Optimization Basics

Design Optimization
TMKT48

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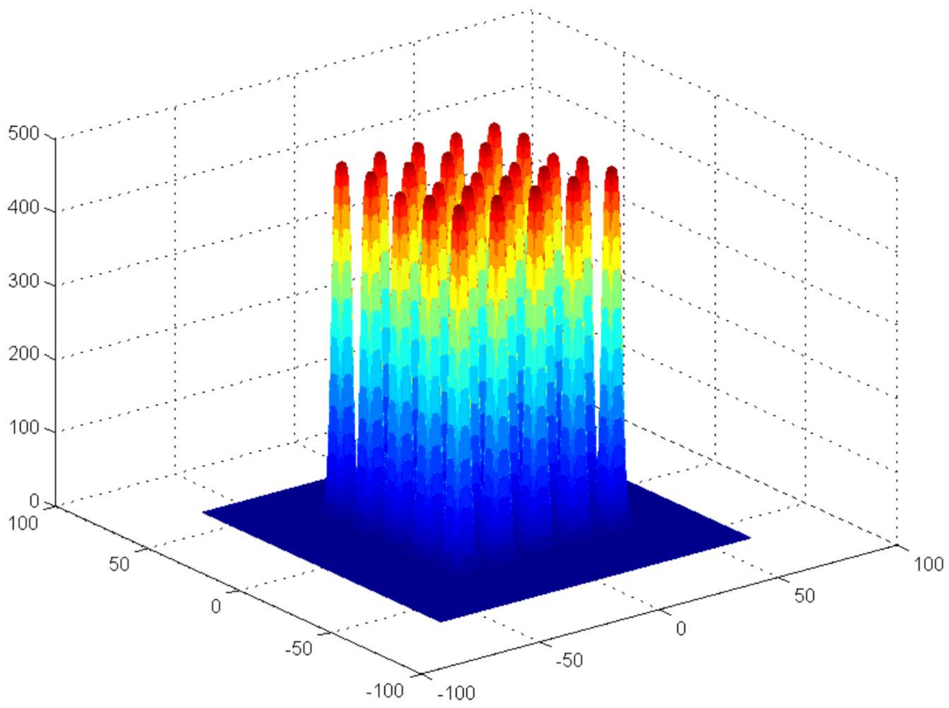
Contents

- Product Development and Optimization
- Elements of Optimization
- Problem Properties
- Solution Approaches



Product Development and Optimization

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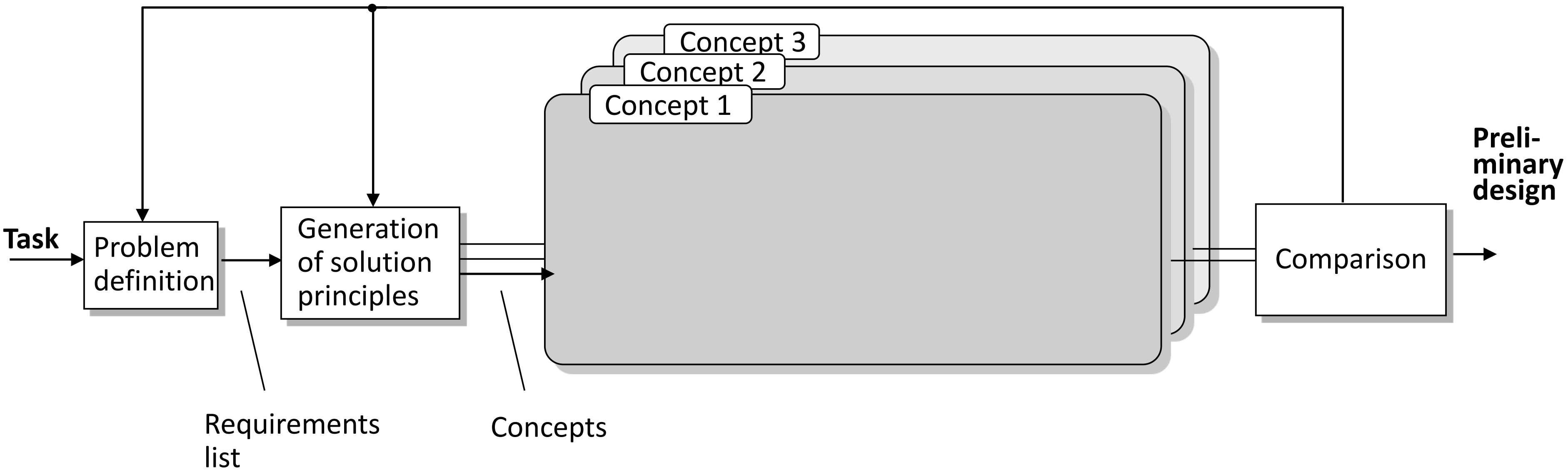
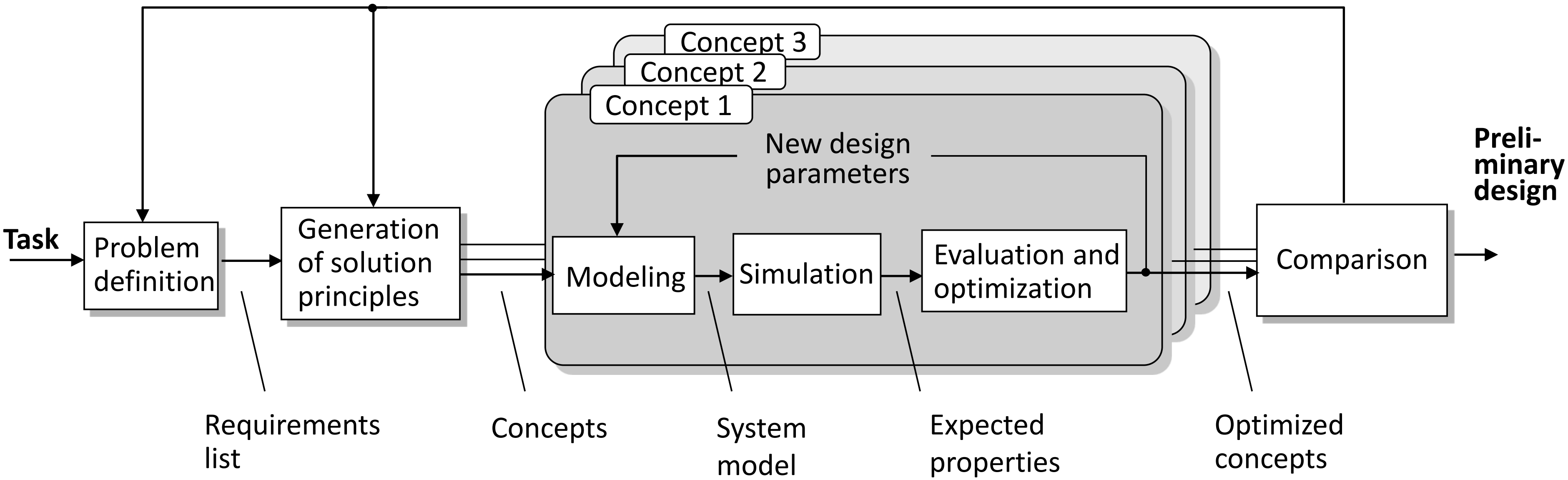
Product Development

- Constant Decision Making
- Sizes
- Costs
- Geometry
- Manufacturing
- …
- The designer wants to find the optimal compromise between the characteristics
- Optimization!!!

General Product Development Process



Optimization in Product Development



What is a model?

A model is a simplified representation of a system intended to enhance our ability to understand, predict and possibly control the behavior of the system.

Neelamkavil, 1987

- A model can be of a mental, verbal, physical or mathematical nature.
- In this course we focus on mathematical models implemented in a computer environment

What is simulation?

