PMATH 440 Notes

Analytic Number Theory

Fall 2025

Based on Professor Michael Rubinstein's Lectures

Contents

Lecture 1, 2025/09/04

1 Introduction

Topics covered in this course

(1). Summation methods (summation by parts, Euler-Maclaurin Summation, Poisson Summation, Dirichlet Hyperbola).

(2). Dirichlet series and Dirichlet divisor problem.

(3). Riemann zeta function ζ. Meromorphic continuation (ζ has a pole at $s = 1$) and functional equation.

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} \quad \text{for} \quad \mathrm{Re}(s) > 1$$

(4). Prime Number Theorem. If $\pi(x) =$ number of prime numbers $\leq x$, then

$$\pi(x) \sim \int_2^x \frac{1}{\log t}\, \mathrm{d}t \sim \frac{x}{\log x}$$

(5). Dirichlet's Theorem. If $0 \neq a, b \in \mathbb{Z}$ and $\gcd(a, b) = 1$, there are infinitely many prime numbers of the form $ak + b$ for $k \in \mathbb{Z}$. For example, there are infinitely many primes of the form $4k + 1$.

(6). More Complex analysis. Gamma function, Weierstrass products and possibly linear fractional transformations and modular forms.

We first introduce some asymptotic notations.

Definition. We say that $f(x) \sim g(x)$ as $x \to \infty$ if

$$\lim_{x\to a} \frac{f(x)}{g(x)} = 1$$

The Prime Number Theorem says $\pi(x) \sim \dfrac{x}{\log x}$ as $x \to \infty$, which is equivalent to

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1$$

Example. By the Stirling's approximation, we know

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad \text{as } n \to \infty$$

Definition. Let f, g be defined on (a subset of) $\mathbb{R}$ and g be a real-valued. We write $f(x) = \mathcal{O}(g(x))$ as $x \to \infty$, where g is real-valued, if there exists $c > 0$ such that $|f(x)| \leq cg(x)$ for all $x > x_0$.

Example. $\sin(x) = \mathcal{O}(1)$ as $x \to \infty$ since sin is bounded.

Example. By the Stirling's formula we have

$$n! = \mathcal{O}\left(\sqrt{n}\left(\frac{n}{e}\right)^n\right) \quad \text{and} \quad n! = \mathcal{O}\left(\frac{n^{n+1}}{e^n}\right)$$

The first one implies the second one because $\sqrt{n} = \mathcal{O}(n)$.

Definition. We write $f(x) = o(g(x))$ as $x \to a$ if

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0$$

In most cases we will take $a = \infty$ or $a = -\infty$. This means "$f(x)$ is much smaller than $g(x)$ near a".

Example. By the Stirling's formula we have

$$\lim_{n \to \infty} \frac{n!}{\frac{n^{n+1}}{e^n}} = \lim_{n \to \infty} \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\frac{n^{n+1}}{e^n}} = \lim_{n \to \infty} \frac{\sqrt{2\pi}}{\sqrt{n}} = 0$$

It follows that $n! = o(n^{n+1}/e^n)$ as $n \to \infty$.

Remark (Vinogradov's notation). We can also write $f(x) = \mathcal{O}(g(x))$ as $f(x) \ll g(x)$.

2 Summation Methods

2.1 Summation by parts

This method is the discrete version of integration by parts.

Theorem 2.1. Let $f : \mathbb{N} \to \mathbb{C}$ and $g : \mathbb{R} \to \mathbb{C}$ be continuously differentiable on $[1, x]$. Then, for all $x \geq 1$ we have

$$\sum_{1 \leq n \leq x} f(n)g(n) = \left(\sum_{1 \leq n \leq x} f(n)\right) g(x) - \int_1^x \sum_{1 \leq n \leq t} f(n)g'(t) \text{ dt} \tag{1}$$

Proof. Consider the term $f(n)g(n)$, we note

$$f(n)g(x) - f(n)\int_n^x g'(t) \text{ dt} = f(n)g(x) - f(n)(g(x) - g(n)) = f(n)g(n) \tag{2}$$

This equality is obtained by looking at the terms that have to do with $f(n)$ in (1). Then summing the equation (2) over $1 \leq n \leq x$ gives us (1). ☐

Example. Consider the harmonic series $\sum_{1\leq n\leq x}\frac{1}{n}$. Take $f(n)=1$ and $g(x)=\frac{1}{x}$. Then by the partial summation formula we have

$$\sum_{1\leq n\leq x}\frac{1}{n}=\left(\sum_{1\leq n\leq x}f(n)\right)g(x)-\int_1^x\sum_{1\leq n\leq t}f(n)g'(t)\ \mathrm{d}t=\frac{\lfloor x\rfloor}{x}+\int_1^x\frac{\lfloor t\rfloor}{t^2}\mathrm{d}t$$

Here note that

$$\lfloor x\rfloor:=\sum_{1\leq n\leq x}1=\text{the largest integer}\leq x$$

and using this we define

$$\{x\}:=x-\lfloor x\rfloor=\text{the fractional part of } x$$

For example $\lfloor\pi\rfloor=3$ and $\{\pi\}=0.1415926535897\cdots$. Therefore

$$\begin{aligned}\sum_{1\leq n\leq x}\frac{1}{n}&=\frac{x-\{x\}}{x}+\int_1^x\frac{t-\{t\}}{t^2}\mathrm{d}t\\&=1-\frac{\{x\}}{x}+\int_1^x\frac{1}{t}-\frac{\{t\}}{t^2}\ \mathrm{d}t\\&=1+\log x-\int_1^x\frac{\{t\}}{t^2}\ \mathrm{d}t+\mathcal{O}\left(\frac{1}{x}\right)\end{aligned}$$

Now we analyze this integral

$$\int_1^x\frac{\{t\}}{t^2}\mathrm{d}t=\underbrace{\int_1^\infty\frac{\{t\}}{t^2}\mathrm{d}t}_{<\infty}-\underbrace{\int_x^\infty\frac{\{t\}}{t^2}\mathrm{d}t}_{\mathcal{O}(\frac{1}{x})}$$

The estimation of second integral is by bounding $\{t\}/t^2$ by $1/t^2$. Therefore

$$\sum_{1\leq n\leq x}\frac{1}{n}=\log x+\underbrace{1-\int_1^\infty\frac{\{t\}}{t^2}\mathrm{d}t}_{:=\gamma}+\mathcal{O}\left(\frac{1}{x}\right)$$

This constant γ is called the Euler's constant, that is,

$$\lim_{x\to\infty}\left(\sum_{1\leq n\leq x}\frac{1}{n}-\log x\right)=\gamma$$

Conjecture: γ is irrational.

Example. Take the log and consider $\log(n!)$. Let $f(n) = 1$ and $g(x) = \log(x)$. By the partial summation formula we have

$$\begin{aligned}
\sum_{1 \le m \le n} \log(m) &= n \log n - \int_1^n \frac{\lfloor t \rfloor}{t} \, \mathrm{d}t \\
&= n \log n - \int_1^n \frac{t - \{t\}}{t} \, \mathrm{d}t \\
&= n \log n - (n-1) + \int_1^n \frac{\{t\}}{t} \, \mathrm{d}t
\end{aligned}$$

Lecture 2, 2025/09/09