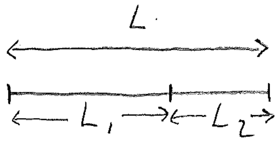
GOLDEN SECTION SEARCH

- One-dimensional method
- Reduces the uncertainty (design space/search interval) by the golden ratio $K = 0.618$ each iteration



$$\frac{L_2}{L_1} = \frac{L_1}{L} \iff \left(\frac{L_2}{L_1}\right)^2 + \frac{L_2}{L_1} = 1$$

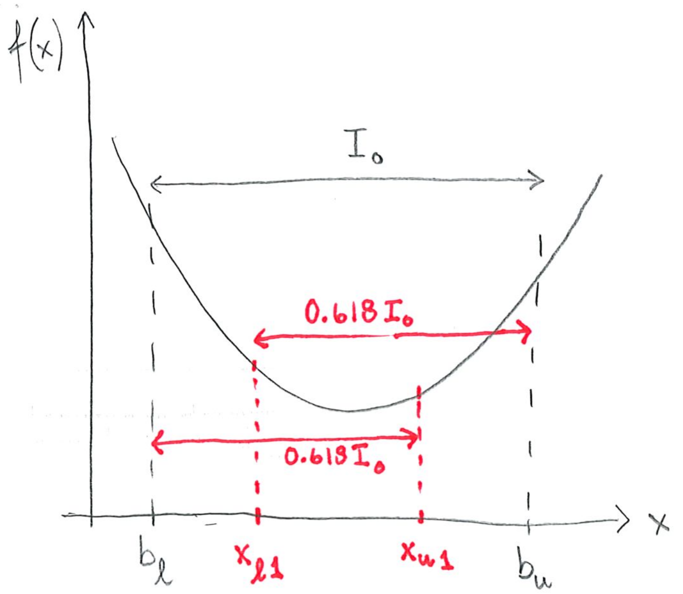
$$\iff \frac{L_2}{L_1} = K \iff K^2 + K = 1$$

$$\Rightarrow K = 0.618$$

The golden section rule: Divide an interval so that the ratio of the smaller part to the larger equals the ratio of the larger to the whole

After n iterations the interval I_0 has been reduced to $I_n = I_0 \cdot K^{n-1} = I_0 \cdot 618^{n-1}$

PROCEDURE:



$$\min f(x)$$
$$\text{s.t. } b_l \le x \le b_u$$

Step 0: Set start interval $I_0 = b_u - b_l$

upper limit

lower limit

Set counter $k = 1$

Step 1: Evaluate $f(x)$ in two points

$$x_{lk} = b_{uk} - 0.618\, I_{k-1}$$
$$x_{uk} = b_{lk} + 0.618\, I_{k-1}$$

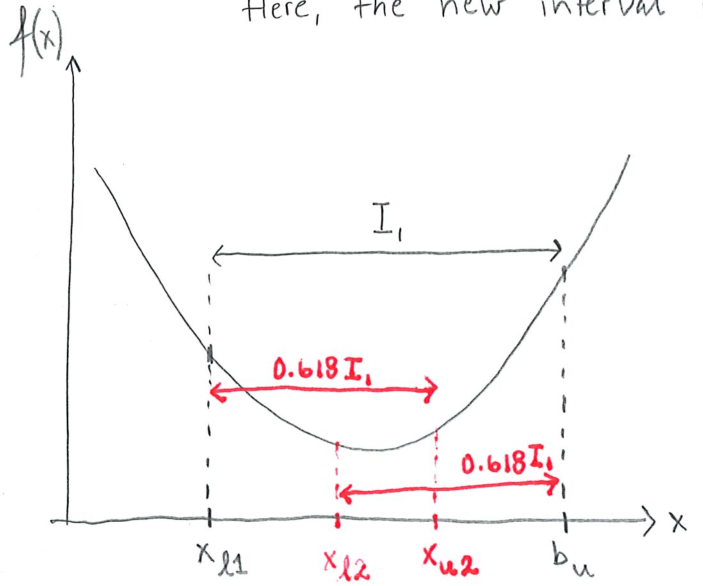
Step 2: Identify the point with the lowest value (here x_{u1})

Step 3: Define a new interval

If $f(x_{\ell k}) > f(x_{uk})$ the new interval is to the right of $x_{\ell k}$.

