Definite Integration

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1 Definition of the Integral and Description of the Set of Integrable Functions

1.1 Introduction

Suppose a point is moving along the real line, with $s(t)$ being its coordinate at time t and $s'(t) = v(t)$ its velocity at the same instant t. Assume that we know the position $S(t_0)$ of the point at time t_0 and that we receive information on its velocity. Having this function, we wish to compute $s(t)$ for any given value of time $t > t_0$.

If we assume that the velocity $v(t)$ varies continuously, the displacement of the point over small time interval can be computed approximately as the product $v(\tau)\Delta t$ of the velocity at an arbitrary instant τ belonging to that time interval and the magnitude Δt of the time interval itself. Taking this observation into account, we partition the interval $[t_0, t]$ by marking some times $t_i, i = 0, 1, \cdots, n$ so that $t_0 < t_1 < \cdots < t_n = t$ and so the interval $[t_{i-1}, t_i]$ are small. Let $\Delta t_i = t_i - t_{i-1}$ and $\tau_i \in [t_{i-1}, t_i]$. Then we have the approximation equality

$$s(t) - s(t_0) \approx \sum_{i=1}^{n} v(\tau_i)\Delta t_i$$

The approximation will become more precise if we partition the close interval into smaller and smaller intervals. Thus we must conclude that in the limit as the length λ of the largest of these intervals tends to zero we shall obtain an exact equality

$$\lim_{\lambda \to 0} \sum_{i=1}^{n} v(\tau_i)\Delta t_i = s(t) - s(t_0) \tag{1}$$

Such sums, called **Riemann sums**, are encountered in a wide variety of situations.

Let us attempt, for example, following Archimedes, to find the area under the parabola $y = x^2$ above the closed interval $[0, 1]$.

$$\lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \frac{1}{3}$$

1.2 Definition of the Riemann Integral

a. Partition

Definition 1.1. A **partition** P of a closed interval $[a, b], a < b$, is a finite system of points $x_0, x_1, \cdots, x_n$ of the interval such that $a = t_0 < t_1 < \cdots < t_n = b$.

The intervals $[t_{i-1}, t_i], i = 1, 2, \cdots, n$ are called the intervals of the partitions P. The largest of the lengths of the intervals of the partition P, denoted $\lambda(P)$, is called the **mesh** of the partition.

Definition 1.2. We speak of a partition with distinguished points (P, ξ) on the closed interval $[a, b]$ if we have a partition P of $[a, b]$ and a point $\xi \in [t_{i-1}, t_i]$ has been chosen in each of the intervals of the partition $[x_{i-1}, x_i], i = 1, 2, \cdots, n$.

We denoted the set of point $(\xi_1, \cdots, \xi_n)$ by the single letter ξ.

b. A Base in the Set of Partitions In the set $\mathcal{P}$ of partitions with distinguished points on a given interval $[a, b]$, we consider the following base $\mathcal{B} = \{B_d\}$. The element $B_d, d > 0$, of the base $\mathcal{B}$ consists of all partitions with distinguished points (P, ξ) on $[a, b]$ for which $\lambda(P) < d$.

c. Riemann Sums

Definition 1.3. If a function f is defined on the closed interval $[a, b]$ and (P, ξ) is a partition with distinguished points on this closed interval, the sum

$$\sigma(f; P, \xi) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i, \tag{2}$$

where $\Delta x_i = x_i - x_{i-1}$, is the **Riemann sum** of the function f corresponding to the partition (P, ξ) with distinguished point on $[a, b]$.

Thus, when the function f is fixed, the Riemann sum $\sigma(f; P, \xi)$ is a function $\Phi(p) = \sigma f; \sigma$ on the set $\mathcal{P}$ of all partitions $p = (P, \xi)$ with distinguished point on the closed interval $[a, b]$. Since there is a base $\mathcal{B}$ in $\mathcal{P}$, one can ask about the limit of the function Φp over the base.

d. The Riemann Integral Let f be a function defined on a closed interval $[a, b]$.

