

# **PMATH 351 Notes**

Real Analysis

Winter 2025

Based on Professor Kevin Hare's Lectures

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# 1 Metric Spaces

## 1.1 Normed Vector Spaces

**Definition.** Let  $V$  be a vector space over  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ . We say  $\|\cdot\| : V \rightarrow \mathbb{R}$  is a **norm** if:

- (i). For all  $v \in V$  we have  $\|v\| = 0 \iff v = 0$ .
- (ii). For all  $v \in V$  and  $\lambda \in \mathbb{K}$  we have  $\|\lambda v\| \leq |\lambda| \|v\|$ .
- (iii). For all  $v, w \in V$  we have  $\|v + w\| \leq \|v\| + \|w\|$ .

A vector space, combined with a norm, is called a **normed vector space**.

**Example.** Let  $V = \mathbb{R}^n$ . Define a map  $\|\cdot\|_1 : \mathbb{R}^n \rightarrow \mathbb{R}$  by:

$$\|v\|_1 = \|(x_1, \dots, x_n)\|_1 = |x_1| + \dots + |x_n|$$

Clearly property 1 and 2 holds. To see property 3 we have:

$$\begin{aligned} \|(x_1, \dots, x_n) + (y_1, \dots, y_n)\|_1 &= |x_1 + y_1| + \dots + |x_n + y_n| \\ &\leq |x_1| + |y_1| + \dots + |x_n| + |y_n| \quad (\triangle \text{ inequality in } \mathbb{R}) \\ &= \|(x_1, \dots, x_n)\|_1 + \|(y_1, \dots, y_n)\|_1 \end{aligned}$$

Hence  $\|\cdot\|_1$  defines a norm on  $V = \mathbb{R}^n$ .

**Example.** Let  $V = \mathbb{R}^n$  again. Define  $\|\cdot\|_\infty : \mathbb{R}^n \rightarrow \mathbb{R}$  by:

$$\|v\|_\infty = \|(x_1, \dots, x_n)\|_\infty = \max(|x_1|, \dots, |x_n|)$$

This also defines a norm on  $\mathbb{R}^n$ .

**Example.** What does the unit ball  $B = \{v \in V : \|v\| \leq 1\}$  look like? Take  $V = \mathbb{R}^2$ .

**Note.** It is possible to extend these two norms to infinite dimensional vector spaces if we are being careful. Both of the norms above are examples of  $p$ -norms, for  $1 \leq p \leq \infty$ .

**Example.** Let  $V = \mathbb{R}[x]$  be a vector over  $\mathbb{R}$ . Define  $\|\cdot\|_1$  and  $\|\cdot\|_\infty$  on  $V$  by:

$$\|f\|_1 = \int_0^1 |f(x)| \, dx \quad \text{and} \quad \|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$$

The three properties are satisfied by these two norms. Note these norms can be defined beyond polynomials if we are careful.

**Theorem 1.1 (Minkowski).** Let  $1 \leq p < \infty$  be a real number.

(i). We define:

$$\ell_p = \left\{ (x_n)_{n=1}^{\infty} \subseteq \mathbb{C} : \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} < \infty \right\}$$

Then the map  $\|\cdot\|_p : \ell_p \rightarrow \mathbb{R}$  defined by:

$$\|(x_n)\|_p := \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}$$

defines a norm on  $\ell_p$ . This is called the  **$\ell_p$ -space**.

(ii). Let  $\mathcal{C}[a, b]$  be the set of continuous functions on  $[a, b]$ . Then:

$$\|f\|_p = \left( \int_a^b |f(x)|^p dx \right)^{1/p}$$

defines a norm. Define  $L^p[a, b] = \{f \in \mathcal{C}[a, b] : \|f\|_p < \infty\}$ , called the  **$L^p$ -space**.

**Proof.** Note for  $p \geq 1$ , define a map  $\varphi(x) = |x|^p$  and  $\varphi$  is convex on  $\mathbb{R}$ . We will prove part 2 first. Assume  $f, g \in \mathcal{C}[a, b]$  and  $f, g \neq 0$ . If  $f = 0$  or  $g = 0$  the triangle inequality is easy to prove.

$$\begin{aligned} \|f + g\|_p^p &= \int_a^b |f(x) + g(x)|^p dx = \int_a^b \left| \|f\|_p \cdot \frac{f}{\|f\|_p} + \|g\|_p \cdot \frac{g}{\|g\|_p} \right|^p dx \\ &= (\|f\|_p + \|g\|_p)^p \int_a^b \left| \underbrace{\frac{\|f\|_p}{\|f\|_p + \|g\|_p}}_{\alpha} \cdot \frac{f}{\|f\|_p} + \underbrace{\frac{\|g\|_p}{\|f\|_p + \|g\|_p}}_{1-\alpha} \cdot \frac{g}{\|g\|_p} \right|^p dx \end{aligned}$$

Note that  $\alpha \in [0, 1]$ , we can rewrite the above quantity as:

$$\begin{aligned} I &:= (\|f\|_p + \|g\|_p)^p \int_a^b \left| \alpha \cdot \frac{f}{\|f\|_p} + (1 - \alpha) \cdot \frac{g}{\|g\|_p} \right|^p dx \\ &= (\|f\|_p + \|g\|_p)^p \int_a^b \varphi \left( \alpha \cdot \frac{f}{\|f\|_p} + (1 - \alpha) \cdot \frac{g}{\|g\|_p} \right)^p dx \end{aligned}$$

Recall  $\varphi(x) = |x|^p$  is convex, we have:

$$\begin{aligned} I &\leq (\|f\|_p + \|g\|_p)^p \left( \alpha \int_a^b \left| \frac{f}{\|f\|_p} \right|^p dx + (1 - \alpha) \int_a^b \left| \frac{g}{\|g\|_p} \right|^p dx \right) \\ &= (\|f\|_p + \|g\|_p)^p (\alpha + 1 - \alpha) = (\|f\|_p + \|g\|_p)^p \end{aligned}$$

This proved that:

$$\|f + g\|_p^p \leq (\|f\|_p + \|g\|_p)^p \implies \|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Part 1 ( $\ell_p$ -space) are proved in the similar way by replacing integral with sum.  $\square$

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Lecture 2, 2025/01/08

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## 1.2 Metric Spaces

**Definition.** Let  $X$  be a non-empty set. A **distance (metric)** on  $X$  is a function  $d : X \times X \rightarrow [0, \infty)$  such that:

- (i). For all  $x, y \in X$  we have  $d(x, y) = 0 \iff x = y$ .
- (ii). For all  $x, y \in X$  we have  $d(x, y) = d(y, x)$ .
- (iii). For all  $x, y, z \in X$  we have  $d(x, z) \leq d(x, y) + d(y, z)$ .

The pair  $(X, d)$  is called a **metric space**. We just say  $X$  is a metric space if  $d$  is understood.

**Example.** Let  $(X, \|\cdot\|)$  be a normed vector space, then  $d(x, y) = \|x - y\|$  is a metric on  $X$ . Clearly  $d(x, y) = 0 \iff \|x - y\| = 0 \iff x - y = 0 \iff x = y$ . Property (ii) is also trivial. For property (iii) we have:

$$d(x, z) = \|x - z\| = \|x - y + y - z\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z)$$

**Example (Graph metric).** Let  $(X, E)$  be a graph where  $X$  is the vertex set. The set of **paths** from  $x$  to  $y$  is:

$$P_{xy} = \{(x = x_1, x_2, \dots, x_n = y) : (x_i, x_{i+1}) \in E\}$$

Define a **weight** function  $\omega : E \rightarrow (0, \infty)$ . Then:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ \min\{\omega(x_1, x_2) + \dots + \omega(x_{n-1}, x_n) \text{ for } (x_1, \dots, x_n) \in P_{xy}\} & \text{otherwise} \end{cases}$$

This distance basically measures the shortest path from  $x$  to  $y$ , with weight on the edge.

**Example (Trivial metric).** Let  $X$  be a non-empty set, define:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Exercise: It is easy to verify that this is a distance function on  $X$ .

**Example ( $p$ -adic metric on  $\mathbb{Q}$ ).** Let  $p$  be a fixed prime in  $\mathbb{N}$ . By unique factorization, every  $q \in \mathbb{Q}$  can be uniquely written as:

$$q = p^n \frac{a}{b}$$

where  $n \in \mathbb{Z}$  and  $a \in \mathbb{Z}$  and  $0 \neq b \in \mathbb{N}$  with  $\gcd(a, b) = 1$ . Define the  **$p$ -adic norm** by:

$$|q|_p = \begin{cases} p^{-n} & \text{if } q \neq 0 \text{ and } n \text{ is from above} \\ 0 & \text{if } q = 0 \end{cases}$$

Exercise: For  $q, r \in \mathbb{Q}$  we have:

$$|q + r|_p \leq \max\{|q|_p, |r|_p\} \leq |q|_p + |r|_p$$

Take  $p = 3$  and  $q = 1/6$  and  $r = 2/9$ , then:

$$\begin{aligned} |q|_3 &= \left| 3^{-1} \cdot \frac{1}{2} \right|_3 = 3^{-( -1)} = 3 \\ |r|_3 &= \left| 3^{-2} \cdot \frac{2}{1} \right|_3 = 3^{-( -2)} = 9 \\ |q + r|_3 &= \left| \frac{3+4}{18} \right|_3 = \left| 3^{-2} \cdot \frac{7}{2} \right|_3 = 9 = \max\{3, 9\} \end{aligned}$$

Define the  **$p$ -adic metric** on  $\mathbb{Q}$  by:

$$d_p(q, r) = |q - r|_p$$

Exercise: This clearly defined a metric on  $\mathbb{Q}$ .

**Example.** Consider  $\{0, 1\}^{\mathbb{N}} = \{(b_n)_{n=1}^{\infty} : b_n \in \{0, 1\}\}$ . Take  $b, c \in \mathbb{N}$  then define:

$$d(b, c) := \begin{cases} 0 & \text{if } b = c \\ \frac{1}{2^n} \text{ for } b = \min\{i \in \mathbb{N} : b_i \neq c_i\} & \text{otherwise} \end{cases}$$

Exericse:  $d$  is a metric on  $\{0, 1\}^{\mathbb{N}}$ , we may call this product metric. Now we define:

$$\rho(b, c) = \sum_{n=1}^{\infty} \frac{d(b_n, c_n)}{2^n} \quad (\text{always converges})$$

Fact (Exercise):  $d(b, c) \leq \rho(b, c) \leq 2d(b, c)$ .

**Definition.** Let  $(X, d)$  be a metric space. If  $\emptyset \neq Y \subseteq X$ , we make  $Y$  a metric space by defining  $d_Y : Y \times Y \rightarrow \mathbb{R}$  by  $d_Y(x, y) = d(x, y)$  for  $x, y \in Y$ . [This is just the restriction  $d|_{Y \times Y}$ ] This is called the **relativized metric** on  $Y$ .

**Definition.** Let  $X$  be a non-empty set and  $d_1, d_2$  be metrics on  $X$ . We say  $d_1$  is **equivalent** to  $d_2$  if there exist  $c, C > 0$  such that:

$$cd_1(x, y) \leq d_2(x, y) \leq Cd_1(x, y)$$

for all  $x, y \in X$ . Exercise: This is an equivalence relation on the set of metrics on  $X$ .

**Example.** Let  $X = \mathbb{R}^n$  and  $1 \leq p < \infty$ . Define a metric:

$$d_p(x, y) = \|x - y\|_p = \left( \sum_{k=1}^n |x_k - y_k|^p \right)^{1/p}$$

and define  $d_\infty(x, y) = \max\{|x_k - y_k| : k \in \{1, \dots, n\}\}$ . Let  $x \in \mathbb{R}^n$ , say  $\|x\|_\infty = x_j$  for some  $j$ . Then we note that:

$$\|x\|_\infty = |x_j| = (|x_j|^p)^{1/p} \leq \left( \sum_{k=1}^n |x_k|^p \right)^{1/p} = \|x\|_p \leq \left( \sum_{k=1}^n |x_j|^p \right)^{1/p} = n^{1/p} \|x\|_\infty$$

To summarize we have:

$$\|x\|_\infty \leq \|x\|_p \leq n^{1/p} \|x\|_\infty$$

Hence  $\|\cdot\|_\infty$  and  $\|\cdot\|_p$  are equivalent norms for all  $1 \leq p < \infty$ . By equivalence,  $\|\cdot\|_p$  are all equivalent norms on  $\mathbb{R}^n$  for  $1 \leq p \leq \infty$ .

Lecture 3, 2025/01/10

### 1.3 Topology of Metric Spaces

**Definition.** Let  $(X, d)$  be a metric space. Take  $x \in X$  and  $r > 0$ . Define an **open ball** centered at  $x$  with radius  $r$  to be:

$$B_r(x) := b_r(x) := B(x, r) := \{y \in X : d(x, y) < r\}$$

Similarly we define a **closed ball** as:

$$\overline{B}_r(x) := \bar{b}_r(x) := \overline{B}(x, r) := \{y \in X : d(x, y) \leq r\}$$

**Definition.** Let  $(X, d)$  be a metric space. Let  $N \subseteq X$  with some  $x \in X$ . We say  $N$  is a **neighborhood** of  $x$  if there exists  $r > 0$  such that  $B_r(x) \subseteq N$ .

**Definition.** Let  $(X, d)$  be a metric space. We say  $N \subseteq X$  is **open** if  $N$  is a neighborhood of  $x$  for all  $x \in N$ . We say  $N$  is **closed** if  $X \setminus N$  is open.

**Example.** Let  $X = \mathbb{R}$  with usual Euclidean metric. Then  $(a, b)$  is open for all  $a < b$  in  $\mathbb{R}$ . The empty set  $\emptyset$  and  $\mathbb{R}$  are open.

**Remark.** In general, in a metric space  $(X, d)$ , the set  $X$  is trivially open and the empty set  $\emptyset$  is vacuously open. Note that  $X \setminus X = \emptyset$  and  $X \setminus \emptyset = X$ . Hence  $X, \emptyset$  are both open and closed.

**Example.** Let  $(X, d)$  be a metric space where  $d$  is the discrete metric. Every subset  $N \subseteq X$  is open! Why? Take  $r = 1/2$  and  $x \in N$ , then  $B_{1/2}(x) = \{x\} \subseteq N$ . Similarly every subset is closed.

**Question:** Consider the metric space  $(\mathbb{Q}, d_3)$ , where  $d_3$  is the 3-adic metric. What do the open sets look like?

**Theorem 1.2 (Union of Open sets).** Let  $(X, d)$  be a metric space. Let  $\{X_i\}_{i \in I}$  be a collection of open sets, then  $\bigcup_{i \in I} X_i$  is an open set.

**Proof.** Let  $x \in \bigcup_{i \in I} X_i$ , then  $x \in X_{i_0}$  for some  $i_0 \in I$ . Since  $X_{i_0}$  is open, there is  $r > 0$  such that  $B_r(x) \subseteq X_{i_0}$ . Hence:

$$B_r(x) \subseteq X_{i_0} \subseteq \bigcup_{i \in I} X_i$$

It follows that  $\bigcup_{i \in I} X_i$  is open, as desired.  $\square$

**Corollary 1.3 (Intersection of Closed sets).** Let  $(X, d)$  be a metric space and  $\{X_i\}_{i \in I}$  a collection of closed sets. Then  $\bigcap_{i \in I} X_i$  is closed.

**Proof.** Take complement using De Morgan's Law and apply the above theorem.  $\square$

**Question:** If  $\{X_i\}_{i \in I}$  is a collection of open sets, what can we say about  $\bigcap_{i \in I} X_i$ ?

(i). If  $|I| = n < \infty$ , then this intersection is open. Consider  $\{X_1, \dots, X_n\}$ . Take  $x \in \bigcap_{i=1}^n X_i$ , then  $x \in X_i$  for all  $i$ , so there is  $r_i > 0$  such that  $B(x, r_i) \subseteq X_i$  for all  $i$ . Take  $r = \min\{r_1, \dots, r_n\}$ , then  $B(x, r) \subseteq \bigcap_{i=1}^n X_i$ , hence open.

(ii). If  $|I| > |\mathbb{N}|$ , this may fail. For example, take  $X_n = (\frac{-1}{n}, \frac{1}{n})$ . Then  $\bigcap_{n=1}^{\infty} X_n = \{0\}$ , not open.

**Proposition 1.4.** Finite intersection of open sets is open and finite union of closed sets is closed.

**Definition.** Let  $(X, d)$  be a metric sapce and  $(x_n)_{n=1}^{\infty}$  be a sequence in  $X$ . Let  $x \in X$ . We say the sequence  $(x_n)_{n=1}^{\infty}$  **converges** to  $x$  if  $\lim_{n \rightarrow \infty} d(x, x_n) = 0$ . Equivalently, for all  $\epsilon > 0$  there is  $N \in \mathbb{N}$  such that for all  $n \in \mathbb{N}$ :

$$n \geq N \implies d(x, x_n) < \epsilon$$

In this case we can write  $\lim_{n \rightarrow \infty} x_n = x$ .

**Example.** Let  $X = \mathbb{Q}$  with Euclidean metric. Let  $(x_n)_{n=1}^{\infty}$  be  $x_n = 1/n$ . This converges to 0.

**Example.** Let  $X = \mathbb{Q}$ , consider the sequence defined by:

$$x_n = \text{truncation of } \pi \text{ to the } n\text{-th decimal place}$$

For example  $x_1 = 3$ ,  $x_2 = 3.1$ ,  $x_3 = 3.14$  and so on. This sequence “converges” to  $\pi$ , but  $\pi \notin \mathbb{Q}$  so this sequence does not converge in  $\mathbb{Q}$ ! It converges in  $\mathbb{R}$ .

**Example.** Let  $(X, d)$  with the discrete metric. A sequence  $(a_n)$  is convergent if and only if it is eventually constant. That is, there is  $N \in \mathbb{N}$  such that  $x_n = X_N$  for all  $n \geq N$ . In this case the limit is just  $\lim_{n \rightarrow \infty} x_n = X_N$ .

**Example.** Consider  $(\mathbb{Q}, d_3)$ , the 3-adic metric. Consider the two sequences:

$$(x_n)_{n=1}^{\infty} = \left(\frac{1}{n}\right)_{n=1}^{\infty} \quad \text{and} \quad (y_n)_{n=1}^{\infty} \quad \text{by} \quad y_n = \begin{cases} 2 & \text{if } n = 1 \\ 2 + 3y_{n-1} & \text{if } n \geq 2 \end{cases}$$

For the first sequence  $(x_n)$ , it has a subsequence  $(3^{-k})_{k=1}^{\infty}$  and  $d(0, 3^{-k}) = 3^k \rightarrow \infty$ . Hence  $(x_n)$  does not converge (We defer the actual proof of this when we see Cauchy sequence). For the second sequence, we see that:

$$y_n = 2 + 2 \cdot 3 + 2 \cdot 3^2 + \cdots + 2 \cdot 3^{n-1} = 3^n - 1$$

Hence  $d(-1, y_n) = \| -3^n \|_3 = \frac{1}{3^n} \rightarrow 0$  so that  $\lim_{n \rightarrow \infty} y_n = -1$ .

Lecture 4, 2025/01/13

**Definition.** Let  $(X, d)$  be a metric space and  $A \subseteq X$ . The **closure** of  $A$  in  $X$  is the smallest closed set in  $X$  that contains  $A$ . We denote the closure of  $A$  by  $\text{cl}(A)$  or  $\overline{A}$ . In other word,  $\overline{A}$  is the intersection of all closed sets that contain  $A$ .

**Example.** The closure of a closed set is itself.

**Example.** Consider the metric space  $(\mathbb{R}, d)$  with the usual Euclidean metric. Then  $\overline{\mathbb{Q}} = \mathbb{R}$ .

**Example.** Consider  $\mathbb{R}$  again with discrete metric, then  $\overline{\mathbb{Q}} = \mathbb{Q}$ . (because every set is closed in this topology).

**Example.** Consider  $(\mathbb{Q}, d)$  with the 3-adic metric. We can show that  $\mathbb{Z}$  is not closed. Define a sequence  $(x_n)_{n=1}^{\infty}$  by  $x_n = \sum_{k=0}^n 9^k$ . This sequence has a limit in  $\mathbb{Q}$  but not in  $\mathbb{Z}$ . How do we “guess” the limit of this sequence? Notice that:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{for all } \|x\| < 1$$

In this case  $\|9\|_3 = 1/9 < 1$ , hence plugging in 9 shows the limit of  $(x_n)$  is  $-1/8 \notin \mathbb{Z}$ . [This is NOT a rigorous proof for now! This just allows us to guess the limit and we can then use the  $\epsilon$  thing to prove the limit]. Hence  $\mathbb{Z}$  does not contain a limit point, which means it is not closed (by the theorem below).

**Theorem 1.5.** A closed set contains all of its limit points. That is, if  $A \subseteq X$  is closed and  $(x_n)_{n=1}^\infty$  is a sequence in  $A$ , then whenever  $\lim_{n \rightarrow \infty} x_n = x \in X$  exists, we must have  $x \in A$ .

## 1.4 Continuous Functions

**Definition.** Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces. A function  $f : X \rightarrow Y$  is **continuous at  $x_0 \in X$**  if for all  $\epsilon > 0$  there is  $\delta > 0$  such that for all  $x \in X$  with  $d(x, x_0) < \delta$ , then  $\rho(f(x), f(x_0)) < \epsilon$ . [Equivalently we have  $f(B_\delta(x_0)) \subseteq B_\epsilon(\epsilon)$ .]

**Example.** Let  $(X, d)$  be a metric space with discrete metric. Let  $f : X \rightarrow Y$  with  $(Y, \rho)$  a metric space. Then  $f$  is continuous at every  $x_0 \in X$ . Why? For any  $\epsilon > 0$ , pick  $\delta = 1/2$ . Then  $d(x, x_0) < 1/2$  implies  $d(x, x_0) = 0$  and  $x = x_0$ . Hence  $\rho(f(x), f(x_0)) = 0 < \epsilon$ .

**Definition.** Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces.

- (i). We say  $f : X \rightarrow Y$  is **continuous on  $X$**  if it is continuous at all  $x_0 \in X$ .
- (ii). We say  $f : X \rightarrow Y$  is **uniformly continuous on  $X$**  if for all  $\epsilon > 0$ , there is  $\delta > 0$  such that for all  $x, y \in X$ :

$$d(x, y) < \delta \implies \rho(f(x), f(y)) < \epsilon$$

That is, the choice of  $\delta > 0$  is independent of  $x, y \in X$ . The usual continuity means for any  $x, y \in X$  we can choose a  $\delta > 0$  for them, but in this case there is one choice of  $\delta > 0$  that works for all  $x, y \in X$ .

**Note.** In the Example above, we see that  $f$  is in fact uniformly continuous.

**Example.** Let  $f : (0, 1) \rightarrow (0, \infty)$  with Euclidean metric given by  $f(x) = 1/x$ . This function is continuous but NOT uniformly continuous. To see it is continuous, fix  $x_0 \in (0, 1)$  and let  $\epsilon > 0$ . We then pick  $\delta > 0$  to be:

$$\delta = \min \left\{ \frac{x_0}{2}, \frac{\epsilon x_0^2}{2} \right\}$$

Then, if  $|x - x_0| < \delta$ , we have:

$$\left| \frac{1}{x} - \frac{1}{x_0} \right| = \left| \frac{x_0 - x}{xx_0} \right| < \frac{\epsilon \cdot x_0^2 / 2}{(x_0/2)x_0} = \epsilon$$

It follows that  $f$  is continuous on  $(0, 1)$ . To see it is NOT uniformly continuous, assume it is. Take  $\epsilon = 1$ , then there is  $\delta > 0$  such that  $|x - y| < \delta$  implies  $|f(x) - f(y)| < 1$ . However, pick  $N \in \mathbb{N}$  large enough so that  $1/N - 1/(N + 1) < \delta$ , then:

$$1 > \left| f\left(\frac{1}{N}\right) - f\left(\frac{1}{N+1}\right) \right| = \left| \frac{1}{1/N} - \frac{1}{1/(N+1)} \right| = 1$$

This is a contradiction, hence  $f$  is NOT uniformly continuous.

**Definition.** Let  $X, Y$  be metric spaces. We say  $f : X \rightarrow Y$  is **sequentially continuous at  $x_0 \in X$**  if for all sequence  $(x_n)_{n=1}^{\infty}$  in  $X$  we have:

$$\lim_{n \rightarrow \infty} x_n = x_0 \implies \lim_{n \rightarrow \infty} f(x_n) = f(x_0)$$

We say  $f$  is **sequentially continuous** if it is sequentially continuous at every  $x_0 \in X$ .

**Theorem 1.6.** Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces and  $f : X \rightarrow Y$ . The followings are equivalent:

- (i).  $f$  is continuous.
- (ii). For all open sets  $V \subseteq Y$  we have  $f^{-1}(V)$  is open in  $X$ .
- (iii).  $f$  is sequentially continuous.

**Proof.** We will prove (i)  $\implies$  (ii)  $\implies$  (i) and (i)  $\implies$  (iii)  $\implies$  (i).

**(i)  $\implies$  (ii).** Assume  $f$  is continuous. Let  $V \subseteq Y$  be open. We want to show  $f^{-1}(V)$  is open in  $X$ . If  $f^{-1}(V) = \emptyset$ , done. Otherwise pick  $x_0 \in f^{-1}(V)$ , then  $f(x_0) \in V$ . Then there is  $\epsilon > 0$  such that  $B_\epsilon(f(x_0)) \subseteq V$ . Since  $f$  is continuous at  $x_0$ , there is  $\delta > 0$  such that  $f(B_\delta(x_0)) \subseteq B_\epsilon(f(x_0)) \subseteq V$ . Hence  $B_\delta(x_0) \subseteq f^{-1}(V)$  and therefore  $f^{-1}(V)$  is open in  $X$ .

**(i)  $\implies$  (iii).** Assume  $f$  is continuous at  $x_0$  and  $(x_n)_{n=1}^{\infty}$  is a sequence with  $x_n \rightarrow x_0$ . Pick  $\epsilon > 0$ , since  $f$  is continuous there is  $\delta > 0$  such that  $d(x, x_0) < \delta$  implies  $\rho(f(x), f(x_0)) < \epsilon$ . Now pick  $N \in \mathbb{N}$  so that  $d(x_n, x_0) < \delta$  for  $n \geq N$ . Hence if  $n \geq N$  we have  $\rho(f(x_n), f(x_0)) < \epsilon$ .

