1 One-dimensional optical lattice potential

The contents of this section follow the derivation found in Sec. IV A of [1].

The Hamiltonian for an atom moving in a 1D lattice potential is

$$H_{\text{single,1D}} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_0\cos^2(kx) \tag{1}$$

where $k = 2\pi/\lambda$, and λ is the wavelength of the lattice laser. If V_0 is in units of the recoil energy $E_R = \frac{\hbar^2 k^2}{2m}$, then the Hamiltonian is

$$\begin{aligned} H_{\text{single,1D}} &= -\frac{1}{k^2}\frac{\partial^2}{\partial x^2} + V_0\cos^2(kx) \\ H_{\text{single,1D}} &= -\frac{1}{k^2}\frac{\partial^2}{\partial x^2} + \frac{V_0}{4}(2 + e^{2ikx} + e^{-2ikx}) \end{aligned} \tag{2}$$

The solution to this equation can be found in terms of Bloch states, which are labeled by their quasi-momentum q, and their band index n

$$\psi_q^n(x) = e^{iqx}\sum_{m=-\infty}^{\infty} c_{qm}^n e^{imGx} \tag{3}$$

The lattice translation invariant function that typically accompanies e^{iqx} has been written here as a sum of plane waves (labeled by the integer m) at the reciprocal lattice vectors. This is the Fourier series of any such periodic function and represents no loss of generality. The reciprocal lattice vector $G = \frac{2\pi}{a} = 2k$, where $a = \lambda/2$ is the lattice spacing.

Plugging the Bloch states into the Hamiltonian and then rearranging some of the terms in the infinite sum, we get

$$\begin{aligned} H_{\text{single,1D}}\psi_q(x) &= \sum_m \left[(q/k + 2m)^2 + \frac{V_0}{4}(2 + e^{2ikx} + e^{-2ikx})\right] c_{qm}^n e^{iqx + im2kx} \\ H_{\text{single,1D}}\psi_q(x) &= \sum_m \left[\left((q/k + 2m)^2 + \frac{V_0}{2}\right) c_{qm}^n + \frac{V_0}{4}c_{q,m-1}^n + \frac{V_0}{4}c_{q,m+1}^n\right] e^{iqx + im2kx} \end{aligned} \tag{4}$$

The left hand side of the time-independent Schrodinger equation is simply

$$E_q^n\psi_q(x) = \sum_m E_q c_{qm}^n e^{iqx + imGx} \tag{5}$$

For the coefficients c_{qm}^n to represent an eigenstate of the problem, the Bloch state needs to satisfy $H\psi_q(x) = E_q^n\psi_q(x)$, and since the plane waves are linearly independent functions this means that

$$\left((q/k + 2m)^2 + \frac{V_0}{2}\right) c_{qm}^n + \frac{V_0}{4}c_{q,m-1}^n + \frac{V_0}{4}c_{q,m+1}^n = E_q c_{qm}^n \tag{6}$$

From now on we simply express the quasimomentum in units of k, so that it takes values in $(-1, 1]$. We have a linear system of equations which determines the c_{qm}^n. The number of equations is infinite, but for our practical purposes we will truncate it such that $|m| < \mathcal{N}$. The resulting equations can be written in matrix form, for example if we select $\mathcal{N} = 2$

$$\begin{bmatrix} \frac{1}{2}V_0 + (q-4)^2 & \frac{1}{4}V_0 & 0 & 0 & 0 \\ \frac{1}{4}V_0 & \frac{1}{2}V_0 + (q-2)^2 & \frac{1}{4}V_0 & 0 & 0 \\ 0 & \frac{1}{4}V_0 & \frac{1}{2}V_0 + q^2 & \frac{1}{4}V_0 & 0 \\ 0 & 0 & \frac{1}{4}V_0 & \frac{1}{2}V_0 + (q+2)^2 & \frac{1}{4}V_0 \\ 0 & 0 & 0 & \frac{1}{4}V_0 & \frac{1}{2}V_0 + (q+4)^2 \end{bmatrix} \tag{7}$$

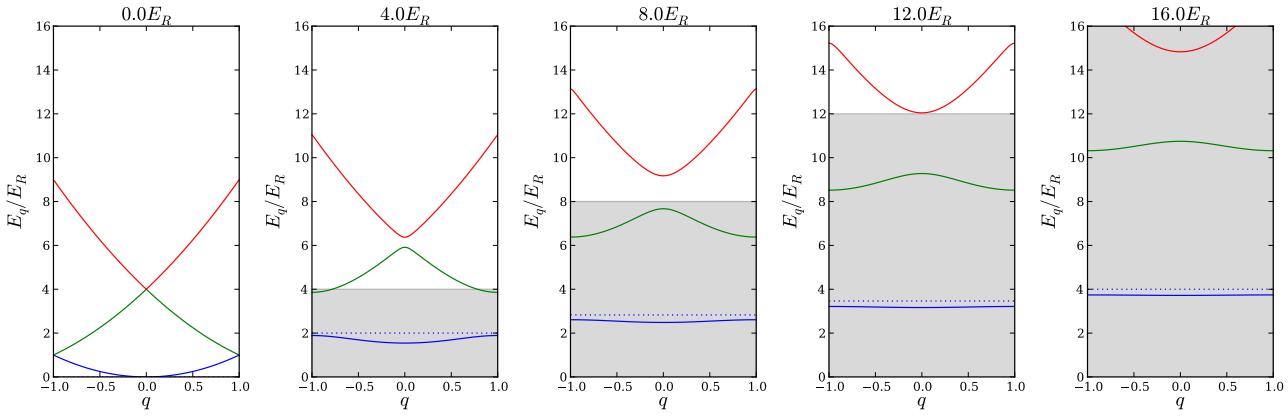

Figure 1: Band structure in a 1D optical lattice. The depth of the lattice is indicated by the shaded area, and the energy of the hamonic oscillator ground state in a single lattice site is shown as a dotted line.

