

## EE-559 – Deep learning

### 5. Losses, optimization, and initialization

François Fleuret

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# Cross-entropy

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As we will see, the criterion of choice for classification is the cross-entropy.

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$$\begin{aligned} \log \mu_W(w \mid \mathcal{D} = \mathbf{d}) &= \log \frac{\mu_{\mathcal{D}}(\mathbf{d} \mid W = w) \mu_W(w)}{\mu_{\mathcal{D}}(\mathbf{d})} \\ &= \log \mu_{\mathcal{D}}(\mathbf{d} \mid W = w) + \log \mu_W(w) - \log Z \\ &= \sum_n \log \mu_{\mathcal{D}}(x_n, y_n \mid W = w) + \log \mu_W(w) - \log Z \\ &= \sum_n \log P(Y = y_n \mid X = x_n, W = w) + \log \mu_W(w) - \log Z' \\ &= \sum_n \log \left( \frac{\exp f_{y_n}(x; w)}{\sum_k \exp f_k(x; w)} \right) + \log \mu_W(w) - \log Z'. \end{aligned}$$

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$$\mathcal{L}(w) = -\frac{1}{N} \sum_{n=1}^N \log \left( \underbrace{\frac{\exp f_{y_n}(x_n; w)}{\sum_k \exp f_k(x_n; w)}}_{P_w(Y=y_n|X=x_n)} \right).$$

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So  $\mathcal{L}$  above is the average of the cross-entropy between the deterministic “true” posterior  $\delta_{y_n}$  and the estimated  $P_w(Y = \cdot | X = x_n)$ .

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>>> f = Variable(Tensor([[ -1,  -3,  4], [ -3,  3, -1]]))
>>> target = Variable(torch.LongTensor([0, 1]))
>>> criterion = torch.nn.CrossEntropyLoss()
>>> criterion(f, target)
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prints

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Variable containing:
  2.5141
[torch.FloatTensor of size 1]
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and indeed

$$-\frac{1}{2} \left( \log \frac{e^{-1}}{e^{-1} + e^{-3} + e^4} + \log \frac{e^3}{e^{-3} + e^3 + e^{-1}} \right) \simeq 2.5141.$$

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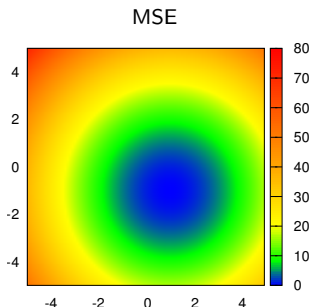
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The range of values is 0 for perfectly classified samples,  $\log(C)$  if the posterior is uniform, and up to  $+\infty$  if the posterior distribution is “worst” than uniform.

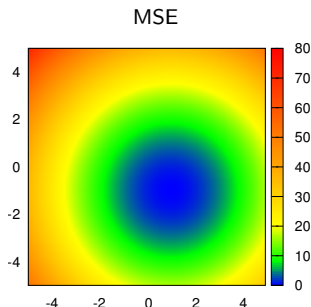


Let's consider the loss for a single sample in a two-class problem, with a predictor with two output values. The  $x$  axis here is the activation of the correct output unit, and the  $y$  axis is the activation of the other one.

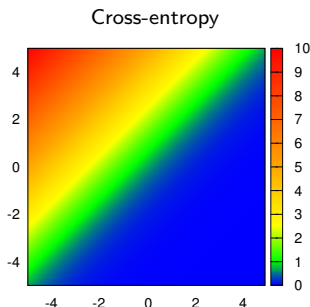


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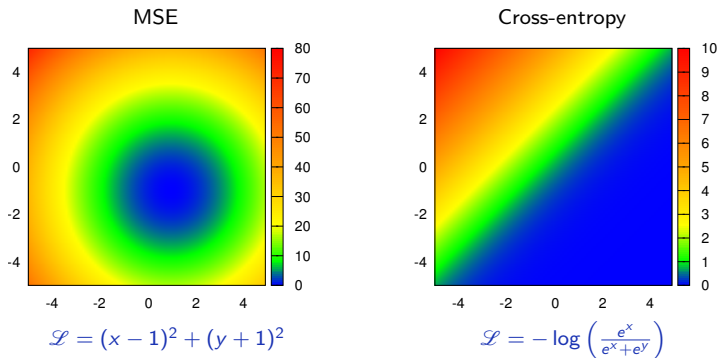


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MSE incorrectly penalizes outputs which are perfectly valid for prediction, contrary to cross-entropy.

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So combining it with the “log soft-max” function

$$(\alpha_1, \dots, \alpha_C) \mapsto \left( \log \frac{\exp \alpha_1}{\sum_k \exp \alpha_k}, \dots, \log \frac{\exp \alpha_C}{\sum_k \exp \alpha_k} \right)$$

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Hence, if a network should compute log-probabilities, it may have a `torch.nn.LogSoftmax` final layer, and be trained with `torch.nn.NLLLoss`.

The mapping

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>>> y = Variable(Tensor([[ -10,  -10,  10,  -5 ],
                          [  3,   0,   0,   0 ],
                          [  1,   2,   3,   4 ]]))
>>> f = torch.nn.Softmax(1)
>>> f(y)
```

prints