**(ii)  $\implies$  (i).** Fix  $x_0 \in X$  and let  $\epsilon > 0$ . Consider the open set  $V = B_\epsilon(f(x_0))$ . Since we are assuming (ii), we know  $f^{-1}(V)$  is open and  $x_0 \in f^{-1}(V)$ . Therefore there is  $\delta > 0$  such that  $B_\delta(x_0) \subseteq f^{-1}(V)$ . Therefore we have  $f(B_\delta(x_0)) \subseteq B_\epsilon(f(x_0))$ , which proved  $f$  is continuous at  $x_0$ .

**(iii)  $\implies$  (i).** We will prove this by contrapositive. Assume  $f$  is not continuous. This means there is  $x_0 \in X$  and  $\epsilon > 0$  such that for all  $\delta > 0$ , there are  $x \in X$  with  $d(x, x_0) < \delta$  but  $\rho(f(x), f(x_0)) \geq \epsilon$ . We are going to construct a sequence  $(x_n)_{n=1}^{\infty} \subseteq X$  using this information that breaks the sequential

continuity. For  $n \in \mathbb{N}$ , we choose  $x_n \in X$  such that  $d(x_0, x_n) < 1/n$  but  $\rho(f(x_0), f(x_n)) \geq \epsilon$ . Then we clearly have  $x_n \rightarrow x_0$  but  $f(x_n)$  does NOT converge to  $f(x_0)$  as they are always at least  $\epsilon$ -away from each other.  $\square$

**Theorem 1.7.** Let  $X, Y, Z$  be metric spaces. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be continuous. Then  $g \circ f : X \rightarrow Z$  is continuous.

**Definition.** Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces. We can define a metric  $d \times \rho$  on  $X \times Y$  by:

$$(d \times \rho)((x_1, y_1), (x_2, y_2)) := d(x_1, x_2) + \rho(y_1, y_2)$$

It is easy to check that this defines a metric.

**Theorem 1.8.** Let  $X, Y, Z, W$  be metric spaces and  $f : X \rightarrow Z$  and  $g : Y \rightarrow W$  be continuous. Then  $f \times g : X \times Y \rightarrow Z \times W$  by  $(f \times g)(x, y) = (f(x), g(y))$  is continuous where  $X \times Y$  and  $Z \times W$  are equipped with the metric defined above.

**Definition.** Let  $f : X \rightarrow Y$  where  $(X, d)$  and  $(Y, \rho)$  are metric spaces. We say  $f$  is an **isometry** if for all  $x_1, x_2 \in X$  we have  $d(x_1, x_2) = \rho(f(x_1), f(x_2))$ .

**Example.** In  $\mathbb{R}^2$ , any rotation, reflection, translation and combination of them are isometries.

**Definition.** Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces. A map  $f : X \rightarrow Y$  is called **Lipschitz** if there exists a constant  $C > 0$  such that:

$$\rho(f(x_1), f(x_2)) \leq Cd(x_1, x_2)$$

for all  $x_1, x_2 \in X$ .

**Definition.** Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces. A map  $f : X \rightarrow Y$  is called **bi-Lipschitz** if there exist constants  $C, c > 0$  such that:

$$cd(x_1, x_2) \leq \rho(f(x_1), f(x_2)) \leq Cd(x_1, x_2)$$

for all  $x_1, x_2 \in X$ .

**Definition.** Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces. We say  $f : X \rightarrow Y$  is a **homeomorphism** if  $f$  is a continuous bijection such that  $f^{-1} : Y \rightarrow X$  is also continuous.

## 1.5 Finite dimensional normed vector spaces

**Definition.** Let  $V$  be a vector space. Two norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  on  $V$  are said to be **equivalent** if there are constants  $c, C > 0$  such that:

$$c\|v\|_2 \leq \|v\|_1 \leq C\|v\|_2$$

for all  $v \in V$ . It is clear that this is an equivalence relation.

**Theorem 1.9.** For  $n \in \mathbb{N}$ , all norms in  $\mathbb{R}^n$  are equivalent. The similar result holds for  $\mathbb{C}^n$ .

**Proof.** It suffices to show all norms  $\|\cdot\|$  are equivalent to the 1-norm  $\|\cdot\|_1$ . Then since equivalence norm is an equivalence relation, all norms are equivalent. A basis for  $\mathbb{R}^n$  is  $\{e_1, \dots, e_n\}$ , the standard basis. Let  $C = \max\{\|e_1\|, \dots, \|e_n\|\}$ . Let  $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ , then:

$$\begin{aligned}\|v\| &= \|v_1e_1 + \dots + v_ne_n\| \\ &\leq |v_1|\|e_1\| + \dots + |v_n|\|e_n\| \quad (\Delta\text{-inequality}) \\ &\leq C(|v_1| + \dots + |v_n|) \\ &= C\|v\|_1\end{aligned}$$

This gives us one inequality. This also shows that  $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}$  is Lipschitz, hence continuous (where  $\mathbb{R}^n$  is equipped with  $\|\cdot\|_1$  norm). Define:

$$S = \{v \in \mathbb{R}^n : \|v\|_1 = 1\}$$

Since  $\|\cdot\|$  is continuous on  $(\mathbb{R}^n, \|\cdot\|_1)$  we have that  $\|\cdot\| : S \rightarrow \mathbb{R}$  obtains its maximum and minimum. Further, the minimum is nonzero. Define  $c = \min_{v \in S} \|v\| > 0$ . Note for all  $0 \neq v \in \mathbb{R}^n$  we have that  $v/\|v\|_1 \in S$ . Hence:

$$\left\| \frac{v}{\|v\|_1} \right\| \geq c \implies \|v\| \geq c\|v\|_1$$

Hence  $c\|v\|_1 \leq \|v\| \leq C\|v\|_1$ , as desired.  $\square$

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## 1.6 Completeness

**Definition.** Let  $(X, d)$  be a metric space and  $(x_n)_{n=1}^\infty$  be a sequence in  $X$ . We say  $(x_n)_{n=1}^\infty$  is a **Cauchy sequence** if for all  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that for all  $n, m \in \mathbb{N}$ :

$$n, m \geq N \implies d(x_n, x_m) < \epsilon$$

**Example.** Let  $(\frac{1}{n})_{n=1}^\infty$  be a sequence in  $\mathbb{Q}$  but with different metrics.

- (i). If  $\mathbb{Q}$  is equipped with the Euclidean metric. This is clearly Cauchy. To see that, let  $\epsilon > 0$  pick  $N > 2/\epsilon$ , then for  $n, m \geq N$  we have:

$$d\left(\frac{1}{n}, \frac{1}{m}\right) = \left| \frac{1}{n} - \frac{1}{m} \right| \leq \left| \frac{1}{n} \right| + \left| \frac{1}{m} \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

- (ii). If  $\mathbb{Q}$  is equipped with the discrete metric, then  $d(1/n, 1/m) = 1$  for all  $n, m \in \mathbb{N}$ . This means this sequence is not Cauchy. (If it is Cauchy, take  $\epsilon = 1/2$  then contradiction)

(iii). If  $\mathbb{Q}$  is equipped with the 3-adic metric, then this is not a Cauchy sequence. Let  $n = 3^k$  and  $m = 3^\ell$  where  $k \neq \ell$ . Then we have:

$$d\left(\frac{1}{n}, \frac{1}{m}\right) = d\left(\frac{1}{3^k}, \frac{1}{3^\ell}\right) = 3^{\min\{k, \ell\}}$$

If we pick  $k, \ell$  large enough, then the distance between them can be arbitrarily large. Hence this is not a Cauchy sequence.

**Theorem 1.10.** Let  $(X, d)$  be a metric space and let  $(x_n)_{n=1}^\infty$  be a convergent sequence, then  $(x_n)_{n=1}^\infty$  is a Cauchy sequence.

**Proof.** Say  $\lim_{n \rightarrow \infty} x_n = x^* \in X$ . Let  $\epsilon > 0$ , pick  $N \in \mathbb{N}$  such that for  $n \in \mathbb{N}$ :

$$n \geq N \implies d(x^*, x_n) < \frac{\epsilon}{2}$$

Now pick  $n, m \in \mathbb{N}$  such that  $n, m \geq N$ , we have:

$$d(x_n, x_m) \leq d(x_n, x^*) + d(x_m, x^*) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore  $(x_n)_{n=1}^\infty$  is a Cauchy sequence.  $\square$

**Example (The Converse is False).** Every convergent sequence is Cauchy but not the other way around. There are cauchy sequences that do not converge.

- (i). Let  $X = \mathbb{Q}$  with the Euclidean metric. Let  $x_n$  = the truncation of  $\pi$  to the  $n$ -th decimal place. For example:  $x_1 = 3, x_2 = 3.1, x_3 = 3.14$  and so on. This is a Cauchy sequence, but the limit does not exist (because its “limit” is  $\pi$ , which is not in  $\mathbb{Q}$ ).
- (ii). Consider the sequence  $(\frac{1}{n})_{n=2}^\infty$  with  $X = (0, 1)$  with Euclidean metric. Then this is Cauchy but not convergent because  $0 \notin X$ .

**Definition.** We say a metric space  $(X, d)$  is **complete** if every Cauchy sequence in  $X$  converges.

**Example (Complete Spaces).**

- (i). The metric space  $(\mathbb{R}, d)$  is complete with Euclidean metric.
- (ii). Any  $X$  with the discrete metric space.

**Definition.** A complete normed vector space is called a **Banach Space**.

**Theorem 1.11.** Let  $(X, d)$  be a complete metric space. Let  $Y \subseteq X$  be a subset. Then  $(Y, d)$  is a complete metric space if and only if  $Y$  is closed in  $X$ .

**Proof.** ( $\Leftarrow$ ). Assume  $Y$  is closed in  $X$ . Let  $(x_n)_{n=1}^\infty \subseteq Y$  be a cauchy sequence in  $Y$ . Hence it is also a cauchy sequence in  $X$ . Therefore  $(x_n)_{n=1}^\infty$  converges to  $x^*$  in  $X$  since  $X$  is complete. However,  $Y$  is closed so it contains its limit point, which means  $x^* \in Y$  and thus  $(x_n)_{n=1}^\infty$  converges in  $Y$ . This proved that  $(Y, d)$  is a complete metric space.

( $\Rightarrow$ ). Assume  $(Y, d)$  is complete. To show  $Y$  is closed in  $X$  it suffices to show it contains all of its limit points. Let  $(x_n)_{n=1}^\infty$  be a convergent sequence with limit  $x^* \in X$ . Since convergent sequences are cauchy, we know  $(x_n)_{n=1}^\infty$  is cauchy. Since  $Y$  is complete, this cauchy sequence converges in  $Y$ ! This means  $x^* \in Y$  and hence  $Y$  is closed.  $\square$

**Theorem 1.12.** Let  $1 \leq p < \infty$ . Then the space  $(\ell_p, \|\cdot\|_p)$  is complete.

**Proof.** An element in  $\ell_p$  is already a sequence, so a sequence of elements in  $\ell_p$  is annoying. We use the following notation.

$$x^{(n)} = \{x_1^{(n)}, x_2^{(n)}, \dots\} = \left(x_k^{(n)}\right)_{k=1}^\infty$$

where  $x^{(n)} \in \ell_p$  is the  $n$ -th term in the sequence  $(x^{(n)})_{n=1}^\infty$ . Let  $(x^{(n)})_{n=1}^\infty$  be a cauchy sequence in  $\ell_p$ . Pick  $\epsilon > 0$ , hence there exists an  $N \in \mathbb{N}$  such that for  $n, m \in \mathbb{N}$ :

$$n, m \geq N \implies d(x^{(n)}, x^{(m)}) = \left( \sum_{k=1}^{\infty} |x_k^{(n)} - x_k^{(m)}|^p \right)^{1/p} < \epsilon \quad (1)$$

Our goal is to find a limit point  $x = (x_k)_{k=1}^\infty \in \ell_p$  of the sequence  $(x^{(n)})_{n=1}^\infty$  and prove it. Fix  $k \in \mathbb{N}$ , we claim that  $(x_k^{(n)})_{n=1}^\infty$  is a cauchy sequence in  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ . Indeed:

$$|x_k^{(n)} - x_k^{(m)}| = \left( |x_k^{(n)} - x_k^{(m)}|^p \right)^{1/p} \leq \left( \sum_{j=1}^{\infty} |x_j^{(n)} - x_j^{(m)}|^p \right)^{1/p}$$

We have seen that the RHS can be arbitrarily small by (1), hence  $(x_k^{(n)})_{n=1}^\infty$  is cauchy in  $\mathbb{K}$ . Since  $\mathbb{K}$  is complete, this limit exists, we define  $x_k = \lim_{n \rightarrow \infty} x_k^{(n)} \in \mathbb{K}$ . We claim that:

$$\lim_{n \rightarrow \infty} x^{(n)} = x \in \ell_p$$

There are two things to prove: the limit is  $x$  and  $x$  lies in  $\ell_p$ .

(i). Pick  $\epsilon > 0$ , there exists an  $N$  such that for all  $n, m \geq N$  we have:

$$d(x^{(n)}, x^{(m)}) = \left( \sum_{k=1}^{\infty} |x_k^{(n)} - x_k^{(m)}|^p \right)^{1/p} < \epsilon$$

For any  $J \in \mathbb{N}$  and for all  $n, m \geq N$  we have:

$$\sum_{k=1}^J |x_k^{(n)} - x_k^{(m)}| \leq \sum_{k=1}^{\infty} |x_k^{(n)} - x_k^{(m)}| < \epsilon^p$$

As this is true for all  $n \geq M$ , it is true as  $m \rightarrow \infty$ . This gives:

$$\lim_{m \rightarrow \infty} \sum_{k=1}^J |x_k^{(n)} - x_k^{(m)}|^p \leq \epsilon^p \implies \sum_{k=1}^J |x_k^{(n)} - x_k|^p < \epsilon^p$$

because  $x_k^{(m)} \rightarrow x_k$  as  $m \rightarrow \infty$ . This result is true for all  $n \geq N$ , independent of the choice of  $J$ . As this is true for all  $J$ , we can take the limit as  $J \rightarrow \infty$ . Hence:

$$\sum_{k=1}^{\infty} |x_k^{(n)} - x_k|^p \leq \epsilon^p \quad (1)$$

It follows that  $(x^{(n)} - x) \in \ell_p$  for all  $n \in \mathbb{N}$ , by definition. We also know  $x^{(n)} \in \ell_p$ , hence:

$$x = (x^{(n)} - x) + x^{(n)} \in \ell_p$$

as  $\ell_p$  is a vector space. Inequality (1) says that for any  $\epsilon > 0$  we can find  $N \in \mathbb{N}$  such that  $n \geq N$  implies:

$$\|x^{(n)} - x\|_p = \left( \sum_{k=1}^{\infty} |x_k^{(n)} - x_k|^p \right)^{1/p} \leq \epsilon$$

This is exactly the definition of  $x^{(n)} \rightarrow x$  in  $\ell_p$ , as desired.  $\square$

## 1.7 Completeness of $\mathbb{R}$

**Definition.** Let  $S \subseteq \mathbb{R}$ . We say  $S$  is **bounded above** if there exists an  $M \in \mathbb{R}$  such that  $s \leq M$  for all  $s \in S$ . Similarly  $S$  is **bounded below** if there is  $N \in \mathbb{R}$  such that  $s \geq N$  for all  $s \in S$ . A set is **bounded** if it is both bounded above and below.

**Example.**  $\mathbb{Z} \subseteq \mathbb{R}$  is not bounded above or below.  $(0, 1) \subseteq \mathbb{R}$  is bounded.

**Definition.** Let  $S \subseteq \mathbb{R}$  be bounded above. Then we say  $M \in \mathbb{R}$  is the **least upper bound** if  $M$  is an upper bound for  $S$  and if  $N \in \mathbb{R}$  is another upper bound for  $S$  we have  $M \leq N$ . We define the **greatest lower bound** similarly. We denote them by  $\sup S$  and  $\inf S$ .

**Theorem 1.13 (Least Upper Bound Property).** Let  $\emptyset \neq S \subseteq \mathbb{R}$  be bounded above, then  $S$  has a least upper bound.

**Proof.** Let  $M \in \mathbb{Z}$  be an upper bound of  $S$ . Consider  $M - 1$ . One of two things is true. Either  $M - 1$  is an upper bound or it is not. If  $M - 1$  is an upper bound, replace  $M$  by  $M - 1$  and repeat this argument. Eventually we will get  $M \in \mathbb{Z}$  such that  $M$  is an upper bound but  $M - 1$  is NOT an upper bound (This process terminates because  $S \neq \emptyset$ ). Divide  $[M - 1, M]$  into 10 subintervals.

$$\left[ M - 1, M - 1 + \frac{1}{10} \right], \dots, \left[ M - 1 + \frac{9}{10}, M \right]$$

We can find some  $k \in \{0, \dots, 9\}$  such that  $M - 1 + \frac{k}{10}$  is not an upper bound and  $M - 1 + \frac{k+1}{10}$  is an upper bound. We construct  $u^*$  as the decimal sequence which is an upper bound (We have to be careful if a ring end point is a least upper bound, as we get a decimal expansion of trailing 9's but this is fine). As desired.  $\square$

**Theorem 1.14 (MCT).** Let  $(x_n)_{n=1}^\infty$  be a bounded, non-decreasing sequence in  $\mathbb{R}$ . Then  $(x_n)_{n=1}^\infty$  converges in  $\mathbb{R}$ .

**Proof.** Let  $x^* = \sup\{x_n : n \in \mathbb{N}\}$ , this exists because  $(x_n)_{n \geq 1}$  is bounded. Let  $\epsilon > 0$ , as  $x^*$  is the least upper bound, there exists  $N \in \mathbb{N}$  such that:

$$x^* - \epsilon < x_N \leq x^*$$

Hence, for  $n \geq N$  we have that  $x_n \geq x_N$ , which means:

$$x^* - \epsilon < x_N \leq x_n \leq x^* < x^* + \epsilon \implies |x^* - x_n| < \epsilon$$

which proved that  $\lim_{n \rightarrow \infty} x_n = x^*$ , as desired.  $\square$

**Theorem 1.15 (Bolzano-Weierstrass).** Every bounded sequence  $(x_n)_{n=1}^\infty$  in  $\mathbb{R}$  has a convergent subsequence (that converges in  $\mathbb{R}$ ).

**Proof.** Just see MATH 147/247 notes, the proof idea is just bisection.  $\square$

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**Lemma 1.16.** Let  $(x_n)_{n=1}^\infty$  be a cauchy sequence in  $\mathbb{R}$ . Then  $(x_n)_{n=1}^\infty$  is bounded.

**Proof.** Pick  $\epsilon = 1$ , there is  $N \in \mathbb{N}$  such that for  $n, m \geq N$  we have  $|x_n - x_m| < 1$ . In particular  $|x_n - x_N| < 1$  for all  $n \geq N$ . Let  $M = \max\{|x_1|, \dots, |x_N| + 1\}$ . Then  $|x_n| \leq M$  for all  $n \in \mathbb{N}$ .  $\square$

**Theorem 1.17.**  $(\mathbb{R}, |\cdot|)$  is a complete normed space.

**Proof.** Let  $(x_n)_{n=1}^\infty$  be a cauchy sequence in  $\mathbb{R}$ . By the above lemma,  $(x_n)_{n=1}^\infty$  is bounded. By Bolzano-Weierstrass,  $(x_n)_{n=1}^\infty$  has a convergent subsequence  $(x_{n_k})_{k=1}^\infty$ . Say  $\lim_{k \rightarrow \infty} x_{n_k} = x^* \in \mathbb{R}$ . We claim that  $\lim_{n \rightarrow \infty} x_n = x^*$  as well. Indeed, let  $\epsilon > 0$ . There is  $N_1 \in \mathbb{N}$  such that:

$$n, m \geq N \implies |x_n - x_m| < \frac{\epsilon}{2}$$

Find  $k \in \mathbb{N}$  such that  $n_k \geq N$  and  $|x_{n_k} - x^*| < \epsilon/2$ . Hence for  $n \geq N$  we have:

$$|x_n - x^*| \leq |x_n - x_{n_k}| + |x_{n_k} - x^*| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore  $\mathbb{R}$  is complete.  $\square$

## 1.8 Limits of continuous functions

**Definition.** Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces. Let  $(f_n)_{n=1}^\infty$  be a sequence of function  $X \rightarrow Y$ . We say  $(f_n)_{n=1}^\infty$  **converges uniformly** to  $f^* : X \rightarrow Y$  if for all  $\epsilon > 0$  there is  $N \in \mathbb{N}$  such that for  $n \in \mathbb{N}$  we have:

$$n \geq N \implies d^*(f_n, f^*) := \sup_{x \in X} \rho(f_n(x), f^*(x)) < \epsilon$$

**Example.** Let  $X = [0, \frac{1}{2}]$  and  $Y = \mathbb{R}$  with Euclidean metrics. Define:

$$f_n(x) = 1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

This  $(f_n)_{n=1}^\infty$  is a sequence of bounded continuous functions from  $X \rightarrow Y$ . We claim that it converges to  $f^*(x) = \frac{1}{1-x}$ . Indeed, for any  $n \in \mathbb{N}$  we have:

$$d^*(f_n, f^*) = \sup_{x \in [0, \frac{1}{2}]} \left| \frac{1}{1-x} - \frac{1 - x^{n+1}}{1-x} \right| = \sup_{x \in [0, \frac{1}{2}]} \frac{x^{n+1}}{|1-x|} \leq \frac{(1/2)^{n+1}}{1/2} = \left(\frac{1}{2}\right)^n$$

where the denominator is at least  $1/2$  and the numerator is at most  $(1/2)^{n+1}$ . As  $n \rightarrow \infty$  this tends to 0, which means  $f_n \rightarrow f^*$  uniformly.

**Theorem 1.18.** Let  $(f_n)_{n=1}^\infty$  be a sequence of continuous functions that converges uniformly to  $f^*$ . Then  $f^*$  is continuous.

**Proof.** Let  $x \in X$  and  $\epsilon > 0$ . Pick  $N \in \mathbb{N}$  such that  $d^*(f^*, f_N) < \epsilon/3$ . Since  $f_N$  is continuous at  $x$ , we can pick  $\delta > 0$  such that:

$$d(x, y) < \delta \implies \rho(f_N(x), f_N(y)) < \frac{\epsilon}{3}$$

Therefore if  $y \in X$  and  $d(x, y) < \delta$ , we have:

$$\begin{aligned} \rho(f^*(x), f^*(y)) &\leq \rho(f^*(x), f_N(x)) + \rho(f_N(x), f_N(y)) + \rho(f_N(y), f^*(y)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

Therefore  $f^*$  is continuous at  $x \in X$ , as desired.  $\square$

**Definition.** Let  $(X, d)$  be a metric space. A subset  $A \subseteq X$  is **bounded** if:

$$\text{diam}(A) := \sup_{x, y \in A} d(x, y) < \infty$$

We say a function  $f : X \rightarrow Y$  is bounded if  $f(X) \subseteq Y$  is bounded.

**Definition.** Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces. Define:

$$\mathcal{C}^b(X, Y) = \{f : X \rightarrow Y \mid f \text{ is continuous and bounded}\}$$

The metric on  $\mathcal{C}^b(X, Y)$  is the metric  $d^*$  defined by:

$$\rho^*(f, g) := \sup_{x \in X} \rho(f(x), g(x))$$

Then  $(\mathcal{C}^b(X, Y), \rho^*)$  is a metric space.

**Theorem 1.19.** Let  $(f_n)$  be a sequence of bounded functions  $f_n \in \mathcal{C}^b(X, \mathbb{K})$  that converges uniformly to  $f^*$ , then  $f^*$  is also bounded.

**Theorem 1.20.** Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces. The metric space  $(\mathcal{C}^b(X, Y), \rho^*)$  is complete if and only if  $(Y, \rho)$  is complete!

**Proof.** See Assignment 2. □

**Theorem 1.21.** Let  $(X, d)$  be a metric space. Then  $\mathcal{C}^b(X, \mathbb{K})$  is complete.

**Proof.** Let  $(f_n)_{n=1}^\infty$  be a cauchy sequence. Construct  $f^* : X \rightarrow \mathbb{K}$  by:

$$f^*(x) := \lim_{n \rightarrow \infty} f_n(x)$$

Why is this well-defined? Note that for all fixed  $x \in \mathbb{K}$ , the sequence  $(f_n(x))_{n=1}^\infty$  is a cauchy sequence in  $\mathbb{K}$ ! Since  $\mathbb{K}$  is complete, this sequence converges. We claim that  $(f_n)_{n=1}^\infty$  converges uniformly to  $f^*$ . Let  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that:

$$n, m \geq N \implies d^*(f_n, f_m) = \sup_{x \in X} |f_n(x) - f_m(x)| < \frac{\epsilon}{2}$$

Let  $n \geq N$  be arbitrary. Let  $x \in X$  be arbitrary as well. Since  $f_n(x) \rightarrow f^*(x)$ , we can find  $M \in \mathbb{N}$  with  $M \geq N$  such that  $|f_n(x) - f^*(x)| < \epsilon/2$ . Then, for  $n \geq N$ :

$$\begin{aligned} |f_n(x) - f^*(x)| &\leq |f_n(x) - f_M(x)| + |f_M(x) - f^*(x)| \\ &\leq d^*(f_n, f_M) + |f_M(x) - f^*(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Since  $x \in X$  is chosen arbitrarily, we have:

$$d^*(f_n, f^*) = \sup_{x \in X} |f_n(x) - f^*(x)| \leq \epsilon$$

Therefore  $f_n \rightarrow f^*$  in the  $d^*$  metric (that is  $f_n \rightarrow f^*$  uniformly in the usual sense). Hence  $f^*$  is continuous and bounded, so  $f^* \in \mathcal{C}^b(X, \mathbb{K})$ .  $\square$

**Theorem 1.22 (Weierstrass M-Test).** Let  $\zeta : X \rightarrow \mathbb{R}$  by  $\zeta(a) = 0$  denote the zero function. Then we let  $(f_n)_{n=1}^\infty$  be a sequence in  $\mathcal{C}^b(X, \mathbb{R})$  such that there exists  $M \in \mathbb{R}$  with:

$$\sum_{n=1}^{\infty} d^*(f_n, \zeta) \leq M < \infty$$

Define  $g_N(x) = \sum_{n=1}^N f_n(x)$ . Then  $(g_N)_{N=1}^\infty$  converges to  $g^* \in \mathcal{C}^b(X)$  in the  $d^*$  metric.

