Annals of Mathematics **00** (XXXX), 1–11
http://dx.doi.org/10.4007/annals.XXXX.00.0.0000

LINMA1731 – Project 2019
Fish schools tracking

By Louis Navarre and Gilles Peiffer

Abstract

In this paper we solve the first part of the project for the class "Stochastic processes: Estimation and prediction" given during the Fall term of 2019. The average speed of each fish in a school of fish is approximated by a gamma-distributed random variable with a shape parameter k and a scale parameter s, and various methods for estimating this quantity are given; a numerical simulation is also included.

Contents

Part 1. Average speed estimation

1. Introduction

For the purpose of this project, we assume that the speed of each fish in a school at time i is a random variable V_i following a Gamma distribution, as suggested in [1]. This distribution is characterized by two parameters: a shape parameter $k > 0$ and a scale parameter $s > 0$. The parameters are the same for every fish and are time invariant. The aim of this first part is to identify these two parameters using empirical observations v_i.

2. Maximum likelihood estimation

Let v_i be i.i.d. realisations of a random variable following a Gamma distribution $\Gamma(k, s)$ (with $i = 1, \dots, N$). We first assume that the shape parameter k is known.

We start by deriving the maximum likelihood estimator of $\theta \coloneqq s$ based on N observations. Since the estimand θ is a deterministic quantity, we use Fisher estimation. In order to do this, let us restate the probability density function of $V_i \sim \Gamma(k, s)$:

$$f_{V_i}(v_i; k, s) = \frac{1}{\Gamma(k) s^k} v_i^{k-1} \mathrm{e}^{-\frac{v_i}{s}}\,, \quad i = 1, \dots, N\,. \tag{2.1}$$

With this in mind, we can find that the likelihood $\mathcal{L}(v_1, \dots, v_N; k, \theta)$ is given by

$$\mathcal{L}(v_1, \dots, v_N; k, \theta) = \prod_{i=1}^{N} f_{V_i}(v_i; k, \theta) = \prod_{i=1}^{N} \frac{1}{\Gamma(k)\theta^k} v_i^{k-1} \mathrm{e}^{-\frac{v_i}{\theta}}\,. \tag{2.2}$$

In order to alleviate notation, we compute instead the log-likelihood, which is generally easier to work with[1]:

$$\begin{aligned}
\ell(v_1, \dots, v_N; k, \theta) &\coloneqq \ln \mathcal{L}(v_1, \dots, v_N; k, \theta) && (2.3)\\
&= \sum_{i=1}^{N} \ln\left(\frac{1}{\Gamma(k)\theta^k} v_i^{k-1} \mathrm{e}^{-\frac{v_i}{\theta}}\right) && (2.4)\\
&= (k-1)\sum_{i=1}^{N} \ln v_i - \sum_{i=1}^{N} \frac{v_i}{\theta} - N\Big(k \ln\theta + \ln\Gamma(k)\Big)\,. && (2.5)
\end{aligned}$$

Now, in order to obtain the maximum likelihood estimate $\hat{\theta}$, we must differentiate the log-likelihood with respect to the estimand θ, and set it equal to

[1] This is possible because the values of θ which maximize the log-likelihood also maximize the likelihood.

zero:

$$\left.\frac{\partial \ell(v_1,\dots,v_N;k,\theta)}{\partial\theta}\right|_{\theta=\hat\theta} = -\frac{kN}{\hat\theta} + \frac{\sum_{i=1}^N v_i}{\hat\theta^2} = 0 \tag{2.6}$$

$$\iff \hat\theta = \frac{\sum_{i=1}^N v_i}{kN} = \frac{\overline{v}}{k}\,. \tag{2.7}$$

This then allows us to find the maximum likelihood estimator $\widehat{\Theta}$, given by

$$\widehat{\Theta} = \frac{\sum_{i=1}^N V_i}{kN} = \frac{\overline{V}}{k}\,. \tag{2.8}$$

3. **Properties of the estimator**

We now wish to show some of the properties of this estimator.

