‘math+econ+code’ masterclass on competitive equilibrium: Walrasian equilibrium with substitutes

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Day 5, May 25 2018: one-to-many matching

Learning objectives: day 5

- One-to-many matching models
- TU case: IP and LP formulation
- Generalized deferred acceptance algorithm

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- Parkes and Ungar (2000). Iterative combinatorial auctions: Theory and practice. *Proc. 17th National Conference on AI*.
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Section 1

The TU case

One limitative feature of the setting seen in the previous lectures was the one-to-one assumption. As for applications, this led us to CEO literature, where one firm needs one CEO. We want, however, move beyond this setting, and understand, for instance, the nature of complementarities of professional athletes in a team, or performers in a cast, or various specialized workers in a company. Although the empirical literature on the topic is still in its infancy, the theoretical knowledge is quite well-developped, in large part thanks to recent developments in combinatorial auction theory, which relies on the same models.

- Consider a finite set of employees $i \in \mathcal{I}$, and a finite set of firms $j \in \mathcal{J}$. We are going to significantly extend the setting we took thus far by allowing firms to hire several employees. However, one will assume one employee may only work for one firm. Let C^j be the set of employees working for firm j; hence we require that

$$C^j \cap C^{j'} = \varnothing \text{ for } j \neq j'$$
$$C^j \in 2^{\mathcal{I}}$$

(where we recall the notation $C^j \in 2^{\mathcal{I}}$ simply means C^j is a subset of $\mathcal{I}$).

- In other words, each firm should be assigned to the element of a partition of the set of workers.

For $j \in \mathcal{J}$ and $C \in 2^{\mathcal{I}}$, introduce $\pi_{Cj} = 1\left\{C = C^j\right\}$. The $\left(\pi_{Cj}\right)$'s offer an alternative, yet equivalent description of the sets C_j, provided that $\pi \in \mathcal{M}_{int}$, where

$$\mathcal{M}_{int} = \left\{ \left(\pi_{Cj}\right)_{C,j} : \begin{pmatrix} \forall j,\ \sum_C \pi_{Cj} \leq 1 \\ \forall i,\ \sum_{C,j} \pi_{Cj} 1\left\{i \in C\right\} \leq 1 \\ \forall C, \forall j,\ \pi_{Cj} \in \mathbb{N} \end{pmatrix} \right\} \tag{1}$$

where the first constraint ensures each firm hires a set of worker (possibly empty) or opts out of the market, and the second constraint ensures that each worker works in at most one coalition; indeed, the second constraint rewrites as

$$\forall i,\ \sum_{C,j} 1\left\{C = C^j\right\} 1\left\{i \in C\right\} \leq 1, \text{ that is } \forall i,\ \sum_j 1\left\{i \in C^j\right\} \leq 1$$

and finally, the third constraint is an integrality condition which expresses that the π_{Cj}'s are actual integer numbers. Due to the integrality condition on π, $\mathcal{M}_{int}$ is called the set of *pure (one-to-many) matchings*.

- One complication, which is absent in the one-to-one case, is that we need to worry about this integrality condition. We saw in the one-to-one case that this integrality condition was not very important, as the value of the optimal matching taken over the set of matchings including or not including the integrality constraint was the same.
- Similarly, a natural question to ask in the current setting is what happens when the integrality condition is relaxed. This prompts us to introduce the set of *fractional (one-to-many) matchings*, which is defined analogously as $\mathcal{M}_{int}$ with the sole difference that π_{jC} may a nonnegative real number:

$$\mathcal{M}_{frac} = \left\{ \left(\pi_{Cj}\right)_{C,j} : \begin{pmatrix} \forall j,\ \sum_C \pi_{Cj} \leq 1 \\ \forall i,\ \sum_{C,j} \pi_{Cj} 1\{i \in C\} \leq 1 \\ \forall C, \forall j,\ \pi_{Cj} \geq 0 \end{pmatrix} \right\} \tag{2}$$

This set is a convex polyhedron. Clearly, $\mathcal{M}_{int} \subset \mathcal{M}_{frac}$ and furthermore, convex combinations of elements of $\mathcal{M}_{int}$ belong in $\mathcal{M}_{frac}$. One may wonder whether conversely, all elements of $\mathcal{M}_{frac}$ are combination of elements of $\mathcal{M}_{int}$? This may seem plausible as in the one-to-one matching case, the Birkhoff-von Neumann theorem ensures that fractional matchings are exactly the set of convex combinations of pure matchings. However, this result does not extend to the one-to-many matching case, as shown in the following example.

