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Properties and methods of estimation for a bivariate exponentiated Fréchet distribution

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Abstract

In this paper, a bivariate extension of exponentiated Fréchet distribution is introduced, namely a bivariate exponentiated Fréchet (BvEF) distribution whose marginals are univariate exponentiated Fréchet distribution. Several properties of the proposed distribution are discussed, such as the joint survival function, joint probability density function, marginal probability density function, conditional probability density function, moments, marginal and bivariate moment generating functions. Moreover, the proposed distribution is obtained by the Marshall-Olkin survival copula. Estimation of the parameters is investigated by the maximum likelihood with the observed information matrix. In addition to the maximum likelihood estimation method, we consider the Bayesian inference and least square estimation and compare these three methodologies for the BvEF. A simulation study is carried out to compare the performance of the estimators by the presented estimation methods. The proposed bivariate distribution with other related bivariate distributions are fitted to a real-life paired data set. It is shown that, the BvEF distribution has a superior performance among the compared distributions using several tests of goodness–of–fit.

Keywords: Copula, exponentiated Fréchet distribution, Maximum likelihood estimators, Fisher information matrix, Bayesian inference, Least squares method.

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1. Introduction

The Fréchet distribution was introduced by a French mathematician named Maurice Fréchet [8] who identified that it is one possible limit distribution for the largest order statistic in [2]. The Fréchet distribution has been shown to be useful for modeling and analysis of several extreme events ranging from accelerated life testing to earthquakes, floods, rain fall, sea currents and wind speeds. Therefore Fréchet distribution is well suited to characterize random variables of large features. Applications of the Fréchet distribution in various fields are given in Kotz and Nadarajah [12]. Also it is widely utilized in finance, actuarial science, hydrology, economics, material sciences, telecommunications, time series modelling, reliability analysis and several other fields of scientific investigation involving extreme events. For informative scholarly work on extreme value distributions and related results the reader is referred to [1], [3] and [13].

Bivariate data analysis is of a major role in interpretation the relations between two variables. As pointed above, there are many bivariate distributions in literature although they are not able to analyze some of paired data. Therefore, there is a need to introduce other bivariate distributions. For this purpose, we introduce a bivariate exponentiated Fréchet (BvEF) distribution, whose marginals are obtained by a similar method used to obtain Marshall-Olkin bivariate exponential model [11, 15] and bivariate generalized exponential model [9]. The proposed distribution is constructed from three independent univariate exponentiated Fréchet distributions, as it shall be shown later. Several properties of this BvEF distribution are established, including the joint moment generating function, hazard rate function, maximum likelihood estimation, least squares estimation (LSE) based on the joint and marginal empirical survival functions, as an alternative procedure to the maximum likelihood, and Bayesian methods of estimation.

To the best of our knowledge the authors are not aware of any bivariate distribution whose marginals are univariate exponentiated Fréchet distribution. Further, the joint probability density function, joint survival function and the hazard rate function of the BvEF distribution have closed forms. The joint probability density and hazard rate functions can take different shapes, which makes the BvEF distribution capable of analyzing bivariate data with different nature. Another motivation is that the distribution may be used as a competing risk model or a shock model based on some of its physical interpretations which will be shown later. A final motivation is that the BvEF distribution can be obtained by using the Marshall-Olkin survival copula with a univariate exponentiated Fréchet distribution as marginals.

Rest of the paper is organized as: In section 2, we define the BvEF distribution and provide many properties. In section 3, we introduce the marginal and joint moment generating function. In sections 4 and 5, we provide the mathematical expectation and moment generating function, respectively. Section 6 deals with the calculation of the monotonically failure rate function. In Section 7 we provide the maximum likelihood estimators of parameters and observed information matrix, the Bayesian and also the least square estimation methods for the unknown parameters of the BvEF distribution. In Section 8, a simulation study is carried out to compare performance of the estimators by the proposed estimation methods. In section 9, we discuss the fitting of the proposed bivariate distribution to a real-life paired data set with other related bivariate distributions and also the estimation procedures. In section 10, we conclude the result.

2. Bivariate Exponentiated Fréchet Distribution

In this section we introduce a modified bivariate distribution with univariate Fréchet marginal distributions. In this sense we start with the definition of the univariate Fréchet distribution

The random variable (r.v) X has an exponentiated Fréchet (EF) distribution with parameters $\beta > 0$ and $\alpha > 0$ if its survival function is given by:

$$S_{EF}(x) = \left(1 - e^{-x^{-\beta}}\right)^{\alpha}, x > 0 \;\; \alpha > 0, \beta > 0. \tag{1}$$

2.1. The Joint Survival Function

Let U_1, U_2 and U_3 be three independent random variables with the exponentiated Fréchet distributions $EF(\beta, \alpha_1)$, $EF(\beta, \alpha_2)$ and $EF(\beta, \alpha_3)$, respectively. Let a random vector (X_1, X_2) be defined as $X_1 = \min(U_1, U_3)$

and $X_2 = \min(U_2, U_3)$. Then the random vector (X_1, X_2) has the bivariate exponentiated Fréchet distribution with parameters β, α_1, α_2 and α_3. We will denote this as $BvEF(\beta, \alpha_1, \alpha_2, \alpha_3)$. We now study the joint distribution of the BvEF for the random vector (X_1, X_2). The following theorem gives the joint survival function of X_1 and X_2 .

Theorem 1. *The survival function of a random vector (X_1, X_2) with the bivariate exponentiated Fréchet distribution with parameters β, α_1, α_2 and α_3 is given by*

$$S_{BvEF}(x_1, x_2) = \left(1 - e^{-x_1^{-\beta}}\right)^{\alpha_1} \left(1 - e^{-x_2^{-\beta}}\right)^{\alpha_2} \left(1 - e^{-z^{-\beta}}\right)^{\alpha_3}, \tag{2}$$

where $z = \max(x_1, x_2)$.

Proof. From the definition of the BvEF distribution, we have that the survival function given as

$$\begin{aligned} S_{BvEF}(x_1, x_2) &= P(\min(U_1, U_3) > x_1, \min(U_2, U_3) > x_2) \\ &= P(U_1 > x_1)P(U_2 > x_2)P(U_3 > \max(x_1, x_2)) \\ &= S_{EF}(x_1, \beta, \alpha_1)S_{EF}(x_2, \beta, \alpha_2)S_{EF}(z, \beta, \alpha_3), \end{aligned} \tag{3}$$

Finally, substituting (1) into result (3), we obtain 2 which completes the proof of theorem. □

From the above theorem we can conclude that the survival function of the random vector with the BvEF distribution with parameters β, α_1, α_2 and α_3, can be rewritten in the form of product of univariate exponentiated Fréchet pdf. Also, we have the following representation.

Corollary 1. *The survival function of the random vector (X_1, X_2) may also be written as*

$$S_{BvEF}(x_1, x_2) = \begin{cases} S_{EF}(x_1, \beta, \alpha_1 + \alpha_3)S_{EF}(x_2, \beta, \alpha_2), & \text{if} \quad x_1 > x_2; \\ S_{EF}(x_1, \beta, \alpha_1)S_{EF}(x_2, \alpha_2 + \alpha_3), & \text{if} \quad x_2 > x_1; \\ S_{EF}(x, \beta, \alpha_1 + \alpha_2 + \alpha_3), & \text{if} \quad x = x_1 = x_2. \end{cases} \tag{4}$$

From the above results we can conclude that the survival function of the random vector (X_1, X_2) is not an absolutely continuous function. It is a mixture of an absolutely continuous component and a singular component. The singular component arises when $x_1 = x_2 = x$ and it is given by $S_{EF}(x, \beta, \alpha_1 + \alpha_2 + \alpha_3)$.

Some physical interpretations for the proposed distribution that may be countered in practical situations are described in the following.

i) Suppose a system is composed of two components, say 1 and 2 and their survival times are X_1 and X_2, respectively. Assuming that there are three different causes of failures, which may affect the system. Due to cause 1, only component 1 can fail and similarly, owing to cause 2 only component 2 can fail. And both components fail at the same time because of cause 3. Let U_1, U_2, U_3 be lifetimes of the different causes, respectively, then the lifetime of the system is a BvEF random variable having the joint survival function defined by (4). Therefore, distribution of the vector (X_1, X_2) is recommended as competing risks model.

ii) Assume that shocks are generated at random from three different sources, say 1, 2, and 3, and the inter-arrival times between the shocks in a source i follow the distribution of $U_i, i = 1, 2, 3$. Suppose these shocks are affecting a system with two components, say 1 and 2. If the shock that is generated from source 1 reaches the system, component 1 fails immediately and similarly, if the shock generated from source 2 arrives the system, component 2 fails. While, if the shock that is generated from source 3 hits the system both the components fail immediately. Let X_1, X_2 denote the survival times of the two components, then survival time of the system follows BvEF distribution. Hence, we have shock model in this case.

iii) For the case $x_1 = x_2 = x$ in Corollary 1, let us consider a series system with two components. Suppose that each component has two subcomponents and one of them is commonly used by each component. Then, $min(X_1, X_2) = min(min(U_1, U_3), min(U_2, U_3))$ is the lifetime of such system and its survival function is $S_{EF}(x, \beta, \alpha_1 + \alpha_2 + \alpha_3)$.

2.2. *Defining **BvEF** through Copula function*

The copula approach allows for the construction of new multivariate distributions based on known marginals.

For specified univariate distribution functions $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$, the copula function $C\Big(F_{X_1}(x_1), F_{X_2}(x_2)\Big)$ results in a bivariate distribution given by

$$F(x_1, x_2) = C\Big(F_{X_1}(x_1), F_{X_2}(x_2)\Big), \tag{5}$$

with corresponding univariate marginal distribution functions $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$. See for more details Sklar [16] and Nelsen [14].