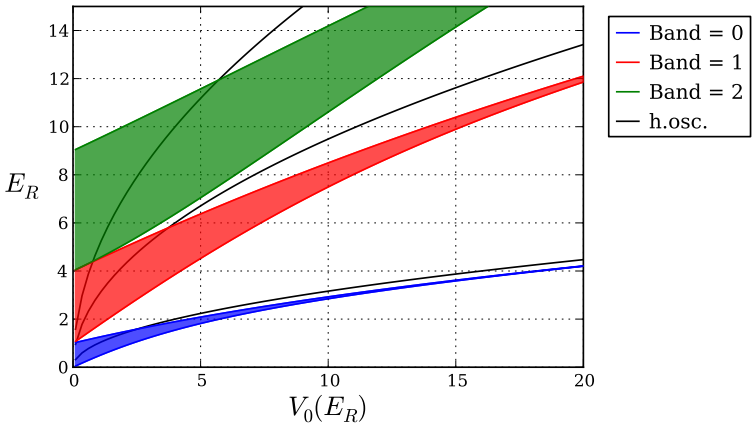

Figure 2: Band structure in a 1D optical lattice. Each band is indicated by the colored area, the harmonic oscillator states in an isolated lattice site are shown as black lines.

In the numerical solution that we implemented we chose $\mathcal{N} = 5$, it turns out that to accurately obtain the dispersion relationship for the n^{th} band you pretty much only need $\mathcal{N} = n + 1$, so using $\mathcal{N} = 5$ is somewhat overkill for us since we will be mostly concentrated on the lowest band and the first excited band.

1.1 Band structure

We can find the solutions for the set of coefficients c^n_{qm} by diagonalizing the matrix shown above. The eigenvalues correspond to the energies E^n_q as a function of quasimomentum q and band index n, this set of solutions is referred to as the band structure, and we show it for a 1D lattice as a function of q in Fig. 1, and also as a function of lattice depth in Fig. 2

1.2 Eigenstates

For each energy eigenvalue we have an associated eigenstate which is defined in terms of the c^n_{qm} by Eq. 3. Typically, numerical diagonalization routines return the normalized eigenvectors of the matrix

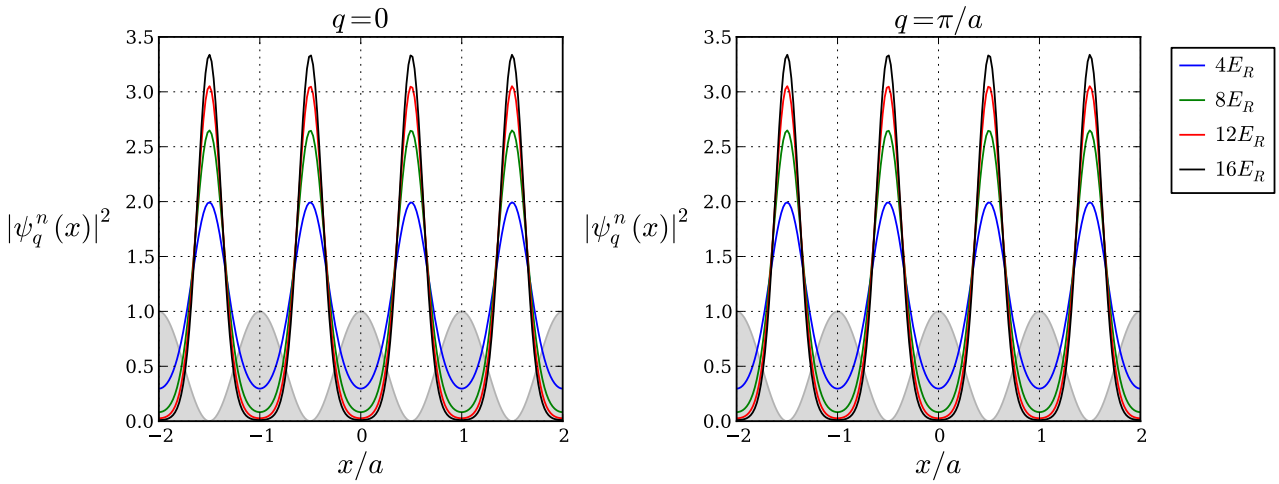

Figure 3: Eigenstates of the Hamiltonian in a 1D optical lattice shown for $q = 0$ (left) and $q = \pi/a$ (right) for various lattice depths. The states are normalized so that the integral of the probability density over one lattice site is equal to one. The gray shaded region is shown to indicate the variation of the lattice potential.

in question, and for us this means that the coefficients c_{qm}^n will satisfy

$$\sum_m |c_{qm}^n|^2 = 1 \tag{8}$$

This has the implication that the states obtained from Eq. 3 will be normalized over a lattice site. In Fig. 3. we show the probability density for a lowest band eigenstate as a function of position in the lattice for various lattice depths. One can see how as the lattice gets deeper the state becomes more localized around the center of each lattice site. Notice that the sites are the positions where x/a is a half-integer, this is due to the fact that we have chosen $V_0 > 0$ and a cosine to represent the lattice potential.