- Consider the case where there are three employees $\mathcal{I} = \{1,2,3\}$ and two firms $\mathcal{J} = \{a,b\}$. Let

$$\pi_{\{1,2\}a} = 1/2,\ \pi_{\{3\}a} = 1/2$$
$$\pi_{\{2,3\}b} = 1/2,\ \pi_{\{1\}b} = 1/2$$

  all other values of the π's being zero.
- Intuitively, this means that:
  - firm a hires either coalition $\{1,2\}$ or $\{3\}$ with probability $1/2$, and that firm b hires either coalition $\{1\}$ or $\{2,3\}$ with probability $1/2$,
  - all workers work either with firm a and firmb with probability $1/2$,
- There is however no way to reconcile these conditional probabilities at the individual level. Indeed, worker 1 has $1/2$ chance to work with firm a or with firm b. If worker 1 works with firm a, then worker 2 works with firm a. If worker 1 works with firm b, then worker 2 works with firm b. Thus this would imply that worker 2 works with firm a with probability one, a contradiction.
- Here, $\pi \in \mathcal{M}_{frac}$, but π cannot be a convex combination of elements of $\mathcal{M}_{int}$.

- Assume that we clone employees and firms so that two employees of each type $\mathcal{I} = \{1, 1^*, 2, 2^*, 3, 3^*\}$ and two firms of each type $\mathcal{J} = \{a, a^*, b, b^*\}$. Let

$$\pi_{\{1,2\}a} = 1, \pi_{\{3\}a^*} = 1 \ \pi_{\{2^*,3^*\}b^*} = 1, \ \pi_{\{1^*\}b} = 1$$

  we have now managed to assign everyone in such a way that respects the prescribed matching frequencies.

- One sees that the problem tends to disappear with many agents of the same type.

- Let u_i be the wage of employee i. Let Φ_{Cj} be the output of firm j with set of employees C. Its profit is

$$V_{Cj}\left(u\right) = \Phi_{Cj} - \sum_{i \in C} u_i.$$

- The firm's problem is to choose the set of workers that maximize profit, that is

$$\arg\max_{C \in 2^{\mathcal{I}}} \left\{ \Phi_{Cj} - \sum_{i \in C} u_i \right\}$$

- The social planner's problem is

$$\mathcal{W}_{int} = \max_{\substack{C^j \cap C^{j'} = \varnothing \\ j \neq j'}} \sum_j \Phi_{C^j j}.$$

- Making use of the π introduced in Expression (1), the expression of $\mathcal{W}_{int}$ can be rewritten as

$$\mathcal{W}_{int} = \max_{\pi \in \mathcal{M}_{int}} \sum_{\substack{j \in \mathcal{J} \\ C \in 2^{\mathcal{I}}}} \pi_{Cj} \Phi_{Cj}. \tag{3}$$

- This problem is an integer programming problem: the constraints induced by $\pi \in \mathcal{M}_{int}$ are linear, but the constraint requiring the integrality constraint $\pi_{Cj} \in \mathbb{N}$ is not. In the case of one-to-one matching, the value of its problem coincides with the value of its linear programming relaxation, i.e. the same program where the integrality constraint has been dropped.

- This prompts us to introduce

$$\mathcal{W}_{LP1} = \max_{\pi \in \mathcal{M}_{frac}} \sum_{\substack{j \in \mathcal{J} \\ C \in 2^{\mathcal{I}}}} \pi_{Cj} \Phi_{Cj}, \tag{4}$$

and, of course

$$\mathcal{W}_{int} \leq \mathcal{W}_{LP1}$$

holds in general.

- Unfortunately, equality does not hold: the inequality may be strict, as shown in the following example: As in the previous example, there are three employees $\mathcal{I} = \{1, 2, 3\}$ and two firms $\mathcal{J} = \{a, b\}$. Assume that firm a has value 3 for $\{1, 2\}$, 2 for $\{3\}$ and 3 for $\{1, 2, 3\}$, while firm b has value 3 for $\{2, 3\}$, 2 for $\{1\}$, and 3 for $\{1, 2, 3\}$. The value of the fractional program above is thus 4, and consists in assigning worker 3 to firm a and worker 1 to firm b. The optimal pure matching has $C^1 = \{3\}$, $C^2 = \{1\}$, thus

$$\mathcal{W}_{int} = 4.$$

The optimal fractional matching has

$$\pi_{\{1,2\}a} = 1/2,\ \pi_{\{3\}a} = 1/2$$
$$\pi_{\{2,3\}b} = 1/2,\ \pi_{\{1\}b} = 1/2$$

all other values of the π's being zero. Thus

$$\mathcal{W}_{LP1} = 5,$$

and in this example

$$\mathcal{W}_{int} < \mathcal{W}_{LP1}.$$

Fortunately, we'll see important cases where the value of these two programs coincides. In particular, note the following result:

