**1. Introduction**

* Spread options are used in various markets as speculation devices and risk management tools.
* This simple relationship between spot and forward prices does not hold in the commodity markets.
* Most of the spread options require only the statistics of the underlying indexes at one single point in time, namely, the time-of-maturity of the option.
* **European Type**: The study focuses on European type spread options, where the payoff is the difference at maturity.
* **Challenges in Pricing**: Emphasizes the challenges in pricing and hedging financial instruments without closed-form solutions.
* **Commodity Markets**: Highlights the differences between commodity and stock markets, particularly seasonality and mean reversion.
* **Models**: Introduction to stochastic differential equations (SDEs) for modeling the dynamics of underlying assets, leading to the use of Monte Carlo methods and the Black-Scholes model.
* Introduce the stochastic differential equations used to model the dynamics of the underlying indexes. The price of a spread option is given by an expectation over the sample paths of the solution of this system of stochastic differential equations.
* The core analysis concerns the pricing of spread options on two correlated geometric Brownian motions.
* Bachelier's model because it is consistent with a model of the spread dynamics based on a single Brownian motion, in the same way Bachelier originally proposed to model the dynamics of the value of a stock by a continuous time process generalizing the notion of random walk.
* Case of a spread option with log-normal underlying indexes and strike K = 0. As in the case of the Bachelier model, it is possible to give a Black-Scholes-type formula for the price of the option.
* The basic problem is the pricing and hedging of the simplest spread option (i.e., a European call option on the difference of two underlying indexes) when the risk-neutral dynamics of the values of the underlying indexes are given by correlated geometric Brownian motions. The results of [10] are based on a systematic analysis of expectations of functions of linear combinations of log-normal random variables.

**2. Zoology of the Spread Option**

* **Definitions**: Spread options in commodity markets can be location spreads (Price different location), calendar spreads (Usually easy to price, same commodity in two different time), processing spreads (Difference in price of input and output), and quality spreads (Different grade of same commodity).
* Examples in Markets:
  + Agricultural Futures Market: Focus on the soybean crush spread.
  + Energy Market: Details on crack spreads (e.g., 3:2:1 crack spread) and spark spreads (e.g., 4:3 spark spread)
* Crack spread options are the subject of many papers attempting to demonstrate the stationarity of the spread time series by means of a statistical quantification of the co-integration properties of the underlying index time series from which the spread is computed.

|  |  |  |
| --- | --- | --- |
| Crush Spread | Crack Spread | Spark Spread |
| Selling the crush means selling Soybean Oil and Soybean Meal and buying Soybeans.  Buying Soybean Meal and Soybean Oil and selling Soybeans. | Buying the crack spread means you “buy” the refined products while selling crude oil. This assumes you expect the crack spread will increase in value.  Selling the crack spread means you “sell” the refined products while buying crude oil. This assumes you expect the crack spread will decrease in value. | Buying the spark spread means buying futures and sell natural gas, profiting if the spark spread increases.  Selling the spark spread means selling futures and buying natural gas, profiting if the spark spread decreases. |

**3. Spread Option Pricing: Mathematical Setup**

* **European Spread Option Pricing**: Discusses pricing using a model of two underlying indexes ​
* We denote by the price at time 0 of this European call option with date of maturity and strike . More generally, we shall denote by its price at time
* Arbitrage must be ruled out in a viable market.
* The Cameron-Martin-Girsanov theory is framework for the switch from the historical probability (quantifying the statistics of the prices observed historically) to the risk-neutral probability (governing the statistics of the derivative prices)

The Black-Scholes formula gives a value for ­ when has a log-normal distribution under the risk-neutral measure.

