Exercise A: Boundedness of trajectories and Lyapunov equation

A1

A2

We assume that $A = J_n(\lambda)$. Hence we can write :

$$x(t) = e^{At}x(0) = e^{J_n(\lambda)t}x(0)$$

We can use the result of question B4 from Homework 1 which develops $e^{J_n(\lambda)}$:

$$\begin{aligned}
x(t) &= e^{J_n(\lambda)t}x(0) \\
&= e^{\lambda t}\left(I + \sum_{k=1}^{n-1}\frac{1}{k!}\big(J_n(0)\big)^k\right)^t x(0) \\
&= e^{\lambda t}e^{J_n(0)t}x(0) \\
&= e^{\lambda t}\left(I + \sum_{k=1}^{n-1}\frac{1}{k!}\big(J_n(0)t\big)^k\right)x(0) \\
&= e^{\lambda t}\left(I + \sum_{k=1}^{n-1}\frac{1}{k!}J_n^k(0)t^k\right)x(0)
\end{aligned}$$

The third and fourth equalities come from the definition of the matrix exponential. We can now develop the terms in parentheses :

$$I + \sum_{k=1}^{n-1}\frac{1}{k!}J_n^k(0)t^k = I + \frac{t}{1!}\begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} 0 & 0 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & & 0 & 0 & 1 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}$$

$$+ \cdots + \frac{t^k}{k!}\begin{pmatrix} \overbrace{0 \; \dots \; 0}^{k \text{ zeros}} & 1 & & \\ \ddots & & \ddots & \ddots \\ & \ddots & & \ddots & 1 \\ & & \ddots & & 0 \\ & & & \ddots & \vdots \\ & & & & 0 \end{pmatrix} + \cdots + \frac{t^{n-1}}{(n-1)!}\begin{pmatrix} \overbrace{0 \; \dots \; 0}^{n-1 \text{ zeros}} & 1 \\ \ddots & 0 \\ \ddots & \vdots \\ & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t/1! & \dots & t/k! & \dots & t^{n-1}/(n-1)! \\ & \ddots & \ddots & & \ddots & \vdots \\ & & 1 & t/1! & & t/k! \\ & & & \ddots & \ddots & \vdots \\ & & & & 1 & t/1! \\ & & & & & 1 \end{pmatrix}$$

We deduce from the previous expression that for $i \in [n]$:

$$x_i(t) = e^{\lambda t}\sum_{j=i}^{n}\frac{1}{(j-i)!}t^{j-i}x_i(0)$$

A3

The minimal polynomial of a matrix $A \in \mathbb{C}^{n\times n}$ is the polynomial

$$m(\lambda) = \prod_i (\lambda - \lambda_i)^{k_i^*},$$

where

$$f(J) = \mathrm{diag}\left\{f\left(J_{k_{i_j}}(\lambda_{i_j})\right)\right\}, \quad k_i^* = \max_{1\leqslant j\leqslant n_i} k_{i_j},$$

and n_i is the number of Jordan blocks with eigenvalue λ_i.

Simple eigenvalues thus have the property that the size of the largest Jordan block with that eigenvalue λ_i is 1 (i.e. $k_i^* = 1$).

A4

A5

Proof. One can show that the matrix P must exist by proving that for $P = I_n$, the statement holds. Indeed, I_n is positive definite and Hermitian. Since D and D^* are diagonal, their sum is also diagonal. In fact,

$$D + D^* = \mathrm{diag}(\lambda_1 + \lambda_1^*, \dots, \lambda_n + \lambda_n^*),$$

where λ_i and λ_i^* are the eigenvalues of D and D^*, respectively. We know that the imaginary parts of these eigenvalues cancel each other out, and that the real parts are the same (and are nonpositive), by virtue of the definition of the conjugate transpose. We thus get

$$D^*P + PD = D^*I_n + I_nD = D^* + D = \mathrm{diag}(\lambda_1 + \lambda_1^*, \dots, \lambda_n + \lambda_n^*) \preceq 0,$$

which proves the existence of such a positive definite Hermitian matrix P. □

A6

A7

A8

Exercise B: Implementation

B1

B2