PMATH 351 Notes

Real Analysis

Winter 2025

Based on Professor Kevin Hare's Lectures

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Lecture 1, 2025/01/06

1 Metric Spaces

1.1 Normed Vector Spaces

Definition. Let V be a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We say $\|\cdot\| : V \to \mathbb{R}$ is a **norm** if:

(i). For all $v \in V$ we have $\|v\| = 0 \iff v = 0$.

(ii). For all $v \in V$ and $\lambda \in \mathbb{K}$ we have $\|\lambda v\| \leq |\lambda|\|v\|$.

(iii). For all $v, w \in V$ we have $\|v + w\| \leq \|v\| + \|w\|$.

A vector space, combined with a norm, is called a **normed vector space**.

Example. Let $V = \mathbb{R}^n$. Define a map $\|\cdot\|_1 : \mathbb{R}^n \to \mathbb{R}$ by:

$$\|v\|_1 = \|(x_1, \cdots, x_n)\|_1 = |x_1| + \cdots + |x_n|$$

Clearly property 1 and 2 holds. To see property 3 we have:

$$\begin{aligned} \|(x_1, \cdots, x_n) + (y_1, \cdots, y_n)\|_1 &= |x_1 + y_1| + \cdots + |x_n + y_n| \\ &\leq |x_1| + |y_1| + \cdots + |x_n| + |y_n| && (\triangle \text{ inequality in } \mathbb{R}) \\ &= \|(x_1, \cdots, x_n)\|_1 + \|(y_1, \cdots, y_n)\|_1 \end{aligned}$$

Hence $\|\cdot\|_1$ defines a norm on $V = \mathbb{R}^n$.

Example. Let $V = \mathbb{R}^n$ again. Define $\|\cdot\|_\infty : \mathbb{R}^n \to \mathbb{R}$ by:

$$\|v\|_\infty = \|(x_1, \cdots, x_n)\|_\infty = \max(|x_1|, \cdots, |x_n|)$$

This also defines a norm on $\mathbb{R}^n$.

Example. What does the unit ball $B = \{v \in V : \|v\| \leq 1\}$ look like? Take $V = \mathbb{R}^2$.

Note. It is possible to extend these two norms to infinite dimensional vector spaces if we are being careful. Both of the norms above are examples of p-norms, for $1 \leq p \leq \infty$.

Example. Let $V = \mathbb{R}[x]$ be a vector over $\mathbb{R}$. Define $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on V by:

$$\|f|_1 = \int_0^1 |f(x)| \, \mathrm{d}x \quad \text{and} \quad \|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$$

The three properties are satisfied by these two norms. Note these norms can be defined beyond polynomials if we are careful.

Theorem 1.1 (Minkowski). Let $1 \leq p < \infty$ be a real number.

(i). We define:

$$\ell_p = \left\{ (x_n)_{n=1}^{\infty} \subseteq \mathbb{C} : \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} < \infty \right\}$$

Then the map $\|\cdot\|_p : \ell_p \to \mathbb{R}$ defined by:

$$\|(x_n)\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}$$

defines a norm on ℓ_p. This is called the ℓ_p**-space**.

(ii). Let $\mathcal{C}[a,b]$ be the set of continuous functions on $[a,b]$. Then:

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, \mathrm{d}x \right)^{1/p}$$

defines a norm. Define $L^p[a,b] = \{f \in \mathcal{C}[a,b] : \|f\|_p < \infty\}$, called the L^p**-space**.

Proof. Note for $p \geq 1$, define a map $\varphi(x) = |x|^p$ and φ is convex on $\mathbb{R}$. We will prove part 2 first. Assume $f, g \in \mathcal{C}[a,b]$ and $f, g \neq 0$. If $f = 0$ or $g = 0$ the triangle inequality is easy to prove.

$$\begin{aligned}
\|f+g\|_p^p = \int_a^b |f(x)+g(x)|^p \, \mathrm{d}x &= \int_a^b \left| \|f\|_p \cdot \frac{f}{\|f\|_p} + \|g\|_p \cdot \frac{g}{\|g\|_p} \right|^p \mathrm{d}x \\
&= (\|f\|_p + \|g\|_p)^p \int_a^b \left| \underbrace{\frac{\|f\|_p}{\|f\|_p + \|g\|_p}}_{\alpha} \cdot \frac{f}{\|f\|_p} + \underbrace{\frac{\|g\|_p}{\|f\|_p + \|g\|_p}}_{1-\alpha} \cdot \frac{g}{\|g\|_p} \right|^p \mathrm{d}x
\end{aligned}$$

Note that $\alpha \in [0,1]$, we can rewrite the above quantity as:

$$\begin{aligned}
I &:= (\|f\|_p + \|g\|_p)^p \int_a^b \left| \alpha \cdot \frac{f}{\|f\|_p} + (1-\alpha) \cdot \frac{g}{\|g\|_p} \right|^p \mathrm{d}x \\
&= (\|f\|_p + \|g\|_p)^p \int_a^b \varphi\left(\alpha \cdot \frac{f}{\|f\|_p} + (1-\alpha) \cdot \frac{g}{\|g\|_p} \right)^p \mathrm{d}x
\end{aligned}$$

Recall $\varphi(x) = |x|^p$ is convex, we have:

$$\begin{aligned}
I &\leq (\|f\|_p + \|g\|_p)^p \left(\alpha \int_a^b \left| \frac{f}{\|f\|_p} \right|^p \mathrm{d}x + (1-\alpha) \int_a^b \left| \frac{g}{\|g\|_p} \right|^p \mathrm{d}x \right) \\
&= (\|f\|_p + \|g\|_p)^p (\alpha + 1 - \alpha) = (\|f\|_p + \|g\|_p)^p
\end{aligned}$$

This proved that:

$$\|f+g\|_p^p \leq (\|f\|_p + \|g\|_p)^p \implies \|f+g\|_p \leq \|f\|_p + \|g\|_p$$

Part 1 (ℓ_p-space) are proved in the similar way by replacing integral with sum. $\square$

Lecture 2, 2025/01/08

1.2 Metric Spaces

Definition. Let X be a non-empty set. A **distance (metric)** on X is a function $d : X \times X \to [0, \infty)$ such that:

(i). For all $x, y \in X$ we have $d(x, y) = 0 \iff x = y$.

(ii). For all $x, y \in X$ we have $d(x, y) = d(y, x)$.

(iii). For all $x, y, z \in X$ we have $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (X, d) is called a **metric space**. We just say X is a metric space if d is understood.

Example. Let $(X, \|\cdot\|)$ be a normed vector space, then $d(x, y) = \|x - y\|$ is a metric on X. Clearly $d(x, y) = 0 \iff \|x - y\| = 0 \iff x - y = 0 \iff x = y$. Property (ii) is also trivial. For property (iii) we have:

$$d(x, z) = \|x - z\| = \|x - y + y - z\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z)$$

Example (Graph metric). Let (X, E) be a graph where X is the vertex set. The set of **paths** from x to y is:

$$P_{xy} = \{(x = x_1, x_2, \cdots, x_n = y) : (x_i, x_{i+1}) \in E\}$$

Define a **weight** function $\omega : E \to (0, \infty)$. Then:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ \min\{\omega(x_1, x_2) + \cdots + \omega(x_{n-1}, x_n) \text{ for } (x_1, \cdots, x_n) \in P_{xy}\} & \text{otherwise} \end{cases}$$

This distance basically measures the shortest path from x to y, with weight on the edge.

Example (Trivial metric). Let X be a non-empty set, define:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Exercise: It is easy to verify that this is a distance function on X.

Example (p-adic metric on $\mathbb{Q}$). Let p be a fixed prime in $\mathbb{N}$. By unique factorization, every $q \in \mathbb{Q}$ can be uniquely written as:

$$q = p^n \frac{a}{b}$$

where $n \in \mathbb{Z}$ and $a \in \mathbb{Z}$ and $0 \neq b \in \mathbb{N}$ with $\gcd(a, b) = 1$. Define the **p-adic norm** by:

$$|q|_p = \begin{cases} p^{-n} & \text{if } q \neq 0 \text{ and } n \text{ is from above} \\ 0 & \text{if } q = 0 \end{cases}$$

Exercise: For $q, r \in \mathbb{Q}$ we have:

$$|q + r|_p \leq \max\{|q|_p,\ |r|_p\} \leq |q|_p + |r|_p$$

Take $p = 3$ and $q = 1/6$ and $r = 2/9$, then:

$$\begin{aligned} |q|_3 &= \left|3^{-1} \cdot \frac{1}{2}\right|_3 = 3^{-(-1)} = 3 \\ |r|_3 &= \left|3^{-2} \cdot \frac{2}{1}\right|_3 = 3^{-(-2)} = 9 \\ |q + r|_3 &= \left|\frac{3+4}{18}\right|_3 = \left|3^{-2} \cdot \frac{7}{2}\right|_3 = 9 = \max\{3, 9\} \end{aligned}$$

Define the **p-adic metric** on $\mathbb{Q}$ by:

$$d_p(q, r) = |q - r|_p$$

Exercise: This clearly defined a metric on $\mathbb{Q}$.

Example. Consider $\{0,1\}^{\mathbb{N}} = \{(b_n)_{n=1}^{\infty} : b_n \in \{0,1\}\}$. Take $b, c \in \mathbb{N}$ then define:

$$d(b, c) := \begin{cases} 0 & \text{if } b = c \\ \frac{1}{2^n} \text{ for } b = \min\{i \in \mathbb{N} : b_i \neq c_i\} & \text{otherwise} \end{cases}$$

Exericse: d is a metric on $\{0,1\}^{\mathbb{N}}$, we may call this product metric. Now we define:

$$\rho(b, c) = \sum_{n=1}^{\infty} \frac{d(b_n, c_n)}{2^n} \qquad \text{(always converges)}$$

Fact (Exercise): $d(b, c) \leq \rho(b, c) \leq 2d(b, c)$.

Definition. Let (X, d) be a metric space. If $\emptyset \neq Y \subseteq X$, we make Y a metric space by defining $d_Y : Y \times Y \to \mathbb{R}$ by $d_Y(x, y) = d(x, y)$ for $x, y \in Y$. [This is just the restriction $d|_{Y \times Y}$] This is called the **relativized metric** on Y.

Definition. Let X be a non-empty set and d_1, d_2 be metrics on X. We say d_1 is **equivalent** to d_2 if there exist $c, C > 0$ such that:

$$cd_1(x,y) \le d_2(x,y) \le Cd_1(x,y)$$

for all $x, y \in X$. Exercise: This is an equivalence relation on the set of metrics on X.

Example. Let $X = \mathbb{R}^n$ and $1 \le p < \infty$. Define a metric:

$$d_p(x,y) = \|x-y\|_p = \left(\sum_{k=1}^{n} |x_i - y_i|^p\right)^{1/p}$$

and define $d_\infty(x,y) = \max\{|x_k - y_k| : k \in \{1, \cdots, n\}\}$. Let $x \in \mathbb{R}^n$, say $\|x\|_\infty = x_j$ for some j. Then we note that:

$$\|x\|_\infty = |x_j| = (|x_j|^p)^{1/p} \le \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} = \|x\|_p \le \left(\sum_{k=1}^{n} |x_j|^p\right)^{1/p} = n^{1/p}\|x\|_\infty$$

To summarize we have:

$$\|x\|_\infty \le \|x\|_p \le n^{1/p}\|x\|_\infty$$

Hence $\|\cdot\|_\infty$ and $\|\cdot\|_p$ are equivalent norms for all $1 \le p < \infty$. By equivalence, $\|\cdot\|_p$ are all equivalent norms on $\mathbb{R}^n$ for $1 \le p \le \infty$.

Lecture 3, 2025/01/10

1.3 Topology of Metric Spaces

Definition. Let (X,d) be a metric space. Take $x \in X$ and $r > 0$. Define an **open ball** centered at x with radius r to be:

$$B_r(x) := b_r(x) := B(x,r) := \{y \in X : d(x,y) < r\}$$

Similarly we define a **closed ball** as:

$$\overline{B}_r(x) := \overline{b}_r(x) := \overline{B}(x,r) := \{y \in X : d(x,y) \le r\}$$

Definition. Let (X,d) be a metric space. Let $N \subseteq X$ with some $x \in X$. We say N is a **neighborhood** of x if there exists $r > 0$ such that $B_r(x) \subseteq N$.

Definition. Let (X,d) be a metric space. We say $N \subseteq X$ is **open** if N is a neighborhood of x for all $x \in N$. We say N is **closed** if $X \setminus N$ is open.

Example. Let $X = \mathbb{R}$ with usual Euclidean metric. Then (a, b) is open for all $a < b$ in $\mathbb{R}$. The empty set $\emptyset$ and $\mathbb{R}$ are open.

Remark. In general, in a metric space (X, d), the set X is trivially open and the empty set $\emptyset$ is vacuously open. Note that $X \setminus X = \emptyset$ and $X \setminus \emptyset = X$. Hence $X, \emptyset$ are both open and closed.

Example. Let (X, d) be a metric space where d is the discrete metric. Every subset $N \subseteq X$ is open! Why? Take $r = 1/2$ and $x \in N$, then $B_{1/2}(x) = \{x\} \subseteq N$. Similarly every subset is closed.

Question: Consider the metric space $(\mathbb{Q}, d_3)$, where d_3 is the 3-adic metric. What do the open sets look like?

Theorem 1.2 (Union of Open sets). Let (X, d) be a metric space. Let $\{X_i\}_{i \in I}$ be a collection of open sets, then $\bigcup_{i \in I} X_i$ is an open set.

Proof. Let $x \in \bigcup_{i \in I} X_i$, then $x \in X_{i_0}$ for some $i_0 \in I$. Since X_{i_0} is open, there is $r > 0$ such that $B_r(x) \subseteq X_{i_0}$. Hence:

$$B_r(x) \subseteq X_{i_0} \subseteq \bigcup_{i \in I} X_i$$

It follows that $\bigcup_{i \in I} X_i$ is open, as desired. □

Corollary 1.3 (Intersection of Closed sets). Let (X, d) be a metric space and $\{X_i\}_{i \in I}$ a collection of closed sets. Then $\bigcap_{i \in I} X_i$ is closed.

Proof. Take complement using De Morgan's Law and apply the above theorem. □

Question: If $\{X_i\}_{i \in I}$ is a collection of open sets, waht can we say about $\bigcap_{i \in I} X_i$?

(i). If $|I| = n < \infty$, then this intersection is open. Consider $\{X_1, \cdots, X_n\}$. Take $x \in \bigcap_{i=1}^n X_i$, then $x \in X_i$ for all i, so there is $r_i > 0$ such that $B(x, r_i) \subseteq X_i$ for all i. Take $r = \min\{r_1, \cdots, r_n\}$, then $B(x, r) \subseteq \bigcap_{i=1}^n X_i$, hence open.