Forming an image of the behavior and properties of a designed product by reasoning and/or testing models

Roozenburg and Eekels, 1995

- This is an excellent definition of simulation, although somewhat broader than used in this course.
- In this course simulation refers to the execution of a model in order to predict the properties of a design proposal.

Elements of Optimization

Elements of optimization: Design variables

- Entities that the designer can change

- Design Variables could be:
  - Continuous: Free to assume any value
  - Discrete: Can assume only fixed values
  - Integer: Can only be integer values

Elements of optimization: Objective Function

- The objective function prescribes the criterion that guides the search for the ”best solution”

ObjValue=f(x) — ObjValue is the quantity that should be maximized or minimized

$x=(x_1,x_2,..,x_n)$ represents the design variables

- Examples
  - Minimize cost
  - Maximize efficiency
  - Minimize Weight

Elements of optimization: Objective Function

- The choice of the objective function is crucial
  - Different objective functions produces different optima

- Problems can be single objective or multi objective
  - Multiple objectives are often conflicting

Elements of optimization: Constraints

- Optimization constraints are numerical values of identified conditions that must be satisfied in order to achieve a feasible solution to a given problem.

- $h(x)=0$ Equality constraint
- $g(x)\leq 0$ Inequality constraint

- Constraints could be inactive or active at the optimum
  - Active => $g(x)=0$. Equality constraints are always active

Elements of optimization: Design Space

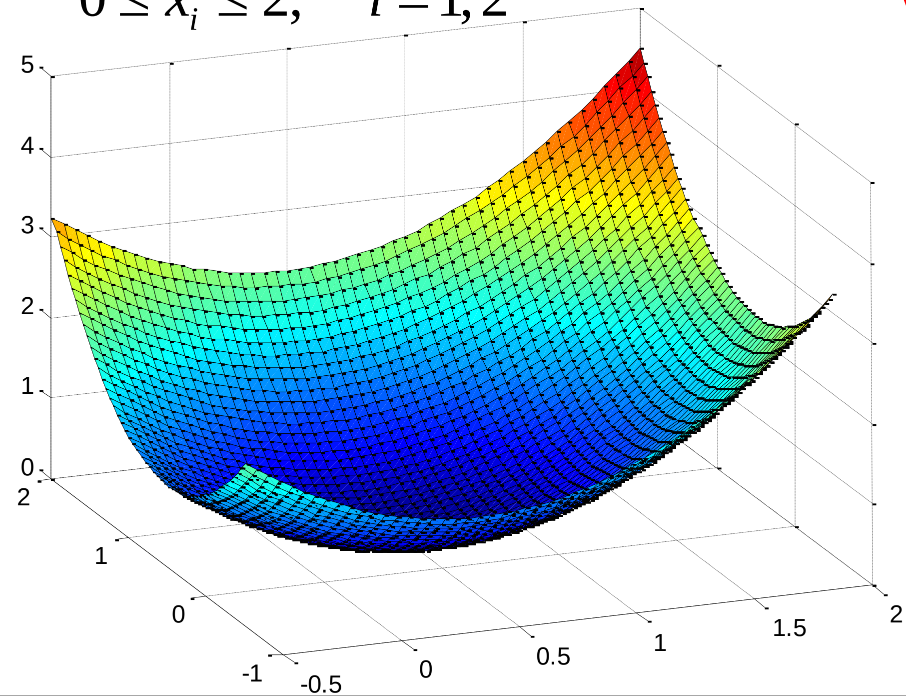
- The imaginary space where possible solutions can be found
- Usually created by assigning minimum and maximum values for each design variable.

Elements of optimization: Example

$$\min_{\mathbf{x}} f(\mathbf{x}) = (x_1 - 0.5)^2 + (x_2 - 0.5)^2$$ Objective Function

