

# Root-Squaring for Root-Finding

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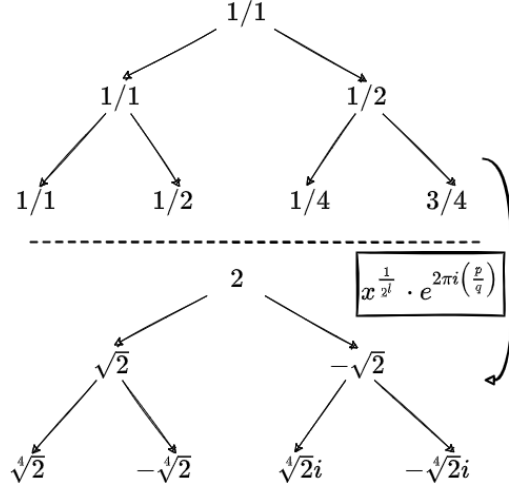


Figure 1: The upper tree depicts the steps of `CIRCLE_ROOTS_RATIONAL_FORM(p,q,l)` in Alg.1 for  $l = 2$ ,  $p = 1$ , and  $q = 1$ . The lower tree depicts the steps of `Roots(r, t, u, l)` in Alg.2 for  $r = 2$ ,  $l = 2$ ,  $p = 1$ , and  $q = 1$

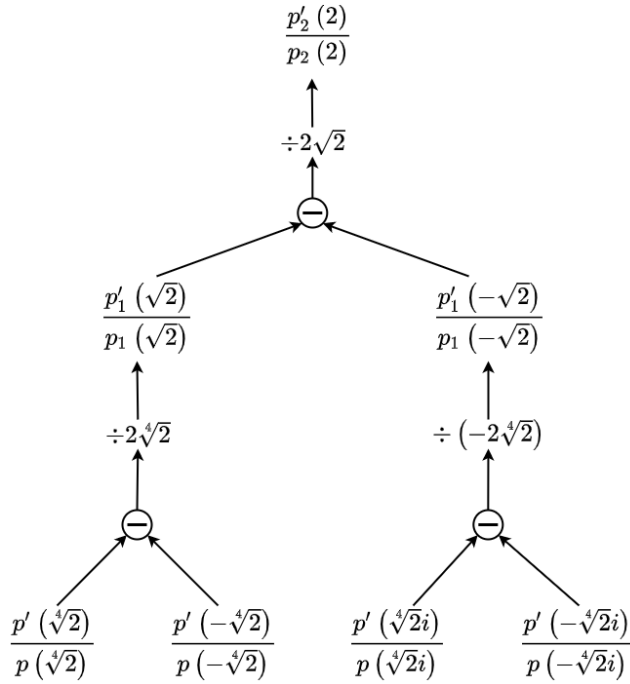


Figure 2: The steps of `DLG_RATIONAL_RORM(p, p', r, t, u, l)` in Alg.3 for  $r = 2$ ,  $l = 2$ ,  $t = 1$ , and  $u = 1$ .

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## Algorithm 1 `CIRCLE_ROOTS_RATIONAL_FORM(p, q, l)`

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```

if  $p \% q == 0$  then
     $r, s := (1, 1)$ 
else
     $r, s := (p, 2q)$ 
end if
if  $r \% s == 0$  then
     $t, u := (1, 2)$ 
else
     $t, u := (2r + s, 2s)$ 
end if
if  $l == 1$  then
    return  $[(r, s), (t, u)]$ 
else if  $l != 0$  then
     $\text{left} := \text{CIRCLE\_ROOTS\_RATIONAL\_FORM}(r, s, l - 1)$ 
     $\text{right} := \text{CIRCLE\_ROOTS\_RATIONAL\_FORM}(t, u, l - 1)$ 
    return  $\text{left} \cup \text{right}$ 
else
    return  $[(p, q)]$ 
end if

```

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## Algorithm 2 `Roots(r, t, u, l)`

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```

 $\text{root\_tree} = \text{CIRCLE\_ROOTS\_RATIONAL\_FORM}(p, q, l)$ 
 $\text{circ\_root} = [\exp(2 \cdot \pi \cdot i \cdot \frac{r}{s}) \text{ for } r, s \text{ in } \text{root\_tree}]$ 
 $\text{roots} = [\sqrt[l]{r} \cdot \text{root} \text{ for } \text{root} \text{ in } \text{circ\_root}]$ 

```

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## ABSTRACT

We revisited the classical root-squaring formula of Dandelin-Lobachevsky-Graeffe for polynomials and found new interesting applications to root-finding.

## CCS CONCEPTS

• Computing methodologies → Hybrid symbolic-numeric methods.

## KEYWORDS

symbolic-numeric computing, root finding, polynomial algorithms, computer algebra

## ACM Reference Format:

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**Algorithm 3** DLG\_RATIONAL\_FORM( $p, p', r, t, u, l$ )

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```

root := ROOTS( $r, t, u, l$ )
for  $r_i \in \text{root}$  do
  base_step[ $i$ ] :=  $\frac{p'(r_i)}{p(r_i)}$ 
end for
diff[0] := base_step
for  $i \leq l$  do
  for  $j \leq 2^{l-i-1}$  do
    diff[ $i+1$ ][ $j$ ] :=  $\frac{1}{2} \frac{\text{diff}[i][2j] - \text{diff}[i][2j+1]}{\text{root}[2j]}$ 
    root = roots( $r, t, u, l-1-i$ )
  end for
end for
return derivs[ $l$ ][0]

```

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**Algorithm 4** DLG( $p, p', l, x, \epsilon$ )

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```

angle :=  $\frac{1}{2\pi i} \log(x)$ 
Set  $t, u := \text{ANGLE\_AS\_INTEGER\_RATIO}(\text{angle}, \epsilon)$ 
 $r := |x|$ 
return DLG_RATIONAL_FORM( $p, p', r, t, u, l$ )

```

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- 1 INTRODUCTION
- 2 RELATED WORKS
- 3 BACKGROUND
- 4 MOTIVATING EXAMPLE

## 5 ALGORITHM DESIGN

## 6 THEORETICAL ANALYSIS

LEMMA 6.1. *The (relative) condition number operator satisfies the following properties:*

- (1)  $\kappa(f(x)) = |x \log'(f(x))|$
- (2)  $\kappa(f(x) - g(x)) = |x \frac{\kappa(f(x)) - \kappa(g(x))}{f(x) - g(x)}|$
- (3)  $\kappa(\frac{f(x)}{g(x)}) = ||\kappa(f(x))| - |\kappa(g(x))||$
- (4)  $\kappa(f(g(x))) = ||\kappa(f(x)) \cdot \kappa(g(x))||$

THEOREM 6.2. *The condition number for  $\frac{p'_l}{p_l}$  using*

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## 7 EXPERIMENTAL RESULTS

## 8 CONCLUSION

## REFERENCES