(ii). If $|I| > |\mathbb{N}|$, this may fail. For example, take $X_n = (\frac{-1}{n}, \frac{1}{n})$. Then $\bigcap_{n=1}^\infty X_n = \{0\}$, not open.

Proposition 1.4. Finite intersection of open sets is open and finite union of closed sets is closed.

Definition. Let (X, d) be a metric sapce and $(x_n)_{n=1}^\infty$ be a sequence in X. Let $x \in X$. We say the sequence $(x_n)_{n=1}^\infty$ **converges** to x if $\lim_{n \to \infty} d(x, x_n) = 0$. Equivalently, for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$:

$$n \geq N \implies d(x, x_n) < \epsilon$$

In this case we can write $\lim_{n \to \infty} x_n = x$.

Example. Let $X = \mathbb{Q}$ with Euclidean metric. Let $(x_n)_{n=1}^{\infty}$ be $x_n = 1/n$. This converges to 0.

Example. Let $X = \mathbb{Q}$, consider the sequence defined by:

$$x_n = \text{truncation of } \pi \text{ to the } n\text{-th decimal place}$$

For example $x_1 = 3,\ x_2 = 3.1,\ x_3 = 3.14$ and so on. This sequence "converges" to π, but $\pi \notin \mathbb{Q}$ so this sequence does not converge in $\mathbb{Q}$! It converges in $\mathbb{R}$.

Example. Let (X, d) with the discrete metric. A sequence (a_n) is convergent if and only if it is eventually constant. That is, there is $N \in \mathbb{N}$ such that $x_n = X_N$ for all $n \geq N$. In this case the limit is just $\lim\limits_{n\to\infty} x_n = X_N$.

Example. Consider $(\mathbb{Q}, d_3)$, the 3-adic metric. Consider the two sequences:

$$(x_n)_{n=1}^{\infty} = \left(\frac{1}{n}\right)_{n=1}^{\infty} \quad \text{and} \quad (y_n)_{n=1}^{\infty} \quad \text{by} \quad y_n = \begin{cases} 2 & \text{if } n = 1 \\ 2 + 3y_{n-1} & \text{if } n \geq 2 \end{cases}$$

For the first sequence (x_n), it has a subsequence $(3^{-k})_{k=1}^{\infty}$ and $d(0, 3^{-k}) = 3^k \to \infty$. Hence (x_n) does not converge (We defer the actual proof of this when we see Cauchy sequence). For the second sequence, we see that:

$$y_n = 2 + 2 \cdot 3 + 2 \cdot 3^2 + \cdots + 2 \cdot 3^{n-1} = 3^n - 1$$

Hence $d(-1, y_n) = \| -3^n\|_3 = \frac{1}{3^n} \to 0$ so that $\lim\limits_{n\to\infty} y_n = -1$.

Lecture 4, 2025/01/13

Definition. Let (X, d) be a metric space and $A \subseteq X$. The **closure** of A in X is the smallest closed set in X that contains A. We denote the closure of A by $\mathrm{cl}(A)$ or $\overline{A}$. In other word, $\overline{A}$ is the intersection of all closed sets that contain A.

Example. The closure of a closed set is itself.

Example. Consider the metric space $(\mathbb{R}, d)$ with the usual Euclidean metric. Then $\overline{\mathbb{Q}} = \mathbb{R}$.

Example. Consider $\mathbb{R}$ again with discrete metric, then $\overline{\mathbb{Q}} = \mathbb{Q}$. (because every set is closed in this topology).

Example. Consider $(\mathbb{Q}, d)$ with the 3-adic metric. We can show that $\mathbb{Z}$ is not closed. Define a sequence $(x_n)_{n=1}^{\infty}$ by $x_n = \sum_{k=0}^{n} 9^k$. This sequence has a limit in $\mathbb{Q}$ but not in $\mathbb{Z}$. How do we "guess" the limit of this sequence? Notice that:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{for all} \quad \|x\| < 1$$

In this case $\|9\|_3 = 1/9 < 1$, hence plugging in 9 shows the limit of (x_n) is $-1/8 \notin \mathbb{Z}$. [This is NOT a rigorous proof for now! This just allows us to guess the limit and we can then use the ϵ thing to prove the limit]. Hence $\mathbb{Z}$ does not contain a limit point, which means it is not closed (by the theorem below).

Theorem 1.5. A closed set contains all of its limit points. That is, if $A \subseteq X$ is closed and $(x_n)_{n=1}^{\infty}$ is a sequence in A, then whenever $\lim_{n\to\infty} x_n = x \in X$ exists, we must have $x \in A$.

1.4 Continuous Functions

Definition. Let (X,d) and (Y,ρ) be metric spaces. A function $f : X \to Y$ is **continuous at** $x_0 \in X$ if for all $\epsilon > 0$ there is $\delta > 0$ such that for all $x \in X$ with $d(x,x_0) < \delta$, then $\rho(f(x), f(x_0)) < \epsilon$. [Equivalently we have $f(B_\delta(x_0)) \subseteq B_\epsilon(\epsilon)$.]

Example. Let (X,d) be a metric space with discrete metric. Let $f : X \to Y$ with (Y,ρ) a metric space. Then f is continuous at every $x_0 \in X$. Why? For any $\epsilon > 0$, pick $\delta = 1/2$. Then $d(x,x_0) < 1/2$ implies $d(x,x_0) = 0$ and $x = x_0$. Hence $\rho(f(x), f(x_0)) = 0 < \epsilon$.

Definition. Let (X,d) and (Y,ρ) be metric spaces.

(i). We say $f : X \to Y$ is **continuous on** X if it is continuous at all $x_0 \in X$.

(ii). We say $f : X \to Y$ is **uniformly continuous on** X if for all $\epsilon > 0$, there is $\delta > 0$ such that for all $x, y \in X$:

$$d(x,y) < \delta \implies \rho(f(x), f(y)) < \epsilon$$

That is, the choice of $\delta > 0$ is independent of $x, y \in X$. The usual continuity means for any $x, y \in X$ we can choose a $\delta > 0$ for them, but in this case there is one choice of $\delta > 0$ that works for all $x, y \in X$.

Note. In the Example above, we see that f is in fact uniformly continuous.

Example. Let $f : (0,1) \to (0,\infty)$ with Euclidean metric given by $f(x) = 1/x$. This function is continuous but NOT uniformly continuous. To see it is continuous, fix $x_0 \in (0,1)$ and let $\epsilon > 0$. We then pick $\delta > 0$ to be:

$$\delta = \min\left\{\frac{x_0}{2}, \frac{\epsilon x_0^2}{2}\right\}$$

Then, if $|x - x_0| < \delta$, we have:

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \left|\frac{x_0 - x}{x x_0}\right| < \frac{\epsilon \cdot x_0^2/2}{(x_0/2)x_0} = \epsilon$$

It follows that f is continuous on $(0,1)$. To see it is NOT uniformly continuous, assume it is. Take $\epsilon = 1$, then there is $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x)-f(y)| < 1$. However, pick $N \in \mathbb{N}$ large enough so that $1/N - 1/(N+1) < \delta$, then:

$$1 > \left| f\left(\frac{1}{N}\right) - f\left(\frac{1}{N+1}\right) \right| = \left| \frac{1}{1/N} - \frac{1}{1/(N+1)} \right| = 1$$

This is a contradiction, hence f is NOT uniformly continuous.

Definition. Let X, Y be metric spaces. We say $f : X \to Y$ is **sequentially continuous at** $x_0 \in X$ if for all sequence $(x_n)_{n=1}^{\infty}$ in X we have:

$$\lim_{n\to\infty} x_n = x_0 \implies \lim_{n\to\infty} f(x_n) = f(x_0)$$

We say f is **sequentially continuous** if it is sequentially continuous at every $x_0 \in X$.

Theorem 1.6. Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$. The followings are equivalent:

(i). f is continuous.

(ii). For all open sets $V \subseteq Y$ we have $f^{-1}(V)$ is open in X.

(iii). f is sequentially continuous.

Proof. We will prove (i) $\implies$ (ii) $\implies$ (i) and (i) $\implies$ (iii) $\implies$ (i).

(i) $\implies$ (ii). Assume f is continuous. Let $V \subseteq Y$ be open. We want to show $f^{-1}(V)$ is open in X. If $f^{-1}(V) = \emptyset$, done. Otherwise pick $x_0 \in f^{-1}(V)$, then $f(x_0) \in V$. Then is $\epsilon > 0$ such that $B_\epsilon(f(x_0)) \subseteq V$. Since f is continuous at x_0, there is $\delta > 0$ such that $f(B_\delta(x_0)) \subseteq B_\epsilon(f(x_0)) \subseteq V$. Hence $B_\delta(x_0) \subseteq f^{-1}(V)$ and therefore $f^{-1}(V)$ is open in X.

(i) $\implies$ (iii). Assume f is continuous at x_0 and $(x_n)_{n=1}^{\infty}$ is a sequence with $x_n \to x_0$. Pick $\epsilon > 0$, since f is continuous there is $\delta > 0$ such that $d(x, x_0) < \delta$ implies $\rho(f(x), f(x_0)) < \epsilon$. Now pick $N \in \mathbb{N}$ so that $d(x_n, x_0) < \delta$ for $n \geq N$. Hence if $n \geq N$ we have $\rho(f(x_n), f(x_0)) < \epsilon$.

Lecture 5, 2025/01/15

(ii) $\implies$ (i). Fix $x_0 \in X$ and let $\epsilon > 0$. Consider the open set $V = B_\epsilon(f(x_0))$. Since we are assuming (ii), we know $f^{-1}(V)$ is open and $x_0 \in f^{-1}(V)$. Therefore there is $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(V)$. Therefore we have $f(B_\delta(x_0)) \subseteq B_\epsilon(f(x_0))$, which proved f is continuous at x_0.

(iii) $\implies$ (i). We will prove this by contrapositive. Assume f is not continuous. This means there is $x_0 \in X$ and $\epsilon > 0$ such that for all $\delta > 0$, there are $x \in X$ with $d(x, x_0) < \delta$ but $\rho(f(x), f(x_0)) \geq \epsilon$. We are going to construct a sequence $(x_n)_{n=1}^{\infty} \subseteq X$ using this information that breaks the sequential continuity. For $n \in \mathbb{N}$, we choose $x_n \in X$ such that $d(x_0, x_n) < 1/n$ but $\rho(f(x_0), f(x_n)) \geq \epsilon$. Then we clearly have $x_n \to x_0$ but $f(x_n)$ does NOT converge to $f(x_0)$ as they are always at least ϵ-away from each other. $\square$

Theorem 1.7. Let X, Y, Z be metric spaces. Let $f : X \to Y$ and $g : Y \to Z$ be continuous. Then $g \circ f : X \to Z$ is continuous.

Definition. Let (X, d) and (Y, ρ) be metric spaces. We can define a metric $d \times \rho$ on $X \times Y$ by:

$$(d \times \rho)((x_1, y_1), (x_2, y_2)) := d(x_1, x_2) + \rho(y_1, y_2)$$

It is easy to check that this defines a metric.

Theorem 1.8. Let X, Y, Z, W be metric spaces and $f : X \to Z$ and $g : Y \to W$ be continuous. Then $f \times g : X \times Y \to Z \times W$ by $(f \times g)(x, y) = (f(x), g(y))$ is continuous where $X \times Y$ and $Z \times W$ are equipped with the metric defined above.

Definition. Let $f : X \to Y$ where (X, d) and (Y, ρ) are metric spaces. We say f is an **isometry** if for all $x_1, x_2 \in X$ we have $d(x_1, x_2) = \rho(f(x_1), f(x_2))$.

Example. In $\mathbb{R}^2$, any rotation, reflection, translation and combination of them are isometries.

Definition. Let (X, d) and (Y, ρ) be metric spaces. A map $f : X \to Y$ is called **Lipschitz** if there exists a constant $C > 0$ such that:

$$\rho(f(x_1), f(x_2)) \leq C d(x_1, x_2)$$

for all $x_1, x_2 \in X$.

Definition. Let (X, d) and (Y, ρ) be metric spaces. A map $f : X \to Y$ is called **bi-Lipschitz** if there exist constants $C, c > 0$ such that:

$$c d(x_1, x_2) \leq \rho(f(x_1), f(x_2)) \leq C d(x_1, x_2)$$

for all $x_1, x_2 \in X$.

Definition. Let (X, d) and (Y, ρ) be metric spaces. We say $f : X \to Y$ is a **homeomorphism** if f is a continuous bijection such that $f^{-1} : Y \to X$ is also continuous.

1.5 Finite dimensional normed vector spaces

Definition. Let V be a vector space. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are said be to **equivalent** if there are constants $c, C > 0$ such that:

$$c\|v\|_2 \leq \|v\|_1 \leq C\|v\|_2$$

for all $v \in V$. It is clear that this is an equivalence relation.

Theorem 1.9. For $n \in \mathbb{N}$, all norms in $\mathbb{R}^n$ are equivalent. The similar result holds for $\mathbb{C}^n$.

Proof. It suffices to show all norms $\|\cdot\|$ are equivalent to the 1-norm $\|\cdot\|_1$. Then since equivalence norm is an equivalence relation, all norms are equivalent. A basis for $\mathbb{R}^n$ is $\{e_1, \cdots, e_n\}$, the standard basis. Let $C = \max\{\|e_1\|, \cdots, \|e_n\|\}$. Let $v = (v_1, \cdots, v_n) \in \mathbb{R}^n$, then:

$$\begin{aligned} \|v\| &= \|v_1 e_1 + \cdots + v_n e_n\| \\ &\leq |v_1|\|e_1\| + \cdots + |v_n|\|e_n\| && (\triangle\text{-inequality}) \\ &\leq C(|v_1| + \cdots + |v_n|) \\ &= C\|v\|_1 \end{aligned}$$

This gives us one inequality. This also shows that $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz, hence continuous (where $\mathbb{R}^n$ is equipped with $\|\cdot\|_1$ norm). Define:

$$S = \{v \in \mathbb{R}^n : \|v\|_1 = 1\}$$

Since $\|\cdot\|$ is continuous on $(\mathbb{R}^n, \|\cdot\|_1)$ we have that $\|\cdot\| : S \to \mathbb{R}$ obtains its maximum and minimum. Further, the minimum is nonzero. Define $c = \min_{v \in S} \|v\| > 0$. Note for all $0 \neq v \in \mathbb{R}^n$ we have that $v/\|v\|_1 \in S$. Hence:

$$\left\| \frac{v}{\|v\|_1} \right\| \geq c \implies \|v\| \geq c\|v\|_1$$

Hence $c\|v\|_1 \leq \|v\| \leq C\|v\|_1$, as desired. $\square$

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1.6 Completeness

Definition. Let (X, d) be a metric space and $(x_n)_{n=1}^\infty$ be a sequence in X. We say $(x_n)_{n=1}^\infty$ is a **Cauchy sequence** if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$:

$$n, m \geq N \implies d(x_n, x_m) < \epsilon$$

Example. Let $(\frac{1}{n})_{n=1}^\infty$ be a sequence in $\mathbb{Q}$ but with different metrics.