**Example.** The series of function  $f(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{2^n}$  is well-defined and is continuous on  $\mathbb{R}$ .

## 2 More Metric Topology

### 2.1 Compactness

**Definition.** Let  $(X, d)$  be a metric space and  $A \subseteq X$ . We say  $\{U_i\}_{i \in I}$  is an **open cover** of  $A$  if each  $U_i$  is open and  $A \subseteq \bigcup_{i \in I} U_i$ .

**Definition.** Let  $(X, d)$  be a metric space and  $A \subseteq X$ . We say  $A$  is **compact** if for every open cover  $\{U_i\}_{i \in I}$  there is a finite subset  $I_0 \subseteq I$  with  $A \subseteq \bigcup_{i \in I_0} U_i$ . This  $\{U_i\}_{i \in I_0}$  is called a **finite subcover**.

**Example.** Let  $A = \{x_1, \dots, x_n\}$  be a finite set, then  $A$  is compact. Why? Let  $\{U_i\}$  be an open cover of  $A$ . For each  $j \in \mathbb{N}$  there is  $i_j \in I$  such that  $a \in U_{i_j}$ . Hence we have:

$$A \subseteq U_{i_1} \cup \dots \cup U_{i_n}$$

This is a finite subcover! Hence  $A$  is compact.

**Example.** Let  $A = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\} \subseteq \mathbb{R}$ . We claim that  $A$  is compact. Let  $\{U_i\}_{i \in I}$  be an open cover. There exists an open set  $U_0$  such that  $0 \in U_0$ . Hence there is  $N \in \mathbb{N}$  large enough such that  $0 \in B_\epsilon(0) \subseteq U_0$ , where  $\epsilon = 1/N$ . This means:

$$\left\{ \frac{1}{n} : n \geq N + 1 \right\} \cup \{0\} \subseteq U_0$$

Then there are only finitely many points left, so we can use finitely many  $U_i$  to cover  $\{\frac{1}{n} : n \geq N + 1\}$ . This gives a finite subcover of  $A$ .

**Example.** Let  $A, B$  be compact sets. Then  $A \cup B$  is compact. Indeed, any open cover of  $A \cup B$  gives an open cover for  $A, B$ . This gives a finite subcover for  $A, B$ , respectively. The union of these two finite subcovers gives a finite subcover of  $A \cup B$ .

**Example.** The set  $A = \{\frac{1}{n} : n \in \mathbb{N}\}$  is not compact! Consider the open cover:

$$\left\{ \left( \frac{1}{n}, 1 + \frac{1}{n} \right) : n \in \mathbb{N} \right\}$$

This has no finite subcover. Indeed, suppose we have a finite subcollection of open sets indexed by  $n_1, \dots, n_r$ . WLOG we may assume  $n_1 < \dots < n_r$ . Then the union of these  $U_i$  is:

$$\left( \frac{1}{n_r}, 1 + \frac{1}{n_1} \right)$$

This clearly does not cover  $A$ .

**Example.** Let  $A = \mathbb{R}$ . Then  $A$  is not compact. The open cover  $\{(-n, n) : n \in \mathbb{N}\}$  has no finite subcover. Similarly  $A = \mathbb{Z}$  is not compact as well.

**Proposition 2.1.** Let  $(X, d)$  be a metric space. If  $A \subseteq X$  is compact, then  $A$  is closed and bounded.

**Proof.** Assume  $A$  is not closed. There exists a subsequence  $(a_n)_{n=1}^{\infty}$  in  $A$  with  $a_n \rightarrow a^*$  and  $a^* \notin A$ . Consider the following open cover:

$$U_n = X \setminus \overline{B_{d(a_n, a^*)}(a^*)}$$

This cannot have a finite subcover, since  $a^*$  is a limit point of  $(a_n)$ . Therefore  $A$  is closed. Similarly suppose  $A$  is not bounded. Fix  $a \in A$ . For all  $N \in \mathbb{N}$  such that there exists an  $a_N \in A$  such that:

$$d(a_N, a) > N$$

Consider the open cover  $\{B_N(a) : N \in \mathbb{N}\}$  of  $A$ . Given a finite subset  $\{N_1 < \dots < N_r\}$ , the union of these is  $B_{N_r}(a)$ . However, for  $N = N_r + 1$  there is  $a_N \in A$  such that  $d(a_N, a) > N$  so  $a_N \notin B_{N_r}(a)$ , but  $a_N \in A$ . Hence this open cover has no finite subcover! Hence  $A$  is bounded.  $\square$

**Definition.** Let  $(X, d)$  be a metric space and  $A \subseteq X$ . We say  $A$  is **sequentially compact** if for every sequence  $(a_n)_{n=1}^{\infty}$  of  $A$ , there is a convergent subsequence  $(a_{n_k})_{k=1}^{\infty}$  with  $a_{n_k} \rightarrow a^* \in A$ .

**Example.** Let  $A$  be a finite set. This is sequentially compact. Why? For any infinite sequence of  $A$ , there exists  $a \in A$  that appears infinitely many times in this sequence. Take this subsequence that only consists of  $a$ . This is a convergent subsequence.

**Example.** Let  $A = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ . This is sequentially compact. Any sequence in  $A$  either has a convergent subsequence that goes to 0 or the sequence only takes on finitely many values.

**Definition.** Let  $(X, d)$  be a metric space. Let  $A \subseteq X$  be a subset. Then  $(A, d_A)$  is a metric space, where  $d_A : A \times A \rightarrow \mathbb{R}$  is the restriction of  $d$  on  $A$ . This is called the **induced metric space**. A subset  $U \subseteq A$  is called **relatively open** if there exists an open set  $U' \subseteq X$  such that  $U = U' \cap A$ .

**Remark.** As a metric space, the open balls of  $(A, d_A)$  are of the form:

$$B_A(a, r) = \{x \in A : d_A(x, a) < r\} = \{x \in X \cap A : d(x, a) < r\} = B_X(a, r) \cap A$$

Therefore, an open set in  $(A, d_A)$  is of the form  $U' \cap A$  for open sets  $U'$  in  $X$ .

**Definition.** A metric space  $(X, d)$  is **compact** if every open cover of  $X$  has a finite subcover. That is, for every open cover  $\{U_i : i \in I\}$ , there is a finite subset  $I_0 \subseteq I$  such that:

$$X = \bigcup_{i \in I_0} U_i$$

Note that this is an equality, not a subset. This is because  $X$  is our whole space, it does not sit in any bigger space.

**Remark.** Note that there are two notions of compactness for a subset  $A \subseteq X$ .

- (i).  $A$  is compact as a subset of  $X$ . [This is the definition we saw above.]
- (ii).  $A$  is compact as a metric space. [Note that for an open cover of  $A$ , the open sets are open sets in  $A$ ! These open sets are different from the open sets in  $X$ .]

In fact, these two notions coincide. Suppose (ii) is true, we want to show (i) is true. Let  $\{U_i : i \in I\}$  be an open cover of  $A$ , where  $U_i$  is an open set of  $X$  for all  $i$ . Then:

$$\{U_i \cap A : i \in I\}$$

is an open cover of the metric space  $(A, d_A)$ , where each  $U_i \cap A$  is an open set in  $A$ . Since  $(A, d_A)$  is compact, there is a finite set  $I_0 \subseteq I$  such that:

$$A = \bigcup_{i \in I_0} (U_i \cap A)$$

Then clearly we have  $A \subseteq \bigcup_{i \in I_0} U_i$ , a finite subcover of  $A$  (as a subset of  $X$ .)

Conversely suppose (i) is true. Let  $\{U_i : i \in I\}$  be an open cover of  $(A, d_A)$ , then for each  $i \in I$  there is an open set  $U'_i \subseteq X$  of  $X$  such that  $U_i = U'_i \cap A$ . Hence  $\{U'_i : i \in I\}$  is an open cover of  $A \subseteq X$ . Since  $A$  is a compact subset of  $X$ , there is a finite  $I_0 \subseteq I$  with  $A \subseteq \bigcup_{i \in I_0} U'_i$ . By taking the intersection with  $A$ , we have:

$$A = \bigcup_{i \in I_0} (U'_i \cap A) = \bigcup_{i \in I_0} U_i$$

Therefore  $(A, d_A)$  is compact and (ii) is true.

**Definition.** Let  $(X, d)$  be a metric space. A collection  $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\} \subseteq X$  is said to have the **finite intersection property (FIP)** if for every finite subset  $\Lambda_0 \subseteq \Lambda$  we have  $\bigcap_{\lambda \in \Lambda_0} F_\lambda \neq \emptyset$ .

**Example.** Let  $X = \mathbb{R}$ . Consider the collection  $\{\mathbb{R} \setminus \{a\} : a \in \mathbb{R}\}$ . This clearly satisfies the FIP. However, the infinite intersection:

$$\bigcap_{a \in \mathbb{R}} (\mathbb{R} \setminus \{a\}) = \emptyset$$

is empty! As we will see, this actually tells us  $\mathbb{R}$  is not compact!

**Definition.** Let  $(X, d)$  be a metric space. A subset  $A \subseteq X$  is called **cauchy** if every cauchy sequence in  $A$  converges to a point in  $A$ .

**Definition.** Let  $(X, d)$  be a metric space. We say  $A \subseteq X$  is **totally bounded** if for all  $\epsilon > 0$  there exists a finite set  $F_\epsilon \subseteq X$  (called an  $\epsilon$ -net) such that:

$$A \subseteq \bigcup_{f \in F_\epsilon} B_\epsilon(f)$$

Note that totally boundedness implies boundedness.

**Remark.** Note that if  $A$  is totally bounded, we may assume  $F_\epsilon \subseteq A$  for all  $\epsilon > 0$ . Suppose for  $\epsilon > 0$  we have an  $\epsilon$ -net  $F = \{x_1, \dots, x_n\} \subseteq X$  of  $A$ , so:

$$A \subseteq \bigcup_{i=1}^n B_{\epsilon/2}(x_i)$$

We may assume  $B_\epsilon(x_i) \cap A \neq \emptyset$  for all  $i$ . (If the intersection is empty we can just remove it from the  $\epsilon$ -net.) Hence we may choose  $y_i \in A \cap B_{\epsilon/2}(x_i)$  for all  $i$ . Note that:

$$A \subseteq \bigcup_{i=1}^n B_\epsilon(y_i)$$

by the triangle inequality. Indeed, for any  $x \in A$  we can choose  $i \in \{1, \dots, n\}$  such that  $x \in B_{\epsilon/2}(x_i)$ . Then we have that:

$$d(x, y_i) \leq d(x, x_i) + d(x_i, y_i) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This proved that  $x \in B_\epsilon(y_i)$ . Hence  $\{y_1, \dots, y_n\} \subseteq A$  is an  $\epsilon$ -net for  $A$ .

Recall in  $\mathbb{R}^n$ , a subset is compact if and only if it is closed and bounded (Heine-Borel). We will now see that for metric spaces, there are also some easier ways to characterize compactness, and the Heine-Borel theorem for  $\mathbb{R}^n$  is a special case of it.

**Theorem 2.2 (Borel-Lebesgue).** Let  $(X, d)$  be a metric space and  $A \subseteq X$ . Then the followings are equivalent:

- (i).  $A$  is compact (either as a subset or a metric space, these two notions are equivalent.)
- (ii). If  $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\}$  is an collection of closed sets in  $(A, d_A)$  with FIP, then  $\bigcap_{\lambda \in \Lambda} F_\lambda \neq \emptyset$ .
- (iii).  $A$  is sequentially compact.
- (iv).  $A$  is complete and totally bounded.

**Example.** Consider  $A = \mathbb{Q} \cap [0, 1]$  and  $B = \mathbb{Z}$  as induced metric spaces from  $(\mathbb{R}, d)$ . By the Borel-Lebesgue theorem, we can show that  $A, B$  are not compact in four different ways.

- (i). For  $A$ , we define the open cover:

$$\left\{ \mathbb{R} \setminus \overline{B_{1/n}(1/\pi)} : n \in \mathbb{N} \right\}$$

This does not have a finite subcover. For  $B$ , the open cover  $\{B_{1/2}(n) : n \in \mathbb{Z}\}$  does not have a fintie subcover as well. Hence  $A, B$  are not compact by definition.

(ii). We need to find a collection of closed sets that FIP but the intersection is empty. Let:

$$\begin{aligned} \{A \cap \overline{B_{1/n}(1/\pi)} : n \in \mathbb{N}\} &\subseteq A \\ \{[n, \infty) \cap B : n \in \mathbb{N}\} &\subseteq B \end{aligned}$$

These two have FIP but the intersection over all  $n \in \mathbb{N}$  is empty.

- (iii). Let  $(a_n)_{n=1}^\infty$  be the sequence in  $A$  such that  $a_n$  is the truncation of the decimal expansion of  $1/\pi$  at the  $n$ -th place. Then  $a_n \rightarrow 1/\pi$  in  $\mathbb{R}$ , which means any convergent subsequence of  $(a_n)$  converges to  $1/\pi \notin A$ . For  $B$ , the sequence  $(b_n)_{n=1}^\infty$  by  $b_n = n$  is a sequence in  $B$  that does not have a convergent subsequence.
- (iv). Let  $(a_n)_{n=1}^\infty$  be the same sequence in (iii), this is cauchy but does not converge in  $A$ . For  $B$ , consider  $\epsilon = 1/2$ . Then  $B = \mathbb{Z}$  does not have a  $\epsilon$ -net. Therefore  $A$  is not complete and  $B$  is not totally bounded.

**Proof of Theorem 2.2.** (i)  $\Rightarrow$  (ii). Assume  $(A, d_A)$  is a compact metric space. Let  $\{F_\lambda : \lambda \in \Lambda\}$  be a collection of closed sets in  $A$  satisfying FIP. Assume for a contradiction that  $\bigcap_{\lambda \in \Lambda} F_\lambda = \emptyset$ . Consider the following collection of open sets in  $A$ :

$$\{U_\lambda := A \setminus F_\lambda : \lambda \in \Lambda\}$$

Note that this is an open cover for  $A$ . Since  $A$  is compact, there exists a finite subset  $\Lambda_0 \subseteq \Lambda$  such that  $\bigcup_{\lambda \in \Lambda_0} U_\lambda = A$ . However, this implies that:

$$\bigcap_{\lambda \in \Lambda_0} F_\lambda = A \setminus \bigcup_{\lambda \in \Lambda_0} U_\lambda = A \setminus A = \emptyset$$

Since  $\Lambda_0$  is finite, this contradicts to our assumption that  $\{F_\lambda : \lambda \in \Lambda\}$  has FIP!

(ii)  $\Rightarrow$  (iii). Assume (ii) is true. We want to show  $A$  is sequentially compact. Let  $(a_n)_{n=1}^\infty$  be a sequence in  $A$ . For each  $k \geq 1$  we define  $S_k = \{a_n : n \geq k\}$  and define the closed set:

$$F_k = \overline{S_k} = \overline{\{a_n : n \geq k\}} \subseteq A$$

to be the closure of a tail of  $(a_n)_{n=1}^\infty$ . Note that  $F_{k+1} \subseteq F_k$  for all  $k \geq 1$ . Define  $\mathcal{F} = \{F_k : k \geq 1\}$ . Then  $\mathcal{F}$  is a collection of closed sets in  $A$  that has FIP. It satisfies FIP because for a finite set  $\{k_1 < \dots < k_r\}$  we have:

$$F_{k_1} \cap \dots \cap F_{k_r} = F_{k_1} \neq \emptyset$$

By our assumption we have  $\bigcap_{k=1}^\infty F_k \neq \emptyset$ . Let's pick  $a^* \in \bigcap_{k=1}^\infty F_k$ . We claim that we can find a subsequence of  $(a_n)$  that converges to  $a^*$ . First we note that:

$$B_r(a^*) \cap S_k \neq \emptyset$$

for all  $r > 0$  and  $k \geq 1$ . This is because each  $a^* \in F_k$  is closed so  $a^*$  is a limit point for every  $S_k$ . In other word, for any  $r > 0$  and  $k \geq 1$  we can find some  $a_i$  such that  $d(a_i, a^*) < r$  and  $i \geq k$ . For  $r = 1$  we can find  $n_1 \geq 1$  with  $d(a_{n_1}, a^*) < 1$ . Inductively suppose we have defined  $n_1, \dots, n_r$ , we can find  $n_{r+1} > n_r$  such that  $d(a_{n_r}, a^*) < 1/(r+1)$ . Hence  $(a_{n_r})_{r=1}^\infty$  is a subsequence that converges to  $a^* \in A$ . Therefore  $(A, d_A)$  is sequentially compact.

**(iii)  $\implies$  (iv).** Assume  $(A, d_A)$  is sequentially compact. We first show that  $A$  is complete (as a subset of  $X$ .) Let  $(a_n)_{n=1}^\infty$  be a cauchy sequence in  $A$ . There exist a convergent subsequence  $(a_{n_k})_{k=1}^\infty$  that converges to  $a^* \in A$ . Since  $(a_n)_{n=1}^\infty$  is cauchy, we must have  $a_n \rightarrow a^*$  as well. Hence  $A$  is complete. Now let us show that  $A$  is totally bounded. Let  $\epsilon > 0$  be arbitrary. Suppose it is not, then there is  $\epsilon > 0$  such that there does not exist a  $\epsilon$ -net for  $A$ . First note that in the case,  $A$  must be infinite. (Any finite set is clearly totally bounded.) Let  $a_1 \in A$  be arbitrary. Hence  $\{a_1\}$  is not an  $\epsilon$ -net. This means there exists  $a_2 \in A$  such that  $d(a_1, a_2) \geq \epsilon$ . Now, inductively suppose we have found  $a_1, \dots, a_r$  for  $r \geq 1$ . Then  $\{a_1, \dots, a_r\}$  is not an  $\epsilon$ -net. We can then find  $a_{r+1} \in A$  such that:

$$d(a_{r+1}, a_i) \geq \epsilon \text{ for all } i \in \{1, \dots, r\}$$

This gives us a sequence  $(a_n)_{n=1}^\infty$  in  $A$  that has no convergent subsequence! (since for all  $n, m$  we have  $d(a_n, a_m) \geq \epsilon$ .) This is a contradiction, so  $A$  is totally bounded.

**(iv)  $\implies$  (i).** Assume (iv) is true, we want to show  $A$  is compact. Suppose for a contradiction that  $A$  is not compact as a metric space. Let  $\mathcal{U} = \{U_i : i \in I\}$  be an open cover of  $A$  that does not have a finite subcover (in this case  $U_i \subseteq X$  is open for all  $i$ ). Since  $A$  is totally bounded, for all  $n \geq 1$  there exists a  $\frac{1}{n}$ -net in  $A$ :

$$F_n = \{x_{n,1}, \dots, x_{n,m_n}\}$$

such that:

$$A = \bigcup_{f \in F_n} B_{1/n}(f) = \bigcup_{f \in F_n} \overline{B_{1/n}(f)}$$

Let  $n = 1$ . Note that if all  $\overline{B_1(f)}$  can be covered by finitely many  $U_i$ 's, then  $A$  can be covered by finitely many  $U_i$ 's, which is impossible. Hence there is  $i_1 \in \{1, \dots, m_1\}$  such that  $\overline{B_1(x_{i_1})}$  does not have a finite subcover of  $\mathcal{U}$ . Let  $y_1 = x_{i_1}$ . Inductively suppose we have chosen  $y_1, \dots, y_k$  so that:

$$X_k := \bigcap_{i=1}^k \overline{B_{1/i}(y_i)}$$

has no finite subcover. Consider the sets:

$$X_{k,i} = X_k \cap \overline{B_{1/(k+1)}(x_{k+1,i})} \text{ for } 1 \leq i \leq m_{k+1}$$

Suppose for a contradiction that each of them has a finite subcover. However:

$$\bigcup_{i=1}^{m_{k+1}} X_{k,i} = \bigcup_{i=1}^{m_{k+1}} X_k \cap \overline{B_{1/(k+1)}(x_{k+1,i})} = X_k \cap \bigcup_{i=1}^{m_{k+1}} \overline{B_{1/(k+1)}(x_{k+1,i})} = X_k \cap A = X_k$$

This means  $X_k$  has a finite subcover, which is impossible! Hence there is  $i_{k+1}$  such that  $\overline{B_{1/(k+1)}(x_{k+1,i_{k+1}})}$  does not have a finite subcover. Let  $y_{k+1} = x_{k+1,i_{k+1}}$ .

Note that  $(y_n)_{n=1}^\infty$  is cauchy in  $A$ . Indeed, let  $\epsilon > 0$  we choose  $N > 2/\epsilon$ . For all  $n \geq m \geq N$  we have:

$$X_n \subseteq \overline{B_{1/m}(y_m)} \cap \overline{B_{1/n}(y_n)} \neq \emptyset$$

and  $X_n$  is non-empty set. We can pick  $x \in X_n$ . Then:

$$d(y_n, y_m) \leq d(y_n, x) + d(y_m, x) \leq \frac{1}{n} + \frac{1}{m} \leq \frac{2}{N} < \epsilon$$

Since  $A$  is complete,  $y_n \rightarrow y^* \in A$  for some  $y^* \in A$ . For any  $m \in \mathbb{N}$  we have:

$$d(y_m, y^*) = \lim_{n \rightarrow \infty} d(y_m, y_n) \leq \lim_{n \rightarrow \infty} \left( \frac{1}{m} + \frac{1}{n} \right) = \frac{1}{m}$$

Since  $\mathcal{U}$  is a cover of  $A$ , there is  $i_0 \in I$  such that  $y^* \in U_{i_0}$ . Since  $U_{i_0}$  is open, then is  $r > 0$  such that  $B_r(y^*) \subseteq U_{i_0}$ . Choose  $m > 2/r$ , then for any  $x \in X_m \subseteq \overline{B_{1/m}(y_m)}$  we have:

$$d(x, y^*) \leq d(x, y_m) + d(y_m, y^*) \leq \frac{2}{m} < r$$

Hence  $X_m \subseteq U_{i_0}$ . This means  $X_m$  does have a finite subcover, contradicting our construction! Therefore  $A$  is compact.  $\square$

**Remark.** Totally bounded is not same as bounded. There exist sets that are closed, bounded but not compact. Consider  $X = \{0, 1\}^{\mathbb{N}}$  with  $\ell^\infty$  norm. It is clearly bounded since  $\|x\|_\infty \leq 1$  for all  $x \in X$ . However, we claim that it is not totally bounded. Suppose it has an  $\frac{1}{2}$ -net:

$$F = \{x_1, \dots, x_n\}$$

Then  $B_{1/2}(x_i) = \{x_i\}$  because  $\|x\|_\infty \in \{0, 1\}$  for any  $x \in X$ . Hence:

$$\bigcup_{i=1}^n B_{1/2}(x_i) = \{x_1, \dots, x_n\} \neq \{0, 1\}^{\mathbb{N}}$$

Therefore this is not a  $\frac{1}{2}$ -net, contradiction. Hence  $(X, \|\cdot\|_\infty)$  is bounded but NOT totally bounded!

**Corollary 2.3 (Heine-Borel).** A subset of  $\mathbb{R}^n$  is compact if and only if it is closed and bounded.

**Proof.** Since  $\mathbb{R}^n$  is compact,  $A$  is closed  $\iff$  it is complete. Moreover, we claim that in  $\mathbb{R}^n$ , bounded implies totally bounded. Let  $\epsilon \in \mathbb{N}$ , we claim that there is also an  $\epsilon$ -net of a bounded set  $A$ . Since  $A$  is bounded, we know  $A \subseteq [-r, r]^n$  for some  $r > 0$ . We can cover  $[-r, r]^n$  with finitely many boxes of side length  $\frac{\epsilon}{2}$ . Any such box can be covered by an  $\epsilon$ -ball. Hence we can use finitely many  $\epsilon$ -balls to cover  $A$ . Therefore  $A$  is totally bounded. Hence  $A$  is bounded  $\iff$  it is totally bounded. The result follows from (iv) of Borel-Lebesgue.  $\square$

## 2.2 Countable and Uncountable Sets

**Definition.** A set  $X$  is **countable** if there is a injection  $f : X \rightarrow \mathbb{N}$ . A set is **denumerable** if there is a bijection  $f : X \rightarrow \mathbb{N}$ . We say a set is **uncountable** if it is not countable.

**Example.** The integers  $\mathbb{Z}$  is countable because  $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$ .

**Example.** The rationals  $\mathbb{Q} \cap [0, 1]$  is also countable because:

$$\mathbb{Q} = \left\{ \frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \dots \right\}$$

Informally: Write  $\frac{p}{q} \in \mathbb{Q} \cap [0, 1]$  with  $q$  in increasing order and  $p \in \{1, \dots, q\}$  such that  $\gcd(p, q) = 1$ . We require coprimeness so that there is no element appearing twice in the list.

**Example.** The set  $\{0, 1\}^{\mathbb{N}}$  is uncountable. Suppose for a contradiction that it is countable. Then:

$$\{0, 1\}^{\mathbb{N}} = \{(x_{1,k})_{k=1}^{\infty}, (x_{2,k})_{k=1}^{\infty}, \dots\}$$

Define a sequence  $(x_n)_{n=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$  by:

$$x_n = \begin{cases} 0 & \text{if } x_{n,n} = 1 \\ 1 & \text{if } x_{n,n} = 0 \end{cases}$$

Then  $(x_n)_{n=1}^{\infty}$  is different from  $(x_{n,k})_{k=1}^{\infty}$  at the  $n$ -th place for all  $n \geq 1$ . This is a new element in  $\{0, 1\}^{\mathbb{N}}$ , contradiction! Hence  $\{0, 1\}^{\mathbb{N}}$  is uncountable. This method is called the **diagonal argument**: If we list out all the given  $(x_{n,k})_{k=1}^{\infty}$  row by row, then our new element  $(x_n)_{n=1}^{\infty}$  is constructed by changing the diagonal entries.

**Example.** Let  $A, B$  be sets and  $f : A \rightarrow B$  be a bijection, then  $A$  is countable if and only if  $B$  is countable.

**Example.** Let  $A \subseteq B$ . If  $A$  is uncountable then so is  $B$ . If  $B$  is countable then so is  $A$ .

**Example.** We claim  $\mathbb{R}$  is uncountable. Let  $X = \{0, 1, \dots, 9\}^{\mathbb{N}}$ . By the same argument we can show that  $X$  is uncountable. Define  $f : X \rightarrow \mathbb{R}$  by:

$$f((x_n)_{n=1}^{\infty}) = \sum_{k=1}^{\infty} \frac{x_k}{10^k}$$

Then  $f : X \rightarrow f(X)$  is a bijection. Since  $f(X) \subseteq \mathbb{R}$ , we know  $\mathbb{R}$  is uncountable.