$$x_1 + x_2 - 2 \le 0$$ Constraint

$$0 \le x_i \le 2, \quad i = 1, 2$$ Variable Limits

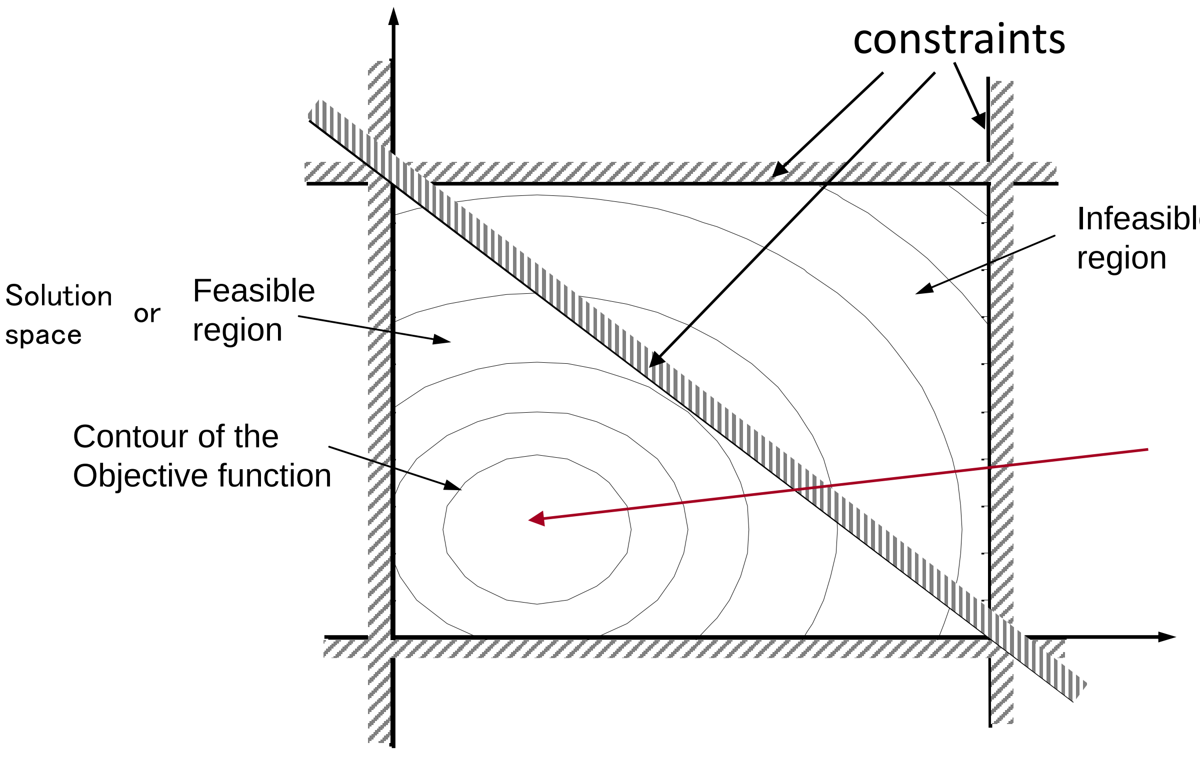


Elements of optimization: Example

$$\min_{\mathbf{x}} f(\mathbf{x}) = (x_1 - 0.5)^2 + (x_2 - 0.5)^2$$

$$x_1 + x_2 - 2 \le 0$$

$$0 \le x_i \le 2, \quad i = 1, 2$$

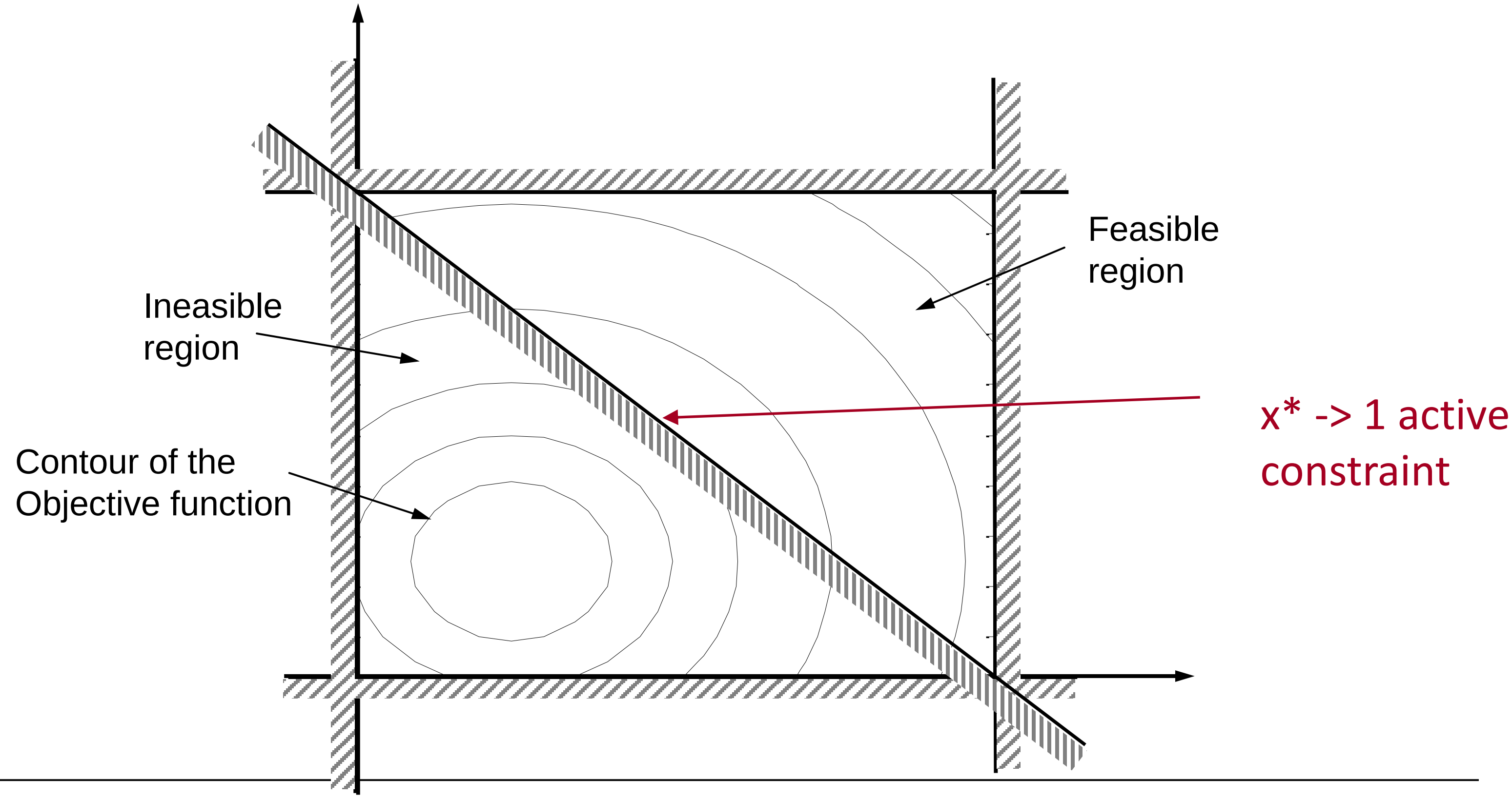


Elements of optimization: Example

$$\min_{\mathbf{x}} f(\mathbf{x}) = (x_1 - 0.5)^2 + (x_2 - 0.5)^2$$

$$x_1 + x_2 - 2 \geq 0$$

$$0 \leq x_i \leq 2, \quad i = 1, 2$$



The Optimization Problem

Design characteristics

$$\min \mathbf{F}(\mathbf{x}) = f\left(DC_1(\mathbf{x}), DC_2(\mathbf{x}), \ldots, DC_m(\mathbf{x})\right)$$

Objective function

$$\mathrm{x}_\mathrm{i}^\mathrm{l} \le x_i \le \mathrm{x}_\mathrm{i}^\mathrm{u} \qquad i = 1, 2, \ldots, n$$

Variable limits

$$\mathrm{g}_\mathrm{j}(\mathbf{x}) \le 0 \qquad j = 1, 2, \ldots, r$$

Constraints

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$$

Design variables

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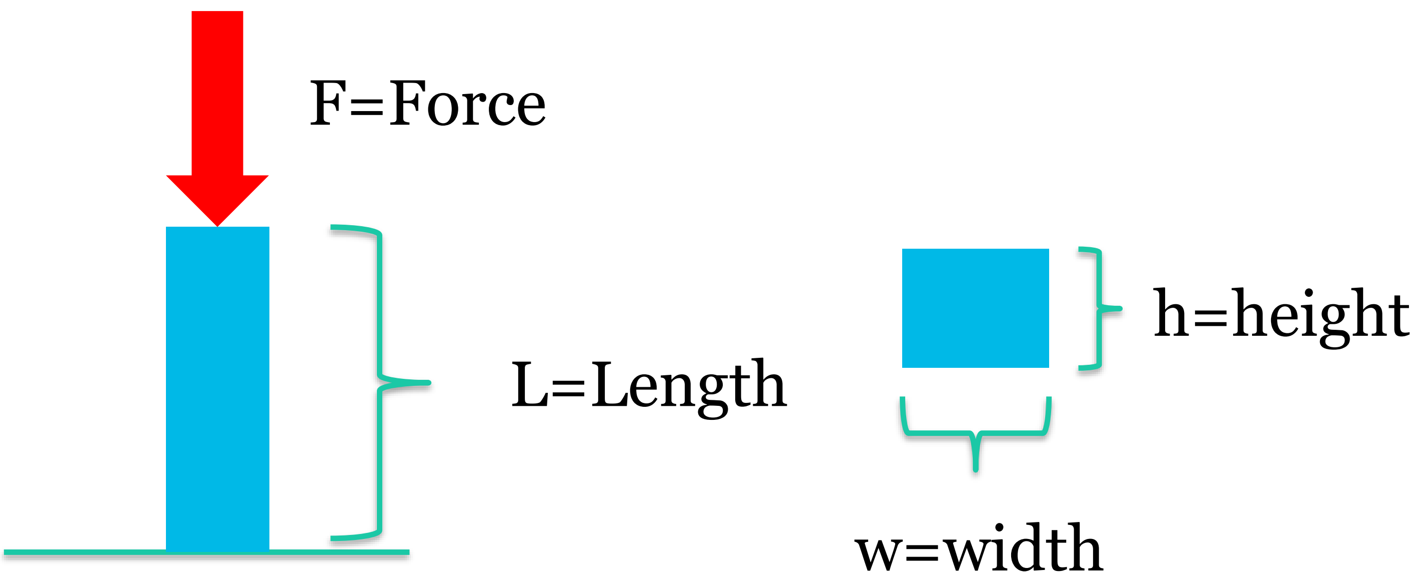
Example - Column

- Design the cross-section so that the rod can handle the load without breaking.

