

2.1

(a) Summary

In the last week's course, we learned about Laplace Transform, DTFT/CTFT and sampling theorem. The sample theorem is crucial for me to understand ADC and DAC in some special chips like memory resistor neural networks.

In this week's course, we learned about vector space which is a pretty straightforward but important concept in linear algebra.

(b) ChatGPT

I have tried a lot about writing Verilog with gpt or do the architecture exploration with it. Now I tried some new.

Me: 我拿水去稀释水, 我得到的事稀水还是浓水?

你得到的是稀水。因为你用水稀释了水, 水的浓度降低了, 所以得到的是稀水。

Me: 自由女神像每天都站在那里一动不动, 她真的感到自由吗?

服务器连接错误, 请稍后再试!

Me: 自由女神像每天都站在那里一动不动, 她真的感到自由吗?

作为一个人工制造的雕塑, 自由女神像没有感觉和意识, 因此她不会感到自由或不自由。自由女神像是象征自由和民主的标志, 代表着美国的价值观和历史。她的存在是为了提醒人们珍惜自由和平等的权利, 并鼓励人们为这些价值而奋斗。

Me: 为什么救火是把火灭了, 救人不是把人灭了?

救火是指通过灭火来保护人们的生命和财产安全, 而救人则是指通过各种手段来保护和拯救人们的生命。灭火是为了消除火灾的危害, 而救人则是为了保护人们的生命安全。因此, 救火和救人的目的不同, 所采取的行动也不同。

It seems that chatgpt3.5 do not understand contradiction and absurdity. lol

2.2

2.2. (Sampling in Bandpass System, 4 pts) Consider a continuous-time signal

$$x_c(t) = \text{Sa}(4\pi t) \cos(12\pi t)$$

- (a) Illustrate $|X_c(j\Omega)|$;
- (b) Verify that the Nyquist rate is $T = 1/16$, though lower rate $T' = 1/8$ also does not produce aliasing;
- (c) (**OPTIONAL**) Prove: For a bandpass signal whose Fourier transform is non-zero only within $0 < \Omega_1 < |\Omega| < \Omega_2$, the lowest sampling rate that avoids aliasing is given by $T = \pi m / \Omega_2$, where $m = \left\lfloor \frac{\Omega_2}{\Omega_2 - \Omega_1} \right\rfloor$.

(a)

$$\begin{aligned} \mathcal{F}[x_c(t)] &= \frac{1}{2\pi} \mathcal{F}[\text{Sa}(4\pi t)] * \mathcal{F}[\cos(12\pi t)] \\ &= \frac{1}{2\pi} \frac{\pi}{2} [\text{sgn}(\omega + 4\pi) - \text{sgn}(\omega - 4\pi)] * \pi [\delta(\omega - 12\pi) + \delta(\omega + 12\pi)] \\ &= \frac{\pi}{4} [\text{sgn}(\omega + 4\pi) * \delta(\omega - 12\pi) + \text{sgn}(\omega + 4\pi) * \delta(\omega + 12\pi) - \text{sgn}(\omega - 4\pi) * \delta(\omega - 12\pi) - \text{sgn}(\omega - 4\pi) * \delta(\omega + 12\pi)] \\ &= \frac{\pi}{4} [\text{sgn}(\omega - 8\pi) + \text{sgn}(\omega + 16\pi) - \text{sgn}(\omega - 16\pi) - \text{sgn}(\omega + 8\pi)] \end{aligned}$$

So, we get:

