2.1

(a) Summary

In the last week's course, we learned about Laplace Transform, DTFT/CTFT and sampling theorem. The sample theorem is crutial for me to understand ADC and DAC in some special chips like memory resistor nerual networks.

In this week's course, we learned about vector space which is a pretty straightforward but important concept in linear algebra.

(b) ChatGPT

2.2

2.2. (Sampling in Bandpass System, 4 pts) Consider a continuous-time signal

$$x_c(t) = \mathrm{Sa}(4\pi t)\cos(12\pi t)$$

(a) Illustrate $|X_c(j\Omega)|$;

(b) Verify that the Nyguist rate is $T = 1/16$, though lower rate $T' = 1/8$ also does not produce aliasing;

(c) (**OPTIONAL**) Prove: For a bandpass signal whose Fourier transform is non-zero only within $0 < \Omega_1 < |\Omega| < \Omega_2$, the lowest sampling rate that avoids aliasing is given by $T = \pi m/\Omega_2$, where $m = \left\lfloor \frac{\Omega_2}{\Omega_2 - \Omega_1} \right\rfloor$.

(a)

$$\begin{aligned}
\mathcal{F}[x_c(t)] &= \frac{1}{2\pi}\mathcal{F}[Sa(4\pi t)] * \mathcal{F}[cos(12\pi t)] \\
&= \frac{1}{2\pi}\frac{\pi}{2}[sgn(\omega+4\pi) - sgn(\omega-4\pi)] * \pi[\delta(\omega-12\pi)+\delta(\omega+12\pi)] \\
&= \frac{\pi}{4}[sgn(\omega+4\pi) * \delta(\omega-12\pi) + sgn(\omega+4\pi) * \delta(\omega+12\pi) - sgn(\omega-4\pi) * \delta(\omega-12\pi) - sgn(\omega
\end{aligned}$$

$$= \frac{\pi}{4}[sgn(\omega-8\pi) + sgn(\omega+16\pi) - sgn(\omega-16\pi) - sgn(\omega+8\pi)]$$

So, we get:

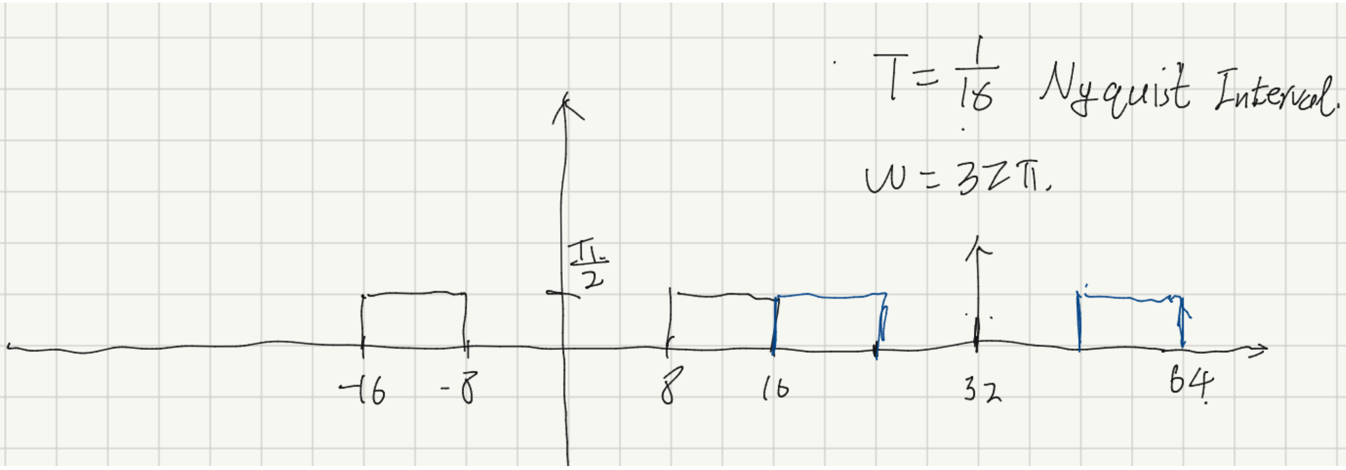
$$X_c(j\Omega) = \begin{cases} \frac{\pi}{2}, \Omega \in (-16\pi, -8\pi], (8\pi, 16\pi] \\ 0, \Omega \in others \end{cases}$$

