

# Definite Integration

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## 1 Definition of the Integral and Description of the Set of Integrable Functions

### 1.1 Introduction

Suppose a point is moving along the real line, with  $s(t)$  being its coordinate at time  $t$  and  $s'(t) = v(t)$  its velocity at the same instant  $t$ . Assume that we know the position  $S(t_0)$  of the point at time  $t_0$  and that we receive information on its velocity. Having this function, we wish to compute  $s(t)$  for any given value of time  $t > t_0$ .

If we assume that the velocity  $v(t)$  varies continuously, the displacement of the point over small time interval can be computed approximately as the product  $v(\tau)\Delta t$  of the velocity at an arbitrary instant  $\tau$  belonging to that time interval and the magnitude  $\Delta t$  of the time interval itself. Taking this observation into account, we partition the interval  $[t_0, t]$  by marking some times  $t_i, i = 0, 1, \dots, n$  so that  $t_0 < t_1 < \dots < t_n = t$  and so the interval  $[t_{i-1}, t_i]$  are small. Let  $\Delta t_i = t_i - t_{i-1}$  and  $\tau_i \in [t_{i-1}, t_i]$ . Then we have the approximation equality

$$s(t) - s(t_0) \approx \sum_{i=1}^n v(\tau_i) \Delta t_i$$

The approximation will become more precise if we partition the close interval into smaller and smaller intervals. Thus we must conclude that in the limit as the length  $\lambda$  of the largest of these intervals tends to zero we shall obtain an exact equality

$$\lim_{\lambda \rightarrow 0} \sum_{i=1}^n v(\tau_i) \Delta t_i = s(t) - s(t_0) \quad (1)$$

Such sums, called **Riemann sums**, are encountered in a wide variety of situations.

Let us attempt, for example, following Archimedes, to find the area under the parabola  $y = x^2$  above the closed interval  $[0, 1]$ .

$$\lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i = \frac{1}{3}$$

## 1.2 Definition of the Riemann Integral

### a. Partition

**Definition 1.1.** A **partition**  $P$  of a closed interval  $[a, b]$ ,  $a < b$ , is a finite system of points  $x_0, x_1, \dots, x_n$  of the interval such that  $a = t_0 < t_1 < \dots < t_n = b$ .

The intervals  $[t_{i-1}, t_i]$ ,  $i = 1, 2, \dots, n$  are called the intervals of the partitions  $P$ . The largest of the lengths of the intervals of the partition  $P$ , denoted  $\lambda(P)$ , is called the **mesh** of the partition.

**Definition 1.2.** We speak of a partition with distinguished points  $(P, \xi)$  on the closed interval  $[a, b]$  if we have a partition  $P$  of  $[a, b]$  and a point  $\xi \in [t_{i-1}, t_i]$  has been chosen in each of the intervals of the partition  $[x_{i-1}, x_i]$ ,  $i = 1, 2, \dots, n$ .

We denoted the set of point  $(\xi_1, \dots, \xi_n)$  by the single letter  $\xi$ .

**b. A Base in the Set of Partitions** In the set  $\mathcal{P}$  of partitions with distinguished points on a given interval  $[a, b]$ , we consider the following base  $\mathcal{B} = \{B_d\}$ . The element  $B_d$ ,  $d > 0$ , of the base  $\mathcal{B}$  consists of all partitions with distinguished points  $(P, \xi)$  on  $[a, b]$  for which  $\lambda(P) < d$ .

### c. Riemann Sums

**Definition 1.3.** If a function  $f$  is defined on the closed interval  $[a, b]$  and  $(P, \xi)$  is a partition with distinguished points on this closed interval, the sum

$$\sigma(f; P, \xi) = \sum_{i=1}^n f(\xi_i) \Delta x_i, \quad (2)$$

where  $\Delta x_i = x_i - x_{i-1}$ , is the **Riemann sum** of the function  $f$  corresponding to the partition  $(P, \xi)$  with distinguished point on  $[a, b]$ .