3.1. *Asymptotically unbiased.*

Definition 3.1 (Unbiased estimator). The Fisher estimator $\widehat{\Theta} = g(Z)$ of θ is *unbiased* if

$$m_{\widehat{\Theta};\theta} := \mathbb{E}\left[g(Z);\theta\right] = \theta\,, \quad \text{for all } \theta\,. \tag{3.1}$$

PROPERTY 3.1. *The maximum likelihood estimator derived in* (2.8) *is asymptotically unbiased, that is,*

$$\lim_{N\to+\infty} \mathbb{E}\left[g(V_1,\dots,V_N);\theta\right] = \theta\,. \tag{3.2}$$

Proof. We wish to prove that $\lim_{N\to+\infty}\mathbb{E}\left[\frac{\overline{V}}{k}\right] = \theta$. We recall that $\mathbb{E}\left[V_i\right] = k\theta$ for $V_i \sim \Gamma(k,\theta)$ and that the expected value operator is linear to obtain

$$\mathbb{E}\left[\frac{\overline{V}}{k}\right] = \frac{\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N V_i\right]}{k} = \frac{\frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[V_i\right]}{k} = \frac{\frac{1}{N}Nk\theta}{k} = \theta\,. \tag{3.3}$$

This proves that the maximum likelihood estimator of (2.8) is unbiased, hence it is also asymptotically unbiased. □

3.2. *Efficiency.*

THEOREM 3.2 (Cramr–Rao inequality). *If* $Z = (Z_1,\dots,Z_N)^T$ *with i.i.d. random variables* Z_k *and if its probability density function given by* $f_Z(z;\theta) = \prod_{k=1}^N f_{Z_k}(z_k;\theta)$ *satisfies certain regularity conditions, then the covariance of any unbiased estimator* $\widehat{\Theta}$ *satisfies the* Cramr–Rao inequality

$$\operatorname{cov}\widehat{\Theta} \succeq \mathcal{I}^{-1}(\theta)\,, \tag{3.4}$$

where $\mathcal{I}(\theta)$ *is the* $p\times p$ Fisher information matrix, *defined by*

$$\left[\mathcal{I}(\theta)\right]_{i,j} := -\mathbb{E}\left[\frac{\partial^2 \ln f_Z(z;\theta)}{\partial\theta_i\partial\theta_j}\right]\,. \tag{3.5}$$

Definition 3.2 (Efficient estimator). An unbiased estimator is said to be *efficient* if it reaches the Cramr–Rao bound for all values of θ, that is,

$$\operatorname{cov}\widehat{\Theta} = \mathcal{I}^{-1}(\theta)\,, \quad \forall\theta\,. \tag{3.6}$$

PROPERTY 3.3. *The maximum likelihood estimator derived in* (2.8) *is efficient.*

Proof. We use the fact that the random variables are independent to simplify the computations. Since θ is a scalar parameter, the Fisher information matrix is a scalar, equal to

$$\mathcal{I}(\theta) = -N\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\left((k-1)\ln v_1 - \frac{v_1}{\theta} - \big(k\ln\theta + \ln\Gamma(k)\big)\right)\right] \tag{3.7}$$

$$= N\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\Big(\frac{v_1}{\theta} + k\ln\theta\Big)\right] = \frac{kN}{\theta^2}\,. \tag{3.8}$$

We must also compute the variance of the ML estimator $\widehat{\Theta}$, which is given by

$$\mathbb{V}\left[\widehat{\Theta}\right] = \mathbb{V}\left[\frac{\overline{V}}{k}\right] = \frac{\theta^2}{kN}\,. \tag{3.9}$$

The Cramr–Rao lower bound is thus reached for all values of θ, which concludes the proof. □

3.3. *Best asymptotically normal.*

Definition 3.3 (Best asymptotically normal). A sequence $\{\widehat{\Theta}_N(Z)\}_{N\in\mathbb{N}}$ of consistent estimators of θ is called *best asymptotically normal* if

$$\sqrt{N}\left(\widehat{\Theta}_N(Z) - \theta\right) \xrightarrow[N\to+\infty]{\mathcal{D}} \mathcal{N}(0,\Sigma)\,, \tag{3.10}$$

for some minimal positive definite matrix Σ.