Thus:

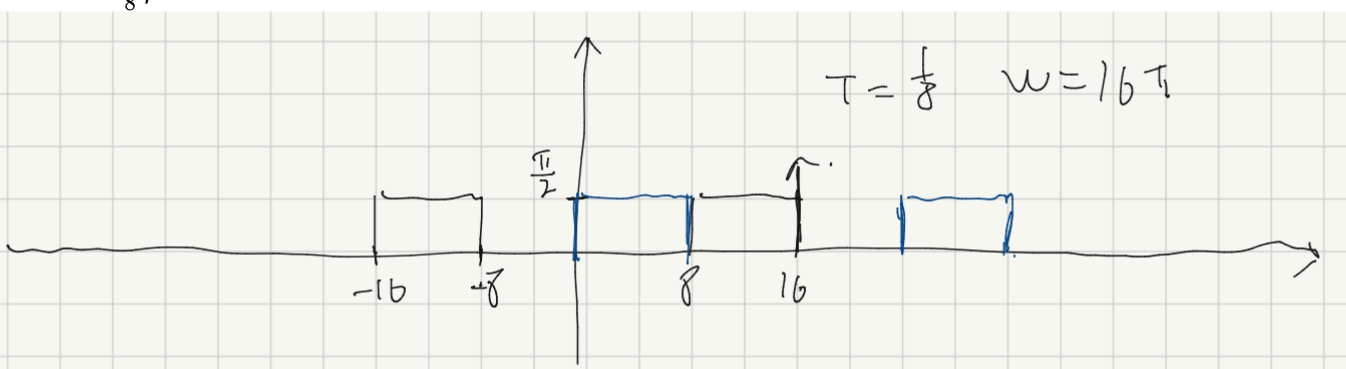
$$|X_c(j\Omega)| = \begin{cases} \frac{\pi}{2}, \Omega \in (-16\pi, -8\pi], (8\pi, 16\pi] \\ 0, \Omega \in others \end{cases}$$

(b)

For T = $\frac{1}{16}$, which is the Nyquist Interval, there is no alias . (The blue signal is the sampled signal).



For T = $\frac{1}{8}$, there is no alias too.



(c)

let's assume T_m is the minimal time interval and $\omega_m = \frac{2\pi}{T_m}$.
The Sampling is basicalling shifting the original signal right and left for ω_m in FD.
So,

$$-\omega_1 \le -\omega_2 + \omega_m \; and \; -\omega_1 + n\omega_m \le \omega_1, n \in \mathbb{N}$$
$$\omega_2 - \omega_1 \le \omega_m \le \frac{2}{n}\omega_1, n \in \mathbb{N}$$

if we have $\omega_1 \le \omega_2 \le (1 + \frac{2}{n})\omega_1$, now $\omega_m = \omega_2 - \omega_1$.So,

$$T_m = \frac{\pi}{\omega_2 - \omega_1}, \quad \frac{1}{1 + \frac{2}{n}} \le \frac{\omega_1}{\omega_2},, n \in \mathbb{N}$$

We have:

$$\lfloor\frac{\omega_2}{\omega_2-\omega_1}\rfloor=\lfloor\frac{1}{1-\frac{\omega_1}{\omega_2}}\rfloor\geq\lfloor\frac{1}{1-\frac{1}{(1+\frac{2}{n})}}\rfloor\geq\lfloor 1.5\rfloor=2, n\in\mathbb{N}$$

i.e.

