Exercise A: Boundedness of trajectories and Lyapunov equation

A1

The speed can by derived from the equality y(t) = T x(t):

$$\begin{aligned} \dot{y(t)} &= T\dot{x(t)} \\ &= TAx(t) \\ &= TAT^{-1}y(t) \\ &= diag\{J_1(\lambda_1), ..., J_r(\lambda_r)\}y(t) \end{aligned}$$

A2

We assume that $A = J_n(\lambda)$. Hence we can write :

$$x(t) = e^{At}x(0) = e^{J_n(\lambda)t}x(0)$$

We can use the result of question B4 from Homework 1 which develops $e^{J_n(\lambda)}$:

$$\begin{aligned} x(t) &= e^{J_n(\lambda)t}x(0) \\ &= e^{\lambda t}\left(I + \sum_{k=1}^{n-1}\frac{1}{k!}\big(J_n(0)\big)^k\right)^t x(0) \\ &= e^{\lambda t}e^{J_n(0)t}x(0) \\ &= e^{\lambda t}\left(I + \sum_{k=1}^{n-1}\frac{1}{k!}\big(J_n(0)t\big)^k\right)x(0) \\ &= e^{\lambda t}\left(I + \sum_{k=1}^{n-1}\frac{1}{k!}J_n^k(0)t^k\right)x(0) \end{aligned}$$

The third and fourth equalities come from the definition of the matrix exponential. We can now develop the terms in parentheses :

$$I + \sum_{k=1}^{n-1}\frac{1}{k!}J_n^k(0)t^k = I + \frac{t}{1!}\begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} 0 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & 0 & 1 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

$$+ \dots + \frac{t^k}{k!}\begin{pmatrix} \overbrace{0 \ \dots \ 0}^{k \text{ zeros}} & 1 & & \\ \ddots & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & \ddots & 0 \\ & & \ddots & \vdots \\ & & & 0 \end{pmatrix} + \dots + \frac{t^{n-1}}{(n-1)!}\begin{pmatrix} \overbrace{0 \ \dots \ 0}^{n-1 \text{ zeros}} & 1 \\ \ddots & 0 \\ \ddots & \vdots \\ & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t/1! & \dots & t/k! & \dots & t^{n-1}/(n-1)! \\ & \ddots & \ddots & & \ddots & \vdots \\ & & 1 & t/1! & & t/k! \\ & & & \ddots & \ddots & \vdots \\ & & & & 1 & t/1! \\ & & & & & 1 \end{pmatrix}$$

We deduce from the previous expression that for $i \in [n]$:

$$x_i(t) = e^{\lambda t}\sum_{j=i}^{n}\frac{1}{(j-i)!}t^{j-i}x_j(0)$$

A3

The minimal polynomial of a matrix $A \in \mathbb{C}^{n\times n}$ is the polynomial

$$m(\lambda) = \prod_i (\lambda - \lambda_i)^{k_i^*},$$

where

$$f(J) = \operatorname{diag}\left\{ f\left(J_{k_{i_j}}(\lambda_{i_j})\right)\right\}, \quad k_i^* = \max_{1\leqslant j\leqslant n_i} k_{i_j},$$

and n_i is the number of Jordan blocks with eigenvalue λ_i.

Simple eigenvalues thus have the property that the size of the largest Jordan block with that eigenvalue λ_i is 1 (i.e. $k_i^* = 1$).

A4

=>
By A1, we know there exists a change of coordinates $y = Tx$ such that $y(t)$ can be decomposed as follows: $y(t) = [y_1(t)^T, ..., y_r(t)^T]^T$,where each $y_i(t)$, $i \in [r]$, is the trajectory of a continuous-time linear dynamical system with a Jordan block as transition matrix.
By A2, we have an expression for the time course of the components $y_i(t)$.
If λ_i is such that $Re(\lambda_i) < 0$, then from A2 we clearly see that the expression $y_i(t)$ tends to zero. If λ_i is such that $Re(\lambda_i) < 0$, by A3 we know the size of the largest Jordan block with that eigenvalue is 1, hence the expression of $y_i(t)$ simplifies to $y_i(t) = e^{\lambda t} y_j(0)$ which tends to zero when t tends to infinity.
We've shown $y_i(t)$ and hence $\|y(t)\| = \|Tx(t)\|$ tends to zero.

