Derivadas das funções trigonométricas, exponenciais e logarítmicas

Mostra-se que:

- $\operatorname{sen}' x = \cos x$
- $\cos' x = -\operatorname{sen} x$
- $(a^x)' = a^x \ln a$
- Em particular $(e^x)' = e^x \underbrace{\ln e}_{1}$, isto é, $(e^x)' = e^x$
- $\log_a' x = \dfrac{1}{x \ln a}$. Em particular $\ln' x = \log_e' x = \dfrac{1}{x \ln e}$. Como $\ln e = 1$, obtemos $\ln' x = \dfrac{1}{x}$

Exemplo 1. Calcular $\operatorname{tg}' x$.

$$\begin{aligned}\operatorname{tg}' x &= \left(\frac{\operatorname{sen} x}{\cos x}\right) = \frac{\operatorname{sen}' x \cos x - \operatorname{sen} x \cos' x}{\cos^2 x}\\ &= \frac{\cos x \cos x - \operatorname{sen} x(-\operatorname{sen} x)}{\cos^2 x} = \frac{\cos^2 x + \operatorname{sen}^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x} = \left(\frac{1}{\cos x}\right)^2 = \sec^2 x\end{aligned}$$

Exemplo 2. Calcular $\sec' x$.

$$\begin{aligned}\sec' x &= \left(\frac{1}{\cos x}\right) = \frac{1' \cos x - 1 \cos' x}{\cos^2 x} = \frac{0 \cos x - 1(-\operatorname{sen} x)}{\cos^2 x}\\ &= \frac{\operatorname{sen} x}{\cos^2 x} = \frac{1}{\cos x}\frac{\operatorname{sen} x}{\cos x} = \sec x \operatorname{tg} x\end{aligned}$$

Exemplo 3.

$(x \cos x)' = x' \cos x + x \cos' x = 1 \cdot \cos x = x \cdot (-\operatorname{sen} x) = \cos x - x \operatorname{sen} x$

Exemplo 4. $(xe^x)' = x'e^x + x(e^x)' = e^x + xe^x = (1+x)e^x$

Exemplo 5.

$(e^x \operatorname{sen} x)' = (e^x)' \operatorname{sen} x + e^x \operatorname{sen}' x = e^x \operatorname{sen} x + e^x \cos x = (\operatorname{sen} x + \cos x)e^x$

Exemplo 6. $(x \ln x)' = x' \ln x + x \ln' x = \ln x + x \cdot \dfrac{1}{x} = \ln x + 1$

Exemplo 7.

$(x^2 \ln x)' = (x^2)' \ln x + x^2 \ln' x = 2x \ln x + x^2 \dfrac{1}{x} = 2x \ln x + x = x(2 \ln x + 1)$

Exemplo 8.

$(2^x x^x)' = (2^x)' x^3 + 2^x (x^3)' = 2^x \ln 2 \cdot x^3 + 2^x \cdot 3x^2 = 2^x x^2 (\ln 2 \cdot x + 3)$

Exemplo 9.

$$\begin{aligned}\left(\frac{e^x}{x+1}\right)' &= \frac{(e^x)'(x+1) - e^x(x+1)'}{(x+1)^2} = \frac{e^x(x+1) - e^x \cdot 1}{(x+1)^2}\\ &= \frac{xe^x + e^x - e^x}{(x+1)^2} = \frac{e^x}{(x+1)^2}\end{aligned}$$

Exemplo 10.

$$\begin{aligned}\left(\frac{e^x}{\cos x}\right)' &= \frac{(e^x)' \cos x - e^x \cos' x}{\cos^2 x} = \frac{e^x \cos x - e^x(-\operatorname{sen} x)}{\cos^2 x}\\ &= \frac{e^x \cos x + e^x \operatorname{sen} x}{\cos^2 x} = \frac{e^x(\cos x + \operatorname{sen} x)}{\cos^2 x}\end{aligned}$$

Exemplo 11.

