Aprendizagem 2021/22

Homework I – Group XXX

I. Pen-and-paper

1)

	y_1	y_2	C
x_1	A	0	1
x_2	B	1	1
x_3	A	1	1
x_4	A	0	1
x_5	B	0	0
x_6	B	0	0
x_7	A	1	0
x_8	B	1	0

$$d(x_1, x_2) = \text{Hamming}(x_1, x_2) + \frac{1}{2}$$

$$= (\underbrace{1}_{A \neq B} + \underbrace{1}_{1 \neq 0}) + 1/2$$

$$= \frac{5}{2}$$

...

$$\text{Weighted mode} = \frac{1}{d_1} + \frac{1}{d_2} + \ldots$$

$d(x_1, x_2)$	x_1 (+)	x_2 (+)	x_3 (+)	x_4 (+)	x_5 (−)	x_6 (−)	x_7 (−)	x_8 (−)			
x_1		5/2	3/2	1/2	3/2	3/2	3/2	5/2	+	$\boxed{x_1 +}$ $\frac{2}{3} + \frac{2}{1} = \frac{8}{3}$	$\boxed{x_1 -}$ $\frac{2}{3} + \frac{2}{3} + \frac{2}{3} = 2$
x_2	5/2		3/2	5/2	3/2	3/2	3/2	1/2	−	$\boxed{x_2 +}$ $\frac{2}{3}$	$\boxed{x_2 -}$ $\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{1}{3} = \frac{7}{3}$
x_3	3/2	3/2		1/2	5/2	5/2	1/2	3/2	−	$\boxed{x_3 +}$ $\frac{2}{3} + \frac{2}{3} + \frac{2}{3} = 2$	$\boxed{x_3 -}$ $2 + 2/3 = 8/3$
x_4	1/2	5/2	3/2		3/2	3/2	3/2	5/2	+	$\boxed{x_4 +}$ $2 + 2/3 = 8/3$	$\boxed{x_4 -}$ $2/3 + 2/3 + 2/3 = 2$
x_5	3/2	3/2	5/2	3/2		1/2	5/2	3/2	−	$\boxed{x_5 +}$ $2/3 + 2/3 + 2/3 = 2$	$\boxed{x_5 -}$ $2 + 2/3 = 8/3$
x_6	3/2	3/2	5/2	3/2	1/2		5/2	3/2	−	$\boxed{x_6 +}$ $2/3 + 2/3 + 2/3 = 2$	$\boxed{x_6 -}$ $2 + 2/3 = 8/3$
x_7	3/2	3/2	1/2	3/2	5/2	5/2		3/2	+	$\boxed{x_7 +}$ $2/3 + 2/3 + 2 + 2/3 = 4$	$\boxed{x_7 -}$ $2/3$
x_8	5/2	1/2	3/2	5/2	3/2	3/2	3/2		+	$\boxed{x_8 +}$ $2 + 2/3 = 8/3$	$\boxed{x_8 -}$ $2/3 + 2/3 + 2/3 = 2$

$\Downarrow$

$$\text{recall} = \frac{TP}{TP + FN} = \frac{\overbrace{2}^{x_1, x_4}}{2 + \underbrace{2}_{x_7, x_8}} = \frac{1}{2}$$

2)

	y_1	y_2	y_3	C
x_1	A	0	1.2	1
x_2	B	1	0.8	1
x_3	A	1	0.5	1
x_4	A	0	0.9	1
x_5	B	0	1	0
x_6	B	0	0.9	0
x_7	A	1	1.2	0
x_8	B	1	0.8	0
x_9	B	0	0.8	1

- $\{y_1, y_2\} \perp\!\!\!\perp \{y_3\} \Rightarrow P(y_1, y_2, y_3 | C) = P(y_1, y_2 | C) \times P(y_3 | C)$
- Bayesian Theorem: $P(c|y) = \dfrac{P(y|c)P(c)}{P(y)}$

