Exercise A: Boundedness of trajectories and Lyapunov equation

A1

Proof. From the equality $y(t) = Tx(t) \iff x(t) = T^{-1}y(t)$, we can write

$$\begin{aligned}\dot{y}(t) &= T\dot{x}(t)\\ &= TAx(t)\\ &= TAT^{-1}y(t)\\ &= \operatorname{diag}\{J_1(\lambda_1), \dots, J_r(\lambda_r)\}\, y(t).\end{aligned}$$

This is the equation of a continuous-time linear system, and each $y_i(t)$ has the Jordan block $J_i(\lambda_i)$ as transition matrix. $\square$

A2

We assume that $A = J_n(\lambda)$, hence we can write

$$x(t) = e^{At}x(0) = e^{J_n(\lambda)t}x(0)$$

We can use the result of question B4 from Homework 1, which develops $e^{J_n(\lambda)}$, to obtain

$$\begin{aligned}x(t) &= e^{J_n(\lambda)t}x(0)\\ &= e^{\lambda t}\left(I + \sum_{k=1}^{n-1}\frac{1}{k!}\big(J_n(0)\big)^k\right)^t x(0)\\ &= e^{\lambda I t}e^{J_n(0)t}x(0)\\ &= e^{\lambda t}\left(I + \sum_{k=1}^{n-1}\frac{1}{k!}\big(J_n(0)t\big)^k\right)x(0)\\ &= e^{\lambda t}\left(I + \sum_{k=1}^{n-1}\frac{1}{k!}\big(J_n(0)\big)^k t^k\right)x(0),\end{aligned}$$

where the third and fourth equalities come from the definition of the matrix exponential. We then develop the terms in parentheses:

$$I + \sum_{k=1}^{n-1}\frac{1}{k!}\big(J_n(0)\big)^k t^k = I + \frac{t}{1!}\begin{pmatrix}0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 & \\ & & & 0 & 1\\ & & & & 0\end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix}0 & 0 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & & 0 & 0 & 1\\ & & & & 0 & 0\\ & & & & & 0\end{pmatrix}$$

$$+ \dots + \frac{t^k}{k!}\begin{pmatrix}\overbrace{0 \;\; \dots \;\; 0}^{k \text{ zeros}} & 1 & & \\ \ddots & \ddots & \ddots & \\ & \ddots & \ddots & 1\\ & & \ddots & 0\\ & & & \vdots\\ & & & 0\end{pmatrix} + \dots + \frac{t^{n-1}}{(n-1)!}\begin{pmatrix}\overbrace{0 \;\; \dots \;\; 0}^{n-1 \text{ zeros}} & 1\\ \ddots & 0\\ \ddots & \vdots\\ & 0\end{pmatrix}$$

$$= \begin{pmatrix}1 & t/1! & \dots & t^k/k! & \dots & t^{n-1}/(n-1)!\\ & \ddots & \ddots & & \ddots & \vdots\\ & & 1 & t/1! & & t^k/k!\\ & & & \ddots & \ddots & \vdots\\ & & & & 1 & t/1!\\ & & & & & 1\end{pmatrix}.$$

We deduce from the previous expression that for $i \in [n]$,

$$x_i(t) = e^{\lambda t} \sum_{j=i}^{n} \frac{1}{(j-i)!} t^{j-i} x_j(0).$$

A3

The minimal polynomial of a matrix $A \in \mathbb{C}^{n\times n}$ is the polynomial

$$m(\lambda) = \prod_i (\lambda - \lambda_i)^{k_i^*},$$

where

$$f(J) = \operatorname{diag}\left\{ f\left(J_{k_{i_j}}(\lambda_{i_j}) \right) \right\}, \quad k_i^* = \max_{1\leqslant j \leqslant n_i} k_{i_j},$$

and n_i is the number of Jordan blocks with eigenvalue λ_i.

Simple eigenvalues thus have the property that the size of the largest Jordan block with that eigenvalue λ_i is 1 (i.e. $k_i^* = 1$).

A4

Proof.

- $1 \implies 2$.
- $2 \implies 1$.

  From A1, we know there exists a change of coordinates $y = Tx$ such that $y(t)$ can be decomposed as follows

  $$y(t) = \left[y_1(t)^\top, \ldots, y_r(t)^\top \right]^\top,$$

  where each $y_i(t)$, $i \in [r]$, is the trajectory of a continuous-time linear dynamical system with a Jordan block as transition matrix.

  From A2, one can then obtain an expression for the components $y_i(t)$. If λ_i is such that $\Re(\lambda_i) < 0$, then from A2, we clearly see that the expression $y_i(t)$ tends to zero, as the exponential decreases more rapidly than the polynomial. If λ_i is purely imaginary, i.e. $\Re(\lambda_i) = 0$, then we can assume it is simple and hence by A3 we know the size of the largest Jordan block with that eigenvalue is 1. The expression of $y_i(t)$ then simplifies to $y_i(t) = e^{\lambda t} y_j(0)$, which tends to zero as t goes to infinity.

