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Semiclassical description of coherence effects in spontaneous emission and superradiance

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A semiclassical radiation theory is developed in which the spontaneous emission field of a two-level atom is a quantum entity represented by atomic operators acting in the Hilbert space of the atomic states. The theory is applied to an investigation of the coherence properties of spontaneous emission. It is found that incoherent and coherent components can be identified in the emission field, the former dominating in the early stages of the decay from the excited state and the latter dominating in the later stages or in the emission from a weakly excited atom. The single-atom coherence properties are shown to lead to well-known superradiance phenomena in the spontaneous emission from an assembly of similar atoms. The paper does not use the theory of quantum electrodynamics; it builds on knowledge at the undergraduate level of classical radiation theory and the quantum mechanics of atoms.

I. INTRODUCTION

The spontaneous emission of light from an excited atom has proved itself a critical testing ground for rival theories of electrodynamics. The theory of quantum electrodynamics (QED), in which both the atomic system and the optical field with which it interacts are quantized by postulate, is able to predict correctly the measurable properties of spontaneous emission, but in a way which highlights the difficulties, such as the appearance of divergent integrals which currently beset the theory. Semiclassical theories of electrodynamics, in which only the atomic system is quantized, have proved extremely successful in a wide range of stimulated absorption and stimulated emission processes, but most attempts have failed to give a satisfactory account of spontaneous emission. The basic difficulty is centered on the fact that an atom in an excited state has no dipole moment; or more precisely, the expectation of the dipole moment operator in an excited state is zero. Conventional semiclassical radiation theories take the expectation to be a classical radiation field source.[1] Such theories therefore predict that an atom prepared in an excited state will be unable to radiate, and so will remain stable (or rather metastable) against spontaneous decay. The experimental evidence is against this prediction[2] and in support of the exponential decay predicted by QED. However, some recent developments of semiclassical radiation theory have indicated a new approach which is able to make predictions of spontaneous emission in agreement with those of QED. In this approach, the observable properties of the radiation field are represented by functions of the atomic operators acting on the Hilbert space of the quantized atom.[3] The radiation field is thus quantized, not by postulate as in QED, but by virtue of the quantization of its atomic source. In this paper we shall describe one way in which such a theory can be developed along heuristic lines, and we apply it to a study of the coherence properties of the spontaneous emission from a single two-level atom and the superradiant emission from an assembly of such atoms. While our conclusions will be identical to those of QED, our treatment will be a good deal simpler both conceptually and algebraically, and accessible to anyone with an elementary knowledge of the quantum theory of the atom.

Coherence is a much overworked word in physics, so it is necessary to state the sense in which we intend to use it. We shall call an emitted radiation field coherent if the phase of its oscillations can be predicted with certainty once the quantum state of the emitting system is known. This means, for example, that when the quantum state of the emitting system is itself prepared by interaction with an incident light beam, any coherent emission can produce systematic interference effects, such as dispersion, with the incident beam. It also means that coherent emissions from an assembly of similar emitting atoms can produce systematic interference effects among themselves, such as superradiance, though we hasten to add that coherence, in the sense that we are using the term, is not a necessary condition for this to occur. We shall discuss this point in Sec. IV.

Section II develops the semiclassical emission theory for a two-level atom. Section III applies this to a study of coherence in the spontaneous emission from a single atom. We shall find that the spontaneous emission from a weakly excited two-level atom is coherent in agreement with the predictions of conventional semiclassical theory,[4] while that from a highly excited atom is incoherent; and for intermediate degrees of excitation both a coherent and and incoherent component can be identified.[5] Section IV shows how these coherence properties can lead in a simple way to superradiant emission from a system of similar two-level atoms.

II. FIELD AND INTENSITY OPERATORS FOR SPONTANEOUS EMISSION

We begin with a statement of some relevant results from the classical Maxwellian theory of electric dipole radiation. These will serve as a guide in choosing the functions of atomic operators that are to represent the field and intensity of the spontaneous emission from a quantized atom.

A. Classical description of electric dipole radiation

The wave equation for the vector potential field $\mathbf{A}(\mathbf{R},t)$ in the presence of a current density field $\mathbf{J}(\mathbf{R},t)$ can be written as

$$\nabla^2\mathbf{A}(\mathbf{R},t) - \frac{1}{c^2}\frac{\partial}{\partial t^2}\mathbf{A}(\mathbf{R},t) = -\mu_0\mathbf{J}(\mathbf{R},t). \qquad (1)$$

Two particular integrals of Eq. (1) can be found:

$$\mathbf{A}_{\pm}(\mathbf{R},t) = \frac{\mu_0}{4\pi}\int_{\substack{\text{all}\\\text{space}}} \frac{\mathbf{J}(\mathbf{R}',t \pm |\mathbf{R}-\mathbf{R}'|/c)}{|\mathbf{R}-\mathbf{R}'|}\,d^3\mathbf{R}'. \quad (2)$$

