$$\psi\rangle = E\ \psi\rangle$$

Computational Techniques for Quantum Lattice Models

Andreas Läuchli,
"New states of quantum matter"
MPI für Physik komplexer Systeme - Dresden

http://www.pks.mpg.de/~aml
aml@pks.mpg.de

Lecture Notes at http:www.pks.mpg.de/~aml/

"INSTANS Summer School 2010" - Benasque - 29/6/2010

Computational Techniques for Quantum Lattice Models

Exact Diagonalization
(Lecture 1, Tuesday)

$$|\psi\rangle$$

DMRG et al.
(Lecture 2, Wednesday)

$$|\psi\rangle$$

Classical & Quantum Monte Carlo + ALPS
(Lecture 3, Thursday)

ALPS

$$\psi\rangle = E\ \psi\rangle$$

First Lecture:
Introduction to Exact Diagonalization

Andreas Läuchli,
"New states of quantum matter"
MPI für Physik komplexer Systeme - Dresden

http://www.pks.mpg.de/~aml
aml@pks.mpg.de

Lecture Notes at http:www.pks.mpg.de/~aml/

"INSTANS Summer School 2010" - Benasque - 29/6/2010

Outline

$$\psi\rangle = E\,\psi\rangle$$

- Main Idea
- Examples of typical applications
- Structure of an Exact Diagonalization package
  - Hilbertspace, Symmetries
  - Hamiltonian
  - Linear Algebra
  - Observables
- Applications

Exact Diagonalization: Main Idea

- Solve the Schrödinger equation of a quantum many body system numerically

$$\psi\rangle = E\ \psi\rangle$$

- Sparse matrix, but for quantum many body systems the vector space dimension grows exponentially!

- Some people will tell you that's all there is.

- But if you want to get a maximum of physical information out of a finite system there is a lot more to do and the reward is a powerful:

Quantum Mechanics Toolbox

Exact Diagonalization: Applications

- Quantum Magnets: nature of novel phases, critical points in 1D, dynamical correlation functions in 1D & 2D
- Fermionic models (Hubbard/t-J): gaps, pairing properties, correlation exponents, etc
- Fractional Quantum Hall states: energy gaps, overlap with model states, entanglement spectra
- Quantum dimer models or other constrained models (anyon chains, ...)
- Full Configuration Interaction in Quantum Chemistry

Exact Diagonalization: Present Day Limits

- Spin S=1/2 models:
  40 spins square lattice, 39 sites triangular, 42 sites Honeycomb lattice
  64 spins or more in elevated magnetization sectors
  up to 1.5 billion(=10^9) basis states with symmetries, up to 4.5 billion without

- t-J models:
  32 sites checkerboard with 2 holes
  32 sites square lattice with 4 holes
  up to 2.8 billion basis states

- Fractional quantum hall effect
  different filling fractions ν, up to 16-20 electrons
  up to 3.5 billion basis states

- Hubbard models
  20 sites square lattice at half filling, 21 sites triangular lattice
  22-25 sites in ultracold atoms setting w.o. spatial symmetries
  up to 160 billion basis states

low-lying eigenvalues, not full diagonalization

Structure of an Exact Diagonalization code

Ingredients

- Hilbert space
  - Basis represention, Lookup techniques
  - Symmetries
- Hamiltonian Matrix
  - Sparse Matrix representation (memory/disk)
  - Matrix recalculation on the fly (matrix-free)
- Linear Algebra : Eigensolver / Time propagation
  - LAPACK full diagonalization
  - Lanczos type diagonalization (needs only $|v\rangle = H|u\rangle$ operations)
  - More exotic eigensolver techniques, real oder imaginary-time propagation,
- Observables
  - Static quantities (multipoint correlation functions, correlation density matrices,...)
  - Dynamic observables (spectral functions, density of states,...)
  - Real-time evolution

Hilbert Space

Basis representation

- States of the Hilbert space need to be represented in the computer.
- Choose a representation which makes it simple to act with the Hamiltonian or other operators on the states, and to localize a given state in the basis
- Simple example: ensemble of S=1/2 sites in binary coding

$$|\uparrow\uparrow\downarrow\uparrow\rangle \rightarrow [1\ 1\ 0\ 1]_2 = 13$$

detection of up or down spin can be done with bit-test.
transverse exchange $S^+S^- + S^-S^+$ can be performed by an XOR operation:

$$\underbrace{[1\ 1\ 0\ 1]_2}_{\text{initial config}}\ \mathrm{XOR}\ \underbrace{[0\ 1\ 1\ 0]_2}_{\text{bit 1 at the two sites coupled}} = \underbrace{[1\ 0\ 1\ 1]_2}_{\text{final config}}$$

- For S=1, one bit is obviously not sufficient. Use ternary representation or simply occupy two bits to label the 3 states.

