‘MATH+ECON+CODE’ MASTERCLASS ON COMPETITIVE EQUILIBRIUM: WALRASIAN EQUILIBRIUM WITH SUBSTITUTES

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Day 2, May 22, 2018: lattices and order
Block 4. Lattice, isotonicity and supermodularity

Learning objectives: block 3

- Lattices
- Supermodularity and quasi-supermodularity
- Veinott's strong set order
- Topkis' theorem
- Multivocal gross substitutes
- Tarski's fixed point theorem

- Topkis (1976). The structure of sublattice of the product of lattices. *Pacific Journal of Mathematics*.
- Topkis (1978). Minimizing a submodular function on a lattice. *Operations research*.
- Veinott (1989). *Lattice programming*. Lecture notes.
- Milgrom, Shannon (1994). "Monotone comparative statics." *Econometrica*.
- Galichon, Samuelson (2018). Mutivocal gross substitutes. Working paper.

Section 1

INTRODUCTION

- Recall the hedonic model with exogenous demand considered in lecture 1. The prices were given by

$$P(l) = \arg\max_{p} F_{\sigma}(p, l), \text{ where}$$

$$F_{\sigma}(p, l) = \sum_{z \in \mathcal{Z}} l_z p_z - \sum_{y \in \mathcal{Y}} m_y \sigma \log \left(1 + \sum_{z \in \mathcal{Z}} \exp\left(\frac{p_z - c_{yz}}{\sigma} \right) \right)$$

with $\sigma > 0$ measures the unobserved heterogeneity among suppliers. (In lecture 1, $\sigma = 1$).

- We would like to understand what is happening when $\sigma \to 0$. Recall that $\sigma \log \left(e^{a/\sigma} + e^{b/\sigma} \right) \to \max(a, b)$, so the problem becomes

$$\left(W^S \right)^* (l) = \max_{p} \left\{ \sum_{z \in \mathcal{Z}} l_z p_z - W^S(p) \right\}, \text{ where}$$

$$W^S(p) = \sum_{y \in \mathcal{Y}} m_y \max_{z \in \mathcal{Z}} \{ p_z - c_{yz}, 0 \}$$

which is neither smooth (because of the max) nor strictly concave (because it is piecewise affine).

- Consider an extreme situation where $c_{yz} = 0$ for all y and z and $p_z = 0$ for all z. Then any supplier is indifferent between any alternative. Depending on how they break ties, this can lead to any vector of demand $l_z \geq 0$ such that $\sum_{z \in \mathcal{Z}} l_z \leq \sum_{y \in \mathcal{Y}} m_y$.
- The equilibrium prices are defined by first order conditions

$$l \in \partial W^S(p)$$

where the (set-valued) demand function is given by the subdifferential of W^S at p, defined by

$$\partial W^S(p) = \left\{ l : W^S(p) + \left(W^S\right)^*(s) = \sum_{z \in \mathcal{Z}} p_z s_z \right\}$$

hence the inverse demand correspondence is given by

$$p = \partial \left(W^S\right)^*(l).$$

- We are led to formulate the set-valued supply as the set of l sucht that $F(p,l) \geq F(p',l)$ for any other p', that

$$s(p) = \left\{ \begin{array}{l} l \in \mathbb{R}_+^{\mathcal{Z}} : \exists \mu_{yz} \geq 0 \ s.t. \\ \left\{ \begin{array}{l} \sum_{z \in \mathcal{Z}} \mu_{yz} \leq m_y \\ \sum_{y \in \mathcal{Y}} \mu_{yz} = l_z \\ \mu_{yz} > 0 \implies z \in \arg\max_{z \in \mathcal{Z}} \{p_z - c_{yz}, 0\} \end{array} \right. \end{array} \right\}$$

or in other words

$$s(p) = \left\{ \begin{array}{l} l \in \mathbb{R}_+^{\mathcal{Z}} : \exists \mu_{yz} \geq 0 \ s.t. \\ \left\{ \begin{array}{l} \sum_{z \in \mathcal{Z}} \mu_{yz} \leq m_y \\ \sum_{y \in \mathcal{Y}} \mu_{yz} = l_z \\ \sum_{yz} \mu_{yz} (p_z - c_{yz}) = \sum_y m_y \max_{z \in \mathcal{Z}} \{p_z - c_{yz}, 0\} \end{array} \right. \end{array} \right\}$$

- This makes quite a bit of sense. A vector of quality distribution $(l_z) \geq 0$ rationalizes market prices p if there is a coupling μ_{yz} of distributions m and l such that if $\mu_{yz} > 0$, then z is the optimal alternative of y.