A5

Proof. One can show that the matrix P must exist by proving that for $P = I_n$, the statement holds. Indeed, I_n is positive definite and Hermitian. Since D and D^* are diagonal, their sum is also diagonal. In fact,

$$D + D^* = \operatorname{diag}(\lambda_1 + \lambda_1^*, \ldots, \lambda_n + \lambda_n^*),$$

where λ_i and λ_i^* are the eigenvalues of D and D^*, respectively. We know that the imaginary parts of these eigenvalues cancel each other out, and that the real parts are the same (and are nonpositive), by virtue of the definition of the conjugate transpose. We thus get

$$D^*P + PD = D^*I_n + I_nD = D^* + D = \operatorname{diag}(\lambda_1 + \lambda_1^*, \ldots, \lambda_n + \lambda_n^*) \preceq 0,$$

which proves the existence of such a positive definite Hermitian matrix P. □

A6

First, we develop $B = I \otimes A^* + A^T \otimes I$:

$$B = \begin{pmatrix} A^* & & \\ & \ddots & \\ & & A^* \end{pmatrix} + \begin{pmatrix} a_{11}I & \ldots & a_{1n}I \\ \vdots & & \vdots \\ a_{n1}I & \ldots & a_{nn}I \end{pmatrix}$$

We know $A \in \mathbb{C}^{n\times n}$ is a Jordan block with eigenvalue λ:

$$\begin{aligned}
B &= \begin{pmatrix} J_n(\lambda)^* & & \\ & \ddots & \\ & & J_n(\lambda)^* \end{pmatrix} + \begin{pmatrix} \lambda I & & & \\ I & \ddots & & \\ & \ddots & \ddots & \\ & & I & \lambda I \end{pmatrix} \\
&= \begin{pmatrix} J_n(\lambda)^* + \lambda I & & & \\ I & \ddots & & \\ & \ddots & \ddots & \\ & & I & J_n(\lambda)^* + \lambda I \end{pmatrix} \\
&= \begin{pmatrix} J_n(\lambda + \lambda^*)^* & & & \\ I & \ddots & & \\ & \ddots & \ddots & \\ & & I & J_n(\lambda + \lambda^*)^* \end{pmatrix}
\end{aligned}$$

The matrix B is a lower triangular and therefore its eigenvalues are the diagonal elements : $\lambda + \lambda^*$. Moreover, we know $Re(\lambda) > 0$ and so $\lambda + \lambda^* = 2Re(\lambda) > 0$. From this we deduce that B is positive definite and hence invertible. It follows that the system $B\,\mathrm{vec}(P) = -\,\mathrm{vec}(Q)$ has a unique solution and consequently we conclude that P exists and is unique.
Next we show that if $P \in \mathbb{C}^{n\times n}$ satisfies $A^*P + PA = -Q$ then P^* also satisfies $A^*P^* + P^*A = -Q$. This can be simply proven by taking the transpose conjuguate of both sides of the equation $A^*P + PA = -Q$ that P satisfies taking into account that Q is hermitian:

$$\begin{aligned}
(A^*P + PA)^* &= (-Q)^* \\
P^*A + A^*P^* &= -Q
\end{aligned}$$

This shows that P^* satisfies $A^*P^* + P^*A = -Q$.
Combining the two last results, we deduce that there always exists a unique hermitian matrix P satisfying $B\,\mathrm{vec}(P) = -\,\mathrm{vec}(Q)$.
Finally, we want to show that P is hermitian. We consider a trajectory $x(t)$ starting from $x(0)$. If we define $V(x(t)) = x(t)^*Px(t)$, we observe that:

$$\begin{aligned}
\dot{V}(x(t)) &= \dot{x}(t)^*Px(t) + x(t)^*P\dot{x}(t) \\
&= x(t)^*A^*Px(t) + x(t)^*PAx(t) \\
&= x(t)^*(A^*P + PA)x(t) \\
&= x(t)^*(-Q)x(t) \\
&\leq 0
\end{aligned}$$

Consequently, we find that for $t > 0$ we have: $x(t)^*Px(t) < x(0)^*Px(0)$.

