1 Statement

Let $\widehat{\Theta}_{\theta,1}$ and $\widehat{\Theta}_{\theta,2}$ be two unbiased estimators for a real parameter θ (deterministic). Let us define $\widehat{\Theta}_\theta = \alpha\widehat{\Theta}_{\theta,1} + \beta\widehat{\Theta}_{\theta,2}$, with $\alpha, \beta \in \mathbb{R}$.

- For which value(s) of α and β is the estimator $\widehat{\Theta}_\theta$ unbiased? Explain.
- For which value(s) of α and β is the estimator $\widehat{\Theta}_\theta$ found above of minimal variance? We assume that $\widehat{\Theta}_{\theta,1}$ and $\widehat{\Theta}_{\theta,2}$ are independent and that $\mathbb{V}\left[\widehat{\Theta}_{\theta,1}\right] = \mathbb{V}\left[\widehat{\Theta}_{\theta,2}\right] = \sigma^2$. Explain.

2 Solution

- Since the parameter is deterministic, we are in the case of Fisher estimation. We can write $\widehat{\Theta}_\theta = g(Z)$, for some random variable Z, hence we have that the estimator is unbiased if

$$\mathbb{E}\left[g(Z);\theta\right] := \int_z g(z)\mathsf{f}_Z(z;\theta)\,\mathrm{d}z = \theta\,.$$

  We know that $\widehat{\Theta}_{\theta,i}$ is unbiased for $i = 1,2$, hence we have $\widehat{\Theta}_{\theta,i} = g_i(Z) = \theta$. We also have that $g(Z) = \alpha g_1(Z) + \beta g_2(Z)$, hence

$$\mathbb{E}\left[g(Z);\theta\right] = \alpha \underbrace{\int_z g_1(z)\mathsf{f}_Z(z;\theta)\,\mathrm{d}z}_{\mathbb{E}[g_1(Z);\theta]=\theta} + \beta \underbrace{\int_z g_2(z)\mathsf{f}_Z(z;\theta)\,\mathrm{d}z}_{\mathbb{E}[g_2(Z);\theta]=\theta} = (\alpha+\beta)\theta\,.$$

  As we want the estimator to be unbiased, we want this quantity to be equal to θ, hence the values of α and β which yield an unbiased estimator are

$$\boxed{\alpha \in \mathbb{R}\,, \quad \beta = 1-\alpha\,.} \tag{1}$$

- In order to decide when an unbiased estimator is of minimal variance, we must compute its variance[1]. The variance of $\widehat{\Theta}_\theta$ is given by

$$\mathbb{V}\left[\widehat{\Theta}_\theta\right] = \mathbb{E}\left[(\widehat{\Theta}_\theta - \theta)^2\right]\,,$$

  where we simplified the expression from the definition thanks to the fact that θ is real and scalar. Developing and using (1), we find

$$\begin{aligned}\mathbb{V}\left[\widehat{\Theta}_\theta\right] &= \mathbb{E}\left[\left(\alpha\widehat{\Theta}_{\theta,1} + (1-\alpha)\widehat{\Theta}_{\theta,2}\right)^2\right]\\ &= \mathbb{E}\left[\alpha^2\widehat{\Theta}_{\theta,1}^2 + 2\alpha(1-\alpha)\widehat{\Theta}_{\theta,1}\widehat{\Theta}_{\theta,2} + (1-\alpha)^2\widehat{\Theta}_{\theta,2}^2\right]\\ &= \mathbb{E}\left[\alpha^2\widehat{\Theta}_{\theta,1}^2\right] + \mathbb{E}\left[2\alpha(1-\alpha)\widehat{\Theta}_{\theta,1}\widehat{\Theta}_{\theta,2}\right] + \mathbb{E}\left[(1-\alpha)^2\widehat{\Theta}_{\theta,2}^2\right]\,.\end{aligned}$$

  The first and third terms are simply multiples of the variances of $\widehat{\Theta}_{\theta,i} = \sigma^2$, for $i = 1,2$. The middle term is a multiple of the cross-covariance between the estimators, which is zero seen as they are independent. Thus, we get

$$\mathbb{V}\left[\widehat{\Theta}_\theta\right] = \alpha^2\sigma^2 + (1-\alpha)^2\sigma^2\,.$$

  We want to minimize this quantity, hence we set its derivative equal to zero:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\mathbb{V}\left[\widehat{\Theta}_\theta\right]\right) &= \big(2\alpha - 2(1-\alpha)\big)\sigma^2 := 0\\ \iff 2\alpha &= 2(1-\alpha)\\ \iff &\boxed{\alpha = \beta = \frac{1}{2}\,.}\end{aligned} \tag{2}$$

[1] Yes, really.

We also need to verify that these values do in fact constitute a minimum. This requires computing the second derivative, which is equal to

$$\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\left(\mathbb{V}\left[\widehat{\Theta}_\theta\right]\right) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\Big(\big(2\alpha - 2(1-\alpha)\big)\sigma^2\Big) = 4\sigma^2 \geq 0\,. \tag{3}$$

As mentioned in (3), this quantity is positive, hence the values of α and β found above minimize the variance, as expected. This can also be observed on Figure 1.

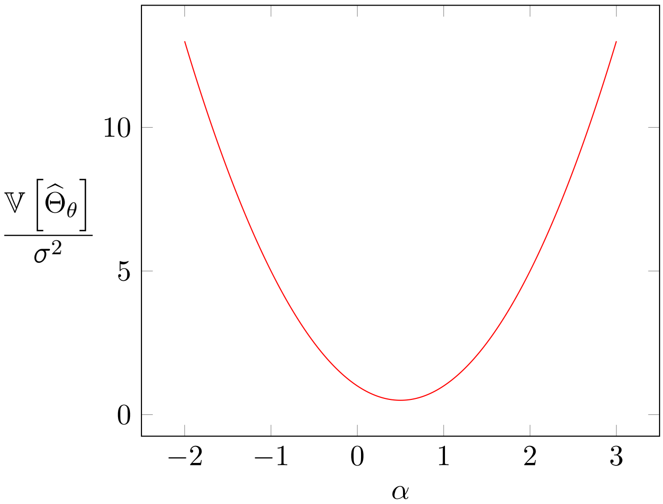

Figure 1: Plot of $\mathbb{V}\left[\widehat{\Theta}_\theta\right]\big/\sigma^2$ as a function of α. The variance is minimized when $\alpha = 1/2$, as expected.

Taking both requirements into account, we find that

$$\boxed{\widehat{\Theta}_\theta = \frac{\widehat{\Theta}_{\theta,1} + \widehat{\Theta}_{\theta,2}}{2}\,.} \tag{4}$$