‘MATH+ECON+CODE’ MASTERCLASS ON COMPETITIVE EQUILIBRIUM: WALRASIAN EQUILIBRIUM WITH SUBSTITUTES

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Day 2, May 22, 2018: lattices and order
Block 4. Lattice, isotonicity and supermodularity

- Lattices
- Supermodularity and quasi-supermodularity
- Veinott's strong set order
- Topkis' theorem
- Multivocal gross substitutes
- Tarski's fixed point theorem

- Topkis (1976). The structure of sublattice of the product of lattices. *Pacific Journal of Mathematics*.
- Topkis (1978). Minimizing a submodular function on a lattice. *Operations research*.
- Veinott (1989). *Lattice programming*. Lecture notes.
- Milgrom, Shannon (1994). "Monotone comparative statics." *Econometrica*.
- Galichon, Samuelson (2018). Mutivocal gross substitutes. Working paper.

Section 1

Introduction

- Recall the hedonic model with exogenous demand considered in lecture 1. The prices were given by

$$P(l) = \arg\max_p F_\sigma(p, l), \text{ where}$$

$$F_\sigma(p, l) = \sum_{z \in \mathcal{Z}} l_z p_z - \sum_{y \in \mathcal{Y}} m_y \sigma \log\left(1 + \sum_{z \in \mathcal{Z}} \exp\left(\frac{p_z - c_{yz}}{\sigma}\right)\right)$$

  with $\sigma > 0$ measures the unobserved heterogeneity among suppliers. (In lecture 1, $\sigma = 1$).
- We would like to understand what is happening when $\sigma \to 0$. Recall that $\sigma \log\left(e^{a/\sigma} + e^{b/\sigma}\right) \to \max(a, b)$, so the problem becomes

$$\left(W^S\right)^*(l) = \max_p \left\{\sum_{z \in \mathcal{Z}} l_z p_z - W^S(p)\right\}, \text{ where}$$

$$W^S(p) = \sum_{y \in \mathcal{Y}} m_y \max_{z \in \mathcal{Z}} \{p_z - c_{yz}, 0\}$$

  which is neither smooth (because of the max) nor strictly concave (because it is piecewise affine).

- Consider an extreme situation where $c_{yz} = 0$ for all y and z and $p_z = 0$ for all z. Then any supplier is indifferent between any alternative. Depending on how they break ties, this can lead to any vector of demand $l_z \geq 0$ such that $\sum_{z \in \mathcal{Z}} l_z \leq \sum_{y \in \mathcal{Y}} m_y$.
- The equilibrium prices are defined by first order conditions

$$l \in \partial W^S(p)$$

where the (set-valued) demand function is given by the subdifferential of W^S at p, defined by

$$\partial W^S(p) = \left\{ l : W^S(p) + \left(W^S \right)^* (s) = \sum_{z \in \mathcal{Z}} p_z s_z \right\}$$

hence the inverse demand correspondence is given by

$$p = \partial \left(W^S \right)^* (l).$$

- We are led to formulate the set-valued supply as the set of l sucht that $F(p,l) \geq F(p',l)$ for any other p', that

$$s(p) = \left\{ \begin{array}{l} l \in \mathbb{R}_+^{\mathcal{Z}} : \exists \mu_{yz} \geq 0 \ s.t. \\ \left\{ \begin{array}{l} \sum_{z \in \mathcal{Z}} \mu_{yz} \leq m_y \\ \sum_{y \in \mathcal{Y}} \mu_{yz} = l_z \\ \mu_{yz} > 0 \implies z \in \arg\max_{z \in \mathcal{Z}} \{p_z - c_{yz}, 0\} \end{array} \right. \end{array} \right\}$$

or in other words

$$s(p) = \left\{ \begin{array}{l} l \in \mathbb{R}_+^{\mathcal{Z}} : \exists \mu_{yz} \geq 0 \ s.t. \\ \left\{ \begin{array}{l} \sum_{z \in \mathcal{Z}} \mu_{yz} \leq m_y \\ \sum_{y \in \mathcal{Y}} \mu_{yz} = l_z \\ \sum_{yz} \mu_{yz} (p_z - c_{yz}) = \sum_y m_y \max_{z \in \mathcal{Z}} \{p_z - c_{yz}, 0\} \end{array} \right. \end{array} \right\}$$

- This makes quite a bit of sense. A vector of quality distribution $(l_z) \geq 0$ rationalizes market prices p if there is a coupling μ_{yz} of distributions m and l such that if $\mu_{yz} > 0$, then z is the optimal alternative of y.

