# Some basic properties of stimulated and spontaneous emission: A semiclassical approach

A. V. Durrant

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# Some basic properties of stimulated and spontaneous emission: A semiclassical approach

A. V. Durrant
*Department of Physics, The Open University, Milton Keynes MK7 6AA, England*
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The coherence and directional properties of stimulated and spontaneous emission can be understood without a knowledge of quantum electrodynamics. This paper presents a semiclassical description of the interaction between a free atom and a weak optical field under conditions of either a stimulated emission or a stimulated absorption resonance. It is shown that, under certain conditions, the stimulated or spontaneous radiation fields emitted from a sample of atoms will propagate in the same direction as the incident beam, will be coherent with the incident beam, and will have the correct phase to provide amplification or attenuation.

## I. INTRODUCTION

The coherence and directional properties of stimulated and spontaneous emission are usually introduced to the student in the context of the laser, and the following assertions, or something like them, are frequently made: (a) stimulated emission is coherent with the stimulating light, but spontaneous emission is not; (b) stimulated emission enters and amplifies the mode of the stimulating field. What these assertions actually mean, and the extent to which they are generally true outside the context of an optical cavity, are not usually made clear in elementary discussions. Most physicists agree that the proper theory for the treatment of such questions is quantum electrodynamics (QED). However, many undergraduate programs do not include a study of QED, and any appeal to concepts like photon creation operators and vacuum fluctuations etc, taken outside the context of a full QED treatment, is not likely to be satisfying or convincing. Now it is evident that only the wave aspects of light are referred to in the assertions, and so one is led to consider a semiclassical description; that is to say, a description in which the atoms are quantized while the optical fields are treated as classical electromagnetic waves.[1] An obvious advantage of a semiclassical treatment in the teaching context is that its basic concepts are within the grasp of students who have had first courses in electromagnetic theory and in quantum mechanics. In this paper we present a semiclassical investigation of the basic interactions that occur when a plane harmonic electromagnetic wave falls on an optically thin sample of atoms under conditions of an absorption or an emission resonance (see Fig. 1). Our conclusions will include precise statements of the coherence and directional properties of the emitted light. Our main restriction consists in taking the interaction perturbation to first order only. This corresponds to cases where the light source is weak, and leads to a linear response for the atoms. We shall make use of phenomenological decay constants to describe the spontaneous decay of excited atomic states; this involves using a non-Hermitian Hamiltonian to describe the radiating atom. This procedure is widely used and its validity well established.[2]

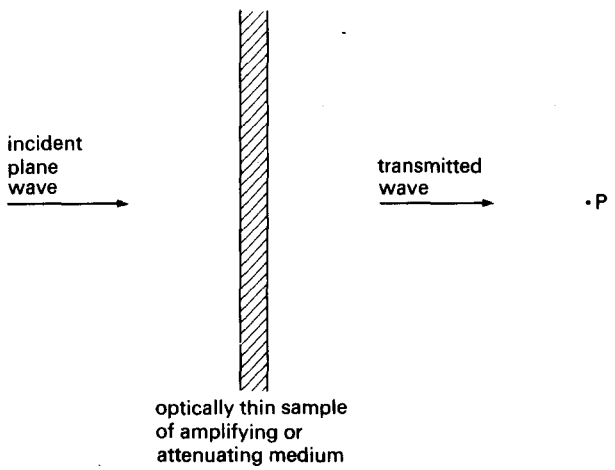


Fig. 1. A plane wave of optical radiation is incident on a plane slab of atoms. The transmitted wave is a superposition of the incident plane wave and the waves emitted from the atoms in the sample. The emitted waves may consist of stimulated or spontaneous emissions depending upon the detailed mechanisms of the interaction (see Fig. 2).

## II. STATE OF SINGLE ATOM

We begin by calculating the quantum state of a single atom placed in an optical field whose electric vector is

$$\mathbf{E}_i(\mathbf{r}, t) = \mathbf{e}E_0 \cos(\omega t - \mathbf{k}\cdot\mathbf{r}). \tag{1}$$

Here $\mathbf{e}$ is a unit polarization vector, $\mathbf{k}$ is the propagation vector, and $\omega$ the angular frequency. $\mathbf{e}$ and $\mathbf{k}$ define the mode of the incident field. We consider first the case where the atom is in an absorption resonance with the field. The analysis may then be simply extended to the case of an emission resonance.