Section 2

Monotone comparative statics

- In the hedonic models, given exogenous demand l, the equlibrium prices solve
$$P(l) = \arg\max_p F(p, l)$$
where as in lecture 1,
$$F(p, l) = \sum_{z \in \mathcal{Z}} l_z p_z - \sum_{y \in \mathcal{Y}} m_y \sigma \log \left(1 + \sum_{z \in \mathcal{Z}} \exp\left(\frac{p_z - c_{yz}}{\sigma} \right) \right).$$
- In the sequel, we shall give several arguments to show that $P(l)$ is isotone, namely, if $l \leq l'$, then $P(l) \leq P(l')$, where $\leq$ is the componentwise partial order on $\mathbb{R}^{\mathcal{Z}}$.

- BGH's theorem seen in Block 2 applies. Indeed, letting

$$D_z(p) = \sum_{y \in \mathcal{Y}} m_y \frac{\exp\left(\frac{p_z - c_{yz}}{\sigma}\right)}{1 + \sum_{z' \in \mathcal{Z}} \exp\left(\frac{p_{z'} - c_{yz'}}{\sigma}\right)}$$

$$D_0(p) = \sum_{y \in \mathcal{Y}} m_y \frac{1}{1 + \sum_{z' \in \mathcal{Z}} \exp\left(\frac{p_{z'} - c_{yz'}}{\sigma}\right)}$$

  is such that $\sum_{z \in \mathcal{Z}_0} D_z(p) = 0$, $D_z(p)$ is a decrasing function of p_z for $z \neq z'$, and $D_0(p)$ is strictly decreasing in any p_z for $z \in \mathcal{Z}$.
- Therefore $D(p) \leq D(p')$ implies $p \leq p'$.

- One way to understand this is that $P(l)$ is given implicitly by first order conditions
$$\nabla_p F(P(l), l) = 0,$$
and thus
$$DP(l) = -\left(D^2_{pp}F\right)^{-1} D^2_{pl}F$$
- One can show that $l \to P(l)$ by showing that the terms of its Jacobian $DP(l)$ are nonnegative:
    - It is easy to check that $-D^2_{pp}F$ has off-diagonal nonpositive terms (because F is supermodular in p) and that it is symmetric, positive definite (because F is smooth and strricty concave). Thus $-D^2_{pp}F$ is a Stieltjes matrix, which implies that the terms of its inverse are positive.
    - Similarly, the terms of $D^2_{pl}F$ are nonnegative, because this matrix happens to be the identity matrix.
    - As a result, the terms of $DP(l)$ are nonnegative, which implies that $l \to P(l)$ is isotone.

- Note that the first argument assumes that $F\left(p.l\right)$ should be strictly concave in p, so that the argmax, namely $P\left(l\right)$, should be uniquely defined; the second argument assumes that $F\left(p,l\right)$ should be twice differentiable.
- What if neither of these hold? as in the intro, consider in particular the case $\sigma \rightarrow 0$. In this case, recall that

$$F_0\left(p,l\right) = \sum_{z\in\mathcal{Z}} l_z p_z - \sum_{y\in\mathcal{Y}} m_y \max_{z\in\mathcal{Z}} \left\{p_z - c_{yz}, 0\right\}.$$

- Topkis theorem requires simply that $F_0\left(p,l\right)$ should be *supermodular* in p (which implies that the domain of $F_0\left(.,l\right)$ should be a *lattice*), and should satisfy *increasing differences* in $\left(p,l\right)$. As a result, the set valued demand $l \rightarrow \arg\max F_0\left(.,l\right)$ will be increasing in *Veinott's order*. In particular, the set of equilibrium prices $\arg\max F_0\left(.,l\right)$ will be a *lattice*.
- Of course, we haven't defined the terms in italics. We are about to do that.

