'MATH+ECON+CODE' MASTERCLASS ON COMPETITIVE EQUILIBRIUM: WALRASIAN EQUILIBRIUM WITH SUBSTITUTES

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Day 4, May 24 2018: matching with general transfers (2)
Block 10. Matching function equilibria

- Matching functions and matching function equilibria
- Iterated projection algorithm
- Applications:
  - Gravity equation
  - Shimer and Smith model of matching with search frictions
  - BLP's random coefficient logit model
  - Choo and Siow's model
  - Dagsvik-Menzel's model

References for block 10

- [BLP] Berry, Levinsohn, Pakes (1995). Automobile Prices in Market Equilibrium Steven Berry, *Econometrica*.
- [D] Dagsvik (2000). Aggregation in matching markets. *International Economic Review*.
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- [M] Menzel (2015) Large matching market as two-sided demand systems. *Econometrica*.
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Section 1

MATCHING FUNCTION EQUILIBRIA: MOTIVATIONS AND DEFINITION

- A matching function is a function $M_{xy}\left(\mu_{x0}, \mu_{0y}\right)$ which is isotone and such that
$$\mu_{xy} = M_{xy}\left(\mu_{x0}, \mu_{0y}\right).$$
- A matching function equilibrium (MFE) is the set of equations
$$\begin{cases} n_x = \mu_{x0} + \sum_{y \in \mathcal{Y}} M_{xy}\left(\mu_{x0}, \mu_{0y}\right) \\ m_y = \mu_{0y} + \sum_{x \in \mathcal{X}} M_{xy}\left(\mu_{x0}, \mu_{0y}\right) \end{cases}$$
- Today, we will first provide examples of models that reformulate as MFEs; then we will discuss existence, computation, and uniqueness of a MFE, and then we'll discuss comparative statics.

- This model appeared in [CS]. Consider a labor market where $\mathcal{X}$ are the types of the workers and $\mathcal{Y}$ are the types of the firms. There are n_x workers of type x, and m_y firms of type y.
- Let w_{xy} be the equilibrium salary of a worker of type x working for a firm y. Assume that worker $x \in \mathcal{X}$ has utility for matching with firm of type y equal to
$$\alpha_{xy} + w_{xy} + \varepsilon_y$$
and ε_0 if remains unemployed, where the random utility vector ε is a vector of i.i.d. Gumbel distributions drawn by each worker, and α is a term that captures job amenity.
- Similarly, the profit of the firm is
$$\gamma_{xy} - w_{xy} + \eta_y$$
and η_0 if it does not hire, where the random utility vector η is a vector of i.i.d. Gumbel distributions drawn by each worker, and γ is a term that captures job productivity.

- The quantities (α, γ, n, m) as well as the distributions of ε and η are exogenous: they are primitives of the model. The equilibrium quantities are the matching patterns (μ_{xy}), as well as the equilibrium wages w_{xy}.
- The conditional choice probabilities (CCPs) on the side of workers is $\mu_{y|x} = \mu_{xy} / n_x$, and the CCPs on the side of firms is $\mu_{x|y} = \mu_{xy} / m_y$. Because we are in a logit model, we have by the log-odds ratio formula that
$$\alpha_{xy} + w_{xy} = \ln \frac{\mu_{xy}}{\mu_{x0}} \text{ and } \gamma_{xy} - w_{xy} = \ln \frac{\mu_{xy}}{\mu_{0y}},$$
so by summation, $\alpha_{xy} + \gamma_{xy} = 2 \ln \mu_{xy} - \ln \mu_{x0} - \ln \mu_{0y}$, thus $\mu_{xy} = \sqrt{\mu_{x0}\mu_{0y}} K_{xy}$, where $K_{xy} = \exp\left(\frac{\alpha_{xy}+\gamma_{xy}}{2}\right)$, that is
$$M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = \sqrt{\mu_{x0}\mu_{0y}} K_{xy}. \tag{1}$$
- The Choo-Siow model and more general models of matching with random heterogeneity in preferences will be studied in Lecture 11.