1.3 Wannier states

It is useful to define a basis of states that are localized around a single lattice site. We will see later on that when using such a basis the Hamiltonian for the Hubbard model takes its most familiar form. The localized states, centered around the site at X_j, can be constructed as the following superposition of eigenstates of the Hamiltonian

$$w^n(x - X_j) = \frac{1}{N_\mathrm{s}} \sum_q e^{-iqX_j} \psi_q^n(x) \tag{9}$$

Here we have considered a finite sized lattice with a total of N_s sites, which implies that the quasimomentum is only defined for the discrete values $q = \frac{n}{N_\mathrm{s}} \frac{\pi}{a}$ where $n \in \{-N_\mathrm{s} + 1, -N_\mathrm{s} + 2, \ldots, N_\mathrm{s} - 1, N_\mathrm{s}\}$. Care must be taken when summing up the eigenstates to obtain the Wannier states, since the $\psi_q^n(x)$ are defined by Eq. 3 up to an overall phase factor. We want the Wannier function to be real valued, and to obtain such a result we must choose a phase factor for all the eigenstates such that $\psi_q^n(x = x_j)$ is a real number, where x_j denotes the center of any lattice site. Using this prescription for the phase, and then adding up all the eigenstates, as indicated in Eq. 9, one obtains the Wannier states which are shown in Fig. 4. It is seen in the figure that the Wannier state will be real valued only if x_j is the center of a lattice site.

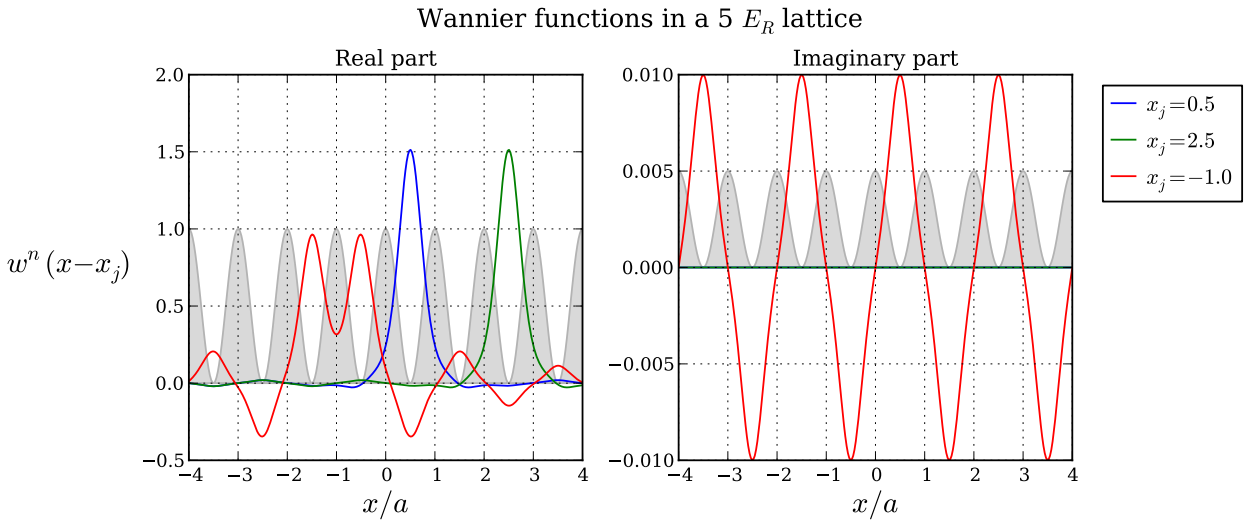

Figure 4: Localized Wannier states in a 1D optical lattice with $V_0 = 5E_R$. With the definition given in Eq. 9, the Wannier states will be real valued if x_j is at the center of a lattice site. The gray shaded region is shown to indicate the spatial variation of the lattice potential.

In some treatments (for instance [2]) the Wannier function is defined with a normalization factor of $\sqrt{N_s}$ rather than N_s as shown here. This is considering eigenfunctions $\psi_q^n(x)$ which are normalized when integrating over the full extent in the lattice. We stick to the N_s normalization factor, without the square root, since the eigenfunctions that are obtained numerically come out normalized over a lattice site, as was explained in the previous section.

As the lattice depth is increased, the Wannier states become more localized. This leads to less overlap betweeen Wannier states in adjacent sites, which results in a reduction of the amplitude for a particle to tunnel from one site to the next one. More localized states also imply that the on-site interaction will be larger, since two particles in the same site will be closer to each other on average. The Wannier states as a function of lattice depth are shown in Fig. 5.

2 Three-dimensional optical lattice potential

The hamiltonian for an atom moving in a 3D lattice can be separated in the three spatial coordinates. So we can use the solutions that were obtained in the previous section for the 1D lattice and obtain the band structure and the Wannier states for the 3D lattice. The band structure is shown in Fig. 6.