**Definition.** Let  $(X, d)$  be a metric space. We say  $A \subseteq X$  is **dense** in  $X$  if  $\overline{A} = X$ .

**Definition.** We say a metric space  $(X, d)$  is **separable** if there is a countable subset  $A \subseteq X$  such that  $A$  is dense in  $X$ .

**Example.** The reals  $\mathbb{R}$  with the usual metric is separable because  $\overline{\mathbb{Q}} = \mathbb{R}$  and  $\mathbb{Q}$  is countable.

**Example.** Let  $(X, d)$  with the discrete metric. Then  $X$  is separable if and only if  $X$  is countable. This is because every subset is closed (equal to their own closure), so the only dense subset is  $X$  itself. Hence  $X$  is countable if and only if  $X$  is separable.

**Proposition 2.4.** Let  $(X, d)$  be a metric space. If  $(X, d)$  is totally bounded, then  $X$  is separable.

**Proof.** For each  $n \in \mathbb{N}$  there is an  $\frac{1}{n}$ -net of  $X$ , call it  $F_n$ . Define  $F = \bigcup_{n=1}^{\infty} F_n$ . Note that  $F$  is countable, being a countable union of finite sets. We claim that  $F$  is dense. Let  $x \in X$  be and  $\epsilon > 0$  be arbitrary. There is  $N \geq 1$  such that  $1/N < \epsilon$ . Since  $F_N$  is an  $\frac{1}{N}$ -net, there is  $f \in F_N$  such that  $d(f, x) < 1/N < \epsilon$ . Since  $f \in F$ , we proved that  $F$  is dense in  $X$ .  $\square$

### 2.3 Compactness and Continuity

**Proposition 2.5.** Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces. If  $f : X \rightarrow Y$  is continuous and  $X$  is compact, then  $f(X)$  is compact.

**Proof.** Let  $\{U_i : i \in I\}$  be an open cover of  $f(X)$  in  $Y$ . Since  $f$  is continuous, each  $f^{-1}(U_i)$  is open. Since  $f^{-1}(Y) = X$ , we know  $\{f^{-1}(U_i) : i \in I\}$  is an open cover of  $X$ . Since  $X$  is compact, there is a finite subcover  $\{i_1, \dots, i_n\}$ . Hence:

$$f(X) \subseteq \bigcup_{k=1}^n U_{i_k}$$

Therefore  $f(X)$  is compact.  $\square$

**Proposition 2.6.** Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces. If  $f : X \rightarrow Y$  is continuous and  $X$  is compact, then  $f$  is uniformly continuous.

**Proof.** Let  $\epsilon > 0$ . For each  $x \in X$  we can find  $\delta_x > 0$  such that for all  $y \in X$ :

$$d(y, x) < \delta_x \implies \rho(f(y), f(x)) < \frac{\epsilon}{2} \quad (1)$$

Now note that  $\{B_{\delta_x/2}(x) : x \in X\}$  is an open cover of  $X$ . Since  $X$  is compact, we know:

$$X = B_{\delta_{x_1}/2}(x_1) \cup \dots \cup B_{\delta_{x_n}/2}(x_n)$$

for some  $x_1, \dots, x_n \in X$ . Now define  $\delta = \frac{1}{2} \min\{\delta_{x_1}, \dots, \delta_{x_n}\} > 0$ . Let  $x, y \in X$  be arbitrary with  $d(x, y) < \delta$ . Say  $y \in B_{\delta_{x_b}}(x_b)$  for some  $x_b \in \{x_1, \dots, x_n\}$ . However:

$$d(x, x_b) \leq d(x, y) + d(y, x_b) < \delta + \frac{1}{2}\delta_{x_b} < \frac{1}{2}\delta_{x_b} + \frac{1}{2}\delta_{x_b} = \delta_{x_b}$$

Since  $d(y, x_b) < \delta_{x_b}$  as well, by (1) we have:

$$\rho(f(x), f(y)) \leq \rho(f(x), f(x_b)) + \rho(f(x_b), f(y)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence  $f$  is uniformly continuous.  $\square$

## 2.4 Cantor Set

**Construction 2.7 (Ver 1).** Let  $C_0 = [0, 1]$ . Recursively,  $C_{i+1}$  is constructed by removing the middle third from each intervals in  $C_i$ . First we see that:

$$\begin{aligned} C_1 &= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \\ C_2 &= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right] \end{aligned}$$

We see  $\{C_n\}_{n=0}^{\infty}$  has the finite intersection property and they are all compact sets. Define:

$$C^* = \bigcap_{n=0}^{\infty} C_n$$

We call  $C^*$  the **(middle-third) cantor set**. Clearly  $0, 1 \in C^*$ . In fact any endpoint of any  $C_n$  is in  $C^*$ . For example  $1/3, 1/9, 2/27 \in C^*$ . We have  $C^*$  is compact (as it is closed and bounded in  $\mathbb{R}$ ).

**Construction 2.8 (Ver 2).** Equivalently we can define:

$$C^* = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{3^n} : a_n \in \{0, 2\} \right\}$$

It is the set of all real numbers that CAN be written in ternary expansion without using 1. [For example  $0.1 \in C^*$  because it CAN be written as  $0.222\dots$ ] This shows that  $C^*$  has an uncountable number of points.

**Construction 2.9 (Ver 3).** The cantor set  $C^*$  is the unique non-empty compact set satisfying:

$$C^* = f_1(C^*) \cup f_2(C^*)$$

where  $f_1(x) = x/3$  and  $f_2(x) = x/3 + 2/3$ .

**Theorem 2.10.** Let  $(X, d)$  be a compact metric space. There is an continuous map  $f : C^* \rightarrow X$  that is surjective.

**Proof.** The idea is to construct  $s_n : C^* \rightarrow X$  such that  $(s_n)$  is cauchy and each  $s_n$  is continuous. As  $n \rightarrow \infty$  we have  $s_n(C^*)$  better approximate  $X$  [produce an  $\epsilon$ -net for smaller  $\epsilon$ .]

For  $n = 1$ , construct a 1-net for  $X$ . That is, a finite set  $F_1$  such that  $X = \bigcup_{f \in F_1} B_1(f)$  [This exists since  $X$  is totally bounded.] We can assume wlog that  $|F_1| = 2^{k_1}$  for some  $k_1$ . [If not power of 2, adding more points if necessary.] Now consider  $C_{k_1}$ , a union of  $2^{k_1}$  intervals containing  $C^*$ . For each  $c \in C^*$ , we know  $c$  is in some subinterval of  $C_{k_1}$ . We map each subinterval in  $C_{k_1}$  to a different  $f \in F_1$ . Let  $s_1$  be this map. Then  $s_1$  is continuous as it is locally constant.

For  $n = 2$ , construct a  $1/2$ -net for each of each  $\overline{B}_1(f_i)$ , where  $\{f_i\} = F_1$  from the construction of  $s_1$ . As before, we can assume that this set is a power of 2, and the same powers of 2. Say  $2^{k_2}$  in size. For each subinterval  $I_i$  used to construct  $s_1$ , subdivide it into  $2^{k_2}$  subintervals. As before,  $s_2$  is continuous. We further notice  $d(s_1(c), s_2(c))$  is not huge. In fact  $d(s_1(c), s_2(c)) \leq 1 + \frac{1}{2}$ .

We continue in this fashion, we get that:

$$d(s_n(c), s_{n+1}(c)) \leq \frac{1}{2^n}$$

We can make this arbitrarily small. Hence for all  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that for  $n, m \geq N$ :

$$d^*(s_n, s_m) = \sup_{c \in C^*} d(s_n(c), s_m(c)) < \epsilon$$

Therefore  $(s_n)$  is a cauchy sequence. As  $C^*$  is compact and  $X$  is complete so  $\mathcal{C}^b(C^*, X) = \mathcal{C}(C^*, X)$  is complete. Hence  $s_n \rightarrow s^* \in \mathcal{C}(C^*, X)$ . We need to show  $s^*(C^*) = X$ , that is,  $s^*$  is onto. Take a point  $x$  in  $X$ . This point will be distance 1 from some point in  $F_1$ . This gives us a subinterval in  $C^*$ . There exists a point in  $F_2$  whose distance is  $1/2$  from  $x$  and  $1 + 1/2$  from  $f_1$ . This gives a smaller subinterval. Repeating this process we get nested subintervals with non-trivial intersection with  $C^*$ . The infinite intersection is in  $C^*$ , and this intersection has  $s^*(c^*) = x$ , as required.  $\square$

## 2.5 Compact sets in $\mathcal{C}(X)$

**Definition.** Let  $(X, d)$  be a compact metric space, we denote:

$$\mathcal{C}(X) := \mathcal{C}(X, \mathbb{R}) = \{f : X \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$$

Here  $\mathbb{R}$  is a metric space with the usual metric. For  $f \in \mathcal{C}(X)$  we define the **uniform norm** by:

$$\|f\|_\infty := \sup\{|f(x)| : x \in X\}$$

Since  $X$  is compact, by the extreme value theorem this supremum can be achieved. So we can equivalently define it as:

$$\|f\|_\infty = \max\{|f(x)| : x \in X\}$$

Note that  $(\mathcal{C}(X), \|\cdot\|_\infty)$  is a normed vector space. In fact, since  $\mathbb{R}$  is complete we knew that  $\mathcal{C}(X)$  is also complete. Therefore  $(\mathcal{C}(X), \|\cdot\|_\infty)$  is a Banach space. Also note that  $f_n \rightarrow f$  uniformly (as functions) is the same as  $f_n \rightarrow f$  as sequences in the normed space  $(\mathcal{C}(X), \|\cdot\|_\infty)$ .

**Remark.** By Borel-Lebesgue we know that:

$$\begin{aligned} K \subseteq \mathcal{C}(X) \text{ is compact} &\iff K \text{ is complete and totally bounded} \\ &\iff K \text{ is closed and totally bounded} \end{aligned}$$

since closed subsets of a complete space are complete.

**Example.** Let  $K = \{f_n(x) = x^n : n \in \mathbb{N}\} \subseteq \mathcal{C}([0, 1])$ . Note that every subsequence of  $(f_n)$  converges pointwise to the function  $f : [0, 1] \rightarrow \mathbb{R}$  defined by:

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Since  $f$  is not continuous, the sequence  $(f_n)$  does not converge in  $\mathcal{C}([0, 1])$ . Therefore  $K$  is not sequentially compact despite being closed and bounded.

**Definition.** Let  $(X, d)$  be complete. A subset  $F \subseteq \mathcal{C}(X)$  is called **equicontinuous at  $x \in X$**  if for all  $\epsilon > 0$  there is  $\delta > 0$  so that for all  $y \in X$ :

$$d(x, y) < \delta \implies |f(x) - f(y)| < \epsilon \text{ for all } f \in F$$

We know  $F \subseteq \mathcal{C}(X)$  is **equicontinuous** if it is equicontinuous at every  $x \in X$ . We say a subset  $F \subseteq \mathcal{C}(X)$  is **uniformly equicontinuous** if for all  $\epsilon > 0$  there is  $\delta > 0$  such that for all  $x, y \in X$ :

$$d(x, y) < \delta \implies |f(x) - f(y)| < \epsilon \text{ for all } f \in F$$

That is, the choice of  $\delta > 0$  does not depend on  $x \in X$ .

**Remark.** Clearly uniformly equicontinuous  $\implies$  equicontinuous.

**Lemma 2.11.** Let  $(X, d)$  be compact. If  $K \subseteq \mathcal{C}(X)$  is compact, then  $K$  is uniformly equicontinuous.

**Proof.** Let  $\epsilon > 0$ . Since  $K$  is compact, it is totally bounded and thus has a  $\frac{\epsilon}{3}$ -net. Say it is  $F = \{f_1, \dots, f_n\} \subseteq K$ . Each  $f_i$  is continuous, thus uniformly continuous (since  $X$  is compact). For each  $i$  there is  $\delta_i > 0$  such that for all  $x, y \in X$ :

$$d(x, y) < \delta_i \implies |f_i(x) - f_i(y)| < \frac{\epsilon}{3}$$

Let  $\delta = \min\{b_1, \dots, b_n\}$ . Now let  $x, y \in X$  with  $d(x, y) < \delta$  and let  $f \in K$  be arbitrary. We can find  $i$  such that  $\|f - f_i\| < \epsilon/3$  (because  $F$  is an  $\epsilon/3$ -net!) Therefore we have:

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| \\ &\leq \|f - f_i\|_\infty + \frac{\epsilon}{3} + \|f - f_i\|_\infty \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

Therefore  $K$  is uniformly equicontinuous.  $\square$

**Lemma 2.12.** Let  $(X, d)$  be compact. Suppose  $F \subseteq \mathcal{C}(X)$  is equicontinuous. Then  $F$  is uniformly equicontinuous.

**Proof.** Let  $\epsilon > 0$ . For each  $x \in X$  there is  $\delta_x > 0$  so that for all  $y \in X$ :

$$d(x, y) < \delta_x \implies |f(x) - f(y)| < \frac{\epsilon}{2} \text{ for all } f \in F$$

Then the collection  $\{B_{\delta_x/2}(x) : x \in X\}$  is an open cover of  $X$ . Since  $X$  is compact, it has a finite subcover, indexed by  $\{x_1, \dots, x_n\}$ . Let  $\delta = \min\{\delta_{x_1}/2, \dots, \delta_{x_n}/2\}$ . Suppose  $y_1, y_2 \in X$  and  $d(y_1, y_2) < \delta$ . Pick  $i$  so that  $d(y_1, x_i) < \delta_{x_i}/2$ . Then:

$$d(y_2, x_i) \leq d(y_2, y_1) + d(y_1, x_i) < \delta + \frac{\delta_{x_i}}{2} \leq \delta_{x_i}$$

Now we know  $d(y_1, x_i) < \delta_{x_i}$  and  $d(y_2, x_i) < \delta_{x_i}$ . By the choice of  $\delta_{x_i}$ , for all  $f \in F$  we have:

$$|f(y_1) - f(y_2)| \leq |f(y_1) - f(x_i)| + |f(y_2) - f(x_i)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore  $F$  is uniformly equicontinuous.  $\square$

**Theorem 2.13 (Arzela-Ascoli).** Let  $(X, d)$  be a compact metric space. A subset  $K \subseteq \mathcal{C}(X)$  is compact if and only if  $K$  is closed, bounded and equicontinuous.

**Proof.** ( $\Rightarrow$ ). If  $K$  is compact then it is closed and bounded by Proposition 2.1. Also we know that  $K$  is equicontinuous by the lemma above.

( $\Leftarrow$ ). Suppose  $K$  is closed, bounded and equicontinuous. Note that  $\mathcal{C}(X)$  is complete and  $K$  is closed, so  $K$  is complete. It remains to show  $K$  is totally bounded. Let  $\epsilon > 0$ . Since  $K$  is equicontinuous, it is uniformly equicontinuous by the lemma above. There is  $\delta > 0$  such that for all  $f \in K$  and  $x, y \in X$  we have:

$$d(x, y) < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{4} \tag{*}$$

Since  $X$  is compact, there is a  $\delta$ -net:

$$F_X = \{x_1, \dots, x_n\} \subseteq X \text{ and } X \subseteq \bigcup_{i=1}^n B_\delta(x_i) \tag{\dagger}$$

Define  $T : K \rightarrow (\mathbb{R}^n, \|\cdot\|_\infty)$  by:

$$T(f) = (f(x_1), \dots, f(x_n))$$

Note that  $\|T(f)\|_\infty = \max\{|f(x_i)| : 1 \leq i \leq n\} \leq \|f\|_\infty$ . [Here is a bit of abusing of notation. The two  $\|\cdot\|$ -norm are on two different spaces.] This implies that  $T(K)$  is bounded in  $\mathbb{R}^n$  since  $K$  is bounded in  $\mathcal{C}(X)$ . This means  $T(K)$  is totally bounded, thus  $\overline{T(K)}$  is compact in  $\mathbb{R}^n$ . This means that there exists a  $\epsilon/4$ -net of  $T(K)$ :

$$F_T = \{T(f_1), \dots, T(f_m)\} \subseteq T(K) \quad \text{and} \quad T(K) \subseteq \bigcup_{i=1}^m B_{\epsilon/4}(f_i) \quad (\dagger\dagger)$$

Here each  $f_i \in K$ . We claim that  $F_K = \{f_1, \dots, f_m\}$  is a  $\epsilon$ -net for  $K$ . Indeed, let  $f \in K$  be arbitrary. We can find some  $j \in \{1, \dots, m\}$  such that  $\|T(f) - T(f_j)\|_\infty < \epsilon/4$  by  $(\dagger\dagger)$ . Now we let  $y \in X$ , we can find  $i \in \{1, \dots, n\}$  such that  $d(x_i, y) < \delta$  by  $(\dagger)$ . Then:

$$\begin{aligned} |f(y) - f_j(y)| &\leq \underbrace{|f(y) - f(x_i)|}_{< \frac{\epsilon}{4} \text{ by } (*)} + \underbrace{|f(x_i) - f_j(x_i)|}_{\leq \|T(f) - T(f_j)\|_\infty < \frac{\epsilon}{4}} + \underbrace{|f_j(x_i) - f_j(y)|}_{< \frac{\epsilon}{4} \text{ by } (*)} &< \frac{3\epsilon}{4} \end{aligned}$$

Since  $y \in X$  is arbitrary, we have  $\|f - f_j\|_\infty \leq \frac{3\epsilon}{4} < \epsilon$ . This proved that  $K \subseteq \bigcup_{i=1}^m B_\epsilon(f_i)$ . Hence  $K$  is totally bounded.  $\square$

## 2.6 Connectedness

**Definition.** Let  $(X, d)$  be a metric space. We say  $A \subseteq X$  is **disconnected** if there exist two open sets  $U, V$  of  $X$  such that  $A \subseteq U \cup V$  and  $U \cap V = \emptyset$  and  $U \cap A \neq \emptyset$  and  $V \cap A \neq \emptyset$ . We say  $A$  is **connected** if it is not disconnected.

**Example.** Let  $(X, d)$  be a metric space. Any finite subset  $A = \{x_1, \dots, x_n\}$  with at least two elements is disconnected. Let  $r = \frac{1}{2} \min\{d(x_i, x_j) : i \neq j\} > 0$ . We define open sets:

$$U = B_r(x_1) \quad \text{and} \quad V = B_r(x_2) \cup \dots \cup B_r(x_n)$$

Then  $A \subseteq U \cup V$  and  $U \cap V = \emptyset$  by our choice of  $r$ . Moreover  $U \cap A = \{x_1\}$  and  $V \cap A = \{x_2, \dots, x_n\}$  are not empty. Therefore  $A$  is disconnected.

**Example.** Let  $X$  be a set with  $|X| \geq 2$ . Let  $d$  be the discrete metric on  $X$ . Then  $(X, d)$  is disconnected. Indeed, let  $x_0 \in X$ . Then  $U = \{x_0\}$  is open and  $V = X \setminus \{x_0\}$  is also open.

**Example.** The middle third cantor set is disconnected.

**Example.** The interval  $[0, 1] \subseteq \mathbb{R}$  is connected. Assume it is disconnected by open sets  $U, V$  of  $\mathbb{R}$ . WLOG we may assume  $0 \in U$ . Let  $C = \{c \in \mathbb{R} : [0, c) \subseteq U\}$ . Since  $U$  is open, there is  $\epsilon > 0$  so that  $B_\epsilon(0) \subseteq U$ . Since  $C$  is nonempty, we let  $c^* = \sup C$ . There are two cases.

- (i). If  $c^* \in U$ . Then as  $U$  is open, there is  $\epsilon > 0$  such that  $B_\epsilon(c^*) \subseteq U$ . This means  $c^* + \epsilon \in U$ , so we have  $c^* + \epsilon \in C$ . Contradiction.
- (ii). If  $c^* \in V$ . There is  $\epsilon > 0$  with  $B_\epsilon(c^*) \subseteq V$ . This means  $B_\epsilon(c^*) \cap U = \emptyset$ . However, by the definition of supremum we know  $c^* - \epsilon \in C$ , so  $[0, c^* - \epsilon) \subseteq U$ . This means  $c^* - \frac{\epsilon}{2} \in U$ , but we know  $c^* - \frac{\epsilon}{2} \in V$  as well. Contradiction.

**Theorem 2.14.** Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces. Suppose  $(X, d)$  is connected. If  $f : X \rightarrow Y$  is continuous, then  $f(X)$  is connected.

**Proof.** Assume  $f(X)$  is disconnected, say by open sets  $U, V$  of  $(Y, \rho)$ . It is easy to see that  $f^{-1}(U)$  and  $f^{-1}(V)$  are open sets that separate  $X$ . Contradiction.  $\square$

**Theorem 2.15.** Any connected subsets of  $\mathbb{R}$  are intervals.

**Proof.** Let  $C$  be a connected set. We define:

$$a = \inf C \in \mathbb{R} \cup \{-\infty\} \quad \text{and} \quad b = \sup C \in \mathbb{R} \cup \{\infty\}$$

If  $c \in \mathbb{R}$  and  $a < c < b$  we must have  $c \in C$ . Otherwise:

$$C \subseteq \underbrace{(-\infty, c)}_U \cup \underbrace{(c, \infty)}_V$$

This gives a separation of  $C$ , contradiction. Hence we have  $(a, b) \subseteq C \subseteq [a, b]$ . This means  $C$  is an interval in  $\mathbb{R}$ .  $\square$

**Definition.** Let  $(X, d)$  be a metric space. We can define an equivalence relation on  $X$  by  $x \sim y$  if and only if there is a connected set  $C$  containing both  $x, y$ . The equivalence classes of this relation are called **connected components**. Let  $x_0 \in X$ . the equivalence class that  $x_0$  lies in is called the connected component of  $x_0$  and it is equal to the union of all connected sets containing  $x_0$ .

**Example.** Let  $X = [0, 1] \cup [2, 3]$  be the metric space with induced Euclidean metric. Then  $[0, 1]$  and  $[2, 3]$  are the connected components of  $X$ .

**Definition.** Let  $(X, d)$  be a metric space. We say  $X$  is **totally disconnected** if every connected component is a singleton set.

**Example.** Finite sets are totally disconnected.

**Definition.** Let  $(X, d)$  be a metric space. We say  $(X, d)$  is **path-connected** if for all  $x, y \in X$  there exists a continuous function  $f : [0, 1] \rightarrow X$  such that  $f(0) = x$  and  $f(1) = y$ .

**Example.** Let  $(V, \|\cdot\|)$  be a normed space. Any convex set  $C \subseteq V$  is path connected. For  $x, y \in C$  we can define  $f(t) = (1 - t)x + ty \in C$ .

**Proposition 2.16.** Let  $(X, d)$  be a metric space. If  $X$  is path-connected then  $X$  is connected.

**Proof.** Suppose  $X = U \cup V$  is disconnected. Pick  $x \in U$  and  $y \in V$ . There is a path  $f : [0, 1] \rightarrow X$  such that  $f(0) = x$  and  $f(1) = y$ . Now:

$$[0, 1] = f^{-1}(X) = f^{-1}(U) \cup f^{-1}(V)$$

Note that  $0 \in f^{-1}(U)$  and  $1 \in f^{-1}(V)$ . It is easy to check  $f^{-1}(U)$  and  $f^{-1}(V)$  give a separation of  $[0, 1]$ . This is a contradiction!  $\square$

**Example.** The converse of this is not true. There exists connected spaces that is not path-connected. We define the following set:

$$X = \left\{ \left( x, \sin\left(\frac{1}{x}\right) \right) : x > 0 \right\} \cup \{(0, 0)\}$$

Then  $X \subseteq \mathbb{R}^2$  is connected but not path connected.

## 2.7 Bonus Cantor Set Stuff

**Definition.** Let  $n \geq 2$  and  $A \subseteq \{0, 1, \dots, n-1\}$  be a finite set. We define the **linear Cantor set**:

$$C_{A,n} = \left\{ \sum_{k=1}^{\infty} \frac{a_k}{n^k} : a_k \in A \right\}$$

**Definition.** Let  $A \subseteq \mathbb{R}$ . We define  $N_\epsilon(A)$  to be the minimal number of  $\epsilon$ -balls needed to cover  $A$ . The **box-counting dimension** of  $A$  is defined as:

$$\dim_B(A) = \lim_{\epsilon \rightarrow 0} \frac{-\log N_A(\epsilon)}{\log \epsilon}$$

if the limit exists. If the limit does not exist, we can take the limsup or liminf to define the **upper box dimension** and **lower box dimension**.

**Example.** Consider the middle third Cantor set. For  $3^{-n} \leq \epsilon < 3^{-(n-1)}$ , we need  $2^n$  intervals of length  $1/3^n$  to cover  $C$ . Hence:

$$\dim_B(C) = \lim_{n \rightarrow \infty} \frac{-\log 2^n}{\log 3^{-n}} = \frac{\log 2}{\log 3}$$

The box-counting dimension of  $C$  is  $\log_3(2)$ .

**Definition.** Let  $(X, \rho)$  be a metric space. For any  $U \subseteq X$  we let  $\text{diam}(U)$  or  $|U|$  denote its diameter. Let  $S \subseteq X$  and let  $\delta > 0$  and  $d \in [0, \infty)$ . We define:

$$H_\delta^d(S) = \inf \left\{ \sum_{i \in I} |U_i|^d : S \subseteq \bigcup_{i \in I} U_i, |U_i| < \delta, |I| \leq |\mathbb{N}| \right\}$$

Then we define:

$$H^d(S) = \lim_{\delta \rightarrow 0} H_\delta^d(S)$$

to be the  **$d$ -dimensional Hausdorff measure of  $S$** .

**Theorem 2.17.** Let  $(X, \rho)$  be a metric space and  $0 \leq s < t < \infty$ . For  $A \subseteq X$  we have:

- (i). If  $H^s(A) < \infty$  then  $H^t(A) = 0$ .
- (ii). If  $H^t(A) > 0$  then  $H^s(A) = \infty$ .

