

Module 3 Review Assignment

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April 21, 2017

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$$\begin{aligned}a_n &= \frac{(1+n)!}{n!} \\a_n &= \frac{(n+1)(n)(n-1)\cdots}{(n)(n-1)\cdots} \\a_n &= n+1 \\ \lim_{\infty} [n+1] &= \infty\end{aligned}$$

2

$$\begin{aligned}& \sum_{n=0}^{\infty} \frac{(-2)^{n+3}}{5^{n+2}} \\& \sum_{n=0}^{\infty} \frac{(-2)^n (-2)^3}{5^n (5)^2} \\& \frac{(-2)^3}{(5)^2} \sum_{n=0}^{\infty} \left(\frac{-2}{5}\right)^n \\& -\frac{8}{25} \sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{5}\right)^n \\& -\frac{8}{25} \cdot \frac{5}{7} = \boxed{-\frac{40}{175}}\end{aligned}$$

3

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n(n+2)} &\rightarrow \frac{A}{n} + \frac{B}{n+2} = \frac{1}{n(n+2)} \rightarrow A(n+2) + Bn = 1 \Rightarrow A = \frac{1}{2} \quad B = -\frac{1}{2} \\ \frac{1}{2} \lim_{N \rightarrow \infty} \left[\sum_{n=0}^N \left(\frac{1}{n+2} - \frac{1}{n} \right) \right] &= \frac{1}{2} \left(1 + \frac{1}{2} + \lim_{N \rightarrow \infty} \left[\frac{1}{N+2} \right] \right) = \frac{3}{4}\end{aligned}$$

4

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2 \cdot 3^n)}{2^{2n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n(-1)(n+2) \cdot 3^n}{2^{2n} \cdot 2} = \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n \frac{(n+2) \cdot 3^n}{2^{2n}}$$

$$\lim_{n \rightarrow \infty} \frac{(n+2 \cdot 3^n)}{2^{2n}} = 0$$

$$\frac{(1+2)3}{2^2} \frac{(2+2)3^2}{2^4}$$

$$\frac{9}{4} \frac{4 \cdot 9}{4 \cdot 4} \quad \frac{(3+2)3^3}{2^6} = \frac{5 \cdot 27}{64} = \frac{135}{64}$$

$$b_1 \quad b_2 \quad b_3$$

$$a_n = \frac{(-1)^{n+1}(n+2)3^n}{2^{2n}}$$

$$b_n = \frac{(n+2)3^n}{2^{2n}}$$

a_1 cancels a_2 and from a_3 on, b_n converges.

5

$$a) \sum_{n=1}^{\infty} (\sqrt{2})^n$$

geometric series
 $r = \sqrt{2}$
 $|r| > 1$
the series diverges by the nth term test

$$b) \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + \ln(n)}$$

$$\lim_{n \rightarrow \infty} \left[\frac{n^{\frac{1}{2}}}{n^2} : \frac{1}{n^2} \right]$$

$$\lim_{n \rightarrow \infty} \left[\frac{n^{\frac{1}{2}+2}}{n^2} \right] \rightarrow \text{By the limit comparison test, both converge and by direct comparison test}$$

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + \ln(n)} \text{ converges absolutely.}$$

$$\sum_{n=1}^{\infty} e^{-1}n! = \sum_{n=1}^{\infty} \frac{n!}{e^n}$$

$$\lim_{n \rightarrow \infty} \left[\sqrt[n]{\frac{n!}{e^n}} \right] = \infty$$

by the Root test $\sum_{n=1}^{\infty} e^{-1}n!$ diverges.

6

$$a) \sum_{n=1}^{\infty} \left(\frac{1}{3} \right)^n$$

converges
 $|r| < 1$

$$b) \sum_{n=1}^{\infty} \frac{1}{\ln n} \left[\frac{1}{\ln n} > \frac{1}{n} \right]$$

by direct comparison test, $a_n > b_n$ the series diverges

$$c) \sum_{n=1}^{\infty} \frac{2n}{n+1} \quad \frac{2(n+1)}{(n+1)+1} : \frac{2n}{n+1} \rightarrow \frac{2(n+1)^2}{2n(n+2)} = \frac{(n+1)^2}{n^2+2n}$$

$$\sum_{n=1}^{\infty} \frac{2n}{n+1} \text{ diverges by the nth term test}$$

$$\lim_{n \rightarrow \infty} \left[\frac{(n+1)^2}{n^2+2n} \right] = 1 \quad 1 \neq 0$$

- d) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\ln(n)}$ by the alternating series test, the series converge conditionally.
- e) $\sum_{n=1}^{\infty} \frac{1}{n(2n+1)}$ Converges by direct comparison to $\sum_{n=1}^{\infty} \frac{1}{n^2}$

7

$$\sum_{n=1}^4 (-1)^{n+1} \left(\frac{1}{2}\right)^n = S_n$$

- a) $S_4 = +\left(\frac{1}{2}\right) - \left(\frac{1}{4}\right) + \left(\frac{1}{8}\right) - \left(\frac{1}{16}\right) = \frac{4+1}{8} - \frac{4+1}{16} = \frac{10-5}{16} = \frac{5}{16}$
- b) $\frac{5}{16} < S_n < \frac{11}{32}$
- c) $\frac{\frac{1}{2}}{1 - \left(\frac{1}{2}\right)} \cdot \frac{1}{2} = \frac{1}{2}$
- d) yes

