Exercise A: Minimal polynomial and Smith normal form

A1

We wish to prove that $\delta_k\big(P(\lambda)\big) = \delta_k\big(Q(\lambda)\big)$, where $Q(\lambda) = M(\lambda)P(\lambda)N(\lambda)$.

Proof. We start by observing that $P(\lambda)$ can be decomposed into its Smith normal form $R(\lambda)D(\lambda)S(\lambda)$, where $R(\lambda), S(\lambda)$ are unimodular matrices, and $D(\lambda)$ is a diagonal matrix with monic diagonal elements such that $d_i(\lambda) \mid d_{i+1}(\lambda)$ for all i.

By Cauchy–Binet, we then observe that $\det P(\lambda) = \det R(\lambda) \det D(\lambda) \det S(\lambda)$, where the unimodular matrices have a nonzero constant determinant, by definition. We choose to denote the product of their determinants by α. We then know that

$$\det P(\lambda) = \alpha \prod_{i=1}^{n} d_i(\lambda).$$

Considering we are ultimately interested in the monic greatest common divisor of minors, we do not have to take the constant factor α into account. We then observe that each monic k-minor is simply a product of k terms chosen from $d_1, \ldots, d_n$. Bearing in mind that $d_i(\lambda) \mid d_{i+1}(\lambda)$ for all i, it is trivial to find that the greatest common divisor of all k-minors is $\prod_{i=1}^{k} d_i(\lambda)$. We can then write

$$\delta_k\big(P(\lambda)\big) = \prod_{i=1}^{k} d_i(\lambda).$$

Applying a similar reasoning to $Q(\lambda)$, we find that

$$\det Q(\lambda) = \det M(\lambda) \det P(\lambda) \det N(\lambda) = \det M(\lambda) \det R(\lambda) \det D(\lambda) \det S(\lambda) \det N(\lambda).$$

As, in this last expression, $M(\lambda), R(\lambda), S(\lambda), N(\lambda)$ are all unimodular, the product of their determinants is simply a nonzero constant β:

$$\det Q(\lambda) = \beta \prod_{i=1}^{n} d_i(\lambda).$$

The same observations that were made for $P(\lambda)$ concerning monicity and divisibility still hold in this case, yielding

$$\delta_k\big(Q(\lambda)\big) = \prod_{i=1}^{k} d_i(\lambda).$$

This concludes our proof that $\delta_k(P(\lambda)) = \delta_k(Q(\lambda))$. $\square$

A2

. We now prove that for all $k \in \{1, ..., n\}$, $\delta_k(P(\lambda)) = \Pi_{i=1}^{k} d_i(\lambda)$ where $d_i(\lambda)$ are the diagonal entries of $D(\lambda)$ and $M(\lambda)D(\lambda)N(\lambda)$ is a Smith decomposition of $P(\lambda)$.

Proof. From A1, we know that $\delta_k(P(\lambda)) = \delta_k(D(\lambda))$ and therefore we will consider $\delta_k(D(\lambda))$.
We first note that all sub-matrices of $D(\lambda)$ are either triangular inferior, triangular superior or null. The determinants of these sub-matrices are hence given by the product of the diagonal elements. The only sub-matrices for which the determinant is non-zero are these where the indices of the removed rows are identical to the indices of the removed columns. Because 0 has infinitely many divisors, we should therefore only consider the case where this specific case.
We denote by $\mathbf{j} = (j_1, ..., j_k)$, where $j_1 < ... < j_k$, the indices of the rows and columns that are kept. The determinant associated to $\mathbf{j}$ can be written as:

$$\det D_{\mathbf{j}}(\lambda) = d_{j_1}(\lambda)...d_{j_k}(\lambda)$$

We now use the property that $d_i(\lambda)$ divides $d_{i+1}(\lambda)$ for all $i \in \{1, ..., n-1\}$.
If we consider the ith factor $d_{j_i}(\lambda)$, we observe that $d_i(\lambda)$ divides $d_{j_i}(\lambda)$ as j_i is either equal to i or larger

than i (due to the strict inequality $j_1 < ... < j_k$). Hence $d_{j_i}(\lambda)$ can be rewritten as $a_{j_i}(\lambda)d_i(\lambda)$ for a certain $a_{j_i}(\lambda) \in \mathbb{C}[\lambda]$. The determinant can thus be developed as:

$$\det D_{\mathbf{j}}(\lambda) = a_{j_1}(\lambda)d_1(\lambda)...a_{j_k}(\lambda)d_k(\lambda) \tag{1}$$