The Wannier states in a 3D lattice are simply products of the Wannier states in each of the three spatial coordinates. They are defined as

$$w^n(\boldsymbol{r} - \boldsymbol{R}_j) = \frac{1}{N_\mathrm{s}^3} \sum_{\boldsymbol{q}} e^{-i\boldsymbol{q}\cdot\boldsymbol{R}_j} \prod_{u=x,y,z} \psi_{q_u}^{n_u}(u) \tag{10}$$

where N_s^3 is the total number of sites in the lattice.

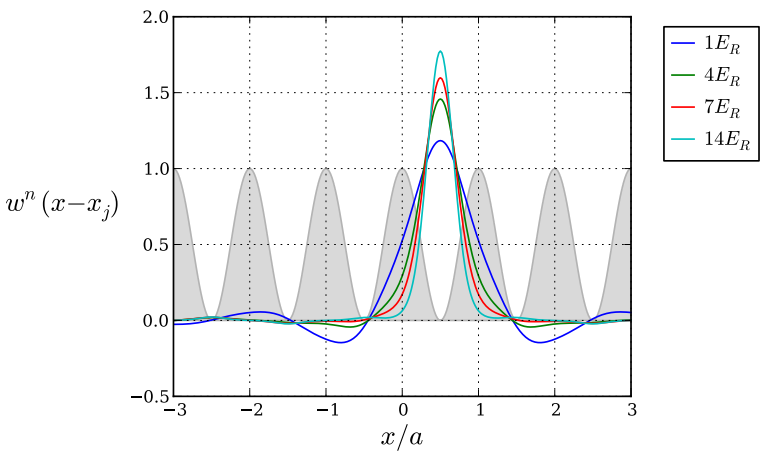

Figure 5: Localized Wannier states in a 1D optical lattice for various lattice depths. The gray shaded region is shown to indicate the spatial variation of the lattice potential.

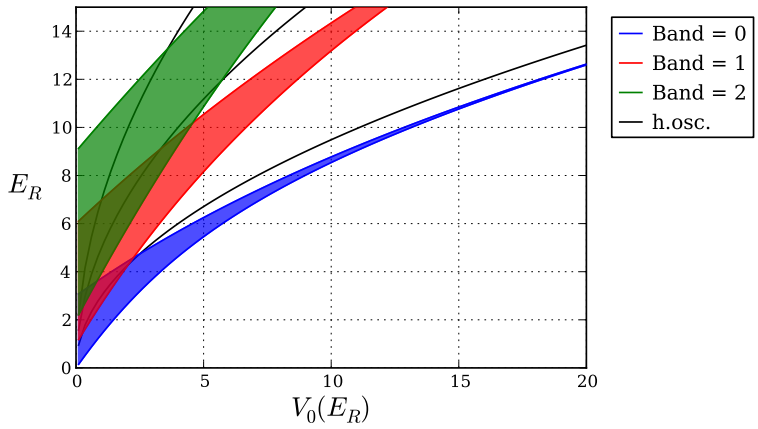

Figure 6: Band structure in a 3D optical lattice. Each band is indicated by the colored area, the harmonic oscillator states in an isolated lattice site are shown as black lines.

3 Hubbard Hamiltonian

The hamiltonian for a single atom in a 3D optical lattice is given by

$$H_{\text{single,3D}} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)+V_0\left(\cos^2(kx)+\cos^2(ky)+\cos^2(kz)\right) \tag{11}$$

and when N particles are considered, along with their interactions the hamiltonian takes a more complicated form

$$\begin{aligned} H =& \sum_{l}^{N}\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_l^2}+\frac{\partial^2}{\partial y_l^2}+\frac{\partial^2}{\partial z_l^2}\right)+V_0\left(\cos^2(kx_l)+\cos^2(ky_l)+\cos^2(kz_l)\right)\right] \\ &+\frac{1}{2}\sum_{l,m,l\neq m}^{N} V_{\text{int}}(x_l,x_m \\ =& H_0+H_{\text{int}} \end{aligned} \tag{12}$$

where the particles are labeled by indices l,m, and V_{int} is the potential energy of interaction between two partcles. In the last line we have defined the more concise notation that splits the Hamiltonian into the non-interacting (H_0) and the interacting (H_{int}) parts. Solving this problem is a daunting task primarily for two reasons:

1. The Bose or Fermi statistics of the identical particles under consideration require the wavefunctions to be symmetrized or antysymmetrized products of single-particle wavefunctions.
2. The interactions between the particles prevent a straightforward reformulation of the problem as a collection of easier-to-solve single particle hamiltonians.

The formalism of many-body theory encapsulates a series of methods to deal with the two issues mentioned above. First, the reformulation of the Schrodinger equation in the language of second quantization provides the advantage that the statistics are automatically taken into account by the notation, so one can essentially forget about the the (anti)symmetrization of the many-particle wave functions. The small price to pay is that one needs to be very careful and consistent about the order in which operators show up in the notation, since the symmetry properties of the resulting states are contained in the commutation relations defined between the operators. Furthermore, second quantization makes it easy to consider the extended Hilbert space where the number of particles is not fixed, this is known as Fock space.

For weak interactions, many-body theory provides a solution to the problem in terms of perturbation expansions for the physical quantities of interest. The theoretical formalism also reduces most of the important physical quantities in terms of certain matrix elements (Green's functions) which allows the user to concentrate on obtaining such matrix elements which serve as a starting point for the exploration of the properties of any system. The complication arises when the interactions are not weak, and the perturbative approach of the many-body formalism breaks down. For this reason, the Hubbard model with strong interactions (we will quantify the definition of strong later on) has been a major challenge for theoretical physicists over the last four decades.