(i). If $\mathbb{Q}$ is equipped with the Euclidean metric. This is clearly Cauchy. To see that, let $\epsilon > 0$ pick $N > 2/\epsilon$, then for $n, m \geq \mathbb{N}$ we have:

$$d\left(\frac{1}{n}, \frac{1}{m}\right) = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

(ii). If $\mathbb{Q}$ is equipped with the discrete metric, then $d(1/n, 1/m) = 1$ for all $n, m \in \mathbb{N}$. This means this sequence is not Cauchy. (If it is Cauchy, take $\epsilon = 1/2$ then contradiction) (iii). If $\mathbb{Q}$ is equipped with the 3-adic metric, then this is not a Cauchy sequence. Let $n = 3^k$ and $m = 3^\ell$ where $k \neq \ell$. Then we have:

$$d\left(\frac{1}{n}, \frac{1}{m}\right) = d\left(\frac{1}{3^k}, \frac{1}{3^\ell}\right) = 3^{\min\{k,\ell\}}$$

If we pick k, ℓ large enoguh, then the distance between them can be arbitrarily large. Hence this is not a Cauchy sequence.

Theorem 1.10. Let (X, d) be a metric space and let $(x_n)_{n=1}^\infty$ be a convergent sequence, then $(x_n)_{n=1}^\infty$ is a Cauchy sequence.

Proof. Say $\lim_{n\to\infty} x_n = x^* \in X$. Let $\epsilon > 0$, pick $N \in \mathbb{N}$ such that for $n \in \mathbb{N}$:

$$n \geq N \implies d(x^*, x_n) < \frac{\epsilon}{2}$$

Now pick $n, m \in \mathbb{N}$ such that $n, m \geq \mathbb{N}$, we have:

$$d(x_n, x_m) \leq d(x_n, x^*) + d(x_m, x^*) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore $(x_n)_{n=1}^\infty$ is a Cauchy sequence. □

Example (The Converse is False). Every convergent sequence is Cauchy but not the other way around. There are cauchy sequences that do not converge.

(i). Let $X = \mathbb{Q}$ with the Euclidean metric. Let $x_n =$ the truncation of π to the n-th decimal place. For example: $x_1 = 3$, $x_2 = 3.1$, $x_3 = 3.14$ and so on. This is a Cauchy sequence, but the limit does not exist (because its "limit" is π, which is not in $\mathbb{Q}$).

(ii). Consider the sequence $(\frac{1}{n})_{n=2}^\infty$ with $X = (0, 1)$ with Euclidean metric. Then this is Cauchy but not convergent because $0 \notin X$.

Definition. We say a metric space (X, d) is **complete** if every Cauchy sequence in X converges.

Example (Complete Spaces).

(i). The metric space $(\mathbb{R}, d)$ is complete with Euclidean metric.

(ii). Any X with the discrete metric space.

Definition. A complete normed vector space is called a **Banach Space**.

Theorem 1.11. Let (X, d) be a complete metric space. Let $Y \subseteq X$ be a subset. Then (Y, d) is a complete metric space if and only if Y is closed in X.

Proof. ($\Leftarrow$). Assume Y is closed in X. Let $(x_n)_{n=1}^{\infty} \subseteq Y$ be a cauchy sequence in Y. Hence it is also a cauchy sequence in X. Therefore $(x_n)_{n=1}^{\infty}$ converges to x^* in X since X is complete. However, Y is closed so it contains its limit point, which means $x^* \in Y$ and thus $(x_n)_{n=1}^{\infty}$ converges in Y. This proved that (Y, d) is a complete metric space.

($\Rightarrow$). Assume (Y, d) is complete. To show Y is closed in X it suffices to show it contains all of its limit points. Let $(x_n)_{n=1}^{\infty}$ be a convergent sequence with limit $x^* \in X$. Since convergent sequences are cauchy, we know $(x_n)_{n=1}^{\infty}$ is cauchy. Since Y is complete, this cauchy sequence converges in Y! This means $x^* \in Y$ and hence Y is closed. □

Theorem 1.12. Let $1 \le p < \infty$. Then the space $(\ell_p, \|\cdot\|_p)$ is complete.

Proof. An element in ℓ_p is already a sequence, so a sequence of elements in ℓ_p is annoying. We use the following notation.

$$x^{(n)} = \{x_1^{(n)},\ x_2^{(n)},\ \cdots\} = \left(x_k^{(n)}\right)_{k=1}^{\infty}$$

where $x^{(n)} \in \ell_p$ is the n-th term in the sequence $(x^{(n)})_{n=1}^{\infty}$. Let $(x^{(n)})_{n=1}^{\infty}$ be a cauchy sequence in ℓ_p. Pick $\epsilon > 0$, hence there exists an $N \in \mathbb{N}$ such that for $n, m \in \mathbb{N}$:

$$n, m \ge N \implies d(x^{(n)}, x^{(m)}) = \left(\sum_{k=1}^{\infty} |x_k^{(n)} - x_k^{(m)}|^p\right)^{1/p} < \epsilon \tag{1}$$

Our goal is to find a limit point $x = (x_k)_{k=1}^{\infty} \in \ell_p$ of the sequence $(x^{(n)})_{n=1}^{\infty}$ and prove it. Fix $k \in \mathbb{N}$, we claim that $(x_k^{(n)})_{n=1}^{\infty}$ is a cauchy sequence in $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Indeed:

$$|x_k^{(n)} - x_k^{(m)}| = \left(|x_k^{(n)} - x_k^{(m)}|^p\right)^{1/p} \le \left(\sum_{j=1}^{\infty} |x_j^{(n)} - x_j^{(m)}|^p\right)^{1/p}$$

We have seen that the RHS can be arbitrarily small by (1), hence $(x_k^{(n)})_{n=1}^{\infty}$ is cauchy in $\mathbb{K}$. Since $\mathbb{K}$ is complete, this limit exists, we define $x_k = \lim_{n\to\infty} x_k^{(n)} \in \mathbb{K}$. We claim that:

$$\lim_{n\to\infty} x^{(n)} = x \in \ell_p$$

There are two things to prove: the limit is x and x lies in ℓ_p.

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(i). Pick $\epsilon > 0$, there exists an N such that for all $n, m \ge N$ we have:

$$d(x^{(n)}, x^{(m)}) = \left(\sum_{k=1}^{\infty} |x_k^{(n)} - x_k^{(m)}|^p\right)^{1/p} < \epsilon$$

For any $J \in \mathbb{N}$ and for all $n, m \geq N$ we have:

$$\sum_{k=1}^{J} |x_k^{(n)} - x_k^{(m)}| \leq \sum_{k=1}^{\infty} |x_k^{(n)} - x_k^{(m)}| < \epsilon^p$$

As this is true for all $n \geq M$, it is true as $m \to \infty$. This gives:

$$\lim_{m \to \infty} \sum_{k=1}^{J} |x_k^{(n)} - x_k^{(m)}|^p \leq \epsilon^p \implies \sum_{k=1}^{J} |x_k^{(n)} - x_k|^p < \epsilon^p$$

because $x_k^{(m)} \to x_k$ as $m \to \infty$. This result is true for all $n \geq N$, independent of the choice of J. As this is true for all J, we can take the limit as $J \to \infty$. Hence:

$$\sum_{k=1}^{\infty} |x_k^{(n)} - x_k|^p \leq \epsilon^p \tag{1}$$

It follows that $(x^{(n)} - x) \in \ell_p$ for all $n \in \mathbb{N}$, by definition. We also know $x^{(n)} \in \ell_p$, hence:

$$x = (x^{(n)} - x) + x^{(n)} \in \ell_p$$

as ℓ_p is a vector space. Inequality (1) says that for any $\epsilon > 0$ we can find $N \in \mathbb{N}$ such that $n \geq N$ implies:

$$\|x^{(n)} - x\|_p = \left(\sum_{k=1}^{\infty} |x_k^{(n)} - x_k|^p \right)^{1/p} \leq \epsilon$$

This is exactly the definition of $x^{(n)} \to x$ in ℓ_p, as desired. $\square$

1.7 Completeness of $\mathbb{R}$

Definition. Let $S \subseteq \mathbb{R}$. We say S is **bounded above** if there exists an $M \in \mathbb{R}$ such that $s \leq M$ for all $s \in S$. Similarly S is **bounded below** if there is $N \in \mathbb{R}$ such that $s \geq N$ for all $s \in S$. A set is **bounded** if it is both bounded above and below.

Example. $\mathbb{Z} \subseteq \mathbb{R}$ is not bounded above or below. $(0, 1) \subseteq \mathbb{R}$ is bounded.

Definition. Let $S \subseteq \mathbb{R}$ be bounded above. Then we say $M \in \mathbb{R}$ is the **least upper bound** if M is an upper bound for S and if $N \in \mathbb{R}$ is another upper bound for S we have $M \leq N$. We define the **greatest lower bound** similarly. We denote them by $\sup S$ and $\inf S$.

Theorem 1.13 (Least Upper Bound Property). Let $\emptyset \neq S \subseteq \mathbb{R}$ be bounded above, then S has a least upper bound.

Proof. Let $M \in \mathbb{Z}$ be an upper bound of S. Consider $M-1$. One of two things is true. Either $M-1$ is an upper bound or it is not. If $M-1$ is an upper bound, replace M by $M-1$ and repeat this argument. Eventually we will get $M \in \mathbb{Z}$ such that M is an upper bound but $M-1$ is NOT an upper bound (This process terminates because $S \neq \emptyset$). Divide $[M-1, M]$ into 10 subintervals.

$$\left[M-1, M-1+\frac{1}{10}\right], \cdots, \left[M-1+\frac{9}{10}, M\right]$$

We can find some $k \in \{0, \cdots, 9\}$ such that $M-1+\frac{k}{10}$ is not an upper bound and $M-1+\frac{k+1}{10}$ is an upper bound. We construct u^* as the decimal sequence which is an upper bound (We have to be careful if a ring end point is a least upper bound, as we get a decimal expansion of trailing 9's but this is fine). As desired. $\square$

Theorem 1.14 (MCT). Let $(x_n)_{n=1}^{\infty}$ be a bounded, non-decreasing sequence in $\mathbb{R}$. Then $(x_n)_{n=1}^{\infty}$ converges in $\mathbb{R}$.

Proof. Let $x^* = \sup\{x_n : n \in \mathbb{N}\}$, this exists because $(x_n)_{n\geq 1}$ is bounded. Let $\epsilon > 0$, as x^* is the least upper bound, there exists $N \in \mathbb{N}$ such that:

$$x^* - \epsilon < x_N \leq x^*$$

Hence, for $n \geq N$ we have that $x_n \geq x_N$, which means:

$$x^* - \epsilon < x_N \leq x_n \leq x^* < x^* + \epsilon \implies |x^* - x_n| < \epsilon$$

which prvoed that $\lim\limits_{n\to\infty} x_n = x^*$, as desired. $\square$

Theorem 1.15 (Bolzano-Weierstrass). Every bounded sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}$ has a convergent subsequence (that converges in $\mathbb{R}$).

Proof. Just see MATH 147/247 notes, the proof idea is just bisection. $\square$

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Lemma 1.16. Let $(x_n)_{n=1}^{\infty}$ be a cauchy sequence in $\mathbb{R}$. Then $(x_n)_{n=1}^{\infty}$ is bounded.

Proof. Pick $\epsilon = 1$, there is $N \in \mathbb{N}$ such that for $n, m \geq N$ we have $|x_n - x_m| < 1$. In particular $|x_n - x_N| < 1$ for all $n \geq N$. Let $M = \max\{|x_1|, \cdots, |x_N| + 1\}$. Then $|x_n| \leq M$ for all $n \in \mathbb{N}$. $\square$

Theorem 1.17. $(\mathbb{R}, |\cdot|)$ is a complete normed space.

Proof. Let $(x_n)_{n=1}^{\infty}$ be a cauchy sequence in $\mathbb{R}$. By the above lemma, $(x_n)_{n=1}^{\infty}$ is bounded. By Bolzano-Weierstrass, $(x_n)_{n=1}^{\infty}$ has a convergent subsequence $(x_{n_k})_{k=1}^{\infty}$. Say $\lim\limits_{k\to\infty} x_{n_k} = x^* \in \mathbb{R}$. We claim that $\lim_{n\to\infty} x_n = x^*$ as well. Indeed, let $\epsilon > 0$. There is $N_1 \in \mathbb{N}$ such that:

$$n, m \geq N \implies |x_n - x_m| < \frac{\epsilon}{2}$$

Find $k \in \mathbb{N}$ such that $n_k \geq N$ and $|x_{n_k} - x^*| < \epsilon/2$. Hence for $n \geq N$ we have:

$$|x_n - x^*| \leq |x_n - x_{n_k}| + |x_{n_k} - x^*| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore $\mathbb{R}$ is complete. □

1.8 Limits of continuous functions

Definition. Let (X, d) and (Y, ρ) be metric spaces. Let $(f_n)_{n=1}^{\infty}$ be a sequence of function $X \to Y$. We say $(f_n)_{n=1}^{\infty}$ **converges uniformly** to $f^* : X \to Y$ if for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for $n \in \mathbb{N}$ we have:

$$n \geq N \implies d^*(f_n, f^*) := \sup_{x \in X} \rho(f_n(x), f^*(x)) < \epsilon$$

Example. Let $X = [0, \frac{1}{2}]$ and $Y = \mathbb{R}$ with Euclidean metrics. Define:

$$f_n(x) = 1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

This $(f_n)_{n=1}^{\infty}$ is a sequence of bounded continuous functions from $X \to Y$. We claim that it converges to $f^*(x) = \frac{1}{1-x}$. Indeed, for any $n \in \mathbb{N}$ we have:

$$d^*(f_n, f^*) = \sup_{x \in [0, \frac{1}{2}]} \left| \frac{1}{1-x} - \frac{1 - x^{n+1}}{1 - x} \right| = \sup_{x \in [0, \frac{1}{2}]} \frac{x^{n+1}}{|1 - x|} \leq \frac{(1/2)^{n+1}}{1/2} = \left(\frac{1}{2}\right)^n$$

where the denominator is at least $1/2$ and the numerator is at most $(1/2)^{n+1}$. As $n \to \infty$ this tends to 0, which means $f_n \to f^*$ uniformly.

