Exercise A: Least square problems

A1

A2

A3

Exercise B: Low-rank approximation

B1

B2

Let $x \in \mathbb{R}^{m\times n}$ be such that $|X_{ij}| \leqslant \varepsilon$ for all $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,n\}$. Let $\|X\|_2$ be the 2-norm of X and let $\|X\|_F$ be its Frobenius norm. We show that $\|X\|_2 \leqslant \|X\|_F \leqslant \sqrt{mn}\varepsilon$.

Proof. First, we show the first inequality. We know from the lecture notes that

$$\|X\|_2 = \sigma_{\max},$$
$$\|X\|_F = \left[\sum_i \sigma_i\right]^{1/2},$$

where σ_i are the singular values of X. From this, it is immediately clear that $\|X\|_2 \leqslant \|X\|_F$.

Next, we use an equivalent form of the Frobenius norm to show the second inequality:

$$\|X\|_F = \left[\sum_{i,j} |X_{ij}|^2\right]^{1/2}.$$

Knowing that $|X_{ij}| \leqslant \varepsilon$, it is immediate that $\|X\|_F \leqslant \left[\sum_{i,j}\varepsilon^2\right]^{1/2} = \left[mn\varepsilon^2\right]^{1/2} = \sqrt{mn}\varepsilon$. This concludes the proof. □

We also give an example where these bounds are tight. Indeed, consider the matrix $X = I_1 \in \mathbb{R}^{1\times 1}$. Clearly, we have $|X_{ij}| \leqslant \varepsilon = 1$ for all i, j (only one value is possible for each). We know that the only singular value of this matrix is 1, and hence

$$\|X\|_2 = \|X\|_F = \sqrt{1\cdot 1}\varepsilon = 1.$$

B3

Exercise C: Low-rank approximation

Discussion

Bonus question