$$\lfloor\frac{\omega_2}{\omega_2-\omega_1}\rfloor\geq 2$$

Let $m'=\frac{\omega_2}{\omega_2-\omega_1}$,

$$T_m=\frac{\pi\omega_2}{(\omega_2-\omega_1)\omega_2}=\frac{m'\pi}{\omega_2}$$

because $n\in\mathbb{N}$, m' can not get every value. So, let's say $m=\lfloor m'\rfloor$:

$$T_m=\frac{m\pi}{\omega_2}$$

if $(1+\frac{2}{n})\omega_1\leq 3\omega_1<\omega_2$, now $\omega_m=\omega_2+\omega_1$. So,

$$T_m=\frac{\pi}{\omega_2+\omega_1},\ \ \frac{1}{3}<\frac{\omega_1}{\omega_2}$$

we have:

$$\lfloor\frac{\omega_2}{\omega_2-\omega_1}\rfloor=\lfloor\frac{1}{1-\frac{\omega_1}{\omega_2}}\rfloor<\lfloor\frac{1}{1-\frac{1}{3}}\rfloor=\lfloor 1.5\rfloor=2$$
$$and$$
$$\lfloor\frac{\omega_2}{\omega_2-\omega_1}\rfloor=\lfloor\frac{1}{1-\frac{\omega_1}{\omega_2}}\rfloor>\lfloor\frac{1}{1-0}\rfloor=1$$

i.e.

$$1<\frac{\omega_2}{\omega_2-\omega_1}<1.5\Rightarrow\lfloor\frac{\omega_2}{\omega_2-\omega_1}\rfloor=1$$

So,

$$T=\frac{\pi m}{\omega_2}=\frac{\pi}{\omega_2}=pi$$

which T becomes Nyquist interval.

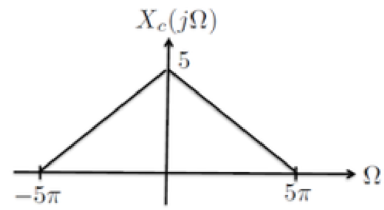
So, for both situation,

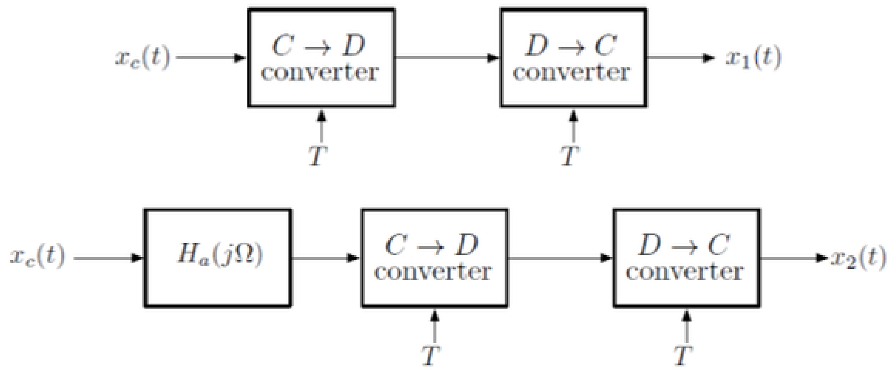
$$T_m=\frac{m\pi}{\omega_2}, m=\lfloor\frac{\omega_2}{\omega_2-\omega_1}\rfloor$$

QED.

2.3

2.3. (Anti-Aliasing, 4 pts) Suppose that a continuous-time signal $x_c(t)$ has the spectrum shown below:





The signal $x_c(t)$ is used as the input to two different systems, one without an anti-aliasing filter and one with an anti-aliasing filter. Above, sampling rate $T = 1/4$, C→D and D→C converters are all ideal, and ideal anti-aliasing filter $H_a(j\Omega)$ is described as:

$$H_a(j\Omega) = \begin{cases} 1, & |\Omega| \leq 4\pi \\ 0, & |\Omega| > 4\pi \end{cases}$$

(a) Calculate the energy of the reconstruction errors in both cases, i.e. $\|x_c(t) - x_1(t)\|_2^2$ and $\|x_c(t) - x_2(t)\|_2^2$ where

