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ORIGINAL ARTICLE

Relation between the lifting surface theory and the lifting line theory in the design of an optimum screw propeller

Koichi Koyama

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Abstract A theory on an optimum screw propeller is described. The optimum means optimum efficiency of a propeller, that is, maximizing thrust horse power for a given shaft horse power. The theory is based on the propeller lifting surface theory. Circulation density (lift density) of the blade is determined by the lifting surface theory on a specified condition in general. However, it is shown that, in the case of optimum condition, the circulation density is not determined by the lifting surface theory, although the circulation distribution which is the chordwise integral of the circulation density is determined. The reason is that the governing equation of the optimization by the lifting surface theory is reduced to that by the lifting line theory. This theoretical deduction is the main part of this paper. The importance of the lifting line theory in the design of the optimum propeller is made clear. Numerical calculations support the conclusion from the deduction. This is shown in the case of freely running propellers and in the case of wake adapted propellers.

Keywords Optimum screw propeller · Propeller lifting surface theory · Propeller lifting line theory

1 Introduction

It goes without saying that solving for the optimum propeller from the efficiency point of view is the one of the important of the work to be done in propeller design. Optimization of the propeller operated in the non-uniform flow field of the wake behind a ship's hull is the one of the important themes. The propeller fulfilling the optimum condition is called the wake adapted propeller. Van Manen [1], Lerbs [2], Hanaoka [3] studied the theme based on the propeller lifting line theory. Studies after them were reviewed in references [5, 6]. Recently a study on the optimization using the numerical lifting surface theory was presented [7]. The author tried to obtain the optimum propeller based on the propeller lifting surface theory [8, 9]. But the trial did not succeed and, on the contrary, the importance of the lifting line theory did became clear. It became clear that the trial was impossible in principle. These are shown in this paper. In other words, this paper shows the importance of the lifting line theory in the design of the optimum propeller in the uniform inflow and in the non-uniform inflow.

In this paper the optimum propeller is considered based on the lifting surface theory which treats a vortex surface system including lifting surface and trailing vortex in the potential flow. This problem is to solve the circulation density of the bound vortex representing the blade for the given condition. Expressions of the propeller lifting surface theory based on helical coordinates are shown first. Using the expressions, the energy equation of the propeller is shown. Optimum condition using the energy equation is transformed into the one of the calculus of variation. It is shown theoretically that the calculus of variation based on the propeller lifting surface theory is reduced to the calculus of variation based on the propeller lifting line theory. And it is shown how the theoretical result appears in the numerical calculation. These are shown for the constant hydrodynamic pitch model for freely running propellers in the first chapter and for the variable hydrodynamic pitch model for wake adapted propellers in the next chapter.

K. Koyama (✉)
Retired, National Maritime Research Institute,
1-5-17 Iwadokita, Komae, Tokyo 2010004, Japan
e-mail: koyama_koichi@hotmail.com

2 A propeller in uniform inflow

Before going to the discussion on the optimum propeller in non-uniform inflow (propeller in a hull wake) we consider the case of the optimum propeller in uniform inflow (propeller in open water) in this chapter.

In this case the model that the hydrodynamic pitch is constant radially is useful. The optimum propeller based on the model is discussed in this chapter.

$$h = h' \tag{1}$$

where $h = h(r)$, $h' = h(r')$, and $2\pi h$ denotes the hydrodynamic pitch. r, r' denote radius. The model is used for the case of the linear theory in which the elements of the free vortex sheet emanated from the blade stay on the generated position due to negligible weak induced velocity. The model is also used for the case of the non-linear theory based on the constant hydrodynamic pitch in which the free vortex sheet is deformed by the induced velocity but the pitch of the sheet is assumed to be constant radially.

In this chapter we consider the case that a propeller advances with a constant velocity V and rotates with a constant angular speed Ω in still water. Axial and tangential inflow velocity at the radius r on the propeller rotating disc are denoted by V, Ωr, respectively.

The bound vortex and the trailing vortex are assumed to be on a helical surface. The pitch of the helical surface is given by $2\pi h(r)$, where

$$\frac{h(r)}{r} = \tan\varepsilon_1 = \frac{V + w_a}{\Omega r + w_t} \tag{2}$$

and w_a, w_t denote axial and tangential induced velocity at the lifting line representing the blade (Fig. 1). The pitch $2\pi h(r)$ is called hydrodynamic pitch. It is assumed that the pitch doesn't vary in the axial direction. In general $h(r)$ varies in radial direction or $h(r)$ is a function of r. But in this chapter we assume that $2\pi h$ is constant in radial direction using the average value of $h(r)$, which is shown by Eq. 1.

2.1 Propeller lifting surface theory using helical coordinates with constant pitch

We use the helical coordinates (τ, σ, μ) with the constant pitch $2\pi h$ which relates to the cylindrical coordinates (x, r, θ) as follows (Fig. 2)

$$\tau = \theta + x/h, \quad \sigma = \theta - x/h, \quad \mu = r/h = 1/\tan\varepsilon_1 \tag{3}$$

$$\tau' = \theta' + x'/h, \quad \sigma' = \theta' - x'/h, \quad \mu' = r'/h = 1/\tan\varepsilon'_1 \tag{4}$$

where the dash$'$ denotes that these quantities are related to the point on the blade.

Normal line to the helical surface is assumed approximately to be the normal line to both the radius vector r and the helix, then the normal derivative is expressed by

$$\begin{aligned}\frac{\partial}{\partial n} &= -\cos\varepsilon_1 \frac{\partial}{\partial x} + \sin\varepsilon_1 \frac{\partial}{r\partial\theta} \\ &= \frac{1}{h\sqrt{1+\mu^2}}\left\{\left(\mu + \frac{1}{\mu}\right)\frac{\partial}{\partial\sigma} - \left(\mu - \frac{1}{\mu}\right)\frac{\partial}{\partial\tau}\right\}.\end{aligned} \tag{5}$$

The segment of the helix of $\mu = $ const., $\sigma = $ const. is expressed by

$$ds = \frac{1}{2}h\sqrt{1+\mu^2}d\tau. \tag{6}$$

Then the perturbation velocity potential for the flow around the propeller is expressed by

$$\Phi = \frac{1}{4\pi}\sum_{m=0}^{l-1}\int_{r_b}^{r_o} dr' \int_{s_1}^{s_2} \gamma ds' \int_{s'}^{\infty} \frac{\partial}{\partial n''}\frac{1}{R} ds'' \tag{7}$$

where l denotes the number of blades, integral means on the helical surface, r_o, r_b denote propeller radius and boss radius, s_1, s_2 denotes the s coordinates of the leading edge and the trailing edge of blade section, and $\gamma = \gamma(s', r')$ denotes circulation density on the blade, and

$$ds' = \frac{1}{2}h\sqrt{1+\mu'^2}d\tau' \tag{8}$$

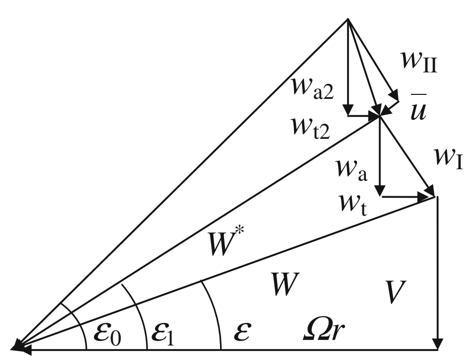

Fig. 1 Inflow and induced velocity vector

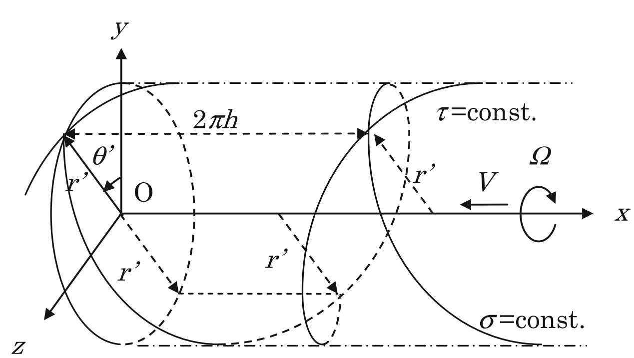

Fig. 2 Helical coordinates

$$\mathrm{d}s'' = \frac{1}{2}h\sqrt{1+\mu'^2}\mathrm{d}T' \tag{9}$$

$$\begin{aligned}\frac{\partial}{\partial n''} &= -\cos\varepsilon'_1\frac{\partial}{\partial x'} + \sin\varepsilon'_1\frac{\partial}{r'\partial\theta'} \\ &= \frac{1}{h\sqrt{1+\mu'^2}}\left\{\left(\mu' + \frac{1}{\mu'}\right)\frac{\partial}{\partial\sigma'} - \left(\mu' - \frac{1}{\mu'}\right)\frac{\partial}{\partial T'}\right\}\end{aligned} \tag{10}$$

$$\begin{aligned}R &= \sqrt{(x-x')^2 + r^2 + r'^2 - 2rr'\cos(\theta - \theta' - 2\pi m/l)}\Bigg| \\ &= h\sqrt{\left(\frac{\tau-\sigma}{2} - \frac{\tau'-\sigma'}{2}\right)^2 + \mu^2 + \mu'^2 - 2\mu\mu'\cos\left(\frac{\tau+\sigma-\tau'-\sigma'}{2} - \frac{2\pi m}{l}\right)}\Bigg|_{\tau'=T'}\end{aligned} \tag{11}$$

As $\sigma = \sigma'$ on the helical lifting surface representing blades, upwash on the blade $w = w(s, r)$ is expressed by

$$w = \frac{\partial\Phi}{\partial n}\bigg|_{\sigma=\sigma'} = \frac{1}{4\pi}\sum_{m=0}^{l-1}\int_{r_b}^{r_o}\mathrm{d}r'\int_{s_1}^{s_2}\gamma\mathrm{d}s'\int_{s'}^{\infty}\frac{\partial^2}{\partial n\partial n''}\frac{1}{R}\bigg|_{\sigma=\sigma'}\mathrm{d}s''. \tag{12}$$

Substituting Eqs. 5 and 8–11 into Eq. 12 we obtain

$$w = -\int_{r_b}^{r_o}\frac{2}{h^2\sqrt{1+\mu'^2}}\mathrm{d}r'\int_{s_1}^{s_2}\gamma K\left(\frac{\tau-\tau'}{2};\mu,\mu'\right)\mathrm{d}s' \tag{13}$$

$$K\left(\frac{\tau-\tau'}{2};\mu,\mu'\right) = -\frac{1}{4\pi}\frac{h^2\sqrt{1+\mu'^2}}{2}\sum_{m=0}^{l-1}\int_{s'}^{\infty}\frac{\partial^2}{\partial n\partial n''}\frac{1}{R}\bigg|_{\sigma=\sigma'}\mathrm{d}s'' \tag{14}$$

$$K(v;\mu,\mu') = \frac{-1}{8\pi}\frac{\sqrt{1+\mu'^2}}{\sqrt{1+\mu^2}}\sum_{m=0}^{l-1}\int_{-\infty}^{v}\times\left\{\frac{\mu\mu' + \cos v'_m}{\bar{R}^3} - \frac{3(\mu v' - \mu'\sin v'_m)(\mu' v' - \mu\sin v'_m)}{\bar{R}^5}\right\}\mathrm{d}v' \tag{15}$$

$$v = (\tau - \tau')/2, \quad v' = (T - \tau')/2, \quad v'_m = v' - 2m\pi/l, \quad \mathrm{d}v' = \frac{1}{2}\mathrm{d}T \tag{16}$$

$$\bar{R} = \sqrt{v'^2 + \mu^2 + \mu'^2 - 2\mu\mu'\cos v'_m} \tag{17}$$

As the upwash w consists of the lifting line term w_{I} and the lifting surface term w_{II}, and kernel function $K(v;\ \mu,\ \mu')$ has a singularity of the 2nd order pole [3], so we get the following results.

$$\begin{aligned}-w &= -w_{\mathrm{I}} - w_{\mathrm{II}} = \int_{\mu_b}^{\mu_o}\mathrm{d}\mu'\int_{\tau_1}^{\tau_2}\gamma K\left(\frac{\tau-\tau'}{2};\mu,\mu'\right)\mathrm{d}\tau' \\ &= \int_{r_b}^{r_o}\frac{2}{\sqrt{1+\mu'^2}}\frac{1}{(r-r')^2}\mathrm{d}r'\int_{s_1}^{s_2}\gamma\bar{K}\left(\frac{\tau-\tau'}{2};\mu,\mu'\right)\mathrm{d}s'\end{aligned} \tag{18}$$

$$\begin{aligned}-w_{\mathrm{I}} &= \int_{\mu_b}^{\mu_o}\mathrm{d}\mu'\int_{\tau_1}^{\tau_2}\gamma K(0;\mu,\mu')\mathrm{d}\tau' \\ &= \int_{r_b}^{r_o}\frac{2}{\sqrt{1+\mu'^2}}\frac{\bar{K}(0;\mu,\mu')}{(r-r')^2}\int_{s_1}^{s_2}\gamma\mathrm{d}s'\mathrm{d}r' \\ &= \int_{r_b}^{r_o}\frac{2}{\sqrt{1+\mu'^2}}\frac{\bar{K}(0;\mu,\mu')}{(r-r')^2}\Gamma\mathrm{d}r'\end{aligned} \tag{19}$$

$$\begin{aligned}-w_{\mathrm{II}} &= \int_{\mu_b}^{\mu_o}\mathrm{d}\mu'\int_{\tau_1}^{\tau_2}\gamma K^{(0)}\left(\frac{\tau-\tau'}{2};\mu,\mu'\right)\mathrm{d}\tau' \\ &= \int_{r_b}^{r_o}\frac{2}{\sqrt{1+\mu'^2}}\frac{1}{(r-r')^2}\mathrm{d}r'\int_{s_1}^{s_2}\gamma\bar{K}^{(0)}\left(\frac{\tau-\tau'}{2};\mu,\mu'\right)\mathrm{d}s'\end{aligned}, \tag{20}$$

where

$$\begin{aligned}K(v;\mu,\mu') &= \frac{\bar{K}(v;\mu,\mu')}{(\mu-\mu')^2} = \frac{h^2}{(r-r')^2}\bar{K}(v;\mu,\mu') \\ &= \frac{h^2}{(\eta-\eta')^2\bar{r}^2}\bar{K}(v;\mu,\mu')\end{aligned} \tag{21}$$

$$K^{(0)}(v;\mu,\mu') = K(v;\mu,\mu') - K(0;\mu,\mu') = \frac{\bar{K}^{(0)}(v;\mu,\mu')}{(\mu-\mu')^2} \tag{22}$$

$$K^{(0)}(v;\mu,\mu') = \frac{-1}{8\pi}\frac{\sqrt{1+\mu'^2}}{\sqrt{1+\mu^2}}\sum_{m=0}^{l-1}\int_{0}^{v}\left\{\frac{\mu\mu' + \cos v'_m}{\bar{R}^3} - \frac{3(\mu v' - \mu'\sin v'_m)(\mu' v' - \mu\sin v'_m)}{\bar{R}^5}\right\}\mathrm{d}v' \tag{23}$$

$$\Gamma = \int \gamma \mathrm{d}s. \tag{24}$$

In the case of the lifting line theory, w_{II} (Eq. 20) becomes [3]

$$-w_{\mathrm{II}} = \frac{1}{2\pi}\int_{s_1}^{s_2} \frac{\gamma(s',r)}{s-s'}\mathrm{d}s'. \tag{25}$$

This is deduced from analyzing the singularity of the integrand of w_{II}.