In reliability analysis it is more convenient to express a joint survival function as a copula of its marginal survival functions. If $S_{X_1}(x_1)$ and $S_{X_2}(x_2)$ are the survival functions of the random variables X_1 and X_2, respectively, then the bivariate survival function

$$S(x_1, x_2) = C^*\Big(S_{X_1}(x_1), S_{X_2}(x_2)\Big), \tag{6}$$

yields a survival function for the random vector $(X_1,\ X_2)$. The function C^* is a copula referred to as the survival copula.

An example of survival copula used in reliability analysis is the Marshall and Olkin copula (see Embrecht *et al.*, [7] and Dobrowolski and Kumar [4]). The Marshall-Olkin survival copula (MOC) is a function defined as

$$C^*(u, v) = u\,v\,min\Big(u^{-\theta}, v^{-\theta}\Big), \tag{7}$$

where $u = S_{X_1}(x_1)$ and $v = S_{X_2}(x_2)$ with $0 < \theta < 1$.

Note that with MOC we can also obtain the $BvEF$ distribution as follows.

Using the copula in (7) with marginal survival functions u and v and the result in the Sklar's theorem then we have

$$S(x_1, x_2) = S_{BvEF}(x_1, \alpha_1 + \alpha_3, \beta) S_{BvEF}(x_2, \alpha_2 + \alpha_3, \beta)\, min\Big(\big(S_{BvEF}(x_1, \alpha_1 + \alpha_3, \beta)\big)^{-\theta}, \big(S_{BvEF}(x_2, \alpha_2 + \alpha_3, \beta)\big)^{-\theta}\Big). \tag{8}$$

To derive the survival function (6) based on the MOC (7), consider the following algebraic operations. From (7), we have

$$\begin{aligned}
C^*(u,\ v) &= \Big(1 - \mathrm{e}^{-\mathrm{x}_1^{-\beta}}\Big)^{\alpha_1+\alpha_3} \Big(1 - \mathrm{e}^{-\mathrm{x}_2^{-\beta}}\Big)^{\alpha_2+\alpha_3} \mathrm{min}\Big(\big(\mathrm{S}_{\mathrm{BvEF}}(\mathrm{x}_1, \alpha_1 + \alpha_3, \beta)\big)^{-\theta},\ \big(\mathrm{S}_{\mathrm{BvEF}}(\mathrm{x}_2, \alpha_2 + \alpha_3, \beta)\big)^{-\theta}\Big) \\
&= S_{BvEF}(x_1, \alpha_1, \beta) S_{BvEF}(x_2, \alpha_2, \beta) \Big(1 - \mathrm{e}^{-\mathrm{x}_1^{-\beta}}\Big)^{\alpha_3} \Big(1 - \mathrm{e}^{-\mathrm{x}_2^{-\beta}}\Big)^{\alpha_3} \\
&\quad \times min\Big(\big(S_{BvEF}(x_1, \alpha_1 + \alpha_3, \beta)\big)^{-\theta}, \big(S_{BvEF}(x_2, \alpha_2 + \alpha_3, \beta)\big)^{-\theta}\Big).
\end{aligned} \tag{9}$$

Now, the algebraic expression

$$\Big(1 - \mathrm{e}^{-\mathrm{x}_1^{-\beta}}\Big)^{\alpha_3} \Big(1 - \mathrm{e}^{-\mathrm{x}_2^{-\beta}}\Big)^{\alpha_3} \mathrm{min}\Big(\big(\mathrm{S}_{\mathrm{BvEF}}(\mathrm{x}_1, \alpha_1 + \alpha_3, \beta)\big)^{-\theta}, \big(\mathrm{S}_{\mathrm{BvEF}}(\mathrm{x}_2, \alpha_2 + \alpha_3, \beta)\big)^{-\theta}\Big)$$

in (9) can be rewritten as

$$\left(1-\mathrm{e}^{-x_1^{-\beta}}\right)^{\alpha_3}\left(1-\mathrm{e}^{-x_2^{-\beta}}\right)^{\alpha_3}\min\left(\left(\mathrm{S_{BvEF}}(x_1,\alpha_1+\alpha_3,\beta)\right)^{-\theta},\left(\mathrm{S_{BvEF}}(x_2,\alpha_2+\alpha_3,\beta)\right)^{-\theta}\right)$$

$$=\begin{cases}\left(1-\mathrm{e}^{-x_1^{-\beta}}\right)^{\alpha_3}\left(1-\mathrm{e}^{-x_2^{-\beta}}\right)^{\alpha_3}\left(1-\mathrm{e}^{-x_1^{-\beta}}\right)^{-\theta(\alpha_1+\alpha_3)}, & x_1<x_2\\ \left(1-\mathrm{e}^{-x_1^{-\beta}}\right)^{\alpha_3}\left(1-\mathrm{e}^{-x_2^{-\beta}}\right)^{\alpha_3}\left(1-\mathrm{e}^{-x_2^{-\beta}}\right)^{-\theta(\alpha_2+\alpha_3)}, & x_1>x_2\end{cases}$$

$$=\begin{cases}\left(1-\mathrm{e}^{-x_1^{-\beta}}\right)^{\alpha_3-\theta(\alpha_1+\alpha_3)}\left(1-\mathrm{e}^{-x_2^{-\beta}}\right)^{\alpha_3}, & x_1<x_2\\ \left(1-\mathrm{e}^{-x_1^{-\beta}}\right)^{\alpha_3}\left(1-\mathrm{e}^{-x_2^{-\beta}}\right)^{\alpha_3-\theta(\alpha_2+\alpha_3)}, & x_1>x_2\end{cases}$$

$$=\begin{cases}\left(1-\mathrm{e}^{-x_2^{-\beta}}\right)^{\alpha_3}, & x_1<x_2\\ \left(1-\mathrm{e}^{-x_1^{-\beta}}\right)^{\alpha_3}, & x_1>x_2\end{cases}=S_{BvEF}(z,\alpha_3,\beta)\,,$$

where θ is chosen accordingly $max(x_1,x_2)=x_2$ or $max(x_1,x_2)=x_1$. In the first case we choose $\theta=\frac{\alpha_3}{\alpha_1+\alpha_3}$ and in the second one $\theta=\frac{\alpha_3}{\alpha_2+\alpha_3}$, with $0<\theta<1$. Hence, we obtain the survival function $S(x_1,x_2)=$ $S_{BvEF}(x_1,\alpha_1,\beta)S_{BvEF}(x_2,x_2,\beta)S_{BvEF}(z,\alpha_3,\beta)$, which is verified in equation (3) .

2.3. The Joint Probability Density Function

The following theorem gives the joint probability density function of the BvEF distribution.

Theorem 2. *If the joint survival function of* (X_1,X_2) *is as in equation* (1)*, the joint pdf of* (X_1,X_2) *is given by*

$$f(x_1,x_2)=\begin{cases}f_1(x_1,x_2) & if \quad x_1>x_2;\\ f_2(x_1,x_2) & if \quad x_1<x_2;\\ f_3(x,x) & if \quad x=x_1=x_2\,,\end{cases}\tag{10}$$

where

$$f_1(x_1,x_2)=\mathrm{e}^{-x_1^{-\beta}-x_2^{-\beta}}\left(1-\mathrm{e}^{-x_1^{-\beta}}\right)^{-1+\alpha_1}\left(1-\mathrm{e}^{-x_2^{-\beta}}\right)^{-1+\alpha_2+\alpha_3}x_1^{-1-\beta}x_2^{-1-\beta}\alpha_2(\alpha_1+\alpha_3)\beta^2\,,\tag{11}$$

$$f_2(x_1,x_2)=\mathrm{e}^{-x_1^{-\beta}-x_2^{-\beta}}\left(1-\mathrm{e}^{-x_1^{-\beta}}\right)^{-1+\alpha_1+\alpha_3}\left(1-\mathrm{e}^{-x_2^{-\beta}}\right)^{-1+\alpha_2}x_1^{-1-\beta}x_2^{-1-\beta}\alpha_1(\alpha_2+\alpha_3)\beta^2\,,\tag{12}$$

$$f_3(x,x)=\mathrm{e}^{-x^{-\beta}}\left(1-\mathrm{e}^{-x^{-\beta}}\right)^{-1+\alpha_1+\alpha_2+\alpha_3}x^{-1-\beta}\alpha_3\beta\,.\tag{13}$$

Proof. Now $f_1(x_1,x_2)$ as in equation (11) and $f_2(x_1,x_2)$ as in equation (12) can easily be obtained by taking second order partial differentiation that is $f_1(x_1,x_2)=\frac{\partial^2 S_1(x_1,x_2)}{\partial x_1\partial x_2}$ and $f_2(x_1,x_2)=\frac{\partial^2 S_2(x_1,x_2)}{\partial x_1\partial x_2}$ of bivariate survival functions

1st condition: $x_1<x_2$

$$\begin{aligned}S_1(x_1,x_2)&=S_{EF}(x_1;\alpha_1+\alpha_3,\beta)S_{EF}(x_1,\alpha_2,\beta)\\&=\left(1-\mathrm{e}^{-x_1{}^{-\beta}}\right)^{\alpha_1+\alpha_3}\left(1-\mathrm{e}^{-x_2{}^{-\beta}}\right)^{\alpha_2}\end{aligned}$$

