Exercise A: Least square problems

A1

A2

A3

Exercise B: Low-rank approximation

B1

For every matrix $A \in \mathbb{R}^{m\times n}$, there exist unitary transformations $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ such that

$$A = U\Sigma V^*, \quad \text{where} \quad \Sigma = \left[\begin{array}{ccc|c} \sigma_1 & & 0 & \\ & \ddots & & 0_{r\times(n-r)} \\ 0 & & \sigma_r & \\ \hline & 0_{(m-r)\times r} & & 0_{(m-r)\times(n-r)} \end{array}\right],$$

with real positive singular values $\sigma_1 \geqslant \cdots \geqslant \sigma_r > 0$.

These singular values are unique: the intuition to see this is that the singular value decomposition is computed inductively, and that the unitary matrices preserve the norm. By taking the property that $\|X\|_2 = \sigma_1$, this means that at every step of the decomposition, no matter what unitary transformations are chosen, the norm (and thus the maximal singular value of the submatrix we are working on) is the same.

Next, we show that the rank of a matrix is equal to its number of nonzero singular values.

Proof. We know that the rank of a diagonal matrix is equal to the number of its nonzero entries. We also note that in the decomposition $A = U\Sigma V$, U and V are of full rank. Therefore, $\operatorname{rank}(A) = \operatorname{rank}(\Sigma) = r$. $\square$

B2

Let $x \in \mathbb{R}^{m\times n}$ be such that $|X_{ij}| \leqslant \varepsilon$ for all $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$. Let $\|X\|_2$ be the 2-norm of X and let $\|X\|_F$ be its Frobenius norm. We show that $\|X\|_2 \leqslant \|X\|_F \leqslant \sqrt{mn}\varepsilon$.

Proof. First, we show the first inequality. We know from the lecture notes that

$$\begin{aligned} \|X\|_2 &= \sigma_{\max}, \\ \|X\|_F &= \left[\sum_i \sigma_i\right]^{1/2}, \end{aligned}$$

where σ_i are the singular values of X. From this, it is immediately clear that $\|X\|_2 \leqslant \|X\|_F$.

Next, we use an equivalent form of the Frobenius norm to show the second inequality:

$$\|X\|_F = \left[\sum_{i,j} |X_{ij}|^2\right]^{1/2}.$$

Knowing that $|X_{ij}| \leqslant \varepsilon$, it is immediate that $\|X\|_F \leqslant \left[\sum_{i,j} \varepsilon^2\right]^{1/2} = \left[mn\varepsilon^2\right]^{1/2} = \sqrt{mn}\varepsilon$. This concludes the proof. $\square$

We also give an example where these bounds are tight. Indeed, consider the matrix $X = I_1 \in \mathbb{R}^{1\times 1}$. Clearly, we have $|X_{ij}| \leqslant \varepsilon = 1$ for all i, j (only one value is possible for each). We know that the only singular value of this matrix is 1, and hence

$$\|X\|_2 = \|X\|_F = \sqrt{1\cdot 1}\varepsilon = 1.$$

B3

Exercise C: Low-rank approximation

Discussion

Bonus question