Theorem 1.18. Let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous functions that converges uniformly to f^*. Then f^* is continuous.

Proof. Let $x \in X$ and $\epsilon > 0$. Pick $N \in \mathbb{N}$ such that $d^*(f^*, f_N) < \epsilon/3$. Since f_N is continuous at x, we can pick $\delta > 0$ such that:

$$d(x, y) < \delta \implies \rho(f_N(x), f_N(y)) < \frac{\epsilon}{3}$$

Therefore if $y \in X$ and $d(x, y) < \delta$, we have:

$$\begin{aligned}
\rho(f^*(x), f^*(y)) &\leq \rho(f^*(x), f_N(x)) + \rho(f_N(x), f_N(y)) + \rho(f_N(y), f^*(y)) \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon
\end{aligned}$$

Therefore f^* is continuous at $x \in X$, as desired. □

Definition. Let (X, d) be a metric space. A subset $A \subseteq X$ is **bounded** if:

$$\operatorname{diam}(A) := \sup_{x,y \in A} d(x, y) < \infty$$

We say a function $f : X \to Y$ is bounded if $f(X) \subseteq Y$ is bounded.

Definition. Let (X, d) and (Y, ρ) be metric spaces. Define:

$$\mathcal{C}^b(X, Y) = \{f : X \to Y \mid f \text{ is continuous and bounded}\}$$

The metric on $\mathcal{C}^b(X, Y)$ is the metric d^* defined by:

$$\rho^*(f, g) := \sup_{x \in X} \rho(f(x), g(x))$$

Then $(\mathcal{C}^b(X, Y), \rho^*)$ is a metric space.

Theorem 1.19. Let (f_n) be a sequence of bounded functions $f_n \in \mathcal{C}^b(X, \mathbb{K})$ that converges uniformly to f^*, then f^* is also bounded.

Theorem 1.20. Let (X, d) and (Y, ρ) be metric spaces. The metric space $(\mathcal{C}^b(X, Y), \rho^*)$ is complete if and only if (Y, ρ) is complete!

Proof. See Assignment 2. □

Theorem 1.21. Let (X, d) be a metric space. Then $\mathcal{C}^b(X, \mathbb{K})$ is complete.

Proof. Let $(f_n)_{n=1}^{\infty}$ be a cauchy sequence. Construct $f^* : X \to \mathbb{K}$ by:

$$f^*(x) := \lim_{n \to \infty} f_n(x)$$

Why is this well-defined? Note that for all fixed $x \in \mathbb{K}$, the sequence $(f_n(x))_{n=1}^{\infty}$ is a cauchy sequence in $\mathbb{K}$! Since $\mathbb{K}$ is complete, this sequence converges. We claim that $(f_n)_{n=1}^{\infty}$ converges uniformly to f^*. Let $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that:

$$n, m \geq N \implies d^*(f_n, f_m) = \sup_{x \in X} |f_n(x) - f_m(x)| < \frac{\epsilon}{2}$$

Let $n \geq N$ be arbitrary. Let $x \in X$ be arbitrary as well. Since $f_n(x) \to f^*(x)$, we can find $M \in \mathbb{N}$ with $M \geq N$ such that $|f_n(x) - f^*(x)| < \epsilon/2$. Then, for $n \geq N$:

$$\begin{aligned}
|f_n(x) - f^*(x)| &\leq |f_n(x) - f_M(x)| + |f_M(x) - f^*(x)| \\
&\leq d^*(f_n, f_M) + |f_M(x) - f^*(x)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Since $x \in X$ is chosen arbitrarily, we have:

$$d^*(f_n, f^*) = \sup_{x \in X} |f_n(x) - f^*(x)| \leq \epsilon$$

Therefore $f_n \to f^*$ in the d^* metric (that is $f_n \to f^*$ uniformly in the usual sense). Hence f^* is continuous and bounded, so $f^* \in \mathcal{C}^b(X, \mathbb{K})$. $\square$

Theorem 1.22 (Weierstrass M-Test). Let $\zeta : X \to \mathbb{R}$ by $\zeta(a) = 0$ denote the zero function. Then we let $(f_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{C}^b(X, \mathbb{R})$ such that there exists $M \in \mathbb{R}$ with:

$$\sum_{n=1}^{\infty} d^*(f_n, \zeta) \leq M < \infty$$

Define $g_N(x) = \sum_{n=1}^{N} f_n(x)$. Then $(g_N)_{N=1}^{\infty}$ converges to $g^* \in \mathcal{C}^b(X)$ in the d^* metric.

Example. The series of function $f(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{2^n}$ is well-defined and is continuous on $\mathbb{R}$.

2 More Metric Topology

2.1 Compactness

Definition. Let (X, d) be a metric space and $A \subseteq X$. We say $\{U_i\}_{i\in I}$ is an **open cover** of A if each U_i is open and $A \subseteq \bigcup_{i\in I} U_i$.

Definition. Let (X, d) be a metric space and $A \subseteq X$. We say A is **compact** if for every open cover $\{U_i\}_{i\in I}$ there is a finite subset $I_0 \subseteq I$ with $A \subseteq \bigcup_{i\in I_0} U_i$. This $\{U_i\}_{i\in I_0}$ is called a **finite subcover**.

Example. Let $A = \{x_1, \cdots, x_n\}$ be a finite set, then A is compact. Why? Let $\{U_i\}$ be an open cover of A. For each $j \in \mathbb{N}$ there is $i_j \in I$ such that $a \in U_{i_j}$. Hence we have:

$$A \subseteq U_{i_1} \cup \cdots \cup U_{i_n}$$

This is a finite subcover! Hence A is compact.

Example. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\} \subseteq \mathbb{R}$. We claim that A is compact. Let $\{U_i\}_{i\in I}$ be an open cover. There exists an open set U_0 such that $0 \in U_0$. Hence there is $N \in \mathbb{N}$ large enough such that $0 \in B_\epsilon(0) \subseteq U_0$, where $\epsilon = 1/N$. This means:

$$\left\{\frac{1}{n} : n \geq N + 1\right\} \cup \{0\} \subseteq U_0$$

Then there are only finitely many points left, so we can use finitely many U_i to cover $\{\frac{1}{n} : n \geq N+1\}$. This gives an finite subcover of A.

Example. Let A, B be compact sets. Then $A \cup B$ is compact. Indeed, any open cover of $A \cup B$ gives an open cover for A, B. This gives a finite subcover for A, B, respectively. The union of these two finite subcovers giives a finite subcover of $A \cup B$.

Example. The set $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ is not compact! Consider the open cover:

$$\left\{\left(\frac{1}{n}, 1 + \frac{1}{n}\right) : n \in \mathbb{N}\right\}$$

This has no finite subcover. Indeed, suppose we have a finite subcollection of open sets indexed by $n_1, \cdots, n_r$. WLOG we may assume $n_1 < \cdots < n_r$. Then the union of these U_i is:

$$\left(\frac{1}{n_r}, 1 + \frac{1}{n_1}\right)$$

This clearly does not cover A.

Example. Let $A = \mathbb{R}$. Then A is not compact. The open cover $\{(-n, n) : n \in \mathbb{N}\}$ has no finite subcover. Similarly $A = \mathbb{Z}$ is not compact as well.

Proposition 2.1. Let (X, d) be a metric space. If $A \subseteq X$ is compact, then A is closed and bounded.

Proof. Assume A is not closed. There exists a subsequence $(a_n)_{n=1}^{\infty}$ in A with $a_n \to a^*$ and $a^* \notin A$. Consider the following open cover:

$$U_n = X \setminus \overline{B_{d(a_n, a^*)}(a^*)}$$

This cannot have a finite subcover, since a^* is a limit point of (a_n). Therefore A is closed. Similarly suppose A is not bounded. Fix $a \in A$. For all $N \in \mathbb{N}$ such that there exists an $a_N \in A$ such that:

$$d(a_N, a) > N$$

Consider the open cover $\{B_N(a) : N \in \mathbb{N}\}$ of A. Given a finite subset $\{N_1 < \cdots < N_r\}$, the union of these is $B_{N_r}(a)$. However, for $N = N_r + 1$ there is $a_N \in A$ such that $d(a_N, a) > N$ so $a_N \notin B_{N_r}(a)$, but $a_N \in A$. Hence this open cover has no finite subcover! Hence A is bounded. $\square$

Definition. Let (X, d) be a metric space and $A \subseteq X$. We say A is **sequentially compact** if for every sequence $(a_n)_{n=1}^{\infty}$ of A, there is a convergent subsequence $(a_{n_k})_{k=1}^{\infty}$ with $a_{n_k} \to a^* \in A$.

Example. Let A be a finite set. This is sequentially compact. Why? For any infinite sequence of A, there exists $a \in A$ that appears infinitely many times in this sequence. Take this subsequence that only consists of a. This is a convergent subsequence.

Example. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. This is sequentially compact. Any sequence in A either has a convergent subsequence that goes to 0 or the sequence only takes on finitely many values.

Definition. Let (X, d) be a metric space. Let $A \subseteq X$ be a subset. Then (A, d_A) is a metric space, where $d_A : A \times A \to \mathbb{R}$ is the restriction of d on A. This is called the **induced metric space**. A subset $U \subseteq A$ is called **relatively open** if there exists an open set $U' \subseteq X$ such that $U = U' \cap A$.

Remark. As a metric space, the open balls of (A, d_A) are of the form:

$$B_A(a, r) = \{x \in A : d_A(x, a) < r\} = \{x \in X \cap A : d(x, a) < r\} = B_X(a, r) \cap A$$

Therefore, an open set in (A, d_A) is of the form $U' \cap A$ for open sets U' in X.

Definition. A metric space (X, d) is **compact** if every open cover of X has a finite subcover. That is, for every open cover $\{U_i : i \in I\}$, there is a finite subset $I_0 \subseteq I$ such that:

$$X = \bigcup_{i \in I_0} U_i$$

Note that this is an equality, not a subset. This is because X is our whole space, it does not sit in any bigger space.

Remark. Note that there are two notions of compactness for a subset $A \subseteq X$.

(i). A is compact as a subset of X. [This is the definition we saw above.]

(ii). A is compact as a metric space. [Note that for an open cover of A, the open sets are open sets in A! These open sets are different from the open sets in X.]

In fact, these two notions concide. Suppose (ii) is true, we want to show (i) is true. Let $\{U_i : i \in I\}$ be an open cover of A, where U_i is an open set of X for all i. Then:

$$\{U_i \cap A : i \in I\}$$

is an open cover of the metric space (A, d_A), where each $U_i \cap A$ is an open set in A. Since (A, d_A) is compact, there is a finite set $I_0 \subseteq I$ such that:

$$A = \bigcup_{i \in I_0} (U_i \cap A)$$

Then clearly we have $A \subseteq \bigcup_{i \in I_0} U_i$, an finite subcover of A (as a subset of X.)

Conversely suppose (i) is true. Let $\{U_i : i \in I\}$ be an open cover of (A, d_A), then for each $i \in I$ there is an open set $U_i' \subseteq X$ of X such that $U_i = U_i' \cap A$. Hence $\{U_i' : i \in I\}$ is an open cover of $A \subseteq X$. Since A is a compact subset of X, there is a finite $I_0 \subseteq I$ with $A \subseteq \bigcup_{i \in I_0} U_i'$. By taking the intersection with A, we have:

$$A = \bigcup_{i \in I_0} (U_i' \cap A) = \bigcup_{i \in I_0} U_i$$

Therefore (A, d_A) is compact and (ii) is true.

Definition. Let (X, d) be a metric space. A collection $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\} \subseteq X$ is said to have the **finite intersection property (FIP)** if for every finite subset $\Lambda_0 \subseteq \Lambda$ we have $\bigcap_{\lambda \in \Lambda_0} F_\lambda \neq \emptyset$.

Example. Let $X = \mathbb{R}$. Consider the collection $\{\mathbb{R} \setminus \{a\} : a \in \mathbb{R}\}$. This clearly satisfies the FIP. However, the infinite intersection:

$$\bigcap_{a \in \mathbb{R}} (\mathbb{R} \setminus \{a\}) = \emptyset$$

is empty! As we will see, this actually tells us $\mathbb{R}$ is not compact!

Definition. Let (X, d) be a metric space. A subset $A \subseteq X$ is called **cauchy** if every cauchy sequence in A converges to a point in A.

Definition. Let (X, d) be a metric space. We say $A \subseteq X$ is **totally bounded** if for all $\epsilon > 0$ there exists a finite set $F_\epsilon \subseteq X$ (called an **ϵ-net**) such that:

$$A \subseteq \bigcup_{f \in F_\epsilon} B_\epsilon(f)$$

Note that totally boundedness implies boundedness.

Remark. Note that if A is totally bounded, we may assume $F_\epsilon \subseteq A$ for all $\epsilon > 0$. Suppose for $\epsilon > 0$ we have an ϵ-net $F = \{x_1, \cdots, x_n\} \subseteq X$ of A, so:

$$A \subseteq \bigcup_{i=1}^{n} B_{\epsilon/2}(x_i)$$

We may assume $B_\epsilon(x_i) \cap A \neq \emptyset$ for all i. (If the intersection is empty we can just remove it from the ϵ-net.) Hence we may chooise $y_i \in A \cap B_{\epsilon/2}(x_i)$ for all i. Note that:

$$A \subseteq \bigcup_{i=1}^{n} B_\epsilon(y_i)$$

by the triangle inequality. Indeed, for any $x \in A$ we can choose $i \in \{1, \cdots, n\}$ such that $x \in B_{\epsilon/2}(x_i)$. Then we have that:

$$d(x, y_i) \leq d(x, x_i) + d(x_i, y_i) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This proved that $x \in B_\epsilon(y_i)$. Hence $\{y_1, \cdots, y_n\} \subseteq A$ is an ϵ-net for A.

Recall in $\mathbb{R}^n$, a subset is compact if and only if it is closed and bounded (Heine-Borel). We will now see that for metric spaces, there are also some easier ways to characterize compactness, and the Heine-Borel theorem for $\mathbb{R}^n$ is a special case of it.

Theorem 2.2 (Borel-Lebesgue). Let (X, d) be a metric space and $A \subseteq X$. Then the followings are equivalent:

(i). A is compact (either as a subset or a metric space, these two notions are equivalent.)

(ii). If $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\}$ is an collection of closed sets in (A, d_A) with FIP, then $\bigcap_{\lambda \in \Lambda} F_\lambda \neq \emptyset$.

(iii). A is sequentially compact.

(iv). A is complete and totally bounded.

Example. Consider $A = \mathbb{Q} \cap [0, 1]$ and $B = \mathbb{Z}$ as induced metric spaces from $(\mathbb{R}, d)$. By the Borel-Lebesgue theorem, we can show that A, B are not compact in four different ways.