- We consider a dynamic model of matching with search frictions where a pair of employers and employees drawn from the population of unmatched agents decide whether to match or not.
- As before, the number of xy pairs is μ_{xy}, the number of unassigned workers of type x is μ_{x0}, and the number of unassigned firms of type y is μ_{0y}.
- The continuation value of an unassigned worker of type x (resp. firm of type y) is U_{x0} (resp. V_{0y}). If x and y match, then x gets $U_{xy} + \varepsilon_y$, and y gets $V_{xy} + \eta_x$. Note that U and V are endogenous.
- Let $a_{xy} = \Pr\left(U_{xy} + \varepsilon_y \geq U_{x0} \text{ and } V_{xy} + \eta_x \geq V_{0y}\right)$ be the probability that if x and y meet, they decide to match.
  - In [ShSm]'s original model, $\varepsilon = 0$ and $\eta = 0$, so that $a_{xy} = 1\left\{U_{xy} + \varepsilon_y \geq U_{x0} \text{ and } V_{xy} + \eta_x \geq V_{0y}\right\}$.
  - In general, a_{xy} is an increasing function of both $U_{xy} - U_{x0}$ and $V_{xy} - V_{0y}$. See e.g. [GJR].

- There is an an exogenous destruction of xy matches with intensity δ, so that the flow of dissolutions of xy matches is

$$\delta\mu_{xy}.$$

- The flow of new xy matches created is equal to

$$\lambda\mu_{x0}\mu_{0y}a_{xy}$$

- At steady state, the flow of matched created is equal to the flow of matches destructed, thus

$$\delta\mu_{xy} = \lambda\mu_{x0}\mu_{0y}a_{xy}$$

which leads to matching function

$$M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = \mu_{x0}\mu_{0y}K_{xy},$$

where $K_{xy} = \left(\lambda/\delta\right) a_{xy}$.

- The previous analysis is incomplete as U_{xy}, U_{x0}, V_{xy} and V_{0y} are endogenous. To close the model, one shall need:
  - feasibility: relate U_{xy} and V_{xy}, typically by $U_{xy} + V_{xy} = \Phi_{xy}$
  - bargaining: e.g. if fair (Nash), $U_{xy} - U_{x0} = V_{xy} - V_{0y}$
  - Bellman equations relating U_{x0} and U_{xy}; for instance

$$U_{x0} = \rho \sum_y a_{xy} \left(U_{xy} - U_{x0} \right) \mu_{0y}$$

$$V_{0y} = \rho \sum_x a_{xy} \left(V_{xy} - V_{0y} \right) \mu_{x0}$$

- Full equilibrium therefore involves $\mu_{x0}, \mu_{0y}, \mu_{xy}, U_{x0}, U_{xy}, V_{0y}, V_{xy}$ as unknowns, and an equal number of equations.

- Dagsvik-Menzel's model ([D], [M]) is a model of matching without transfers and with logit heterogeneity. If worker i of type $x \in \mathcal{X}$ matches with firm j of type $y \in \mathcal{X}$, then worker and firm get respectively
$$\alpha_{xy} + \varepsilon_{ij} \text{ and } \gamma_{xy} + \eta_{ij}$$
where ε_{ij} and η_{ij} are iid Gumbel. If unassigned, get $\alpha_{x0} + \varepsilon_{i0}$ and $\gamma_{0y} + \eta_{0j}$.
- Wages are decided exogenously: there are no possible equilibrium adjustments of wage. Hence, worker i chooses preferred firm among firms who find her acceptable. Let u_i (resp. v_j) be the payoffs of worker i (resp. firm j) at equilibrium. One has
$$u_i = \max\left(\max_j \left\{\alpha_{x_i y_j} + \varepsilon_{ij} : v_j \leq \gamma_{x_i y_j} + \eta_{ij}\right\}, \alpha_{x0} + \varepsilon_{i0}\right)$$
$$v_j = \max\left(\max_i \left\{\gamma_{x_i y_j} + \eta_{ij} : u_i \leq \alpha_{x_i y_j} + \varepsilon_{ij}\right\}, \gamma_{0y} + \eta_{0j}\right).$$