Thus, when the function  $f$  is fixed, the Riemann sum  $\sigma(f; P, \xi)$  is a function  $\Phi(p) = \sigma f; \sigma$  on the set  $\mathcal{P}$  of all partitions  $p = (P, \xi)$  with distinguished point on the closed interval  $[a, b]$ . Since there is a base  $\mathcal{B}$  in  $\mathcal{P}$ , one can ask about the limit of the function  $\Phi p$  over the base.

**d. The Riemann Integral** Let  $f$  be a function defined on a closed interval  $[a, b]$ .

**Definition 1.4.** The number  $I$  is the **Riemann integral** of the function  $f$  on the closed interval  $[a, b]$  if for every  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$\left| I - \sum_{i=1}^n f(\xi_i) \Delta x_i \right| < \epsilon$$

for any partition  $(P, \xi)$  with distinguished points on  $[a, b]$  whose mesh  $\lambda(P)$  is less than  $\delta$ .

Since the partition  $p = (P, \xi)$  for which  $\lambda(P) < \delta$  form the element  $B_\delta$  of the base  $\mathcal{B}$  introduced above in the set  $\mathcal{P}$  of partitions with distinguished points, the above definition is equivalent to

$$I = \lim_{\mathcal{B}} \Phi(p)$$

The integral of  $f(x)$  over  $[a, b]$  is denoted

$$\int_a^b f(x) \, dx,$$

in which the number  $a$  and  $b$  are called respectively the lower and upper limits of integration. The function  $f$  is called the integrand,  $f(x)dx$  is called the differential form, and  $x$  is the variable of integration. Thus

$$\int_a^b f(x) \, dx = \lim_{\lambda(P) \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i \quad (3)$$

**Definition 1.5.** A function  $f$  is Riemann integrable on the closed interval  $[a, b]$  if the limit of the Riemann sums in Eq. 3 exists as  $\lambda(P) \rightarrow 0$  (that is, the Riemann integral of  $f$  is defined).

The set of Riemann-integrable functions on a closed interval  $[a, b]$  will be denoted  $\mathcal{R}[a, b]$ .

### 1.3 The Set of Integrable Functions

The integrability or non-integrability of a function  $f$  on  $[a, b]$  depends on the existence of the limit below

$$\lim_{\lambda(P) \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i$$

By the Cauchy criterion, this limit exists if and only if for every  $\epsilon > 0$  there exists an element  $B_\delta \in \mathcal{B}$  in the base such that

$$|\Phi(p') - \Phi(p'')| < \epsilon$$

for any two points  $p', p'' \in B_\delta$ .

In more detailed notation, what has just been said means that for any  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$|\sigma(f; P', \xi') - \sigma(f; P'', \xi'')| < \epsilon$$

or, what is the same,

$$\left| \sum_{i=1}^{n'} f(\xi'_i) \Delta x'_i - \sum_{i=1}^{n''} f(\xi''_i) \Delta x''_i \right| < \epsilon$$

for any partition  $(P', \xi')$  and  $(P'', \xi'')$  with distinguished points on the interval  $[a, b]$  with  $\lambda(P') < \delta$  and  $\lambda(P'') < \delta$ .

#### a. A Necessary Condition for Integrability.

**Proposition 1.1.** A necessary condition for a function  $f$  defined on a closed interval  $[a, b]$  to be Riemann integrable on  $[a, b]$  is that  $f$  be bounded on  $[a, b]$ .

**b. A Sufficient Condition for Integrability and the Most Important Classes of Integrable Functions** We begin with some notation and remarks that will be used in the explanation to follow.

We agree that when a partition  $P$

$$a = x_0 < x_1 < \cdots < x_n = b$$

is given on the interval  $[a, b]$ , we shall use the symbol  $\Delta_i$  to denote the interval  $[x_{i-1}, x_i]$  along with  $\Delta x_i$  as a notation for the difference  $x_i - x_{i-1}$ . If a partition  $\tilde{P}$  of the closed interval  $[a, b]$  is obtained from a partition  $P$  by the jointing new points to  $P$ , we call  $\tilde{P}$  a refinement of  $P$ . When a refinement  $\tilde{P}$  of a partition  $P$  is constructed, some of the closed intervals  $\Delta_i = [x_{i-1}, x_i]$  of the partition  $P$  themselves undergo partitioning:

$$x_{i-1} = x_{i0} < x_{i1} < \cdots < x_{in_i} = x_i.$$

**Proposition 1.2.** A sufficient condition for a bounded function  $f$  to be integrable on a closed interval  $[a, b]$  is that for every  $\epsilon > 0$  there exist a number  $\delta > 0$  such that

$$\sum_{i=0}^n \omega(f; \Delta_i) \Delta x_i < \epsilon$$

for any partition  $P$  of  $[a, b]$  with mesh  $\lambda(P) < \delta$ .