Otherwise it is to the left of x_u

Here, the new interval is $I_1 = b_u - x_{\ell 1}$



Step 4: Converged? It has converged if $I_k \leq \varepsilon$

ε is set before the optimization starts

If not converged: $k = k + 1$

go to step 1

(Here, $x_{\ell 2}$ and x_{u2} are created)

Step 5: Output the solution as the best of $x_{\ell k}$ and x_{uk} of the final iteration

Derivative Methods (based on line-search)

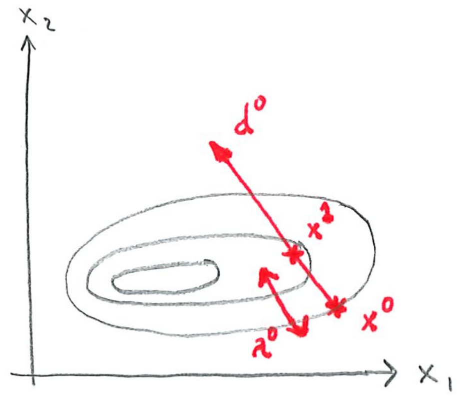
- Are classified as first or second order depending on the order of the used derivatives

- They are iterative search methods involving movement from one point x^k to another x^{k+1} along a line

$$x^{k+1} = x^k + \lambda^k d^k$$

d = search direction
λ = step size
k = counter/iteration

STEEPEST DESCENT: procedure



Step 0: Enter starting point $\bar{x}^0$

* Enter termination criterion ε (smallest gradient magnitude)

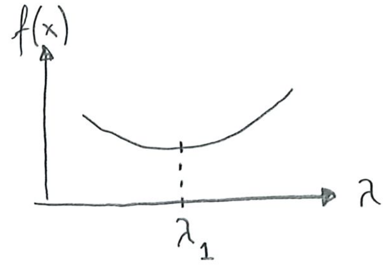
* Set $k = 0$

Step 1: Calculate $\nabla f(\bar{x}^k)$

If $\|\nabla f(\bar{x}^k)\| \leq \varepsilon$ stop

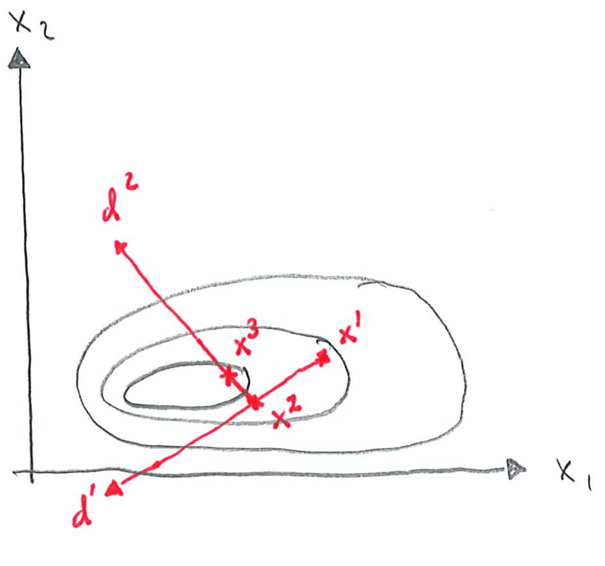
Step 2: Perform line search in the $-\nabla f(\bar{x}^k)$ direction $\Rightarrow \lambda^k$

1D-optimization problem:



