A Category View of Product Topology

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This article introduces a category view of the product topology of two given topological spaces.

1 Categories

This section will introduce what category is and some basic categories.

Definition. A category $\mathscr{C}$ consists of:

C1 A class $\mathbf{ob}\ \mathscr{C}$ of objects like A, B, C, D.

C2 For each ordered pair of objects (A, B), a set $\mathbf{hom}_{\mathscr{C}}(A, B)$ of morphisms like f, g, h.
A class $\mathbf{mor}\ \mathscr{C}$ of morphisms like $f \in \mathbf{hom}_{\mathscr{C}}(A, B)$, $g \in \mathbf{hom}_{\mathscr{C}}(B, C)$, $h \in \mathbf{hom}_{\mathscr{C}}(C, D)$.

C3 For each ordered triple of objects (A, B, C), a map as an operation on $\mathbf{mor}\ \mathscr{C}$
$\bullet_{\mathscr{C}} \colon \mathbf{hom}_{\mathscr{C}}(A, B) \times \mathbf{hom}_{\mathscr{C}}(B, C) \to \mathbf{hom}_{\mathscr{C}}(A, C)$; $(f, g) \mapsto g \bullet_{\mathscr{C}} f$.

which satisfying:

A1 If $(A, B) \neq (C, D)$, then $\mathbf{hom}_{\mathscr{C}}(A, B) \cap \mathbf{hom}_{\mathscr{C}}(C, D) = \{\}$.

A2 Associativity: $(h \bullet_{\mathscr{C}} g) \bullet_{\mathscr{C}} f = h \bullet_{\mathscr{C}} (g \bullet_{\mathscr{C}} f)$, thus $h \bullet_{\mathscr{C}} g \bullet_{\mathscr{C}} f$ is valid.

A3 Identity: There exists $1_B \in \mathbf{hom}_{\mathscr{C}}(B, B)$ such that $1_B \bullet_{\mathscr{C}} f = f$, $g \bullet_{\mathscr{C}} 1_B = g$.

Here, dots and subscripts $\mathscr{C}$ can be omitted if it is clear.

It is worthwhile to point out that although one can still consider the class, object, and morphism as a set, element, and map separably, they are not. They are more abstract forms or just symbols. Additionally, commutative diagrams still work for representing propositions and helping thinking but no longer carry the meaning of mapping or something similar. It is better to view morphism as some relation between two objects, but their order needs to be distinguished. To understand such an abstract thing, many examples are needed.

Example. Several categories are listed below (easy to verify all conditions):

Sets (not necessarily with special structures) should be the easiest.

E1 $\mathscr{Set}$, the category of sets. Here **ob** $\mathscr{Set}$ is the class of all sets, $\mathbf{hom}_{\mathscr{Set}}(A, B)$ is the set of all maps from A to B, and $\bullet_{\mathscr{Set}}$ is the composition of maps.

Now consider sets with an operation (magmas) and maps preserving it (homomorphisms).

E2 $\mathscr{Grp}$, the category of groups. Here **ob** $\mathscr{Grp}$ is the class of all groups, $\mathbf{hom}_{\mathscr{Grp}}(G, H)$ is the set of all (group) homomorphisms from G to H, and $\bullet_{\mathscr{Grp}}$ is the composition of homomorphisms.

E3 $\mathscr{Ab}$, the category of abelian groups. Details are similar to $\mathscr{Grp}$.

Now consider structures with more than one operation.

E4 $\mathscr{Rng}$, the category of rings. Here **ob** $\mathscr{Rng}$ is the class of all rings, $\mathbf{hom}_{\mathscr{Rng}}(R, S)$ is the set of all (ring) homomorphisms from R to S, and $\bullet_{\mathscr{Rng}}$ is the composition of homomorphisms.

E5 $\mathscr{Ring}$, the category of rings with 1 (multiplication identity). Here $\mathbf{hom}_{\mathscr{Ring}}(R, S)$ is the set of all (ring) homomorphisms from R to S mapping 1_R to 1_S. Other details are omitted.

Now, consider some other examples.

E6 $\mathscr{Top}$, the category of topological spaces. Here **ob** $\mathscr{Top}$ is the class of all topological spaces, $\mathbf{hom}_{\mathscr{Top}}(X, Y)$ is the set of all continuous maps from X to Y, and $\bullet_{\mathscr{Top}}$ is the composition of continuous maps.

2 Products

Constructions in different areas of mathematics can unify in the language of category. In particular, this subsection will introduce products.

