# AN ANALOGUE OF GREENBERG'S PSEUDO-NULL CONJECTURE FOR CM FIELDS

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Abstract. Let $K$ be a CM field and $K^+$ be the maximal totally real subfield of $K$. Assume that primes above $p$ in $K^+$ all split in $K$. Let $\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{P}_s, \tilde{\mathfrak{P}}_1, \tilde{\mathfrak{P}}_2, \cdots, \tilde{\mathfrak{P}}_s$ be prime ideas in $K$ above $p$, where $\tilde{\mathfrak{P}}_i$ is the complex conjugation of $\mathfrak{P}_i$. We show that there is $\mathbb{Z}_p$-extension of $K$ such that $\tilde{\mathfrak{P}}_1, \tilde{\mathfrak{P}}_2, \cdots, \tilde{\mathfrak{P}}_s$ are unramified. Such $\mathbb{Z}_p$-extension is unique if Leopoldt's conjecture holds. We try to illustrate the idea that such $\mathbb{Z}_p$-extension for CM field has similar properties as cyclotomic $\mathbb{Z}_p$-extension of a totally real field. Greenberg proved some criterion for Iwasawa invariant$\mu = \lambda = 0$ for cyclotomic $\mathbb{Z}_p$-extension of a totally real field. We will prove analogous results. We also give an analogous numerical criterion for $\mu = \lambda = 0$ by Fukuda and Komatsu.

## 1. Introduction

Let $F$ be any number field and $p$ be an odd prime. Let $F \subset F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots \subset F_\infty = \bigcup_n F_n$ be a $\mathbb{Z}_p$-extension of $F$, that is, $\mathrm{Gal}(F_n/F) = \mathbb{Z}/p^n\mathbb{Z}$ and $\mathrm{Gal}(F_\infty/F) = \mathbb{Z}_p$. Let $A_n$ be the $p$ primary part of the class group of $F_n$. Iwasawa [Iwa73] proved that there are three constants $\lambda$, $\mu$, and $\nu$ such that

$$|A_n| = p^{\mu p^n + \lambda n + \nu}$$

when $n$ is sufficiently large. These are the so called Iwasawa invariants for the $\mathbb{Z}_p$-extension $F_\infty/F$.

For each number field $F$, there is one obvious $\mathbb{Z}_p$-extension. Let $\zeta_{p^n}$ be the primitive $p^n$-th root of unity. Then $\mathbb{Q}(\zeta_{p^{n+1}})/\mathbb{Q}$ has a unique degree $p^n$ sub-extension of $\mathbb{Q}$ denoted as $\mathbb{Q}_n$. Put $\mathbb{Q}_\infty = \bigcup_n \mathbb{Q}_n$ and $F_\infty = F\mathbb{Q}_\infty$. Then $F_\infty/F$ is a $\mathbb{Z}_p$-extension of $F$ which we call the cyclotomic $\mathbb{Z}_p$-extension of $F$. Let $F_n$ be the $n$-th layer of the $\mathbb{Z}_p$-extension of $F_\infty/F$.

When $F$ is a totally real field, Greenberg [Gre76] conjectured that $\mu = \lambda = 0$ for the cyclotomic $\mathbb{Z}_p$-extension of $F$. In other words, the order of the groups $A_n$ should be bounded as $n \to \infty$ for the cyclotomic $\mathbb{Z}_p$-extension of a totally real field.

Much research has been done to study this well-known conjecture, for example, [FK86][Tay96][McC01]. Iwasawa conjectured that $\mu$ should be zero for cyclotomic $\mathbb{Z}_p$-extension for any number field, and it has been proved by Ferrero and Washington [FW79] that $\mu = 0$ for the cyclotomic $\mathbb{Z}_p$-extension for abelian number field.

### 1.1. results of the paper.

Now, instead of considering the cyclotomic $\mathbb{Z}_p$-extension of a totally real field, we will consider a certain $\mathbb{Z}_p$-extension of a CM field. Throughout the paper $p$ will be an odd prime. Let $K$ be a CM field and $K^+$ be the maximal subfield fixed by complex conjugation. Let $S^+ = \{\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_s\}$

*Date*: July 28, 2024.

Thanks to Lawrence C. Washington for interest in the topics. The authors would like to thank Jie Yang and Preston Wake for reading an early draft of the paper.

be a set containing primes of $K^+$ above $p$, and assume that $\mathcal{P}_i$ splits in $K$ as $\mathcal{P}_i\mathcal{O}_K = \mathfrak{P}_i\tilde{\mathfrak{P}}_i$ for $1 \leq i \leq s$, where $\tilde{\mathfrak{P}}_i$ is the complex conjugation of $\mathfrak{P}_i$. We write

$$S = \{\mathfrak{P}_1, \mathfrak{P}_2, \dots, \mathfrak{P}_s\}.$$

In Section 2, we prove the following theorem.

**Theorem 1.1.** *Assume that primes above $p$ in $K^+$ split in the CM field $K$. Then there is a $\mathbb{Z}_p$-extension $K_\infty/K$ unramified outside $S$. If the Leopoldt's conjecture holds, then such $\mathbb{Z}_p$-extension is unique.*

We will refer to the $\mathbb{Z}_p$-extension $K_\infty/K$ in Theorem 1.1 as the $S$-ramified $\mathbb{Z}_p$-extension of $K$. Theorem1.1 is equivalent to Theorem 2.1 in the paper. Theorem 1.1 is a generalization of Section 2 of Goto [Got06]. The difference in this paper is that we do not assume $p$ splits completely in $K$, nor do we assume $K$ is abelian. The theorem can also be viewed as an analogue of the fact that there is only one $\mathbb{Z}_p$-extension of a totally real field.

The $S$-ramified $\mathbb{Z}_p$-extensions of a CM field $K$ is similar to the cyclotomic $\mathbb{Z}_p$-extenion of $K^+$ in certain cases. For instance Fukuda and Komatsu [FK02],[FK14] give numerical evidence that the $\lambda$-invariant vanishes for $S$-ramified $\mathbb{Z}_p$-extensions of imaginary quadratic fields. Hence, we propose the following analogy of Greenberg's conjecture.

**Conjecture 1.2.** *Let $K_\infty/K$ be the $S$-ramified $\mathbb{Z}_p$-extension defined as Theorem 2.1. Then $A_n$ will be bounded as $n \to \infty$. In other words, the Iwasawa invariant $\mu = \lambda = 0$.*

We can't prove the conjecture. However, we can prove that the conjecture holds under some assumptions. From now on, we assume that $p$ is an odd prime and primes above $p$ in $K^+$ split in $K$ and all primes in $S$ are totally ramified in the $S$-ramified $\mathbb{Z}_p$-extension. They correspond to the Assumption (1) (4),(2) in the paper.

Let $i_{n,m} : \mathrm{Cl}(K_n) \to \mathrm{Cl}(K_m)$ for $m \geq n$ be the natural map between class groups induced by the inclusion. Let $H_n = \cup_{m\geq n}\mathrm{Ker}\ (i_{n,m})$.

**Theorem 1.3.** *Assume that $p$ is inert in $K^+/\mathbb{Q}$ and Leopoldt's conjecture holds for $K$. Then the following are equivalent.*

- *(a) $A_0 = H_0$.*
- *(b) $|A_n|$ is bounded as $n \to \infty$.*

The above theorem is Theorem 4.1 in the paper. Let $B_n$ be the subgroup of $A_n$ fixed by $\mathrm{Gal}(K_n/K)$. Let $D_n$ be the subgroup of $A_n$ generated by prime ideals above $S$.

**Theorem 1.4.** *Assume $p$ splits completely in $K^+/\mathbb{Q}$ and Leopoldt's conjecture holds for $K$. Then the following are equivalent.*

- *(a) $B_n = D_n$ for all sufficiently large $n$.*
- *(b) $|A_n|$ is bounded as $n \to \infty$.*

The above theorem is the Theorem 5.1 in the paper. In fact, the above two theorems are analogous to Greenberg's results [Gre76] for cyclotomic $\mathbb{Z}_p$-extension of totally real fields. The proof is also analogous to Greenberg's proof.

Let $E(L) := \mathcal{O}_L^*$ be the group of units of $\mathcal{O}_L$ for a number field $L$. Let $K$ be a CM field. Next, we compare $S$-ramified $\mathbb{Z}_p$-extension $K_\infty/K$ and cyclotomic $\mathbb{Z}_p$-extension $K_\infty^+/K^+$. Let $K_n$ be the $n$th layer of $K_\infty/K$ and $K_n^+$ be the $n$th layer of $K_\infty^+/K^+$. Let $\mathrm{N}_n$ be the norm map from field $K_n$ to $K$ or $K_n^+$ to $K^+$.

**Proposition 1.5.** *Assume $p$ splits completely in $K/\mathbb{Q}$ and Leopoldt's conjecture holds for $K$.*

$$E(K)/(N_n(K_n^*) \cap E(K)) \cong E(K^+)/(N_n((K_n^+)^*) \cap E(K^+))$$

The proposition is Proposition 6.4 in the paper. The proposition is interesting because $K_n$ and $K_n^+$ are globally unrelated, but locally they are similar. That's the key idea for us to prove the proposition.