PROPERTY 3.4. *The maximum likelihood estimator of* (2.8) *is best asymptotically normal.*

Proof. In our case, we can show using the Cramr–Rao lower bound that Σ is minimal if it is equal to $\mathcal{I}^{-1}(\theta)$. To alleviate notations, we will write $\ell(\theta)$ instead of $\ell(v_1,\ldots,v_N;k,\theta)$. By definition, since $\hat\theta = \arg\max_\theta \ell(\theta)$, we know that $\ell'(\hat\theta) = 0$. Let θ_0 be the true value of the parameter θ. We can then use Taylor expansion on $\ell'(\hat\theta)$ around $\hat\theta = \theta_0$ to obtain

$$\ell'(\hat\theta) = \ell'(\theta_0) + \frac{\ell''(\theta_0)}{1!}(\hat\theta - \theta_0) + \mathcal{O}\left((\hat\theta-\theta_0)^2\right)\,. \tag{3.11}$$

We know the expression on the left is zero, hence

$$\ell'(\theta_0) = -\ell''(\theta_0)(\hat\theta - \theta_0) + \mathcal{O}\left((\hat\theta-\theta_0)^2\right)\,. \tag{3.12}$$

Rearranging and multiplying by $\sqrt{n}$, we get

$$\sqrt{n}(\hat{\theta} - \theta_0) = \frac{\ell'(\theta_0)/\sqrt{n}}{-\ell''(\theta_0)/n + \mathcal{O}\Big((\hat{\theta} - \theta_0)/n\Big)}\,. \tag{3.13}$$

Next, we need to show that $\ell'(\theta_0)/\sqrt{n} \sim \mathcal{N}\big(0, \mathcal{I}(\theta_0)\big)$. This is done using the Lindeberg–Lvy central limit theorem, in Appendix A.1. We know that $1/N\ell''(\theta_0) = \mathcal{I}(\theta_0)$. Finally, we can rewrite

$$\sqrt{N}(\hat{\theta} - \theta_0) \sim \frac{\mathcal{N}(0, \mathcal{I}(\theta_0))}{\mathcal{I}(\theta_0)} = \mathcal{N}(0, \mathcal{I}^{-1}(\theta_0))\,, \tag{3.14}$$

where we didn't take into account the remainder of the Taylor series, which goes to zero. This proves that the ML estimator is best asymptotically normal. □

3.4. *Consistent.*

Definition 3.4 (Consistent estimator). A sequence $\{\widehat{\Theta}_N(Z)\}_{N\in\mathbb{N}}$ of estimators of θ is called *consistent* if

$$\operatorname*{plim}_{N\to+\infty} \widehat{\Theta}_N(Z) = \theta\,. \tag{3.15}$$

Equivalently, we can show that the MSE of the estimator converges to zero as N goes to infinity.

PROPERTY 3.5. *The maximum likelihood estimator of* (2.8) *is consistent.*

Proof. We have shown that the estimator is unbiased, hence its MSE is equal to its variance. Since the estimator is efficient by Property 3.3, we know that its variance is equal to the Cramr–Rao lower bound, $\operatorname{cov}\widehat{\Theta} = \mathcal{I}^{-1}(\theta)$. We found this lower bound to be equal to $\theta^2/(kN)$ in (3.8). We have

$$\lim_{N\to+\infty} \operatorname{cov}\widehat{\Theta} = \lim_{N\to+\infty} \frac{\theta^2}{kN} = 0\,. \tag{3.16}$$

This proves that the variance (and hence the mean square error) of the estimator goes to zero as N goes to infinity, hence the estimator is consistent. □

4. Joint maximum likelihood estimation

We now consider $V_i \sim \Gamma(k, s)$ (for $i = 1, \ldots, N$) with both k and s unknown. Before, we assumed k was known, so we could maximize the log-likelihood function with respect to s. Now, we have to maximize this function with respect to s and k at the same time. We know the maximum likelihood estimator of s, $\hat{s} = f(k)$. Therefore, in the log-likelihood function, we can replace all the occurrences of s by the estimator we found, $\hat{s}$. We then get a function of k only, which we can differentiate and set its derivative to zero, solving to find the maximum likelihood estimator of k. We use the log-likelihood as given in (2.5). We abusively write $\ell(v)$ instead of $\ell(v_1, \ldots, v_N; \theta)$ Substituting in the estimator $\hat{s}$ instead of s, one finds

$$\ell(v) = (k-1)\sum_{i=1}^{N} \ln v_i - \sum_{i=1}^{N} \frac{kv_i}{\overline{v}} - Nk\ln\overline{v} + Nk\ln k - N\ln\Gamma(k)\big) . \tag{4.1}$$