Moneyness:

If > 1 ITM

Else OTM

**and for density and cumulative distribution function of the standard normal distribution**

Payout

Put Parity

**The price o four spread option is given by the risk-neutral expectation.**

* 1. **Black-Scholes Paradigm:**
* **PDE Solution**: Derivation using parabolic PDEs and the risk-neutral expectation.
* **Double Integral Pricing**: When **pricing a spread option, the only thing we need is the joint density of the couple of random variables under the risk-neutral measure**. This density is usually called the state price density. So, ignoring momentarily the dynamics of the underlying indexes, we can **write the price of a spread option as a double integral**. In simpler terms, the price of a spread option can be calculated by taking the integral of a function of two variables with respect to a bivariate Gaussian distribution. This means that we can calculate the price of the spread option by computing a double integral over the joint density of the two underlying indexes.
* **Close formulas**: Simple and easy to implement, always gives the same result (Different from Monte Carlo), partial derivatives of the price with respect to the parameters give the sensitivity of the price with respect to those parameters. (Greeks)

**Key Mathematical Frameworks**

* 1. **Markovian Models: General setup pricing spread options.**

Where denotes the couple and and are independent standard real-valued Wiener processes also called processes of Brownian motions.

The intuitive interpretation of equations (3.8) is as follows: at each time , the infinitesimal changes in the return on are normally distributed with means and variance of **giving the instantaneous correlation between these two conditionally normal random variables**. We also assume that the coefficients , , and are smooth enough for the existence and uniqueness of a strong solution. (**Interest rates are constant** )

**3.3 Geometric Brownian Motion Model**

For simple and trackable model are constant.

Where are two Brownian motions with correlation .

In risk-neutral .

**3.4 Numeric**

Numerical methods arise in the absence of explicit close formulae, valuing the option can be done by solving PDE.

**Trinomial Trees**

A trinomial tree is a tree-based **numerical scheme that spans two directions (Two underlying process), where each node leads to nine new nodes at the next time step.** This method is used to price and hedge financial instruments **in the absence of explicit formulae in closed forms**. It is an explicit finite difference method that is used to solve a pricing PDE. Trinomial trees are a generalization of the binomial tree method, which is used to price American options.

**Monte Carlo**

The idea is to generate many sample paths of the process over the interval for each of these sample paths to compute the value of the function of the path whose expectation we wish to evaluate, and then to average these values over the sample paths. The only **difficulty is in quantifying and controlling the error.** Various methods of random sampling (stratification being one of them) and variance reduction.

The **situation is much simpler when we assume that all the coefficients are deterministic**. Indeed, if the coefficients are constant, the couple of indexes at maturity can be written in the form.

Where U and V are two independent standard Gaussian random variables. The simulation of samples of is quite easy.

**Quasi-Monte Carlo**

With Monte Carlo methods we generate uniform random points in [0, l]. Quasi-Monte Carlo methods, however, use nonrandom points to have a more nicely uniform distribution. **It is now clear that, at least for low-dimension problems, quasi-Monte Carlo methods should be favored.**

**Conclusion**

**Monte Carlo Methods for Option Pricing**

1. **Approximation Accuracy**: Monte Carlo methods can provide accurate price approximations.
2. **Sensitivity Computations**: Calculating Greeks (sensitivities) using Monte Carlo methods requires careful handling.
3. **Challenges**:
   * 1. Repeatedly recomputing prices with slight parameter variations is computationally intensive.
     2. Important for constructing hedging portfolios and calculating Value at Risk (VaR).

**Greeks in Option Pricing**

* **First-Order Greeks**: Partial derivatives with respect to the current values of the underlying index.
* **Gamma**: Second derivative of the option price with respect to the underlying index.
  + For single asset options, Black-Scholes PDE simplifies Gamma calculation using first-order Greeks.
  + **For spread options, the PDE involves three second-order derivatives, complicating the computation**.

**Numerical Methods and Convergence Issues**

* **Second-Order Derivatives**: Critical for spread options, often result in poor approximation and instability.
* **Closed-Form Formulae**: Generally, more stable and efficient for deriving Greeks.

**Trinomial Tree Method**

* **Advantages**: Computes partial derivatives along with the price.
* **Disadvantages**:
  + Slow convergence rate.
  + Exponential growth in complexity with the number of assets.