Section 2

Monotone comparative statics

- In the hedonic models, given exogenous demand l, the equlibrium prices solve

$$P(l) = \arg\max_p F(p, l)$$

where as in lecture 1,

$$F(p, l) = \sum_{z \in \mathcal{Z}} l_z p_z - \sum_{y \in \mathcal{Y}} m_y \sigma \log \left(1 + \sum_{z \in \mathcal{Z}} \exp \left(\frac{p_z - c_{yz}}{\sigma} \right) \right).$$

- In the sequel, we shall give several arguments to show that $P(l)$ is isotone, namely, if $l \leq l'$, then $P(l) \leq P(l')$, where $\leq$ is the componentwise partial order on $\mathbb{R}^{\mathcal{Z}}$.

- BGH's theorem seen in Block 2 applies. Indeed, letting

$$D_z(p) = \sum_{y \in \mathcal{Y}} m_y \frac{\exp\left(\frac{p_z - c_{yz}}{\sigma}\right)}{1 + \sum_{z' \in \mathcal{Z}} \exp\left(\frac{p_{z'} - c_{yz'}}{\sigma}\right)}$$

$$D_0(p) = \sum_{y \in \mathcal{Y}} m_y \frac{1}{1 + \sum_{z' \in \mathcal{Z}} \exp\left(\frac{p_{z'} - c_{yz'}}{\sigma}\right)}$$

  is such that $\sum_{z \in \mathcal{Z}_0} D_z(p) = 0$, $D_z(p)$ is a decrasing function of p_z for $z \neq z'$, and $D_0(p)$ is strictly decreasing in any p_z for $z \in \mathcal{Z}$.
- Therefore $D(p) \leq D(p')$ implies $p \leq p'$.

- One way to understand this is that $P(l)$ is given implicitly by first order conditions
$$\nabla_p F(P(l), l) = 0,$$
and thus
$$DP(l) = -\left(D^2_{pp}F\right)^{-1} D^2_{pl}F$$
- One can show that $l \to P(l)$ by showing that the terms of its Jacobian $DP(l)$ are nonnegative:
  - It is easy to check that $-D^2_{pp}F$ has off-diagonal nonpositive terms (because F is supermodular in p) and that it is symmetric, positive definite (because F is smooth and stricty concave). Thus $-D^2_{pp}F$ is a Stieltjes matrix, which implies that the terms of its inverse are positive.
  - Similarly, the terms of $D^2_{pl}F$ are nonnegative, because this matrix happens to be the identity matrix.
  - As a result, the terms of $DP(l)$ are nonnegative, which implies that $l \to P(l)$ is isotone.

- Note that the first argument assumes that $F\left(p.l\right)$ should be strictly concave in p, so that the argmax, namely $P\left(l\right)$, should be uniquely defined; the second argument assumes that $F\left(p,l\right)$ should be twice differentiable.
- What if neither of these hold? as in the intro, consider in particular the case $\sigma \rightarrow 0$. In this case, recall that

$$F_0\left(p,l\right) = \sum_{z\in\mathcal{Z}} l_z p_z - \sum_{y\in\mathcal{Y}} m_y \max_{z\in\mathcal{Z}} \left\{p_z - c_{yz}, 0\right\}.$$

- Topkis theorem requires simply that $F_0\left(p,l\right)$ should be *supermodular* in p (which implies that the domain of $F_0\left(.,l\right)$ should be a *lattice*), and should satisfy *increasing differences* in $\left(p,l\right)$. As a result, the set valued demand $l \rightarrow \arg\max F_0\left(.,l\right)$ will be increasing in *Veinott's order*. In particular, the set of equilibrium prices $\arg\max F_0\left(.,l\right)$ will be a *lattice*.
- Of course, we haven't defined the terms in italics. We are about to do that.

Section 3

SUPERMODULARITY

- The *partial order* on $\mathbb{R}^n$, also called *componentwise order*, and denoted $\geq$, is the binary relation defined by $x \geq y$ if and only if $x_i \geq y_i$ for all $i \in \{1, ..., n\}$.
- We consider two sets $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ endowed with the componentwise order.
    - A map $f : X \longmapsto Y$ is *isotone* if $x \leq y$ implies $f(x) \leq f(y)$.
    - A map $f : X \longmapsto Y$ is *antitone* if $x \leq y$ implies $f(x) \geq f(y)$.
    - A map $f : X \longmapsto Y$ is *inverse isotone* if $f(x) \leq f(y)$ implies $x \leq y$.