Taking the derivative of this function with respect to k, we get

$$\left.\frac{\partial \ell(v)}{\partial k}\right|_{k=\hat{k}} = \sum_{i=1}^{N} \ln v_i - N - N\ln\overline{v} + N\ln\hat{k} + \frac{N\hat{k}}{\hat{k}} - N\frac{\Gamma(\hat{k})}{\Gamma(\hat{k})}\psi^{(0)}(\hat{k}) \tag{4.2}$$

$$= \sum_{i=1}^{N} \ln v_i - N\ln\sum_{i=1}^{N} v_i + N\ln\hat{k} + N\ln N - N\psi^{(0)}(\hat{k}) , \tag{4.3}$$

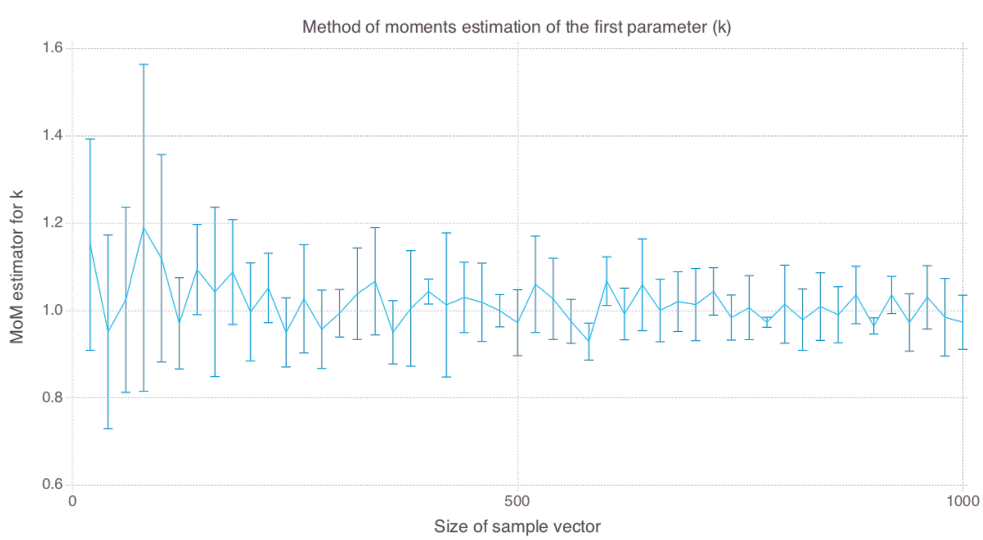
where $\psi^{(0)}$ is the digamma function, i.e. the logarithmic derivative of the gamma function. We now look for a root of this equation:

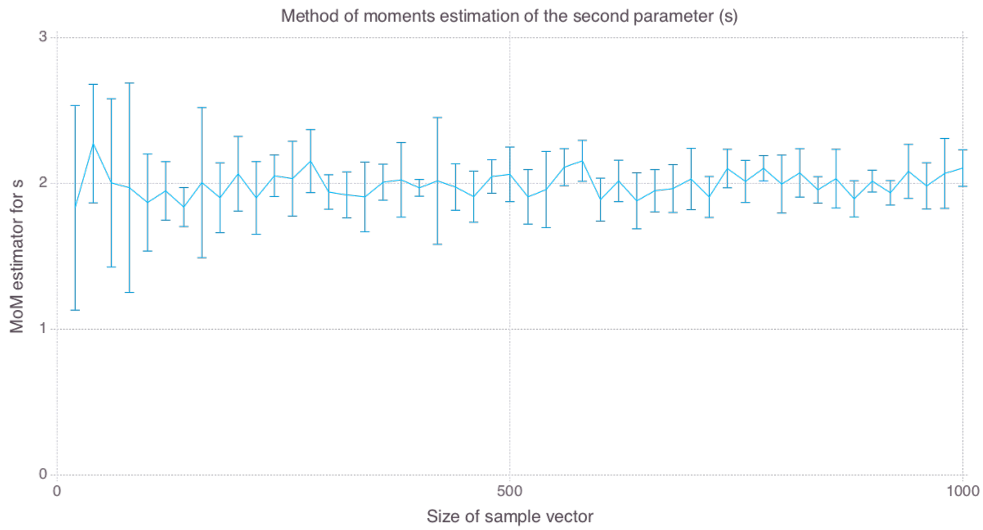
$$\ln\hat{k} - \psi^{(0)}(\hat{k}) = \ln\left(\sum_{i=1}^{N} v_i\right) - \ln N - \frac{\sum_{i=1}^{N}\ln v_i}{N} \tag{4.4}$$

