Module 3 Review Assignment

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1

$$
\begin{aligned}
a_n =& \frac{(1+n)!}{n!} \\
a_n =& \frac{(n+1)(n)(n-1)\cdots}{(n)(n-1)\cdots} \\
a_n =& n+1 \\
& \lim_{\infty}[n+1] = \infty
\end{aligned}
$$

2

$$
\begin{aligned}
& \sum_{n=0}^{\infty} \frac{(-2)^{n+3}}{5^{n+2}} \\
& \sum_{n=0}^{\infty} \frac{(-2)^n (-2)^3}{5^n (5)^2} \\
\frac{(-2)^3}{(5)^2} & \sum_{n=0}^{\infty} \left(\frac{-2}{5}\right)^n \\
-\frac{8}{25} & \sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{5}\right)^n \\
-\frac{8}{25} & \cdot \frac{5}{7} = \boxed{-\frac{40}{175}}
\end{aligned}
$$

3

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n(n+2)} \to \frac{A}{n} + \frac{B}{n+2} = \frac{1}{n(n+2)} \to A(n+2) + Bn = 1 \ \Rightarrow \ A = \frac{1}{2} \ B = -\frac{1}{2} \\
\frac{1}{2} \lim_{N\to\infty} \left[\sum_{n=0}^{N} \left(\frac{1}{n+2} - \frac{1}{n}\right)\right] = \frac{1}{2}\left(1 + \frac{1}{2} + \lim_{N\to\infty}\left[\frac{1}{N+2}\right]\right) = \frac{3}{4}
\end{aligned}
$$

4

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2\cdot 3^n)}{2^{2n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n(-1)(n+2)\cdot 3^n}{2^{2n}\cdot 2} = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n \frac{(n+2)\cdot 3^n}{2^{2n}}$$

$$\lim_{n\to\infty} \frac{(n+2\cdot 3^n)}{2^{2n}} = 0$$

$$\frac{(1+2)3}{2^2} \quad \frac{(2+2)3^2}{2^4}$$

$$\underset{b_1}{\frac{9}{4}} \quad \underset{b_2}{\frac{\cancel{4}\cdot 9}{\cancel{4}\cdot 4}} \qquad \frac{(3+2)3^3}{2^6} = \frac{5\cdot 27}{64} = \underset{b_3}{\frac{135}{64}}$$

$$a_n = \frac{(-1)^{n+1}(n+2)\,3^n}{2^{2n}}$$

$$b_n = \frac{(n+2)\,3^n}{2^{2n}}$$

a_1 cancels a_2 and from a_3 on, b_n converges.

5

a) $\sum_{n=1}^{\infty} \left(\sqrt{2}\right)^n$

geometric series

$r = \sqrt{2}$

$|r| > 1$

the series diverges by the nth term test

b) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+\ln(n)}$

$\lim_{n\to\infty} \left[\frac{n^{\frac{1}{2}}}{n^2} : \frac{1}{n^2}\right]$

$\lim_{n\to\infty} \left[\frac{n^{\frac{1}{2}+2}}{n^2}\right]$ → By the limit comparison test, both converge and by direct comparison test $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+\ln(n)}$ converges absolutely.

$\sum_{n=1}^{\infty} e^{-1}n! = \sum_{n=1}^{\infty} \frac{n!}{e^n}$

$\lim_{n\to\infty} \left[\sqrt[n]{\left|\frac{n!}{e^n}\right|}\right] = \infty$

by the Root test $\sum_{n=1}^{\infty} e^{-1}n!$ diverges.

6

a) $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$

converges

$|r| < 1$

b) $\sum_{n=1}^{\infty} \frac{1}{\ln n} \boxed{\frac{1}{\ln n} > \frac{1}{n}}$

by direct comparison test, $a_n > b_n$ the series diverges

$$c)\ \sum_{n=1}^{\infty} \frac{2n}{n+1} \qquad \frac{2(n+1)}{(n+1)+1} : \frac{2n}{n+1} \to \frac{\cancel{2}(n+1)^2}{\cancel{2}n(n+2)} = \frac{(n+1)^2}{n^2+2n}$$

$\sum_{n=1}^{\infty} \frac{2n}{n+1}$ diverges by the nth term test

$$\lim_{n\to\infty} \left[\frac{(n+1)^2}{n^2+2n}\right] = 1 \;\; 1 \neq 0$$

$d)$ $\sum\limits_{n=1}^{\infty}(-1)^n \frac{1}{\ln(n)}$ by the alternating series test, the series converge conditionally.
$e)$ $\sum\limits_{n=1}^{\infty}\frac{1}{n(2n+1)}$ Converges by direct comparison to $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$

7

$$\sum_{n=1}^{4}(-1)^{n+1}\left(\frac{1}{2}\right)^n = S_n$$

$a)$ $S_4 = +\left(\frac{1}{2}\right)-\left(\frac{1}{4}\right)+\left(\frac{1}{8}\right)-\left(\frac{1}{16}\right)=\frac{4+1}{8}-\frac{4+1}{16}=\frac{10-5}{16}=\frac{5}{16}$
$b)$ $\frac{5}{16}<S_n<\frac{11}{32}$
$c)$ $\frac{\frac{1}{2}}{1-\left(\frac{1}{2}\right)}\cdot\frac{1}{2}=\frac{1}{2}$
$d)$ yes

