'math+econ+code' masterclass on competitive equilibrium: Walrasian equilibrium with substitutes

Alfred Galichon (NYU)

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Day 4, May 24 2018: matching with general transfers (2)
Block 11. Matching models and collective models

Learning objectives: block 11

- Galois connections, distance-to-frontier function
- Collective models, sharing rule, Pareto weights

- [OTME], Ch. 10.4
- Browning, Chiappori, Weiss (2014). *Family Economics*. Princeton.
- Nöldeke and Samuelson (2017). The implementation duality. Mimeo.
- G, Kominers and Weber (2017). Costly concessions. An Empirical Framework for Matching with Imperfectly Transferable Utility. Mimeo.
- Dupuy, G, Jaffe and Kominers (2017). Taxation in matching markets. Mimeo.

Section 1

Motivation 1: matching with taxes

- Recall the interpretation of Optimal Transport as a model of the labor market. A population of *workers* is characterized by their type $x \in \mathcal{X}$. There is a mass n_x of workers of type x.
- A population of *firms* is characterized by their types $y \in \mathcal{Y}$. There is a mass m_y of firms of type y.
- Each worker must work for one firm; each firm must hire one worker. Let μ_{xy} be the mass of a matched (x, y) pair. μ should have marginal less than n and m, which is denoted

$$\mu \in \mathcal{M}(n, m).$$

- The equilibrium assignment is determined by an important quantity: the **wages**. Let w_{xy} be the wage of employee x working for firm of type y.
- Let the indirect surpluses of worker x and firm y be respectively

$$u_x = \max_y \{\alpha_{xy} + w_{xy}\}$$
$$v_y = \max_x \{\gamma_{xy} - w_{xy}\}$$

so that (μ, w) is an equilibrium when

$$u_x + v_y \geq \Phi_{xy} \text{ with equality if } \mu_{xy} > 0$$
$$u_x \geq 0 \text{ with equality if } \mu_{x0} > 0$$
$$v_y \geq 0 \text{ with equality if } \mu_{0y} > 0.$$

- In this case, μ and (u, v) are determined by the optimal transport problem

$$\max_{\mu \in \mathcal{M}(P,Q)} \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \mu_{xy} \Phi_{xy}.$$

- Consider the same setting as above, but introduce (possibly nonlinear) taxes.
- Instead of assuming that workers' and firm's payoffs are linear in wages, assume

$$u_x = \max_y \{\alpha_{xy} + N(w_{xy}), 0\}$$
$$v_y = \max_x \{\gamma_{xy} - w_{xy}, 0\}$$

where $N(w)$ is indecreasing and continuous, interpreted as the net wage if w if the gross wage.
- Of course, OT is recovered when $N(w) = w$ (no tax).
- Linear taxes: $N(w) = (1-\theta) w$, where $\theta \in (0,1)$ is the (flat) tax rate.
- Progressive tax schedule:

$$N(w) = \min_{k \in \{0,1,\ldots,K\}} \{(1-\theta_k)(w - w_k) + n_k\},$$

where $n_0 = 0 < \ldots < n_k$, are the net income at the start of bracket k, $\theta_0 = 0 < \theta_1 < \ldots < \theta_k$ are the marginal tax rates in bracket k. $[n^{k+1} = n^k + \left(1-\theta^k\right)\left(w^{k+1} - w^k\right).]$

- Given a worker x and a firm y, the payoffs $\left(u_x, v_y\right)$ are feasible if and only if

$$D_{xy}\left(u_x, v_y\right) \leq 0,$$

where $D_{xy}\left(u, v\right) = \max_k D^k\left(u, v\right)$ with

$$D^k\left(u, v\right) = \frac{u - \alpha_{xy} - n^k - \left(1 - \theta^k\right)\left(\gamma_{xy} - v - w^k\right)}{2 - \theta^k},$$

and as a result

$$D\left(u, v\right) = \max_{k=1,\ldots,K}\left\{\frac{u - \alpha_{xy} - n^k - \left(1 - \theta^k\right)\left(\gamma_{xy} - v - w^k\right)}{2 - \theta^k}\right\}.$$