Section 3

SUPERMODULARITY

- The *partial order* on $\mathbb{R}^n$, also called *componentwise order*, and denoted $\geq$, is the binary relation defined by $x \geq y$ if and only if $x_i \geq y_i$ for all $i \in \{1, ..., n\}$.
- We consider two sets $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ endowed with the componentwise order.
  - A map $f : X \longmapsto Y$ is *isotone* if $x \leq y$ implies $f(x) \leq f(y)$.
  - A map $f : X \longmapsto Y$ is *antitone* if $x \leq y$ implies $f(x) \geq f(y)$.
  - A map $f : X \longmapsto Y$ is *inverse isotone* if $f(x) \leq f(y)$ implies $x \leq y$.

- Let $(L, \leq)$ be a set endowed with partial order. L is a lattice whenever there exist two operations $\vee$ called "join" and $\wedge$ "meet" such that for $x, x' \in L$, $x \vee x'$ and $x \wedge x'$ are elements of L such that $y \leq x$ and $y \leq x'$ implies $y \leq x \wedge x'$, and $y \geq x$ and $y \geq x'$ implies $y \geq x \vee x'$.
- $\mathbb{R}^n$ endowed with the componentwise order is a lattice, with $(x \vee x')_i = \max(x_i, x_i')$, and $(x \wedge x')_i = \min(x_i, x_i')$.
- A subset $X \subseteq \mathbb{R}^n$ is a sublattice (of $\mathbb{R}^n$) when $x, x' \in X$ implies $x \wedge x' \in X$ and $x \vee x' \in X$. A sublattice X is complete if $\inf B$ and $\sup B$ exist whenever $B \subseteq X$.

Definition. A function $R_{ij} : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ is called a *rent function* associated to ij if the following four conditions are met: (i) if $i \neq j$, $R_{ij}(x)$ is nonincreasing in x_i, (ii) if $i \neq j$, $R_{ij}(x)$ is nondecreasing in x_j, (iii) $R_{ij}(x)$ does not depend on x_k for $k \notin \{i, j\}$, and (iv) for all $t \in \mathbb{R}$,
$R_{ij}(x + t(e_j - e_i)) = R_{ij}(x) + t$.
The following theorem was first published in Topkis (1976).
Theorem. X is a sublattice of $\mathbb{R}^n$ if and only if it there are rent functions R_{ij} where $1 \leq i, j \leq n$, such that

$$X = \bigcap_{1 \leq i,j \leq n} \{x \in \mathbb{R}^n : R_{ij}(x) \leq 0\}. \tag{1}$$

Note that if for $i \in \{1, ..., n\}$, one defines $X_i = \{z \in \mathbb{R} : R_{ii}(x) = -\infty\}$, (recalling that R_{ii} can only take values $-\infty$ and $+\infty$), X_i is a chain and $X \subseteq X_1 \times X_2 \times ... \times X_n$.

The proof given here follows in Veinott's (1989) lecture notes.
Proof. The proof of the "if" statement is easy. Conversely, let X be a lattice and show that it has representation (1). To this end, define the canonical rent function associated to the lattice as

$$R_{ij}^{X}(x) = \inf_{y \in X} \{\max(y_i - x_i, x_j - y_j)\}, \quad (2)$$

which takes value in $\mathbb{R} \cup \{-\infty\}$ for $i \neq j$, while $R_{ii}^{X}(x) = \inf_{y \in X} \{|y_i - x_i|\} \in \mathbb{R}_+$. Clearly, R_{ij} is a rent function. Also, if $x \in X$, then $R_{ij}^{X}(x) \leq 0$, thus

$$X \subseteq \bigcap_{1 \leq i,j \leq n} \{x \in \mathbb{R}^n : R_{ij}(x) \leq 0\}.$$