2nd condition: $x_2<x_1$

$$\begin{aligned}S_2(x_1,x_2)&=S_{EF}(x_1;\alpha_1,\beta)S_{EF}(x_1,\alpha_2+\alpha_3,\beta)\\&=\left(1-\mathrm{e}^{-x_1{}^{-\beta}}\right)^{\alpha_1}\left(1-\mathrm{e}^{-x_2{}^{-\beta}}\right)^{\alpha_2+\alpha_3}\end{aligned}$$

$$\int_0^\infty \int_0^{x_2} f_1(x_1,x_2)dx_1dx_2 + \int_0^\infty \int_0^{x_1} f_2(x_1,x_2)dx_1dx_2 + \int_0^\infty f_3(x,x)\mathrm{d}x = 1$$

$$I_1 + I_2 + \int_0^\infty f_3(x,x)\mathrm{d}x = 1.$$

First taking

$$\begin{aligned}
I_1 &= \int_0^\infty \int_0^{x_2} f_1(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 \\
&= \int_0^\infty \int_0^{x_2} \mathrm{e}^{-x_1^{-\beta}-x_2^{-\beta}} \left(1-\mathrm{e}^{-x_1^{-\beta}}\right)^{-1+\alpha_1+\alpha_3} \left(1-\mathrm{e}^{-x_2^{-\beta}}\right)^{-1+\alpha_2} x_1^{-1-\beta}x_2^{-1-\beta}\alpha_2(\alpha_1+\alpha_3)\beta^2\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= \int_0^\infty \mathrm{e}^{-x_2^{-\beta}} \left(1-\mathrm{e}^{-x_2^{-\beta}}\right)^{-1+\alpha_2} x_2^{-1-\beta}\alpha_2\,(\alpha_1+\alpha_3)\beta^2dx_1 \int_0^{x_2} \mathrm{e}^{-x_1^{-\beta}} \left(1-\mathrm{e}^{-x_1^{-\beta}}\right)^{-1+\alpha_1+\alpha_3} x_1^{-1-\beta}\mathrm{d}x_2 \\
&= \int_0^\infty \mathrm{e}^{-x_2^{-\beta}} \left(1-\mathrm{e}^{-x_2^{-\beta}}\right)^{-1+\alpha_1+\alpha_2+\alpha_3} x_2^{-1-\beta}\alpha_2\,\beta\,\mathrm{d}x_2.
\end{aligned} \tag{14}$$

Similarly

$$I_2 = \int_0^\infty \mathrm{e}^{-x_1{}^{-\beta}} \left(1-\mathrm{e}^{-x_1{}^{-\beta}}\right)^{-1+\alpha_1+\alpha_2+\alpha_3} x_1{}^{-1-\beta}\alpha_1\beta dx_1. \tag{15}$$

From (14) and (15) we get

$$f_3(x,x) = \mathrm{e}^{-x^{-\beta}} \left(1-\mathrm{e}^{-x^{-\beta}}\right)^{-1+\alpha_1+\alpha_2+\alpha_3} x^{-1-\beta}\alpha_3\beta.$$

This complete the proof. □

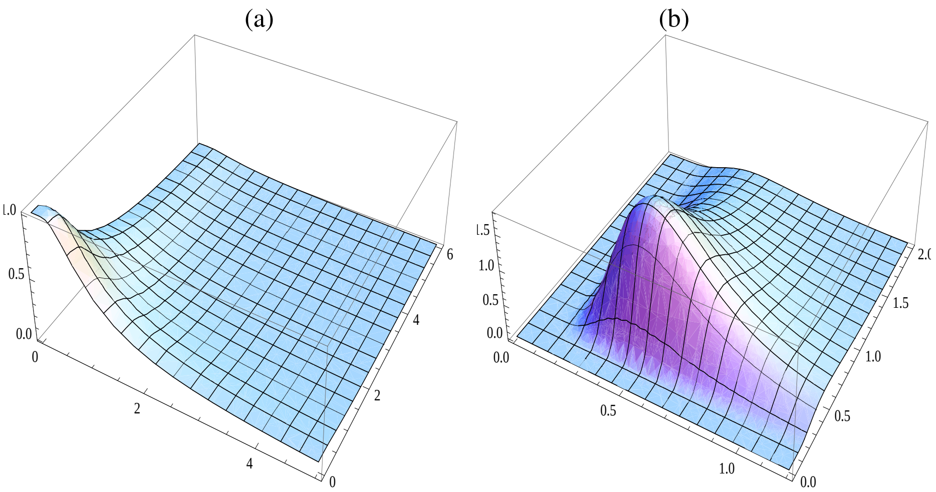

Figure 1: Plots of the BSF (a) and the BPDF (b). (a) $\beta = 1.2; \alpha_1 = 0.8; \alpha_2 = 1; \alpha_3 = 0.5$, (b) $\beta = 1.2; \alpha_1 = 2.3; \alpha_2 = 4; \alpha_3 = 0.05$.

Corollary 2. *The joint pdf of X_1 and X_2 as provided in equation* (10) *can also be expressed in the following form for $z = max(x_1,x_2)$ and for $f_1(.,.)$, $f_2(.,.)$ same as defined in* (11) *and* (12) *for*

$$f(x_1,x_2) = \frac{\alpha_1+\alpha_2}{\alpha_1+\alpha_2+\alpha_3} f_\theta(x_1,x_2) + \frac{\alpha_1+\alpha_2}{\alpha_1+\alpha_2+\alpha_3} f_\phi(x_1,x_2)$$

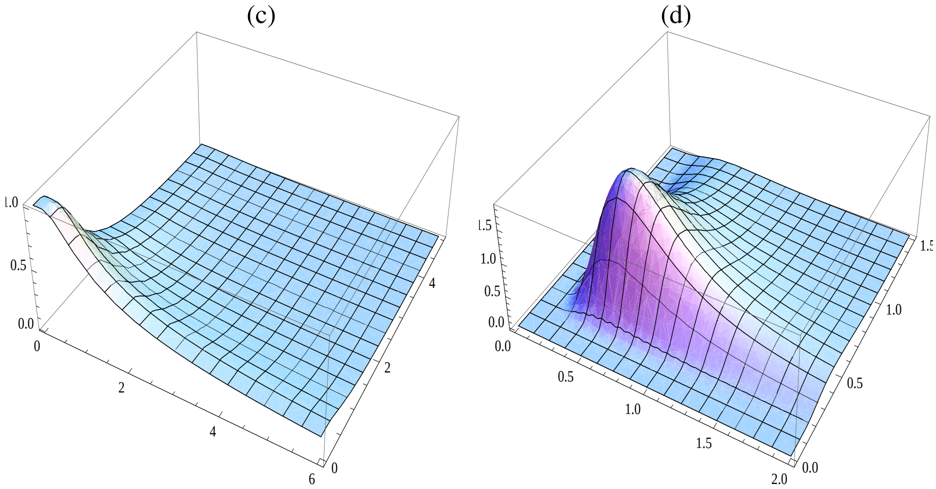

Figure 2: Plots of the BPDF (c) and (d). (c) $\beta = 1.2; \alpha_1 = 3.1; \alpha_2 = 3.5; \alpha_3 = 3$, (d) $\beta = 1.2; \alpha_1 = 2.3; \alpha_2 = 3; \alpha_3 = 1.2$.

$$f_\theta(x_1, x_2) = \frac{\alpha_1 + \alpha_2 + \alpha_3}{\alpha_1 + \alpha_2} \begin{cases} f_{EF}(x_1; (\alpha_1 + \alpha_3), \beta) f_{EF}(x_2; \alpha_2, \beta) & \text{if} \quad x_1 < x_2; \\ f_{EF}(x_1; \alpha_1, \beta) f_{EF}(x_2; (\alpha_2 + \alpha_3), \beta) & \text{if} \quad x_2 < x_1; \end{cases}$$

and

$$f_\phi(x) = f_{EF}(x, \beta, \alpha_1 + \alpha_2 + \alpha_3).$$

Clearly, here $f_\theta(x_1, x_2)$ and $f_\phi(x)$ are the absolute continuous part and singular part, respectively.

3. Marginal and Conditional Probability Density Functions:

In this section we derive the marginal density function of X_i and the conditional density functions of $X_i|X_j$, $(i \neq j = 1, 2)$.

3.1. *Marginal Probability Density Function*

The marginal pdf of BvEF distribution can be easily obtained by using the following theorem.

Theorem 3. *The marginal pdf of X_i $(i = 1, 2)$ is given by*

$$f_{X_i}(x_i) = \mathrm{e}^{-x_i^{-\beta}} \left(1 - \mathrm{e}^{-x_i^{-\beta}}\right)^{-1+\alpha_3+\alpha_i} x_i^{-1-\beta} (\alpha_3 + \alpha_i)\beta \tag{16}$$

Proof. The marginal pdf of X_i can be derived form the marginal survival function of X_i, say $S_{X_i}(x_i)$, as follows

$$S_{X_i}(x_i) = P(X_i > x_i)$$

since U_i is independent of U_3, we simply have

$$S_{X_i}(x_i) = \left(1 - \mathrm{e}^{-x_i^{-\beta}}\right)^{\alpha_i + \alpha_3}$$

from which we readily derive the pdf of X_i, $f(x_i) = -\frac{\partial S(x_i)}{\partial x_i}$, given in (16). □

Corollary 3. *Let $(X_1, X_2) \sim BvEF(\alpha_1 + \alpha_2 + \alpha_3, \beta,)$,then*
(1) $X_1 \sim EF(x_1; \alpha_1 + \alpha_3, \beta,)$
(2) $X_2 \sim EF(x_2; \alpha_2 + \alpha_3, \beta,)$
(3) $X_3 = max(X_1, X_2) \sim EF(x_3; \alpha_1 + \alpha_2 + \alpha_3, \beta,)$

3.2. *Conditional Probability Density Function*

Using the marginal probability density functions of X_1 and X_2 we can now derive the conditional probability density functions as presented in the following theorem.