PROPOSITION

If for all j, the indirect utility of firm j defined by the map

$$u \to \max_{C \in 2^{\mathcal{I}}} \left(\Phi_{Cj} - \sum_{i \in C} u_i \right)$$

is submodular, then

$$\mathcal{W}_{int} = \mathcal{W}_{LP1}.$$

THEOREM

The dual expression of $\mathcal{W}_{LP1}$ is

$$\mathcal{W}_{LP1} = \min_{u,v \geq 0} \left\{ \sum_{i \in \mathcal{I}} u_i + \sum_{j \in \mathcal{J}} v_j \right\} \tag{5}$$

$$s.t. \ \sum_{i \in C} u_i + v_j \geq \Phi_{Cj} \ \forall j \in \mathcal{J}, C \in 2^{\mathcal{I}} \tag{6}$$

Equivalently

$$\mathcal{W}_{LP1} = \min_{u \geq 0} \sum_{i \in \mathcal{I}} u_i + \sum_{j \in \mathcal{J}} \max_{C \in 2^{\mathcal{I}}} \left(\Phi_{Cj} - \sum_{i \in C} u_i \right).$$

PROOF.

This follows by standard linear programming duality, but we recall it for clarity. Start from the primal problem

$$\max_{\pi_{Cj} \geq 0} \sum_{C,j} \pi_{Cj} \Phi_{Cj}$$

$$s.t. \sum_{C} \pi_{Cj} \leq 1 \left[v_j \right]$$

$$\sum_{C,j} \pi_{Cj} 1 \left\{ i \in C \right\} \leq 1 \left[u_i \right]$$

and rewrite it as a min-max problem

$$\max_{\pi_{Cj \geq 0}} \min_{v_j, u_i \geq 0} \left(\begin{array}{c} \sum_{C,j} \pi_{Cj} \Phi_{Cj} - \sum_{C,j} v_j \pi_{Cj} + \sum_j v_j \\ - \sum_{i,C,j} u_i \pi_{Cj} 1 \left\{ i \in C \right\} + \sum_i u_i \end{array} \right)$$

Rearranging the desired dual program. □

As in the one-to-one matching case, this duality result has an important interpretation in terms of stability. Condition

$$\sum_{i \in C} u_i + v_j \geq \Phi_{Cj}$$

is a stability condition: it expresses that if a coalition is not working for a given firm, there is no way for this firm to hire the coalition and share the surplus generated in a way that would make everybody better off. Hence, as before, u_i can be interpreted as the equilibrium payoff of worker i, while v_j can be interpreted as the equilibirum payoff of worker j.

The question is whether these stable payoffs u_i and v_j are actually feasible, i.e. if there is a matching that can produce them. It turns out that this issue is directly related to the issue of whether $\mathcal{W}_{int} = \mathcal{W}_{LP1}$. Indeed, assume this equality holds. Then there exists a matching C^j such that

$$\sum_{j \in \mathcal{J}} \Phi_{C^j j} = \sum_{i \in \mathcal{I}} u_i + \sum_{j \in \mathcal{J}} v_j = \sum_{j \in \mathcal{J}} \left(v_j + \sum_{i \in C^j} u_i \right)$$

thus payoffs u_i and v_j are feasible, i.e. if j is matched to coalition C^j, there is a way to share the surplus created in such a way that

$$\Phi_{C^j j} = v_j + \sum_{i \in C^j} u_i .$$

Conversely, if there are stable feasible payoffs u_i and v_j, then equality

$$\sum_{j\in\mathcal{J}} \Phi_{C^j j} = \sum_{i\in\mathcal{I}} u_i + \sum_{j\in\mathcal{J}} v_j = \mathcal{W}_{LP1}$$

but $\sum_{j\in\mathcal{J}} \Phi_{C^j j} \leq \mathcal{W}_{int}$thus $\mathcal{W}_{LP1} = \mathcal{W}_{int}$. This leads to:

THEOREM

(i) There exist at least one stable one-to-many matching if and only if

$$\mathcal{W}_{int} = \mathcal{W}_{LP1}.$$

(ii) The set of stable one-to-many matchings coincides with the (possibly empty) set of integral solutions of (4). (iii) Whenever they exist, stable matchings maximize social welfare (3). (iv) Whenever a stable matching exists, employees' and workers' outcome payoffs u_i and v_j are solution to the dual program to (5).