Expressions shown above are for the theory of constant pitch helical surface. It should be noted that the radial component of the flow is ignored as the effect is considered to be small in this paper.

2.2 Work done by a propeller and energy loss of a propeller

Work done against the surrounding fluid by a propeller in a unit time is expressed by

$$P = -l\rho \iint \frac{\partial\Phi}{\partial n}(W^* + \bar{u})\gamma \mathrm{d}r\mathrm{d}s \tag{26}$$

where the integral domain is the lifting surface corresponding to a blade of the propeller, ρ is the density of the fluid, $W^* + \bar{u}$ denotes the velocity at the blade (Fig. 1)

$$W^* = \sqrt{(V + w_{\mathrm{a}})^2 + (\Omega r + w_{\mathrm{t}})^2}. \tag{27}$$

$\bar{u}$ denotes the velocity component due to the correction term for the lifting surface. $\rho(W^* + \bar{u})\gamma$ implies lift density. $\partial\Phi/\partial n$ is the upwash on the lifting surface given by Eq. 12.

As shown in Appendix 3, we have the relation

$$\begin{aligned} -(W^* + \bar{u})\frac{\partial\Phi}{\partial n} &= \Omega r\{V + w_{\mathrm{a}} + w_{\mathrm{a2}}\} \\ &\quad - V\{\Omega r + w_{\mathrm{t}} + w_{\mathrm{t2}}\}. \end{aligned} \tag{28}$$

Then the expression

$$\begin{aligned} P &= l\rho \iint \gamma\Omega r\{V + w_{\mathrm{a}} + w_{\mathrm{a2}}\}\mathrm{d}r\mathrm{d}s \\ &\quad - l\rho \iint \gamma V\{\Omega r + w_{\mathrm{t}} + w_{\mathrm{t2}}\}\mathrm{d}r\mathrm{d}s \end{aligned} \tag{29}$$

is obtained. The torque and the thrust working on the small area drds of the lifting surface are denoted by dQ, dS,

$$\begin{aligned} \mathrm{d}Q &= l\rho\gamma r\{(W^* + \bar{u})\sin\varepsilon_1 - w_{\mathrm{II}}\cos\varepsilon_1\}\mathrm{d}r\mathrm{d}s \\ &= l\rho\gamma r\{V + w_{\mathrm{a}} + w_{\mathrm{a2}}\}\mathrm{d}r\mathrm{d}s \end{aligned} \tag{30}$$

$$\begin{aligned} \mathrm{d}S &= l\rho\gamma\{(W^* + \bar{u})\cos\varepsilon_1 + w_{\mathrm{II}}\sin\varepsilon_1\}\mathrm{d}r\mathrm{d}s \\ &= l\rho\gamma\{\Omega r + w_{\mathrm{t}} + w_{\mathrm{t2}}\}\mathrm{d}r\mathrm{d}s. \end{aligned} \tag{31}$$

Then Eq. 29 becomes

$$P = \Omega\iint \mathrm{d}Q - V\iint \mathrm{d}S. \tag{32}$$

The first term and the second term of the right hand side of Eq. 29 imply the shaft horse power and the thrust horse power, respectively, so P implies the energy loss of the propeller. The above transformation shows that the work done against surrounding fluid by the propeller (Eq. 26) is equal to the energy loss of the propeller (Eq. 32).

Velocity component due to the correction term for the lifting surface $\bar{u}$ is small, so it will be neglected below. P and P_{I} (the lifting line component of P) and P_{II} (the correction component of P for lifting surface) are shown as follows

$$P = P_{\mathrm{I}} + P_{\mathrm{II}} \tag{33}$$

$$\begin{aligned} P &= -l\rho \int\int \gamma W^* \frac{\partial\Phi}{\partial n}\mathrm{d}r\mathrm{d}s \\ &= l\rho\int_{r_{\mathrm{b}}}^{r_{\mathrm{o}}} W^*\mathrm{d}r\int_{s_1}^{s_2}\gamma ds\int_{r_{\mathrm{b}}}^{r_{\mathrm{o}}}\frac{2}{h^2\sqrt{1+\mu'^2}}\mathrm{d}r'\int_{s_1}^{s_2}\gamma' K\left(\frac{\tau-\tau'}{2};\mu,\mu'\right)\mathrm{d}s' \end{aligned} \tag{34}$$

$$\begin{aligned} P_{\mathrm{I}} &= -l\rho\iint \gamma W^*\frac{\partial\Phi_{\mathrm{I}}}{\partial n}\mathrm{d}r\mathrm{d}s \\ &= l\rho\int_{r_{\mathrm{b}}}^{r_{\mathrm{o}}} W^*\mathrm{d}r\int_{s_1}^{s_2}\gamma\mathrm{d}s\int_{r_{\mathrm{b}}}^{r_{\mathrm{o}}}\frac{2}{h^2\sqrt{1+\mu'^2}}\mathrm{d}r'\int_{s_1}^{s_2}\gamma' K(0;\mu,\mu')\mathrm{d}s' \\ &= l\rho\int_{r_{\mathrm{b}}}^{r_{\mathrm{o}}} W^*\Gamma\mathrm{d}r\int_{r_{\mathrm{b}}}^{r_{\mathrm{o}}}\frac{2}{h^2\sqrt{1+\mu'^2}}\Gamma' K(0;\mu,\mu')\mathrm{d}r' \end{aligned} \tag{35}$$

$$\begin{aligned} P_{\mathrm{II}} &= -l\rho\iint\gamma W^*\frac{\partial\Phi_{\mathrm{II}}}{\partial n}\mathrm{d}r\mathrm{d}s \\ &= l\rho\int_{r_{\mathrm{b}}}^{r_{\mathrm{o}}} W^*\mathrm{d}r\int_{s_1}^{s_2}\gamma\mathrm{d}s\int_{r_{\mathrm{b}}}^{r_{\mathrm{o}}}\frac{2}{h^2\sqrt{1+\mu'^2}}\mathrm{d}r'. \\ &\quad\times\int_{s_1}^{s_2}\gamma' K^{(0)}\left(\frac{\tau-\tau'}{2};\mu,\mu'\right)\mathrm{d}s' \end{aligned} \tag{36}$$

2.3 Optimum propeller in uniform inflow

Our task is to seek the optimum solution so that the value P is minimum for the given shaft horse power. So P should be a minimum on condition that the first term of P (Eq. 29) is specified. In that case we can get the solution by making the following functional F minimum.

$$\begin{aligned} F &= l\rho\iint k\gamma\Omega r(V + w_{\mathrm{a}} + w_{\mathrm{a2}})\mathrm{d}r\mathrm{d}s \\ &\quad - l\rho\iint\gamma V(\Omega r + w_{\mathrm{t}} + w_{\mathrm{t2}})\mathrm{d}r\mathrm{d}s \end{aligned} \tag{37}$$

$$\delta F = 0 \tag{38}$$

where k is the multiplier of Lagrange. This is the formulation from the propeller lifting surface theory. The expression will be transformed to the expression by the lifting line theory below.

First of all we have to investigate P_{II}. Using Eq. 36 we can expand as follows

$$\begin{aligned} P_{\mathrm{II}} &= l\rho \int_{r_b}^{r_o} W^* \mathrm{d}r \int_{s_1}^{s_2} \gamma \mathrm{d}s \int_{r_b}^{r_o} \frac{2}{h^2\sqrt{1+\mu'^2}} \mathrm{d}r' \\ &\quad \times \int_{s_1}^{s_2} \gamma' K^{(0)}\left(\frac{\tau-\tau'}{2}; \mu, \mu'\right) \mathrm{d}s' \\ &= l\rho \int_{r_b}^{r_o} \frac{W^*}{\sqrt{1+\mu^2}} \mathrm{d}r \int_{s_1}^{s_2} \gamma \mathrm{d}s \int_{r_b}^{r_o} \frac{2}{h^2} \sqrt{\frac{1+\mu^2}{1+\mu'^2}} \mathrm{d}r' \\ &\quad \times \int_{s_1}^{s_2} \gamma' K^{(0)}\left(\frac{\tau-\tau'}{2}; \mu, \mu'\right) \mathrm{d}s' \\ &= l\rho \int_{r_b}^{r_o} (V+w_a) \mathrm{d}r \int_{s_1}^{s_2} \gamma \mathrm{d}s \int_{r_b}^{r_o} \frac{2}{h^2} \sqrt{\frac{1+\mu^2}{1+\mu'^2}} \mathrm{d}r' \\ &\quad \times \int_{s_1}^{s_2} \gamma' K^{(0)}\left(\frac{\tau-\tau'}{2}; \mu, \mu'\right) \mathrm{d}s' \\ &= l\rho (V+w_a) \int_{r_b}^{r_o} \mathrm{d}r \int_{s_1}^{s_2} \gamma(\tau,\mu) \mathrm{d}s \int_{r_b}^{r_o} \frac{2}{h^2} \sqrt{\frac{1+\mu^2}{1+\mu'^2}} \mathrm{d}r' \\ &\quad \times \int_{s_1}^{s_2} \gamma'(\tau',\mu') K^{(0)}\left(\frac{\tau-\tau'}{2}; \mu, \mu'\right) \mathrm{d}s' \end{aligned} \tag{39}$$

Here it has been assumed

$$V + w_a = \text{constant.} \tag{40}$$

Next, the relation for the kernel function (cf. Appendix 2)

$$\sqrt{\frac{1+\mu^2}{1+\mu'^2}} K^{(0)}(v;\mu,\mu') = -\sqrt{\frac{1+\mu'^2}{1+\mu^2}} K^{(0)}(-v;\mu',\mu) \tag{41}$$

is used. When in the final expression of Eq. 39 for P_{II} the order of integration (drds) and (dr'ds') is changed, and then the notations (r, s) and (r', s') are also changed for each of the others, the expression P_{II} becomes the same form as before but changes its sign. So the value must be zero.

$$P_{\mathrm{II}} = 0 \tag{42}$$

Then from Eqs. 33 and 42

$$P = P_{\mathrm{I}}. \tag{43}$$

These results shows Munk's theorems [3].

Munk's theorem II (Eq. 42): energy loss from two bound vortex elements vanishes for each other.

Munk's theorem I (Eq. 43): energy loss is not altered when the bound vortex elements are displaced in blade moving direction.

The situation is similar to the wing going straight on. However, the difference between the wing and the propeller blade should be noticed. In case of the wing, the minimum induced drag does not change when one of the lift element shifts in the moving direction according to Munk's theorem I. In case of the propeller, it is not thrust (or torque) but energy loss P is unchanged when one of the lift elements shifts in the moving direction. It is expected at this moment that the optimum lift distribution not only radial but also in a chordwise direction is determined for a given torque. These results are the same as the results already shown by Hanaoka [3, 4].

Next we have to investigate F (Eq. 37). The ratio of the lifting surface correction term of thrust to the lifting line term of thrust is developed as follows.

$$\begin{aligned} S_s/S_l &= l\rho \iint \gamma(w_{t2}) \mathrm{d}r\mathrm{d}s \Big/ l\rho \iint \gamma(\Omega r + w_t) \mathrm{d}r\mathrm{d}s \\ &= \iint \gamma(w_{\mathrm{II}} \sin\varepsilon_1) \mathrm{d}r\mathrm{d}s \Big/ \iint \gamma\mu(V+w_a) \mathrm{d}r\mathrm{d}s \\ &= \iint \gamma\left(w_{\mathrm{II}} \frac{1}{\sqrt{1+\mu^2}}\right) \mathrm{d}r\mathrm{d}s \Big/ (V+w_a) \iint \gamma\mu \mathrm{d}r\mathrm{d}s \end{aligned} \tag{44}$$

In the meantime the ratio of the lifting surface correction term of torque to the lifting line term of torque is developed as follows.

$$\begin{aligned} Q_s/Q_l &= l\rho \iint \gamma r(w_{a2}) \mathrm{d}r\mathrm{d}s \Big/ l\rho \iint \gamma r(V+w_a) \mathrm{d}r\mathrm{d}s \\ &= \iint \gamma r(-w_{\mathrm{II}} \cos\varepsilon_1) \mathrm{d}r\mathrm{d}s \Big/ \iint \gamma r(V+w_a) \mathrm{d}r\mathrm{d}s \\ &= \iint \gamma r\left(w_{\mathrm{II}} \frac{-\mu}{\sqrt{1+\mu^2}}\right) \mathrm{d}r\mathrm{d}s \Big/ (V+w_a) \iint \gamma r \mathrm{d}r\mathrm{d}s \\ &= \iint \gamma\left(w_{\mathrm{II}} \frac{-\mu^2}{\sqrt{1+\mu^2}}\right) \mathrm{d}r\mathrm{d}s \Big/ (V+w_a) \iint \gamma\mu \mathrm{d}r\mathrm{d}s \end{aligned} \tag{45}$$

Here we assume Eq. 40 in Eqs. 44, 45.

From Eq. 42

$$\begin{aligned} P_{\mathrm{II}} &= -l\rho \iint \gamma W^* w_{\mathrm{II}} \mathrm{d}r\mathrm{d}s \\ &= -l\rho(V+w_a) \iint \gamma\sqrt{1+\mu^2} w_{\mathrm{II}} \mathrm{d}r\mathrm{d}s = 0. \end{aligned} \tag{46}$$

Then

$$\iint \gamma \frac{1+\mu^2}{\sqrt{1+\mu^2}} w_{\mathrm{II}} \mathrm{d}r\mathrm{d}s = 0. \tag{47}$$

Using Eq. 47, it is easy to show Eqs. 44 and 45 are the same.

$$S_{\mathrm{s}}/S_{\mathrm{l}} = Q_{\mathrm{s}}/Q_{\mathrm{l}} = \alpha \tag{48}$$

Then Eq. 37 can be developed as follows.

$$\begin{aligned} F &= l\rho(1+\alpha)\iint k\gamma\Omega r(V+w_{\mathrm{a}})\mathrm{d}r\mathrm{d}s - l\rho(1+\alpha) \\ &\quad\times \iint \gamma V(\Omega r + w_{\mathrm{t}})\mathrm{d}r\mathrm{d}s \\ &= l\rho(1+\alpha)\int k\Omega r(V+w_{\mathrm{a}})\left(\int \gamma \mathrm{d}s\right)\mathrm{d}r - l\rho(1+\alpha) \\ &\quad\times \int V(\Omega r + w_{\mathrm{t}})\left(\int \gamma \mathrm{d}s)\mathrm{d}r\right) \\ &= l\rho(1+\alpha)\int k\Gamma\Omega r(V+w_{\mathrm{a}})\mathrm{d}r - l\rho(1+\alpha) \\ &\quad\times \int \Gamma V(\Omega r + w_{\mathrm{t}})\mathrm{d}r \end{aligned} \tag{49}$$

The optimum problem for the lifting surface (Eq. 37) is transformed to the optimum problem for the lifting line (Eq. 49). The solution of Eq. 38 is drawn not from the lifting surface formulation but from the lifting line formulation. So it becomes clear that it is impossible to determine the chordwise distribution of lift from the optimum energy condition.