Definition. Let $\mathcal{C}$ be a category. For any $A_1, A_2 \in \mathbf{ob}\ \mathcal{C}$, their product (in category $\mathcal{C}$) is defined as a triple (A, p_1, p_2) where $A \in \mathbf{ob}\ \mathcal{C}$ and $p_i \in \mathbf{hom}_{\mathcal{C}}(A, A_i)$ for $i = 1, 2$ such that:

For any $B \in \mathbf{ob}\ \mathcal{C}$ and $f_i \in \mathbf{hom}_{\mathcal{C}}(B, A_i)$ for $i = 1, 2$, there exists a unique $f \in \mathbf{hom}_{\mathcal{C}}(B, A)$ such that $p_i \bullet_{\mathcal{C}} f = f_i$ for $i = 1, 2$. That is, the following diagram commutes for $i = 1, 2$:

$$\begin{tikzcd} A \arrow[d, "p_i"] & B \arrow[l, "f"'] \arrow[dl, "f_i"] \\ A_i & \end{tikzcd}$$

Proposition. Product is unique up to unique isomorphism in $\mathcal{C}$. That is, for any two products (A, p_1, p_2) and (A', p'_1, p'_2) of A_1, A_2, there exists a unique isomorphism $h \in \mathbf{hom}_{\mathcal{C}}(A, A')$ such that $p_i = p'_i \bullet_{\mathcal{C}} h$ for $i = 1, 2$.

Proof. Leave as an exercise. □

We may extend this definition to the product of arbitrary many objects.

Definition. Let $\mathcal{C}$ be a category and I be an index set. For $A_\alpha \in \mathbf{ob}\ \mathcal{C}, \alpha \in I$, their product (in category $\mathcal{C}$) is a tuple (A, p_α) where $A \in \mathbf{ob}\ \mathcal{C}$ and $p_\alpha \in \mathbf{hom}_{\mathcal{C}}(A, A_\alpha)$ for $\alpha \in I$ such that:

For any $B \in \mathbf{ob}\ \mathcal{C}$ and $f_\alpha \in \mathbf{hom}_{\mathcal{C}}(B, A_\alpha)$, $\alpha \in I$, there exists a unique $f \in \mathbf{hom}_{\mathcal{C}}(B, A)$ such that $p_\alpha \bullet_{\mathcal{C}} f = f_\alpha$ for $\alpha \in I$. That is, the following diagram commutes for $\alpha \in I$:

$$\begin{tikzcd} A \arrow[d, "p_\alpha"] & B \arrow[l, "f"'] \arrow[dl, "f_\alpha"] \\ A_\alpha & \end{tikzcd}$$

Proposition. Product is unique up to unique isomorphism in $\mathcal{C}$. That is, for any two products (A, p_α) and (A', p'_α) of $A_\alpha, \alpha \in I$, there exists a unique isomorphism $h \in \mathbf{hom}_{\mathcal{C}}(A, A')$ such that $p_\alpha = p'_\alpha \bullet_{\mathcal{C}} h$ for $\alpha \in I$.

Proof. Leave as an exercise. □

It can be shown that direct products of groups are products in $\mathcal{G}rp$, and Cartesian products are products in $\mathcal{S}et$.

3 Product Topology

Let us first look at some basics of topology.

Definition. Given a set X, a topology on X is a pair of X and a family $\mathscr{T}$ of subsets of X (i.e. $\mathscr{T} \subseteq 2^X$) satisfying the following axioms:

A0 Subsets (of X) in $\mathscr{T}$ are called open sets (subsets).

A1 Unions of open sets are open:

For $U_\alpha \in \mathscr{T}, \alpha \in A$, $\bigcup_{\alpha \in A} U_\alpha \in \mathscr{T}$.

A2 Finite intersections of open sets are open:

For $U_\alpha \in \mathscr{T}, i = 1, 2, \cdots, n$, $\bigcap_{i=1}^{n} U_i \in \mathscr{T}$.

A3 Universal sets and empty set are open: $X \in \mathscr{T}$, $\emptyset \in \mathscr{T}$.

Definition. Given a set X with two topologies $\mathscr{T}_1$ and $\mathscr{T}_2$, $\mathscr{T}_1$ is said to be finer (stronger) than $\mathscr{T}_2$ if $\mathscr{T}_1 \supseteq \mathscr{T}_2$. In this case, $\mathscr{T}_2$ is said to be coarser (weaker) than $\mathscr{T}_1$.

Definition. Let X be a set and $\mathscr{F}$ be a family of subsets of X. Then, we define:

1. $\overline{\mathscr{F}}_{\mathcal{T}}$, the topology generated by $\mathscr{F}$, is defined as the coarsest (weakest) topology contains $\mathscr{F}$. Equivalently, $\overline{\mathscr{F}}_{\mathcal{T}} := \bigcap_{\mathscr{T} \supseteq \mathscr{B}} \mathscr{T}$. One can consider it as "closure" in the sense of topology.
2. $\overline{\mathscr{F}}_{\mathcal{U}}$, the set of unions generated by $\mathscr{F}$, is defined as the set of arbitrary unions of sets in $\mathscr{F}$. That is, $\overline{\mathscr{F}}_{\mathcal{U}} := \{U \mid U \text{ is a union of subsets (of } X\text{) in } \mathscr{F}\}$.