3.1 Second quantization

The contents of this section follow the treatment in the books by Fetter and Walecka [3] and Schwabl [4].

Let's start with a complete orthonormal set of single particle states $\{|i\rangle\} = \{|1\rangle, |2\rangle, \ldots\}$, using these states we can write the basis states for the N-particle system as

$$|i_1, \ldots i_\alpha, \ldots i_N\rangle \tag{13}$$

which represents a state in which particle 1 is in state $i_1 \in \{|i\rangle\}$, particle α is in state i_α and so on. These product states are not eigenstates of the permutation operator P_{ij} which interchanges particles i and j. However, starting from the product states we can obtain the symmetrized (bosons) and antisymmetrized (fermions) normalized basis states.

For bosons the normalized symmetrized states are

$$|n_1, n_2, \ldots\rangle = \frac{1}{\sqrt{N! n_1! n_2! \ldots}} \sum_P P|i_1, i_2, \ldots i_N\rangle \tag{14}$$

where the sum over P runs over all $N!$ elements of the permutation group for N objects. In this expression, n_1 is the number of times that the state $|1\rangle$ occurs among the N particles, or in other words, n_1 is the number of particles in state $|1\rangle$. The sum of all occupation numbers n_i must equal the total number of particles, but otherwise there is no restriction in the occupation number for bosons.

For fermions the normalized antysymmetrized states are written in the form of Slater determinants:

$$\begin{aligned} |n_1, n_2, \ldots\rangle =& \frac{1}{\sqrt{N!}} \sum_P (-1)^P P|i_1, i_2, \ldots i_N\rangle \\ =& \frac{1}{\sqrt{N!}} \begin{vmatrix} |i_1\rangle_1 & |i_1\rangle_2 & \cdots & |i_1\rangle_N \\ \vdots & \vdots & \ddots & \vdots \\ |i_N\rangle_1 & |i_N\rangle_2 & \cdots & |i_N\rangle_N \end{vmatrix} \end{aligned} \tag{15}$$

In this case, the product states are multiplied by -1 for odd permutations, and the occupation numbers n_i can only take the values 0 or 1.

For both bosons and fermions, we can combine the states for $N = 0, 1, 2, \ldots$ particles to obtain a complete orthonormal set of states for arbitrary particle number. This set of states, which are referred to as number states, spans what is called the Fock space.

We now define the creation operators for bosons, which allow us to take a state from the subspace of N particles, to the subspace of $N + 1$ particles.

$$a_i^\dagger |\ldots, n_i, \ldots\rangle = \sqrt{n_i + 1}|\ldots, n_i + 1, \ldots\rangle \tag{16}$$

It follows that the adjoint of the creation operator is the annihilation operator and satisfies

$$a_i |\ldots, n_i, \ldots\rangle = \begin{cases} \sqrt{n_i}|\ldots, n_i - 1, \ldots\rangle & \text{if } n_i \geq 0, \\ 0 & \text{if } n_i = 0 \end{cases} \tag{17}$$

The creation and annihilation operators are defined such that one can create any state starting from the vacuum state $|0\rangle \equiv |0, 0, \ldots\rangle$ in which there are no particles at all. In more formal terms

$$|n_1, n_2, \ldots\rangle = \frac{1}{\sqrt{n_1! n_2! \ldots}} (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} \ldots |0\rangle \tag{18}$$

The boson creation and annihilation operators satisfty the Bose communtation relations

$$[a_i, a_j] = 0 \qquad [a_i^\dagger, a_j^\dagger] = 0 \qquad [a_i, a_j^\dagger] = \delta_{ij} \tag{19}$$

In the case of fermions we want to also define creation operators such that the number states can be written as in Eq. 18. A subtlety arises in the case of fermions since the order in which the creation operators are applied affects the resulting number state[1]. Take the number state defined in Eq. 15. If we interchange the state labels 1 and 2 we get

$$\begin{aligned} |n_2, n_1, \ldots\rangle =& \frac{1}{\sqrt{N!}} \sum_P (-1)^P P |i_2, i_1, \ldots i_N\rangle \\ =& -\frac{1}{\sqrt{N!}} \sum_P (-1)^P P |i_1, i_2, \ldots i_N\rangle \\ =& -|n_1, n_2, \ldots\rangle \end{aligned} \tag{20}$$

where the minus sign in the second line is a result of the properties of determinants, namely you get a minus sign if you exchange two columns.

If we adopt as a definition of the creation operators the following expression for the number state

$$|n_1, n_2, \ldots\rangle = (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} \ldots |0\rangle \quad , \qquad n_i = 0, 1 \tag{21}$$

then using the result of Eq. 20 above, the fermion creation operators must satisfy the following anti-commutation relation

$$a_i^\dagger a_j^\dagger + a_j^\dagger a_i^\dagger \equiv [a_i^\dagger, a_j^\dagger]_+ = 0 \tag{22}$$

Notice that this also contains the necessary implication that $(a_i^\dagger)^2 = 0$, which is a manifestation of the Pauli exclusion principle. It follows from here that the creation and annihilation operators for fermions satisfy

$$\begin{aligned} a_i^\dagger |\ldots, n_i, \ldots\rangle =& (1 - n_i)(-1)^{\sum_{k<i} n_k} |\ldots, n_i + 1, \ldots\rangle \\ a_i |\ldots, n_i, \ldots\rangle =& n_i (-1)^{\sum_{k<i} n_k} |\ldots, n_i - 1, \ldots\rangle \end{aligned} \tag{23}$$

and also that they satisfy the Fermi anticommutation relations

$$[a_i, a_j]_+ = 0 \qquad [a_i^\dagger, a_j^\dagger]_+ = 0 \qquad [a_i, a_j^\dagger]_+ = \delta_{ij} \tag{24}$$

From now on we shall focus on the case of Fermions, since this is the most relevant for our experiment.

