Exercise A: Boundedness of trajectories and Lyapunov equation

A1

Proof. We use the transformation $y(t) = Tx(t) \iff x(t) = T^{-1}y(t)$, where T is the transformation matrix used in the Jordan decomposition of A. We then show that it necessarily leads to $y_i(t)$ having a Jordan block as transition matrix.

$$\begin{aligned}
\dot{y}(t) &= T\dot{x}(t) \\
&= TAx(t) \\
&= TAT^{-1}y(t) \\
&= \operatorname{diag}\{J_1(\lambda_1), \ldots, J_r(\lambda_r)\}\, y(t).
\end{aligned}$$

For the last equality, we used the definition of the Jordan form, applicable because of our particular choice of T. This is the equation of a continuous-time linear system, and each $y_i(t)$ has $J_i(\lambda_i)$ as transition matrix. □

A2

We assume that $A = J_n(\lambda)$, hence we can write

$$x(t) = e^{At}x(0) = e^{J_n(\lambda)t}x(0).$$

We can use the result of question B4 from Homework 1, which develops $e^{J_n(\lambda)}$, to obtain

$$\begin{aligned}
x(t) &= e^{J_n(\lambda)t}x(0) \\
&= e^{\lambda t}\left(I + \sum_{k=1}^{n-1} \frac{1}{k!}\left(J_n(0)\right)^k\right)^t x(0) \\
&= e^{\lambda I t}e^{J_n(0)t}x(0) \\
&= e^{\lambda t}\left(I + \sum_{k=1}^{n-1} \frac{1}{k!}\left(J_n(0)t\right)^k\right) x(0) \\
&= e^{\lambda t}\left(I + \sum_{k=1}^{n-1} \frac{1}{k!}\left(J_n(0)\right)^k t^k\right) x(0),
\end{aligned}$$

where the third and fourth equalities come from the definition of the matrix exponential. We then develop the terms in parentheses:

$$I + \sum_{k=1}^{n-1} \frac{1}{k!}\big(J_n(0)\big)^k t^k = I + \frac{t}{1!}\begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} 0 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & 0 & 1 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

$$+ \cdots + \frac{t^k}{k!}\begin{pmatrix} \overbrace{0 \; \dots \; 0}^{k \text{ zeros}} & 1 & & \\ \ddots & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & \ddots & 0 \\ & & \ddots & \vdots \\ & & & 0 \end{pmatrix} + \cdots + \frac{t^{n-1}}{(n-1)!}\begin{pmatrix} \overbrace{0 \; \dots \; 0}^{n-1 \text{ zeros}} & 1 \\ \ddots & 0 \\ \ddots & \vdots \\ & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t/1! & \dots & t^k/k! & \dots & t^{n-1}/(n-1)! \\ & \ddots & \ddots & & \ddots & \vdots \\ & & 1 & t/1! & & t^k/k! \\ & & & \ddots & \ddots & \vdots \\ & & & & 1 & t/1! \\ & & & & & 1 \end{pmatrix}.$$

We deduce from the previous expression that for $i \in [n]$,

$$x_i(t) = e^{\lambda t} \sum_{j=i}^{n} \frac{1}{(j-i)!} t^{j-i} x_j(0).$$

A3

The minimal polynomial of a matrix $A \in \mathbb{C}^{n \times n}$ is the polynomial

$$m(\lambda) = \prod_i (\lambda - \lambda_i)^{k_i^*},$$

where

$$f(J) = \operatorname{diag}\left\{ f\left(J_{k_{i_j}}(\lambda_{i_j}) \right) \right\}, \quad k_i^* = \max_{1 \leqslant j \leqslant n_i} k_{i_j},$$

and n_i is the number of Jordan blocks with eigenvalue λ_i.

Simple eigenvalues thus have the property that the size of the largest Jordan block with that eigenvalue λ_i is 1 (i.e. $k_i^* = 1$).

A4

Proof.

- $1 \implies 2$.