(i). For A, we define the open cover:

$$\left\{\mathbb{R} \setminus \overline{B_{1/n}(1/\pi)} : n \in \mathbb{N}\right\}$$

This does not have a finite subcover. For B, the open cover $\{B_{1/2}(n) : n \in \mathbb{Z}\}$ does not have a fintie subcover as well. Hence A, B are not compact by definition.

(ii). We need to find a collection of closed sets that FIP but the intersection is empty. Let:

$$\left\{A \cap \overline{B_{1/n}(1/\pi)} : n \in \mathbb{N}\right\} \subseteq A$$
$$\{[n, \infty) \cap B : n \in \mathbb{N}\} \subseteq B$$

These two have FIP but the intersection over all $n \in \mathbb{N}$ is empty.

(iii). Let $(a_n)_{n=1}^{\infty}$ be the sequence in A such that a_n is the truncation of the decimal expansion of $1/\pi$ at the n-th place. Then $a_n \to 1/\pi$ in $\mathbb{R}$, which means any convergent subsequence of (a_n) converges to $1/\pi \notin A$. For B, the sequence $(b_n)_{n=1}^{\infty}$ by $b_n = n$ is a sequence in B that does not have a convergent subsequence.

(iv). Let $(a_n)_{n=1}^{\infty}$ be the same sequence in (iii), this is cauchy but does not converge in A. For B, consider $\epsilon = 1/2$. Then $B = \mathbb{Z}$ does not have a ϵ-net. Therefore A is not complete and B is not totally bounded.

Proof of Theorem 2.2. (i) $\Longrightarrow$ (ii). Assume (A, d_A) is a compact metric space. Let $\{F_\lambda : \lambda \in \Lambda\}$ be a collection of closed sets in A satisfying FIP. Assume for a contradiction that $\bigcap_{\lambda \in \Lambda} F_\lambda = \emptyset$. Consider the following collection of open sets in A:

$$\{U_\lambda := A \setminus F_\lambda : \lambda \in \Lambda\}$$

Note that this is an open cover for A. Since A is compact, there exists a finite subset $\Lambda_0 \subseteq \Lambda$ such that $\bigcup_{\lambda \in \Lambda_0} U_\lambda = A$. However, this implies that:

$$\bigcap_{\lambda \in \Lambda_0} F_\lambda = A \setminus \bigcup_{\lambda \in \Lambda_0} U_\lambda = A \setminus A = \emptyset$$

Since Λ_0 is finite, this contradicts to our assumption that $\{F_\lambda : \lambda \in \Lambda\}$ has FIP!

(ii) $\Longrightarrow$ (iii). Assume (ii) is true. We want to show A is sequentially compact. Let $(a_n)_{n=1}^{\infty}$ be a sequence in A. For each $k \geq 1$ we define $S_k = \{a_n : n \geq k\}$ and define the closed set:

$$F_k = \overline{S_k} = \overline{\{a_n : n \geq k\}} \subseteq A$$

to be the closure of a tail of $(a_n)_{n=1}^{\infty}$. Note that $F_{k+1} \subseteq F_k$ for all $k \geq 1$. Define $\mathcal{F} = \{F_k : k \geq 1\}$. Then $\mathcal{F}$ is a collection of closed sets in A that has FIP. It satisfies FIP because for a finite set $\{k_1 < \cdots < k_r\}$ we have:

$$F_{k_1} \cap \cdots \cap F_{k_r} = F_{k_1} \neq \emptyset$$

By our assumption we have $\bigcap_{k=1}^{\infty} F_k \neq \emptyset$. Let's pick $a^* \in \bigcap_{k=1}^{\infty} F_k$. We claim that we can find a subsequence of (a_n) that converges to a^*. First we note that:

$$B_r(a^*) \cap S_k \neq \emptyset$$

for all $r > 0$ and $k \geq 1$. This is because each $a^* \in F_k$ is closed so a^* is a limit point for every S_k. In other word, for any $r > 0$ and $k \geq 1$ we can find some a_i such that $d(a_i, a^*) < r$ and $i \geq k$. For $r = 1$ we can find $n_1 \geq 1$ with $d(a_{n_1}, a^*) < 1$. Inductively suppose we have defined $n_1, \cdots, n_r$, we can find $n_{r+1} > n_r$ such that $d(a_{n_r}, a^*) < 1/(r+1)$. Hence $(a_{n_r})_{r=1}^{\infty}$ is a subsequence that converges to $a^* \in A$. Therefore (A, d_A) is sequentially compact.

(iii) $\Longrightarrow$ (iv). Assume (A, d_A) is sequentially compact. We first show that A is complete (as a subset of X.) Let $(a_n)_{n=1}^{\infty}$ be a cauchy seqeunce in A. There exist a convergent subsequence $(a_{n_k})_{k=1}^{\infty}$ that converges to $a^* \in A$. Since $(a_n)_{n=1}^{\infty}$ is cauchy, we must have $a_n \to a^*$ as well. Hence A is complete. Now let us show that A is totally bounded. Let $\epsilon > 0$ be arbitrary. Suppose it is not, then there is $\epsilon > 0$ such that there does not exist a ϵ-net for A. First note that in the case, A must be infinite. (Any finite set is clearly totally bounded.) Let $a_1 \in A$ be arbitrary. Hence $\{a_1\}$ is not an ϵ-net. This means there exists $a_2 \in A$ such that $d(a_1, a_2) \geq \epsilon$. Now, inductively suppose we have found $a_1, \cdots, a_r$ for $r \geq 1$. Then $\{a_1, \cdots, a_r\}$ is not an ϵ-net. We can then find $a_{r+1} \in A$ such that:

$$d(a_{r+1}, a_i) \geq \epsilon \text{ for all } i \in \{1, \cdots, r\}$$

This gives us a sequence $(a_n)_{n=1}^{\infty}$ in A that has no convergent subsequence! (since for all n, m we have $d(a_n, a_m) \geq \epsilon$.) This is a contradiction, so A is totally bounded.

(iv) $\Longrightarrow$ (i). Assume (iv) is true, we want to show A is compact. Suppose for a contradiction that A is not compact as a metric space. Let $\mathcal{U} = \{U_i : i \in I\}$ be an open cover of A that does not have a finite subcover (in this case $U_i \subseteq X$ is open for all i). Since A is totally bounded, for all $n \geq 1$ there exists a $\frac{1}{n}$-net in A:

$$F_n = \{x_{n,1}, \cdots, x_{n,m_n}\}$$

such that:

$$A = \bigcup_{f \in F_n} B_{1/n}(f) = \bigcup_{f \in F_n} \overline{B_{1/n}(f)}$$

Let $n = 1$. Note that if all $\overline{B_1(f)}$ can be covered by finitely many U_i's, then A can be covered by finitely many U_i's, which is impossible. Hence there is $i_1 \in \{1, \cdots, m_n\}$ such that $\overline{B_1(x_{i_1})}$ does not have a finite subcover of $\mathcal{U}$. Let $y_1 = x_{i_1}$. Inductively suppose we have chosen $y_1, \cdots, y_k$ so that:

$$X_k := \bigcap_{i=1}^{k} \overline{B_{1/i}(y_i)}$$

has no finite subcover. Consider the sets:

$$X_{k,i} = X_k \cap \overline{B_{1/(k+1)}(x_{k+1,i})} \;\text{ for } 1 \leq i \leq m_{n+1}$$

Suppose for a contradiction that each of them has a finite subcover. However:

$$\bigcup_{i=1}^{m_{n+1}} X_{n,i} = \bigcup_{i=1}^{m_{n+1}} X_k \cap \overline{B_{1/(k+1)}(x_{k+1,i})} = X_k \cap \bigcup_{i=1}^{m_{n+1}} \overline{B_{1/(k+1)}(x_{k+1,i})} = X_k \cap A = X_k$$

This means X_k has a finite subcover, which is impossible! Hence there is i_{k+1} such that $\overline{B_{1/(k+1)}(x_{k+1,i_{k+1}})}$ does not have a finite subcover. Let $y_{k+1} = x_{k+1,i_{k+1}}$.

Note that $(y_n)_{n=1}^{\infty}$ is cauchy in A. Indeed, let $\epsilon > 0$ we choose $N > 2/\epsilon$. For all $n \geq m \geq N$ we have:

$$X_n \subseteq \overline{B_{1/m}(y_m)} \cap \overline{B_{1/n}(y_n)} \neq \emptyset$$

and X_n is non-empty set. We can pick $x \in X_n$. Then:

$$d(y_n, y_m) \leq d(y_n, x) + d(y_m, x) \leq \frac{1}{n} + \frac{1}{m} \leq \frac{2}{N} < \epsilon$$

Since A is complete, $y_n \to y^* \in A$ for some $y^* \in A$. For any $m \in \mathbb{N}$ we have:

$$d(y_m, y^*) = \lim_{n\to\infty} d(y_m, y_n) \leq \lim_{n\to\infty} \left(\frac{1}{m} + \frac{1}{n}\right) = \frac{1}{m}$$

Since $\mathcal{U}$ is a cover of A, there is $i_0 \in I$ such that $y^* \in U_{i_0}$. Since U_{i_0} is open, then is $r > 0$ such that $B_r(y^*) \subseteq U_{i_0}$. Choose $m > 2/r$, then for any $x \in X_m \subseteq \overline{B_{1/m}(y_m)}$ we have:

$$d(x, y^*) \leq d(x, y_m) + d(y_m, y^*) \leq \frac{2}{m} < r$$

Hence $X_m \subseteq U_{i_0}$. This means X_m does have a finite subcover, contradicting our construction! Therefore A is compact. □

Remark. Totally bounded is not same as bounded. There exist sets that are closed, bounded but not compact. Consider $X = \{0,1\}^{\mathbb{N}}$ with ℓ^∞ norm. It is clearly bounded since $\|x\|_\infty \leq 1$ for all $x \in X$. However, we claim that it is not totally bounded. Suppose it has an $\frac{1}{2}$-net:

$$F = \{x_1, \cdots, x_n\}$$

Then $B_{1/2}(x_i) = \{x_i\}$ because $\|x\|_\infty \in \{0,1\}$ for any $x \in X$. Hence:

$$\bigcup_{i=1}^{n} B_{1/2}(x_i) = \{x_1, \cdots, x_n\} \neq \{0,1\}^{\mathbb{N}}$$

Therefore this is not a $\frac{1}{2}$-net, contradiction. Hence $(X, \|\cdot\|_\infty)$ is bounded but NOT totally bounded!

Corollary 2.3 (Heine-Borel). A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof. Since $\mathbb{R}^n$ is compact, A is closed $\iff$ it is complete. Moreover, we claim that in $\mathbb{R}^n$, bounded implies totally bounded. Let $\epsilon \in \mathbb{N}$, we claim that there is also an ϵ-net of a bounded set A. Since A is bounded, we know $A \subseteq [-r, r]^n$ for some $r > 0$. We can cover $[-r, r]^n$ with finitely many boxes of side length $\frac{\epsilon}{2}$. Any such box can be covered by an ϵ-ball. Hence we can use finitely many ϵ-balls to cover A. Therefore A is totally bounded. Hence A is bounded $\iff$ it is totally bounded. The result follows from (iv) of Borel-Lebesgue. $\square$

2.2 Countable and Uncountable Sets

Definition. A set X is **countable** if there is a injection $f : X \to \mathbb{N}$. A set is **denumerable** if there is a bijection $f : X \to \mathbb{N}$. We say a set is **uncountable** if it is not countable.

Example. The integers $\mathbb{Z}$ is countable because $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \cdots\}$.

Example. The rationals $\mathbb{Q} \cap [0, 1]$ is also countable because:

$$\mathbb{Q} = \left\{\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \cdots\right\}$$

Informally: Write $\frac{p}{q} \in \mathbb{Q} \cap [0, 1]$ with q in increasing order and $p \in \{1, \cdots, q\}$ such that $\gcd(p, q) = 1$. We require coprimeness so that there is no element appearing twice in the list.

Example. The set $\{0, 1\}^{\mathbb{N}}$ is uncountable. Suppose for a contradiction that it is countable. Then:

$$\{0, 1\}^{\mathbb{N}} = \{(x_{1,k})_{k=1}^{\infty}, (x_{2,k})_{k=1}^{\infty}, \cdots\}$$

Define a sequence $(x_n)_{n=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ by:

$$x_n = \begin{cases} 0 & \text{if } x_{n,n} = 1 \\ 1 & \text{if } x_{n,n} = 0 \end{cases}$$

Then $(x_n)_{n=1}^{\infty}$ is different from $(x_{n,k})_{k=1}^{\infty}$ at the n-th place for all $n \geq 1$. This is a new element in $\{0, 1\}^{\mathbb{N}}$, contradiction! Hence $\{0, 1\}^{\mathbb{N}}$ is uncountable. This method is called the **diagonal argument**: If we list out all the given $(x_{n,k})_{k=1}^{\infty}$ row by row, then our new element $(x_n)_{n=1}^{\infty}$ is constructed by changing the diagonal entries.

Example. Let A, B be sets and $f : A \to B$ be a bijection, then A is countable if and only if B is countable.

Example. Let $A \subseteq B$. If A is uncountable then so is B. If B is countable then so is A.

Example. We claim $\mathbb{R}$ is uncountable. Let $X = \{0, 1, \cdots, 9\}^{\mathbb{N}}$. By the same argument we can show that X is uncountable. Define $f : X \to \mathbb{R}$ by:

$$f((x_n)_{n=1}^{\infty}) = \sum_{k=1}^{\infty} \frac{x_k}{10^k}$$

Then $f : X \to f(X)$ is a bijection. Since $f(X) \subseteq \mathbb{R}$, we know $\mathbb{R}$ is uncountable.

Definition. Let (X, d) be a metric space. We say $A \subseteq X$ is **dense** in X if $\overline{A} = X$.

Definition. We say a metric space (X, d) is **separable** if there is a countable subset $A \subseteq X$ such that A is dense in X.

Example. The reals $\mathbb{R}$ with the usual metric is separable because $\overline{\mathbb{Q}} = \mathbb{R}$ and $\mathbb{Q}$ is countable.

Example. Let (X, d) with the discrete metric. Then X is separable if and only if X is countable. This is because every subset is closed (equal to their own closure), so the only dense subset is X itself. Hence X is countable if and only if X is separable.

Proposition 2.4. Let (X, d) be a metric space. If (X, d) is totally bounded, then X is separable.