3.2 Operators in second quantization

So far two great leaps have been taken:

1. We have swept antisymmetrization under the rug by introducing the number states, defined from the vacuum in terms of creation operators wich satisfy the Fermi anticommutation reations.
2. We started from an N particle hamiltonian, but we have now defined states that can handle the description of systems with an arbitrary number of particles

The two ideas mentioned are related to the states used to describe the system, now we will turn to the problem of the observables and see how they are handled in the second quantization.

Let us consider the following sum over particles $\sum_\alpha |i\rangle_\alpha \langle j|_\alpha$ and apply it to the number states as defined in Eq. 15

$$\left(\sum_\alpha |i\rangle_\alpha \langle j|_\alpha \right) |n_1, n_2, \ldots\rangle = \frac{1}{\sqrt{N!}} \sum_P (-1)^P P \left(\sum_\alpha |i\rangle_\alpha \langle j|_\alpha |i_1, i_2, \ldots i_N\rangle \right) \tag{25}$$

[1]This is not a problem in bosons which is seen by looking at Eq. 18 and recalling that the Bose commutation relations say that all creation operators commute

On the left hand side, for the term in parenthesis not to vanish, there must be one particle in state $|j\rangle$, so we must have $n_j = 1$ in the initial state. Also, since these are fermions, there can be no particles in state $|i\rangle$ in the intitial state, so $n_i = 0$, or else the Slater determinant operator will make the state vanish after applying the $|i\rangle\langle j|$. If the particle initially in state $|j\rangle$ is labeled as β then the two mentioned conditions can be embodied as

$$\begin{aligned}\left(\sum_\alpha |i\rangle_\alpha\langle j|_\alpha\right)|n_1, n_2, \ldots\rangle =& n_j(1-n_i)\frac{1}{\sqrt{N!}}\sum_P (-1)^P P\left(|i\rangle_\beta \underbrace{|i_1, i_2, \ldots i_N\rangle}_{\text{wihtout } |j\rangle_\beta}\right)\\ =& n_j(1-n_i)\frac{1}{\sqrt{N!}}\begin{vmatrix} |i_1\rangle_1 & |i_1\rangle_2 & \cdots & |i_1\rangle_N \\ \vdots & \vdots & & \vdots \\ |i\rangle_1 & |i\rangle_2 & \cdots & |i\rangle_N \\ \vdots & \vdots & & \vdots \\ |i_N\rangle_1 & |i_N\rangle_2 & \cdots & |i_N\rangle_N \end{vmatrix}\end{aligned} \tag{26}$$

In the determinant of the left the state $|i\rangle$ appears in the j^{th} row, so a few rows need to be exchanged to put it in the correct place according to our sign convention for the number states.

$$\begin{aligned}\left(\sum_\alpha |i\rangle_\alpha\langle j|_\alpha\right)|n_1, n_2, \ldots\rangle =& n_j(1-n_i)|\ldots, n_i+1, \ldots, n_j-1, \ldots\rangle \times \begin{cases}(-1)^{\sum_{k<j} n_k + \sum_{k<i} n_k} & \text{if } i \le j,\\ (-1)^{\sum_{k<j} n_k + \sum_{k<i} n_k - 1} & \text{if } i > j\end{cases}\\ =& a_i^\dagger a_j |n_1, n_2, \ldots\rangle\end{aligned} \tag{27}$$

where the last equality can be obtained by examining the definition of the creating and annihilation operators given above.

After this last step we can establish the important relation

$$\sum_\alpha |i\rangle_\alpha\langle j|_\alpha = a_i^\dagger a_j \tag{28}$$

We now turn our attention to the operators in the N-particle system. Consider an operator T that is a sum over single particle operators

$$T = \sum_\alpha t_\alpha \tag{29}$$

If we insert the completness relation for the single particle states twice in this sum we have

$$\begin{aligned}T =& \sum_\alpha \left(\sum_i |i\rangle_\alpha\langle i|_\alpha\right) t_\alpha \left(\sum_j |j\rangle_\alpha\langle j|_\alpha\right)\\ =& \sum_{ij}\langle i|t|j\rangle \sum_\alpha |i\rangle_\alpha\langle j|_\alpha\\ =& \sum_{ij}\langle i|t|j\rangle a_i^\dagger a_j \equiv \sum_{ij} t_{ij} a_i^\dagger a_j\end{aligned} \tag{30}$$

This is the other big leap provided by the second quantization: the operators which were written as a sum over particles now are written as a sum of creation and annihilation operators over single particle states.