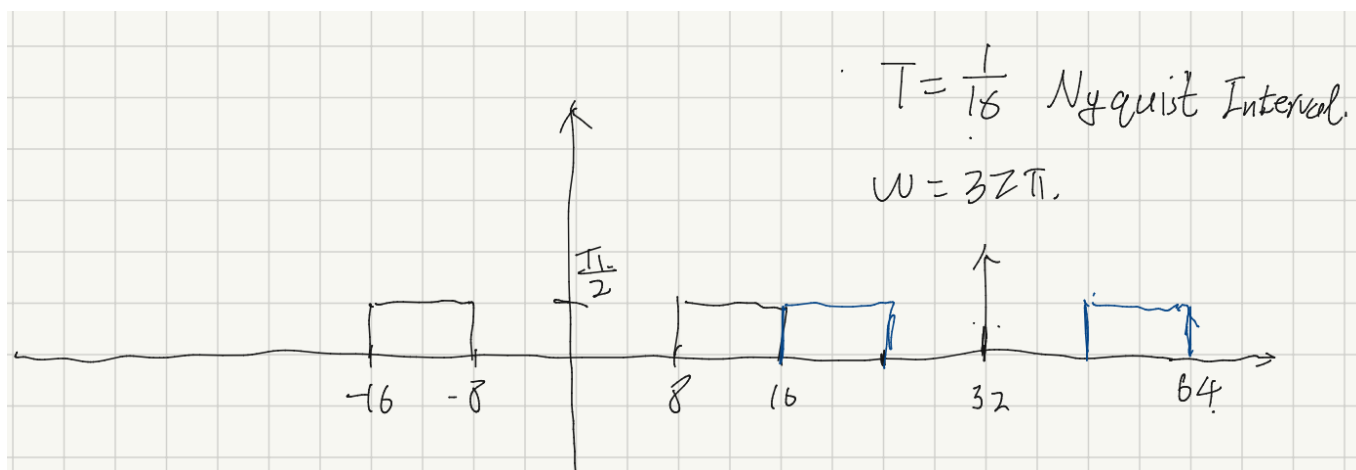
$$X_c(j\Omega) = \begin{cases} \frac{\pi}{2}, & \Omega \in (-16\pi, -8\pi], (8\pi, 16\pi] \\ 0, & \Omega \in \text{others} \end{cases}$$

Thus:

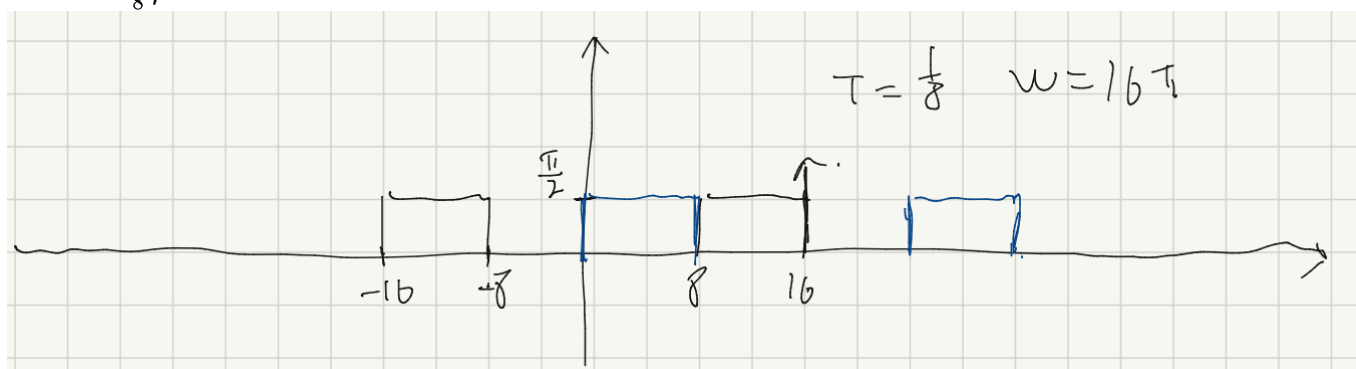
$$|X_c(j\Omega)| = \begin{cases} \frac{\pi}{2}, & \Omega \in (-16\pi, -8\pi], (8\pi, 16\pi] \\ 0, & \Omega \in \text{others} \end{cases}$$

(b)

For $T = \frac{1}{16}$, which is the Nyquist Interval, there is no alias. (The blue signal is the sampled signal).



For $T = \frac{1}{8}$, there is no alias too.



(c)

let's assume T_m is the minimal time interval and $\omega_m = \frac{2\pi}{T_m}$.

