

## Exercise A: The Kronecker product

### A1

The Kronecker product of two matrices  $A \in \mathbb{F}^{m \times n}$  and  $B \in \mathbb{F}^{p \times q}$  is the matrix of size  $mp \times nq$  whose elements are all possible products between the elements of  $A$  and  $B$  arranged in the following way:

$$A \otimes B := \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix},$$

$$\text{with } A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & \cdots & b_{1q} \\ \vdots & \ddots & \vdots \\ b_{p1} & \cdots & b_{pq} \end{bmatrix}.$$

### A2

The Kronecker product is associative. Let  $C \in \mathbb{F}^{s \times t}$  be a third matrix. We show that  $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ , with  $C = \begin{bmatrix} c_{11} & \cdots & c_{1t} \\ \vdots & \ddots & \vdots \\ c_{s1} & \cdots & c_{st} \end{bmatrix}$ .

*Proof.*

$$\begin{aligned} (A \otimes B) \otimes C &= \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \otimes C \\ &= \begin{bmatrix} a_{11}b_{11}C & \cdots & a_{11}b_{1q}C & \cdots & a_{1n}b_{11}C & \cdots & a_{1n}b_{1q}C \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{11}b_{p1}C & \cdots & a_{11}b_{pq}C & \cdots & a_{1n}b_{p1}C & \cdots & a_{1n}b_{pq}C \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1}b_{11}C & \cdots & a_{m1}b_{1q}C & \cdots & a_{mn}b_{11}C & \cdots & a_{mn}b_{1q}C \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1}b_{p1}C & \cdots & a_{m1}b_{pq}C & \cdots & a_{mn}b_{p1}C & \cdots & a_{mn}b_{pq}C \end{bmatrix} \\ &= A \otimes (B \otimes C). \end{aligned}$$

□

The Kronecker product is non-commutative; we show that  $A \otimes B \neq B \otimes A$

*Proof.* We show a counterexample to the claim of commutativity. Let

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}.$$

In that case, we have

$$A \otimes B = \begin{bmatrix} 0 & -2 & 0 & -3 \\ -2 & 2 & -3 & 3 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad B \otimes A = \begin{bmatrix} 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & -1 \\ -2 & -3 & 2 & 3 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

We see that  $A \otimes B \neq B \otimes A$ .

□

Finally, the set  $\mathbb{F}^{n \times n}$  equipped with the Kronecker product is not a group as it does not satisfy the closure property. Indeed, one can easily see that the Kronecker product of two matrices in  $\mathbb{F}^{n \times n}$  is an element of  $\mathbb{F}^{n^2 \times n^2}$ .

**A3**

Let  $A \in \mathbb{F}^{m \times n}$ ,  $B \in \mathbb{F}^{p \times q}$ ,  $C \in \mathbb{F}^{n \times r}$ , and  $D \in \mathbb{F}^{q \times s}$ ,

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

*Proof.* We simply verify that

$$\begin{aligned} (A \otimes B)(C \otimes D) &= \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \begin{bmatrix} c_{11}D & \cdots & c_{1r}D \\ \vdots & \ddots & \vdots \\ c_{n1}D & \cdots & c_{nr}D \end{bmatrix} \\ &= \begin{bmatrix} \sum_{k=1}^n a_{1k}c_{k1}BD & \cdots & \sum_{k=1}^n a_{1k}c_{kr}BD \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^n a_{mk}c_{k1}BD & \cdots & \sum_{k=1}^n a_{mk}c_{kr}BD \end{bmatrix} \\ &= AC \otimes BD. \end{aligned}$$

□

This allows us to say that (if  $A \in \mathbb{F}^{n \times n}$  and  $B \in \mathbb{F}^{m \times m}$  are nonsingular)

$$(A \otimes B)(A^{-1} \otimes B^{-1}) = AA^{-1} \otimes BB^{-1} = I_n \otimes I_m = I_{nm},$$

and hence that

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$$

**A4**

We first show the first property, P1:

$$A^{\otimes k} B^{\otimes k} = (AB)^{\otimes k}.$$

*Proof.* By induction. The base case is trivial:

$$A^{\otimes 1} B^{\otimes 1} = AB = (AB)^{\otimes 1}.$$

Next, we assume the property holds for  $k$ , and we prove it for  $k+1$ :

$$\begin{aligned} A^{\otimes k+1} B^{\otimes k+1} &= (A^{\otimes k} \otimes A)(B^{\otimes k} \otimes B) \\ &\stackrel{\text{A3}}{=} (A^{\otimes k} B^{\otimes k}) \otimes AB \\ &= (AB)^{\otimes k} \otimes AB \\ &= (AB)^{\otimes k+1}. \end{aligned}$$