The problem is solved by using Eqs. 38 and 49 and a specified shaft horse power

$$l\rho\iint \gamma\Omega r(V + w_{\mathrm{a}} + w_{\mathrm{a2}})\mathrm{d}r\mathrm{d}s = \Omega_0^Q \tag{50}$$

for the unknowns k and γ. It should be noticed that the condition (Eq. 50) is by the lifting surface theory.

2.4 Numerical calculation

Our assignment is to seek the function γ which satisfies the minimum condition of the functional F.

$$\begin{aligned} F = l\rho \int_{r_{\mathrm{b}}}^{r_{\mathrm{o}}}\int_{s_1}^{s_2} &\gamma[k\Omega r\{V - \cos\varepsilon_1(w_{\mathrm{I}} + w_{\mathrm{II}})\} \\ &- V\{\Omega r + \sin\varepsilon_1(w_{\mathrm{I}} + w_{\mathrm{II}})\}]\mathrm{d}s\mathrm{d}r \end{aligned} \tag{51}$$

Using Birnbaum series for the mode function, circulation density γ of the bound vortex is expressed as follows

$$\begin{aligned} \frac{\gamma(\xi,\eta)}{W^*(\eta)} &= \sum_{N=0}^{n-1} A^{(N)}(\eta)\lambda_N(\xi) \\ &= A^{(0)}(\eta)\sqrt{\frac{1-\xi}{1+\xi}} + A^{(1)}(\eta)\sqrt{1-\xi^2} \\ &\quad + A^{(2)}(\eta)\xi\sqrt{1-\xi^2} + A^{(3)}(\eta)\xi^2\sqrt{1-\xi^2}, \end{aligned} \tag{52}$$

where ξ denotes chordwise variable, η denotes radial variable, $\lambda_N(\xi)$ denote Birnbaum series, and $A^{(N)}(\eta)$ denote unknowns. Substitution of Eq. 52 into Eq. 51 gives

$$\begin{aligned} F &= l\rho \int_{r_{\mathrm{b}}}^{r_{\mathrm{o}}} W^* \sum_{N=0}^{n-1} A^{(N)}(\eta) \\ &\quad\times \int_{s_1}^{s_2} \lambda_N(\xi)[k\Omega r\{V - \cos\varepsilon_1(w_{\mathrm{I}} + w_{\mathrm{II}})\} \\ &\quad\times - V\{\Omega r + \sin\varepsilon_1(w_{\mathrm{I}} + w_{\mathrm{II}})\}]\mathrm{d}s\mathrm{d}r. \end{aligned} \tag{53}$$

First variation of Eq. 53 is

$$\begin{aligned} \delta F &= l\rho \int_{r_{\mathrm{b}}}^{r_{\mathrm{o}}} W^* \sum_{N=0}^{n-1} \delta A^{(N)}(\eta) \\ &\quad\times \int_{s_1}^{s_2} \lambda_N(\xi)[k\Omega r\{V - \cos\varepsilon_1(w_{\mathrm{I}} + w_{\mathrm{II}})\} \\ &\quad - V\{\Omega r + \sin\varepsilon_1(w_{\mathrm{I}} + w_{\mathrm{II}})\}]\mathrm{d}s\mathrm{d}r + \delta F_2 \end{aligned} \tag{54}$$

$$\begin{aligned} \delta F_2 &= l\rho \int_{r_{\mathrm{b}}}^{r_{\mathrm{o}}}\int_{s_1}^{s_2} W^* \sum_{N=0}^{n-1} \delta A^{(N)}(\eta)\lambda_N(\xi) \\ &\quad\times \left[\int_{r_{\mathrm{b}}}^{r_{\mathrm{o}}} \mathrm{d}r' \int_{s_1}^{s_2} W^{*\prime} \sum_{N'=0}^{n-1} A^{(N')}(\eta')\lambda_{N'}(\xi')\right. \\ &\quad\times (k\Omega r\cos\varepsilon_1 + V\sin\varepsilon_1)' \frac{2}{h^2\sqrt{1+\mu'^2}} \\ &\quad\times \left\{\sqrt{\frac{1+\mu^2}{1+\mu'^2}} K(0;\mu,\mu')\right. \\ &\quad \left.\left. - \sqrt{\frac{1+\mu^2}{1+\mu'^2}} K^{(0)}\left(\frac{\tau-\tau'}{2};\mu,\mu'\right)\right\}\mathrm{d}s'\right]\mathrm{d}s\mathrm{d}r. \end{aligned} \tag{55}$$

In the above expressions $w_{\mathrm{I}} + w_{\mathrm{II}}$ is a function of γ, so variation term from $w_{\mathrm{I}} + w_{\mathrm{II}}$ gives δF_2. The detail is shown in Appendix 5 as the Galerkin calculation method. Sometimes δF_2 is neglected in the calculation, in which case we call it as being by Method I in this paper. We call it by Method II for the case which includes δF_2.

Using Eq. 54 we get the equation $\delta F = 0$ in which $\delta A^{(N)}(\eta)$ can be arbitrary, so the following expression is obtained.

$$\int_{s_1}^{s_2} \lambda_N(\xi)\left[k\Omega r\{V-\cos\varepsilon_1(w_{\mathrm{I}}+w_{\mathrm{II}})\}\right.$$
$$-V\{\Omega r+\sin\varepsilon_1(w_{\mathrm{I}}+w_{\mathrm{II}})\}.\mathrm{d}s$$
$$+\int_{s_1}^{s_2}\lambda_N(\xi)\left[\int_{r_b}^{r_o}\mathrm{d}r'\int_{s_1}^{s_2}W^{*'}\sum_{N'=0}^{n-1}A^{(N')}(\eta')\lambda_{N'}(\xi')\right.$$
$$\times(k\Omega r\cos\varepsilon_1+V\sin\varepsilon_1)'\frac{2}{h^2\sqrt{1+\mu'^2}}\left\{\sqrt{\frac{1+\mu^2}{1+\mu'^2}}K(0;\mu,\mu')\right.$$
$$\left.\left.-\sqrt{\frac{1+\mu^2}{1+\mu'^2}}K^{(0)}\left(\frac{\tau-\tau'}{2};\mu,\mu'\right)\right\}\mathrm{d}s'\right]\mathrm{d}s=0 \quad \text{on } N,r \tag{56}$$

Equation 56 is the integral equations for γ, as $w_{\mathrm{I}}+w_{\mathrm{II}}$ is given by the integral of the function including γ (Eqs. 19, 20). Equation 56 is the equation for Method II. In case of Method I the second term of the left hand side of the equation is neglected.

Equation 56 can be rearranged as follows

$$\int_{s_1}^{s_2}\lambda_N(\xi)[(k-1)\Omega rV]\mathrm{d}s$$
$$+\int_{s_1}^{s_2}\lambda_N(\xi)[k\Omega r\{-\cos\varepsilon_1(w_{\mathrm{I}})\}-V\{\sin\varepsilon_1(w_{\mathrm{I}})\}]\mathrm{d}s$$
$$+\int_{s_1}^{s_2}\lambda_N(\xi)[k\Omega r\{-\cos\varepsilon_1(w_{\mathrm{II}})\}-V\{\sin\varepsilon_1(w_{\mathrm{II}})\}]\mathrm{d}s$$
$$+\int_{s_1}^{s_2}\lambda_N(\xi)\left[\int_{r_b}^{r_o}\mathrm{d}r'\int_{s_1}^{s_2}W^{*'}\sum_{N'=0}^{n-1}A^{(N')}(\eta')\lambda_{N'}(\xi')\right.$$
$$\times(k\Omega r\cos\varepsilon_1+V\sin\varepsilon_1)'\frac{2}{h^2\sqrt{1+\mu'^2}}$$
$$\times\left\{\sqrt{\frac{1+\mu^2}{1+\mu'^2}}K(0;\mu,\mu')\right\}\mathrm{d}s']\mathrm{d}s$$
$$+\int_{s_1}^{s_2}\lambda_N(\xi)\left[\int_{r_b}^{r_o}\mathrm{d}r'\int_{s_1}^{s_2}W^{*'}\sum_{N'=0}^{n-1}A^{(N')}(\eta')\lambda_{N'}(\xi')\right.$$
$$\times(k\Omega r\cos\varepsilon_1+V\sin\varepsilon_1)'\frac{2}{h^2\sqrt{1+\mu'^2}}$$
$$\times\left\{-\sqrt{\frac{1+\mu^2}{1+\mu'^2}}K^{(0)}\left(\frac{\tau-\tau'}{2};\mu,\mu'\right)\right\}\mathrm{d}s']\mathrm{d}s=0 \text{ on } N,r. \tag{57}$$

The second term of the left hand side of Eq. 57 implies the lifting line term of Method I, the third term implies the lifting surface correction term of Method I, the fourth term implies the lifting line term of the addition for Method II, and the fifth term implies the lifting surface correction term of the addition for Method II.

On the other hand, the first variation of Eq. 51 is

$$\delta F=l\rho\int_{r_b}^{r_o}\int_{s_1}^{s_2}\delta\gamma[k\Omega r\{V-\cos\varepsilon_1(w_{\mathrm{I}}+w_{\mathrm{II}})\}$$
$$-V\{\Omega r+\sin\varepsilon_1(w_{\mathrm{I}}+w_{\mathrm{II}})\}]\mathrm{d}s\mathrm{d}r+\delta F_2. \tag{58}$$

This is shown as the collocation calculation method in Appendix 5. Using Eq. 58 we get the equation $\delta F=0$ in which $\delta\gamma$ can be arbitrary, so the following expressions are obtained in case of Method I.

$$k\Omega r\{V-\cos\varepsilon_1(w_{\mathrm{I}}+w_{\mathrm{II}})\}-V\{\Omega r+\sin\varepsilon_1(w_{\mathrm{I}}+w_{\mathrm{II}})\}$$
$$=0 \text{ on } s,r \tag{59}$$

In case of the lifting line theory $w_{\mathrm{II}}=0$ and using Eq. 2 the expression (Eq. 59) reduces to

$$k=\frac{V\{\Omega r+w_{\mathrm{I}}\sin\varepsilon_1\}}{\Omega r\{V-w_{\mathrm{I}}\cos\varepsilon_1\}}=\frac{V}{\Omega h(r)}. \tag{60}$$

Therefore, we get

$$h(r)=\frac{V}{\Omega k}=\text{const.} \tag{61}$$

This is the linearized Betz condition [6].

Numerical calculation was carried out by using Eq. 56 for two propellers. Propeller A is a conventional propeller and Propeller B is a highly skewed propeller. The blade contour is shown in Fig. 3. The propeller is a four bladed propeller. Particulars of the propeller and the design condition are shown in Table 1. The optimum propeller is solved on condition that the advance coefficient $J=0.6456$, and the torque coefficient $K_{\mathrm{q}}=0.0366$.

Figures 4 and 5 show the results for propeller A in the uniform inflow. The circulation distribution Γ is shown in Fig. 4, and hydrodynamic pitch $2\pi h$ is shown in Fig. 5. The value $2\pi h$ shown in the figure is the value calculated by Eq. 19. The radial mean of the value is used for the constant pitch.

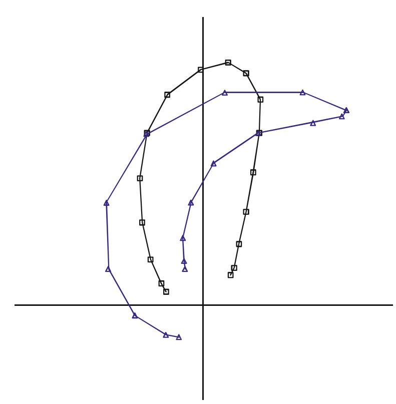

Fig. 3 Projected contour of propeller A, B

Table 1 Particulars and design condition

Diameter	D (m)	0.253
Boss ratio	br	0.167
Expanded area ratio	A_E/A_O	0.475
Number of blades	Z	4
Advance velocity	V (m/s)	1.638
Number of revolutions	n (rps)	10.03
Advance ratio	J	0.646
Torque coefficient	K_q	0.0366

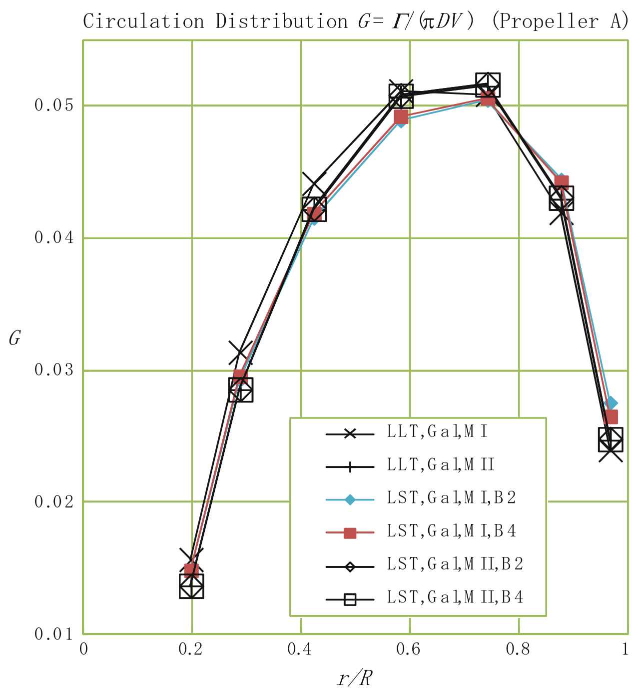

Fig. 4 Circulation distribution of CP (propeller A)

The marks LST and LLT show the value calculated by the lifting surface theory and the lifting line theory, respectively. The marks MI and MII show the value by Method I and Method II, respectively. The mark B2 and B4 show the value by the calculation using the 2nd term and the 4th term of the Birnbaum series (Eq. 52), respectively. If more than one term of the Birnbaum series are selected, the equation cannot be solved, because the determinant is nearly zero.

All the results are almost the same. There is a small discrepancy between the results by Method I and the results by Method II. Only the calculated hydrodynamic pitch by Method I of the lifting line theory shows a constant exactly. The results by Method II of the lifting surface theory coincide with the results by Method II of the lifting line theory, which verifies the theory shown in Sect. 2.3.