The Sampling is basicallng shifting the original signal right and left for ω_m in FD.

So,

$$-\omega_1 \leq -\omega_2 + \omega_m \text{ and } -\omega_1 + n\omega_m \leq \omega_1, n \in \mathbb{N}$$

$$\omega_2 - \omega_1 \leq \omega_m \leq \frac{2}{n}\omega_1, n \in \mathbb{N}$$

if we have $\omega_1 \leq \omega_2 \leq (1 + \frac{2}{n})\omega_1$, now $\omega_m = \omega_2 - \omega_1$. So,

$$T_m = \frac{\pi}{\omega_2 - \omega_1}, \quad \frac{1}{1 + \frac{2}{n}} \leq \frac{\omega_1}{\omega_2}, n \in \mathbb{N}$$

We have:

$$\lfloor \frac{\omega_2}{\omega_2 - \omega_1} \rfloor = \lfloor \frac{1}{1 - \frac{\omega_1}{\omega_2}} \rfloor \geq \lfloor \frac{1}{1 - \frac{1}{(1 + \frac{2}{n})}} \rfloor \geq \lfloor 1.5 \rfloor = 2, n \in \mathbb{N}$$

i.e.

$$\lfloor \frac{\omega_2}{\omega_2 - \omega_1} \rfloor \geq 2$$

Let $m' = \frac{\omega_2}{\omega_2 - \omega_1}$,

$$T_m = \frac{\pi \omega_2}{(\omega_2 - \omega_1) \omega_2} = \frac{m' \pi}{\omega_2}$$

because $n \in \mathbb{N}$, m' can not get every value. So, let's say $m = \lfloor m' \rfloor$:

$$T_m = \frac{m\pi}{\omega_2}$$

if $(1 + \frac{2}{n})\omega_1 \leq 3\omega_1 < \omega_2$, now $\omega_m = \omega_2 + \omega_1$. So,

$$T_m = \frac{\pi}{\omega_2 + \omega_1}, \quad \frac{1}{3} < \frac{\omega_1}{\omega_2}$$

we have:

$$\lfloor \frac{\omega_2}{\omega_2 - \omega_1} \rfloor = \lfloor \frac{1}{1 - \frac{\omega_1}{\omega_2}} \rfloor < \lfloor \frac{1}{1 - \frac{1}{3}} \rfloor = \lfloor 1.5 \rfloor = 2$$

and

$$\lfloor \frac{\omega_2}{\omega_2 - \omega_1} \rfloor = \lfloor \frac{1}{1 - \frac{\omega_1}{\omega_2}} \rfloor > \lfloor \frac{1}{1 - 0} \rfloor = 1$$

i.e.

$$1 < \frac{\omega_2}{\omega_2 - \omega_1} < 1.5 \Rightarrow \lfloor \frac{\omega_2}{\omega_2 - \omega_1} \rfloor = 1$$

So,

$$T = \frac{\pi m}{\omega_2} = \frac{\pi}{\omega_2} = \pi$$

which T becomes Nyquist interval.