  We start by applying A1, to obtain a trajectory $y(t)$. If all trajectories are bounded, then that means (from applying A2 on the Jordan transition matrix for $y(t)$) that

  $$\sup_{t \geqslant 0} \|y_i(t)\| = \sup_{t \geqslant 0} \left\| e^{\lambda t} \sum_{j=i}^{n} \frac{t^{j-i}}{(j-i)!} y_j(0) \right\| < \infty.$$

  Clearly, there are two ways to satisfy this condition: either $\Re(\lambda) < 0$ (in which case the exponential dominates any polynomial to its right), or $\Re(\lambda) = 0$, and $n = i$ (otherwise, the sum becomes an unbounded polynomial which is *not* dominated by the exponential as the latter simply performs a rotation in the complex plane). We thus require that $n = i$, as in this case the trajectory keeps its length (also known as marginal stability). We finally observe that requiring $n = i$ corresponds to having the Jordan block for λ be of dimension 1×1, and thus λ must be simple.

- $2 \implies 1$.

  From A1, we know there exists a change of coordinates $y = Tx$ such that $y(t)$ can be decomposed as follows
  $$y(t) = \left[y_1(t)^\top, \ldots, y_r(t)^\top\right]^\top,$$
  where each $y_i(t)$, $i \in [r]$, is the trajectory of a continuous-time linear dynamical system with a Jordan block as transition matrix.

  From A2, one can then obtain an expression for the components $y_i(t)$. If λ_i is such that $\Re(\lambda_i) < 0$, then from A2, we clearly see that the expression $y_i(t)$ tends to zero, as the exponential decreases more rapidly than the polynomial. If λ_i is purely imaginary, i.e. $\Re(\lambda_i) = 0$, then we can assume it is simple and hence by A3 we know the size of the largest Jordan block with that eigenvalue is 1. The expression of $y_i(t)$ then simplifies to $y_i(t) = e^{\lambda t} y_i(0)$, which remains bounded as t goes to infinity.

  This shows that if all eigenvalues are either in the open left-hand plane or imaginary and simple, then $y_i(t)$ and hence $\|y(t)\| = \|Tx(t)\|$ remains bounded. Finally, by virtue of the inversibility of T, this shows that $\sup_{t \geqslant 0} \|x(t)\| < \infty$. □

A5

Proof. One can show that the matrix P must exist by proving that for $P = I_n$, the statement holds. Indeed, I_n is positive definite and Hermitian. Since D and D^* are diagonal, their sum is also diagonal. In fact,

$$D + D^* = \operatorname{diag}(\lambda_1 + \lambda_1^*, \ldots, \lambda_n + \lambda_n^*),$$

where λ_i and λ_i^* are the eigenvalues of D and D^*, respectively. We know that the imaginary parts of these eigenvalues cancel each other out, and that the real parts are the same (and are nonpositive), by virtue of the definition of the conjugate transpose. We thus get

$$D^* P + PD = D^* I_n + I_n D = D^* + D = \operatorname{diag}(\lambda_1 + \lambda_1^*, \ldots, \lambda_n + \lambda_n^*) \preceq 0,$$

which proves the existence of such a positive definite Hermitian matrix P. □

A6

Proof. First, we develop $B = I \otimes A^* + A^\top \otimes I$:

$$B = \begin{pmatrix} A^* & & \\ & \ddots & \\ & & A^* \end{pmatrix} + \begin{pmatrix} a_{11} I & \ldots & a_{1n} I \\ \vdots & \ddots & \vdots \\ a_{n1} I & \ldots & a_{nn} I \end{pmatrix}.$$

We know $A \in \mathbb{C}^{n \times n}$ is a Jordan block with eigenvalue λ, thus

$$\begin{aligned} B &= \begin{pmatrix} \left(J_n(\lambda)\right)^* & & \\ & \ddots & \\ & & \left(J_n(\lambda)\right)^* \end{pmatrix} + \begin{pmatrix} \lambda I & & & \\ I & \ddots & & \\ & \ddots & \ddots & \\ & & I & \lambda I \end{pmatrix} \\ &= \begin{pmatrix} \left(J_n(\lambda)\right)^* + \lambda I & & & \\ I & \ddots & & \\ & \ddots & \ddots & \\ & & I & \left(J_n(\lambda)\right)^* + \lambda I \end{pmatrix}. \end{aligned}$$

This can then be rewritten as

$$B = \begin{pmatrix} \big(J_n(\lambda+\lambda^*)\big)^* & & & \\ I & \ddots & & \\ & \ddots & \ddots & \\ & & I & \big(J_n(\lambda+\lambda^*)\big)^* \end{pmatrix}.$$

