$$s(x_3) = \frac{b(x_3)}{a(x_3)} - 1 = \frac{2}{\sqrt{5}} - 1 \simeq -0.1056$$

$$= \frac{0 - 0.1056}{2} = -0.0528$$

Therefore, the silhouette of the largest cluster is $s(c_2) = -0.0528$ //