

Exercise A: Minimal polynomial and Smith normal form

A1

First we prove a lemma.

Lemma. *If $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$ then for $k = 1, \dots, n$:*

- $\delta_k(A)$ divides $\delta_k(AB)$
- $\delta_k(B)$ divides $\delta_k(AB)$

Proof. If we denote by $(AB)_{\mathbf{IJ}}$ the k -by- k sub-matrix of AB taking the rows \mathbf{I} and the columns \mathbf{J} (hence $|\mathbf{I}| = |\mathbf{J}| = k$). We can write and develop the associated k -minor:

$$\begin{aligned} \det(AB)_{\mathbf{IJ}} &= \det(A_{\mathbf{I}*} B_{*\mathbf{J}}) \\ &= \sum_{\mathbf{j}_k} A_{\mathbf{I}*} \binom{k}{\mathbf{j}_k} B_{*\mathbf{J}} \binom{\mathbf{j}_k}{k} \\ &= \sum_{\mathbf{j}_k} \det A_{\mathbf{I}\mathbf{j}_k} \det B_{\mathbf{j}_k\mathbf{J}} \end{aligned}$$

The second equality is derived from the Binet-Cauchy theorem. We observe that the k -minors of AB can be written as linear combinations of the k -minors of A . Thus, every common divisor of the k -minors of A must also be a common divisor of the k -minors of AB . As it is true for the greatest common divisor, we have that $\delta_k(A)$ divides $\delta_k(AB)$. The same reasoning is valid taking B instead of A and therefore we prove that $\delta_k(B)$ divides $\delta_k(AB)$. □

We now prove that $\delta_k(P(\lambda)) = \delta_k(Q(\lambda))$, where $Q(\lambda) = M(\lambda)P(\lambda)N(\lambda)$.

Proof. Using the lemma with $A = M(\lambda)P(\lambda)$ and $B = N(\lambda)$, we have that $\delta_k(M(\lambda)P(\lambda))$ divides $\delta_k(Q(\lambda))$. Then applying once more the lemma with $A = \delta_k(M(\lambda))$ and $B = \delta_k(P(\lambda))$, we obtain that $\delta_k(P(\lambda))$ divides $\delta_k(M(\lambda)P(\lambda))$. We deduce that $\delta_k(P(\lambda))$ divides $\delta_k(Q(\lambda))$.

We can apply the exact same reasoning to $P(\lambda) = N^{-1}(\lambda)Q(\lambda)M^{-1}(\lambda)$. Doing so, we have that $\delta_k(Q(\lambda))$ divides $\delta_k(P(\lambda))$. Consequently, we conclude that $\delta_k(Q(\lambda))$ is equal to $\delta_k(P(\lambda))$. □

A2

. We now prove that for all $k \in \{1, \dots, n\}$, $\delta_k(P(\lambda)) = \prod_{i=1}^k d_i(\lambda)$ where $d_i(\lambda)$ are the diagonal entries of $D(\lambda)$ and $M(\lambda)D(\lambda)N(\lambda)$ is a Smith decomposition of $P(\lambda)$.

Proof. From A1, we know that $\delta_k(P(\lambda)) = \delta_k(D(\lambda))$ and therefore we will consider $\delta_k(D(\lambda))$.

We first note that all sub-matrices of $D(\lambda)$ are either triangular inferior, triangular superior or null. The determinants of these sub-matrices are hence given by the product of the diagonal elements. The only sub-matrices for which the determinant is non-zero are these where the indices of the removed rows are identical to the indices of the removed columns. Because 0 has infinitely many divisors, we should therefore only consider the case where this specific case.

We denote by $\mathbf{j} = (j_1, \dots, j_k)$, where $j_1 < \dots < j_k$, the indices of the rows and columns that are kept. The determinant associated to \mathbf{j} can be written as:

$$\det D_{\mathbf{j}}(\lambda) = d_{j_1}(\lambda) \dots d_{j_k}(\lambda)$$

We now use the property that $d_i(\lambda)$ divides $d_{i+1}(\lambda)$ for all $i \in \{1, \dots, n-1\}$.