- Let $(L, \leq)$ be a set endowed with partial order. L is a lattice whenever there exist two operations $\vee$ called "join" and $\wedge$ "meet" such that for $x, x' \in L$, $x \vee x'$ and $x \wedge x'$ are elements of L such that $y \leq x$ and $y \leq x'$ implies $y \leq x \wedge x'$, and $y \geq x$ and $y \geq x'$ implies $y \geq x \vee x'$.
- $\mathbb{R}^n$ endowed with the componentwise order is a lattice, with $\left(x \vee x'\right)_i = \max\left(x_i, x_i'\right)$, and $\left(x \wedge x'\right)_i = \min\left(x_i, x_i'\right)$.
- A subset $X \subseteq \mathbb{R}^n$ is a sublattice (of $\mathbb{R}^n$) when $x, x' \in X$ implies $x \wedge x' \in X$ and $x \vee x' \in X$. A sublattice X is complete if $\inf B$ and $\sup B$ exist whenever $B \subseteq X$.

Definition. A function $R_{ij} : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ is called a *rent function* associated to ij if the following four conditions are met: (i) if $i \neq j$, $R_{ij}(x)$ is nonincreasing in x_i, (ii) if $i \neq j$, $R_{ij}(x)$ is nondecreasing in x_j, (iii) $R_{ij}(x)$ does not depend on x_k for $k \notin \{i, j\}$, and (iv) for all $t \in \mathbb{R}$, $R_{ij}(x + t(e_j - e_i)) = R_{ij}(x) + t$.
The following theorem was first published in Topkis (1976).
Theorem. X is a sublattice of $\mathbb{R}^n$ if and only if it there are rent functions R_{ij} where $1 \leq i, j \leq n$, such that

$$X = \bigcap_{1 \leq i,j \leq n} \{x \in \mathbb{R}^n : R_{ij}(x) \leq 0\}. \tag{1}$$

Note that if for $i \in \{1, ..., n\}$, one defines $X_i = \{z \in \mathbb{R} : R_{ii}(x) = -\infty\}$, (recalling that R_{ii} can only take values $-\infty$ and $+\infty$), X_i is a chain and $X \subseteq X_1 \times X_2 \times ... \times X_n$.

The proof given here follows in Veinott's (1989) lecture notes.
Proof. The proof of the "if" statement is easy. Conversely, let X be a lattice and show that it has representation (1). To this end, define the canonical rent function associated to the lattice as

$$R_{ij}^{X}(x) = \inf_{y \in X} \{\max(y_i - x_i, x_j - y_j)\}, \tag{2}$$

which takes value in $\mathbb{R} \cup \{-\infty\}$ for $i \neq j$, while $R_{ii}^{X}(x) = \inf_{y \in X} \{|y_i - x_i|\} \in \mathbb{R}_+$. Clearly, R_{ij} is a rent function. Also, if $x \in X$, then $R_{ij}^{X}(x) \leq 0$, thus

$$X \subseteq \bigcap_{1 \leq i,j \leq n} \{x \in \mathbb{R}^n : R_{ij}(x) \leq 0\}.$$

Conversely, assume $R_{ij}^{X}(x) \leq 0$ for all $1 \leq i, j \leq n$ and show that $x \in X$. By definition, this implies the existence of $y^{ij} \in X$ such that $y_i^{ij} \leq x_i$, $y_j^{ij} \geq x_j$ for $i \neq j$, $y^{ii} = x_i$. Set $y^i = \bigvee_{1 \leq j \leq n} y^{ij} \in X$. One has $y_i^i = x_i$, and $y_j^i \geq x_j$ for $j \neq i$. Therefore, $x = \bigwedge_{1 \leq i \leq n} y^i$, and therefore $x = \bigwedge_{1 \leq i \leq n} \bigvee_{1 \leq j \leq n} y^{ij} \in X$.

Example. Let (c_{ij}) be a $n \times n$ matrix. The set of x such that $x_j - x_i \leq c_{ij}$ for all $i \neq j$ is a lattice called the *dual transportation polyhedron*. More on this tomorrow.

Example. More generally, let A be a $m \times n$ matrix, and $c \in \mathbb{R}^m$. The set

$$X = \{x \in \mathbb{R}^n : Ax \leq c\}$$

is a lattice if and only if every row of A has at most two nonzero elements, and does not have two nonzero elements of the same sign. The matrix A is sometimes called a *node-incidence matrix with gains*.