The minus sign gives the so-called retarded solution for which the disturbance at $(\mathbf{R},t)$ is correlated with the current density at earlier space–time points $(\mathbf{R}',t_r)$ where t_r is the retarded time $t - |\mathbf{R}-\mathbf{R}'|/c$. We shall assume that this solution represents the situation we intend to study; namely, that of an emission field, i.e., a vector potential field generated by a current density source field. We assume that there are no other fields present so that the advanced solution, obtained by taking the plus sign in Eq. (2), and the free-field solutions of the homogeneous wave equation, will not be needed. We thus take the solution of Eq. (1) to be

$$\mathbf{A}(\mathbf{R},t) = \frac{\mu_0}{4\pi}\int_{\substack{\text{all}\\\text{space}}} \frac{\mathbf{J}(\mathbf{R}',t_r)}{|\mathbf{R}-\mathbf{R}'|}\,d^3\mathbf{R}', \quad (3)$$

where the minus subscript on $\mathbf{A}$ has been dropped for convenience.

We shall want to specialize Eq. (3) to the case where the source is a point charge $-e$ in orbit around a massive slowly moving central point at $\mathbf{R}_c$. Introducing the coordinate $\mathbf{r}$ of the point charge relative to $\mathbf{R}_c$ as shown in Fig. 1, we have

$$\mathbf{J}(\mathbf{R}',t) = -e\frac{d\mathbf{r}}{dt}\,\delta(\mathbf{R}'-\mathbf{R}_c-\mathbf{r}),$$

with

$$\left|\frac{d\mathbf{r}}{dt}\right| \gg \left|\frac{d\mathbf{R}_c}{dt}\right|. \quad (4)$$

This particular specialization is intended to give the classical counterpart of a one-electron atom undergoing radiative transitions. If we want to refer to one electric dipole transition in the atom, without radiative damping or other perturbations, then we further specialize to a simple harmonic oscillation of small amplitude $\mathbf{r}_0$ at the frequency $\omega/2\pi$ of the transition:

$$d\mathbf{r}/dt = -\omega\mathbf{r}_0\sin(\omega t+\phi)$$

with

$$|\mathbf{r}_0| \ll \lambda = 2\pi c/\omega. \quad (5)$$

If to Eqs. (4) and (5) we add the condition $|\mathbf{R}-\mathbf{R}_c| \gg \lambda$, which restricts us to the far field or radiation field of the dipole, then Eq. (3) can be integrated to give

$$\mathbf{A}(\mathbf{R},t) = -\frac{\mu_0\omega}{4\pi|\mathbf{R}-\mathbf{R}_c|}\mathbf{d}_0\sin(\omega t_r+\phi), \quad (6)$$

where $\mathbf{d}_0 = -e\mathbf{r}_0$ is the amplitude of the electric dipole moment of the charge $-e$ about $\mathbf{R}_c$.

For the purpose of calculating the electric and magnetic field vectors of the radiation field, and the cycle-averaged Poynting vector, only the transverse θ° component (Fig. 1) of the vector potential is needed. For convenience we shall refer to this component as the *radiation field F*.

$$F \equiv \boldsymbol{\theta}^\circ\cdot\mathbf{A}(\mathbf{R},t) = -\frac{\mu_0\omega\sin\theta}{4\pi|\mathbf{R}-\mathbf{R}_c|}|\mathbf{d}_0|\sin(\omega t_r+\phi). \quad (7)$$

Equation (7) relates the radiation field F to the magnitude of the retarded dipole moment $d = |\mathbf{d}_0|\cos(\omega t_r+\phi)$.

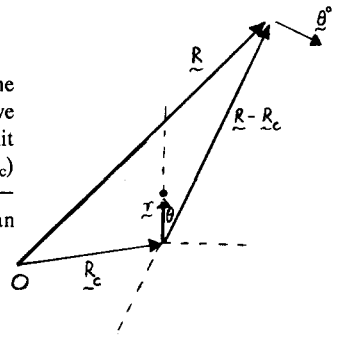

Fig. 1. $\mathbf{r}$ is the position vector of the moving charge relative to the massive point at $\mathbf{R}_c$. θ°, the transverse unit vector, is directed normal to $(\mathbf{R}-\mathbf{R}_c)$ and lies in the plane of $\mathbf{r}$ and $(\mathbf{R}-\mathbf{R}_c)$. $\mathbf{R}$ and $\mathbf{R}_c$ are measured from an arbitrary fixed origin 0.

We shall call the cycle-averaged value of F^2 the *intensity* I. So defined, the intensity I is simply related to the cycle-averaged magnitude of the Poynting vector $\mathbf{S}$:

$$I = (F^2)_{av} = (\mu_0 c/\omega^2)(|\mathbf{S}|)_{av}.$$

We shall want to decompose the retarded dipole moment and the field F into pairs of complex conjugate terms.

$$d = d^{(+)} + d^{(-)}, \quad (8)$$

$$F = F^{(+)} + F^{(-)}, \quad (9)$$

where

$$d^{(\pm)} = \frac{1}{2}|\mathbf{d}_0|\exp[\mp i(\omega t_r+\phi)], \quad (10)$$

and

$$F^{(\pm)} = \mp i(\mu_0\omega\sin\theta/4\pi|\mathbf{R}-\mathbf{R}_c|)d^{(\pm)}. \quad (11)$$

The intensity can now be expressed as

$$I = 2F^{(-)}F^{(+)} = 2(\mu_0\omega\sin\theta/4\pi|\mathbf{R}-\mathbf{R}_c|)^2 d^{(-)}d^{(+)}. \quad (12)$$

With the radiation field and intensity variables, $F^{(\pm)}$ and I, expessed as functions of the source variables $d^{(\pm)}$, we are now ready to make the transition to quantum mechanics.

