Lecture Notes

Optimization Basics

Modeling, Simulation and Optimization can be used to find the optimal potential of each concept

Elements of Optimization

Design variables are entities that the designer can change

The objective function describes how "good" a solution is

Constraints are limits that specifies if a solution is valid / feasible or not

The design space is the imaginary space where possible solutions can be found

Elements of Optimization: Example

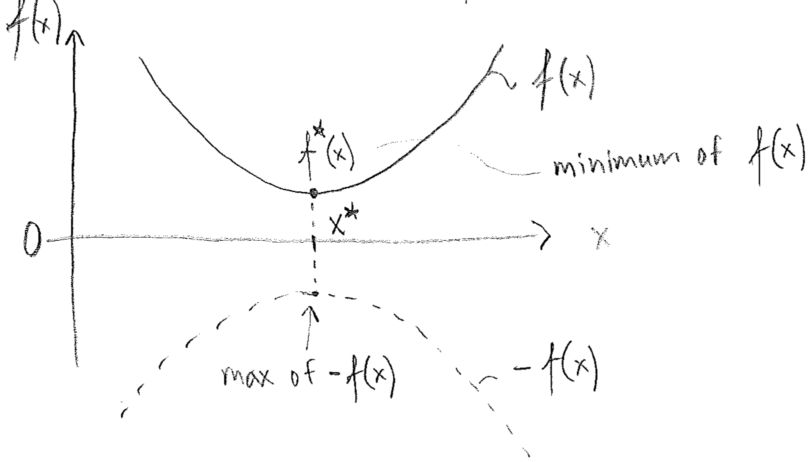
$\min f(x) = (x_1 - 0{,}5)^2 + (x_2 - 0{,}5)^2$ Objective Function

$x_1 + x_2 - 2 \leq 0$ Constraint

$0 \leq x_i \leq 2, \quad i = 1, 2$ Variable Limits

Minimization or Maximization

$$\min f(x) = -\max(-f(x))$$



$$\max f(x) = -\min(-f(x))$$

Conditions For Optimality

Definition
$$f(\bar{x}^*) < f(\bar{x}^* + \Delta\bar{x}) \quad , \quad 0 < \|\Delta\bar{x}\| < \delta$$

$f(\bar{x}^*)$ is smaller than all points in its surrounding

Taylor series expansion
$$f(x+\Delta x) = f(x) + \Delta x^t \nabla f(x) + \frac{1}{2}\Delta x^t \nabla^2 f(x)\cdot \Delta x + \dots$$

Consider first order terms
$$f(x^* + \Delta x) = f(x^*) + \Delta x^t \nabla f(x^*) \iff f(x^*) = f(x^* + \Delta x) - \Delta x^t \nabla f(x^*)$$
$$\text{IF } f(x^*) \leq f(x^* + \Delta x) \Rightarrow \nabla f(x^*) = 0$$

To guarantee a strict optimum consider 2nd order terms
$$f(x^* + \Delta x) = f(x^*) + \frac{1}{2}\Delta x^t \nabla^2 f(x^*)\cdot \Delta x$$
$$\Rightarrow \quad f(x^*) = f(x^* + \Delta x) - \frac{1}{2}\Delta x^t \nabla^2 f(x^*)\Delta x$$

x^* is an optimum : $f(x^* + \Delta x) > f(x^*) \Rightarrow$
$$f(x^*) + \frac{1}{2}\Delta x^t \nabla^2 f(x^*)\Delta x > f(x^*) \quad \Rightarrow \nabla^2 f(x^*) > 0$$

$\nabla f(x^*) = 0$ stationary point

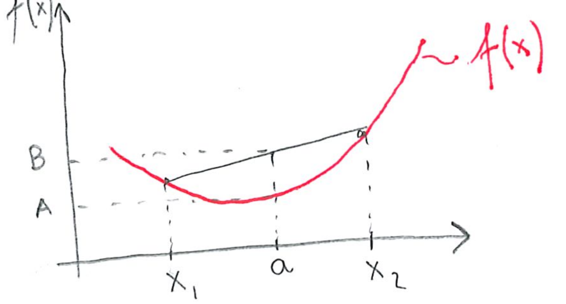
$\nabla^2 f(x^*) > 0$ for minimum

$\nabla^2 f(x^*) < 0$ for maximum

Convexity

Definition

$$f(\theta \bar{x}_1 + (1-\theta)\bar{x}_2) \leq \theta f(\bar{x}_1) + (1-\theta) f(\bar{x}_2) \quad , \quad 0 \leq \theta \leq 1$$



The left side of the inequality is a point on the curve $f(x)$

The right side is a point on the line between $f(\bar{x}_1)$ & $f(\bar{x}_2)$

The definition says that $f(x)$ is convex if the curve is always below the line

If we change $\leq$ to $\geq$ we get the definition for concave

A local minimum is the global minimum for a convex function

A local maximum is the global maximum for a concave function

It is usually difficult to prove that the objective function and solution space are convex