Conversely, assume $R_{ij}^{X}(x) \leq 0$ for all $1 \leq i, j \leq n$ and show that $x \in X$. By definition, this implies the existence of $y^{ij} \in X$ such that $y_i^{ij} \leq x_i$, $y_j^{ij} \geq x_j$ for $i \neq j$, $y^{ii} = x_i$. Set $y^i = \bigvee_{1 \leq j \leq n} y^{ij} \in X$. One has $y_i^i = x_i$, and $y_j^i \geq x_j$ for $j \neq i$. Therefore, $x = \bigwedge_{1 \leq i \leq n} y^i$, and therefore $x = \bigwedge_{1 \leq i \leq n} \bigvee_{1 \leq j \leq n} y^{ij} \in X$.

Example. Let (c_{ij}) be a $n \times n$ matrix. The set of x such that $x_j - x_i \leq c_{ij}$ for all $i \neq j$ is a lattice called the *dual transportation polyhedron*. More on this tomorrow.

Example. More generally, let A be a $m \times n$ matrix, and $c \in \mathbb{R}^m$. The set

$$X = \{x \in \mathbb{R}^n : Ax \leq c\}$$

is a lattice if and only if every row of A has at most two nonzero elements, and does not have two nonzero elements of the same sign. The matrix A is sometimes called a *node-incidence matrix with gains*.

- Let L be a sublattice of $\mathbb{R}^d$. Then $f : \mathbb{L} \to \mathbb{R}$ is supermodular if $f(x \wedge x') + f(x \vee x') \geq f(x) + f(x')$
- If f is C^2 and $L = \mathbb{R}^d$ is a rectangle, this is equivalently expressed by

$$\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \geq 0 \ \forall i \neq j$$

- f is submodular iff $-f$ is supermodular.

- Consider $f : X \times Y \to \mathbb{R}$ where X and Y are partially ordered sets. Then f has increasing differences in (x, y) iff whenever $x \leq x'$ and $y \leq y'$, then $f(x', y') - f(x, y') \geq f(x', y) - f(x, y)$.
- If $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, this is equivalently expressed by

$$\frac{\partial^2 f}{\partial x_i \partial y_j} \geq 0 \ \forall i \in \{1, ..., n\}, j \in \{1, ..., m\}.$$

- Consider X and X' two subsets of $\mathbb{R}^d$. Then X' dominates X in Veinott's strong set order, denoted $X \leq_v X'$, if $x \in X$ and $x' \in X'$ implies $x \wedge x' \in X$ and $x \vee x' \in X'$.
- In particular, $X \leq_v X$ if and only if X is a sublattice of $\mathbb{R}^d$.

The following theorem first appeared in Topkis (1976).

Theorem. (a) The following statements are equivalent:

(i) X and X' are complete sublattices of $\mathbb{R}^d$ such that $X \leq_v X'$.

(ii) There exist a complete sublattice Y of $\mathbb{R}^d$ such that $X = Y \cap \{x : x \leq \sup X\}$, and $X' = Y \cap \{x : x \geq \inf X'\}$.

(b) Further, if X and X' are complete sublattices such that $X \leq_v X'$, then $X \cup X'$ is a complete sublattice.

Section 4

TOPKIS THEOREM

Theorem (Topkis, 1978). Let X be a lattice, T a partially ordered set, and assume $f(., t)$ is supermodular for all t and $(x, t) \to f(x, t)$ has increasing differences. Then

$$t \to \arg\max f(., t)$$

is isotone in Veinott's strong set order.