```
Variable containing:
  2.0612e-09  2.0612e-09  1.0000e+00  3.0590e-07
  8.7005e-01  4.3317e-02  4.3317e-02  4.3317e-02
  3.2059e-02  8.7144e-02  2.3688e-01  6.4391e-01
[torch.FloatTensor of size 3x4]
```



PyTorch provides many other criteria, among which

- `torch.nn.MSELoss`
- `torch.nn.CrossEntropyLoss`
- `torch.nn.NLLLoss`
- `torch.nn.L1Loss`
- `torch.nn.NLLLoss2d`
- `torch.nn.MultiMarginLoss`

## Stochastic gradient descent

So far, to minimize a loss of the form

$$\mathcal{L}(w) = \sum_{n=1}^N \underbrace{\ell(f(x_n; w), y_n)}_{\ell_n(w)}$$

we have considered the gradient-descent algorithm

$$w_{t+1} = w_t - \eta \nabla \mathcal{L}(w_t).$$

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- It takes time to compute (more exactly **all our time!**).
- It is an empirical estimation of an hidden quantity, and any partial sum would similarly be an unbiased empirical estimate, although more noisy.
- It is computed incrementally

$$\nabla \mathcal{L}(w_t) = \sum_{n=1}^N \nabla \ell_n(w_t),$$

and when we compute  $\ell_n$ , we have already computed  $\ell_1, \dots, \ell_{n-1}$ , and we could have a better estimate of  $w^*$  than  $w_t$ .

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So instead of summing over all the samples and moving by  $\eta$ , we can visit only  $M$  samples and move by  $K\eta$ , which would cut the computation by  $K$ .

Although this is an ideal case, there is redundancy in practice that results in similar behaviors.

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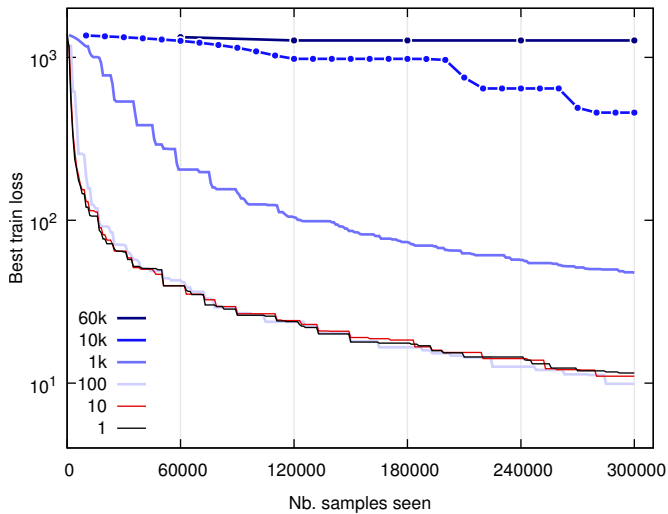
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**The stochastic behavior of this procedure helps evade local minima.**

Mini-batch size and loss reduction (MNIST)

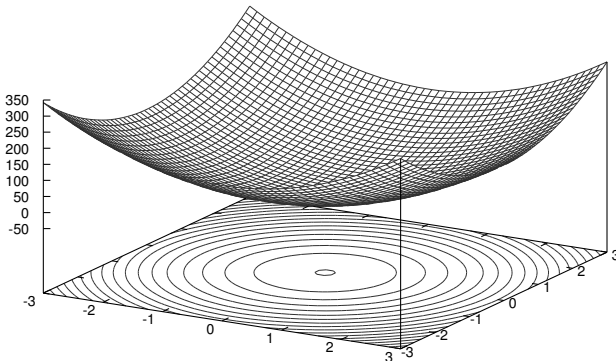




## Limitation of the gradient descent

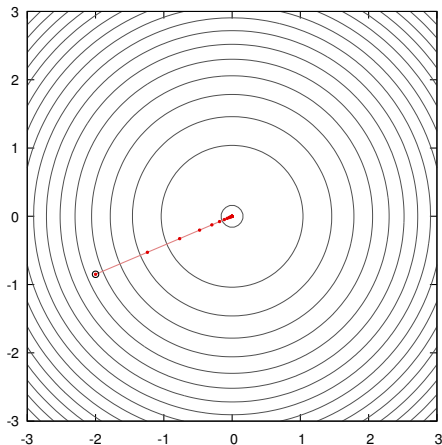
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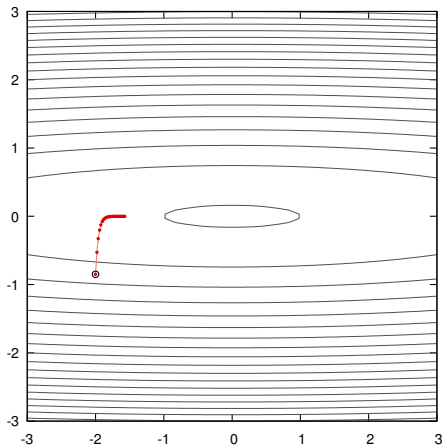
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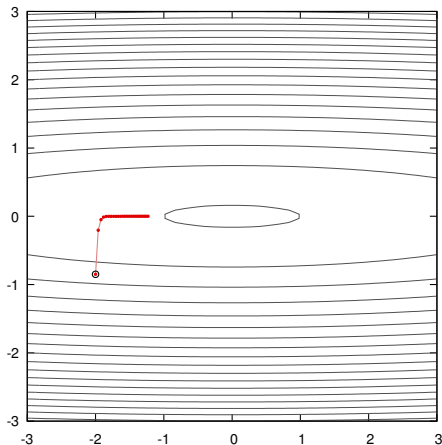
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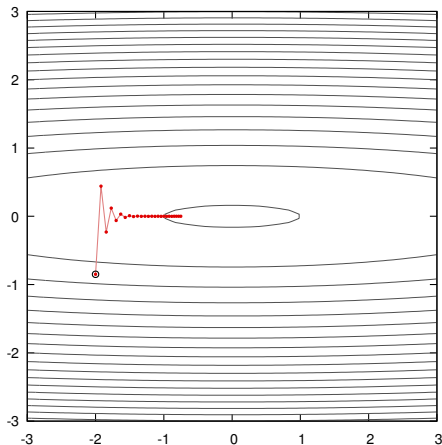
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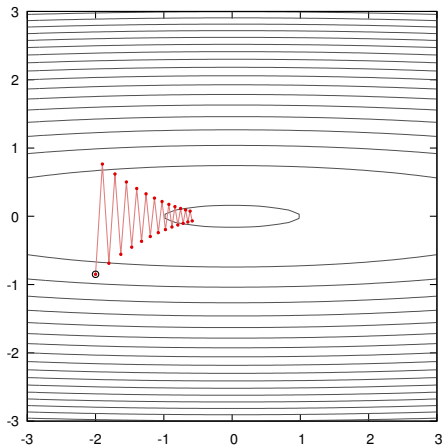
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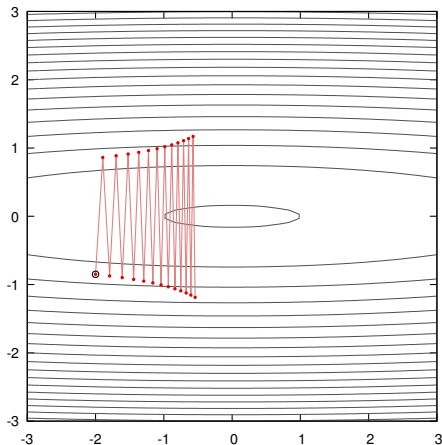
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Deep-learning generally relies on a smarter use of the gradient, using statistics over its past values to make a “smarter step” with the current one.

## Momentum and moment estimation

The “vanilla” mini-batch stochastic gradient descent (SGD) consists of

$$w_{t+1} = w_t - \eta g_t,$$

where

$$g_t = \sum_{b=1}^B \nabla \ell_{n(t,b)}(w_t)$$

is the gradient summed over a mini-batch.

The first improvement is the use of a “momentum” to add inertia in the choice of the step direction

$$\begin{aligned}u_t &= \gamma u_{t-1} + \eta g_t \\w_{t+1} &= w_t - u_t.\end{aligned}$$

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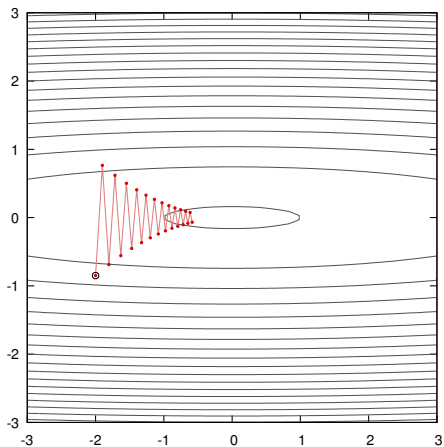
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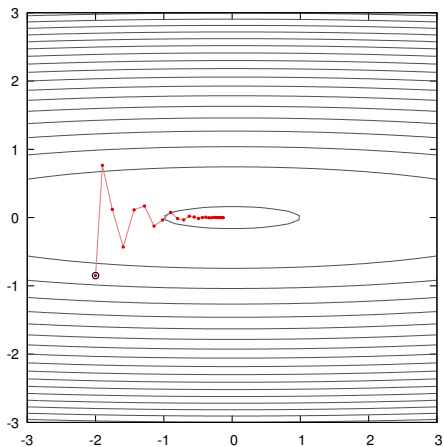
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- it dampens oscillations in narrow valleys.

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Another class of methods exploits the statistics over the previous steps to compensate for the anisotropy of the mapping.

The Adam algorithm uses moving averages of each coordinate and its square to rescale each coordinate separately.

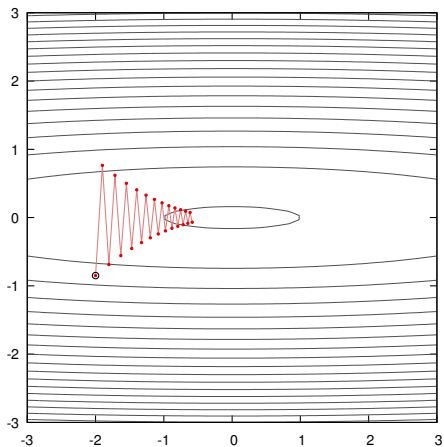
The update rule is, **on each coordinate separately**

$$\begin{aligned}m_t &= \beta_1 m_{t-1} + (1 - \beta_1) g_t \\ \hat{m}_t &= \frac{m_t}{1 - \beta_1} \\ v_t &= \beta_2 v_{t-1} + (1 - \beta_2) g_t^2 \\ \hat{v}_t &= \frac{v_t}{1 - \beta_2} \\ w_{t+1} &= w_t - \frac{\eta}{\sqrt{\hat{v}_t} + \epsilon} \hat{m}_t\end{aligned}$$

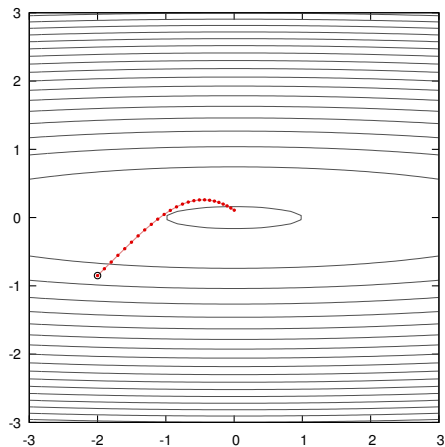
(Kingma and Ba, 2014)

This can be seen as a combination of momentum, with  $\hat{m}_t$ , and a per-coordinate re-scaling with  $\hat{v}_t$ .

$$\eta = 5.0e - 2$$



Adam,  $\beta_1 = 0.9, \beta_2 = 0.999, \epsilon = 1e-8, \eta = 1.0e-1$





These two core strategies have been used in multiple incarnations:

- Nesterov's accelerated gradient,
- Adagrad,
- Adadelata,
- RMSprop,
- AdaMax,
- Nadam ...

`torch.optim`

PyTorch includes the many variants of the stochastic gradient descent.

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We can implement the standard SGD as follows

```
for e in range(25):
    for b in range(0, train_input.size(0), mini_batch_size):
        output = model(train_input.narrow(0, b, mini_batch_size))
        loss = criterion(output, train_target.narrow(0, b, mini_batch_size))
        model.zero_grad()
        loss.backward()
        for p in model.parameters(): p.data -= eta * p.grad.data
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```

which can be re-written with the `torch.optim` package

```
optimizer = torch.optim.SGD(model.parameters(), lr = eta)

for e in range(25):
    for b in range(0, train_input.size(0), mini_batch_size):
        output = model(train_input.narrow(0, b, mini_batch_size))
        loss = criterion(output, train_target.narrow(0, b, mini_batch_size))
        model.zero_grad()
        loss.backward()
        optimizer.step()
```

An optimizer has an internal state to keep quantities such as moving averages, and operates on an iterator over `Parameter`s.

Values specific to the optimizer can be specified to its constructor, and the `step` method updates the internal state according to the `grad` attributes of the `Parameter`s, and updates the latter according to the internal state.

- `torch.optim.SGD` (momentum, and Nesterov's algorithm),
- `torch.optim.Adam`
- `torch.optim.Adadelta`
- `torch.optim.Adagrad`
- `torch.optim.RMSprop`
- `torch.optim.LBFGS`
- ...

An optimizer can also operate on several iterators, each corresponding to a group of `Parameter`s that should be handled similarly. For instance, different layers may have different learning rates or momentums.

So to use Adam with its default setting instead of vanilla SGD, we just have to change

```
optimizer = optim.SGD(model.parameters(), lr = eta)
```

into

```
optimizer = optim.Adam(model.parameters(), lr = eta)
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The learning rate may have to be different if the functional was not properly scaled.



An example putting all this together

We now have the tools to define a deep network:

- fully connected layers,
- convolutional layers,
- pooling layers,
- ReLU.

And we have the tools to optimize it:

- Loss,
- back-propagation,
- stochastic gradient descent.

The only piece missing is the policy to initialize the parameters.

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The only piece missing is the policy to initialize the parameters.

PyTorch initializes parameters with default rules when modules are created. They normalize weights according to the layer sizes (Glorot and Bengio, 2010) and behave usually very well. We will come back to this.