There are many kinds of numerical criterion for $\mu = \lambda = 0$ for cyclotomic $\mathbb{Z}_p$-extension for a real quadratic field. We give a similar numerical criterion for the $S$-ramified $\mathbb{Z}_p$-extension of imaginary biquadratic fields. Let $m, d \in \mathbb{Z}^+$ that are squarefree and coprime. Denote $k = \mathbb{Q}(\sqrt{-m})$, $F = \mathbb{Q}(\sqrt{d})$, $K = Fk$, and $\varepsilon$ to be the fundamental unit for $K$. Suppose that $p$ splits completely in $K$, with $p\mathcal{O}_k = \mathfrak{p}\tilde{\mathfrak{p}}$ and $\mathfrak{p}\mathcal{O}_K = \mathfrak{P}\bar{\mathfrak{P}}$. Take $S = \{\mathfrak{P}, \bar{\mathfrak{P}}\}$.

**Theorem 1.6.** *Suppose $p$ doesn't divide the class number of $K$, and that $r$ is the smallest positive integer such that*

$$\varepsilon^{p-1} \equiv 1 \pmod{\bar{\mathfrak{P}}^r}.$$

*If $N_{r-1}(E_{r-1}) = E_0$, then $\mu = \lambda = 0$ for the $S$-ramified $\mathbb{Z}_p$-extension of $K_\infty/K$.*

The theorem is Theorem 6.7 in the paper. It can be viewed as an analogous numerical criterion of Fukuda and Komatsu [FK86].

Here is the structure of the paper. In the section 2, we prove Theorem 2.1. In the section 3, we prove some properties for $S$-ramified $\mathbb{Z}_p$-extension and introduce some assumptions. In section 4, we deal with the case when $p$ is inert in $K^+/\mathbb{Q}$. In the section 5, we deal with the case when $p$ splits completely in $K^+/\mathbb{Q}$. In the section 6, we compare ambiguous class group between $S$-ramified $\mathbb{Z}_p$-extension of $K$ and cyclotomic $\mathbb{Z}_p$-extension of $K^+$. We also give a numerical criterion for $\mu = \lambda = 0$ for biquadratic fields in the section.

1.2. **Potential directions.** There are other research directions to study the Conjecture 1.2.

(i) There are many other criterion developed to study Greenberg's conjecture. In particular, there are many simple criterion relating to real quadratic fields [Tay96][FK86]. It would be great to generalize these criterion to study the conjecture 1.2 for the biquadratic field case.

(ii) Papers [FK02] [FK14] only calculate the examples for the $S$-ramified $\mathbb{Z}_p$-extension of $K_\infty/K$ defined by theorem 2.1 for imaginary quadratic field $K$. All calculated examples have $\lambda = 0$. Could we calculate more examples for the general CM field $K$?

(iii) Papers [Gil87] and [Sch87] proved that $\mu = 0$ for such $S$-ramified $\mathbb{Z}_p$-extension of $K_\infty/K$ when $K$ is imaginary quadratic field. Ferrero-Washington [FW79] proved that $\mu = 0$ for cyclotomic $\mathbb{Z}_p$-extension of the abelian number field. Could we adapt their argument to prove that $\mu = 0$ for $S$-ramified $\mathbb{Z}_p$-extension when CM field $K$ is abelian?

Finally, we provide another analogous result in this subsection. Let $F$ be a totally real field and $F_\infty/F$ be the cyclotomic $\mathbb{Z}_p$-extension of $F$. Let $F_n$ be the $n$-th layer of $F_\infty/F$. Let $M$ be the maximal pro-$p$ abelian extension of $F$ unramified outside primes above $p$. We know that $\mathrm{Gal}(M/F_\infty)$ is a finite group if Leopoldt's conjecture holds.

We say $a \sim b$ if two numbers $a, b$ have the same $p$ adic valuation. The following theorem is in the appendix of [Coa77].

**Theorem 1.7** (Coates [Coa77]). *Under the assumption of Leopoldt's conjecture, we have*

$$\# Gal(M/F_\infty) \sim \frac{w_1(F(\mu_p))h_F R_p(F)\prod_{\mathcal{P}|p}(1-(N\mathcal{P})^{-1})}{\sqrt{\Delta_{F/\mathbb{Q}}}}$$

*Here $\mu_p$ is the group of $p$th roots of unity, $w_1(F(\mu_p))$ is the number of roots of unity of $F(\mu_p)$ and $h_F$ is the class number of $F$ and $R_p(F)$ is the $p$-adic regulator of $F$ and $N\mathcal{P}$ is the absolute norm of $\mathcal{P}$ and $\Delta_{F/\mathbb{Q}}$ is the discriminant of $F$.*

Now, we begin to state the analogous result of Coates. Let $F$ be a totally real field with $[F:\mathbb{Q}] = d$, $k$ an imaginary quadratic field, and $p \geq 3$ a prime that splits as $p\mathcal{O}_k = \mathfrak{p}\tilde{\mathfrak{p}}$. Assume $p$ doesn't divide the class number of $k$. Let $k \subseteq k_1 \subseteq \cdots \subseteq k_\infty$ be the unique non-cyclotomic $\mathbb{Z}_p$-extension of $k$ unramified outside $\mathfrak{p}$. Denote $K = kF$ and $K \cap k_\infty = k_e$. Define $K_n = k_{n+e}F$, $K_\infty = k_\infty F$. Hence, $K_n$ is the $n$-th layer of the $\mathbb{Z}_p$ extension $K_\infty/K$. We know that $K_\infty/K$ is the $S$-ramified $\mathbb{Z}_p$-extension, where $S$ is the set of prime above $\mathfrak{p}$ in $K$.

Let $M$ be the maximal pro-$p$ abelian extension of $K$ unramified outside $S$.

**Theorem 1.8** (Coates and Wiles [CW77a]). *Under the assumption of Leopoldt's conjecture, we have*

$$\# Gal(M/K_\infty) \sim \frac{p^{e+1}h_K R_p(K)\prod_{\mathfrak{P}|\mathfrak{p}}(1-(N\mathfrak{P})^{-1})}{\nu_K\sqrt{\Delta_{K/k}}}$$

*Here $e$ is an integer defined by $K \cap k_\infty = k_e$, $h_K$ is the class number of $K$, $R_p(K)$ is the $p$-adic regulator of $K$, $N\mathfrak{P}$ is the absolute norm of $\mathfrak{P}$, $\nu_K$ is the order of the group of $p$ power root of unity in $K$ and $\Delta_{K/k}$ is the relative discriminant of $K$ over $k$.*

*Remark* 1.9. The value on the right-hand side of both theorems can be interpreted as a $p$-adic residue for a function derived from the characteristic polynomial of Iwasawa module $X_\infty$ in a natural way. See the appendix of [Coa77].

*Remark* 1.10. Coates and Wiles proved the above theorem from a different motivation. The method of proof was also used in their paper about the conjecture of Birch and Swinnerton-Dyer [CW77b].

We hope comparing these two theorems can give the reader a new view.

## 2. Uniqueness of $S$-Ramified $\mathbb{Z}_p$-Extensions

Let $K$ be a CM field and $K^+$ be the maximal subfield fixed by complex conjugation. Let $S^+ = \{\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_s\}$ be the set of primes of $K^+$ above $p$. Assume that each of the primes above $p$ in $K^+$ split in $K$. Write $\mathcal{P}_i\mathcal{O}_K = \mathfrak{P}_i\tilde{\mathfrak{P}}_i$ for $1 \leq i \leq s$, where $\tilde{\mathfrak{P}}_i$ is the complex conjugation of $\mathfrak{P}_i$, and set $S = \{\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_s\}$. The following theorem can be viewed as the analogue of the fact that there is a unique $\mathbb{Z}_p$-extension of a totally real field for which Leopoldt's conjecture holds[Was97, Theorem 13.4]. The method of the proof is similar to the proof of Theorem 13.4 in Washington's book [Was97].