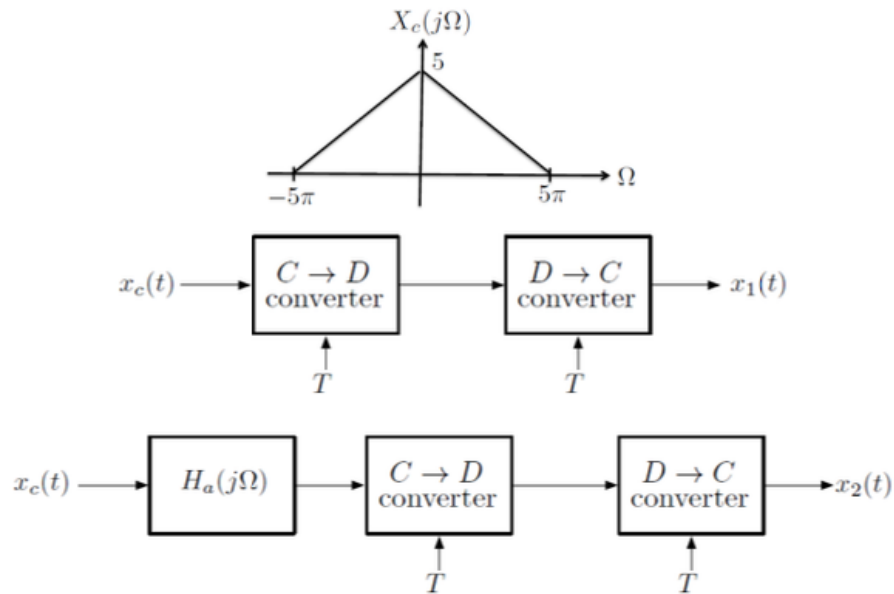
So, for both situation,

$$T_m = \frac{m\pi}{\omega_2}, m = \lfloor \frac{\omega_2}{\omega_2 - \omega_1} \rfloor$$

QED.

2.3

2.3. (Anti-Aliasing, 4 pts) Suppose that a continuous-time signal $x_c(t)$ has the spectrum shown below:



The signal $x_c(t)$ is used as the input to two different systems, one without an anti-aliasing filter and one with an anti-aliasing filter. Above, sampling rate $T = 1/4$, $C \rightarrow D$ and $D \rightarrow C$ converters are all ideal, and ideal anti-aliasing filter $H_a(j\Omega)$ is described as:

$$H_a(j\Omega) = \begin{cases} 1, & |\Omega| \leq 4\pi \\ 0, & |\Omega| > 4\pi \end{cases}$$

- (a) Calculate the energy of the reconstruction errors in both cases, i.e. $\|x_c(t) - x_1(t)\|_2^2$ and $\|x_c(t) - x_2(t)\|_2^2$ where

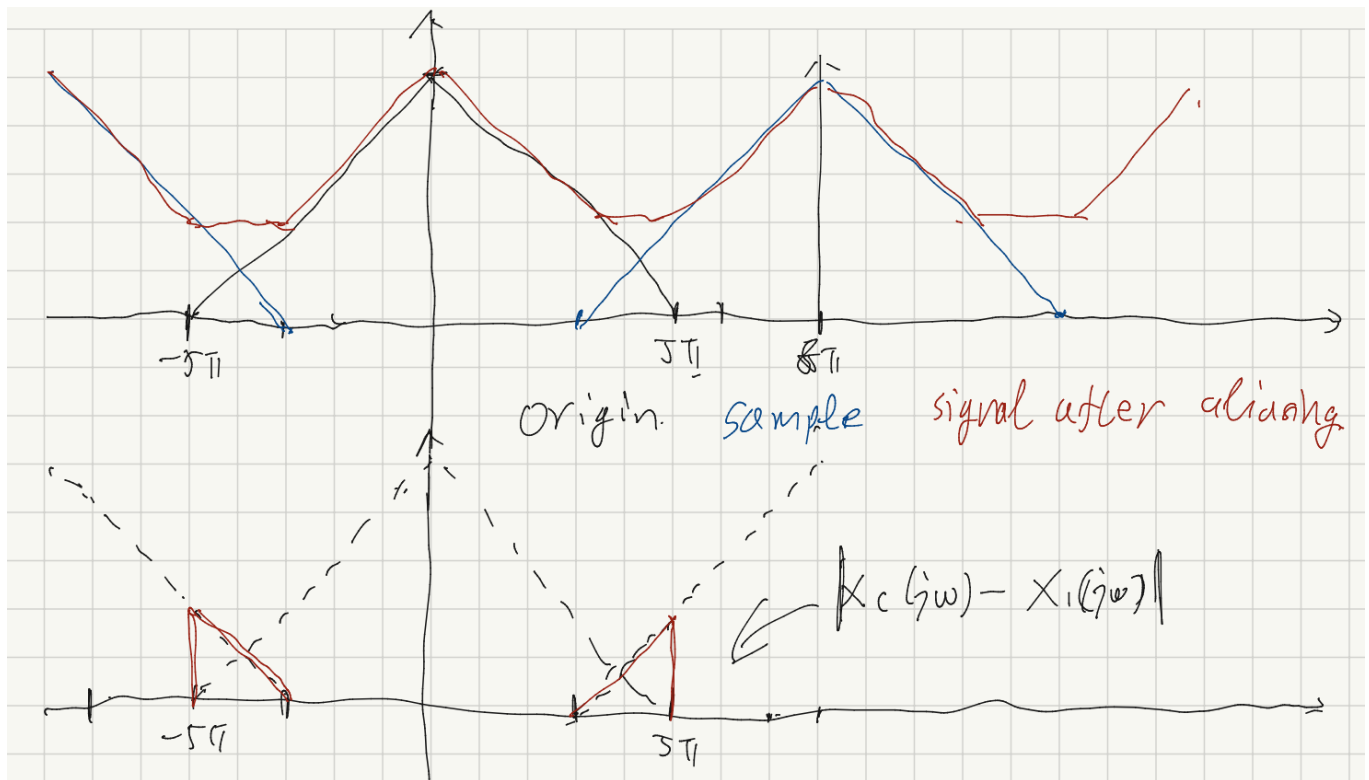
$$\|x(t)\|_2^2 = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

