4)

Design Matrix :

$$
X = \begin{pmatrix}
1 & x_1^1 & x_1^2 & x_1^3 \\
1 & x_2^1 & x_2^2 & x_2^3 \\
1 & x_3^1 & x_3^2 & x_3^3 \\
1 & x_4^1 & x_4^2 & x_4^3 \\
1 & x_5^1 & x_5^2 & x_5^3
\end{pmatrix}
=
\begin{pmatrix}
1 & 0.8 & 0.64 & 0.512 \\
1 & 1 & 1 & 1 \\
1 & 1.2 & 1.44 & 1.728 \\
1 & 1.4 & 1.96 & 2.744 \\
1 & 1.6 & 2.56 & 4.096
\end{pmatrix}
$$

Bias ↓ (on the first column of 1's)

$$
X^T = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
0.8 & 1 & 1.2 & 1.4 & 1.6 \\
0.64 & 1 & 1.44 & 1.96 & 2.56 \\
0.512 & 1 & 1.728 & 2.744 & 4.096
\end{pmatrix}
$$

$4 \times 5 \qquad 5 \times 4$

$$
X^T X = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
0.8 & 1 & 1.2 & 1.4 & 1.6 \\
0.64 & 1 & 1.44 & 1.96 & 2.56 \\
0.512 & 1 & 1.728 & 2.744 & 4.096
\end{pmatrix}
\begin{pmatrix}
1 & 0.8 & 0.64 & 0.512 \\
1 & 1 & 1 & 1 \\
1 & 1.2 & 1.44 & 1.728 \\
1 & 1.4 & 1.96 & 2.744 \\
1 & 1.6 & 2.56 & 4.096
\end{pmatrix}
$$

$$
= \begin{pmatrix}
5 & 6 & 7.6 & 10.08 \\
6 & 7.6 & 10.08 & 13.8784 \\
7.6 & 10.08 & 13.8784 & 19.68 \\
10.08 & 13.8784 & 19.68 & 28.55488
\end{pmatrix}
$$

$$\left(x^T x + \lambda I\right)^{-1} = \left[\begin{pmatrix} 5 & 6 & 7.6 & 10.08 \\ 6 & 7.6 & 10.08 & 13.8784 \\ 7.6 & 10.08 & 13.8784 & 19.68 \\ 10.08 & 13.8784 & 19.68 & 28.55488 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\right]^{-1}$$

$$= \begin{pmatrix} 7 & 6 & 7.6 & 10.08 \\ 6 & 9.6 & 10.08 & 13.8784 \\ 7.6 & 10.08 & 15.8784 & 19.68 \\ 10.08 & 13.8784 & 19.68 & 30.55488 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 0.3417 & -0.1214 & -0.0749 & -0.0093 \\ -0.1214 & 0.3892 & -0.0967 & -0.0745 \\ -0.0749 & -0.0967 & 0.3726 & -0.1714 \\ -0.0093 & -0.0745 & -0.1714 & 0.18 \end{pmatrix}$$

$$\left(x^T x + \lambda I\right)^{-1} x^T = \begin{pmatrix} 0.3417 & -0.1214 & -0.0749 & -0.0093 \\ -0.1214 & 0.3892 & -0.0967 & -0.0745 \\ -0.0749 & -0.0967 & 0.3726 & -0.1714 \\ -0.0093 & -0.0745 & -0.1714 & 0.18 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0.8 & 1 & 1.2 & 1.4 & 1.6 \\ 0.64 & 1 & 1.44 & 1.96 & 2.56 \\ 0.512 & 1 & 1.728 & 2.744 & 4.096 \end{pmatrix}$$

$$= \begin{pmatrix} 0.1918 & 0.136 & 0.072 & -0.0007 & -0.082 \\ 0.9 & 0.0966 & 0.0777 & 0.0296 & -0.051 \\ -0.001 & 0.0296 & 0.0495 & 0.05 & 0.0223 \\ -0.086 & -0.075 & -0.034 & 0.0444 & 0.1701 \end{pmatrix}$$

$$w = \left(x^T x + \lambda I\right)^{-1} x^T Z = \begin{pmatrix} 0.1918 & 0.136 & 0.072 & -0.0007 & -0.082 \\ 0.9 & 0.0966 & 0.0777 & 0.0296 & -0.051 \\ -0.001 & 0.0296 & 0.0495 & 0.05 & 0.0223 \\ -0.086 & -0.075 & -0.034 & 0.0444 & 0.1701 \end{pmatrix} \begin{pmatrix} 24 \\ 20 \\ 10 \\ 13 \\ 12 \end{pmatrix}$$

$$= \begin{pmatrix} 7.0451 \\ 4.6409 \\ 1.9673 \\ -1.301 \end{pmatrix}$$

a) $\hat{z}(x, w) = w_0 + w_1 x + w_2 x^2 + w_3 x^3$

$$= 7.0451 + 4.6409 \, x + 1.9673 x^2 - 1.301 x^3$$

$\hat{z}_0 = \hat{z}(x_0, w) = 7.0451 + 4.6409 \times 0.8 + 1.9673 \times 0.8^2 - 1.301 \times 0.8^3$

