MATRIX THEORY: HOMEWORK 3 (v1), 3 November 2020

This homework deals with the stability of linear dynamical systems. Consider a continuous-time linear dynamical system:

$$\dot{x}(t) = Ax(t), \quad x(t) \in \mathbb{C}^n, \quad t \geq 0, \tag{1}$$

where $A \in \mathbb{C}^{n\times n}$. Note that the solution of (1) is given by $x(t) = \mathrm{e}^{At}x(0)$, where the matrix exponential has been defined in Homework 1.

Exercise A: Boundedness of trajectories and Lyapunov equation

In this section, you will be asked to show the equivalence between the following statements:

1. Every trajectory of (1) is bounded:

$$\forall\, x(0) \in \mathbb{C}^n, \quad \sup_{t\geq 0} \|x(t)\| < \infty.$$

2. All the eigenvalues of A are in the open left-hand plane, or are on the imaginary axis and are simple[1].

3. There is a positive definite Hermitian matrix satisfying the *Lyapunov equation* with A:

$$\exists\, P \succ 0 \in \mathbb{C}^{n\times n} \quad \text{s.t.} \quad A^*P + PA \preceq 0.$$

(A1) Suppose A has r Jordan blocks. Using the Jordan decomposition theorem, show that there is a change of coordinates $y = Tx$, with $T \in \mathbb{C}^{n\times n}$ invertible, such that $y(t)$ can be decomposed as follows: $y(t) = [y_1^\top(t), \dots, y_r^\top(t)]^\top$, where each $y_i(t)$, $i \in [r]$, is the trajectory of a continuous-time linear dynamical system with a Jordan block as transition matrix.

(A2) From now on, we assume that A in (1) is a Jordan block with eigenvalue λ. Using the results from Homework 1, give an explicit expression for $x_i(t)$, $i \in [n]$, as a function of $x_i(0)$, $i \in [n]$, and λ.

(A3) Define the minimal polynomial of a matrix $A \in \mathbb{C}^{n\times n}$.

We say that an eigenvalue λ of A is *simple* if it is a simple root of the minimal polynomial of A. Give a characterization of simple eigenvalues in terms of the Jordan decomposition of A.

(A4) Based on A1, A2 and A3, show the equivalence of the statements 1 and 2 in the introduction of Exercise A.

[1] See A3 for the definition of *simple* eigenvalues.

(A5) Let $D \in \mathbb{C}^{n\times n}$ be a diagonal matrix with all its eigenvalues in the complex left-hand plane (i.e., all real parts are nonpositive). Show that there exists a positive definite Hermitian matrix $P \in \mathbb{C}^{n\times n}$ such that[2] $D^*P + PD \preceq 0$.

In Theorem 5.11 in the lecture notes, it is shown that the above also holds when D is any matrix with eigenvalues in the open left-hand plane (i.e., has negative real part). We will ask you to provide an alternative proof of this result; this proof is useful for the numerical computations that we will ask you in Exercise B.

(A6) Let $A \in \mathbb{C}^{n\times n}$ be a Jordan block with eigenvalue λ in the open left-hand plane. Let $Q \in \mathbb{C}^{n\times n}$ be any positive definite Hermitian matrix. Let $B = I \otimes A^* + A^\top \otimes I$. From Homework 1, you know that $P \in \mathbb{C}^{n\times n}$ satisfies the Lyapunov equation $A^*P + PA = -Q$ if and only if if satisfies the system of equations $B\,\mathrm{vec}(P) = -\mathrm{vec}(Q)$, where $\mathrm{vec}(\cdot)$ is the vectorization operator. Show that the eigenvalues of B are equal to $\lambda^* + \lambda$. What can you deduce about the existence and uniqueness of P?

Show that if $P \in \mathbb{C}^{n\times n}$ satisfies $A^*P + PA = -Q$, then P^* also satisfies $A^*P^* + P^*A = -Q$. Deduce that there always exists a Hermitian matrix $P \in \mathbb{C}^{n\times n}$ satisfying $B\,\mathrm{vec}(P) = -\mathrm{vec}(Q)$.

Finally, to show that any Hermitian matrix $P \in \mathbb{C}^{n\times n}$ satisfying $A^*P + PA = -Q$ is positive definite, let $x(0) \in \mathbb{C}^n$ and consider the trajectory $x(t)$ of (1) starting from $x(0)$. What can you say about the comparison between $x(t)^*Px(t)$ and $x(0)^*Px(0)$? Using A2, what can you deduce about the positive definiteness of P?