```

from torch import cuda, nn, optim
from torch.nn import functional as F
from torch.autograd import Variable
from torchvision import datasets

class Net(nn.Module):
    def __init__(self):
        super(Net, self).__init__()
        self.conv1 = nn.Conv2d(1, 32, kernel_size=5)
        self.conv2 = nn.Conv2d(32, 64, kernel_size=5)
        self.fc1 = nn.Linear(256, 200)
        self.fc2 = nn.Linear(200, 10)

    def forward(self, x):
        x = F.relu(F.max_pool2d(self.conv1(x), kernel_size=3, stride=3))
        x = F.relu(F.max_pool2d(self.conv2(x), kernel_size=2, stride=2))
        x = F.relu(self.fc1(x.view(-1, 256)))
        x = self.fc2(x)
        return x

```

```

train_set = datasets.MNIST('./data/mnist/', train = True, download = True)
train_input = Variable(train_set.train_data.view(-1, 1, 28, 28).float())
train_target = Variable(train_set.train_labels)

model, criterion = Net(), nn.CrossEntropyLoss()

if cuda.is_available():
    model.cuda()
    criterion.cuda()
    train_input, train_target = train_input.cuda(), train_target.cuda()

mu, std = train_input.data.mean(), train_input.data.std()
train_input.data.sub_(mu).div_(std)

lr, nb_epochs, batch_size = 1e-1, 10, 100

optimizer = optim.SGD(model.parameters(), lr = lr)

for k in range(nb_epochs):
    for b in range(0, train_input.size(0), batch_size):
        output = model(train_input.narrow(0, b, batch_size))
        loss = criterion(output, train_target.narrow(0, b, batch_size))
        model.zero_grad()
        loss.backward()
        optimizer.step()

```

## $L_2$ and $L_1$ penalties

We have motivated the use of a loss with a Bayesian formulation combining the probability of the data given the model and the probability of the model

$$\log \mu_W(w \mid \mathcal{D} = \mathbf{d}) = \log \mu_{\mathcal{D}}(\mathbf{d} \mid W = w) + \log \mu_W(w) - \log Z.$$

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$$\lambda \|w\|_2^2.$$



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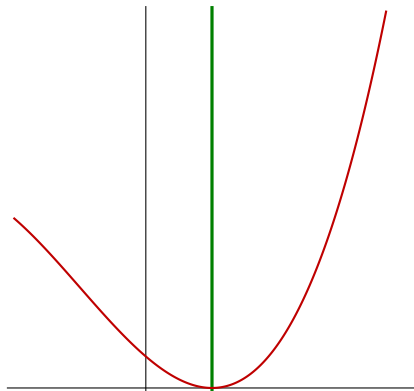
$$\lambda \|w\|_2^2.$$

Since this penalty is convex, its sum with a convex functional is convex.

This is called the  $L_2$  regularization, or “weight decay” in the artificial neural network community.

Increasing the  $\lambda$  parameter moves the optimal closer to 0, and away from the optimal for the loss alone.

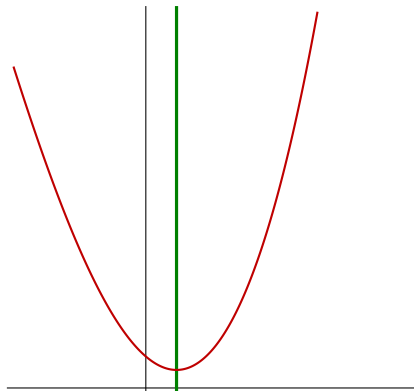
Since the derivative of  $\|x\|_2^2$  is zero at zero, the optimal will never move there if it was not already there.



$$(x - 1)^2 + \frac{1}{6}(x - 1)^3$$

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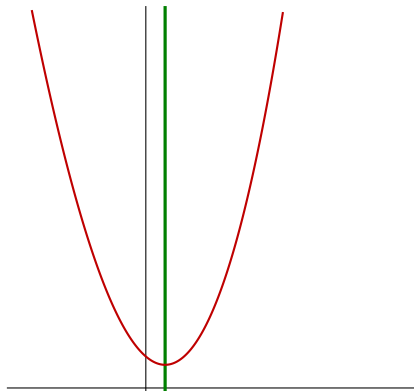
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$$(x - 1)^2 + \frac{1}{6}(x - 1)^3 + x^2$$

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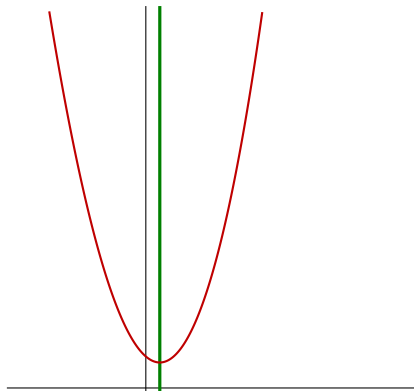
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$$(x - 1)^2 + \frac{1}{6}(x - 1)^3 + 2x^2$$

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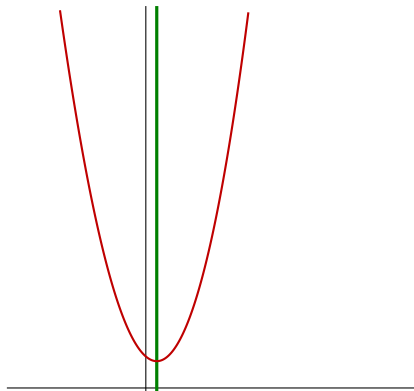
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$$(x-1)^2 + \frac{1}{6}(x-1)^3 + 3x^2$$

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$$(x - 1)^2 + \frac{1}{6}(x - 1)^3 + 4x^2$$

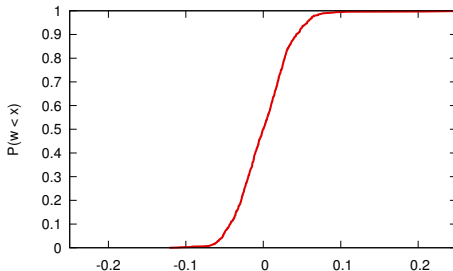
Convnet trained on MNIST with 1,000 samples and a  $L_2$  penalty.

$\lambda$	Error	
	Train	Test
0.000	0.000	0.064
0.001	0.000	0.063
0.002	0.000	0.064
0.004	0.005	0.065
0.010	0.022	0.075
0.020	0.048	0.101

```
output = model(train_input.narrow(0, b, bs))
loss = criterion(output, train_target.narrow(0, b, bs))

for p in model.parameters():
    loss += lambda * p.pow(2).sum()

model.zero_grad()
loss.backward()
optimizer.step()
```



$\lambda = 0.000$

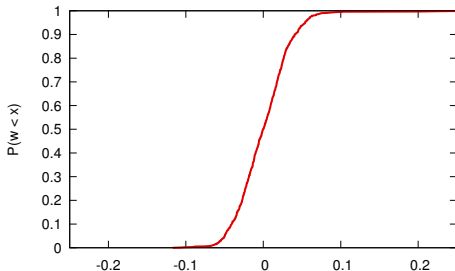
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model.zero_grad()
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```



$\lambda = 0.001$



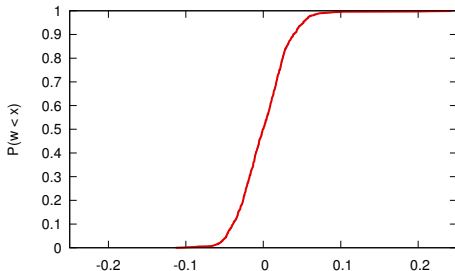
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output = model(train_input.narrow(0, b, bs))
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for p in model.parameters():
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model.zero_grad()
loss.backward()
optimizer.step()
```



$\lambda = 0.002$

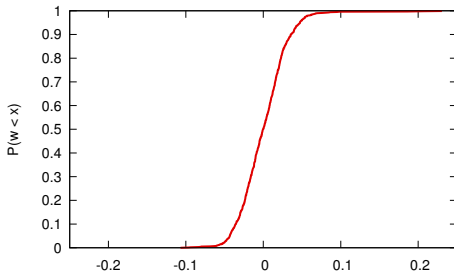
Convnet trained on MNIST with 1,000 samples and a  $L_2$  penalty.

$\lambda$	Error	
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output = model(train_input.narrow(0, b, bs))
loss = criterion(output, train_target.narrow(0, b, bs))

for p in model.parameters():
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model.zero_grad()
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```



$\lambda = 0.004$

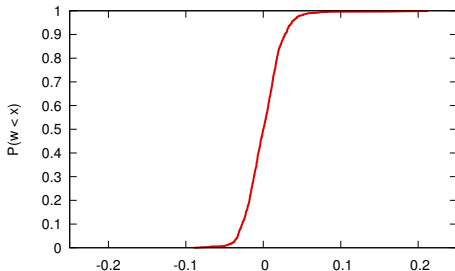
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$\lambda = 0.010$

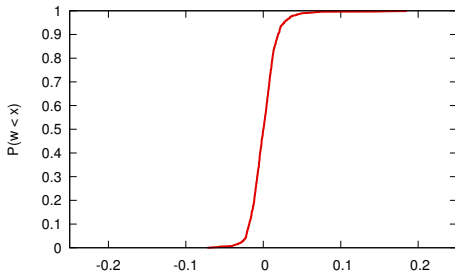
Convnet trained on MNIST with 1,000 samples and a  $L_2$  penalty.

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for p in model.parameters():
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model.zero_grad()
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```



$\lambda = 0.020$

We can apply the exact same scheme with a Laplace prior

$$\begin{aligned}\mu(w) &= \frac{1}{(2b)^D} \exp\left(-\frac{\|w\|_1}{b}\right) \\ &= \frac{1}{(2b)^D} \exp\left(-\frac{1}{b} \sum_{d=1}^D |w_d|\right),\end{aligned}$$

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which results in a penalty term of the form

$$\lambda \|w\|_1.$$

This is the  $L_1$  regularization. As for the  $L_2$ , this penalty is convex, and its sum with a convex functional is convex.

An important property of the  $L_1$  penalty is that, if  $\mathcal{L}$  is convex, and

$$w^* = \underset{w}{\operatorname{argmin}} \mathcal{L}(w) + \lambda \|w\|_1$$

then

$$\forall d, \left| \frac{\partial \mathcal{L}}{\partial w_d}(w^*) \right| < \lambda \Rightarrow w_d^* = 0.$$



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then

$$\forall d, \left| \frac{\partial \mathcal{L}}{\partial w_d}(w^*) \right| < \lambda \Rightarrow w_d^* = 0.$$

In practice it means that this penalty pushes some of the variables to zero, but contrary to the  $L_2$  penalty they actually move and remain there.

The  $\lambda$  parameter controls the sparsity of the solution.