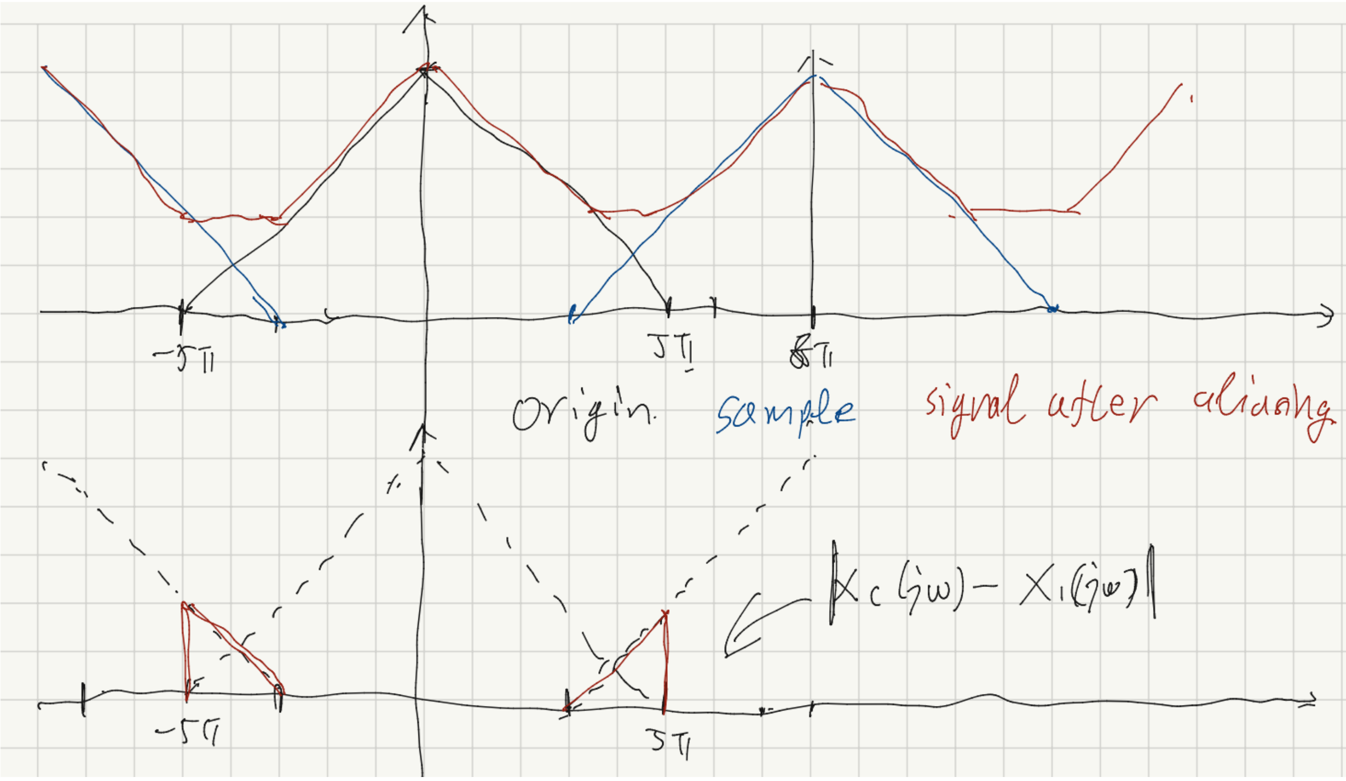
$$\|x(t)\|_2^2 = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

(b) Prove (rigorously) that there is no other signal that could come out of the D→C converter that is closer to $x_c(t)$ than $x_2(t)$. Here closer means that the energy in the error is smaller.

Hint: use Parseval Theorem in both (a) and (b).

(a)

For $x_1(t)$, the sampling interval is less than Nyquist interval. So aliasing happened, which is shown in the graph below.



The $|X_c(j\omega) - X_1(j\omega)|$ is shown in the graph too, which is :

$$|X_c(j\omega) - X_1(j\omega)| = \begin{cases} -\frac{1}{\pi}\omega - 3, \omega \in (-5\pi, -3\pi) \\ \frac{1}{\pi}\omega - 3, \omega \in (3\pi, 5\pi) \\ 0, \omega \in others \end{cases}$$

$$|X_c(j\omega) - X_1(j\omega)|^2 = \begin{cases} \frac{1}{\pi}\omega^2 + \frac{6}{\pi}\omega + 9, \omega \in (-5\pi, -3\pi) \\ \frac{1}{\pi}\omega^2 - \frac{6}{\pi}\omega + 9, \omega \in (3\pi, 5\pi) \\ 0, \omega \in others \end{cases}$$

