

# Root-Squaring for Root-Finding

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## ABSTRACT

We revisited the classical root-squaring formula of Dandelin-Lobachevsky-Graeffe for polynomials and found new interesting applications to root-finding.

## CCS CONCEPTS

• Computing methodologies → Hybrid symbolic-numeric methods.

## KEYWORDS

symbolic-numeric computing, root finding, polynomial algorithms, computer algebra

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## 1 INTRODUCTION

We revisit the famous methods simultaneously discovered by Dandelin, Lobachevsky, and Graeffe (See [1] for a history of this problem) and make further progress by applying this identity to the problem of approximating the root radius of a polynomial.

## 2 RELATED WORKS

## 3 BACKGROUND

## 4 MOTIVATING EXAMPLE

## 5 ALGORITHM DESIGN

## 6 THEORETICAL ANALYSIS

**THEOREM 6.1.** *Algorithm. 3 performs  $q \log q$  floating point subtractions, divisions, and multiplications and  $q \log q$  applications of sin and cos, where  $q = 2^l$ ; furthermore, Algorithm. 3 performs at most  $q \log q$  integer additions, “multiplications-by-2”, and  $2^e$  (i.e., mod  $2^e$ ) operations.*

## 7 EXPERIMENTAL RESULTS

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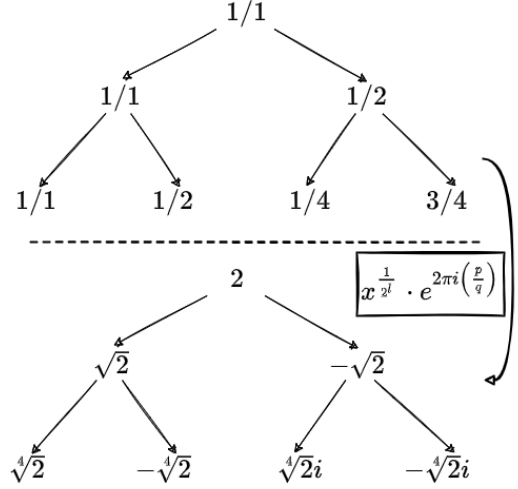


Figure 1: The upper tree depicts the steps of `CIRCLE_ROOTS_RATIONAL_FORM`( $p, q, l$ ) in Alg.1 for  $l = 2$ ,  $p = 1$ , and  $q = 1$ . The lower tree depicts the steps of `Roots`( $r, t, u, l$ ) in Alg.?? for  $r = 2$ ,  $l = 2$ ,  $p = 1$ , and  $q = 1$

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### Algorithm 1 `CIRCLE_ROOTS_RATIONAL_FORM`( $p, q, l$ )

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```

if  $p \% q == 0$  then
     $r, s := (1, 1)$ 
else
     $r, s := (p, 2q)$ 
end if
if  $r \% s == 0$  then
     $t, u := (1, 2)$ 
else
     $t, u := (2r + s, 2s)$ 
end if
if  $l == 1$  then
    return  $[(r, s), (t, u)]$ 
else if  $l != 0$  then
    left := CIRCLE_ROOTS_RATIONAL_FORM( $r, s, l - 1$ )
    right := CIRCLE_ROOTS_RATIONAL_FORM( $t, u, l - 1$ )
    return left  $\cup$  right
else
    return  $[(p, q)]$ 
end if

```

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## 8 CONCLUSION

## REFERENCES

- [1] Alston S. Householder. 1959. Dandelin, Lobachevskii, or Graeffe. *The American Mathematical Monthly* 66, 6 (1959), 464–466. <http://www.jstor.org/stable/2310626>

**Table 1: Experimental Data for Chebyshev.**

DEGREE	$l$	$e = -\log( x )$	MPMATH PRECISION	RELATIVE ERROR $r_d$	RELATIVE ERROR $r_1$	RUNTIME	MPSOLVE ROOT RADIUS
20	4	616	332	0.15	0.09	0.17	[0.00982, 1.0]
40							
80							
160							
320							

**Algorithm 3** DLG\_RATIONAL\_FORM( $p, p', r, t, u, l$ )

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```

root := ROOTS( $r, t, u, l$ )
for  $r_i \in \text{root}$  do
    base_step[ $i$ ] :=  $\frac{p'(r_i)}{p(r_i)}$ 
end for
diff[0] := base_step
for  $i \leq l$  do
    for  $j \leq 2^{l-i-1}$  do
        diff[ $i+1$ ][ $j$ ] :=  $\frac{1}{2} \frac{\text{diff}[i][2j] - \text{diff}[i][2j+1]}{\text{root}[2j]}$ 
        root = roots( $r, t, u, l-1-i$ )
    end for
end for
return diff[ $l$ ][0]

```

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**Algorithm 4** DLG( $p, p', l, x, \epsilon$ )

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```

angle :=  $\frac{1}{2\pi i} \log(x)$ 
 $u := 2^\epsilon$ 
 $t := (\text{angle} \cdot u) \% 1$ 
 $r := |x|$ 
return DLG_RATIONAL_FORM( $p, p', r, t, u, l$ )

```

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