We also observe that when $\mathbf{j} = (1, ..., k)$, we have:

$$\det D_{\mathbf{j}}(\lambda) = \Pi_{i=1}^{k} d_i(\lambda) \tag{2}$$

which is a monic polynomial as it is the product of monic polynomials.
Because $\Pi_{i=1}^{k} d_i(\lambda)$ is a divisor of (1) for any $\mathbf{j}$ and that $\delta_k(D(\lambda))$ cannot contain any additional factor due to (2), we conclude that the monic greatest common divisor of all k-minors of $D(\lambda)$ is $\Pi_{i=1}^{k} d_i(\lambda)$. Therefore, we have:

$$\delta_k(P(\lambda)) = \delta_k(D(\lambda)) = \Pi_{i=1}^{k} d_i(\lambda) \tag{3}$$

□

The diagonal entries $d_1(\lambda), ..., d_n(\lambda)$ of $D(\lambda)$ are unique and depend only on the sequence $\delta_1(P(\lambda))$,...,$\delta_n(P(\lambda))$.

Proof. We first consider the first entry $d_1(\lambda)$. From (3), we know that $d_1(\lambda) = \delta_1(\lambda)$. The notion of monic greatest common divisor leads to the unicity of $\delta_1(\lambda)$ and hence we deduce $d_1(\lambda)$ is unique as well. This constitutes the base case of the induction proof.
Next, we prove that if $d_k(\lambda)$ is unique then $d_{k+1}(\lambda)$ is also unique. Again from (3), we have that $d_{k+1}(\lambda) = \delta_{k+1}(P(\lambda))/d_k(\lambda)$. By the unicity of the notion of monic greatest common divisor and the unicity $d_k(\lambda)$, we conclude that $d_{k+1}(P(\lambda))$ is unique.
Clearly the diagonal entries $d_1(\lambda), ..., d_n(\lambda)$ can be computed using the sequence $\delta_1(P(\lambda)), ..., \delta_n(P(\lambda))$. First, we have $d_1(\lambda) = \delta_1(P(\lambda))$ and then we use $d_{k+1}(\lambda) = \delta_{k+1}(P(\lambda))/d_k(\lambda)$ for the next entries. □

There are unimodular matrices $M(\lambda), N(\lambda) \in \mathbb{C}^{n\times n}[\lambda]$ such that $Q(\lambda) = M(\lambda)P(\lambda)N(\lambda)$ if and only if $\delta_k(P(\lambda)) = \delta_k(Q(\lambda))$ for all $k \in \{1, ..., n\}$.

Proof. $\Rightarrow$ This was shown in A1.
$\Leftarrow$ From the previous proved statement we know that the diagonal entries of $D(\lambda)$ of the Smith decomposition of $P(\lambda)$ are unique and depend only on the sequence $\delta_1(P(\lambda)), ..., \delta_n(P(\lambda))$. As this sequence is the same for $P(\lambda)$ and $Q(\lambda)$, we deduce $D(\lambda)$ is the same for their Smith decompositions:

$$P(\lambda) = M_1(\lambda)D(\lambda)N_1(\lambda)$$
$$Q(\lambda) = M_2(\lambda)D(\lambda)N_2(\lambda)$$

where $M_1(\lambda), M_1(\lambda), M_1(\lambda), M_1(\lambda) \in \mathbb{C}^{n\times n}[\lambda]$ are unimodular and $D(\lambda) \in \mathbb{C}^{n\times n}[\lambda]$ diagonal.
From the first equation, we can write $D(\lambda) = M_1(\lambda)^{-1}P(\lambda)N_1(\lambda)^{-1}$ as $M_1(\lambda)$ and $N_1(\lambda)$ are unimodular and hence invertible. Next, we inject this expression of $D(\lambda)$ in the second equation and we obtain:

$$Q(\lambda) = M_2(\lambda)M_1(\lambda)^{-1}P(\lambda)N_1(\lambda)^{-1}N_2(\lambda)$$