*Proof.* Let  $P$  be a partition of  $[a, b]$  and  $\tilde{P}$  a refinement of  $P$ . Let us estimate the difference between the Riemann sums  $\sigma(f; \tilde{P}, \tilde{\xi}) - \sigma(f; P, \xi)$ . Using the notation introduced above, we can write

$$\begin{aligned}
\left| \sigma(f; \tilde{P}, \tilde{\xi}) - \sigma(f; P, \xi) \right| &= \left| \sum_{i=1}^n \sum_{j=1}^{n_j} f(\xi_{ij}) \Delta x_{ij} - \sum_{i=1}^n f(\xi_i) \Delta x_i \right| \\
&= \left| \sum_{i=1}^n \sum_{j=1}^{n_j} f(\xi_{ij}) \Delta x_{ij} - \sum_{i=1}^n \sum_{j=1}^{n_j} f(\xi_i) \Delta x_{ij} \right| \\
&= \left| \sum_{i=1}^n \sum_{j=1}^{n_j} (f(\xi_{ij}) - f(\xi_i)) \Delta x_{ij} \right| \leq \sum_{i=1}^n \sum_{j=1}^{n_j} |f(\xi_{ij}) - f(\xi_i)| \Delta x_{ij} \\
&= \sum_{i=1}^n \sum_{j=1}^{n_j} \omega(f; \Delta_i) \Delta x_{ij} = \sum_{i=1}^n \omega(f; \Delta x_i) \Delta x_i.
\end{aligned}$$

It follows from the estimation for the difference of the Riemann sums that if the function satisfies the sufficient condition given in the statement of the proposition, then for each  $\epsilon > 0$ , we can find  $\delta > 0$  such that

$$|\sigma(f; \tilde{P}, \tilde{\xi}) - \sigma(f; P, \xi)| < \frac{\epsilon}{2}$$

Now if  $(P', \xi')$  and  $(P'', \xi'')$  are arbitrary partitions with distinguished points on  $[a, b]$  whose meshes satisfy  $\lambda(P') < \delta$  and  $\lambda(P'') < \delta$ , then, by what has been proved, the partition  $\tilde{P} = P' \cup P''$ , we have

$$|\sigma(f; \tilde{P}, \tilde{\xi}) - \sigma(f; P', \xi')| < \frac{\epsilon}{2}$$

$$|\sigma(f; \tilde{P}, \tilde{\xi}) - \sigma(f; P'', \xi'')| < \frac{\epsilon}{2}$$

It follows that

$$|\sigma(f; P', \xi') - \sigma(f; P'', \xi'')| < \epsilon$$

provided that  $\lambda(P') < \delta$ ,  $\lambda(P'') < \delta$ . Therefore, by the Cauchy criterion, the limit of the Riemann sums exists:

$$\lim_{\lambda(P) \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i,$$

that is  $f \in \mathcal{R}[a, b]$ . □

**Corollary 1.1.**  $(f \in C[a, b]) \Rightarrow (f \in \mathcal{R}[a, b])$ , that is, every continuous function on a closed interval is integrable on that close interval.

**Corollary 1.2.** If a bounded function  $f$  on a closed interval  $[a, b]$  is continuous everywhere except at a finite set of points, then  $f \in \mathcal{R}[a, b]$ .