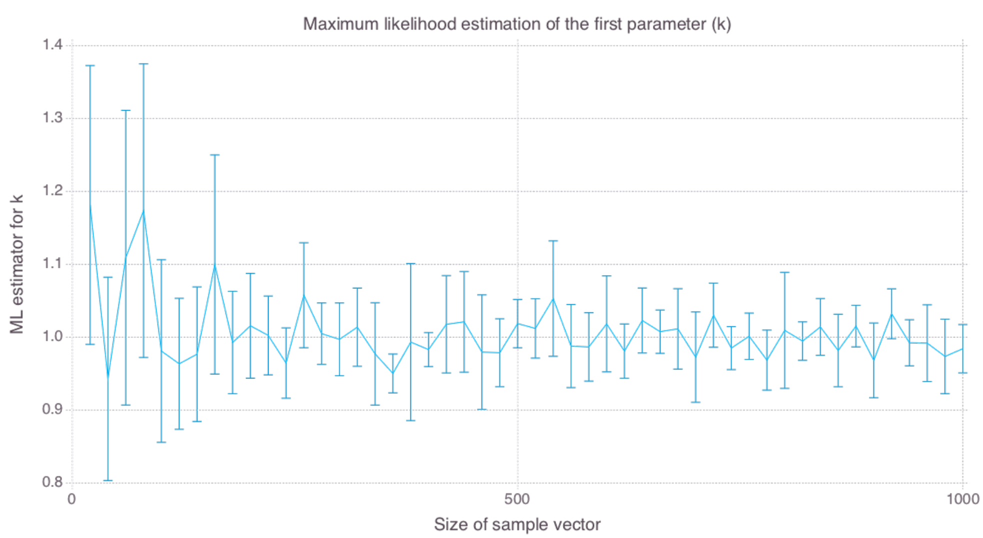
$$\iff \ln\hat{k} - \psi^{(0)}(\hat{k}) = \ln\left(\frac{\sum_{i=1}^{N} v_i}{N}\right) - \frac{\sum_{i=1}^{N}\ln v_i}{N} . \tag{4.5}$$

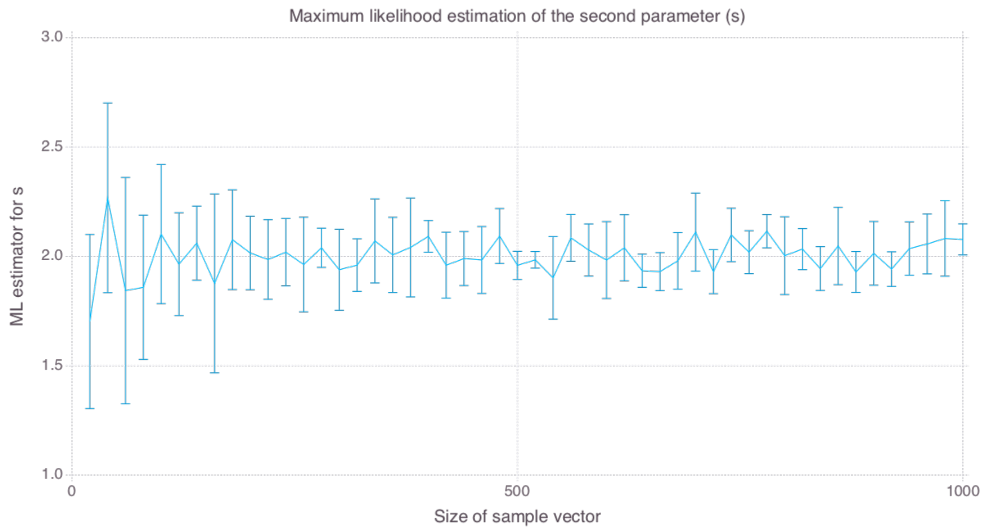
This equation has no closed-form solution for k, but can be approximated using numerical methods since the function is very well-behaved.