So,

$$\begin{aligned} \int_{-\infty}^{\infty} |X_c(j\omega) - X_1(j\omega)|^2 d\omega &= 2\int_{3\pi}^{5\pi} (\frac{1}{\pi}\omega^2 - \frac{6}{\pi}\omega + 9) d\omega \\ &= 2(\frac{1}{3\pi}\omega^3 - \frac{6}{2\pi}\omega^2 + 9\omega)|_{3\pi}^{5\pi} \\ &= \frac{196}{3}\pi^2 - 60\pi \end{aligned}$$

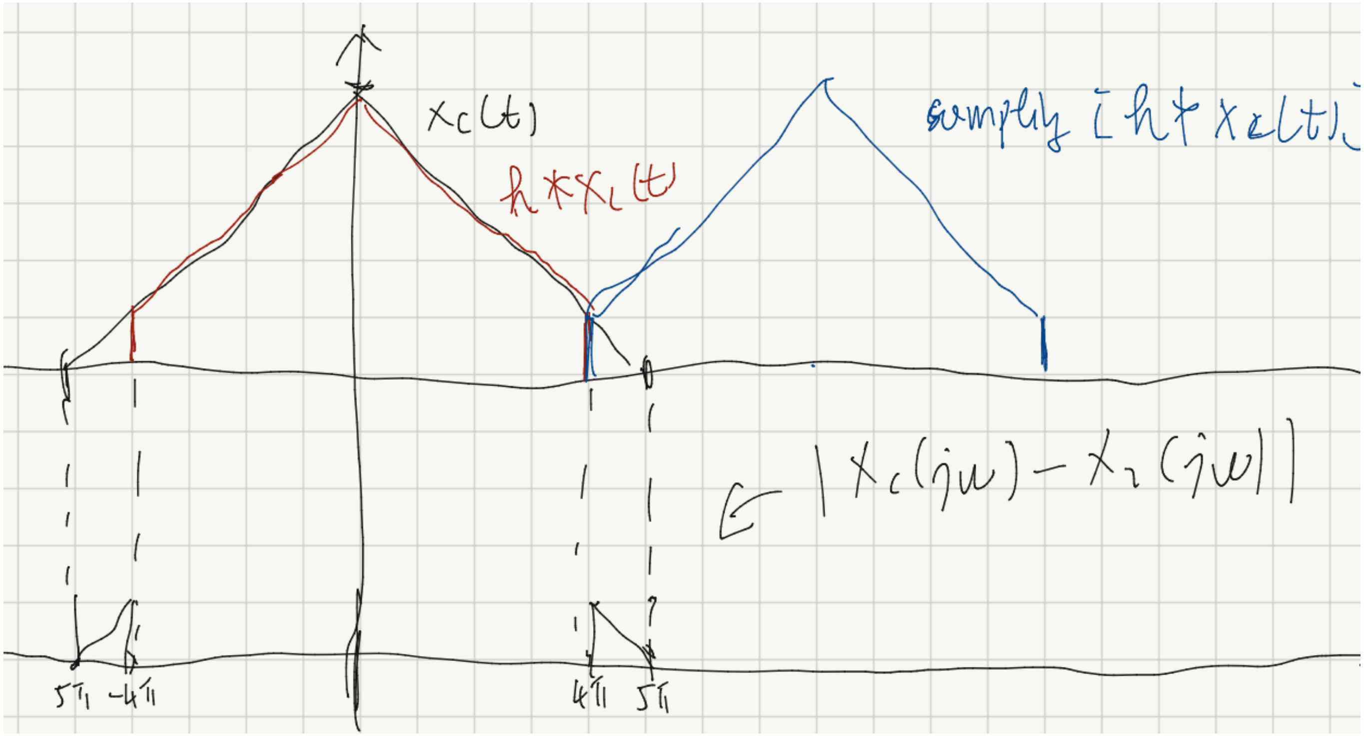
Because of Parseval theorem:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 dt$$

So,

$$\int_{-\infty}^{\infty}|x_c(t)-x_1(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|X_c(j\omega)-X_1(j\omega)|^2dt=\frac{98}{3}\pi-30$$

For $x_2(t)$, the sampling interval is smaller than Nyquist interval after convert with h(t). So, the energy reduction comes from the cut of H(jw), which is shown in the graph.



