

1 Statement

Let $\hat{\Theta}_{\theta,1}$ and $\hat{\Theta}_{\theta,2}$ be two unbiased estimators for a real parameter θ (deterministic). Let us define $\hat{\Theta}_\theta = \alpha\hat{\Theta}_{\theta,1} + \beta\hat{\Theta}_{\theta,2}$, with $\alpha, \beta \in \mathbb{R}$.

- For which value(s) of α and β is the estimator $\hat{\Theta}_\theta$ unbiased? Explain.
- For which value(s) of α and β is the estimator $\hat{\Theta}_\theta$ found above of minimal variance? We assume that $\hat{\Theta}_{\theta,1}$ and $\hat{\Theta}_{\theta,2}$ are independent and that $\mathbb{V}[\hat{\Theta}_{\theta,1}] = \mathbb{V}[\hat{\Theta}_{\theta,2}] = \sigma^2$. Explain.

2 Solution

- Since the parameter is deterministic, we are in the case of Fisher estimation. We can write $\hat{\Theta}_\theta = g(Z)$, for some random variable Z , hence we have that the estimator is unbiased if

$$\mathbb{E}[g(Z); \theta] := \int_z g(z) f_Z(z; \theta) dz = \theta.$$

We know that $\hat{\Theta}_{\theta,i}$ is unbiased for $i = 1, 2$, hence we have $\hat{\Theta}_{\theta,i} = g_i(Z) = \theta$. We also have that $g(Z) = \alpha g_1(Z) + \beta g_2(Z)$, hence

$$\mathbb{E}[g(Z); \theta] = \underbrace{\alpha \int_z g_1(z) f_Z(z; \theta) dz}_{\mathbb{E}[g_1(Z); \theta] = \theta} + \underbrace{\beta \int_z g_2(z) f_Z(z; \theta) dz}_{\mathbb{E}[g_2(Z); \theta] = \theta} = (\alpha + \beta)\theta.$$

As we want the estimator to be unbiased, we want this quantity to be equal to θ , hence the values of α and β which yield an unbiased estimator are

$$\boxed{\alpha \in \mathbb{R}, \quad \beta = 1 - \alpha.} \quad (1)$$

- In order to decide when an unbiased estimator is of minimal variance, we must compute its variance¹. The variance of $\hat{\Theta}_\theta$ is given by

$$\mathbb{V}[\hat{\Theta}_\theta] = \mathbb{E}[(\hat{\Theta}_\theta - \theta)^2],$$

where we simplified the expression from the definition thanks to the fact that θ is real and scalar. Developing and using (1), we find (with the subscript “c” indicating a centered variable)

$$\begin{aligned} \mathbb{V}[\hat{\Theta}_\theta] &= \mathbb{E}\left[\left(\alpha\hat{\Theta}_{\theta,1,c} + (1-\alpha)\hat{\Theta}_{\theta,2,c}\right)^2\right] \\ &= \mathbb{E}\left[\alpha^2\hat{\Theta}_{\theta,1,c}^2 + 2\alpha(1-\alpha)\hat{\Theta}_{\theta,1,c}\hat{\Theta}_{\theta,2,c} + (1-\alpha)^2\hat{\Theta}_{\theta,2,c}^2\right] \\ &= \mathbb{E}\left[\alpha^2\hat{\Theta}_{\theta,1,c}^2\right] + \mathbb{E}\left[2\alpha(1-\alpha)\hat{\Theta}_{\theta,1,c}\hat{\Theta}_{\theta,2,c}\right] + \mathbb{E}\left[(1-\alpha)^2\hat{\Theta}_{\theta,2,c}^2\right]. \end{aligned}$$

The first and third terms are simply multiples of the variances of $\hat{\Theta}_{\theta,i} = \sigma^2$, for $i = 1, 2$. The middle term is a multiple of the cross-covariance between the estimators, which is zero seen as they are independent. Thus, we get

$$\mathbb{V}[\hat{\Theta}_\theta] = \alpha^2\sigma^2 + (1-\alpha)^2\sigma^2.$$

We want to minimize this quantity, hence we set its derivative equal to zero:

$$\begin{aligned} \frac{d}{d\alpha} \left(\mathbb{V}[\hat{\Theta}_\theta] \right) &= (2\alpha - 2(1-\alpha))\sigma^2 := 0 \\ \iff 2\alpha &= 2(1-\alpha) \\ \iff \boxed{\alpha = \beta = \frac{1}{2}.} \end{aligned} \quad (2)$$

¹Yes, really.

We also need to verify that these values do in fact constitute a minimum. This requires computing the second derivative, which is equal to

$$\frac{d^2}{d\alpha^2} \left(\mathbb{V} [\hat{\Theta}_\theta] \right) = \frac{d}{d\alpha} \left((2\alpha - 2(1 - \alpha))\sigma^2 \right) = 4\sigma^2 \geq 0. \quad (3)$$

As mentioned in (3), this quantity is positive, hence the values of α and β found above minimize the variance, as expected. This can also be observed on Figure 1.

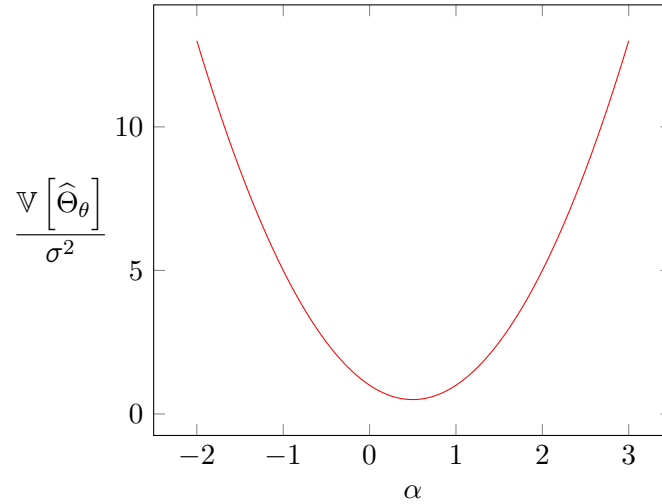


Figure 1: Plot of $\mathbb{V} [\hat{\Theta}_\theta] / \sigma^2$ as a function of α . The variance is minimized when $\alpha = 1/2$, as expected.

Taking both requirements into account, we find that

$$\boxed{\hat{\Theta}_\theta = \frac{\hat{\Theta}_{\theta,1} + \hat{\Theta}_{\theta,2}}{2}}. \quad (4)$$