- Because of the Gumbel assumption, if i is of type x,

$$\left\{\begin{array}{l} u_i = e^{\alpha_{x0}} + \sum_j e^{\alpha_{xy_j}} 1\left\{v_j \leq \gamma_{xy_j} + \eta_{ij}\right\} + \hat{\varepsilon}_i \\ v_j = e^{\gamma_{0y}} + \sum_i e^{\gamma_{x_i y}} 1\left\{u_i \leq \alpha_{x_i y} + \varepsilon_{ij}\right\} + \hat{\eta}_j \end{array}\right.$$

where ϵ_i and η_j are standard Gumbel.

- Hence $u_i = u_x + \hat{\varepsilon}_i$ and $v_j = v_y + \hat{\eta}_j$, where

$$\left\{\begin{array}{l} u_x = \log\left(e^{\alpha_{x0}} + \sum_y e^{\alpha_{xy}} m_y \Pr\left(v_y + \hat{\eta}_j \leq \gamma_{xy} + \eta_{ij}\right)\right) \\ v_y = \log\left(e^{\gamma_{0y}} + \sum_x e^{\gamma_{xy}} n_x \Pr\left(u_x + \hat{\varepsilon}_i \leq \alpha_{xy} + \varepsilon_{ij}\right)\right). \end{array}\right.$$

- Because $\hat{\eta}_j$ and η_{ij} are Gumbel and (nearly) independent, in the large population limit,

$$\Pr\left(v_y + \hat{\eta}_j \leq \gamma_{xy} + \eta_{ij}\right) = \frac{e^{\gamma_{xy}}}{e^{v_y} + e^{\gamma_{xy}}} \simeq e^{\gamma_{xy} - v_y}$$

$$\Pr\left(u_x + \hat{\varepsilon}_i \leq \alpha_{xy} + \varepsilon_{ij}\right) = \frac{e^{\alpha_{xy}}}{e^{u_x} + e^{\alpha_{xy}}} \simeq e^{\alpha_{xy} - u_x}$$

- Hence,

$$\begin{cases} e^{u_x} = e^{\alpha_{x0}} + \sum_y m_y e^{\alpha_{xy} + \gamma_{xy} - v_y} \\ e^{v_y} = e^{\gamma_{0y}} + \sum_x n_x e^{\alpha_{xy} + \gamma_{xy} - u_x} \end{cases}$$

- Thus

$$\begin{cases} n_x = n_x e^{\alpha_{x0} - u_x} + \sum_y n_x m_y e^{\alpha_{xy} + \gamma_{xy} - u_x - v_y} \\ m_y = m_y e^{\gamma_{0y} - v_y} + \sum_x n_x m_y e^{\alpha_{xy} + \gamma_{xy} - u_x - v_y} \end{cases}$$

- Setting $\mu_{x0} = n_x e^{\alpha_{x0} - u_x}$ and $\mu_{0y} = m_y e^{\gamma_{0y} - v_y}$ thus yields $\mu_{xy} = M_{xy}(\mu_{x0}, \mu_{0y})$, where

$$M_{xy}(\mu_{x0}, \mu_{0y}) = \mu_{x0}\mu_{0y}K_{xy}$$

  with $K_{xy} = \exp(\alpha_{xy} + \gamma_{xy})$.
- Note the resemblances and the differences with Choo-Siow model:
  - In both models, $\alpha + \gamma$ is a primitive of the model.
  - In the Choo-Siow model, M is positive homogenous of degree one (constant returns to scale), while in the Dagsvik-Menzel model, is positive homogenous of degree 2 (increasing returns to scale). Why?

