Question:

$$Evaluate: \int_0^\infty \frac{\sin(x)\sin(2x)\sin(3x)}{x^3}dx.$$

Rewrite as:

$$Evaluate \int_0^\infty \frac{(\sin x \sin 2x \sin 3x)dx}{x^3}.$$

First, apply the trigonometric identity for the product of three sine functions:

$$\sin a \sin b \sin c = \frac{1}{4}\left[\sin(a+b-c) + \sin(b+c-a) + \sin(c+a-b) - \sin(a+b+c)\right].$$

Substituting $a = x$, $b = 2x$, $c = 3x$:

$$\sin x \sin 2x \sin 3x = \frac{1}{4}\left[\sin(0 \cdot x) + \sin(4x) + \sin(2x) - \sin(6x)\right] = \frac{1}{4}\left(\sin 2x + \sin 4x - \sin 6x\right).$$

Thus, the integral becomes:

$$\int_0^\infty \frac{(\sin x \sin 2x \sin 3x)dx}{x^3} = \frac{1}{4}\int_0^\infty \frac{(\sin 2x + \sin 4x - \sin 6x)dx}{x^3}.$$

Express $1/x^3$ using its integral representation derived from Gamma function:

$$\frac{1}{x^3} = \frac{1}{2}\int_0^\infty t^2 e^{-tx}\,dt, \quad x > 0.$$

Substitute this into the integral:

$$\frac{1}{4}\int_0^\infty \left(\frac{1}{2}\int_0^\infty t^2 e^{-tx}\,dt\right)(\sin 2x + \sin 4x - \sin 6x)\,dx.$$

Interchange the order of integration
(justified by Fubini's theorem given convergence):

$$\frac{1}{8}\int_0^\infty t^2\left[\int_0^\infty e^{-tx}(\sin 2x + \sin 4x - \sin 6x)\,dx\right]dt.$$

The inner integral is the Laplace transform:

$$\mathcal{L}\{\sin mx\}(t) = \frac{m}{t^2+m^2}.$$

Thus:

$$\int_0^\infty e^{-tx}\sin 2x\,dx = \frac{2}{t^2+4}, \quad \int_0^\infty e^{-tx}\sin 4x\,dx = \frac{4}{t^2+16}, \quad \int_0^\infty e^{-tx}\sin 6x\,dx = \frac{6}{t^2+36}.$$

The expression simplifies to:

$$\frac{1}{8}\int_0^\infty t^2\left(\frac{2}{t^2+4}+\frac{4}{t^2+16}-\frac{6}{t^2+36}\right)dt.$$

Rewrite each term:

$$\frac{t^2}{t^2+b}=1-\frac{b}{t^2+b}.$$

So:

$$2\left(1-\frac{4}{t^2+4}\right)+4\left(1-\frac{16}{t^2+16}\right)-6\left(1-\frac{36}{t^2+36}\right)=2+4-6-\frac{8}{t^2+4}-\frac{64}{t^2+16}+\frac{216}{t^2+36}$$

$$=-\frac{8}{t^2+4}-\frac{64}{t^2+16}+\frac{216}{t^2+36}.$$

Now integrate:

$$\int_0^\infty \frac{1}{t^2+a^2}\,dt=\frac{\pi}{2a}.$$

Thus:

$$\int_0^\infty\left(-\frac{8}{t^2+4}-\frac{64}{t^2+16}+\frac{216}{t^2+36}\right)dt=-8\cdot\frac{\pi}{4}-64\cdot\frac{\pi}{8}+216\cdot\frac{\pi}{12}=-2\pi-8\pi+18\pi=8\pi.$$

Finally:

$$\frac{1}{8}\cdot 8\pi=\pi.$$