1 Statement

Let X be a continuous random variable whose probability density function is given by

$$\mathsf{T}_X(x) = \begin{cases} c\,, & -1 \le x \le 1, \\ 0\,, & \text{otherwise.} \end{cases} \tag{1}$$

1. What is the value of c?
2. Derive the expression of the cumulative distribution function of X.
3. Derive the expression of the characteristic function of X.
4. Let Y be the continuous random variable defined as $Y = 3\mathrm{e}^X + 1$. Derive the expression of the probability density function of Y.

2 Solution

1. In order to find the value of c, we must recall that, by the definition of the probability density function,

$$\int_{-\infty}^{+\infty} \mathsf{T}_X(x)\,\mathrm{d}x = 1\,. \tag{2}$$

With this in mind, we find that

$$\int_{-1}^{1} c\,\mathrm{d}x = 1\,. \tag{3}$$

Solving the integral on the left yields

$$\Big[cx\Big]_{x=-1}^{x=1} = 1 \iff \boxed{c = \frac{1}{2}}\,. \tag{4}$$

2. The cumulative distribution function $\mathsf{F}_X(x)$ is defined as

$$\mathsf{F}_X(x) = \int_{-\infty}^{x} \mathsf{T}_X(\xi)\,\mathrm{d}\xi \implies \boxed{\mathsf{F}_X(x) = \begin{cases} 0\,, & x < -1\,, \\ (x+1)/2\,, & -1 \le x \le 1\,, \\ 1\,, & x > 1\,. \end{cases}} \tag{5}$$

3. The characteristic function $\phi_X(q)$ is defined as

$$\phi_X(q) = \mathsf{E}\left[\mathrm{e}^{\mathrm{j}qX}\right] = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{j}qx}\mathsf{T}_X(x)\,\mathrm{d}x\,. \tag{6}$$

This means that in our case,

$$\phi_X(q) = \int_{-1}^{1} \mathrm{e}^{\mathrm{j}qx}\frac{1}{2}\,\mathrm{d}x \iff \phi_X(q) = \frac{1}{2}\left[\frac{1}{\mathrm{j}q}\mathrm{e}^{\mathrm{j}qx}\right]_{x=-1}^{x=1}\,. \tag{7}$$

Substituting the values of x in the previous equation then yields

$$\phi_X(q) = \frac{1}{\mathrm{j}2q}(\mathrm{e}^{\mathrm{j}q} - \mathrm{e}^{-\mathrm{j}q}) \implies \boxed{\phi_X(q) = \frac{\sinh(\mathrm{j}q)}{\mathrm{j}q}}\,. \tag{8}$$

4. Since the transformation $u\colon x \mapsto 3\mathrm{e}^x + 1$ is strictly increasing, u has an inverse, $u^{-1}\colon y \mapsto \ln\left(\frac{y-1}{3}\right)$. Thus, we have

$$
\begin{aligned}
\mathsf{F}_Y(y) &= \Pr\left\{u^{-1}\left(3\mathrm{e}^X + 1\right) \le u^{-1}(y)\right\} && (9)\\
&= \Pr\left\{X \le \ln\left(\frac{y-1}{3}\right)\right\} && (10)\\
&= \mathsf{F}_X\left(\ln\left(\frac{y-1}{3}\right)\right) && (11)\\
&= \begin{cases} 0\,, & y < 3/\mathrm{e} + 1\,,\\ \left(\ln\left(\frac{y-1}{3}\right) + 1\right)\Big/ 2\,, & 3/\mathrm{e} + 1 \le y \le 3\mathrm{e} + 1\,,\\ 1\,, & y > 3\mathrm{e} + 1\,. \end{cases} && (12)
\end{aligned}
$$

We then finally take the derivative of $\mathsf{F}_Y(y)$ in order to find $\mathsf{T}_Y(y)$:

$$
\boxed{\mathsf{T}_Y(y) = \begin{cases} \frac{1}{2(y-1)}\,, & 3/\mathrm{e} + 1 \le y \le 3\mathrm{e} + 1,\\ 0\,, & \text{otherwise.} \end{cases}} \tag{13}
$$