- Let L be a sublattice of $\mathbb{R}^d$. Then $f : \mathbb{L} \to \mathbb{R}$ is supermodular if $f(x \wedge x') + f(x \vee x') \geq f(x) + f(x')$
- If f is C^2 and $L = \mathbb{R}^d$ is a rectangle, this is equivalently expressed by

$$\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \geq 0 \; \forall i \neq j$$

- f is submodular iff $-f$ is supermodular.

- Consider $f : X \times Y \to \mathbb{R}$ where X and Y are partially ordered sets. Then f has increasing differences in (x, y) iff whenever $x \leq x'$ and $y \leq y'$, then $f(x', y') - f(x, y') \geq f(x', y) - f(x, y)$.
- If $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, this is equivalently expressed by

$$\frac{\partial^2 f}{\partial x_i \partial y_j} \geq 0 \ \forall i \in \{1, ..., n\}, j \in \{1, ..., m\}.$$

- Consider X and X' two subsets of $\mathbb{R}^d$. Then X' dominates X in Veinott's strong set order, denoted $X \leq_v X'$, if $x \in X$ and $x' \in X'$ implies $x \wedge x' \in X$ and $x \vee x' \in X'$.
- In particular, $X \leq_v X$ if and only if X is a sublattice of $\mathbb{R}^d$.

The following theorem first appeared in Topkis (1976).
Theorem. (a) The following statements are equivalent:
(i) X and X' are complete sublattices of $\mathbb{R}^d$ such that $X \leq_v X'$.
(ii) There exist a complete sublattice Y of $\mathbb{R}^d$ such that
$X = Y \cap \{x : x \leq \sup X\}$, and $X' = Y \cap \{x : x \geq \inf X'\}$.
(b) Further, if X and X' are complete sublattices such that $X \leq_v X'$, then
$X \cup X'$ is a complete sublattice.

Section 4

TOPKIS THEOREM

Theorem (Topkis, 1978). Let X be a lattice, T a partially ordered set, and assume $f\left(.,t\right)$ is supermodular for all t and $\left(x,t\right)\rightarrow f\left(x,t\right)$ has increasing differences. Then

$$t\rightarrow \arg\max f\left(.,t\right)$$

is isotone in Veinott's strong set order.

Proof. Let $t \leq t'$ and take $x \in \arg\max f(., t)$ and $x' \in \arg\max f(., t')$. Then $f(x \wedge x', t) \leq f(x, t)$ and $f(x \vee x', t') \leq f(x', t')$. Summing yields

$$f\left(x \wedge x', t\right) + f\left(x \vee x', t'\right) \leq f(x, t) + f\left(x', t'\right)$$

but by supermodularity $f(x, t) + f(x', t) - f(x \vee x', t) \leq f(x \wedge x', t)$, hence

$$f(x, t) + f\left(x', t\right) - f\left(x \vee x', t\right) + f\left(x \vee x', t'\right) \leq f(x, t) + f\left(x', t'\right)$$

but by increasing differences,
$-f(x', t) + f(x', t') \leq -f(x \vee x', t) + f(x \vee x', t')$, hence

$$f(x, t) + f\left(x', t\right) \leq f(x, t) + f\left(x', t'\right).$$

As we have reached an equality, all the intermediates inequalities, and thus

$$f\left(x \wedge x', t\right) \leq f(x, t) \text{ and } f\left(x \vee x', t'\right) \leq f\left(x', t'\right), \text{ } QED.$$

Corollary 1. If $f : \mathbb{R}^d \to \mathbb{R}$ is supermodular, then $\arg\min f$ is a lattice.

Corollary 2. Under the assumptions of Topkis' theorem,

$$t \to \inf \arg\min f\left(., t\right) \text{ and } t \to \sup \arg\min f\left(., t\right)$$

are isotone.

The function given by

$$P(l) = \arg\max_p F_\sigma(p, l), \text{ where}$$

$$F_\sigma(p, l) = \sum_{z \in \mathcal{Z}} l_z p_z - \sum_{y \in \mathcal{Y}} m_y \sigma \log \left(1 + \sum_{z \in \mathcal{Z}} \exp\left(\frac{p_z - c_{yz}}{\sigma} \right) \right)$$

follows the assumptions of Topkis theorem (exercise).