**Theorem 2.1.** *Let $T$ be the maximum abelian extension of $K$ unramified outside $S$. Then there is a surjective homomorphism $Gal(T/K) \to \mathbb{Z}_p^{1+\delta}$ with finite kernel, where $\delta$ is the Leopoldt defect (see [Was97, Theorem 13.4 page 266]). In particular, if Leopolt's conjecture holds for $K$, then $\delta = 0$ and there is a unique $\mathbb{Z}_p$-extension contained in $T$.*

*Proof.* Let $T$ be the maximal abelian extension of $K$ which is unramified outside $S$. Let $\mathbb{A}_K^*$ be the group of idèles of $K$. By class field theory, there is a closed subgroup $R \subset \mathbb{A}_K^*$ such that

$$\mathrm{Gal}(T/K) \cong \mathbb{A}_K^*/R$$

Let $U_v$ be the local unit group at a place $v$ of $K$ and $U_v = K_v^*$ if $v$ is an archimedean place. Define

$$U' = \prod_{i=1}^{s} U_{\mathfrak{P}_i}, \qquad U'' = \prod_{v \notin S} U_v, \qquad U = U' \times U''$$

We can view $U'$ as a subgroup of $\mathbb{A}_K^*$ by placing a 1 in each component outside of $S$. We can view $U''$ as a subgroup of $\mathbb{A}_K^*$ in a similar way. Let $W = \overline{K^*U''}$ be the closure of $K^*U''$ inside $\mathbb{A}_K^*$. Since $T$ is unramified outside $S$, we have $W \subset R$. Since $T$ is maximal, we must have $W = R$. Thus $\mathrm{Gal}(T/K) \cong \mathbb{A}_K^*/W$. Let $H$ be the Hilbert class field of $K$. Then a similar argument shows

$$\mathrm{Gal}(H/K) = \mathbb{A}_K^*/(K^*U).$$

We have $\mathrm{Gal}(T/H) = K^*U/W = U'W/W \cong U'/(U' \cap W)$. Let $U_{1,\mathfrak{P}_i}$ be the group of local units congruent to 1 modulo $\mathfrak{P}_i$. Put $U_1 = \prod_{i=1}^{s} U_{1,\mathfrak{P}_i}$. Then

$$U' = U_1 \times (\text{finite group})$$

Hence

$$\mathrm{Gal}(T/H)/(\text{finite group}) \cong U_1(U' \cap W)/(U' \cap W) = U_1/(U_1 \cap W)$$

Let $E_1$ be the group of units in $K$ congruent to 1 modulo the primes in $S$. Then we can embed $E_1$ in $\mathbb{A}_K^*$ by

$$\varphi : E_1 \hookrightarrow U_1 \subset \mathbb{A}_K^*.$$

In a moment we will prove Lemma 2.2, which implies

$$U_1 \cap W = U_1 \cap \overline{K^*U''} = \overline{\varphi(E_1)}.$$

Let $E_1(K^+)$ be the group of units in $K^+$ congruent to 1 modulo the primes in $S^+$. We have

$$E_1(K^+) \subset E_1 \subset \mathcal{O}_K^*$$

Since $\mathrm{Rank}_{\mathbb{Z}}\mathcal{O}_K^* = \mathrm{Rank}_{\mathbb{Z}}E_1(K^+) = [K:\mathbb{Q}]/2-1$, the index of $E_1(K^+)$ in $E_1$ is finite. Hence, the index of $\overline{E_1(K^+)}$ in $\overline{E_1}$ is finite. Assume that $\mathrm{Rank}_{\mathbb{Z}_p}(\overline{E_1(K^+)}) = [K:\mathbb{Q}]/2-1-\delta$ and $\delta \geq 0$. Hence

$$\overline{\varphi(E_1)} \cong \mathbb{Z}_p^{[K:\mathbb{Q}]/2-1-\delta} \times (\text{finite group}).$$

Recall that [Was97, Page 75] Leopoldt's conjecture predicts that $\delta = 0$. By [Was97, Prop 5.7], we know

$$U_1 \cong (\text{finite group}) \times \mathbb{Z}_p^{[K:\mathbb{Q}]/2}.$$

Therefore

$$U_1/(U_1 \cap W) = (\text{finite group}) \times \mathbb{Z}_p^{1+\delta}.$$

Hence

$$\mathrm{Gal}(T/H) = (\text{finite group}) \times \mathbb{Z}_p^{1+\delta}.$$

Since $\mathrm{Gal}(H/K) \cong \mathrm{Cl}(K)$ is a finite group,

$$\mathrm{Gal}(T/K)/\mathbb{Z}_p^{1+\delta} \cong (\text{finite group}).$$

We will also prove Lemma 2.3, which shows there is a finite group such that

$$\mathrm{Gal}(T/K)/(\text{finite group}) \cong \mathbb{Z}_p^{1+\delta}.$$

Let $\tilde{K}$ be the compositum of all $\mathbb{Z}_p$-extension of $K$ unramified outside $S$. The fixed field of this finite group must be $\tilde{K}$ and $\mathrm{Gal}(\tilde{K}/K) \cong \mathbb{Z}_p^{1+\delta}$. If Leoplodt's conjecture holds for $K$, then Leoplodt's conjecture holds for $K^+$. Hence $\delta = 0$, which implies there is a unique $\mathbb{Z}_p$-extension of $K$ unramified outside $S$.

□

**Lemma 2.2.** $U_1 \cap W = U_1 \cap \overline{K^*U''} = \overline{\varphi(E_1)}$

*Proof.* Take $\varepsilon \in E_1$. Then $\varphi(\varepsilon) \in U_1$ and

$$\varphi(\varepsilon) = \varepsilon\frac{\varphi(\varepsilon)}{\varepsilon}.$$

We have $\varepsilon \in K^*$ and $\frac{\varphi(\varepsilon)}{\varepsilon} \in U''$. Hence $\overline{\varphi(E_1)} \subset U_1 \cap W$.

Recall that in a topological space, the closure generated by a set $V$ is the intersection of closed subsets containing $V$. Define $U_{n,\mathfrak{P}_i} = \{x \in U_{\mathfrak{P}_i} | x \equiv 1 \mod \mathfrak{P}_i^n\}$ and $U_n = \prod_{i=1}^s U_{n,\mathfrak{P}_i}$. Then

$$W = \overline{K^*U''} = \bigcap_n K^*U''U_n$$

$$\varphi(E_1) = \bigcap_n \varphi(E_1)U_n$$

It suffices to show that

$$U_1 \cap K^*U''U_n \subset \varphi(E_1)U_n$$

Take $x \in K^*$, $u'' \in U''$ and $u \in U_n$ such that $xu''u \in U_1$. Then $xu'' \in U_1$. At primes of $S$, we have $u'' = 1$, so $x$ is a local unit. At the primes outside of $S$, we have that $u''$ is a unit, hence $x$ is also a local unit. Therefore, $x$ is a local unit everywhere which implies $x$ is a global unit. At the primes of $S$ we have $xu'' = x$, and at the primes outside of $S$ we have $xu'' = 1$. This exactly means that $\varphi(x) = xu''$. Hence

$$xu''u \in \varphi(E_1)U_n.$$

This completes the proof of the lemma. □

**Lemma 2.3.** *Let $J$ be a profinite abelian group and $\mathbb{Z}_p^r \subset J$. Assume that $J/\mathbb{Z}_p^r$ is a finite group. Then there exists a finite group $T$ such that*

$$J/T \cong \mathbb{Z}_p^r$$

*Proof.* Write $G = J/\mathbb{Z}_p^r$ and denote $N$ to be the size of $G$. Then

$$N\mathbb{Z}_p^r \subset NJ \subset \mathbb{Z}_p^r$$

Hence $NJ \cong \mathbb{Z}_p^r$. Putting $J[N] = \{x \in J | Nx = 0\}$, we have

$$J/J[N] \cong NJ \cong \mathbb{Z}_p^r$$

$$x \to Nx$$

In fact, $\#J[N] \leq N$ is finite by the snake lemma.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & J[N] \arrow[r] \arrow[d] & G[N] \arrow[d] & \\
0 \arrow[r] & \mathbb{Z}_p^r \arrow[r] \arrow[d, "N"] & J \arrow[r] \arrow[d, "N"] & G \arrow[r] \arrow[d, "N"] & 0 \\
0 \arrow[r] & \mathbb{Z}_p^r \arrow[r] & J \arrow[r] & G \arrow[r] & 0
\end{tikzcd}$$

□

## 3. Analogue of Greenberg's Criterion For $S$-Ramified $\mathbb{Z}_p$-Extensions of CM Fields

Let $K$ be a CM field. Throughout the paper, we will assume that

(1) the primes above $p$ in $K^+$ all split in $K$.

(2) assume $p$ is an odd prime.

and we consider the $\mathbb{Z}_p$-extension $K_\infty/K$ unramified outside $S$ which exists by Theorem 2.1. It is unique if the Leopoldt's conjecture holds for $K$. We call such noncyclotomic $\mathbb{Z}_p$-extension of CM field $K$ as $S$-ramified $\mathbb{Z}_p$-extension.

We will show that such $S$-ramified $\mathbb{Z}_p$-extensions $K_\infty/K$ for CM field $K$ have similar properties as cyclotomic $\mathbb{Z}_p$-extensions of totally real fields. We will demonstrate this idea by providing a series of analogous results of Greenberg [Gre76], the proofs of which closely follow the arguments found in that paper.

In the case of the cyclotomic $\mathbb{Z}_p$-extension of totally real fields, any prime above $p$ will become totally ramified starting from some higher layer. However, we do not know whether the prime inside $S$ is ramified in the $S$-ramified $\mathbb{Z}_p$ extension $K_\infty/K$. We make the following assumption for the whole paper.

