Normal Distribution Derivation

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May 24, 2017

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1 Intuition

First, to derive this incredibly useful formula, we start by the intuition that it is symmetric and that it is similar to the derivative of the logistic equation, with that intuition, we can come up with the following differential equation:

$$\frac{df}{dx} = -k(x-\mu)f(x) \tag{1}$$

Where k is the constant that defines the rate at which it decreases, $(x-\mu)$ describes the center, and x describes the rate at which the frequencies fall off proportionally to the distance of the score from the mean, and $f(x)$ to the frequencies themselves.

2 Solution to the Intuition

With a differential equation, we can now find a solution:

$$\frac{df}{f} = -k(x-\mu)dx \tag{2}$$

$$\int \frac{1}{f} df = -k \int (x-\mu)dx \tag{3}$$

$$\ln(f) = \left[\frac{-k(x-\mu)^2}{2} + C\right] \tag{4}$$

$$f = e^{-k\frac{(x-\mu)^2}{2} + C} \tag{5}$$

$$f = e^{-k\frac{(x-\mu)^2}{2}} e^C \tag{6}$$

$$f = e^{-k\frac{(x-\mu)^2}{2}} C \tag{7}$$

$$f(x) = Ce^{-k\frac{(x-\mu)^2}{2}}$$

3 Specific Solution

3.1 Solve for C

Now we know a general solution to the differential equation, but that by itself is not helpful when trying to find the probability of an event that follows a normal distribution.

To find this specific solution, we start by taking the definite integral from $-\infty$ to ∞ since we know that the area under the curve is 1.

$$C \int_{-\infty}^{\infty} e^{-\frac{k}{2}(x-\mu)^2} \; dx = 1 \tag{8}$$

3.1.1 U-Substitution

To integrate this expression, u-substitution is really useful:

$$u = \sqrt{\frac{k}{2}}(x-\mu) \tag{9}$$

$$\frac{k}{2}(x-\mu^2) = u^2 \tag{10}$$

$$du = \sqrt{\frac{k}{2}} dx \Longrightarrow dx = \sqrt{\frac{2}{k}} du \tag{11}$$

Therefore:

$$C \int_{-\infty}^{\infty} e^{-\frac{k}{2}(x-\mu)^2} dx = C\sqrt{\frac{2}{k}} \int_{-\infty}^{\infty} e^{-u^2} du = 1 \tag{12}$$

3.1.2 Preparation for Polar Coordinates

If we square the expression we obtain the following:

$$\left(C\sqrt{\frac{2}{k}} \int_{-\infty}^{\infty} e^{-u^2} du \right)^2 = 1^2 \tag{13}$$

$$\frac{2C^2}{k} \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right) \left(\int_{-\infty}^{\infty} e^{-y^2} dy \right) = 1 \tag{14}$$

When the two integrals are separated, we can use different variables for each one. Then, by Fubini's Theorem, we obtain the following:

$$\frac{2C^2}{k} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\left[x^2+y^2\right]} dx\ dy = 1 \tag{15}$$

3.1.3 Translate to Polar Coordinates

$$\frac{2C^2}{k} \int_{0}^{2\pi} \int_{0}^{\infty} re^{-r^2} dr\ d\theta = 1 \tag{16}$$

3.1.4 U-Substitution and Solving

$v = -r^2$ $\qquad$ $dv = -2r\ dr$

$$\frac{C^2}{k} \int_{0}^{2\pi} \int_{0}^{\infty} -e^{-v} dv\ d\theta = 1 \tag{17}$$

$$\frac{C^2}{k} \int_{0}^{2\pi} 1\ d\theta = 1 \tag{18}$$

$$\frac{C^2}{k} 2\pi = 1 \tag{19}$$

$$C^2 = \frac{k}{2\pi} \tag{20}$$

$$C = \sqrt{\frac{k}{2\pi}} \tag{21}$$

3.2 Solving for k

Now we can substitute C with can solve for k

$$Ce^{-\frac{k}{2}(x-\mu)^2} \longrightarrow f(x) = \sqrt{\frac{k}{2\pi}} e^{-\frac{k}{2}(x-\mu)^2} \tag{22}$$

3.2.1 Intuition

Since k defines the spread of the distribution $f(x)$, we can calculate the Expected value to find the variance σ^2, the expected value can be computed in this case through $E[X] = \int\limits_{-\infty}^{\infty} xf(x)dx$, it is necessary to manipulate the distribution in order to find σ

$$x - \mu = v$$
$$dx = dv$$

$$E(v) = \sqrt{\frac{k}{2\pi}} \int\limits_{-\infty}^{\infty} ve^{-\frac{k}{2}v^2} dv \tag{23}$$

$$w = -\frac{k}{2}v^2$$
$$dw = -kv \; dv$$
$$v \; dv = -\frac{1}{k}dw$$

$$E(v) = \sqrt{\frac{1}{2\pi k}} \int\limits_{-\infty}^{\infty} e^w dw = \sqrt{\frac{1}{2\pi k}}[0 - 0] = 0 \tag{24}$$

$$E(x) = E(v) + \mu = \mu$$

$$\sigma^2 = \int\limits_{-\infty}^{\infty} (x-\mu)^2 f(x) \; dx = \sqrt{\frac{k}{2\pi}} \int\limits_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{k}{2}(x-\mu)^2} dx \tag{25}$$

$$w = x - \mu \qquad dx = dw$$

$$\sigma^2 = \sqrt{\frac{k}{2\pi}} \int\limits_{-\infty}^{\infty} w^2 e^{-\frac{k}{2}w^2} dw \tag{26}$$

3.2.2 Integration by Parts

$$u = w \qquad v = -\frac{1}{k}e^{-\frac{k}{2}w^2}$$
$$du = dw \qquad dv = we^{-\frac{k}{2}w^2} dw$$

$$\sqrt{\frac{k}{2\pi}}\left[-\frac{ve^{-\frac{k}{2}v^2}}{k}\right]_{-\infty}^{\infty}+\frac{1}{k}\sqrt{\frac{k}{2\pi}}\int\limits_{-\infty}^{\infty} e^{-\frac{k}{2}w^2}dw \tag{27}$$

The first part simply reduces to 0 and in the second one, the integral part,since it is the area under a bell shaped curve ends up being equal to 1 when multiplied by $\sqrt{\frac{k}{2\pi}}$, leaving $\sigma^2 = \frac{1}{k}$ or solving for k, $k = \frac{1}{\sigma^2}$

4 Gaussian Distribution Probability Density Function

And Finally, the Normal Distribution Curve formula:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{28}$$