Proof. For each $n \in \mathbb{N}$ there is an $\frac{1}{n}$-net of X, call it F_n. Define $F = \bigcup_{n=1}^{\infty} F_n$. Note that F is countable, being a countable union of finite sets. We claim that F is dense. Let $x \in X$ be and $\epsilon > 0$ be arbitrary. There is $N \geq 1$ such that $1/N < \epsilon$. Since F_N is an $\frac{1}{N}$-net, there is $f \in F_N$ such that $d(f, x) < 1/N < \epsilon$. Since $f \in F$, we proved that F is dense in X. $\square$

2.3 Compactness and Continuity

Proposition 2.5. Let (X, d) and (Y, ρ) be metric spaces. If $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact.

Proof. Let $\{U_i : i \in I\}$ be an open cover of $f(X)$ in Y. Since f is continuous, each $f^{-1}(U_i)$ is open. Since $f^{-1}(Y) = X$, we know $\{f^{-1}(U_i) : i \in I\}$ is an open cover of X. Since X is compact, there is a finite subcover $\{i_1, \cdots, i_n\}$. Hence:

$$f(X) \subseteq \bigcup_{k=1}^{n} U_{i_k}$$

Therefore $f(X)$ is compact. $\square$

Proposition 2.6. Let (X, d) and (Y, ρ) be metric spaces. If $f : X \to Y$ is continuous and X is compact, then f is uniformly continuous.

Proof. Let $\epsilon > 0$. For each $x \in X$ we can find $\delta_x > 0$ such that for all $y \in X$:

$$d(y, x) < \delta_x \implies \rho(f(y), f(x)) < \frac{\epsilon}{2} \tag{1}$$

Now note that $\{B_{\delta_x/2}(x) : x \in X\}$ is an open cover of X. Since X is compact, we know:

$$X = B_{\delta_{x_1}/2}(x_1) \cup \cdots \cup B_{\delta_{x_n}/2}(x_n)$$

for some $x_1, \cdots, x_n \in X$. Now define $\delta = \frac{1}{2}\min\{\delta_{x_1}, \cdots, \delta_{x_n}\} > 0$. Let $x, y \in X$ be arbitray with $d(x,y) < \delta$. Say $y \in B_{\delta_{x_b}}(x_b)$ for some $x_b \in \{x_1, \cdots, x_n\}$. However:

$$d(x, x_b) \leq d(x,y) + d(y, x_b) < \delta + \frac{1}{2}\delta_{x_b} < \frac{1}{2}\delta_{x_b} + \frac{1}{2}\delta_{x_b} < \delta_{x_b}$$

Since $d(y, x_b) < \delta_{x_b}$ as well, by (1) we have:

$$\rho(f(x), f(y)) \leq \rho(f(x), f(x_b)) + \rho(f(x_b), f(y)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence f is uniformly continuous. $\square$

2.4 Cantor Set

Construction 2.7 (Ver 1). Let $C_0 = [0,1]$. Recursively, C_{i+1} is constructed by removing the middle third from each intervals in C_i. First we see that:

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$
$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$

We see $\{C_n\}_{n=0}^{\infty}$ has the finite intersection property and they are all compact sets. Define:

$$C^* = \bigcap_{n=0}^{\infty} C_n$$

We call C^* the **(middle-third) cantor set**. Clearly $0, 1 \in C^*$. In fact any endpoint of any C_n is in C^*. For example $1/3, 1/9, 2/27 \in C^*$. We have C^* is compact (as it is closed and bounded in $\mathbb{R}$).

Construction 2.8 (Ver 2). Equivalently we can define:

$$C^* = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{3^n} : a_n \in \{0, 2\} \right\}$$

It is the set of all real numbers that CAN be written in ternary expansion wiwthout using 1. [For example $0.1 \in C^*$ because it CAN be written as $0.222\cdots$] This shows that C^* has an uncountable number of points.

Construction 2.9 (Ver 3). The cantor set C^* is the unique non-empty compact set satisfying:

$$C^* = f_1(C^*) \cup f_2(C^*)$$

where $f_1(x) = x/3$ and $f_2(x) = x/3 + 2/3$.

Theorem 2.10. Let (X, d) be a compact metric space. There is an continuous map $f : C^* \to X$ that is surjective.

Proof. The idea is to construct $s_n : C^* \to X$ such that (s_n) is cauchy and each s_n is continuous. As $n \to \infty$ we have $s_n(C^*)$ better approximate X [produce an ϵ-net for smaller ϵ.]

For $n = 1$, construct a 1-net for X. That is, a finite set F_1 such that $X = \bigcup_{f \in F_1} B_1(f)$ [This exists since X is totally bounded.] We can assume wlog that $|F_1| = 2^{k_1}$ for some k_1. [If not power of 2, adding more points if necessary.] Now consider C_{k_1}, a union of 2^{k_1} intervals containing C^* For each $c \in C^*$, we know c is in some subinterval of C_{k_1}. We map each subinterval in C_{k_1} to a different $f \in F_1$. Let s_1 be this map. Then s_1 is continuous as it is locally constant.

For $n = 2$, construct a 1/2-net for each of each $\overline{B}_1(f_i)$, where $\{f_i\} = F_1$ from the construction of s_1. As before, we can assume that this set is a power of 2, and the same powers of 2. Say 2^{k_2} in size. For each subinterval I_i used to construct s_1, subdivide it into 2^{k_2} subintervals. As before, s_2 is continuous. We further notice $d(s_1(c), s_2(c))$ is not huge. In fact $d(s_1(c), s_2(c)) \leq 1 + \frac{1}{2}$.

We continue in this fashion, we get that:

$$d(s_n(c), s_{n+1}(c)) \leq \frac{1}{2^n}$$

We can make this arbitrarily small. Hence for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n, m \geq N$:

$$d^*(s_n, s_m) = \sup_{c \in C^*} d(s_n(c), s_m(c)) < \epsilon$$

Therefore (s_n) is a cauchy sequence. As C^* is compact and X is complete so $\mathcal{C}^b(C^*, X) = \mathcal{C}(C^*, X)$ is complete. Hence $s_n \to s^* \in \mathcal{C}(C^*, X)$. We need to show $s^*(C^*) = X$, that is, s^* is onto. Take a point x in X. This point will be distance 1 from some point in F_1. This gives us a subinterval in C^*. There exists a point in F_2 whose distance is $1/2$ from x and $1 + 1/2$ from f_1. This gives a smaller subinterval. Repeating this process we get nested subintervals with non-trivial intersection with C^*. The infinite intersection is in C^*, and this intersection has $s^*(c^*) = x$, as required. ☐

2.5 Compact sets in $\mathcal{C}(X)$

Definition. Let (X, d) be a compact metric space, we denote:

$$\mathcal{C}(X) := \mathcal{C}(X, \mathbb{R}) = \{f : X \to \mathbb{R} \mid f \text{ is continuous}\}$$

Here $\mathbb{R}$ is a metric space with the usual metric. For $f \in \mathcal{C}(X)$ we define the **uniform norm** by:

$$\|f\|_\infty := \sup\{|f(x)| : x \in X\}$$

Since X is compact, by the extreme value theorem this supremum can be achieved. So we can equivalently define it as:

$$\|f\|_\infty = \max\{|f(x)| : x \in X\}$$

Note that $(\mathcal{C}(X), \|\cdot\|_\infty)$ is a normed vector space. In fact, since $\mathbb{R}$ is complete we knew that $\mathcal{C}(X)$ is also complete. Therefore $(\mathcal{C}(X), \|\cdot\|_\infty)$ is a Banach space. Also note that $f_n \to f$ uniformly (as functions) is the same as $f_n \to f$ as sequences in the normed space $(\mathcal{C}(X), \|\cdot\|_\infty)$.

Remark. By Borel-Lebesgue we know that:

$$\begin{aligned} K \subseteq \mathcal{C}(X) \text{ is compact } &\iff K \text{ is complete and totally bounded} \\ &\iff K \text{ is closed and totally bounded} \end{aligned}$$

since closed subsets of a complete space are complete.

Example. Let $K = \{f_n(x) = x^n : n \in \mathbb{N}\} \subseteq \mathcal{C}([0,1])$. Note that every subsequence of (f_n) converges pointwise to the function $f : [0,1] \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Since f is not continuous, the sequence (f_n) does not converge in $\mathcal{C}([0,1])$. Therefore K is not sequentially compact despite being closed and bounded.

Definition. Let (X, d) be complete. A subset $F \subseteq \mathcal{C}(X)$ is called **equicontinuous at** $x \in X$ if for all $\epsilon > 0$ there is $\delta > 0$ so that for all $y \in X$:

$$d(x,y) < \delta \implies |f(x) - f(y)| < \epsilon \text{ for all } f \in F$$

We know $F \subseteq \mathcal{C}(X)$ is **equicontinuous** if it is equicontinuous at every $x \in X$. We say a subset $F \subseteq \mathcal{C}(X)$ is **uniformly equicontinuous** if for all $\epsilon > 0$ there is $\delta > 0$ such that for all $x, y \in X$:

$$d(x,y) < \delta \implies |f(x) - f(y)| < \epsilon \text{ for all } f \in F$$

That is, the choice of $\delta > 0$ does not depend on $x \in X$.

Remark. Clearly uniformly equicontinuous $\implies$ equicontinuous.

Lemma 2.11. Let (X, d) be compact. If $K \subseteq \mathcal{C}(X)$ is compact, then K is uniformly equicontinuous.

Proof. Let $\epsilon > 0$. Since K is compact, it is totally bounded and thus has a $\frac{\epsilon}{3}$-net. Say it is $F = \{f_1, \cdots, f_n\} \subseteq K$. Each f_i is continuous, thus uniformly continuous (since X is compact). For each i there is $\delta_i > 0$ such that for all $x, y \in X$:

$$d(x,y) < \delta_i \implies |f_i(x) - f_i(y)| < \frac{\epsilon}{3}$$

Let $\delta = \min\{b_1, \cdots, b_n\}$. Now let $x, y \in X$ with $d(x,y) < \delta$ and let $f \in K$ be arbitrary. We can find i such that $\|f - f_i\| < \epsilon/3$ (because F is an $\epsilon/3$-net!) Therefore we have:

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| \\ &\leq \|f - f_i\|_\infty + \frac{\epsilon}{3} + \|f - f_i\|_\infty \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

Therefore K is uniformly equicontinuous. □

Lemma 2.12. Let (X, d) be compact. Suppose $F \subseteq \mathcal{C}(X)$ is equicontinuous. Then F is uniformly equicontinuous.

Proof. Let $\epsilon > 0$. For each $x \in X$ there is $\delta_x > 0$ so that for all $y \in X$:

$$d(x,y) < \delta_x \implies |f(x) - f(y)| < \frac{\epsilon}{2} \quad \text{for all } f \in F$$

Then the collection $\{B_{\delta_x/2}(x) : x \in X\}$ is an open cover of X. Since X is compact, it has a finite subcover, indexed by $\{x_1, \cdots, x_n\}$. Let $\delta = \min\{\delta_{x_1}/2, \cdots, \delta_{x_n}/2\}$. Suppose $y_1, y_2 \in X$ and $d(y_1, y_2) < \delta$. Pick i so that $d(y_1, x_i) < \delta_{x_i}/2$. Then:

$$d(y_2, x_i) \leq d(y_2, y_1) + d(y_1, x_i) < \delta + \frac{\delta_{x_i}}{2} \leq \delta_{x_i}$$

Now we know $d(y_1, x_i) < \delta_{x_i}$ and $d(y_2, x_i) < \delta_{x_i}$. By the choice of δ_{x_i}, for all $f \in F$ we have:

$$|f(y_1) - f(y_2)| \leq |f(y_1) - f(x_i)| + |f(y_2) - f(x_i)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore F is uniformly equicontinuous. □

Theorem 2.13 (Arzela-Ascoli). Let (X, d) be a compact metric space. A subset $K \subseteq \mathcal{C}(X)$ is compact if and only if K is closed, bounded and equicontinuous.

Proof. ($\Rightarrow$). If K is compact then it is closed and bounded by Proposition 2.1. Also we know that K is equicontinuous by the lemma above.

($\Leftarrow$). Suppose K is closed, bounded and equicontinuous. Note that $\mathcal{C}(X)$ is complete and K is closed, so K is complete. It remains to show K is totally bounded. Let $\epsilon > 0$. Since K is equicontinuous, it is uniformly equicontinuous by the lemma above. There is $\delta > 0$ such that for all $f \in K$ and $x, y \in X$ we have:

$$d(x,y) < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{4} \tag{$*$}$$

Since X is compact, there is a δ-net:

$$F_X = \{x_1, \cdots, x_n\} \subseteq X \quad \text{and} \quad X \subseteq \bigcup_{i=1}^{n} B_\delta(x_i) \tag{$\dagger$}$$

Define $T : K \to (\mathbb{R}^n, \|\cdot\|_\infty)$ by:

$$T(f) = (f(x_1), \cdots, f(x_n))$$

Note that $\|T(f)\|_\infty = \max\{|f(x_i)| : 1 \leq i \leq n\} \leq \|f\|_\infty$. [Here is a bit of abusing of notation. The two $\|\cdot\|$-norm are on two different spaces.] This implies that $T(K)$ is bounded in $\mathbb{R}^n$ since K is bounded in $\mathcal{C}(X)$. This means $T(K)$ is totally bounded, thus $\overline{T(K)}$ is compact in $\mathbb{R}^n$. This means that there exists a $\epsilon/4$-net of $T(K)$:

$$F_T = \{T(f_1), \cdots, T(f_m)\} \subseteq T(K) \quad \text{and} \quad T(K) \subseteq \bigcup_{i=1}^{m} B_{\epsilon/4}(f_i) \tag{††}$$

Here each $f_i \in K$. We claim that $F_K = \{f_1, \cdots, f_m\}$ is a ϵ-net for K. Indeed, let $f \in K$ be arbitrary. We can find some $j \in \{1, \cdots, m\}$ such that $\|T(f) - T(f_j)\|_\infty < \epsilon/4$ by (††). Now we let $y \in X$, we can find $i \in \{1, \cdots, n\}$ such that $d(x_i, y) < \delta$ by (†). Then:

$$|f(y) - f_j(y)| \leq \underbrace{|f(y) - f(x_i)|}_{< \frac{\epsilon}{4} \text{ by } (*)} + \underbrace{|f(x_i) - f_j(x_i)|}_{\leq \|T(f) - T(f_j)\|_\infty < \frac{\epsilon}{4}} + \underbrace{|f_j(x_i) - f_j(y)|}_{< \frac{\epsilon}{4} \text{ by } (*)} < \frac{3\epsilon}{4}$$

Since $y \in X$ is arbitrary, we have $\|f - f_j\|_\infty \leq \frac{3\epsilon}{4} < \epsilon$. This proved that $K \subseteq \bigcup_{i=1}^m B_\epsilon(f_j)$. Hence K is totally bounded. $\square$

2.6 Connectedness

Definition. Let (X, d) be a metric space. We say $A \subseteq X$ is **disconnected** if there exist two open sets U, V of X such that $A \subseteq U \cup V$ and $U \cap V = \emptyset$ and $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$. We say A is **connected** if it is not disconnected.