The $|X_c(j\omega)-X_2(j\omega)|$ is shown in the graph too, which is :

$$|X_c(j\omega)-X_2(j\omega)|=\begin{cases}\frac{1}{\pi}\omega+5,\omega\in(-5\pi,-4\pi)\\-\frac{1}{\pi}\omega+5,\omega\in(4\pi,5\pi)\\0,\omega\in others\end{cases}$$

$$|X_c(j\omega)-X_2(j\omega)|^2=\begin{cases}\frac{1}{\pi}\omega^2+\frac{10}{\pi}\omega+25,\omega\in(-5\pi,-4\pi)\\\frac{1}{\pi}\omega^2-\frac{10}{\pi}\omega+25,\omega\in(3\pi,5\pi)\\0,\omega\in others\end{cases}$$

So,

$$\begin{aligned}\int_{-\infty}^{\infty}|X_c(j\omega)-X_2(j\omega)|^2d\omega&=2\int_{4\pi}^{5\pi}(\frac{1}{\pi}\omega^2+\frac{10}{\pi}\omega+25)d\omega\\&=2(\frac{1}{3\pi}\omega^3+\frac{5}{\pi}\omega^2+25\omega)|_{4\pi}^{5\pi}\\&=2(\frac{61}{3}\pi^2+70\pi)\end{aligned}$$

Because of Parseval theorem:

$$\int_{-\infty}^{\infty}|x(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|^2dt$$

So,

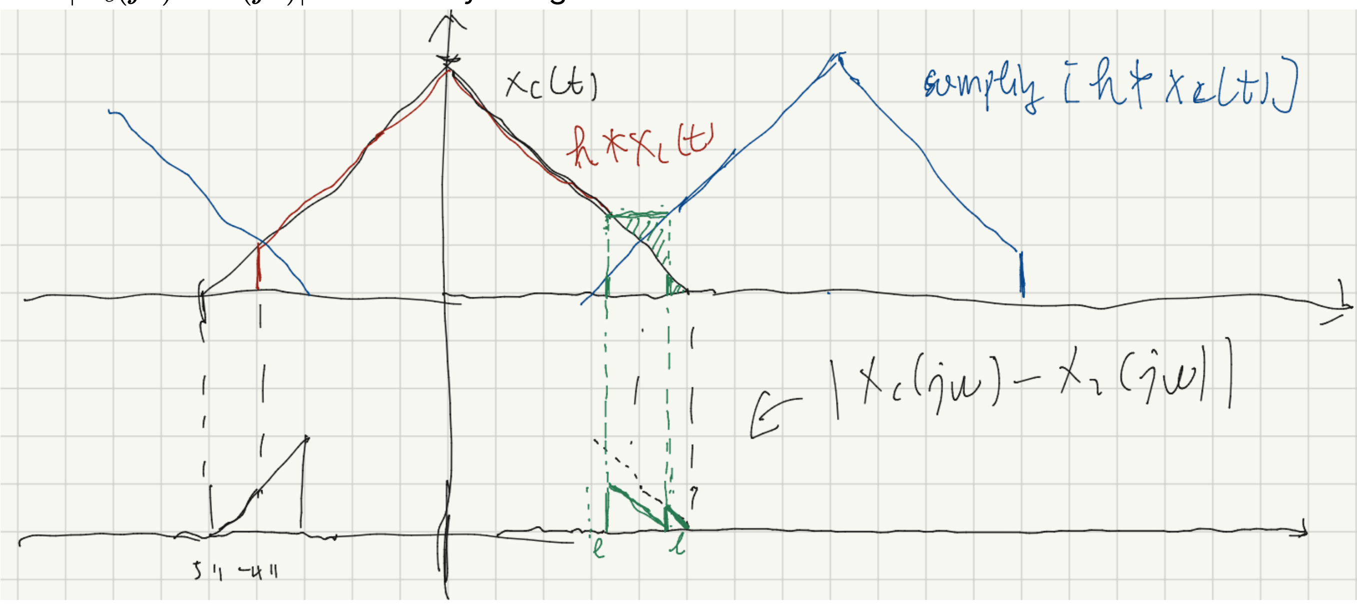
$$\int_{-\infty}^{\infty}|x_c(t)-x_2(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|X_c(j\omega)-X_2(j\omega)|^2dt=\frac{61}{3}\pi+70$$