Finally, as the inverse of a unimodular matrix is unimodular and the product of unimodular matrices is unimodular, we have $M(\lambda) = M_2(\lambda)M_1(\lambda)^{-1}$ and $N(\lambda) = N_1(\lambda)^{-1}N_2(\lambda)$ which are unimodular matrices such that $Q(\lambda) = M(\lambda)P(\lambda)N(\lambda)$.

□

A3

Let $A \in \mathbb{C}^{n\times n}$ be a Jordan block with eigenvalue λ_1. The elementary polynomials of $\lambda I - A$ are equal to: $d_i(\lambda) = 1$ for $i = 1, ..., n-1$ and $d_n(\lambda) = (\lambda - \lambda_1)^n$.

Proof. We first write the matrix $\lambda I - A$ that we call $P(\lambda)$:

$$P(\lambda) = \lambda I - A = \begin{pmatrix} \lambda - \lambda_1 & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & \lambda - \lambda_1 \end{pmatrix}$$

As we have seen the sequence $d_1(\lambda), ..., d_n(\lambda)$ can be derived by the sequence $\delta_1(P(\lambda)), ..., \delta_n(P(\lambda))$, we first derive the latter sequence. When computing each $\delta_k(P(\lambda))$, we observe that for $k < n$, a valid sub-matrix to take into account is the one which has all the -1 entries on its diagonal. We illustrate here the case where $k = n - 1$:

$$P(\lambda) = \begin{pmatrix} \lambda - \lambda_1 & \boxed{\begin{matrix} -1 & & \\ \ddots & \ddots & \\ & \ddots & -1 \end{matrix}} \\ & \lambda - \lambda_1 \end{pmatrix}$$

As this sub-matrix is triangular inferior, its determinant is equal to the product of its diagonal entries. The determinant of such matrices is therefore 1 or -1. As 1 and -1 have only one monic divisor which is 1, we conclude that $\delta_k(P(\lambda)) = 1$ for $k \in \{1, ..., n-1\}$. From the recursive formula proposed in A2, we deduce that $d_k(\lambda) = 1$ for $k \in \{1, ..., n-1\}$.
When $k = n$, the only possible sub-matrix to consider is the whole matrix which determinant is equal to the product of the diagonal entries, hence $(\lambda - \lambda_1)^n$. We deduce that $\delta_n(P(\lambda)) = (\lambda - \lambda_1)^n$ and therefore using the recursive formula proposed in A2 we obtain $d_n(\lambda) = (\lambda - \lambda_1)^n$. $\square$

A4

A5

Exercise B: Implementation

B1

Using the Jordan normal form is not numerically stable because taking limits does not commute with forming the Jordan canonical form. A simple example is the matrix $A = I_2$, approximated by $A_\varepsilon = \left[\begin{smallmatrix} 1 & \varepsilon \\ 0 & 1 \end{smallmatrix}\right]$, the latter having Jordan canonical form $J_\varepsilon = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. However, the Jordan form of A is simply $J = I_2$. We thus have

$$\lim_{\varepsilon \to 0} A_\varepsilon = A, \quad \text{but} \quad \lim_{\varepsilon \to 0} J_\varepsilon \neq J.$$

Similarly, computing the minimal polynomials yields $p_{A_\varepsilon}(\lambda) = (\lambda - 1)^2 \neq \lambda - 1 = p_A(\lambda)$.

B2