Section 2

Motivation 2: family economics

- Consider a model of family economics in which the utilities of a man x and a woman y who match and decide on a public good $g \in G$ are given by
$$u_x = \tilde{\alpha}_{xy}^{g} + \tau \log c_x \text{ and } v_y = \tilde{\gamma}_{xy}^{g} + \tau \log c_y$$
where c_x and c_y are the private consumptions, subject to budget constraint
$$c_x + c_y = b_{xy}^{g}$$
- Utilities u_x and v_y are feasible whenever $D_{xy}\left(u_x, v_y\right) \leq 0$, where

$$D_{xy}\left(u, v\right) = \min_{g \in G} D_{xy}^{g}\left(u, v\right)$$
$$D_{xy}^{g}\left(u, v\right) = \tau \log \left(\frac{\exp\left(\frac{u - \alpha_{xy}^{g}}{\tau}\right) + \exp\left(\frac{v - \gamma_{xy}^{g}}{\tau}\right)}{2} \right),$$

where $\alpha_{xy}^{g} = \tilde{\alpha}_{xy}^{g} + \tau \log\left(b_{xy}^{g}/2\right)$ and $\gamma_{xy}^{g} = \tilde{\alpha}_{xy}^{g} + \tau \log\left(b_{xy}^{g}/2\right)$.

Section 3

Equilibrium transport

- Let $\mathcal{A}=\mathcal{X}\times\mathcal{Y}\cup\mathcal{X}\times\{0\}\cup\{0\}\times\mathcal{Y}$, and normalize $u_0=v_0=0$. We have therefore that (μ,u,v) is an equilibrium outcome when

$$\left\{\begin{array}{l}(PF):\mu\in\mathcal{M}(n,m)\\(DF):D_{xy}(u_x,v_y)\geq 0\ \forall xy\in\mathcal{A},\\(NC):\mu_{xy}>0x\Longrightarrow D_{xy}(u_x,v_y)=0.\end{array}\right.$$

- This is an *equilibrium transport problem*, as seen yesterday.

- [GKW] introduce the *distance-to frontier* (DTF) function: if $\mathcal{F}_{xy}$ is the feasible set of utilities that x and y can achieve by matching, then for $(u, v) \in \mathbb{R}^2$, let

$$D_{xy}(u, v) = \min\left\{t \in \mathbb{R} : (u - t, v - t) \in \mathcal{F}_{xy}\right\}$$

  which is the distance along the diagonal between (u, v) and the frontier of $\mathcal{F}_{xy}$, with a minus sign if (u, v) is in the set.
- Economic intepretation: what is the quantity of utility that we can give or remove to x and y *in the same amount* such that they reach the efficient frontier?
- This object has nice properties:
  - $D_{xy}(u, v) \leq 0$ iff $(u, v) \in \mathcal{F}_{xy}$
  - $D_{xy}(u, v) < 0$ iff $(u, v) \in \mathcal{F}_{xy}^0$
  - $D_{xy}(u + t, v + t) = D_{xy}(u, v) + t$
- Note that in the case of OT,

$$D_{xy}(u, v) = \frac{u + v - (\alpha_{xy} + \gamma_{xy})}{2}.$$

- More generally, the following operations on DTF functions correspond to geometric operations on feasible sets:
  - $\max\left\{D^1, D^2\right\}$: intersection
  - $\min\left\{D^1, D^2\right\}$: union
  - $D\left(u-\alpha, v-\gamma\right)$: translation
  - $T\Psi\left(u/T, v/T\right)$: homothety
  - $\lambda\Psi^1+\left(1-\lambda\right)D^2$: intepolation
- These operations are exploited in the TraME project (https://github.com/TraME-Project/), a software for flexible computation of equilibrium transportation problems.

Section 4

THE EMPIRICAL FRAMEWORK

- Assume that there are groups, or clusters of men and women who share similar observable characteristics, called *types*. There are n_x men of type x, and m_y women of type y.
- Let $\mu_{xy} \geq 0$ be the number of men of type x matched to women of type y. This quantity satisfies

$$\sum_y \mu_{xy} \leq n_x$$
$$\sum_x \mu_{xy} \leq m_y$$

- We shall denote μ_{x0} and μ_{0y} the number of single men of type x and single women of type y.

