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LINMA1731 – Project 2019
Fish schools tracking

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Abstract

In this paper we solve the first part of the project for the class "Stochastic process: Estimation and prediction" given during the Fall term of 2019. The average speed of each fish in a school of fish is approximated by a gamma-distributed random variable with a shape parameter k and a scale parameter s, and various methods for estimating this quantity are given; a numerical simulation is also included.

Contents

Part 1. **Average speed estimation**

1. **Introduction**

For the purpose of this project, we assume that the speed of each fish in a school at time i is a random variable V_i following a Gamma distribution, as suggested in [1]. This distribution is characterized by two parameters: a shape parameter $k > 0$ and a scale parameter $s > 0$. The parameters are the same for every fish and are time invariant. The aim of this first part is to identify these two parameters using empirical observations v_i.

2. **Maximum likelihood estimation**

Let v_i be i.i.d. realisations of a random variable following a Gamma distribution $\Gamma(k, s)$ (with $i = 1, \ldots, N$). We first assume that the shape parameter k is known.

We start by deriving the maximum likelihood estimator of $\theta \coloneqq s$ based on N observations. Since the estimand θ is a deterministic quantity, we use Fisher estimation. In order to do this, let us restate the probability density function of $V_i \sim \Gamma(k, s)$:

$$f_{V_i}(v_i; k, s) = \frac{1}{\Gamma(k) s^k} v_i^{k-1} \mathrm{e}^{-\frac{v_i}{s}}\,, \quad i = 1, \ldots, N\,. \tag{2.1}$$

With this in mind, we can find that the likelihood $\mathcal{L}(v_1, \ldots, v_N; k, \theta)$ is given by

$$\begin{aligned} \mathcal{L}(v_1, \ldots, v_N; k, \theta) &= \prod_{i=1}^{N} f_{V_i}(v_i; k, \theta) && (2.2) \\ &= \prod_{i=1}^{N} \frac{1}{\Gamma(k) \theta^k} v_i^{k-1} \mathrm{e}^{-\frac{v_i}{\theta}}\,. && (2.3) \end{aligned}$$

In order to alleviate notation, we compute instead the log-likelihood, which is generally easier to work with[1]:

$$\ell(v_1,\dots,v_N;k,\theta) \coloneqq \ln \mathcal{L}(v_1,\dots,v_N;k,\theta) \tag{2.4}$$

$$= \ln\left(\prod_{i=1}^{N} \frac{1}{\Gamma(k)\theta^k} v_i^{k-1} \mathrm{e}^{-\frac{v_i}{\theta}}\right) \tag{2.5}$$

$$= \sum_{i=1}^{N} \ln\left(\frac{1}{\Gamma(k)\theta^k} v_i^{k-1} \mathrm{e}^{-\frac{v_i}{\theta}}\right) \tag{2.6}$$

$$= (k-1)\sum_{i=1}^{N} \ln v_i - \sum_{i=1}^{N} \frac{v_i}{\theta} - N\big(k\ln\theta + \ln\Gamma(k)\big)\,. \tag{2.7}$$

Now, in order to obtain the maximum likelihood estimate $\widehat{\theta}$, we must differentiate the log-likelihood with respect to the estimand θ, and set it equal to zero:

$$\left.\frac{\partial \ell(v_1,\dots,v_N;k,\theta)}{\partial\theta}\right|_{\theta=\widehat{\theta}} = -\frac{kN}{\widehat{\theta}} + \frac{\sum_{i=1}^{N} v_i}{\widehat{\theta}^2} = 0 \tag{2.8}$$

$$\iff \widehat{\theta} = \frac{\sum_{i=1}^{N} v_i}{kN} = \frac{\overline{v}}{k}\,. \tag{2.9}$$

This then allows us to find the maximum-likelihood estimator $\widehat{\Theta}$, given by

$$\widehat{\Theta} = \frac{\sum_{i=1}^{N} V_i}{kN} = \frac{\overline{V}}{k}\,. \tag{2.10}$$

3. **Properties of the estimator**

We now wish to show some of the properties of this estimator.