$$\begin{aligned}\left(\frac{1+\operatorname{sen} x}{1+\cos x}\right)' &= \frac{(1+\operatorname{sen} x)'(1+\cos x) - (1+\operatorname{sen} x)(1+\cos x)'}{(1+\cos x)^2}\\ &= \frac{(1' + \operatorname{sen}' x)(1+\cos x) - (1+\operatorname{sen} x)(1' + \cos' x)}{(1+\cos x)^2}\\ &= \frac{(0+\cos x)(1+\cos x) - (1+\operatorname{sen} x)(0 - \operatorname{sen} x)}{(1+\cos x)^2}\\ &= \frac{\cos x(1+\cos x) + (1+\operatorname{sen} x)\operatorname{sen} x}{(1+\cos x)^2}\\ &= \frac{\cos x + \cos^2 x + \operatorname{sen} x + \operatorname{sen}^2 x}{(1+\cos x)^2}\\ &= \frac{\overbrace{\cos^2 x + \operatorname{sen}^2 x}^{1} + \cos x + \operatorname{sen} x}{(1+\cos x)^2}\\ &= \frac{1 + \cos x + \operatorname{sen} x}{(1+\cos x)^2}\end{aligned}$$

Exemplo 12.

$$\left(\frac{2}{x^3} - \frac{1}{2x^4}\right)' = \left(2x^{-3} - \frac{x^{-4}}{2}\right)' = -6x^{-4} + \frac{4x^{-5}}{2} = -\frac{6}{x^4} + \frac{2}{x^5}$$

Exercícios de revisão

1 Derivar e simplificar ao máximo:

(1) $y = t^3 \operatorname{sen} t$

(2) $f(x) = \dfrac{\operatorname{sen} x}{1+\cos x}$

(3) $g(x) = \dfrac{1-\operatorname{sen} x}{1-\cos x}$

(4) $h(x) = \dfrac{1-\cos x}{1+\cos x}$

(5) $y = x \operatorname{sen} x + \cos x$

(6) $y = \operatorname{sen} x \operatorname{cotg} x$

(7) $y = \dfrac{\operatorname{tg} x - 1}{\sec x}$

(8) $y = \dfrac{\operatorname{tg} a}{1+a^2}$

(9) $f(x) = 3^x + \log_3 x$

(10) $f(x) = e^x + \dfrac{\ln x}{x}$

(11) $f(x) = x^3 \ln x - \dfrac{x^3}{3} + e^{\pi/3}$

(12) $f(x) = \dfrac{\operatorname{sen} x + \cos x}{\operatorname{sen} x - \cos x}$

(13) $f(x) = (x^2 - 2x + 2)e^x$

(14) $f(x) = \dfrac{x - \operatorname{sen} x}{1+\cos x}$

(15) $f(t) = 2t \operatorname{sen} t - (t^2 - 2)\cos t$

(16) $f(x) = \dfrac{x^2}{2}\ln x - \dfrac{x^2}{4} + x(\ln x - 1) + 3$

Respostas

1

(1) $\dfrac{dy}{dt} = t^2(t\cos t + 3\operatorname{sen} t)$

(2) $\dfrac{df}{dx} = \dfrac{1}{1+\cos x}$

(3) $\dfrac{dg}{dx} = \dfrac{1-\operatorname{sen} x - \cos x}{(1-\cos x)^2}$

(4) $\dfrac{dh}{dx} = \dfrac{2\operatorname{sen} x}{(1+\cos x)^2}$

(5) $y' = x\cos x$

(6) $y' = -\operatorname{sen} x$

(7) $y' = \dfrac{1+\operatorname{tg} x}{\sec x}$

(8) $y' = \dfrac{(1+a^2)\sec^2 a - 2a\operatorname{tg} a}{(1+a^2)^2}$

(9) $y' = 3^x \ln 3 + \dfrac{1}{x\ln 3}$

(10) $f'(x) = e^x + \dfrac{1-\ln x}{x^2}$

(11) $f'(x) = 3x^2 \ln x$

(12) $f'(x) = \dfrac{-2}{(\operatorname{sen} x - \cos x)^2}$

(13) $f'(x) = x^2 e^x$

(14) $f'(x) = \dfrac{x\operatorname{sen} x}{(1+\cos x)^2}$

(15) $f'(x) = t^2 \operatorname{sen} t$

(16) $f'(x) = (x+1)\ln x$