□

Next, we show the second property, P2:

$$(A^{\otimes k})^T = (A^T)^{\otimes k}.$$

*Proof.* We start by proving an auxiliary lemma, L1.

$$(A \otimes B)^T = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}^T = \begin{bmatrix} a_{11}B^T & \cdots & a_{m1}B^T \\ \vdots & \ddots & \vdots \\ a_{1n}B^T & \cdots & a_{mn}B^T \end{bmatrix} = A^T \otimes B^T.$$

We then proceed by induction. The base case is trivial as before:

$$(A^{\otimes 1})^T = A^T = (A^T)^{\otimes 1}.$$

Next, we assume the property holds for  $k$ , and we prove it for  $k+1$ :

$$\begin{aligned} (A^{\otimes k+1})^T &= (A^{\otimes k} \otimes A)^T \\ &\stackrel{\text{L1}}{=} (A^{\otimes k})^T \otimes A^T \\ &= (A^T)^{\otimes k} \otimes A^T \\ &= (A^T)^{\otimes k+1}. \end{aligned}$$

□

Finally, we show the following: for any vector  $v \in \mathbb{R}^n$ ,

$$\|v^{\otimes k}\| = \|v\|^k,$$

where  $\|v\| = \sqrt{v^\top v}$ .

*Proof.*

$$\begin{aligned} \|v^{\otimes k}\| &= \sqrt{(v^{\otimes k})^\top v^{\otimes k}} \\ &\stackrel{\text{P2}}{=} \sqrt{(v^\top)^{\otimes k} v^{\otimes k}} \\ &\stackrel{\text{P1}}{=} \sqrt{(v^\top v)^{\otimes k}} \\ &= \sqrt{(v^\top v)^k} \\ &= \left(\sqrt{v^\top v}\right)^k \\ &= \|v\|^k, \end{aligned}$$

where the fourth equality follows from a simplification of the Kronecker product for scalars, and the fifth equality is a property of the square root.  $\square$

## A5

The determinant of a square matrix  $A \in \mathbb{F}^{n \times n}$  is given by

$$\det(A) = \sum_{\mathbf{j}} (-1)^{t(\mathbf{j})} a_{1j_1} \cdot a_{2j_2} \cdots a_{nj_n},$$

where the index vector  $\mathbf{j}$  constitutes a permutation of  $\{1, 2, \dots, n\}$ , and  $t(\mathbf{j})$  denotes the parity of each quasi-diagonal.

First, we show that  $\det(A \otimes I_m) = \det(A)^m$ .

*Proof.* Laplace's theorem states that for a matrix  $B \in \mathbb{F}^{n \times n}$  and a  $p$ -tuple of rows  $\mathbf{i}_p$ , we have:

$$\det(B) = \sum_{\mathbf{j}_p} B \begin{pmatrix} \mathbf{i}_p \\ \mathbf{j}_p \end{pmatrix} B^c \begin{pmatrix} \mathbf{i}_p \\ \mathbf{j}_p \end{pmatrix}.$$

We apply this theorem to the matrix  $B = A \otimes I_m$  with the  $n$ -tuple  $\mathbf{i}_n = (1, m+1, 2m+1, \dots, (n-1)m+1)$  (if  $n = 1$  then the tuple is just  $(1)$ ). For every  $n$ -tuple  $\mathbf{j}_n$  that contains another index than those present in  $\mathbf{i}_n$ , the minor  $B \begin{pmatrix} \mathbf{i}_n \\ \mathbf{j}_n \end{pmatrix}$  is zero. Indeed, if we consider a new matrix  $B'$  only containing the rows whose indices are in  $\mathbf{i}_n$ , we have:

$$B' = \begin{bmatrix} a_{11} & 0 & \cdots & 0 & a_{12} & 0 & \cdots & 0 & \cdots & a_{1n} & 0 & \cdots & 0 \\ a_{21} & 0 & \cdots & 0 & a_{22} & 0 & \cdots & 0 & \cdots & a_{2n} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & 0 & \cdots & 0 & a_{n2} & 0 & \cdots & 0 & \cdots & a_{nn} & 0 & \cdots & 0 \end{bmatrix}.$$