- **Assumption 1**: Assume that if man i of type x and woman j of type y match, then they respectively get

$$u_i = U_i + \varepsilon_{iy}$$
$$v_j = V_j + \eta_{jx}$$

where the systematic part of the utilities U_i and V_j satisfy the feasibility equation

$$D_{x_i y_j}\left(U_i, V_j\right) \leq 0$$

- **Assumption 2**: there are a large number of invididuals per group and the ε and the η's random vectors with a nonvanishing density.

- Thus, we can rewrite the feasibility constraint as (for i in x and j in y)

$$\mu_{ij} = 1 \Longrightarrow D_{xy}\left(u_i - \varepsilon_{iy}, v_j - \eta_{jx}\right) \leq 0.$$

and stability

$$\forall i \in x, j \in y,\ D_{xy}\left(u_i - \varepsilon_{iy}, v_j - \eta_{jx}\right) \geq 0.$$

- This allows to define

$$U_{xy} = \min_{i:x_i = x}\left\{u_i - \varepsilon_{iy}\right\} \text{ and } V_{xy} = \min_{j:y_j = y}\left\{v_j - \eta_{xj}\right\}$$

so that

$$D_{xy}\left(U_{xy}, V_{xy}\right) = 0$$

and that μ_{xy} is related to U and V by

$$\mu_{xy} = \sum_{i:x_i = x}\sum_{j:y_j = y} 1\left\{u_i = U_{xy} + \varepsilon_{iy}\right\} = \sum_{i:x_i = x}\sum_{j:y_j = y} 1\left\{v_j = V_{xy} + \eta_{xj}\right\}.$$

Theorem 1 (Galichon, Kominers and Weber). Under Assumptions 1, 2 and 3 above, at equilibrium, there exist functions U_{xy} and V_{xy} such that the systematic part of the utilities of i and j if they are matched only depends on the observable types of these partners, that is $U_i = U_{x_i y_j}$ and $V_j = V_{x_i y_j}$.

This theorem extends to the general ITU case a result which was known in the TU case (Choo and Siow, Chiappori, Salanié and Weiss, Galichon and Salanié).

Implication of this theorem: the matching problem now embeds two sets of discrete choice problems. Indeed, man i and woman j (of types x and y) solve respectively

$$\max_y \left\{ U_{xy} + \varepsilon_{iy}, \varepsilon_{i0} \right\}$$
$$\max_x \left\{ V_{xy} + \eta_{jx}, \eta_{j0} \right\}$$

which are standard discrete choice problems; thus the log-odds ratio formula applies, and

$$\ln \frac{\mu_{xy}}{\mu_{x0}} = U_{xy}$$
$$\ln \frac{\mu_{xy}}{\mu_{0y}} = V_{xy}$$

But remember that $D_{xy}\left(U_{xy}, V_{xy}\right) = 0$, thus

$$D_{xy}\left(\ln \frac{\mu_{xy}}{\mu_{x0}}, \ln \frac{\mu_{xy}}{\mu_{0y}}\right) = 0.$$

Theorem 2 (GKW). Equilibrium in the ITU problem with logit heterogeneities is fully characterized by the set of nonlinear equations in μ_{xy}, μ_{x0} and μ_{0y}

$$
\begin{aligned}
D_{xy}\left(\ln\frac{\mu_{xy}}{\mu_{x0}}, \ln\frac{\mu_{xy}}{\mu_{0y}}\right) &= 0 \\
\sum_y \mu_{xy} + \mu_{x0} &= n_x \\
\sum_x \mu_{xy} + \mu_{0y} &= m_y
\end{aligned}
$$

Under very mild conditions on D it exists; under mild conditions on D it is also unique.