Recall

$$\mathcal{W}_{int} = \max_{\pi} \sum_{j,C} \pi_{Cj} \Phi_{Cj}$$

subject to

$$\sum_{C} \pi_{Cj} \leq 1 \; \forall j \tag{7}$$

$$\sum_{C,j} \pi_{Cj} 1 \{ i \in C \} \leq 1 \; \forall i \tag{8}$$

$$\pi_{Cj} \in \mathbb{N} \; \forall C, \forall j. \tag{9}$$

One of the reasons of the fact that this Integer Programming problem fails to coincide with its linear programming relaxation is the fact that the linear programming relaxation fails to ensure that mixtures of partitions will be selected, as clear from the example above. As a result, the social optimum will failed to be obtained as an outcome of the decentralized equilibrium; in fact, there may be no decentralized equilibrium, because of complementarities between workers may lead to ambiguous marginal productivity. As will be emphasized below, this can expressed as the fact that the core of the individualistic game may be empty.

In order to remedy this problem, let us introduce an *union*, with coercive powers over workers. The union will have the prerogative to group workers into teams, and it will have the responsibility of bargaining with the firm over the surplus left to the firms, and the surplus of the union. One may think that the union will then share its own surplus among the workers, but this will not be relevant for us as the union does not have to worry about workers' participation and incentive compatibility.

For now, we shall assume that the union is firm-neutral, i.e. has the power to assign workers among coalitions, i.e. form partitions of workers, and bargains with firms. Let us compute the stable set, if it exists.
We have to introduce the set of partitions of $\mathcal{I}$, which we denote $\Pi(\mathcal{I})$. For $\pi \in \Pi$, denote $C \in \pi$ to indicate C is part of partition π. Consider the relaxed problem

$$\mathcal{W}_{LP2} = \max_{\pi_{Cj}, \lambda_\pi \geq 0} \sum_{j,C} \pi_{Cj} \Phi_{Cj}$$

subject to

$$\sum_C \pi_{Cj} \leq 1 \; \forall j \tag{10}$$

$$\sum_j \pi_{Cj} \leq \sum_{\pi \in \Pi(\mathcal{I})} \lambda_\pi 1\{C \in \pi\} \; \forall C \tag{11}$$

$$\sum_{\pi \in \Pi(\mathcal{I})} \lambda_\pi \leq 1 \tag{12}$$

Note that if integrality of π and λ is imposed, then the corresponding integer programming problem has the same value of $\mathcal{W}_{int}$. Indeed, condition (12) ensures that at most one partition will be employed. Condition (11) expresses that if a coalition is a member of the employed partition, then this coalition may be hired by at most one firm; while if it is not a member of the employed partition, then it may not be hired by any firm. Clearly, conditions (12) and (11) imply (8); indeed, from inequality (11),

$$\begin{aligned}\sum_C \sum_j \pi_{Cj} 1\{i \in C\} &\leq \sum_C \sum_{\pi \in \Pi(\mathcal{I})} \lambda_\pi 1\{C \in \pi\} 1\{i \in C\} \\ &\leq \sum_{\pi \in \Pi(\mathcal{I})} \lambda_\pi \sum_C 1\{C \in \pi\} 1\{i \in C\} \\ &\leq \sum_{\pi \in \Pi(\mathcal{I})} \lambda_\pi\end{aligned}$$

which is less than 1 by condition (12).

Theorem

The dual expression of $\mathcal{W}_{LP2}$ is given by

$$\min_{U,v_j\geq 0}\left\{U+\sum_{j\in\mathcal{J}}v_j\right\} \tag{13}$$
$$s.t.\ u_C+v_j\geq\Phi_{Cj}\ \forall j\in\mathcal{J},C\in 2^{\mathcal{I}}$$
$$U\geq\sum_{C\in\pi}u_C\ \forall\pi\in\Pi\left(\mathcal{I}\right)$$

equivalently

$$\min_{u_C,v_j\geq 0}\left\{\max_{\pi\in\Pi(\mathcal{I})}\left(\sum_{C\in\pi}u_C\right)^{+}+\sum_{j\in\mathcal{J}}v_j\right\} \tag{14}$$
$$s.t.\ u_C+v_j\geq\Phi_{Cj}\ \forall j\in\mathcal{J},C\in 2^{\mathcal{I}}$$

THEOREM (CTD)

Yet equivalently

$$\min_{u_C \geq 0} \left\{ \max_{\pi \in \Pi(\mathcal{I})} \left(\sum_{C \in \pi} u_C \right)^{+} + \sum_{j \in \mathcal{J}} \max_{C \in 2^{\mathcal{I}}} \left(\Phi_{Cj} - u_C \right) \right\}. \tag{15}$$

Before giving the proof, let us make some comments on this result:
1. In the solution of the dual, v_j is the equilibrium payoff to firm j, U is the total payoff to the union, and $U_{\bar{C}^j j}$ is the payoff going to employees of firm j. Note that this program does not specify the salaries of the individuals, as the individualistic game may not have a stable solution.
2. The dual program introduces payoffs made to a coalition C, to be shared among its members. If $\mathcal{W}_{int} = \mathcal{W}_{LP2}$, this can be interpreted as the Walrasian price of coalition C, irrespective of which firm this coalition works for. If $\mathcal{W}_{int} < \mathcal{W}_{LP2}$, coalitions will not have a Walrasian price irrespective of the firm they work with. Their Walrasian price will depend on the firm they work with.
3. The combinatorial structure of this problem is much more involved than problem $\mathcal{W}_{LP1}$, as there is a number equal to the cardinality of $\Pi(\mathcal{I})$ more inequalities to check. Hence, conditions ensuring $\mathcal{W}_{int} \leq \mathcal{W}_{LP2} \leq \mathcal{W}_{LP1}$ will save us from this extra computational burden.