$\underbrace{\hspace{10em}}_{m \text{ columns}}$

Thus if  $\mathbf{j}_n$  contains any index not present in  $\mathbf{i}_n$ , a column of zeros is included leading to a zero minor. We are left with

$$\det(A \otimes I_m) = \det(B) = B \begin{pmatrix} \mathbf{i}_n \\ \mathbf{i}_n \end{pmatrix} B^c \begin{pmatrix} \mathbf{i}_n \\ \mathbf{i}_n \end{pmatrix} \quad (1)$$

$$= \det(A) \det(A \otimes I_{m-1}). \quad (2)$$

The first term of (2) can be easily found by inspecting  $B'$  and taking only columns whose indices are in  $\mathbf{i}_n$ . The second term is derived by removing the rows and columns of  $A \otimes I_m$  whose indices are in  $\mathbf{i}_n$ , which gives the following matrix:

$$\begin{bmatrix} a_{11}I_{m-1} & \cdots & a_{1n}I_{m-1} \\ \vdots & \ddots & \vdots \\ a_{n1}I_{m-1} & \cdots & a_{nn}I_{m-1} \end{bmatrix}.$$

From (2), we then conclude that  $\det(A \otimes I_m) = \det(A)^m$ .  $\square$

Similarly, we can also prove that  $\det(I_m \otimes A) = \det(A)^m$ .

*Proof.* Indeed, if we take  $B = I_m \otimes A$  and  $\mathbf{i}_n = (1, 2, \dots, n)$ , then we have :

$$B' = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & \cdots & 0 \end{bmatrix}.$$

(m-1)n columns

With the same argument as before, if  $\mathbf{j}_n$  contains any index not present in  $\mathbf{i}_n$ , a column of zeros is included leading to a zero minor.

We are left with

$$\begin{aligned} \det(I_m \otimes A) &= \det(B) = B \begin{pmatrix} \mathbf{i}_n \\ \mathbf{i}_n \end{pmatrix} B^c \begin{pmatrix} \mathbf{i}_n \\ \mathbf{i}_n \end{pmatrix} \\ &= \det(A) \det(I_{m-1} \otimes A), \end{aligned}$$

which leads us to  $\det(I_m \otimes A) = \det(A)^m$ . □

From this, we can deduce that for  $A \in \mathbb{F}^{n \times n}$  and  $B \in \mathbb{F}^{m \times m}$ ,  $\det(A \otimes B) = \det(A)^m \det(B)^n$ .

*Proof.* We can write

$$\begin{aligned} A \otimes B &= (AI_n) \otimes (I_m B) \\ &\stackrel{A3}{=} (A \otimes I_m)(I_n \otimes B). \end{aligned}$$

Taking the determinant on both sides, and using the fact that  $\det(AB) = \det(A)\det(B)$  (Exercise 1.18 in the lecture notes), we then get

$$\begin{aligned} \det(A \otimes B) &= \det(A \otimes I_m) \det(I_n \otimes B) \\ &= \det(A)^m \det(B)^n. \end{aligned} \quad \square$$

## A6



The rank of a matrix  $A \in \mathbb{F}^{m \times n}$  is equal to the largest size of its nonzero minors. From this, we prove the following property:  $\text{rank}(A \otimes B) = \text{rank}(A) \text{rank}(B) = \text{rank}(B \otimes A)$ .

*Proof.* Let  $A \in \mathbb{F}^{m \times n}$ ,  $B \in \mathbb{F}^{p \times q}$ .

First, we note that  $B \otimes A$  can be obtained by permuting rows and columns of  $A \otimes B$ . As elementary operations do not affect the rank of a matrix, we deduce that  $\text{rank}(A \otimes B) = \text{rank}(B \otimes A)$ .