Example. Let (X, d) be a metric space. Any finite subset $A = \{x_1, \cdots, x_n\}$ with at least two elements is disconnected. Let $r = \frac{1}{2}\min\{d(x_i, x_j) : i \neq j\} > 0$. We define open sets:

$$U = B_r(x_1) \quad \text{and} \quad V = B_r(x_2) \cup \cdots \cup B_r(x_n)$$

Then $A \subseteq U \cup V$ and $U \cap V = \emptyset$ by our choice of r. Moreover $U \cap A = \{x_1\}$ and $V \cap A = \{x_2, \cdots, x_n\}$ are not empty. Therefore A is disconnected.

Example. Let X be a set with $|X| \geq 2$. Let d be the discrete metric on X. Then (X, d) is disconnected. Indeed, let $x_0 \in X$. Then $U = \{x_0\}$ is open and $V = X \setminus \{x_0\}$ is also open.

Example. The middle third cantor set is disconnected.

Example. The interval $[0, 1] \subseteq \mathbb{R}$ is connected. Assume it is disconnected by open sets U, V of $\mathbb{R}$. WLOG we may assume $0 \in U$. Let $C = \{c \in \mathbb{R} : [0, c) \subseteq U\}$. Since U is open, there is $\epsilon > 0$ so that $B_\epsilon(0) \subseteq U$. Since C is nonempty, we let $c^* = \sup C$. There are two cases.

(i). If $c^* \in U$. Then as U is open, there is $\epsilon > 0$ such that $B_\epsilon(c^*) \subseteq U$. This means $c^* + \epsilon \in U$, so we have $c^* + \epsilon \in C$. Contradiction.

(ii). If $c^* \in V$. There is $\epsilon > 0$ with $B_\epsilon(c^*) \subseteq V$. This means $B_\epsilon(c^*) \cap U = \emptyset$. However, by the definition of supremum we know $c^* - \epsilon \in C$, so $[0, c^* - \epsilon) \subseteq U$. This means $c^* - \frac{\epsilon}{2} \in U$, but we know $c^* - \frac{\epsilon}{2} \in V$ as well. Contradiction.

Theorem 2.14. Let (X, d) and (Y, ρ) be metric spaces. Suppose (X, d) is connected. If $f : X \to Y$ is continuous, then $f(X)$ is connected.

Proof. Assume $f(X)$ is disconnected, say by open sets U, V of (Y, ρ). It is easy to see that $f^{-1}(U)$ and $f^{-1}(V)$ are open sets that separate X. Contradiction. $\square$

Theorem 2.15. Any connected subsets of $\mathbb{R}$ are intervals.

Proof. Let C be a connected set. We define:

$$a = \inf C \in \mathbb{R} \cup \{-\infty\} \quad \text{and} \quad b = \sup C \in \mathbb{R} \cup \{\infty\}$$

If $c \in \mathbb{R}$ and $a < c < b$ we must have $c \in C$. Otherwise:

$$C \subseteq \underbrace{(-\infty, c)}_{U} \cup \underbrace{(c, \infty)}_{V}$$

This gives a separation of C, contradiction. Hence we have $(a, b) \subseteq C \subseteq [a, b]$. This means C is an interval in $\mathbb{R}$. $\square$

Definition. Let (X, d) be a metric space. We can define an equivalence relation on X by $x \sim y$ if and only if there is a connected set C containing both x, y. The equivalence classes of this relation are called **connected components**. Let $x_0 \in X$. the equivalence class that x_0 lies in is called the connected component of x_0 and it is equal to the union of all connected sets containing x_0.

Example. Let $X = [0, 1] \cup [2, 3]$ be the metric space with induced Euclidean metric. Then $[0, 1]$ and $[2, 3]$ are the connected components of X.

Definition. Let (X, d) be a metric space. We say X is **totally disconnected** if every connected component is a singleton set.

Example. Finite sets are totally disconnected.

Definition. Let (X, d) be a metric space. We say (X, d) is **path-connected** if for all $x, y \in X$ there exists a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.

Example. Let $(V, \|\cdot\|)$ be a normed space. Any convex set $C \subseteq V$ is path connected. For $x, y \in C$ we can define $f(t) = (1-t)x + ty \in C$.

Proposition 2.16. Let (X, d) be a metric space. If X is path-connected then X is connected.

Proof. Suppose $X = U \cup V$ is disconnected. Pick $x \in U$ and $y \in V$. There is a path $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$. Now:

$$[0,1] = f^{-1}(X) = f^{-1}(U) \cup f^{-1}(V)$$

Note that $0 \in f^{-1}(U)$ and $1 \in f^{-1}(V)$. It is easy to check $f^{-1}(U)$ and $f^{-1}(V)$ give a separation of $[0,1]$. This is a contradiction! $\square$

Example. The converse of this is not true. There exists connected spaces that is not path-connected. We define the following set:

$$X = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) : x > 0 \right\} \cup \{(0,0)\}$$

Then $X \subseteq \mathbb{R}^2$ is connected but not path connected.

2.7 Bonus Cantor Set Stuff

Definition. Let $n \geq 2$ and $A \subseteq \{0, 1, \cdots, n-1\}$ be a finite set. We define the **linear Cantor set**:

$$C_{A,n} = \left\{ \sum_{k=1}^{\infty} \frac{a_k}{n^k} : a_k \in A \right\}$$

Definition. Let $A \subseteq \mathbb{R}$. We define $N_\epsilon(A)$ to be the minimal number of ϵ-balls needed to cover A. The **box-counting dimension** of A is defined as:

$$\dim_B(A) = \lim_{\epsilon \to 0} \frac{-\log N_A(\epsilon)}{\log \epsilon}$$

if the limit exists. If the limit does not exist, we can take the limsup or liminf to define the **upper box dimension** and **lower box dimension**.

Example. Consider the middle third Cantor set. For $3^{-n} \leq \epsilon < 3^{-(n-1)}$, we need 2^n intervals of length $1/3^n$ to cover C. Hence:

$$\dim_B(C) = \lim_{n \to \infty} \frac{-\log 2^n}{\log 3^{-n}} = \frac{\log 2}{\log 3}$$

The box-counting dimension of C is $\log_3(2)$.

Definition. Let (X, ρ) be a metric space. For any $U \subseteq X$ we let $\operatorname{diam}(U)$ or $|U|$ denote its diameter. Let $S \subseteq X$ and let $\delta > 0$ and $d \in [0, \infty)$. We define:

$$H^d_\delta(S) = \inf\left\{\sum_{i\in I} |U_i|^d : S \subseteq \bigcup_{i\in I},\ |U_i| < \delta,\ |I| \leq |\mathbb{N}|\right\}$$

Then we define:

$$H^d(S) = \lim_{\delta\to 0} H^d_\delta(S)$$

to be the **d-dimensional Hausdorff measure of S.**

Theorem 2.17. Let (X, ρ) be a metric space and $0 \leq s < t < \infty$. For $A \subseteq X$ we have:

(i). If $H^s(A) < \infty$ then $H^t(A) = 0$.

(ii). If $H^t(A) > 0$ then $H^s(A) = \infty$.

Proof. It suffices to prove (i) since (ii) is just the contrapositive of (i). We have:

$$\begin{aligned}
H^t_\delta(A) &= \inf\left\{\sum_{i\in I} |U_i|^t : A \subseteq \bigcup_{i\in I} U_i,\ |U_i| < \delta,\ |I| \leq |\mathbb{N}|\right\} \\
&= \inf\left\{\sum_{i\in I} |U_i|^{t-s}|U_i|^s : A \subseteq \bigcup_{i\in I} U_i,\ |U_i| < \delta,\ |I| \leq |\mathbb{N}|\right\} \\
&\leq \inf\left\{\sum_{i\in I} \delta^{t-s}|U_i|^s : A \subseteq \bigcup_{i\in I} U_i,\ |U_i| < \delta,\ |I| \leq |\mathbb{N}|\right\} \\
&= \delta^{t-s} \inf\left\{\sum_{i\in I} |U_i|^s : A \subseteq \bigcup_{i\in I} U_i,\ |U_i| < \delta,\ |I| \leq |\mathbb{N}|\right\} \\
&= \delta^{t-s} H^s_\delta
\end{aligned}$$

Suppose $H^s(A) < \infty$, we then have:

$$H^t(A) = \lim_{\delta\to 0} \delta^{t-s} H^s_\delta = H_\delta \lim_{\delta\to 0} \delta^{t-s} = 0$$

As desired. $\square$

Corollary 2.18. There is at most one $d \in [0, \infty)$ with $0 < H^d(A) < \infty$.

Definition. Same setting as above. We define the **Hausdorff dimension** of A to be:

$$\dim_H(A) = \sup\{d \in [0, \infty) : H^d(A) = \infty\} = \inf\{d \in [0, \infty) : H^d(A) = 0\}$$

Example. Let $A = \mathbb{Q} \cap [0,1]$. We need $\lceil \epsilon^{-1} \rceil$ many ϵ-balls to cover A, as $\mathbb{Q}$ is dense in $\mathbb{R}$. Hence:

$$\dim_B(A) = \lim_{\epsilon \to 0} \frac{-\log \lceil \epsilon^{-1} \rceil}{\log \epsilon} = 1$$

However, we claim the Hausdorff dimension is 0. Consider:

$$\begin{aligned} H^0_\delta(A) &= \inf \left\{ \sum_{i \in I} |U_i|^0 : A \subseteq \bigcup_{i \in I} U_i,\ |U_i| < \delta,\ |I| \leq |\mathbb{N}| \right\} \\ &= \inf \left\{ \sum_{i \in I} |U_i|^t : A \subseteq \bigcup_{i \in I} U_i,\ |U_i| < \delta,\ I \text{ finite} \right\} \\ &= \inf \left\{ |I| : A \subseteq \bigcup_{i \in I} U_i,\ |U_i| < \delta,\ I \text{ finite} \right\} \\ &= \left\lceil \frac{1}{\delta} \right\rceil \end{aligned}$$

Then we have $H^0(A) = \lim_{\delta \to 0} H^0_\delta(A) = \infty$. Let $d > 0$, we wish to show that $H^d(A) = 0$. To do this it suffices to show for all $\epsilon > 0$ and $\delta > 0$ we have $H^d_\delta(A) \leq \epsilon$. Since A is countable, we can enumerate $A = \{r_n : n \geq 1\}$. For each $n \geq 1$ let:

$$\epsilon_n = \min \left\{ \delta,\ \frac{1}{2} \left(\frac{\epsilon}{2^n} \right)^{1/d} \right\} > 0$$

Then let $U_n = B_{\epsilon_n}(r_n)$ and $|U_n| \leq \left(\frac{\epsilon}{2^n}\right)^{1/d}$. Hence we have:

$$\sum_{n=1}^{\infty} |U_n|^d \leq \sum_{n=1}^{\infty} \left(\left(\frac{\epsilon}{2^n} \right)^{1/d} \right)^d = \epsilon$$

Hence $H^d(A) = 0$ for all $d > 0$, so $\dim_H(A) = \inf\{d \geq 0 : H^d(A) = 0\} = 0$.

Proposition 2.19. For any linear Cantor set $C_{A,n}$ we have $\dim_B(C_{A,n}) = \dim_H(C_{A,n})$.

Proposition 2.20. Let $A, B \subseteq \mathbb{R}$, then:

$$\dim_H(A \cup B) = \max\{\dim_H(A), \dim_H(B)\}$$

Proposition 2.21. Let $A, B \subseteq \mathbb{R}$, then:

$$\dim_H(A + B) \leq \dim_H(A) + \dim_H(B)$$

Proposition 2.22. Let $\emptyset \neq A \subseteq \mathbb{R}^n$, then $0 \leq \dim_H(A) \leq n$.

Example. From A4 we saw that $C+C=[0,2]$, where C is the middle-third Cantor set. That is:

$$C_{\{0,2\},3}+C_{\{0,2\},3}=[0,2]$$

We know the box counting dimension is $\log_3(2)$, so $\dim_H(C_{\{0,2\},3})=\log_3(2)$ as well.

Example. What is the dimension of $C_{\{0,3\},4}$ and the dimension of $C_{\{0,3\},4}+C_{\{0,3\},4}$? In general, we need 2^n intervals of length 4^{-n} to cover $C_{\{0,3\},4}$, so:

$$\dim_B(C_{\{0,3\},4})=\lim_{n\to\infty}\frac{\log 2^n}{\log 4^{-n}}=\frac{1}{2}=\dim_H(C_{\{0,3\},4})$$

What does $C_{\{0,3\},4}+C_{\{0,3\},4}$ looks like?

$$\begin{aligned}C_{\{0,3\},4}+C_{\{0,3\},4}&=\left\{\sum_{k=1}^{\infty}\frac{a_k+b_k}{4^k}:a_k,b_k\in\{0,3\}\right\}=\left\{\sum_{k=1}^{\infty}\frac{c_k}{4^k}:c_k\in\{0,3,6\}\right\}\\&=\left\{\sum_{k=1}^{\infty}\frac{c_k}{4^k}:c_k\in\left\{0,\frac{3}{2},3\right\}\right\}+\left\{\sum_{k=1}^{\infty}\frac{c_k}{4^k}:c_k\in\left\{0,\frac{3}{2},3\right\}\right\}\\&=2C_{\{0,\frac{3}{2},3\},4}\end{aligned}$$

We see that:

$$\dim_H(C_{\{0,\frac{3}{2},3\},4})=\dim_B(C_{\{0,\frac{3}{2},3\},4})=\frac{\log 3}{\log 4}<1$$

Theorem 2.23. Let $C_{A,n}$ be a linear Cantor set. If $\dim_H(C_{A,n})<\frac{1}{2}$ then $C_{A,n}+C_{A,n}\neq[0,2]$.

Proof. By Proposition 2.21 we have:

$$\dim_H(C_{A,n}+C_{A,n})\leq\dim_H(C_{A,n})+\dim_H(C_{A,n})<1$$

However $\dim_H([0,2])=1$. Hence $C_{A,n}+C_{A,n}\neq[0,2]$. $\square$

Example. Let $C\subseteq\mathbb{R}^n$ be a perfect and totally disconnected set with $\dim_H(C)<\frac{1}{2}$. Then $C+C$ is a perfect and totally disconnected set.

Theorem 2.24. Let $C_{A,n}$ be a linear Cantor set, then:

$$C_{A,n}=\bigcup_{a\in A}S_a(C_{A,n})$$

where $S_a:\mathbb{R}\to\mathbb{R}$ is defined by $S_a(x)=\frac{x+a}{n}$.