Figures 6 and 7 show the results for propeller B in the uniform inflow. The results by Method II of the lifting surface theory coincide with the results by Method II of the

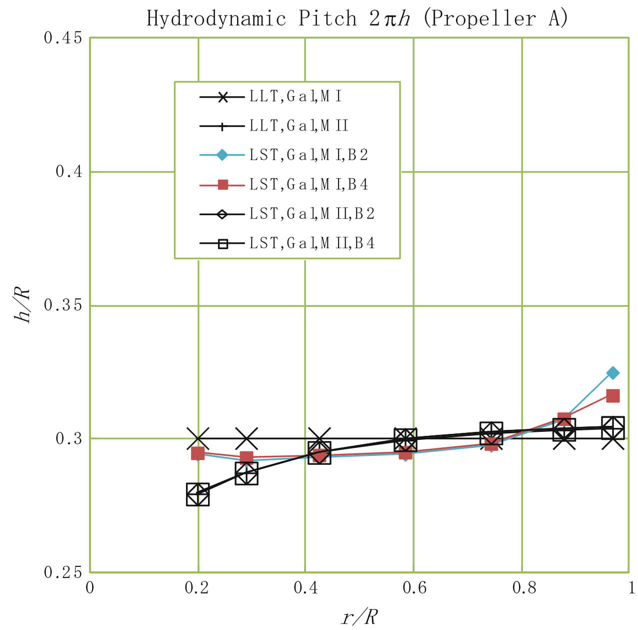

Fig. 5 Hydrodynamic pitch distribution of CP

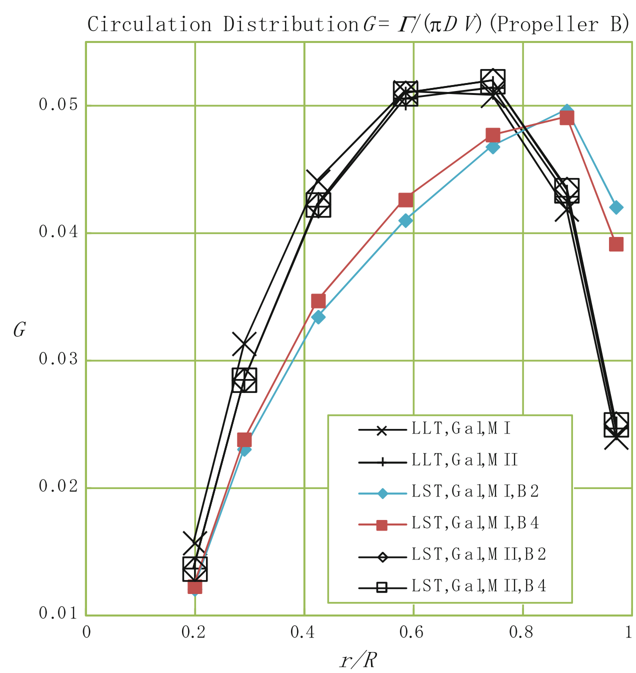

Fig. 6 Circulation distribution of HS (propeller B)

lifting line theory also in this case. The results by Method I of the lifting surface theory don't coincide with the others. The reason for the different performance of Method I is explained as follows. In the case of Propeller B the order of the third term and the fifth term of the left hand side of Eq. 57 becomes large, and the two terms are canceled by each other in case of Method II. So the contribution of the third term in Method I is the reason.

Thrust coefficient K_t, torque coefficient K_q, and propeller efficiency $\eta_p = VS/(\Omega Q) = JK_t/(2\pi K_q)$ are shown in Table 2. The results by the lifting surface theory K_t(LST), K_q(LST), η_p(LST) are from the integral (Eqs. 30, 31), whereas the results by the lifting line theory K_t(LLT), K_q(LLT), η_p(LLT) are from the integral (Eqs. 30, 31), where the lifting surface terms w_{t2},w_{a2} are excluded. Table 2 shows the comparison between the lifting line calculation and the lifting surface calculation. In case of CP (propeller A), K_t, K_q by the lifting surface calculation are smaller than those by the lifting line calculation by about 0.2 %. In case of HS (propeller B) the former is smaller than the latter by about 1 %. The reduction rate is almost the same for K_t and K_q. As a result, there is no discrepancy between η_p of propeller A and η_p of propeller B.

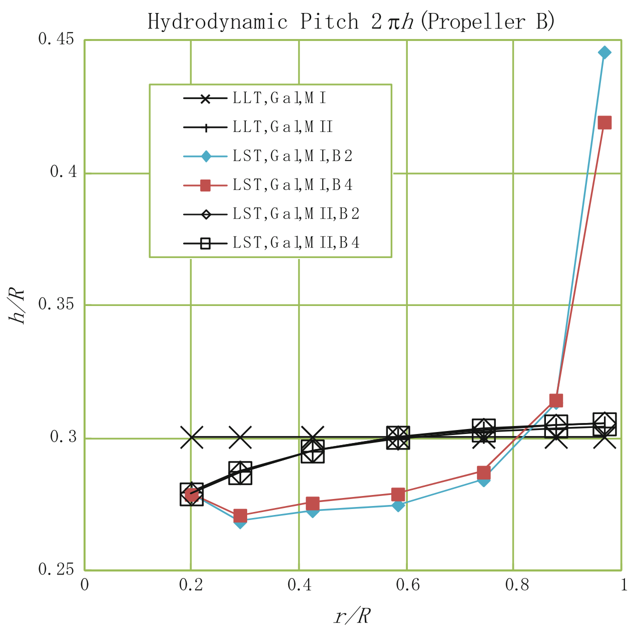

Fig. 7 Hydrodynamic pitch distribution of HS

3 A propeller in non-uniform inflow

In the previous chapter we discussed the optimum propeller in uniform inflow. In this chapter we will discuss the optimum propeller in non-uniform inflow. So in this chapter we assume that the hydrodynamic pitch is not constant in the radial direction.

$$h \neq h' \tag{62}$$

This model is useful for the general case of a propeller, although the model of constant hydrodynamic pitch used in the previous chapter is useful for some special cases.

A propeller advances with a constant velocity V_s and rotates with a constant angular speed Ω in a non-uniform flow field. Axial and tangential inflow velocity at the radius r on the propeller rotating disc are denoted by V_1, $\Omega_1 r$,

$$V_1 = V_s\{1 - w(r)\}, \quad \Omega_1 r = \Omega r\{1 - \omega(r)\} \tag{63}$$

where $w(r)$, $\omega(r)$ are the axial and the tangential wake fraction. We ignore the circumferential variation of the inflow in this paper. The special case of $w(r) = \omega(r) = 0$ corresponds to the case of an open propeller in the uniform inflow. The pitch of the helical surface on which the bound vortex and the trailing vortex are located is assumed to be given by $2\pi h(r)$, where

$$\frac{h(r)}{r} = \tan \varepsilon_1 = \frac{V_1 + w_a}{\Omega_1 r + w_t} \tag{64}$$

Table 2 Designed thrust, torque, and efficiency

	Kt(LLT)	ratio	Kt(LST)	ratio	Kq(LLT)	ratio	Kq(LST)	ratio	η_p(LLT)	ratio	η_p(LST)	ratio
LLT (Method I)	0.2436	1.0005			0.03660	1.0008			0.6838	0.9997		
LLT (Method II)	0.2434	1.0000			0.03657	1.0000			0.6840	1.0000		
CP (Method I)	0.2439	1.0017	0.2434	0.9999	0.03666	1.0025	0.03659	1.0005	0.6835	0.9993	0.6836	0.9994
CP (Method II)	0.2441	1.0025	0.2436	1.0005	0.03669	1.0033	0.03662	1.0014	0.6834	0.9992	0.6834	0.9992
HS (Method I)	0.2419	0.9936	0.2401	0.9862	0.03684	1.0074	0.03657	1.0000	0.6746	0.9863	0.6745	0.9862
HS (Method II)	0.2456	1.0089	0.2430	0.9980	0.03700	1.0118	0.03660	1.0008	0.6820	0.9971	0.6820	0.9971

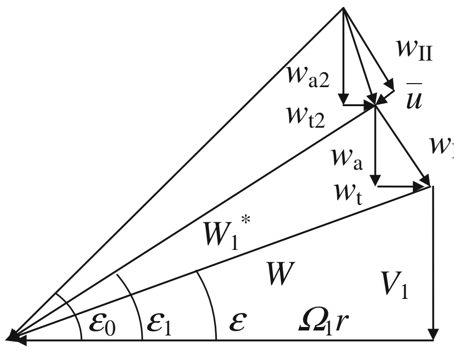

Fig. 8 Inflow and induced velocity vector

and w_a, w_t denote axial and tangential induced velocity at the lifting line representing the blade (Fig. 8). The pitch $2\pi h(r)$ is called the hydrodynamic pitch. The pitch is a function of r in general. It is assumed that the pitch doesn't vary in the axial direction.

3.1 Propeller lifting surface theory using helical coordinates with variable pitch

We use the helical coordinates (τ, σ, μ) with the variable pitch $2\pi h(r)$ which relates to the cylindrical coordinates (x, r, θ) as follows

$$\tau = \theta + x/h, \quad \sigma = \theta - x/h, \quad \mu = r/h = 1/\tan\varepsilon_1 \tag{65}$$

$$\tau' = \theta' + x'/h', \quad \sigma' = \theta' - x'/h', \quad \mu' = r'/h' = 1/\tan\varepsilon'_1 \tag{66}$$

where the dash$'$ denotes that these quantities are related to the point on the blade.

The normal line to the helical surface is assumed approximately to be the normal line to both the radius vector r and the helix, then the normal derivative is expressed by

$$\begin{aligned}\frac{\partial}{\partial n_1} &= -\cos\varepsilon_1 \frac{\partial}{\partial x} + \sin\varepsilon_1 \frac{\partial}{r\partial\theta} \\ &= \frac{1}{h\sqrt{1+\mu^2}}\left\{\left(\mu + \frac{1}{\mu}\right)\frac{\partial}{\partial\sigma} - \left(\mu - \frac{1}{\mu}\right)\frac{\partial}{\partial\tau}\right\}.\end{aligned} \tag{67}$$

The segment of the helix of $\mu = \text{const.}$, $\sigma = \text{const.}$ is expressed by

$$\mathrm{d}s = \frac{1}{2}h\sqrt{1+\mu^2}\,\mathrm{d}\tau. \tag{68}$$

Then the perturbation velocity potential for the flow around the propeller is expressed by

$$\Phi = \frac{1}{4\pi}\sum_{m=0}^{l-1}\int_{r_b}^{r_o}\mathrm{d}r'\int_{s_1}^{s_2}\gamma\,\mathrm{d}s'\int_{s'}^{\infty}\frac{\partial}{\partial n''_1}\frac{1}{R}\mathrm{d}s'' \tag{69}$$

where l denotes the number of blades, integral means on the helical surface, r_o, r_b denote propeller radius and boss radius, s_1, s_2 denotes the s coordinates of the leading edge and the trailing edge of blade section, and $\gamma = \gamma(s', r')$ denotes the circulation density on the blade, and

$$\mathrm{d}s' = \frac{1}{2}h'\sqrt{1+\mu'^2}\,\mathrm{d}\tau' \tag{70}$$

$$\mathrm{d}s'' = \frac{1}{2}h'\sqrt{1+\mu'^2}\,\mathrm{d}T' \tag{71}$$

$$\begin{aligned}\frac{\partial}{\partial n''_1} &= -\cos\varepsilon'_1\frac{\partial}{\partial x'} + \sin\varepsilon'_1\frac{\partial}{r'\partial\theta'} \\ &= \frac{1}{h'\sqrt{1+\mu'^2}}\left\{\left(\mu' + \frac{1}{\mu'}\right)\frac{\partial}{\partial\sigma'} - \left(\mu' - \frac{1}{\mu'}\right)\frac{\partial}{\partial T'}\right\}\end{aligned} \tag{72}$$

$$\begin{aligned}R &= \sqrt{(x - x'^2) + r^2 + r'^2 - 2rr'\cos(\theta - \theta' - 2\pi m/l)}\Big| \\ &= \sqrt{\left(\frac{\tau-\sigma}{2}h - \frac{\tau'-\sigma'}{2}h'\right)^2 + (\mu h)^2 + (\mu' h'^2) - 2\mu h\mu' h'\cos\left(\frac{\tau+\sigma-\tau'-\sigma'}{2} - \frac{2\pi m}{l}\right)}\Bigg|_{\tau'=T'}.\end{aligned} \tag{73}$$

Upwash on the blade $w = w(s,r)$ is expressed by

$$\begin{aligned}w = \frac{\partial\Phi}{\partial n_1}\bigg|_{\text{on blade}} &= \frac{1}{4\pi}\sum_{m=0}^{l-1}\int_{r_b}^{r_o}\mathrm{d}r'\int_{s_1}^{s_2}\gamma\,\mathrm{d}s'\int_{s'}^{\infty}\frac{\partial^2}{\partial n_1\partial n''_1}\frac{1}{R}\mathrm{d}s''\bigg|_{\text{on blade}} \\ &= \int_{r_b}^{r_o}\mathrm{d}r'\int_{s_1}^{s_2}\gamma h'\sqrt{1+\mu'^2}F^1\,\mathrm{d}s'\end{aligned} \tag{74}$$

$$F_1 = \frac{1}{8\pi}\sum_{m=0}^{l-1}\int_{\tau'}^{\infty}\frac{\partial^2}{\partial n_1\partial n''_1}\frac{1}{R}\mathrm{d}T'. \tag{75}$$

And using the notations

$$F^1 = \int_{\tau'}^{\infty} f\,\mathrm{d}T' \tag{76}$$

$$f = \frac{1}{8\pi}\sum_{m=0}^{l-1}\frac{\partial^2}{\partial n_1 \partial n_1''}\frac{1}{R} \tag{77}$$

we get

$$\begin{aligned} f = & \frac{1}{8\pi}\sum_{m=0}^{l-1}\frac{1}{\sqrt{1+\mu^2}\sqrt{1+\mu'^2}} \\ & \times \left[\frac{1}{R^3}\left\{\mu\mu' + \cos\left(\frac{\tau+\sigma-T'-\sigma'}{2} - \frac{2\pi m}{l}\right)\right\}\right. \\ & + \frac{-3}{R^5}\left\{\mu\left(\frac{\tau-\sigma}{2}h - \frac{T'-\sigma'}{2}h'\right)\right. \\ & \left. - \mu' h' \sin\left(\frac{\tau+\sigma-T'-\sigma'}{2} - \frac{2\pi m}{l}\right)\right\} \\ & \times\left\{\mu'\left(\frac{\tau-\sigma}{2}h - \frac{T'-\sigma'}{2}h'\right)\right. \\ & \left.\left. -\mu h \sin\left(\frac{\tau+\sigma-T'-\sigma'}{2} - \frac{2\pi m}{l}\right)\right\}\right] \end{aligned} \tag{78}$$

where σ, σ' are given from the value on the blade.