Definition 1.4. uran The number I is the **Riemann integral** of the function f on the closed interval $[a, b]$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left| I - \sum_{i=1}^{n} f(\xi_i) \Delta x_i \right| < \epsilon$$

for any partition (P, ξ) with distinguished points on $[a, b]$ whose mesh $\lambda(P)$ is less than δ.

Since the partition $p = (P, \xi)$ for which $\lambda(P) < \delta$ form the element B_δ of the base $\mathcal{B}$ introduced above in the set $\mathcal{P}$ of partitions with distinguished points, the above definition is equivalent to

$$I = \lim_{\mathcal{B}} \Phi(p)$$

The integral of $f(x)$ over $[a, b]$ is denoted

$$\int_a^b f(x) \, \mathrm{d}x,$$

in which the number a and b are called respectively the lower and upper limits of integration. The function f is called the integrand, $f(x)\mathrm{d}x$ is called the differential form, and x is the variable of integration. Thus

$$\int_a^b f(x) \, \mathrm{d}x = \lim_{\lambda(P) \to 0} \sum_{i=1}^{n} f(\xi_i) \Delta x_i \tag{3}$$

Definition 1.5. A function f is Riemann integrable on the closed interval $[a, b]$ if the limit of the Riemann sums in Eq. 3 exists as $\lambda(P) \to 0$(that is, the Riemann integral of f is defined).

The set of Riemann-integrable functions on a closed interval $[a, b]$ will be denoted $\mathcal{R}[a, b]$.

1.3 The Set of Integrable Functions

The integrability or non-integrability of a function f on $[a, b]$ depends on the existence of the limit below

$$\lim_{\lambda(P) \to 0} \sum_{i=1}^{n} f(\xi_i) \Delta x_i$$

By the Cauchy criterion, this limit exists if and only if for every $\epsilon > 0$ there exists an element $B_\delta \in \mathcal{B}$ in the base such that

$$|\Phi(p') - \Phi(p'')| < \epsilon$$

for any two points $p', p'' \in B_\delta$.

In more detailed notation, what has just been said means that for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|\sigma(f; P', \xi') - \sigma(f; P'', \xi'')| < \epsilon$$

or, what is the same,

$$\left|\sum_{i=1}^{n'} f(\xi_i')\Delta x_i' - \sum_{i=1}^{n''} f(\xi_i'')\Delta x_i''\right| < \epsilon$$

for any partition (P', ξ') and (P'', ξ'') with distinguished points on the interval $[a, b]$ with $\lambda(P') < \delta$ and $\lambda(P'') < \delta$.

a. A Necessary Condition for Integrability.

Proposition 1.1. A necessary condition for a function f defined on a closed interval $[a, b]$ to be Riemann integrable on $[a, b]$ is that f be bounded on $[a, b]$.

b. A Sufficient Condition for Integrability and the Most Important Classes of Integrable Functions We begin with some notation and remarks that will be used in the explanation to follow.

We agree that when a partition P

$$a = x_0 < x_1 < \cdots < x_n = b$$

is given on the interval $[a, b]$, we shall use the symbol Δ_i to denote the interval $[x_{i-1}, x_i]$ along with Δx_i as a notation for the difference $x_i - x_{i-1}$. If a partition $\widetilde{P}$ of the closed interval $[a, b]$ is obtained from a partition P by the jointing new points to P, we call $\widetilde{P}$ a refinement of P. When a refinement $\widetilde{P}$ of a partition P is constructed, some of the closed intervals $\Delta_i = [x_{i-1}, x_i]$ of the partition P themselves undergo partitioning:

$$x_{i-1} = x_{i0} < x_{i1} < \cdots < x_{in_i} = x_i.$$

Proposition 1.2. A sufficient condition for a bounded function f to be integrable on a closed interval $[a, b]$ is that for every $\epsilon > 0$ there exist a number $\delta > 0$ such that

$$\sum_{i=0}^{n} \omega(f; \Delta_i)\Delta x_i < \epsilon$$

for any partition P of $[a, b]$ with mesh $\lambda(P) < \delta$.