Basis representation

- For t-J models at low doping it is useful to factorize hole positions and spin configurations on the occupied sites.
- For Hubbard models one can factorize the Hilbert space in up and down electron configurations.
- For constrained models - such as dimer models - the efficient generation of all basis states requires some thought.
- One of the key challenges for a fast ED code is to find the index of the new configuration in the list of all configurations (index f in $H_{i,f}$).
- Let us look at the example of S=1/2 spins at fixed S^z

Basis lookup procedures (Lin tables)

- One of the key problems for a fast ED code is to find the index of the new configuration in the list of all configurations (index f in $H_{i,f}$).

$$[1\ 0\ 1\ 1]_2 = 11_{10}$$

- But is 11 the index of this configuration in a list of all S^z=3/2 states ? no !

- Use Lin tables to map from binary number to index in list of allowed states:
  (generalization of this idea works for arbitrary number of additive quantum numbers)

- Two tables with $2^{(N/2)}$ [=sqrt(2^N)] entries, one for MSBs and one for LSBs

$$\begin{array}{lcl} [0\ 0] & = & X \\ [0\ 1] & = & 0 \\ [1\ 0] & = & 1 \\ [1\ 1] & = & 2 \end{array}$$

MSB

$$\begin{array}{lcl} [0\ 0] & = & X \\ [0\ 1] & = & 0 \\ [1\ 0] & = & 1 \\ [1\ 1] & = & 0 \end{array}$$

LSB

$$\begin{array}{lcl} \mathrm{Ind}([0\ 1\ 1\ 1]) = 0 + 0 & = & 0 \\ \mathrm{Ind}([1\ 0\ 1\ 1]) = 1 + 0 & = & 1 \\ \mathrm{Ind}([1\ 1\ 0\ 1]) = 2 + 0 & = & 2 \\ \mathrm{Ind}([1\ 1\ 1\ 0]) = 2 + 1 & = & 3 \end{array}$$

Basis lookup procedures (Lin tables)

- Lookup can therefore be done with two direct memory reads. This is a time and memory efficient approach (at least in many interesting cases).
- An alternative procedure is to build a hash list [const access time] or to perform a binary search [log access time].
- This becomes somewhat more involved when using spatial symmetries...

Symmetries

- Consider a XXZ spin model on a lattice. What are the symmetries of the problem ?

$$H = \sum_{i,j} J^{xy}_{i,j}(S^x_i S^x_j + S^y_i S^y_j) + J^z_{i,j} S^z_i S^z_j$$

- The Hamiltonian conserves total S^z, we can therefore work within a given S^z sector This easily implemented while constructing the basis, as we discussed before.
- The Hamiltonian is invariant under the space group, typically a few hundred elements. (in many cases = Translations x Pointgroup). Needs some technology to implement...
- At the Heisenberg point, the total spin is also conserved. It is however very difficult to combine the SU(2) symmetry with the lattice symmetries in a computationally useful way (non-sparse and computationally expensive matrices).
- At S^z=0 one can use the spin-flip (particle-hole) symmetry which distinguishes even and odd spin sectors at the Heisenberg point. Simple to implement.

Spatial Symmetries

- Spatial symmetries are important for reduction of Hilbert space
- Symmetry resolved eigenstates teach us a lot about the physics at work, dispersion of excitations, symmetry breaking tendencies, topological degeneracy, ...

40 sites square lattice
$T \otimes PG$ =40 x 4 elements

Icosidodecahedron (30 vertices)
I_h:120 elements

Spatial Symmetries

- Symmetries are sometimes not easily visible, use graph theoretical tools to determine symmetry group [nauty, grape].
- In an ED code a spatial symmetry operation is a site permutation operation.
  (could become more complicated with spin-orbit interactions and multiorbital sites)

40 sites square lattice
$T \otimes PG$ =40 x 4 elements

Icosidodecahedron (30 vertices)
I_h:120 elements

Spatial Symmetries: Building the basis

- Build a list of all allowed states satisfying the “diagonal” constraints, like particle number, total S^z, ...