With the  $L_1$  penalty, the update rule becomes

$$w_{t+1} = w_t - \eta g_t - \lambda \operatorname{sign}(w_t),$$

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where  $\text{sign}$  is applied per-component. This is almost identical to

$$\begin{aligned} w'_t &= w_t - \eta g_t \\ w_{t+1} &= w'_t - \lambda \text{sign}(w'_t). \end{aligned}$$

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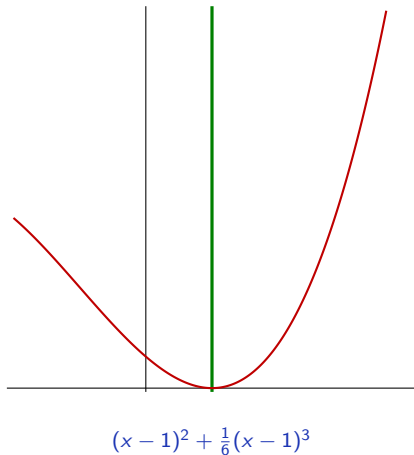
While this is not a problem in principle, since  $w_t$  will fluctuate around zero, it can be an issue if the zeroed weights are handled in a specific manner (e.g. sparse coding to reduce memory footprint or computation).

The **proximal operator** takes care of preventing parameters from “crossing zero”, by adapting  $\lambda$  when it is too large

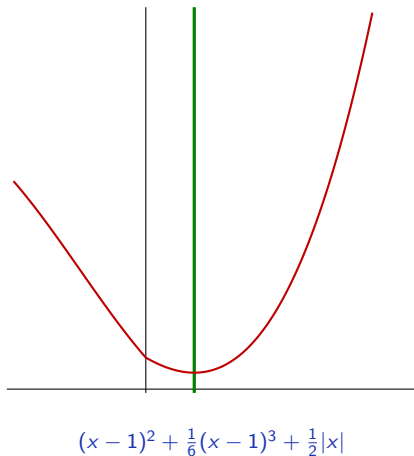
$$\begin{aligned}w'_t &= w_t - \eta g_t \\w_{t+1} &= w'_t - \min(\lambda, |w'_t|) \odot \text{sign}(w'_t).\end{aligned}$$

where  $\min$  is component-wise, and  $\odot$  is the Hadamard component-wise product.

Increasing the  $\lambda$  parameter moves the optimal closer to 0, and away from the optimal for the loss without penalty.

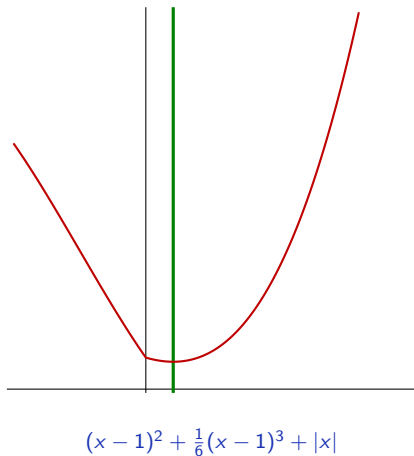


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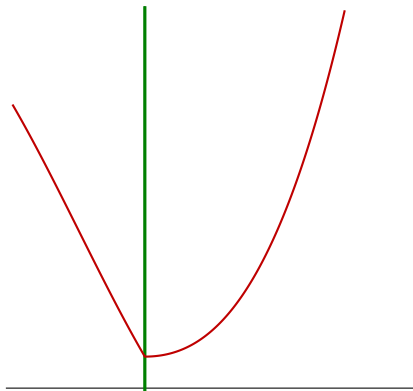




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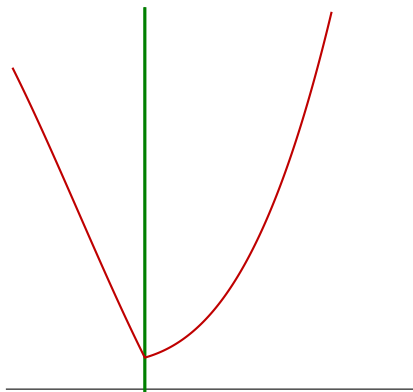


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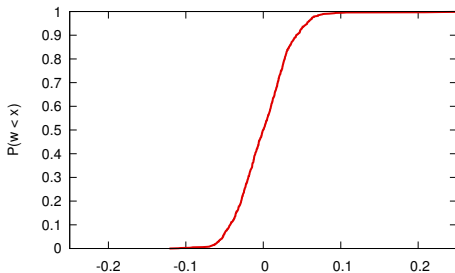
Convnet trained on MNIST with 1,000 samples and a  $L_1$  penalty.

$\lambda$	Error	
	Train	Test
0.00000	0.000	0.064
0.00001	0.000	0.063
0.00002	0.000	0.067
0.00005	0.004	0.068
0.00010	0.087	0.128
0.00020	0.057	0.101
0.00050	0.496	0.516

```
output = model(train_input.narrow(0, b, bs))
loss = criterion(output, train_target.narrow(0, b, bs))

model.zero_grad()
loss.backward()
optimizer.step()

for p in model.parameters():
    p.data -= p.data.sign() * p.data.abs().clamp(max = lambda_1)
```



$\lambda = 0.00000$

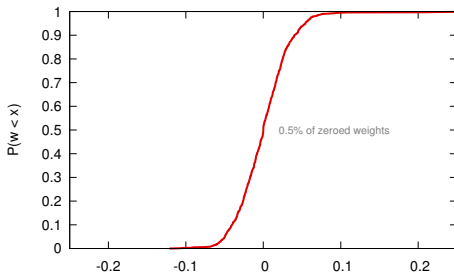
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$\lambda = 0.00001$

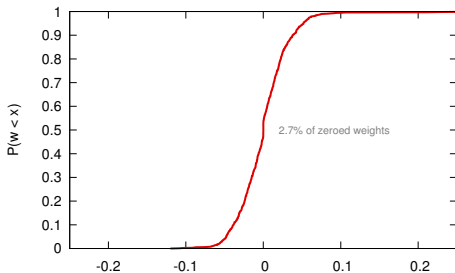
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0.00050	0.496	0.516

```
output = model(train_input.narrow(0, b, bs))
loss = criterion(output, train_target.narrow(0, b, bs))

model.zero_grad()
loss.backward()
optimizer.step()

for p in model.parameters():
    p.data -= p.data.sign() * p.data.abs().clamp(max = lambda_1)
```



$\lambda = 0.00002$

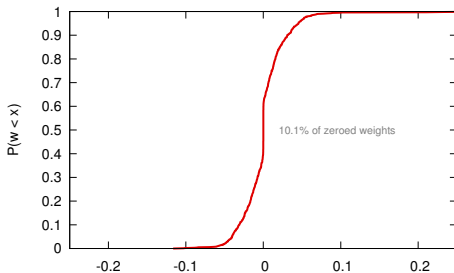
Convnet trained on MNIST with 1,000 samples and a  $L_1$  penalty.

$\lambda$	Error	
	Train	Test
0.00000	0.000	0.064
0.00001	0.000	0.063
0.00002	0.000	0.067
0.00005	0.004	0.068
0.00010	0.087	0.128
0.00020	0.057	0.101
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$\lambda = 0.00005$

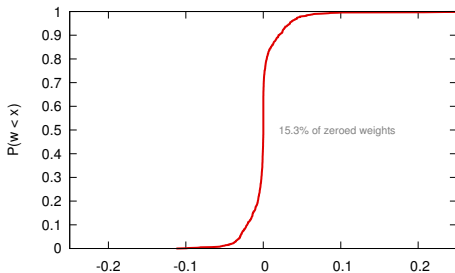
Convnet trained on MNIST with 1,000 samples and a  $L_1$  penalty.

$\lambda$	Error	
	Train	Test
0.00000	0.000	0.064
0.00001	0.000	0.063
0.00002	0.000	0.067
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    p.data -= p.data.sign() * p.data.abs().clamp(max = lambda_1)
```



$\lambda = 0.0001$



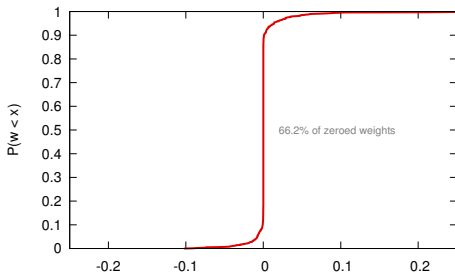
Convnet trained on MNIST with 1,000 samples and a  $L_1$  penalty.

$\lambda$	Error	
	Train	Test
0.00000	0.000	0.064
0.00001	0.000	0.063
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optimizer.step()

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```



$\lambda = 0.0002$

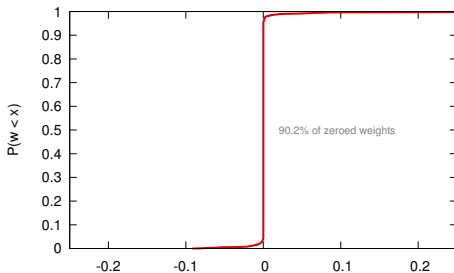
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optimizer.step()

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    p.data -= p.data.sign() * p.data.abs().clamp(max = lambda_1)
```



$\lambda = 0.0005$

Penalties on the weights may be useful when dealing with small models and small data-sets and are still standard when data is scarce.

While they have a limited impact for large-scale deep learning, they may still provide the little push needed to beat baselines.