(b)

Let's say:

$$H(j\Omega)=\begin{cases}1, |\Omega|<5\pi-l\\0, |\Omega|>5\pi-l\end{cases} \quad ,l\in[0,\pi]$$

The $|X_c(j\omega)-X(j\omega)|$ is basically the green shadow



which is:

$$|X_c(j\omega)-X(j\omega)|=\begin{cases}\frac{1}{\pi}\omega+5, \omega\in(-5\pi,-5\pi+l)\\ \frac{1}{\pi}\omega-(5-\frac{1}{\pi}l), \omega\in(-5\pi+l,-3\pi-l)\\ -\frac{1}{\pi}\omega+(5-\frac{1}{\pi}l), \omega\in(3\pi+l,5\pi-l)\\ -\frac{1}{\pi}\omega=5, \omega\in(5\pi-l,5\pi)\\ 0, \omega\in others\end{cases}$$

$$|X_c(j\omega)-X(j\omega)|^2=\begin{cases}\frac{1}{\pi}\omega^2+\frac{10}{\pi}\omega+25, \omega\in(-5\pi,-5\pi+l)\\ \frac{1}{\pi}\omega^2+\frac{2(5-\frac{1}{\pi}l)}{\pi}\omega+(5-\frac{1}{\pi}l)^2, \omega\in(-5\pi+l,-3\pi-l)\\ \frac{1}{\pi}\omega^2+\frac{2(5-\frac{1}{\pi}l)}{\pi}\omega+(5-\frac{1}{\pi}l)^2, \omega\in(3\pi+l,5\pi-l)\\ \frac{1}{\pi}\omega^2-\frac{10}{\pi}\omega+25, \omega\in(5\pi-l,5\pi)\\ 0, \omega\in others\end{cases}$$

So,

$$\int_{-\infty}^{\infty} |X_c(j\omega) - X_2(j\omega)|^2 d\omega = 2(\int_{5\pi-l}^{5\pi} (\frac{1}{\pi}\omega^2 + \frac{10}{\pi}\omega + 25) d\omega + \int_{3\pi+l}^{5\pi-l} (\frac{1}{\pi}\omega^2 + \frac{2(5-\frac{1}{\pi}l)}{\pi}\omega + (5 - \frac{1}{\pi}l)^2)($$
$$= a\ huge\ function\ about\ l$$

It is not hard to see that (i.e. I am tierd of solving this intrgral equation), when $l = \pi$, the function about l will get to its minimum point.

2.4

2.4. (Multi-Path Phenomenon, **OPTIONAL**) Consider a communication system that for any send signal $s(t)$, the received signal $r(t)$ is mixed through two possible paths:

$$r(t) = s(t) + as(t - t_0)$$

where $0 < a < 1$ and $t_0 > 0$ are known constants.

(a) A receiver processes $r(t)$ with his filter $h(t) = \delta(t) - a\delta(t - t_0)$. Show that Multi-Path phenomenon can be alleviated.

(b) Find a filter $h^*(t)$ that can (asymptotically) recover original signal $s(t)$ using $r(t)$.
Hint: $\frac{1}{1+ae^{-j\Omega t_0}} = \sum_{k=0}^{\infty}(-a)^k e^{-j\Omega k t_0}$.

(a)

$$\begin{aligned} r(t) * h(t) &= [s(t) + as(t - t_0)] * [\delta(t) - a\delta(t - t_0)] \\ &= s(t) * \delta(t) - s(t) * a\delta(t - t_0) + as(t - t_0) * \delta(t) + as(t - t_0) * a\delta(t - t_0) \\ &= s(t) - as(t - t0) + as(t - t0) - a^2 s(t - 2t_0) \\ &= s(t) - a^2 s(t - 2t_0) \end{aligned}$$

The multi-path phenomenon is alleviated because:

1. $0 < a < 1$, so $a^2 < a < 1$. The noise will be less than before
2. The noise change from $s(t - t_0)$ to $s(t - 2t_0)$, which has less depentancy across time.

(b)