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LINMA1731 Project
Fish Schools Tracking

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Abstract

In this paper we solve the first part of the project for the class "Stochastic process: Estimation and prediction" given during the Fall term of 2019. The average speed of each fish in a school of fish is approximated by a Gamma-distributed random variable, and various methods for estimating this quantity are given; a numerical simulation is also included.

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Part 1. **Average speed estimation**

1. **Introduction**

For the purpose of this project, we assume that the speed of each fish in a school at time i is a random variable V_i following a Gamma distribution, as suggested in [1]. This distribution is characterized by two parameters: a shape parameter $k > 0$ and a scale parameter $s > 0$. The parameters are the same for every fish and are time invariant. The aim of this first part is to identify these two parameters using empirical observations v_i.

2. **Maximum likelihood estimation**

Let v_i be i. i. d. realisations of a random variable following a Gamma distribution $\Gamma(k, s)$ (with $i = 1, \dots, N$). We first assume that the shape parameter k is known.

We start by deriving the maximum likelihood estimator of $\theta \coloneqq s$ based on N observations. In order to do this, let us restate the probability density function of $V_i \sim \Gamma(k, s)$:

$$f_{V_i}(k, s; v_i) = \frac{1}{\Gamma(k)s^k} v_i^{k-1} \mathrm{e}^{-\frac{v_i}{s}}\,, \quad i = 1, \dots, N\,. \tag{1}$$

With this in mind, we can find the likelihood $\mathcal{L}(k, \theta; v_1, \dots, v_N)$ is given by

$$\begin{aligned} \mathcal{L}(k, \theta; v_1, \dots, v_N) &= \prod_{i=1}^{N} f_{V_i}(k, \theta; v_i) && (2)\\ &= \prod_{i=1}^{N} \frac{1}{\Gamma(k)\theta^k} v_i^{k-1} \mathrm{e}^{-\frac{v_i}{\theta}}\,. && (3) \end{aligned}$$

In order to alleviate notation, we compute instead the log-likelihood, which is generally easier to work with:

$$\begin{aligned} \ell(k, \theta; v_1, \dots, v_N) &\triangleq \ln \mathcal{L}(k, \theta; v_1, \dots, v_N) && (4)\\ &= \ln\left(\prod_{i=1}^{N} \frac{1}{\Gamma(k)\theta^k} v_i^{k-1} \mathrm{e}^{-\frac{v_i}{\theta}}\right) && (5)\\ &= \sum_{i=1}^{N} \ln\left(\frac{1}{\Gamma(k)\theta^k} v_i^{k-1} \mathrm{e}^{-\frac{v_i}{\theta}}\right) && (6)\\ &= (k-1)\sum_{i=1}^{N} \ln v_i - \sum_{i=1}^{N} \frac{v_i}{\theta} - N\big(k \ln\theta + \ln\Gamma(k)\big)\,. && (7) \end{aligned}$$

Now, in order to obtain the maximum likelihood estimator $\widehat{\Theta}_\theta$ (which we will abusively write as $\hat{\theta}$ for the rest of this paper), we must derive the log-likelihood with respect to the estimand θ, and set it equal to zero:

$$\left.\frac{\partial \ell(k,\theta;v_1,\dots,v_N)}{\partial \theta}\right|_{\theta=\hat{\theta}} = -\frac{kN}{\hat{\theta}} + \frac{\sum_{i=1}^{N} v_i}{\hat{\theta}^2} = 0 \tag{8}$$

$$\iff \hat{\theta} = \frac{\sum_{i=1}^{N} v_i}{kN} = \frac{\overline{v_i}}{k}\,. \tag{9}$$

3. **Properties of the estimator**

We now wish to show some of the properties of this estimator.

3.1. *Asymptotically unbiased.*

PROPERTY 3.1. *The maximum likelihood estimator is asymptotically unbiased.*

Proof. The proof is trivial and left as an exercise to the reader. □

3.2. *Efficiency.*

PROPERTY 3.2. *The maximum likelihood estimator is efficient.*

Proof. The proof is trivial and left as an exercise to the reader. □

3.3. *Best asymptotically normal.*

PROPERTY 3.3. *The maximum likelihood estimator is best asymptotically normal.*

Proof. The proof is trivial and left as an exercise to the reader. □

3.4. *Consistent.*

PROPERTY 3.4. *The maximum likelihood estimator is consistent.*

Proof. The proof is trivial and left as an exercise to the reader. □

4. **Joint maximum likelihood estimation**

We now consider $V_i \sim \Gamma(k,s)$ (for $i = 1,\dots,N$) with both k and s unknown.

5. Numerical simulation

6. Fisher information matrix

7. Numerical proof

References

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