If we consider the i th factor $d_{j_i}(\lambda)$, we observe that $d_i(\lambda)$ divides $d_{j_i}(\lambda)$ as j_i is either equal to i or larger than i (due to the strict inequality $j_1 < \dots < j_k$). Hence $d_{j_i}(\lambda)$ can be rewritten as $a_{j_i}(\lambda)d_i(\lambda)$ for a certain $a_{j_i}(\lambda) \in \mathbb{C}[\lambda]$. The determinant can thus be developed as:

$$\det D_{\mathbf{j}}(\lambda) = a_{j_1}(\lambda)d_1(\lambda) \dots a_{j_k}(\lambda)d_k(\lambda) \tag{1}$$

We also observe that when $\mathbf{j} = (1, \dots, k)$, we have:

$$\det D_{\mathbf{j}}(\lambda) = \prod_{i=1}^k d_i(\lambda) \quad (2)$$

which is a monic polynomial as it is the product of monic polynomials.

Because $\prod_{i=1}^k d_i(\lambda)$ is a divisor of (1) for any \mathbf{j} and that $\delta_k(D(\lambda))$ cannot contain any additional factor due to (2), we conclude that the monic greatest common divisor of all k -minors of $D(\lambda)$ is $\prod_{i=1}^k d_i(\lambda)$. Therefore, we have:

$$\delta_k(P(\lambda)) = \delta_k(D(\lambda)) = \prod_{i=1}^k d_i(\lambda) \quad (3)$$

□

The diagonal entries $d_1(\lambda), \dots, d_n(\lambda)$ of $D(\lambda)$ are unique and depend only on the sequence $\delta_1(P(\lambda)), \dots, \delta_n(P(\lambda))$.

Proof. We first consider the first entry $d_1(\lambda)$. From (3), we know that $d_1(\lambda) = \delta_1(\lambda)$. The notion of monic greatest common divisor leads to the unicity of $\delta_1(\lambda)$ and hence we deduce $d_1(\lambda)$ is unique as well. This constitutes the base case of the induction proof.

Next, we prove that if $d_k(\lambda)$ is unique then $d_{k+1}(\lambda)$ is also unique. Again from (3), we have that $d_{k+1}(\lambda) = \delta_{k+1}(P(\lambda))/d_k(\lambda)$. By the unicity of the notion of monic greatest common divisor and the unicity $d_k(\lambda)$, we conclude that $d_{k+1}(P(\lambda))$ is unique.

Clearly the diagonal entries $d_1(\lambda), \dots, d_n(\lambda)$ can be computed using the sequence $\delta_1(P(\lambda)), \dots, \delta_n(P(\lambda))$. First, we have $d_1(\lambda) = \delta_1(P(\lambda))$ and then we use $d_{k+1}(\lambda) = \delta_{k+1}(P(\lambda))/d_k(\lambda)$ for the next entries. □

There are unimodular matrices $M(\lambda), N(\lambda) \in \mathbb{C}^{n \times n}[\lambda]$ such that $Q(\lambda) = M(\lambda)P(\lambda)N(\lambda)$ if and only if $\delta_k(P(\lambda)) = \delta_k(Q(\lambda))$ for all $k \in \{1, \dots, n\}$.

Proof. \Rightarrow This was shown in A1.

\Leftarrow From the previous proved statement we know that the diagonal entries of $D(\lambda)$ of the Smith decomposition of $P(\lambda)$ are unique and depend only on the sequence $\delta_1(P(\lambda)), \dots, \delta_n(P(\lambda))$. As this sequence is the same for $P(\lambda)$ and $Q(\lambda)$, we deduce $D(\lambda)$ is the same for their Smith decompositions:

$$\begin{aligned} P(\lambda) &= M_1(\lambda)D(\lambda)N_1(\lambda) \\ Q(\lambda) &= M_2(\lambda)D(\lambda)N_2(\lambda) \end{aligned}$$

where $M_1(\lambda), M_1(\lambda), M_1(\lambda), M_1(\lambda) \in \mathbb{C}^{n \times n}[\lambda]$ are unimodular and $D(\lambda) \in \mathbb{C}^{n \times n}[\lambda]$ diagonal.