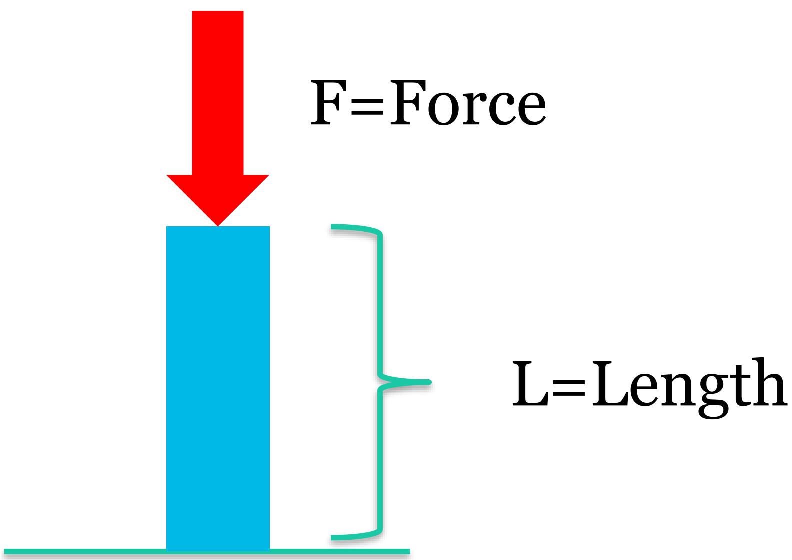


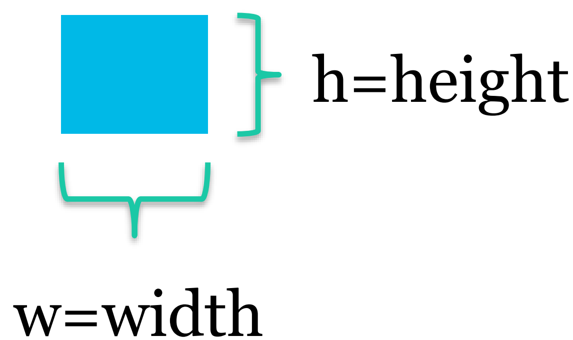
Example - Column

- Design the cross-section so that the rod can handle the load without breaking.

$V=L*Area=L*w*h$

$\sigma=F/Area=F/(w*h)$

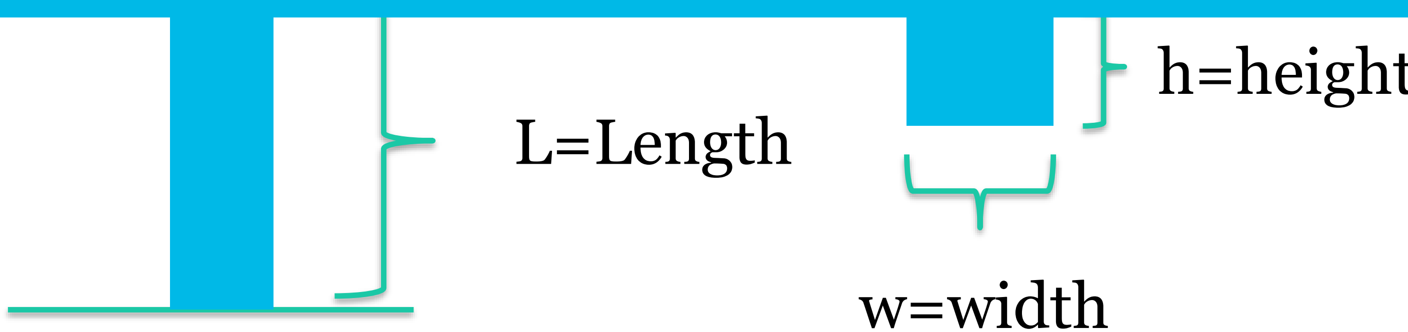




Example - Column

- Design the cross-section so that the rod can handle

Discuss with the person next to you:
What is the objective?
Which are the design variables?
What is the constraint?



Example - Column

- Design the cross-section so that the rod can handle the load without breaking.

$V = L*Area = L*w*h$

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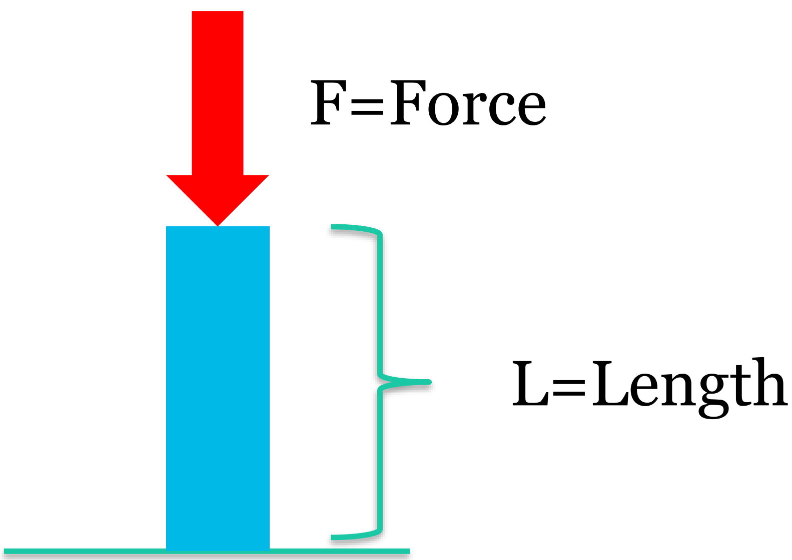
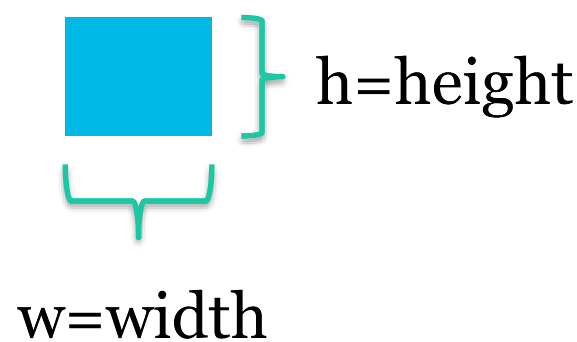


