

1 Statement

Let $E(t)$ be a white noise with autocovariance function given by $C_E(\tau) = 4\delta(\tau)$. This process passes through an LTI system \mathcal{L} (also called *innovation filter*) and we get the real stochastic process $Y(t)$ as output. The power spectral density of $Y(t)$, denoted by $\gamma_Y(\omega)$, is given by

$$\gamma_Y(\omega) = \frac{64\omega^2 + 64}{4\omega^4 + 65\omega^2 + 16}.$$

1. What is the power spectral density of the white noise, i.e., $\gamma_E(\omega)$?
2. Give the expressions of $\gamma_E(s)$ and $\gamma_Y(s)$, using the appropriate change of variable.
3. Compute the transfer function $L(s)$ of the LTI system \mathcal{L} . Make sure the filter is minimum-phase.

2 Solution

1. For the first exercise, we must recall that the power spectral density is defined as the Fourier transform of the autocovariance function. We thus have

$$\begin{aligned}\gamma_E(\omega) &= \mathcal{F}_\tau [C_E(\tau)](\omega) \\ &= \mathcal{F}_\tau [4\delta(\tau)](\omega) \\ &= \int_{-\infty}^{+\infty} 4\delta(\tau)e^{-j\omega\tau} d\tau.\end{aligned}$$

Computing this integral, we find that

$$\boxed{\gamma_E(\omega) = 4.} \quad (1)$$

2. Our current power spectral density expressions are functions of the imaginary variable $j\omega$. In order to express them as functions of $s = \sigma + j\omega$, we must bear in mind that $\omega^2 = -s^2$ and that $\omega^4 = s^4$. With these conversions, we find that

$$\boxed{\gamma_Y(s) = \frac{-64s^2 + 64}{4s^4 - 65s^2 + 16},} \quad (2)$$

$$\boxed{\gamma_E(s) = 4.} \quad (3)$$

3. In order to obtain a causal, stable filter, we must verify that the Paley–Wiener condition is satisfied, otherwise no such solution can exist. Using [WolframAlpha](#), we find that

$$\int_{-\infty}^{+\infty} \frac{\left| \ln \left(\frac{64\omega^2 + 64}{4\omega^4 + 65\omega^2 + 16} \right) \right|}{1 + \omega^2} d\omega \approx 2.90319 < +\infty,$$

which means the condition is satisfied. Since $Y(t)$ is the output of an LTI system with transfer function $L(s)$ to which we feed a white noise $E(t)$, we know that

$$\gamma_Y(s) = L(s)L(-s)\gamma_E(s).$$

We then apply the spectral factorization procedure.

$$\begin{aligned}\gamma_Y(s) &= L(s)L(-s)\gamma_E(s) \\ &= \underbrace{\left(2 \frac{s+1}{(s+4)(s+1/2)} \right)}_{L(s)} \underbrace{\left(2 \frac{-s+1}{(-s+4)(-s+1/2)} \right)}_{L(-s)} \gamma_E(s).\end{aligned}$$

We thus obtain the following expression for the transfer function of the innovation filter \mathcal{L} :

$$\boxed{L(s) = 2 \frac{s+1}{(s+4)(s+1/2)}.} \quad (4)$$

In order to verify whether this filter is minimum-phase, we must check that the poles and zeroes are all in the open left-half plane, that is, $\text{Re}(p_i) < 0$ and $\text{Re}(z_i) < 0$, for all i . The poles are $p_1 = -4, p_2 = -1/2$ and the only zero is $z_1 = -1$. This implies that the filter is minimum-phase, as required.