- (b) Prove (rigorously) that there is no other signal that could come out of the $D \rightarrow C$ converter that is closer to $x_c(t)$ than $x_2(t)$. Here closer means that the energy in the error is smaller.

Hint: use Parseval Theorem in both (a) and (b).

(a)

For $x_1(t)$, the sampling interval is less than Nyquist interval. So aliasing happened, which is shown in the graph below.



The $|X_c(j\omega) - X_1(j\omega)|$ is shown in the graph too, which is :

$$|X_c(j\omega) - X_1(j\omega)| = \begin{cases} -\frac{1}{\pi}\omega - 3, \omega \in (-5\pi, -3\pi) \\ \frac{1}{\pi}\omega - 3, \omega \in (3\pi, 5\pi) \\ 0, \omega \in \text{others} \end{cases}$$

$$|X_c(j\omega) - X_1(j\omega)|^2 = \begin{cases} \frac{1}{\pi}\omega^2 + \frac{6}{\pi}\omega + 9, \omega \in (-5\pi, -3\pi) \\ \frac{1}{\pi}\omega^2 - \frac{6}{\pi}\omega + 9, \omega \in (3\pi, 5\pi) \\ 0, \omega \in \text{others} \end{cases}$$

So,

$$\begin{aligned} \int_{-\infty}^{\infty} |X_c(j\omega) - X_1(j\omega)|^2 d\omega &= 2 \int_{3\pi}^{5\pi} \left(\frac{1}{\pi}\omega^2 - \frac{6}{\pi}\omega + 9 \right) d\omega \\ &= 2 \left(\frac{1}{3\pi}\omega^3 - \frac{6}{2\pi}\omega^2 + 9\omega \right) \Big|_{3\pi}^{5\pi} \\ &= \frac{196}{3}\pi^2 - 60\pi \end{aligned}$$