**Proof.** It suffices to prove (i) since (ii) is just the contrapositive of (i). We have:

$$\begin{aligned} H_\delta^t(A) &= \inf \left\{ \sum_{i \in I} |U_i|^t : A \subseteq \bigcup_{i \in I} U_i, |U_i| < \delta, |I| \leq |\mathbb{N}| \right\} \\ &= \inf \left\{ \sum_{i \in I} |U_i|^{t-s} |U_i|^s : A \subseteq \bigcup_{i \in I} U_i, |U_i| < \delta, |I| \leq |\mathbb{N}| \right\} \\ &\leq \inf \left\{ \sum_{i \in I} \delta^{t-s} |U_i|^s : A \subseteq \bigcup_{i \in I} U_i, |U_i| < \delta, |I| \leq |\mathbb{N}| \right\} \\ &= \delta^{t-s} \inf \left\{ \sum_{i \in I} |U_i|^s : A \subseteq \bigcup_{i \in I} U_i, |U_i| < \delta, |I| \leq |\mathbb{N}| \right\} \\ &= \delta^{t-s} H_\delta^s \end{aligned}$$

Suppose  $H^s(A) < \infty$ , we then have:

$$H^t(A) = \lim_{\delta \rightarrow 0} \delta^{t-s} H_\delta^s = H_\delta \lim_{\delta \rightarrow 0} \delta^{t-s} = 0$$

As desired. □

**Corollary 2.18.** There is at most one  $d \in [0, \infty)$  with  $0 < H^d(A) < \infty$ .

**Definition.** Same setting as above. We define the **Hausdorff dimension** of  $A$  to be:

$$\dim_H(A) = \sup\{d \in [0, \infty) : H^d(A) = \infty\} = \inf\{d \in [0, \infty) : H^d(A) = 0\}$$

**Example.** Let  $A = \mathbb{Q} \cap [0, 1]$ . We need  $\lceil \epsilon^{-1} \rceil$  many  $\epsilon$ -balls to cover  $A$ , as  $\mathbb{Q}$  is dense in  $\mathbb{R}$ . Hence:

$$\dim_B(A) = \lim_{\epsilon \rightarrow 0} \frac{-\log \lceil \epsilon^{-1} \rceil}{\log \epsilon} = 1$$

However, we claim the Hausdorff dimension is 0. Consider:

$$\begin{aligned} H_\delta^0(A) &= \inf \left\{ \sum_{i \in I} |U_i|^0 : A \subseteq \bigcup_{i \in I} U_i, |U_i| < \delta, |I| \leq |\mathbb{N}| \right\} \\ &= \inf \left\{ \sum_{i \in I} |U_i|^t : A \subseteq \bigcup_{i \in I} U_i, |U_i| < \delta, I \text{ finite} \right\} \\ &= \inf \left\{ |I| : A \subseteq \bigcup_{i \in I} U_i, |U_i| < \delta, I \text{ finite} \right\} \\ &= \left\lceil \frac{1}{\delta} \right\rceil \end{aligned}$$

Then we have  $H^0(A) = \lim_{\delta \rightarrow 0} H_\delta^0(A) = \infty$ . Let  $d > 0$ , we wish to show that  $H^d(A) = 0$ . To do this it suffices to show for all  $\epsilon > 0$  and  $\delta > 0$  we have  $H_\delta^d(A) \leq \epsilon$ . Since  $A$  is countable, we can enumerate  $A = \{r_n : n \geq 1\}$ . For each  $n \geq 1$  let:

$$\epsilon_n = \min \left\{ \delta, \frac{1}{2} \left( \frac{\epsilon}{2^n} \right)^{1/d} \right\} > 0$$

Then let  $U_n = B_{\epsilon_n}(r_n)$  and  $|U_n| \leq \left( \frac{\epsilon}{2^n} \right)^{1/d}$ . Hence we have:

$$\sum_{n=1}^{\infty} |U_n|^d \leq \sum_{n=1}^{\infty} \left( \left( \frac{\epsilon}{2^n} \right)^{1/d} \right)^d = \epsilon$$

Hence  $H^d(A) = 0$  for all  $d > 0$ , so  $\dim_H(A) = \inf\{d \geq 0 : H^d(A) = 0\} = 0$ .

**Proposition 2.19.** For any linear Cantor set  $C_{A,n}$  we have  $\dim_B(C_{A,n}) = \dim_H(C_{A,n})$ .

**Proposition 2.20.** Let  $A, B \subseteq \mathbb{R}$ , then:

$$\dim_H(A \cup B) = \max\{\dim_H(A), \dim_H(B)\}$$

**Proposition 2.21.** Let  $A, B \subseteq \mathbb{R}$ , then:

$$\dim_H(A + B) \leq \dim_H(A) + \dim_H(B)$$

**Proposition 2.22.** Let  $\emptyset \neq A \subseteq \mathbb{R}^n$ , then  $0 \leq \dim_H(A) \leq n$ .

**Example.** From A4 we saw that  $C + C = [0, 2]$ , where  $C$  is the middle-third Cantor set. That is:

$$C_{\{0,2\},3} + C_{\{0,2\},3} = [0, 2]$$

We know the box counting dimension is  $\log_3(2)$ , so  $\dim_H(C_{\{0,2\},3}) = \log_3(2)$  as well.

**Example.** What is the dimension of  $C_{\{0,3\},4}$  and the dimension of  $C_{\{0,3\},4} + C_{\{0,3\},4}$ ? In general, we need  $2^n$  intervals of length  $4^{-n}$  to cover  $C_{\{0,3\},4}$ , so:

$$\dim_B(C_{\{0,3\},4}) = \lim_{n \rightarrow \infty} \frac{\log 2^n}{\log 4^{-n}} = \frac{1}{2} = \dim_H(C_{\{0,3\},4})$$

What does  $C_{\{0,3\},4} + C_{\{0,3\},4}$  looks like?

$$\begin{aligned} C_{\{0,3\},4} + C_{\{0,3\},4} &= \left\{ \sum_{k=1}^{\infty} \frac{a_k + b_k}{4^k} : a_k, b_k \in \{0, 3\} \right\} = \left\{ \sum_{k=1}^{\infty} \frac{c_k}{4^k} : c_k \in \{0, 3, 6\} \right\} \\ &= \left\{ \sum_{k=1}^{\infty} \frac{c_k}{4^k} : c_k \in \left\{0, \frac{3}{2}, 3\right\} \right\} + \left\{ \sum_{k=1}^{\infty} \frac{c_k}{4^k} : c_k \in \left\{0, \frac{3}{2}, 3\right\} \right\} \\ &= 2C_{\{0, \frac{3}{2}, 3\}, 4} \end{aligned}$$

We see that:

$$\dim_H(C_{\{0, \frac{3}{2}, 3\}, 4}) = \dim_B(C_{\{0, \frac{3}{2}, 3\}, 4}) = \frac{\log 3}{\log 4} < 1$$

**Theorem 2.23.** Let  $C_{A,n}$  be a linear Cantor set. If  $\dim_H(C_{A,n}) < \frac{1}{2}$  then  $C_{A,n} + C_{A,n} \neq [0, 2]$ .

**Proof.** By Proposition 2.21 we have:

$$\dim_H(C_{A,n} + C_{A,n}) \leq \dim_H(C_{A,n}) + \dim_H(C_{A,n}) < 1$$

However  $\dim_H([0, 2]) = 1$ . Hence  $C_{A,n} + C_{A,n} \neq [0, 2]$ . □

**Example.** Let  $C \subseteq \mathbb{R}^n$  be a perfect and totally disconnected set with  $\dim_H(C) < \frac{1}{2}$ . Then  $C + C$  is a perfect and totally disconnected set.

**Theorem 2.24.** Let  $C_{A,n}$  be a linear Cantor set, then:

$$C_{A,n} = \bigcup_{a \in A} S_a(C_{A,n})$$

where  $S_a : \mathbb{R} \rightarrow \mathbb{R}$  is defined by  $S_a(x) = \frac{x+a}{n}$ .

**Proof.** Note that we have:

$$\begin{aligned}
 C_{A,n} &= \left\{ \frac{a_1}{n} + \sum_{k=2}^{\infty} \frac{a_k}{n^k} : a_k \in A \right\} \\
 &= \bigcup_{a \in A} \left\{ \frac{a}{n} + \left\{ \sum_{k=2}^{\infty} \frac{a_k}{n^k} : a_k \in A \right\} \right\} \\
 &= \bigcup_{a \in A} \left\{ \frac{a}{n} + \frac{1}{n} \left\{ \sum_{k=1}^{\infty} \frac{a_k}{n^k} : a_k \in A \right\} \right\} \\
 &= \bigcup_{a \in A} \frac{a}{n} + \frac{1}{n} C_{A,n} \\
 &= \bigcup_{a \in A} S_a(C_{A,n})
 \end{aligned}$$

As desired.  $\square$

**Theorem 2.25.** Let  $A \subseteq \{0, \dots, n-1\}$  and  $0, n-1 \in A$ . Define:

$$B := A + A = \{0 = b_0 < b_1 < \dots < b_k = 2n-2\}$$

Then  $C_{A,n} + C_{A,n} = [0, 2]$  if and only if  $b_i - b_{i-1} \leq 2$  for all  $1 \leq i \leq k$ .

**Proof.** Note that we have:

$$C_{A,n} + C_{A,n} = \left\{ \sum_{r=1}^{\infty} \frac{a_r + c_r}{n^r} : a_r, c_r \in A \right\} = \left\{ \sum_{r=1}^{\infty} \frac{b_r}{n^r} : b_r \in B \right\} = C_{B,n}$$

Then  $b_i - b_{i-1} \leq 2$  for all  $i$  if and only if  $[0, 2] = \bigcup_{i=0}^k S_{b_i}(C_{B,n}) = C_{B,n}$ .  $\square$

**Definition.** A **Cantorval** is a compact subset of  $\mathbb{R}$  with non-empty interior such that none of its connected components are isolated.

**Fact.** Let  $A \subseteq \{0, \dots, n-1\}$  and  $0, n-1 \in A$ . Exactly one of the followings is true:

1.  $C_{A,n} + C_{A,n} = [0, 2]$ .
2.  $C_{A,n} + C_{A,n}$  is a totally disconnected and perfect set.
3.  $C_{A,n} + C_{A,n}$  is a Cantorval.

## 3 Completeness

### 3.1 Baire Category Theorem

**Definition.** Let  $(X, d)$  be a metric space. We say  $A \subseteq X$  is **nowhere dense** if  $\text{int}(\overline{A}) = \emptyset$ .

**Example.** Consider  $(\mathbb{R}, d)$  with Euclidean metric. A singleton is nowhere dense. The integers  $\mathbb{Z}$  is nowhere dense. Rationals  $\mathbb{Q}$  is NOT nowhere dense, as  $\overline{\mathbb{Q}} = \mathbb{R}$ . The Cantor set is nowhere dense.

**Example.** Consider the metric space  $(X, d)$  where  $d$  is the discrete metric. Any non-empty set is NOT nowhere dense because every  $A \subseteq X$  is both open and closed, so:

$$\text{int}(\overline{A}) = \text{int}(A) = A \neq \emptyset$$

The only nowhere dense subset of  $(X, d)$  is  $\emptyset$ .

**Lemma 3.1.** Let  $(X, d)$  be a metric space. If  $A \subseteq X$  is nowhere, then  $X \setminus \overline{A}$  is open and dense.

**Proof.** Since  $\overline{A}$  is closed, clearly  $X \setminus \overline{A}$  is open. Suppose  $x \notin X \setminus \overline{A}$  and let  $\epsilon > 0$ . We want to find  $y \in X \setminus \overline{A}$  such that  $y \in B_\epsilon(x)$ , which proves that  $X \setminus \overline{A}$  is dense in  $X$ . Since  $x \notin X \setminus \overline{A}$ , we know  $x \in \overline{A}$ . Since  $A$  is nowhere dense,  $\text{int}(\overline{A}) = \emptyset$ . Hence we can find  $y \notin \overline{A}$  such that  $y \in B_\epsilon(x)$ , which means  $y \in X \setminus \overline{A}$ , as desired.  $\square$

**Definition.** Let  $(X, d)$  be a metric space. We say  $A \subseteq X$  is **first category (meagre)** if we can write  $A$  as a countable union of nowhere dense sets. That is:

$$A = \bigcup_{n=1}^{\infty} K_n$$

where each  $K_n \subseteq X$  is nowhere dense. When  $X$  is first category as a set, then we also say  $(X, d)$  is first category. Otherwise we say  $A$  is **second category**.

**Example.** Consider  $(\mathbb{R}, d)$  with the usual metric. Any nowhere dense set is first category. The rationals  $\mathbb{Q}$  is first category because it is the countable union of  $q \in \mathbb{Q}$ .

**Question:** Is  $\mathbb{R}$ , with the usual metric, first category?

**Answer:** It is not first category (not obvious) by the Baire Category Theorem.

**Theorem 3.2 (Baire Category Theorem).** Any non-empty complete metric space  $(X, d)$  is second category.

**Example.** The reals  $\mathbb{R}$  is not first category. The  $\ell^p$  spaces for  $1 \leq p < \infty$  are not first category. This does not apply to  $(\mathbb{Q}, d)$  with the Euclidean metric since it is not complete.

**Corollary 3.3.** Let  $(X, d)$  be a non-empty complete metric space with  $X = \bigcup_{n=1}^{\infty} K_n$ , then there is  $n \geq 1$  such that  $\text{int}(\overline{K}_n) \neq \emptyset$ .

**Proof.** We know  $X$  is not first category by the BCT, so one of  $K_n$  is not nowhere dense.  $\square$

**Proof of BCT.** Assume  $(K_n)_{n=1}^{\infty}$  is a sequence of nowhere dense sets, we want to show  $X \neq \bigcup_{n=1}^{\infty} K_n$  by constructing  $x^* \in X$  such that  $x^* \notin K_n$  for all  $n \geq 1$ . Pick any  $x_0 \in X$  and  $r_0 > 0$ . Consider  $\overline{B}_{r_0}(x_0)$ . Since  $K_1$  is nowhere dense, we can find  $x_1 \in \overline{B}_{r_0}(x_0)$  and  $r_1 < r_0/2$  such that:

$$\overline{B}_{r_1}(x_1) \cap K_1 = \emptyset \quad \text{and} \quad \overline{B}_{r_1}(x_1) \subseteq \overline{B}_{r_0}(x_0)$$

We repeat this process. Suppose we have defined  $x_n$  and  $r_n$ , we find  $x_{n+1}$  and  $r_{n+1}$  such that  $x_{n+1} \in \overline{B}_{r_n}(x_n)$  and  $r_n < r_{n-1}/2$  with:

$$\overline{B}_{r_{n+1}}(x_{n+1}) \cap K_{n+1} = \emptyset \quad \text{and} \quad \overline{B}_{r_{n+1}}(x_{n+1}) \subseteq \overline{B}_{r_n}(x_n)$$

We claim that  $(x_n)_{n=1}^{\infty}$  is cauchy and its limit  $x^*$  satisfies our desired property. Let  $m > n$ , notice:

$$d(x_n, x_m) \leq r_n < \frac{r_{n-1}}{2} < \cdots < \frac{r_0}{2^n}$$

Therefore  $(x_n)_{n=1}^{\infty}$  is cauchy. Since  $(X, d)$  is complete, we let  $\lim_{n \rightarrow \infty} x_n = x^* \in X$ . Note that  $(x_n)_{n=k}^{\infty}$  is a sequence in  $\overline{B}_{r_k}(x_k)$  for all  $k \geq 1$  and each such closed ball is closed. Therefore:

$$x^* = \lim_{n \rightarrow \infty} x_n \in \overline{B}_{r_k}(x_k)$$

Hence  $x^* \in \overline{B}_{r_n}(x_n)$  for all  $n \geq 1$ . Hence  $x^* \notin K_n$  for all  $n \geq 1$ , as desired.  $\square$

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**Definition.** Let  $(X, d)$  be a metric space. We say  $A \subseteq X$  is a  $G_{\delta}$  set if there exist a countable sequence of open sets  $U_n \subseteq X$  such that  $A = \bigcap_{n=1}^{\infty} U_n$ .

**Example.** Any open set is a  $G_{\delta}$  set by definition.

**Example.** The irrational numbers are a  $G_{\delta}$  set. Note that  $\mathbb{Q}$  is countable, so:

$$\mathbb{R} \setminus \mathbb{Q} = \bigcap_{r \in \mathbb{Q}} (\mathbb{R} \setminus \{r\})$$

**Definition.** Let  $(X, d)$  be a metric space. We say  $A \subseteq X$  is an  $F_{\sigma}$  set if there is a countable sequence of closed sets  $C_n \subseteq X$  such that  $A = \bigcup_{n=1}^{\infty} C_n$ .

**Remark.** Note that  $A$  is  $G_\delta$  if and only if  $A^c$  is  $F_\sigma$ .

**Example.** Any closed set is a  $F_\sigma$  set.

**Example.** The interval  $A = (0, 1)$  is an  $F_\sigma$  set because  $(0, 1) = \bigcup_{n=2}^{\infty} \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$ .

**Example.** Note that  $\mathbb{Q} = \bigcup_{r \in \mathbb{Q}} \{r\}$  is  $F_\sigma$ . However, we claim that  $\mathbb{Q}$  is NOT a  $G_\delta$  set! Assume for a contradiction that  $\mathbb{Q}$  is a  $G_\delta$  set, say:

$$\mathbb{Q} = \bigcap_{n=1}^{\infty} U_n$$

where each  $U_n \subseteq \mathbb{R}$  is an open set. This means  $\mathbb{Q} \subseteq U_n$  for all  $n \geq 1$ . Hence each  $U_n$  is an open dense set. Then  $\mathbb{R} \setminus U_n$  is closed and nowhere dense. This means:

$$\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n=1}^{\infty} (\mathbb{R} \setminus U_n)$$

is a union of nowhere dense sets! Hence  $\mathbb{R} \setminus \mathbb{Q}$  is first category. Since  $\mathbb{Q}$  is first category, we have:

$$\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$$

is first category, being the union of two sets that are first category. Since  $\mathbb{R}$  is complete, it is second category by BCT. Contradiction.

## 3.2 Nowhere Differentiable Functions

For this section we consider the space:

$$\mathcal{C}[0, 1] = \{f : [0, 1] \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$$

We will show that “most” functions  $f \in \mathcal{C}[0, 1]$  are nowhere differentiable!

**Definition.** Let  $f \in \mathcal{C}[0, 1]$ . We say  $f$  is **Lipschitz** at  $x_0 \in X$  if there is  $K \in \mathbb{R}$  (dependent on  $x_0$ ) such that for all  $x \in [0, 1]$  we have:

$$|f(x_0) - f(x)| \leq K|x_0 - x|$$

We say  $f$  is **Lipschitz** if the choice of  $K$  is independent of  $x_0$ .

**Lemma 3.4.** Let  $f \in \mathcal{C}[0, 1]$  and  $x_0 \in [0, 1]$ . Assume  $f'(x_0)$  exists, then  $f$  is Lipschitz at  $x_0$ .

**Proof.** Let  $c_1 = |f'(x_0)| \geq 0$ . This implies that:

$$c_1 = \lim_{x \rightarrow x_0} \left| \frac{f(x) - f(x_0)}{x - x_0} \right|$$

There exists  $\delta > 0$  (small enough such that  $(x_0 - \delta, x_0 + \delta) \subseteq [0, 1]$ ) such that for all  $x \in (x_0 - \delta, x_0 + \delta)$  we have that:

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| \leq c_1 + 1$$

Consider the function  $h(x) = \frac{f(x) - f(x_0)}{x - x_0}$  on the set  $[0, x_0 - \delta] \cup [x_0 + \delta, 1]$ . Note that  $h$  is continuous on this compact set, hence it is bounded on it. Let  $c_2 \in \mathbb{R}$  such that for all  $x \in [0, x_0 - \delta] \cup [x_0 + \delta, 1]$  we have that:

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| \leq c_2$$

Let  $K = \max\{c_1 + 1, c_2\} > 0$ , then for all  $x \in [0, 1]$  we have:

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| \leq K \implies |f(x) - f(x_0)| \leq K|x - x_0|$$

As desired.  $\square$

**Lemma 3.5.** Let  $f \in \mathcal{C}[0, 1]$  be Lipschitz at  $x_0 \in [0, 1]$  with constant  $K$ . Then for all  $a, b \in [0, 1]$  with  $a \leq x_0 \leq b$  we have:

$$|f(a) - f(b)| \leq K|a - b|$$

**Proof.** Since  $a \leq x_0 \leq b$  we have  $|a - x_0| + |x_0 - b| = |a - b|$ . Then:

$$|f(a) - f(b)| \leq |f(a) - f(x_0)| + |f(x_0) - f(b)| \leq K(|a - x_0| + |x_0 - b|) = K|a - b|$$

As desired.  $\square$

**Example.** Define a function  $f : [0, 1] \rightarrow \mathbb{R}$  by  $f(x) = \sum_{n=1}^{\infty} 2^{-n} \cos(\pi 10^n x)$ . We claim that  $f \in \mathcal{C}[0, 1]$  but nowhere differentiable! It suffices to show it is the limit of a sequence of continuous functions. For each  $N \geq 1$  let:

$$f_N(x) = \sum_{n=1}^N 2^{-n} \cos(\pi 10^n x)$$

Then each  $f_N \in \mathcal{C}[0, 1]$ . We claim that  $(f_N)_{N=1}^{\infty}$  is cauchy. Let  $N > M$ , we have:

$$\|f_N - f_M\|_{\infty} = \sup_{x \in [0, 1]} \left| \sum_{n=M+1}^N 2^{-n} \cos(\pi 10^n x) \right| \leq \sum_{n=M+1}^N 2^{-n} \rightarrow 0$$

because the series  $\sum_{n=1}^{\infty} 2^{-n} = 1$  converges, its tail goes to 0. Therefore  $(f_N)_{N=1}^{\infty}$  is cauchy and since  $\mathcal{C}[0, 1]$  is complete, it converges to  $f \in \mathcal{C}[0, 1]$ . To show it is nowhere differentiable, it suffices to show it is not Lipschitz at any  $x_0 \in [0, 1]$ . Write  $x_0 = \sum_{k=1}^{\infty} \frac{a_k}{10^k}$  in base 10. Suppose  $f$  is Lipschitz at  $x_0$

with constant  $K \in \mathbb{R}$ . Let  $N \geq 1$  (to be chosen later), we define:

$$x_L = \sum_{k=1}^N \frac{a_k}{10^k} \quad \text{and} \quad x_R = x_L + \frac{1}{10^N}$$

We consider the difference between  $f(x_R)$  and  $f(x_L)$ . Note that:

$$\cos(x) - \epsilon \leq \cos(x + \epsilon) \leq \cos(x) + \epsilon \quad (1)$$

for  $\epsilon > 0$  small. By (1), for  $1 \leq k \leq N$  we have:

$$\cos(\pi 10^k(x_L + 10^{-N})) - \cos(\pi 10^k x_L) = \pi 10^{-N+k}$$

For  $k > N$ , note that  $10^k x_L$  and  $10^k x_R$  are integers so:

$$\cos(\pi 10^k(x_L + 10^{-N})) - \cos(\pi 10^k x_L) = 0$$

With some work, this gives:

$$|f(x_L) - f(x_R)| \geq (5^N + \text{small stuff})|x_L - x_R|$$

If we pick  $N$  so that  $5^N > K$  then this gives a contradiction.

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Lecture 21, 2025/03/03

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**Theorem 3.6.** Consider  $\mathcal{C}[0, 1]$ . The set of  $f \in \mathcal{C}[0, 1]$  that are Lipschitz at at least one point are first category.

**Proof.** For each  $k \geq 1$  we define:

$$A_k = \{f \in \mathcal{C}[0, 1] : f \text{ is Lipschitz somewhere with constant } k\}$$

We see that  $A_k \subseteq A_{k+1}$  for all  $k \geq 1$ . The set of functions that are Lipschitz somewhere is:

$$L = \bigcup_{k=1}^{\infty} A_k$$

We want to show  $L$  is first category. It suffices to show every  $A_k$  is nowhere dense. We first claim that  $A_k$  is closed for all  $k \geq 1$ . Let  $(f_n)_{n=1}^{\infty}$  be a cauchy sequence in  $A_k$ . Since  $C[0, 1]$  is complete, we know  $f_n \rightarrow f^*$  in  $\mathcal{C}[0, 1]$ . We need to show  $f^* \in A_k$ . Let  $(x_n)_{n=1}^{\infty}$  be a sequence such that  $f_n$  is Lipschitz with some constant  $k$  at  $x_n$ . As  $[0, 1]$  is compact, it will have a convergent subsequence. Say  $(x_{n_i})_{i=1}^{\infty}$  converges to  $x^*$  in  $[0, 1]$ . We claim that  $f^*$  is Lipschitz at  $x^*$  with constant  $k$ . For any  $x \in [0, 1]$  and  $i \geq 1$  we have:

$$\begin{aligned} |f^*(x) - f^*(x^*)| &= |f^*(x) - f_{n_i}(x) + f_{n_i}(x) - f_{n_i}(x_{n_i}) + f_{n_i}(x_{n_i}) - f_{n_i}(x^*) + f_{n_i}(x^*) - f^*(x^*)| \\ &\leq |f^*(x) - f_{n_i}(x)| + |f_{n_i}(x) - f_{n_i}(x_{n_i})| + |f_{n_i}(x_{n_i}) - f_{n_i}(x^*)| + |f_{n_i}(x^*) - f^*(x^*)| \\ &< \|f^* - f_{n_i}\|_{\infty} + k|x - x_{n_i}| + k|x_{n_i} - x^*| + \|f_{n_i} - f^*\|_{\infty} \end{aligned}$$

By taking the limit as  $i \rightarrow \infty$  we know  $\|f^* - f_{n_i}\|_\infty \rightarrow 0$  and  $\|x_{n_i} - x^*\| \rightarrow 0$ , so we have:

$$|f^*(x) - f^*(x^*)| \leq k|x - x^*|$$

Since  $x \in [0, 1]$  is arbitrary, this proved that  $f^*$  is Lipschitz at  $x^*$  with constant  $k$ . Therefore  $f^* \in A_k$  and  $A_k$  is closed. To show  $A_k$  is nowhere dense, it suffices to show  $\text{int}(A_k) = \emptyset$  as  $A_k$  is closed. This means for all  $\epsilon > 0$  and  $f \in A_k$  we can find some  $g \notin A_k$  with  $\|f - g\|_\infty < \epsilon$ . In fact, we can find  $g$  that is nowhere differentiable with  $\|f - g\|_\infty < \epsilon$ .