From the first equation, we can write $D(\lambda) = M_1(\lambda)^{-1}P(\lambda)N_1(\lambda)^{-1}$ as $M_1(\lambda)$ and $N_1(\lambda)$ are unimodular and hence invertible. Next, we inject this expression of $D(\lambda)$ in the second equation and we obtain:

$$Q(\lambda) = M_2(\lambda)M_1(\lambda)^{-1}P(\lambda)N_1(\lambda)^{-1}N_2(\lambda)$$

Finally, as the inverse of a unimodular matrix is unimodular and the product of unimodular matrices is unimodular, we have $M(\lambda) = M_2(\lambda)M_1(\lambda)^{-1}$ and $N(\lambda) = N_1(\lambda)^{-1}N_2(\lambda)$ which are unimodular matrices such that $Q(\lambda) = M(\lambda)P(\lambda)N(\lambda)$. □

A3

Let $A \in \mathbb{C}^{n \times n}$ be a Jordan block with eigenvalue λ_1 . The elementary polynomials of $\lambda I - A$ are equal to: $d_i(\lambda) = 1$ for $i = 1, \dots, n-1$ and $d_n(\lambda) = (\lambda - \lambda_1)^n$.

Proof. We first write the matrix $\lambda I - A$ that we call $P(\lambda)$:

$$P(\lambda) = \lambda I - A = \begin{pmatrix} \lambda - \lambda_1 & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & \lambda - \lambda_1 \end{pmatrix}$$

As we have seen the sequence $d_1(\lambda), \dots, d_n(\lambda)$ can be derived by the sequence $\delta_1(P(\lambda)), \dots, \delta_n(P(\lambda))$, we first derive the latter sequence. When computing each $\delta_k(P(\lambda))$, we observe that for $k < n$, a valid sub-matrix to

take into account is the one which has all the -1 entries on its diagonal. We illustrate here the case where $k = n - 1$:

$$P(\lambda) = \begin{pmatrix} \lambda - \lambda_1 & & & & \\ & \boxed{\begin{matrix} -1 & & & \\ & \ddots & & \\ & & -1 & \\ & & & -1 \end{matrix}} & & \\ & & & \lambda - \lambda_1 & \\ & & & & \lambda - \lambda_1 \end{pmatrix}$$

As this sub-matrix is triangular inferior, its determinant is equal to the product of its diagonal entries. The determinant of such matrices is therefore 1 or -1. As 1 and -1 have only one monic divisor which is 1, we conclude that $\delta_k(P(\lambda)) = 1$ for $k \in \{1, \dots, n - 1\}$. From the recursive formula proposed in A2, we deduce that $d_k(\lambda) = 1$ for $k \in \{1, \dots, n - 1\}$.

When $k = n$, the only possible sub-matrix to consider is the whole matrix which determinant is equal to the product of the diagonal entries, hence $(\lambda - \lambda_1)^n$. We deduce that $\delta_n(P(\lambda)) = (\lambda - \lambda_1)^n$ and therefore using the recursive formula proposed in A2 we obtain $d_n(\lambda) = (\lambda - \lambda_1)^n$. \square

A4

Let $J_1 \in \mathbb{C}^{n_1 \times n_1}$ and $J_2 \in \mathbb{C}^{n_2 \times n_2}$ be two Jordan blocks with respective eigenvalues λ_i and size n_i , $i = 1, 2$. Let $A \in \mathbb{C}^{n \times n}$ with $n = n_1 + n_2$ be the Jordan matrix consisting of the two Jordan blocks J_1 and J_2 . Then $\lambda I - A$ can be reduced to the form

$$M(\lambda)(\lambda I - A)N(\lambda) = \text{diag}\{\underbrace{1, \dots, 1}_{n_1-1}, (\lambda - \lambda_1)^{n_1}, \underbrace{1, \dots, 1}_{n_2-1}, (\lambda - \lambda_2)^{n_2}\}$$

With unimodular matrices $M(\lambda)$, $N(\lambda) \in \mathbb{C}^{n \times n}[\lambda]$.

