

1 Statement

Let $X(t)$ be a continuous WSS stochastic process. The covariance function of $X(t)$ is given by $C_X(\tau) = e^{-\tau^2/2}$ and its mean by $\mathbf{m}_X = 1$.

$X(t)$ is filtered by an LTI system of impulse response $h(t) = e^{-3t}u(t)$ to produce a new process $Y(t)$.

1. Derive the expression of the power spectral density of $X(t)$.
2. Derive the mean of $Y(t)$.
3. Derive the expression of the power spectral density of $Y(t)$.

2 Solution

1. For the first exercise, we must recall that the power spectral density is defined as the Fourier transform of the autocovariance function. We thus have

$$\begin{aligned}\gamma_X(\omega) &= \mathcal{F}_\tau [C_X(\tau)](\omega) \\ &= \mathcal{F}_\tau [e^{-\tau^2/2}](\omega) \\ &= \int_{-\infty}^{+\infty} e^{-\tau^2/2} e^{-j\omega\tau} d\tau.\end{aligned}$$

Computing this integral, we find that

$$\boxed{\gamma_X(\omega) = \sqrt{2\pi} e^{-\omega^2/2}}. \quad (1)$$

2. In order to compute the mean of $Y(t)$, we remember that the output of an LTI system that receives a WSS input is also WSS:

$$\begin{aligned}\mathbf{m}_Y &= \mathbb{E}[Y(t)] \\ &= \int_{-\infty}^{+\infty} y(\tau) d\tau \\ &= \mathbf{m}_X \int_{-\infty}^{+\infty} h(\tau) d\tau \\ &= \int_{-\infty}^{+\infty} e^{-3\tau} u(\tau) d\tau \\ &= \int_0^{+\infty} e^{-3\tau} d\tau.\end{aligned}$$

Computing this integral yields

$$\boxed{\mathbf{m}_Y = \frac{1}{3}}. \quad (2)$$

3. In order to compute the power spectral density of $Y(t)$, we must recall the Wiener–Khinchin theorem: since $Y(t)$ is the output of an LTI system to which a WSS random process is fed,

$$\gamma_Y(\omega) = |H(\omega)|^2 \gamma_X(\omega). \quad (3)$$

In order to use the result of (1), we must compute the Fourier transform of the impulse response, $H(\omega)$:

$$\begin{aligned}H(\omega) &= \mathcal{F}_t [h(t)](\omega) \\ &= \mathcal{F}_t [e^{-3t}u(t)](\omega) \\ &= \int_{-\infty}^{+\infty} e^{-3\tau} u(\tau) e^{j\omega\tau} d\tau \\ &= \int_0^{+\infty} e^{-3\tau} e^{j\omega\tau} d\tau \\ &= \frac{1}{j\omega + 3}.\end{aligned} \quad (4)$$

With the result in (4), we can easily compute $\gamma_Y(\omega)$ using (3) and (1):

$$\boxed{\gamma_Y(\omega) = \sqrt{2\pi} \frac{1}{|\mathrm{j}\omega + 3|^2} \mathrm{e}^{-\omega^2/2}}. \quad (5)$$