Learn a Bayesian Classifier $\Rightarrow$ Compute it's parameters

$P(C)$

$P(C=1) = 5/9 \quad P(C=0) = 4/9$

$P(y_1, y_2 | C)$

$P(y_1 = A, y_2 = 0 | C = 1) = 2/5 = 0.4$
$P(y_1 = A, y_2 = 1 | C = 1) = 1/5 = 0.2$
$P(y_1 = B, y_2 = 0 | C = 1) = 1/5 = 0.2$
$P(y_1 = B, y_2 = 1 | C = 1) = 1/5 = 0.2$

$P(y_1 = A, y_2 = 0 | C = 0) = 0$
$P(y_1 = A, y_2 = 1 | C = 0) = 1/4 = 0.25$
$P(y_1 = B, y_2 = 0 | C = 0) = 2/4 = 0.5$
$P(y_1 = B, y_2 = 1 | C = 0) = 1/4 = 0.25$

$P(y_3 | C)$

$\mu_1 / \sigma_1 \rightarrow$ mean/standard deviation of observations where $C=1$
$\mu_0 / \sigma_0 \rightarrow$ mean/standard deviation of observations where $C=0$

$$\mu_1 = \frac{1.2 + 0.8 + 0.5 + 0.9 + 0.8}{5} = 0.84 \qquad \sigma_1 = \sqrt{\frac{(1.2-0.84)^2 + (0.8-0.84)^2 + (0.5-0.84)^2 + (0.9-0.84)^2 + (0.8-0.84)^2}{5-1}} \simeq 0.251$$

$$\mu_0 = \frac{1 + 0.9 + 1.2 + 0.8}{4} = 0.975 \qquad \sigma_0 = \ldots \simeq 0.171$$

$$P(y_3 | C=1) \sim N(y_3 | \mu_1, \sigma_1^2) = \frac{1}{0.251\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2 \times 0.063}(y_3 - 0.84)^2\right)$$

$$\simeq 1.5894\, e^{-7.9365(y_3 - 0.84)^2}$$

$$P(y_3 | C=0) \sim N(y_3 | \mu_0, \sigma_0^2) = \frac{1}{0.171\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2 \times 0.0292}(y_3 - 0.975)^2\right)$$

$$\simeq 2.333\, e^{-17.1(y_3 - 0.975)^2}$$

3) under MAP assumption, $P(c=1 \mid \theta_i) = \dfrac{P(\theta_i \mid c=1) P(c=1)}{P(\theta_i)}$

$\theta_1 = (A, 1, 0.8)$ $\theta_2 = (B, 1, 1)$ $\theta_3 = (B, 0, 0.9)$

$$P(\theta_1 \mid c=1) = P(A, 1 \mid c=1)\, P(0.8 \mid c=1)$$
$$= 0.2 \times N(0.8 \mid \mu_1, \sigma_1^2)$$
$$= 0.2 \times 1.56934$$
$$= 0.31387$$

$$P(\theta_1 \mid c=0) = P(A, 1 \mid c=0)\, P(0.8 \mid c=0)$$
$$= 0.25 \times N(0.8 \mid \mu_0, \sigma_0^2)$$
$$\simeq 0.25 \times 1.38191$$
$$= 0.34548$$

$$P(\theta_2 \mid c=1) = P(B, 1 \mid c=1)\, P(1 \mid c=1)$$
$$= 0.2 \times N(1 \mid \mu_1, \sigma_1^2)$$
$$= 0.2 \times 1.29717$$
$$= 0.25943$$

$$P(\theta_2 \mid c=0) = P(B, 1 \mid c=0)\, P(1 \mid c=0)$$
$$= 0.25 \times N(1 \mid \mu_0, \sigma_0^2)$$
$$\simeq 0.25 \times 2.3082$$
$$= 0.577$$

$$P(\theta_3 \mid c=1) = P(B, 0 \mid c=1)\, P(0.9 \mid c=1)$$
$$= 0.2 \times N(0.9 \mid \mu_1, \sigma_1^2)$$
$$= 0.2 \times 1.54463$$
$$= 0.30893$$

