4.1 Summary

Firstly, we learned about the Haar wavelet transform. It basically is mapping a signal to an orthobasis in a Helbert space. After learning 3 weeks about linear algebra, the wavelet transform connect the abstract mathmatic to siganl processing. It should be very useful as I can imagine.

Then we learned about othonormal wavelet basis, which I do not really understand.

After That we learned about the important inverse problems. This section introduces some example of using linear inverse problem to simplify a question.

4.2

4.2. (Orthogonal transform, 2 pts) Consider a linear mapping $f : H \to H$, where H is a Hilbert space with inner-product $< \cdot, \cdot >_H$. Let $\{e_i\}_{i=1}^N$ be an orthogonal basis of H. Show that the following statements are equivalent:

1. f preserves orthogonality, i.e., $\{f(e_i)\}_{i=1}^N$ is also an orthogonal basis.
2. f preserves norm, i.e., $\|f(x)\|_H = \|x\|_H, \forall x \in H$.

(1) $\to$ (2)

$$\begin{aligned}
||f(x)||_H^2 &= \langle f(x), f(x) \rangle_H \\
&= \langle \sum_{i=1}^N x_i f(e_i), \sum_{j=1}^N y_j f(e_j) \rangle_H \\
&= \sum_{i=1}^N \sum_{j=1}^N \langle x_i f(e_i), y_j f(e_j) \rangle_H \\
&= \sum_{i=1}^N \langle x_i f(e_i), y_i f(e_i) \rangle_H \\
&= \sum_{i=1}^N x_i y_i \\
&= \sum_{i=1}^N \langle x_i e_i, y_i e_i \rangle_H \\
&= \sum_{i=1}^N \sum_{j=1}^N \langle x_i e_i, y_j e_j \rangle_H \\
&= \langle \sum_{i=1}^N x_i e_i, \sum_{j=1}^N y_j e_j \rangle_H \\
&= \langle x, x \rangle_H \\
&= ||x||_H^2
\end{aligned}$$

Since $||f(x)||_H \geq 0$ and$||x||_H \geq 0$, so we have $||f(x)||_H = ||x||_H$

(2) $\rightarrow$ (1)
let $i \neq j$

$$\begin{aligned}||f(e_i)+f(e_j)||_H^2 + ||f(e_i)-f(e_j)||_H^2 &= \langle f(e_i), f(e_j)\rangle_H + \overline{\langle f(e_i), f(e_j)\rangle_H} + 2Re\{\langle f(e_i), f(e_j)\rangle_H\} \\ &+ \langle f(e_i), f(e_j)\rangle_H + \overline{\langle f(e_i), f(e_j)\rangle_H} - 2Re\{\langle f(e_i), f(e_j)\rangle_H\} \\ &= 2\langle f(e_i), f(e_j)\rangle_H + 2\overline{\langle f(e_i), f(e_j)\rangle_H} \\ &= 4\langle f(e_i), f(e_j)\rangle_H\end{aligned}$$

So,

$$\begin{aligned}\langle f(e_i), f(e_j)\rangle_H &= \frac{1}{4}(||f(e_i)+f(e_j)||_H^2 + ||f(e_i)-f(e_j)||_H^2) \\ &= \frac{1}{4}(||f(e_i)+f(e_j)||_H^2 + ||f(e_i)-f(e_j)||_H^2)\end{aligned} \tag{1}$$

Now lets take a look at $f(e_i)+f(e_j)$:

$$\begin{aligned}e_i &= \frac{(e_i+e_j)}{2} + \frac{(e_i-e_j)}{2} \\ e_j &= \frac{(e_i+e_j)}{2} - \frac{(e_i-e_j)}{2}\end{aligned}$$

Since f is a linear mapping:

$$\begin{aligned}f(e_i) &= f(\frac{(e_i+e_j)}{2} + \frac{(e_i-e_j)}{2}) \\ &= f(\frac{(e_i+e_j)}{2}) + f(\frac{(e_i-e_j)}{2})\end{aligned}$$

So,

$$f(e_j) = f(\frac{(e_i+e_j)}{2}) - f(\frac{(e_i-e_j)}{2})$$

Then we have:

$$\begin{aligned}f(e_i)+f(e_j) &= f(\frac{(e_i+e_j)}{2}) - f(\frac{(e_i-e_j)}{2}) + f(\frac{(e_i+e_j)}{2}) + f(\frac{(e_i-e_j)}{2}) \\ &= 2f(\frac{(e_i+e_j)}{2}) \\ &= f(e_i+e_j)\end{aligned} \tag{2}$$