### A. Case (i): Absorption resonance [Fig. 2(a)]

The simplest case to consider is absorption from the ground state. Let the ket $|a\rangle$ represent the ground state of the atom and let us take the zero of energy at this level. The first excited state $|b\rangle$ then has energy $\hbar\omega_b$. Using the electric dipole approximation, we can write the interaction Hamiltonian as

$$\hat{H}_D = -\mathbf{E}_i(\mathbf{r}', t)\cdot\hat{\mathbf{D}}, \tag{2}$$

where $\hat{\mathbf{D}}$ is the electric dipole moment operator for the atom situated at position $\mathbf{r}'$. We shall assume that state $|b\rangle$ is the only state in resonance with the field,[3] and that $|a\rangle$ is the only state below $|b\rangle$ into which spontaneous

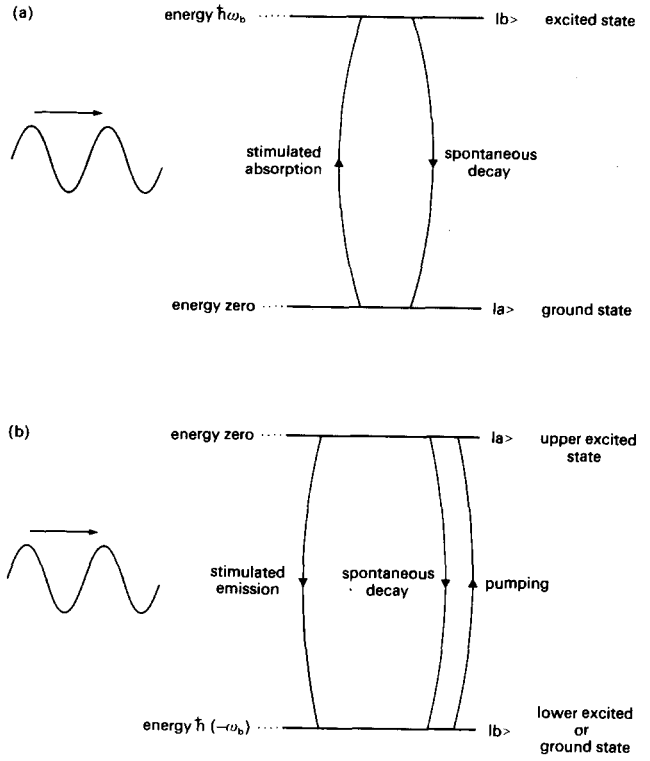


Fig. 2. (a) Two-level model for an absorption resonance. The incident optical field stimulates the absorption transition from the ground state $|a\rangle$ of an atom to an excited state $|b\rangle$. In the steady state the stimulated absorption to $|b\rangle$ is balanced by spontaneous decay of $|b\rangle$ back to $|a\rangle$. Note that for a weak incident field the stimulated decay of $|b\rangle$ is negligible compared with the spontaneous decay; the stimulated decay does not appear in the first-order treatment of the absorption process. (b) Two-level model for an emission resonance. The incident optical field stimulates the emission transition from the upper excited state $|a\rangle$ to a lower state $|b\rangle$. In the steady state the population of state $|a\rangle$ is maintained against the combined effects of spontaneous and stimulated decay by a strong externally applied pumping mechanism.

decay from $|b\rangle$ can occur. With these assumptions we are effectively dealing with a two-level atom, and a perturbation expansion for the state of the atom under the influence of (2) need contain only two terms.

Before we can apply perturbation theory we must first specify the unperturbed, or zero-order, states of the atom. We cannot use the states $|a\rangle$ and $|b\rangle$ as they stand because we know that, even in the absence of an applied perturbation such as (2), any excited state will suffer spontaneous decay. This decay has its origin in a much stronger interaction than the one that we are considering. This, in the language of QED, is the interaction with the vacuum fluctuations of the field; or, in semiclassical language, it is the interaction with the so-called self-field, or radiation-reaction field, of the atom.[4] For our purpose we can take account of this strong perturbation in a purely phenomenological manner. This we base on the observation that a population of excited atoms decays exponentially. Thus, if a single atom is prepared at time zero in the state $|b\rangle$, then in the absence of any externally applied fields, the probability of finding the atom in state $|b\rangle$ at time $t$ is $\exp(-\Gamma_b t)$, where the decay constant $\Gamma_b$ is just the Einstein $A$ coefficient for the transition. The most general expression for the probability amplitude of state $|b\rangle$ at time $t$ is therefore $\exp[-(\Gamma_b t/2 + i\delta_b)]$. Thus we take account of the spontaneous decay of state $|b\rangle$ by replacing the usual time-dependent factor $\exp(-i\omega_b t)$ by the modified time dependence $\exp\{-i[(\omega_b - i\Gamma_b/2)t + \delta_b]\}$ [the phase factor $\exp(-i\delta_b)$, which is of course unimportant when it appears as an overall multiplying factor in a quantum state, can have physical significance when it appears as a relative phase between two components in a superposition state]. In a similar way we could take account of the corresponding growth in the population of the ground state $|a\rangle$. This will not prove necessary however since we are restricting the analysis to weak optical fields so that, to first order, the ground-state population will not be significantly disturbed by the optical perturbation.

