Question 1: Search

The following is a variation on the search model done in class, where agents grow at different rates in different states. Each firm is differentiated by a productivity, z, and a state $i \in \{\ell, h\}$. Furthermore, assume that they discount the future as some infinitesimal interest rate $r > 0$ and die/exit at some infinitesimal rate $\rho > 0$. While operating, they make flow profits of $\pi(z) \equiv e^z$, independent of their i state.

If the firm is in state h, then it grows at rate $\gamma > 0$. If it is in state ℓ, it doesn't grow. They switch between i states are rate $\alpha > 0$, so that they have a $\alpha\Delta$ probability of a switch in any discrete time period. If we assume logs, in a discrete time approximation for some small time interval $\Delta > 0$ then,

$$z_{t+1} = \begin{cases} z_t & \text{if } i = \ell \\ z_t + \Delta\gamma & \text{if } i = h \end{cases} \tag{1}$$

In any small Δ, they discount the future at rate $r\Delta$, and have a $\rho\Delta$ probability of dying.

Writing down the Bellman equation for a firm in the ℓ state, we add in the z state compared to the model in class, with a probability of

$$\begin{aligned} V_\ell(t,z) &= \Delta\pi(z) + \underbrace{(1-\rho\Delta)}_{\text{Survive}}\underbrace{(1-r\Delta)}_{\text{Discount}}\underbrace{\mathbb{E}\left[V_i(t+\Delta, z_{t+\Delta})\right]}_{\text{Find Expected } z \text{ and } i} + \underbrace{\rho\Delta(1-r\Delta)\times 0}_{\text{No value at Death}} & (2) \\ &= \Delta e^z + (1-\rho\Delta)(1-r\Delta)\left[\underbrace{\Delta\alpha}_{\text{Switch } i}\ V_h(t+\Delta, z) + \underbrace{(1-\Delta\alpha)}_{\text{No Switch}}V_\ell(t+\Delta, z)\right] & (3) \end{aligned}$$

Agents grow if entering in h state but are otherwise identical

$$V_h(t,z) = \Delta e^z + (1-\rho\Delta)(1-r\Delta)\left[\Delta\alpha V_\ell(t+\Delta, \underbrace{z+\gamma\Delta}_{\text{Grow at } h}) + \underbrace{(1-\Delta\alpha)}_{\text{No Switch}}V_h(t+\Delta, z+\gamma\Delta)\right] \tag{4}$$

1.1 Derive the Continuous Time Bellman Equations

Take equations 3 and 4, and take a $\Delta \to 0$ limit, similar to what we did in class, making the same assumption of a stationary equilibrium where $\frac{\partial V_i(t,z)}{\partial t} = 0$. For this, you can use the limit:

$$\lim_{\Delta\to 0} \frac{v(t, z+\gamma\Delta) - v(t,z)}{\Delta} = \gamma\frac{\partial v(t,z)}{\partial z} \tag{5}$$

This limit is identical for $v(t+\Delta, z+\gamma\Delta) - v(t+\Delta, z+\gamma\Delta)$, etc. in the numerator You can check with L'Hopitals rule for variations. Similarly, $\lim_{\Delta\to 0}\frac{v(t+\Delta,z)-v(t,z)}{\Delta} = \frac{\partial v(t,z)}{\partial t}$. If you assume the stationarity and drop the time subscript accordingly, then the two Bellman equations should have an equation in $v_\ell(z), v_h(z), v_h'(z)$, etc. where the discount rate is modified to include the sum of both r and ρ.

1.2 Solve the Continuous Time Bellman Equations

Without a great deal of effort or knowledge of differential equations, you should be able to solve the system of 2 ODEs from the previous part. The easiest is probably guess-and-verify. The hint is that $v_\ell(z) = C_\ell e^z$ and $v_h(z) = C_h e^z$ for some undetermined constants C_ℓ and C_h. You should be able to solve a system of equations numerically or analytically.

1.3 Simulating Paths

Let $r = .1, \alpha = .6, \rho = .1$, and $\gamma = .05$. Assume that any firm entering starts with at time $t = 0$ with $z = 0$ and starts in the low state $i = \ell$.

- Simulate 100 stochastic paths of z and i, starting from the $z = 0, i = \ell$ state for $t = 0$ to $t = 10$. To do this, you will have to choose a small $\Delta > 0$, and simulate the switch probabilities between ℓ and h, and then evolve z according to equation 1 depending on the i state. You should then have a total of around $10/\Delta$ time periods when discretized.
- Don't forget the arrival rate of exit ρ, and which point z disappears and the process stops.
- Display a histogram of the distribution of z states at $t = .1$, $t = 2$, and $t = 10$. Keep in mind that as the maximum growth rate is γ in any infinitesimal time period, the distribution of z at time t is in $[0, \gamma t]$
- Using your solution from section 3 for the $v_i(z)$ and the given constants to simulate the path of $v_i(z)$ for the paths of z and i you previously calculated. This will give you a distribution of value functions starting from entry. Display the distribution as a histogram for $t = 0.1$, $t = 2.0$, and $t = 10.0$

Question 2: Analyzing DSGE Models

This question relies on modifying the code for the An-Schorfheide model on GitHub `https://github.com/jlperla/ECON546_2018/blob/master/julia%20tutorial/DSGE%20example.i` If you have been working exclusively on JuliaBox and do not have direct access to DSGE.jl, then please email me to discuss.

1. Plot the IRFs for some 3 values: $\tau_\ell < \tau < \tau_h$ where τ is the inverse of the intertemporal elasticity of substitution in the existing code. Interpret the differences, and choose τ_ℓ and τ_h in order to accentuate the differences
2. Plot the IRFs for some 3 values: $\kappa_\ell < \kappa < \kappa_h$ where κ is the NK coefficient on output parameter in the existing code. Interpret the differences, and choose κ_ℓ and κ_h in order to accentuate the differences

Question 3: (Optional)Analyzing Larger DSGE Models

Instead of the An and Schorfheide (2006) model, plot the IRFs Smets and Wouters (2007), FRBNY DSGE model (version 990.2), or the 1002.9 model. These are the actual DSGE models used by the NY Fed.

This may take some work, require looking at the underlying papers to understand the parameters, sleuthing with `m.exogenous_shocks, m.observables, m.endogenous_states` and `m.parameters`, etc.

Try to adapt the existing An-Schorfheide code in our example notebook, while looking at the underlying properties of the model/etc. to find the relevant shocks, states, etc. The most important simplification is to try to run the model at the parameter values in the underlying code, and without re-estimating. The Bayesian estimation of the parameters from the data is very neat, but extremely time-consuming.[1]

[1] In particular, looking at `http://frbny-dsge.github.io/DSGE.jl/latest/running_existing_model.html` and other sample code, you do not want to run `estimate(m)` and `compute_moments(m)`. If you are having a lot of trouble, I can put you in touch with the developers at the Federal Reserve Bank of New York.