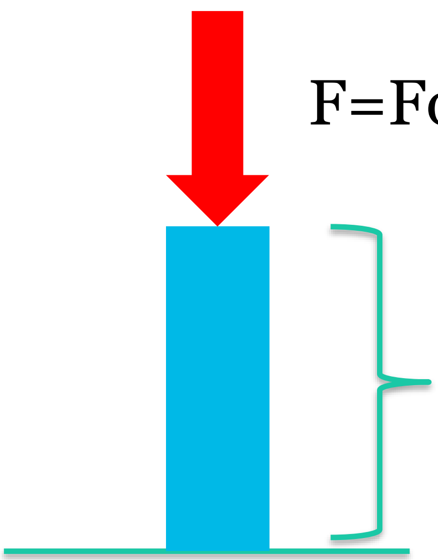


Example - Column

- Objective: Minimize V
- Design variables: w, h
- Constraints: σ

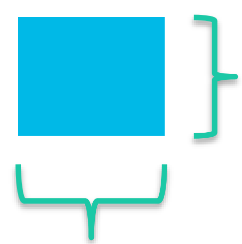
Min $V = L * w * h$

s.t. $\sigma = F/(w * h) \leq \sigma_{max}$



$V = L * Area = L * w * h$

$\sigma = F/Area = F/(w * h)$



Example - Column

$\text{Min } V = L * w * h$

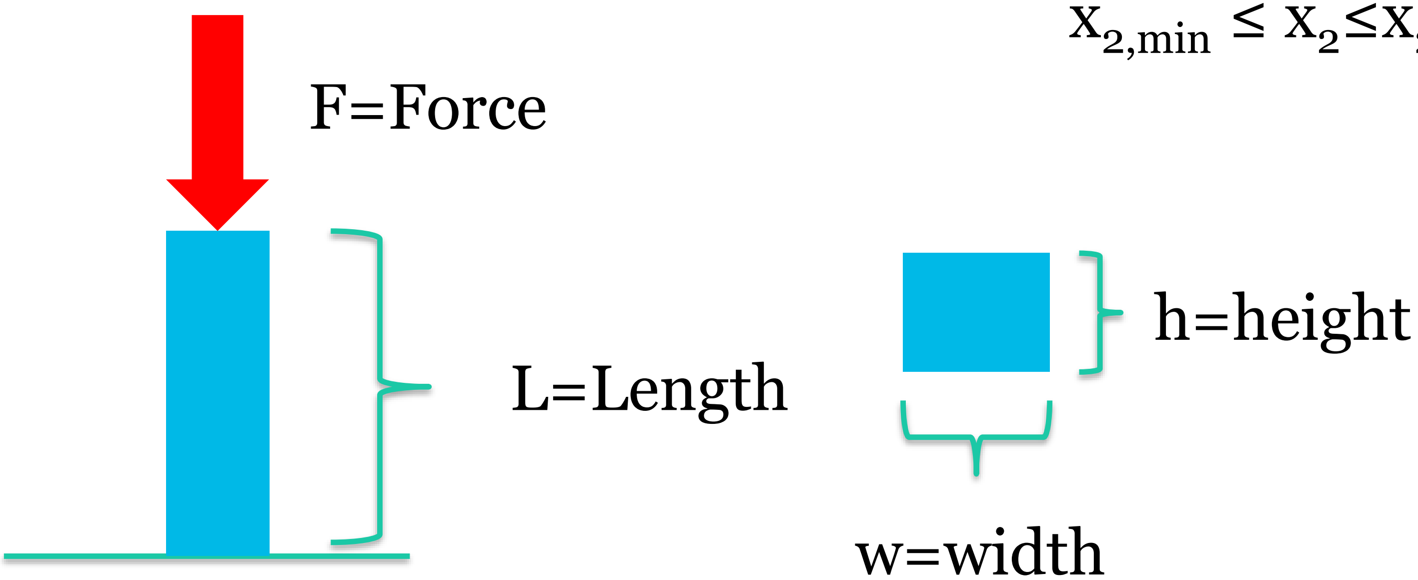
$\text{s.t.} \quad \sigma = F/(w*h) \leq \sigma_{max}$

$\text{Min } f(x) = L * x1 * x2$

$\text{s.t.} \quad \sigma = F/(x1*x2) \leq \sigma_{max}$

$x_{1,min} \leq x_1 \leq x_{1,max}$

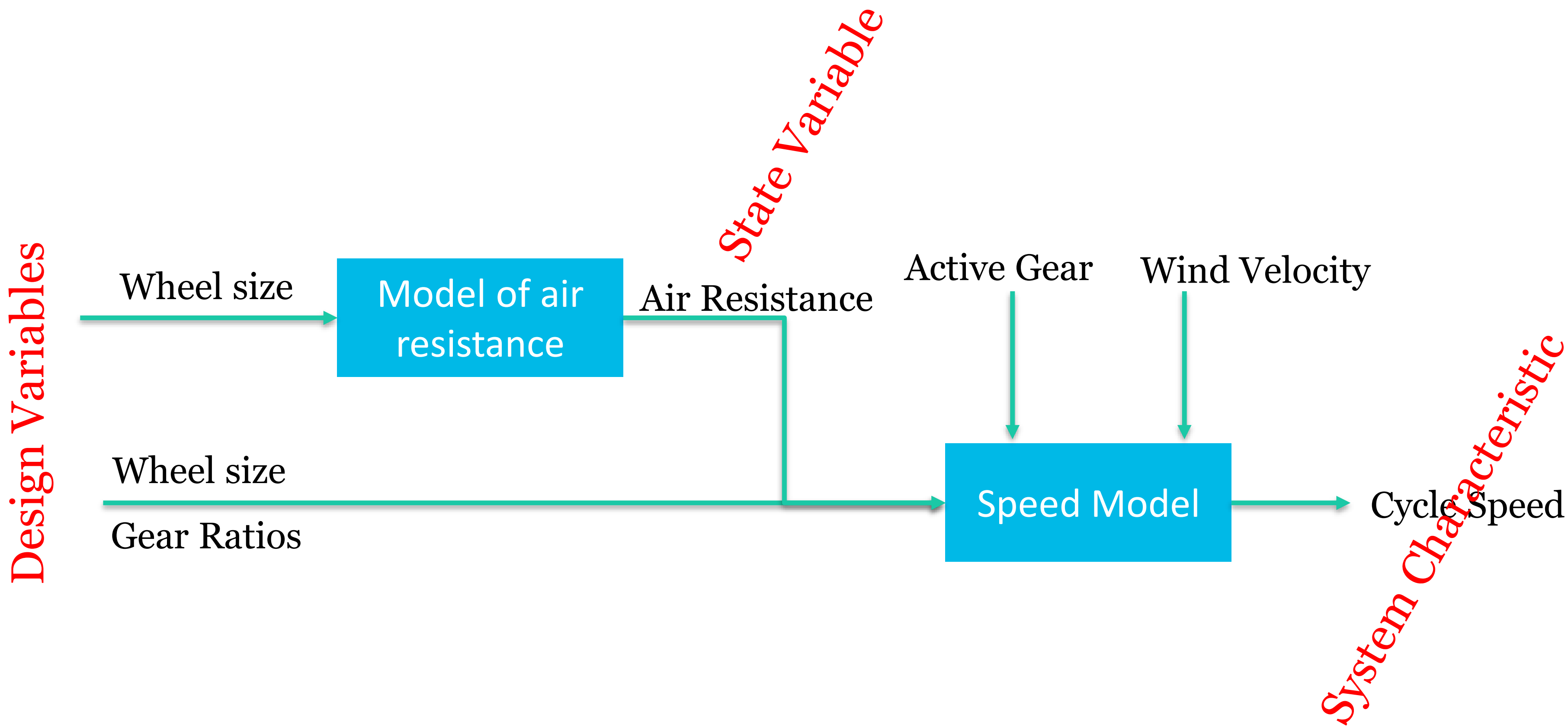
$x_{2,min} \leq x_2 \leq x_{2,max}$



More definitions…

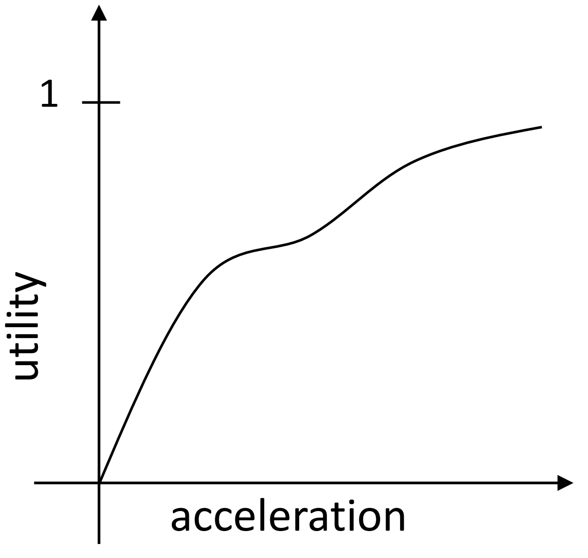
Definition	Description	Examples
Design Variables	Entities that the designer can vary	Sizes, material
System Characteristics	Properties or performance of the system	Cost, speed, horse power, weight
State Variables	Internal Variable	Stress, pressure, current
Environmental Variables	Entities that you can vary in your model/simulation but not IRL	Weather phenomena (temperature, humidity)
Operational Variables	May be changed by the operator after the design has been built	Duty cycle, load conditions

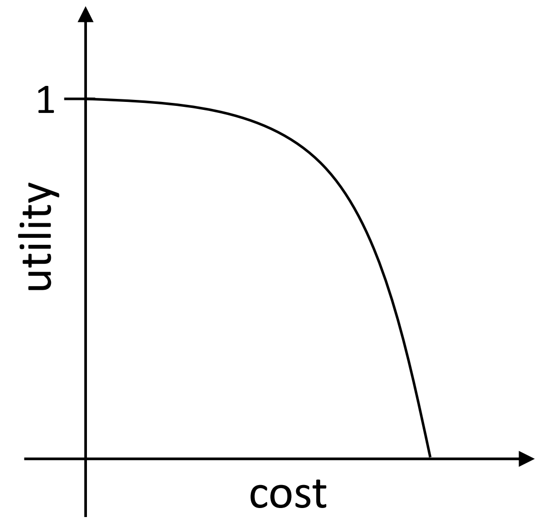
Example – Bicycle model



The concept of value

- Value is the measure of what is good and desirable about a design





"The more the better"