We now write a perturbation expansion for the state of the atom at time $t$:

$$|t\rangle = \alpha_a(t)|a\rangle + \alpha_b(t)|b\rangle \exp\{-i[(\omega_b - i\Gamma_b/2)t + \delta_b]\}, \quad (3)$$

with the condition $|\alpha_a(t)|^2 \approx 1$ imposed by our assumption of a weak optical field. We could of course subsume the phase factor $\exp(-i\delta_b)$ in the unknown amplitude $\alpha_b(t)$. By displaying this factor explicitly we are merely emphasizing the fact that at this stage in the argument we cannot specify the relative phase of the two components in the superposition. Our problem now is to find the induced probability amplitude $\alpha_b(t)$ of the excited state, for it is this that will determine the optical field radiated by the atom. The state $|t\rangle$ satisfies the Schrödinger equation for the Hamiltonian $(\hat{H}_0 + \hat{H}_D)$, where $\hat{H}_D$ is the interaction Hamiltonian given by (2), and $\hat{H}_0$ is the Hamiltonian of the atom in the absence of the applied optical field. $\hat{H}_0$ is assumed to contain the effects of spontaneous decay as well as the internal interactions within the atom. With our phenomenological description $\hat{H}_0$ will be non-Hermitian with eigenvalues 0 and $\hbar(\omega_b - i\Gamma_b/2)$ for the two states.

We now proceed to use time-dependent perturbation theory to find the excited-state amplitude $\alpha_b(t)$ to first order. To do this we substitute the state $|t\rangle$ of Eq. (3) into the Schrödinger equation for $(\hat{H}_0 + \hat{H}_D)$. Knowing the eigenstates and eigenvalues of $\hat{H}_0$ we obtain directly

$$\frac{d}{dt'}\alpha_b(t') = \frac{iE_0 D_{ba}}{2\hbar}\alpha_a(t') \times (\exp\{i[(\omega_b - \omega - i\Gamma_b/2)t' + \mathbf{k}\cdot\mathbf{r}']\} + \exp\{i[(\omega_b + \omega - i\Gamma_b/2)t' - \mathbf{k}\cdot\mathbf{r}']\})\exp(i\delta_b)$$

at any time $t'$. Here $D_{ba} = \langle b|\mathbf{e}\cdot\hat{\mathbf{D}}|a\rangle$. We now put $\alpha_a(t') = 1$ for all $t'$ and integrate from $t' = -\infty$ to $t' = t$ to obtain the steady-state solution at time $t$:

$$\alpha_b(t) = \frac{E_0 D_{ba}}{2\hbar}\left(\frac{\exp\{i[(\omega_b - \omega - i\Gamma_b/2)t + \mathbf{k}\cdot\mathbf{r}']\}}{\omega_b - \omega - i\Gamma_b/2} + \frac{\exp\{i[(\omega_b + \omega - i\Gamma_b/2)t - \mathbf{k}\cdot\mathbf{r}']\}}{\omega_b + \omega - i\Gamma_b/2}\right)\exp(i\delta_b). \quad (4)$$

Since the frequencies $\omega_b$ and $\omega$ are essentially positive we can strike out the second "nonresonant" term. Then, substituting (4) into (3) we obtain the steady state of the atom:

$$|t\rangle = |a\rangle + \frac{E_0 D_{ba}}{2\hbar}|b\rangle \frac{\exp[-i(\omega t - \mathbf{k}\cdot\mathbf{r}')]}{\omega_b - \omega - i\Gamma_b/2}. \quad (5)$$

We note that the angular frequency of oscillation of the excited state amplitude is equal to the optical angular frequency $\omega$, and that the phase $\delta_b$ no longer appears. This means that the oscillations have a definite phase relation with respect to the electric vector of the incident light at the location $\mathbf{r}'$ of the atom. This result will have important consequences when we come to consider the radiation field emitted by the atom in Sec. III.

We emphasize that Eq. (5) is a steady-state solution in which the rate of optical excitation to state $|b\rangle$ is balanced by the rate of spontaneous decay of $|b\rangle$ back to $|a\rangle$. The rate of stimulated decay from $|b\rangle$ to $|a\rangle$ is very much smaller than the rate of spontaneous decay and does not appear in this first-order treatment. An experimental situation corresponding closely to our model is the irradiation of mercury vapor with a steady beam of 2537-Å resonance radiation which excites the transition $^1S_0 \rightarrow {}^3P_1$. For an even isotope of mercury the ground state is nondegenerate as assumed in our model. The steady state is maintained by the spontaneous decay of the excited state $^3P_1 \rightarrow {}^1S_0$ with the emission of a field of resonance fluorescence, as is well known.