Using the notations

$$u' = \frac{\tau+\sigma-T'-\sigma'}{2}, \quad du' = -\frac{1}{2}dT' \tag{79}$$

$$u = \frac{\tau+\sigma-\tau'-\sigma'}{2} \tag{80}$$

$$\begin{aligned} v' = & \frac{\tau-\sigma}{2}h - \frac{T'-\sigma'}{2}h' = \frac{\tau-\sigma}{2}h - \frac{\tau'-\sigma'}{2}h' \\ & - uh' + u'h' = \frac{\tau-\sigma}{2}h - \frac{\tau'-\sigma'}{2}h' - (u-u') \\ & \times \frac{h+h'}{2} + (u-u')\frac{h-h'}{2} \end{aligned} \tag{81}$$

we get

$$F^1 = \int_{\tau'}^{\infty} f dT' = \int_{u}^{-\infty} f(-2du') = \int_{-\infty}^{u} -2f du' \tag{82}$$

$$\begin{aligned} f = & \frac{1}{8\pi}\sum_{m=0}^{l-1}\frac{1}{\sqrt{1+\mu^2}\sqrt{1+\mu'^2}}\left[\frac{1}{R^3}\left\{\mu\mu' + \cos\left(u' - \frac{2\pi m}{l}\right)\right\}\right. \\ & + \frac{-3}{R^5}\left\{\mu v' - \mu' h' \sin\left(u' - \frac{2\pi m}{l}\right)\right\} \\ & \left.\times\left\{\mu' v' - \mu h \sin\left(u' - \frac{2\pi m}{l}\right)\right\}\right] \end{aligned} \tag{83}$$

$$R = \sqrt{v'^2 + (\mu h)^2 + (\mu' h'^2) - 2\mu h \mu' h' \cos\left(u' - \frac{2\pi m}{l}\right)}. \tag{84}$$

Then upwash on the blade (Eq. 74) is expressed as follows

$$w = w_{\mathrm{I}} + w_{\mathrm{II}} \tag{85}$$

$$\begin{aligned} w_{\mathrm{I}} = & \left.\frac{\partial \Phi_{\mathrm{I}}}{\partial n_1}\right|_{\mathrm{on\,blade}} = \int_{r_b}^{r_o} dr' \int_{s_1}^{s_2} \gamma(s', r') h' \sqrt{1+\mu'^2} F_{\mathrm{I}}^1 ds' \\ & = \int_{r_b}^{r_o} \Gamma(r') h' \sqrt{1+\mu'^2} F_{\mathrm{I}}^1 dr' \end{aligned} \tag{86}$$

$$w_{\mathrm{II}} = \left.\frac{\partial \Phi_{\mathrm{II}}}{\partial n_1}\right|_{\mathrm{on\,blade}} = \int_{r_b}^{r_o} dr' \int_{s_1}^{s_2} \gamma(s', r') h' \sqrt{1+\mu'^2} F_{\mathrm{II}}^1 ds' \tag{87}$$

$$F^1 = F_{\mathrm{I}}^1 + F_{\mathrm{II}}^1 = \int_{-\infty}^{u} 2f du' \tag{88}$$

$$F_{\mathrm{I}}^1 = \int_{-\infty}^{0} 2f du' \tag{89}$$

$$F_{\mathrm{II}}^1 = \int_{0}^{u} 2f du' \tag{90}$$

$$\Gamma(r) = \int_{s_1}^{s_2} \gamma(s, r) ds. \tag{91}$$

And w_{I} corresponds to lifting line term (the contribution from the free vortex of lifting line theory) and w_{II} corresponds to the correction term for lifting surface (residual term w–w_{I} for the lifting surface theory).

The expressions shown above are for the theory of variable pitch helical surface. If we set $h = h'$ in the above expressions, the expressions are reduced to that for the constant pitch helical surface which is shown in Sect. 2.1. The relation between the expressions in Sect. 2.1 and in Sect. 3.1 is shown in Appendix 1. It is to be noted that the definition of the variable v' is different between Sect. 2 and in Sect. 3. The radial component of the flow is ignored as in Sect. 2.

3.2 Work done by a propeller and energy loss of a propeller

Work done against the surrounding fluid by a propeller in a unit time is expressed by

$$P = -l\rho \iint \frac{\partial \Phi}{\partial n_1}\left(W_{\mathrm{I}}^* + \bar{u}\right)\gamma dr ds \tag{92}$$

where the integral domain is the lifting surface corresponding to a blade of the propeller, ρ is the density of the fluid, $W_{\mathrm{I}}^* + \bar{u}$ denotes the velocity at the blade (Fig. 8)

$$W_{\mathrm{I}}^* = \sqrt{(V_1 + w_a)^2 + (\Omega_1 r + w_t)^2}. \tag{93}$$

$\bar{u}$ denotes the velocity component due to the correction term for the lifting surface. $\rho(W_{\mathrm{I}}^* + \bar{u})\gamma$ implies lift density. $\partial\Phi/\partial n_1$ is the upwash on the lifting surface given by Eq. 74.

Similar to the case of Sect. 2.2, we have the relation

$$-(W_1^* + \bar{u})\frac{\partial \Phi}{\partial n_1} = \Omega_1 r\{V_1 + w_{\mathrm{a}} + w_{\mathrm{a2}}\} - V_1\{\Omega_1 r + w_{\mathrm{t}} + w_{\mathrm{t2}}\}. \quad (94)$$

Then the expression

$$P = l\rho \iint \gamma \Omega_1 r\{V_1 + w_{\mathrm{a}} + w_{\mathrm{a2}}\} \mathrm{d}r\mathrm{d}s - l\rho \iint \gamma V_1\{\Omega_1 r + w_{\mathrm{t}} + w_{\mathrm{t2}}\}\mathrm{d}r\mathrm{d}s \quad (95)$$

is obtained. The torque and the thrust worked on the small area drds of the lifting surface are denoted by dQ, dS,

$$\mathrm{d}Q = l\rho\gamma r\left\{(W_1^* + \bar{u})\sin\varepsilon_1 - w_{\mathrm{II}}\cos\varepsilon_1\right\}\mathrm{d}r\mathrm{d}s = l\rho\gamma r\{V_1 + w_{\mathrm{a}} + w_{\mathrm{a2}}\}\mathrm{d}r\mathrm{d}s \quad (96)$$

$$\mathrm{d}S = l\rho\gamma\left\{(W_1^* + \bar{u})\cos\varepsilon_1 + w_{\mathrm{II}}\sin\varepsilon_1\right\}\mathrm{d}r\mathrm{d}s = l\rho\gamma\{\Omega_1 r + w_{\mathrm{t}} + w_{\mathrm{t2}}\}\mathrm{d}r\mathrm{d}s. \quad (97)$$

Then Eq. 95 becomes

$$P = \iint \Omega_1 \mathrm{d}Q - \iint V_1 \mathrm{d}S. \quad (98)$$

The first term and the second term of the right hand side of Eq. 95 imply the shaft horse power and the thrust horse power, respectively, so P implies the energy loss of the propeller. The above transformation shows that the work done against surrounding fluid by the propeller (Eq. 92) is equal to the energy loss of the propeller (Eq. 98).

The velocity component due to the correction term for the lifting surface $\bar{u}$ is small, so it will be neglected below. P and P_{I} (the lifting line component of P) and P_{II} (the correction component of P for the lifting surface) are shown as follows

$$P = P_{\mathrm{I}} + P_{\mathrm{II}} \quad (99)$$

$$P = -l\rho \iint \gamma W_1^* \frac{\partial \Phi}{\partial n_1}\mathrm{d}r\mathrm{d}s = -l\rho \int_{r_{\mathrm{b}}}^{r_{\mathrm{o}}} \mathrm{d}r \int_{s_2}^{s_2} W_1^*\gamma \mathrm{d}s \int_{r_{\mathrm{b}}}^{r_{\mathrm{o}}} \mathrm{d}r' \int_{s_1}^{s_2} W_1^{*'}\gamma'\mathrm{d}s' \frac{h'\sqrt{1+\mu'^2}}{W_1^{*'}} \times \int_{-\infty}^{u} 2f\mathrm{d}u' \quad (100)$$

$$P_{\mathrm{I}} = -l\rho \iint \gamma W_1^* \frac{\partial \Phi_{\mathrm{I}}}{\partial n_1}\mathrm{d}r\mathrm{d}s = -l\rho \int_{r_{\mathrm{b}}}^{r_{\mathrm{o}}} \mathrm{d}r \int_{s_1}^{s_2} W_1^*\gamma \mathrm{d}s \int_{r_{\mathrm{b}}}^{r_{\mathrm{o}}} \mathrm{d}r' \int_{s_1}^{s_2} W_1^{*\prime}\gamma'\mathrm{d}s' \frac{h'\sqrt{1+\mu'^2}}{W_1^{*'}} \int_{-\infty}^{0} 2f\mathrm{d}u' \quad (101)$$

$$P_{\mathrm{II}} = -l\rho \iint \gamma W_1^* \frac{\partial \Phi_{\mathrm{II}}}{\partial n_1}\mathrm{d}r\mathrm{d}s = -l\rho \int_{r_{\mathrm{b}}}^{r_{\mathrm{o}}} \mathrm{d}r \int_{s_1}^{s_2} W_1^*\gamma \mathrm{d}s \int_{r_{\mathrm{b}}}^{r_{\mathrm{o}}} \mathrm{d}r' \int_{s_1}^{s_2} W_1^{*'}\gamma'\mathrm{d}s' \frac{h'\sqrt{1+\mu'^2}}{W_1^{*'}} \times \int_{0}^{u} 2f\mathrm{d}u' \quad (102)$$

3.3 Optimum propeller in non-uniform inflow

Our task is to seek the optimum solution so that the value P is minimum for the given shaft horse power. So P should be a minimum on condition that the first term of P (Eq. 95) is specified. In that case, we can get the solution by making the following functional F minimum.

$$F = l\rho \iint k\gamma\Omega_1 r\{V_1 - \cos\varepsilon_1(w_{\mathrm{I}} + w_{\mathrm{II}})\}\mathrm{d}r\mathrm{d}s - l\rho \iint \gamma V_1\{\Omega_1 r + \sin\varepsilon_1(w_{\mathrm{I}} + w_{\mathrm{II}})\}\mathrm{d}r\mathrm{d}s \quad (103)$$

$$\delta F = 0 \quad (104)$$

where k is the multiplier of Lagrange. This is the formulation based on the propeller lifting surface theory. The expression will be transformed to the expression by the lifting line theory below.

Using the expressions

$$F = k(\Omega_L^Q + \Omega_S^Q) - (V_L^S + V_S^S) \quad (105)$$

$$V_L^S = l\rho \iint \gamma V_1(\Omega_1 r + \sin\varepsilon_1(w_{\mathrm{I}}))\mathrm{d}r\mathrm{d}s \quad (106)$$

$$V_S^S = l\rho \iint \gamma V_1 \cdot \sin\varepsilon_1 \cdot w_{\mathrm{II}}\mathrm{d}r\mathrm{d}s \quad (107)$$

$$\Omega_L^Q = l\rho \iint \gamma\Omega_1 r(V_1 - \cos\varepsilon_1(w_{\mathrm{I}}))\mathrm{d}r\mathrm{d}s \quad (108)$$

$$\Omega_S^Q = l\rho \iint \gamma\Omega_1 r(-\cos\varepsilon_1 \cdot w_{\mathrm{II}})\mathrm{d}r\mathrm{d}s \quad (109)$$

we get

$$\Omega_L^Q - V_L^S = -l\rho \iint \gamma W_1^* w_{\mathrm{I}}\mathrm{d}r\mathrm{d}s = P_{\mathrm{I}} \quad (110)$$

$$\Omega_S^Q - V_S^S = -l\rho \iint \gamma W_1^* w_{\mathrm{II}}\mathrm{d}r\mathrm{d}s = P_{\mathrm{II}}. \quad (111)$$

At first the characteristics of P_{II} (Eq. 102) will be investigated.

In the expression P_{II} the order of integral $\int_{r_{\mathrm{b}}}^{r_{\mathrm{o}}}\mathrm{d}r \int_{s_1}^{s_2}\mathrm{d}s$ and $\int_{r_{\mathrm{b}}}^{r_{\mathrm{o}}}\mathrm{d}r' \int_{s_1}^{s_2}\mathrm{d}s'$ is changed, and then the notation (τ, σ, μ) and (τ', σ', μ') are changed for each other.

$$P_{\mathrm{II}} = -l\rho \int_{r_b}^{r_o} \mathrm{d}r \int_{s_2}^{s_2} W_1^* \gamma \mathrm{d}s \frac{h\sqrt{1+\mu^2}}{W_1^*} \int_{r_b}^{r_o} \mathrm{d}r' \times \int_{s_2}^{s_2} W_1^{*'} \gamma' \mathrm{d}s' \int_0^{-u} 2f' \mathrm{d}u' \tag{112}$$

where

$$f' = \frac{1}{8\pi} \sum_{m=0}^{l-1} \frac{1}{\sqrt{1+\mu'^2}\sqrt{1+\mu^2}} \times \left[\frac{1}{R^3} \left\{ \mu'\mu + \cos\left(u' - \frac{2\pi m}{l}\right) \right\} + \frac{-3}{R^5} \left\{ \mu' v'' - \mu h \sin\left(u' - \frac{2\pi m}{l}\right) \right\} \times \left\{ \mu v'' - \mu' h' \sin\left(u' - \frac{2\pi m}{l}\right) \right\} \right] \tag{113}$$

$$v'' = \frac{\tau' - \sigma'}{2} h' - \frac{\tau - \sigma}{2} h - (-u - u') \frac{h' + h}{2}. \tag{114}$$

Here the condition (A)

$$v_R \equiv (u - u') \frac{h - h'}{2} = 0 \tag{115}$$

is assumed for v' (Eq. 81), then

$$v' = \frac{\tau - \sigma}{2} h - \frac{\tau' - \sigma'}{2} h' - (u - u') \frac{h + h'}{2}. \tag{116}$$

This assumption is also used in the expression (Eq. 114).

The sign of v' changes when replacing (τ, σ, h)–(τ', σ', h') for each other in v' and changing the sign of the integral variable u'. In this change the function f (Eq. 83) does not change. Also, the sign of m is also changed as the sign of u'. In the above change the sign of u changes. As a result we have

$$\int_0^{-u} 2f' \mathrm{d}u' = -\int_0^{u} 2f \mathrm{d}u'. \tag{117}$$

The detail is shown in Appendix 4. Then we get

$$P_{\mathrm{II}} = +l\rho \int_{r_b}^{r_o} \mathrm{d}r \int_{s_1}^{s_2} W_1^* \gamma \mathrm{d}s \frac{h\sqrt{1+\mu^2}}{W_1^*} \int_{r_b}^{r_o} \mathrm{d}r' \times \int_{s_1}^{s_2} W_1^{*'} \gamma' \mathrm{d}s' \int_0^{u} 2f \mathrm{d}u'. \tag{118}$$

Next, if the condition (B)

$$\bar{\Omega} = \frac{W_1^*}{h\sqrt{1+\mu^2}} = \frac{W_1^{*'}}{h'\sqrt{1+\mu'^2}} = \bar{\Omega}' = \mathrm{const.} \tag{119}$$

is assumed, the previous Eq. 118 is reduced to

$$P_{\mathrm{II}} = +l\rho \int_{r_b}^{r_o} \mathrm{d}r \int_{s_1}^{s_2} W_1^* \gamma \mathrm{d}s \int_{r_b}^{r_o} \mathrm{d}r' \times \int_{s_1}^{s_2} W_1^{*'} \gamma' \mathrm{d}s' \frac{h'\sqrt{1+\mu'^2}}{W_1^{*'}} \int_0^{u} 2f \mathrm{d}u' \tag{120}$$

where

$$\bar{\Omega} = \frac{W_1^*}{h\sqrt{1+\mu^2}} = \frac{V_1 + w_a}{h} = \frac{\Omega_1 r + w_t}{r}. \tag{121}$$

The expression Eq. 120 for P_{II} is the same form as the original expression Eq. 102 but changes its sign. So we get

$$P_{\mathrm{II}} = 0. \tag{122}$$

From Eqs. 99 and 122

$$P = P_{\mathrm{I}}. \tag{123}$$

These results show Munk's theorems [3] as in Sect. 2.3. Equation 122 implies Munk's theorem II. Equation 123 implies Munk's theorem I.