B. States and operators for dipole radiation

We consider a one-electron atom described by center-of-mass coordinate $\mathbf{R}_c$ and electronic coordinate $\mathbf{r}$ relative to $\mathbf{R}_c$. The wave function of the atom, neglecting spin factors, then takes the form

$$\langle\mathbf{R}_c,\mathbf{r}|\text{atom}\rangle = \langle\mathbf{R}_c|\text{c of m}\rangle\langle\mathbf{r}|\text{elec}\rangle. \quad (13)$$

Our aim is to describe the observable properties of the atom's radiation field by functions of the atomic operators acting on the Hilbert space of the atomic wave functions (13). We shall be guided in our choice of functions by the form of the classical expressions (11) and (12). Thus, the expectations of the radiation field and intensity will be expressions of the form:

$$\langle F^{(\pm)}\rangle = \frac{\mp i\mu_0\omega}{4\pi}\Big\langle \text{c of m}\Big|\Big\langle\text{elec}\Big| \times\left(\frac{\sin\theta d^{(\pm)}}{|\mathbf{R}-\mathbf{R}_c|}\right)_{op}\Big|\text{elec}\Big\rangle\Big|\text{c of m}\Big\rangle, \quad (14)$$

$$\langle F\rangle = \langle F^{(+)}\rangle + \langle F^{(-)}\rangle,$$

$$\langle I\rangle = 2\left(\frac{\mu_0\omega}{4\pi}\right)^2 \left\langle \text{c of m}\middle|\left\langle \text{elec}\right| \times \left(\frac{\sin^2\theta d^{(-)}d^{(+)}}{|\mathbf{R}-\mathbf{R}_c|^2}\right)_{op}\middle|\text{elec}\right\rangle\middle|\text{c ofm}\right\rangle, \quad (15)$$

where $(\quad)_{op}$ indicates the operator corresponding to the classical quantity enclosed. The aim of this section is to specify what these operators are. Some simplification can be obtained by making the assumption that the center-of-mass wave function is particle-like, i.e., the function $\langle \mathbf{R}_c|\text{c of m}\rangle$ is highly located at the point $R_c = R_0$. The center-of-mass motion can then be treated classically, and Eqs. (14) and (15) reduced to

$$\langle F^{(\pm)}\rangle = \frac{\mp i\mu_0\omega\sin\theta}{4\pi|\mathbf{R}-\mathbf{R}_0|}\langle \text{elec}|(d^{(\pm)})_{op}|\text{elec}\rangle, \qquad \langle F\rangle = \langle F^{(+)}\rangle + \langle F^{(-)}\rangle, \quad (16)$$

$$\langle I\rangle = 2\left(\frac{\mu_0\omega\sin\theta}{|\mathbf{R}-\mathbf{R}_0|}\right)^2\langle \text{elec}|(d^{(-)}d^{(+)})_{op}|\text{elec}\rangle. \quad (17)$$

The electronic motion will be simplified by assuming a two-state atom spanned by the ground state $|g\rangle$ and an excited state $|e\rangle$ of the electronic Hamiltonian, so that a general electronic state at time t will be a linear superposition $|t\rangle$:

$$|\text{elec}\rangle \equiv |t\rangle = c_g(t)|g\rangle + c_e(t)|e\rangle, \quad (18)$$

with

$$|c_g(t)|^2 + |c_e(t)|^2 = 1.$$

As indicated, we shall use the Schrödinger picture in which all the time dependence is carried by the electronic states. Our problem now is to decide on the Schrödinger time-independent operators that represent the complex classical dipole components $d^{(\pm)}$ and the product $(d^{(-)}d^{(+)})$ appearing in Eqs. (16) and (17).