Similarly, the gravity model of trade which is given by

$$\min_p \sum_y m_y p_y - \sum_x n_x p_x + \sum_{xy} \exp\left(p_y - p_x - c_{xy}\right)$$

also satisfies the assumptions of Topkis theorem, upon some changes of signs (exercise).

Section 5

Multivocal gross substitute property

- Let's recapitulate the results that we have seen so far to show inverse isotonicity of demand. Here are the results that can be applied

	$E(p)$ **point-valued**	$E(p)$ **set-valued**
$E(p)$ **is a subdifferential**	Topkis and BGH	Topkis only
$E(p)$ **not a subdifferential**	BGH only	??

- As one sees, when is the subdifferential of a convex function, that is, when we can reformulate the equilibrium problem as an optimization problem, Topkis' theorem applies – both in the point-valued and the set-valued case.
- When $E(p)$ is a map, Berry, Gandhi and Haile's theorem applies–both when the equilibrium can be formulated as an optimization problem and when it cannot.
- When $E(p)$ is a correspondence which is not a subdifferential, we need a new tool. This tool is provided by a recent theorem by a recent theorem by Galichon and Samuelson (2018).

Galichon and Samuelson (2018) define the following property:
Definition. Let E be a partially ordered set and L a lattice. A correspondence: $L \to E$ is said to satisfy the *multivocal gross substitutes (MGS) property* if for any $s \in E\left(p\right)$ and $s' \in E\left(p'\right)$, there exists $s^{\vee} \in E\left(p \vee p'\right)$ and $s^{\wedge} \in E\left(p \wedge p'\right)$ such that for all $z \in \mathcal{Z}_0$

$$\left\{ \begin{array}{l} p_z \leq p'_z \implies s_z \leq s^{\wedge}_z \text{ and } s^{\vee}_z \leq s'_z \\ p_z > p'_z \implies s'_z \leq s^{\wedge}_z \text{ and } s^{\vee}_z \leq s_z \end{array} \right. .$$

Let us show that this is the right generalization of the classical condition. Indeed, consider s and s' such that $s_z > s'_z$ for some z and $p_{z'} = p'_{z'}$ for $z' \neq z$. We have $p \wedge p' = p'$, thus $s^{\wedge} = \sigma\left(p \wedge p'\right) = s'$. Therefore the condition expresses as $p_{z'} \leq p'_{z'}$ for all $z' \neq z$.

The following theorem is due to Galichon and Samuelson (2018).
Theorem. Assume that a correspondence $E : L \rightarrow \mathbb{R}^{\mathcal{Z}_0}$ satisfies MGS and is such that for any $s \in E(p)$, then $\sum_{z \in \mathcal{Z}_0} s_z = 1$. Then $s \rightarrow E^{-1}(s)$ is isotone with respect to Veinott's strong set order, i.e. $s_z \leq s'_z$ for all $z \in \mathcal{Z}$, $s \in E(p)$ and $s' \in E(p')$ imply $s \in E(p \wedge p')$ and $s' \in E(p \vee p')$.

We have by assumption that

$$z \in \mathcal{Z}_0^{\leq} \implies s_z \leq s_z^{\wedge}, \tag{3}$$

and for $z \in \mathcal{Z}_0^{>}$, $s'_z \leq s_z^{\wedge}$, thus as $0 \notin \mathcal{Z}_0^{>}$, we get

$$z \in \mathcal{Z}_0^{>} \implies s_z \leq s'_z \leq s_z^{\wedge}, \tag{4}$$

by summation of (3) and (4), we get

$$1 = \sum_{z \in \mathcal{Z}_0} s_z \leq \sum_{z \in \mathcal{Z}_0} s_z^{\wedge} = 1, \tag{5}$$

hence all the inequalities involved are equalities, and therefore

$$z \in \mathcal{Z}_0^{\leq} \implies s_z = s_z^{\wedge}, \text{ and}$$
$$z \in \mathcal{Z}_0^{>} \implies s_z = s'_z = s_z^{\wedge},$$

thus $s = s^{\wedge}$, that is $s \in E(p \wedge p')$. A similar argument shows that $s' \in E(p \vee p')$.