From Eqs. 111 and 122

$$\Omega_S^Q = V_S^S. \tag{124}$$

From Eqs. 106, 108 and 64 the lifting line terms are

$$V_L^S = l\rho \iint \gamma V_1 (\Omega_1 r + \sin \varepsilon_1 (w_{\mathrm{I}})) \mathrm{d}r\mathrm{d}s = l\rho \iint \gamma V_1 \mu (V_1 - \cos \varepsilon_1 (w_{\mathrm{I}})) \mathrm{d}r\mathrm{d}s \tag{125}$$

$$\Omega_L^Q = l\rho \iint \gamma \Omega_1 r (V_1 - \cos \varepsilon_1 (w_{\mathrm{I}})) \mathrm{d}r\mathrm{d}s = l\rho \iint \gamma \Omega_1 h (\Omega_1 r + \sin \varepsilon_1 (w_{\mathrm{I}})) \mathrm{d}r\mathrm{d}s. \tag{126}$$

Here, if the condition (C)

$$\frac{V_1}{\Omega_1 h} = \frac{V_1}{\Omega_1 r} \frac{2\pi r}{2\pi h} = \frac{V_1}{\Omega_1 r} \frac{\Omega_1 r + w_t}{V_1 + w_a} = k' = \mathrm{const.} \tag{127}$$

is assumed, we get

$$V_L^S = k' \Omega_L^Q. \tag{128}$$

Using the above results, we obtain the final result. On condition (A) (B) (C), Eq. 105 is transformed as follows

$$F = k(\Omega_L^Q + \Omega_S^Q) - (V_L^S + V_S^S) = k\Omega_L^Q \left(1 + \frac{\Omega_S^Q}{\Omega_L^Q}\right) - V_L^S \left(1 + \frac{V_S^S}{V_L^S}\right) = k\Omega_L^Q (1 + \alpha) - V_L^S \left(1 + \frac{\alpha}{k'}\right) = \left(1 + \frac{\alpha}{k'}\right) \left[k\Omega_L^Q \frac{1+\alpha}{1+\alpha/k'} - V_L^S\right] = \left(1 + \frac{\alpha}{k'}\right) [K\Omega_L^Q - V_L^S] \tag{129}$$

where

$$\alpha = \Omega_S^Q / \Omega_L^Q \tag{130}$$

$$K = k(1+\alpha)/(1+\alpha/k'). \tag{131}$$

This result means that the optimum problem of the lifting surface (Eq. 105) is reduced to the optimum problem of the lifting line

$$\frac{F}{1+\alpha/k'} = K\Omega_L^Q - V_L^S. \tag{132}$$

In other words, the solution of Eq. 104 is obtained not by the lifting surface problem but by the lifting line problem. This proves that the condition of the minimum energy loss cannot determine the chordwise position of the lifting element.

The above discussion is on the basis of the assumption of conditions (A), (B), (C). In the special case of non-linear theory based on constant hydrodynamic pitch, the conditions (A), (B), (C) are realized exactly and so the above discussion is realized correctly. This was already shown in Sect. 2.3.

In the condition (A) (Eq. 115), $|h - h'| << 1$ means near constant pitch, and $|u - u'| << 1$ means short chord length (large aspect ratio of blade). The condition (B) (Eq. 119) corresponds to the assumption (Eq. 40) used in Eqs. 39, 44 and 45 in Sect. 2.3. The condition (C) (Eq. 127) is from the results of Hanaoka's wake adapted propeller theory [3].

The problem is solved by using Eqs. 104 and 132 and a specified shaft horse power

$$l\rho \iint \gamma\Omega_1 r\{V_1 - \cos\varepsilon_1(w_{\mathrm{I}} + w_{\mathrm{II}})\}\mathrm{d}r\mathrm{d}s = \Omega_0^Q \tag{133}$$

for the unknowns k and γ. It should be noticed that the condition (Eq. 133) is by the lifting surface theory.

3.4 Numerical calculation

Our assignment is to seek the function γ which satisfies the minimum condition of the functional F (Eqs. 103, 163) (cf. Appendix 5). The functional F is also given by another expression (Eq. 165) using the Birnbaum series (Eq. 164) for γ. As shown in Appendix 5, we have two numerical methods. The collocation calculation method is based on Eq. 163 and the Galerkin calculation method is based on Eq. 165. For each method we have two methods, Method I and Method II. The δF_2 term is neglected in Method I, whereas the δF_2 term is included in Method II. Integral equations of Method I and Method II of the collocation method are given by Eqs. 173 and 172, respectively. Integral equations of Method I and Method II of the Galerkin method are given by Eqs. 182 and 181, respectively.

Numerical calculation was carried out by the variable hydrodynamic pitch model for Propeller B in non-uniform inflow of the hull wake. Propeller B is a four bladed highly skewed propeller. The blade contour is shown in Fig. 3. Particulars of the propeller and the design condition are

Table 3 Particulars and design condition

Diameter	D (m)	0.253
Boss ratio	br	0.167
Expanded area ratio	A_E/A_O	0.475
Number of blades	Z	4
Ship velocity	V_S (m/s)	2.253
Number of revolutions	n (rps)	10.03
Torque coefficient	K_q	0.0366

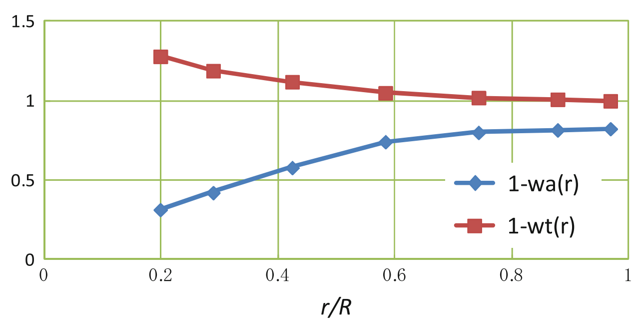

Fig. 9 Radial distribution of inflow velocity

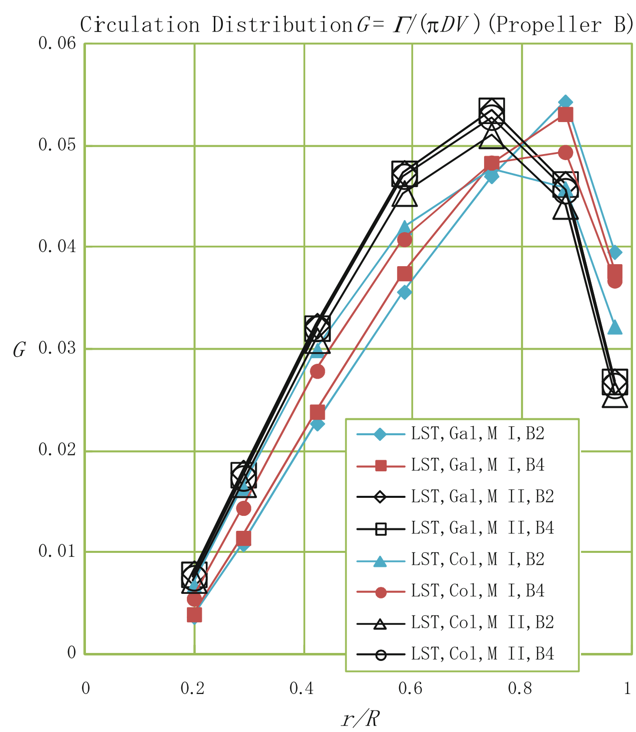

Fig. 10 Circulation distribution of HS (propeller B)

shown in Table 3. The radial distribution of the wake is shown in Fig. 9. Figures 10 and 11 show the results.

Results shown are the calculation of the equation $\delta F = 0$ using the expression 177 or 167, where only one term of the Birnbaum series Eq. 164 was selected in general. If more than one term of the series are selected, the equation cannot be solved, because the determinant is nearly zero. This means that we can get the solution for the given chordwise distribution of γ, but that we cannot get the chordwise distribution of γ. This is the result from the theory discussed in Sect. 3.3.

A special case is the calculation method I of collocation calculation method Eq. 173. In that case, the solution using more than one term of the Birnbaum series is obtained. The solution γ is considered to be one of the indefinite solutions. From the point of view of the calculus of variation, method I is not complete. Accurate numerical solution Γ should be the same as that of the lifting line calculation. The deviation from the lifting line solution shows the inaccuracy of the numerical solutions.

In Fig. 10 the radial distribution of the circulation is shown. The radial distribution of the hydrodynamic pitch is shown in Fig. 11. All calculations in Figs. 10 and 11 are a lifting surface calculation using one term of the Birnbaum series (Eq. 164). One group denoted by B2 uses the 2nd term of Eq. 164 and another group denoted by B4 uses the 4th term of Eq. 164. The mark MI and MII show the value by Method I (Eqs. 173, 182) and Method II (Eqs. 172, 181), respectively.

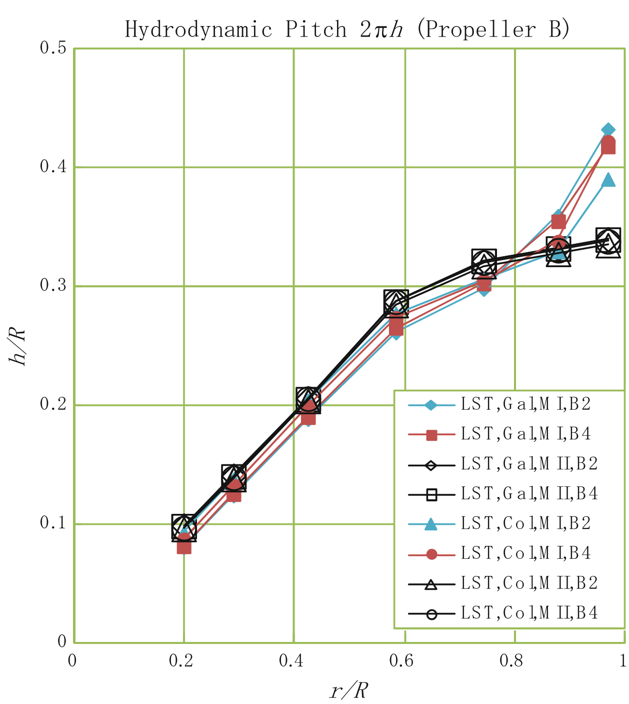

Fig. 11 Hydrodynamic pitch distribution of HS

The results by Method II are almost the same as those by the lifting line theory which shows the high accuracy of Method II. Calculation by the lifting line theory, which is not shown in Figs. 10 and 11, was performed by neglecting the lifting surface correction term in the expressions. The results by Method I are not the same as Method II which shows an unsatisfactory approximation of Method I.

4 Conclusions

We have investigated the relation between the lifting surface theory and the lifting line theory in the design of the optimum propeller.

In case of the theoretical model with constant hydrodynamic pitch, which corresponds to the case of a propeller in open water, it was shown that the governing equation of the optimization by the lifting surface theory is reduced to that by the lifting line theory. So the integral equation with more than one of the terms of the Birnbaum series does not give the solution. But with one term of Birnbaum series it does. This means that the optimum circulation distribution which is the chordwise integral of the circulation density can be determined but that the optimum circulation density can not be determined.

In case of the theoretical model with variable hydrodynamic pitch, which corresponds to the case of a propeller in a hull wake, the same conclusion is obtained provided that the three conditions are assumed.

The numerical calculation supports the conclusion. The conclusion is considered to be useful as the basis of propeller design.

Acknowledgments The author is deeply grateful to the late Dr. T. Hanaoka, whose warm guidance and encouragement have supported the author's life of study on propeller theory.

Appendix 1: Relation between the two expressions for the upwash

Upwash on the blade is expressed by Eqs. 18, 19 and 20 in Sect. 2.1. On the other hand, in Sect. 3.1 upwash on the blade is expressed by Eqs. 85, 86 and 87. The expressions in Sect. 2.1 are for the constant hydrodynamic pitch model and the expressions in Sect. 3.1 are for variable hydrodynamic pitch model. If $h = h'$ is set in the expressions in Sect. 3.1, the expressions are reduced to the expressions in Sect. 2.1. So we obtain the relation between both expressions as follows

$$h'\sqrt{1+\mu'^2}F^1\Big|_{h=h'} = -\frac{2}{h^2\sqrt{1+\mu'^2}}K\left(\frac{\tau-\tau'}{2};\mu,\mu'\right) \tag{134}$$

$$F^1\big|_{h=h'} = -\frac{2}{h^3(1+\mu'^2)}K\left(\frac{\tau-\tau'}{2};\mu,\mu'\right) \quad (135)$$

$$F^1_{\mathrm{I}}\big|_{h=h'} = -\frac{2}{h^3(1+\mu'^2)}K(0;\mu,\mu') \quad (136)$$

$$F^1_{\mathrm{II}}\big|_{h=h'} = -\frac{2}{h^3(1+\mu'^2)}K^{(0)}\left(\frac{\tau-\tau'}{2};\mu,\mu'\right). \quad (137)$$

Appendix 2: Opposite symmetry of kernel function $K^{(0)}(v;\mu,\mu')$

We will prove Eq. 41 using Eq. 23. Writing as

$$f_m(v';\mu,\mu') = \frac{\mu\mu' + \cos v'_m}{\bar{R}^3} - \frac{3(\mu v' - \mu'\sin v'_m)(\mu' v' - \mu\sin v'_m)}{\bar{R}^5} \quad (138)$$

$$K^{(0)}(v;\mu,\mu') = \frac{-1}{8\pi}\frac{\sqrt{1+\mu'^2}}{\sqrt{1+\mu^2}}\int_0^v \sum_{m=0}^{l-1} f_m(v';\mu,\mu')\mathrm{d}v'. \quad (139)$$

Symmetry of μ and μ'

$$f_m(v';\mu,\mu') = f_m(v';\mu',\mu) \quad (140)$$

$$\sqrt{\frac{1+\mu^2}{1+\mu'^2}}K^{(0)}(v;\mu,\mu') = \sqrt{\frac{1+\mu'^2}{1+\mu^2}}K^{(0)}(v;\mu',\mu) \quad (141)$$

is obtained. Next, opposite symmetry of v will be shown.

$$\begin{aligned} f_m(v';\mu,\mu') &= \frac{\mu\mu' + \cos v'_m}{\sqrt{v'^2+\mu^2+\mu'^2-2\mu\mu'\cos v'_m}^{\,3}} - \frac{3\left(\mu v' - \mu'\sin v'_m\right)\left(\mu' v' - \mu\sin v'_m\right)}{\sqrt{v'^2+\mu^2+\mu'^2-2\mu\mu'\cos v'_m}^{\,5}} \\ &= \frac{\mu\mu' + \left(\cos v'\cos\frac{2m\pi}{l} + \sin v'\sin\frac{2m\pi}{l}\right)}{\sqrt{v'^2+\mu^2+\mu'^2-2\mu\mu'\left(\cos v'\cos\frac{2m\pi}{l}+\sin v'\sin\frac{2m\pi}{l}\right)}^{\,3}} \\ &\quad - \frac{3\left\{\mu v' - \mu'\left(\sin v'\cos\frac{2m\pi}{l} - \cos v'\sin\frac{2m\pi}{l}\right)\right\}\left\{\mu' v' - \mu\left(\sin v'\cos\frac{2m\pi}{l} - \cos v'\sin\frac{2m\pi}{l}\right)\right\}}{\sqrt{v'^2+\mu^2+\mu'^2-2\mu\mu'\left(\cos v'\cos\frac{2m\pi}{l}+\sin v'\sin\frac{2m\pi}{l}\right)}^{\,5}} \end{aligned} \quad (142)$$

So

$$f_m(v';\mu,\mu') = f_{-m}(-v';\mu,\mu'). \quad (143)$$

Then

$$f_m(v';\mu,\mu') + f_{-m}(v';\mu,\mu') = f_m(-v';\mu,\mu') + f_{-m}(-v';\mu,\mu'). \quad (144)$$

And when $m=0$,

$$f_0(v';\mu,\mu') = f_0(-v';\mu,\mu'). \quad (145)$$

Then,

$$\sum_{m=0}^{l-1} f_m(v';\mu,\mu') = \sum_{m=0}^{l-1} f_m(-v';\mu,\mu') \quad (146)$$

is obtained. The function is the even function of v'. So the kernel function which is an integral of the function, is an odd function of v.

$$K^{(0)}(v;\mu,\mu') = -K^{(0)}(-v;\mu,\mu') \quad (147)$$

The combination of Eqs. 141 and 147 can prove Eq. 41.