Section 5

Topkis theorem

Proof. Let $t \leq t'$ and take $x \in \arg\max f(., t)$ and $x' \in \arg\max f(., t')$. Then $f(x \wedge x', t) \leq f(x, t)$ and $f(x \vee x', t') \leq f(x', t')$. Summing yields

$$f\left(x \wedge x', t\right) + f\left(x \vee x', t'\right) \leq f(x, t) + f\left(x', t'\right)$$

but by supermodularity $f(x, t) + f(x', t) - f(x \vee x', t) \leq f(x \wedge x', t)$, hence

$$f(x, t) + f\left(x', t\right) - f\left(x \vee x', t\right) + f\left(x \vee x', t'\right) \leq f(x, t) + f\left(x', t'\right)$$

but by increasing differences,
$-f(x', t) + f(x', t') \leq -f(x \vee x', t) + f(x \vee x', t')$, hence

$$f(x, t) + f\left(x', t\right) \leq f(x, t) + f\left(x', t'\right).$$

As we have reached an equality, all the intermediates inequalities, and thus

$$f\left(x \wedge x', t\right) \leq f(x, t) \text{ and } f\left(x \vee x', t'\right) \leq f\left(x', t'\right), \; QED.$$

Corollary 1. If $f : \mathbb{R}^d \to \mathbb{R}$ is supermodular, then $\arg\min f$ is a lattice.

Corollary 2. Under the assumptions of Topkis' theorem,

$$t \to \inf \arg\min f\left(., t\right) \text{ and } t \to \sup \arg\min f\left(., t\right)$$

are isotone.

The function given by

$$P(l) = \arg\max_{p} F_{\sigma}(p, l)\text{, where}$$

$$F_{\sigma}(p, l) = \sum_{z \in \mathcal{Z}} l_z p_z - \sum_{y \in \mathcal{Y}} m_y \sigma \log\left(1 + \sum_{z \in \mathcal{Z}} \exp\left(\frac{p_z - c_{yz}}{\sigma}\right)\right)$$

follows the assumptions of Topkis theorem (exercise).

Similarly, the gravity model of trade which is given by

$$\min_{p} \sum_{y} m_y p_y - \sum_{x} n_x p_x + \sum_{xy} \exp\left(p_y - p_x - c_{xy}\right)$$

also satisfies the assumptions of Topkis theorem, upon some changes of signs (exercise).

Section 6

MULTIVOCAL GROSS SUBSTITUTE PROPERTY

Galichon and Samuelson (2018) define the following property:
Definition. Let E be a partially ordered set and L a lattice. A correspondence: $L \to E$ is said to satisfy the *multivocal gross substitutes (MGS) property* if for any $s \in E(p)$ and $s' \in E(p')$, there exists $s^{\vee} \in E(p \vee p')$ and $s^{\wedge} \in E(p \wedge p')$ such that for all $z \in \mathcal{Z}_0$

$$\begin{cases} p_z \leq p'_z \implies s_z \leq s^{\wedge}_z \text{ and } s^{\vee}_z \leq s'_z \\ p_z > p'_z \implies s'_z \leq s^{\wedge}_z \text{ and } s^{\vee}_z \leq s_z \end{cases}.$$

Let us show that this is the right generalization of the classical condition. Indeed, consider s and s' such that $s_z > s'_z$ for some z and $p_{z'} = p'_{z'}$ for $z' \neq z$. We have $p \wedge p' = p'$, thus $s^{\wedge} = \sigma(p \wedge p') = s'$. Therefore the condition expresses as $p_{z'} \leq p'_{z'}$ for all $z' \neq z$.

The following theorem is due to Galichon and Samuelson (2018).
Theorem. Assume that a correspondence $E : L \to \mathbb{R}^{\mathcal{Z}_0}$ satisfies MGS and is such that for any $s \in E(p)$, then $\sum_{z \in \mathcal{Z}_0} s_z = 1$. Then $s \to E^{-1}(s)$ is isotone with respect to Veinott's strong set order, i.e. $s_z \leq s'_z$ for all $z \in \mathcal{Z}$, $s \in E(p)$ and $s' \in E(p')$ imply $s \in E(U \wedge U')$ and $s' \in E(U \vee U')$.