**Corollary 1.3.** A monotonic function on a closed interval is integrable on that interval.

**Definition 1.6.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a real valued function that is defined and bounded on the closed interval  $[a, b]$ , let  $P$  be a partition of  $[a, b]$ , and let  $\Delta_i (i = 1, 2, \dots, n)$  be the intervals of the partition  $P$ . Let  $m_i = \inf_{x \in \Delta_i} f(x)$  and  $M_i = \sup_{x \in \Delta_i} f(x), i = 1, 2, \dots, n$ .

The sums

$$s(f; P) = \sum_{i=1}^n m_i \Delta x_i$$

and

$$S(f; P) = \sum_{i=1}^n M_i \Delta x_i$$

are called respectively the lower and upper Riemann sums of the function  $f$  on the interval  $[a, b]$  corresponding to the partition  $P$  of the interval. The sums  $s(f; P)$  and  $S(f; P)$  are also called the lower and upper **Darboux** sums corresponding to the partition  $P$  of  $[a, b]$ .

If  $(P, \xi)$  is an arbitrary partition with distinguished points on  $[a, b]$ , then obviously

$$s(f; P) \leq \sigma(f; P, \xi) \leq S(f; P) \quad (4)$$

**Lemma 1.4.**

$$s(f; P) = \inf_{\xi} \sigma(f; P, \xi)$$

$$S(f; P) = \sup_{\xi} \sigma(f; P, \xi)$$

**Proposition 1.3.** A bounded real-valued function  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable on  $[a, b]$  if and only if the following limit exist and are equal to each other:

$$\underline{I} = \lim_{\lambda(P) \rightarrow 0} s(f; P); \bar{I} = \lim_{\lambda(P) \rightarrow 0} S(f; P). \quad (5)$$

When this happens, the common value  $I = \underline{I} = \bar{I}$  is the integral

$$\int_a^b f(x) \, dx$$

**Proposition 1.4.** A necessary and sufficient condition for a function  $f : [a, b] \rightarrow \mathbb{R}$  defined on a closed interval  $[a, b]$  to be **Riemann integrable** on  $[a, b]$  is the following relation:

$$\lim_{\lambda(P) \rightarrow 0} \sum_{i=1}^n \omega(f; \Delta_i) \Delta x_i = 0 \quad (6)$$

### c. The Vector Space $\mathcal{R}[a, b]$

**Proposition 1.5.** If  $f, g \in \mathcal{R}[a, b]$ , then

1.  $(f + g) \in \mathcal{R}[a, b]$ ;
2.  $\alpha f \in \mathcal{R}[a, b]$ , where  $\alpha$  is a numerical coefficient;
3.  $|f| \in \mathcal{R}[a, b]$ ;
4.  $f|_{[c, d]} \in \mathcal{R}[a, b]$  if  $[c, d] \subset [a, b]$ ;
5.  $(f \cdot g) \in \mathcal{R}[a, b]$ .

## 2 Linearity, Additivity and Monotonicity of the Integral

### 2.1 The Integral as a Linear Function on the Space $\mathcal{R}[a, b]$

**Theorem 2.1.** If  $f, g \in \mathcal{R}[a, b]$ , then  $\alpha f + \beta g \in \mathcal{R}[a, b]$ , and

$$\int_a^b (\alpha f + \beta g) \, dx = \alpha \int_a^b f(x) \, dx + \beta \int_a^b g(x) \, dx$$

**Remark.** To avoid any possible confusion, functions defined on functions are usually called *functionals*. Thus we have proved that the integral is a linear functional on the vector space  $\mathcal{R}[a, b]$  of integrable functions.

### 2.2 The Integral as a Additive Function of the Interval of Integration

The value of the integral  $\int_a^b f(x) \, dx = I(f; [a, b])$  depends on both the integrand and the closed interval  $[a, b]$  over which the integral is taken. For example, if  $f \in \mathcal{R}[a, b]$ , then, as we know,  $f|_{[\alpha, \beta]} \in \mathcal{R}[\alpha, \beta]$  if  $[\alpha, \beta] \subset [a, b]$ , that is  $\int_\alpha^\beta f(x) \, dx$  is defined.