Theorem 4. *The conditional pdf of X_i, given $X_j = x_j$ denoted by*

$$f_{X_i|X_j}(x_i|x_j) = \begin{cases} f^{(1)}_{X_i|X_j}(x_i|x_j) & if \quad x_i < x_j \\ f^{(2)}_{X_i|X_j}(x_i|x_j) & if \quad x_j < x_i \\ f^{(3)}_{X_i|X_j}(x_i|x_j) & if \quad x_i = x_j \end{cases}$$

where

$$f^{(1)}_{X_i|X_j}(x_i|x_j) = \frac{\mathrm{e}^{-x_i{}^{-\beta}}\left(1-\mathrm{e}^{-x_i{}^{-\beta}}\right)^{-1+\alpha_1+\alpha_3}\left(1-\mathrm{e}^{-x_j{}^{-\beta}}\right)^{-\alpha_3} x_i{}^{-1-\beta}\alpha_2\left(\alpha_1+\alpha_3\right)\beta}{\alpha_2+\alpha_3}$$

$$f^{(2)}_{X_i|X_j}(x_i|x_j) = \frac{\left(1-\mathrm{e}^{-x_i{}^{-\beta}}\right)^{\alpha_1} x_i{}^{-1-\beta}\alpha_1\beta}{-1+\mathrm{e}^{x_i{}^{-\beta}}}$$

$$f^{(3)}_{X_i|X_j}(x_i|x_j) = \frac{\mathrm{e}^{-x_i{}^{-\beta}+x_j{}^{-\beta}}\left(1-\mathrm{e}^{-x_i{}^{-\beta}}\right)^{-1+\alpha_1+\alpha_2+\alpha_3}\left(1-\mathrm{e}^{-x_j{}^{-\beta}}\right)^{1-\alpha_2-\alpha_3} x_i{}^{-1-\beta} x_j{}^{1+\beta}\alpha_3}{\alpha_2+\alpha_3}$$

Proof. The theorem follows readily upon substituting the joint PDF of (X_1, X_2) in (10), and the marginal pdf of X_i (i = 1, 2) in (16), in the relation

$$f_{X_i|X_j}(x_i|x_j) = \frac{f_{X_i,X_j}(x_i, x_j)}{f_{X_j}(x_j)}.$$

□

4. Mathematical Expectation

Based on the results presented in the last two sections, we can derive the r^{th} moments of X_i (i = 1, 2).

Theorem 5. *The r^{th} moment of X_i (i = 1, 2) is given by:*

$$E\left(x_i{}^r\right) = \sum_{k=0}^{\infty}\binom{-1+\alpha_3+\alpha_i}{k}(-1)^k(\alpha_3+\alpha_i)(1+k)^{-1+\frac{r}{\beta}}\gamma\left(1-\frac{r}{\beta}\right). \tag{17}$$

Proof. We start with the known definition of the r^{th} moment of the random variable X_i $(i = 1, 2)$ with pdf $f_{x_i}(x_i)$ given by

$$E\left(x_i{}^r\right) = \int_0^{\infty} x_i{}^r f_{x_i}(x_i)dx_i \tag{18}$$

substituting for $f_{X_i}(x_i)$ from (16), we get

$$E\left(x_i{}^r\right) = \int_0^{\infty} x_i{}^r\,\mathrm{e}^{-x_i{}^{-\beta}}\left(1-\mathrm{e}^{-x_i{}^{-\beta}}\right)^{-1+\alpha_3+\alpha_i} x_i{}^{-1-\beta}\left(\alpha_3+\alpha_i\right)\beta dx_i$$

Let

$z = x_i^{-\beta}$, hence

$$E\left(x_i^{r}\right) = \int_0^\infty z^{-\frac{r}{\beta}} \mathrm{e}^{-z} \left(1 - \mathrm{e}^{-z}\right)^{-1+\alpha_3+\alpha_i} \left(\alpha_3 + \alpha_i\right) dz$$
$$= \left(\alpha_3 + \alpha_i\right) \int_0^\infty z^{-\frac{r}{\beta}} \mathrm{e}^{-z} \left(1 - \mathrm{e}^{-z}\right)^{-1+\alpha_3+\alpha_i} \, dz$$
$$= \sum_{k=0}^{\infty} \binom{-1 + \alpha_3 + \alpha_i}{k} (-1)^k \left(\alpha_3 + \alpha_i\right) \int_0^\infty z^{-\frac{r}{\beta}} \mathrm{e}^{-(k+1)z} \, dz$$

from this we derive the expression of $E\left(x_i^{r}\right)$ given in (17). □

5. Moment Generating Function

5.1. *Marginal Moment Generating Function*

In this subsection, we present the joint moment generating function of (X_1, X_2) and the marginal moment generating function of X_i (i=1,2)

Lemma 1. *The marginal moment generating function of X_i (i=1,2).*

$$M_{x_i}\left(t_i\right) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{-1 + \alpha_3 + \alpha_i}{k} \frac{(-1)^{1+k+m}(k+1)^m}{(k)!} (\alpha_3 + \alpha_i) t_i^{(1+m)\beta} \beta \Gamma[-(1+m)\beta]. \tag{19}$$

Proof. The marginal moment generating function of X_i (i=1,2) as follows

$$M_{x_i}\left(t_i\right) = E\left(\mathrm{e}^{-t_i x_i}\right) = \int_0^\infty \mathrm{e}^{-t_i x_i} f_{x_i}\left(x_i\right) dx_i \tag{20}$$

and substituting for $f_{X_i}\left(x_i\right)$ from (16) in (20) we have

$$M_{x_i}\left(t_i\right) = \int_0^\infty \mathrm{e}^{-t_i x} \mathrm{e}^{-x^{-\beta}} \left(1 - \mathrm{e}^{-x^{-\beta}}\right)^{-1+\alpha_3+\alpha_i} x^{-1-\beta} \left(\alpha_3 + \alpha_i\right) \beta dx$$
$$= \left(\alpha_3 + \alpha_i\right) \beta \int_0^\infty \mathrm{e}^{-t_i x} \mathrm{e}^{-x^{-\beta}} \sum_{k=0}^{\infty} \binom{-1 + \alpha_3 + \alpha_i}{k} (-1)^k \left(\mathrm{e}^{-x^{-\beta}}\right)^k x^{-1-\beta} dx$$
$$= \sum_{k=0}^{\infty} \binom{-1 + \alpha_3 + \alpha_i}{k} (-1)^k \left(\alpha_3 + \alpha_i\right) \beta \int_0^\infty \mathrm{e}^{-t_i x} \left(\mathrm{e}^{-x^{-\beta}}\right)^{k+1} x^{-1-\beta} dx.$$

Let $z = x_i^{-\beta}$, hence

$$M_{x_i}\left(t_i\right) = \sum_{k=0}^{\infty} \binom{-1 + \alpha_3 + \alpha_i}{k} (-1)^k \left(\alpha_3 + \alpha_i\right) \int_0^\infty \mathrm{e}^{-t_i z^{\frac{-1}{\beta}}} \left(\mathrm{e}^{-z}\right)^{k+1} \mathrm{d}z$$
$$= \sum_{k=0}^{\infty} \binom{-1 + \alpha_3 + \alpha_i}{k} (-1)^k \left(\alpha_3 + \alpha_i\right) \int_0^\infty \mathrm{e}^{-t_i z^{\frac{-1}{\beta}}} \sum_{k=0}^{\infty} \frac{(-1)^m (z)^m (k+1)^m}{(k)!} \mathrm{d}z$$
$$= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{-1 + \alpha_3 + \alpha_i}{k} \frac{(-1)^{k+m}(k+1)^m}{(k)!} \left(\alpha_3 + \alpha_i\right) \int_0^\infty \mathrm{e}^{-t_i z^{\frac{-1}{\beta}}} (z)^m \mathrm{d}z.$$

from which we readily derive the expression of $M_{x_i}\left(t_i\right)$ given in (19). □

5.2. ***Joint Moment Generating Function***

The joint MGF of (X_1, X_2) can be derived as follows:

Theorem 6. *If $(X_1, X_2) \sim BvEF(\beta, \alpha_1, \alpha_2, \alpha_3)$, then the bivariate MGF of (X_1, X_2) is given by*

$$
\begin{aligned}
M(t_1, t_2) = &\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\beta\frac{(-1)^{1+p+q+r+s}(p+1)^q(r+1)^s}{(q+1)!m!}\gamma(-(2+q+s)\beta)\Bigg[\begin{pmatrix} -1+\alpha_3+\alpha_1 \\ h \end{pmatrix}\begin{pmatrix} -1+\alpha_2 \\ r \end{pmatrix} \\
&\times {}_1F_1\left(-\beta(q+1); -\beta(q+1)+1; -x_2t_1\right)\alpha_2\left(\alpha_1+\alpha_3\right)t_2{}^{(2+q+s)\beta} + \begin{pmatrix} -1+\alpha_3+\alpha_2 \\ p \end{pmatrix}\begin{pmatrix} -1+\alpha_1 \\ r \end{pmatrix} \\
&\times {}_1F_1\left(-\beta(q+1); -\beta(q+1)+1; -x_1t_2\right)\alpha_1\left(\alpha_2+\alpha_3\right)t_1{}^{(2+q+s)\beta}\Bigg] + \sum_{p=0}^{\infty}\sum_{j=0}^{\infty}\begin{pmatrix} -1+\alpha_1+\alpha_2+\alpha_3 \\ p \end{pmatrix} \\
&\times \frac{(-1)^{p+q}(p+1)^q}{q!}\ \alpha_3\beta\ \gamma(1+j\beta)\times(t_1+t_2)^{-1-j\beta}. \qquad (21)
\end{aligned}
$$

Proof. We know that

$$
\begin{aligned}
M(t_1, t_2) &= E\left(e^{-t_1x_1-t_2x_2}\right) \\
&= \int_0^{\infty}\int_0^{\infty} e^{-(t_1x_1+t_2x_2)} f_{X_1,X_2}(x_1, x_2)\, dx_1 dx_2 \\
&= \int_0^{\infty}\int_0^{x_1} e^{-(t_1x_1+t_2x_2)} f_1(x_1, x_2)\, dx_1 dx_2 + \int_0^{\infty}\int_0^{x_2} e^{-(t_1x_1+t_2x_2)} f_2(x_1, x_2)\, dx_2 dx_1 \\
&\quad + \int_0^{\infty} e^{-(t_1+t_2)x_3} f_3(x_1, x_2)\, dx_3. \qquad (22)
\end{aligned}
$$

