1 Statement

Let $X(t)$ be a continuous WSS stochastic process. The covariance function of $X(t)$ is given by $C_X(\tau) = \mathrm{e}^{-\tau^2/2}$ and its mean by $\mathsf{m}_X = 1$.

$X(t)$ is filtered by an LTI system of impulse response $h(t) = \mathrm{e}^{-3t}u(t)$ to produce a new process $Y(t)$.

1. Derive the expression of the power spectral density of $X(t)$.
2. Derive the mean of $Y(t)$.
3. Derive the expression of the power spectral density of $Y(t)$.

2 Solution

1. For the first exercise, we must recall that the power spectral density is defined as the Fourier transform of the autocovariance function. We thus have

$$\begin{aligned}\gamma_X(\omega) &= \mathscr{F}_\tau\left[C_X(\tau)\right](\omega)\\ &= \mathscr{F}_\tau\left[\mathrm{e}^{-\tau^2/2}\right](\omega)\\ &= \int_{-\infty}^{+\infty} \mathrm{e}^{-\tau^2/2}\mathrm{e}^{-\mathrm{j}\omega\tau}\,\mathrm{d}\tau\,.\end{aligned}$$

Computing this integral, we find that

$$\boxed{\gamma_X(\omega) = \sqrt{2\pi}\mathrm{e}^{-\omega^2/2}\,.} \tag{1}$$

2. In order to compute the mean of $Y(t)$, we remember that the output of an LTI system that receives a WSS input is also WSS:

$$\begin{aligned}\mathsf{m}_Y &= \mathsf{E}\left[Y(t)\right]\\ &= \int_{-\infty}^{+\infty} y(\tau)\,\mathrm{d}\tau\\ &= \mathsf{m}_X\int_{-\infty}^{+\infty} h(\tau)\,\mathrm{d}\tau\\ &= \int_{-\infty}^{+\infty} \mathrm{e}^{-3\tau}u(\tau)\,\mathrm{d}\tau\\ &= \int_{0}^{+\infty} \mathrm{e}^{-3\tau}\,\mathrm{d}\tau\,.\end{aligned}$$

Computing this integral yields

$$\boxed{\mathsf{m}_Y = \frac{1}{3}\,.} \tag{2}$$

3. In order to compute the power spectral density of $Y(t)$, we must recall the Wiener–Khinchin theorem: since $Y(t)$ is the output of an LTI system to which a WSS random process is fed,

$$\gamma_Y(\omega) = \left|H(\omega)\right|^2\gamma_X(\omega)\,. \tag{3}$$

In order to use the result of (1), we must compute the Fourier transform of the impulse response, $H(\omega)$:

$$\begin{aligned}H(\omega) &= \mathscr{F}_t\left[h(t)\right](\omega)\\ &= \mathscr{F}_t\left[\mathrm{e}^{-3t}u(t)\right](\omega)\\ &= \int_{-\infty}^{+\infty} \mathrm{e}^{-3\tau}u(\tau)\mathrm{e}^{\mathrm{j}\omega\tau}\,\mathrm{d}\tau\\ &= \int_{0}^{+\infty} \mathrm{e}^{-3\tau}\mathrm{e}^{\mathrm{j}\omega\tau}\,\mathrm{d}\tau\\ &= \frac{1}{\mathrm{j}\omega+3}\,.\end{aligned} \tag{4}$$

With the result in (4), we can easily compute $\gamma_Y(\omega)$ using (3) and (1):

$$\boxed{\gamma_Y(\omega) = \frac{\sqrt{2\pi}}{\omega^2 + 9}\mathrm{e}^{-\omega^2/2}\,.} \tag{5}$$