Pick  $f \in A_k$  and  $\epsilon > 0$ . There is a polynomial function  $p(x)$  such that  $\|f - p\|_\infty < \frac{\epsilon}{2}$ . Such a polynomial exists by the Stone-Weierstrass theorem that we will see later. By the example we did last lecture, we have a function that is differentiable nowhere, call it  $h$ . Recall that the  $h$  we constructed last time has  $\|h\|_\infty \leq 1$ . Now  $p + \frac{\epsilon}{2}h$  is differentiable nowhere and:

$$\left\| f - \left( p + \frac{\epsilon}{2}h \right) \right\|_\infty \leq \|f - p\|_\infty + \frac{\epsilon}{2} \|h\|_\infty < \epsilon$$

This proved that  $\text{int}(A_k) = \emptyset$ , thus  $L$  is first category. □

**Corollary 3.7.** The set of functions that are differentiable somewhere is first category in  $\mathcal{C}[0, 1]$ .

### 3.3 Contraction Mapping Principle

**Theorem 3.8 (Contraction Mapping Principle).** Let  $(X, d)$  be a complete metric space. Let  $f : X \rightarrow X$  be Lipschitz with constant  $K < 1$  (such function is called a **contraction**). Then:

- (i). There is a unique  $x^* \in X$  with  $f(x^*) = x^*$ . [Existence and Uniqueness of fixed point]
- (ii). For any  $x_0 \in X$ , we can construct a sequence  $(x_n)_{n=1}^\infty$  by  $x_{n+1} = f(x_n)$ . Then  $(x_n)_{n=1}^\infty$  is cauchy and we have  $x_n \rightarrow x^*$ .

**Example.** Let  $X = \mathbb{R}$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  by  $f(x) = \frac{1}{2}x + \frac{1}{2}$ . Note that  $x^* = 1$  is a fixed point. Let  $x_0 = 2$  then  $x_1 = 1 + 1/2 = 3/2$  and  $x_2 = 1 + 1/4 = 5/4$ . In general for  $n \geq 0$  we have  $x_n = 1 + 1/2^n$  and it is easy to see that  $x_n \rightarrow 1$ .

**Proof.** Pick  $x_0 \in X$  and define  $(x_n)_{n=1}^\infty$  by  $x_{n+1} = f(x_n)$  for  $n \geq 0$ . We claim that  $(x_n)_{n=1}^\infty$  is Cauchy. Let  $d(x_0, x_1) = c \geq 0$ . Then:

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq K \cdot d(x_n, x_{n-1}) \leq \dots \leq K^n \cdot d(x_1, x_0) = K^n \cdot c$$

In general, if  $m > n$  we have:

$$d(x_m, x_n) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq c \sum_{i=n}^{m-1} K^i \leq c \sum_{i=n}^{\infty} K^i$$

Since  $K < 1$ , this is a tail of a convergent series  $\sum_{i=1}^{\infty} K^i$ . Hence this  $d(x_m, x_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore  $(x_n)_{n=1}^{\infty}$  is cauchy, as desired. Since  $X$  is complete,  $x_n \rightarrow x^*$  in  $X$ . Hence:

$$f(x^*) = f\left(\lim_{n \rightarrow \infty} x_n\right) = \lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = x^*$$

Assume we have two fixed points  $x^*$  and  $y^*$  in  $X$ , then:

$$d(x^*, y^*) = d(f(x^*), f(y^*)) \leq K \cdot d(x^*, y^*)$$

Hence  $d(x^*, y^*) = 0$ , so the fixed point is unique.  $\square$

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Lecture 22, 2025/03/05

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**Example (Logistic Equation).** For  $\lambda \in [0, 4]$  we define  $f_{\lambda} : [0, 1] \rightarrow \mathbb{R}$  by  $f_{\lambda}(x) = \lambda x(1 - x)$ . This is used in population dynamics. Here  $\lambda$  represents the birth rate and  $x$  represents the current population and  $1 - x$  represents the impact of limited resources.

Let  $\lambda \in [0, 1)$ , then we have:

$$\begin{aligned} |f_{\lambda}(x) - f_{\lambda}(y)| &= |\lambda x(1 - x) - \lambda y(1 - y)| \\ &= |\lambda x(1 - x - y) - \lambda y(1 - x - y)| \\ &= \lambda|x - y||1 - x - y| \\ &< \lambda|x - y| \quad (|1 - x - y| \in [0, 1]) \end{aligned}$$

This means  $f_{\lambda}$  is Lipschitz with constant  $\lambda < 1$ . Hence it has a unique attractive fixed point satisfies  $x^* = \lambda x^*(1 - x^*)$ . In fact  $x^* = 0$ . This species is heading for extinction.

**Theorem 3.9.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  with continuous derivative. Suppose  $p \in \mathbb{R}$  is a fixed point of  $f$  such that  $|f'(p)| < 1$ . Then there exists  $a, b \in \mathbb{R}$  with  $a < p < b$  such that  $f : [a, b] \rightarrow [a, b]$  is Lipschitz on  $[a, b]$  with constant  $K < 1$ .

**Proof.** There exists an interval  $[a, b]$  such that  $|f'(x)| \leq K < 1$  (with  $|f'(p)| < K < 1$ ). This is because we have a continuous derivative. We see for all  $x, y \in [a, b]$  there is  $c \in [x, y]$  by the mean value theorem that:

$$\left| \frac{f(x) - f(y)}{x - y} \right| = |f'(c)| \leq K$$

Hence  $f$  is Lipschitz on  $[a, b]$  with constant  $K$ .  $\square$

**Example.** Consider the Logistic equation again. Consider  $f_{\lambda}(x) = \lambda x(1 - x)$  with  $\lambda \in (1, 3)$ . This has two fixed points,  $x = 0$  or  $x = 1 - \lambda^{-1}$ . We have  $f'_{\lambda}(x) = \lambda - 2\lambda x$ . At  $x = 0$  we have  $f'_{\lambda}(x) = \lambda > 1$ , so 0 is NOT an attractive fixed point. At  $x^* = 1 - 1/\lambda$  then we have:

$$f'(x^*) = \lambda - 2\lambda \left(1 - \frac{1}{\lambda}\right) = 2 - \lambda < 1$$

Hence there is  $a, b \in \mathbb{R}$  with  $a < 1 - \lambda^{-1} < b$  such that for all  $x_0 \in [a, b]$  we have  $x_n \rightarrow 1 - \lambda^{-1}$ , with  $x_n = f_\lambda(x_{n-1})$ . This means we have a stable population.

**Definition.** Let  $K$  be the set of all compacts sets in  $\mathbb{R}^n$ . For  $A, B \in K$  we define:

$$\begin{aligned} d_H(A, B) &= \inf\{\epsilon > 0 : A \subseteq B + B_\epsilon(0), B \subseteq A + B_\epsilon(0)\} \\ &= \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\} \end{aligned}$$

This is known as the **Hausdorff metric** on  $K$ .

**Example.** Let  $A = [0, 1]$  and  $B = [\frac{1}{3}, \frac{3}{2}]$ . Note that:

$$A \subseteq B + B_{1/3+\epsilon}(0) \quad \text{and} \quad B \subseteq A + B_{1/2+\epsilon}(0)$$

for all  $\epsilon > 0$ . Hence the  $d_H(A, B) = \frac{1}{2}$ .

**Remark.** This indeed gives us a metric. [Exercise]

**Construction 3.10.** Let  $n = 1$  and  $K$  denote the set of compact sets in  $\mathbb{R}$ . Consider the following map  $S : K \rightarrow K$  defined by:

$$S(A) = \frac{1}{3}A \cup \left( \frac{1}{3}A + \frac{2}{3} \right)$$

If  $A = [1, 2]$  then  $S(A) = [\frac{1}{3}, \frac{2}{3}] \cup [1, \frac{4}{3}]$ . If we let  $A_0 = A$  and define  $A_n = S(A_{n-1})$ , then  $A_n$  converges to the cantor set in  $(K, d_H)$ . In fact, we can prove  $S$  is Lipschitz with constant  $1/3$  and thus by the contraction mapping principle, it has a unique fixed point  $C$ , the cantor set.

Let  $A, B \in K$  be arbitrary. Say  $d_H(A, B) = c > 0$  (if  $A = B$  then trivial). This means  $A \subseteq B + B_c(0)$  and  $B \subseteq A + B_c(0)$ . Note that:

$$S(A) = \frac{1}{3}A \cup \left( \frac{1}{3}A + \frac{2}{3} \right) \quad \text{and} \quad S(B) = \frac{1}{3}B \cup \left( \frac{1}{3}B + \frac{2}{3} \right)$$

Since  $A \subseteq B + B_c(0)$  we have:

$$\frac{1}{3}A \subseteq \frac{1}{3}(B + B_c(0)) = \frac{1}{3}B + B_{c/3}(0) \quad \text{and} \quad \frac{1}{3}B \subseteq \frac{1}{3}A + B_{c/3}(0)$$

Similarly we have:

$$\begin{aligned} \frac{1}{3}B &\subseteq \frac{1}{3}A + B_{c/3}(0) \\ \frac{1}{3}A + \frac{2}{3} &\subseteq \frac{1}{3}B + \frac{2}{3} + B_{c/3}(0) \\ \frac{1}{3}B + \frac{2}{3} &\subseteq \frac{1}{3}A + \frac{2}{3} + B_{c/3}(0) \end{aligned}$$

Hence we have  $d_H(S(A), S(B)) = \frac{c}{3} = \frac{1}{3}d_H(A, B)$ . With some work, we can show  $K$  is complete. Therefore the map  $S$  has a unique attractive fixed point, which is the Cantor set!

### 3.4 Newton's Method

We see from the previous result that if  $g$  has a fixed point  $g(p) = p$  and  $|g'(p)| = \lambda < 1$ , then  $p$  is an attractive fixed point, within a interval around  $p$ . Moreover:

$$|g^{(n)}(x) - p| \leq \lambda^n |x - p| \quad (\text{approximately})$$

We see that the smaller  $\lambda$  is, the faster the convergence is. This implies that  $\lambda = 0$  is ideal. This is what is explained by Newton's method.

**Theorem 3.11.** Let  $f$  be twice continuously differentiable such that  $f(p) = 0$  and  $f'(p) \neq 0$  for some  $p \in \mathbb{R}$ . Define  $g$  by:

$$g(x) = x - \frac{f(x)}{f'(x)}$$

Then  $g(p) = p$  and  $g'(p) = 0$ .

**Proof.** It is easy to see that  $g(p) = p$ . Moreover,

$$g'(p) = 1 - \frac{f'(p)f'(p) - f''(p)f(p)}{(f'(p))^2} = 1 - 1 = 0$$

As desired. □

**Corollary 3.12.** If we start sufficiently close to  $p$ , then we are attracted to  $p$ .

Consider the Taylor polynomial of  $g$  around  $x = p$ , we have:

$$\begin{aligned} g(x) &= g(p) + g'(p)(x - p) + \frac{g''(p)}{2!}(x - p)^2 + \dots \\ &= p + 0 + C(x - p)^2 + \dots \end{aligned}$$

That is, if  $x = p + \epsilon$ , then  $g(x) \approx p + C\epsilon^2$ . [Quadratic convergence]

### 3.5 Metric Completion

**Definition.** Let  $(X, d)$  be a metric space. We say  $(Y, \rho)$  is a **completion** of  $(X, d)$  if  $(Y, \rho)$  is a complete space and there exists an **isometry**  $J : X \rightarrow Y$  (that is,  $\rho(Jx, Jy) = d(x, y)$  for all  $x, y \in X$ ) such that  $\overline{JX} = Y$  ( $JX$  is dense in  $Y$ ).

**Remark.** Our goal is to show that for a metric space  $(X, d)$ ,

1. The completion of  $(X, d)$  always exists.

2. The completion of  $(X, d)$  is unique (up to isometric isomorphism).
3. Show the completion of  $\mathbb{Q}$  is  $\mathbb{R}$ .
4. Discuss the completion of  $\mathbb{Q}$  with  $p$ -adic metric.

**Theorem 3.13.** Every metric space has a completion.

**Proof.** Let  $(X, d)$  be a metric space. Recall that:

$$\mathcal{C}^b(X) = \{f : X \rightarrow \mathbb{R} \mid f \text{ continuous and bounded}\}$$

is a complete metric space (in fact normed) with norm  $\|f\|_\infty = \sup_{x \in X} |f(x)|$ . We will find a closed subset  $Y \subseteq \mathcal{C}^b(X)$  (then  $Y$  is complete) and an isometry  $J : X \rightarrow Y$  such that  $JX$  is dense in  $Y$ .

Fix  $x_0 \in X$ . We define  $J : X \rightarrow \mathcal{C}^b(X)$  by  $J(x) = f_x$ , where:

$$f_x(y) = d(x, y) - d(x_0, y)$$

We claim that these functions have the desired property. Clearly each  $f_x$  is continuous, as  $d(x, \cdot)$  and  $d(x_0, \cdot)$  are both continuous. To see they are bounded, we note that:

$$f_x(y) = d(x, y) - d(x_0, y) \leq d(x, x_0)$$

by the triangle inequality. For fixed  $x \in X$ , we know  $d(x, x_0)$  is a constant. Hence  $f_x$  is bounded above. Similarly we have:

$$f_x(y) = d(x, y) - d(x_0, y) \geq -d(x, x_0)$$

by the triangle inequality again. Hence  $f_x \in \mathcal{C}^b(X)$  for all  $x \in X$ . Now we claim  $J : x \mapsto f_x$  is an isometry. Indeed, let  $x, z \in X$  we have:

$$\begin{aligned} \|f_x - f_z\|_\infty &= \sup_{y \in X} |d(x, y) - d(x_0, y) - d(z, y) + d(x_0, y)| \\ &= \sup_{y \in X} |d(x, y) - d(z, y)| \\ &\leq \sup_{y \in X} d(x, z) = d(x, z) \end{aligned}$$

This bound can be achieved at  $y = x$ , so we get  $\|f_x - f_z\|_\infty = d(x, z)$ . Let  $Y = \overline{JX}$  in  $\mathcal{C}^b(X)$ , so  $Y$  is complete, being a closed subset of a complete space. By construction we have  $\overline{JX} = Y$ , so  $JX$  is dense in  $Y$ . Since  $J$  is an isometry,  $(Y, \|\cdot\|_\infty)$  is a completion of  $X$ .  $\square$

**Example.** Let  $X = \mathbb{Q} \cap [0, 1]$  with usual metric  $d$ . Let  $x_0 = \frac{1}{2}$ . We can approach  $\pi/10$  with a cauchy sequence  $(x_n)_{n=1}^\infty$  in  $(X, d)$ .

**Construction 3.14.** Let  $(X, d)$  be a metric space. We define:

$$Z = \{\text{cauchy sequences in } X\}$$

We define a psuedo-metric  $\tilde{\rho}$  on  $Z$  by:

$$\tilde{\rho}((x_n), (y_n)) = \lim_{d \rightarrow \infty} d(x_n, y_n)$$

for  $(x_n)_{n=1}^{\infty}$  and  $(y_n)_{n=1}^{\infty}$  in  $Z$ . This is possibily NOT a metric because (for example the distance between  $(1, 0, \dots)$  and  $(0, 0, \dots)$  is zero but they are different).

We say two Cauchy sequences  $(x_n)_{n=1}^{\infty}$  and  $(y_n)_{n=1}^{\infty}$  are **equivalent** if  $\tilde{\rho}((x_n), (y_n)) = 0$ . Let:

$$Y = Z/\sim = \{\text{equivalence classes of cauchy sequences in } X\}$$

Let  $J : X \rightarrow Y$  by  $J(x) = [(x_n)_{n=1}^{\infty}]$  where  $x_n = x$  for all  $n \geq 1$ . Then  $J$  and  $Y$  have all the desired properties. Hence  $Y$  is a completion of  $X$ .

**Theorem 3.15.** Let  $(X, d)$  be a metric space with completion  $(Y, \rho)$  and  $J : X \rightarrow Y$ . Let  $(Z, \sigma)$  be a complete metric space and  $f : X \rightarrow Z$  is uniformly continuous. Then there is a unique uniformly continuous map  $\tilde{f} : Y \rightarrow Z$  with  $\tilde{f}(J(x)) = f(x)$  for all  $x \in X$ .

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ J \downarrow & \nearrow \tilde{f} & \\ Y & & \end{array}$$

**Corollary 3.16.** Let  $(X, d)$  be a metric space with completion  $(Y, \rho)$  and  $(Z, \sigma)$  given by  $J_Y : X \rightarrow Y$  and  $J_Z : X \rightarrow Z$ , respectively. Then  $J_Y$  and  $J_Z$  can be extended to isometries  $\tilde{J}_Y : Z \rightarrow Y$  and  $\tilde{J}_Z : Y \rightarrow Z$  such that  $\tilde{J}_Z$  and  $\tilde{J}_Y$  are inverses of each other.

**Proof.** By Theorem 3.15, we have these two diagrams:

$$\begin{array}{ccc} X & \xrightarrow{J_Z} & Z \\ J_Y \downarrow & \nearrow \tilde{J}_Z & \\ Y & & \end{array} \quad \begin{array}{ccc} X & \xrightarrow{J_Y} & Y \\ J_Z \downarrow & \nearrow \tilde{J}_Y & \\ Z & & \end{array}$$

Hence  $\tilde{J}_Z$  and  $\tilde{J}_Y$  exist. Since  $J_Z = \tilde{J}_Z \circ J_Y$  and  $J_Y = \tilde{J}_Y \circ J_Z$ , it is not hard to see  $\tilde{J}_Y$  and  $\tilde{J}_Z$  are inverses of each other. Now we want to show  $\tilde{J}_Y$  and  $\tilde{J}_Z$  are isometries. We first show  $\tilde{J}_Z : Y \rightarrow Z$

is an isometry. Let  $a, b \in Y$  be arbitrary. We want to show that:

$$\rho(a, b) = \sigma(\tilde{J}_Z(a), \tilde{J}_Z(b))$$

Since  $J_Y(X)$  is dense in  $Y$ , we can find sequences  $(a_n)_{n=1}^\infty$  and  $(b_n)_{n=1}^\infty$  in  $X$  such that:

$$a = \lim_{n \rightarrow \infty} J_Y(a_n) \quad \text{and} \quad b = \lim_{n \rightarrow \infty} J_Y(b_n)$$

Now, by continuity we have that:

$$\begin{aligned} \rho(a, b) &= \lim_{n \rightarrow \infty} \rho(J_Y(a_n), J_Y(b_n)) = \lim_{n \rightarrow \infty} d(a_n, b_n) = \lim_{n \rightarrow \infty} \sigma(J_Z(a_n), J_Z(b_n)) \\ &= \lim_{n \rightarrow \infty} \sigma(\tilde{J}_Z(J_Y(a_n)), \tilde{J}_Z(J_Y(b_n))) = \sigma(\tilde{J}_Z(a), \tilde{J}_Z(b)) \end{aligned}$$

Hence  $\tilde{J}_Z$  is an isometry. By the same argument,  $\tilde{J}_Y$  is an isometry.  $\square$

**Proof of Theorem 3.15.** **Step 1.** Let  $(x_n)_{n=1}^\infty$  be a cauchy sequence in  $X$ , we claim  $(f(x_n))_{n=1}^\infty$  is a cauchy sequence in  $Z$ . Let  $\epsilon > 0$ . Since  $f : X \rightarrow Z$  is uniformly continuous, there is  $\delta > 0$  such that for  $x, y \in X$ :

$$d(x, y) < \delta \implies \sigma(f(x), f(y)) < \epsilon$$

Since  $(x_n)_{n=1}^\infty$  is cauchy, we can find  $N \geq 1$  such that for all  $n, m \geq N$  we have  $d(x_n, x_m) < \delta$ . Hence for  $n, m \geq N$  we have  $\sigma(f(x_n), f(x_m)) < \epsilon$ . Therefore  $(f(x_n))_{n=1}^\infty$  is cauchy in  $(Z, \sigma)$ .

**Step 2.** Let  $y \in Y$ . Since  $JX$  is dense in  $Y$ , we can find a cauchy sequence  $(x_n)_{n=1}^\infty$  such that  $\lim_{n \rightarrow \infty} J(x_n) = y$ . We define:

$$\tilde{f}(y) = \lim_{n \rightarrow \infty} f(x_n)$$

Since  $Z$  is complete, this limit exists (shown in Step 1 that  $(f(x_n))_{n=1}^\infty$  is cauchy). Now we need to check this definition is well-defined. That is, we need to show this definition does not depend on the choice of the cauchy sequences. If we chose two different cauchy sequences  $(x_n)_{n=1}^\infty$  and  $(y_n)_{n=1}^\infty$  in  $X$  such that:

$$y = \lim_{n \rightarrow \infty} J(x_n) = \lim_{n \rightarrow \infty} J(y_n)$$

Construct a new Cauchy sequence  $(z_n)_{n=1}^\infty = (x_1, y_1, x_2, y_2, \dots)$ . We see that:

$$\tilde{f}(y) = \lim_{n \rightarrow \infty} f((z_n)_{n=1}^\infty) = \lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} f(y_n)$$

Therefore  $\tilde{f}$  does not depend on the choices of cauchy sequences. Hence  $\tilde{f}$  is well-defined. Moreover, for  $x \in X$  we can pick the constant cauchy sequence  $(x)_{n=1}^\infty$  with  $y = J(x)$ . Then we have:

$$\tilde{f}(y) = \lim_{n \rightarrow \infty} f(x) = f(x)$$

Therefore we have  $\tilde{f}(J(x)) = f(x)$  for all  $x \in X$ .

**Step 3.** We need to show  $\tilde{f}$  is uniformly continuous. Pick  $\epsilon > 0$ . Pick  $\delta > 0$  such that for  $x, y \in X$  with  $d(x, y) < \delta$  we have  $\sigma(f(x), f(y)) < \epsilon$ . Pick points in  $Y$  with  $\rho(y_1, y_2) < \delta$ . Find cauchy sequence  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  in  $X$  such that  $J(a_n) \rightarrow y_1$  nad  $J(b_n) \rightarrow y_2$ . We can pick  $N \geq 1$  sufficiently large so that  $d(a_n, b_n) < \delta$  for  $n > N$ . Hence  $\sigma(f(a_n), f(b_n)) < \epsilon$  for  $n > N$ . Therefore:

$$\sigma(\tilde{f}(y_1), \tilde{f}(y_2)) \leq \epsilon$$

by taking the limit. Therefore  $\tilde{f}$  is uniformly continuous.  $\square$

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### 3.6 The Real Numbers

**Definition.** A **field** is a bunch of things you can add and mulitply and subtract and divide (by nonzero elements).

**Example.** Rational, real and complex numbers are fields.  $\mathbb{Z}/p\mathbb{Z}$  is a field for prime  $p$ . The rational functions  $\mathbb{R}(x)$  is a field.

**Definition.** We say a field  $\mathbb{F}$  is an **ordered field** if we can write  $\mathbb{F}$  as a disjoint union:

$$\mathbb{F} = \mathbb{P} \sqcup \{0\} \sqcup (-\mathbb{P})$$

such that  $a, b \in \mathbb{P}$  impleis  $a + b \in \mathbb{P}$  and  $ab \in \mathbb{P}$ . Think of  $\mathbb{P}$  as the set of positive elements in  $\mathbb{F}$ .

**Lemma 3.17.** If  $\mathbb{F}$  is an ordered field, then  $1 \in \mathbb{P}$ .

**Proof.** If  $1 \in \mathbb{P}$  then we are done. If  $1 \in -\mathbb{P}$ , then  $-1 \in \mathbb{P}$ . Hence  $1 = (-1)(-1) \in \mathbb{P}$ . This is a contradiction.  $\square$

**Example.** The reals is an ordered field. Let  $\mathbb{P} = \{x \in \mathbb{R} : x > 0\}$ .

**Example.** By the same logic, the rationals  $\mathbb{Q}$  is an ordered field with  $\mathbb{P} = \{x \in \mathbb{Q} : x > 0\}$ .

**Example.** The complex numbers  $\mathbb{C}$  cannot be made into an order field! Suppose there is  $\mathbb{P}$ . Assume that  $i \in \mathbb{P}$ , then  $-1 = i \cdot i \in \mathbb{P}$  and thus  $1 \in -\mathbb{P}$ . This contradicts the previous Lemma. Similarly if  $i \in -\mathbb{P}$  we would get another contradiction.

**Example.**  $\mathbb{Z}/p\mathbb{Z}$  is not an ordered field. Assume it is  $1 \in \mathbb{P}$ . Then  $p \cdot 1 = 1 + \dots + 1 = 0 \in \mathbb{P}$ . This is a contradiction.

**Example.**  $\mathbb{R}(x)$  is an ordered field! We define:

$$\mathbb{P} = \left\{ \frac{p(x)}{q(x)} \in \mathbb{R}(x) : \text{there is } T \in \mathbb{R} \text{ such that } \frac{p(t)}{q(t)} > 0 \text{ for all } t \geq T \right\}$$

Not hard to check that  $\mathbb{P}$  has the desired property.

**Definition.** Let  $\mathbb{F}$  be an ordered field with  $\mathbb{F} = \mathbb{P} \sqcup \{0\} \sqcup (-\mathbb{P})$ . We define  $a < b$  if  $b - a \in \mathbb{P}$ . We can define  $a \leq b$  if  $a = b$  or  $a < b$ . It is easy to check  $<$  defines a total order of  $\mathbb{F}$ .

**Example.** In  $\mathbb{R}(x)$  with  $\mathbb{P}$  above, we have  $\frac{1}{x} < 1$  because  $1 - \frac{1}{x} > 0$  for  $x \geq 2$ .

**Definition.** We say an ordered field  $\mathbb{F}$  has the **least upper bound property (LUBP)** if for all  $\emptyset \neq S \subseteq \mathbb{F}$  that has an upper bound (there is  $M \in \mathbb{F}$  with  $s \leq M$  for all  $s \in S$ ), there exists a **least upper bound**  $x \in \mathbb{F}$  in the sense that if  $y < x$  then  $y$  is NOT an upper bound of  $S$ .

**Example.** We have seen that  $\mathbb{R}$  has LUBP by Theorem 1.13

**Example.**  $\mathbb{Q}$  does not have LUBP. Take  $S = \{x \in \mathbb{Q} : x^2 < 2\}$ . This does not have a supremum.

**Example.** The set of rational functions  $\mathbb{R}(x)$  does NOT have the LUBP. Take:

$$S = \left\{ \frac{a}{x} : a \in \mathbb{R} \right\}$$

This has an upper bound but does not have a least upper bound.

**Notation.** Let  $\mathbb{F}$  be an ordered field. For  $n \in \mathbb{N}$  we define:

$$n := \underbrace{1 + \cdots + 1}_{n \text{ times}} \in \mathbb{P}$$

**Definition.** We say an ordered field  $\mathbb{F}$  is **Archimedean** if for any  $x > 0$  there exists  $n \in \mathbb{N}$  such that  $0 < \frac{1}{n} < x$ .