- for each state we apply all symmetry operations and keep the state as a representative if it has the smallest integer representation among all generated states in the orbit.
  Example: 4 site ring with cyclic translation T, S^z=3/2 sector

$$T^0([0\ 1\ 1\ 1]) \rightarrow [0\ 1\ 1\ 1]$$
$$T^1([0\ 1\ 1\ 1]) \rightarrow [1\ 0\ 1\ 1]$$
$$T^2([0\ 1\ 1\ 1]) \rightarrow [1\ 1\ 0\ 1]$$
$$T^3([0\ 1\ 1\ 1]) \rightarrow [1\ 1\ 1\ 0]$$

keep state

$$T^0([1\ 0\ 1\ 1]) \rightarrow [1\ 0\ 1\ 1]$$
$$T^1([1\ 0\ 1\ 1]) \rightarrow [1\ 1\ 0\ 1]$$
$$T^2([1\ 0\ 1\ 1]) \rightarrow [1\ 1\ 1\ 0]$$
$$T^3([1\ 0\ 1\ 1]) \rightarrow [0\ 1\ 1\ 1]$$

discard state

...

Spatial Symmetries: Building the basis

- For one-dimensional representations χ of the spatial symmetry group:
  - "Bloch" state $$|\tilde{r}\rangle = \frac{1}{\mathcal{N}\sqrt{|G|}} \sum_{g \in G} \chi(g)|g(r)\rangle$$
  - Norm of the state is given as: $$\mathcal{N} = \sqrt{\sum_{g \in G, g(r)=r} \chi(r)}$$
  - The norm (and therefore the state itself) can vanish if it has a nontrivial stabilizer combined with a nontrivial representation χ.
- Example: 4 site S=1/2 ring with cyclic translations:

	$K=0$	$K=\pm\pi/2$	$K=\pi$	**2^4=16**
S^z=2	$\|1\ 1\ 1\ 1\rangle, \mathcal{N}=2$			1+1
S^z=1	$\|0\ 1\ 1\ 1\rangle, \mathcal{N}=1$	$\|0\ 1\ 1\ 1\rangle, \mathcal{N}=1$	$\|0\ 1\ 1\ 1\rangle, \mathcal{N}=1$	4+4
S^z=0	$\|0\ 1\ 0\ 1\rangle, \mathcal{N}=\sqrt{2}$		$\|0\ 1\ 0\ 1\rangle, \mathcal{N}=\sqrt{2}$	2
	$\|0\ 0\ 1\ 1\rangle, \mathcal{N}=1$	$\|0\ 0\ 1\ 1\rangle, \mathcal{N}=1$	$\|0\ 0\ 1\ 1\rangle, \mathcal{N}=1$	4

The Hamiltonian Matrix

The Hamiltonian Matrix

- Now that we have a list of representatives and their norms, can we calculate the matrix elements of the Hamiltonian ? $\langle \tilde{s}|H|\tilde{r}\rangle = ?$

- Let us look at an elementary, non-branching term in the Hamiltonian:

$$h^{\alpha}|r\rangle = h^{\alpha}(r)|s\rangle$$

- We can now calculate the matrix element $\langle \tilde{s}|h^{\alpha}|\tilde{r}\rangle$ without double expanding the Bloch states:

$$\langle \tilde{s}|h^{\alpha}|\tilde{r}\rangle = \frac{\mathcal{N}_s}{\mathcal{N}_r}\chi(g^*)h^{\alpha}(r)$$

- key algorithmic problem: given a possibly non-representative $|s\rangle$, how do we find the associated representative $|\tilde{s}\rangle$, as well as a symmetry element g^* relating $|s\rangle$ to $|\tilde{s}\rangle$?

The Hamiltonian Matrix

- key algorithmic problem: given a possibly non-representative $|s\rangle$, how do we find the associated representative $|\tilde{s}\rangle$, as well as a symmetry element g^* relating $|s\rangle$ to $|\tilde{s}\rangle$?

  - Brute force: loop over all symmetry operations applied on $|s\rangle$ and retain $|\tilde{s}\rangle$ and g^*. This is however often not efficient (many hundred symmetries).

  - Prepare a lookup list, relating each allowed configuration with the index of its representative, and also the associated group element linking the two. Gives fast implementation, but needs a list of the size of the non spatially-symmetrized Hilbert space.

  - For specific lattices and models (Hubbard models) clever tricks exist which factorize the symmetry group into a sublattice conserving subgroup times a sublattice exchange. They give $|\tilde{s}\rangle$ fast, then a hash or binary search is needed to locate $|\tilde{s}\rangle$ in the list of representatives in order to get its index.