(3) For any $S$-ramified $\mathbb{Z}_p$-extension $K_\infty/K$, assume that all primes in $S$ are ramified in $K_\infty/K$

Let $K_n$ be the $n$th layer of the $S$-ramified $\mathbb{Z}_p$-extension $K_\infty/K$. Let $A_n$ denote the $p$-primary part of the class group of $K_n$, and let $\sigma$ be a topological generator of $\mathrm{Gal}(K_\infty/K) \cong \mathbb{Z}_p$. Recall that for each prime $\mathcal{P}_i$ of $K^+$ we assume $\mathcal{P}_i\mathcal{O}_K = \mathfrak{P}_i\tilde{\mathfrak{P}}_i$ and denote $S = \{\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_s\}$. Define

$$B_n := \{c \in A_n \mid c^\sigma = c\} = A_n^\sigma$$

as in [Gre76]. Greenberg[Gre76, Proposition 1] showed that $|B_n|$ is bounded for the cyclotomic $\mathbb{Z}_p$-extension of a totally real field assuming Leopoldt's conjecture. We have the similar result.

**Proposition 3.1.** *Suppose that Leopoldt's conjecture holds for $K$. Then $|B_n|$ is bounded as $n \to \infty$ for the unique $S$-ramified $\mathbb{Z}_p$-extension as in Theorem 2.1.*

*Proof.* Let $T$ be the maximal abelian extension of $K$ unramified outside of $S$. By Theorem 2.1 we have that $\mathrm{Gal}(T/K_\infty) < \infty$. Let $L'_n$ be the maximal pro-$p$ abelian extension of $K_n$ that is unramified over $K_n$. Then $A_n \cong \mathrm{Gal}(L'_n/K_n)$. Let $L_n$ be the maximal pro-$p$ abelian extension of $K$ that is unramified over $K_n$. Then $A_n^{\sigma-1} \cong \mathrm{Gal}(L'_n/L_n)$. Then $[L_n : K_n] = [A_n : A_n^{\sigma-1}] = |B_n|$. When $n$ is large enough, $K_\infty/K_n$ is totally ramified. Thus $L_n \cap K_\infty = K_n$. Hence when $n$ is large enough, we have $|B_n| = [L_n : K_n] = [L_nK_\infty : K_\infty] \leq [T : K_\infty] \leq \infty$. $\square$

Let $D_n$ be the subgroup of $A_n$ which consists of ideal classes that contain a product of prime ideals above the primes of $S$.

*Remark* 3.2. In [Gre76], Greenberg defines $D_n$ to be the subgroup of $A_n$ which consists of ideal classes that contain a product of prime ideals above $p$. In that situation, all primes above $p$ are ramified in the cyclotomic $\mathbb{Z}_p$-extension. In our case, only the primes in $S$ are ramified in the $S$-ramified $\mathbb{Z}_p$-extension.

Denote $e$ to be the smallest positive integer such that the primes of $K_e$ above the primes of $S$ are totally ramified in $K_\infty/K_e$.

**Corollary 3.3.** *Let $[\alpha] \in D_0$. Then the ideal $\alpha$ will become principal in $K_m$ when $m$ is large enough.*

*Proof.* Take $\mathfrak{P}_i \in S$. Let $\Omega_{i,n}$ be the product of prime above $\mathfrak{P}_i$ in $K_n$. Then $\Omega_{i,n}\mathcal{O}_{K_m} = \Omega_{i,m}^{p^{m-n}}$ when $m \geq n \geq e$ since $K_m/K_n$ is totally ramified at primes above $S$. Assume $\alpha = \prod_i \mathfrak{P}_i^{s_i}$ for some integer $s_i$. Then $\alpha\mathcal{O}_{K_n}$ is a product of $\Omega_{i,n}$. We have $\alpha\mathcal{O}_{K_m} = b_m^{p^{m-n}}$ for some ideal $b_m$ in $K_m$ and $b_m$ is a product of $\Omega_{i,m}$. Hence, $b_m \in B_m$. By Proposition 3.1, we know $B_m$ is bounded. Hence $\alpha\mathcal{O}_{K_m}$ become principal when $m$ is large enough. $\square$

The same argument can show the following result when $K_\infty/K$ is totally ramified at all primes in $S$.

**Corollary 3.4.** *Let $[\alpha] \in D_n$ and $e = 0$. Then the ideal $\alpha$ will become principal in $K_m$ when $m \geq n$ is large enough.*

*Proof.* Let $\mathfrak{P}_{i,n}$ be the prime of $K_n$ that lies above $\mathfrak{P}_i \in S$. Then $\mathfrak{P}_{i,n}\mathcal{O}_{K_m} = \mathfrak{P}_{i,m}^{p^{m-n}}$. We may assume that $\alpha = \prod_i \mathfrak{P}_{i,n}^{s_i}$ for some integer $s_i$. Then $\alpha\mathcal{O}_{K_m} = \prod_i \mathfrak{P}_{i,n}^{s_i}\mathcal{O}_{K_m} = \prod_i \mathfrak{P}_{i,m}^{s_ip^{(m-n)}}$ for $m \geq n$. On the other hand, $[\prod_i \mathfrak{P}_{i,m}^{s_i}] \subset B_m$ and the size $B_m$ is bounded as $m \to \infty$. Hence $\alpha\mathcal{O}_{K_m}$ becomes principal when $m$ is large enough. $\square$

Let $i_{n,m} : \mathrm{Cl}(K_n) \to \mathrm{Cl}(K_m)$ for $m > n$ be the map induced by natural inclusion. Let $\mathrm{N}_{m,n} : \mathrm{Cl}(K_m) \to \mathrm{Cl}(K_n)$ for $m > n$ be the norm map. Put $H_{n,m} = \mathrm{Ker}\ (i_{n,m})$. Because $\mathrm{N}_{m,n} \circ i_{n,m} = p^{m-n}$, we have that $H_{n,m} \subset A_n$. Put $H_n = \bigcup_{m \geq n} H_{n,m}$. Corollary 3.3 says that $D_0 \in H_0$. Corollary 3.4 says that $D_n \subset H_n$ when $e = 0$. Greenberg [Gre76, Proposition 2] proved that the size of $A_n$ is bounded as $n \to \infty$ for the cyclotomic $\mathbb{Z}_p$-extension of a totally real field if and only if $H_n = A_n$ for all $n$. We have the following similar result.

**Proposition 3.5.** *We have that $|A_n|$ is bounded if and only if $H_n = A_n$ for all $n$ for the $S$-ramified $\mathbb{Z}_p$-extension defined as Theorem 2.1.*

*Proof.* Iwasawa proved that $|H_n|$ is bounded [Iwa73, Theorem 10 on page 264], so $|A_n|$ is bounded if $A_n = H_n$ for all $n$.

Now, assume $|A_n|$ is bounded. Let $L'_n$ be the maximal pro-$p$ abelian extension of $K_n$ that is unramified over $K_n$. Then $A_n \cong \mathrm{Gal}(L'_n/K_n)$. When $m \geq n \geq e$, the field extension $K_m/K_n$ is totally ramified at primes above $S$. It implies that $K_m \cap L'_n = K_n$. Hence the restriction map from $\mathrm{Gal}(L'_m/K_m) \to \mathrm{Gal}(L'_n/K_n)$ is surjective. Therefore, the norm map $\mathrm{N}_{m,n} : A_m \to A_n$ is surjective for $m \geq n \geq e$. Since we assume $|A_n|$ is bounded as $n \to \infty$, we have $\mathrm{N}_{m,n}$ is an isomorphism when $m \geq n \geq n_0$ for some sufficiently large integer $n_0$.

Take $c \in A_n$. Define $c_r := i_{n,r}(c)$ where $r \geq n_0$. Take $m$ large enough such that $c_r^{p^{m-r}} = 1$. Then $\mathrm{N}_{m,r}(i_{r,m}(c_r)) = c_r^{p^{m-r}} = 1$. We know $\mathrm{Ker}\ \mathrm{N}_{m,r} = 1$ since $m \geq r \geq n_0$. It implies that $i_{n,m}(c) = i_{r,m}(c_r) = 1$. Hence $c \in H_n$ by definition. □

To simplify the argument in the paper, we made one more assumption in the following sections.

$$\text{Assume that } K_\infty/K \text{ is totally ramified at all primes in } S. \tag{4}$$

In other words, we further assume $e = 0$. The assumption (4) is equivalent to $e = 0$ and assumption (3). One big advantage of the assumption (4) is that $D_n \subset B_n$ under the assumption.

## 4. The Case Where $p$ is Inert in $K^+$

In this section, we assume that $p$ remains prime in $K^+$. In other words, $p\mathcal{O}_K^+ = \mathcal{P}$ is a prime ideal. Hence there are only two primes $\mathfrak{P}$ and $\tilde{\mathfrak{P}}$ above $p$ in $K$, and $S = \{\mathfrak{P}\}$. We still keep the assumption (1), (4), (2) as in previous section.