Appendix 3: Proof of equation 28

Equation 28 is proved as follows.

$$\begin{aligned} -(W^* + \bar{u})\frac{\partial\Phi}{\partial n} &= -(W^* + \bar{u})(w_{\mathrm{I}} + w_{\mathrm{II}}) \\ &= -(W^* + \bar{u})w_{\mathrm{I}} - W^* w_{\mathrm{II}} - \bar{u}w_{\mathrm{II}} \end{aligned} \quad (148)$$

The third term of the final expression can be neglected due to smallness and from Fig. 1

$$W^* = \Omega r\cos\varepsilon_1 + V\sin\varepsilon_1 \quad (149)$$

$$-w_{\mathrm{I}} = \Omega r\sin\varepsilon_1 - V\cos\varepsilon_1. \quad (150)$$

Then, we may manipulate Eq. 148 as follows

$$\begin{aligned} -(W^* + \bar{u})\frac{\partial\Phi}{\partial n} &= (W^* + \bar{u})(\Omega r\sin\varepsilon_1 - V\cos\varepsilon_1) \\ &\quad - w_{\mathrm{II}}(\Omega r\cos\varepsilon_1 + V\sin\varepsilon_1) \\ &= \Omega r\{(W^* + \bar{u})\sin\varepsilon_1 - w_{\mathrm{II}}\cos\varepsilon_1\} \\ &\quad - V\{(W^* + \bar{u})\cos\varepsilon_1 + w_{\mathrm{II}}\sin\varepsilon_1\} \\ &= \Omega r\{V + w_{\mathrm{a}} + w_{\mathrm{a}2}\} - V\{\Omega r + w_{\mathrm{t}} + w_{\mathrm{t}2}\} \end{aligned} \quad (151)$$

where

$$\begin{array}{ll} w_{\mathrm{a}} = -w_{\mathrm{I}} \cos \varepsilon_1, & w_t = w_{\mathrm{I}} \sin \varepsilon_1 \\ w_{\mathrm{a2}} = \bar{u} \sin \varepsilon_1 - w_{\mathrm{II}} \cos \varepsilon_1, & w_{t2} = \bar{u} \cos \varepsilon_1 + w_{\mathrm{II}} \sin \varepsilon_1 \end{array}. \tag{152}$$

Appendix 4: Proof of equation 117

Equation 117 is proved as follows. Using the notation

$$\begin{aligned} f_m(u', v_0'; \mu, \mu') = {} & \frac{1}{8\pi} \frac{1}{\sqrt{1+\mu^2}\sqrt{1+\mu'^2}} \\ & \times \left[\frac{1}{R^3} \left\{ \mu\mu' + \cos\left(u' - \frac{2\pi m}{l}\right) \right\} \right. \\ & + \frac{-3}{R^5} \left\{ \mu v' - \mu' h' \sin\left(u' - \frac{2\pi m}{l}\right) \right\} \\ & \left. \times \left\{ \mu' v' - \mu h \sin\left(u' - \frac{2\pi m}{l}\right) \right\} \right] \end{aligned} \tag{153}$$

$$v' = v_0' + u' \frac{h + h'}{2} \tag{154}$$

$$v_0' = \frac{\tau - \sigma}{2} h - \frac{\tau' - \sigma'}{2} h' - u \frac{h + h'}{2} \tag{155}$$

we have from Eqs. 83 and 113

$$f = \sum_{m=0}^{l-1} f_m(u', v_0'; \mu, \mu') \tag{156}$$

$$f' = \sum_{m=0}^{l-1} f_m(u', -v_0'; \mu', \mu). \tag{157}$$

It is easy to know the relations.

$$f_m(u', v_0'; \mu, \mu') = f_{-m}(-u', -v_0'; \mu', \mu) \tag{158}$$

$$\begin{aligned} & f_m(u', v_0'; \mu, \mu') + f_{-m}(u', v_0'; \mu, \mu') \\ & \quad = f_{-m}(-u', -v_0'; \mu', \mu) + f_m(-u', -v_0'; \mu', \mu) \end{aligned} \tag{159}$$

$$f_0(u', v_0'; \mu, \mu') = f_0(-u', -v_0'; \mu', \mu) \tag{160}$$

So

$$\sum_{m=0}^{l-1} f_m(u', v_0'; \mu, \mu') = \sum_{m=0}^{l-1} f_m(-u', -v_0'; \mu', \mu) \tag{161}$$

Then,

$$\begin{aligned} \int_0^u 2f \mathrm{d}u' & = \int_0^u 2 \sum_{m=0}^{l-1} f_m(u', v_0'; \mu, \mu') \mathrm{d}u' \\ & = \int_0^u 2 \sum_{m=0}^{l-1} f_m(-u', -v_0'; \mu', \mu) \mathrm{d}u' \\ & = \int_0^{-u} 2 \sum_{m=0}^{l-1} 2f_m(u'', -v_0'; \mu', \mu)(-\mathrm{d}u'') \\ & = -\int_0^{-u} 2f' \mathrm{d}u' \end{aligned} \tag{162}$$

Appendix 5: Numerical method

Our assignment is to seek the function γ which satisfies the minimum condition of the functional F.

$$\begin{aligned} F = l\rho \int_{r_\mathrm{b}}^{r_\mathrm{o}} \int_{s_1}^{s_2} & \gamma [k\Omega_1 r \{V_1 - \cos\varepsilon_1 (w_{\mathrm{I}} + w_{\mathrm{II}})\} \\ & - V_1 \{\Omega_1 r + \sin\varepsilon_1 (w_{\mathrm{I}} + w_{\mathrm{II}})\}] \mathrm{d}s \mathrm{d}r \end{aligned} \tag{163}$$

Using the Birnbaum series for the mode function, circulation density γ of the bound vortex is expressed as follows

$$\begin{aligned} \frac{\gamma(\xi, \eta)}{W_1^*(\eta)} & = \sum_{N=0}^{n-1} A^{(N)}(\eta) \lambda_N(\xi) \\ & = A^{(0)}(\eta) \sqrt{\frac{1-\xi}{1+\xi}} + A^{(1)}(\eta) \sqrt{1-\xi^2} \\ & + A^{(2)}(\eta) \xi \sqrt{1-\xi^2} + A^{(3)}(\eta) \xi^2 \sqrt{1-\xi^2} \end{aligned} \tag{164}$$

where ξ denotes chordwise variable, η denotes radial variable, $\lambda_{\mathrm{N}}(\xi)$ denote Birnbaum series, and $A^{(\mathrm{N})}(\eta)$ denote unknowns. Substitution of Eq. 164 into Eq. 163 gives

$$\begin{aligned} F = l\rho \int_{r_\mathrm{b}}^{r_\mathrm{o}} & W_1^* \sum_{N=0}^{n-1} A^{(N)}(\eta) \\ & \times \int_{s_1}^{s_2} \lambda_N(\xi) [k\Omega_1 r \{V_1 - \cos\varepsilon_1 (w_{\mathrm{I}} + w_{\mathrm{II}})\} \\ & - V_1 \{\Omega_1 r + \sin\varepsilon_1 (w_{\mathrm{I}} + w_{\mathrm{II}})\}] \mathrm{d}s \mathrm{d}r \end{aligned}. \tag{165}$$

Numerical calculation methods for $\delta F = 0$ are shown in this Appendix. These expressions shown here are for a propeller in non-uniform inflow which is discussed in Sect. 3. If V_1, Ω_1, h' in the expressions are changed to V, Ω, h, the expressions become those for a propeller in uniform inflow discussed in Sect. 2. Here we show two numerical methods which are by Eq. 163 and by Eq. 165. We call the calculation using Eq. 163 as being the collocation calculation method, and we call the calculation using Eq. 165 as the Galerkin calculation method in this paper.

At first the collocation calculation method based on Eq. 163 will be shown.

As w_{I} and w_{II} are given by Eqs. 86 and 87, respectively, expression for F by Eq. 163 is

$$\begin{aligned} F = l\rho \int_{r_\mathrm{b}}^{r_\mathrm{o}} \int_{s_1}^{s_2} & \gamma [(k-1)\Omega_1 r V_1 \\ & - \{k\Omega_1 r \cos\varepsilon_1 + V_1 \sin\varepsilon_1\} \int_{r_\mathrm{b}}^{r_\mathrm{o}} \mathrm{d}r' \int_{s_1}^{s_2} \gamma' h' \sqrt{1+\mu'^2} \\ & \times \left\{ \int_{-\infty}^{0} 2f \mathrm{d}u' + \int_0^u 2f \mathrm{d}u' \right\} \mathrm{d}s' \Bigg] \mathrm{d}s \mathrm{d}r \end{aligned} \tag{166}$$

where f is given by Eq. 83 of the main body. The first variation of Eq. 166 is

$$\delta F = l\rho \int_{r_b}^{r_o}\int_{s_1}^{s_2} \delta\gamma[k\Omega_1 r\{V_1 - \cos\varepsilon_1(w_I + w_{II})\} - V_1\{\Omega_1 r + \sin\varepsilon_1(w_I + w_{II})\}]\mathrm{d}s\mathrm{d}r + \delta F_2 \tag{167}$$

$$\delta F_2 = l\rho \int_{r_b}^{r_o}\int_{s_1}^{s_2} \gamma[-\{k\Omega_1 r\cos\varepsilon_1 + V_1\sin\varepsilon_1\} \times \int_{r_b}^{r_o}\mathrm{d}r' \int_{s_1}^{s_2} \delta\gamma' h'\sqrt{1+\mu'^2} \times \left\{\int_{-\infty}^{0} 2f\mathrm{d}u' + \int_0^u 2f\mathrm{d}u'\right\}\mathrm{d}s'\Bigg]\mathrm{d}s\mathrm{d}r \tag{168}$$

The order of integral $\int_{r_b}^{r_o} dr \int_{s_1}^{s_2} ds$ and $\int_{r_b}^{r_o} dr' \int_{s_1}^{s_2} ds'$ is changed for δF_2.

$$\delta F_2 = l\rho \int_{r_b}^{r_o}\int_{s_1}^{s_2} \delta\gamma' h'\sqrt{1+\mu'^2} \times \left[\int_{r_b}^{r_o}\mathrm{d}r \int_{s_1}^{s_2} \{-\gamma(k\Omega_1 r\cos\varepsilon_1 + V_1\sin\varepsilon_1)\} \times \left\{\int_{-\infty}^{0} 2f\mathrm{d}u' + \int_0^u 2f\mathrm{d}u'\right\}\mathrm{d}s\right]\mathrm{d}s'\mathrm{d}r' \tag{169}$$

The notations (τ, σ, μ) and (τ', σ', μ') are changed for each other.

$$\delta F_2 = l\rho \int_{r_b}^{r_o}\int_{s_1}^{s_2} \delta\gamma\left[-\int_{r_b}^{r_o}\mathrm{d}r' \int_{s_1}^{s_2} \gamma'\{k\Omega_1 r\cos\varepsilon_1 + V_1\sin\varepsilon_1\}' h\sqrt{1+\mu^2} \times \left\{\int_{-\infty}^{0} 2f'\mathrm{d}u' + \int_0^{-u} 2f'\mathrm{d}u'\right\}\mathrm{d}s'\right]\mathrm{d}s\mathrm{d}r \tag{170}$$

where f' is given by Eq. 113 of the main body. As the result we get

$$\begin{aligned}\delta F = l\rho \int_{r_b}^{r_o}\int_{s_1}^{s_2} \delta\gamma\Bigg[&(k-1)\Omega_1 r V_1 \\ &- \{k\Omega_1 r\cos\varepsilon_1 + V_1\sin\varepsilon_1\}\int_{r_b}^{r_o}\mathrm{d}r' \int_{s_2}^{s_2} \gamma h'\sqrt{1+\mu'^2} \\ &\times \left\{\int_{-\infty}^{0} 2f\mathrm{d}u' + \int_0^u 2f\mathrm{d}u'\right\}\mathrm{d}s' \\ &- \int_{r_b}^{r_o}\mathrm{d}r' \int_{s_2}^{s_2} \gamma\{k\Omega_1 r\cos\varepsilon_1 + V_1\sin\varepsilon_1\}' h\sqrt{1+\mu^2} \\ &\times \left\{\int_{-\infty}^{0} 2f'\mathrm{d}u' + \int_0^{-u} 2f'\mathrm{d}u'\right\}\mathrm{d}s'\Bigg]\mathrm{d}s\mathrm{d}r\end{aligned} \tag{171}$$

Then we get the equation $\delta F = 0$ in which $\delta\gamma$ can be arbitrary, so the following expressions are obtained.

$$\begin{aligned}0 = {}&(k-1)\Omega_1 r V_1 \\ &- \{k\Omega_1 r\cos\varepsilon_1 + V_1\sin\varepsilon_1\}\int_{r_b}^{r_o}\mathrm{d}r' \int_{s_1}^{s_2} \gamma h'\sqrt{1+\mu'^2} \\ &\times \left\{\int_{-\infty}^{0} 2f\mathrm{d}u' + \int_0^u 2f\mathrm{d}u'\right\}\mathrm{d}s' \\ &- \int_{r_b}^{r_o}\mathrm{d}r' \int_{s_1}^{s_2} \gamma\{k\Omega_1 r\cos\varepsilon_1 + V_1\sin\varepsilon_1\}' h\sqrt{1+\mu^2} \\ &\times \left\{\int_{-\infty}^{0} 2f'\mathrm{d}u' + \int_0^{-u} 2f'\mathrm{d}u'\right\}\mathrm{d}s' \quad \text{on } s, r\end{aligned} \tag{172}$$

Equation 172 is the integral equation for γ which we call Method II. If the δF_2 term is neglected in the calculation for Eq. 172

$$\begin{aligned}0 = {}&(k-1)\Omega_1 r V_1 \\ &- \{k\Omega_1 r\cos\varepsilon_1 + V_1\sin\varepsilon_1\}\int_{r_b}^{r_o}\mathrm{d}r' \int_{s_1}^{s_2} \gamma h'\sqrt{1+\mu'^2} \\ &\times \left\{\int_{-\infty}^{0} 2f\mathrm{d}u' + \int_0^u 2f\mathrm{d}u'\right\}\mathrm{d}s' \quad \text{on } s, r\end{aligned} \tag{173}$$

This Eq. 173 is the integral equation for γ which we call Method I. The final equation for the numerical calculation is given by substitution of the Birnbaum series Eq. 164 into Eq. 172 or Eq. 173.

