1 Statement

Let $X(t)$ be a WSS stochastic process with autocovariance function $\mathsf{C}_X(\tau)$. Let us consider the two following continuous-time stochastic processes,

$$Y(t) = X(t)\cos(\omega t + \phi) \quad \text{and} \quad Z(t) = X(t)\cos[(\omega+\delta)t + \phi]\,, \tag{1}$$

where ω and δ are constant and ϕ is a random variable independent of $X(t)$ and uniformly distributed on $[0, 2\pi]$. Their autocovariance functions are

- $\mathsf{C}_Y(\tau) = \frac{1}{2}\mathsf{C}_X(\tau)\cos(\omega\tau)$;
- $\mathsf{C}_Z(\tau) = \frac{1}{2}\mathsf{C}_X(\tau)\cos[(\omega+\delta)\tau]$.

Discuss the wide sense stationarity of process $H(t) = Y(t) + Z(t)$.

2 Solution

In order for $H(t)$ to be WSS, the signal should respect two conditions:

- The mean should not depend on the time.
- The covariance between two times t and t' should only depend on their difference, $\tau = t - t'$.

2.1 Condition on the mean

We thus compute

$$\begin{aligned}
\mathsf{m}_H(t) &= \mathsf{E}\left[H(t)\right] && (2)\\
&= \mathsf{E}\left[X(t)[\cos(\omega t+\phi) + \cos((\omega+\delta)t+\phi)]\right] && (3)\\
&= \mathsf{E}\left[X(t)\right]\mathsf{E}\left[\cos(\omega t+\phi) + \cos((\omega+\delta)t+\phi)\right] && (4)\\
&= \frac{\mathsf{m}_X}{2\pi}\int_0^{2\pi}\left[\cos(\omega t+\phi) + \cos((\omega+\delta)t+\phi)\right]\mathrm{d}\phi && (5)\\
&= \frac{\mathsf{m}_X}{2\pi}\left(\int_{\omega t}^{2\pi+\omega t}\cos\phi'\,\mathrm{d}\phi' + \int_{(\omega+\delta)t}^{2\pi+(\omega+\delta)t}\cos\phi'\,\mathrm{d}\phi'\right) && (6)\\
&= 0\,. && (7)
\end{aligned}$$

Hence the mean of $Y(t)$, $Z(t)$ and $H(t)$ does not depend on t.

2.2 Condition on the autocovariance

Next, we look at the autocovariance. We compute (using the previous result)

$$\mathsf{C}_H(t,t') = \mathsf{E}\left[H_\mathrm{c}(t)H_\mathrm{c}^\mathrm{H}(t')\right] \tag{8}$$

where $^\mathrm{H}$ denotes the transjugate matrix,

$$\begin{aligned}
&= \mathsf{E}\left[H(t)H^\mathrm{H}(t')\right] && (9)\\
&= \mathsf{E}\left[(Y(t)+Z(t))(Y^\mathrm{H}(t')+Z^\mathrm{H}(t'))\right]\,, && (10)\\
&= \mathsf{E}\left[Y(t)Y^\mathrm{H}(t')\right] + \mathsf{E}\left[Z(t)Z^\mathrm{H}(t')\right] + \mathsf{E}\left[Y(t)Z^\mathrm{H}(t')\right] + \mathsf{E}\left[Z(t)Y^\mathrm{H}(t')\right] && (11)\\
&= \mathsf{C}_Y(\tau) + \mathsf{C}_Z(\tau) + 2\mathsf{E}\left[X(t)X^\mathrm{H}(t')\cos(\omega t+\phi)\cos((\omega+\delta)t'+\phi)\right] && (12)\\
&= \mathsf{C}_Y(\tau) + \mathsf{C}_Z(\tau) + 2\left(\mathsf{C}_X(\tau) + \mathsf{m}_X\mathsf{m}_X^\mathrm{H}\right)\mathsf{E}\left[\cos(\omega t+\phi)\cos((\omega+\delta)t'+\phi)\right]\,. && (13)
\end{aligned}$$

We then only have to determine whether the expectation in that last term depends only on τ.

$$\begin{aligned}
\mathsf{E}\left[\cos(\omega t+\phi)\cos((\omega+\delta)t'+\phi)\right] &= \mathsf{E}\left[\cos(\omega\tau - \delta t') + \cos(\omega(t+t') + \delta t' + 2\phi)\right] && (14)\\
&= \cos(\omega\tau - \delta t') + \frac{1}{2\pi}\int_0^{2\pi}\cos(\omega(t+t') + \delta t' + 2\phi)\,\mathrm{d}\phi && (15)\\
&= \cos(\omega\tau - \delta t')\,. && (16)
\end{aligned}$$

This final term should only depend on τ in order for the autocovariance to only depend on τ, and hence make $H(t)$ a WSS stochastic process. The only way for this to be true, is to have $\delta = 0$. Since there was no previous constraint on $H(t)$ because of the mean, we can conclude that $\boxed{H(t) \text{ is a WSS process if and only if } \delta = 0.}$