### B. Case (ii): Emission resonance [Fig. 2(b)]

We now consider the case where the atom is in an excited state and the applied optical perturbation, Eq. (2), induces a small oscillating probability amplitude for a lower state. We shall assume, as in case (i), a two-level system driven in the steady state. In this case though, the steady-state requirement demands that we postulate a pumping mechanism, as indicated in Fig. 2(b), which maintains the upper state population against the effects of spontaneous decay and the applied optical perturbation. We have not given the details of the pumping process in Fig. 2(b). Optical pumping via a third higher state not shown in the figure, or excitation by collisions with energetic electrons in a discharge, or resonant exchange of excitation by collision with an excited foreign gas atom as occurs in the He–Ne laser, are possible examples of a suitable pumping mechanism. We shall make two assumptions. First, that the effect of the pumping is to produce an exponential decay of any population of atoms prepared in the lower state; and second, that the rate constant for pumping from the lower to the upper state is very much greater than the rate constant for spontaneous decay of the upper state, so the probability of finding the atom in the upper state is always nearly unity. We shall then ignore the effects of spontaneous decay on the populations of the two states. We now choose to use the symbols $|a\rangle$ and $|b\rangle$ to represent the upper and lower states, respectively, of our two-level system. With this notation we can take over Eq. (3) as the two-state perturbation expansion for the atom under the influence of the applied optical perturbation, except that $\omega_b$ in the complex exponential must now be replaced by $-\omega_b$ (with $\omega_b$ positive) if we keep the convention that energy is to be measured from state $|a\rangle$. Thus we write

$$|t\rangle = \alpha_a(t)|a\rangle + \alpha_b(t)|b\rangle \exp\{i[(\omega_b + i\Gamma_b/2)t - \delta_b]\},$$

with $|\alpha_a(t)|^2 \approx 1$ as before, but here this condition expresses the strong pumping from $|b\rangle$ to $|a\rangle$. The phenomenological decay constant $\Gamma_b$ here describes the effect of the pumping mechanism in removing population from state $|b\rangle$; thus if a population of atoms were prepared in state $|b\rangle$ it would decay with rate constant $\Gamma_b$. We have put the phenomenological decay constant of state $|a\rangle$, $\Gamma_a$, equal to zero. This expresses our assumption that the population of state $|a\rangle$ is maintained against spontaneous decay by the strong pumping. We note that this phenomenological treatment hides many of the details of the pumping cycle which are not of interest to us; in particular the spontaneous decay of state $|a\rangle$ and the corresponding spontaneous emission field cannot appear in our analysis which is concerned only with the stimulated decay of $|a\rangle$ under the influence of the applied optical perturbation. We shall return to this point in Sec. III.

The perturbation calculation will now proceed exactly as in case (i) except for the change in the sign of $\omega_b$ which makes the second term in Eq. (4) resonant and the first nonresonant. Equation (5) which represents the steady state of the atom for an absorption resonance may now be generalised to include the case of emission resonance:

$$|t\rangle = |a\rangle \pm \frac{E_0 D_{ba}}{2\hbar}|b\rangle \frac{\exp[\mp i(\omega t - \mathbf{k}\cdot\mathbf{r}')]}{\omega_b - \omega \mp i\Gamma_b/2}. \quad (6)$$

Here the upper signs are to be taken for an absorption resonance from a ground state $|a\rangle$; and the lower signs for an emission resonance from a pumped excited state $|a\rangle$. The two cases differ only in the phase of the oscillation relative to that of the applied field, and in the interpretations of $|a\rangle$, $|b\rangle$, and $\Gamma_b$.

## III. RADIATION FIELD FROM A SINGLE ATOM

The expectation value of the electric dipole moment operator $\hat{\mathbf{D}}$ in the state $|t\rangle$ of (6) is readily found to be

$$\langle t|\hat{\mathbf{D}}|t\rangle = \pm e E_0 X \cos(\omega t - \mathbf{k}\cdot\mathbf{r}' - \alpha), \quad (7)$$

with

$$X = \frac{|D_{ba}|^2/\hbar}{[(\omega_b - \omega)^2 + \Gamma_b^2/4]^{1/2}}$$

and

$$\tan\alpha = \frac{\Gamma_b/2}{\omega_b - \omega}.$$

Equation (7) describes forced simple harmonic oscillations of the induced electric dipole moment of the atom, with a frequency-dependent resonance factor $X$ and relative phase $\alpha$. We note that at the peak of the resonance, $\omega = \omega_b$, the phase of the induced moment lags the applied field by $\pi/2$ when the + sign is taken (absorption resonance), and leads by $\pi/2$ when the − sign is taken (emission resonance).