8

$a)$ F $b)$ F $c)$ T $d)$ T

9

$a)$ $\sum\limits_{n=0}^{\infty}\frac{x^n}{3n+1}$ $\quad \frac{\frac{x^{n+1}}{3(n+1)+1}}{\frac{x^n}{3n+1}}$

$$\lim_{n\to\infty}\left|\frac{x^{n+1}(3n+1)}{(3(n+1))x^n}\right|=\lim_{n\to\infty}\left|\frac{x(3n+1)}{3n+4}\right|=\lim_{n\to\infty}\left[|x|\frac{3+\frac{1}{n}}{3+\frac{4}{n}}\right]=|x| \quad -1<x<1$$

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$$a)\ \sum_{n=1}^{\infty}\frac{1}{2+3^n}\Rightarrow\lim_{n\to\infty}\left|\frac{\frac{1}{2+3^{n+1}}}{\frac{1}{2+3^n}}\right|=\lim_{n\to\infty}\left|\frac{2+3^n}{2+3^{n+1}}\right|=0$$

$$b)\ \sum_{n=0}^{\infty}\frac{n}{n^3+4}\Rightarrow\lim_{n\to\infty}\left|\frac{\frac{n+1}{(n+1)^3+4}}{\frac{n}{n^3+4}}\right|=1$$

$$\lim_{n\to\infty}\left[\frac{\frac{n}{n^3+4}}{\frac{1}{n^2}}\right]\Rightarrow\lim_{n\to\infty}\left[\frac{n^3}{n^3+4}\right]=\lim_{n\to\infty}\left[\frac{1}{1+\frac{4}{n^3}}\right]=0$$

$$c)\ \sum_{n=0}^{\infty}(-1)^{n+1}\frac{n^2}{3^n}\Rightarrow\lim_{n\to\infty}\left|\frac{\frac{(n+1)^2}{3^{(n+1)}}(-1)^{n+2}}{\frac{n^2}{3^n}(-1)^{n+1}}\right|=\lim_{n\to\infty}\left[\frac{\cancel{3}^n(n+1)^2(-1)^{\cancel{n}+2}}{3\cdot\cancel{3}^n\cdot n^2(-1)^{\cancel{n}+1}}\right]=\lim_{n\to\infty}\left[(-1)^2\frac{(n+1)^2}{n^2}\right]=1$$

11

$a)$ $\ln(1+x)$

$$f(0) = 0 \qquad f(x) = \ln(1+x)$$
$$f^{'}(0) = 1 \qquad f^{'}(x) = \frac{1}{x+1}$$
$$f^{''}(0) = -1 \qquad f^{''}(x) = -\frac{1}{(x+1)^2}$$
$$f^{'''}(0) = 2 \qquad f^{'''}(x) = \frac{2}{(x+1)^3}$$
$$f^{(4)}(0) = -6 \qquad f^{(4)}(x) = -\frac{6}{(x+1)^4}$$
$$f^{(5)}(0) = 24 \qquad f^{(5)}(x) = \frac{24}{(x+1)^5}$$

$0 + \frac{1}{1}(x)^1 - \frac{1}{2}(x)^2 + \frac{1}{3}(x)^3 - \frac{1}{4}(x)^4$

$\boxed{\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} (x)^n}$

$b)$ $\left|\frac{M}{(n+1)!}(x-c)^{n+1}\right|$ where $M \geq \left|f^{n+1}(x)\right| \to \frac{M(x)^5}{120}$

$c)$ $\ln(1+x)$ about 1 4^{th}

$$\ln(2) + \frac{\frac{1}{2}}{1}(x-1)^1 - \frac{\frac{1}{4}}{2}(x-1)^2 + \frac{\frac{2}{8}}{6}(x-1)^3 - \frac{\frac{6}{16}}{24}(x-1)^4$$

$d)$ $x^2 - \frac{1}{2}(x)^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8$ $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \left(x^2\right)^n$

$e)$ $x\ln\left(x^2+1\right) = \ln(1+a) \to e^{x\ln\left(1+x^2\right)} = e^{\ln(1+x)} \to \left(e^{\ln\left(1+x^2\right)}\right)^x = 1+a \to \boxed{a = \left(1+x^2\right)^x - 1}$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \left(\left(1+x^2\right)^x - 1\right)^n$$

$$\left(\left(1+x^2\right)^x - 1\right) - \frac{1}{2}\left(\left(1+x^2\right)^x - 1\right)^2 + \frac{1}{3}\left(\left(1+x^2\right)^x - 1\right)^3 - \frac{1}{4}\left(\left(1+x^2\right)^x - 1\right)^4$$

$f)$ $\frac{d}{dx}\left[\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \left(x^2\right)^n\right]$

$$\sum_{n=0}^{\infty} \frac{d}{dx}\left[(-1)^{n+1} \frac{1}{n} \left(x^2\right)^n\right] = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \cdot \frac{d}{dx}\left[x^2\right]^n = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2\cancel{n}}{\cancel{n}} x^{2n-1} = \sum_{n=1}^{\infty} (-1)^{n+1} 2x^{2n-1}$$

$g)$ $\int\limits_0^1 \ln\left(1+x^2\right) dx \to \int\limits_0^1 \sum\limits_{n=0}^{\infty} (-1)^{n+1} \frac{1}{n} \left(x^2\right)^n dx$

$\ln(2) \approx 0.2694 \qquad |E_n(x)| \leq \frac{0.2694\,|x|^{n+1}}{(n+1)!}$

$0.01 \leq \frac{0.2694}{(n+1)!}$ for $n = 4$ $\quad \frac{0.2694}{5!} \approx 0.0058$

$$\int\limits_0^1 \left(x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8\right) dx = \left[\frac{x^3}{3} - \frac{x^5}{10} + \frac{x^7}{21} - \frac{x^9}{36}\right]_0^1 = \frac{1}{3} - \frac{1}{10} + \frac{1}{21} - \frac{1}{36} = \frac{319}{1260} \approx 0.25$$