PROOF OF THEOREM 4.

Again, this follows from standard duality in linear programming. One has

$$\mathcal{W}_{LP2} = \max_{\pi_{Cj}, \lambda_\pi \geq 0} \sum_{j,C} \pi_{Cj} \Phi_{Cj}$$

subject to

$$\sum_C \pi_{Cj} \leq 1 \;\; [v_j]$$

$$\sum_j \pi_{Cj} \leq \sum_{\pi \in \Pi(\mathcal{I})} \lambda_\pi 1\{C \in \pi\} \;\; [u_C]$$

$$\sum_{\pi \in \Pi(\mathcal{I})} \lambda_\pi \leq 1 \, [U]$$

□

PROOF.

The program rewrites as a min-max problem as

$$\max_{\pi_{Cj},\lambda_\pi \geq 0} \min_{v_j,u_C,U\geq 0} \begin{pmatrix} \sum_{j,C} \pi_{Cj}\Phi_{Cj} + \sum_j v_j + U \\ -\sum_{Cj} \pi_{Cj} v_j - \sum_{Cj} \pi_{Cj} u_C \\ +\sum_{C,\pi} u_C \lambda_\pi 1\{C \in \pi\} - \sum_\pi \lambda_\pi U \end{pmatrix}$$

hence the dual program is

$$\min_{v_j,u_C,U\geq 0} \left(\sum_j v_j + U \right)$$
$$s.t.\ u_C + v_j \geq \Phi_{Cj}\ \forall C, j$$
$$U \geq \sum_\pi u_C 1\{C \in \pi\}\ \forall \pi.$$

□

Clearly, $\mathcal{W}_{int} \leq \mathcal{W}_{LP2} \leq \mathcal{W}_{LP1}$, where the latter inequality comes from the fact that the set of constraints in the dual expression of $\mathcal{W}_{LP2}$ is larger than in the set of constraints in the dual expression of $\mathcal{W}_{LP1}$. But in fact, any of these inequalities can be strict.

- It can be easily seen that in the case of the example above, one recovers $\mathcal{W}_{int} = \mathcal{W}_{LP2} < \mathcal{W}_{LP1}$.
- However, there are also cases where $\mathcal{W}_{int} < \mathcal{W}_{LP2} = \mathcal{W}_{LP1}$.

The following necessary condition to equality can be proven:

THEOREM (PARKES AND UNGAR)

When the functions $C \rightarrow \Phi_{Cj}$ *are submodular for all* j*, then* $\mathcal{W}_{int} = \mathcal{W}_{LP2}$.

(Remember, a set function $f(C)$ is submodular whenever $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$).

A consequence of the fact that the strict inequality $\mathcal{W}_{int} < \mathcal{W}_{LP2}$ is that if unions are firm-neutral, a stable payoff may not need to exist, and hence the core of the game may be empty. This means that unions need to worry not only about how to group agents into team, but also also about which firms they match with.

We see that if we want to hope for the existence of a core of the game, we need to strenghten the prerogatives of the union. Instead of being firm-neutral, the union will worry about which team works with which firm: such union will be called *firm-specific*. We shall see that in the corresponding bargaining game with firms, there is now a stable payoff, thus the core of the game is now non-empty.
In mathematical terms, this amounts to provide a reinforced formulation of $\mathcal{W}_{int}$ which is equal to its linear programming relaxation. In order to do that, we need to introduce the set of labelled partitions $\Pi^{l}\left(\mathcal{I}\right)$: this is the set of $\bar{C} = \left(\bar{C}^{j}\right)_{j\in\mathcal{J}}$ such that $\bar{C}^{j} \in 2^{\mathcal{I}}$ for every j and such that

$$\bar{C}^{j} \cap \bar{C}^{j'} = \varnothing \text{ for } j \neq j'.$$

We consider reformulate $\mathcal{W}_{int}$ as

$$\mathcal{W}_{int} = \max_{\pi_{Cj},\lambda_{\bar{C}}} \sum_{j,C} \pi_{Cj} \Phi_{Cj}$$