(A7) Putting together A3, A5 and A6, prove that the statement 2 implies the statement 3 in the introduction of Exercise A. Finally, prove that the statement 3 implies the statement 1 in the introduction of Exercise A.

(A8) Show that, when A is real ($A \in \mathbb{R}^{n\times n}$), the statement 3 in the introduction of Exercise A is equivalent to the following statement:

3'. There is a *real* positive definite symmetric matrix satisfying the *Lyapunov equation* with A:
$$\exists\, P \succ 0 \in \mathbb{R}^{n\times n} \quad \text{s.t.} \quad A^\top P + PA \preceq 0.$$

Exercise B: Implementation

(B1) As mentioned in A6, in the case of $A \in \mathbb{C}^{n\times n}$ having all its eigenvalues in the open left-hand plane, the resolution of the Lyapunov equation $A^*P + PA = -Q$ can be approached by solving the system of equations $(I \otimes A^* + A^\top \otimes I)\mathrm{vec}(P) = -\mathrm{vec}(Q)$. The complexity of solving this system is in $\mathcal{O}(n^6)$. This complexity can be improved by considering a Schur decomposition of A.

Let $A = USU^*$ be a Schur decomposition of A. Show that $I \otimes A^* + A^\top \otimes I = V(I \otimes S^* + S^\top \otimes I)V^*$ where $V = U^\top \otimes U$. Show that V is unitary and $I \otimes S^* + S^\top \otimes I$ is lower triangular. Propose a

[2] We use the notation $Q \preceq 0$ to denote that Q is Hermitian and negative semidefinite.

method to solve $(I \otimes A^* + A^\top \otimes I)\text{vec}(P) = -\text{vec}(Q)$ in $\mathcal{O}(n^4)$ operations[3], assuming that you have the Schur decomposition of A (as you will see later in the course, the Schur decomposition of a matrix $A \in \mathbb{R}^{n\times n}$ can be computed in $\mathcal{O}(n^3)$ operations, using the QR algorithm).

(B2) You are asked to (try to) prove the boundedness of the trajectories of the systems defined by the following three matrices, using the approaches in the statements 2 and 3 in the introduction of exercise A. For the approach based on the Lyapunov equation (statement 3), you have to use the efficient algorithm that you proposed in B1 to solve the system of equations. When possible, use the identity matrix for the right-hand side term of the Lyapunov equation (i.e., for the matrix "Q"). When this is not possible, explain the difficulties you are encountering.

$$A_1 = \begin{bmatrix} 10.5 & -9.5 & -6.5 & 4.5 \\ 13.0 & -12.0 & -7.5 & 5.5 \\ 9.0 & -6.0 & -6.0 & 2.0 \\ 13.0 & -10.0 & -7.5 & 3.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 14.5 & -13.5 & -8.5 & 6.5 \\ 16.0 & -15.0 & -9.0 & 7.0 \\ 13.0 & -10.0 & -8.0 & 4.0 \\ 16.0 & -13.0 & -9.0 & 5.0 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 16.0 & -15.0 & -10.0 & 8.0 \\ 18.0 & -17.0 & -11.0 & 9.0 \\ 14.0 & -11.0 & -9.0 & 5.0 \\ 18.0 & -15.0 & -11.0 & 7.0 \end{bmatrix}.$$

For the implementation, you can use any coding language (e.g. Matlab, Julia, Python,...) you want. You can use the built-in functions of the language of your choice to compute the eigenvalues and the Schur decomposition of the matrices.

Which difficulties did you encounter in the computations for the matrices A_2 and A_3? Were you able to certify the boundedness of the trajectories of the system for those matrices, taking into account numerical inaccuracy of floating-point arithmetics?

In the next homework, we will see how the question of boundedness of trajectories of linear systems can be tackled from a third approach, namely using the Smith decomposition of polynomial matrices.

Practical information

The homework solution should be written in English.
Please, send it by email to zheming.wang@uclouvain.be, julien.calbert@uclouvain.be and guillaume.berger@uclouvain.be with as object: `[LINMA2380] - Homework 3 - Group XX`, and with the same name for the pdf file containing your solution *(failure to respect these guidelines may induce negative points)*.

Deadline for turning in the homework: Monday 23 November 2020 (11:59pm).

It is expected that each group makes the homework individually. If your group has problems or questions, you are welcome to contact the teaching assistants (see emails above).

[3]In fact, by using the symmetry of the problem, the complexity can be reduced to $\mathcal{O}(n^3)$, but this is beyond the scope of this homework.