MGS property of set-valued excess supply

- Consider $S_z(p)$ be the set of excess supply vectors associated with price vector p. One has:

$$S_z(p) = \left\{ (s_z), \exists \mu_{yz} \geq 0 : \left\{ \begin{array}{l} s_z = \sum_{y \in \mathcal{Y}} \mu_{yz} - l_z, \ \sum_{z \in \mathcal{Z}_0} \mu_{yz} = m_y \\ \mu_{yz} > 0 \implies z \in \arg\max_{z \in \mathcal{Z}_0} \{ \mathcal{U}_{yz}(p_z) \} \end{array} \right\} \right\}.$$

- Slightly reformulate as:

$$S_z(p) = \left\{ (s_z), \exists \mu_{yz}, U_y \geq 0 : \left\{ \begin{array}{l} s_z = \sum_{y \in \mathcal{Y}} \mu_{yz} - l_z, \ \sum_{z \in \mathcal{Z}_0} \mu_{yz} = m_y \\ U_y \geq \mathcal{U}_{yz}(p_z), \ \mu_{yz} > 0 \implies U_y = \mathcal{U}_{yz}(p \end{array} \right. \right.$$

- One has clearly $\sum_{z \in \mathcal{Z}_0} S_z(p) = \sum_{z \in \mathcal{Z}_0} l_z - \sum_{y \in \mathcal{Y}} m_y$. Let us show that S satisfies the MGS property.

- Take $s \in E(p)$ and $s' \in E(p')$. Then there exists μ and $\mu' \geq 0$ such that

$$\begin{cases} s_z = \sum_{y \in \mathcal{Y}} \mu_{yz} - l_z, \sum_{z \in \mathcal{Z}_0} \mu_{yz} = m_y \\ U_y \geq \mathcal{U}_{yz}(p_z), \ \mu_{yz} > 0 \implies U_y = \mathcal{U}_{yz}(p_z) \end{cases}$$

$$\begin{cases} s'_z = \sum_{y \in \mathcal{Y}} \mu'_{yz} - l_z, \sum_{z \in \mathcal{Z}_0} \mu'_{yz} = m_y \\ U'_y \geq \mathcal{U}_{yz}(p'_z), \ \mu'_{yz} > 0 \implies U'_y = \mathcal{U}_{yz}(p'_z) \end{cases}$$

- One has $U_y \vee U'_y \geq \mathcal{U}_{yz}(p_z \vee p'_z)$.
- Lemma: $1\left\{U_y > U'_y\right\} \mu_{yz} \leq 1\left\{p_z > p'_z\right\} \mu_{yz}$ and $1\left\{U_y \leq U'_y\right\} \mu'_{yz} \leq 1\left\{p_z \leq p'_z\right\} \mu'_{yz}$. Indeed:
  - $\mu_{yz} > 0$ and $U_y > U'_y$ implies $\mathcal{U}_{yz}(p_z) = U_y > U'_y \geq \mathcal{U}_{yz}(p'_z)$.
  - $\mu'_{yz} > 0$ and $U_y \leq U'_y$ implies $\mathcal{U}_{yz}(p_z) \leq U_y \leq U'_y = \mathcal{U}_{yz}(p'_z)$

- Define $\mu^{\vee}_{yz} = 1\left\{U_y > U'_y\right\}\mu_{yz} + 1\left\{U_y \leq U'_y\right\}\mu'_{yz}$. One has
$$\mu^{\vee}_{yz} > 0 \implies U_y \vee U'_y = \mathcal{U}_{yz}\left(p_z \vee p'_z\right)$$
- One has $\sum_{z\in\mathcal{Z}_0}\mu^{\vee}_{yz} =$ $1\left\{U_y > U'_y\right\}\sum_{z\in\mathcal{Z}_0}\mu_{yz} + 1\left\{U_y \leq U'_y\right\}\sum_{z\in\mathcal{Z}_0}\mu'_{yz} = m_y$.
- Finally, let $s^{\vee}_z = \sum_{y\in\mathcal{Y}}\mu^{\vee}_{yz} - l_z \leq 1\left\{p_z > p'_z\right\}s_z + 1\left\{p_z \leq p'_z\right\}s'_z$. One has therefore
$$\begin{cases} p_z \leq p'_z \implies s_z \leq s^{\wedge}_z \text{ and } s^{\vee}_z \leq s'_z \\ p_z > p'_z \implies s'_z \leq s^{\wedge}_z \text{ and } s^{\vee}_z \leq s_z \end{cases}.$$

Section 6

TARSKI'S FIXED POINT THEOREM

Theorem (Tarski). If L is a complete lattice and and if $F : L \to L$ is isotone, then the set of fixed points of f in L is also a complete lattice.

Kleene's fixed point theorem provide conditions for a constructive argument following which the lattice bounds can be obtained by iterative applications of F.