The classical expression for the electric vector of the radiation field at a point $\mathbf{r}$ a distance $R >> \lambda$ from an oscillating electric dipole moment $\mathbf{D}(t)$ situated at $\mathbf{r}'$ is

$$\mathbf{E}(\mathbf{r}, t) = (\omega^2/4\pi\epsilon_0 c^2 R) \times [\mathbf{D}(t - R/c) - \mathbf{n}\cdot\mathbf{D}(t - R/c)\mathbf{n}], \quad (8)$$

where $\mathbf{n}$ is the unit vector $(\mathbf{r} - \mathbf{r}')/|\mathbf{r} - \mathbf{r}'|$, $R = |\mathbf{r} - \mathbf{r}'|$, and $\omega$ is the angular frequency of the dipole. If we now replace $\mathbf{D}(t)$ in Eq. (8) by the expectation value of the operator $\hat{\mathbf{D}}$ given by Eq. (7), we obtain a radiation field[5]

$$\mathbf{E}(\mathbf{r}, t) = \pm(\omega^2/4\pi\epsilon_0 c^2 R)E_0[\mathbf{e} - (\mathbf{n}\cdot\mathbf{e})\mathbf{n}] \times X\cos(\omega t - \mathbf{k}\cdot\mathbf{r}' - \alpha - kR), \quad (9)$$

with $k = |\mathbf{k}| = \omega/c$. We shall refer to this radiation field as the *induced field*.

### A. Coherence properties of induced field

If we know the position vector $\mathbf{r}'$ of the atom, then the phase of the induced field is well defined. The induced field is therefore coherent with the incident field. Now this conclusion applies whether we take the + sign or the − sign in (9). In the first case we are dealing with an absorption resonance from the ground state. In our first-order theory the excited atoms return to the ground state by spontaneous decay, and the induced field represents the corresponding field of spontaneous emission. In the case of mercury vapor irradiated by 2537-Å resonance radiation we would refer to this field as the resonance fluorescence. In the second case we are dealing with an emission resonance from an excited state, and the induced field represents the field of stimulated emission associated with the stimulated decay of the excited state to the lower state. The distinction between spontaneous and stimulated emission appears here in the relative phase rather than in the fundamental coherence properties of the two processes. In what sense then are we to understand assertion (a) in Sec. I which refers to the incoherent nature of spontaneous emission? To answer this question we must examine some of the assumptions of our model.

### B. Incoherent emissions[11]

Consider first the case of absorption resonance. We have assumed in our two-state model that the ground state $|a\rangle$ is the only lower state accessible from $|b\rangle$ by spontaneous decay. If we remove this restriction and allow other such states $|d\rangle$, etc., which may or may not be degenerate with $|a\rangle$, then we must include terms like

$$\alpha_d(t)|d\rangle \exp\{-i[\omega_d - i\Gamma_d/2)t + \delta_d]\}$$

in our assumed superposition state (3). Continuing with our assumption of a weak incident field we suppose that the steady-state probabilities $|\alpha_d(t)|^2$, etc., are, like $|\alpha_a(t)|^2$, substantially unchanged by the perturbation, so that condition $|\alpha_a(t)|^2 \approx 1$ of Eq. (3) is replaced by $|\alpha_a(t)|^2 + |\alpha_d(t)|^2 + \cdots \approx 1$. The calculated expectation value of $\hat{\mathbf{D}}$, and therefore the emitted radiation field, will now contain additional terms representing the spontaneous emission from the upper state $|b\rangle$ to the lower states $|d\rangle$, etc. Now these additional terms will contain phase factors $\exp(-i\delta_d)$, etc. These phase factors have their origins in the detailed mechanisms of the relaxation processes responsible for maintaining the steady-state populations, and must therefore be regarded as unknown random variables as far as our analysis is concerned. We conclude that the phases of the additional spontaneous emissions are unrelated to the phase of the incident optical field, i.e., these spontaneous emissions are incoherent. This conclusion is supported by the QED treatment.[6] It is usual to refer to the incoherent spontaneous emission process as Raman scattering, while the coherent spontaneous emission in Eq. (9) is called Rayleigh scattering. We note that in our analysis the coherent Rayleigh-type scattering can occur at or near a resonance as well as in the optical frequency region well below a resonance where the familiar $\omega^4$ intensity law applies.