Our starting point is the Hermitian electric dipole moment operator:

$$\hat{\mathbf{d}} = -e\hat{\mathbf{r}},$$

where $\hat{\mathbf{r}}$ is the electron position operator corresponding to the coordinate $\mathbf{r}$. For any atomic system we can decompose the magnitude of $\hat{\mathbf{d}}$ into non-Hermitian components connecting pairs of states. For this purpose, we introduce the identity operator $\Sigma_i|i\rangle\langle i|$, where the summation is over the complete set of states $|i\rangle$ of the Hamiltonian. Thus,

$$\hat{\mathbf{d}} = \sum_i\sum_j |i\rangle\langle i|\hat{\mathbf{d}}|j\rangle\langle j| = \sum_i\sum_j \mathbf{d}_{ij}|i\rangle\langle j|.$$

For our assumed two-state system this reduces to

$$\hat{\mathbf{d}} = \mathbf{d}_{ge}|g\rangle\langle e| + \mathbf{d}_{eg}|e\rangle\langle g|.$$

We shall suppose that the relative phase of the two states has been chosen such that the matrix elements are positive real:

$$\mathbf{d}_{ge} = \mathbf{d}_{eg} = \mathbf{d}'.$$

It is convenient to introduce the notation

$$\hat{\eta} \equiv |g\rangle\langle e|; \qquad \hat{\eta}^\dagger \equiv |e\rangle\langle g|,$$

for the so-called downwards and upwards transition operators, so that we can write, for the magnitude of $\hat{\mathbf{d}}$,

$$\hat{d} = \hat{\eta}d' + \hat{\eta}^\dagger d'.$$

We now postulate the following correspondence. The classical complex time-retarded components of d, $d^{(\pm)}$ of Eq. (10), are to be represented by the non-Hermitian transition components of d, $\hat{\eta}d'$, and $\hat{\eta}^\dagger d'$, acting on the time-retarded electronic states. Thus,

$$\left.\begin{aligned}(d^{(+)})_{op} &= \hat{\eta}d',\\ (d^{(-)})_{op} &= \hat{\eta}^\dagger d',\end{aligned}\right\} \text{ acting on } |t_r\rangle. \quad (19)$$

The field operators are to have the form suggested by the classical relationships, Eq. (11):

$$\left.\begin{aligned}\hat{F}^{(+)} &= -iF_0\hat{\eta}\\ \hat{F}^{(-)} &= iF_0\hat{\eta}^\dagger\end{aligned}\right\} \text{ acting on } |t_r\rangle \qquad \hat{F} = \hat{F}^{(+)} + \hat{F}^{(-)} \quad (20)$$

with

$$F_0 = \mu_0\omega d'\sin\theta/4\pi|\mathbf{R}-\mathbf{R}_0|.$$

The field expectation values are then

$$\langle F^{(+)}\rangle = -iF_0\langle t_r|\hat{\eta}|t_r.\rangle, \qquad \langle F^{(-)}\rangle = iF_0\langle t_r|\hat{\eta}^\dagger|t_r\rangle, \qquad \langle F\rangle = \langle F^{(+)}\rangle + \langle F^{(-)}\rangle. \quad (21)$$

For the purpose of evaluating these, we note the following nonzero matrix elements:

$$\langle g|\hat{\eta}|e\rangle = \langle g||g\rangle\langle e||e\rangle = 1, \qquad \langle e|\hat{\eta}^\dagger|g\rangle = \langle e||e\rangle\langle g||g\rangle = 1. \quad (22)$$

The other matrix elements are zero. We note also the following products:

$$\hat{F}^{(+)}\hat{F}^{(-)} = F_0^2\hat{\eta}\hat{\eta}^\dagger = F_0^2|g\rangle\langle g|,$$

$$\hat{F}^{(-)}\hat{F}^{(+)} = F_0^2\hat{\eta}^\dagger\hat{\eta} = F_0^2|e\rangle\langle e|,$$

and the only nonzero matrix elements of these:

$$\langle g|\hat{\eta}\hat{\eta}^\dagger|g\rangle = \langle e|\hat{\eta}^\dagger\hat{\eta}|e\rangle = 1. \quad (23)$$

Before we can evaluate expectations of the intensity, Eq. (17), we need to make an additional postulate to resolve the ambiguity in the ordering of $\hat{\eta}^\dagger$ and $\hat{\eta}$. We shall take the correspondence

$$(d^{(-)}d^{(+)})_{op} = \hat{\eta}^\dagger\hat{\eta}(d')^2 \text{ acting on } |t_r.\rangle.$$

The intensity operator for emission is then

$$\hat{I} = 2F_0^2\hat{\eta}^\dagger\hat{\eta} \text{ acting on } |t_r\rangle, \quad (24)$$

or, directly in terms of the field operators,

$$\hat{I} = 2\hat{F}^{(-)}\hat{F}^{(+)} \text{ acting on } |t_r\rangle,$$

which corresponds to the classical relationship Eq. (12). The expectation, Eq. (17), becomes

$$\langle I\rangle = 2F_0^2\langle t_r|\hat{\eta}^\dagger\hat{\eta}|t_r\rangle. \quad (25)$$

The choice of the order $\hat{\eta}^\dagger\hat{\eta}$ with retarded states corresponds to the choice of a retarded solution of Eq. (1) in the classical description of emission. A corresponding choice is made in QED when a normal ordering of the field annihilation and creation operators is taken for an emission process.

The operators, Eqs. (20) and (24), form the basis of our description of the spontaneous emission from a two-level atom. The description is semiclassical in the sense that the field is not quantized independently of the atom as it is in QED, but the description is quantum in the sense that the

emission field is a quantized entity with field and intensity operators acting on the Hilbert space of the atom. In this respect the treatment differs from the conventional semiclassical ones where the emission field is a classical entity generated by the expectation of the dipole moment.