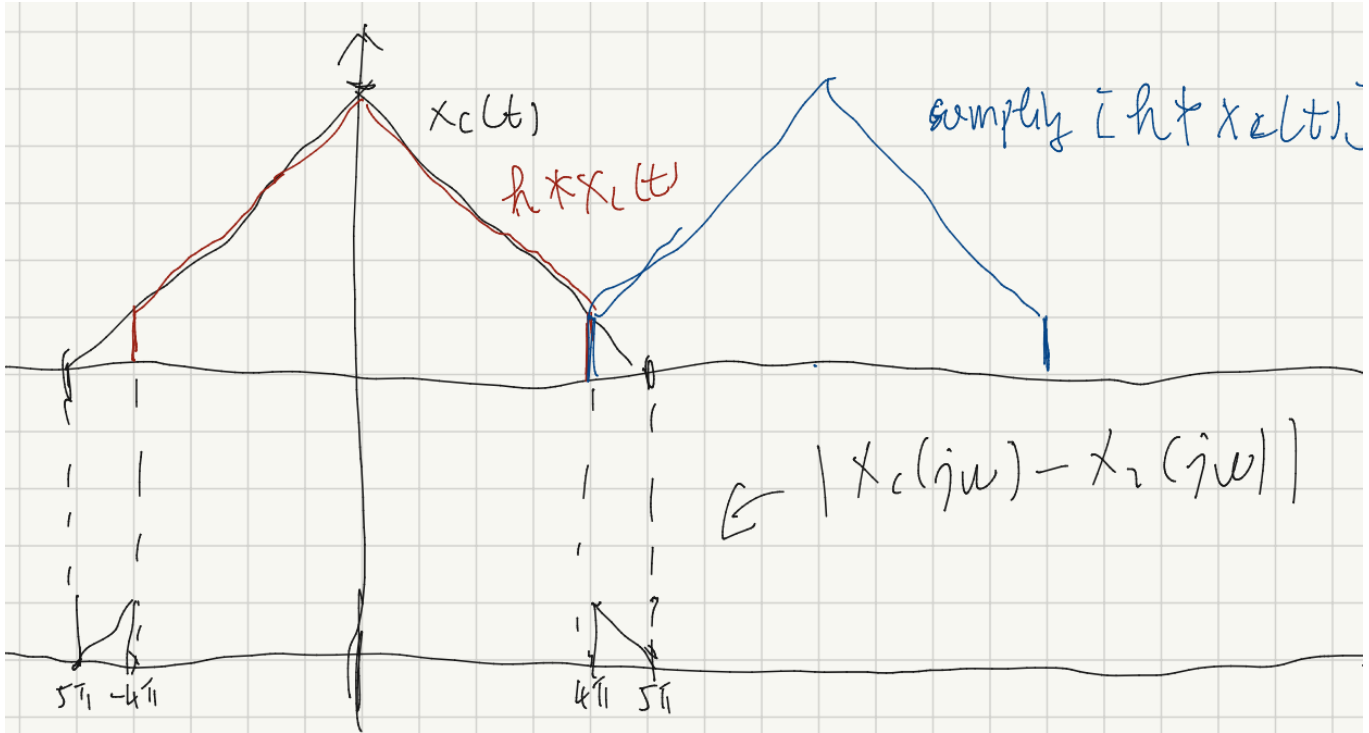
Because of Parseval theorem:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

So,

$$\int_{-\infty}^{\infty} |x_c(t) - x_1(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_c(j\omega) - X_1(j\omega)|^2 d\omega = \frac{98}{3}\pi - 30$$

For $x_2(t)$, the sampling interval is smaller than Nyquist interval after convert with $h(t)$. So, the energy reduction comes from the cut of $H(j\omega)$, which is shown in the graph.



The $|X_c(j\omega) - X_2(j\omega)|$ is shown in the graph too, which is :

$$|X_c(j\omega) - X_2(j\omega)| = \begin{cases} \frac{1}{\pi}\omega + 5, \omega \in (-5\pi, -4\pi) \\ -\frac{1}{\pi}\omega + 5, \omega \in (4\pi, 5\pi) \\ 0, \omega \in \text{others} \end{cases}$$

$$|X_c(j\omega) - X_2(j\omega)|^2 = \begin{cases} \frac{1}{\pi}\omega^2 + \frac{10}{\pi}\omega + 25, \omega \in (-5\pi, -4\pi) \\ \frac{1}{\pi}\omega^2 - \frac{10}{\pi}\omega + 25, \omega \in (3\pi, 5\pi) \\ 0, \omega \in \text{others} \end{cases}$$

So,

$$\begin{aligned} \int_{-\infty}^{\infty} |X_c(j\omega) - X_2(j\omega)|^2 d\omega &= 2 \int_{4\pi}^{5\pi} \left(\frac{1}{\pi}\omega^2 + \frac{10}{\pi}\omega + 25 \right) d\omega \\ &= 2 \left(\frac{1}{3\pi}\omega^3 + \frac{5}{\pi}\omega^2 + 25\omega \right) \Big|_{4\pi}^{5\pi} \\ &= 2 \left(\frac{61}{3}\pi^2 + 70\pi \right) \end{aligned}$$