Proof. Let P be a partition of $[a,b]$ and $\widetilde{P}$ a refinement of P. Let us estimate the difference between the Riemann sums $\sigma(f;\widetilde{P},\widetilde{\xi}) - \sigma(f;P,\xi)$. Using the notation introduced above, we can write

$$\begin{aligned}
\left|\sigma(f;\widetilde{P},\widetilde{\xi}) - \sigma(f;P,\xi)\right| &= \left|\sum_{i=1}^{n}\sum_{j=1}^{n_j} f(\xi_{ij})\Delta x_{ij} - \sum_{i=1}^{n} f(\xi_i)\Delta x_i\right| \\
&= \left|\sum_{i=1}^{n}\sum_{j=1}^{n_j} f(\xi_{ij})\Delta x_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{n_j} f(\xi_i)\Delta x_{ij}\right| \\
&= \left|\sum_{i=1}^{n}\sum_{j=1}^{n_j} (f(\xi_{ij}) - f(\xi_i))\Delta x_{ij}\right| \leq \sum_{i=1}^{n}\sum_{j=1}^{n_j} |f(\xi_{ij}) - f(\xi_i)|\Delta x_{ij} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n_j} \omega(f;\Delta_i)\Delta x_{ij} = \sum_{i=1}^{n} \omega(f;\Delta x_i)\Delta x_i.
\end{aligned}$$

It follows from the estimation for the difference of the Riemann sums that if the function satisfies the sufficient condition given in the statement of the proposition, then for each $\epsilon > 0$, we can find $\delta > 0$ such that

$$|\sigma(f;\widetilde{P},\widetilde{\xi}) - \sigma(f;P,\xi)| < \frac{\epsilon}{2}$$

Now if (P',ξ') and (P'',ξ'') are arbitrary partitions with distinguished points on $[a,b]$ whose meshes satisfy $\lambda(P') < \delta$ and $\lambda(P'') < \delta$, then, by what has been proved, the partition $\widetilde{P} = P' \cup P''$, we have

$$\left|\sigma(f;\widetilde{P},\widetilde{\xi}) - \sigma(f;P',\xi')\right| < \frac{\epsilon}{2}$$

$$\left|\sigma(f;\widetilde{P},\widetilde{\xi}) - \sigma(f;P'',\xi'')\right| < \frac{\epsilon}{2}$$

It follows that

$$|\sigma(f;P',\xi') - \sigma(f;P'',\xi'')| < \epsilon$$

provided that $\lambda(P') < \delta, \lambda(P'') < \epsilon$. Therefore, by the Cauchy criterion, the limit of the Riemann sums exists:

$$\lim_{\lambda(P)\to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i,$$

that is $f \in \mathcal{R}[a,b]$. ☐

Corollary 1.1. $(f \in C[a,b]) \Rightarrow (f \in \mathcal{R}[a,b])$, that is, every continuous function on a closed interval is integrable on that close interval.

Corollary 1.2. If a bounded function f on a closed interval $[a, b]$ is continuous everywhere except at a finite set of points, then $f \in \mathcal{R}[a, b]$.

Corollary 1.3. A monotonic function on a closed interval is integrable on that interval.

Definition 1.6. Let $f : [a, b] \to \mathbb{R}$ be a real valued function that is defined and bounded on the closed interval $[a, b]$, let P be a partition of $[a, b]$, and let $\Delta_i (i = 1, 2, \cdots, n)$ be the intervals of the partition P. Let $m_i = \inf_{x \in \Delta_i} f(x)$ and $M_i = \sup_{x \in \Delta_i} f(x), i = 1, 2, \cdots, n$.

The sums

$$s(f; P) = \sum_{i=1}^{n} m_i \Delta x_i$$

and

$$S(f; P) = \sum_{i=1}^{n} M_i \Delta x_i$$

are called respectively the lower and upper Riemann sums of the function f on the interval $[a, b]$ corresponding to the partition P of the interval. The sums $s(f; P)$ and $S(f; P)$ are also called the lower and upper **Darboux** sums corresponding to the partition P of $[a, b]$.