**Lemma 2.2.** if  $a < b < c$  and  $f \in \mathcal{R}[a, c]$ , then  $f|_{[a, b]} \in \mathcal{R}[a, b]$ ,  $f|_{[b, c]} \in \mathcal{R}[b, c]$ , and the following equality holds:

$$\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx$$

From the definition of integral, we have: if  $a > b$  then,

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

In this connection, it is also natural to set

$$\int_a^a f(x) dx = 0$$

**Theorem 2.3.** Let  $a, b, c \in \mathbb{R}$  and let  $f$  be a function integrable over the largest closed interval having two of these points as endpoints. Then the restriction of  $f$  to each of the other closed interval is also integrable over those intervals and the following equality holds:

$$\int_a^b f(x) dx + \int_b^c f(x) dx + \int_c^a f(x) dx = 0$$

**Definition 2.1.** Suppose that to each  $(\alpha, \beta)$  of points  $\alpha, \beta \in [a, b]$  a number  $I(\alpha, \beta)$  is assigned so that

$$I(\alpha, \gamma) = I(\alpha, \beta) + I(\beta, \gamma)$$

for any triple point  $\alpha, \beta, \gamma$ . Then the function  $I(\alpha, \beta)$  is called an additive(oriented) interval function defined on intervals contained in  $[a, b]$ .

If  $f \in [A, B]$ , and  $a, b, c \in [A, B]$ , then, setting  $I(a, b) = \int_a^b f(x) dx$ , we conclude that

$$\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx.$$

that is, the integral is an additive interval function on the interval of integration.

## 2.3 Estimation of the Integral, Monotonicity of the Integral, and the Mean-Value Theorem

### 2.3.1 A General Estimation of the Integral.

**Theorem 2.4.** If  $a \leq b$  and  $f \in \mathcal{R}[a, b]$ , then  $|f| \in \mathcal{R}[a, b]$  and the following inequality holds

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f|(x) dx$$



If  $|f|(x) \leq C$  on  $[a, b]$  then

$$\int_a^b |f| \, dx \leq C(b - a)$$

### 2.3.2 Monotonicity of the Integral and the First Mean-Value Theorem

**Theorem 2.5.** If  $a \leq b$ ,  $f_1, f_2 \in \mathcal{R}[a, b]$ ,  $f_1(x) \leq f_2(x)$ ,  $\forall x \in [a, b]$ , then

$$\int_a^b f_1(x) \, dx \leq \int_a^b f_2(x) \, dx$$

**Corollary 2.6.** If  $a \leq b$ ,  $f \in \mathcal{R}[a, b]$ ,  $m \leq f(x) \leq M$ ,  $\forall x \in [a, b]$ , then

$$m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a)$$

**Corollary 2.7.** If  $a \leq b$ ,  $f \in \mathcal{R}[a, b]$ ,  $m = \inf_{x \in [a, b]} f(x)$ ,  $M = \sup_{x \in [a, b]} f(x)$ , then there exists a number  $\mu \in [m, M]$  such that

$$\int_a^b f(x) \, dx = \mu(b - a)$$

**Corollary 2.8.** If  $f \in C[a, b]$ , there exists a point  $\xi \in [a, b]$  such that

$$\int_a^b f(x) \, dx = f(\xi)(b - a) \tag{7}$$

**Remark.** The equality Equation(7) is often called **the first mean-value theorem**. We, however, reserve that name for the following somewhat more general proposition.

**Theorem 2.9 ((First Mean-Value Theorem)).** Let  $f, g \in \mathcal{R}[a, b]$ ,  $m = \inf_{x \in [a, b]} f(x)$ ,  $M = \sup_{x \in [a, b]} f(x)$ . If  $g$  is nonnegative (or nonpositive) on  $[a, b]$ , then

$$\int_a^b (fg) \, dx = \mu \int_a^b g(x) \, dx$$

where  $\mu \in [m, M]$ . If, in addition, it is known that  $f \in C[a, b]$ , then there exists a point  $\xi \in [a, b]$  such that

