5.1 Summary

We first introduce the concept of eigenvalue. Then we talked about eigenvalue decomposition. Knowing such technique can help us solve. In the case of y = Ax, when A is sym + def, we can use the eigenvalue decomposition to soleve the problem elegently.

In practice, the sym+def matrices are relatively rare to find. The singular value decomposition (SVD) takes apart an arbitrary MxN. We can use the SVD to tackle the general system of linear equations.

Then we learned about pseudo inverse.

After that we learned about stable reconstruction with the Tikhonov regularization.

5.2 SVD

5.2. (SVD, 2 pts) Derive the pseudo inverse of matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and use it to give the least-square solution to $Ax = [1, 0]^T$.

Solve:

$$\begin{aligned} det(\lambda I - A) &= det(\begin{bmatrix} \lambda - 1 & -1 \\ -1 & \lambda - 1 \end{bmatrix}) \\ &= (\lambda_1 - 1)^2 - 1 \end{aligned}$$

We get,

$$\lambda_1 = 2$$
$$\lambda_2 = 0$$

So,

$$\Sigma = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

$$A^+ = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Now we solve:

$$
\begin{aligned}
(X^TX)V &= V\Lambda \\
\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} V &= V \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \\
\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix} &= \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}
$$

we have:

$$
V = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}
$$

since V is symmetric,

$$
U = V = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}
$$

So, the SVD of A is:

$$
A = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}
$$

So the pseudo inverse of A is:

$$
A^+ = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}
$$

We have:

$$
\begin{aligned}
r &= V\beta - V\Sigma V^T V\alpha \\
&= V(\beta - \Sigma\alpha) \\
&= VV^T(y - \Sigma x) \\
&= y - \Sigma x
\end{aligned}
$$

so:

$$
||r||_2^2 = ||y - \Sigma x||_2^2
$$

so, x is

$$
\begin{aligned}
x &= V\Sigma^{-1}V^Ty \\
&= A^+y \\
&= \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}
\end{aligned}
$$

5.3 Convolution Theorem

5.3. (Convolution Theorem, 3pts) An $N \times N$ Circulant Matrix H has form

$$H = \begin{bmatrix} h[0] & h[N-1] & h[N-2] & \cdots & h[1] \\ h[1] & h[0] & h[N-1] & \cdots & h[2] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h[N-1] & h[N-2] & h[N-3] & \cdots & h[0] \end{bmatrix}$$

(a) Show that discrete Fourier vectors

$$f_k = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 \\ e^{j2\pi k/N} \\ e^{j2\pi 2k/N} \\ \vdots \\ e^{j2\pi(N-1)k/N} \end{bmatrix}.$$

for $k = 0, 1, \cdots, N-1$ are eigenvectors of H.

(b) The circular convolution of a signal x and impulse response h is given by:

$$y[n] = x[n] \circledast h[n] = \sum_{k=0}^{N-1} x[k]_N h[n-k]_N$$

where we use $[\cdot]_N = [\cdot \mathrm{mod} N]$ to denote a periodic indexing, e.g., $h[0]_N = h[0], h[-1]_N = h[N-1], h[N]_N = h[0]$. Explain why the circular convolution in time domain is multiplication in DFT domain.

Hint: The circular convolution can be written in matrix form $y = Hx$. H is a circulant matrix and have diagonal decomposition $H = FDF^H$.

a

We have:

$$\begin{aligned} Hf_k &= \begin{bmatrix} h[0] + h[N-1]e^{j2\pi k/N} + h[N-2]e^{j2\pi 2k/N} + \cdots + h[1]e^{j2\pi 2(N-1)k/N} \\ h[1] + h[0]e^{j2\pi k/N} + h[N-1]e^{j2\pi 2k/N} + \cdots + h[2]e^{j2\pi 2(N-1)k/N} \\ \vdots \\ h[N-1] + h[N-2]e^{j2\pi k/N} + h[N-1]e^{j2\pi 2k/N} + \cdots + h[0]e^{j2\pi 2(-1)k/N} \end{bmatrix} \\ &= (h[0] + h[N-1]e^{j2\pi 2k/N} + \cdots + h[1]e^{j2\pi 2(N-1)k/N})f_k \\ &= \lambda_k f_k \end{aligned}$$

So f_k is eigenvector.

b

$$y = Hx = FDF^H x$$

We find that F is made up by fk and D is a diagonal matrix with eigenvalues like $(h[0] + h[N-1]e^{j2\pi 2k/N} + \cdots + h[1]e^{j2\pi 2(N-1)k/N})$. We knows that f_k is discrete Fourier vector, so actually $F^H x$ is actually doing DFT and Fx is doing IDFT. While Dx is like doing the multiplication.
So, $FDF^H x$ is like do a DFT first, and then do a multiplication and do an IDFT again.
reference:https://brianmcfee.net/dstbook-site/content/ch10-convtheorem/ConvolutionTheorem.html

5.4 SVD Regularization

5.4. (SVD Regularization, 3 pts) The pseudo inverse reconstruction for least square problem can be unstable when small singular value occurs. Write a script to verify that more stable result can be achieved by truncated SVD or Tikhonov regularization.

We will do this with a de-convolution problem as example. The data can be found in `blocks.mat`, which include:

- `x`, a 512×1 signal.
- `h`, a 30×1 filter.
- `y`, the result of convolution `x*h`.
- `yn`, a noisy observation of `y`. (i.i.d. Gaussian noise with standard deviation 0.01)

(a) Construct a matrix A that gives $y = Ax$.

(b) Find the reconstruction error of pseudo inverse solution. How do the reconstruction error be related to singular value of A?

(c) Find the reconstruction error of truncated SVD solution. How do you decide the rank of low-rank approximation?

(d) Find the reconstruction error of Tikhonov regularization solution. How do you decide the regularization weight δ?

a.

$$A = \begin{bmatrix} h[0] & 0 & 0 & \cdots & 0 \\ h[1] & h[0] & 0 & \cdots & 0 \\ h[2] & h[1] & h[0] & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h[29] \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 & \cdots & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \cdots & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{3} \end{bmatrix} \in \mathbb{R}^{541 \times 512}$$

Kind of like a 30-in-width belt under diagonal.

b.

```
A = zeros(541, 512);

for i = 1:512

for j = 0 : 29

A(i+j, i) = 1/3;

end

end

[U, S, V] = svd(A);

A_ = pinv(A);

x_bar = A_ * yn;

norm(x_bar - x, 2)
```

So, $||\hat{x} - x||_2 = 50.183$

c.