**Implied Parameters**

* **Implied Volatility and Correlation**: Values that best match market prices.
* **Inversion Challenge**: Numerical methods struggle to invert pricing algorithms to retrieve implied parameters efficiently.
* **Closed-Form Formulae**: Typically provide better capabilities for estimating implied parameters through inversion.

**Computational Methods**

* **Monte Carlo and Trinomial Tree**: Both have limitations in higher dimensions and sensitivity calculations.
* **Closed-Form Formulae**: Preferred for their efficiency in computing implied parameters and stability in deriving Greeks.

**Conclusion**

* Practical Implications:
  + Closed-form solutions are preferred for their efficiency and consistency.
  + Importance of partial derivatives (the Greeks) for risk management and trading.

**4. The Bachelier Model for Spread Options**

**ABM Instead of GBM for dynamics spreads. Prices of options can be derived by computing Gaussian Integrals leading to closed formulas.**

Most applications, **underlying instruments are modeled by means of log-normal distributions as prescribed by Geometric Brownian Motion Model, and the underlying price are inherently positive**. **This does not apply to spreads themselves, that can be negative.**

This simple remark is the starting point of a series of papers **proposing the use of Arithmetic Brownian motion (ABM) (as opposed to the Geometric Brownian Motion (GBM) leading to the log normal distribution) for the dynamics of spreads**. In so doing, **prices of options can be derived by computing Gaussian integrals leading to simple closed form formulae.**

The premise of the pricing formula proposed in this section is to assume that the risk-neutral dynamics of the spread S(t) is given by a stochastic differential equation.

Equation (4.1) is appropriate when the spread is defined as for some coefficients and , and when the dynamics of the individual component indexes are given by stochastic differential equations of the form.

**with positive constants and and two Brownian motions and with correlation .**

However, **such a rationale cannot be taken very seriously** because it implies dynamics for and that are totally unrealistic since their marginal distributions are Gaussian, and therefore  **can be negative with positive probability**.

**Spreads**: Contrast with GBM since it assumes only positive values, that does not hold in spreads.

**Geometric Brownian Motion (GBM)**: Used to model underlying instruments that have positive price and fallow log-normal distributions.

**Arithmetic Brownian Motion for Spreads (ABM)**: Assume ABM instead of GBM, allowing negative values.

**Stochastic Differential Equation for Spreads (SDE)**: , Spread fallow ABM.

**Spread**: where determine the spread composition

**Component Indexes:**  **fallow their own SDE. Where are positive and constant and are Brownian motions with correlation** .

**Challenges with ABM**: **Unrealistic Marginal Distributions are Gaussian under ABM, implying that can be negative, which is unrealistic for many assets**.

**Modified Parameters for Spread**: The standard deviation for the spread is,

A new Brownian motion can be defined as **In sum Using ABM for spreads addresses the issue of allowing negative values**. **However, it leads to unrealistic assumptions for the individual component indexes . The pricing formula derived from ABM involves computing Gaussian integrals, which lead to simple closed-form solutions**.

**4.1 Pricing Formulae**

Arithmetic Brownian Motion for Spread Pricing (ABM): Leads to closed form formula (BS) and it is appropriate for modeling spread, which can be negative. (Unlike process modeled by GBM).

Dynamics of the Underlying Instruments: For individual instruments the dynamics is given by,

Assuming constant drift and .

**Spread Dynamics**: , as Gaussian random variable.

**Option Price**

If the value of the spread at maturity is assumed to have a Gaussian distribution with the correct first two moments, then the price of a spread option with maturity and strike is given by,

**Price at time of a spread option with strike price is given by: that can be expressed in term of the mean and standard deviation of ,**

**Where is standard normal random variable**. The price can be simplified by,

**Solving SDE**

Equation 4.1 can be solved by, , where is a Gaussian random variable with mean and variance, .

**Time-Dependent Volatility**

If we allow time-dependent volatility the dynamics are , witch becomes and variance . For compatibility with the earlier variance in 4.1 we must take

**Conclusion**

**ABM for Spread Pricing:** More suitable for modeling spreads which can be negative.

**Dynamics:** Individual instruments follow GBM while spreads follow ABM.

**Closed-Form Formula:** Like the Black-Scholes formula but adapted for spreads.

**Gaussian Distribution:** is Gaussian with explicitly computable mean and variance.

**Time-Dependent Volatility:** Allows for adjusting the model to ensure realistic dynamics and consistency in variance calculation.

