Notes on Multigrid Preconditioned Conjugate Gradient Method

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1 Introduction

For high dimensional(2D or 3D) Variable Coefficients Elliptic PDEs, we need to solve a large sparse linear equation. FFT is no longer valid due to the coefficients is dependent on coordinates.

Conjugate Gradient method is good for such linear system, especially with a proper preconditioner. People have proved multigrid method could be a good choice. The resulting algorithm is then called Multigrid Preconditioned Conjugate Gradient Method (MGCG).

2 MGCG

We have to notice that the preconditioner for CG have some constraints: positive definite and symmetric. So, not all multigrid method could be used.

2.1 Smoother

the smoother need to be symmetric, which means Gauss-Seidel and SOR is excluded. We could use damped Jacobi, Red-Black Symmetric Gauss Seidel or Multi-Color Symmetric Successive Over-Relaxation.

I choose RBSGS for it is easy to parallelism. The key ideal is to decompose the grid into two set: one is label as 'red', the other is label as 'black'.

For 2D elliptic equation:

$$-\nabla\epsilon\nabla u + k^2 u = f \tag{1}$$

with ϵ a scalar, it is discretized as :

$$k_{i,j}u_{i,j} + \frac{\epsilon_{i-,j}(u_{i,j}-u_{i-1,j}) + \epsilon_{i+,j}(u_{i,j}-u_{i+1,j})}{h_x^2} + \frac{\epsilon_{i,j-}(u_{i,j}-u_{i,j-1}) + \epsilon_{i,j+}(u_{i,j}-u_{i,j+1})}{h_y^2} = f_{i,j} \tag{2}$$

If (i,j) is 'red', then $(i-1,j),(i+1,j),(i,j-1),(i,j+1)$ is 'black'. So that, the linear equation can be written as:

$$\begin{pmatrix} D_r & C \\ C^T & D_b \end{pmatrix}\begin{pmatrix} u_r \\ u_b \end{pmatrix} = \begin{pmatrix} f_r \\ f_b \end{pmatrix} \tag{3}$$

where D_r and D_b are diagonal matrix, C is sparse square matrix.

RBSGS has three steps:

(1) , $D_r u_r^{(n+1/2)} = f_r - C u_b^{(n)}$

(2) , $D_b u_b^{(n+1)} = f_b - C^T u_r^{(n+1/2)}$

(3) , $D_r u_r^{(n+1)} = f_r - C u_b^{(n+1)}$

In matrix form:

$$\begin{pmatrix} D_r & C \\ C^T & D_b + C^T D_r^{-1} C \end{pmatrix} \begin{pmatrix} u_r^{(n+1)} \\ u_b^{(n+1)} \end{pmatrix} = \begin{pmatrix} f_r \\ f_b \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & C^T D_r^{-1} C \end{pmatrix} \begin{pmatrix} u_r^{(n)} \\ u_b^{(n)} \end{pmatrix} \quad (4)$$

2.2 Projection

To ensure the preconditioner is symmetric, we could first require the restriction operator r from fine grid to coarse grid is the transpose(adjoint) of the prolongation operator p from coarse grid to fine grid: $r = bp^T$ where b is a constant to ensure $r * p = 1$.

2.3 MG circle

We could use V-circle or W-circle. The resulting MGCG satisfy the condition we need.
Full-circle????NOT SURE

3 Advantage

The CG iteration steps is independent with grid size.