$$P(\theta_3 \mid c=0) = P(B, 0 \mid c=0)\, P(0.9 \mid c=0)$$
$$= 0.5 \times N(0.9 \mid \mu_0, \sigma_0^2)$$
$$\simeq 0.5 \times 2.1190$$
$$= 1.0595$$

$$P(c=1 \mid \theta_1) = \frac{P(\theta_1 \mid c=1) P(c=1)}{P(\theta_1)} = \frac{0.31387 \times 5/9}{P(\theta_1)} = \frac{0.17434}{P(\theta_1)}$$

$$P(c=0 \mid \theta_1) = \frac{P(\theta_1 \mid c=0) P(c=0)}{P(\theta_1)} = \frac{0.34548 \times 4/9}{P(\theta_1)} = \frac{0.15355}{P(\theta_1)}$$

$$P(c=1 \mid \theta_1) = \frac{0.17434}{0.17434 + 0.15355}$$
$$= 0.53170$$

$$P(c=1\mid\theta_2) = \frac{P(\theta_2\mid c=1)P(c=1)}{P(\theta_2)} \qquad P(c=0\mid\theta_2) = \frac{P(\theta_2\mid c=0)P(c=0)}{P(\theta)}$$

$$= \frac{0.25943 \times 5/9}{P(\theta_2)} \qquad = \frac{0.577 \times 4/9}{P(\theta_2)}$$

$$= \frac{0.14413}{P(\theta_2)} \qquad = \frac{0.25644}{P(\theta_2)}$$

$$P(c=1\mid\theta_2) = \frac{0.14413}{0.14413 + 0.25644}$$

$$= 0.35981$$

$$P(c=1\mid\theta_3) = \frac{P(\theta_3\mid c=1)P(c=1)}{P(\theta_3)} \qquad P(c=0\mid\theta_3) = \frac{P(\theta_3\mid c=0)P(c=0)}{P(\theta_3)}$$

$$= \frac{0.30893 \times 5/9}{P(\theta_3)} \qquad = \frac{1.0595 \times 4/9}{P(\theta_3)}$$

$$= \frac{0.17163}{P(\theta_3)} \qquad = \frac{0.47089}{P(\theta_3)}$$

$$P(c=1\mid\theta_3) = \frac{0.17163}{0.17163 + 0.47089}$$

$$= 0.26712$$

Therefore, the posterior probabilities of $c = \text{Positive}$ (given $\theta_1, \theta_2, \theta_3$) are:

$$P(c=1\mid\theta_1) = 0.53170$$
$$P(c=1\mid\theta_2) = 0.35981$$
$$P(c=1\mid\theta_3) = 0.26712$$

4) $\theta = 0.5$

$P(c=1 \mid \theta_1) = 0.53170 > 0.5 \Rightarrow$ Classify as Positive ✓

$P(c=1 \mid \theta_2) = 0.35981 < 0.5 \Rightarrow$ Classify as Negative ✗

$P(c=1 \mid \theta_3) = 0.26712 < 0.5 \Rightarrow$ Classify as Negative ✓

Accuracy $= 2/3$

$\theta = 0.3$

$P(c=1 \mid \theta_1) = 0.53170 > 0.3 \Rightarrow$ Classify as Positive ✓

$P(c=1 \mid \theta_2) = 0.35981 > 0.3 \Rightarrow$ Classify as Positive ✓

$P(c=1 \mid \theta_3) = 0.26712 < 0.3 \Rightarrow$ Classify as Negative ✓

Accuracy $= 1$

$\theta = 0.7$

$P(c=1 \mid \theta_1) = 0.53170 < 0.7 \Rightarrow$ Classify as Negative ✗

$P(c=1 \mid \theta_2) = 0.35981 < 0.7 \Rightarrow$ Classify as Negative ✗

$P(c=1 \mid \theta_3) = 0.26712 < 0.7 \Rightarrow$ Classify as Negative ✓

Accuracy $= 1/3$

Therefore, the decision threshold of $\theta = 0.3$ optimizes test accuracy.