A7

2 => 3
Jordan form of A such that simple eigenvalues in block A_1 and other eigenvalues in block A_2:

$$\begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix}$$

By A3 we find P_1 such that $P_1A_1 + A_1^*P_1 = -Q1$ and by A6 we find P_2 such that $P_2A_2 + A_2^*P_2 = -Q2$

$$\begin{pmatrix} P_1 & \\ & P_2 \end{pmatrix}\begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix} + \begin{pmatrix} A_1^* & \\ & A_2^* \end{pmatrix}\begin{pmatrix} P_1 & \\ & P_2 \end{pmatrix} = -\begin{pmatrix} Q_1 & \\ & Q_2 \end{pmatrix} \preceq 0$$

A8

Exercise B: Implementation

B1

First, $I \otimes A^* + A^T \otimes I = V(I \otimes S^* + S^T \otimes I)V^*$ has to be demonstrated:

$$\begin{aligned}
V(I \otimes S^* + S^T \otimes I)V^* &= (U^* \otimes U)(I \otimes S^* + S^T \otimes I)(U^* \otimes U)^* \\
&= [(U^* \otimes U)(I \otimes S^*) + (U^* \otimes U)(S^T \otimes I)](U \otimes U^*) \\
&= [(U^* I \otimes US^*) + (U^* S^T \otimes UI)](U \otimes U^*) \\
&= (U^* I \otimes US^*)(U \otimes U^*) + (U^* S^T \otimes UI)(U \otimes U^*) \\
&= (U^* IU \otimes US^* U^*) + (U^* S^T U \otimes UIU^*) \\
&= (I \otimes U(US)^*) + (U^*(U^T S)^T \otimes I) \\
&= (I \otimes (USU^*)^*) + ((U^T SU^{*T})^T \otimes I) \\
&= (I \otimes A^*) + (A^T \otimes I)
\end{aligned}$$

Secondly, the demonstration of V being unitary:

$$\begin{aligned}
VV^* &= I \\
&= (U^* \otimes U)(U^* \otimes U)^* \\
&= (U^* \otimes U)(U \otimes U^*) \\
&= U^* U \otimes UU^* \\
&= I \otimes I \\
&= I
\end{aligned}$$

Thirdly, $I \otimes S^* + S^T \otimes I$ is lower triangular because S is upper triangular, thus S^T and S^* and lower triangular. Their Kronecker products with the identity matrix remain lower triangular matrices and finally their sum remains a lower triangular matrix.

Before presenting the method to solve $(I \otimes A^* + A^T \otimes I)vec(P) = -vec(Q)$, let's introduce a lemme Consider two matrices C,D partitioned in blocks of size $n_1 x n_p, n_2 x n_p, ..., n_{p-1} x n_p$ according to

$$P = \begin{pmatrix} P_1 \\ \vdots \\ P_{p-1} \end{pmatrix} \in R^{N x n_r}, D = \begin{pmatrix} D_1 \\ \vdots \\ D_{p-1} \end{pmatrix} \in R^{N x n_p}$$

where $C, D_j \in R^{n_j x n_p}$ and $N = \sum_{j=1}^{p-1} n_j$. Let $T \in R^{nxn}$ be a block triangular matrix partitioned as P and D. For any $R_2 2 \in R^{n_p x n_p}$, if P satisfies the equation $D = TP + PR_{22}^T$, then P_j, p-1, p-2, ..., 1 satisfy

$$T_{jj} P_j + P_j R_{22}^T = D_j^{\sim} \tag{1}$$

where $D_j^{\sim} = D_j - \sum_{i=j+1}^{p-1} T_{ji} P_i$.

The method to solve $(I \otimes A^* + A^T \otimes I)vec(P) = -vec(Q)$ is the following:
Since all matrices have a Schur decomposition: there exists matrices U and S such that:$A = USU^*$ where U is an orthogonal matrix and $S \in R^{nxn}$ a block-triangular matrix where

$$S = \begin{pmatrix} S_{11} & \dots & T_{1,r} \\ & \ddots & \vdots \\ & & T_{r,r} \end{pmatrix} \tag{2}$$

and $S_{jj} \in R^{n_j x n_j}$, $n_j \in \{1, 2\}$, j = 1, ..., r and $\sum_{j=1}^{r} n_j = n$.
The Lyapunov equation is multiplied from the right and left with U and U^* respectively,