**Example.**  $\mathbb{R}$  and  $\mathbb{Q}$  are archimedean.

**Example.** The rational functions  $\mathbb{R}(x)$  is NOT archimedean. Note that  $\frac{1}{x} < \frac{1}{n}$  for all  $n \in \mathbb{N}$ .

**Theorem 3.18.** Let  $\mathbb{F}$  be an ordered field. Then:

- (a). There is a nonzero field homomorphism  $\phi : \mathbb{Q} \rightarrow \mathbb{F}$ . [This means  $\phi$  is injective, so  $\mathbb{F}$  contains a copy of  $\mathbb{Q}$ ] and  $\mathbb{Q} \cap \mathbb{P} = \{x \in \mathbb{Q} : x > 0\}$ .
- (b). If  $\mathbb{F}$  has the LUBP then  $\mathbb{F}$  is archimedean.
- (c). If  $\mathbb{F}$  is archimedean and  $x < y$ , then there exists  $\frac{m}{n} < \mathbb{Q}$  such that  $x < \frac{m}{n} < y$ .

**Proof. (a).** For  $m, n \in \mathbb{Q}$  with  $m, n > 0$  we define:

$$\phi\left(\frac{m}{n}\right) = (\underbrace{1 + \cdots + 1}_{m \text{ times}})(\underbrace{1 + \cdots + 1}_{n \text{ times}})^{-1}$$

For  $q \in \mathbb{Q}$  with  $q < 0$  we just define  $\phi(-q) = -\phi(q)$ . Easy to check  $\phi$  is nonzero (since  $\phi(1) = 1 \neq 0$ ). Hence  $\phi$  is injective and  $\mathbb{F}$  contains a copy of  $\mathbb{Q}$ .

**(b).** Assume  $\mathbb{F}$  has the LUBP. We define:

$$J = \{x \in \mathbb{P} : nx < 1 \text{ for all } n \in \mathbb{N}\}$$

If  $J = \emptyset$  then for all  $x \in \mathbb{P}$  there is  $n \in \mathbb{N}$  such that  $nx > 1$ , so  $x > 1/n$  as required. Suppose  $J$  is not empty, then  $J$  is bounded above by 1. By the LUBP, it has a least upper bound  $y$ . Pick  $x_1, x_2 \in J$ . Then  $2nx_1, 2nx_2 < 1$  for all  $n \in \mathbb{N}$ . Hence:

$$n(x_1 + x_2) < 1$$

for all  $n \in \mathbb{N}$ . This means  $x_1 + x_2 \in J$ . Therefore  $x_1 + x_2 \leq y$  and  $x_1 \leq y - x_2$  for all  $x_1 \in J$ . Hence  $x_1$  is a better upper bound for  $J$ , meaning  $y$  is not the least upper bound. This is a contradiction, so  $J = \emptyset$ . Therefore  $\mathbb{F}$  is archimedean.  $\square$

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**Definition.** Let  $\mathbb{F}$  and  $\mathbb{K}$  be ordered fields. A map  $\gamma : \mathbb{F} \rightarrow \mathbb{K}$  is an **embedding** if  $\gamma$  is a field homomorphism and preserves order. That is,  $\gamma(a) \leq \gamma(b)$  whenever  $a \leq b$ .

**Theorem 3.19.** Let  $\mathbb{F}$  be an Archimedean ordered field and  $\mathbb{K}$  is a complete ordered field. Then there is an embedding from  $\mathbb{F}$  to  $\mathbb{K}$ .

**Example.** Both  $\mathbb{Q}$  and  $\mathbb{R}$  are Archimedean and  $\mathbb{R}, \mathbb{C}, \mathbb{R}(x)$  are complete.

**Proof of Theorem 3.19.** Both  $\mathbb{F}$  and  $\mathbb{K}$  are ordered fields. Hence they contain a copy of  $\mathbb{Q}$ . Let us call them  $\mathbb{Q}_\mathbb{F}$  and  $\mathbb{Q}_\mathbb{K}$ , respectively. We define:

$$\gamma_0 : \mathbb{Q}_\mathbb{F} \rightarrow \mathbb{Q}_\mathbb{K} \text{ by } \gamma_0(r_\mathbb{F}) = r_\mathbb{K}$$

For  $f \in \mathbb{F}$  we define  $S_f = \{r \in \mathbb{Q}_\mathbb{F} : r < f\}$ . Note that  $S_f$  is bounded above, hence  $\gamma_0(S_f)$  is bounded above. Now we define:

$$\gamma(f) := \sup\{\gamma_0(r) : \gamma_0(r) \in \gamma_0(S_f)\}$$

here the supremum is taken in  $\mathbb{K}$ . As  $\mathbb{K}$  is complete we have  $\gamma(f) \in \mathbb{K}$ . This defined a map  $\gamma : \mathbb{F} \rightarrow \mathbb{K}$ . Let  $f_1, f_2 \in \mathbb{F}$  with  $f_1 < f_2$ . We know there exists  $r \in \mathbb{Q}_\mathbb{F}$  such that  $f_1 < r < f_2$ . This

tells us that:

$$\gamma_0(s) < \gamma_0(r) \text{ for all } s \in S_{f_1} \text{ and } \gamma_0(r) < \gamma_0(s) \text{ for some } s \in S_{f_2}$$

Hence  $\gamma(f_1) < \gamma(f_2)$ . That is,  $\gamma$  preserves order. With loss of generality, assume  $0 < f_1, f_2$ . Now:

$$\begin{aligned} S_{f_1} + S_{f_2} &= \{r_1 < f_1 : r_1 \in \mathbb{Q}_{\mathbb{F}}\} + \{r_2 < f_2 : r_2 \in \mathbb{Q}_{\mathbb{F}}\} \\ &= \{r_1 + r_2 : r_1 < f_1, r_2 < f_2, r_1, r_2 \in \mathbb{Q}_{\mathbb{F}}\} \\ &= \{r_3 : r_3 < f_1 + f_2, r_3 \in \mathbb{Q}_{\mathbb{F}}\} \\ &= S_{f_1+f_2} \end{aligned}$$

This gives  $\gamma(f_1) + \gamma(f_2) = \gamma(f_1 + f_2)$ . Now:

$$\begin{aligned} S_{f_1} \cdot S_{f_2} &= (\{0 \leq r_1 < f_1 : r_1 \in \mathbb{Q}_{\mathbb{F}} \cap \mathbb{P}\} \cup (-\mathbb{P})) \cdot (\{0 \leq r_2 < f_2 : r_2 \in \mathbb{Q}_{\mathbb{F}} \cap \mathbb{P}\} \cup (-\mathbb{P})) \\ &= \{0 \leq r_1 \cdot r_2 : r_1 < f_1, r_2 < f_2, r_1, r_2 \in \mathbb{Q}_{\mathbb{F}} \cap \mathbb{P}\} \cup \{0\} \cup (-\mathbb{P}) \\ &= \{0 \leq r_1 \cdot r_2 < f_1 \cdot f_2 : r_1, r_2 \in \mathbb{Q}_{\mathbb{F}} \cap (\mathbb{P} \cup \{0\})\} \cup (-\mathbb{P}) \\ &= S_{f_1 f_2} \end{aligned}$$

Hence we have  $\gamma(f_1 f_2) = \gamma(f_1)\gamma(f_2)$ . This defined a field homomorphism  $\mathbb{F} \rightarrow \mathbb{K}$ . Since this is nonzero, it is an injection (an embedding).  $\square$

**Corollary 3.20.** There is a unique complete Archimedean ordered field, up to isomorphism.

**Proof.** Assume  $\mathbb{K}$  and  $\mathbb{F}$  are both Archimedean complete ordered field. By Theorem 3.19, there are order preserving homomorphisms:

$$\gamma_0 : \mathbb{K} \rightarrow \mathbb{F} \text{ and } \gamma_1 : \mathbb{F} \rightarrow \mathbb{K}$$

Since  $\gamma_0$  is identity on  $\mathbb{Q}_{\mathbb{K}}$  and  $\gamma_1$  is identity on  $\mathbb{Q}_{\mathbb{F}}$ , we know  $\gamma_0 \circ \gamma_1$  is identity on  $\mathbb{Q}_{\mathbb{F}}$ . By the completeness of  $\mathbb{F}$ , the map  $\gamma_0 \circ \gamma_1$  is identity on  $\mathbb{F}$ .  $\square$

**Definition.** We call this unique complete Archimedean ordered field  $\mathbb{R}$ , the **real numbers**.

**Remark.** This only proved the uniqueness of such complete archimedean ordered field, but we have not constructed such field yet. Now we are going to provide a detailed construction of real numbers.

**Definition.** We say  $\emptyset \neq C \subseteq \mathbb{Q}$  is a **cut** if for all  $x \in C$  we have  $y \in C$  for all  $y < x$ . Further, we requires that  $C \neq \mathbb{Q}$ .

**Example.**  $C = \{x \in \mathbb{Q} : x < 7\}$  is a cut. [This represents the real number 7.]

**Example.**  $C = \{x \in \mathbb{Q} : x \leq 0 \text{ or } x^2 < 2\}$  is a cut. [This represents the real number  $\sqrt{2}$ .]

**Definition.** Let  $C_1, C_2$  be cuts. We define  $C_1 < C_2$  if  $C_1 \subsetneq C_2$ . We define  $C_1 \leq C_2$  if  $C_1 \subseteq C_2$ .

**Theorem 3.21.** Let  $\mathcal{R} = \{C \subseteq \mathbb{Q} : C \text{ is a cut}\}$ . Then  $(\mathcal{R}, \leq)$  has the least upper bound property.

**Proof.** Let  $\mathcal{S} \subseteq \mathcal{R}$ . Suppose there is  $P \in \mathcal{R}$  such that  $R < P$  for all  $S \in \mathcal{S}$ . We claim that  $\mathcal{S}$  has a least upper bound. We define:

$$E = \bigcup_{S \in \mathcal{S}} S$$

We see that if  $y \in E$  then  $y \in S$  for some  $S \in \mathcal{S}$ . If  $x < y$  then  $x \in S \subseteq E$ . This proved that  $E$  is a cut. As  $S \subseteq E$  for all  $S \in \mathcal{S}$ , hence  $E$  is an upper bound of  $\mathcal{S}$ . We claim that  $E$  is the least upper bound. Suppose for a contradiction that  $F < E$  is also an upper bound. Since  $F < E$  there is  $r \in E \setminus F$ . Then as  $r \in E$ , there exists  $S \in \mathcal{S}$  such that  $r \in S$ . But  $S \not\subseteq F$ , which means  $S \not\leq F$ . This means  $F$  is not an upper bound, contradiction. Therefore  $E$  is the least upper bound.  $\square$

**Construction 3.22.** We need to show how we can write  $\mathcal{R}$  as a field, and show it is complete. For cuts  $C_1, C_2 \in \mathcal{R}$ , we define:

$$C_1 + C_2 := \{c_1 + c_2 : c_1 \in C_1, c_2 \in C_2\}$$

It is easy to check that  $C_1 + C_2$  is a cut. If  $0 \in C_1, C_2$  we define:

$$C_1 \cdot C_2 := \{c_1 c_2 : c_1, c_2 \geq 0, c_1 \in C_1, c_2 \in C_2\} \cup (-\mathbb{Q})$$

The other cases are similar to define. [The idea of dedekind is that we want to define a real number  $\alpha$  as the cut  $\{r \in \mathbb{Q} : r < \alpha\}$ . The multiplication is intuitive but tricky to write down.] With some work we can show  $\mathcal{R}$  is a complete Archimedean ordered field.

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We define a metric on  $\mathcal{R}$  by:

$$d(C_1, C_2) := \max \left( \sup_{c_1 \in C_1} \inf_{c_2 \in C_2} |c_1 - c_2|, \sup_{c_2 \in C_2} \inf_{c_1 \in C_1} |c_1 - c_2| \right)$$

Consider the map  $\gamma : \mathbb{Q} \rightarrow \mathcal{R}$  by  $\gamma(q) = \{r \in \mathbb{Q} : r < q\}$ . Then  $\gamma$  is an embedding of  $\mathbb{Q}$  into  $\mathcal{R}$  such that  $\gamma(\mathbb{Q})$  is dense in  $\mathcal{R}$ . Hence  $\mathcal{R}$  is a completion of  $\mathbb{Q}$ .

### 3.7 The $p$ -adic Numbers

Let  $p$  be a prime number. We know  $(\mathbb{Z}, |\cdot|_p)$  and  $(\mathbb{Q}, |\cdot|_p)$  are not complete. We can get this by either the Baire category theorem or a counting argument.

**Definition.** We call the completion of  $(\mathbb{Z}, |\cdot|_p)$  the  **$p$ -adic integers**.

**Definition.** We call the completion of  $(\mathbb{Q}, |\cdot|_p)$  the  **$p$ -adic numbers**.

**Theorem 3.23.** Let  $(x_n)_{n=1}^\infty$  be a Cauchy sequence in  $(\mathbb{Z}, |\cdot|_p)$ . Either  $\lim x_n = 0$  in  $p$ -adic norm or for all  $k \geq 1$ , the sequence  $(a_n \pmod{p^k})_{n=1}^\infty$  is an eventually constant sequence in  $\mathbb{Z}/p^k\mathbb{Z}$ .

**Proof.** If  $\lim x_n = 0$  then we are done. Suppose not. Pick  $k \geq 1$  arbitrary. Then  $|\cdot|_p : (\mathbb{Q}, |\cdot|_p) \rightarrow \mathbb{R}$  is a continuous function. Hence  $(|x_n|_p)_{n=1}^\infty$  is a cauchy sequence in  $\mathbb{R}$  that does not converge to 0. We know that  $|\cdot|_p$  takes on values of the form  $p^r$  for  $r \in \mathbb{Z}$ . Therefoe  $(|x_n|_p)_{n=1}^\infty$  is eventually constant in  $\mathbb{R}$ . Say  $|x_n|_p = p^{-N}$  for  $n$  large enough and some  $N \geq 0$ . Let  $\epsilon < p^{-k}$ . There exists  $N_0$  such that  $|x_n - x_m|_p < p^{-k}$  for all  $n, m \geq N_0$ . This means  $p^k | (x_n - x_m)$  so  $x_n \equiv x_m \pmod{p^k}$ .  $\square$

**Theorem 3.24.** For all  $a \in \mathbb{Z}_p$  there exists  $a_0 \in \{0, 1, \dots, p-1\}$  such that  $|a - a_0|_p \leq \frac{1}{p}$ .

**Proof.** We know  $\mathbb{Z}$  is dense in  $\mathbb{Z}_p$ . Pick  $k \in \mathbb{Z}$  such that  $|k - a| \leq \frac{1}{p}$ . Pick  $a_0 \in \{0, \dots, p-1\}$  such that  $k \equiv a_0 \pmod{p}$ . This gives:

$$|a - a_0|_p \leq |a - a_k|_p + |k - a_0|_p \leq \frac{1}{p} + \frac{1}{p} = \frac{2}{p}$$

If  $p \geq 3$ , then this gives that  $|a - a_0| \leq \frac{1}{p}$  (as the norm is 1 or less than  $1/p$ ). If  $p = 2$  we need to do more work, but it is still easy.  $\square$

**Corollary 3.25.** Let  $a \in \mathbb{Z}_p$ . There exists  $a_0, a_1, \dots, a_n \in \{0, \dots, p-1\}$  such that:

$$|a - (a_0 + a_1p + \dots + a_np^n)|_p \leq \frac{1}{p^{n+1}}$$

**Remark.** This means we can write  $a = \sum_{n=0}^\infty a_np^n$  with  $a_n \in \{0, \dots, n-1\}$  for  $a \in \mathbb{Z}_p$ .

**Corollary 3.26.** For each  $a \in \mathbb{Z}_p$  there is a sequence  $(a_n)_{n=0}^\infty$  with  $a_n \in \{0, \dots, p-1\}$  such that:

$$a = \sum_{n=0}^\infty a_np^n = \lim_{N \rightarrow \infty} x_N$$

where  $x_N := \sum_{n=0}^N a_np^n$  and  $(x_N)_{n=0}^\infty$  is cauchy in  $|\cdot|_p$ .

**Example.** Let  $p = 3$ . Note that:

$$-1 = \sum_{n=0}^\infty 2 \cdot 3^n = 2 + 2 \cdot 3 + 2 \cdot 3^2 + \dots$$

What about  $\alpha = \sum_{n=0}^\infty 2 \cdot 3^n + \sum_{n=0}^\infty 2 \cdot 3^n$ ? The  $N$ -th partial sum is

$$1 + \sum_{n=1}^N 2 \cdot 3^n + 3^{N+1}$$

Taking  $N \rightarrow \infty$  shows that the associated sequence is  $(1, 2, 2, 2, \dots)$ .

**Example.** Let  $p = 3$  again. Consider the multiplication  $\alpha = (\sum_{n=0}^{\infty} 2 \cdot 3^n)(\sum_{n=0}^{\infty} 2 \cdot 3^n)$ . The product of the  $N$ -partial sums is equal to:

$$1 + (3^{N+1} - 2) \cdot 3^{N+1} \equiv 1 \pmod{3^{N+1}}$$

Hence the associated sequence is  $(1, 0, 0, 0, \dots)$ .

**Theorem 3.27.**  $\mathbb{Z}_p$  is compact for all prime  $p$ .

**Proof.** We know that  $\mathbb{Z}_p$  is complete, as it is the completion of  $(\mathbb{Z}, |\cdot|_p)$ . We also know  $\mathbb{Z}$  is totally bounded by an assignment. Pick  $\epsilon > 0$  and  $k \geq 0$  such that  $p^{-k} < \epsilon$ . For every  $a \in \mathbb{Z}_p$  we can find  $a_i \in \{0, \dots, p-1\}$  such that:

$$|a - (a_0 + a_1p + \dots + a_kp^k)|_p \leq \frac{1}{p^{k+1}} < \epsilon$$

There are  $p^{k+1}$  choices for  $a_0, \dots, a_k$ , hence this gives a  $\epsilon$ -net.  $\square$

**Remark.** Addition and multiplication on  $\mathbb{Q}_p$  are similar because we can always write  $\alpha \in \mathbb{Q}_p$  as:

$$\alpha = \sum_{n=-N}^{\infty} a_n p^n$$

for  $a_n \in \{0, \dots, p-1\}$  and some  $N \geq 0$ .

**Remark.** By an assignment we showed how to invert an element in  $\mathbb{Z}_p$  with  $a_0 \neq 0$  (and hence  $\mathbb{Q}_p$ ). Hence  $\mathbb{Q}_p$  is a field. We also show that  $\sqrt{-2} \in \mathbb{Q}_3$  and more generally  $\sqrt{1-p} \in \mathbb{Q}_p$  for  $p \geq 3$ . This means that  $\mathbb{Q}_p$  is NOT an ordered field. Note that  $p = 2$  is a special case to consider, and is typically a special case for any  $p$ -adic problems.

## 4 Approximation Theory

### 4.1 Polynomial Approximation

When we proved that the set of functions that are differentiable somewhere was first category, we first approximated a random function by a differentiable function and then added a small non-differentiable function to it. We can find such a function if we can approximate it by a polynomial. There are three methods we will discuss, but the first two do not work.

**Method (Taylor Polynomials).** Say  $f$  is some function, then we know that:

$$f(x) \approx \sum_{n=0}^N \frac{f^{(n)}(c)}{n!} (x - c)^n$$

Here  $f^{(n)}(c)$  is the  $n$ -th derivative of  $f$  at  $c$ . This has the following problems:

- (1). We need  $f$  to have lots of derivatives. This means it is useless for non-differentiable functions.
- (2). This only really converges inside its disk of convergence. Take  $f(x) = (1 + x^2)^{-1}$ . Its Taylor series converges for  $|x| < 1$ . Also, take:

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

This only converges at  $x = 0$ , so its Taylor series is useless.

**Method (Lagrange Polynomials).** Let  $f$  be some function and  $x_0, x_1, \dots, x_n$  be a collection of distinct points (in the domain of  $f$ ). Define:

$$P_k(x) = \prod_{i \neq k} \left( \frac{x - x_i}{x_k - x_i} \right)$$

This is a polynomial in  $x$ . Note that  $P_k(x_j) = \delta_{kj}$  for all  $k, j$ . Define:

$$P(x) = \sum_{i=0}^n f(x_i) P_i(x)$$

This has the property that  $P(x_i) = f(x_i)$  for  $i = 0, 1, \dots, n$ . One would hope that the more points one uses, the better the approximation. Consider:

$$f : [0, 1] \rightarrow \mathbb{R} \quad \text{by} \quad f(x) = \frac{1}{1 + 25x^2}$$

In this case the Lagrange polynomial does not approximate  $f$  at all.

**Theorem 4.1 (Weierstrass Approximation Theorem).** Let  $f : [0, 1] \rightarrow \mathbb{R}$  be continuous. For every  $\epsilon > 0$  there exists a polynomial  $p(x) \in \mathbb{R}[x]$  such that:

$$\|f - p\|_\infty = \sup_{x \in [0,1]} |f(x) - p(x)| < \epsilon$$

**Proof.** Assume WLOG that  $f(0) = f(1) = 0$ . If  $f(0) = a$  and  $f(1) = b$  we could consider the function  $g(x) = f(x) - a + (a - b)x$ . Consider:

$$Q_n(x) = \begin{cases} (1 - x^2)^n c_n & \text{if } x \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}$$

where for each  $n \geq 1$  we define:

$$c_n = \left( \int_{-1}^1 (1 - x^2)^n dx \right)^{-1}$$

Hence we have  $\int_{-1}^1 Q_n(x) dx = 1$ . Now we define functions  $q_n$  by:

$$q_n(x) = \int_{-1}^1 f(x+t) Q_n(t) dt$$

where  $f(z) = 0$  if  $z \notin [0, 1]$ . We claim that  $q_n(x)$  is a polynomial in  $x$  and  $\|q_n - f\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$ . Our plan to prove this claim is as follows:

1. Estimate  $c_n$  for each  $n \geq 1$ .
2. For  $\delta > 0$  we have  $\lim_{n \rightarrow \infty} \int_{-\delta}^{\delta} Q_n(x) dx = 0$ . This implies  $\int_{-1}^{-\delta} Q_n(x) dx \rightarrow 0$  since  $Q_n$  is even.
3. Show  $q_n(x)$  is a polynomial.

**Step 1.** By trig substitution with  $x = \sin(u)$  we have  $dx = \cos(u) du$ . Then:

$$\begin{aligned} \int_{-1}^1 (1 - x^2)^n dx &= \int_{-\pi/2}^{\pi/2} (1 - \sin^2(u))^n \cos u du = \int_{-\pi/2}^{\pi/2} \cos^{2n+1}(u) du \\ &= 2 \int_0^{\pi/2} \cos^{2n+1}(u) du = 2 \left( \frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdots (2n-1)(2n+1)} \right) \geq \frac{2}{2n+1} \end{aligned}$$

This gives the estimation that  $c_n \leq \frac{2n+1}{2} \leq 2n+1$ .

**Step 2.** Fix  $\delta > 0$ . Then we have:

$$I_n = \int_{-\delta}^1 (1 - x^2)^n c_n dx \leq \int_{-\delta}^1 (1 - \delta^2)^n (2n+1) dx \leq (1 - \delta^2)^n (2n+1)$$

We see that  $(1 - \delta^2)^n(2n + 1)$  goes to 0 as  $n \rightarrow \infty$  (using ratio test). Hence  $I_n \rightarrow 0$  as  $n \rightarrow \infty$ .

**Step 3.** By substitution with  $u = x + t$  we have:

$$q_n(x) = \int_{-1}^1 f(x+t)Q_n(t) dt = \int_{-1+x}^{1+x} f(u)Q_n(u-x) du$$

We are assuming that  $f(u) = 0$  for all  $u \notin [0, 1]$ , hence:

$$q_n(x) = \int_0^1 f(u)Q(u-x) du$$

This is a polynomial! (Consider what happens to individual  $x^j$  term in the polynomial  $Q(u-x)$ ). For example, we have that:

$$\int_0^1 f(u) \sum_{i,j} a_{ij} x^i u^j du = \sum_i \left( \int_0^1 \sum_j a_{ij} f(u) u^j du \right) x^i$$

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Since  $f \in \mathcal{C}[0, 1]$  is continuous on  $[0, 1]$ , it is uniformly continuous. Pick  $\delta > 0$  such that for all  $x, y \in [0, 1]$  we have:

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{2}$$

We also know that  $f$  is bounded. There is  $M > 0$  such that  $|f(x)| < M$  for all  $x \in [0, 1]$ . Now pick  $n$  sufficiently large so that:

$$\int_{-1}^{-\delta} 2MQ_n(t) dt + \int_{-\delta}^1 2MQ_n(t) dt < \frac{\epsilon}{2} \quad (*)$$

Pick  $x \in [0, 1]$ . Notice that:

$$|q_n(x) - f(x)| = \left| \int_{-1}^1 f(t+x)Q_n(t) dt - f(x) \int_{-1}^1 Q_n(t) dt \right| = \left| \int_{-1}^1 (f(x+t) - f(x))Q_n(t) dt \right|$$

Now we estimate two integrals.

$$A = \left| \int_{-\delta}^{\delta} (f(x+t) - f(x))Q_n(t) dt \right| \leq \int_{-\delta}^{\delta} |f(x+t) - f(x)|Q_n(t) dt \leq \frac{\epsilon}{2} \int_{-\delta}^{\delta} Q_n(t) dt = \frac{\epsilon}{2}$$

By the choice of  $n$  in equation  $(*)$  we have:

$$B = \left| \int_{-1}^{-\delta} \underbrace{(f(x+t) - f(x))}_{\leq 2M} Q_n(t) dt + \int_{\delta}^1 \underbrace{(f(x+t) - f(x))}_{\leq 2M} Q_n(t) dt \right| < \frac{\epsilon}{2}$$

Now, by the triangle inequality we obtain that:

$$|q_n(x) - f(x)| = \left| \int_{-1}^1 (f(x+t) - f(x))Q_n(t) dt \right| \leq A + B \leq \epsilon$$

Since  $x \in [0, 1]$  is arbitrary, we have  $\|q_n - f\| \leq \epsilon$ . As desired.  $\square$

## 4.2 Stone-Weierstrass Theorem

**Definition.** Let  $X$  be a compact metric space. Then  $\mathcal{C}(X) = \{f : X \rightarrow \mathbb{R} \mid f \text{ continuous}\}$  is a vector space over  $\mathbb{R}$ . We say  $\mathcal{A} \subseteq \mathcal{C}(X)$  is an **algebra** if  $\mathcal{A}$  is a vector subspace of  $\mathcal{C}(X)$  and for all  $f, g \in \mathcal{A}$  we have  $fg \in \mathcal{A}$ .