Let

$$M(t_1, t_2) = I_1 + I_2 + I_3, \qquad (23)$$

where

$$
\begin{aligned}
I_1 &= \int_0^{\infty}\int_0^{x_2} e^{-(t_1x_1+t_2x_2)} f_1(x_1, x_2)\, dx_1 dx_2 \\
&= \int_0^{\infty}\int_0^{x_2} e^{-(t_1x_1+t_2x_2)} e^{-x_1{}^{-\beta}-x_2{}^{-\beta}}\left(1-e^{-x^{-\beta}}\right)^{-1+\alpha_1+\alpha_3}\left(1-e^{-x_2{}^{-\beta}}\right)^{-1+\alpha_2} x_1{}^{-1-\beta}x_2{}^{-1-\beta}\alpha_2\left(\alpha_1+\alpha_2\right)\beta^2 dx_1 dx_2 \\
&= \sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-1)^{m+p+q}(k+1)^q(t_1)^m}{m!q!}\begin{pmatrix} -1+\alpha_3+\alpha_1 \\ p \end{pmatrix}\int_0^{\infty} e^{-t_2x_2}e^{-x_2{}^{-\beta}}\left(1-e^{-x_2{}^{-\beta}}\right)^{-1+\alpha_2} x_2{}^{-1-\beta} \\
&\quad \times (-1)^p\alpha_2\left(\alpha_1+\alpha_2\right)\beta^2\int_0^{x_2}(x_1)^m(x_1)^{-q\beta}(x_1)^{-1-\beta} dx_1 dx_2 \\
&= x_2{}^{-\beta}\sum_{p=0}^{\infty}\begin{pmatrix} -1+\alpha_3+\alpha_1 \\ p \end{pmatrix}\alpha_2\left(\alpha_1+\alpha_2\right)\beta^2\int_0^{\infty} e^{-t_2x_2}e^{-x_2{}^{-\beta}}\sum_{r=0}^{\infty}\begin{pmatrix} -1+\alpha_2 \\ r \end{pmatrix}(-1)^r\left(e^{-x_2{}^{-\beta}}\right)^r\ x_2{}^{-1-\beta}dx_2 \\
&\quad \times (-1)^p\sum_{q=0}^{\infty}\frac{(-1)^q(h+1)^q x_2{}^{-j\beta}}{q!}\left\{\frac{1}{-\beta(q+1)} + \frac{(-z_2t_1)}{(-\beta(q+1)+1)1!} + \frac{(-z_2t_1)^2}{(-\beta(q+1)+2)2!} + ...\right\} \\
&= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\begin{pmatrix} -1+\alpha_3+\alpha_1 \\ p \end{pmatrix}\begin{pmatrix} -1+\alpha_2 \\ r \end{pmatrix}\frac{(-1)^{1+p+q+r}(p+1)^q}{(q+1)!}\int_0^{\infty} e^{-t_2z_2}\left(e^{-x_2{}^{-\beta}}\right)^{r+1} x_2{}^{-1-2\beta-j\beta}dx_2 \\
&\quad \times {}_1F_1\left(-\beta(q+1); -\beta(q+1)+1; -x_2t_1\right)\alpha_2\left(\alpha_1+\alpha_3\right)\beta \\
&= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\begin{pmatrix} -1+\alpha_3+\alpha_1 \\ p \end{pmatrix}\begin{pmatrix} -1+\alpha_2 \\ r \end{pmatrix}(r+1)^s\frac{(-1)^{1+p+q+r+s}(p+1)^q}{(q+1)!s!} \\
&\quad \times {}_1F_1\left(-\beta(q+1); -\beta(q+1)+1; -x_2t_1\right)\alpha_2\left(\alpha_1+\alpha_3\right)\beta\int_0^{\infty} e^{-t_2x_2}x_2{}^{-1-2\beta-q\beta-s\beta}dx_2.
\end{aligned}
$$

$$I_1 = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{-1+\alpha_3+\alpha_1}{p}\binom{-1+\alpha_2}{r}{}_1F_1\left(-\beta(q+1);-\beta(q+1)+1;-x_2t_1\right)\alpha_2\left(\alpha_1+\alpha_3\right)\beta$$
$$\times (r+1)^s\frac{(-1)^{1+p+q+r+s}(p+1)^q}{(q+1)!s!}\gamma(-(2+q+s)\beta)\ t_2{}^{(2+q+s)\beta}. \tag{24}$$

Similarly

$$I_2 = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{-1+\alpha_3+\alpha_2}{p}\binom{-1+\alpha_1}{r}{}_1F_1\left(-\beta(q+1);-\beta(q+1)+1;-x_1t_2\right)\alpha_1\left(\alpha_2+\alpha_3\right)\beta$$
$$\times (r+1)^s\frac{(-1)^{1+p+q+r+s}(p+1)^q}{(q+1)!s!}\gamma(-(2+q+s)\beta)\ t_1{}^{(2+q+s)\beta}, \tag{25}$$

and

$$I_3 = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\binom{-1+\alpha_1+\alpha_2+\alpha_3}{p}\frac{(-1)^{p+q}(p+1)^q}{q!}\alpha_3\beta\gamma(1+q\beta)(t_1+t_2)^{-1-q\beta}. \tag{26}$$

Using (24), (25) and (26) in (23), we get the required result given in (21). □

6. Monotonically Failure Rate Function

Let (X_1, X_2) is a bivariate r.v with the joint $f(x_1, x_2)$, then the bivariate hazard rate function (BHzF) $h(x_1, x_2)$ is defined as

$$h(x_1, x_2) = \frac{f_i(x_1, x_2)}{S_i(x_1, x_2)}, \quad i = 1, 2. \tag{27}$$

Using (2) and (10) in (27), we obtain the bivariate hazard rate function

$$h(x_1, x_2) = \begin{cases} h_1(x_1, x_2), & \text{if} \quad x_1 < x_2 \\ h_2(x_1, x_2), & \text{if} \quad x_2 < x_1 \end{cases}, \tag{28}$$

where

$$h_1(x_1, x_2) = \frac{x_1^{-1-\beta}x_2^{-1-\beta}\alpha_2(\alpha_1+\alpha_3)\beta^2}{\left(-1+e^{x_1^{-\beta}}\right)\left(-1+e^{x_2^{-\beta}}\right)}$$
$$h_2(x_1, x_2) = \frac{x_1^{-1-\beta}x_2^{-1-\beta}\alpha_1(\alpha_2+\alpha_3)\beta^2}{\left(-1+e^{x_1^{-\beta}}\right)\left(-1+e^{x_2^{-\beta}}\right)}.$$

For the both cases $x_1 < x_2$ and $x_2 < x_1$, plots of the BHzF are given in Figures 3 and 4.

7. Parameter Estimation

To estimate the unknown parameters of the BvEF distribution, we use three methods of estimation: maximum likelihood, Bayesian inference and least squares estimation.

Consider the following notations and sets for the rest of the paper; $I_1 = \{(x_{1i}, x_{2i}), x_{1i} > x_{2i}, i = 1, \ldots, n\}$, $I_2 = \{(x_{1i}, x_{2i}), x_{1i} < x_{2i}, i = 1, \ldots, n\}$, $I_3 = \{(x_{1i}, x_{2i}), x_{1i} = x_{2i}, i = 1, \ldots, n\}$, $I = I_1 \cup I_2 \cup I_3$, $n_1 = |I_1|$, $n_2 = |I_2|$ and $n_3 = |I_3|$ with $n_1 + n_2 + n_3 = n$.

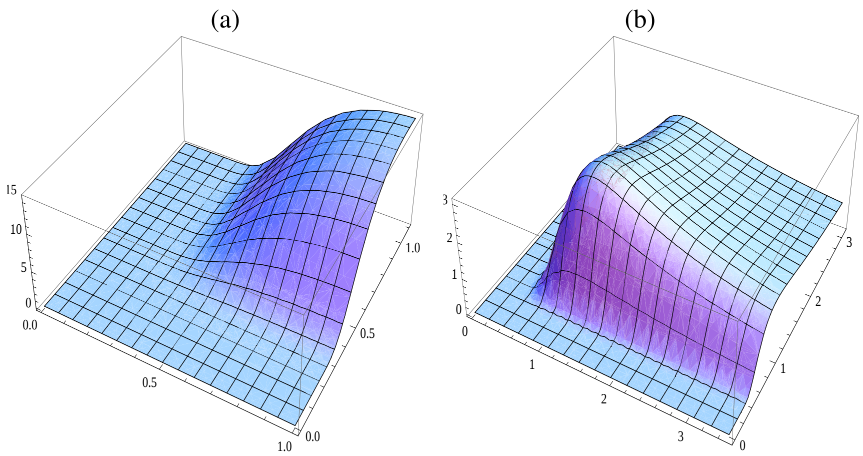

Figure 3: Plots of the BHzF (a)and (b).(a)$\beta = 1.8; \alpha_1 = 2.8; \alpha_2 = 2.52; \alpha_3 = 2.5$, (b) $\beta = 2.3; \alpha_1 = 2.7; \alpha_2 = 0.52; \alpha_3 = 0.5$.

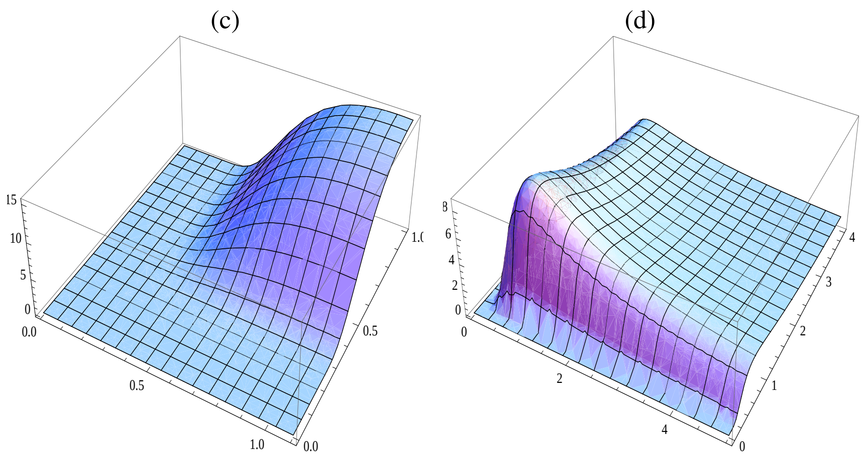

Figure 4: Plots of the BHzF (c) and (d). (c)$\beta = 1.23; \alpha_1 = 0.8; \alpha_2 = 1; \alpha_3 = 0.5$, (d) $\beta = 1.24; \alpha_1 = 2.3; \alpha_2 = 4; \alpha_3 = 0.05$

.