Hamiltonian Matrix Storage

- Different possibilities exist:
  - Store hamiltonian matrix elements in memory in a sparse matrix format
    Fast matrix vector multiplies, but obviously limited by available memory.
  - Store hamiltonian matrix elements on disk in a sparse matrix format.
    In principle possible due to the vast disk space available, but I/O speed is much slower than main memory access times. Difficult to parallelize.
  - Recalculate the hamiltonian matrix elements in each iterations “on the fly”.
    Needed for the cutting edge simulations, where the whole memory is used by the Lanczos vectors. Can be parallelized on most architectures.

The Linear Algebra Backend

Linear Algebra:
The most popular: Lanczos Algorithm

- Lanczos Algorithm (C. Lanczos, 1950)

Three vector recursion

$$
\begin{aligned}
|\phi'\rangle &= H|\phi_n\rangle - \beta_n|\phi_{n-1}\rangle\,, \\
\alpha_n &= \langle\phi_n|\phi'\rangle\,, \\
|\phi''\rangle &= |\phi'\rangle - \alpha_n|\phi_n\rangle\,, \\
\beta_{n+1} &= ||\phi''|| = \sqrt{\langle\phi''|\phi''\rangle}\,, \\
|\phi_{n+1}\rangle &= |\phi''\rangle/\beta_{n+1}\,,
\end{aligned}
$$

$$
\tilde{H}_N = \begin{bmatrix}
\alpha_0 & \beta_1 & 0 & \cdots\cdots\cdots & & 0 \\
\beta_1 & \alpha_1 & \beta_2 & 0 & \cdots\cdots & 0 \\
0 & \beta_2 & \alpha_2 & \beta_3 & 0 & 0 \\
 & & \ddots & \ddots & \ddots & \\
0 & \cdots & 0 & \beta_{N-2} & \alpha_{N-2} & \beta_{N-1} \\
0 & \cdots\cdots\cdots & & 0 & \beta_{N-1} & \alpha_{N-1}
\end{bmatrix}
$$

- Eigenvalues of H_N converge rapidly towards eigenvalues of H.

- Once desired eigenvalue is converged, restart recursion and assemble the eigenvector.

very quick convergence for extremal eigenvalues !

Linear Algebra: Lanczos Algorithm

- Once the ground state has converged, the vectors in the recursion tend to lose their orthogonality. As a consequence fake new eigenvalues show up in the approximate spectrum. These can be removed by heuristic techniques

- Degeneracies of eigenvalues can not be resolved by construction. For this task one would need a band lanczos or the (Jacobi-)Davidson technique. However multiply degenerate eigenvalues are converged.

- Checkpointing is useful when performing large-scale simulations.

Full Diagonalization: Thermodynamics

- Lapack / Householder complete diagonalization of the spectrum.
- Calculate partition function and all the thermodynamic quantities you want, often the only pedestrian method available for frustrated systems.
- Symmetries are also very important, because the computational requirements scale as $O(D^3)$, where D is the dimension of the block Hilbert space. Typical D's for a workstation are a few 1'000, up to a few 100'000 on supercomputers.

F. Heidrich-Meisner, A. Honecker, T. Vekua, Phys. Rev. B **74**, 020403(R) (2006).

Observables

Observables

- In principle once can calculate any correlation function, since one has access to the full many body wave functions. When using spatial symmetries, the correlation functions need to be properly symmetrized too.

- Complicated correlation functions occur in frustrated systems:

Dimer-dimer correlations

Spin current correlations

Frequency Dynamics

- $$G_A(\omega + i\eta) = \langle\psi|A^\dagger \frac{1}{E_0 + \omega + i\eta - H} A|\psi\rangle \quad A = S^\alpha(\mathbf{q}), c_\mathbf{k}, \ldots$$

- Generate Krylov space of $A|\psi\rangle$
  Use continued fraction used to invert $(E_0 + \omega + i\eta - H)$

- Triangular Lattice Spin Dynamics in zero field

AML unpublished

Exact Diagonalization
Real-Time Dynamics

- It is expensive to obtain the full propagator $\exp[-itH]$

- Krylov methods exist to approximate the propagator for a given state $|\psi(0)\rangle$
  One can get the time propagated state $|\psi(t)\rangle$ with only $|v\rangle = H|u\rangle$ operations.

- Example: time evolution of a strongly correlated quantum systems after an abrupt change in the parameters in the Hamiltonian. Revivals and Relaxation.