$$
\begin{aligned}
-UQU^* &= UA^*PU^* + UPAU^* && (3)\\
&= UA^*U^*UPU^* + UPU^*UAU^* && (4)\\
&= SY + YS^* && (5)
\end{aligned}
$$

where $Y = U^*PU^*$.
Matrices and corresponding blocks are introduced such that

$$
-UQU^* = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, Y = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z22 \end{pmatrix}, S = \begin{pmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{pmatrix} \tag{6}
$$

where the blocks are such that $Z_{22}, C_{22}, S_{rr} = R_{22} \in R^{n_r x n_r}$ (the size of the last block of S).
This triangularized problem can be solved with backward substitution.
The following equations are obtained by separating thr blocks in the equation (5)

$$
\begin{aligned}
C_{11} &= R_{11}Z_{11} + R_{12}Z_{21} + Z_{11}R_{11}^T + Z_{12}R_{12}^T && (7)\\
C_{12} &= R_{11}Z_{12} + R_{12}Z_{22} + Z_{12}R_{22}^T && (8)\\
C_{21} &= R_{22}Z_{21} + Z_{21}R_{11}^T + Z_{22}R_{12}^T && (9)\\
C_{22} &= R_{22}Z_{22} + Z_{22}R_{22}^T && (10)
\end{aligned}
$$

The last equation of size $n_r x n_r$ can be solved explicitly since $n_r \in \{1, 2\}$ thanks to the choice of block sizes:
if $n_r = 1$, Z_{22} is scalar: $Z_{22} = \frac{C_{22}}{2R_{22}}$
if $n_r = 2$, the equation 10 can be solved easily because it is a 2x2 Lyapunov equation. It can be solved with:$vec(Z_{22} = (I \otimes R_{22} + R_{22} \otimes I)^{-1} vec(C_{22})$.
Thanks to the lemme , the now known matrix Z_{22} can be insert in the others:

$$
\begin{aligned}
C_{12}^{\sim} &= C_{12} - R_{12}Z_{22} = R_{11}Z_{12} + Z_{12}R_{22}^T && (11)\\
C_{21}^{\sim} &= C_{21}^T - R_{12}Z_{21}^T + Z_{21}^T R_{22}^T && (12)
\end{aligned}
$$

P_j can be computed explicitly from a small linear system $vec(P_j) = (I \otimes S_{jj} + R_{22} \otimes I)^{-1} vec(-Q_j^{\sim})$. By solving for j= p-1, ..., 1 for both equations of (11) and (**??**), solutions for Z_{12} and Z_{21} are obtained. Insertion of Z_{12}, Z_{21} and Z_{22} into (7) gives a new Lyapunov equation of size $n - n_p$ and the process can be repeated for the smaller matrix.

Algorithm 1:

Compute the real Schur decomposition [U, S] = schur(A) and establish $n_1, ..., n_r, S_{12}, S_{1r}, ..., S_{rr}$ with partitioning according to
Set $C = U(-Q)U^*$
Set m = n
for *k = r, ..., 1 do* **do**
  Set $m = m - n_k$
  Partition the matrix C with $C_{11}, C_{12}, C_{21}, C_{22}$ according to C = $\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$ with $C_{22} \in R^{n_k x n_k}$
  Set $R_{22} = S_{kk}$ and $R_{11} = \begin{pmatrix} S_{11} & \dots & S_{1,k-1} \\ & \ddots & \vdots \\ & & S_{k-1,k-1} \end{pmatrix}$, $R_{12} = \begin{pmatrix} S_{1,k} \\ \vdots \\ S_{k-1,k} \end{pmatrix}$
  Solve 10 for $Z_{22} \in R^{n_k x n_k}$ using or
  Compute $C_{12}^{\sim}, C_{21}^{\sim}$ using 11 and 12
  Solve 11 and 12 for $Z_{12} \in R^{m x n_k}$ and $Z_{21} \in R^{n_k x m}$ using the lemme and with p=j
  Store Y(1:m,m+(1:n_k))=Z_{12}
  Store Y(m(1:n_k), 1:m) = Z_{21}
  Store Y(m+(1:n_k), m+(1:n_k))=Z_{22}
  Set $C = C_{11} - R_{12}Z_{21} - Z_{12}R_{12}^T$
end
end **return** *solution* $P = UYU^*$

B2