**4.2 Experiment Comparison – Geometric Brownian Motion (GBM) Versus Bachelier Model Approximation**

(Gas) | (Electricity) | Option Type: Spark Spread

Grid: ,

Gráfico, Gráfico de superfície

Descrição gerada automaticamente

Gráfico, Gráfico de superfície

Descrição gerada automaticamente

**At short maturity (60 Days / 252): Good agreement between models, independent of correlation and strike being very negative.**

**At longer maturity (1.5 Years):**

* **Error Trend**: Normal approximation underestimates option value.
* **Error Dependence**: Error increases with strike and decreases with correlation.
* **Impact of Maturity**: Increasing time-to-maturity increases error significantly for out-of-the-money options with large strikes.
* **Increased Volatility**:
  + Impact: Higher volatility in electricity prices worsens normal approximation performance for log-normal indexes

**4.3 Consistency with the forward curve**

* **Forward Curve Calculation: Theoretical values of forward prices should match observed forward prices (Market) if the model is consistent**.
* **This is done by computing the theoretical values of the forward curve from the model and reconciling the physical commodity market models with its equity relatives**.
* **Solve anomalies, arbitrage, storage, yields added to the stochastic factors**.
* Deterministic Interest Rates: Assumes interest rates are constant and convenience yield is not stochastic.
* **Risk-Neutral Expectation: Forward prices are computed as the conditional expectation of future spot prices under the risk-neutral measure**.
* The old economic belief states that forward prices are the best predictors of future spot prices under the historical measure. However, forward prices under the risk-neutral measure differ from this assumption.

Forward Contract Pricing: , receive the commodity with price and pay forward price . buy the commodity now for and borrow from the bank to be sure to have at time . Here , represent the risk-neutral condition given the past information up to .

* **This practice involves adding parameters to the model to replicate market prices, ensuring the model's consistency.**

**Example**

, and forward prices for two different commodities are used to calibrate the model. (Interpolation or smooth to create continuous forward)

**Mathematical Model**

**Verify if the pricing model supports the observed forward curve.** Stochastic Integral for forward price is,

Indicates an exponential forward curve starting from the current value of the spread, which is often unrealistic.

**Time-Dependent Drift**

To match the observed forward curve , use a time-dependent drift . The adjusted SDE is,

The forward price becomes,

To match , set, . **In other words, by choosing the time-dependent drift coefficient as the logarithmic derivative of the observed forward curve, the model becomes consistent in the sense that the current forward curve computed out of this model is exactly the current market curve. This implies new expressions for the mean and variance of .**

**Mean and Variance of the Spread**

**Adjusted Volatility to be consistent with 4.4.**

Conclusion

* **Model Calibration**: Adjust the model parameters to align with observed forward prices.
* **Risk-Neutral Expectation**: Forward prices calculated using risk-neutral measures.
* **Time-Dependent Drift**: Modify drift to ensure the model matches the market forward curve.
* **New Expressions**: Derive mean and variance for spreads, ensuring consistency with market data.
* **Volatility Adjustment**: Ensure volatility term aligns with the observed forward curve, maintaining model consistency.

**5. The case of or Option to Exchange**

**The spread option with strike simplifies to an exchange option, meaning the payoff allows the holder to exchange for if outperforms .**

**Margrave Formula**

**Assumptions**: The underlying fallow GBM. The indices are correlated through GBM with parameter and the risk-neutral dynamics are considered constant .