**Theorem 4.1.** *Suppose that the odd prime $p$ is inert in $K^+/\mathbb{Q}$ and Leopoldt's conjecture holds. Let $K_\infty/K$ be the $S$-ramified $\mathbb{Z}_p$-extension defined by theorem 2.1. Assume $K_\infty/K$ is totally ramified at all prime in $S$. With the same notation as before, the following statements are equivalent:*

*(a)* $A_0 = H_0$.
*(b)* $|A_n|$ *is bounded as* $n \to \infty$.

*Remark* 4.2. Greenberg's Theorem 1 in [Gre76] states a similar criterion for cyclotomic $\mathbb{Z}_p$-extension of a totally real field when $p$ remains prime.

Before we prove the Theorem 4.1, let us recall some well-known formulas for the order of the ambiguous and strong ambiguous class groups. Let $L/F$ be a cyclic extension of number fields, and $\sigma$ a generator of the Galois group $\mathrm{Gal}(L/F)$. We call an ideal class $[c] \in \mathrm{Cl}(L)$ an ambiguous class if $[c]^\sigma = [c]$. We call an ideal class $[c] \in \mathrm{Cl}(L)$ a strongly ambiguous class if $c^\sigma = c$, that is

$$\mathrm{Am}(L/F) := \{[c] \in \mathrm{Cl}(L) | [c]^\sigma = [c]\}$$

$$\mathrm{Am}(L/F)_{st} := \{[c] \in \mathrm{Cl}(L) | c^\sigma = c\}.$$

In other words, the group $\mathrm{Am}(L/F)$ consists of ideal classes that are fixed by the Galois group $\mathrm{Gal}(L/F)$, and the group $\mathrm{Am}_{st}(L/F)$ consists of ideal classes that contain an ideal that is fixed by the Galois group $\mathrm{Gal}(L/F)$. The order of these groups are given by Chevalley [Che34]:

$$\begin{aligned} |\mathrm{Am}(L/F)| &= \frac{h_F \prod_v e_v}{[L:F][\mathcal{O}_F^* : \mathcal{O}_F^* \cap \mathrm{N}(L^*)]} \\ |\mathrm{Am}(L/F)_{st}| &= \frac{h_F \prod_v e_v}{[L:F][\mathcal{O}_F^* : \mathrm{N}(\mathcal{O}_L^*)]} \end{aligned} \tag{5}$$

where the product is taken over all places $v$ of $L$, $e_v$ is the ramification degree of $v$ in $L/F$, and $h_F$ is the class number of $F$. For any abelian group $M$, we use $M[p^\infty]$ to denote the $p$-part of $M$. In other words, $M[p^\infty] = \{x \in M | p^n x = 0 \text{ for some } n\}$

*Proof of Theorem 4.1.* Proposition 3.5 gives the implication $(b) \implies (a)$.

Now assume that $A_0 = H_0$. Let $\sigma$ be the generator of $\mathrm{Gal}(K_\infty/K)$. Hence $\sigma$ is also the generator of Galois group $\mathrm{Gal}(K_n/K)$ by restriction. In our case, there are only two primes $\mathfrak{P}$ and $\tilde{\mathfrak{P}}$ above $p$ in $K$. Recall we assume that the $\mathbb{Z}_p$-extension $K_\infty/K$ is totally ramified over $\mathfrak{P}$ and unramified over $\tilde{\mathfrak{P}}$ by definition. By Proposition 3.1, the size of $B_n = \mathrm{Am}(K_n/K)[p^\infty]$ is bounded. Hence the subgroup $\mathrm{Am}(K_n/K)_{st}[p^\infty]$ is bounded. By (5),

$$|\mathrm{Am}(K_n/K)_{st}| = \frac{h_K \prod_v e_v}{[K_n:K][\mathcal{O}_K^* : \mathrm{N}(\mathcal{O}_{K_n}^*)]} = \frac{h_K \cdot p^n}{p^n[\mathcal{O}_K^* : \mathrm{N}(\mathcal{O}_{K_n}^*)]} = \frac{h_K}{[\mathcal{O}_K^* : \mathrm{N}(\mathcal{O}_{K_n}^*)]}$$

and so it must be that $[\mathcal{O}_K^* : \mathrm{N}(\mathcal{O}_{K_n}^*)]$ is bounded.

Suppose $A_n \neq H_n$ for some $n$. Then by the lemma 4.3, there is $c \in A_n$ such that $c \notin H_n$ and $c^{\sigma-1} \in H_n$. Hence there exists $m$ such that $i_{n,m}(c^{\sigma-1}) = 0$. Let $c' = i_{n,m}(c)$, and let $\alpha$ be an ideal of $K_m$ such that $[\alpha] \in c'$. Then $\alpha^{\sigma-1} = (\beta)$ for some $\beta \in K_m^*$, and $\mathrm{N}_{m,0}(\beta) = \varepsilon \in \mathcal{O}_K^*$. Since $[\mathcal{O}_K^* : \mathrm{N}(\mathcal{O}_{K_n}^*)]$ is bounded, we know $\mathrm{N}_{s,0}(\beta) = \varepsilon^{s-m} \in \mathrm{N}_{s,0}(\mathcal{O}_{K_s}^*)$ for $s$ sufficiently larger than $m$. There exists $\eta \in \mathcal{O}_{K_s}^*$ such that $\mathrm{N}_{s,0}(\beta) = \mathrm{N}_{s,0}(\eta)$. Hence, there is $\gamma \in K_s^*$ such that $\beta\eta^{-1} = \gamma^{\sigma-1}$ by Hilbert's Theorem 90. Therefore,

$$(\alpha\mathcal{O}_{K_s})^{\sigma-1} = (\beta) = (\beta\eta^{-1}) = (\gamma^{\sigma-1})$$

hence the ideal class $i_{n,s}(c)$ contains a fractional ideal $\alpha\mathcal{O}_{K_s}(\gamma)^{-1}$ that is invariant under the action of $\mathrm{Gal}(K_s/K)$. In other words, $i_{n,s}(c) \in \mathrm{Am}_{st}(K_s/K)[p^\infty]$. Notice that $\mathrm{Am}_{st}(K_s/K)[p^\infty] = i_{0,s}(A_0)D_s$. We know $i_{0,s}(A_0) \subset H_s$ by assumption and $D_s \subset H_s$ by Corollary 3.4. Hence $i_{n,s}(c) \in \mathrm{Am}_{st}(K_s/K)[p^\infty] \subset H_s$. This contradicts our assumption that $c \notin H_n$. $\square$

**Lemma 4.3.** *Let $\sigma$ be a generator of the cyclic group $G = \mathbb{Z}/p^n\mathbb{Z}$. Let $X \neq \{0\}$ be an abelian $p$-group with an action of $\mathbb{Z}/p^n\mathbb{Z}$ on it. Then there is an element $x \in X$ such that $x \neq 0$ and $x^{\sigma-1} = 0$.*

*Proof.* Consider the following exact sequence,

$$0 \to X^G \to X \xrightarrow{\sigma-1} X \to X/X^{\sigma-1} \to 0$$

$X^G$ is the set of fixed element of $X$ under the action of $G$. Since $G$ and $X$ are $p$-group, $X^G$ can't be a trivial group. Hence, $X/X^{\sigma-1}$ is nontrivial. Next, consider the following exact sequence.

$$0 \to (X^{\sigma-1})^G \to X^{\sigma-1} \xrightarrow{\sigma-1} X^{\sigma-1} \to X^{\sigma-1}/X^{(\sigma-1)^2} \to 0$$

Continue the same analysis, we have a filtration,

$$X \supsetneq X^{\sigma-1} \supsetneq X^{(\sigma-1)^2} \supsetneq \cdots \supsetneq X^{(\sigma-1)^k} = 0$$

for some integer $k$. Take a nontrivial element $x \in X^{(\sigma-1)^{k-1}}$, then $x^{\sigma-1} = 0$. □

## 5. The case $p$ splits completely in $K^+$

In this section, we consider the case that $p$ splits completely in $K^+$. We still keep the assumption (1), (4), (2) as in previous section. Goto [Got06] studies this case for an abelian CM field $K$, but here we do not need to assume $K$ is abelian.

**Theorem 5.1.** *Assume that $p$ splits completely in $K^+$ and Leopoldt's conjecture holds for $K$. Consider the $S$-ramified $\mathbb{Z}_p$-extension $K_\infty/K$ defined by theorem 2.1. The following two statements are equivalent:*

*(a) $B_n = D_n$ for all sufficiently large $n$.*

*(b) $|A_n|$ is bounded as $n \to \infty$.*

*Remark* 5.2. Greenberg [Gre76, Theorem 2] states a similar criterion for the cyclotomic $\mathbb{Z}_p$-extension of a totally real field when $p$ splits completely. The method of proof is also similar to Greenberg's proof.

