Monetary Search and Liquidity

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1 Monetary Search Model

1.1 Continuous Time Search with a Markov Chain

- Payoff $\pi_1(t)$ if in state 1
- Payoff $\pi_2(t)$ if in state 2
- Discount rate $0 < \beta < 1$
- Probability of switching, if in 1, to $2 = p_1$
- Probability of switching, if in 2, to $1 = p_2$

1.2 Bellman Equations

$$V_1(t) = \pi_t(t) + \beta \left[(1 - p_1)V_1(t+1) + p_1 V_2(t+1)\right] \tag{1}$$

$$V_2(t) = \pi_2(t) + \beta \left[p_2 V_1(t+1) + (1 - p_2)V_2(t+1)\right] \tag{2}$$

1.3 Heuristic Continuous Time Limit

For small $\triangle > 0$ (decreasing time scale):

- Payoffs for flow payoffs $\pi_1(t), \pi_2(t)$ are $\triangle\pi_1(t), \triangle\pi_2(t)$,
  i.e. fixed for small t interval
- Need to adjust $\beta(\triangle)$.
  The adjustment can be done many ways, but need $\lim\limits_{\triangle \to 0} \beta(\triangle) = 1$.

- Let $\beta(\triangle) \equiv 1 - r \cdot \triangle$
  where β is the discount factor, and r is the discount rate.

Finally, the switching probabilities must decrease:

- $p_1(\triangle) \equiv \triangle\alpha_1$ and $p_2(\triangle) \equiv \triangle\alpha_2$

Note that $\alpha_1 > 0$ and $\alpha_2 > 0$ are the <u>intensities</u> and not bounded by 1.
Substitute into the value function in equation (1):

$$V_1(t) = \triangle\pi_1(t) + (1 - \triangle r)(1 - \triangle\alpha_1)V_1(t + \triangle) + (1 - \triangle r)(\triangle\alpha_1)V_2(t) \tag{3}$$

Expand:

$$\begin{aligned} V_1(t) = {} & \triangle\pi_1(t) + V_1(t + \triangle) - \triangle r V_1(t + \triangle) - \triangle\alpha_1 V_1(t + \triangle) \\ & + \triangle\alpha_1 V_2(t + \triangle) + (\triangle^2 \text{ terms}) \end{aligned} \tag{4}$$

Rearrange and divide by $\triangle$:

$$rV_1(t + \triangle) = \pi_1(t) + \alpha_1 \left[V_2(t + \triangle) - V_1(t + \triangle)\right] + \frac{V_1(t + \triangle) - V_1(t)}{\triangle} + (\triangle \text{ terms}) \tag{5}$$

Take the limit as $\triangle \to 0$: no-arbitrage condition

$$rV_1(t + \triangle) = \pi_1(t) + \alpha_1 \left[V_2(t + \triangle) - V_1(t + \triangle)\right] + \boldsymbol{\partial}_t V_1(t) \tag{6}$$

$$rV_1(t) = \pi_1(t) + \alpha_1 \left[V_2(t) - V_1(t)\right] + \boldsymbol{\partial}_t V_1(t) \tag{7}$$

where:

- $rV_1(t)$: cost per unit of asset in foregone interest
- $\pi_1(t)$: flow profits
- α_1: intensity
- $V_2(t + \triangle) - V_1(t + \triangle)$: change in value
- $\boldsymbol{\partial}_t V_1(t)$: capital gains

- Same working for the Bellman equation $V_2(t)$ in equation (2).
- With a time-varying $\pi_1(t)$ and $\pi_2(t)$, and appropriate boundary values, we can solve the system of ODEs.

1.4 Stationary

Let $\pi_1(t) = \pi_1$ and $\pi_2(t) = \pi_2$.
Unless boundary values are changing (e.g., exit decision), then the values are stationary: $V_1(t) = V_1$ and $V_2(t) = V_2$,
i.e. no capital gains: $\Rightarrow \boldsymbol{\partial}_t V_1 = \boldsymbol{\partial}_t V_2 = 0$.
Then:

$$rV_1 = \pi_1 + \alpha_1(V_2 - V_1) \tag{8}$$
$$rV_2 = \pi_2 + \alpha_2(V_1 - V_2) \tag{9}$$

This is a system of 2 equations in V_1 and V_2. We can add in max's, etc. to decisions.

2 Kiyotaki-Wright (1993) Style

2.1 Approach

- Money will smooth functions in barter.
- The agent may want to trade for money even if there is no direct utility.
- Doesn't directly address domination of money by bond holdings, as there are severe limitations on direct storage.
- The basic friction is the <u>double-coincidence of wants</u> in barter.