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a) F

b) F

c) T

d) T

9

$$a) \sum_{n=0}^{\infty} \frac{x^n}{3n+1} \quad \frac{\frac{x^{n+1}}{3(n+1)+1}}{\frac{x^n}{3n+1}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{x^{n+1} (3n+1)}{(3(n+1)) x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x (3n+1)}{3n+4} \right| = \lim_{n \rightarrow \infty} \left[|x| \frac{3 + \frac{1}{n}}{3 + \frac{4}{n}} \right] = |x| \quad -1 < x < 1$$

$$b) \sum_{n=0}^{\infty} \frac{4^n}{n!} (x-2)^n$$

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{4^n}{(n+1)!} (x-2)^{n+1}}{\frac{4^n}{n!} (x-2)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{4^{n+1}}{(n+1)!} (x-2)^{n+1} \cdot \frac{n!}{4^n (x-2)^n} \right| = \boxed{4(x-2) \lim_{n \rightarrow \infty} \left[\frac{1}{n+1} \right] = 0}$$

The series converges in the $(-\infty, \infty)$ interval

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$$\begin{aligned}
 a) \quad & \sum_{n=1}^{\infty} \frac{1}{2+3^n} \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{2+3^{n+1}}}{\frac{1}{2+3^n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{2+3^n}{2+3^{n+1}} \right| = 0 \\
 b) \quad & \sum_{n=0}^{\infty} \frac{n}{n^3+4} \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{\frac{n+1}{(n+1)^3+4}}{\frac{n}{n^3+4}} \right| = 1 \\
 & \lim_{n \rightarrow \infty} \left[\frac{\frac{n}{n^3+4}}{\frac{1}{n^2}} \right] \Rightarrow \lim_{n \rightarrow \infty} \left[\frac{n^3}{n^3+4} \right] = \lim_{n \rightarrow \infty} \left[\frac{1}{1+\frac{4}{n^3}} \right] = 0 \\
 c) \quad & \sum_{n=0}^{\infty} (-1)^{n+1} \frac{n^2}{3^n} \quad \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)^2}{3^{n+1}}}{\frac{n^2}{3^n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2}{3^{n+1}} \cdot \frac{3^n}{n^2} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2}{3n^2} \right| = \boxed{\frac{1}{3}}
 \end{aligned}$$

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$$a) \ln(1+x)$$

$f(0) = 0$	$f(x) = \ln(1+x)$
$f'(0) = 1$	$f'(x) = \frac{1}{x+1}$
$f''(0) = -1$	$f''(x) = -\frac{1}{(x+1)^2}$
$f'''(0) = 2$	$f'''(x) = \frac{2}{(x+1)^3}$
$f^{(4)}(0) = -6$	$f^{(4)}(x) = -\frac{6}{(x+1)^4}$
$f^{(5)}(0) = 24$	$f^{(5)}(x) = \frac{24}{(x+1)^5}$

$$0 + \frac{1}{1}(x)^1 - \frac{1}{2}(x)^2 + \frac{1}{3}(x)^3 - \frac{1}{4}(x)^4$$

$$\boxed{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} (x)^n}$$

$$b) \left| \frac{M}{(n+1)!} (x-c)^{n+1} \right| \quad \text{where } M \geq |f^{n+1}(x)| \rightarrow \frac{M(x)^5}{120}$$

$$c) \ln(1+x) \text{ about } 1 \quad 4^{th}$$

$$\ln(2) + \frac{1}{2}(x-1)^1 - \frac{1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3 - \frac{6}{24}(x-1)^4$$

$$d) x^2 - \frac{1}{2}(x)^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} (x^2)^n$$

$$e) x \ln(x^2+1) = \ln(1+a) \rightarrow e^{x \ln(1+x^2)} = e^{\ln(1+x)} \rightarrow (e^{\ln(1+x^2)})^x = 1+a \rightarrow \boxed{a = (1+x^2)^x - 1}$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \left((1+x^2)^x - 1 \right)^n$$

$$((1+x^2)^x - 1) - \frac{1}{2}((1+x^2)^x - 1)^2 + \frac{1}{3}((1+x^2)^x - 1)^3 - \frac{1}{4}((1+x^2)^x - 1)^4$$

$$f) \frac{d}{dx} \left[\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} (x^2)^n \right]$$

$$\sum_{n=0}^{\infty} \frac{d}{dx} \left[(-1)^{n+1} \frac{1}{n} (x^2)^n \right] = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \cdot \frac{d}{dx} [x^2]^n = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{\cancel{n}} x^{2n-1} = \sum_{n=1}^{\infty} (-1)^{n+1} 2x^{2n-1}$$

$$g) \int_0^1 \ln(1+x^2) dx \rightarrow \int_0^1 \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{n} (x^2)^n dx$$

$$\ln(2) \approx 0.2694 \quad |E_n(x)| \leq \frac{0.2694 |x|^{n+1}}{(n+1)!}$$

$$0.01 \leq \frac{0.2694}{(n+1)!} \text{ for } n=4 \quad \frac{0.2694}{5!} \approx 0.0058$$

$$\int_0^1 \left(x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 \right) dx = \left[\frac{x^3}{3} - \frac{x^5}{10} + \frac{x^7}{21} - \frac{x^9}{36} \right]_0^1 = \frac{1}{3} - \frac{1}{10} + \frac{1}{21} - \frac{1}{36} = \frac{319}{1260} \approx 0.25$$