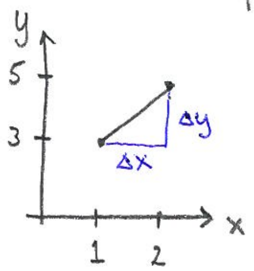
Example 1:

Data:

x	y
1	3
2	5

Fit $y = kx + m$ (straight line) to the data



$$k = \frac{\Delta y}{\Delta x} = \frac{y(2) - y(1)}{x(2) - x(1)} = \frac{5-3}{2-1} = 2$$

$$y = 2x + m$$

$$y(1) = 3 = 2 \cdot 1 + m = 2 + m \iff m = 1$$

$\Longrightarrow y = 2x + 1$

This gives an estimation of y for every x

POLYNOMIAL RESPONSE SURFACES

Example 1: $y = 2x + 1$ is an example for 1 variable and order 1.

Solving it with matrices:

$$\bar{x}\,\bar{\beta} = \bar{y} = \begin{pmatrix} 1 & x(1) \\ 1 & x(2) \end{pmatrix}\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} y(1) \\ y(2) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \underset{r_2 - r_1}{\sim} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 2 \end{array}\right) \Rightarrow \begin{cases} \beta_0 + \beta_1 = 3 \Leftrightarrow \beta_0 = 3 - 2 = 1 \\ \beta_1 = 2 \end{cases}$$

$$\Rightarrow \hat{y} = 1 + 2x$$

Example 2: Fit a second order response surface to the data and find the minimum value

x	y
0	1/4
1	1/4
2	9/4

$$1D \Rightarrow \hat{y} = \beta_0 + \beta_1 x + \beta_{11} x_1^2$$

$$\bar{x}\,\bar{\beta} = \bar{y} = \begin{array}{c} \begin{array}{ccc} & x & x^2 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \end{array} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_{11} \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/4 \\ 9/4 \end{pmatrix}$$

$$\underset{r_3 - r_1}{\overset{r_2 - r_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1/4 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 4 & 2 \end{array}\right) \underset{r_3 - 2r_1}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1/4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 \end{array}\right) \Rightarrow \begin{cases} \beta_0 = 1/4 \\ \beta_1 = -\beta_{11} = -1 \\ \beta_{11} = 1 \end{cases}$$

$$\Rightarrow \hat{y} = \frac{1}{4} - x + x^2$$

Find the optimum analytically or by using an optimization algorithm.

$$\left.\begin{array}{l} \dfrac{\partial \hat{y}}{\partial x} = -1 + 2x = 0 \Leftrightarrow x = -\dfrac{1}{2} \\[2ex] \dfrac{\partial^2 \hat{y}}{\partial x^2} = 2 > 0 \cup \Rightarrow \text{minimum} \end{array}\right\} \quad x = \frac{1}{2},\ y = \frac{1}{4} - \frac{1}{2} + \frac{1}{4} = 0 \text{ is the minimum}$$

Example 3: 2D, Second Order Response Surface

$$2D \Rightarrow \hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \beta_{12} x_1 x_2$$

$6\ \beta_i \Rightarrow$ we need at least 6 experiments/samples

x_1	x_2	y
1	1	0
3	3	4
1	3	8
3	1	4
2	2	1
0	1	1

$$\Rightarrow \begin{array}{c} \begin{array}{cccccc} & x_1 & x_2 & x_1^2 & x_2^2 & x_1 x_2 \end{array} \\ \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 9 & 9 & 9 \\ 1 & 1 & 3 & 1 & 9 & 3 \\ 1 & 3 & 1 & 9 & 1 & 3 \\ 1 & 2 & 2 & 4 & 4 & 4 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}}_{\bar{X}} \end{array} \underbrace{\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_{11} \\ \beta_{22} \\ \beta_{12} \end{pmatrix}}_{\bar{\beta}} = \underbrace{\begin{pmatrix} 0 \\ 4 \\ 8 \\ 4 \\ 1 \\ 1 \end{pmatrix}}_{\bar{y}}$$

$$\hat{\beta} = (\bar{X}^t \bar{X})^{-1} \bar{X}^t \bar{y} = (1\ \ 0\ \ -2\ \ 1\ \ 2\ \ -2)^t$$

$$\Rightarrow \hat{y} = 1 - 2x_2 + x_1^2 + 2x_2^2 - 2x_1 x_2$$

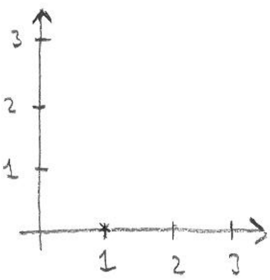
Interpolation

Estimate the value at a new point as a linear combination of the values at known points

$$\hat{y} = \sum_{i=1}^{n} w_i Y(x_i)$$

Example: Estimate the value at $\bar{x} = (2, 1)$

x_1	x_2	y
0	1	1
3	0	3
3	3	6



point	distance to [2 1]	w	y	y·w
1	2	0.30	1	0.30
2	$\sqrt{2}$	0.43	3	1.28
3	$\sqrt{5}$	0.27	6	1.62
sum →	5.65	1		3.20

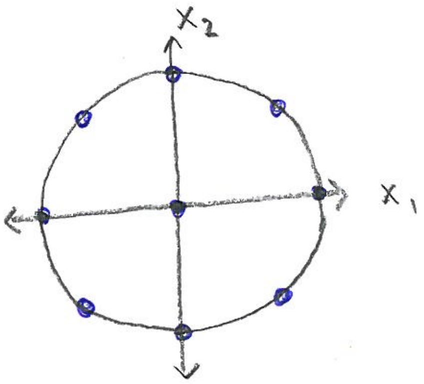
$\Rightarrow \hat{y}(2\ 1) = 3.20$

DESIGN OF EXPERIMENTS

Airplane example: Two variables, x_1 real number
$0 \le x_1 \le 270$

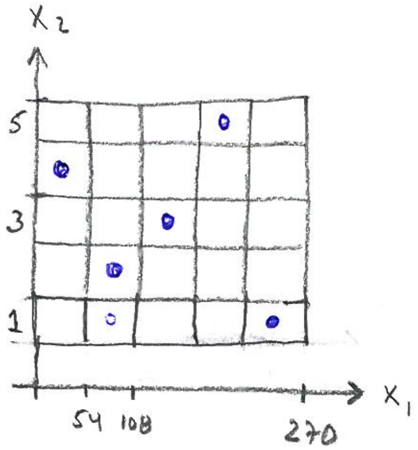
x_2, integer
1, 2, 3, 4, 5

Central Composite Design



x_1	x_2	
135	3	center
0	3	axis runs
270	3	
135	1	
135	5	
200	2	corner runs
200	4	
70	2	
70	4	

LATIN HYPERCUBE SAMPLING



x_1	x_2
27	4
81	2
135	3
189	5
243	1