## Vanishing gradient

Consider the gradient estimation for a standard MLP:

### Forward pass

$$\forall n, x^{(0)} = x, \quad \forall l = 1, \dots, L, \quad \begin{cases} s^{(l)} = w^{(l)}x^{(l-1)} + b^{(l)} \\ x^{(l)} = \sigma(s^{(l)}) \end{cases}$$

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### Forward pass

$$\forall n, \mathbf{x}^{(0)} = \mathbf{x}, \quad \forall l = 1, \dots, L, \quad \begin{cases} \mathbf{s}^{(l)} = \mathbf{w}^{(l)} \mathbf{x}^{(l-1)} + \mathbf{b}^{(l)} \\ \mathbf{x}^{(l)} = \sigma(\mathbf{s}^{(l)}) \end{cases}$$

### Backward pass

$$\begin{cases} \left[ \frac{\partial \ell}{\partial \mathbf{x}^{(L)}} \right] = \nabla_1 \ell(\mathbf{x}^{(L)}) & \left[ \frac{\partial \ell}{\partial \mathbf{s}^{(l)}} \right] = \left[ \frac{\partial \ell}{\partial \mathbf{x}^{(l)}} \right] \odot \sigma'(\mathbf{s}^{(l)}) \\ \text{if } l < L, \left[ \frac{\partial \ell}{\partial \mathbf{x}^{(l)}} \right] = (\mathbf{w}^{(l+1)})^T \left[ \frac{\partial \ell}{\partial \mathbf{s}^{(l+1)}} \right] \end{cases}$$

$$\left[ \left[ \frac{\partial \ell}{\partial \mathbf{w}^{(l)}} \right] \right] = \left[ \frac{\partial \ell}{\partial \mathbf{s}^{(l)}} \right] (\mathbf{x}^{(l-1)})^T \quad \left[ \frac{\partial \ell}{\partial \mathbf{b}^{(l)}} \right] = \left[ \frac{\partial \ell}{\partial \mathbf{s}^{(l)}} \right].$$

We have

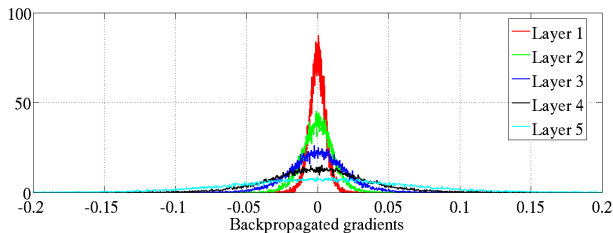
$$\left[ \frac{\partial \ell}{\partial \mathbf{x}^{(l)}} \right] = \left( \mathbf{w}^{(l+1)} \right)^T \left( \sigma'(\mathbf{s}^{(l)}) \odot \left[ \frac{\partial \ell}{\partial \mathbf{x}^{(l+1)}} \right] \right).$$

so the gradient “vanishes” exponentially with the depth if the  $\mathbf{w}$ s are ill-conditioned or the activations are in the saturating domain of  $\sigma$ .

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so the gradient “vanishes” exponentially with the depth if the  $w$ s are ill-conditioned or the activations are in the saturating domain of  $\sigma$ .



(Glorot and Bengio, 2010)



## Weight initialization

The analysis for the weight initialization relies on controlling

$$\mathbb{V}\left(\frac{\partial \ell}{\partial w_{i,j}^{(l)}}\right) \text{ and } \mathbb{V}\left(\frac{\partial \ell}{\partial b_i^{(l)}}\right)$$

so that

- the gradient does not vanish, and
- weights evolve at the same rate across layers during training, and no layer reaches a saturation behavior before others.

We will use that, if  $A$  and  $B$  are independent

$$\mathbb{V}(AB) = \mathbb{V}(A) \mathbb{V}(B) + \mathbb{V}(A) \mathbb{E}(B)^2 + \mathbb{V}(B) \mathbb{E}(A)^2.$$

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$$\mathbb{V}(AB) = \mathbb{V}(A) \mathbb{V}(B) + \mathbb{V}(A) \mathbb{E}(B)^2 + \mathbb{V}(B) \mathbb{E}(A)^2.$$

Notation in the coming slides will drop indexes when variances are identical for all activations or parameters in a layer.

In a standard layer

$$x_i^{(l)} = \sigma \left( \sum_{j=1}^{N_{l-1}} w_{ij}^{(l)} x_j^{(l-1)} + b_i^{(l)} \right)$$

where  $N_l$  is the number of units in layer  $l$ , and  $\sigma$  is the activation function.

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Assuming  $\sigma'(0) = 1$ , and we are in the linear regime

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$$x_i^{(l)} \simeq \sum_{j=1}^{N_{l-1}} w_{i,j}^{(l)} x_j^{(l-1)} + b_i^{(l)}.$$

From which, if both the  $w^{(l)}$ s and  $x^{(l-1)}$ s are centered

$$\begin{aligned} \mathbb{V}(x_i^{(l)}) &\simeq \mathbb{V} \left( \sum_{j=1}^{N_{l-1}} w_{i,j}^{(l)} x_j^{(l-1)} \right) \\ &= \sum_{j=1}^{N_{l-1}} \mathbb{V}(w_{i,j}^{(l)}) \mathbb{V}(x_j^{(l-1)}) \end{aligned}$$

and if the  $b^{(l)}$  are centered, so are the  $x^{(l)}$ s.

So if the  $w_{i,j}^{(l)}$  are sampled i.i.d in each layer, and the  $x_i^{(l)}$  have all the same variance for  $l$  fixed

$$\begin{aligned}\mathbb{V}\left(x^{(l)}\right) &\simeq \sum_{j=1}^{N_{l-1}} \mathbb{V}\left(w^{(l)}\right) \mathbb{V}\left(x^{(l-1)}\right) \\ &= N_{l-1} \mathbb{V}\left(w^{(l)}\right) \mathbb{V}\left(x^{(l-1)}\right) .\end{aligned}$$



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**So we have for the variance of the activations:**

$$\mathbb{V}\left(x^{(l)}\right) \simeq \mathbb{V}\left(x^{(0)}\right) \prod_{q=1}^l N_{q-1} \mathbb{V}\left(w^{(q)}\right).$$

This leads to a first type of initialization

$$\mathbb{V}\left(w^{(l)}\right)=\frac{1}{N_{l-1}}.$$

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In `torch/nn/functions/linear.py`

```
def reset_parameters(self):
    stdv = 1. / math.sqrt(self.weight.size(1))
    self.weight.data.uniform_(-stdv, stdv)
    if self.bias is not None:
        self.bias.data.uniform_(-stdv, stdv)
```

There is a slight mistake here: the standard deviation of  $\mathcal{U}[-\delta, \delta] = \sqrt{3}\delta$ , hence a  $\sqrt{3}$  is missing.

We can look at the variance of the gradient wrt the activations. Since

$$\begin{aligned}\frac{\partial \ell}{\partial x_i^{(l)}} &= \sum_{h=1}^{N_{l+1}} \frac{\partial \ell}{\partial x_h^{(l+1)}} \frac{\partial x_h^{(l+1)}}{\partial x_i^{(l)}} \\ &\simeq \sum_{h=1}^{N_{l+1}} \frac{\partial \ell}{\partial x_h^{(l+1)}} w_{h,i}^{(l+1)}\end{aligned}$$

we get

$$\mathbb{V}\left(\frac{\partial \ell}{\partial \mathbf{x}^{(l)}}\right) \simeq N_{l+1} \mathbb{V}\left(\frac{\partial \ell}{\partial \mathbf{x}^{(l+1)}}\right) \mathbb{V}\left(\mathbf{w}^{(l+1)}\right).$$

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$$\mathbb{V}\left(\frac{\partial \ell}{\partial x^{(l)}}\right) \simeq \mathbb{V}\left(\frac{\partial \ell}{\partial x^{(L)}}\right) \prod_{q=l+1}^L N_q \mathbb{V}\left(w^{(q)}\right).$$

Since

$$x_i^{(l)} \simeq \sum_{j=1}^{N_{l-1}} w_{i,j}^{(l)} x_j^{(l-1)} + b_i^{(l)}$$

we have

$$\begin{aligned} \frac{\partial \ell}{\partial w_{i,j}^{(l)}} &= \frac{\partial \ell}{\partial x_i^{(l)}} \frac{\partial x_i^{(l)}}{\partial w_{i,j}^{(l)}} \\ &\simeq \frac{\partial \ell}{\partial x_i^{(l)}} x_j^{(l-1)} \end{aligned}$$

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and we get the variance of the gradient wrt the weights

$$\begin{aligned} \mathbb{V}\left(\frac{\partial \ell}{\partial w^{(l)}}\right) &\simeq \mathbb{V}\left(\frac{\partial \ell}{\partial x^{(l)}}\right) \mathbb{V}\left(x^{(l-1)}\right) \\ &= \mathbb{V}\left(\frac{\partial \ell}{\partial x^{(L)}}\right) \left(\prod_{q=l+1}^L N_q \mathbb{V}\left(w^{(q)}\right)\right) \mathbb{V}\left(x^{(0)}\right) \left(\prod_{q=1}^l N_{q-1} \mathbb{V}\left(w^{(q)}\right)\right) \\ &= \frac{N_0}{N_l} \left(\prod_{q=1}^L N_q \mathbb{V}\left(w^{(q)}\right)\right) \mathbb{V}\left(x^{(0)}\right) \mathbb{V}\left(\frac{\partial \ell}{\partial x^{(L)}}\right). \end{aligned}$$

Similarly, since

$$x_i^{(l)} \simeq \sum_{j=1}^{N_{l-1}} w_{i,j}^{(l)} x_j^{(l-1)} + b_i^{(l)}$$

we have

$$\begin{aligned} \frac{\partial \ell}{\partial b_i^{(l)}} &= \frac{\partial \ell}{\partial x_i^{(l)}} \frac{\partial x_i^{(l)}}{\partial b_i^{(l)}} \\ &\simeq \frac{\partial \ell}{\partial x_i^{(l)}} \end{aligned}$$



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**so we get the variance of the gradient wrt the biases**

$$\mathbb{V}\left(\frac{\partial \ell}{\partial b^{(l)}}\right) \simeq \mathbb{V}\left(\frac{\partial \ell}{\partial x^{(l)}}\right).$$

So finally, there is nothing we can do to control the variance of the gradient wrt the weights.