The matrix B is lower-triangular and therefore its eigenvalues are the diagonal elements $\lambda + \lambda^*$. Moreover, we know that $\Re(\lambda) < 0$ and so $\lambda + \lambda^* = 2\Re(\lambda) < 0$. From this, we deduce that B is negative definite and hence invertible. It follows that the system $B\operatorname{vec}(P) = -\operatorname{vec}(Q)$ has a unique solution and consequently we conclude that P exists and is unique. $\square$

Proof. Next we show that if $P \in \mathbb{C}^{n\times n}$ satisfies $A^*P + PA = -Q$ then P^* also satisfies $A^*P^* + P^*A = -Q$. This can simply be proven by taking the transpose conjugate of both sides of the equation $A^*P + PA = -Q$ that P satisfies taking into account that Q is Hermitian:

$$\begin{aligned} &\big(A^*P + PA\big)^* = (-Q)^* \\ \iff\ & P^*A + A^*P^* = -Q \end{aligned}$$

This shows that P^* satisfies $A^*P^* + P^*A = -Q$.

Combining the two last results, we deduce that there always exists a unique Hermitian matrix P satisfying $B\operatorname{vec}(P) = -\operatorname{vec}(Q)$. $\square$

Proof. Finally, we want to show that P is positive definite. We consider a trajectory $x(t)$ starting from $x(0)$. If we define $V\big(x(t)\big) = x(t)^*Px(t)$, we observe that:

$$\begin{aligned} \dot V\big(x(t)\big) &= \dot x(t)^*Px(t) + x(t)^*P\dot x(t) \\ &= x(t)^*A^*Px(t) + x(t)^*PAx(t) \\ &= x(t)^*(A^*P + PA)x(t) \\ &= x(t)^*(-Q)x(t) \\ &< 0. \end{aligned}$$

Consequently, we find that for $t > 0$ we have $x(t)^*Px(t) < x(0)^*Px(0)$. We know from A2 that $\lim_{t\to\infty} x(t) = 0$, as it decreases exponentially. We thus have that $V\big(x(t)\big)$ is always positive, and hence P is positive definite. $\square$

A7

Proof.

- $2 \implies 3$. Properties of the eigenvalues are conserved when applying similarity transformations, hence without loss of generality, we can consider A to be in its Jordan normal form, with its simple eigenvalues in block A_1 and other eigenvalues in block A_2:

  $$A = \begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix}.$$

  By A5, we then find a positive definite Hermitian matrix P_1 such that $P_1A_1 + A_1^*P_1 = -Q_1$ and by A6 we find a positive definite Hermitian P_2 such that $P_2A_2 + A_2^*P_2 = -Q_2$, with Q_1, Q_2 both positive definite matrices.

  $$PA + A^*P = \begin{pmatrix} P_1 & \\ & P_2 \end{pmatrix}\begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix} + \begin{pmatrix} A_1^* & \\ & A_2^* \end{pmatrix}\begin{pmatrix} P_1 & \\ & P_2 \end{pmatrix} = -\begin{pmatrix} Q_1 & \\ & Q_2 \end{pmatrix} = -Q \preceq 0.$$

  This is precisely statement 3, as P_1, P_2 being Hermitian guarantees the same property for P.

- $3 \implies 1$.

  We again consider $V\big(x(t)\big)$ from A6. We can write, for any trajectory $x(t)$,

$$V\big(x(t)\big) = V\big(x(0)\big) + \int_0^t \dot{V}\big(x(\tau)\big)\,\mathrm{d}\tau \leqslant V\big(x(0)\big).$$

  This means that the whole trajectory lies in $\Big\{z \mid V(z) \leqslant V\big(x(0)\big)\Big\}$, which corresponds to a bounded ellipsoid. □

A8

Proof. Clearly, if statement 3' is satisfied then so is statement 3 as real positive definite symmetric matrices are positive definite Hermitian matrices.
If statement 3 is satisfied we know we have a positive definite Hermitian matrix P such that $A^*P + PA \succeq 0$. We consider the real matrix $P_r = P + P^*$ and we observe that:

$$\begin{aligned}
A^*P_r + P_rA &= A^*(P + P^*) + (P + P^*)A \\
&= A^*P + A^*P^* + PA + P^*A \\
&= (A^*P + PA) + (A^*P + PA)^* \succeq 0
\end{aligned}$$