*7.1. **Maximum likelihood Estimation***

The likelihood function for the the parameter vector $\boldsymbol{\theta} = (\beta, \alpha_1, \alpha_2, \alpha_3)$. Let (x_1, x_2) be a paired sample of size n from the BvEF distribution obtained as

$$L(\theta | \boldsymbol{x_1}, \boldsymbol{x_2}) = \prod_{i \in I_1} f_1(x_{1i},\ x_{2i}) \ \prod_{i \in I_2} f_2(x_{1i},\ x_{2i}) \ \prod_{i \in I_3} f_3(x_i)\,. \tag{29}$$

By using the equations (11), (12) and (13) in (29), we find that the log-likelihood function is given by

$$
\begin{aligned}
\ell(\boldsymbol{\theta}) &= \sum_{i=1}^{n_1} \ln f_1\left(x_{1i}, x_{2i}\right) + \sum_{i=1}^{n_2} \ln f_2\left(x_{1i}, x_{2i}\right) + \sum_{i=1}^{n_3} \ln f_3\left(x_i, x_i\right) \\
&= n_1 \ln\left(\alpha_2\right) + n_1 \ln\left(\alpha_1+\alpha_3\right) + 2n_1 \ln(\beta) + n_2 \ln\left(\alpha_1\right) + n_2 \ln\left(\alpha_2+\alpha_3\right) + 2n_2 \ln(\beta) + n_3 \ln\left(\alpha_3\right) + n_3 \ln(\beta) \\
&\quad - \sum_{i=1}^{n_1}\left(x_{1i}\right)^{-\beta} - \sum_{i=1}^{n_1}\left(x_{2i}\right)^{-\beta} - \left(1-\alpha_1-\alpha_3\right)\sum_{i=1}^{n_1}\left(1-\mathrm{e}^{-x_{1i}^{-\beta}}\right) - \left(1-\alpha_2\right)\sum_{i=1}^{n_1}\left(1-\mathrm{e}^{-x_{2i}^{-\beta}}\right) - (1+\beta)\sum_{i=1}^{n_1}\ln\left(x_{1i}\right) \\
&\quad - (1+\beta)\sum_{i=1}^{n_1}\ln\left(x_{2i}\right) - \sum_{i=1}^{n_2}\left(x_{1i}\right)^{-\beta} - \sum_{i=1}^{n_2}\left(x_{2i}\right)^{-\beta} - \left(1-\alpha_1\right)\sum_{i=1}^{n_2}\left(1-\mathrm{e}^{-x_{1i}^{-\beta}}\right) - \left(1-\alpha_2-\alpha_3\right)\sum_{i=1}^{n_2}\left(1-\mathrm{e}^{-x_{2i}^{-\beta}}\right) \\
&\quad - (1+\beta)\sum_{i=1}^{n_2}\ln\left(x_{1i}\right) - (1+\beta)\sum_{i=1}^{n_2}\ln\left(x_{2i}\right) - \sum_{i=1}^{n_3}\left(x_{3i}\right)^{-\beta} \\
&\quad - \left(1-\alpha_1-\alpha_2-\alpha_3\right)\sum_{i=1}^{n_3}\left(1-e^{-x_{3i}^{-\beta}}\right) - (1+\beta)\sum_{i=1}^{n_3}\ln\left(x_{3i}\right).
\end{aligned}
$$

The maximum likelihood estimators (MLEs) can be obtained by solving the system of nonlinear equations $\frac{\partial \ell}{\partial \beta} = 0, \frac{\partial \ell}{\partial \alpha_i} = 0, i = 1, 2, 3$ and their values with the second partial derivatives are available upon request. The solution of these nonlinear equations does not have closed form but can be solved numerically. Also, using the second partial derivatives of L with respect to the parameters, we can get 4×4 information matrix that can be used for interval estimation of the parameters and this is verified for real-life data example in Section 9.

7.2. *Bayesian Analysis*

Bayesian inference offers an alternative to Maximum Likelihood and allows us to determine estimators more accurate and easily interpretable.

Assuming independence a priori among the parameters β, α_1, α_2 and α_3, we consider gamma prior distributions for each parameter. Thus, the joint prior distribution for $\theta = (\beta, \alpha_1, \alpha_2, \alpha_3)$ is given by

$$\pi(\boldsymbol{\theta}) = \pi(\beta|\, a_0, b_0)\pi(\alpha_1|\, a_1, b_1)\pi(\alpha_2|\, a_2, b_2)\pi(\alpha_3|\, a_3, b_3) \tag{30}$$

where a_i, b_i, $i = 0, 1, 2, 3$, are known and non-negative values and $\pi(\,\cdot|\, a_i, b_i)$ denotes the density of gamma distribution with mean a_i/b_i and variance a_i/b_i^2.

Usually, we assume the hyperparameters $a_i = b_i = 0.01$, $i = 0, 1, 2, 3$, to provide no prior information.

From (29) and (30), the joint posterior distribution for the vector parameter $\boldsymbol{\theta} = (\beta, \alpha_1, \alpha_2, \alpha_3)$ is given by

$$p(\boldsymbol{\theta}\,|\,\boldsymbol{x}, \boldsymbol{y}) \propto \pi(\boldsymbol{\theta}) L(\boldsymbol{\theta}\,|\,\boldsymbol{x}, \boldsymbol{y}). \tag{31}$$

To get the posterior summaries of interest, we simulate samples from the joint posterior distribution (31) using MCMC methods, see Section 8.

7.3. *Least Squares Estimation*

In this section another method of estimation of the parameters is given which is based on the idea of the joint and marginal empirical survival functions.

Let (X_i, X_j), $i, j = 1, \ldots, n$, be n independent and identically distributed pairs from the BvEF distribution with a common joint survival function $S(x_1, x_2)$ given by (1).

The survival function of the observations can be estimated by the empirical survival function given by

$$S_n(x_i, x_j) = \frac{\sum_{i=1}^{n} I(X_i > x_i, X_j > x_j)}{n}, \tag{32}$$

and $I(\cdot)$ is the indicator function.

Consider the marginal survival functions $S_X(x_i)$ and $S_X(x_j)$. Let

$$\widehat{S}_X(x_i) = \frac{\sum_{i=1}^{n} I(X > x_i)}{n} \text{ and } \widehat{S}_X(x_j) = \frac{\sum_{j=1}^{n} I(X > x_j)}{n} \tag{33}$$

be the empirical marginal survival functions of the variables X_1 and X_2, respectively, evaluated at the observed value of (x_i, x_j).

The proposed Least Squares method involves estimating the parameters β, α_1, α_2 and α_3 by minimizing the function $SQ(\beta, \alpha_1, \alpha_2, \alpha_3)$ defined as

$$SQ(\beta, \alpha_1, \alpha_2, \alpha_3) = \sum_{i=1}^{n}\Big(S(x_i, x_j) - S_n(x_i, x_j)\Big)^2 + \sum_{i=1}^{n}\Big(S_X(x_i) - \widehat{S}_X(x_i)\Big)^2 + \sum_{i=1}^{n}\Big(S_X(x_j) - \widehat{S}_X(x_j)\Big)^2. \tag{34}$$

This proposed methodology is easy to implement and is free from calculus. It optimizes the objective function by searching over a wide range of values and determines the estimator of the parameters.

We propose the confidence interval based on the bootstrapping. The percentile bootstrap method, proposed by Efron [5], is widely used in the literature. To estimate the bootstrap interval, we proceed as follows:

i) Compute $\widehat{\beta}$, $\widehat{\alpha}_1$, $\widehat{\alpha}_2$, $\widehat{\alpha}_3$ from the sample generated (x_i, y_i), $i = 1, \ldots n$, using the MLE.

ii) Generate bootstrap sample (x_i^*, x_j^*) from BvEF, $i = 1, \ldots, n$, using $\widehat{\beta}$, $\widehat{\alpha}_1$, $\widehat{\alpha}_2$, $\widehat{\alpha}_3$. Obtain the bootstrap estimate of β, α_1, α_2, α_3, say $\widehat{\beta}^*$, $\widehat{\alpha}_1^*$, $\widehat{\alpha}_2^*$, $\widehat{\alpha}_3^*$, respectively, using the bootstrap sample.

iii) Repeat Step (ii) a large number B of times.

iv) From $\widehat{\theta}^*_{(1)} \leq \widehat{\theta}^*_{(2)} \leq \ldots \leq \widehat{\theta}^*_{(B)}$, for some value of γ, $(0 < \gamma < 1)$, the bootstrap confidence interval $100(1-\gamma)\%$ for the parameter θ is given by $(\widehat{\boldsymbol{\theta}}^*_{(q_1)}, \widehat{\boldsymbol{\theta}}^*_{(q_2)})$ where $q_1 = [B \times \frac{\alpha}{2}]$ and $q_2 = B - q_1$ and [ ] indicates the smallest integer greater than or equal to the argument. This is done for any parameter $\boldsymbol{\theta} = \beta$, α_1, α_2 and α_3.

8. Simulation studies

In this section, we conduct Monte Carlo simulation studies to investigate the performance of the proposed estimators discussed in this paper.

For each sample size (n) and the specified values of the parameters 10010 data sets are generated from the BvEF distribution with parameters values $(\beta, \alpha_1, \alpha_2, \alpha_3)$=(1.0, 2.0, 3.0, 1.5).

From each data set, the estimates with standard deviation and 95% coverage probabilities of the parameters β, α_1, α_2 and α_3 are obtained by the proposed estimation approaches.