To control the variance of activations, we need

$$\mathbb{V}\left(w^{(l)}\right)=\frac{1}{N_{l-1}},$$

and to control the variance of the gradient wrt activations, and through it the variance of the gradient wrt the biases

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From which we get as a compromise the “Xavier initialization”

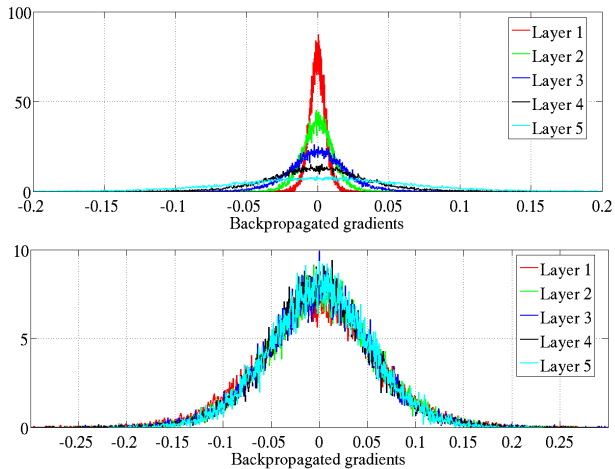
$$\mathbb{V}\left(w^{(l)}\right) = \frac{1}{\frac{N_{l-1} + N_l}{2}} = \frac{2}{N_{l-1} + N_l}.$$

(Glorot and Bengio, 2010)

In torch/nn/init.py

```
def xavier_normal(tensor, gain=1):
    if isinstance(tensor, Variable):
        xavier_normal(tensor.data, gain=gain)
        return tensor

    fan_in, fan_out = _calculate_fan_in_and_fan_out(tensor)
    std = gain * math.sqrt(2.0 / (fan_in + fan_out))
    return tensor.normal_(0, std)
```



(Glorot and Bengio, 2010)

The weights can also be scaled to account for the activation functions.

Remember that we have

$$\begin{aligned}\mathbb{V}(AB) &= \mathbb{V}(A)\mathbb{V}(B) + \mathbb{V}(A)\mathbb{E}(B)^2 + \mathbb{V}(B)\mathbb{E}(A)^2 \\ &= \mathbb{V}(A)\mathbb{E}(B^2) + \mathbb{V}(B)\mathbb{E}(A)^2.\end{aligned}$$

For the forward pass, if

$$\begin{aligned}s_i^{(l)} &= \sum_{j=1}^{N_{l-1}} w_{i,j}^{(l)} \sigma \left( s_j^{(l-1)} \right) + b_i^{(l)} \\ x_i^{(l)} &= \sigma \left( s_i^{(l)} \right),\end{aligned}$$

and  $\mathbb{E} \left( w^{(l)} \right) = 0$ ,  $s^{(l-1)}$  is symmetric, and  $\sigma$  is ReLU, we have

$$\begin{aligned}\mathbb{V} \left( s_i^{(l)} \right) &= N_{l-1} \mathbb{V} \left( w^{(l)} \sigma \left( s^{(l-1)} \right) \right) \\ &= N_{l-1} \mathbb{V} \left( w^{(l)} \right) \mathbb{E} \left( \sigma \left( s^{(l-1)} \right)^2 \right) \\ &= N_{l-1} \mathbb{V} \left( w^{(l)} \right) \frac{1}{2} \mathbb{E} \left( \left( s^{(l-1)} \right)^2 \right) \\ &= \frac{1}{2} N_{l-1} \mathbb{V} \left( w^{(l)} \right) \mathbb{V} \left( s^{(l-1)} \right).\end{aligned}$$

For the backward

$$\begin{aligned}
 \mathbb{V}\left(\frac{\partial \ell}{\partial x_i^{(l)}}\right) &= \sum_{h=1}^{N_{l+1}} \mathbb{V}\left(\underbrace{\sigma'\left(s_h^{(l+1)}\right)}_{0/1} \underbrace{\frac{\partial \ell}{\partial x_h^{(l+1)}} w_{h,i}^{(l+1)}}_{\mathbb{E}(\cdot)=0, \text{ symmetric}}\right) \\
 &= \sum_{h=1}^{N_{l+1}} \mathbb{E}\left(\sigma'\left(s_h^{(l+1)}\right) \left(\frac{\partial \ell}{\partial x_h^{(l+1)}} w_{h,i}^{(l+1)}\right)^2\right) \\
 &= \sum_{h=1}^{N_{l+1}} \frac{1}{2} \mathbb{E}\left(\left(\frac{\partial \ell}{\partial x_h^{(l+1)}} w_{h,i}^{(l+1)}\right)^2\right) \\
 &= \frac{1}{2} \sum_{h=1}^{N_{l+1}} \mathbb{V}\left(\frac{\partial \ell}{\partial x_h^{(l+1)}}\right) \mathbb{V}\left(w_{h,i}^{(l+1)}\right).
 \end{aligned}$$



So ReLU impacts the forward and backward pass as if the weights had half their variances, which motivates multiplying them by a corrective gain of  $\sqrt{2}$ .

(He et al., 2015)

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The same type of reasoning can be applied to other activation functions.

In `torch/nn/init.py`

```
if nonlinearity in linear_fns or nonlinearity == 'sigmoid':  
    return 1  
elif nonlinearity == 'tanh':  
    return 5.0 / 3  
elif nonlinearity == 'relu':  
    return math.sqrt(2.0)
```

## Data normalization

The analysis for the weight initialization relies on keeping the activation variance constant.

For this to be true, not only the variance has to remain unchanged through layers, but it has to be correct for the input too.

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This can be done in several ways. Under the assumption that all the input components share the same statistics, we can do

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mu, std = train_input.mean(), train_input.std()
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```

Thanks to the magic of broadcasting we can normalize component-wise with

```
mu, std = train_input.mean(0), train_input.std(0)
train_input.sub_(mu).div_(std)
test_input.sub_(mu).div_(std)
```

## Choice of the architecture and step size

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- Re-use something “well known, that works”, or at least start from there,



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We will re-visit this list with additional regularization / normalization methods.

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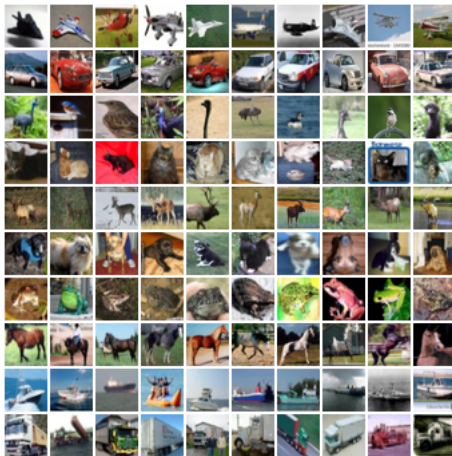
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The practical strategy is to look at the losses and error rates across epochs and pick a learning rate and learning rate adaptation. For instance by reducing it at discrete pre-defined steps, or with a geometric decay.

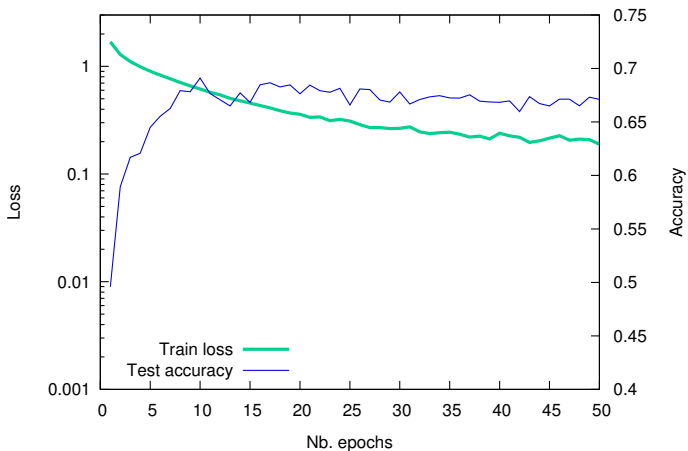
## CIFAR10 data-set



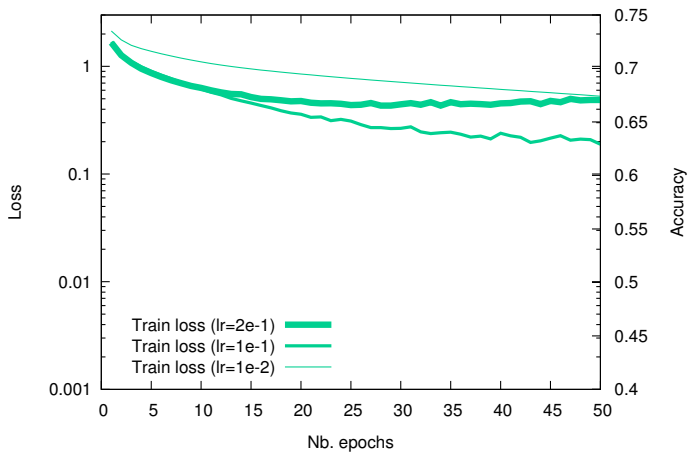
$32 \times 32$  color images, 50,000 train samples, 10,000 test samples.

(Krizhevsky, 2009, chap. 3)

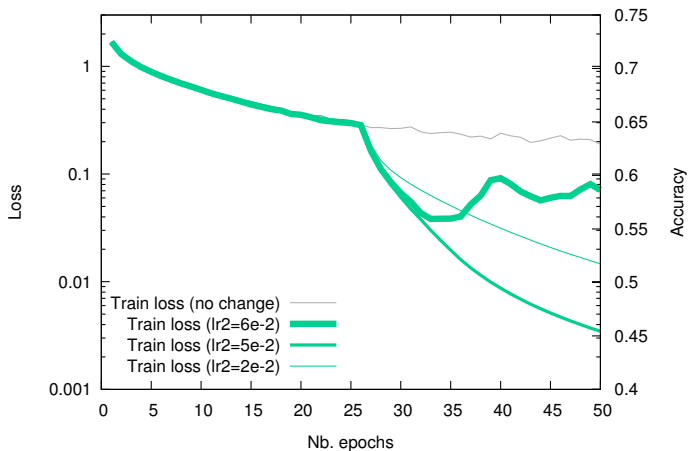
Small convnet on CIFAR10, cross-entropy, batch size 100,  $\eta = 1e-1$ .