Assume: Bilateral barter, for trade:

(1) You need to want what I have, **and**

(2) I need to want what you have

- Unless both of the above hold, no trade occurs, even if it would be efficient (e.g., I know someone else who wants what you have).
- In principle, in a full contract space with perfect observation, we could trade elaborate contingent contracts.

2.2 Key Assumptions

(1) Double-coincidence of wants

(2) No possible enforcement of commitment

(3) Anonymity, so history is not observable

(4) Bilateral barter; decreases possible contract space.

Then:
- Fiat money can be a Pareto improvement, and accepting money for goods optimal.
- Money can enhance specialization of production by decreasing barter frictions.

Strip down the idea See note by Siu and Fernandez-Villaverde, and Walsh 3.4:
- Agents $i \in [0,1]$.
- Goods indivisible, cost to produce: $c \geq 0$
- Cannot store goods (otherwise, falls apart).

Utility:

- For some fraction $x \in (0,1)$ of $i \in [0,1]$ goods: derives utility $U > 0$.
- For own good: derives no utility (could weaken).

Bilateral and Random Meetings

- Randomly match i and j consumers, then:

$$\mathbb{P}\left(i \text{ wants } j\text{'s good}\right) = x \qquad \text{(single coincidence)} \tag{10}$$

$$\mathbb{P}\left(i \text{ wants } j\text{'s good} \,|\, j \text{ wants } i\text{'s good}\right) = y \qquad \text{(double coincidence)} \tag{11}$$

In principle, we could have correlation of preferences, etc., so $y \neq x$ in general.

Money:

- Assume $M \in [0,1]$ fiat money exists.
- Doesn't matter what, just coordination and fixed supply (Gold? Seashells? Paper bills? ...)
- No direct utility of money, just needs to be easy to store or transfer
- Indivisible (can be relaxed)
- Assume money holders cannot produce, and can only hold 1 unit of money.

Why Simplify So Much?

- We want to avoid complicated distributions of money holdings and good holdings as the equilibrium strategy depends on the distributions (Achilles' Heel of this model style?)
- Indivisibility means price set at 0 or 1
- Two states of agents:
  (0) no money
  (1) have money

2.3 Symmetric Equilibrium

The decisions of agents are:

- 0-type : trade good for money?
- 1-type : trade money for good?

We concentrate of symmetric equilibria, with potentially mixed strategies: assume an equilibrium where:

- 0-type : all trade with probability $\Pi_0 \in [0, 1]$
- 1-type : all trade with probability $\Pi_1 \in [0, 1]$
  (inclusive bounds mean pure strategy is included)

For individual agents, choices denoted:

- 0-type : all trade with probability $\pi_0 \in [0, 1]$
- 1-type : all trade with probability $\pi_1 \in [0, 1]$ (if you like the good)

Symmetric Equilibrium: $\pi_0 = \Pi_0,\ \pi_1 = \Pi_1$ (i.e., big K, little k argument).
Hence, with probability $\Pi = \Pi_0\Pi_1$, both want to trade in a 0-1 random match.

2.4 Matching Arrivals and Probabilities

- Let arrival rate of random meetings be $\alpha > 0$.
- Due to indivisibilities: they hold money with M prob., and produce with $1 - M$ prob.

- If you had money (i.e., **1-type**):

  (1) Prob. $1 - M$ meet a producer

  (2) Prob. x they produce what you want

  (3) Prob. Π_0 they are willing to sell to you

  (4) Given all these, set Π_1 that you want to trade.

- If you were a producer (i.e., **0-type**):

  (1) Prob. M meet money holder

  (2) Prob. Π_1 they are willing to trade

  or:

  (1) Prob. $1 - M$ meet producer

  (2) Prob. $x \cdot y$ both like each other's goods (i.e., barter)

3 Value Functions and Decisions

3.1 Money Holders

$$rV_1 = \underbrace{\alpha}_{\text{arrival}} \overbrace{(1-M)}^{\text{producer}} \underbrace{x}_{\substack{\text{want}\\\text{their}\\\text{product}}} \overbrace{\Pi_0}^{\substack{\text{they}\\\text{will}\\\text{trade}}} \cdot \underbrace{\max_{\pi_1 \in [0,1]} \{\pi_1\}}_{\substack{\text{your trade choice}\\\text{of good/money if}\\\text{you like the good}}} \left(\overbrace{u}^{\substack{\text{consume}\\\text{good}}} + \underbrace{V_0 - V_1}_{\text{change type}} \right) \tag{12}$$

This is the under the assumption of not allowing free disposal of money.