Let  $R_1$  and  $Q_1$  be products of elementary transformations such that

$$R_1 B Q_1 = \begin{bmatrix} I_r & 0_{r \times (q-r)} \\ 0_{(p-r) \times r} & 0_{(p-r) \times (q-r)} \end{bmatrix}.$$

By Theorem 1.8 of the lecture notes, we know such matrices exist. The scalar  $r$  is the rank of  $B$ .

Next, we multiply on both sides the matrix  $A \otimes B$  by matrices with  $R_1$  and  $Q_1$  on the diagonal:

$$\begin{aligned} &\begin{bmatrix} R_1 & & \\ & \ddots & \\ & & R_1 \end{bmatrix} \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \begin{bmatrix} Q_1 & & \\ & \ddots & \\ & & Q_1 \end{bmatrix} \\ &= \begin{bmatrix} a_{11}R_1BQ_1 & \cdots & a_{1n}R_1BQ_1 \\ \vdots & & \vdots \\ a_{m1}R_1BQ_1 & \cdots & a_{mn}R_1BQ_1 \end{bmatrix} \\ &= \begin{bmatrix} a_{11}I_r & 0_{r \times (q-r)} & \cdots & a_{1n}I_r & 0_{r \times (q-r)} \\ 0_{(p-r) \times r} & 0_{(p-r) \times (q-r)} & \cdots & 0_{(p-r) \times r} & 0_{(p-r) \times (q-r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1}I_r & 0_{r \times (q-r)} & \cdots & a_{mn}I_r & 0_{r \times (q-r)} \\ 0_{(p-r) \times r} & 0_{(p-r) \times (q-r)} & \cdots & 0_{(p-r) \times r} & 0_{(p-r) \times (q-r)} \end{bmatrix}. \end{aligned}$$

By manipulating the last matrix with permutation matrices, the following matrix is obtained:

$$\left[ \begin{array}{ccc|c} A & & & \\ & \ddots & & \\ & & A & \\ \hline & & 0_{m(p-r) \times nr} & 0_{m(p-r) \times n(q-r)} \end{array} \right]$$

where the matrix  $A$  appears  $r$  times on the diagonal and the rest of the matrix is only zeros. Its rank is still equal to the rank of  $A \otimes B$  as only elementary operations have been applied.

Let  $R_2$  and  $Q_2$  be products of elementary transformations such that:

$$R_2 A Q_2 = \begin{bmatrix} I_s & 0_{s \times (n-s)} \\ 0_{(m-s) \times s} & 0_{(m-s) \times (n-s)} \end{bmatrix}.$$

With  $s$  the rank of  $A$ .

We multiply on both sides the matrix obtained previously by matrices with  $R_2$  and  $Q_2$  on the diagonal :

$$\begin{aligned} & \left[ \begin{array}{ccc|c} R_2 & & & \\ & \ddots & & \\ & & R_2 & \\ \hline & & 0_{m(p-r) \times mr} & 0_{m(p-r) \times m(p-r)} \end{array} \right] \left[ \begin{array}{ccc|c} A & & & \\ & \ddots & & \\ & & A & \\ \hline & & 0_{m(p-r) \times nr} & 0_{m(p-r) \times n(q-r)} \end{array} \right] \\ & \qquad \qquad \qquad \left[ \begin{array}{ccc|c} Q_2 & & & \\ & \ddots & & \\ & & Q_2 & \\ \hline & & 0_{n(q-r) \times nr} & 0_{n(q-r) \times n(q-r)} \end{array} \right] \\ & = \left[ \begin{array}{ccc|c} R_2 A Q_2 & & & \\ & \ddots & & \\ & & R_2 A Q_2 & \\ \hline & & 0_{m(p-r) \times nr} & 0_{m(p-r) \times n(q-r)} \end{array} \right] \\ & = \left[ \begin{array}{ccc|c} I_s & 0_{s \times (n-s)} & & \\ 0_{(m-s) \times s} & 0_{(m-s) \times (n-s)} & & \\ & & \ddots & \\ & & & I_s & 0_{s \times (n-s)} \\ & & & 0_{(m-s) \times s} & 0_{(m-s) \times (n-s)} \\ \hline & & & 0_{m(p-r) \times nr} & 0_{m(p-r) \times n(q-r)} \end{array} \right] \end{aligned}$$

As the identity matrix  $I_s$  appears  $r$  times on the diagonal, we deduce that

$$\text{rank}(A \otimes B) = sr = \text{rank}(A) \text{rank}(B).$$



□

## A7



We show that :  $\text{vec}(AXB) = (B^\top \otimes A) \text{vec}(X)$ .