The sample sizes (n) considered are 20, 50, 100 and 200, in order to cover the small, medium, and large sample sizes encountered in real-life data.

The bias, mean square error (MSE) and coverage probability (CP) criterion are used to compare the maximum likelihood, Bayesian and LSE estimates for different sample sizes.

The generation of random samples from a BvEF with parametric vector $\boldsymbol{\theta}$ =$(\beta, \alpha_1, \alpha_2, \alpha_3)$ can be easily obtained using the following algorithm.

i) Generate U_i using the exponentiated Fréchet $EF(\beta, \alpha_i)$ distribution with known values for the parameters α_i and β, $i = 1, 2, 3$

ii) Take $X = min(U_1, U_3)$ and $Y = min(U_2, U_3)$, therefore (X, Y) follows a bivariate exponentiated Fréchet distribution of Marshall-Olkin type.

The following procedure is adopted to evaluate the performance of the estimates:

Step 1: Generate pseudo random sample from BvEF$(\beta, \alpha_1, \alpha_2, \alpha_3)$ with size n, as given in the algorithm above.

Step 2. For each sample generated in step 1, calculate the estimators $\widehat{\boldsymbol{\theta}} = (\widehat{\beta}, \widehat{\alpha}_1, \widehat{\alpha}_2, \widehat{\alpha}_3)$, standard deviation (sd) and respectives 95% confidence intervals through MLE and Bayesian approaches as shown earlier.

Step 3. Repeat the steps 1 and 2, $N = 1000$ times.

Step 4. Using the $N = 1000$ values of $\widehat{\theta}$ and the true $\boldsymbol{\theta}$ =$(\beta, \alpha_1, \alpha_2, \alpha_3)$, compute the mean estimate $\widehat{\theta_i} = \frac{1}{N}\sum_{j=1}^{N}\widehat{\theta}_{i,j}$, mean $sd = \frac{1}{N}\sum_{j=1}^{N} sd(\widehat{\theta}_{i,j})$, $bias(\widehat{\theta_i}) = \frac{1}{N}\sum_{j=1}^{N}(\widehat{\theta}_{i,j} - \theta_i)$ and the mean square error $MSE(\widehat{\theta_i}) = \frac{1}{N}\sum_{j=1}^{N}(\widehat{\theta}_{i,j} - \theta_i)$, for $i = 1, 2, 3$ and 4 where $\theta_1 = \beta$, $\theta_2 = \alpha_1$, $\theta_3 = \alpha_2$ and $\theta_4 = \alpha_3$.

Also, the empirical coverage probability for each parameter θ_i is obtained from the 95% confidence and credible intervals for θ_i calculated in Step 2 and evaluated now as $CP(\theta_i) = \frac{number\, of\, 95\%\, intervals\, containing\, \theta_i}{N}$, for $i = 1,\ 2,\ 3$ and 4.

It is expected that for both approaches the bias and the MSE are closer to zero and CP is closer to nominal value 95%.

We compute the Bayes estimates using the squared error loss function in all cases.

We also need to appeal to numerical procedures to obtain the summaries of marginal posterior distributions such as Bayes estimates, standard deviations and credible intervals. We then use the MCMC algorithm to obtain a sample of values of β, α_1,α_2 and α_3 from the joint posterior. The chain is run for 105000 iterations with a burn-in period of 5000.

The coding and the analysis were performed using the R programming language.

Figures 5-7 present the plots of bias, MSE and average inteval length from the MLE, Bayesian and LSE estimates of parameters to show how these measures vary with respect to n.

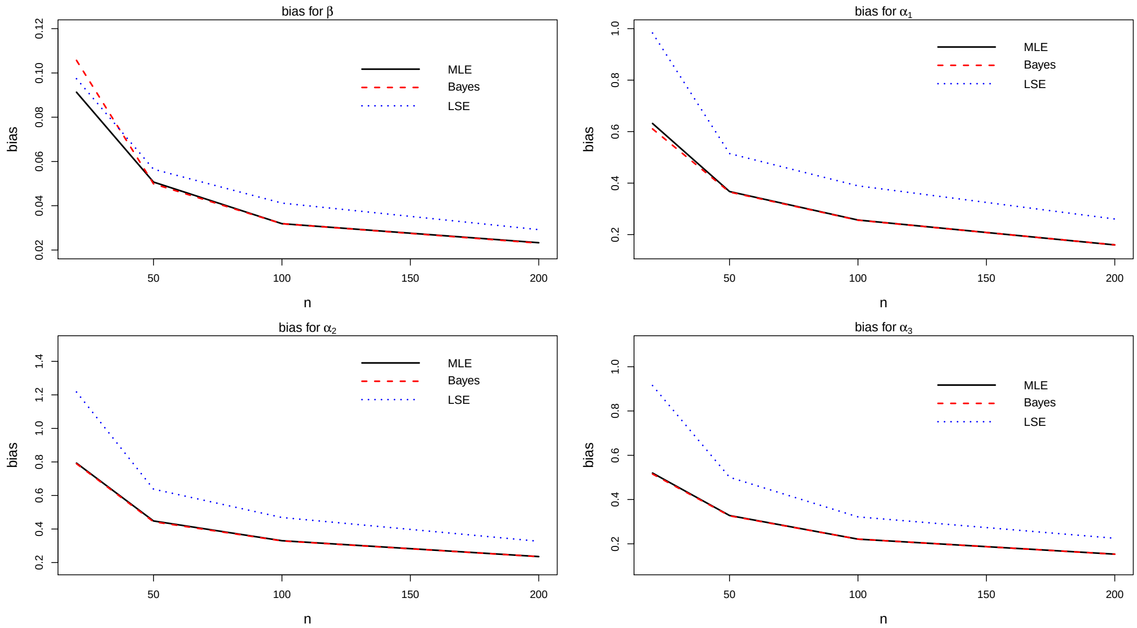

Figure 5: Plot of biases versus n under MLE, Bayesian and LSE approaches.

Some of the points are quite clear from the figures above, and they are:

• The biases and MSE of the MLE, Bayesian and LSE approaches for each parameter β, α_1, α_2 and α_3 decay towards zero as the sample size increases, as expected;

• The empirical coverage probabilities CP are close to nominal coverage level when the sample size increases;

• The biases appear smallest for β in comparison with α_i, i=1,2,3;

• The mean squared errors appear smallest for β while the largest is for α_2;

• The convergence of the biases and MSEs to zero appears very slowly for $n > 20$;

• The performance of the Bayes estimates behaves in a very similar way with the corresponding MLEs exception for CPs.

• The LSE method fails to provide estimates closer to the true value for the parameters by considering small sample data.

9. Illustrative literature data example

To illustrate the proposed procedure in this section, we consider the following example.

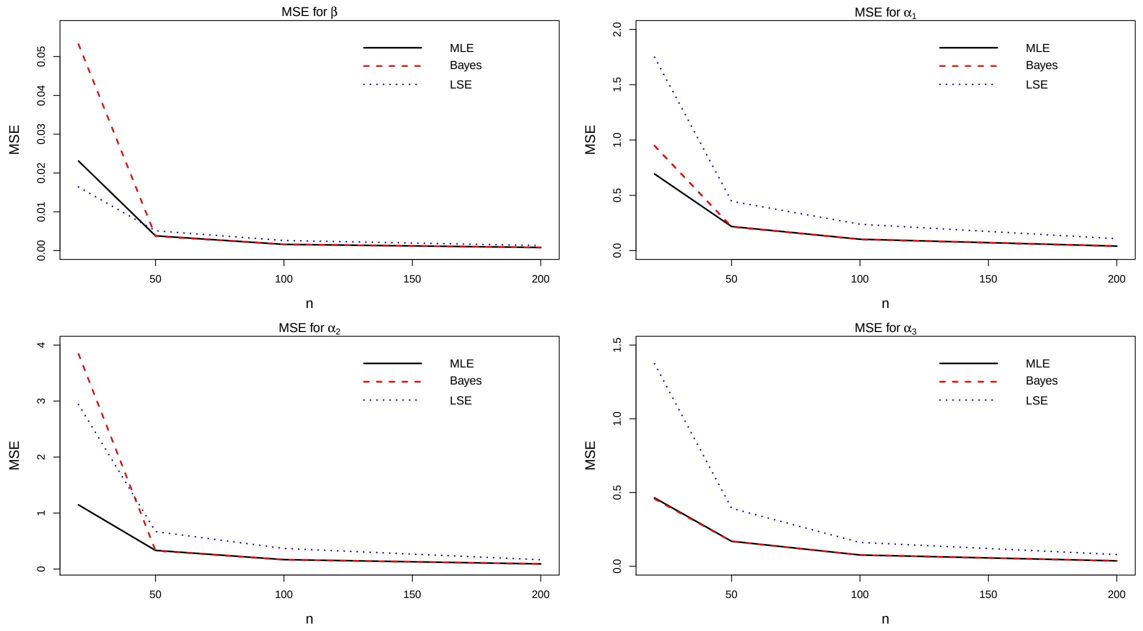

Figure 6: Plot of MSE versus n under MLE, Bayesian and LSE approaches.