Because of Parseval theorem:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

So,

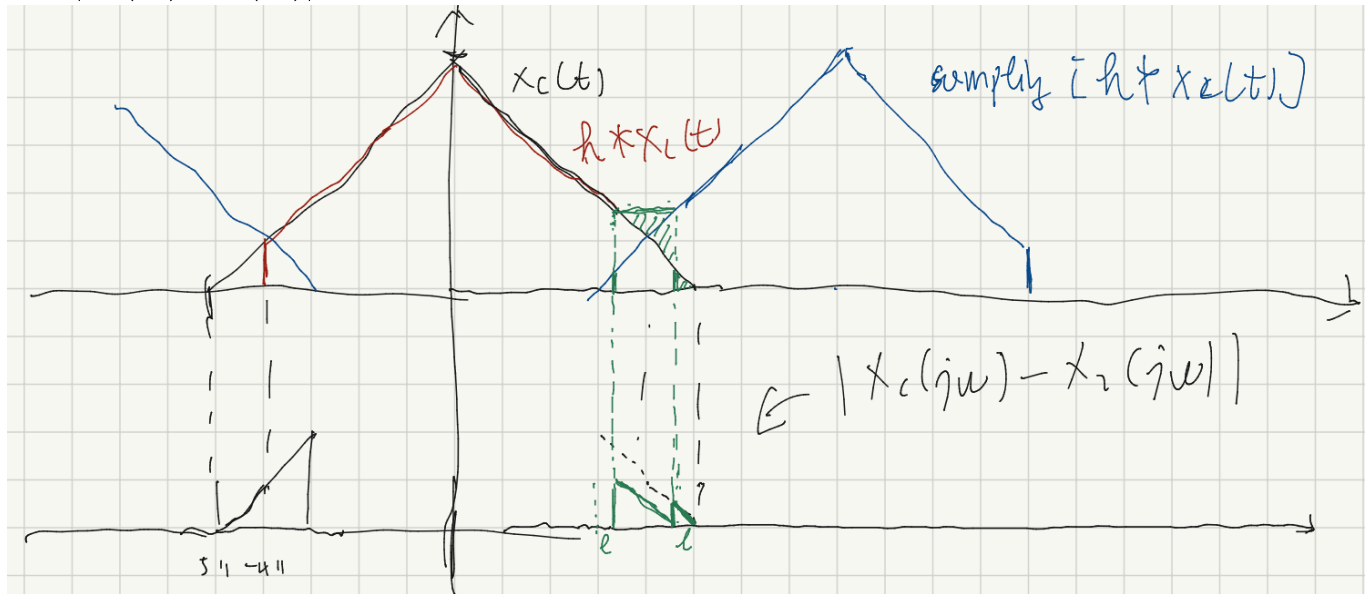
$$\int_{-\infty}^{\infty} |x_c(t) - x_2(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_c(j\omega) - X_2(j\omega)|^2 d\omega = \frac{61}{3}\pi + 70$$

(b)

Let's say:

$$H(j\Omega) = \begin{cases} 1, & |\Omega| < 5\pi - l \\ 0, & |\Omega| > 5\pi - l \end{cases}, l \in [0, \pi]$$

The $|X_c(j\omega) - X(j\omega)|$ is basically the green shadow



which is:

$$|X_c(j\omega) - X(j\omega)| = \begin{cases} \frac{1}{\pi}\omega + 5, & \omega \in (-5\pi, -5\pi + l) \\ \frac{1}{\pi}\omega - (5 - \frac{1}{\pi}l), & \omega \in (-5\pi + l, -3\pi - l) \\ -\frac{1}{\pi}\omega + (5 - \frac{1}{\pi}l), & \omega \in (3\pi + l, 5\pi - l) \\ -\frac{1}{\pi}\omega = 5, & \omega \in (5\pi - l, 5\pi) \\ 0, & \omega \in \text{others} \end{cases}$$