*Proof.* Let  $A \in \mathbb{F}^{m \times n}$ ,  $B \in \mathbb{F}^{p \times q}$ , and  $X \in \mathbb{F}^{n \times p}$ .

We develop the right-hand side of the equality we want to prove:

$$\begin{aligned} (B^\top \otimes A) \text{vec}(X) &= \begin{bmatrix} b_{11}A & b_{21}A & \cdots & b_{p1}A \\ b_{12}A & b_{22}A & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ b_{1q}A & \cdots & \cdots & b_{pq}A \end{bmatrix} \begin{bmatrix} X_{:,1} \\ X_{:,2} \\ \vdots \\ X_{:,p} \end{bmatrix} \\ &= \begin{bmatrix} b_{11}AX_{:,1} + b_{21}AX_{:,2} + \cdots + b_{p1}AX_{:,p} \\ b_{12}AX_{:,1} + \cdots + b_{p2}AX_{:,p} \\ \vdots \\ b_{1q}AX_{:,1} + \cdots + b_{pq}AX_{:,p} \end{bmatrix}. \end{aligned}$$

We recognize the elements of a product  $D$  of three matrices in vectorized form:

$$d_i = \sum_{r=1}^p b_{ri} \sum_{k=1}^n a_{rk} x_{kr},$$

which shows  $D$  is simply  $AXB$ , and hence proves  $\text{vec}(AXB) = (B^\top \otimes A) \text{vec}(X)$ .  $\square$

The proven equality can be used to solve the Sylvester equation:  $AX + XA^\top = B$  where  $X$  is the unknown. Indeed, we can vectorize both sides of the equation:

$$\text{vec}(AX) + \text{vec}(XA^\top) = \text{vec}(B).$$

Then, we use some identity matrices to be able to apply the proven relation:

$$\begin{aligned} \text{vec}(B) &= \text{vec}(AX) + \text{vec}(XA^\top) = \text{vec}(AXI) + \text{vec}(IXA^\top) \\ &= (I \otimes A) \text{vec}(X) + (A \otimes I) \text{vec}(X) \\ &= (I \otimes A + A \otimes I) \text{vec}(X). \end{aligned}$$

The term  $\text{vec}(X)$  can then be isolated:

$$\text{vec}(X) = (I \otimes A + A \otimes I)^{-1} \text{vec}(B).$$

Finally, the matrix  $X$  can be simply reconstructed from  $\text{vec}(X)$ .

## 1 Exercise B: The matrix exponential

### B1

If  $\lambda \in \mathbb{C}$  is an eigenvalue of  $A$  then  $e^\lambda$  is an eigenvalue of  $e^A$ .

*Proof.* We know  $\lambda$  is an eigenvalue of  $A$ . Hence  $Av = \lambda v$  for some eigenvector  $v$ .

$$\begin{aligned} e^A v &= \left( I + \sum_{k=1}^{\infty} \frac{1}{k!} A^k \right) v \\ &= v + \sum_{k=1}^{\infty} \frac{1}{k!} A^k v \\ &= v + \sum_{k=1}^{\infty} \frac{1}{k!} \lambda^k v \\ &= \left( \sum_{k=0}^{\infty} \frac{1}{k!} \lambda^k \right) v \\ &= e^\lambda v. \end{aligned}$$

The third line is derived using the equality  $A^k v = \lambda^k v$  and the last line using the Taylor series of the exponential function.

This proves that  $e^\lambda$  is an eigenvalue of  $e^A$ , with eigenvector  $v$ .  $\square$

### B2

For any matrix  $A \in \mathbb{C}^{n \times n}$ ,  $\text{rank}(e^A) = n$ .

