Exercise A: The Kronecker product

A1

The Kronecker product of two matrices $A \in \mathbb{F}^{m\times n}$ and $B \in \mathbb{F}^{p\times q}$ is the matrix of size $mp \times nq$ whose elements are all possible products between the elements of A and B arranged in the following way:

$$A \otimes B := \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

A2

The Kronecker product is associative. Let $C \in \mathbb{F}^{s\times t}$ be a third matrix. We show that $(A \otimes B) \otimes C = A \otimes (B \otimes C)$.

Proof.

$$\begin{aligned}(A \otimes B) \otimes C &= \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \otimes C \\ &= \begin{bmatrix} a_{11}b_{11}C & \cdots & a_{11}b_{1q}C & \cdots & a_{1n}b_{11}C & \cdots & a_{1n}b_{1q}C \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{11}b_{p1}C & \cdots & a_{11}b_{pq}C & \cdots & a_{1n}b_{p1}C & \cdots & a_{1n}b_{pq}C \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1}b_{11}C & \cdots & a_{m1}b_{1q}C & \cdots & a_{mn}b_{11}C & \cdots & a_{mn}b_{1q}C \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1}b_{p1}C & \cdots & a_{m1}b_{pq}C & \cdots & a_{mn}b_{p1}C & \cdots & a_{mn}b_{pq}C \end{bmatrix} \\ &= A \otimes (B \otimes C).\end{aligned}$$

□

The Kronecker is non-commutative; we show that $A \otimes B \neq B \otimes A$

Proof. We show a counterexample to the claim of commutativity. Let

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}, \quad , B = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}.$$

In that case, we have

$$A \otimes B = \begin{bmatrix} 0 & -2 & 0 & -3 \\ -2 & 2 & -3 & 3 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad B \otimes A = \begin{bmatrix} 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & -1 \\ -2 & -3 & 2 & 3 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

We see that $A \otimes B \neq B \otimes A$. □

Finally, the set $\mathbb{F}^{n\times n}$ equipped with the Kronecker product is a group by virtue of it being a field.

A3

Let $A \in \mathbb{F}^{m\times n}$, $B \in \mathbb{F}^{p\times q}$, $C \in \mathbb{F}^{n\times r}$, and $D \in \mathbb{F}^{q\times s}$

Proof. We simply verify that

$$
\begin{aligned}
(A \otimes B)(C \otimes D) &= \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \begin{bmatrix} c_{11}D & \cdots & c_{1r}D \\ \vdots & \ddots & \vdots \\ c_{n1}D & \cdots & c_{nr}D \end{bmatrix} \\
&= \begin{bmatrix} \sum_{k=1}^n a_{1k}c_{k1}BD & \cdots & \sum_{k=1}^n a_{1k}c_{kr}BD \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^n a_{mk}c_{k1}BD & \cdots & \sum_{k=1}^n a_{mk}c_{kr}BD \end{bmatrix} \\
&= AC \otimes BD.
\end{aligned}
$$

□

This allows us to say that (if $A \in \mathbb{F}^{n\times n}$ and $B \in \mathbb{F}^{m\times m}$ are nonsingular)

$$(A \otimes B)(A^{-1} \otimes B^{-1}) = AA^{-1} \otimes BB^{-1} = I_n \otimes I_m = I_{nm},$$

and hence that

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$$

A4

We first show the first property, P1.

Proof. By induction. The base case is trivial:

$$A^{\otimes 1}B^{\otimes 1} = AB = (AB)^{\otimes 1}.$$

Next, we assume the property holds for $k = n$, and we prove it for $k = n + 1$:

$$
\begin{aligned}
A^{\otimes k+1}B^{\otimes k+1} &= (A^{\otimes k} \otimes A)(B^{\otimes k} \otimes B) \\
&\overset{\text{A3}}{=} (A^{\otimes k}B^{\otimes k}) \otimes AB \\
&= (AB)^{\otimes k} \otimes AB \\
&= (AB)^{\otimes k+1}.
\end{aligned}
$$

□

Next, we show the second property, P2.

Proof. We start by proving an auxiliary lemma, L1.

$$(A \otimes B)^\top = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}^\top = \begin{bmatrix} a_{11}B^\top & \cdots & a_{m1}B^\top \\ \vdots & \ddots & \vdots \\ a_{1n}B^\top & \cdots & a_{mn}B^\top \end{bmatrix} = A^\top \otimes B^\top.$$

We then proceed by induction. The base case is trivial as before:

$$(A^{\otimes 1})^\top = A^\top = (A^\top)^{\otimes 1}.$$

Next, we assume the property holds for $k = n$, and we prove it for $k = n + 1$:

$$
\begin{aligned}
(A^{\otimes k+1})^\top &= (A^{\otimes k} \otimes A)^\top \\
&\overset{\text{L1}}{=} (A^{\otimes k})^\top \otimes A^\top \\
&= (A^\top)^{\otimes k} \otimes A^\top \\
&= (A^\top)^{\otimes k+1}.
\end{aligned}
$$

□

Finally, we show the following:

$$\|v^{\otimes k}\| = \|v\|^k.$$

Proof.

$$\begin{aligned}
\|v^{\otimes k}\| &= \sqrt{(v^{\otimes k})^\top v^{\otimes k}} \\
&\overset{\text{P2}}{=} \sqrt{(v^\top)^{\otimes k} v^{\otimes k}} \\
&\overset{\text{P1}}{=} \sqrt{(v^\top v)^{\otimes k}} \\
&= \sqrt{(v^\top v)^k} \\
&= \left(\sqrt{v^\top v}\right)^k \\
&= \|v\|^k,
\end{aligned}$$

where the fourth equality follows from a simplification of the Kronecker product for scalars, and the fifth equality is a property of the square root. □

A5

The determinant of a square matrix $A \in \mathbb{F}^{n\times n}$ as

$$\det(A) = \sum_{\boldsymbol{j}} (-1)^{t(\boldsymbol{j})} a_{1j_1} \cdot a_{2j_2} \dots a_{nj_n},$$

where the index vector $\boldsymbol{j}$ constitutes a permutation of $\{1, 2, \dots, n\}$, and $t(\boldsymbol{j})$ denotes the parity of each quasi-diagonal.

Next, we show that $\det(A \otimes I_m) = \det(A)^m$.

Proof.

$$\det(A \otimes I_m) = \det\left(\begin{bmatrix} a_{11}I_m & \cdots & a_{1n}I_m \\ \vdots & \ddots & \vdots \\ a_{n1}I_m & \cdots & a_{nn}I_m \end{bmatrix}\right).$$

□

From this, we can deduce that for $A \in \mathbb{F}^{n\times n}$ and $B \in \mathbb{F}^{m\times m}$, $\det(A \otimes B) = \det(A)^m \det(B)^n$.

Proof. We can write

$$\begin{aligned}
A \otimes B &= (AI_n) \otimes (I_m B) \\
&\overset{A3}{=} (A \otimes I_m)(I_n \otimes B).
\end{aligned}$$

Taking the determinant on both sides, and using the fact that $\det(AB) = \det(A)\det(B)$ (exercise 1.18 in the lecture notes), we then get

$$\begin{aligned}
\det(A \otimes B) &= \det(A \otimes I_m)\det(B \otimes I_n) \\
&= \det(A)^m \det(B)^n.
\end{aligned}$$

□

A6

The rank of a matrix $A \in \mathbb{F}^{m\times n}$ is equal to the largest size of its nonzero minors.

Next, we prove the property on the rank.

Proof. □