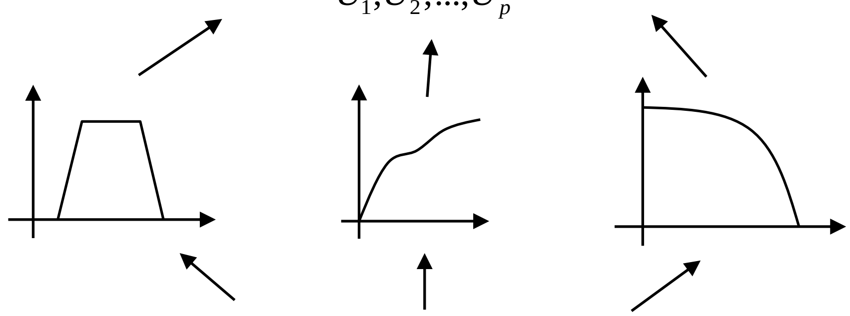
"Less is better"

Hierarchy of Design Variables

Objective function

Value categories

$U_1, U_2, \ldots, U_p$



System characteristics

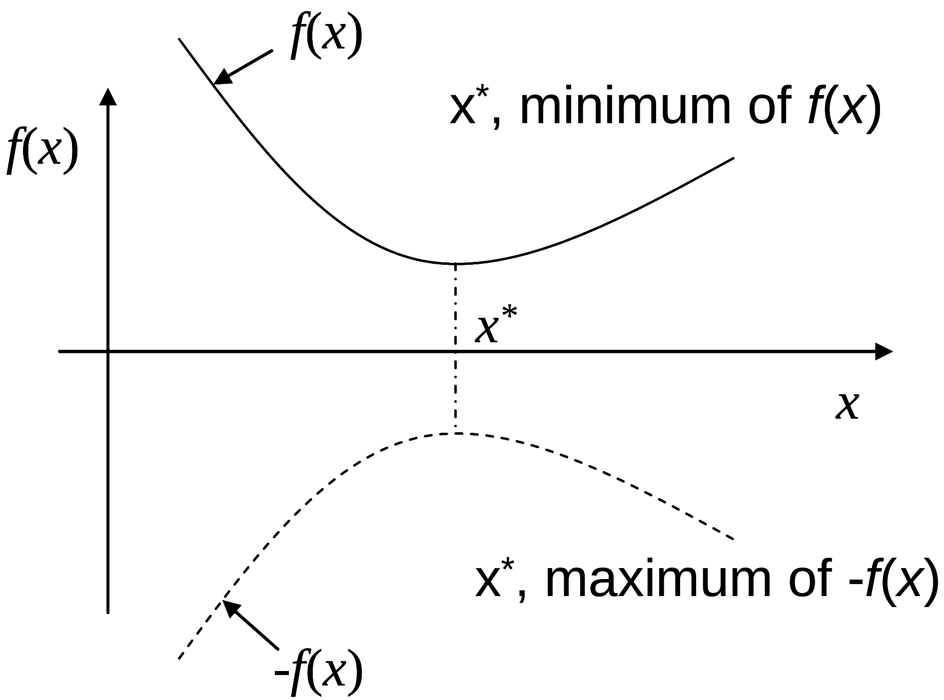
$y_1, y_2, y_3, \ldots, y_m$

Design parameters

$x_1, x_2, x_3, x_4, \ldots, x_n$

Minimization or Maximization?

minimize (f(**x**)) = - max(-f(**x**))
it also gives the same x*



Conditions of optimality

a point $\mathbf{x}^*$ is a strong minimum if there exist a scalar $\delta > 0$ such that:

$$f\left(\mathbf{x}^*\right) < f\left(\mathbf{x}^* + \Delta\mathbf{x}\right) \quad \textit{for all } \Delta\mathbf{x} \qquad 0 < \left\|\Delta\mathbf{x}\right\| < \delta$$

Taylor series expansion

$$f\left(\mathbf{x} + \Delta\mathbf{x}\right) = f\left(\mathbf{x}\right) + \Delta\mathbf{x}^T \nabla f\left(\mathbf{x}\right) + \frac{1}{2}\Delta\mathbf{x}^T \nabla^2 f\left(\mathbf{x}\right)\Delta\mathbf{x} + \ldots$$

1. Necessary conditions for $\mathbf{x}^*$ to be a stationary point $\qquad \nabla f\left(\mathbf{x}^*\right) = 0$

2. Sufficient condition is that the Hessian matrix is positive definite at $\mathbf{x}^*$ $\qquad \nabla^2 f\left(\mathbf{x}^*\right) > 0$

Definition of global optimum

A feasible point $\mathbf{x}^*$ is a global optimum if there exist no other feasible point with a better objective function value, i.e. for a minimization problem

$$f\left(\mathbf{x}^*\right) \leq f\left(\mathbf{x}\right), \forall \mathbf{x} \in S$$

S = Solution space

Definition of local optima

A feasible point **x*** is a local optimum if there in a small surrounding of **x***, δ(**x***), does not exist another feasible point with a better objective function value, i.e. for a minimization problem

$$f\left(\mathbf{x}^*\right) \le f\left(\mathbf{x}\right), \mathbf{x} \in S \text{ and } \mathbf{x} \in \delta\left(\mathrm{x}^*\right)$$

where the surrounding is defined as:

$$\delta\left(\mathbf{x}^*\right) = \left\{\mathbf{x} \middle| \; \left\|\mathbf{x}^* - \mathbf{x}\right\| < \varepsilon, \; \varepsilon > 0\right\}$$

Problem Properties

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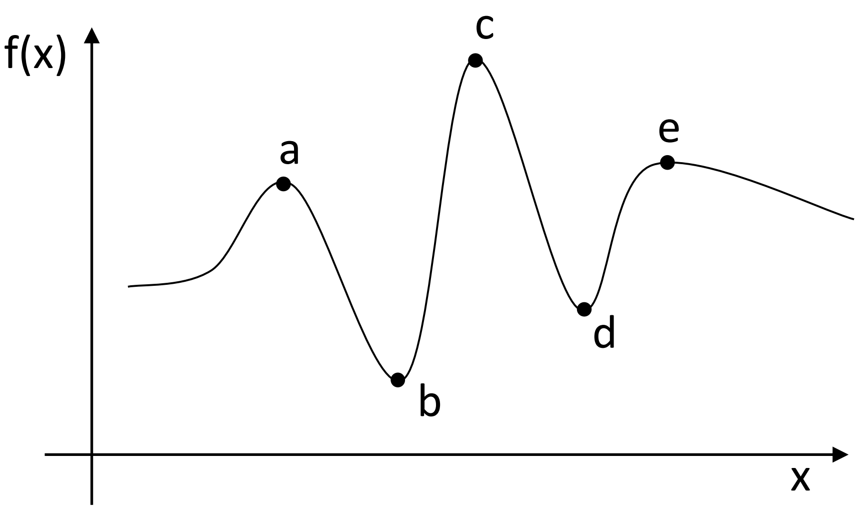
Global and local optima

A function is unimodal if it has just one hump or depression within a defined interval, otherwise it is multi-modal (as below).