If (P, ξ) is an artitrary partition with distinguished points on $[a, b]$, then obviously

$$s(f; P) \leq \sigma(f; P, \xi) \leq S(f; P) \tag{4}$$

Lemma 1.4.

$$s(f; P) = \inf_{\xi} \sigma(f; P, \xi)$$
$$S(f; P) = \sup_{\xi} \sigma(f; P, \xi)$$

Proposition 1.3. A bounded real-valued function $f : [a, b] \to \mathbb{R}$ is Rimann integrable on $[a, b]$ if and only if the following limit exist and are equal to each other:

$$\underline{I} = \lim_{\lambda(P) \to 0} s(f; P); \overline{I} = \lim_{\lambda(P) \to 0} S(f; P). \tag{5}$$

When this happens, the common value $I = \underline{I} = \overline{I}$ is the integral

$$\int_a^b f(x)\, \mathrm{d}x$$

Proposition 1.4. A necessary and sufficient condition for a function $f : [a, b] \to \mathbb{R}$ defined on a closed interval $[a, b]$ to be **Riemann integrable** on $[a, b]$ is the following relation:

$$\lim_{\lambda(P)\to 0} \sum_{i=1}^{n} \omega(f; \Delta_i)\Delta x_i = 0 \tag{6}$$

c. The Vector Space $\mathcal{R}[a, b]$

Proposition 1.5. If $f, g \in \mathcal{R}[a, b]$, then

1. $(f + g) \in \mathcal{R}[a, b]$;
2. $\alpha f \in \mathcal{R}[a, b]$, where α is a numerical coefficient;
3. $|f| \in \mathcal{R}[a, b]$;
4. $f|_{[c,d]} \in \mathcal{R}[a, b]$ if $[c, d] \subset [a, b]$;
5. $(f \cdot g) \in \mathcal{R}[a, b]$.

2 Linearity, Additivity and Monotonicity of the Integral

2.1 The Integral as a Linear Function on the Space $\mathcal{R}[a, b]$

Theorem 2.1. If $f, g \in \mathcal{R}[a, b]$, then $\alpha f + \beta g \in \mathcal{R}[a, b]$, and

$$\int_a^b (\alpha f + \beta g)\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x + \beta \int_a^b g(x)\,\mathrm{d}x$$

Remark. *To avoid any possible confusion, functions defined on functions are usually called functionals. Thus we have proved that the integral is a liner functional on the vector space $\mathcal{R}[a, b]$ of integrable functions.*

2.2 The Integral as a Additive Function of the Interval of Integration

The value of the integral $\int_a^b f(x)\,\mathrm{d}x = I(f; [a, b])$ depends on both the integrand and the closed interval $[a, b]$ over which the integral is taken. For example, if $f \in \mathcal{R}[a, b]$, then, as we know, $f|_{[\alpha,\beta]} \in \mathcal{R}[\alpha, \beta]$ if $[\alpha, \beta] \subset [a, b]$, that is $\int_\alpha^\beta f(x)\,\mathrm{d}x$ is defined.

Lemma 2.2. if $a < b < c$ and $f \in \mathcal{R}[a, c]$, then $f|_{[a,b]} \in \mathcal{R}[a, b], f|_{[b,c]} \in \mathcal{R}[b, c]$, and the following equality holds:

$$\int_a^c f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x$$

From the definition of integral, we have: if $a > b$ then,

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$$

In this connection, it is also natural to set

$$\int_a^a f(x)\,\mathrm{d}x = 0$$

Theorem 2.3. Let $a, b, c \in \mathbb{R}$ and let f be a function integrable over the largest closed interval having two of these points as endpoints. Then the restriction of f to each of the other closed interval is also integrable over those intervals and the following equality holds:

$$\int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x + \int_c^a f(x)\,\mathrm{d}x = 0$$

Definition 2.1. Suppose that to each (α, β) of points $\alpha, \beta \in [a, b]$ a number $I(\alpha, \beta)$ is assigned so that

$$I(\alpha, \gamma) = I(\alpha, \beta) + I(\beta, \gamma)$$

for any triple point α, β, γ. Then the function $I(\alpha, \beta)$ is called an additive(oriented) interval function defined on intervals contained in $[a, b]$.

If $f \in [A, B]$, and $a, b, c \in [A, B]$, then, setting $I(a, b) = \int_a^b f(x)\,\mathrm{d}x$, we conclude that

$$\int_a^c f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x.$$

that is, the integral is an additive interval function on the interval of integration.