   One can consider it as "closure" in the sense of union.

Definition. Given a set X, a family $\mathscr{B}$ of subsets of X is called a basis for a topology if:

1. For any $x \in X$, there exists $B \in \mathscr{B}$ such that $x \in B$. Equivalently, $X = \bigcup_{B \in \mathscr{B}} B$.
2. For any $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$, there exists $B \in \mathscr{B}$ such that $x \in B \subseteq B_1 \cap B_2$.

   Equivalently, $B_1 \cap B_2 = \bigcup_{\substack{B \in \mathscr{B} \\ B \subseteq B_1 \cap B_2}} B$.

Proposition. For a basis $\mathscr{B}$ for a topology on X, $\overline{\mathscr{B}}_{\mathcal{U}}$ is a topology on X, where $\{\}$ should be considered as the empty union that joins no subsets. Moreover, $\overline{\mathscr{B}}_{\mathcal{U}} = \overline{\mathscr{B}}_{\mathcal{T}}$.

For a family $\mathscr{F}$ of subsets of X such that $\overline{\mathscr{F}}_{\mathcal{U}}$ is a topology on X (where $\{\}$ should be considered as the empty union that joins no subsets), $\mathscr{F}$ is a basis for a topology on X. Moreover, $\overline{\mathscr{F}}_{\mathcal{T}} = \overline{\mathscr{F}}_{\mathcal{U}}$.

Proof. The proof is easy but nontrivial. Omit here and leave it as an exercise. $\square$

Then, we can define the product of two topological spaces as the following:

Definition. Given two topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{S})$, their (topological) product is their Cartesian product $X \times Y := \{(x, y) \mid x \in X, y \in Y\}$ with the product topology defined as the topology generated by $\mathscr{B} := \{U \times V \mid U \in \mathcal{T}, V \in \mathcal{S}\}$.

Proposition. The above $\mathscr{B}$ is a basis for a topology on $X \times Y$.
Then, the product topology $\overline{\mathscr{B}}_{\mathcal{T}}$ is $\overline{\mathscr{B}}_{\mathcal{U}}$ in fact.

Proof. Since $X \in \mathcal{T}$ and $Y \in \mathcal{S}$, $X \times Y = \bigcup\limits_{X\times Y \in \mathscr{B}} X \times Y$.

Let $B_i = U_i \times V_i \in \mathscr{B}$ for $i = 1, 2$. Then, $B_1 \cap B_2 = (U_1 \times V_1) \cap (U_2 \times V_2) =$ [1]$(U_1 \cap U_2) \times (V_1 \cap V_2) = U' \times V' \in \mathscr{B}$ since $U' = U_1 \cap U_2 \in \mathcal{T}$ and $V' = V_1 \cap V_2 \in \mathcal{S}$. Thus, $B_1 \cap B_2 = \bigcup\limits_{U'\times V' \in \mathscr{B}} U' \times V'$. $\square$

Theorems. (The reason why the above definition of the product topology is good.)

1. The product topology is the coarsest (weakest) topology such that the projection maps $p : X \times Y \to X; (x, y) \mapsto x$ and $q : X \times Y \to X; (x, y) \mapsto y$ are continuous.

2. For another topological space Z with continuous maps $f : Z \to X$ and $g : Z \to Y$, there exists a unique continuous map $h : Z \to X \times Y$ such that the following diagram commutes:

$$
\begin{tikzcd}
Z \arrow[rrd, bend left, "g"] \arrow[rd, dashed, "h"] \arrow[rdd, bend right, "f"'] & & \\
& X \times Y \arrow[r, "q"'] \arrow[d, "p"'] & Y \\
& X &
\end{tikzcd}
$$

Proof.

1. Let $\mathcal{T}_{X\times Y}$ be a topology on $X \times Y$ such that p, q are continuous. Since p is required to be continuous, $p^{-1}(U) = U \times Y$ must be open in $X \times Y$ for any U open in X. Similarly, $q^{-1}(V) = X \times V$ must be open in $X \times Y$ for any V open in Y. Thus, $U \times Y, X \times V \in \mathcal{T}_{X\times Y}$. Then, $U \times V = (U \times Y) \cap (X \times V) \in \mathcal{T}_{X\times Y}$ for any $U \in \mathcal{T}$ and $V \in \mathcal{S}$. As a result, $\mathscr{B} \subseteq \mathcal{T}_{X\times Y}$. By definition of $\overline{\mathscr{B}}_{\mathcal{T}}$, $\overline{\mathscr{B}}_{\mathcal{T}} \subseteq \mathcal{T}_{X\times Y}$ (also $\overline{\mathscr{B}}_{\mathcal{U}} \subseteq \mathcal{T}_{X\times Y}$). That is, the product topology is the coarsest (weakest) topology such that the projection maps are continuous.