subject to

$$\sum_{C} \pi_{Cj} \leq 1 \ \forall j \tag{16}$$

$$\pi_{Cj} \leq \sum_{\bar{C} \in \Pi^{I}(\mathcal{I})} \lambda_{\bar{C}} 1 \left\{ C = \bar{C}^{j} \right\} \ \forall C \tag{17}$$

$$\sum_{\bar{C} \in \Pi^{I}(\mathcal{I})} \lambda_{\bar{C}} \leq 1 \tag{18}$$

$$\pi_{Cj}, \lambda_{\bar{C}} \in \mathbb{N} \tag{19}$$

Although this reformulation may seem tautological, it does have some insights. Indeed it turns out that the Integer Program $\mathcal{W}_{int}$ formulated so coincides with its corresponding linear programming relaxation, and its dual has interesting interpretation in terms of Walrasian prices.

THEOREM

(i) One has

$$\mathcal{W}_{int} = \mathcal{W}_{LP3}$$

where

$$\mathcal{W}_{LP3} = \max_{\pi_{Cj}, \lambda_{\bar{C}} \geq 0} \sum_{j,C} \pi_{Cj} \Phi_{Cj} \tag{20}$$

subject to

$$\sum_{C} \pi_{Cj} \leq 1 \ \forall j \tag{21}$$

$$\pi_{Cj} \leq \sum_{\bar{C} \in \Pi^{I}(\mathcal{I})} \lambda_{\bar{C}} 1 \left\{ C = \bar{C}^{j} \right\} \ \forall C, j \tag{22}$$

$$\sum_{\bar{C} \in \Pi^{I}(\mathcal{I})} \lambda_{\bar{C}} \leq 1 \tag{23}$$

THEOREM

(ii) the program is equal to its dual formulation

$$\mathcal{W}_{int} = \min_{v_j, u_{Cj,U \geq 0}} U + \sum_j v_j$$

$$u_{Cj} + v_j \geq \Phi_{Cj} \; \forall C \in 2^{\mathcal{I}}, j \in \mathcal{J}$$

$$U \geq \sum_j u_{\bar{C}^j j} \; \forall \bar{C} \in \Pi^I(\mathcal{I})$$

or equivalently

$$\mathcal{W}_{int} = \min_{u_{Cj} \geq 0} \left(\max_{\bar{C} \in \Pi^I(\mathcal{I})} \left(\sum_j u_{\bar{C}^j j} \right) + \sum_j \max_{C,j} \left(\Phi_{Cj} - u_{Cj} \right)^+ \right)$$

Before we give the proof, let us provide an interpretation of the duality result, let us comment on it. Note that the payoffs to coalition C no longer appear in the dual program; instead, as noted above, when $\mathcal{W}_{int} < \mathcal{W}_{LP2}$ it no longer makes sense to speak about the Walrasian price of a coalition without specifying which firm employs the coalition; instead, if one adds the specification of which firm the coalition works with, the "employed coalition" (C, j) now has a Walrasian price, which is given by U_{Cj}.
(i) One has clearly

$$\mathcal{W}_{int} \leq \mathcal{W}_{LP3}.$$

Conversely, consider $\left(\pi^*_{Cj}, \lambda^*_{\bar{C}}\right)$ a fractional solution of program $\mathcal{W}_{LP}$. For $\bar{C} \in \Pi^I(\mathcal{I})$, note that

$$\sum_j \Phi_{\bar{C}^j j} \leq \mathcal{W}_{int}$$

hence

$$\sum_{\bar{C} \in \Pi^I(\mathcal{I})} \lambda^*_{\bar{C}} \sum_j \Phi_{\bar{C}^j j} \leq \mathcal{W}_{int}$$

hence

$$\sum_{\bar{C},j,C} \Phi_{\bar{C}^j j} \lambda^*_{\bar{C}} 1\left\{\bar{C}^j = C\right\} \leq \mathcal{W}_{int}$$

thus, by (22)

$$\mathcal{W}_{LP3} = \sum_{j,C} \pi^*_{Cj} \Phi_{\bar{C}^j j} \leq \mathcal{W}_{int}$$

QED.