Proof. From A3, we have already proved that $\lambda I - J_1$ can be written as

$$\begin{aligned} P_1(\lambda) &:= \lambda I_{n_1} - J_1 = M_1(\lambda)D_1(\lambda)N_1(\lambda) \\ \Leftrightarrow D_1 &= M_1^{-1}(\lambda)P_1(\lambda)N_1^{-1}(\lambda) \\ &= \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & (\lambda - \lambda_1)^{n_1} \end{bmatrix} \end{aligned}$$

Similarly,

$$\begin{aligned} P_2(\lambda) &:= \lambda I_{n_2} - J_2 = M_2(\lambda)D_2(\lambda)N_2(\lambda) \\ \Leftrightarrow D_2 &= M_2^{-1}(\lambda)P_2(\lambda)N_2^{-1}(\lambda) \\ &= \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & (\lambda - \lambda_2)^{n_2} \end{bmatrix} \end{aligned}$$

Note that $M_1^{-1}(\lambda)$, $N_1^{-1}(\lambda)$, $M_2^{-1}(\lambda)$ and $N_2^{-1}(\lambda)$ are unimodular as their inverse are unimodular.

Now, if we consider the diagonal matrix formed by D_1 and D_2 , we have :

$$\begin{aligned}
 D(\lambda) &:= \begin{bmatrix} D_1 & \\ & D_2 \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & (\lambda - \lambda_1)^{n_1} & \\ & & & & & 1 \\ & & & & & & 1 \\ & & & & & & & \ddots \\ & & & & & & & & 1 \\ & & & & & & & & & (\lambda - \lambda_2)^{n_2} \end{bmatrix} \\
 &= \begin{bmatrix} M_1^{-1}(\lambda)P_1(\lambda)N_1^{-1}(\lambda) & & \\ & M_2^{-1}(\lambda)P_2(\lambda)N_2^{-1}(\lambda) & \\ & & \end{bmatrix} \\
 &= \underbrace{\begin{bmatrix} M_1^{-1}(\lambda) & \\ & M_2^{-1}(\lambda) \end{bmatrix}}_{:=M(\lambda)} \underbrace{\begin{bmatrix} P_1(\lambda) & \\ & P_2(\lambda) \end{bmatrix}}_{:=P(\lambda)=(\lambda I - A)} \underbrace{\begin{bmatrix} N_1^{-1}(\lambda) & \\ & N_2^{-1}(\lambda) \end{bmatrix}}_{:=N(\lambda)}
 \end{aligned}$$

$M(\lambda)$ is indeed unimodular as its determinant is the product of the determinants of $M_1^{-1}(\lambda)$ and $M_2^{-1}(\lambda)$ (both unimodular). Thus, the determinant of $M_1(\lambda)$ is equal to the product of 2 nonzero constants, which is a nonzero constant.

Similarly, $N(\lambda)$ is also unimodular. □

Remark : this result can be easily extended to the case of A being composed by more than 2 Jordan blocks with the same reasoning.

$$\underbrace{\begin{bmatrix} D_1 & & \\ & D_2 & \\ & & D_3 \\ & & & \ddots \end{bmatrix}}_{:=D(\lambda)} = \underbrace{\begin{bmatrix} M_1^{-1}(\lambda) & & \\ & M_2^{-1}(\lambda) & \\ & & M_3^{-1}(\lambda) \\ & & & \ddots \end{bmatrix}}_{:=B^{-1}(\lambda)} \underbrace{\begin{bmatrix} P_1(\lambda) & & \\ & P_2(\lambda) & \\ & & P_3(\lambda) \\ & & & \ddots \end{bmatrix}}_{:=P(\lambda)=(\lambda I - A)} \underbrace{\begin{bmatrix} N_1^{-1}(\lambda) & & \\ & N_2^{-1}(\lambda) & \\ & & N_3^{-1}(\lambda) \\ & & & \ddots \end{bmatrix}}_{:=C^{-1}(\lambda)} \quad (4)$$