A point that is the best in its immediate vicinity is a local optima

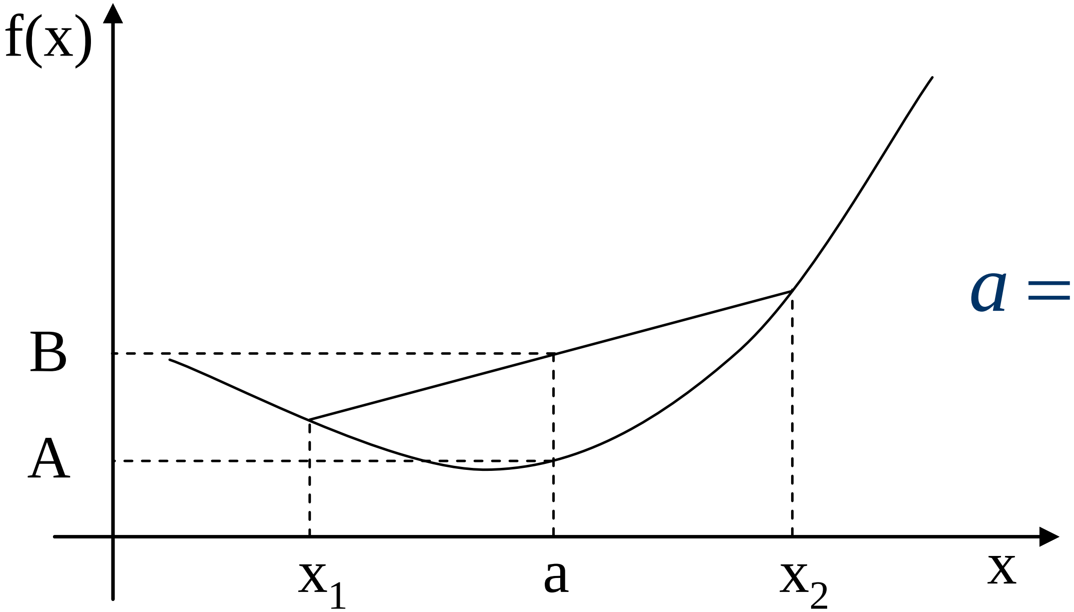
The "best" point of all local optima is the global optima

Convex Functions

Definition

$$f\left(\theta \mathbf{x}_1 + (1-\theta)\mathbf{x}_2\right) \le \theta f\left(\mathbf{x}_1\right) + (1-\theta) f\left(\mathbf{x}_2\right), \quad 0 \le \theta \le 1$$

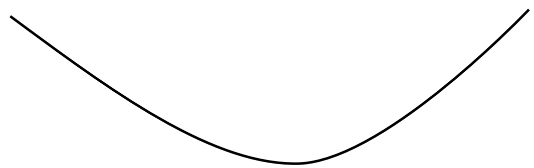
Explanation



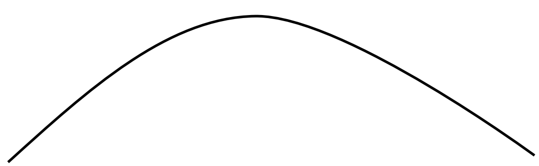
$$a = \theta x_1 + (1-\theta) x_2$$

Convex and Concave Functions

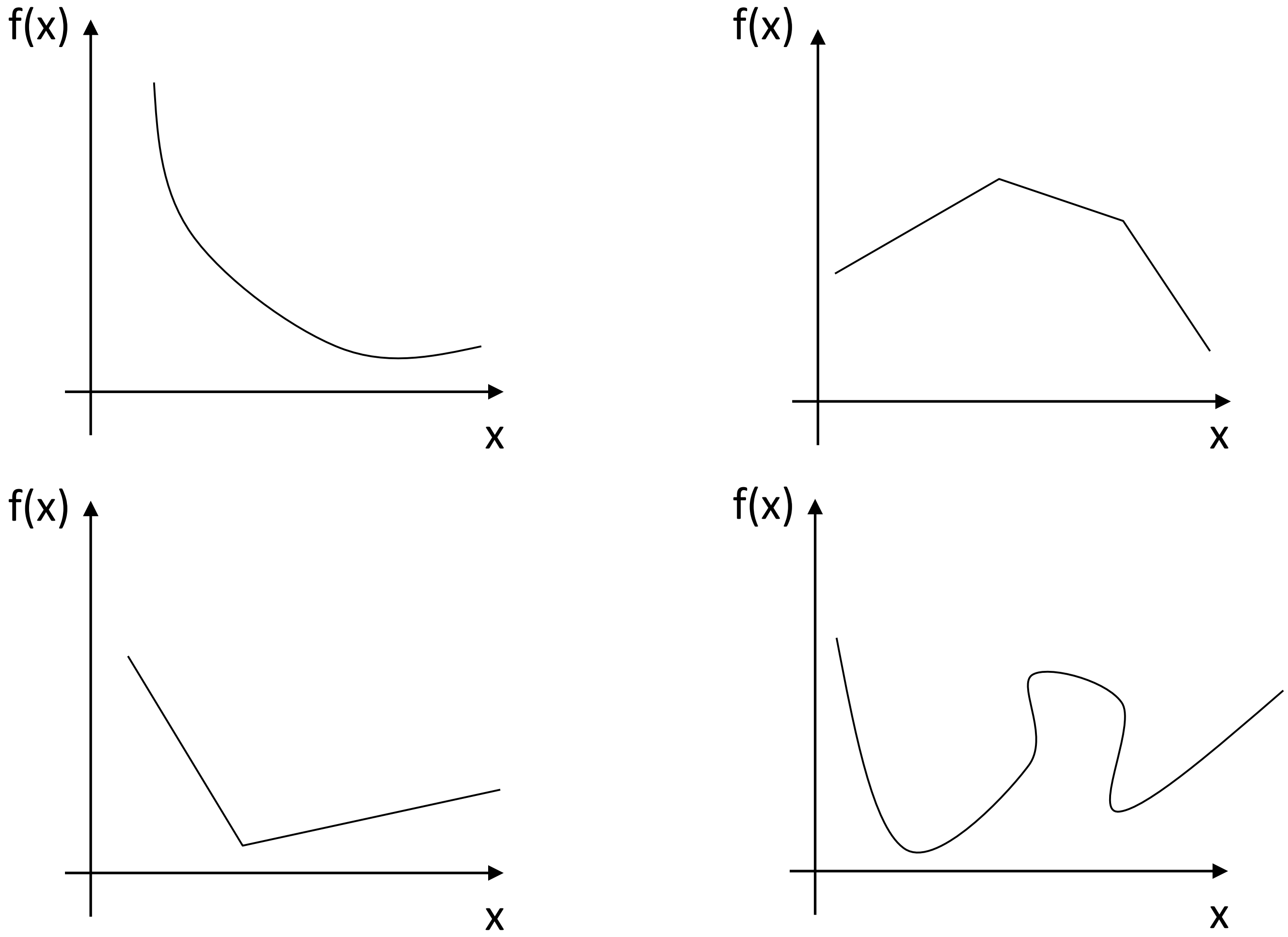
- A function is convex if you can place two points anywhere on the function curve and a line between them always is above the function curve