- In a number of models of international trade, one assume that the flow of goods from country x to country y obeys a distribution

$$\mu_{xy} = A_x B_y e^{\Phi_{xy}}$$

where Φ_{xy} is a measure of the proximity between country x and country y, and A_x and B_y are coefficient that measure the economic importance of the countries.

- In 'naive gravity equations', A_x and B_y are the GDP, or power of the GDP.
- Since Anderson and van Wincoop (2003), A_x and B_y are adjusted by

$$\sum_y \mu_{xy} = n_x \text{ and } \sum_x \mu_{xy} = m_y$$

and thus we are in the framework of a MFE.

- In BLP's random coefficient logit model, a consumer of type x has random utility $\delta_y + \nu_x^\intercal \xi_y + T\eta_y$ for a yoghurt y, where δ_y is the systematic utility associated with yoghurt y (to be identified), ξ_y is the vector of characteristics of yoghurt, ν_x is the vector of valuation of the characteristics by the consumer, and (η_y) is a vector of iid Gumbel preference shocks, which is independent from ν_x.
- Let $X \sim P$ be the random types, and m_y be the market share of yoghurt y. One has

$$\mathbb{E}_P \left[\Pr \left(\delta_y + \nu_X^\intercal \xi_y + T\eta_y \geq \delta_{y'} + \nu_X^\intercal \xi_{y'} + T\eta_{y'} \forall y' \in \mathcal{Y} \right) \right] = m_y.$$

- By independence between ν and η, reformulate this as

$$\mathbb{E}_P \left[\frac{\exp\left(\delta_y + \nu_X^\intercal \xi_y \right)}{\sum_{y' \in \mathcal{Y}} \exp\left(\delta_{y'} + \nu_X^\intercal \xi_{y'} \right)} \right] = m_y.$$

- Now sample X and let $x \in \mathcal{X}$ be the sampled points – a sample of size N. One has

$$\sum_{x \in \mathcal{X}} \frac{1}{N} \frac{\exp\left(\delta_y + \nu_x^\intercal \xi_y\right)}{\sum_{y' \in \mathcal{Y}} \exp\left(\delta_{y'} + \nu_x^\intercal \xi_{y'}\right)} = m_y$$

- Denoting $\mu_{0y} = \exp\left(\delta_y\right)/N$ and $\mu_{x0} = 1/\left(\sum_{y' \in \mathcal{Y}} \exp\left(\delta_{y'} + \nu_x^\intercal \xi_{y'}\right)\right)$ and $M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = \mu_{x0}\mu_{0y} \exp\left(\nu_x^\intercal \xi_y\right)$, one has $\sum_{x \in \mathcal{X}} M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = n_x$ where $n_x := 1$, and thus

$$\begin{cases} \sum_{x \in \mathcal{Y}} M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = n_x \\ \sum_{x \in \mathcal{X}} M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = m_y \end{cases}.$$

- A *matching function equilibrium* is a solution of the following system with unknowns μ_{x0} and μ_{0y}:

$$\begin{cases} \mu_{x0} + \sum_{y \in \mathcal{Y}} M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = n_x \\ \mu_{0y} + \sum_{x \in \mathcal{X}} M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = m_y \end{cases} . \tag{2}$$

- In the sequel we will consider the following questions:
  - Existence of an equilibrium
  - Algorithms for the determination of an equilibrium
  - Uniqueness of an equilibrium

Section 2

Existence of equilibrium

We will assume the following about the matching functions:

ASSUMPTION

M is such that for every $x \in \mathcal{X}$, $y \in \mathcal{Y}$:
(i) Map $M_{xy} : (a, b) \mapsto M_{xy}(a, b)$ is continuous.
(ii) Map $M_{xy} : (a, b) \mapsto M_{xy}(a, b)$ is weakly isotone, i.e. if $a \leq a'$ and $b \leq b'$, then $M_{xy}(a, b) \leq M_{xy}(a', b')$.
(iii) For each $a > 0$, $\lim_{b \to 0^+} M_{xy}(a, b) = 0$, and for each $b > 0$, $\lim_{a \to 0^+} M_{xy}(a, b) = 0$.

