Module 3 Review Assignment

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1

$$
\begin{aligned}
a_n =& \frac{(1+n)!}{n!} \\
a_n =& \frac{(n+1)(n)(n-1)\cdots}{(n)(n-1)\cdots} \\
a_n =& n+1 \\
& \lim_{\infty} [n+1] = \infty
\end{aligned}
$$

2

$$
\begin{aligned}
& \sum_{n=0}^{\infty} \frac{(-2)^{n+3}}{5^{n+2}} \\
& \sum_{n=0}^{\infty} \frac{(-2)^n (-2)^3}{5^n (5)^2} \\
\frac{(-2)^3}{(5)^2} & \sum_{n=0}^{\infty} \left(\frac{-2}{5}\right)^n \\
-\frac{8}{25} & \sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{5}\right)^n \\
-\frac{8}{25} & \cdot \frac{5}{7} = \boxed{-\tfrac{40}{175}}
\end{aligned}
$$

3

$$
\sum_{n=1}^{\infty} \frac{1}{n\,(n+2)} \rightarrow \frac{A}{n} + \frac{B}{n+2} = \frac{1}{n(n+2)} \rightarrow A(n+2) + Bn = 1 \;\Rightarrow\; A = \frac{1}{2} \; B = -\frac{1}{2}
$$

$$
\frac{1}{2} \lim_{N\to\infty} \left[\sum_{n=0}^{N} \left(\frac{1}{n+2} - \frac{1}{n}\right)\right] = \frac{1}{2}\left(1 + \frac{1}{2} + \lim_{N\to\infty}\left[\frac{1}{N+2}\right]\right) = \frac{3}{4}
$$

4

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n+2\cdot 3^n)}{2^{2n+1}}=\sum_{n=1}^{\infty}\frac{(-1)^n(-1)(n+2)\cdot 3^n}{2^{2n}\cdot 2}=\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(n+2)\cdot 3^n}{2^{2n}}$$

$$\lim_{n\to\infty}\frac{(n+2\cdot 3^n)}{2^{2n}}=0$$

$$\frac{(1+2)3}{2^2}\quad\frac{(2+2)3^2}{2^4}$$

$$\frac{9}{4}\quad\frac{\cancel{4}\cdot 9}{\cancel{4}\cdot 4}\qquad\frac{(3+2)3^3}{2^6}=\frac{5\cdot 27}{64}=\frac{135}{64}$$

b_1 b_2 b_3

$$a_n=\frac{(-1)^{n+1}(n+2)\,3^n}{2^{2n}}$$

$$b_n=\frac{(n+2)\,3^n}{2^{2n}}$$

a_1 cancels a_2 and from a_3 on, b_n converges.

5

a) $\sum_{n=1}^{\infty}\left(\sqrt{2}\right)^n$

geometric series

$r=\sqrt{2}$

$|r|>1$

the series diverges by the nth term test

b) $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+\ln(n)}$

$\lim_{n\to\infty}\left[\frac{n^{\frac{1}{2}}}{n^2}:\frac{1}{n^2}\right]$

$\lim_{n\to\infty}\left[\frac{n^{\frac{1}{2}+2}}{n^2}\right]\to$ By the limit comparison test, both converge and by direct comparison test $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+\ln(n)}$ converges absolutely.

$\sum_{n=1}^{\infty}e^{-1}n!=\sum_{n=1}^{\infty}\frac{n!}{e^n}$

$\lim_{n\to\infty}\left[\sqrt[n]{\left|\frac{n!}{e^n}\right|}\right]=\infty$

by the Root test $\sum_{n=1}^{\infty}e^{-1}n!$ diverges.