3.1. *Asymptotically unbiased.*

Definition 3.1 (Unbiased estimator). The Fisher estimator $\widehat{\Theta} = g(Z)$ of θ is *unbiased* if

$$m_{\widehat{\Theta};\theta} \coloneqq \mathbf{E}\left[g(Z);\theta\right] = \theta\,, \quad \text{for all } \theta\,, \tag{3.1}$$

where

$$\mathbf{E}\left[g(Z);\theta\right] \coloneqq \int_{\operatorname{dom} Z} g(Z) f_Z(z;\theta)\,\mathrm{d}z\,. \tag{3.2}$$

[1]This is possible because the values of θ which maximize the log-likelihood also maximize the likelihood.

PROPERTY 3.1. *The maximum likelihood estimator derived in* (2.10) *is asymptotically unbiased, that is,*

$$\lim_{N\to+\infty} \mathbf{E}\left[g(V_1,\ldots,V_n);\theta\right] = \theta\,. \tag{3.3}$$

Proof. We wish to prove that

$$\lim_{N\to+\infty} \mathbf{E}\left[\frac{\overline{V}}{k}\right] = \theta\,. \tag{3.4}$$

We recall that $\mathbf{E}\left[V_i\right] = k\theta$ for $V_i \sim \Gamma(k,\theta)$ and that the expected value operator is linear to obtain that

$$\mathbf{E}\left[\frac{\overline{V}}{k}\right] = \frac{\mathbf{E}\left[\frac{1}{N}\sum_{i=1}^{N} V_i\right]}{k} = \frac{\frac{1}{N}\sum_{i=1}^{N}\mathbf{E}\left[V_i\right]}{k} = \frac{\frac{1}{N}Nk\theta}{k} = \theta\,. \tag{3.5}$$

This proves that the maximum likelihood estimator of (2.10) is unbiased, hence it is also asymptotically unbiased. □

3.2. *Efficiency.*

THEOREM 3.2 (Cramér–Rao inequality). *If $Z = (Z_1,\ldots,Z_N)^T$ with i.i.d. random variables Z_k and if its probability density function given by $f_Z(z;\theta) = \prod_{k=1}^{N} f_{Z_k}(z_k;\theta)$ satisfies the following regularity condition:*

$$\mathbf{E}\left[\frac{\partial f_Z(z;\theta)}{\partial\theta}\right] = \int_{-\infty}^{+\infty} \frac{\partial f_Z(z;\theta)}{\partial\theta} f_Z(z;\theta)\,\mathrm{d}z\,, \quad \forall\theta\,, \tag{3.6}$$

then the covariance of any unbiased estimator $\widehat{\Theta}$ satisfies the Cramér–Rao inequality

$$\operatorname{cov}\widehat{\Theta} \geq \mathcal{I}^{-1}(\theta)\,, \tag{3.7}$$

where $\mathcal{I}(\theta)$ is the $N\times N$ Fisher information matrix, *defined by*

$$\left[\mathcal{I}(\theta)\right]_{i,j} \coloneqq -\mathbf{E}\left[\frac{\partial^2 \ln f_Z(z;\theta)}{\partial\theta_i\partial\theta_j}\right]\,. \tag{3.8}$$

Definition 3.2 (Efficient estimator). An estimator is said to be *efficient* if it reaches the Cramér–Rao bound for all values of θ, that is,

$$\operatorname{cov}\widehat{\Theta} = \mathcal{I}^{-1}(\theta)\,, \quad \forall\theta\,. \tag{3.9}$$

PROPERTY 3.3. *The maximum likelihood estimator derived in* (2.10) *is efficient.*

Proof. □

3.3. *Best asymptotically normal.*

PROPERTY 3.4. *The maximum likelihood estimator is best asymptotically normal.*

Proof. The proof is trivial and left as an exercise to the reader. □

3.4. *Consistent.*

PROPERTY 3.5. *The maximum likelihood estimator is consistent.*

Proof. The proof is trivial and left as an exercise to the reader. □

4. Joint maximum likelihood estimation

We now consider $V_i \sim \Gamma(k, s)$ (for $i = 1, \ldots, N$) with both k and s unknown.

5. Numerical simulation

6. Fisher information matrix

7. Numerical proof

References

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