III. COHERENCE IN THE SPONTANEOUS EMISSION FROM A SINGLE ATOM

In this section we shall evaluate expectations and uncertainties for the field and intensity of spontaneous emission from a single two-level atom. These will be used to define the term coherence in a way that is consistent with the idea that a field is coherent if its phase is uniquely determined by the quantum state of the emitting atom.

The exponential decay of an excited atom will not appear explicitly in our analysis because we have not included the perturbing effect of radiation reaction on the state of the atom. The perturbation is very weak, however, and so the decay constant is very small compared with the resonance frequency of the atom. Because of this it is a good approximation to evaluate all expectation values in the unperturbed atom as we do below, and then, if required, to impose the slow exponential decay on the excited state probability $|c_e|^2$.[6] Our expectation values will appear as explicit functions of c_g and c_e with it understood that these amplitudes are changing slowly and monotonically in time towards the limit $|c_e| \to 0$ as $t \to \infty$.

Consider an atom prepared in the excited state $|e\rangle$ at time zero. The decay will take the atom into a general superposition state $|t\rangle$ of Eq. (18); but it will be instructive to look first of all at the radiation field and intensity at the retarded time zero, i.e., at the time when the observer at **R** "sees" the atom in the excited state $|e\rangle$. Accordingly, we evaluate the expectation values in the state $|e\rangle$ using Eqs. (21) and (25), and the matrix elements, Eqs. (22) and (23):

$$\langle F^{\pm}\rangle_{t_r=0} = \langle F\rangle_{t_r=0} = 0,$$
$$\langle I\rangle_{t_r=0} = 2F_0^2.$$

The vanishing of the field expectations above does not imply the absence of spontaneous emission from the excited atom, for the expectation of the intensity is seen to be nonzero. We shall interpret these results to mean that the emission field is completely indeterminate in phase, so that if measurements of F were to be made on a large number of identically prepared systems the results would yield positive and negative numbers with mean zero.

Further insight is obtained by evaluating the uncertainties, defined for an arbitrary state $|t\rangle$, to be

$$\Delta F_{t_r}^{(\pm)} = F_0\{\langle t_r|\hat{\eta}^\dagger\hat{\eta}|t_r\rangle - |\langle t_r|\hat{\eta}|t_r\rangle|^2\}^{1/2},$$
$$\Delta F_{t_r} = \sqrt{2}\,\Delta F_{t_r}^{(\pm)},$$
$$\Delta I_{t_r} = 2F_0^2\{\langle t_r|(\hat{\eta}^\dagger\hat{\eta})(\hat{\eta}^\dagger\hat{\eta})|t_r\rangle - \langle t_r|\hat{\eta}^\dagger\hat{\eta}|t_r\rangle^2\}^{1/2},$$

where, in $\Delta F_{t_r}^{(\pm)}$ we have again imposed the ordering $\hat{\eta}^\dagger\hat{\eta}$ for the emission process. We note that there is no additional ordering problem in ΔI for $\hat{\eta}^\dagger\hat{\eta}$ is Hermitian and $(\hat{\eta}^\dagger\hat{\eta})(\hat{\eta}^\dagger\hat{\eta}) = |e\rangle\langle e||e\rangle\langle e| = \hat{\eta}^\dagger\hat{\eta}$.
Evaluating these in the state $|e\rangle$ gives

$$\Delta F_{t_r=0}^{(\pm)} = F_0,$$
$$\Delta F_{t_r=0} = \sqrt{2}\,F_0, \tag{26}$$
$$\Delta I_{t_r=0} = 0.$$

The intensity is seen to have zero uncertainty and is therefore a perfectly well-determined quantity of magnitude equal to its expectation value, which was found to be $2F_0^2$. The field uncertainty $\Delta F_{t_r=0} = \sqrt{2}F_0$ gives the rms spread about zero when measurements of F are carried out on a large number of identically prepared systems. These results taken together suggest a picture in which the field at a point is represented by a sinusoidal vibration of definite amplitude $(2I)^{1/2} = 2F_0$ and indefinite phase.[7] Such a field is, according to definitions that we shall introduce later [Eqs. (29) and (30) below], a completely incoherent field.

We now go on to evaluate the expectations and uncertainties at an arbitrary retarded time $t_r \geq 0$ when the atom is in a general superposition state $|t\rangle$. The retarded-time labels on $F_{t_r}^{(+)}$ and $c_e(t_r)$, etc., will be omitted for convenience. We find

$$\langle F^{(+)}\rangle = -iF_0c_g^*c_e; \quad \langle F^{(-)}\rangle = iF_0c_gc_e^*,$$
$$\langle F\rangle = -iF_0(c_g^*c_e - c_gc_e^*), \tag{27}$$
$$\langle I\rangle = 2F_0^2|c_e|^2.$$

$$\Delta F^{(\pm)} = F_0|c_e|^2,$$
$$\Delta F = \sqrt{2}F_0|c_e|^2, \tag{28}$$
$$\Delta I = 2F_0^2|c_g||c_e|.$$