Proof. Note that we have:

$$\begin{aligned}
C_{A,n} &= \left\{ \frac{a_1}{n} + \sum_{k=2}^{\infty} \frac{a_k}{n^k} : a_k \in A \right\} \\
&= \bigcup_{a \in A} \left\{ \frac{a}{n} + \left\{ \sum_{k=2}^{\infty} \frac{a_k}{n^k} : a_k \in A \right\} \right\} \\
&= \bigcup_{a \in A} \left\{ \frac{a}{n} + \frac{1}{n} \left\{ \sum_{k=1}^{\infty} \frac{a_k}{n^k} : a_k \in A \right\} \right\} \\
&= \bigcup_{a \in A} \frac{a}{n} + \frac{1}{n} C_{A,n} \\
&= \bigcup_{a \in A} S_a(C_{A,n})
\end{aligned}$$

As desired. □

Theorem 2.25. Let $A \subseteq \{0, \cdots, n-1\}$ and $0, n-1 \in A$. Define:

$$B := A + A = \{0 = b_0 < b_1 < \cdots < b_k = 2n-2\}$$

Then $C_{A,n} + C_{A,n} = [0,2]$ if and only if $b_i - b_{i-1} \leq 2$ for all $1 \leq i \leq k$.

Proof. Note that we have:

$$C_{A,n} + C_{A,n} = \left\{ \sum_{r=1}^{\infty} \frac{a_r + c_r}{n^r} : a_r, c_r \in A \right\} = \left\{ \sum_{r=1}^{\infty} \frac{b_r}{n^r} : b_r \in B \right\} = C_{B,n}$$

Then $b_i - b_{i-1} \leq 2$ for all i if and only if $[0,2] = \bigcup_{i=0}^{k} S_{b_i}(C_{B,n}) = C_{B,n}$. □

Definition. A **Cantorval** is a compact subset of $\mathbb{R}$ with non-empty interior such that none of its connected components are isolated.

Fact. Let $A \subseteq \{0, \cdots, n-1\}$ and $0, n-1 \in A$. Exactly one of the followings is true:

1. $C_{A,n} + C_{A,n} = [0,2]$.
2. $C_{A,n} + C_{A,n}$ is a totally disconnected and perfect set.
3. $C_{A,n} + C_{A,n}$ ia a Cantorval.

Lecture 19, 2025/02/24

3 Completeness

3.1 Baire Category Theorem

Definition. Let (X, d) be a metric space. We say $A \subseteq X$ is **nowhere dense** if $\operatorname{int}(\overline{A}) = \emptyset$.

Example. Consider $(\mathbb{R}, d)$ with Euclidean metric. A singleton is nowhere dense. The integers $\mathbb{Z}$ is nowhere dense. Rationals $\mathbb{Q}$ is NOT nowhere dense, as $\overline{\mathbb{Q}} = \mathbb{R}$. The Cantor set is nowhere dense.

Example. Consider the metric space (X, d) where d is the discrete metric. Any non-empty set is NOT nowhere dense because every $A \subseteq X$ is both open and closed, so:

$$\operatorname{int}(\overline{A}) = \operatorname{int}(A) = A \neq \emptyset$$

The only nowhere dense subset of (X, d) is $\emptyset$.

Lemma 3.1. Let (X, d) be a metric space. If $A \subseteq X$ is nowhere, then $X \setminus \overline{A}$ is open and dense.

Proof. Since $\overline{A}$ is closed, clearly $X \setminus \overline{A}$ is open. Suppose $x \notin X \setminus \overline{A}$ and let $\epsilon > 0$. We want to find $y \in X \setminus \overline{A}$ such that $y \in B_\epsilon(x)$, which proves that $X \setminus \overline{A}$ is dense in X. Since $x \notin X \setminus \overline{A}$, we know $x \in \overline{A}$. Since A is nowhere dense, $\operatorname{int}(\overline{A}) = \emptyset$. Hence we can find $y \notin \overline{A}$ such that $y \in B_\epsilon(x)$, which means $y \in X \setminus \overline{A}$, as desired. $\square$

Definition. Let (X, d) be a metric space. We say $A \subseteq X$ is **first category (meagre)** if we can write A as a countable union of nowhere dense sets. That is:

$$A = \bigcup_{n=1}^{\infty} K_n$$

where each $K_n \subseteq X$ is nowhere dense. When X is first category as a set, then we also say (X, d) is first category. Otherwise we say A is **second category**.

Example. Consider $(\mathbb{R}, d)$ with the usual metric. Any nowhere dense set is first category. The rationals $\mathbb{Q}$ is first category because it is the countable union of $q \in \mathbb{Q}$.

Question: Is $\mathbb{R}$, with the usual metric, first category?

Answer: It is not first category (not obvious) by the Baire Category Theorem.

Theorem 3.2 (Baire Category Theorem). Any non-empty complete metric space (X, d) is second category.

Example. The reals $\mathbb{R}$ is not first category. The ℓ^p spaces for $1 \leq p < \infty$ are not first category. This does not apply to $(\mathbb{Q}, d)$ with the Euclidean metric sicne it is not complete.

Corollary 3.3. Let (X, d) be a non-empty complete metric space with $X = \bigcup_{n=1}^{\infty} K_n$, then there is $n \geq 1$ such that $\operatorname{int}(\overline{K_n}) \neq \emptyset$.

Proof. We know X is not first category by the BCT, so one of K_n is not nowhere dense. □

Proof of BCT. Assume $(K_n)_{n=1}^{\infty}$ is a sequence of nowhere dense sets, we want to show $X \neq \bigcup_{n=1}^{\infty} K_n$ by constructing $x^* \in X$ such that $x^* \notin K_n$ for all $n \geq 1$. Pick any $x_0 \in X$ and $r_0 > 0$. Consider $\overline{B_{r_0}(x_0)}$. Since K_1 is nowhere dense, we can find $x_1 \in \overline{B_{r_0}(x_0)}$ and $r_1 < r_0/2$ such that:

$$\overline{B_{r_1}(x_1)} \cap K = \emptyset \quad \text{and} \quad \overline{B_{r_1}(x_1)} \subseteq \overline{B_{r_0}(x_0)}$$

We repeat this process. Suppose we have defined x_n and r_n, we find x_{n+1} and r_{n+1} such that $x_{n+1} \in \overline{B_{r_n}(x_n)}$ and $r_n < r_{n-1}/2$ with:

$$\overline{B_{r_{n+1}}(x_{n+1})} \cap K = \emptyset \quad \text{and} \quad \overline{B_{r_{n+1}}(x_{n+1})} \subseteq \overline{B_{r_n}(x_n)}$$

We claim that $(x_n)_{n=1}^{\infty}$ is cauchy and its limit x^* satisfies our desired property. Let $m > n$, notice:

$$d(x_n, x_m) \leq r_n < \frac{r_{n-1}}{2} < \cdots < \frac{r_0}{2^n}$$

Therefore $(x_n)_{n=1}^{\infty}$ is cauchy. Since (X, d) is complete, we let $\lim_{n\to\infty} x_n = x^* \in X$. Note that $(x_n)_{n=k}^{\infty}$ is a sequence in $\overline{B_{r_k}(x_k)}$ for all $k \geq 1$ and each such closed ball is closed. Therefore:

$$x^* = \lim_{n\to\infty} x_n \in \overline{B_{r_k}(x_k)}$$

Hence $x^* \in \overline{B_{r_n}(x_n)}$ for all $n \geq 1$. Hence $x^* \notin K_n$ for all $n \geq 1$, as desired. □

Lecture 20, 2025/02/26

Definition. Let (X, d) be a metric space. We say $A \subseteq X$ is a G_δ **set** if there exist a countable sequence of open sets $U_n \subseteq X$ such that $A = \bigcap_{n=1}^{\infty} U_n$.

Example. Any open set is a G_δ set by definition.

Example. The irrational numbers are a G_δ set. Note that $\mathbb{Q}$ is countable, so:

$$\mathbb{R} \setminus \mathbb{Q} = \bigcap_{r\in\mathbb{Q}} (\mathbb{R} \setminus \{r\})$$

Definition. Let (X, d) be a metric space. We say $A \subseteq X$ is an F_σ **set** it there is a countable sequence of closed sets $C_n \subseteq X$ such that $A = \bigcup_{n=1}^{\infty} C_n$.

Remark. Note that A is G_δ if and only if A^c is F_σ.

Example. Any closed set is a F_σ set.

Example. The interval $A = (0,1)$ is an F_σ set because $(0,1) = \bigcup_{n=2}^{\infty} \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$.

Example. Note that $\mathbb{Q} = \bigcup_{r \in \mathbb{Q}} \{r\}$ is F_σ. However, we claim that $\mathbb{Q}$ is NOT a G_δ set! Assume for a contradiction that $\mathbb{Q}$ is a G_δ set, say:

$$\mathbb{Q} = \bigcap_{n=1}^{\infty} U_n$$

where each $U_n \subseteq \mathbb{R}$ is an open set. This means $\mathbb{Q} \subseteq U_n$ for all $n \geq 1$. Hence each U_n is an open denset set. Then $\mathbb{R} \setminus U_n$ is closed and nowhere dense. This means:

$$\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n=1}^{\infty} (\mathbb{R} \setminus U_n)$$

is a union of nowhere dense sets! Hence $\mathbb{R} \setminus \mathbb{Q}$ is first category. Since $\mathbb{Q}$ is first category, we have:

$$\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$$

is first category, being the union of two sets that are first category. Since $\mathbb{R}$ is complete, it is second category by BCT. Contradiction.

3.2 Nowhere Differentiable Functions

For this section we consider the space:

$$\mathcal{C}[0,1] = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$$

We will show that "most" functions $f \in \mathcal{C}[0,1]$ are nowhere differentiable!

Definition. Let $f \in \mathcal{C}[0,1]$. We say f is **Lipschitz at** $x_0 \in X$ if there is $K \in \mathbb{R}$ (dependent on x_0) such that for all $x \in [0,1]$ we have:

$$|f(x_0) - f(x)| \leq K|x_0 - x|$$

We say f is **Lipschitz** if the choice of K is independent of x_0.

Lemma 3.4. Let $f \in \mathcal{C}[0,1]$ and $x_0 \in [0,1]$. Assume $f'(x_0)$ exists, then f is Lipschitz at x_0.

Proof. Let $c_1 = |f'(x_0)| \geq 0$. This implies that:

$$c_1 = \lim_{x \to x_0} \left| \frac{f(x) - f(x_0)}{x - x_0} \right|$$

There exists $\delta > 0$ (small enough such that $(x_0-\delta, x_0+\delta) \subseteq [0,1]$) such that for all $x \in (x_0-\delta, x_0+\delta)$ we have that:

$$\left|\frac{f(x)-f(x_0)}{x-x_0}\right| \leq c_1 + 1$$

Consider the function $h(x) = \frac{f(x)-f(x_0)}{x-x_0}$ on the set $[0, x_0-\delta] \cup [x_0+\delta, 1]$. Note that h is continuous on this compact set, hence it is bounded on it. Let $c_2 \in \mathbb{R}$ such that for all $x \in [0, x_0-\delta] \cup [x_0+\delta, 1]$ we have that:

$$\left|\frac{f(x)-f(x_0)}{x-x_0}\right| \leq c_2$$

Let $K = \max\{c_1+1, c_2\} > 0$, then for all $x \in [0,1]$ we have:

$$\left|\frac{f(x)-f(x_0)}{x-x_0}\right| \leq K \implies |f(x)-f(x_0)| \leq K|x-x_0|$$

As desired. □

Lemma 3.5. Let $f \in \mathcal{C}[0,1]$ be Lipschitz at $x_0 \in [0,1]$ with constant K. Then for all $a, b \in [0,1]$ with $a \leq x_0 \leq b$ we have:

$$|f(a)-f(b)| \leq K|a-b|$$

Proof. Since $a \leq x_0 \leq b$ we have $|a-x_0| + |x_0-b| = |a-b|$. Then:

$$|f(a)-f(b)| \leq |f(a)-f(x_0)| + |f(x_0)+f(b)| \leq K(|a-x_0|+|x_0-b|) = K|a-b|$$

As desired. □

Example. Define a function $f : [0,1] \to \mathbb{R}$ by $f(x) = \sum_{n=1}^{\infty} 2^{-n}\cos(\pi 10^n x)$. We claim that $f \in \mathcal{C}[0,1]$ but nowhere differentiable! It suffices to show it is the limit of a sequence of continuous functions. For each $N \geq 1$ let:

$$f_N(x) = \sum_{n=1}^{N} 2^{-n}\cos(\pi 10^n x)$$

Then each $f_N \in \mathcal{C}[0,1]$. We claim that $(f_N)_{N=1}^{\infty}$ is cauchy. Let $N > M$, we have:

$$\|f_N - f_M\|_\infty = \sup_{x\in[0,1]} \left|\sum_{n=M+1}^{N} 2^{-n}\cos(\pi 10^n x)\right| \leq \sum_{n=M+1}^{N} 2^{-n} \to 0$$

because the series $\sum_{n=1}^{\infty} 2^{-n} = 1$ converges, its tail goes to 0. Therefore $(f_N)_{N=1}^{\infty}$ is cauchy and since $\mathcal{C}[0,1]$ is complete, it converges to $f \in \mathcal{C}[0,1]$. To show it is nowhere differentiable, it suffices to show it is not Lipschitz at any $x_0 \in [0,1]$. Write $x_0 = \sum_{k=1}^{\infty} \frac{a_k}{10^k}$ in base 10. Suppose f is Lipschitz at x_0

with constant $K \in \mathbb{R}$. Let $N \geq 1$ (to be chosen later), we define:

$$x_L = \sum_{k=1}^{N} \frac{a_k}{10^k} \quad \text{and} \quad x_R = x_L + \frac{1}{10^N}$$

We consider the difference between $f(x_R)$ and $f(x_L)$. Note that:

$$\cos(x) - \epsilon \leq \cos(x + \epsilon) \leq \cos(x) + \epsilon \tag{1}$$

for $\epsilon > 0$ small. By (1), for $1 \leq k \leq N$ we have:

$$\cos(\pi 10^k (x_L + 10^{-N})) - \cos(\pi 10^k x_L) = \pi 10^{-N+k}$$

For $k > N$, note that $10^k x_L$ and $10^k x_R$ are integers so:

$$\cos(\pi 10^k (x_L + 10^{-N})) - \cos(\pi 10^k x_L) = 0$$

With some work, this gives:

$$|f(x_L) - f(x_R)| \geq (5^N + \text{small stuff})|x_L - x_R|$$

If we pick N so that $5^N > K$ then this gives a contradiction.