Algorithm. Step 0. Fix the initial value of μ_{0y} at $\mu_{0y}^{0} = m_y$.
Step $2t+1$. Keep the values μ_{0y}^{2t} fixed. For each $x \in \mathcal{X}$, solve for the value, μ_{x0}^{2t+1}, of μ_{x0} so that

$$\sum_{y \in \mathcal{Y}} M_{xy}(\mu_{x0}, \mu_{0y}^{2t}) + \mu_{x0} = n_x.$$

Step $2t+2$. Keep the values μ_{x0}^{2t+1} fixed. For each $y \in \mathcal{Y}$, solve for which is the value, μ_{0y}^{2t+2}, of μ_{0y} so that

$$\sum_{x \in \mathcal{X}} M_{xy}(\mu_{x0}^{2t+1}, \mu_{0y}) + \mu_{0y} = m_y.$$

Iterate until the updates are below some threshold.

The following theorem from [CCGW], building on [GKW], ensures that the algorithm converges to a matching function equilibrium.

THEOREM

Under Assumptions (i)–(iii) above, there exists a matching function equilibrium which is the limit of $\left(\mu_{x0}^{2t+1}, \mu_{0y}^{2t+2}\right)$ *defined in the algorithm.*

- We show that the construction of μ_{x0}^{2t+1} and μ_{0y}^{2t+2} at each step is well defined. Consider step $2t+1$. For each $x \in \mathcal{X}$, the equation to solve is
$$\sum_{y \in \mathcal{Y}} M_{xy}(\mu_{x0}, \mu_{0y}) + \mu_{x0} = n_x$$
but the right-hand side is a continuous and increasing function of μ_{x0}, tends to 0 when $\mu_{x0} \to 0$ and tends to $+\infty$ when $\mu_{x0} \to +\infty$. Hence μ_{x0}^{2t+1} is well defined and is in $(0, +\infty)$. The map $\left(\mu_{0y}^{2t}\right) \to \left(\mu_{x0}^{2t+1}\right)$ is antitone, meaning that $\mu_{0y}^{2t} \leq \tilde{\mu}_{0y}^{2t}$ for all $y \in \mathcal{Y}$ implies $\tilde{\mu}_{x0}^{2t+1} \leq \mu_{x0}^{2t+1}$ for all $x \in \mathcal{X}$.
- By the same token, the map $\left(\mu_{x0}^{2t+1}\right) \to \left(\mu_{0y}^{2t+2}\right)$ is well defined and antitone. Thus, the map $\left(\mu_{0y}^{2t}\right) \to \left(\mu_{0y}^{2t+2}\right)$ is isotone. But $\mu_{0y}^{2} \leq m_y = \mu_{0y}^{0}$ implies that $\mu_{0.}^{2t+2} \leq \mu_{0.}^{2t}$. Hence $\left(\mu_{0y}^{2t}\right)_{t \in \mathbb{N}}$ is a decreasing sequence, bounded from below by 0. As a result $\left(\mu_{0y}^{2t}\right)$ converges. Letting $(\bar{\mu}_{0y})$ be its limit, and letting $(\bar{\mu}_{x0})$ be the limit of $\left(\mu_{x0}^{2t+1}\right)$, it is not hard to see that $(\bar{\mu}_{0x}, \bar{\mu}_{0y})$ is a solution to (2).