C. Kollath, AML, E. Altman, PRL 2007

Exact Diagonalization: Applications

- Quantum Magnets: nature of novel phases, critical points in 1D, dynamical correlation functions in 1D & 2D

- Fermionic models (Hubbard/t-J): gaps, pairing properties, correlation exponents, etc

- Fractional Quantum Hall states: energy gaps, overlap with model states, entanglement spectra

- Quantum dimer models or other constrained models (anyon chain..)

- Full Configuration Interaction in Quantum Chemistry

“Tower of States” spectroscopy

- What are the finite size manifestations of a continuous symmetry breaking ?
  (eg in superfluids/superconductors, magnetic order, spin nematic order)

- Order parameter is zero on a finite system ! (symmetric partition function)

- So usually one looks into order parameter correlations

“Tower of States” spectroscopy

- What are the finite size manifestations of a continuous symmetry breaking ?
  (eg in superfluids/superconductors, magnetic order, spin nematic order)

- Low-energy dynamics of the order parameter
  Theory: P.W. Anderson 1952, Numerical tool: Bernu, Lhuillier and others, 1992 -

- Dynamics of the free order parameter is visible in the finite size spectrum. Depends on the continuous symmetry group.

- U(1): $(S^z)^2$ SU(2): $S(S+1)$

- Symmetry properties of levels in the Tower states are crucial and constrain the nature of the broken symmetries.

Tower of States
S=1 on triangular lattice

- Bilinear-biquadratic S=1 model on the triangular lattice (model for $NiGaS_4$).

$$H = \sum_{\langle i,j \rangle} \cos(\theta)\ \mathbf{S}_i \cdot \mathbf{S}_j + \sin(\theta)\ \ (\mathbf{S}_i \cdot \mathbf{S}_j)^2$$

AML, F. Mila, K. Penc, PRL '06

Tower of States
S=1 on triangular lattice: Antiferromagnetic phase

- ϑ=0 : coplanar magnetic order,
  120 degree structure

- Breaks translation symmetry. Tree site unit cell
  ⇒ nontrivial momenta must appear in TOS
- non-collinear magnetic structure
  ⇒ SU(2) is completely broken,

  number of levels in TOS increases with S
- Quantum number are identical to the S=1/2 case

Tower of States
S=1 on triangular lattice: Ferroquadrupolar phase

- ϑ=-π/2 : ferroquadrupolar phase, finite quadrupolar moment, no spin order

- No translation symmetry breaking.
  ⇒ only trivial momentum appears in TOS
- Ferroquadrupolar order parameter, only even S
- all directors are collinear
  ⇒ SU(2) is broken down to U(1),

  number of states in TOS is *independent* of S.

Tower of States
S=1 on triangular lattice: Antiferroquadrupolar phase

- ϑ=3π/8 : antiferroquadrupolar phase, finite quadrupolar moment, no spin order, three sublattice structure.

- Breaks translation symmetry. Tree site unit cell ⇒ nontrivial momenta must appear in TOS
- Antiferroquadrupolar order parameter, complicated S dependence. Can be calculated using group theoretical methods.

Exact Diagonalization: Applications

- Quantum Magnets: nature of novel phases, critical points in 1D, dynamical correlation functions in 1D & 2D

- Fermionic models (Hubbard/t-J): gaps, pairing properties, correlation exponents, etc

- Fractional Quantum Hall states: energy gaps, overlap with model states, entanglement spectra

- Quantum dimer models or other constrained models (anyon chain..)

- Full Configuration Interaction in Quantum Chemistry

A “one-dimensional” microscopic approach

Consider sample with lengths $L_1,\ L_2$

Each electron state has fixed area

choose
(‘Landau gauge’)

(any shape is OK)

A “one-dimensional” microscopic approach

Each box is either empty 0 or filled 1

$$\psi_k \sim e^{ik\frac{2\pi}{L_1}y} e^{-(x-k\frac{2\pi}{L_1})^2/2}$$

(Landau gauge)

$1\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0$ A possible state at $\nu = 1/3$

Hamiltonian (ee-interaction) *(eg Coulomb $V(r)=e^2/r$)* No kinetic energy!

$1..0.....0..1 \leftrightarrow 0..1.....1..0$ $V_{k,m}$ *(all ee-terms that preserve position of CM)*

k+m k-m

including

$1.....1$ $V_{k,0}$ *(electrostatic repulsion)*

k

Exact mapping of a single Landau level!