Step 3: $\bar{x}^{k+1} = \bar{x}^k + \lambda^k(-\nabla f(\bar{x}^k))$, $k = k+1$

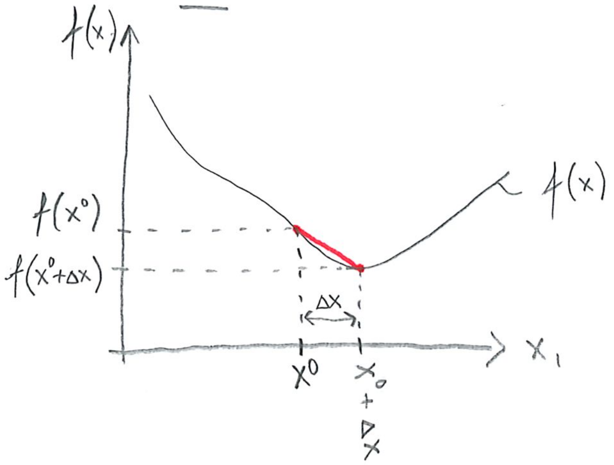
Go to step 1



x2
d2
x3
x1
x2
x1
d1

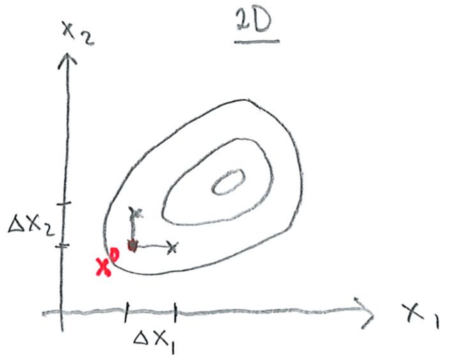
Numerical approximation of derivatives

1D



$$\frac{\partial f}{\partial x_1} = \frac{\Delta f}{\Delta x_1} = \frac{f(x^0+\Delta x) - f(x^0)}{\Delta x}$$

2D



Check in one variable at a time

Newton's (Second Order) method

Perform a 2^{nd} order Taylor series expansion of $f(x)$ at the point $\bar{x}^k$

$$f(\bar{x}^k + \Delta\bar{x}^k) = f(\bar{x}^k) + \Delta\bar{x}^{kt} \cdot \nabla f(\bar{x}^k) + \frac{1}{2}\Delta\bar{x}^{kt}\, \nabla^2 f(\bar{x}^k)\, \Delta\bar{x}^k$$

Replace ∇f and $\nabla^2 f$ with g and H respectively

$$f(\bar{x}^k + \Delta\bar{x}^k) = f(\bar{x}^k) + \Delta\bar{x}^{kt} g + \frac{1}{2}\Delta\bar{x}^{kt} H \,\Delta\bar{x}^k$$

Find the minimum of $f(\bar{x}^k + \Delta\bar{x}^k)$ by differentiation with respect to $\Delta\bar{x}^k$ and setting the derivate to 0

$$\frac{\partial f(\bar{x}^k + \Delta\bar{x}^k)}{\partial \Delta\bar{x}^k} = g + H \cdot \Delta\bar{x}^k = 0 \implies \Delta\bar{x}^k = -H^{-1} g$$

This finds the optimal step $\Delta\bar{x}^k$

Newton's Method: procedure

Step 0: Enter starting point x^0 and termination criterion ε

Set $k = 0$

Step 1: Evaluate the gradient $g(\bar{x}^k) = \nabla f(\bar{x}^k)$

If $\| g(\bar{x}^k) \| \leq \varepsilon$, stop

Step 2: Evaluate Hessian Matrix (2^{nd} order derivatives) H^k and its inverse $(H^{-1})^k$

Step 3: Perform a line search in the search direction $d^k = (H^{-1})^k g^k$ to obtain λ^k that minimizes $f(\bar{x})$ in the d^k direction

Step 4: Set $x^{k+1} = x^k + \lambda^k d^k$

Set $k = k + 1$

Go to step 1