AP Calculus BC Module Three Lesson Four Assignment

A student was presented with the following AP question:

The function f is defined by the power series:

$$f(x) = 1 + (x+1) + (x+1)^2 + \ldots + (x+1)^n + \ldots = \sum_{n=0}^{\infty} (x+1)^n$$

for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

(b) The power series above is the Taylor series for f about $x = -1$. Find the sum of the series for f.

(c) Let g be the function defined by $g(x) = \int_{-1}^{x} f(t)\,dt$. Find the value of $g\left(-\frac{1}{2}\right)$, if it exists, or explain why $g\left(-\frac{1}{2}\right)$ cannot be determined.

(d) Let h be the function defined by $h(x) = f(x^2 - 1)$. Find the first 3 non-zero terms and the general term of the Taylor series for h about $x = 0$, and find the value of $h\left(\frac{1}{2}\right)$.

Look at the student's response to each part below and then using the scoring rubric that follows on the last page, determine the score the student would earn overall.

(a) $-1 < x+1 < 1$ (in order to converge $\sum_{n=1}^{\infty} (x+1)^n$, $x+1$ has to be between -1 and 1)

$\therefore -2 < x < 0$ ✓

(b) $f(x) = 1 + (x+1) + (x+1)^2 + \cdots + (x+1)^n$

$= 1 \times \frac{1}{1-(x+1)} = \frac{1}{-x}$ ✓

(c) $\int_{-1}^{x} f(t)\,dt = g(x)$

$= x + \frac{(x+1)^2}{2} + \frac{(x+1)^3}{3} + \cdots$ ✓

$g\left(-\frac{1}{2}\right) = -\frac{1}{2} + \frac{\left(\frac{1}{2}\right)^2}{2} + \frac{\left(\frac{1}{2}\right)^3}{3} + \cdots$

it can't be determined, because the ~~general term~~ $g(x)$ is $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n+1} \frac{1}{(n+1)}$ which does not converge.

failed to do it $\int_{-1}^{-1/2} -\frac{1}{x}\,dx$ rather than $\int_{-1}^{-1/2} \sum_{n=0}^{\infty} (x+1)^n\,dx$

(d) $h(x)=f(x^2-1)$

$=\cancel{}\ 1+(x^2)+(x^2)^2+\cdots+(x^2)^n$

$h(\frac{1}{2})=$ ~~...~~ $1+(\frac{1}{2})^2+(\frac{1}{2})^4+(\frac{1}{2})^6+\cdots$

$=1\times\frac{1}{1-\frac{1}{4}}=1\times\frac{4}{3}=\frac{4}{3}$

✓

The scoring guide for this question shows:

The function f is defined by the power series:

$$f(x)=1+(x+1)+(x+1)^2+\ldots+(x+1)^n+\ldots=\sum_{n=0}^{\infty}(x+1)^n$$

for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

(b) The power series above is the Taylor series for f about $x=-1$. Find the sum of the series for f.

(c) Let g be the function defined by $g(x)=\int_{-1}^{x}f(t)dt$. Find the value of $g\left(-\frac{1}{2}\right)$, if it exists, or explain why $g\left(-\frac{1}{2}\right)$ cannot be determined.

(d) Let h be the function defined by $h(x)=f(x^2-1)$. Find the first 3 non-zero terms and the general term of the Taylor series for h about $x=0$, and find the value of $h\left(\frac{1}{2}\right)$.

(a) The power series is geometric with ratio $(x+1)$. The series converges if and only if $\lvert x+1\rvert<1$. Therefore, the interval of convergence is $-2<x<0$.	$3:\begin{cases}1: \text{identifies as geometric}\\ 1: \lvert x+1\rvert<1\\ 1: \text{interval of convergence}\end{cases}$
OR	OR
$\lim_{n\to\infty}\left\lvert\frac{(x+1)^{n+1}}{(x+1)^n}\right\rvert=\lvert x+1\rvert<1$ when $-2<x<0$. At $x=-2$, the series is $\sum_{n=0}^{\infty}(-1)^n$, which diverges since the terms do not converge to 0. At $x=0$, the series is $\sum_{n=0}^{\infty}1$, which similarly diverges. Therefore, the interval of convergence is $-2<x<0$.	$3:\begin{cases}1: \text{sets up limit of ratio}\\ 1: \text{radius of convergence}\\ 1: \text{interval of convergence}\end{cases}$
(b) Since the series is geometric, $f(x)=\sum_{n=0}^{\infty}(x+1)^n=\frac{1}{1-(x+1)}=-\frac{1}{x}$ for $-2<x<0$.	1 : answer
(c) $g\left(-\frac{1}{2}\right)=\int_{-1}^{-\frac{1}{2}}-\frac{1}{x}\,dx=-\ln\lvert x\rvert\Big\rvert_{x=-1}^{x=-\frac{1}{2}}=\ln 2$	$2:\begin{cases}1: \text{antiderivative}\\ 1: \text{value}\end{cases}$
(d) $h(x)=f(x^2-1)=1+x^2+x^4+\cdots+x^{2n}+\cdots$ $h\left(\frac{1}{2}\right)=f\left(-\frac{3}{4}\right)=\frac{4}{3}$	$3:\begin{cases}1: \text{first three terms}\\ 1: \text{general term}\\ 1: \text{value of } h\left(\frac{1}{2}\right)\end{cases}$

Use the scoring guide to determine the score the student would earn and explain your answer

~~4~~/4   8/9   The student failed to find the definite integral. Nevertheless, the antiderivative was found