2. Let $h : z \mapsto (f(z), g(z))$, which is the only natural way to define h. Then, it is obvious that h makes the above diagram commutative. Now, it suffices to prove h is continuous and unique. Let $W = U \times V \in \mathscr{B}$. $h^{-1}(W) = h^{-1}(U \times V) = f^{-1}(U) \cap g^{-1}(V)$ open in Z since f, g are continuous and U, V are open. Thus, for any $\bigcap\limits_{\alpha\in I} W_\alpha \in \overline{\mathscr{B}}$, $h^{-1}(\bigcap\limits_{\alpha\in I} W_\alpha) = \bigcap\limits_{\alpha\in I} h^{-1}(W_\alpha)$ is open, and h is continuous. Let h' be another map from Z to $X \times Y$ satisfying all conditions. Then for any $z \in Z$, $p(h'(z)) = f(z)$ and $q(h'(z)) = g(z)$. Thus, $h'(z) = (f(z), g(z)) = h(z)$ and $h' = h$. $\square$

[1]For union, it might Not be true.

As can be seen, for $X, Y \in \textbf{ob}\ \mathscr{T}op$, we have a triple $(X \times Y, p, q)$ where $X \times Y \in \textbf{ob}\ \mathscr{T}op$, $p \in \textbf{hom}_{\mathscr{T}op}(X \times Y, X)$ and $q \in \textbf{hom}_{\mathscr{T}op}(X \times Y, Y)$. For any $Z \in \textbf{ob}\ \mathscr{T}op$, $f \in \textbf{hom}_{\mathscr{T}op}(Z, X)$ and $g \in \textbf{hom}_{\mathscr{T}op}(Z, Y)$, there is a unique $h \in \textbf{hom}_{\mathscr{T}op}(Z, X \times Y)$ such that the above diagram commutative. Thus, the (topological) product of X and Y is a (categorical) product in $\mathscr{T}op$. Hence, it is unique up to unique isomorphisms. That is to say, any (categorical) product of X and Y in $\mathscr{T}op$ is isomorphic to the (topological) product by a unique isomorphism.

Alternatively, we may state this proposition less categorically.

Proposition. Let $X \times Y$ be the product topology of X and Y. Define projection maps $p : X \times Y \to X; (x, y) \mapsto x$ and $q : X \times Y \to X; (x, y) \mapsto y$. For any topological space Z with a map $h : Z \to X \times Y$, h is continuous if and only if $p \circ h$ and $q \circ h$ are continuous.

Proof. "$\Rightarrow$": It is obvious since the product topology makes p, q continuous and compositions of continuous maps are continuous.

"$\Leftarrow$": By the above theorems $\square$

Now, define the product of arbitrary topological spaces.

Definition. Let X be a set, $\mathscr{S}$ be a family of subsets of X, and $\overline{\mathscr{S}}$ be the collection of all unions of finite intersections of elements (subsets of X) in $\mathscr{S}$.

$\mathscr{S}$ is called a subbasis of a topology $\mathcal{T}$ on X if $\mathcal{T} = \overline{\mathscr{S}}$. In this case, $\mathcal{T}$ is called the topology generated by subbasis $\mathscr{S}$.

Example. A subbasis of the product topology of $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ is
$\mathscr{S} := \{p^{-1}(U), q^{-1}(V) \mid U \in \mathcal{T}, V \in \mathcal{S}\}$ (since $U \times V = p^{-1}(U) \cap q^{-1}(V)$).

Definition. Given topological spaces $(X_\alpha, \mathcal{T}_\alpha)$, $\alpha \in I$, the product topology on the Cartesian product $\prod\limits_{\alpha \in I} X_\alpha$, which is the product of sets X_α in $\mathcal{S}et$, is the topology generated by subbasis $\mathscr{S} := \{{p_\alpha}^{-1}(U_\alpha) \mid U_\alpha \in \mathcal{T}_\alpha, \alpha \in I\}$.

Explicitly, open sets in the product topology are unions of $\prod\limits_{\alpha \in I} U_\alpha$ where U_α are open in X_α and all but finite U_α are empty.

Proposition. The Cartesian product with the above product topology is a product in $\mathscr{T}op$.

Proof. Leave as an exercise. $\square$

Remark. The box topology on the Cartesian product is the topology whose open sets are unions of $\prod\limits_{\alpha \in I} U_\alpha$ where U_α is open in X_α. It equals the product topology if $|I| < \infty$. However, if $|I| = \infty$, it is strictly finer (stronger) than the product topology and is NOT a product in $\mathscr{T}op$.

That is all to say.