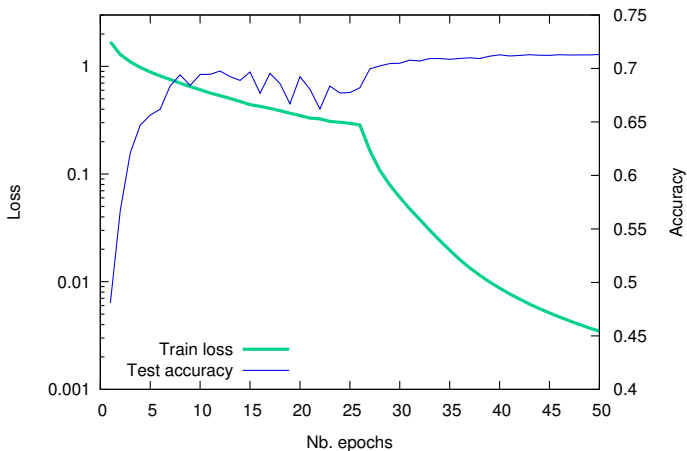
Small convnet on CIFAR10, cross-entropy, batch size 100



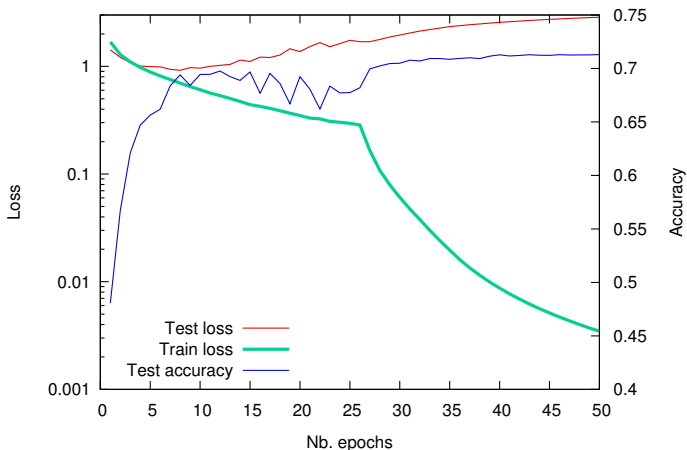
Using  $\eta = 1e - 1$  for 25 epochs, then reducing it.



Using  $\eta = 1e - 1$  for 25 epochs, then  $\eta = 5e - 2$ .



The test loss is a poor performance indicator, as it may increase even more on misclassified examples, and decrease less on the ones getting fixed.

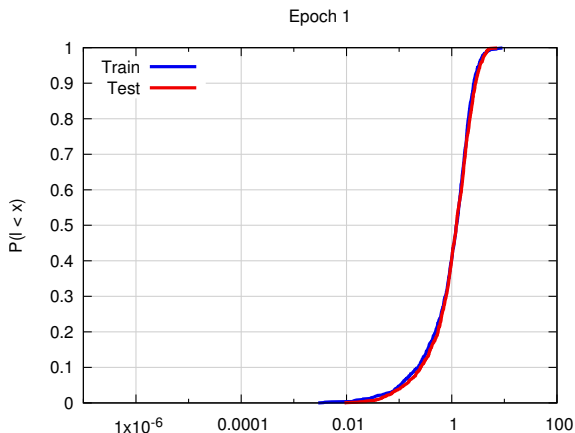




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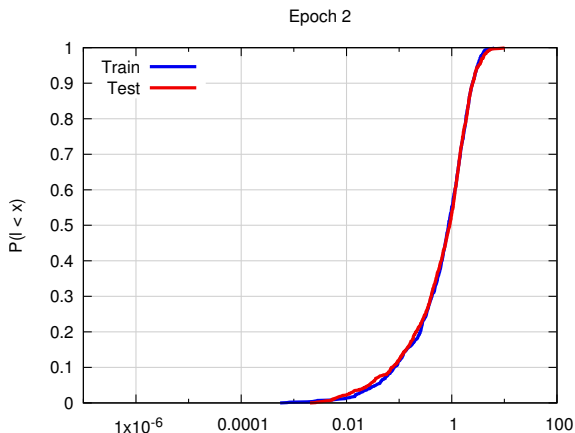
through epochs to visualize the over-fitting.



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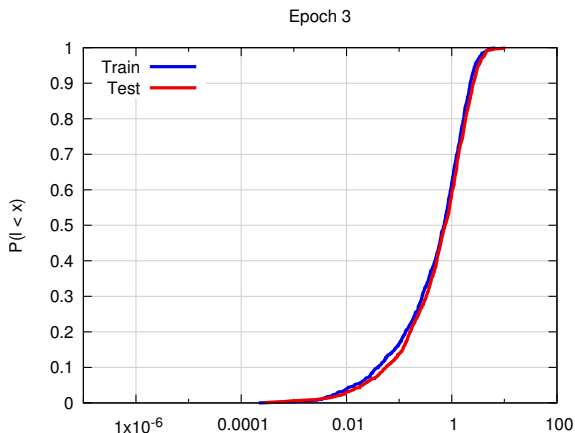
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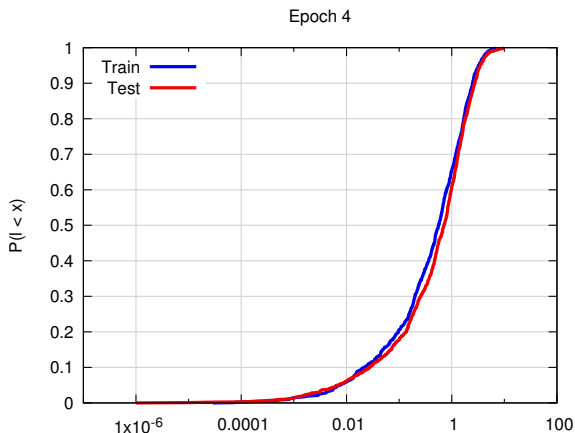
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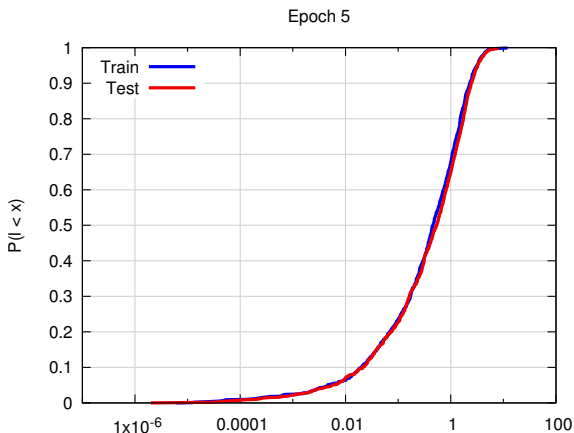
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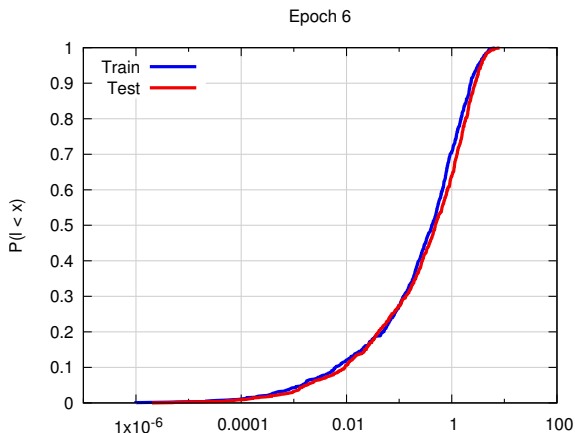
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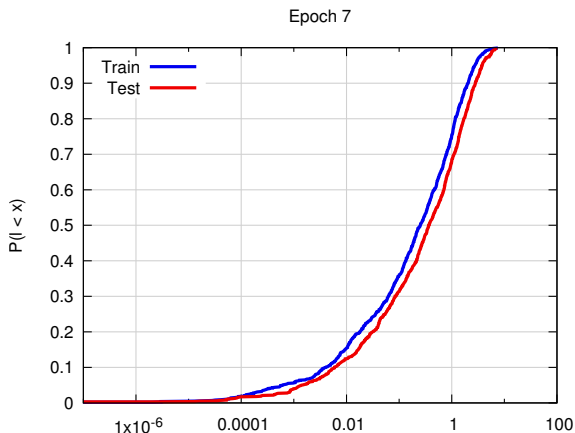
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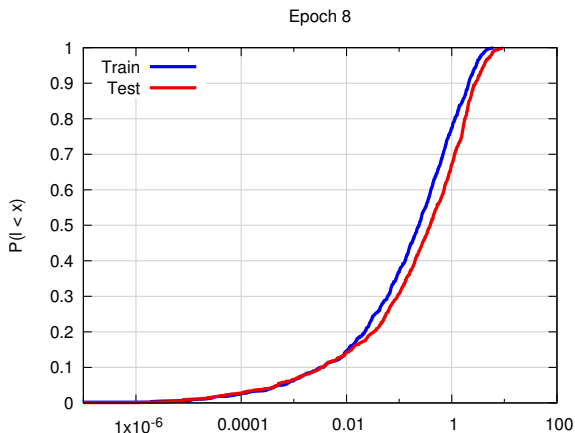
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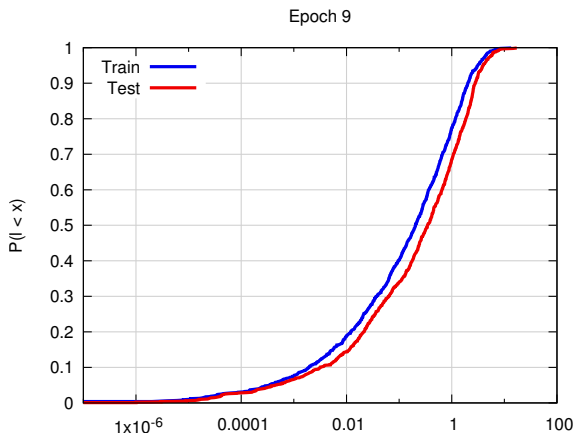