Take (2) into (1):

$$
\begin{aligned}
\langle f(e_i), f(e_j)\rangle_H &= \frac{1}{4}(||f(e_i)+f(e_j)||_H^2 + ||f(e_i)-f(e_j)||_H^2) \\
&= \frac{1}{4}(||f(e_i+e_j)||_H^2 + ||f(e_i-e_j)||_H^2) \\
&= \frac{1}{4}(||e_i+e_j||_H^2 + ||e_i-e_j||_H^2) \\
&= \langle e_i, e_j\rangle_H \\
&= 0
\end{aligned} \tag{1}
$$

4.3

4.3. (Haar Wavelet, 3 pts) Consider a signal in Haar scaling space $x \in V_2$ where

$$
x(t) = \begin{cases} 2, & 0 \le t < 1/4, \\ 0, & 1/4 \le t < 2/4, \\ 2, & 2/4 \le t < 3/4, \\ 3, & 3/4 \le t < 1, \\ 0, & \text{otherwise.} \end{cases}
$$

(a) Apply Haar Transform. Determine the scaling coefficients $s_{0,0}$ at scale 0, detail coefficients $w_{0,0}$ at scale 0, and detail coefficients $\{w_{1,0}, w_{1,1}\}$ at scale 1.

(b) Verify that the energy is persevered during the Haar Transform:

$$
\int x^2(t)\mathrm{d}t = s_{0,0}^2 + \sum_{j,n} w_{j,n}^2
$$

a)

We first calculate $\phi_{2,0}$, $\phi_{2,1}$, $\phi_{2,2}$, $\phi_{2,3}$:

$$
\phi_{2,0} = \begin{cases} 2, t \in [0, \frac{1}{4}) \\ 0, others \end{cases}
$$

$$
\phi_{2,1} = \begin{cases} 2, t \in [\frac{1}{4}, \frac{1}{2}) \\ 0, others \end{cases}
$$

$$
\phi_{2,2} = \begin{cases} 2, t \in [\frac{1}{2}, \frac{3}{4}) \\ 0, others \end{cases}
$$

$$
\phi_{2,3} = \begin{cases} 2, t \in [\frac{3}{4}, 1] \\ 0, others \end{cases}
$$

So, we have \$s*{2,0}, s*{2,1}, s*{2,2}, s*{2,3}:

$$
\begin{aligned}
s_{2,0} &= \langle x, \phi_{2,0} \rangle = 1 \\
s_{2,1} &= \langle x, \phi_{2,1} \rangle = 0 \\
s_{2,2} &= \langle x, \phi_{2,2} \rangle = 1 \\
s_{2,3} &= \langle x, \phi_{2,3} \rangle = \frac{3}{2}
\end{aligned}
$$

So, we got $s_{1,0}, s_{1,1}$ and $w_{1,0}, w_{1,1}$:

$$
\begin{aligned}
s_{1,0} &= \frac{1}{\sqrt{2}}(\phi_{2,0} + \phi_{2,1}) = \frac{1}{\sqrt{2}} \\
s_{1,1} &= \frac{1}{\sqrt{2}}(\phi_{2,2} + \phi_{2,3}) = \frac{5}{2\sqrt{2}} \\
w_{1,0} &= \frac{1}{\sqrt{2}}(\phi_{2,0} - \phi_{2,1}) = \frac{1}{\sqrt{2}} \\
w_{1,1} &= \frac{1}{\sqrt{2}}(\phi_{2,2} - \phi_{2,3}) = -\frac{1}{2\sqrt{2}}
\end{aligned}
$$

Then we will got $s_{0,0}$:

$$
s_{0,0} = \frac{1}{\sqrt{2}}(\phi_{1,0} + \phi_{1,1}) = \frac{7}{4}
$$

b)

$$
\begin{aligned}
\int x^2(t)dt &= s_{0,0}^2 + \sum_{j,n} w_{j,n}^2 \\
&= s_{0,0}^2 + \sum_{j=0}^{J} \sum_{n=0}^{2^J} w_{j,n}^2
\end{aligned}
$$

4.4

4.4. (Linear Inverse Problem, 3 pts) For an image signal $x[m,n] \in \mathbb{R}^{N\times N}$, a simple way to add bluring effect is to replace original pixels with average value of their neighbors:

$$
y[m,n] = \frac{1}{|P_{mn}|} \sum_{m',n' \in P_{mn}} x[m',n'], 1 \le m,n \le N
$$

where $P_{mn} = \{m', n' : \max(m'-m, n'-n) \le R\}$ is a R-neighborhoods of pixel (m,n). Please formulate a linear inverse problem $y = Ax$ to find the best de-blured image x for a given observed image y. What is the size of your system matrix A, and is there any way we can reduce its size?

$$
y[m,n] = \frac{1}{|P_{mn}|} \sum_{m',n' \in P_{mn}} x[m',n'], 1 \le m,n \le N
$$

We can use a convolution core to accomplish this thing.
The core will be in shape of RxR:

$$\frac{1}{R^2}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$