  This shows that if all eigenvalues are either in the open left-hand plane or imaginary and simple, then $y_i(t)$ and hence $\|y(t)\| = \|Tx(t)\|$ tends to zero. Finally, by virtue of the inversibility of T, this shows that $\sup_{t\geqslant 0}\|x(t)\| < \infty$. □

A5

Proof. One can show that the matrix P must exist by proving that for $P = I_n$, the statement holds. Indeed, I_n is positive definite and Hermitian. Since D and D^* are diagonal, their sum is also diagonal. In fact,

$$D + D^* = \operatorname{diag}(\lambda_1 + \lambda_1^*, \ldots, \lambda_n + \lambda_n^*),$$

where λ_i and λ_i^* are the eigenvalues of D and D^*, respectively. We know that the imaginary parts of these eigenvalues cancel each other out, and that the real parts are the same (and are nonpositive), by virtue of the definition of the conjugate transpose. We thus get

$$D^*P + PD = D^*I_n + I_nD = D^* + D = \operatorname{diag}(\lambda_1 + \lambda_1^*, \ldots, \lambda_n + \lambda_n^*) \preceq 0,$$

which proves the existence of such a positive definite Hermitian matrix P. □

A6

Proof. First, we develop $B = I \otimes A^* + A^\top \otimes I$:

$$B = \begin{pmatrix} A^* & & \\ & \ddots & \\ & & A^* \end{pmatrix} + \begin{pmatrix} a_{11}I & \dots & a_{1n}I \\ \vdots & \ddots & \vdots \\ a_{n1}I & \dots & a_{nn}I \end{pmatrix}.$$

We know $A \in \mathbb{C}^{n\times n}$ is a Jordan block with eigenvalue λ, thus

$$\begin{aligned} B &= \begin{pmatrix} \big(J_n(\lambda)\big)^* & & \\ & \ddots & \\ & & \big(J_n(\lambda)\big)^* \end{pmatrix} + \begin{pmatrix} \lambda I & & & \\ I & \ddots & & \\ & \ddots & \ddots & \\ & & I & \lambda I \end{pmatrix} \\ &= \begin{pmatrix} \big(J_n(\lambda)\big)^* + \lambda I & & & \\ I & \ddots & & \\ & \ddots & \ddots & \\ & & I & \big(J_n(\lambda)\big)^* + \lambda I \end{pmatrix} \\ &= \begin{pmatrix} \big(J_n(\lambda+\lambda^*)\big)^* & & & \\ I & \ddots & & \\ & \ddots & \ddots & \\ & & I & \big(J_n(\lambda+\lambda^*)\big)^* \end{pmatrix}. \end{aligned}$$

The matrix B is lower-triangular and therefore its eigenvalues are the diagonal elements $\lambda + \lambda^*$. Moreover, we know that $\Re(\lambda) > 0$ and so $\lambda + \lambda^* = 2\Re(\lambda) > 0$. From this, we deduce that B is positive definite and hence invertible. It follows that the system $B\,\mathrm{vec}(P) = -\,\mathrm{vec}(Q)$ has a unique solution and consequently we conclude that P exists and is unique. □

Proof. Next we show that if $P \in \mathbb{C}^{n\times n}$ satisfies $A^*P + PA = -Q$ then P^* also satisfies $A^*P^* + P^*A = -Q$. This can simply be proven by taking the transpose conjugate of both sides of the equation $A^*P + PA = -Q$ that P satisfies taking into account that Q is Hermitian:

$$\begin{aligned} & \big(A^*P + PA\big)^* = (-Q)^* \\ \iff\ & P^*A + A^*P^* = -Q \end{aligned}$$

This shows that P^* satisfies $A^*P^* + P^*A = -Q$.

Combining the two last results, we deduce that there always exists a unique Hermitian matrix P satisfying $B\,\mathrm{vec}(P) = -\,\mathrm{vec}(Q)$. □

Proof. Finally, we want to show that P is positive definite. We consider a trajectory $x(t)$ starting from $x(0)$. If we define $V\big(x(t)\big) = x(t)^*Px(t)$, we observe that:

$$\begin{aligned} \dot V\big(x(t)\big) &= \dot x(t)^*Px(t) + x(t)^*P\dot x(t) \\ &= x(t)^*A^*Px(t) + x(t)^*PAx(t) \\ &= x(t)^*(A^*P + PA)x(t) \\ &= x(t)^*(-Q)x(t) \\ &< 0. \end{aligned}$$

Consequently, we find that for $t > 0$ we have $x(t)^*Px(t) < x(0)^*Px(0)$. We know from A2 that $\lim_{t\to\infty} x(t) = 0$, as it decreases exponentially. We thus have that $V\big(x(t)\big)$ is always positive, and hence P is positive definite. □

A7

Proof.