We know the last expression is semipositive definite as the conjugate transpose of a positive semidefinite matrix is positive semidefinite and the sum of positive semidefinite matrices is positive semidefinite. Hence, P_r is a real matrix satisfying the Lyapunov equation. □

Exercise B: Implementation

B1

First, we show that $I \otimes A^* + A^\top \otimes I = V(I \otimes S^* + S^\top \otimes I)V^*$, where $V = U^* \otimes U$.

Proof.

$$\begin{aligned}
V(I \otimes S^* + S^\top \otimes I)V^* &= (U^* \otimes U)(I \otimes S^* + S^\top \otimes I)(U^* \otimes U)^* \\
&= \Big[(U^* \otimes U)(I \otimes S^*) + (U^* \otimes U)(S^\top \otimes I)\Big](U \otimes U^*) \\
&= \Big[(U^*I \otimes US^*) + (U^*S^\top \otimes UI)\Big](U \otimes U^*) \\
&= (U^*I \otimes US^*)(U \otimes U^*) + (U^*S^\top \otimes UI)(U \otimes U^*) \\
&= (U^*IU \otimes US^*U^*) + (U^*S^\top U \otimes UIU^*) \\
&= (I \otimes U(US)^*) + (U^*(U^\top S)^\top \otimes I) \\
&= \Big(I \otimes (USU^*)^*\Big) + \Big((U^\top SU^{*\top})^\top \otimes I\Big) \\
&= (I \otimes A^*) + (A^\top \otimes I).
\end{aligned}$$

□

Next, we prove that V is unitary:

.

$$\begin{aligned}
VV^* &= (U^* \otimes U)(U^* \otimes U)^* \\
&= (U^* \otimes U)(U \otimes U^*) \\
&= U^*U \otimes UU^* \\
&= I \otimes I \\
&= I.
\end{aligned}$$

□

We also show that $I \otimes S^* + S^\top \otimes I$ is lower triangular.

Proof. This is apparent because S is upper triangular, thus $S^\top$ and S^* are lower triangular. Their Kronecker products with the identity matrix remain lower triangular matrices and finally the sum of these products remains lower triangular as well. □

This system can then be solved as follows: we want to solve the system

$$(I \otimes A^* + A^\top \otimes I)\,\mathrm{vec}(P) = -\,\mathrm{vec}(Q).$$

By the first proof result of this question, that yields

$$V(I \otimes S^* + S^\top \otimes I)V^* = -\,\mathrm{vec}(Q),$$

which, by virtue of the second result, becomes

$$(I \otimes S^* + S^\top \otimes I)V^* = -V^*\,\mathrm{vec}(Q).$$

The third result of this question tells us that the system on the left is lower triangular. This means it can be solved efficiently by forward substitution, leaving us with $V^*\,\mathrm{vec}(P)$. Finally, we premultiply this by V, and are left with $\mathrm{vec}(P)$.

Next, we show that this method has a computational complexity in $\mathcal{O}(n^4)$.

Proof. A rudimentary complexity analysis of this method goes as follows (some negligible complexities are not considered as they are dominated by another complexity in the same step):

- Compute the Schur decomposition of $A = USU^*$ in $\mathcal{O}(n^3)$.
- Compute $V = U^* \otimes U$ and $C = I \otimes S^* + S^\top \otimes I$ (Kronecker product, sum) in $\mathcal{O}(n^4)$ each.
- Compute $D = -V^*\,\mathrm{vec}(Q)$ (matrix-vector product between $n^2 \times n^2$ matrix and $n^2 \times 1$ vector) in $\mathcal{O}(n^4)$.
- Solve $CX = D$ (forward substitution on $n^2 \times n^2$ system) in $\mathcal{O}(n^4)$.
- Compute $\mathrm{vec}(P) = VX$ (matrix-vector product between $n^2 \times n^2$ matrix and $n^2 \times 1$ vector) in $\mathcal{O}(n^4)$.