We note that the field expectations are, in general, nonzero; and that the intensity has a nonzero uncertainty and is therefore no longer a determinate quantity. We note also, that the field expectation has a phase completely determined by the retarded state of the atom, i.e., by the retarded complex amplitudes c_g and c_e. Accordingly, we are led to define the coherent spontaneous-emission field to be the field expectation $\langle F\rangle$. The intensity of the coherent emission is then

$$I_{\text{coh}} = 2\langle F^{(-)}\rangle\langle F^{(+)}\rangle = 2F_0^2|c_g|^2|c_e|^2. \tag{29}$$

The coherent field is the only spontaneous-emission field predicted by conventional semiclassical radiation theories, and this accounts for many of the shortcomings of such theories; in particular, their failure to predict spontaneous emision from an atom in the excited state. It is clear from our analysis that the coherent field does not give a complete description. If we subtract the intensity of the coherent field from the total expectation $\langle I\rangle$, we obtain, using Eqs. (27) and the normalization condition ($|c_g|^2 + |c_e|^2 = 1$),

$$I_{\text{incoh}} = \langle I\rangle - I_{\text{coh}} = 2F_0^2|c_e|^4, \tag{30}$$

which we define to be the intensity of incoherent spontaneous emission. By comparing this result with Eqs. (28) we find

$$I_{\text{incoh}} = 2\Delta F^{(-)}\Delta F^{(+)},$$

which associates the incoherent field with the field uncertainties, and provides an alternative definition of the incoherent field.

Figure 2 illustrates how the coherent and incoherent parts of the field change as the decay of the excited state proceeds. Initially, when the atom is seen in the excited state $|e\rangle$, the

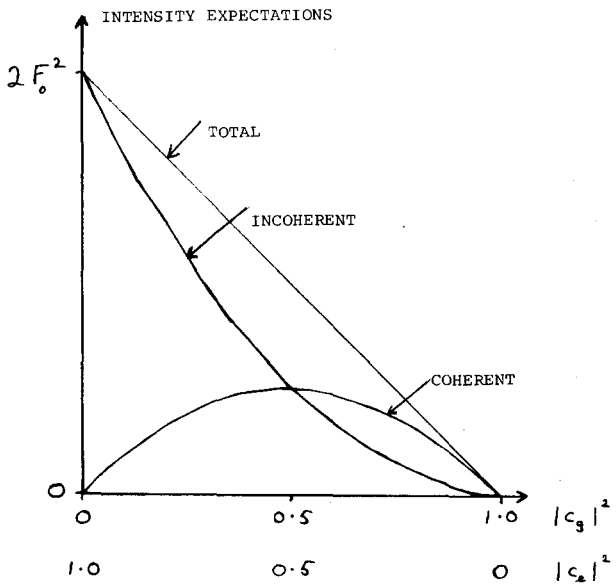

Fig. 2. Expectations of the coherent, the incoherent, and the total intensities of spontaneous emission from a two-level atom, shown as functions of the excited- and ground-state probabilities.

spontaneous emission is totally incoherent. As the decay proceeds and ground-state probability amplitude is generated, the coherent field grows and eventually dominates. In the limit ($|c_e|^2 \to 0$), when the decay is nearly over, the spontaneous-emission field is completely coherent. This type of behavior was first predicted by Senitzky[5] in a QED treatment. We note that it is consistent with the conventional semiclassical prediction that the spontaneous emission from a two-level atom is coherent whenever the excitation is weak as in Rayleigh scattering[4]; but we note also that the field in this limit is not classical in nature, for although the phase is well determined, the intensity is not. The uncertainty in intensity expressed as a fraction of the total expectation is

$$\Delta I/\langle I\rangle = |c_g|/|c_e|, \tag{31}$$

which actually diverges in the weak-excitation limit $|c_e|^2 \to 0$. This is a clear indication of the quantum nature of the field[8]; for the states $|e\rangle$ and $|g\rangle$ are eigenstates of $\hat{I}$ with eigenvalues $2F_0^2$ and zero, respectively, and so a measurement of I will yield the value $2F_0^2$ with probability $|c_e|^2$ or the value zero with probability $|c_g|^2$.

The changing properties of the decaying field characterized as we have seen by the sharpening of phase and the blurring of intensity, can be expressed concisely in terms of an uncertainty-type relation. From Eqs. (27) and (28) we have

$$\Delta F^{(\pm)}/|\langle F^{(\pm)}\rangle| = |c_e|/|c_g|.$$

Combining this result with Eq. (31) gives

$$\frac{\Delta F^{(\pm)}}{|\langle F^{(\pm)}\rangle|} \times \frac{\Delta I}{\langle I\rangle} = 1.$$

IV. SUPERRADIANT EMISSION FROM N ATOMS

We now go on to see how the single-atom coherence properties discussed above give rise to interference effects in the spontaneous emission from a system of N similar two-level atoms. We shall assume that the atoms do not perturb one another and that there are no stimulated emission and absorption processes occurring. We make no additional assumptions concerning the mean separation of the atoms.