*Proof.* Assume that $B_n = D_n$ for all sufficiently large $n$. Since $\mathrm{N}_{m,n} : D_m \to D_n$ is surjective and $B_n$ is bounded under the assumption of Leopoldt's conjecture by Proposition 3.1, we know that $\mathrm{N}_{m,n} : B_m \to B_n$ is isomorphism for all $m \geq n \geq n_0$ for some $n_0$. Let Ker $(\mathrm{N}_{m,n})$ be the kernel of the map $\mathrm{N}_{m,n} : A_m \to A_n$. Then Ker $(\mathrm{N}_{m,n}) \cap B_m = 1$ for all $m \geq n \geq n_0$ for some $n_0$. View Ker $(\mathrm{N}_{m,n})$ as an abelian $p$-group with an action of $\mathrm{Gal}(K_m/K)$. By the general theory of group action, the fixed point of a nontrivial abelian $p$-group by a $p$-group is nontrivial. The fixed point of Ker $(\mathrm{N}_{m,n})$ by $\mathrm{Gal}(K_m/K)$ is Ker $(\mathrm{N}_{m,n}) \cap B_m = 1$. Hence Ker $(\mathrm{N}_{m,n}) = 1$. Therefore, $\mathrm{N}_{m,n} : A_m \to A_n$ is an isomorphism when $m \geq n \geq n_0$, which implies that $|A_n|$ is bounded.

Assume that $|A_n|$ is bounded as $n \to \infty$. We will prove that $B_n = \mathrm{Am}(K_n/K)[p^\infty] = \mathrm{Am}_{st}(K_n/K)[p^\infty]$ when $n$ is large enough. Recall that $\mathrm{Am}_{st}(K_n/K)[p^\infty] = i_{0,n}(A_0)D_n$ and $i_{0,n}(A_0)$ will become trivial when $n$ is large enough by Proposition 3.5. Hence $B_n = D_n$ when $n$ is large enough.

Since $|A_n|$ is bounded, reverse the argument in the first paragraph to get $\mathrm{N}_{m,n} : B_m \to B_n$ is isomorphism when $m \geq n \geq n_0$ for some $n_0$. Let $c \in B_n$ and take $c' \in B_m$ such that $\mathrm{N}_{m,n}(c') = c$. Let $J$ be an ideal of

$K_m$ such that $[J] = c'$ and let $I = \mathrm{N}_{m,n}(J)$. Then $[I] = c$. Let $J^{\sigma-1} = (\beta)$ and $I^{\sigma-1} = (\alpha)$, where $\beta \in K_m^*$ and $\alpha = \mathrm{N}_{m,n}(\beta)$. Put $\varepsilon = \mathrm{N}_{m,0}(\beta) = \mathrm{N}_{n,0}(\alpha)$. Then $\varepsilon \in \mathcal{O}_K^*$.

Let $K_{n,\mathfrak{P}_i}$ be the localization of $K_n$ at $\mathfrak{P}_i$. We have $K_{\mathfrak{P}_i} \cong \mathbb{Q}_p$ because $p$ splits completely in $K$. By local class field theory, a local unit in $\mathcal{O}_{K_{\mathfrak{P}_i}}$ sits in $\mathrm{N}_{m,0}(K^*_{m,\mathfrak{P}_i})$ if and only if it is a $p^m$-th power in $K^*_{\mathfrak{P}_i}$. Hence, $\varepsilon$ is a $p^m$-th power in $K^*_{\mathfrak{P}_i}$ (see Section 6). Let $\mathcal{P}_i$ be the prime ideal in $K^+$ below $\mathfrak{P}_i$. Since $p$ splits completely in $K$, we have $K_{\mathfrak{P}_i} \cong K^+_{\mathcal{P}_i} \cong K_{\tilde{\mathfrak{P}}_i} \cong \mathbb{Q}_p$.

Recall that the subgroup generated by torsion units of $K$ and real units of $K^+$ has index 1 or 2 inside the group of units of $K$ (see Theorem 4.12 in Washington [Was97]). Assume that $\varepsilon^2 = \varepsilon'\varepsilon''$ such that $\varepsilon'$ is a root of unity inside $K$ and $\varepsilon''$ is a unit of $K^+$. Since we assume $p$ splits completely in $K$ and $p$ splits completely in $\mathbb{Q}(\zeta_n)$ if and only if $p \equiv 1 \pmod{n}$, we have the order of $\varepsilon'$ divides $p-1$. Thus $\varepsilon'$ is a Teichmüller representative $\mathbb{F}_p^* \to \mathbb{Z}_p^*$ and obviously is a $p^m$-th power of itself.

Since $\varepsilon''$ is a unit in $K^+$, we have $\varepsilon^2$ is a $p^m$-th power in $K^*_{\mathfrak{P}_i}$ if and only if $\varepsilon''$ is a $p^m$-th power in $K^*_{\mathfrak{P}_i}$ if and only if $\varepsilon''$ is a $p^m$th power in $(K^+)^*_{\mathcal{P}_i}$ if and only if $\varepsilon''$ is a $p^m$-th power in $K^*_{\tilde{\mathfrak{P}}_i}$ if and only if $\varepsilon^2$ is a $p^m$-th power in $K^*_{\tilde{\mathfrak{P}}_i}$. In other words, $\varepsilon^2$ is a $p^m$-th power after localization at any prime above $p$. There are integers $x, y$ such that $2x + p^m y = 1$. Hence $\varepsilon = \varepsilon^{2x+p^m y}$ is a $p^m$-th power after localization at any prime above $p$. By Leopoldt's conjecture and enlarging $m$ if necessary, we may assume $\varepsilon = \eta^{p^n}$ for some $\eta \in \mathcal{O}_K^*$.

Since $\mathrm{N}_{n,0}(\alpha\eta^{-1}) = 1$, there exists $\gamma \in K_n$ such that $\alpha\eta^{-1} = \gamma^{\sigma-1}$. Thus

$$I^{\sigma-1} = (\alpha) = (\alpha\eta^{-1}) = (\gamma^{\sigma-1})$$

So the ideal class $c \in B_n$ contains a fractional ideal $I(\gamma)^{-1}$ that is fixed by $\mathrm{Gal}(K_n/K)$. Hence $c \in \mathrm{Am}_{st}(K_n/K)[p^\infty]$. Therefore, $B_n = \mathrm{Am}_{st}(K_n/K)[p^\infty]$.

□

## 6. The Ambiguous Class Groups

In this section, we compare the $S$-ramified $\mathbb{Z}_p$-extension $K_\infty/K$ of Theorem 2.1 and the cyclotomic $\mathbb{Z}_p$-extension $F^c_\infty/F$ for the totally real field $F$ by computing a certain norm index. Fukuda and Komatsu [FK86] give a numerical criterion to determine if $\lambda = 0$ for cyclotomic $\mathbb{Z}_p$-extension for real quadratic fields. We will give an analogous result for the $S$-ramified $\mathbb{Z}_p$-extension $K_\infty/K$ for imaginary biquadratic fields. First, we will review some needed results from local class field theory (see [Mil20] for more details).

**Theorem 6.1** (Local Artin Reciprocity). *Let $K/F$ be an Abelian Galois extension local fields. Then the local Artin map gives an isomorphism*

$$F^*/N_{K/F}(K^*) \cong Gal(K/F).$$

**Theorem 6.2** (The Local to Global Principal). *Suppose $L/K$ is a cyclic Galois extension. Then if $\gamma \in K$ is a local norm from $L_v$ for all places $v$ of $L$, then $\gamma$ is a global norm from $L$.*

Now, suppose that $p$ splits completely in $K/\mathbb{Q}$, and that $K_n$ is the $n$-th layer in a $\mathbb{Z}_p$-extension of $K$. Then $K_n/K$ is cyclic and is unramified outside of the primes above $p$. If $v$ is a place of $K_n$ unramified in $K_n/K$, then the norm map of the resulting local fields is surjective at the group of local units. Suppose $\mathfrak{P}$

is a prime of $K$ above $p$ which is totally ramified in $K_n$, and $\mathfrak{P}_n$ is a prime of $K_n$ above $\mathfrak{P}$. Let $(K_n)_{\mathfrak{P}_n}$ be the completion of $K_n$ at the prime $\mathfrak{P}_n$ and $K_0 = K$. By Theorem 6.1,

$$K_{\mathfrak{P}}^*/\mathrm{N}_n((K_n)_{\mathfrak{P}_n}^*) \cong \mathbb{Z}/p^n\mathbb{Z}$$

where $\mathrm{N}_n$ is the norm map from $(K_n)_{\mathfrak{P}_n}$ to $K_{\mathfrak{P}}$. Since we assume that $p$ splits completely in $K$, we have $K_{\mathfrak{P}} \cong \mathbb{Q}_p$. Let $\pi$ and $\varpi$ be uniformizers of $K_{\mathfrak{P}}$ and $(K_n)_{\mathfrak{P}_n}$ respectively, such that $\mathrm{N}_n(\varpi) = \pi$. Let $\mathcal{O}_\pi$ and $\mathcal{O}_\varpi$ be the local rings of integers in each field. Let $\mu_{p-1}$ be the group of $(p-1)$-st roots of unity. We have

$$(K_n)_{\mathfrak{P}_n}^* \cong \varpi^{\mathbb{Z}} \times \mu_{p-1} \times (1 + \varpi\mathcal{O}_\varpi)$$

and

$$K_{\mathfrak{P}}^* \cong \pi^{\mathbb{Z}} \times \mathcal{O}_\pi^* \cong \pi^{\mathbb{Z}} \times \mu_{p-1} \times (1 + \pi\mathcal{O}_\pi)$$

Let $\psi : K_{\mathfrak{P}}^* \to \mathcal{O}_\pi^*$ be the projection of the decomposition to the factor $\mathcal{O}_\pi^*$. Now, $\mathrm{N}_n((K_n)_{\mathfrak{P}_n}^*) \cong \pi^{\mathbb{Z}} \times \mu_{p-1} \times (1 + \pi^m\mathcal{O}_\pi)$ for some $m$, and Theorem 6.1 implies $m = n + 1$. Let us now prove the following useful corollary.