$$= 11.35078$$

$\hat{z}_1 = \hat{z}(x_1, w) = 7.0451 + 4.6409 \times 1 + 1.9673 \times 1^2 - 1.301 \times 1^3$

$$= 12.3523$$

$\hat{z}_2 = \hat{z}(x_2, w) = 7.0451 + 4.6409 \times 1.2 + 1.9673 \times 1.2^2 - 1.301 \times 1.2^3$

$$= 13.199$$

$\hat{z}_3 = \hat{z}(x_3, w) = 7.0451 + 4.6409 \times 1.4 + 1.9673 \times 1.4^2 - 1.301 \times 1.4^3$

$$= 13.8283$$

$\hat{z}_4 = \hat{z}(x_4, w) = 7.0451 + 4.6409 \times 1.6 + 1.9673 \times 1.6^2 - 1.301 \times 1.6^3$

$$= 14.178$$

$$RMSE = \sqrt{\sum_{i=0}^{4} \frac{(\hat{z}_i - z_i)^2}{5}}$$

$$= \sqrt{\frac{1}{5}\left((\hat{z}_0 - z_0)^2 + (\hat{z}_1 - z_1)^2 + (\hat{z}_2 - z_2)^2 + (\hat{z}_3 - z_3)^2 + (\hat{z}_4 - z_4)^2 \right)}$$

$$= 6.8433$$

3) **1° • Forward Propagation**

$$\rightarrow W^{[1]} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \rightarrow b^{[1]} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\rightarrow W^{[2]} = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad \rightarrow b^{[2]} = 1$$

$$\rightarrow Z^{[:]} = W^{[:]} X^{[:-1]} + b^{[1]}$$

$$\rightarrow X^{[:]} = f(Z^{[:]}) = e^{0.1 Z^{[:]}}$$

$$X_o^{[0]} = 0.8$$

$$Z_o^{[1]} = W_o^{[1]} X_o^{[0]} + b_o^{[1]}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} 0.8 + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.8 \\ 1.8 \end{bmatrix}$$

$$X_o^{[1]} = f(Z_o^{[1]})$$

$$= \begin{bmatrix} 1.197 \\ 1.197 \end{bmatrix}$$

$$Z_o^{[2]} = W_o^{[2]} X_o^{[1]} + b_o^{[2]}$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1.197 \\ 1.197 \end{bmatrix} + 1$$

$$= 3.394$$

$$X_o^{[2]} = f(Z_o^{[2]})$$

$$= 1.4041$$

$$X_1^{[0]} = 1$$

$$Z_1^{[1]} = W_1^{[1]} X_1^{[0]} + b_1^{[1]}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} 1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$X_1^{[1]} = f(Z_1^{[1]})$$

$$= \begin{bmatrix} 1.221 \\ 1.221 \end{bmatrix}$$

$$Z_1^{[2]} = W_1^{[2]} X_1^{[1]} + b_1^{[2]}$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1.221 \\ 1.221 \end{bmatrix} + 1$$

$$= 3.442$$

$$X_1^{[2]} = f(Z_1^{[2]})$$

$$= 1.411$$

$$X_2^{[0]} = 1.2$$

$$Z_2^{[1]} = W_2^{[1]} X_2^{[0]} + b_2^{[1]}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} 1.2 + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.2 \\ 2.2 \end{bmatrix}$$

$$X_2^{[1]} = f(Z_2^{[1]})$$

$$= \begin{bmatrix} 1.246 \\ 1.246 \end{bmatrix}$$

$$Z_2^{[2]} = W_2^{[2]} X_2^{[1]} + b_2^{[2]}$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1.246 \\ 1.246 \end{bmatrix} + 1$$

$$= 3.492$$

$$X_2^{[2]} = f(Z_2^{[2]})$$

$$= 1.418$$

2° - Backward Propagation

$\rightarrow X_0^{[2]} = 1.4041 \qquad \rightarrow X_1^{[2]} = 1.411 \qquad \rightarrow X_2^{[2]} = 1.418$

$\rightarrow T_0 = 24 \qquad \rightarrow t_1 = 20 \qquad \rightarrow t_2 = 10$

$\rightarrow E_\delta\left(w^{[i]}\right) = \frac{1}{2}\left(x_\delta^{[i]} - t_\delta\right)^2$

$\rightarrow \delta_\delta^{[i]} = \frac{\partial E}{\partial x^{[i]}} \circ \frac{\partial x^{[i]}}{\partial z^{[i]}}$

$\qquad = \left(x_\delta^{[i]} - t_\delta\right) \circ 0.1 e^{0.1 z_\delta^{[i]}}$