Operators like the potential energy, which are a sum over two-particle (or many-particle) operators, can be equally expressed as sums of creation and annihilation operators. For a two-body operator we have the expression

$$\begin{aligned} F =& \frac{1}{2}\sum_{\alpha\neq\beta} f(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) \\ =& \frac{1}{2}\sum_{ijkm} \langle ij|f|km\rangle a_i^\dagger a_j^\dagger a_m a_k \end{aligned} \tag{31}$$

3.3 Second quantized Hubbard hamiltonian

The Hubbard hamiltonian in Eq. 12 is a sum of two single-particle operators and one two-particle operator. These are respectively: the kinetic energy, the energy of the atoms in the lattice potential, and the interactions between the atoms. As a single-particle basis we pick the Wannier states that were derived in Section. 1.3

In the Hubbard hamiltonian the two single-particle operators are grouped together to define the non-interacting part of the hamiltonian

$$\begin{aligned} H_0 =& \sum_l^N -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_l^2}+\frac{\partial^2}{\partial y_l^2}+\frac{\partial^2}{\partial z_l^2}\right) + V_0\left(\cos^2(kx_l)+\cos^2(ky_l)+\cos^2(kz_l)\right) \\ =& \sum_l^N H^l_{\text{single,3D}} \end{aligned} \tag{32}$$

3.3.1 Tunneling matrix element, t

H_0 is a single particle operator, so it's second quantized form is

$$\begin{aligned} H_0 =& \sum_{ij} \langle i|H_{\text{single,3D}}|j\rangle a_i^\dagger a_j \\ =& -\sum_{ij} t_{ij} a_i^\dagger a_j \end{aligned} \tag{33}$$

Note that the sign of t_{ij} was picked rather arbitrarily to follow the usual conventions. We now proceed to find the value of the matrix element. We use the definition of the Wannier states given in Eq. 10 to find

$$\begin{aligned} -t_{ij} =& \frac{1}{N_{\text{s}}^6}\int \mathrm{d}\boldsymbol{r}\ \sum_{\boldsymbol{q}'} e^{i\boldsymbol{q}'\cdot\boldsymbol{R}_i} \prod_{u'=x,y,z} \psi_{q'_{u'}}^{n'_{u'}*}(u')\Big(H_{\text{single,3D}}\Big)\sum_{\boldsymbol{q}} e^{-i\boldsymbol{q}\cdot\boldsymbol{R}_j} \prod_{u=x,y,z}\psi_{q_u}^{n_u}(u) \\ =& \sum_{\boldsymbol{q}\boldsymbol{q}'} \frac{E^n_{\boldsymbol{q}}}{N_{\text{s}}^6} e^{i\boldsymbol{q}'\cdot\boldsymbol{R}_i} e^{-i\boldsymbol{q}\cdot\boldsymbol{R}_j}\int \mathrm{d}\boldsymbol{r} \prod_{u'=x,y,z}\psi_{q'_{u'}}^{n'_{u'}*}(u') \prod_{u=x,y,z}\psi_{q_u}^{n_u}(u) \\ =& \sum_{\boldsymbol{q}\boldsymbol{q}'} \frac{E^n_{\boldsymbol{q}}}{N_{\text{s}}^6} e^{i\boldsymbol{q}'\cdot\boldsymbol{R}_i} e^{-i\boldsymbol{q}\cdot\boldsymbol{R}_j}\delta_{\boldsymbol{q}\boldsymbol{q}'}\delta_{nn'}N_{\text{s}}^3 \\ =& \frac{1}{N_{\text{s}}^3}\sum_{\boldsymbol{q}} E^n_{\boldsymbol{q}} e^{i\boldsymbol{q}\cdot(\boldsymbol{R}_i-\boldsymbol{R}_j)} \end{aligned} \tag{34}$$

We observe that there is no amplitude to go between states that are in two different bands, as is indicated by the appereance of $\delta_{nn'}$. In what follows we will consider only the lowesrt band, $n = 0$, so we will drop the band index altogether. This simplification imposes two important requirements for our system:

1. The temperature needs to be small compared to the energy gap between the lowest and first excited band.
2. The interaction enrgy scale must also be small comapred to the energy gap between the lowest and first excited band.

In the 3D lattice, the energy $E_{\boldsymbol{q}} = \sum_{u=x,y,z} E^u_{q_u}$, and by inserting this into the sum for t_{ij} above we find

$$-t_{ij} = \frac{1}{N_\mathrm{s}^3}\left[\left(\sum_{q_x} E^{1\mathrm{D}}_{q_x} e^{iq_x x_{ij}}\right)\sum_{q_y} e^{iq_y y_{ij}} \sum_{q_z} e^{iq_z z_{ij}} + \sum_{q_x} e^{iq_x x_{ij}} \left(\sum_{q_y} E^{1\mathrm{D}}_{q_y} e^{iq_y x_{ij}}\right)\sum_{q_z} e^{iq_z z_{ij}} + \sum_{q_x} e^{iq_x x_{ij}} \sum_{q_y} e^{iq_y y_{ij}} \left(\sum_{q_z} E^{1\mathrm{D}}_{q_z} e^{iq_z z_{ij}}\right)\right] \tag{35}$$