Turning now to the case of an emission resonance, we obtain similar conclusions when states $|d\rangle$, etc., lower than state $|b\rangle$ are introduced, but in addition to this we must here include the much stronger source of spontaneous emission which occurs from the upper state $|a\rangle$ to all lower states of opposite parity. This emission does not appear explicitly in our analysis, which leads only to the induced field (9), since we used a phenomenological damping constant $\Gamma_a = 0$ for the upper state which expressed the effect of the pumping mechanism in maintaining any population of atoms prepared in state $|a\rangle$. In order to investigate the properties of the spontaneous emission from $|a\rangle$ we must go beyond the phenomenological treatment and consider the details of the pumping cycle in which individual atoms undergo spontaneous emission from state $|a\rangle$ to state $|b\rangle$ and are then pumped back to $|a\rangle$. Now with our assumption of a weak incident field the pumping cycle will be uninfluenced by, and independent of, the incident field. There will therefore be no definite phase relationship between the spontaneous emission field from $|a\rangle$ and the incident field: the fields are incoherent with one another. We note that for the weak incident fields assumed in our analysis the spontaneous emission from state $|a\rangle$, for a single atom, will be very much stronger than the induced emission from state $|a\rangle$; although, as we shall see below, the coherence property of the latter can lead to a "beaming" of the induced emissions from an assembly of atoms so that at large distances the induced field is the dominant one.

### C. Directional properties of the induced field

The induced field propagates outwards from the atom with an angular distribution given by the dipole factor in square brackets in Eq. (9). This is to be compared with the incident field which propagates in a single plane-wave mode of wave vector $\mathbf{k}$. When fields are confined to an indefinitely large volume, or indeed to a volume $V$ of macroscopic dimensions, the mode density for plane waves, $\rho(\omega)$, takes the form of a quasicontinuum: $\rho(\omega) = V\omega^2/(\pi^2 c^3)$, as is well known. This means that if we expand the dipole-like induced field in plane wave modes we will find only an infinitesimal fraction of the power in the particular mode of the incident field. How then can an induced field of stimulated emission lead to amplification of the incident light? The question becomes more acute when we examine the phase of the induced field at points $\mathbf{r}$ in the forward direction from the atom, i.e., for $(\mathbf{r} - \mathbf{r}')$ in the direction of $\mathbf{k}$. We can then put

$\mathbf{n} \cdot \mathbf{e} = 0$ and $\mathbf{k} \cdot \mathbf{r}' + kR = \mathbf{k} \cdot \mathbf{r}$ in Eq. (9) to obtain the forward field:

$$\mathbf{E}_f(\mathbf{r}, t) = \pm (\omega^2/4\pi\epsilon_0 c^2 R) E_0 \mathbf{e} X \cos(\omega t - \mathbf{k} \cdot \mathbf{r} - \alpha). \quad (10)$$

At the peak of an emission resonance we take the minus sign and put $\omega = \omega_b$ to give $\alpha = \pi/2$. The forward emission is therefore not in phase with the incident light; it leads by $\pi/2$ !

To see how amplification of the incident light can occur we shall have to investigate the interference effects between the induced emissions from different atoms in a sample.

## IV. INDUCED FIELD FROM A SLAB

Here we investigate the resultant induced field emitted by the atoms in a thin slab of material placed at right angles to the direction of propagation of an incident plane wave. This is the geometry normally considered in refractive index theory. We shall first consider the idealised case of a plane wave falling on an infinite slab of atoms, and then go on to consider the physically real case of an aperture-limited plane wave falling on a finite slab.

### A. Infinite slab

Consider an optically thin infinite plane slab of noninteracting atoms placed normal to the incident light (Fig. 1). We shall assume that the atoms are stationary and uniformly distributed in the slab. Let $N$ be the number of atoms per unit volume. The induced field at a point $P$ is found by summing the contributions from all atoms in the slab. This leads to a very well-known interference problem which was first solved by Fresnel for scalar waves using his method of half-period zones. We do not give the details here for the method is described in many standard textbooks.[7] The result can be stated as follows: the resultant field at a point $P$ a distance $R >> \lambda$ from the slab of thickness $d$ is[8]

$$\mathbf{E}_p(\mathbf{r}, t) = [i] R N \lambda d \, \mathbf{E}_f(\mathbf{r}, t).$$

$\mathbf{E}_f(\mathbf{r}, t)$ is the field at $P$ produced by forward emission from a single atom in the slab. This is given by Eq. (10). The symbol $[i]$ denotes a phase lag of $\pi/2$. Thus we can write

$$\mathbf{E}_p(\mathbf{r}, t) = \pm \mathbf{e} E_0 (N\pi d/\epsilon_0 \lambda) X \times \cos(\omega t - \mathbf{k} \cdot \mathbf{r} - \alpha - \pi/2). \quad (11)$$

Equation (11) has the form of a plane wave propagating in the same plane wave mode as the incident wave. At the peak of a resonance we have $\alpha = \pi/2$ and the stimulated emission (take the $-$ sign) is now in phase with the incident light. We can now understand how light amplification occurs: the dipole-like induced fields from all atoms in the slab interfere to construct a forward-propagating plane wave of just the right phase to provide amplification of the incident stimulating beam.[9] Similarly for an absorption resonance (take the + sign) we obtain a forward-propagating plane wave of spontaneous emission which, at the peak of the resonance, is in antiphase with the incident light, and this will give rise to attenuation.