We describe the N-atom system by an uncorrelated state, i.e., by a simple product of the unperturbed single-atom states $|i\rangle$:

$$|N \text{ atoms}\rangle = \prod_{i=1}^{N} |i\rangle, \tag{32}$$

with

$$|i\rangle = c_{gi}|g\rangle + c_{ei}|e\rangle.$$

The field operators representing the spontaneous-emission field of the system will be the sum of the corresponding single-atom operators:

$$\begin{aligned} \hat{F}_N^{(+)} &= \sum_i \hat{F}_i^{(+)} = -iF_0 \sum_i \hat{\eta}_i, \\ \hat{F}_N^{(-)} &= \sum_i \hat{F}_i^{(-)} = iF_0 \sum_i \hat{\eta}_i^\dagger, \\ \hat{F}_N &= \hat{F}_N^{(+)} + \hat{F}_N^{(-)}, \end{aligned} \tag{33}$$

where, by taking F_0 as a common factor, we have assumed that the volume containing the N atoms is sufficiently small that the geometric factor $\sin\theta/(\mathbf{R} - \mathbf{R}_0)$ in F_0 is substantially the same for all atoms. The operators Eqs. (33) act on the uncorrelated state Eq. (32) with time-retarded amplitudes c_{gi} and c_{ei}. The labeling of $\hat{\eta}_i$ and $\hat{\eta}_i^\dagger$ serves to emphasize that these are single-atom operators acting only on the single-atom state $|i\rangle$ having the same label.

The expectations of the field operators in the uncorrelated state are readily found to be

$$\begin{aligned} \langle F_N^{(+)}\rangle &= -iF_0 \sum_i (c_{gi}^* c_{ei}), \\ \langle F_N^{(-)}\rangle &= iF_0 \sum_i (c_{gi} c_{ei}^*), \\ \langle F_N\rangle &= -iF_0 \sum_i (c_{gi}^* c_{ei} - c_{gi} c_{ei}^*). \end{aligned} \tag{34}$$

As a natural generalization of our earlier definition, we define the coherent spontaneous-emission field of the system to be the expectation $\langle F_N\rangle$. So defined, the coherent field is simply the sum of the coherent fields of the individual atoms [see Eq. (27)]. The operator representing the total intensity of the spontaneous emission is constructed, as before, from the field operators:

$$\hat{I}_N = 2\hat{F}_N^{(-)}\hat{F}_N^{(+)} = 2F_0^2 \sum_{ij} (\hat{\eta}_i^\dagger \hat{\eta}_j). \tag{35}$$

To evaluate the expectation value it is convenient to separate terms for which $j = i$:

$$\langle \hat{I}_N\rangle = 2F_0^2 \left[\sum_i \langle \hat{\eta}_i^\dagger \hat{\eta}_i\rangle + \sum_{\substack{i \\ j\neq i}} \langle \hat{\eta}_i^\dagger \hat{\eta}_j\rangle \right].$$

Evaluating this in the state Eq. (32) gives

$$\langle \hat{I}_N\rangle = 2F_0^2 \left[\sum_i |c_{ei}|^2 + \sum_{\substack{i \\ j\neq i}} (c_{gi} c_{ei}^* c_{gj}^* c_{ej}) \right]. \tag{36}$$

Using the normalization condition for each pair of amplitudes ($|c_{gi}|^2 + |c_{ei}|^2 = 1$) we can express this as

$$\langle \hat{I}_N \rangle = 2F_0^2 \left[\sum_i |c_{ei}|^4 + \sum_{ij} (c_{gi} c_{ei}^* c_{gj}^* c_{ej}) \right], \quad (37)$$

where there is no longer a restriction on j in the second summation.

Equation (37) has a very simple interpretation. The second summation term is just the intensity of the coherent spontaneous emission of the N atoms [see Eq. (34)], while the first summation term is just the sum of the incoherent intensities of the N atoms [see Eq. (30)].

A special case of interest occurs when the N atoms are all prepared with the same degree of excitation. We then put

$$|c_{gi}|^2 = |c_g|^2; \qquad |c_{ei}|^2 = |c_e|^2, \quad (38)$$

where $|c_g|^2$ and $|c_e|^2$ are the same for all atoms. Equation (37) then becomes

$$\langle \hat{I}_N \rangle = 2F_0^2 \{ N|c_e|^4 + N^2 \Gamma |c_g|^2 |c_e|^2 \}, \quad (39)$$

with

$$\Gamma = |[\exp(-i\delta_i)]_{av}|^2.$$

Here δ_i is the phase of the excited-state amplitude c_{ei} relative to that of the ground-state amplitude c_{gi}, and $[\quad]_{av}$ indicates an average over all atoms.