$$
\begin{aligned}
0 &= (k-1)\Omega_1 r V_1 \\
&- \int_{r_b}^{r_o} \mathrm{d}r' \int_{s_1}^{s_2} W_1^{*'} \sum_{N'=0}^{n-1} A^{(N')}(\eta')\lambda_{N'}(\xi') \\
&\times \{k\Omega_1 r\cos\varepsilon_1 + V_1 \sin\varepsilon_1\} h'\sqrt{1+\mu'^2} \\
&\times \left\{ \int_{-\infty}^{0} 2f\,\mathrm{d}u' + \int_{0}^{u} 2f\,\mathrm{d}u' \right\} \mathrm{d}s' \\
&- \int_{r_b}^{r_o} \mathrm{d}r' \int_{s_1}^{s_2} W_1^{*'} \sum_{N'=0}^{n-1} A^{(N')}(\eta')\lambda_{N'}(\xi') \\
&\times \{k\Omega_1 r\cos\varepsilon_1 + V_1 \sin\varepsilon_1\}' h\sqrt{1+\mu^2} \\
&\times \left\{ \int_{-\infty}^{0} 2f'\,\mathrm{d}u' + \int_{0}^{-u} 2f'\,\mathrm{d}u' \right\} \mathrm{d}s' \quad \text{on } N, r
\end{aligned} \tag{174}
$$

$$
\begin{aligned}
0 &= (k-1)\Omega_1 r V_1 \\
&- \int_{r_b}^{r_o} \mathrm{d}r' \int_{s_1}^{s_2} W_1^{*'} \sum_{N'=0}^{n-1} A^{(N')}(\eta')\lambda_{N'}(\xi') \\
&\times \{k\Omega_1 r\cos\varepsilon_1 + V_1 \sin\varepsilon_1\} h'\sqrt{1+\mu'^2} \\
&\times \left\{ \int_{-\infty}^{0} 2f\,\mathrm{d}u' + \int_{0}^{u} 2f\,\mathrm{d}u' \right\} \mathrm{d}s' \quad \text{on } N, r
\end{aligned} \tag{175}
$$

Next, the Galerkin calculation method, which is based on Eq. 165 using the Birnbaum series will be shown below. From Eq. 165

$$
\begin{aligned}
F = l\rho \int_{r_b}^{r_o}\int_{s_1}^{s_2} & W_1^{*} \sum_{N=0}^{n-1} A^{(N)}(\eta)\lambda_N(\xi)[(k-1)\Omega_1 r V_1 \\
& -\{k\Omega_1 r\cos\varepsilon_1 + V_1 \sin\varepsilon_1\} \\
& \times \int_{r_b}^{r_o} \mathrm{d}r' \int_{s_1}^{s_2} W_1^{*'} \sum_{N'=0}^{n-1} A^{(N')}(\eta')\lambda_{N'}(\xi') h'\sqrt{1+\mu'^2} \\
& \times \left\{ \int_{-\infty}^{0} 2f\,\mathrm{d}u' + \int_{0}^{u} 2f\,\mathrm{d}u' \right\} \mathrm{d}s' \Bigg] \mathrm{d}s\mathrm{d}r
\end{aligned} \tag{176}
$$

First variation of Eq. 176 is

$$
\begin{aligned}
\delta F = l\rho \int_{r_b}^{r_o}\int_{s_1}^{s_2} & W_1^{*} \sum_{N=0}^{n-1} \delta A^{(N)}(\eta)\lambda_N(\xi)[(k-1)\Omega_1 r V_1 \\
& -\{k\Omega_1 r\cos\varepsilon_1 + V_1 \sin\varepsilon_1\}(w_{\mathrm{I}} + w_{\mathrm{II}})]\mathrm{d}s\mathrm{d}r + \delta F_2
\end{aligned} \tag{177}
$$

$$
\begin{aligned}
\delta F_2 = l\rho \int_{r_b}^{r_o}\int_{s_1}^{s_2} & W_1^{*} \sum_{N=0}^{n-1} A^{(N)}(\eta)\lambda_N(\xi)[-\{k\Omega_1 r\cos\varepsilon_1 + V_1 \sin\varepsilon_1\} \\
& \times \int_{r_b}^{r_o} \mathrm{d}r' \int_{s_1}^{s_2} W_1^{*'} \sum_{N'=0}^{n-1} \delta A^{(N')}(\eta')\lambda_{N'}(\xi') h'\sqrt{1+\mu'^2} \\
& \times \left\{ \int_{-\infty}^{0} 2f\,\mathrm{d}u' + \int_{0}^{u} 2f\,\mathrm{d}u' \right\} \mathrm{d}s' \Bigg] \mathrm{d}s\mathrm{d}r
\end{aligned} \tag{178}
$$

By changing the order of integral $\int_{r_b}^{r_o}\mathrm{d}r\int_{s_1}^{s_2}\mathrm{d}s$ and $\int_{r_b}^{r_o}\mathrm{d}r'\int_{s_1}^{s_2}\mathrm{d}s'$, and the notation ($\tau$, σ, μ) and (τ', σ', μ') are changed for each other, we get

$$
\begin{aligned}
\delta F_2 = l\rho \int_{r_b}^{r_o}\int_{s_1}^{s_2} & W_1^{*} \sum_{N=0}^{n-1} \delta A^{(N)}(\eta)\lambda_N(\xi) \\
& \times \Bigg[\int_{r_b}^{r_o} \mathrm{d}r' \int_{s_1}^{s_2} W_1^{*'} \sum_{N'=0}^{n-1} A^{(N')}(\eta')\lambda_{N'}(\xi') \\
& \times \{-(k\Omega_1 r\cos\varepsilon_1 + V_1 \sin\varepsilon_1)'\} h\sqrt{1+\mu^2} \\
& \times \left\{ \int_{-\infty}^{0} 2f'\,\mathrm{d}u' + \int_{0}^{-u} 2f'\,\mathrm{d}u' \right\} \mathrm{d}s' \Bigg] \mathrm{d}s\mathrm{d}r
\end{aligned} \tag{179}
$$

As the result, we get

$$
\begin{aligned}
\delta F = l\rho \int_{r_b}^{r_o}\int_{s_1}^{s_2} & W_1^{*} \sum_{N=0}^{n-1} \delta A^{(N)}(\eta)\lambda_N(\xi)[(k-1)\Omega_1 r V_1 \\
& - \int_{r_b}^{r_o} \mathrm{d}r' \int_{s_1}^{s_2} W_1^{*'} \sum_{N'=0}^{n-1} A^{(N')}(\eta')\lambda_{N'}(\xi') \\
& \times \{k\Omega_1 r\cos\varepsilon_1 + V_1 \sin\varepsilon_1\} h'\sqrt{1+\mu'^2} \\
& \times \left\{ \int_{-\infty}^{0} 2f\,\mathrm{d}u' + \int_{0}^{u} 2f\,\mathrm{d}u' \right\} \mathrm{d}s' \\
& - \int_{r_b}^{r_o} \mathrm{d}r' \int_{s_1}^{s_2} W_1^{*'} \sum_{N'=0}^{n-1} A^{(N')}(\eta')\lambda_{N'}(\xi') \\
& \times \{k\Omega_1 r\cos\varepsilon_1 + V_1 \sin\varepsilon_1\}' h\sqrt{1+\mu^2} \\
& \times \left\{ \int_{-\infty}^{0} 2f'\,\mathrm{d}u' + \int_{0}^{-u} 2f'\,\mathrm{d}u' \right\} \mathrm{d}s' \Bigg] \mathrm{d}s\mathrm{d}r
\end{aligned} \tag{180}
$$

Then we get the equation $\delta F = 0$ in which $\delta A^{(N)}(\eta)$ can be arbitrary, so the following expression is obtained.

$$0=\int_{s_1}^{s_2}\lambda_N(\xi)\Big[(k-1)\Omega_1 rV_1 - \int_{r_b}^{r_o}\mathrm{d}r'\int_{s_1}^{s_2}W_1^{*'}\sum_{N'=0}^{n-1}A^{(N')}(\eta')\lambda_{N'}(\xi') \times\{k\Omega_1 r\cos\varepsilon_1+V_1\sin\varepsilon_1\}h'\sqrt{1+\mu'^2} \times\left\{\int_{-\infty}^{0}2f\,\mathrm{d}u'+\int_0^u 2f\,\mathrm{d}u'\right\}\mathrm{d}s' - \int_{r_b}^{r_o}\mathrm{d}r'\int_{s_1}^{s_2}W_1^{*'}\sum_{N'=0}^{n-1}A^{(N')}(\eta')\lambda_{N'}(\xi') \times\{k\Omega_1 r\cos\varepsilon_1+V_1\sin\varepsilon_1\}'h\sqrt{1+\mu^2} \times\left\{\int_{-\infty}^{0}2f'\,\mathrm{d}u'+\int_0^{-u}2f'\,\mathrm{d}u'\right\}\mathrm{d}s'\Big]\mathrm{d}s \quad \text{on } N,r \tag{181}$$

This is the integral equation for $A^{(N')}(\eta')$ or γ which we call Method II. If the δF_2 term is neglected in the calculation for Eq. 181

$$0=\int_{s_1}^{s_2}\lambda_N(\xi)\Big[(k-1)\Omega_1 rV_1 - \int_{r_b}^{r_o}\mathrm{d}r'\int_{s_1}^{s_2}W_1^{*'}\sum_{N'=0}^{n-1}A^{(N')}(\eta')\lambda_{N'}(\xi') \times\{k\Omega_1 r\cos\varepsilon_1+V_1\sin\varepsilon_1\}h'\sqrt{1+\mu'^2} \times\left\{\int_{-\infty}^{0}2f\,\mathrm{d}u'+\int_0^u 2f\,\mathrm{d}u'\right\}\mathrm{d}s'\Big]\mathrm{d}s \quad \text{on } N,r \tag{182}$$

This Eq. 182 is the integral equation for $A^{(N')}(\eta')$ or γ which we call Method I. The difference between Eqs. 174 and 175 in the collocation calculation method and Eqs. 181 and 182 in the Galerkin calculation method is the effect of the chordwise integral $\int_{s_1}^{s_2}\lambda_N(\xi)\mathrm{d}s$.

Here we have shown two numerical methods, the collocation calculation method based on Eq. 163 and the Galerkin calculation method based on Eq. 165. For each method we have two methods, Method I and Method II. In the above expressions $w_{\mathrm{I}}+w_{\mathrm{II}}$ is a function of γ, so variation term from $w_{\mathrm{I}}+w_{\mathrm{II}}$ gives δF_2. Sometimes δF_2 is neglected in the calculation, in which case we call it by Method I in this paper. We call it by Method II for the case which includes δF_2.

Appendix 6: Singularity of the kernel function

Kernel functions used in Sect. 2.1 of the main body are $K(v;\mu,\mu')$ (Eqs. 15, 21) and $K^{(0)}(v;\mu,\mu')$ (Eqs. 22, 23). From the Ref. [3] the singularity of $\bar{K}(v;\mu,\mu')$ is given as follows

$$\bar{K}(v;\mu,\mu')\cong\begin{cases}-\frac{1}{4\pi}\left\{\sqrt{1+\mu^2}+\frac{(\mu-\mu')^2}{2\sqrt{1+\mu^2}}\ln|\mu-\mu'|\right\}, & v>0\\ -\frac{1}{8\pi}\left\{\sqrt{1+\mu^2}+\frac{(\mu-\mu')^2}{2\sqrt{1+\mu^2}}\ln|\mu-\mu'|\right\}, & v=0\\ 0, & v<0\end{cases} \tag{183}$$

As

$$K^{(0)}(v;\mu,\mu')=K(v;\mu,\mu')-K(0;\mu,\mu'). \tag{184}$$

So

$$\bar{K}^{(0)}(v;\mu,\mu')\cong\begin{cases}-\frac{1}{8\pi}\left\{\sqrt{1+\mu^2}+\frac{(\mu-\mu')^2}{2\sqrt{1+\mu^2}}\ln|\mu-\mu'|\right\}, & v>0\\ 0, & v=0\\ \frac{1}{8\pi}\left\{\sqrt{1+\mu^2}+\frac{(\mu-\mu')^2}{2\sqrt{1+\mu^2}}\ln|\mu-\mu'|\right\}, & v<0\end{cases} \tag{185}$$

From Appendix 1

$$(\mu-\mu')^2F^1=-\frac{2}{h^3(1+\mu'^2)}\bar{K}(v;\mu,\mu') \tag{186}$$

$$(\mu-\mu')^2F^1_{\mathrm{I}}=-\frac{2}{h^3(1+\mu'^2)}\bar{K}(0;\mu,\mu') \tag{187}$$

$$(\mu-\mu')^2F^1_{\mathrm{II}}=-\frac{2}{h^3(1+\mu'^2)}\bar{K}^{(0)}(v;\mu,\mu'). \tag{188}$$

Appendix 7: Numerical method of singular integral (Multhopp matrix)

Upwash on the blade w is given by Eq. 74 of the main body. Function F^1 in the expression has the singularity of the 2nd order pole at $r=r'$ as shown in Appendix 6. The numerical integral can be done using the Multhopp matrix as follows [3].

$$w_v=\int_{r_b}^{r_o}\frac{g}{(r-r')^2}\mathrm{d}r'=\int_{-1}^{1}\frac{g}{(\eta-\eta')^2}\frac{\mathrm{d}\eta'}{\bar{r}}=2\pi\sum_{j=1}^{m}\bar{b}_{vj}\frac{g_{vj}}{\bar{r}} \tag{189}$$

$$\bar{b}_{vj}=\begin{cases}\frac{1-(-1)^{j-v}}{2(m+1)}\frac{\sin\varphi_j}{(\cos\varphi_v-\cos\varphi_j)^2}, & v\neq j\\ -\frac{m+1}{4\sin\varphi_v}, & v=\mathrm{j}\end{cases} \tag{190}$$

$$g = (r - r')^2 \int_{s_1}^{s_2} \gamma h' \sqrt{1 + \mu'^2} F^1 \mathrm{d}s'. \tag{191}$$

According to the definition of μ, μ'

$$\begin{aligned} g &= (h\mu - h'\mu')^2 \int_{s_1}^{s_2} \gamma h' \sqrt{1 + \mu'^2} F^1 \mathrm{d}s' \\ &= (\mu h/h' - \mu')^2 \int_{s_1}^{s_2} \gamma h'^3 \sqrt{1 + \mu'^2} F^1 \mathrm{d}s'. \end{aligned} \tag{192}$$

Referring to Appendix 1

$$\begin{aligned} \lim_{r \to r'} g &= \lim_{r \to r'} (\mu - \mu')^2 \int_{s_1}^{s_2} \gamma h^3 \sqrt{1 + \mu'^2} F^1 \mathrm{d}s' \\ &= \lim_{r \to r'} h^2 (\mu - \mu')^2 \int_{s_1}^{s_2} \gamma \frac{-2}{h^2 \sqrt{1 + \mu'^2}} K\left(\frac{\tau - \tau'}{2}; \mu, \mu'\right) \mathrm{d}s' \\ &= \lim_{r \to r'} \frac{-2}{\sqrt{1 + \mu'^2}} \int_{s_1}^{s_2} \gamma \bar{K}\left(\frac{\tau - \tau'}{2}; \mu, \mu'\right) \mathrm{d}s' \end{aligned} \tag{193}$$

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