$\rightarrow \delta_\delta^{[i]} = w^{[i+1]^T} \cdot \delta_\delta^{[i+1]} \circ 0.1 e^{0.1 z_\delta^{[i]}}$

$\rightarrow \delta^{[i]} = \sum_{\delta=0}^{2} \delta_\delta^{\delta_i}$

$\rightarrow f'(u) = 0.1 e^{0.1 u}$

$\rightarrow x^{[i]} = f(z^{[i]})$

$\Rightarrow \frac{\partial x^{[i]}}{\partial z^{[i]}} = f'(z^{[i]}) \cdot z^{[i]'}$

$\qquad = 0.1 e^{0.1 z^{[i]}}$

$\rightarrow \frac{\partial E}{\partial w^{[i]}} = \sum_{\delta=0}^{2} \delta_\delta^{[i]} \cdot \left(\frac{\partial z_\delta^{[i]}}{\partial w^{[i]}}\right)^T$

$\qquad = \sum_{\delta=0}^{2} \delta_\delta^{[i]} \cdot \left(x_0^{[i-1]}\right)^T$

$\rightarrow w^{[k]} = w^{[k]} - n \frac{\partial E}{\partial w^{[i]}}$

$\rightarrow b^{[k]} = b^{[k]} - n \delta^{[k]}$

Calculations

Level 2

$\delta_0^{[2]} = \left(1.4041 - 24\right) \times 0.1 e^{0.1 \times 3.394}$

$\qquad = -3.427$

$\delta_1^{[2]} = \left(1411 - 20\right) \times 0.1 e^{0.1 \times 3.442}$

$\qquad = -2.823$

$\delta_2^{[2]} = \left(1.418 - 10\right) \times 0.1 e^{0.1 \times 3.492}$

$\qquad = -1.217$

$$W^{[2]} = W^{[2]} - h \boxed{\frac{\partial E}{\partial w^{[2]}}}$$

$$= \sum \delta_i^{[0]} \cdot x_i^{[1]^T}$$

$$= -3.1727 \begin{bmatrix} 1.197 \\ 1.197 \end{bmatrix}^T - 2.623 \begin{bmatrix} 1.221 \\ 1.221 \end{bmatrix}^T - 1.217 \begin{bmatrix} 1.246 \\ 1.246 \end{bmatrix}^T$$

$$= \begin{bmatrix} -8.517 \\ -8.517 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} - 0.1 \begin{bmatrix} -8.517 & -8.517 \end{bmatrix}$$

$$= \begin{bmatrix} 1.8517 & 1.8517 \end{bmatrix}$$

$$b^{[2]} = b^{[2]} - h \delta^{[2]}$$

$$= 1 - 0.1 (-3.1727 - 2.623 - 1.217)$$

$$= 1.7013$$

Level 1

$$\delta_0^{[1]} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \times -3.1727 \quad \odot \quad \begin{bmatrix} 0.1e^{0.1 \times 1.8} \\ 0.1e^{0.1 \times 1.8} \end{bmatrix}$$

$$= \begin{bmatrix} -0.38 \\ -0.38 \end{bmatrix}$$

$$\delta_1^{[1]} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \times -2.623 \quad \odot \quad \begin{bmatrix} 0.1e^{0.1 \times 2} \\ 0.1e^{0.1 \times 2} \end{bmatrix}$$

$$= \begin{bmatrix} -0.32 \\ -0.32 \end{bmatrix}$$

$$\delta_2^{[1]} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \times -1.217 \quad \odot \quad \begin{bmatrix} 0.1e^{0.1 \times 2.2} \\ 0.1e^{0.1 \times 2.2} \end{bmatrix}$$

$$= \begin{bmatrix} -0.15 \\ -0.15 \end{bmatrix}$$

$$w^{[1]} = w^{[1]} - h \boxed{\frac{\partial E}{\partial w^{[1]}}}$$

$$'' $$

$$\sum \delta^{[1]} \cdot x^{[0]T}$$

$$= \begin{bmatrix} -0.38 \\ -0.38 \end{bmatrix} \times 0.8 + \begin{bmatrix} -0.32 \\ -0.32 \end{bmatrix} \times 1 + \begin{bmatrix} -0.15 \\ -0.15 \end{bmatrix} \times 1.2$$

$$= \begin{bmatrix} -0.804 \\ -0.804 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 0.1 \begin{bmatrix} -0.804 \\ -0.804 \end{bmatrix}$$

$$= \begin{bmatrix} 1.0804 \\ 1.0804 \end{bmatrix}$$

$$b^{[1]} = b^{[1]} - h \, \delta^{[1]}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 0.1 \left(\begin{bmatrix} -0.38 \\ -0.38 \end{bmatrix} + \begin{bmatrix} -0.32 \\ -0.32 \end{bmatrix} + \begin{bmatrix} -0.15 \\ -0.15 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1.0805 \\ 1.0805 \end{bmatrix}$$