As these computations are done sequentially, their total complexity is simply the dominant complexity of all the steps, i.e. $\mathcal{O}(n^4)$. □

B2

We want to test the boundednesss of trajectories defined by systems with three different matrices: A_1, A_2 and A_3 (see the statement). For this purpose, we use statement 2 and 3 of the introduction of exercise A.

Statement 2

To check whether statement 2 is satisfied or not, we looked at the eigenvalues Λ_i and Jordan form J_i of the matrices:

$$\Lambda_1 = \begin{pmatrix} -0.5 \\ -0.5 \\ -1 \\ -2 \end{pmatrix} J_1 = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -0.5 & 1 \\ 0 & 0 & 0 & -0.5 \end{pmatrix}$$

$$\Lambda_2 = \begin{pmatrix} 0 \\ -0.5 \\ -1 \\ -2 \end{pmatrix} J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\Lambda_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ -2 \end{pmatrix} J_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We observe that A_1 has eigenvalues such that $\Re(\lambda) < 0$ and hence satisfies statement 2. The matrix A_2 has eigenvalues such $\Re(\lambda) < 0$ except for one ($\lambda = 0$) which is simple (largest Jordan block associated to $\lambda = 0$ has size 1) so the statement is again verified. The matrix A_3 has an eigenvalue (λ=0) such that $\Re(\lambda) = 0$ and associated to a Jordan block of size 2, hence not simple, therefore it does not satisfied the statement. We conclude that the trajectory will remain bounded only for the two first systems.

Statement 3

The use of statement 3 is less straightforward than the previous one. To know whether it exists a positive definite Hermitian matrix P satisfying the Lyapunov equation with A_i, we can use the method described in the previous subsection to obtain P. To do so, we need to choose a matrix Q. We first start by choosing Q as being the identity matrix.
For the matrix A_1, we obtain:

$$P_1 = \begin{pmatrix} 735.2222 & -696.2622 & -398.0000 & 377.9289 \\ -696.2622 & 661.8022 & 377.2533 & -359.0156 \\ -398.0000 & 377.2533 & 216.1111 & -205.1422 \\ 377.9289 & -359.0156 & -205.1422 & 195.3400 \end{pmatrix}$$

The eigenvalues of matrix P_1 are: 1806.1, 1.4, 0.8 and 0.2. Hence the matrix P_1 is a positive definite Hermitian matrix and the statement 3 is fulfilled.
Doing the exact same thing with the matrices A_2 and A_3 does not achieve good results due to the zero eigenvalue. For example, the matrix P obtained with A_2 is:

$$P_2 = 10^{15} \begin{pmatrix} 6.3488 & -6.3488 & -3.1744 & 3.1744 \\ -6.3488 & 6.3488 & 3.1744 & -3.1744 \\ -3.1744 & 3.1744 & 1.5872 & -1.5872 \\ 3.1744 & -3.1744 & -1.5872 & 1.5872 \end{pmatrix}$$

with eigenvalues: 0 and $1.5872 \cdot 10^{16}$.
Computing $A_2^* P_2 + P_2 A_2$ gives:

$$A_2^* P_2 + P_2 A_2 = \begin{pmatrix} 88 & -40 & -24 & 8 \\ -24 & -28 & -8 & 18 \\ -24 & -8 & -8 & 6 \\ 8 & 16 & 2 & -16 \end{pmatrix} \neq -I$$

As P_2 is not positive definite and the computations are prone to numerical errors, we cannot conclude that statement 3 is satisfied for A_2 using the identity matrix as Q.
By theorem 5.11 of the course notes, we know that for every positive definite matrix Q we can find a positive definite Hermitian matrix P satisfying the Lyapunov equation if A has its eigenvalues in the open left-hand plane. This was the case for A_1. However, both A_2 and A_3 have a zero eigenvalue which prevents to use the theorem. Hence, it is not guaranteed that for all semidefinite positive Q we can find a positive definite Hermitian matrix P satisfying the Lyapunov equation. Finding a semidefinite positive matrix Q such that there is an appropriate solution P is tedious. On top of that, as we have seen with the computations using $Q = I$, numerical errors make the work even harder. We tried to generate random positive definite matrices by generating a random matrix Z and setting $Q = ZZ^*$ but that did not yield good results. Finally we note that such a matrix Q does not even exists for A_3 as it does not fulfill statement 2.