5. Numerical simulation




Method of moments estimation of the second parameter (s)
MoM estimator for s
0
1
2
3
0
500
1000
Size of sample vector


Maximum likelihood estimation of the first parameter (k)
ML estimator for k
0.8
0.9
1.0
1.1
1.2
1.3
1.4
0
500
1000
Size of sample vector



6. Fisher information matrix

We can compute the Fisher information matrix. The entry (i, j) of this matrix is given by equation 3.5. Since we have two estimators, this matrix is a 2×2 matrix. Remember that

$$\ell(x) = (k-1)\sum_{i=1}^{N} \ln v_i - \sum_{i=1}^{N} \frac{v_i}{s} - N\Big(k \ln s + \ln \Gamma(k)\Big) \tag{6.1}$$

Therefore, before calculating the entries of the information matrix, we have to compute the partial derivatives before taking the expectation of it. Two examples of derivation are given in equations 6.4 and 6.5. The procedure is similar for the other entries and the calculations can be found in Appendix **??**

$$\frac{\partial h(z;\theta)}{\partial s} = \frac{1}{s^2}\sum_{i=1}^{N} v_i - \frac{Nk}{s} \tag{6.2}$$

$$\frac{\partial h(z;\theta)}{\partial k} = \sum_{i=1}^{N} \ln v_i - N \ln s - N\psi_0(k) \tag{6.3}$$

$$\frac{\partial^2 h(z;\theta)}{\partial s^2} = -\frac{2}{s^3}\sum_{i=1}^{N} v_i + \frac{Nk}{s^2} \tag{6.4}$$

$$\frac{\partial^2 h(z;\theta)}{\partial k^2} = -N\psi_1(k). \tag{6.5}$$

$$\frac{\partial^2 h(z;\theta)}{\partial k \partial s} = -\frac{N}{s}. \tag{6.6}$$

Then, we take the expectation of these computed values.

$$\mathcal{I}_{00} = -\mathbb{E}\left\{\frac{\partial^2 h(z;\theta)}{\partial s^2}\right\} = -\mathbb{E}\left\{-\frac{2}{s^3}\sum_{i=1}^{N} v_i + \frac{Nk}{s^2}\right\} \tag{6.7}$$

$$= \frac{2}{s^3}Nsk - \frac{NK}{s^2} = \frac{NK}{s^2}. \tag{6.8}$$

$$\mathcal{I}_{01} = \mathcal{I}_{10} = -\mathbb{E}\left\{\frac{\partial^2 h(z;\theta)}{\partial k \partial s}\right\} = \frac{N}{s}. \tag{6.9}$$

$$\mathcal{I}_{11} = -\mathbb{E}\left\{\frac{\partial^2 h(z;\theta)}{\partial k^2}\right\} = -N\psi_1(k). \tag{6.10}$$

Finally, the Fisher information matrix is

$$\mathcal{I}(\theta) = N\begin{pmatrix} \frac{K}{s^2} & \frac{1}{s} \\ \frac{1}{s} & \psi_1(k) \end{pmatrix}$$

Then we have

$$\mathcal{I}^{-1}(\theta) = \frac{1}{N}\begin{pmatrix} \psi_1(k) & -\frac{1}{s} \\ -\frac{1}{s} & \frac{K}{s^2} \end{pmatrix} \cdot \frac{s^2}{K\psi_1(k) - 1}$$

7. Numerical proof

Figure FIGURE shows the matrix norm induced by the vector 2-norm. We see that, the more N increase, the more we get close to 2 (which is the norm for a matrix 2x2 composed of 1). Figure FIGURE shows also the empirical covariance matrices, the inverse of the Fisher information matrix, and the matrix composed of the the ratio between the coefficients of the first two for a value of $N = 3000$. Wee directly see that these coefficients are very close to one, showing that the covariance matrix tends to the Cramer-Rao lower bound, resulting in the proof that our estimators are efficient.