(Topological) Entanglement Entropy

- Let us look at reduced density matrices, and their entanglement entropies

$$\rho = \mathrm{Tr}_E |\psi\rangle\langle\psi|$$

$$S(\rho) = \mathrm{Tr}[-\rho \log \rho]$$

For topologically ordered phases:

Perimeter/Area Law

$$S(\rho) = \alpha L - \gamma + \cdots$$

Topological entanglement entropy

$$\gamma = \log \mathcal{D}$$

$\mathcal{D}$ Total quantum dimension

Kitaev & Preskill PRL '06
Levin & Wen PRL '06

Topological entanglement entropy for FQH states on the sphere

- FQH model states are known to have topological order

- Can one extract γ based on the entanglement entropy ?

Haque, Zozulya & Schoutens, PRL '07

- Feasible, but tricky on the sphere.
  Complications due to varying length as a function of latitude

How to do this on the torus

- The torus can be tuned continuously by varying L_1 and L_2 ($L_1 L_2 = 2\pi N_s$).

- We study orbital partitioning, which is expected to correspond to real space partitioning

Area law at constant L_1

- Degenerate wavefunctions ⇒ averaging of entropies yields smooth curve !

AML, Bergholtz & Haque, arXiv:1003.5656

Area law at constant L_1

- Increasing N_s (and thus L_2) at constant L_1 ⇒ Saturation at large l_A

AML, Bergholtz & Haque, arXiv:1003.5656

Entanglement entropy $S(L_1)$

- For large enough N_s, $S(L_1)$ converges for each L_1

$$S(L_1) = 2\alpha L_1 - 2\gamma + \ldots$$

Boundary entropy density (α)

AML, Bergholtz & Haque, arXiv:1003.5656

Extracting the topological entanglement entropy

- Use a running γ extraction, and monitor L_1 convergence

$$S(L_1) = 2\alpha L_1 - 2\gamma + \ldots$$

- 2γ converges towards expected Ln(3) !
  Most accurate numerical determination for FQH states to date.

AML, Bergholtz & Haque, arXiv:1003.5656

FQH using DMRG ?

- $S(L) \sim L_1$
- $m \sim \exp[S] \sim \exp[a\, L_1]$, exponential effort in the physical width of the system (see DMRG talk tomorrow for more details on this)
- DMRG will not work easily for large 2D FQH samples
- fermionic MERA/PEPS in suitable gauge ?

Entanglement Spectra (Li & Haldane PRL '08)

- The entanglement entropy is a single number !

- Is there more one can extract from the reduced density matrix ?

- One can always write $\rho =: \exp[-H_{\mathrm{Entanglement}}]$

Entanglement Spectrum:

$$H_{\mathrm{Entanglement}}|\psi\rangle = \xi|\psi\rangle \qquad S = \sum_i \xi_i \exp(-\xi_i)$$

- Assuming that the entanglement Hamiltonian and the physical Hamiltonian are "similar" (e.g. as in free fermionic systems), then one expects to see features related to the open boundary structure in the spectrum of the reduced density matrix

- FQH states have interesting edge physics, visible in entanglement spectrum ?

Moore-Read state on the sphere (Li & Haldane, PRL '08)

- Entanglement spectrum has dispersive structure

- Degeneracy at large momenta follows CFT counting rule (edge theory of the Pfaffian is U(1)+Majorana)

Wen, PRL '93

Entanglement Spectra of FQH States: Torus

- The torus offers several advantages, e.g. a partition has two contributing edges. Chiral Luttinger liquid(s) in Entanglement Spectrum ?

$\nu = 1/3$, Laughlin WF

AML, E. Bergholtz, J. Suorsa & M. Haque
PRL **104** 156404 (2010)

Exact Diagonalization Literature

- N. Laflorencie & D. Poilblanc,

  "Simulations of pure and doped low-dimensional spin-1/2 gapped systems"

  Lecture Notes in Physics **645**, 227 (2004).

- R.M. Noack & S. Manmana,

  "*Diagonalization- and Numerical Renormalization-Group-Based Methods for Interacting Quantum Systems*",

  AIP Conf. Proc. **789**, 93 (2005).

- A. Weisse, H. Fehske

  "Exact Diagonalization Techniques"

  Lecture Notes in Physics 739, 529 (2008).

- A. Läuchli

  "Numerical Simulations of Frustrated Systems"

  to appear in "Highly Frustratred Magnetism", Eds. Lacroix Mendels, Mila, (2010).

Thank you !