MATRIX THEORY: HOMEWORK 1 (v1), 22 September 2020

This homework focuses on the algebraic properties of matrices. This topic is partially covered in the first chapter of the lecture notes of the course. You may use any tools and results from this chapter, and only from that one (except for the definition of eigenvalues in Question B).

Exercise A: The Kronecker product

The Kronecker product is an operation on two matrices of arbitrary size, generalizing the outer product of two vectors. It is used in different applications, like tensor calculus, signal processing, dynamical systems, etc. In the first part of this homework, we will study some of the properties of the Kronecker product.

In this question, we let $\mathbb{F}$ be an arbitrary field (e.g., $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$).[1]

(A1) Recall the definition of the course for the Kronecker product of two matrices $A \in \mathbb{F}^{m\times n}$ and $B \in \mathbb{F}^{p\times q}$.

(A2) Is the Kronecker product associative? Is it commutative? Is the set $\mathbb{F}^{n\times n}$, equipped with the Kronecker product, a group? Justify your answers.

(A3) Show that the following property holds for all matrices A, B, C, D with compatible sizes: $(A \otimes B)(C \otimes D) = (AC \otimes BD)$. Deduce an expression for the inverse of the Kronecker product of two invertible matrices.

(A4) We define the Kronecker power $A^{\otimes k}$ by:

$$\begin{aligned} A^{\otimes 1} &:= A \\ A^{\otimes (k+1)} &:= A^{\otimes k} \otimes A = A \otimes A^{\otimes k}, \quad k = 1, 2, \ldots \end{aligned}$$

Show that $A^{\otimes k} B^{\otimes k} = (AB)^{\otimes k}$ and $(A^{\otimes k})^\top = (A^\top)^{\otimes k}$.

Using the above, show that for any vector $v \in \mathbb{R}^n$, $\|v^{\otimes k}\| = \|v\|^k$, where $\|v\| = \sqrt{v^\top v}$.

(A5) Recall the definition of the course for the determinant of a square matrix $A \in \mathbb{F}^{n\times n}$.

Using the results from the first chapter of the course, show that $\det(A \otimes I_m) = \det(A)^m$, where I_m is the identity matrix in $\mathbb{F}^{m\times m}$. Deduce $\det(A \otimes B)$.

(A6) Recall the definition of the course for the rank of a matrix $A \in \mathbb{F}^{m\times n}$. Deduce the following property: $\operatorname{rank}(A \otimes B) = \operatorname{rank}(A)\operatorname{rank}(B) = \operatorname{rank}(B \otimes A)$.

[1] See Appendix 2 in the lecture notes for the definitions of algebraic structures relevant for this course.

(A7) The *vectorization* of a matrix $A \in \mathbb{F}^{m \times n}$ is vector of mn elements obtained by stacking the columns of A:

$$\mathrm{vec}(A) = \begin{bmatrix} A_{:,1} \\ A_{:,2} \\ \vdots \\ A_{:,m} \end{bmatrix}$$

Show that $\mathrm{vec}(AXB) = (B^\top \otimes A)\,\mathrm{vec}(X)$.

Deduce a method to solve the *Sylvester* equation: $AX + XA^\top = B$ where X is the unknown.

Exercise B: The matrix exponential

Similar to the exponential of numbers, the exponential of matrices can also be defined by

$$e^A = I + \sum_{k=1}^{\infty} \frac{1}{k!} A^k$$

where $A \in \mathbb{C}^{n \times n}$. Unfortunately, not all familiar properties of the scalar exponential function carry over to the matrix exponential. For instance, $e^A e^B = e^{A+B}$ does not always hold for any $A, B \in \mathbb{C}^{n \times n}$. However, if A and B commute, meaning $AB = BA$, then we have $e^A e^B = e^B e^A = e^{A+B}$.

(B1) Suppose $\lambda \in \mathbb{C}$ is an eigenvalue of A. Show that e^λ is an eigenvalue of e^A.

(B2) Prove that for any $A \in \mathbb{C}^{n \times n}$, $\mathrm{rank}(e^A) = n$.

(B3) A matrix $A \in \mathbb{C}^{n \times n}$ is skew-Hermitian if $A = -A^*$, where A^* denotes the conjugate transpose of A. Prove that if A is skew-Hermitian, then e^A is unitary, i.e., $e^A(e^A)^* = I$.

(B4) Let $J_n(\lambda)$ be an $n \times n$ matrix that has zeros everywhere except along the diagonal and the superdiagonal, with each element of the diagonal consisting of a common complex number λ and each element of the superdiagonal consisting of a 1. Such a matrix is known as a *Jordan* block. Verify the following equations:

$$(J_n(0))^n = 0$$

and

$$e^{J_n(\lambda)} = e^\lambda \left(I + \sum_{k=1}^{n-1} \frac{1}{k!} (J_n(0))^k \right).$$

(B5) Show that for any matrices $A, B \in \mathbb{C}^{n \times n}$, $e^{A \otimes I + I \otimes B} = e^A \otimes e^B$.

Practical information

The homework solution should be written in English.
Please, send it by email to zheming.wang@uclouvain.be, julien.calbert@uclouvain.be and guillaume.berger@uclouvain.be with as object: `[LINMA2380] - Homework 1 - Group XX`, and with the same name for the pdf file containing your solution *(failure to respect these guidelines may induce negative points).*

Deadline for turning in the homework: Monday 12 October 2020 (11:59pm).

It is expected that each group makes the homework individually. If your group has problems or questions, you are welcome to contact the teaching assistants (see emails above).