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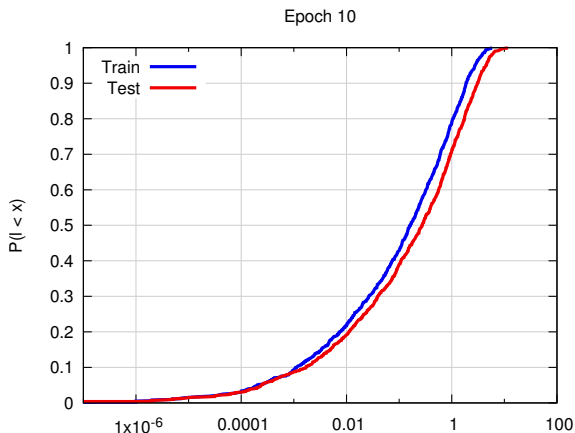
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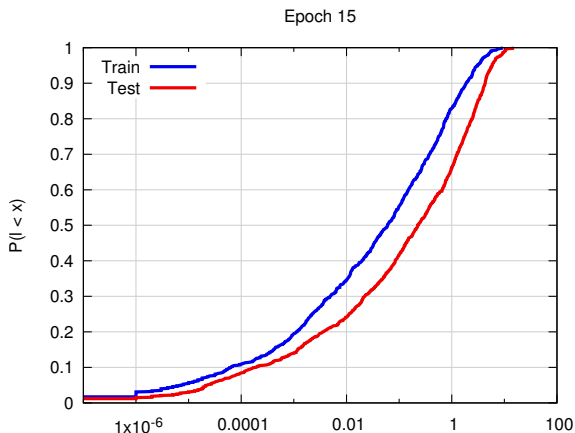
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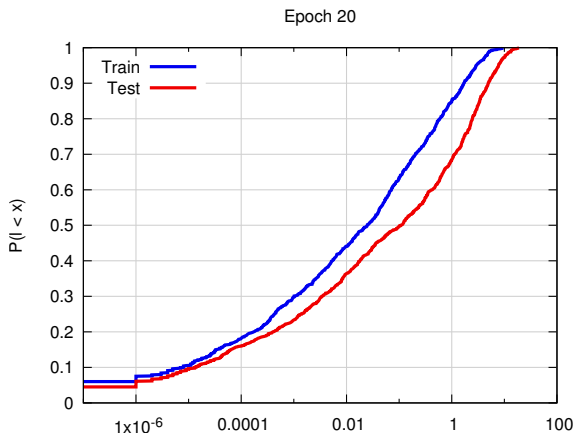
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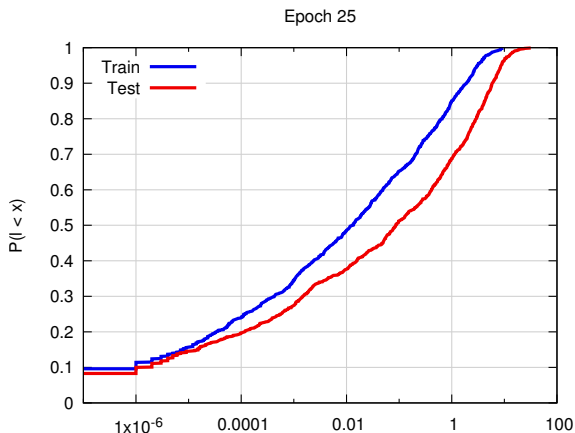
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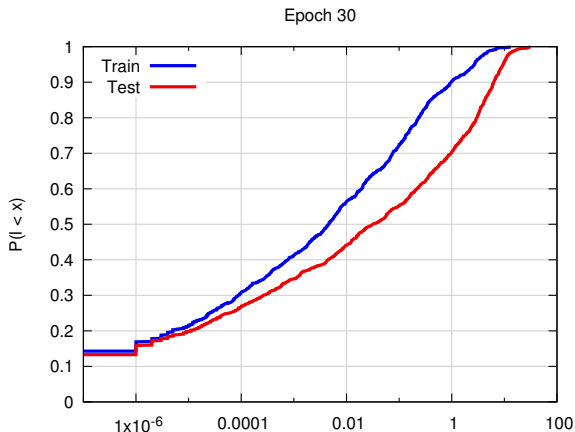
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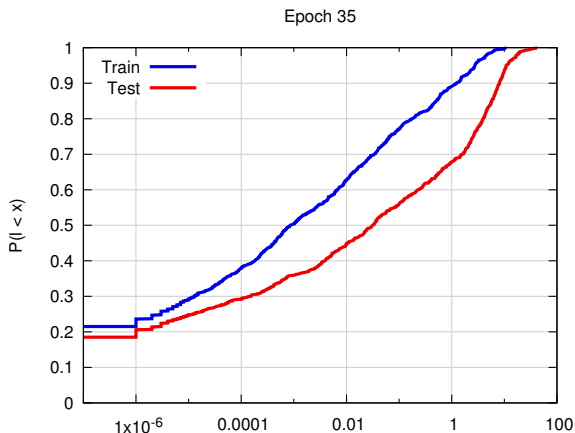
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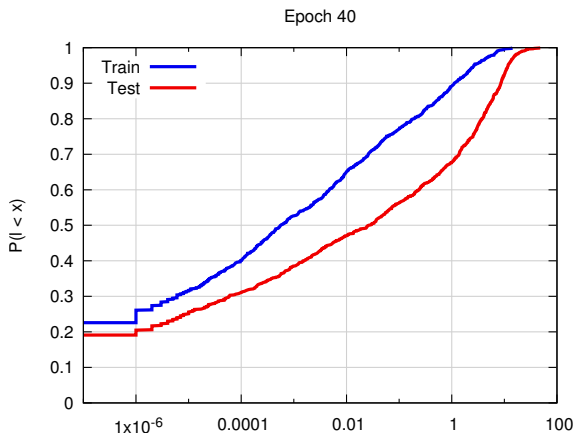
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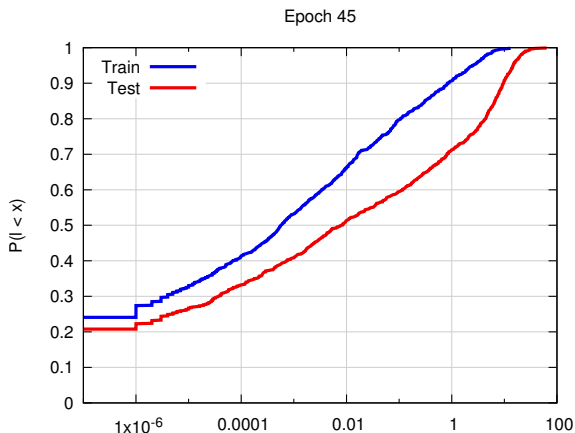




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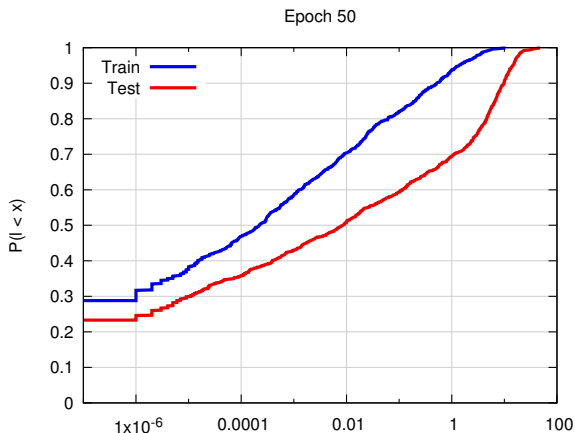
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Writing a `torch.autograd.Function`

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This is achieved by writing sub-classes of `torch.autograd.Function`, which have to implement two static methods:

- `forward(...)` takes as argument a context to store information needed for the backward pass, and the quantities it should process, which are `Tensors` for the differentiable ones, but can also be any other types. It should return one or several `Tensors`.

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Evaluating such a `Function` is done through its `apply(...)` method, which takes as many arguments as `forward(...)`, context excluded.

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This machinery is hidden to you and this level of details should not be required for normal operations.

Consider a function to set to zero the first  $n$  components of a tensor.

```
class KillHead(Function):
    @staticmethod
    def forward(ctx, input, n):
        ctx.n = n
        result = input.clone()
        result[:, 0:ctx.n] = 0
        return result

    @staticmethod
    def backward(ctx, grad_output):
        result = grad_output.clone()
        result[:, 0:ctx.n] = 0
        return result, None

killhead = KillHead.apply
```

## It can be used for instance

```
y = Variable(Tensor(3, 8).normal_())
x = Variable(y.data.new(y.size()).normal_(), requires_grad = True)

criterion, eta = nn.MSELoss(), 1e-0

for k in range(10):
    r = killhead(x, 2)
    loss = criterion(r, y)
    if k > 0: x.grad.data.zero_()
    loss.backward()
    print(k, loss.data[0])
    x.data -= eta * x.grad.data
```

## prints

```
0 0.9134871959686279
1 0.7928655743598938
2 0.6915099620819092
3 0.6063430905342102
4 0.5347792506217957
5 0.474645733833313
6 0.424116849899292
7 0.3816585838794708
8 0.3459818661212921
9 0.3160035014152527
```

The `torch.autograd.gradcheck(...)` function checks numerically that the backward function is correct, *i.e.*

$$\forall i, j, \left| \frac{f_i(x_1, \dots, x_j + \epsilon, \dots, x_D) - f_i(x_1, \dots, x_j - \epsilon, \dots, x_D)}{2\epsilon} - (J_f(x))_{i,j} \right| \leq \alpha$$

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from torch.autograd import gradcheck

input = (Variable(torch.DoubleTensor(10, 20).uniform_(-1, 1), requires_grad=True), 4)

if gradcheck(KillHead.apply, input, eps = 1e-6, atol = 1e-4):
    print('All good captain.')
else:
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It is advisable to use `DoubleTensor`s for such a check.

Consider a function that takes two similar sized `Variable` and apply component-wise

$$(u, v) \mapsto |uv|.$$



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```
class Something(Function):
    @staticmethod
    def forward(ctx, input1, input2):
        ctx.save_for_backward(input1, input2)
        return (input1 * input2).abs()

    @staticmethod
    def backward(ctx, grad_output):
        input1, input2 = ctx.saved_variables
        return grad_output * input1.sign() * input2.abs(), \
            grad_output * input1.abs() * input2.sign()

something = Something.apply
```



The method `save_for_backward` is used to save `Tensor`s but the fields `saved_variables` are `Variable`s.

Practical session:

<https://fleuret.org/dlc/dlc-practical-5.pdf>

The end

## References

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