3.2 Producers

$$
\begin{aligned}
rV_0 = & \overbrace{\underbrace{\alpha}_{\text{arrival}} \overbrace{x}^{\substack{\text{like}\\\text{good}}} \underbrace{y}_{\substack{\text{they}\\\text{like}\\\text{mine}}} \overbrace{(1-M)}^{\substack{\text{meet}\\\text{producer}}} (\underbrace{u}_{\text{consume}} - \underbrace{c}_{\text{produce}})}^{\text{barter; no real choice}} \\
& \underbrace{+ \alpha \underbrace{M}_{\substack{\text{meet}\\\text{money}\\\text{holder}}} \overbrace{x}^{\substack{\text{they}\\\text{like}\\\text{mine}}} \underbrace{\Pi_1}_{\substack{\text{trade}\\\text{prob.}}} \max_{\pi_0 \in [0,1]} \{\pi_0\} \left(\overbrace{V_1 - V_0}^{\text{change type}} - \underbrace{c}_{\text{produce}} \right)}_{\text{produce for money}}
\end{aligned}
\tag{13}
$$

3.3 Strategies

- $\pi_1 = \begin{cases} 0 & \text{if } u + V_0 - V_1 \equiv \Omega_1 < 0 \\ \text{mixed} & u + V_0 - V_1 = 0 \\ 1 & \text{if } u + V_0 - V_1 > 0 \end{cases}$

- $\pi_0 = \begin{cases} 0 & \text{if } V_1 - V_0 - c \equiv \Omega_0 < 0 \\ \text{mixed} & V_1 - V_0 - c = 0 \\ 1 & \text{if } V_1 - V_0 - c > 0 \end{cases}$

- We will ignore mixed equilibria (not interesting / stable)

3.4 Pure Strategy Equilibrium

- We will conjecture $\pi_1 = \Pi_1 = 1$ (i.e., always trade money for good you like).
- We won't prove, but can show this is optimal strategy for any $\Pi_0 \in [0, 1]$, (i.e., with/without money).
- Also, note that $V_0 - V_1 > 0$
- Then, $\Pi = \Pi_0\Pi_1$ and $\Pi = \pi_0$ due to $\Pi_1 = 1$:
$\Rightarrow$ if $\Pi > 0$, say we have a monetary equilibrium.
Define:

$$\Omega_0 \equiv V_1 - V_0 - c, \text{ (i.e., surplus from production for money)} \tag{14}$$

$$\Omega_1 \equiv V_0 - V_1 - u, \text{ (i.e., surplus from money for good you like)} \tag{15}$$

Can show that $\Omega_1 > 0$.
Then, given guess that $\pi_0 = \Pi_0 = \Pi$, what are the optimal choices if $\Omega_0 \gtreqless 0$?
Using equations (12) to (15), we can show that:

$$\Omega_0 \gtreqless 0 \qquad \textbf{if and only if} \qquad \frac{\alpha x(1-M)(\Pi - y)(u-c)}{r} \gtreqless c \tag{16}$$

where c is the cost of production.

3.5 Pure Strategies

(1) There is always a $\Pi = 0$ equilibrium as the sign becomes negative. This is confirmed by taking $\Omega_0 < 0$.

(2) There is a $\Pi = 1$ equilibrium **if and only if:** $\alpha x(1-M)(\Pi - y)(u-c) > rc$.
This is the condition for possibility of monetary equilibrium.

What makes it **less likely** (i.e. smaller region) to achieve equilibrium?

(1) $r \uparrow:$ more impatient

(2) $y \uparrow:$ more double-coincidence

(3) $M \uparrow:$ less scarce money

(4) $\alpha \downarrow:$ less meetings are chance to trade.

Normalizing $\alpha x = 1$, without loss of generality, and set $c = 0$, then the welfare becomes:

$$\begin{aligned} W &= \underbrace{M}_{\substack{\text{with}\\\text{money}}} V_1 + \underbrace{(1-M)}_{\substack{\text{without}\\\text{money}}} V_0 && (17)\\ &= \frac{u}{r}(1-M)\left[(1-M)y + M\Pi\right] && (18) \end{aligned}$$

Maximizing welfare by choosing money supply from equation (18) by taking the FOC w.r.t $\boldsymbol{\partial}_M$:

$$\begin{aligned} M &= \frac{\Pi - 2y}{2\Pi - 2y} && (19)\\ &= \begin{cases} 0 & \text{if } \Pi = 0 \\ \frac{1-2y}{2-2y} & \text{if } \Pi = 1 \end{cases} && (20) \end{aligned}$$