2.3 Estimation of the Integral, Monotonicity of the Integral, and the Mean-Value Theorem

2.3.1 A General Estimation of the Integral.

Theorem 2.4. If $a \leq b$ and $f \in \mathcal{R}[a, b]$, then $|f| \in \mathcal{R}[a, b]$ and the following inequality holds

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f|(x)\,\mathrm{d}x$$

If $|f|(x) \leq C$ on $[a, b]$ then

$$\int_a^b |f| \, \mathrm{d}x \leq C(b-a)$$

2.3.2 Monotonicity of the Integral and the First Mean-Value Theorem

Theorem 2.5. If $a \leq b, f_1, f_2 \in \mathcal{R}[a, b], f_1(x) \leq f_2(x), \forall x \in [a, b]$, then

$$\int_a^b f_1(x) \, \mathrm{d}x \leq \int_a^b f_2(x) \, \mathrm{d}x$$

Corollary 2.6. If $a \leq b, f \in \mathcal{R}[a, b], m \leq f(x) \leq M, \forall x \in [a, b]$, then

$$m(b-a) \leq \int_a^b f(x) \, \mathrm{d}x \leq M(b-a)$$

Corollary 2.7. If $a \leq b, f \in \mathcal{R}[a, b], m = \int_{x \in [a,b]} f(x), M = \sup_{x \in [a,b]} f(x)$, then there exists a number $\mu \in [m, M]$ such that

$$\int_a^b f(x) \, \mathrm{d}x = \mu(b-a)$$

Corollary 2.8. If $f \in C[a, b]$, there exists a point $\xi \in [a, b]$ such that

$$\int_a^b f(x) \, \mathrm{d}x = f(\xi)(b-a) \tag{7}$$

Remark. *The equality Equation(7) is often called **the first mean-value theorem**. We, however, reserve that name for the following somewhat more general proposition.*

Theorem 2.9 ((**First Mean-Value Theorem**))**.** Let $f, g \in \mathcal{R}[a, b], m = \inf_{x \in [a,b]} f(x), M = \sup_{x \in [a,b]} f(x)$. If g is nonnegative (or nonpositive) on $[a, b]$, then

$$\int_a^b (fg) \, \mathrm{d}x = \mu \int_a^b g(x) \, \mathrm{d}x$$

where $\mu \in [m, M]$ If, in addition, it is known that $f \in C[a, b]$, then there exits a point $\xi \in [a, b]$ such that

$$\int_a^b (fg) \, \mathrm{d}x = f(\xi) \int_a^b g(x) \, \mathrm{d}x$$

Abel's Transformation Let $A_k = \sum\limits_{i=1}^{k} a_i, A_0 = 0$, then

$$\begin{aligned}
\sum_{i=1}^{n} a_i b_i &= \sum_{i=1}^{n}(A_i - A_{i-1})b_i = \sum_{i=1}^{n} A_i b_i - \sum_{i=1}^{n} A_{i-1} b_i \\
&= \sum_{i=1}^{n} A_i b_i - \sum_{i=0}^{n-1} A_i b_{i+1} = A_n b_n - A_0 b_1 + \sum_{i=1}^{n-1} A_i (b_i - b_{i-1}) \\
&= A_n b_n - A_0 b_1 + \sum_{i=1}^{n-1} A_i (b_i - b_{i-1}) \\
&= A_n b_n + \sum_{i=1}^{n-1} A_i (b_i - b_{i-1})
\end{aligned}$$

Lemma 2.10. If the numbers $A_k = \sum_{i=1}^{k} a_i (k = 1, 2, \cdots, n)$ satisfy the inequality $m \le A_k \le M$ and the numbers $b_i, i = 1, 2, \cdots, n$ are nonnegative and $b_i \ge b_{i+1}$ for $i = 1, 2, \cdots, n-1$, then,

$$mb_1 \le \sum_{i=1}^{n} a_i b_i \le M b_1$$

Lemma 2.11. If $f \in \mathcal{R}[a, b]$, then for any $x \in [a, b]$ the function

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

is defined and $F(x) \in C[a, b]$.