- $2 \implies 3$.

  Jordan form of A such that simple eigenvalues in block A_1 and other eigenvalues in block A_2:

$$\begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix}$$

  By A3 we find P_1 such that $P_1A_1 + A_1^*P_1 = -Q1$ and by A6 we find P_2 such that $P_2A_2 + A_2^*P_2 = -Q2$

$$\begin{pmatrix} P_1 & \\ & P_2 \end{pmatrix}\begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix} + \begin{pmatrix} A_1^* & \\ & A_2^* \end{pmatrix}\begin{pmatrix} P_1 & \\ & P_2 \end{pmatrix} = -\begin{pmatrix} Q_1 & \\ & Q_2 \end{pmatrix} \preceq 0$$

- $3 \implies 1$.

□

A8

Proof.

- $3 \implies 3'$.
- $3' \implies 3$.

□

Exercise B: Implementation

B1

First, we show that $I \otimes A^* + A^\top \otimes I = V(I \otimes S^* + S^\top \otimes I)V^*$, where $V = U^* \otimes U$.

Proof.

$$\begin{aligned} V(I \otimes S^* + S^\top \otimes I)V^* &= (U^* \otimes U)(I \otimes S^* + S^\top \otimes I)(U^* \otimes U)^* \\ &= \Big[(U^* \otimes U)(I \otimes S^*) + (U^* \otimes U)(S^\top \otimes I)\Big](U \otimes U^*) \\ &= \Big[(U^*I \otimes US^*) + (U^*S^\top \otimes UI)\Big](U \otimes U^*) \\ &= (U^*I \otimes US^*)(U \otimes U^*) + (U^*S^\top \otimes UI)(U \otimes U^*) \\ &= (U^*IU \otimes US^*U^*) + (U^*S^\top U \otimes UIU^*) \\ &= (I \otimes U(US)^*) + (U^*(U^\top S)^\top \otimes I) \\ &= \Big(I \otimes (USU^*)^*\Big) + \Big((U^\top SU^{*\top})^\top \otimes I\Big) \\ &= (I \otimes A^*) + (A^\top \otimes I). \end{aligned}$$

□

Next, we prove that V is unitary:

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$$\begin{aligned} VV^* &= (U^* \otimes U)(U^* \otimes U)^* \\ &= (U^* \otimes U)(U \otimes U^*) \\ &= U^*U \otimes UU^* \\ &= I \otimes I \\ &= I. \end{aligned}$$

□

We also show that $I \otimes S^* + S^\top \otimes I$ is lower triangular.

Proof. This is apparent because S is upper triangular, thus $S^\top$ and S^* are lower triangular. Their Kronecker products with the identity matrix remain lower triangular matrices and finally the sum of these products remains lower triangular as well. □

This system can then be solved as follows: we want to solve the system

$$(I \otimes A^* + A^\top \otimes I)\,\mathrm{vec}(P) = -\,\mathrm{vec}(Q).$$

By the first proof result of this question, that yields

$$V(I \otimes S^* + S^\top \otimes I)V^* = -\,\mathrm{vec}(Q),$$

which, by virtue of the second result, becomes

$$(I \otimes S^* + S^\top \otimes I)V^* = -V^*\,\mathrm{vec}(Q).$$

The third result of this question tells us that the system on the left is lower triangular. This means it can be solved efficiently by forward substitution, leaving us with $V^*\,\mathrm{vec}(P)$. Finally, we premultiply this by V, and are left with $\mathrm{vec}(P)$.

Next, we show that this method has a computational complexity in $\mathcal{O}(n^4)$.

Proof. A rudimentary complexity analysis of this method goes as follows (some negligible complexities are not considered as they are dominated by another complexity in the same step):

- Compute the Schur decomposition of $A = USU^*$ in $\mathcal{O}(n^3)$.
- Compute $V = U^* \otimes U$ and $C = I \otimes S^* + S^\top \otimes I$ (Kronecker product, sum) in $\mathcal{O}(n^4)$ each.
- Compute $D = -V^*\,\mathrm{vec}(Q)$ (matrix-vector product between $n^2 \times n^2$ matrix and $n^2 \times 1$ vector) in $\mathcal{O}(n^4)$.
- Solve $CX = D$ (forward substitution on $n^2 \times n^2$ system) in $\mathcal{O}(n^4)$.
- Compute $\mathrm{vec}(P) = VX$ (matrix-vector product between $n^2 \times n^2$ matrix and $n^2 \times 1$ vector) in $\mathcal{O}(n^4)$.

As these computations are done sequentially, their total complexity is simply the dominant complexity of all the steps, i.e. $\mathcal{O}(n^4)$. □

B2