Note that first equation defines implicitely μ_{xy} as a function of μ_{x0} and μ_{0y}, which can be written as a matching function

$$\mu_{xy} = M_{xy}\left(\mu_{x0}, \mu_{0y}\right) := \exp\left(-D_{xy}\left(-\ln \mu_{x0}, \ln \mu_{0y}\right)\right)$$

hence we can restate the previous result as:

Theorem 2' (GKW). Equilibrium in the ITU problem with logit heterogeneities is fully characterized by the set of nonlinear equations in μ_{x0} and μ_{0y}

$$\sum_y M_{xy}\left(\mu_{x0}, \mu_{0y}\right) + \mu_{x0} = n_x$$
$$\sum_x M_{xy}\left(\mu_{x0}, \mu_{0y}\right) + \mu_{0y} = m_y.$$

With general random utilities, the welfare of men and women is equal too

$$G(U) = \sum_{x \in \mathcal{X}} n_x \mathbb{E} \max_y \{U_{xy} + \varepsilon_{iy}, \varepsilon_{i0}\}$$
$$H(V) = \sum_{y \in \mathcal{Y}} m_y \mathbb{E} \max_x \{V_{xy} + \eta_{jx}, \eta_{j0}\}$$

thus, by the Daly-Zachary-Williams theorem, the demand for match xy by men and women is respectively

$$\mu_{xy} = \partial G(U) / \partial U_{xy} \text{ and } \mu_{xy} = \partial H(V) / \partial V_{xy}$$

which inverts into

$$U_{xy} = \partial G^*(\mu) / \partial \mu_{xy} \text{ and } V_{xy} = \partial H^*(\mu) / \partial \mu_{xy}.$$

But remember that $D_{xy}(U_{xy}, V_{xy}) = 0$, thus the equilibrium μ is determined by

$$D_{xy}(\partial G^*(\mu) / \partial \mu_{xy}, \partial H^*(\mu) / \partial \mu_{xy}) = 0.$$

Theorem 3 (GKW). Equilibrium in the ITU problem with general heterogeneities is fully characterized by the set of nonlinear equations in μ_{xy}, μ_{x0} and μ_{0y}

$$\begin{aligned} D_{xy}\left(\partial G^*\left(\mu\right)/\partial\mu_{xy}, \partial H^*\left(\mu\right)/\partial\mu_{xy}\right) &= 0 \\ \sum_y \mu_{xy} &\leq n_x \\ \sum_x \mu_{xy} &\leq m_y \end{aligned}$$

Under very mild conditions on D it exists; under mild conditions on D it is also unique.

Equilibrium characterization (general heterogeneity)

One can show that the solutions (u, v) of $D_{xy}(u, v) = 0$ can be represented as $u = \mathcal{U}_{xy}(w)$ and $v = \mathcal{V}_{xy}(w)$, where $w = u - v$ and $\mathcal{U}_{xy}(.)$ is continuous and nondecreasing, while $\mathcal{V}_{xy}(.)$ is continuous and nonincreasing. In this case, the equilibrium conditions can be restated in terms of finding a vector (W_{xy}) such that

$$Z(W) = 0,$$

where

$$Z_{xy}(W) = \partial H(\mathcal{V}(W)) / \partial V_{xy} - \partial G(\mathcal{U}(W)) / \partial U_{xy}.$$

W plays the role of a price, Z of an excess demand function. Satisfies

$$\begin{aligned}
\partial Z_{xy} / \partial W_{xy} &\leq 0 \\
\partial Z_{xy} / \partial W_{xy'} &\geq 0 \text{ and } \partial Z_{xy} / \partial W_{x'y} \geq 0,\ x' \neq x, y' \neq y \\
\partial Z_{xy} / \partial W_{x'y'} &= 0,\ x' \neq x, y' \neq y.
\end{aligned}$$

In particular, Z satisfies "Gross Subsitutes".

One has:

Theorem 3' (GKW). Equilibrium in the ITU problem with general heterogeneities is equivalent to looking for a set of Walrasian prices W such that

$$Z\left(W\right)=0.$$

Gross substitutes ensures coordinate descent algorithm (nonlinear Jacobi) converges:

- Start with an initial vector of prices $\left(W_{xy}^{0}\right)$ for which excess demand is negative, that is such that $Z\left(W^{0}\right)\leq 0$.
- At step t, set W_{xy}^{t}, the price of match xy at time t such that $Z\left(W_{xy}^{t},W_{-xy}^{t-1}\right)=0$, where $\left(W_{xy}^{t},W_{-xy}^{t-1}\right)$ denotes the price vector which coincides with W^{t-1} on all entries except on the xy entry, and which sets price W_{xy}^{t} to the xy entry.

Note that $\left(W_{xy}^{t}\right)$ converges monotonically toward the equilibrium $\left(W_{xy}\right)$.