**Lemma 6.3.** *Assume $p$ splits completely in $K$ and primes ramified in the $\mathbb{Z}_p$-extension of $K_\infty/K$ are totally ramified. Keep the same notation as above and let $\gamma \in \mathcal{O}_K$. Then $\gamma \in N_n(K_n^*)$ if and only if $\psi(\gamma)$ is a $p^n$-th power modulo $\mathfrak{P}$ for all primes $\mathfrak{P}$ of $K$ above $p$ which are totally ramified in $K_n/K$. In particularly, assume $\gamma \in \mathcal{O}_K^*$, then $\gamma \in N_n(K_n^*)$ if and only if*

$$\gamma^{p-1} \equiv 1 \mod \mathfrak{P}^{n+1}$$

*for all $\mathfrak{P}$ ramified in $K_n/K$.*

*Proof.* Suppose that $\gamma \in \mathcal{O}_K$ and let $S$ be the set of primes that ramify in $K_n/K$. By the local to global principal 6.2, $\gamma \in \mathrm{N}_n(K_n^*)$ if and only if $\gamma$ is local norm for all primes $\mathfrak{P} \in S$. We assumed that $K_n/K$ is totally ramified at $\mathfrak{P} \in S$, we have $\mathrm{N}_n((K_n)_{\mathfrak{P}_n}) = \pi^{\mathbb{Z}} \times \mu_p \times (1 + \pi^{n+1}\mathcal{O}_\pi)$ by above argument. Hence $\gamma \in \mathrm{N}_n(K_n^*)$ if and only if $\psi(\gamma)^{p-1} \in 1 + \pi^{n+1}\mathcal{O}_\pi$ if and only if $\psi(\gamma)^{p-1} \equiv 1 \mod \mathfrak{P}^{n+1}$. Since we assume that $p$ split in $K$, $K_{\mathfrak{P}} \cong \mathbb{Q}_p$. Hence, an element in $1 + \pi^{n+1}\mathcal{O}_\pi$ if and it is a $p^n$-th power. The conclusion follows. ☐

For a number field $L$ we denote $E(L)$ to be the group of units of $\mathcal{O}_L$ and $W(L)$ to be the roots of unity in $L$. Note that if an odd prime $p$ splits completely in $L$ then the order of $W(L)$ is coprime to $p$. Indeed, if $W(L)$ contained a primitive $p$-th root of unity, then $p$ would be ramified in $L/\mathbb{Q}$.

Let $p$ be an odd prime, and $K$ a CM field satisfying Leopoldt's conjecture with $K^+$ its maximal totally real subfield. Take $F = K^+$. Further, suppose that $p$ splits completely in $K/\mathbb{Q}$. We define $S$ and $S^+$ as we did in the previous sections, for example, see the beginning of section 2. Let $K_\infty/K$ be the $\mathbb{Z}_p$-extension $K \subseteq K_1 \subseteq \cdots \subseteq K_\infty$ that is unramified outside of $S$, and let $F \subseteq F_1^c \subseteq \cdots \subseteq F_\infty^c$ be the cyclotomic $\mathbb{Z}_p$-extension of $F$. We assume that any ramified primes in $K_\infty/K$ or $F_\infty^c/F$ are totally ramified. We still keep the assumption (1), (4), (2) as in previous section.

**Proposition 6.4.** *With the above set up,*

$$E(K)/(N_n(K_n^*) \cap E(K)) \cong E(F)/(N_n((F_n^c)^*) \cap E(F))$$

*Proof.* For convenience we write $H_n(K) = \mathrm{N}_n(K_n^*) \cap E(K)$ and $H_n(F) = \mathrm{N}_n((F_n^c)^*) \cap E(F)$. Consider the map $\Theta : E(F) \to E(K)/H_n(K)$, which is the inclusion $E(F) \hookrightarrow E(K)$ followed by the quotient map $E(K) \to E(K)/H_n(K)$. Notice that $W(K) \subseteq H_n(K)$, since the order of $W(K)$ is coprime to $p$, and $E(K)/H_n(K)$ is a $p$-group. Thus, we have the containment $E(F)W(K) \subseteq E(F)H_n(K) \subseteq E(K)$. Now, by Theorem 4.12 of Washington [Was97] we have that $[E(K) : E(F)W(K)] \leq 2$, so it must be that $[E(K) : E(F)H_n(K)] \leq 2$. But $H_n(K) \subseteq E(F)H_n(K) \subseteq E(K)$, and $E(K)/H_n(K)$ is a $p$-group. This forces $E(F)H_n(K) = E(K)$ (recall that we are assuming $p$ is odd). The image of $E(F)$ under $\Theta$ is $E(F)H_n(K)/H_n(K)$, so the above argument shows that $\Theta$ is surjective.

Now suppose that $\beta \in \mathrm{Ker}\ \Theta$. Then we have that $\beta \in H_n(K)$, and so by Lemma 6.3, for any prime $\mathfrak{P}$ of $K$ above $\mathfrak{p}$ we have

$$\beta^{p-1} \equiv 1 \pmod{\mathfrak{P}^{n+1}}.$$

Let $\mathcal{P} = \mathfrak{P} \cap \mathcal{O}_F$ and notice $\beta \in \mathcal{O}_F$, so this implies that $\beta^{p-1} \equiv 1 \pmod{\mathcal{P}^{n+1}}$ for all primes $\mathcal{P}$ of $F$ above $p$. Hence $\beta \in H_n(F)$. Thus, $\mathrm{Ker}\ \Theta \subseteq H_n(F)$. Suppose that $\beta \in H_n(F)$. Then By Lemma 6.3, we have that

$$\beta^{p-1} \equiv 1 \pmod{\mathcal{P}^{n+1}}$$

for all primes $\mathcal{P}$ of $F$ above $p$. For any prime $\mathcal{P}$ of $F$ above $p$, we have $\mathcal{P}\mathcal{O}_K = \mathfrak{P}\bar{\mathfrak{P}}$, and

$$\beta^{p-1} \equiv 1 \pmod{\mathfrak{P}^{n+1}} \qquad \text{and } \beta^{p-1} \equiv 1 \pmod{\bar{\mathfrak{P}}^{n+1}}$$

since $\beta \in E(F)$. Therefore, $\beta \in H_n(K)$ by Lemma 6.3. This shows that $\mathrm{Ker}\ \Theta = H_n(F)$ so that $E(F)/H_n(F) \cong E(K)/H_n(K)$. ☐

*Remark* 6.5. Notice that $F_n^c \not\subset K_n$. Proposition 6.4 is interesting because we don't have a direct relation between $K_n$ and $F_n^c$. Though $K_n$ and $F_n^c$ are globally different and unrelated, they are similar locally.

Given an extension $L/M$ of number fields, let $\mathrm{Am}_p(L/M)$ denote the $p$-ambiguous class group, that is

$$\mathrm{Am}_p(L/M) = A(L)^{\mathrm{Gal}(L/M)}$$

where $A(L)$ is the $p$-class group of $L$.

**Corollary 6.6.** *Assume that $p$ splits completely in $K$ and ramified primes in $K_\infty/K$ or $F_\infty^c/F$ are totally ramified. With the above setup, we have*

$$\frac{|Am_p(K_n/K)|}{|A(K)|} = \frac{|Am_p(F_n^c/F)|}{|A(F)|}.$$

*Proof.* Chevalley's formula (5) has that

$$|\mathrm{Am}_p(K_n/K)| = |A(K)| \frac{\prod_{\mathcal{P}} e(\mathfrak{P}_n/\mathfrak{P})}{[K_n : K][E(K) : \mathrm{N}_n(K_n^*) \cap E(K)]}$$

where the product ranges over primes $\mathfrak{P}$ of $K$ above $\mathfrak{p}$, and $e(\mathfrak{P}_n/\mathfrak{P})$ denotes the ramification index. If $\mathfrak{P}$ lies above $\mathfrak{p}$, then $e(\mathfrak{P}_n/\mathfrak{P}) = p^n$. Similarly,

$$|\mathrm{Am}_p(F_n^c/F)| = |A(F)| \frac{\prod_{\mathcal{P}} e(\mathcal{P}_n/\mathcal{P})}{[F_n^c : F][E(F) : \mathrm{N}_n((F_n^c)^*) \cap E(F)]}.$$

Now, if $\mathcal{P}$ ramifies in $F_n^c$, then $e(\mathcal{P}_n/\mathcal{P}) = p^n$, and the number of ramified primes in $F_n^c/F$ is the same as the number of ramified primes in $K_n/K$. This together with the previous proposition proves the Corollary. ☐

6.1. **The $S$-Ramified $\mathbb{Z}_p$-Extensions of Imaginary Biquadratic Fields.** As an application, we prove results analogous to those of Fukuda Komatsu [FK86] for the $S$-ramified extension of an imaginary biquadratic field.