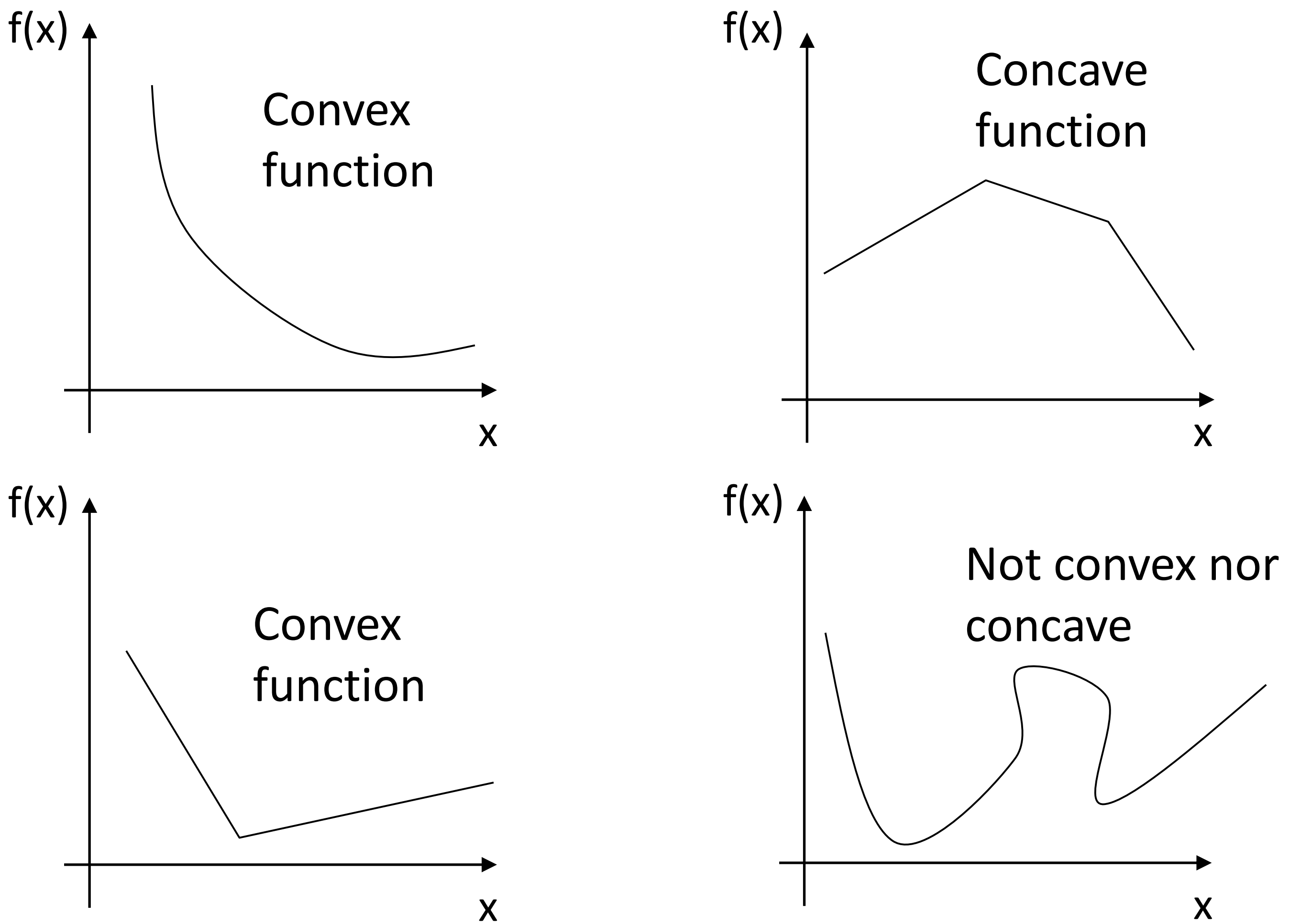
- A function is concave if you can place two points anywhere on the function curve and a line between them always is below the function curve



Convex and Concave Functions



Convex and Concave Functions



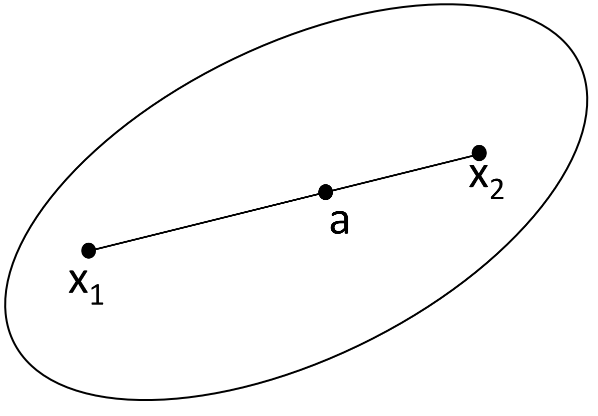
Convex Sets

Definition

A set $S \subseteq R^n$ is convex if there for every choice of points $\mathbf{x}_1$ and $\mathbf{x}_2 \subseteq S$ and $0 \leq \lambda \leq 1$ it is true that:

$$\mathbf{x} = \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \subseteq X$$

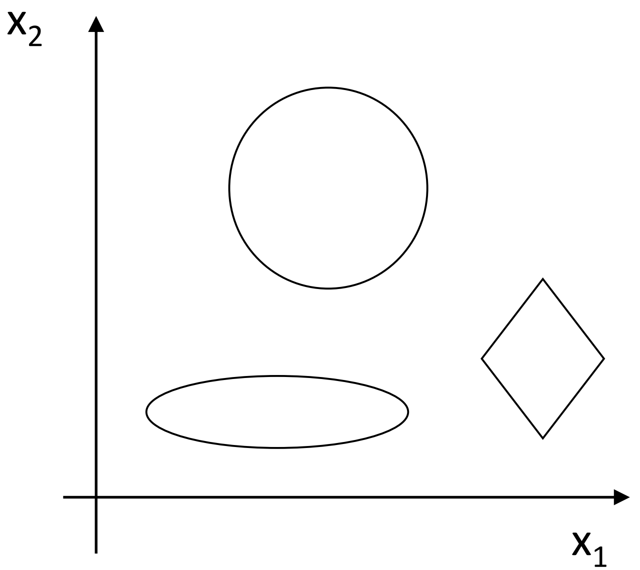
Explanation



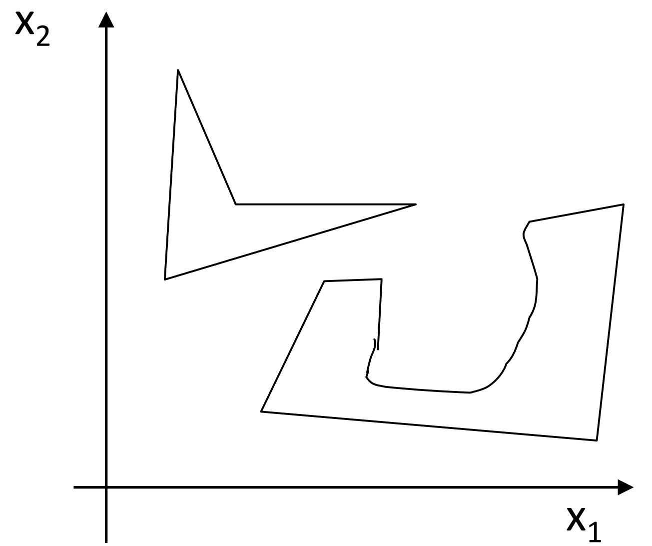
A set is convex if you can place two points anywhere in the set and a line between them always is inside the set

Convex Sets

Convex sets



Non convex sets



If we have a convex objective function and a convex solution space we know that we have found the true optimum

Classification of Optimization Problems

- Constraint
  - Constraint or unconstraint optimization
- Variable
  - Single-variable or multivariable
  - Continuous, discrete or integer optimization
- Objectives: Single or multiobjective optimization
- Linearity: Linear or nonlinear opitmization
- Time: Dynamic or static optimization
- Data: Deterministic or stochastic optimization

Solution approaches in optimization

- Graphical method
  - The objective function is plotted in terms of the design variables. Only possible for low-dimensional problems.

- Analytical technique
  - Based on differential calculus. The objective function is differentiated and set to zero. Can only be used on analytical mathematical models that are differentiable.
- Numerical technique
  - Iterative search processes that make use of information from past iterations. Might use gradient information or not
- Experimental technique
  - Does not require a mathematical model of the physical system

Concluding Remarks

Crucial Questions

- Which are the design parameters that changes during the optimization?
  - Determine design variables
- What shall we optimize for?
  - Objective function?
- How shall we formulate the problem?
  - Constraints that needs to be fulfilled
- Which optimization method shall we use?

Crucial Questions

- Is an optimization needed?
- Is it possible to formulate and solve the problem as an optimization problem?
- How much does the optimization cost?

Consider this

- Optimization does not replace technical know-how and sound judgement
- Requirements and constraints are usually very vague
- The objective function formulation (how to calculate the value of a design) is crucial

Questions?