Sensitivity Analysis

How much a change in a design variable affect the design characteristics.

$$\Delta DC = S \cdot \Delta X$$

change in design variable

change in design characteristic

Sensitivity Matrix

$$\begin{bmatrix} \Delta DC_1 \\ \Delta DC_2 \\ \vdots \\ \Delta DC_m \end{bmatrix} = \begin{bmatrix} \frac{\Delta DC_1}{\Delta X_1} & \cdots & \frac{\Delta DC_1}{\Delta X_n} \\ \frac{\Delta DC_2}{\Delta X_1} & \ddots & \frac{\Delta DC_2}{\Delta X_n} \\ \vdots & & \vdots \\ \frac{\Delta DC_m}{\Delta X_1} & \cdots & \frac{\Delta DC_m}{\Delta X_n} \end{bmatrix} \begin{bmatrix} \Delta X_1 \\ \Delta X_2 \\ \vdots \\ \Delta X_n \end{bmatrix}$$

$$\Delta DC_i = \sum_{j=1}^{n} \frac{\Delta DC_i}{\Delta X_j} \Delta x_j = \sum_{j=1}^{n} S_{ij} \Delta x_j$$

S_{ij} is often calculated using partial derivatives

$$S_{ij} = \frac{\Delta DC_i}{\Delta x_j} = \frac{\Delta y_i}{\Delta x_j} = \frac{\partial y_i}{\partial x_j} = \frac{y_i(\bar{x}+h) - y_i(\bar{x})}{h}$$

Example 1 :

$$\begin{cases} y_1(\bar{x}) = x_1 + 2x_2 \\ y_2(\bar{x}) = 3 \cdot x_2 - x_3^2 \end{cases} \quad , \text{ sensitivities for the point } \bar{x} = [1 \;\; 1 \;\; 3/2]$$

This problem can be calculated with analytical derivatives.

$$\frac{\partial y_1}{\partial x_1} = 1 \qquad \frac{\partial y_1}{\partial x_2} = 2 \qquad \frac{\partial y}{\partial x_3} = 0$$

$$\frac{\partial y_2}{\partial x_1} = 0 \qquad \frac{\partial y_2}{\partial x_2} = 3 \qquad \frac{\partial y_2}{\partial x_3} = -2x_3 = / x_3 = \tfrac{3}{2} / = -3$$

If we do not know the analytical functions (e.g. the electric motorcycle)

$$\frac{\partial y_1}{\partial x_1} = \frac{y_1(\bar{x}+h) - y_1(\bar{x})}{h} = \frac{y_1(1.01 \;\; 1 \;\; \frac{3}{2}) - y_1(1 \;\; 1 \;\; \frac{3}{2})}{0.01} = \frac{2.51 - 2.5}{0.01} = \frac{0.01}{0.01} = 1$$

$$\Rightarrow \begin{bmatrix} \Delta y_1 \\ \Delta y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -3 \end{bmatrix}}_{S} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

A change of 1 unit in x_2 results in a change of 2 units in y_1 and 3 units in y_2

Normalized sensitivities :

$$S_{norm,ij} = S_{ij} \cdot \frac{\partial x_{j0}}{\partial y_{i0}} = \frac{\partial y_i}{\partial x_j} \cdot \frac{\partial x_{j0}}{\partial y_{i0}}$$

$$\Rightarrow \quad S_{norm,11} = S_{11} \cdot \frac{x_{10}}{y_{10}} = 1 \cdot \frac{1}{1+2} = \frac{1}{3}$$

$$S_{norm,12} = S_{12} \cdot \frac{x_{20}}{y_{10}} = 2 \cdot \frac{1}{1+2} = \frac{2}{3}$$

$$\Rightarrow \quad S_{norm} = \begin{bmatrix} 1/3 & 2/3 & 0 \\ 0 & 4 & -6 \end{bmatrix}$$

A change of 1% in x_2 results in a change of 0.67% in y_1 and 4% in y_2

Normalized sensitivities

$$S_{norm,ij} = \frac{\Delta DC_i}{\Delta x_j} \cdot \frac{x_{j0}}{DC_{i0}}$$

x_{j0} ~ reference x_j-value

DC_{i0} ~ reference DC_i-value

Values far from 0 have high impacts on the DC's.

Ex:

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ S_{norm} = & \multicolumn{3}{c}{\begin{bmatrix} 0.1 & 0.8 & -0.2 \\ -0.2 & 0.3 & -0.7 \end{bmatrix}} \end{array}$$

- higher x_2 is most important to increase y_1
- higher x_3 is most important to decrease y_2

x_1 is not so important since it barely affects y_1 and y_2

$\Rightarrow$ Put extra focus on x_2 and x_3 when you design your product