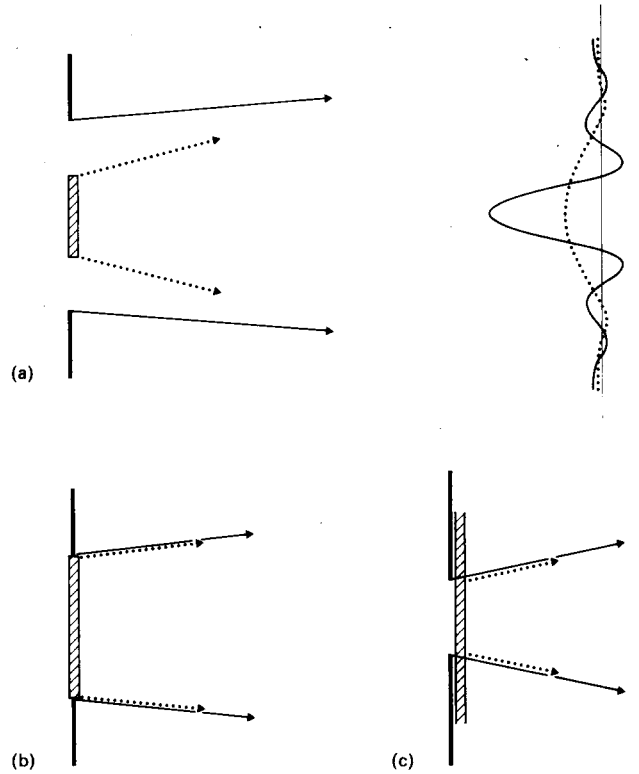


Fig. 3. Apertures and finite slabs with forward-diffraction cones indicated. In (a) the forward emission from the slab enters an angular range greater than that covered by the diffracted incident wave. The amplitude and phase variations across the emitted wave front do not match those of the incident wave, as indicated in the far-field amplitude patterns (drawn for stimulated emission), and so the amplification is distorted. In (b) and (c) the diffraction cones (nearly) coincide, and undistorted amplification or attenuation can occur.

These considerations lead in an obvious way to a refractive index for the medium.

### B. Finite slab

In physically realizable situations we have to deal with finite slabs and aperture-limited plane waves as in Fig. 3(a). The interference problem for the emitting atoms then becomes, essentially, a Fresnel diffraction problem determined by the boundaries of the slab, and the idea of a forward direction for emission has to be replaced by the idea of a diffraction cone of forward directions for the slab. Similarly, the incident plane wave is spread over a range of plane wave modes due to diffraction at the apertures, and the idea of a forward direction for the incident wave has to be replaced by the idea of a diffraction cone covering a finite range of plane wave modes. Undistorted amplification (or attenuation) of the diffraction-limited incident wave can now occur only if the two diffraction problems coincide; that is to say, if the forward diffraction cone for the slab coincides with the forward diffraction cone defined by the apertures. See Figs. 3(b) and 3(c). Such coincidence is found in a laser when the amplifying medium fills the whole of the optical cavity.

### C. Incoherence in the induced field

Within the assumptions of this section the induced field is completely coherent and propagates in the forward

modes. One of these assumptions, which was made implicitly in the acceptance of Eq. (10), concerns the description of the emitting medium: we took the atomic density to be uniform and continuous. In view of the discrete nature of any material medium, and the random atomic motions inherent in the state of thermal equilibrium, we should regard the atomic density as a random function of position and time, and the symbol $N$ that we have used, as the ensemble average of this function. The fluctuations about the ensemble average give rise to an incoherent component of the induced field,[10] in addition to the coherent one that we have calculated from the ensemble average. The incoherent component is not "beamed" into the forward modes but appears in lateral directions with a power which is proportional to the total number of atoms present in the sample, and which, for an absorption resonance, provides the power balance to compensate for the attenuation of the incident beam. From a microscopic point of view the origin of the incoherence can be found from inspection of Eq. (9) which reveals that for emission directions $\mathbf{n}$ outside the forward direction, the phase is not independent of the random position $\mathbf{r}'$ of the emitting atom. From a macroscopic point of view we are dealing with the scattering of the incident light by density fluctuations as we have indicated. This is a well-developed subject which we do not need to pursue here, but which is relevant to our conclusions; for we have now identified a source of incoherence in addition to the Raman scattering and related processes at the quantum level which we discussed in Sec. III.