$$\int_a^b (fg) \, dx = f(\xi) \int_a^b g(x) \, dx$$

**Abel's Transformation** Let  $A_k = \sum_{i=1}^k a_i$ ,  $A_0 = 0$ , then

$$\begin{aligned}
\sum_{i=1}^n a_i b_i &= \sum_{i=1}^n (A_i - A_{i-1}) b_i = \sum_{i=1}^n A_i b_i - \sum_{i=1}^n A_{i-1} b_i \\
&= \sum_{i=1}^n A_i b_i - \sum_{i=0}^{n-1} A_i b_{i+1} = A_n b_n - A_0 b_1 + \sum_{i=1}^{n-1} A_i (b_i - b_{i+1}) \\
&= A_n b_n - A_0 b_1 + \sum_{i=1}^{n-1} A_i (b_i - b_{i-1}) \\
&= A_n b_n + \sum_{i=1}^{n-1} A_i (b_i - b_{i-1})
\end{aligned}$$

**Lemma 2.10.** If the numbers  $A_k = \sum_{i=1}^k a_i$  ( $k = 1, 2, \dots, n$ ) satisfy the inequality  $m \leq A_k \leq M$  and the numbers  $b_i$ ,  $i = 1, 2, \dots, n$  are nonnegative and  $b_i \geq b_{i+1}$  for  $i = 1, 2, \dots, n-1$ , then,

$$mb_1 \leq \sum_{i=1}^n a_i b_i \leq Mb_1$$

**Lemma 2.11.** If  $f \in \mathcal{R}[a, b]$ , then for any  $x \in [a, b]$  the function

$$F(x) = \int_a^x f(t) dt$$

is defined and  $F(x) \in C[a, b]$ .

**Lemma 2.12.** If  $f, g \in \mathcal{R}[a, b]$  and  $g$  is a non-negative non-increasing function on  $[a, b]$  then there exists a point  $\xi \in [a, b]$  such that

$$\int_a^b (fg) dx = g(a) \int_a^\xi f(x) dx.$$

**Theorem 2.13** (Second mean-value theorem for the integral). *If  $f, g \in \mathcal{R}[a, b]$  and  $g$  is a monotonic function on  $[a, b]$ , then there exists a point  $\xi \in [a, b]$  such that*

$$\int_a^b (fg) dx = g(a) \int_a^\xi f(x) dx + g(b) \int_\xi^b f(x) dx$$

### 3 作業

#### 3.1 證明題

1. 證明：若分割 $\tilde{P}$  是分割 $P$ 增加若干分點得到的分割，則有：

$$\sum_{\tilde{P}} \omega'_i \Delta x'_i \leq \sum_P \omega_i \Delta x_i$$

2. 證明：若 $f$ 在 $[a, b]$ 上可積， $[\alpha, \beta] \subset [a, b]$ ，則 $f$ 在 $[\alpha, \beta]$ 上也可積。
3. 設 $f, g$ 均為定義在 $[a, b]$ 上的有界函數，僅在有限個點處 $f(x) \neq g(x)$ ，證明：若 $f$ 在 $[a, b]$ 上可積，則 $g$ 在 $[a, b]$ 上也可積，且有：

$$\int_a^b f(x) dx = \int_a^b g(x) dx$$

4. 設 $f$ 在 $[a, b]$ 上有界， $\{a_n\} \subset [a, b]$ ， $\lim_{n \rightarrow \infty} a_n = c$ ，證明：若 $f$ 在 $[a, b]$ 上只有 $a_n, n = 1, 2, \dots$ 為其間斷點，則 $f$ 在 $[a, b]$ 上可積。
5. 證明：若 $f \in C[a, b]$  且 $f(x) \geq 0, \forall x \in [a, b]$  則以下結果成立：

- (a) 如果函數 $f(x)$ 存在一點 $f(x_0) > 0, x_0 \in [a, b]$ ，則有：

$$\int_a^b f(x) dx > 0$$

- (b) 若 $\int_a^b f(x) = 0$ ，則有 $f(x) \equiv 0$

6. 證明若 $f \in C[a, b], f(x) \geq 0, \forall x \in [a, b]$ ，且 $M = \max_{[a, b]} f(x)$ ，則

$$\lim_{n \rightarrow \infty} \left( \int_a^b f^n(x) dx \right)^{\frac{1}{n}} = M$$