**Example.** Let  $X$  be any compact space. Then  $\mathcal{A} = \{\text{constant functions}\}$  is an algebra.

**Example.** The set of polynomials is an algebra in  $\mathcal{C}[0, 1]$ . Moreover, even degree Polynomials are an algebra. Odd degree polynomials do not form an algebra because  $x \cdot x$  is even.

**Example.** Differentiable functions form an algebra.

**Example.** Let  $X = [0, 1]$ . Then  $\{f \in \mathcal{C}[0, 1] : f(0) = f(1)\}$  is an algebra.

**Example.** Even polynomials (polynomials that are also even functions) is an algebra.

**Definition.** For two functions  $f, g : X \rightarrow \mathbb{R}$  we define  $f \vee g : X \rightarrow \mathbb{R}$  and  $f \wedge g : X \rightarrow \mathbb{R}$  by:

$$(f \vee g)(x) := \max(f(x), g(x)) \quad \text{and} \quad (f \wedge g)(x) = \min(f(x), g(x))$$

We also write  $f \vee g = \max(f, g)$  and  $f \wedge g = \min(f, g)$ .

**Definition.** Let  $X$  be compact and  $\mathcal{A} \subseteq \mathcal{C}(X)$  be an algebra. We say  $\mathcal{A}$  is a **vector lattice** if for all  $f, g \in \mathcal{A}$  we have  $f \vee g \in \mathcal{A}$  and  $f \wedge g \in \mathcal{A}$ .

**Example.** Constant functions is a vector lattice.

**Example.** Let  $P = \text{polynomials on } [0, 1]$ . This is not a vector lattice. Note  $x \vee 0 = |x|$  is not a polynomial. However  $\overline{P} = \mathcal{C}[0, 1]$  by the Weierstrass approximation theorem. Hence  $\overline{P}$  is a lattice.

**Definition.** Let  $X$  be compact and  $\mathcal{A} \subseteq \mathcal{C}(X)$  is an algebra. We say  $\mathcal{A}$  **separates points** if for all  $x, y \in X$  with  $x \neq y$  there is  $f \in \mathcal{A}$  such that  $f(x) \neq f(y)$ .

**Example.** Constant functions do not separate points.

**Example.** Polynomials separate points because  $x$  separates points.

**Example.** Even functions in  $\mathcal{C}[-1, 1]$  do not separate points.

**Definition.** Let  $X$  be compact and  $\mathcal{A} \subseteq \mathcal{C}(X)$  be an algebra. We say  $\mathcal{A}$  vanishes at  $x \in X$  if for all  $f \in \mathcal{A}$  we have  $f(x) = 0$ .

**Example.** Constant functions  $X \rightarrow \mathbb{R}$  do not vanish anywhere (because 1 does not vanish anywhere).

**Example.** The algebra  $\mathcal{A} = \text{span}_{\mathbb{R}}\{x^{2n} : n \geq 1\}$  vanishes at 0.

**Example.** Let  $P = \text{set of polynomials on } [0, 1]$ . Then  $P$  is an algebra and  $\overline{P}$  is a vector lattice. Also  $P$  separates points and  $P$  does not vanish anywhere.

**Theorem 4.2 (Stone-Weierstrass).** Let  $X$  be a compact metric space. Let  $\mathcal{A} \subseteq \mathcal{C}(X)$  be an algebra that separates points and does not vanish anywhere. Then  $\mathcal{A}$  is dense in  $\mathcal{C}(X)$ .

Our plan of the proof is the followings:

1. If  $\mathcal{A}$  is an algebra, then  $\mathcal{A}$  in  $\mathcal{C}(X)$  is a vector lattice.
2. If  $\mathcal{A}$  separates points then for all  $x, y \in X$  with  $x \neq y$  and  $a, b \in \mathbb{R}$ , there exists  $f \in \mathcal{A}$  such that  $f(x) = a$  and  $f(y) = b$ .
3. For each  $a \in X$  and  $\epsilon > 0$ , we can find  $g_a \in \overline{\mathcal{A}}$  such that  $g_a(x) > f(x) - \epsilon$  and  $g_a(a) = f(a)$ .
4. Using these  $g_a$ , we can find  $g$  such that  $f(x) + \epsilon > g(x) > f(x) - \epsilon$  for all  $x \in X$ .

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**Lemma 4.3.** Let  $X$  be compact and  $\mathcal{A} \subseteq \mathcal{C}(X)$  be an algebra. Then  $\overline{\mathcal{A}}$  is a closed algebra and a vector lattice.

**Proof.** Clearly  $\overline{\mathcal{A}}$  is closed and it is easy to check it is an algebra. Recall that  $\overline{\mathcal{A}}$  is a vector lattice if for all  $f, g \in \overline{\mathcal{A}}$  we have  $f \vee g$  and  $f \wedge g \in \overline{\mathcal{A}}$ . We will first show that if  $f \in \overline{\mathcal{A}}$  then  $|f| \in \overline{\mathcal{A}}$ . Now take  $f \in \overline{\mathcal{A}}$ , we know  $L := \|f\|_{\infty} < \infty$ . By the Weierstrass approximation theorem we know polynomials are dense in  $\mathcal{C}[-L, L]$ . Since  $g(x) = |x| \in \mathcal{C}[-L, L]$ , there is a sequence of polynomials  $(p_n)_{n=1}^{\infty}$  such that  $p_n \rightarrow g$  uniformly on  $[-L, L]$ . Notice  $p_n(0) \rightarrow 0$  as  $n \rightarrow \infty$ . Let  $q_n = p_n - p_n(0)$ , then we have  $q_n \rightarrow g$  as well. Note that  $q_n(f) \in \mathcal{A}$  and  $q_n(f) \rightarrow |f|$ . It follows that  $|f| \in \overline{\mathcal{A}}$ . Now:

$$\begin{aligned} \max(f, g) &= \frac{f+g}{2} + \frac{|f-g|}{2} \in \overline{\mathcal{A}} \\ \min(f, g) &= \frac{f+g}{2} - \frac{|f-g|}{2} \in \overline{\mathcal{A}} \end{aligned}$$

This proved that  $\overline{\mathcal{A}}$  is a vector lattice. □

**Lemma 4.4.** Let  $X$  be compact and  $\mathcal{A} \subseteq \mathcal{C}(X)$  be an algebra. Further, assume  $\mathcal{A}$  separates points and vanishes nowhere. For all  $x, y \in X$  with  $x \neq y$  and any  $c, d \in \mathbb{R}$  we can find  $f \in \mathcal{A}$  such that  $f(x) = c$  and  $f(y) = d$ .

**Proof.** Let  $x, y \in X$  with  $x \neq y$ . Let  $g \in \mathcal{A}$  such that  $g(x) \neq g(y)$ . Write  $g(x) = a$  and  $g(y) = b$  and  $a \neq b$ . Hence they are not both 0. WLOG assume  $b \neq 0$ .

**Case 1.** Assume  $a = 0$ . Since  $\mathcal{A}$  does not vanish at  $x$ , there exists  $h \in \mathcal{A}$  such that  $h(x) \neq 0$ . Set:

$$f(z) = \frac{c}{h(x)}h(z) + \left( \frac{d}{g(y)} - \frac{c \cdot h(y)}{h(x)g(y)} \right)g(z)$$

Evaluate this function  $f$  at  $x$  we get  $f(x) = c$  and  $f(y) = d$ .

**Case 2.** Assume  $a \neq 0$ . Consider the function:

$$\tilde{g}(z) = g(z) - \frac{g(z)^2}{g(x)}$$

Then  $\tilde{g}(x) = 0$  and  $\tilde{g}(y) \neq 0$ . Now apply case 1.  $\square$

**Proof of Theorem 4.2.** Let  $f \in \mathcal{C}(X)$  be arbitrary and  $\epsilon > 0$ . Fix  $a \in X$ . For each  $a \neq x \in X$  there is a function  $h_x \in \mathcal{A}$  such that:

$$h_x(a) = f(a) \text{ and } h_x(x) = f(x)$$

by Lemma 4.4 applying to  $c = f(a)$  and  $d = f(x)$ . Define:

$$U_x = \{z \in X : h_x(z) > f(z) - \epsilon\}$$

Notice that  $x \in U_x$  and  $a \in U_x$ . Also note that  $U_x$  is open because:

$$U_x = (h_x - f)^{-1}((- \epsilon, \infty))$$

Note that  $\{U_x\}_{x \in X}$  is an open cover of  $X$ . As  $X$  is compact, we have a finite subcover:

$$\{U_{x_1}, \dots, U_{x_n}\}$$

Take  $g_a = \max(h_{x_1}, \dots, h_{x_n}) \in \overline{\mathcal{A}}$ , as  $\overline{\mathcal{A}}$  is a vector lattice. Notice that:

$$g_a(a) = f(a) \text{ and } g_a(z) > f(z) - \epsilon \text{ for all } z \in X$$

Let  $V_a = \{z \in X : g_a(z) < f(z) + \epsilon\}$ . We see that  $a \in V_a$  and each  $V_a$  is open. As before  $\{V_a\}_{a \in X}$  is an open cover. We have a finite subcover  $\{V_{a_1}, \dots, V_{a_k}\}$ . Take  $g = \min(g_{a_1}, \dots, g_{a_k})$ . We see that  $g(z) > f(z) - \epsilon$  for all  $z \in X$  by the properties of  $g_{a_i}$ . Further  $g(z) < f(z) + \epsilon$  by properties of  $U_{a_i}$ . Hence  $\|g - f\|_\infty < \epsilon$ . Since  $f \in \mathcal{C}(X)$  and  $\epsilon > 0$  are arbitrary, we proved  $\overline{\mathcal{A}} = \mathcal{C}(X)$ .  $\square$

### 4.3 Best Approximation

**Notation.** For  $n \geq 0$  let  $\mathbb{P}_n[x]$  denote the polynomials of degree at most  $n$ .

**Definition.** For  $X$  compact and  $f \in \mathcal{C}(X)$  and  $n \geq 1$  we define:

$$E_n(f) = \inf_{p \in \mathbb{P}_n[x]} \|f - p\|_\infty$$

Note that  $T : \mathbb{P}_n[x] \rightarrow \mathbb{R}$  by  $p \mapsto \|f - p\|_\infty$  is a continuous function. Consider  $S \subseteq \mathbb{P}_n[x]$  such that:

$$S = \{p \in \mathbb{P}_n[x] : \|p\|_\infty \leq 4\|f\|_\infty\}$$

Then  $S \subseteq \mathbb{P}_n[x]$  is compact and the polynomial  $0 \in S$ . By restriction,  $T : S \rightarrow \mathbb{R}$  is continuous on a compact set! Hence there exists  $p^* \in S$  such that:

$$\|p^* - f\|_\infty = \inf_{p \in S} T(p) = \inf_{p \in S} \|p - f\|_\infty \leq \|f\|_\infty$$

If  $p \in \mathbb{P}_n[x] \setminus S$  then we have  $\|p - f\|_\infty \geq 2\|f\|_\infty$ . Hence:

$$\|p^* - f\|_\infty = \inf_{p \in \mathbb{P}_n[x]} \|f - p\|_\infty = E_n(f) \quad (*)$$

We say  $p^*$  is a **best approximation** of  $f$  of degree  $n$ .

**Definition.** We say a function  $g \in \mathcal{C}[a, b]$  satisfies the **equioscillation property** of degree  $n$  if there exists  $(n+2)$  points  $x_1 < \dots < x_{n+2}$  in  $[a, b]$  with:

$$g(x_i) = (-1)^i \|g\|_\infty \text{ or } g(x_i) = (-1)^{i+1} \|g\|_\infty$$

for all  $i \in \{1, \dots, n+2\}$ .

**Theorem 4.5.** Let  $n \geq 1$  and  $f \in \mathcal{C}[a, b]$ . Assume  $p \in \mathbb{P}_n[x]$  such that  $g := f - p$  satisfies the equioscillation property of degree  $n$ . Then  $p$  is a best approximation of  $f$ , that is,  $\|f - p\|_\infty = E_n(f)$ .

**Proof.** Assume  $p$  is not a best approximation. There exists another polynomial  $r(x)$  that gives a better approximation. We know  $q = r - p$  is a polynomial of degree at most  $n$ . Let  $x_1, \dots, x_{n+2}$  such that  $g(x_i) = (-1)^i \|g\|_\infty$ . We see that  $g(x_i)$  and  $q(x_i)$  must have the same sign, otherwise:

$$|g(x_i) - q(x_i)| > |g(x_i)| = \|g\|_\infty$$

which is a contradiction since  $|g(x_i) - q(x_i)| < \|g\|_\infty$  as it is a better approximation. That is,  $q(x)$  has  $(n+2)$  sign changes. As  $\deg(q) \leq n$  we have  $q(x) = 0$ . This proves  $p(x)$  is a best approximation.  $\square$

**Theorem 4.6.** If  $p \in \mathbb{P}_n[x]$  is a best approximation of  $f$ , then  $g = f - p$  satisfies the equioscillation property of degree  $n$ .

**Theorem 4.7.** Best approximations are unique.

## 5 Differential Equations

**Example.** Consider the differential equation  $y' = x^2 + 1$ . This is a boring differential equation. We can just find an anti-derivative for  $x^2 + 1$ .

**Example.** Consider  $(y')^2 + 1 = 0$ , an first order DE with no real solutions.  $[y' = \pm i]$

**Example.** Consider the second order DE  $y'' = -y$ . We see that  $y(x) = a \sin(x)$  and  $y(x) = b \cos(x)$  are both solutions. In fact  $\text{span}\{\sin(x), \cos(x)\}$  is the set of all solutions to this DE.

**Goal:** We wish to use the contraction mapping principle to show that certain families of first order DEs have a solution, and that the solution is unique.

**Example.** Consider the differential equation  $y' = -1 + y/2$  on  $x \in [-1, 1]$  with  $y(0) = 1$ . How do we solve it? We have:

$$\int_0^x y'(t) dt = \int_0^x -1 + \frac{y(t)}{2} dt$$

By the FTC this gives us:

$$y(x) = y(0) - x + \int_0^x \frac{y(t)}{2} dt = 1 - x + \int_0^x \frac{y(t)}{2} dt$$

Consider  $T : \mathcal{C}[-1, 1] \rightarrow \mathcal{C}[-1, 1]$  given by  $Tf(x) = 1 - x + \int_0^x f(t)/2 dt$ . We claim that  $T$  is Lipschitz with constant  $< 1$ . Indeed, we have:

$$\begin{aligned} \|Tf - Tg\|_\infty &= \sup_{x \in [-1, 1]} \left| \frac{1}{2} \int_0^x (f(t) - g(t)) dt \right| \\ &\leq \sup_{x \in [-1, 1]} \left| \frac{1}{2} \int_0^x |f(t) - g(t)| dt \right| \\ &\leq \frac{1}{2} \sup_{x \in [-1, 1]} \int_0^x \|f - g\|_\infty dt \\ &= \frac{1}{2} \|f - g\|_\infty \end{aligned}$$

By the contraction mapping principle,  $T$  has a unique fixed point. Moreover, for all  $f \in \mathcal{C}[-1, 1]$  the sequence  $(T^n f)_{n=0}^\infty$  converges to this unique fixed point! Let  $f_0 = 0$ , then:

$$\begin{aligned} f_1(x) &= (Tf_0)(x) = 1 - x + \frac{1}{2} \int_0^x 0 dt = 1 - x \\ f_2(x) &= (Tf_1)(x) = 1 - x + \frac{1}{2} \int_0^x (1-t) dt = 1 - \frac{x}{2} - \frac{x^2}{4} \end{aligned}$$

By some computation we can see that:

$$f_3(x) = (Tf_2)(x) = 1 - \frac{x}{2} - \frac{x^2}{4} - \frac{x^3}{8}$$

$$f_4(x) = (Tf_3)(x) = 2 - \left(1 + \frac{x}{2} + \frac{(x/2)^2}{2!} + \frac{(x/2)^3}{3!}\right) - \frac{x^4}{192}$$

If we continue, we obtain  $f^*(x) = 2 - e^{x/2}$  is the fixed point and this is a solution to our DE.

**Remark.** The key observation is we could construct  $T : \mathcal{C}(X) \rightarrow \mathcal{C}(X)$  that was Lipschitz and contractive. We will show for a large family of first order DE, something “like this” will happen!

**Definition.** Let  $I_1$  and  $I_2$  be intervals. We say  $\varphi : I_1 \times I_2 \rightarrow \mathbb{R}$  is **Lipschitz in  $y$**  if there is  $L \geq 0$  such that for any fixed  $x \in I_1$  and all  $y_1, y_2 \in I_2$ :

$$|\varphi(x, y_1) - \varphi(x, y_2)| \leq L|y_1 - y_2|$$

**Example.** For  $y' = 1 + x^2$  we can take  $\varphi(x, y) = 1 + x^2$ . This is clearly Lipschitz in  $y$  with constant  $L = 0$ . This is because  $|\varphi(x, y_1) - \varphi(x, y_2)| = 0$  for any  $y_1, y_2$ .

**Example.** For  $y' = -1 + y/2$  we let  $\varphi(x, y) = -1 + y/2$ . Then  $\varphi$  is Lipschitz in  $y$  with constant  $1/2$ .

**Lemma 5.1.** Let  $I_1 = [a, b]$ . Let  $\varphi : I_1 \times \mathbb{R} \rightarrow \mathbb{R}$  be Lipschitz in  $y$  with constant  $L$ . Let  $c \in [a, b]$  and define:

$$T : \mathcal{C}[a, b] \rightarrow \mathcal{C}[a, b] \quad \text{by} \quad Tf(x) = c_0 + \int_0^x \varphi(t, f(t)) dt$$

where  $y(c) = c_0$  is the initial condition. Let  $f, g \in \mathcal{C}[a, b]$ . If there exists  $M$  and  $k$  such that:

$$|f(x) - g(x)| \leq \frac{M|x - c|^k}{k!} \quad \text{for all } x \in [a, b]$$

Then for all  $x \in [a, b]$  we have:

$$|Tf(x) - Tg(x)| \leq \frac{LM|x - c|^{k+1}}{(k+1)!}$$

**Note.** For all  $f, g \in \mathcal{C}[a, b]$  there is such constant  $M = \|f - g\|_\infty$  and  $k = 0$ , so:

$$|f(x) - g(x)| \leq \sup_{x \in [a, b]} |f(x) - g(x)| = \|f - g\|_\infty = \frac{M|x - c|^0}{0!}$$

By iterating this we get that:

$$|T^k f(x) - T^k g(x)| \leq \frac{L^k |x - c|^k}{k!}$$

Taking  $k \rightarrow \infty$  we get  $\frac{L^k |x - c|^k}{k!} \rightarrow 0$ . Hence there is  $k_0$  large enough such that:

$$L_0 := \frac{L^{k_0} |x - c|^{k_0}}{k_0!} < 1$$

It follows that  $T^{k_0}$  is a contraction with constant  $L_0 < 1$ .

## 5.1 Global Solutions of ODEs

**Proof of Lemma 5.1.** Assume (1) holds. Then:

$$\begin{aligned}
 |Tf(x) - Tg(x)| &= \left| c_0 + \int_c^x \varphi(t, f(t)) dt - c_0 - \int_c^x \varphi(t, g(t)) dt \right| \\
 &= \left| \int_c^x \varphi(t, f(t)) - \varphi(t, g(t)) dt \right| \\
 &\leq \int_c^x L|f(t) - g(t)| dt \tag{Lipschitz in $y$} \\
 &\leq \int_c^x \frac{LM|t - c|^k}{k!} dt \tag{by (1)} \\
 &= \frac{LM|x - c|^{k+1}}{(k+1)!}
 \end{aligned}$$

As desired.  $\square$

**Theorem 5.2 (Global Picard Theorem).** If  $\varphi : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$  is Lipschitz in  $y$  and  $c \in [a, b]$  then there exists a unique solution to  $y'(x) = \varphi(x, y(x))$  with  $y(c) = c_0$  in  $\mathcal{C}[a, b]$ .

**Proof.** Use  $T : \mathcal{C}[a, b] \rightarrow \mathcal{C}[a, b]$  as in the previous lemma. We know from Lemma 5.1 that:

- (1) Take  $k = 0$  and  $M = \|f - g\|_\infty$ , it satisfies the condition of the previous lemma. Hence:

$$|Tf(x) - Tg(x)| \leq L\|f - g\|_\infty|x - c|$$

- (2) By a different corollary there exists  $k_0$  such that  $T^{(k_0)}$  is a contraction on  $\mathcal{C}[a, b]$ . This gives a unique fixed point to  $T$ . Hence we have a unique solution to the DE.  $\square$

## 5.2 Local Solutions

There are solutions where  $\varphi$  is not Lipschitz in  $y$ , where  $\varphi$  is “nice enough” that we can still do something to find a unique solution. The problem occurs as  $y \in \mathbb{R}$  and  $\mathbb{R}$  is big.

**Example.** Consider  $y' = -2xy^2$  with  $y(0) = 1$ . This has solution  $y(x) = \frac{1}{1+x^2}$ . The associated function here is  $\varphi(x, y) = -2xy^2$ , which is not Lipschitz in  $y$  for  $x \neq 0$ . Let  $[a, b] = [-1/4, 1/4]$  and:

$$\mathcal{C}'[a, b] = \left\{ f \in \mathcal{C}[a, b] : |f(x) - 1| \leq \frac{1}{2}, x \in [a, b] \right\} \subseteq \mathcal{C}[a, b]$$

Let  $c = 0$  then  $c_0 = y(0) = 1$ . Let  $T : \mathcal{C}'[a, b] \rightarrow \mathcal{C}'[a, b]$  by:

$$Tf(x) = 1 + \int_0^x \varphi(t, f(t)) dt = 1 - 2 \int_0^x tf(t)^2 dt$$

We claim this is well-defined, that is,  $Tf \in \mathcal{C}'[a, b]$  for all  $f \in \mathcal{C}[a, b]$ . We also need to show  $T$  is Lipschitz. Assume  $|f(x) - 1| \leq 1/2$  for all  $x \in [a, b]$ . This implies  $|f(x)| \leq 3/2$ . Then:

$$|Tf(x) - 1| = \left| 1 - 2 \int_0^x t f(t)^2 dt - 1 \right| \leq \int_0^{1/4} 2t \left( \frac{3}{2} \right)^2 dt = \frac{9}{64} < \frac{1}{2}$$

Hence  $Tf \in \mathcal{C}'[a, b]$ . Consider the Lipschitz constant on  $\varphi$ . For fixed  $x$  we have:

$$\left| \frac{\partial}{\partial y} \varphi(x, y) \right| = 4|xy| \leq \frac{3}{2}$$

as  $x \in [-1/4, 1/4]$  and  $y \in [1/2, 3/2]$ . Using the same trick before there is  $k_0$  such that  $T^{(k_0)}$  is a contraction.

**Definition.** We say  $\varphi : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$  is **locally Lipschitz** in  $y$  is for all  $(x_0, y_0) \in [a, b] \times \mathbb{R}$  there exists  $h > 0$  such that  $\varphi$  is Lipschitz on  $[x_0 - h, x_0 + h] \times [y_0 - h, y_0 + h]$  in  $y$ .

**Lemma 5.3.** Let  $\varphi : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$  be locally Lipschitz in  $y$  on a convex compact set  $K$ . Then  $\varphi$  is Lipschitz on  $K$ .

**Proof.** For every  $(x, y)$  we can find a neighborhood where  $\varphi$  is Lipschitz in  $y$  (on this neighborhood). This gives an open cover of  $K$ . We can find a finite subcover. Pick  $L$  to be the worst constant from this finite set. With some work we can finish the proof.  $\square$

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**Theorem 5.4 (Local Picard's Theorem).** Suppose  $\varphi : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous and locally Lipschitz on  $[a, b] \times [c_0 - R, c_0 + R]$ . Then the DE  $y' = \varphi(x, y)$  with  $y(a) = c_0$  has a solution on  $[a, a + h]$  with  $h = \min(b - a, R/\|\varphi\|)$ , where:

$$\|\varphi\| := \sup_{\substack{x \in [a, b] \\ y \in [c_0 - R, c_0 + R]}} |\varphi(x, y)|$$

**Proof.** Take  $T : \mathcal{C}[a, a + h] \rightarrow \mathcal{C}[a, a + h]$  by:

$$Tf(x) = c_0 + \int_a^x \varphi(t, f(t)) dt$$

As before, take  $\mathcal{C}' \subseteq \mathcal{C}[a, a + h]$  by:

$$\mathcal{C}' = \{f \in \mathcal{C}[a, a + h] : \|f - c_0\|_\infty \leq R\}$$

We need to show  $T : \mathcal{C}' \rightarrow \mathcal{C}'$  and  $T$  is Lipschitz in  $y$  on  $\mathcal{C}'$ . We see  $\mathcal{C}'$  is a compact set. As  $\varphi$  and  $T$  are locally Lipschitz in  $y$  on a compact and convex set,  $\varphi$  and  $T'$  are Lipschitz on  $\mathcal{C}'$ . To see that

$T : \mathcal{C}' \rightarrow \mathcal{C}'$ , note for  $f \in \mathcal{C}'$  we have:

$$\begin{aligned} |Tf(x) - c_0| &= \left| c_0 + \int_a^x \varphi(t, f(t)) dt - c_0 \right| \\ &= \left| \int_a^x \varphi(t, f(t)) dt \right| \\ &\leq h \cdot \|\varphi\| \leq R \end{aligned}$$

Hence  $T : \mathcal{C}' \rightarrow \mathcal{C}'$ . Using the same trick as before there is  $k_0$  such that  $T^{k_0}$  is contractive. Hence there exists a solution.  $\square$

**Remark.** We can often use the solution on  $[a, a+h]$ , and use local Picard to extend this (using  $[a+h, b]$  and  $y(a+h) = c_0$  for the DE). Sometimes this blows up, but often it is for a good reason.

**Remark.** We did this analysis for  $y : \mathbb{R} \rightarrow \mathbb{R}$ . We could have done something similar for  $y : \mathbb{R} \rightarrow \mathbb{R}^n$ .

**Remark.** We can modify higher order DEs to look like first order DEs with more parts.