*Proof.* The matrix  $e^A$  is invertible and its inverse is  $e^{-A}$ . In fact,

$$e^A e^{-A} = e^{-A} e^A = e^{A-A} = I.$$

The second equality is valid since  $A$  and  $-A$  commute. Since  $e^A$  is non-singular, its determinant is nonzero. Theorem 1.9 of the course notes states the rank of  $e^A$  is equal to the largest size of its nonzero minors which in this case is the whole matrix. We deduce  $\text{rank}(e^A) = n$ .  $\square$

### B3

If  $A \in \mathbb{C}^{n \times n}$  is skew-Hermitian, i.e.  $A = -A^*$ , then  $e^A$  is unitary, i.e.  $e^A(e^A)^* = I$ .

*Proof.* We have:

$$\begin{aligned}
 (e^A)^* &= \left( I + \sum_{k=1}^{\infty} \frac{1}{k!} A^k \right)^* \\
 &= \left( I^* + \sum_{k=1}^{\infty} \frac{1}{k!} (A^k)^* \right) \\
 &= \left( I + \sum_{k=1}^{\infty} \frac{1}{k!} (A^*)^k \right) \\
 &= e^{A^*} \\
 &= e^{-A},
 \end{aligned}$$

where the second and third lines are derived using the following properties:

- $(A + B)^* = A^* + B^*$ ;
- $(A^k)^* = (A^*)^k$ .

The fourth line comes from the definition of the exponential matrix and the last line from the fact that  $A$  is skew-Hermitian.

Finally can write :

$$\begin{aligned}
 e^A(e^A)^* &= e^A(e^{-A}) \\
 &= e^{A-A} \\
 &= I.
 \end{aligned}$$

The second line is valid since  $A$  and  $-A$  commute. □

### B4

We start by showing that  $(J_n(0))^n = 0$ .

*Proof.* One trivially notes that  $A := J_n(0) \in \mathbb{C}^{n \times n}$  is strictly upper triangular, for any  $n \geq 1$ . We want to prove by induction on  $k \geq 1$  that  $a_{ij}^k = 0$  whenever  $i \leq j + k - 1$ , with  $A := (J_n(0))^k := [a_{ij}^k]$ .

- When  $k = 1$ , this is trivially true, as this precisely states that  $A$  is strictly upper triangular.
- For the induction step, we fix  $k \geq 1$  and we suppose that  $a_{ij}^k = 0$  when  $i \leq j + k - 1$ .

Fix  $i, j$  and suppose that  $i \leq j + (k + 1) - 1 = j + k$ . In that case,

$$a_{ij}^{k+1} = \sum_{\ell=0}^n a_{i\ell}^k a_{\ell j}.$$

However, regarding the base case, we know that  $a_{\ell j} = 0$  whenever  $\ell \leq j$ , so that

$$a_{ij}^{k+1} = \sum_{\ell=j+1}^n a_{i\ell}^k a_{\ell j}.$$

Moreover, keeping in mind  $i \leq j + k$ , we observe that if  $j + 1 \leq \ell \leq n$ , then

$$i \leq j + k = (j + 1) + k - 1 \leq \ell + k - 1$$

so that  $a_{i\ell}^k = 0$  by the induction hypothesis. This allows us to conclude that  $a_{ij}^{k+1} = 0$  when  $i \leq j + (k + 1) - 1$ , as desired.

Finally, the result is obtained by taking the case  $k = n$ , which tells us that  $a_{ij}^n = 0$  whenever  $i \leq j + n - 1$ , but this is always true as  $i \leq n$  and  $j + n - 1 \geq n$  for all  $1 \leq i, j \leq n$ . This shows that  $A = (J_n(0))^n = 0$ .  $\square$

Next, we show that

$$e^{J_n(\lambda)} = e^\lambda \left( I + \sum_{k=1}^{n-1} \frac{1}{k!} (J_n(0))^k \right).$$