Proof. By induction, let's consider $A = \begin{bmatrix} A_1 & \\ & J_p \end{bmatrix}$. With $A_1 \in \mathbb{C}^{n \times n}$ a block diagonal matrix with $(p-1)$ Jordan blocks and $J_p \in \mathbb{C}^{m \times m}$ the p th Jordan block. Let's assume that :

$$\underbrace{\begin{bmatrix} D_1 & & \\ & D_2 & \\ & & \ddots \\ & & & D_{p-1} \end{bmatrix}}_{:=D_{A_1}(\lambda)} = \underbrace{\begin{bmatrix} M_1^{-1}(\lambda) & & \\ & M_2^{-1}(\lambda) & \\ & & \ddots \\ & & & M_{p-1}^{-1}(\lambda) \end{bmatrix}}_{:=M_{A_1}^{-1}(\lambda)} \underbrace{\begin{bmatrix} P_1(\lambda) & & \\ & P_2(\lambda) & \\ & & \ddots \\ & & & P_{p-1}(\lambda) \end{bmatrix}}_{:=P_{A_1}(\lambda)=(\lambda I - A_1)} \underbrace{\begin{bmatrix} N_1^{-1}(\lambda) & & \\ & N_2^{-1}(\lambda) & \\ & & N_3^{-1}(\lambda) \\ & & & \ddots \end{bmatrix}}_{:=N_{A_1}^{-1}(\lambda)}$$

Then,

$$\begin{aligned}
 \lambda I_{m+n} - A &= \begin{bmatrix} \lambda I_n - A_1 & \\ & \lambda I_m - J_p \end{bmatrix} \\
 &= \begin{bmatrix} M_{A_1}(\lambda)D_{A_1}(\lambda)N_{A_1}(\lambda) & \\ & \lambda I_m - J_p \end{bmatrix} \\
 &= \begin{bmatrix} M_{A_1}(\lambda)D_{A_1}(\lambda)N_{A_1}(\lambda) & \\ & M_{J_p}(\lambda)D_{J_p}(\lambda)N_{J_p}(\lambda) \end{bmatrix} \\
 &= \underbrace{\begin{bmatrix} M_{A_1}(\lambda) & \\ & M_{J_p}(\lambda) \end{bmatrix}}_{B(\lambda)} \underbrace{\begin{bmatrix} D_{A_1}(\lambda) & \\ & D_{J_p}(\lambda) \end{bmatrix}}_{D(\lambda)} \underbrace{\begin{bmatrix} N_{A_1}(\lambda) & \\ & N_{J_p}(\lambda) \end{bmatrix}}_{C(\lambda)}
 \end{aligned}$$

From the first equation to the second, we used our assumption and from the second to the third, we used A3. \square

Now, we want to find the elementary polynomials of $P(\lambda) = (\lambda I - A)$. For that, we will use the property proved in A1 s.t. $\delta_k(P(\lambda)) = \delta_k(D(\lambda))$.

Indeed, it is clear that for $k = 1, 2, \dots, (n_1 + n_2 - 2)$, we can always choose a submatrix containing only ones on the diagonal, which means that $\delta_k(P(\lambda)) = 1$. And by A2, we have that the monic $d_k(\lambda)$ are all equal to 1 for $k = 1, 2, \dots, (n_1 + n_2 - 2)$.