Lemma 2.12. If $f, g \in \mathcal{R}[a, b]$ and g is a non-negative non-increasing function on $[a, b]$ then there exists a point $\xi \in [a, b]$ such that

$$\int_a^b (fg)\,\mathrm{d}x = g(a) \int_a^{\xi} f(x)\,\mathrm{d}x.$$

Theorem 2.13 (Second mean-value theorem for the integral)**.** *If $f, g \in \mathcal{R}[a, b]$ and g is a monotonic function on $[a, b]$, then there exists a point $\xi \in [a, b]$ such that*

$$\int_a^b (fg)\,\mathrm{d}x = g(a) \int_a^{\xi} f(x)\,\mathrm{d}x + g(b) \int_{\xi}^{b} f(x)\,\mathrm{dx}$$

3 The Integral and the Derivative

3.1 The Integral and the Primitive

Let f be a Riemann-integrable function on a closed interval $[a, b]$. On this interval let us consider the function

$$F(x) = \int_a^x f(t)\,\mathrm{d}t \tag{8}$$

often called an integral with variable upper bound limit.

Since $f \in \mathcal{R}[a,b]$, if follows that $f|_{[a,x]} \in \mathcal{R}[a,x]$ if $[a,x] \subset [a,b]$, therefore the function

$$x \to F(x)$$

is unambiguously defined for $x \in [a,b]$.

Lemma 3.1. If $f \in \mathcal{R}[a,b]$ and the function f is continuous at a point $x \in [a,b]$, then the function F defined on $[a,b]$ by Equation(8) is differentiable at the point x, and the following equality holds:

$$F'(x) = f(x)$$

Theorem 3.2. Every continuous function $f : [a,b] \to \mathbb{R}$ on the closed interval $[a,b]$ has a primitive, and every primitive of f on $[a,b]$ has the form

$$\mathcal{F}(x) = \int_a^x f(t)\,\mathrm{d}t + C$$

Definition 3.1. A continuous function $x \to F(x)$ on an interval of the real line is called a primitive (or generalized primitive) of the function $x \to f(x)$ defined on the same interval if the relation $\mathcal{F}'(x) = f(x)$ holds at all points of the interval, with only a finite number of exceptions.

3.2 The Newton-Leibniz Formula

Theorem 3.3. If $f : [a,b] \to \mathbb{R}$ is a bounded function with a finite number of points of discontinuity, then $f \in \mathcal{R}[a,b]$ and

$$\int_a^b f(x)\,\mathrm{d}x = \mathcal{F}(b) - \mathcal{F}(a) \tag{9}$$

where $\mathcal{F} : [a,b] \to \mathbb{R}$ is any primitive of f on $[a,b]$.

Remark. *Relation(9), which is fundamental for all of analysis, is called the Newton-Leibniz formula(or fundamental theorem of calculus.)*

3.3 Integration by Parts in the Definite Integral and Taylor's Formula

Proposition 3.1. If the function $u(x)$ and $v(x)$ are continuously differentiable on a closed interval with endpoints a and b, then

$$\int_a^b (u(x)v'(x))\,\mathrm{d}x = (uv)|_a^b - \int_a^b (v(x)u'(x))\,\mathrm{d}x$$

or

$$\int_a^b u(x)\,\mathrm{d}v(x) = (uv)|_a^b - \int_a^b v(x)\,\mathrm{d}u(x)$$

As a corollary we now obtain the Taylor formula with integral form of the remainder. Suppose on the closed interval with endpoints a and x the function $t \to f(t)$ has n continuous derivatives, we have

$$
\begin{aligned}
f(x) - f(a) &= \int_a^x f'(t)\,\mathrm{d}t = -\int_a^x \int_a^x f'(t)(x-t)'\,\mathrm{d}t \\
&= -f'(t)(x-t)|_a^x + \int_a^x f''(t)(x-t)\,\mathrm{d}t \\
&= f'(a)(x-a) - \frac{1}{2}\int_a^x f''(t)[(x-t)^2]'\,\mathrm{d}t \\
&= f'(a)(x-a) - \frac{1}{2}f''(t)(x-t)^2|_a^x + \frac{1}{2}\int_a^x f'''(t)(x-t)^2\,\mathrm{d}t \\
&= f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 - \frac{1}{2\cdot 3}\int_a^x f'''(t)[(x-t)^3]'\,\mathrm{d}t \\
&= \cdots \\
&= f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots + \frac{1}{(n-1)!}f^{(n-1)}(a)(x-a)^{n-1} + r_{n-1}(a;x)
\end{aligned}
$$