Let $m, d \in \mathbb{Z}^+$ that are squarefree and coprime. Denote $k = \mathbb{Q}(\sqrt{-m})$, $F = \mathbb{Q}(\sqrt{d})$, $K = Fk$, and $\varepsilon$ to be the fundamental unit for $K$. Suppose that $p > 2$ is a prime that splits completely in $K$, with $p\mathcal{O}_k = \mathfrak{p}\tilde{\mathfrak{p}}$ and $\mathfrak{p}\mathcal{O}_K = \mathfrak{P}\bar{\mathfrak{P}}$. Suppose that

$$k \subseteq k_1 \subseteq k_2 \subseteq \cdots \subset \bigcup_n k_n = k_\infty$$

is the unique $\mathbb{Z}_p$-extension of $k$ unramified outside $\mathfrak{p}$. Put $K_n = Fk_n$ and $K_\infty = Fk_\infty$. Then $K_\infty/K$ is a $S$-ramified $\mathbb{Z}_p$-extension of $K$, where $S$ is the set of primes above $\mathfrak{p}$ in $K$.

Denote $E_n$ to be the units of $\mathcal{O}_{K_n}$, and $\mathfrak{P}_n$ to be the prime of $\mathcal{O}_{K_n}$ that lies above $\mathfrak{P}$. Let $\mathrm{N}_{n,m} : K_n \to K_m$ be the norm map, and $\mathrm{N}_n : K_n \to K$ norm map from $K_n$ to $K$. Let $h_K$, $h_F$ and $h_k$ be the class numbers for $K$, $F$ and $k$ respectively.

The following is an analogue of the main theorem in [FK86]:

**Theorem 6.7.** *Suppose $p \nmid h_K$, and that $r$ is the smallest positive integer such that*

$$\varepsilon^{p-1} \equiv 1 \pmod{\bar{\mathfrak{P}}^r}.$$

*If $N_{r-1}(E_{r-1}) = E_0$ then $\mu = \lambda = 0$ for the $S$-ramified $\mathbb{Z}_p$-extension of $K_\infty/K$ defined above.*

*Remark* 6.8. [Yok65, Proposition 1] tells us that $p \nmid h_K$ implies $p \nmid h_F$ and $p \nmid h_k$. Hence primes ramified in the $\mathbb{Z}_p$-extension $k_\infty/k$ is totally ramified. Thus the primes ramified in the $\mathbb{Z}_p$-extension $K_\infty/K$ is totally ramified. Since $[F : \mathbb{Q}] = 2$ and $p \geq 3$, primes ramified in the cyclotomic $\mathbb{Z}_p$-extension $F_\infty^c/F$ is also totally ramified. Hence the proposition in the previous subsection holds for our case.

*Proof.* Suppose that $\mathrm{N}_{r-1}(E_{r-1}) = E_0$. Then there is $\beta \in E_{r-1}$ such that $\mathrm{N}_{r-1}(\beta) = \varepsilon$, and so for any $n \geq r-1$ we have $\varepsilon^{p^{n-r+1}} \in \mathrm{N}_n(E_n)$. Thus, $|E_0/\mathrm{N}_n(E_n)| \leq p^{n-r+1}$. Now, $|E_0/(\mathrm{N}_n(K_n^*) \cap E_0)| = |E(F)/(\mathrm{N}_n((F_n^c)^*) \cap E(F))|$ by Proposition 6.4. Meanwhile, Fukuda and Komatsu [FK86, Lemma 2] calculate that $|E(F)/(\mathrm{N}_n((F_n^c)^*) \cap E(F))| = p^{n-r+1}$, hence $|E_0/\mathrm{N}_n(E_n)| \leq p^{n-r+1} = |E_0/(\mathrm{N}_n(K_n^*) \cap E_0)| \leq |E_0/\mathrm{N}_n(E_n)|$. By Chevalley's formula, $B_n = D_n$ for all $n \geq r-1$ so that $\mu = \lambda = 0$ by Theorem 5.1. □

**Lemma 6.9.** *Suppose that $p$ splits in $K$. Let $\mathfrak{p}$ be a prime above $p$ in $F = K^+$. Let $\mathfrak{P}$ be a prime above $\mathfrak{p}$ in $K$. Let $\varepsilon_K$ be a fundamental unit of $E(K)$, and $\varepsilon_F$ a fundamental unit of $F$. Then*

$$\varepsilon_K^{p-1} \equiv 1 \pmod{\mathfrak{P}^n} \qquad \Longleftrightarrow \qquad \varepsilon_F^{p-1} \equiv 1 \pmod{\mathfrak{p}^n}.$$

*Proof.* Let $W(K)$ be the group of roots of unity in $K$. Since $p$ splits completely in $K$, we have $\#W(K)$ divides $p-1$. Recall that $[E(K) : W(K)E(F)] \leq 2$ by [Was97, Theorem 2.13]. We have $\varepsilon_K^2 = \varepsilon_F^q\zeta$ for some $q \in \mathbb{Z}$ and $\zeta \in W(K)$. Since $E(F) \subset E(K)$, we have $\varepsilon_F = \varepsilon_K^r\eta$ for some $r \in \mathbb{Z}$ and $\eta \in W(K)$. Let $F_\mathfrak{p}$ be the completion of $F$ at $\mathfrak{p}$ and $K_\mathfrak{P}$ be the completion of $K$ at $\mathfrak{P}$. Since $p$ splits completely in $K$, we have $F_\mathfrak{p} \cong K_\mathfrak{P} \cong \mathbb{Q}_p$. Then $\varepsilon_F^{p-1} \equiv 1 \pmod{\mathfrak{p}^n}$ if and only if $\varepsilon_F^{p-1}$ is a $p^{n-1}$-th power if and only if $\varepsilon_K^{p-1}$ is a $p^{n-1}$-th power if and only if $\varepsilon_K^{p-1} \equiv 1 \pmod{\mathfrak{P}^n}$. □

**Corollary 6.10** (Analogue of Lemma 4 in [FK86])**.** *Let $\varepsilon$ be a fundamental unit of $E(K)$. Suppose that $p \nmid h_K$, and that*

$$\varepsilon^{p-1} \equiv 1 \pmod{\bar{\mathfrak{P}}^2} \qquad \textit{but} \qquad \varepsilon^{p-1} \not\equiv 1 \pmod{\bar{\mathfrak{P}}^3}.$$

*Write $\mathfrak{P}^{h_K} = (\alpha)$, and suppose that*

$$\alpha^{p-1} \equiv 1 \pmod{\bar{\mathfrak{P}}} \qquad \textit{but} \qquad \alpha^{p-1} \not\equiv 1 \pmod{\bar{\mathfrak{P}}^2}$$

*Then $\mu = \lambda = 0$.*

*Proof.* We again follow the proof of Fukuda and Komatsu in [FK86]. Under the assumptions Lemma 6.3 implies that $E(K) = \mathrm{N}_1(K_1^*) \cap E(K)$. Therefore,

$$[B_1 : D_1] = [\mathrm{N}_1(K_1^*) \cap E(K) : \mathrm{N}_1(E_1)] = [E(K) : \mathrm{N}_1(E_1)].$$

Therefore, by Theorem 6.7, if $B_1 = D_1$ then $\lambda = 0$. By Lemma 6.9, we have that

$$\varepsilon_F^{p-1} \equiv 1 \pmod{\mathfrak{p}^2} \qquad \text{but} \qquad \varepsilon_F^{p-1} \not\equiv 1 \pmod{\mathfrak{p}^3}.$$

So by Proposition 1 in [FK86] combined with Corollary 6.6, we have $|B_1| = p$. Let $\mathfrak{P}_1$ be the prime of $K_1$ above $\mathfrak{P}$ in $K$. Then the class of $\mathfrak{P}_1^{h_K}$ generates $D_1$ (here we are using the assumption that $p \nmid h_K$). We will show that $\mathfrak{P}_1^{h_K}$ is not principle. Indeed, suppose that $\mathfrak{P}_1^{h_K} = (\alpha_1)$. Then

$$\mathrm{N}_1(\alpha_1) = \alpha\varepsilon^t$$

for some $t \in \mathbb{Z}$. Now, Lemma 6.3 implies that $\mathrm{N}_1(\alpha_1)^{p-1} \equiv 1 \pmod{\bar{\mathfrak{P}}^2}$. Thus $\alpha^{p-1} \equiv 1 \pmod{\bar{\mathfrak{P}}^2}$, which contradicts our assumptions on $\varepsilon$ and $\alpha$. □

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