(ii) Write the primal problem

$$\mathcal{W}_{LP3} = \max_{\pi_{Cj},\lambda_{\bar{C}}} \sum_{j,C} \pi_{Cj}\Phi_{Cj}$$

subject to

$$\sum_{C} \pi_{Cj} \leq 1 \ \ [v_j]$$

$$\pi_{Cj} \leq \sum_{\bar{C}\in\Pi^I(\mathcal{I})} \lambda_{\bar{C}} 1\left\{ C = \bar{C}^j \right\} \ \ \left[u_{Cj}\right]$$

$$\sum_{\bar{C}\in\Pi^I(\mathcal{I})} \lambda_{\bar{C}} \leq 1 \, [U]$$

Hence the dual formulation of the problem is

$$\min_{v_j, u_{Cj,U\geq 0}} U + \sum_j v_j$$

$$u_{Cj} + v_j \geq \Phi_{Cj} \ \forall C \in 2^{\mathcal{I}}, j \in \mathcal{J}$$

$$U \geq \sum_j u_{\bar{C}^j j} \ \forall \bar{C} \in \Pi^I\left(\mathcal{I}\right).$$

We are going to model the fact that agents are gross substitutes for firms in the following sense: Assume agents' wages are u_i. In that case, the coalition of workers $C^d(u)$ demanded by a firm with productivity Φ_C satisfies

$$C^d(u) \in \arg\max_C \left(\Phi_C - \sum_{i \in C} u_i \right).$$

Assume that there is an increase in the wages of a subset of workers, but the wages of the rest of the workers remains the same. Call u' the new vector of wages, so that $u' \geq u$. Then the Gross Substitutes condition expresses that if some worker i was in demand under wage system u, and if the wage of this worker has remained unchanged $u_i = u'_i$, then this worker is still in demand under the new wage system u'.

DEFINITION

Workers are gross substitutes for production function Φ_C if for $u' \geq u$,

$$i \in C^d(u) \text{ and } u_i = u'_i \text{ implies } i \in C^d\left(u'\right),$$

where C^d is the demand associated to Φ_C.

This expresses the fact that workers are *gross substitutes*, which expresses the idea of substitutability in the classical sense: when the price of one input rises, the demand of the firm for products that are substitute increases. On the contrary, if a set of inputs are complement, if the price of one increases then the demand for the whole set decreases.

Introducing $1_C = \left(1_{\{i \in C\}}\right)_i$, one has

$$1_{C^d}(u) \in \arg\max_{1_C} \left(\Phi_C - u.1_C\right)$$

where $f.g = \sum_i f_i g_i$. Thus, introducing the firm's indirect utility $\hat{\Phi}(u)$ as

$$\hat{\Phi}(u) = \max_C \left(\Phi_C - \sum_{i \in C} u_i\right)$$

one sees that $\hat{\Phi}$ is convex, it is the Legendre transform of $1_C \to -\Phi_C$ evaluated at $-u$, and one has

$$-1_{C^d}(u) \in \partial\hat{\Phi}(u)$$

where $\partial\hat{\Phi}(u)$ is the subgradient. Hence, requesting that $1^i_{C^d}(u)$ be nondecreasing with respect to u^j, $j \neq i$ means that

$$\forall i \neq j,\ \frac{\partial^2 \hat{\Phi}}{\partial u^i \partial u^j}(u) \leq 0$$

whenever this quantity exists, that is $\hat{\Phi}$ is submodular.

Hence:

THEOREM

Workers are gross substitute if and only if indirect utility function $\hat{\Phi}$ is submodular.

Recall that Φ is monotone if $C \subset C'$ implies that $\Phi_C \subset \Phi_{C'}$. We have the following result:

PROPOSITION

Assume Φ monotone. Then if workers are gross substitutes under Φ, then $C \to \Phi_C$ is submodular.

Still assuming Φ monotone, the following characterization is very useful in practice:

PROPOSITION

Assume Φ is monotone. Then the two following properties are equivalent:
- Φ has the Gross Substitute property
- Φ has the Single Improvement (SI) property: if $C \notin C^d(u)$, then there exists C' such that $|C \backslash C'| \leq 1$, $|C' \backslash C| \leq 1$ and
$\Phi_C - \sum_{i \in C} u_i < \Phi_{C'} - \sum_{i \in C'} u_i$.

In other words, Φ has the SI property if and only if any suboptimal coalition can be improved by a coalition obtained by adding or removing at most one worker (possibly both).

- One has

$$\max \sum_{i\in\mathcal{I},B\in 2^{\mathcal{J}}} \pi_{iB\,iB}$$
$$s.t. \sum_{B} \pi_{iB} = 1$$
$$\sum_{iB} \pi_{iB} 1\left\{j\in B\right\} = 1$$

which has dual

$$\min_{\substack{u_i\geq 0\\ v_j\geq 0}} \sum_{i\in\mathcal{I}} u_i + \sum_{j\in\mathcal{J}} v_j$$
$$s.t.\ u_i + \sum_{j\in B} v_j \geq \Phi_{iB} = \alpha_{iB} + \gamma_{ij}$$