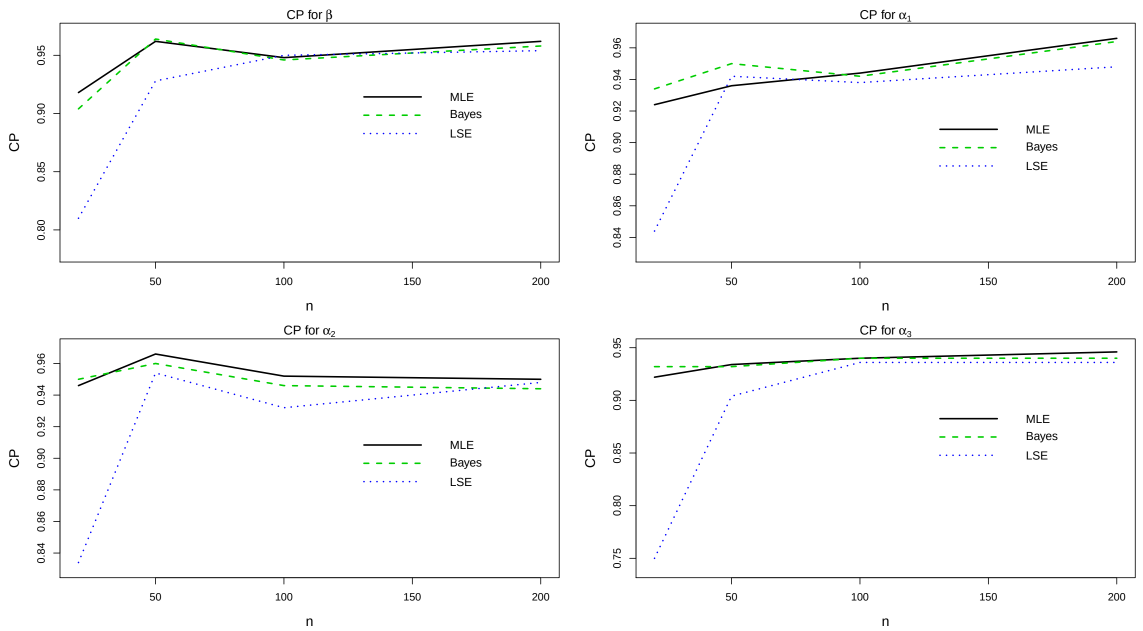

Figure 7: Plot of coverage probabilities versus n under MLE, Bayesian and LSE approaches.

By using MLEs method, we fit the parameters of the bivariate exponentiated Gompertez (BEG) [9], bivariate exponentaited modified Weibull extension (BEMWE) [6]) and bivariate generalized Gompertz (BGG)[10] distributions to the National Football League (NFL) data [17]. This bivariate sample data is based on the next algorithm

• Step 1: generate V_i using the BvEF distribution with parameter α_i for $i = 1, 2, 3$.
• Step 2: take $Z_1 = max\{V_1, V_3\}$, $Z_2 = max\{V_2, V_3\}$ and then bivariate vector $(Z_1, Z_2) \sim BvEF.(\beta, \alpha_1, \alpha_2, \alpha_3)$

Table 1: American Football (National Football League) data

X_1	X_2	X_1	X_2	X_1	X_2
2.05	3.98	5.78	25.98	10.40	10.25
9.05	9.05	13.80	49.75	2.98	2.98
0.85	0.85	7.25	7.25	3.88	6.43
3.34	3.43	4.25	4.25	0.75	0.75
7.78	7.78	1.65	1.65	11.63	17.37
10.57	14.82	6.42	15.08	1.38	1.38
7.05	7.05	4.22	9.48	10.53	10.53
2.58	2.58	15.53	15.53	12.13	12.13
7.23	9.68	2.90	2.90	14.58	14.58
6.85	34.58	7.02	7.02	11.82	11.82
32.45	42.35	6.42	6.42	5.52	11.27
8.53	15.57	8.98	8.98	19.65	10.71
31.13	49.88	10.15	10.15	17.83	17.83
14.58	20.57	8.87	8.87	10.85	38.07

Here, we prove the capability of the BvEF distribution by the help of a real-life data set using MLE approach. By using MLE method, we fit the BEG, A. BEMWE and BGG distributions to the NFL data given in Table 1.

Table 2 lists the MLEs of the parameters and Table 3 gives the values of goodness of fit statistics for the mentioned fitted models above. As it can be observed from Table 3 that BvEF distribution has the smaller values of these statistics as compared to other distributions in the table, the newly defined BvEF distribution is a competitive model for lifetime data analysis.

Table 2: MLEs of the parameters

Distributions	Estimates			
BEG$(\hat\alpha_1, \hat\alpha_2, \hat\alpha_3, \hat\lambda)$	$\hat\alpha_1 = 0.043$	$\hat\alpha_2 = 0.528$	$\hat\alpha_3 = 1.037$	$\hat\lambda = 0.787$
BGG$(\hat\lambda_1, \hat\lambda_2, \hat\lambda_3, \hat\alpha, 0.1)$	$\hat\lambda_1 = 0.024$	$\hat\lambda_2 = 0.150$	$\hat\lambda_3 = 0.310$	$\hat\alpha = 0.0044$
BEMWE$(\hat\beta_1, \hat\beta_2, \hat\beta_3, \hat\lambda, 0.1, 0.42)$	$\hat\beta_1 = 0.212$	$\hat\beta_2 = 1.315$	$\hat\beta_3 = 2.645$	$\hat\lambda = 0.096$
BvEF$(\hat\alpha_1, \hat\alpha_2, \hat\alpha_3, \hat\beta)$	$\hat\alpha_1 = 0.0933$	$\hat\alpha_2 = 0.0313$	$\hat\alpha_3 = 0.0505$	$\hat\beta = 2.583$

Table 3: Goodness–of–fit statistics for the NFL data

Distributions	$-\ell$	AIC	$CAIC$	BIC
BEG$(\hat\alpha_1, \hat\alpha_2, \hat\alpha_3, \hat\lambda)$	370.41	748.82	749.90	755.771
BGG$(\hat\lambda_1, \hat\lambda_2, \hat\lambda_3, \hat\alpha, 0.1)$	260.5	529	530	539.688
BEMWE$(\hat\beta_1, \hat\beta_2, \hat\beta_3, \hat\lambda, 0.1, 0.42)$	239.86	487.36	488.39	502.146
BvEF$(\hat\alpha_1, \hat\alpha_2, \hat\alpha_3, \hat\beta)$	**169.816**	**345.824**	**346.906**	**354.583**

Further, we estimate the parameters $(\beta,\ \alpha_1,\ \alpha_2, \alpha_3)$ by using the MLE, Bayesian and LSE approaches proposed in previous sections for the NFL data described above.

Table 4 presents the MLE, Bayes and LSE estimates for the parameters and their respective standard deviations. We observe that the point estimates of the parameters are similar. Table 5 gives the credible intervals and the confidence interval for the parameters of the BvEF distribution.

From Table 4, it is observed that the estimates and respective standard deviations for the parameters obtained by the MLE and Bayesian approaches are quite similar, which is confirmed previously by simulation.

Table 4: MLE, Bayesian and LSE approaches for the estimation of the parameters of the BvEF distribution

parameter	β	α_1	α_2	α_3
MLE	2.583 (0.9386)	0.0933 (0.0120)	0.0313 (0.0300)	0.0505 (0.0730)
Bayes	2.9734 (0.9102)	0.0153 (0.0119)	0.0696 (0.0280)	0.1883 (0.0650)
LSE	3.5442 (1.2101)	0.0288 (0.0250)	0.0101 (0.0139)	0.0925 (0.0641)

Table 5: 95% confidence and credible intervals for the parameters of the BvEF distribution

parameter	β	α_1	α_2	α_3
MLE	(0.8394, 4.5189)	(-0.0080, 0.0390)	(0.0121, 0.1299)	(0.0481, 0.3345)
Bayes	(1.5752, 5.0682)	(0.0014, 0.0457)	(0.0299, 0.1393)	(0.0926, 0.3421)
LSE	(1.3252, 5.6306)	(0.0082, 0.1048)	(-0.0012, 0.0488)	(0.0581, 0.2822)

Note also that in Table 5 there are intervals with negative lower confidence limits for the parameters α_1 and α_2 using MLE and LSE approaches, respectively, which does not happen with the Bayesian estimation. This occurs because both methods used are not good enough when there is not enough data.

Figure 8 shows the marginal posterior densities for the parameters of the BvEF distribution.

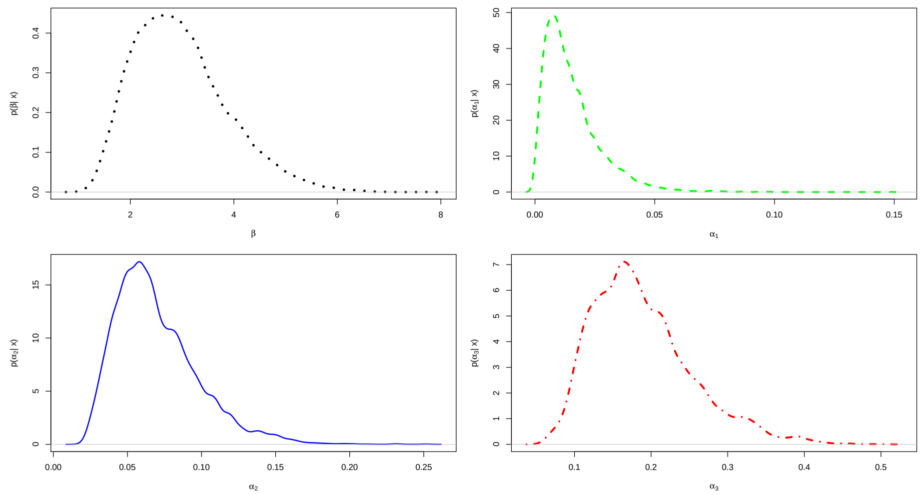

Figure 8: Plots of marginal posterior densities for the parameters β, α_1, α_2 and α_3

10. Conclusion

In this paper we introduce a flexible bivariate distribution based on a convenient tool for studying the dependency structure with applications to data pairs with ties. This distribution is obtained from Marsall-Olkin method and its marginals are also exponentiated Fréchet. It is observed that the proposed bivariate distribution can be obtained using the Marshall-Olkin survival copula with univariate exponentiated Fréchet distribution as marginals.

We derive several probabilistics properties, such as, joint density, marginal distributions, conditional densities, moments, and others.

We also discuss three estimation methods to estimate the parameters of the BvEF distribution. Based on a simulation study, we show that the biases, MSE and coverage probabilities of the parameters obtained from the MLE and Bayesian approaches are quite similars and from the LSE method is just reasonable.

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