$$M = X + W \sim \mathcal{N}(0, v) \tag{1}$$

where

$$X \sim p_X(x) = \rho \mathcal{N}(x|0, \sigma_X^2) + (1-\rho)\delta(x) \tag{2}$$

The posterior distribution is given by

$$p(x|m) = \frac{p_X(x)p(m|x)}{\int p_X(x)p(m|x)\mathrm{d}x} \tag{3}$$

$$= \frac{[\rho \mathcal{N}(x|0, \sigma_X^2) + (1-\rho)\delta(x)]\mathcal{N}(x|m, v)}{\int [\rho \mathcal{N}(x|0, \sigma_X^2) + (1-\rho)\delta(x)]\mathcal{N}(x|m, v)\mathrm{d}x} \tag{4}$$

$$\int [\rho \mathcal{N}(x|0, \sigma_X^2) + (1-\rho)\delta(x)]\mathcal{N}(x|m, v)\mathrm{d}x \tag{5}$$

$$= \rho \mathcal{N}(0|m, v + \sigma_X^2) + (1-\rho)\mathcal{N}(0|m, v) \tag{6}$$

$$\mathbb{E}[X] = \frac{\rho \frac{m\sigma_X^2}{v+\sigma_X^2} \mathcal{N}(0|m, v + \sigma_X^2)}{(1-\rho)\mathcal{N}(0|m, v) + \rho \mathcal{N}(0|m, v + \sigma_X^2)} \tag{7}$$

$$\mathbb{E}[X^2] = \frac{\rho \left[\frac{\sigma_X^2 v}{v+\sigma_X^2} + \left| \frac{m\sigma_X^2}{v+\sigma_X^2} \right|^2 \right] \mathcal{N}(0|m, v + \sigma_X^2)}{(1-\rho)\mathcal{N}(0|m, v) + \rho \mathcal{N}(0|m, v + \sigma_X^2)} \tag{8}$$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - |\mathbb{E}[X]|^2 \tag{9}$$

Define

$$C = \frac{\rho \mathcal{N}(0|m, v + \sigma_X^2)}{\rho \mathcal{N}(0|m, v + \sigma_X^2) + (1-\rho)\mathcal{N}(0|m, v)} \tag{10}$$

we have

$$\mathbb{E}[X] = C \cdot \frac{m\sigma_X^2}{v + \sigma_X^2} \tag{11}$$

$$\mathrm{Var}[X] = C \cdot \left[\frac{\sigma_X^2 v}{v + \sigma_X^2} + \left| \frac{m\sigma_X^2}{v + \sigma_X^2} \right|^2 \right] - |\mathbb{E}[X]|^2 \tag{12}$$

$$p(m) = \int p_X(x)\mathcal{N}(x|m,v)\mathrm{d}x = (1-\rho)\mathcal{N}(0|m,v) + \rho\mathcal{N}(0|m,v+\sigma_X^2) \tag{13}$$

$$\begin{aligned}
\text{MSE} &= \int |x-\hat{x}|p(x,m)\mathrm{d}x\mathrm{d}m & (14)\\
&= \int |x|^2 p_X(x)\mathrm{d}x - \int |\hat{x}|^2 p(m)\mathrm{d}m & (15)\\
&= \rho\sigma_X^2 - \int |\hat{x}|^2 p(m)\mathrm{d}m & (16)
\end{aligned}$$

$$\begin{aligned}
\int |\hat{x}|^2 p(m)\mathrm{d}m &= \int \left| C \cdot \frac{m\sigma_X^2}{v+\sigma_X^2} \right|^2 p(m)\mathrm{d}m & (17)\\
&= \int C \cdot |m|^2 \left(\frac{\sigma_X^2}{v+\sigma_X^2}\right)^2 \rho\mathcal{N}(0|m,v)\mathrm{d}m & (18)\\
&= \left(\frac{\rho\sigma_X^2}{v+\sigma_X^2}\right)^2 \int |m|^2 \frac{|\mathcal{N}(0|m,v+\sigma_X^2)|^2}{(1-\rho)\mathcal{N}(0|m,v)+\rho\mathcal{N}(0|m,v+\sigma_X^2)}\mathrm{d}m & (19)
\end{aligned}$$