where

$$
r_{n-1}(a;x) = \frac{1}{(n-1)!}\int_a^x f^{(n)}(t)(x-t)^{n-1}\,\mathrm{d}t. \tag{10}
$$

Proposition 3.2. If the function $t \to f(t)$ has continuous derivatives up to order n inclusive on the closed interval with endpoints a and x, then Taylor's formula holds:

$$
f(x) = f(a) + \frac{1}{1!}f'(a)(x-a) + \cdots + \frac{1}{(n-1)!}f^{(n-1)}(a)(x-a)^{n-1} + r_{n-1}(a;x),
$$

with remainder term $r_{n-1}(a;x)$ represented in the integral form (10).

We note that from the First Mean-Value Theorem, we can derive at the Lagrange remainder.

3.4 Change of Variable in an Integral

Proposition 3.3. If $\varphi : [\alpha, \beta] \to [a, b]$ is a continuously differentiable mapping of the closed interval $[\alpha, \beta]$ into the closed interval $[a, b]$ such that $\varphi(\alpha) = a$ and $\varphi(\beta) = b$, then for any continuous function $f(x)$ on $[a, b]$ the function $f(\varphi(t))\varphi'(t)$ is continuous on the closed interval $[\alpha, \beta]$, and

$$
\int_a^b f(x)\,\mathrm{d}x = \int_\alpha^\beta f(\varphi(t))\varphi'(t)\,\mathrm{d}t. \tag{11}
$$

3.5 Some Examples

Examples 1.

$$\int_{-1}^{1} \sqrt{1 - x^2}\,\mathrm{d}x$$

Examples 2.

$$\int_{-\pi}^{\pi} \sin mx \cos nx\,\mathrm{d}x = 0,$$

$$\int_{-\pi}^{\pi} \sin^2 mx, \mathrm{d}x = \pi$$

$$\int_{-\pi}^{\pi} \cos^2 mx, \mathrm{d}x = \pi,$$

$m, n \in \mathbb{N}$.

4 作業

4.1 證明題

1. 證明：若分割$\widetilde{P}$ 是分割P增加若干分點得到的分割，則有：

$$\sum_{\widetilde{P}} \omega_i' \Delta x_i' \le \sum_{P} \omega_i \Delta x_i$$

2. 證明：若f在$[a,b]$上可積，$[\alpha,\beta] \subset [a,b]$, 則$f$在$[\alpha,\beta]$上也可積。

3. 設f,g均為定義在$[a,b]$上的有界函數，僅在有限個點處$f(x) \neq g(x)$, 證明：若f在$[a,b]$上可積，則g在$[a,b]$上也可積，且有：

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b g(x)\,\mathrm{d}x$$

4. 設f在$[a,b]$上有界，$\{a_n\} \subset [a,b]$, $\lim\limits_{n\to\infty} a_n = c$, 證明：若$f$在$[a,b]$上只有$a_n, n = 1,2,\cdots$為其間斷點，則$f$在$[a,b]$ 上可積。

5. 證明：若$f \in C[a,b]$ 且$f(x) \ge 0, \forall x \in [a,b]$ 則以下結果成立：

   (a) 如果函數$f(x)$存在一點$f(x_0) > 0, x_0 \in [a,b]$，則有：

   $$\int_a^b f(x)\,\mathrm{d}x > 0$$

   (b) 若$\int_a^b f(x) = 0$，則有$f(x) \equiv 0$

6. 證明若$f \in C[a,b], f(x) \ge 0, \forall x \in [a,b]$, 且$M = \max\limits_{[a,b]} f(x)$,則

$$\lim_{n\to\infty} \left(\int_a^b f^n(x)\,\mathrm{d}x \right)^{\frac{1}{n}} = M$$