## V. SUMMARY

Stimulated and spontaneous emissions have the following properties.[11] The stimulated emission from a single atom is coherent with the incident stimulating beam. The spontaneous emission from an atom in absorption resonance with an incident field is also coherent with the incident field provided the atom returns to its original quantum state. Other spontaneous emissions are incoherent. For electric dipole transitions the coherent emission from a single atom, whether stimulated or spontaneous, has a dipole-like angular distribution. For coherent emission from a macroscopic sample, the dipole-like angular distribution can be drastically modified by interference effects. In particular, when the diffraction limiting of the incident light coincides with the boundaries of the sample, the coherent emissions enter the mode or modes of the incident light allowing amplification by stimulated emission or attenuation by absorption and spontaneous emission. Thermal density fluctuations in the sample degrade the coherence of the emitted light and give rise to a lateral dipole-like field which is incoherent with the incident beam.

[1]Semiclassical theory has of course been widely used in the theory of the laser as in, for example, the classic paper by W. E. Lamb, Jr., Phys. Rev. **134,** A1429 (1964). In most of these works the medium is described by a macroscopic polarization obeying Maxwell's equations, and the detailed mechanisms by which coherence and directional properties are established are hidden in the formalism.

[2]The use of damping Hamiltonians is discussed by R. Loudon, *The Quantum Theory of Light* (Clarendon, Oxford, 1973), Chap. 8.; and by W. Louisell, *Radiation and Noise in Quantum Electronics* (McGraw-Hill, New York, 1964) Chap. 7.

[3]Magnetic degeneracy of the upper level does not cause difficulty for we can regard the state $|b\rangle$ as that linear superposition of the degenerate magnetic substates which can be excited from the ground state by the polarized electric vector of the incident light.

[4]The self-field of an accelerating electron is discussed in a number of textbooks on classical electrodynamics, for example, W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley, Reading, PA, 1964), Chap. 21. The mechanism by which the self-field can cause spontaneous decay of excited atomic states had been investigated by G. W. Series, in *Optical Pumping and Atomic Line Shape,* edited by T. Skalinsky (Panstwowe Wydawnictwo Namkowe, Warszwa, 1969), pp. 25–41. See also papers by W. L. Lama and L. Mandel and by R. K. Bullough, in *Coherence and Quantum Optics,* edited by L. Mandel and E. Wolf (Plenum, New York, 1973).

[5]Semiclassical electrodynamics has traditionally used a number of techniques, based on the correspondence principle, for calculating the radiation field emitted by an atom. See, for example, E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University, Cambridge, 1963), Secs. 4 and 5. Our use of the expectation value of $\hat{\mathbf{D}}$ has a strong heuristic appeal and, within the assumptions of this section, leads to the same results as the other techniques. This topic is the subject of current research in the field of semiclassical electrodynamics. See papers under the heading: "Quantum Electrodynamics and Alternative Theories, I, II, and III," in *Coherence and Quantum Optics,* edited by L. Mandel and E. Wolf (Plenum, New York, 1973).

[6]See W. Heitler, *The Quantum Theory of Radiation* (Clarendon, Oxford, 1966), Chap. V.

[7]See, for example, R. W. Ditchburn, *Light* (Blackie, London, 1963), Chap. 6. See also R. de L. Kronig, J. Opt. Soc. Am. **12,** 547 (1926).

[8]This result can be justified for vector wave fields by direct integration of the induced fields from all atoms in the slab.

[9]The interference problem that we have described appears also in the QED treatment where the corresponding matrix element is summed over the contributions of the different atoms before it is squared to obtain the total transition rate. The procedure is illustrated in R. Loudon, *The Quantum Theory of Light* (Clarendon, Oxford, 1973), pp. 319–321. Although his treatment refers specifically to third-harmonic generation, the procedure for summing over all atomic positions is quite general for coherent emission processes, provided the appropriate expression for $\Delta\mathbf{k}$ (his notation) is used.

[10]This is the scattering of light by density fluctuations; see, for example, N. G. van Kampen, *Quantum Optics,* edited by R. J. Glauber (Academic, New York, 1969).

[11]At the risk of repetition, we emphasize that our treatment of absorption from a ground state $|a\rangle$ is restricted to weakly excited systems ($|\alpha_a(t)|^2 \approx 1$). When the optical excitation is strong, the coherent field that we have calculated using $\langle t|\mathbf{D}|t\rangle$ does not give the complete spontaneous emission field. I. R. Senitsky [Phys. Rev. **111,** 3 (1958)] has shown that the spontaneous emission from a strongly excited two-level system contains an incoherent component as well. In the limit of weak excitation, however, the intensity of this incoherent component is negligible compared with that of the coherent field.