We have by assumption that

$$j \in \mathcal{J}_0^{\leq} \implies s_j \leq s_j^{\wedge}, \tag{3}$$

thus by summation

$$-\sum_{j \in \mathcal{J}_0^{\leq}} s_j \geq -\sum_{j \in \mathcal{J}_0^{\leq}} s_j^{\wedge}. \tag{4}$$

Similarly, we have by assumption that

$$j \in \mathcal{J}_0^{>} \implies s_j' \leq s_j^{\wedge}, \tag{5}$$

hence we get by summation that

$$-\sum_{j \in \mathcal{J}_0^{>}} s_j' \geq -\sum_{j \in \mathcal{J}_0^{>}} s_j^{\wedge}. \tag{6}$$

Therefore, one has

$$\sum_{j\in\mathcal{J}_0^{>}} s'_j \geq \sum_{j\in\mathcal{J}_0^{>}} s_j = 1 - \sum_{j\in\mathcal{J}_0^{\leq}} s_j \geq 1 - \sum_{j\in\mathcal{J}_0^{\leq}} s_j^{\wedge} = \sum_{j\in\mathcal{J}_0^{>}} s_j^{\wedge}$$

$$\sum_{j\in\mathcal{J}_0^{\leq}} s'_j = 1 - \sum_{j\in\mathcal{J}_0^{>}} s'_j \geq 1 - \sum_{j\in\mathcal{J}_0^{>}} s_j^{\wedge} = \sum_{j\in\mathcal{J}_0^{\leq}} s_j^{\wedge}$$

thus, by summation, of the latter two lines

$$1 = \sum_{j\in\mathcal{J}_0} s'_j \geq \sum_{j\in\mathcal{J}_0} s_j = 1$$

hence all inequalities which have been summed are equalities. As a result, we get that:
(a) $j \in \mathcal{J}_0^{>}$ implies $s'_j = s_j$
(b) Inequality (4) holds as an equality, which implies that all the elementary inequalities (3) hold as equalities, that is $j \in \mathcal{J}_0^{\leq}$ implies $s_j = s_j^{\wedge}$.
(c) Inequality (6) holds as an equality, which implies that all the elementary inequalities (5) hold as equalities, that is $j \in \mathcal{J}_0^{>}$ implies $s'_j = s_j^{\wedge}$.

Combining (a) and (c) yields $j \in \mathcal{J}_0^{>}$ implies $s_j = s_j^{\wedge}$
Combining further with (b) yields $s_j = s_j^{\wedge}$ for all $j \in \mathcal{J}_0$.
A similar argument shows that $s' = s^{\vee}$, as follows. We have by assumption that

$$j \in \mathcal{J}_0^{\leq} \implies s'_j \geq s_j^{\vee}, \tag{7}$$

thus by summation

$$\sum_{j \in \mathcal{J}_0^{\leq}} s'_j \geq \sum_{j \in \mathcal{J}_0^{\leq}} s_j^{\vee}. \tag{8}$$

Similarly, we have by assumption that

$$j \in \mathcal{J}_0^{>} \implies s_j \geq s_j^{\vee}, \tag{9}$$

hence we get by summation that (with the first inequality holding by construction)

$$\sum_{j \in \mathcal{J}_0^{>}} s'_j \geq \sum_{j \in \mathcal{J}_0^{>}} s_j \geq \sum_{j \in \mathcal{J}_0^{>}} s_j^{\vee}. \tag{10}$$

Therefore, adding these two strings, one has

$$1 = \sum_{j \in \mathcal{J}_0} s'_j \geq \sum_{j \in \mathcal{J}_0} s_j^{\vee} = 1$$

and hence all inequalities which have been summed are equalities. We can then combine (9) and the middle equality in (10) to conclude that for all $j \in \mathcal{J}_0^{>}$, we have $s_j = s_j^{\vee}$, and then use observation a from the previous step to conclude that for all $j \in \mathcal{J}_0^{>}$, we have $s'_j = s_j^{\vee}$. Next, (7) and the equality in (8) ensure that for all $j \in \mathcal{J}_0^{\leq}$, we have $s'_j = s_j^{\vee}$, giving the result.

Section 7

Tarski's fixed point theorem

Theorem (Tarski). If L is a complete lattice and and if $F : L \to L$ is isotone, then the set of fixed points of f in L is also a complete lattice.

Kleene's fixed point theorem provide conditions for a constructive argument following which the lattice bounds can be obtained by iterative applications of F.