- In order to implement this on Gurobi, let P be the power set matrix such that the rows are coalitions B and the column are workers j, so that $P_{B,j} = 1\left\{j\in B\right\}$. The constraint $u_i + \sum_{j\in B} v_j \geq \Phi_{iB} = \alpha_{iB} + \gamma_{ij}$ rewrites in a matrix way as

$$\left(1_{2^J}\otimes I_I \;\; P\otimes 1_I\right)\begin{pmatrix} u \\ v \end{pmatrix}$$

Section 2

Deferred acceptance

- Consider a set of firms $i \in \mathcal{I}$ and a set of workers $j \in \mathcal{J}$. Worker j's wage is v_j. If a firm i hires a bundle of workers $B \subseteq \mathcal{J}$, then firm i's utility is
$$u_i = {}_{iB} - \sum_{j \in B} v_j$$
- Let $B_i = \arg\max_{B \subseteq \mathcal{J}} \left\{ \Phi_{iB} - \sum_{j \in B} v_j \right\}$ be the optimal choice of firm i. Stability requires that:
$$\begin{cases} \Phi_{iB_i} - \sum_{j \in B_i} v_j \geq \Phi_{iB} - \sum_{j \in B} v_j \ \forall i \in \mathcal{I}, \forall B \subseteq \mathcal{J} \\ v_j \geq 0 \ \forall j \in \mathcal{J} \\ B_i \cap B_{i'} = \emptyset \ \forall i \neq i' \end{cases}$$

- In order to generalize Gale and Shapley's deferred acceptance algorithm, Kelso and Crawford propose to discretize the set of possible wages. Worker j can only receive a wage included in a finite set $\left\{v_j^w, w \in \mathcal{W}\right\}$ of possible wage. A *contract* is the joint information of a firm $i \in \mathcal{I}$, a worker $j \in \mathcal{J}$ and a wage $w \in \mathcal{W}$; hence, the set of contracts is $\mathcal{K} = \mathcal{I} \times \mathcal{J} \times \mathcal{W}$.
- One may assume that $\mathcal{W} = \{w_1, ..., w_K\}$, where $0 = w_1 < ... < w_K = +\infty$. We define a map $\text{next} : \mathcal{W} \backslash \{w_K\} \to \mathcal{W} \backslash \{w_1\}$ such that $\text{next}\left(w_k\right) = w_{k+1}$.
- For a contract k, we denote i^k, j^k and w^k the firm, worker and wage that compse this contract. For $\mathcal{X} \subseteq \mathcal{K}$ a set of contracts, we shall denote

$$\mathcal{X}\left(i\right) = \left\{k \in \mathcal{X} : i^k = i\right\} \text{ and } \mathcal{X}\left(j\right) = \left\{k \in \mathcal{X} : j^k = j\right\}$$

which are the contracted that involve firm i (resp. worker j).

Kelso and Crawford's algorithm in a nutshell

- Initially, all the contracts are available to the firms.
- At each phase:
  - firms pick their favorite combinations of contracts among the available ones, and propose them to workers, making sure they don't offer two contract to the same worker.
  - workers who receive several contract offers retain their most preferred ones, making sure they don't retain more than one.
  - The rejected contracts (i.e. those that have not been retained by some worker) become unavailable.
- The algorithm continues until no contract is rejected any longer.

Let $W_t(i,j) \in \mathcal{W}$ be the current wage of worker j for firm $i \in \mathcal{I}$ at step t. Define $\mathcal{P}_t(i)$ as the set of workers whom firm i has proposed to hire to at step t; define $\mathcal{P}_t^{-1}(j) = \{i \in \mathcal{I} : j \in \mathcal{P}_t(i)\}$; define $\mathcal{E}_t(j)$ as the set of firms j has not turned down at the end of step t and $\mathcal{E}_t^{-1}(i)$ accordingly.

- Algorithm (Kelso-Crawford).
  - Set $W_t(i,j) = 0$. (Initially, all the workers are available to all firms at zero wage)
  - At step $t \geq 0$, assume $W_t(i,j)$ has been defined and set

$$\begin{cases} \mathcal{P}_t(i) = \arg\max_{B \subseteq \mathcal{J}} \left\{_{iB} - \sum_{j \in B} W_t(i,j)\right\} & \text{(firms propose)} \\ \mathcal{E}_t(j) = \arg\max_i \left\{W_t(i,j) : i \in \mathcal{P}_t^{-1}(j)\right\} & \text{(workers dispose)} \end{cases}$$

    and update the available offers

$$\begin{aligned} W_{t+1}(i,j) &= \text{next}(W_t(i,j) \text{ if } j \in \mathcal{P}_t(i) \backslash \mathcal{E}_t^{-1}(i) \\ &= W_t(i,j) \end{aligned}$$

    When $W_{t+1} = W_t$, stop.
- In the sequel it will be useful to define u_i^t and v_j^t as the value of the maximization problems above.