1 Statement

Let $E(t)$ be a white noise with autocovariance function given by $C_E(\tau) = 4\delta(\tau)$. This process passes through an LTI system $\mathcal{L}$ (also called *innovation filter*) and we get the real stochastic process $Y(t)$ as output. The power spectral density of $Y(t)$, denoted by $\gamma_Y(\omega)$, is given by

$$\gamma_Y(\omega) = \frac{64\omega^2 + 64}{4\omega^4 + 65\omega^2 + 16}\,.$$

1. What is the power spectral density of the white noise, i.e., $\gamma_E(\omega)$?
2. Give the expressions of $\gamma_E(s)$ and $\gamma_Y(s)$, using the appropriate change of variable.
3. Compute the transfer function $L(s)$ of the LTI system $\mathcal{L}$. Make sure the filter is minimum-phase.

2 Solution

1. For the first exercise, we must recall that the power spectral density is defined as the Fourier transform of the autocovariance function. We thus have

$$\begin{aligned}\gamma_E(\omega) &= \mathscr{F}_\tau\left[C_E(\tau)\right](\omega)\\ &= \mathscr{F}_\tau\left[4\delta(\tau)\right](\omega)\\ &= \int_{-\infty}^{+\infty} 4\delta(\tau)\mathrm{e}^{-\mathrm{j}\omega\tau}\,\mathrm{d}\tau\,.\end{aligned}$$

Computing this integral, we find that

$$\boxed{\gamma_E(\omega) = 4\,.} \tag{1}$$

2. Our current power spectral density expressions are functions of the imaginary variable $\mathrm{j}\omega$. In order to express them as functions of $s = \sigma + \mathrm{j}\omega$, we must bear in mind that $\omega^2 = -s^2$ and that $\omega^4 = s^4$. With these conversions, we find that

$$\gamma_Y(s) = \frac{-64s^2 + 64}{4s^4 - 65s^2 + 16}\,, \tag{2}$$

$$\gamma_E(s) = 4\,. \tag{3}$$

3. In order to obtain a causal, stable filter, we must verify that the Paley–Wiener condition is satisfied, otherwise no such solution can exist. Using WolframAlpha, we find that

$$\int_{-\infty}^{+\infty} \frac{\left|\ln\left(\frac{64\omega^2+64}{4\omega^4+65\omega^2+16}\right)\right|}{1+\omega^2}\,\mathrm{d}\omega \approx 2.90319 < +\infty\,,$$

which means the condition is satisfied. Since $Y(t)$ is the output of an LTI system with transfer function $L(s)$ to which we feed a white noise $E(t)$, we know that

$$\gamma_Y(s) = L(s)L(-s)\gamma_E(s)\,.$$

We then apply the spectral factorization procedure.

$$\begin{aligned}\gamma_Y(s) &= L(s)L(-s)\gamma_E(s)\\ &= \underbrace{\left(2\frac{s+1}{(s+4)(s+1/2)}\right)}_{L(s)}\underbrace{\left(2\frac{-s+1}{(-s+4)(-s+1/2)}\right)}_{L(-s)}\gamma_E(s)\,.\end{aligned}$$

We thus obtain the following expression for the transfer function of the innovation filter $\mathcal{L}$:

$$\boxed{L(s) = 2\frac{s+1}{(s+4)(s+1/2)}\,.} \tag{4}$$

In order to verify whether this filter is minimum-phase, we must check that the poles and zeroes are all in the open left-half plane, that is, $\mathrm{Re}(p_i) < 0$ and $\mathrm{Re}(z_i) < 0$, for all i. The poles are $p_1 = -4, p_2 = -1/2$ and the only zero is $z_1 = -1$. This implies that the filter is minimum-phase, as required.