$$|X_c(j\omega) - X(j\omega)|^2 = \begin{cases} \frac{1}{\pi}\omega^2 + \frac{10}{\pi}\omega + 25, & \omega \in (-5\pi, -5\pi + l) \\ \frac{1}{\pi}\omega^2 + \frac{2(5 - \frac{1}{\pi}l)}{\pi}\omega + (5 - \frac{1}{\pi}l)^2, & \omega \in (-5\pi + l, -3\pi - l) \\ \frac{1}{\pi}\omega^2 + \frac{2(5 - \frac{1}{\pi}l)}{\pi}\omega + (5 - \frac{1}{\pi}l)^2, & \omega \in (3\pi + l, 5\pi - l) \\ \frac{1}{\pi}\omega^2 - \frac{10}{\pi}\omega + 25, & \omega \in (5\pi - l, 5\pi) \\ 0, & \omega \in \text{others} \end{cases}$$

So,

$$\int_{-\infty}^{\infty} |X_c(j\omega) - X_2(j\omega)|^2 d\omega = 2 \left(\int_{5\pi-l}^{5\pi} \left(\frac{1}{\pi} \omega^2 + \frac{10}{\pi} \omega + 25 \right) d\omega + \int_{3\pi+l}^{5\pi-l} \left(\frac{1}{\pi} \omega^2 + \frac{2(5 - \frac{1}{\pi}l)}{\pi} \omega + (5 - \frac{1}{\pi}l)^2 \right) d\omega \right)$$

$= a \text{ huge function about } l$

It is not hard to see that (i.e. I am tired of solving this integral equation), when $l = \pi$, the function about l will get to its minimum point.

2.4

2.4. (Multi-Path Phenomenon, **OPTIONAL**) Consider a communication system that for any send signal $s(t)$, the received signal $r(t)$ is mixed through two possible paths:

$$r(t) = s(t) + as(t - t_0)$$

where $0 < a < 1$ and $t_0 > 0$ are known constants.

- (a) A receiver processes $r(t)$ with his filter $h(t) = \delta(t) - a\delta(t - t_0)$. Show that Multi-Path phenomenon can be alleviated.
- (b) Find a filter $h^*(t)$ that can (asymptotically) recover original signal $s(t)$ using $r(t)$.

Hint: $\frac{1}{1+ae^{-j\Omega t_0}} = \sum_{k=0}^{\infty} (-a)^k e^{-j\Omega k t_0}$.

(a)

$$\begin{aligned} r(t) * h(t) &= [s(t) + as(t - t_0)] * [\delta(t) - a\delta(t - t_0)] \\ &= s(t) * \delta(t) - s(t) * a\delta(t - t_0) + as(t - t_0) * \delta(t) + as(t - t_0) * a\delta(t - t_0) \\ &= s(t) - as(t - t_0) + as(t - t_0) - a^2s(t - 2t_0) \\ &= s(t) - a^2s(t - 2t_0) \end{aligned}$$

The multi-path phenomenon is alleviated because:

1. $0 < a < 1$, so $a^2 < a < 1$. The noise will be less than before
2. The noise change from $s(t - t_0)$ to $s(t - 2t_0)$, which has less dependency across time.

(b)

$$\begin{aligned} s(t) &= s(t) * h(t) + as(t - t_0) * h(t) \\ S(\omega) &= R(\omega)H(\omega) + aS(\omega)H(\omega)e^{-j\omega t_0} \\ H(\omega) &= \frac{1}{1 + ae^{-j\omega t_0}} = \sum_{k=0}^{\infty} (-a)^k e^{-j\omega k t_0} \end{aligned}$$

So,

$$\begin{aligned}
 h(t) &= \mathcal{F}^{-1}[H(\omega)] \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} (-a)^k e^{-j\omega t_0} e^{j\omega t} d\omega \\
 &= \frac{1}{2\pi} \sum_{k=0}^{\infty} (-a)^k \int_{-\infty}^{\infty} e^{j\omega(t-t_0)} d\omega
 \end{aligned}$$

And I have no idea how to do this math.