6

a) $\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$

converges

$|r|<1$

b) $\sum_{n=1}^{\infty}\frac{1}{\ln n}\boxed{\frac{1}{\ln n}>\frac{1}{n}}$

by direct comparison test, $a_n>b_n$ the series diverges

$$c)\ \sum_{n=1}^{\infty}\frac{2n}{n+1}\qquad\frac{2(n+1)}{(n+1)+1}:\frac{2n}{n+1}\to\frac{\cancel{2}(n+1)^2}{\cancel{2}n(n+2)}=\frac{(n+1)^2}{n^2+2n}$$

$\sum_{n=1}^{\infty}\frac{2n}{n+1}$ diverges by the nth term test

$$\lim_{n\to\infty}\left[\frac{(n+1)^2}{n^2+2n}\right]=1\ 1\neq 0$$

$d)\ \sum\limits_{n=1}^{\infty}(-1)^n\frac{1}{\ln(n)}$ by the alternating series test, the series converge conditionally.
$e)\ \sum\limits_{n=1}^{\infty}\frac{1}{n(2n+1)}$ Converges by direct comparison to $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$

7

$$\sum_{n=1}^{4}(-1)^{n+1}\left(\frac{1}{2}\right)^n = S_n$$

$a)\ S_4 = +\left(\frac{1}{2}\right) - \left(\frac{1}{4}\right) + \left(\frac{1}{8}\right) - \left(\frac{1}{16}\right) = \frac{4+1}{8} - \frac{4+1}{16} = \frac{10-5}{16} = \frac{5}{16}$
$b)\ \frac{5}{16} < S_n < \frac{11}{32}$
$c)\ \frac{\frac{1}{2}}{1-\left(\frac{1}{2}\right)} \cdot \frac{1}{2} = \frac{1}{2}$
$d)$ yes

8

$a)$ F $b)$ F $c)$ T $d)$ T

9

$a)\ \sum\limits_{n=0}^{\infty}\frac{x^n}{3n+1} \qquad \frac{\frac{x^{n+1}}{3(n+1)+1}}{\frac{x^n}{3n+1}}$

$$\lim_{n\to\infty}\left|\frac{x^{n+1}(3n+1)}{(3(n+1))x^n}\right| = \lim_{n\to\infty}\left|\frac{x(3n+1)}{3n+4}\right| = \lim_{n\to\infty}\left[|x|\frac{3+\frac{1}{n}}{3+\frac{4}{n}}\right] = |x| \quad -1 < x < 1$$

$b)\ \sum\limits_{n=0}^{\infty}\frac{4^n}{n!}(x-2)^n$

$$\lim_{n\to\infty}\left|\frac{\frac{4^n}{(n+1)!}(x-2)^{n+1}}{\frac{4^n}{n!}(x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{4^{n+1}}{(n+1)!}(x-2)^{n+1}\cdot\frac{n!}{4^n(x-2)^n}\right| = \boxed{4(x-2)\lim_{n\to\infty}\left[\frac{1}{n+1}\right] = 0}$$

The series converges in the $(-\infty,\infty)$ interval

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$$a)\ \sum_{n=1}^{\infty}\frac{1}{2+3^n}\Rightarrow \lim_{n\to\infty}\left|\frac{\frac{1}{2+3^{n+1}}}{\frac{1}{2+3^n}}\right|=\lim_{n\to\infty}\left|\frac{2+3^n}{2+3^{n+1}}\right|=0$$

$$b)\ \sum_{n=0}^{\infty}\frac{n}{n^3+4}\Rightarrow \lim_{n\to\infty}\left|\frac{\frac{n+1}{(n+1)^3+4}}{\frac{n}{n^3+4}}\right|=1$$

$$\lim_{n\to\infty}\left[\frac{\frac{n}{n^3+4}}{\frac{1}{n^2}}\right]\Rightarrow \lim_{n\to\infty}\left[\frac{n^3}{n^3+4}\right]=\lim_{n\to\infty}\left[\frac{1}{1+\frac{4}{n^3}}\right]=0$$

$$c)\ \sum_{n=0}^{\infty}(-1)^{n+1}\frac{n^2}{3^n}\quad \lim_{n\to\infty}\left|\frac{\frac{(n+1)^2}{3^{n+1}}}{\frac{n^2}{3^n}}\right|=\lim_{n\to\infty}\left|\frac{(n+1)^2}{3^{\cancel{n}+1}}\cdot\frac{\cancel{3^n}}{n^2}\right|=\lim_{n\to\infty}\left|\frac{(n+1)^2}{3n^2}\right|=\boxed{\frac{1}{3}}$$