Appendix A. Omitted proofs

A.1. *Proof of normality using the central limit theorem.*

Proof. We want to prove that $\ell'(\theta_0)/\sqrt{N} \sim \mathcal{N}\big(0, \mathcal{I}(\theta_0)\big)$. We simplify notation by writing $f(v_i)$ instead of $f_{V_i}(v_i; k, \theta)$ and using Euler's notation for derivatives. First, we show that the expected value of $\ell'(\theta_0)/\sqrt{N}$ is zero.

$$\mathbb{E}\left[\frac{\ell'(\theta_0)}{\sqrt{N}}\right] = \int_{-\infty}^{+\infty} \partial_{\theta_0}\left(\sum_{i=1}^{N} \frac{\ln f(v_i)}{\sqrt{N}}\right) f(v_i)\,\mathrm{d}v_i \tag{A.1}$$

$$= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \int_{-\infty}^{+\infty} \frac{\partial_\theta f(v_i)}{f(v_i)} f(v_i)\,\mathrm{d}v_i \tag{A.2}$$

$$= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \int_{-\infty}^{+\infty} \frac{\partial_\theta f(v_i)}{f(v_i)} f(v_i)\,\mathrm{d}v_i \tag{A.3}$$

$$= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \int_{-\infty}^{+\infty} \partial_\theta f(v_i)\,\mathrm{d}v_i \tag{A.4}$$

$$= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \partial_\theta \int_{-\infty}^{+\infty} f(v_i)\,\mathrm{d}v_i \tag{A.5}$$

$$= 0\,. \tag{A.6}$$

Next, we compute the variance of $\ell'(\theta_0)/\sqrt{N}$. First, we compute (for $i = 1, \ldots, N$)

$$\mathbb{E}\left[\left(\partial_{\theta_0} \ln f(v_i)\right)^2\right] = \int_{-\infty}^{+\infty} \partial_{\theta_0} \ln f(v_i) \frac{\partial_{\theta_0} f(v_i)}{f(v_i)} f(v_i)\,\mathrm{d}v_i \tag{A.7}$$

$$= \int_{-\infty}^{+\infty} \partial_{\theta_0} \ln f(v_i) \partial_{\theta_0} f(v_i)\,\mathrm{d}v_i\,. \tag{A.8}$$

Using the product rule, we can then find

$$\begin{aligned} = &-\int_{-\infty}^{+\infty} \partial_{\theta_0\theta_0} \ln f(v_i) f(v_i)\,\mathrm{d}v_i \\ &+ \int_{-\infty}^{+\infty} \partial_{\theta_0}\left(\partial_{\theta_0} \ln f(v_i) f(v_i)\right)\mathrm{d}v_i\,. \end{aligned} \tag{A.9}$$

$$= -\mathbb{E}\left[\partial_{\theta_0\theta_0} \ln f(v_i)\right] + \partial_{\theta_0} \int_{-\infty}^{\infty} \frac{\partial_{\theta_0} f(v_i)}{f(v_i)} f(v_i)\,\mathrm{d}v_i \tag{A.10}$$

$$= \mathcal{I}(\theta)\,, \tag{A.11}$$

where the last expression can be shown to be zero by a similar argument as the one used above for the expected value. Knowing this, one easily finds

$$\mathbb{V}\left[\frac{\ell'(\theta_0)}{\sqrt{N}}\right] = \frac{1}{N}\mathbb{V}\left[\sum_{i=1}^{N} \partial_{\theta_0} \ln f(v_i)\right] = \mathcal{I}(\theta_0)\,, \tag{A.12}$$

since the random variables are i.i.d.. Using the Lindeberg–Lvy CLT, we thus have

$$\frac{\ell'(\theta_0)}{\sqrt{N}} \sim \mathcal{N}(0, \mathcal{I}(\theta_0))\,. \tag{A.13}$$

□

References

Universit catholique de Louvain, Ottignies-Louvain-la-Neuve, Belgium
E-mail: navarre.louis@student.uclouvain.be

Universit catholique de Louvain, Ottignies-Louvain-la-Neuve, Belgium
E-mail: gilles.peiffer@student.uclouvain.be