AP Calculus BC **Module 2 Lesson 4 Assignment**

A student was presented with the following AP question:

Let g be the function given by $g(x) = \dfrac{1}{\sqrt{x}}$.

(a) Find the average value of g on the closed interval $[1, 4]$.

(b) Let S be the solid generated when the region bounded by the graph of $y = g(x)$, the vertical lines $x = 1$ and $x = 4$, and the x-axis is revolved about the x-axis. Find the volume of S.

(c) For the solid S, given in part (b), find the average value of the areas of the cross sections perpendicular to the x-axis.

(d) The average value of a function f on the unbounded interval $[a, \infty)$ is defined to be $\lim_{b\to\infty}\left[\dfrac{\int_a^b f(x)\,dx}{b-a}\right]$. Show that the improper integral $\int_4^\infty g(x)\,dx$ is divergent, but the average value of g on the interval $[4, \infty)$ is finite.

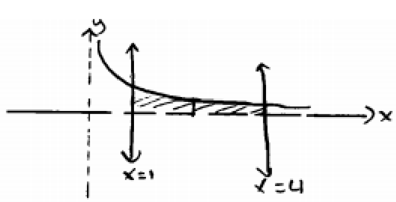
Look at the student's response to each part below and then using the scoring rubric that follows on the last page, determine the score the student would earn overall.

Work for problem 5(a)

$$\text{Avg } g(x) = \frac{1}{4-1}\int_1^4 \frac{1}{\sqrt{x}}\,dx = \frac{1}{3}\int_1^4 x^{-1/2}\,dx = \frac{1}{3}\left[2\sqrt{x}\right]_1^4 = \frac{1}{3}\left[2\sqrt{4} - 2\sqrt{1}\right]$$

$$= \frac{1}{3}\left[2(2) - 2\right] = \frac{2}{3}$$

Work for problem 5(b)



$$V = \pi\int_1^4 R^2(x) - r^2(x)\,dx$$

$$R(x) = \frac{1}{\sqrt{x}} \Rightarrow R^2(x) = \frac{1}{x}$$

$$r(x) = 0$$

$$= \pi\int_1^4 \frac{1}{x}\,dx$$

$$= \pi\left[\ln x\right]_1^4 = \pi \ln 4 \text{ units}^3$$

Work for problem 5(c)

~~l~~ length of cross sections perpendicular to x-axis $= \frac{1}{\sqrt{x}}$ $\quad A = \left(\frac{1}{\sqrt{x}}\right)^2 = \frac{1}{x}$

$$\text{avg} = \frac{1}{4-1}\int_1^4 \left(\frac{1}{\sqrt{x}}\right)^2 dx = \frac{1}{3}\int_1^4 \frac{1}{x}\,dx = \frac{1}{3}\ln x\Big]_1^4 = \frac{1}{3}\ln 4$$

Work for problem 5(d)

$$\int_4^\infty g(x)\,dx = \lim_{a\to\infty}\int_4^a \frac{1}{\sqrt{x}}\,dx = \lim_{a\to\infty} 2\sqrt{x}\Big]_4^a = \lim_{a\to\infty} 2\sqrt{a} - 2\sqrt{4} = \infty$$

$\Rightarrow$ the improper integral $\int_4^\infty g(x)\,dx$ diverges.

$$\lim_{b\to\infty}\frac{\int_a^b f(x)\,dx}{b-a} \Rightarrow \text{average value of } g = \lim_{b\to\infty}\frac{\int_a^b g(x)\,dx}{b-a}$$

$$= \lim_{b\to\infty}\frac{\int_4^b g(x)\,dx}{b-4}$$

$$= \lim_{b\to\infty} g(x) = 0$$

$\Rightarrow$ avg value of g on $[4,\infty)$ is finite.

****Scoring Guide on Next Page****

Use the scoring guide below to determine the score the student would earn and explain your answer.

Let g be the function given by $g(x) = \dfrac{1}{\sqrt{x}}$.

(a) Find the average value of g on the closed interval $[1, 4]$.

(b) Let S be the solid generated when the region bounded by the graph of $y = g(x)$, the vertical lines $x = 1$ and $x = 4$, and the x-axis is revolved about the x-axis. Find the volume of S.

(c) For the solid S, given in part (b), find the average value of the areas of the cross sections perpendicular to the x-axis.

(d) The average value of a function f on the unbounded interval $[a, \infty)$ is defined to be $\displaystyle\lim_{b\to\infty}\left[\frac{\int_a^b f(x)\,dx}{b-a}\right]$. Show that the improper integral $\displaystyle\int_4^\infty g(x)\,dx$ is divergent, but the average value of g on the interval $[4, \infty)$ is finite.

Solution	Scoring
(a) $\displaystyle\frac{1}{3}\int_1^4 \frac{1}{\sqrt{x}}\,dx = \frac{1}{3}\cdot 2\sqrt{x}\Big\vert_1^4 = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$	2 : { 1 : integral; 1 : antidifferentiation and evaluation
(b) Volume $\displaystyle = \pi\int_1^4 \frac{1}{x}\,dx = \pi \ln x\Big\vert_1^4 = \pi\ln 4$	2 : { 1 : integral; 1 : antidifferentiation and evaluation
(c) The cross section at x has area $\pi\left(\dfrac{1}{\sqrt{x}}\right)^2 = \dfrac{\pi}{x}$ Average value $\displaystyle = \frac{1}{3}\int_1^4 \frac{\pi}{x}\,dx = \frac{1}{3}\pi\ln 4$	1 : answer
(d) $\displaystyle\int_4^\infty g(x)\,dx = \lim_{b\to\infty}\int_4^b \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty}\left(2\sqrt{b} - 4\right) = \infty$ This limit is not finite, so the integral is divergent. $\displaystyle\frac{\int_4^b g(x)\,dx}{b-4} = \frac{1}{b-4}\int_4^b \frac{1}{\sqrt{x}}\,dx = \frac{2\sqrt{b}-4}{b-4}$ $\displaystyle\lim_{b\to\infty}\frac{2\sqrt{b}-4}{b-4} = 0$	4 : { 1 : $\int_4^b g(x)\,dx = 2\sqrt{b} - 4$; 1 : indicates integral diverges; 1 : $\dfrac{1}{b-4}\int_4^b g(x)\,dx = \dfrac{2\sqrt{b}-4}{b-4}$; 1 : finite limit as $b \to \infty$