Then, we need to distinguish 2 cases :

- When $\lambda_1 \neq \lambda_2$,

$$\begin{aligned} \delta_{n_1+n_2-1}(P(\lambda)) &= 1 & \Leftrightarrow d_{n_1+n_2-1}(\lambda) &= 1 \\ \delta_{n_1+n_2}(P(\lambda)) &= (\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} & \Leftrightarrow d_{n_1+n_2}(\lambda) &= (\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \end{aligned}$$

- When $\lambda_1 = \lambda_2$,

$$\begin{aligned} \delta_{n_1+n_2-1}(P(\lambda)) &= (\lambda - \lambda_1)^{\min(n_1, n_2)} & \Leftrightarrow d_{n_1+n_2-1}(\lambda) &= (\lambda - \lambda_1)^{\min(n_1, n_2)} \\ \delta_{n_1+n_2}(P(\lambda)) &= (\lambda - \lambda_1)^{n_1+n_2} & \Leftrightarrow d_{n_1+n_2}(\lambda) &= \frac{(\lambda - \lambda_1)^{n_1+n_2}}{(\lambda - \lambda_1)^{\min(n_1, n_2)}} = (\lambda - \lambda_1)^{\max(n_1, n_2)} \end{aligned}$$

We can observe that the minimal polynomial of A is equal to $d_{n_1+n_2}(\lambda)$.

A5

Let $A \in \mathbb{C}^{n \times n}$. There are unimodular matrices $M(\lambda), N(\lambda) \in \mathbb{C}^{n \times n}[\lambda]$ s.t.

$$\lambda I - A = M(\lambda)E(\lambda)N(\lambda)$$

where $E(\lambda)$ is diagonal with elements being either 1's or polynomials of the form $(\lambda - \lambda_i)^{n_i}$ where λ_i are the eigenvalues of A and n_i the size of the corresponding Jordan block in the Jordan decomposition of A .

Proof. First, let us write the Jordan decomposition of A .

$$T^{-1}AT = J \quad \Leftrightarrow \quad A = TJT^{-1}$$

With J , the Jordan matrix and T being a similarity transformation (this implies that T is invertible, its determinant is nonzero, which means that it is a unimodular matrix).

We can now derive :

$$\begin{aligned} \lambda I - A &= \lambda I - TJT^{-1} \\ &= T(\lambda I - J)T^{-1} \\ &= \underbrace{TB(\lambda)}_{:=M(\lambda)} \underbrace{D(\lambda)}_{:=E(\lambda)} \underbrace{C(\lambda)T^{-1}}_{:=N(\lambda)} \end{aligned}$$

The last equality is obtained with the results in A4 (cfr Equation). $M(\lambda)$ and $N(\lambda)$ are indeed unimodular as they are products of 2 unimodular matrices. \square

The matrices A and J have the same eigenvalues which means that they have the same minimal polynomial as well. In A4, we have already observed that a Jordan matrix with 2 Jordan blocks has a minimal polynomial equal to its last elementary polynomial. It is trivial to see that this result can be extended to a Jordan matrix with more Jordan blocks. Hence the minimal polynomial of A is equal to the last elementary polynomial of J (obtained with $D(\lambda)$) which is also the last elementary polynomial of A (obtained with $E(\lambda) = D(\lambda)$).

Exercise B: Implementation

B1

Using the Jordan normal form is not numerically stable because taking limits does not commute with forming the Jordan canonical form. A simple example is the matrix $A = I_2$, approximated by $A_\varepsilon = \begin{bmatrix} 1 & \varepsilon \\ 0 & 1 \end{bmatrix}$, the latter having Jordan canonical form $J_\varepsilon = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. However, the Jordan form of A is simply $J = I_2$. We thus have

$$\lim_{\varepsilon \rightarrow 0} A_\varepsilon = A, \quad \text{but} \quad \lim_{\varepsilon \rightarrow 0} J_\varepsilon \neq J.$$

Similarly, computing the minimal polynomials yields $p_{A_\varepsilon}(\lambda) = (\lambda - 1)^2 \neq \lambda - 1 = p_A(\lambda)$.

B2