*Proof.* We note that  $J_n(\lambda) = J_n(0) + \lambda I$ . Furthermore,  $J_n(0)$  and  $\lambda I$  commute as the identity matrix multiplied by a scalar commutes with every matrix. Thus, we can write:

$$\begin{aligned} e^{J_n(\lambda)} &= e^{\lambda I + J_n(0)} \\ &= e^{\lambda I} e^{J_n(0)}. \end{aligned}$$

If we develop the last right-hand side and use the Taylor series expansion of the exponential, we obtain

$$\begin{aligned} e^{J_n(0)} e^{\lambda I} &= \sum_{j=0}^{\infty} \frac{1}{j!} (\lambda I)^j \left( \sum_{k=0}^{\infty} \frac{1}{k!} (J_n(0))^k \right) \\ &= I \sum_{j=0}^{\infty} \frac{1}{j!} \lambda^j \sum_{k=0}^{\infty} \frac{1}{k!} (J_n(0))^k \\ &= e^\lambda \left( \sum_{k=0}^{\infty} \frac{1}{k!} (J_n(0))^k \right) \\ &= e^\lambda \left( I + \sum_{k=1}^{n-1} \frac{1}{k!} (J_n(0))^k \right). \end{aligned}$$

For the last equality, we used the nilpotent property of  $J_n(0)$  that we proved previously. Putting the left-hand side in its original form, this is precisely what we were trying to prove.  $\square$

## B5

We want to show that for any matrices  $A, B \in \mathbb{C}^{n \times n}$ ,

$$e^{A \otimes I + I \otimes B} = e^A \otimes e^B.$$

*Proof.* We first note that  $A \otimes I$  and  $I \otimes B$  commute. Indeed, by using the mixed product property, we observe that

$$(A \otimes I)(I \otimes B) = (AI) \otimes (IB) = (I \otimes B)(A \otimes I).$$

Hence we can write

$$e^{A \otimes I + I \otimes B} = e^{A \otimes I} e^{I \otimes B}.$$



Next, we develop the last equality using the definition of the matrix exponential:



$$\begin{aligned}
 e^{A \otimes I} e^{I \otimes B} &= \sum_{k=0}^{\infty} \frac{1}{k!} (A \otimes I)^k \sum_{j=0}^{\infty} \frac{1}{j!} (I \otimes B)^j \\
 &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{k!} \frac{1}{j!} (A \otimes I)^k (I \otimes B)^j \\
 &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{k!} \frac{1}{j!} (A \otimes I) \underbrace{\cdots}_{k-2 \text{ times}} (A \otimes I) (I \otimes B) \underbrace{\cdots}_{j-2 \text{ times}} (I \otimes B) \\
 &\stackrel{A3}{=} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{k!} \frac{1}{j!} (A^2 \otimes I^2) \underbrace{\cdots}_{k-3 \text{ times}} (A \otimes I) (I^2 \otimes B^2) \underbrace{\cdots}_{j-3 \text{ times}} (I \otimes B) \\
 &\stackrel{A3}{=} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{k!} \frac{1}{j!} (A^k \otimes I^k) (I^j \otimes B^j) \\
 &\stackrel{A3}{=} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{k!} \frac{1}{j!} (A^k I^j) \otimes (I^k B^j) \\
 &= \sum_{k=0}^{\infty} \frac{1}{k!} A^k \otimes \sum_{j=0}^{\infty} \frac{1}{j!} B^j \\
 &= e^A \otimes e^B.
 \end{aligned}$$

The second to last equality comes from the distributivity of the Kronecker product.

Indeed, let us take  $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ ,  $B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix}$ .

$$\begin{aligned}
 (A + B) \otimes C &= \begin{bmatrix} (a_{11} + b_{11})C & \cdots & (a_{1n} + b_{1n})C \\ \vdots & \ddots & \vdots \\ (a_{m1} + b_{m1})C & \cdots & (a_{mn} + b_{mn})C \end{bmatrix} \\
 &= \begin{bmatrix} a_{11}C + b_{11}C & \cdots & a_{1n}C + b_{1n}C \\ \vdots & \ddots & \vdots \\ a_{m1}C + b_{m1}C & \cdots & a_{mn}C + b_{mn}C \end{bmatrix} \\
 &= A \otimes C + B \otimes C
 \end{aligned}$$

The left distributivity can also be proved the same way.

□