**Pricing Formula**

The price of a spread option with strike and maturity is given by,

**Derivation of Margrabe Formula**

To prove formula 5.1, we define two new independent Brownian motions and ns by,

Where . and are well-defined as long as . **The risk-neutral valuation rule gives us the price of a zero-exercise price spread option as fallows,**

**Change of measure refers to changing from one probability measure to another**. This is often done to simplify calculations or to move from the **real-world probability measure to the risk-neutral measure .** The Girsanov theorem helps us understand how Brownian motions (the mathematical model for random movement) change under this new measure. Girsanov's theorem provides the way to change the measure. It involves the Radon-Nikodym derivative, which in this context can be seen as a factor that adjusts probabilities from one measure to another.

**We use the subscript to emphasize the fact that the expectations are computed under the risk-neutral probability measure .**

**Chage of Measures**

New probability measure , using Girsanov’s theorem with Radon-Nikodýim derivative with respect to given on the -algebra .

**Essentially, it's the adjustment factor at time** . **This exponential term is the specific adjustment factor, involving volatility and a Brownian motion .** Under , and are Brownian motions, so, . Where is a geometric Brownian motion under with volatility.

Using this fact, the last expression can be viewed as the price of a European call option with no interest rate, strike 1, and volatility . The value is thus given by the classical Black-Scholes formula (3.4) with the appropriate parameters.

Initial price of two assets

Dividend yield of two assets

Cumulative distribution function

**The price of option is given by,**

**It is interesting to note that these formulae are independent of the risk-free rate . The above closed form formula is very nice, but unfortunately the case where cannot be treated with the same success.**

**6. Pricing Options on the Spread of GBM**

**Risk-Free Bank Account**: Constant interest rate and two assets’ prices .

General Spread Option Pricing Formula (3.7)

This integral represents the price as a function of two variables under a bivariate Gaussian distribution.

**Normalized Notation**

Where are real constants and and are jointly Gaussian random variables with correlation . Approximate the price of a spread option provided we set,

, , , and

**Pricing Formula**

**Approximate Pricing**

The authors derived a family of upper and lower bounds for the price . Close to the (3.4)

**This approximation is equal to the true price when (Margrave) or (BS) or .**

**6.2 Hedging and the Computation of the Greeks**

Problems of the numerical approximations of the price of the spread option when computing Greeks. **Hedging strategies can be computed using 6.3 giving .**

*The portfolio formed at time by*

*units of the underlying assets and respectively, is a subhedge for the option. In other words, its value at the time of maturity T is almost surely a lower bound for the payoff.*

Let and denote sensitivities of the price functional (6.3) with respect to the volatilities of each asset, let be the sensitivity with respect to the correlation parameter , let be the sensitivity with respect to the strike price and let be the sensitivity with respect to the maturity time . They are given by the formula.

**6.3 Kirk’s Formula**

Proposed a closed form approximation for the price of a spread option. It reads as follows:

We start with two assets prices , the is adjusted for time value multiplying by witch discount for present value. The volatility consider both the individual volatility of two assets and their correlation. The term is weighted that adjust the influences of volatilities based on the relative size of to the adjusted price. The natural logarithmic measures the relative difference between the price of two assets and the combined value of asset 1 and the adjusted . The is used to calculate probabilities of certain outcome under assumption of normal distribution.

The first term represents the expected value of assets 2 being above the adjusted . The second term subtract the adjusted and assets 1 value multiplied by the probability of it being ITM.

**Kirk's formula approximates the price of a spread option by considering the prices and volatilities of two underlying assets and their correlation. It uses the adjusted strike price to account for the time value of money and applies the standard normal distribution to determine the probabilities of the spread being profitable. The formula then combines these probabilities with the respective asset prices to estimate the option's price.**

**6.4 Comparison of the Three Approximations**

**Performance**

* Numerical experiments show that **Bachelier's model generally underestimates prices compared to the approximation**
* **Kirk’s formula and the approximation ​ perform well in terms of hedging efficiency, often outperforming Bachelier’s model.**
* Hedging strategies derived from provide better tracking of the option payoff at maturity.
* **The pricing and hedging of spread options on geometric Brownian motions involve advanced formulas and approximations that consider the dynamics of the underlying assets, correlations, and various sensitivities (Greeks).**
* **Kirk’s formula and the approximation ​ offer practical and efficient solutions for both pricing and hedging these options.**