1 Statement

Let $X(n)$ be a real WSS process whose correlation function is given by $R_X(k) = \mathrm{e}^{-k^2}$. This signal is amplified by a factor $c \in \mathbb{R}$ and corrupted by an additive white noise $N(n)$ of variance σ_N^2. We hence observe a signal $Y(n)$ given by $Y(n) = cX(n) + N(n)$.

One would like to design a Wiener filter of order two (i.e. with two coefficients) in order to optimally compute an estimate of $X(n)$ based on observations $Y(n)$.

- Based on the orthogonality principle, give the linear system of equations to solve in order to find the expression of the two coefficients.
- Give the expressions of the two coefficients of the filter.
- Give the expression of the transfer function $W(z)$ of the Wiener filter.

2 Solution

- The orthogonality principle says that $\mathbb{E}\left[\left(X(n) - \hat{X}(n)\right)\left(cX(n) + N(n)\right)\right] = 0$. From this, one can deduce the Wiener–Hopf equations, given in matrix form by

$$R_Y w = R_{XY}\,.$$

One can compute these matrices[1]:

$$\begin{aligned} R_Y(k) &= c^2 R_X(k) + R_N(k) = c^2 \mathrm{e}^{-k^2} + \sigma_N^2 \delta(k)\,,\\ R_{XY}(k) &= c^2 \mathrm{e}^{-k^2}\,. \end{aligned}$$

Knowing that the filter is of order two, one then finds the following Wiener–Hopf equations:

$$\begin{aligned} &\begin{pmatrix} R_Y(0) & R_Y(-1) \\ R_Y(1) & R_Y(0) \end{pmatrix} \begin{pmatrix} w(0) \\ w(1) \end{pmatrix} = \begin{pmatrix} R_{XY}(0) \\ R_{XY}(1) \end{pmatrix} \\ \implies &\boxed{\begin{pmatrix} c^2 + \sigma_N^2 & c^2/\mathrm{e} \\ c^2/\mathrm{e} & c^2 + \sigma_N^2 \end{pmatrix} \begin{pmatrix} w(0) \\ w(1) \end{pmatrix} = \begin{pmatrix} c^2 \\ c^2/\mathrm{e} \end{pmatrix}.} \end{aligned} \tag{1}$$

- Rearranging the system in (1) to isolate the optimal filter, one finds

$$\begin{pmatrix} w_{\mathrm{opt}}(0) \\ w_{\mathrm{opt}}(1) \end{pmatrix} = \begin{pmatrix} c^2 + \sigma_N^2 & c^2/\mathrm{e} \\ c^2/\mathrm{e} & c^2 + \sigma_N^2 \end{pmatrix}^{-1} \begin{pmatrix} c^2 \\ c^2/\mathrm{e} \end{pmatrix}.$$

Solving this equation yields

$$\begin{pmatrix} w_{\mathrm{opt}}(0) \\ w_{\mathrm{opt}}(1) \end{pmatrix} = \frac{\mathrm{e}}{(\mathrm{e}^2 - 1)c^4 + 2\mathrm{e}^2 c^2 \sigma_N^2 + \mathrm{e}^2 \sigma_N^4} \begin{pmatrix} \mathrm{e}c^2 + \mathrm{e}\sigma_N^2 & -c^2 \\ -c^2 & \mathrm{e}c^2 + \mathrm{e}\sigma_N^2 \end{pmatrix} \begin{pmatrix} c^2 \\ c^2/\mathrm{e} \end{pmatrix}.$$

This finally evaluates to

$$\boxed{\begin{pmatrix} w_{\mathrm{opt}}(0) \\ w_{\mathrm{opt}}(1) \end{pmatrix} = \frac{\mathrm{e}c^2}{(\mathrm{e}^2 - 1)c^4 + 2\mathrm{e}^2 c^2 \sigma_N^2 + \mathrm{e}^2 \sigma_N^4} \begin{pmatrix} \mathrm{e}c^2 + \mathrm{e}\sigma_N^2 - c^2/\mathrm{e} \\ \sigma_N^2 \end{pmatrix}.} \tag{2}$$

- The transfer function $W(z)$ of the filter can then be found to be

$$\boxed{W(z) = \frac{\mathrm{e}c^2}{(\mathrm{e}^2 - 1)c^4 + 2\mathrm{e}^2 c^2 \sigma_N^2 + \mathrm{e}^2 \sigma_N^4} \left(\mathrm{e}c^2 + \mathrm{e}\sigma_N^2 - c^2/\mathrm{e} + z^{-1}\sigma_N^2\right),} \tag{3}$$

from (2).

[1] Using properties of the covariance which also apply to the correlation due to its quadratic nature, as well as using the fact that X and N are uncorrelated.