- We illustrate the algorithm on Dagsvik-Menzel's model. Recall that in this model $M_{xy}\left(\mu_{x0},\mu_{0y}\right)=\mu_{x0}\mu_{0y}K_{xy}$. Loop consists in:

$$\left\{\begin{array}{l}\mu_{x0}^{2t+1}=\frac{n_x}{\sum_{y\in\mathcal{Y}}K_{xy}\mu_{0y}^{2t}+1}\\ \mu_{0y}^{2t+2}=\frac{m_y}{\sum_{x\in\mathcal{X}}\mu_{x0}^{2t+1}K_{xy}+1}\end{array}\right. .$$

- In R, this is implemented as:
  - mux0 = n / (K %*% mu0y + 1)
  - mu0y = m / (t(K) %*% mux0 + 1)
- This algorithm has various names: Iterated Proportional Fitting Procedure (IPFP); Sinkhorn's algorithm; RAS algorithm. Extremely fast to code and run. Although it is known since the 1940s, it has been rediscovered recently and widely used for machine learning applications.

Section 3

Uniqueness of equilibrium

THEOREM

Under Assumptions (i)–(iii) above, and under the additional assumption that the domain of M_{xy} is $\mathbb{R}^2_+$ for each x and y, the matching function equilibrium is unique.

The proof of this theorem (proven more generally in [GKW]) is based on reformulating the equilibrium as a demand system and applying the result of Berry, Gandhi and Haile (2013).

- Let $\mathcal{Z} = \mathcal{X} \cup \mathcal{Y}$ be the set of goods, and let $p_z = \mu_{z0}$ if $z \in \mathcal{X}$, and $p_z = -\mu_{0z}$ if $z \in \mathcal{Y}$. Consider the map

$$E_x(p) = p_x + \sum_{y \in \mathcal{Y}} M_{xy}(p_x, -p_y) - n_x$$

$$E_y(p) = m_y + p_y - \sum_{x \in \mathcal{Y}} M_{xy}(p_x, -p_y)$$

- Let 0 be a zero good defined by

$$E_0(p) = 1 + \sum_{x \in \mathcal{Y}} n_x - \sum_{y \in \mathcal{Y}} m_y - \sum_{y \in \mathcal{Y}} p_y - \sum_{x \in \mathcal{Y}} p_x.$$

and let

$$\mathcal{Z}_0 = \mathcal{Z} \cup \{0\}.$$

- We now show that, under minimal assumptions on M_{xy}, the map E satisfies the assumptions in this lecture, guaranteeing the existence and uniqueness of the equilibrium matching μ.

- Recall the assumptions of Berry, Gandhi and Haile (2013), described in day 1:
  - Assumption 1: the domain of E is a Catesian product
  - Assumption 2 (weak substitutes): $\forall z \in \mathcal{Z}_{\emptyset}$, $\forall z' \in \mathcal{Z} \setminus \{z\}$, $E_z(u)$ is weakly decreasing in $u_{z'}$.
  - Assumption 3 (connected strict substitutes): $\forall z \in \mathcal{Z}$, there is a path $z_0 = \emptyset$, $z_2, ..., z_{k-1}, z_k = z$ such that for every $i = 0, ..., k$, $E_{z_{i-1}}(p)$ is strictly increasing in p_{z_i}.
- Let us verify these assumptions:
  - Assumption 1 is trivially verified given the fact that we have assumed the domain of M_{xy} was $\mathbb{R}_+^2$.
  - Assumption 2 is trivially verified.
  - Assumption 3 is trivially verified too, as $E_{\emptyset}(p)$ is strictly decreasing in any p_z.

EXERCISE

Write down the Jacobian on the map on the left handside of equations (2). When can one interpret these equations as the first order conditions to a convex minimization problem?

EXERCISE

Consider the case $M_{xy}(\mu_{x0}, \mu_{0y}) = \sqrt{\mu_{x0}\mu_{0y}} K_{xy}$*, where* $K_{xy} > 0$*. [This is the Choo-Siow model, which will be discussed in class]. Write an algorithm for determining the equilibrium.*

EXERCISE

In the setting of the previous exercise, assume that $K_{xy} = \exp\left(\frac{\Phi_{xy}}{2T}\right)$*, for* $T \to 0$*. Show that the solution* μ_{xy} *converges when* $T \to 0$ *to the solution of a linear programming problem. Interpret. [NB: it is useful to have solved the two previous exercises for this one].*