The specialization, Eq. (38), is often taken to describe the result of preparing the system by excitation with a short resonant plane-wave pulse, and the result, Eq. (39), is in agreement with the well-known QED result.[9,10] For the purpose of comparison, it is convenient to express Eq. (39) slightly differently. By making use of the normalization condition ($|c_g|^2 + |c_e|^2 = 1$) in Eq. (39), or by using Eq. (38) in Eq. (37), we can obtain

$$\langle \hat{I}_N \rangle = 2F_0^2 \{ N|c_e|^2 + (N^2\Gamma - N)|c_g|^2 |c_e|^2 \}$$

$$= NF_0^2 \{ 1 - \cos\beta + (N\Gamma - 1)\frac{1}{2}\sin^2\beta \},$$

with $\sin^2(\beta/2) = |c_e|^2$.

In this form, our result agrees with Eq. (4.5) of Rehler and Eberly[9] who use a QED approach. See also, Eq. (78) of Dicke.[10] The second term in Eq. (39) contains the factor N^2 which characterizes superradiance. The actual magnitude of this term will depend critically through the factor Γ on the details of the interference, i.e., on the spatial geometry of the emitting system and the angular position of the observer relative to the propagation direction of the exciting plane-wave pulse. Radiation patterns for various geometries, and discussions of the validity of Eq. (39) can be found in the literature. The time dependence of the emissions can also be deduced from the form of Eq. (39) using a power-balance argument.[9]

We have mentioned that the predictions of our theory agree with those of QED. The point of agreement can be traced back from the final result, Eq. (39), to our expression for the expectation of the intensity,

$$\langle \hat{I} \rangle = 2F_0^2 \left\langle N \text{ atoms} \left| \sum_{ij} (\hat{\eta}_i^\dagger \hat{\eta}_j) \right| N \text{ atoms} \right\rangle.$$

Examples of equivalent expressions can be found in the QED treatments.[11] In our treatment this expression is essentially a postulate based on correspondence with classical radiation theory. In QED it is a deduced result removed by many algebraic steps from the basic postulate of field quantization. From this point of view our treatment may be regarded as a heuristic shortcut to the QED results.

Finally, we must point out that our use of the term coherent, though common with that of many other authors, is not consistent with Dicke's use of the term in his pioneering 1954 paper.[10] Dicke considers the spontaneous emission from correlated states of the N-atom system, as opposed to the uncorrelated states that we have used.[12] Now it is easy to show that the coherent component of the emission field $\langle F_N \rangle$ is zero in any such correlated state of the system. The superradiant intensity predicted by Dicke's Eq. (27) must therefore be completely incoherent in our sense of the word, though Dicke refers to it as coherent radiation. It is clear that he is using the term coherent in a different sense. We could attempt to clarify the situation by saying that the atoms in a Dicke-correlated state radiate coherently in the sense that the emitted radiations all have the same phase, but that this common phase is indeterminate. The indeterminacy of the common phase makes $\langle F_N \rangle$ zero, and so the radiation is incoherent in our sense. The situation is further discussed in the literature.[13]

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[1]For a survey of conventional semiclassical radiation theory, also called neoclassical radiation theory, see E. T. Jaynes, in *Coherence and Quantum Optics,* edited by E. Mandel and L. Wolf (Plenum, New York, 1973). See also, R. K. Nesbet, Phys. Rev. A **4,** 259 (1971).

[2]See experiments reported by H. M. Gibbs, in *Coherence and Quantum Optics, ibid.;* also, in Phys. Rev. A **8,** 456 (1973).

[3]See R. K. Nesbet, Phys. Rev. Lett. **27,** 553 (1971); and G. W. Series, *Proceedings of the International Conference on Optical Pumping and Atomic Line Shape,* Warsaw (1968).

[4]See, for example, A. V. Durrant, Am. J. Phys. **44,** 630 (1976).

[5]Coherent and incoherent components in spontaneous emission have been discussed in a QED treatment, by I. R. Senitzky, Phys. Rev. **111,** 3 (1958).

[6]The exponential decay constant can be determined by a power balance argument. See Ref. (3) above.

[7]This behavior is similar to that of a single-mode photon-number state of the quantized field in QED. See Fig. 7.1 and the relevant text in R. Loudon, *The Quantum Theory of Light* (Clarendon, Oxford, 1973).

[8]This behavior is similar to that of the single-mode phase states of the quantized field in QED. See Fig. 7.2 and discussion, in R. Loudon, *ibid.* Ref. 7.

[9]N. E. Rehler and J. H. Eberly, Phys. Rev. A **3,** 1735 (1971). See also, J. H. Eberly, Am. J. Phys. **40,** 1374 (1972).

[10]R. H. Dicke, Phys. Rev. **93,** 99 (1954).

[11]See, for example, Eq. (28) of Ref. 10, or Eq. (3.30) of Ref. 9. See also, Eq. (4.12) of Bonifacio, Schwendimann, and Haake, Phys. Rev. A **4,** 302 (1971).

[12]For a discussion of the quantum mechanical description of a system of two-level atoms, see, F. T. Arecchi *et al.,* Phys. Rev. A **6,** 2211 (1972).

[13]See Ref. 5. Also, G. S. Agarwal, in *Coherence and Quantum Optics,* edited by E. Mandel and L. Wolf (Plenum, New York, 1973).