We make use of the identity $\sum_{q_x} e^{iq_x(x_i - x_j)} = N_\mathrm{s}\delta_{x_i x_j}$, and similarly for y, z to obtain

$$-t_{ij} = \frac{1}{N_\mathrm{s}}\left[\left(\sum_{q_x} E^{1\mathrm{D}}_{q_x} e^{iq_x x_{ij}}\right)\delta_{y_i y_j}\delta_{z_i z_j} + \left(\sum_{q_y} E^{1\mathrm{D}}_{q_y} e^{iq_y x_{ij}}\right)\delta_{x_i x_j}\delta_{z_i z_j} + \left(\sum_{q_z} E^{1\mathrm{D}}_{q_z} e^{iq_z z_{ij}}\right)\delta_{x_i x_j}\delta_{y_i y_j}\right] \tag{36}$$

We see that if $i = j$ we have

$$-t_{ii} = \frac{3}{N_\mathrm{s}}\sum_q E^{1\mathrm{D}}_q \tag{37}$$

Considering this term just adds an overall energy (proportional to the number of particles in the system) to the hamiltonian, so we just ignore it for convenience. If on the other hand $i \neq j$, then we see that we can only consider the posibility that one of the x_{ij}, y_{ij}, z_{ij} is non-zero, otherwise all three of the terms in the expression above will vanish due to the Kronnecer delta terms.

It is interesting to see that there is no matrix elemnt to tunnel diagonally between sites in the 3D lattice. This is in contrast to the actual situation in real materials, where localized atomic orbitals will adopt functional forms that have nothing to do with the Wannier state. In such cases there is generally a matrix element for an electron to tunnel between orbitals that are localized in any two lattice sites. Tunneling beyond nearest neighbors is typically neglected in what is called the tight-binding limit.

For the 3D lattice the expression for the tunneling matrix elements simplifies to

$$t_{ij} = -\frac{1}{N_\mathrm{s}}\sum_q E^{1\mathrm{D}}_q e^{iq\Delta_{ij}} \tag{38}$$

where Δ_{ij} is the distance between sites at $\boldsymbol{R}_i$ and $\boldsymbol{R}_j$, and R_{ij} must be purely directed along x, y, or z. Also, as in the definition of the 1D Wannier states, the sum over q runs over the discrete values $q = \frac{n}{N_\mathrm{s}}\frac{\pi}{a}$ where $n \in \{-N_\mathrm{s}+1, -N_\mathrm{s}+2, \ldots, N_\mathrm{s}-1, N_\mathrm{s}\}$.

We see that t_{ij} and $E^{1\mathrm{D}}_q$ are the Fourier series of one another, or in other words that the relation just shown can be inverted to obtain

$$E^{1\mathrm{D}}_q = -\sum_{\Delta_{ij}} t_{ij} e^{-iq\Delta_{ij}} \tag{39}$$

In the tight-binding approximation $\Delta_{ij} \in \{-a, a\}$ and $t_{ij} \equiv t$, so we have

$$E^{1\mathrm{D}}_{q,\mathrm{tb}} = -2t\cos(qa) \tag{40}$$

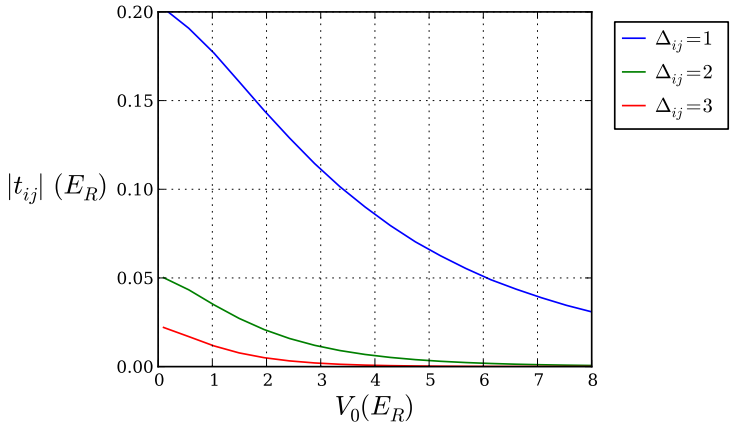

Figure 7: Tunneling matrix element in an optical lattice as a function of lattice depth. Nearest-neighbor and beyond nearest neighbor matrix elements are shown to illustrate the range of lattice depths for which the tight-binding limit is a good approximation. X_{ij} corresponds to the distance between intial and final site in the tunneling matrix element.

It is useful to find out the range of lattice depths for which the tight-binding approximation is valid in the optical lattice potential. To do this we just need to look at the tunneling matrix elements for beyond nearest-neighbor tunneling, this is shown in Fig. 7, which shows that for lattice depths $\gtrsim 5E_R$ we can safely ignore tunneling beyond nearest neighbors.

Finally, we have the second quantized form of H_0 in the tight-binding limit

$$H_0 = -t \sum_{\langle ij \rangle} a_i^\dagger a_j \tag{41}$$

where the $\langle\rangle$ denote nearest-neighbors, and the creation operator $a_i^\dagger$ create particles in the Wannier state localized at site i.

Notice that up to now we have ignored the spin part of the wavefunction. We can include it easilly by noticing that H_0 does not act on the spin at all, so the states $|i\rangle$ and $|j\rangle$ that we have used in the derivation above need to have the same spin. Including the spin our basis set is now larger, so we include it in the sum.

$$H_0 = -t \sum_{\langle ij \rangle, \sigma=\uparrow\downarrow} a_{i\sigma}^\dagger a_{j\sigma} \tag{42}$$

3.3.2 On-site interaction energy, U

3.4 On-site interactions

4 Local density approximation

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