11

$a)\ \ln(1+x)$

$$f(0)=0 \qquad f(x)=\ln(1+x)$$
$$f^{'}(0)=1 \qquad f^{'}(x)=\frac{1}{x+1}$$
$$f^{''}(0)=-1 \qquad f^{''}(x)=-\frac{1}{(x+1)^2}$$
$$f^{'''}(0)=2 \qquad f^{'''}(x)=\frac{2}{(x+1)^3}$$
$$f^{(4)}(0)=-6 \qquad f^{(4)}(x)=-\frac{6}{(x+1)^4}$$
$$f^{(5)}(0)=24 \qquad f^{(5)}(x)=\frac{24}{(x+1)^5}$$

$0+\frac{1}{1}(x)^1-\frac{1}{2}(x)^2+\frac{1}{3}(x)^3-\frac{1}{4}(x)^4$

$\boxed{\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}(x)^n}$

$b)\ \left|\frac{M}{(n+1)!}(x-c)^{n+1}\right|$ where $M\geq\left|f^{n+1}(x)\right|\to\frac{M(x)^5}{120}$

$c)\ \ln(1+x)$ about 1 $\quad 4^{th}$

$$\ln(2)+\frac{\frac{1}{2}}{1}(x-1)^1-\frac{\frac{1}{4}}{2}(x-1)^2+\frac{\frac{2}{8}}{6}(x-1)^3-\frac{\frac{6}{16}}{24}(x-1)^4$$

$d)\ x^2-\frac{1}{2}(x)^4+\frac{1}{3}x^6-\frac{1}{4}x^8\ \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\left(x^2\right)^n$

$e)\ x\ln\left(x^2+1\right)=\ln(1+a)\to e^{x\ln\left(1+x^2\right)}=e^{\ln(1+x)}\to\left(e^{\ln\left(1+x^2\right)}\right)^x=1+a\to\boxed{a=\left(1+x^2\right)^x-1}$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\left(\left(1+x^2\right)^x-1\right)^n$$

$$\left(\left(1+x^2\right)^x-1\right)-\frac{1}{2}\left(\left(1+x^2\right)^x-1\right)^2+\frac{1}{3}\left(\left(1+x^2\right)^x-1\right)^3-\frac{1}{4}\left(\left(1+x^2\right)^x-1\right)^4$$

$f)\ \frac{d}{dx}\left[\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\left(x^2\right)^n\right]$

$$\sum_{n=0}^{\infty}\frac{d}{dx}\left[(-1)^{n+1}\frac{1}{n}\left(x^2\right)^n\right]=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\cdot\frac{d}{dx}\left[x^2\right]^n=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2\,\cancel{n}}{\cancel{n}}x^{2n-1}=\sum_{n=1}^{\infty}(-1)^{n+1}\,2x^{2n-1}$$

$g)\ \int\limits_0^1 \ln\left(1+x^2\right)dx \rightarrow \int\limits_0^1\sum\limits_{n=0}^{\infty}(-1)^{n+1}\frac{1}{n}\left(x^2\right)^n dx$

$\ln(2)\approx 0.2694 \qquad |E_n(x)|\leq\frac{0.2694\,|x|^{n+1}}{(n+1)!}$

$0.01\leq\frac{0.2694}{(n+1)!}$ for $n=4 \quad \frac{0.2694}{5!}\approx 0.0058$

$$\int\limits_0^1\left(x^2-\frac{1}{2}x^4+\frac{1}{3}x^6-\frac{1}{4}x^8\right)dx=\left[\frac{x^3}{3}-\frac{x^5}{10}+\frac{x^7}{21}-\frac{x^9}{36}\right]_0^1=\frac{1}{3}-\frac{1}{10}+\frac{1}{21}-\frac{1}{36}=\frac{319}{1260}\approx 0.25$$