Algebraic Diagonals and Asymptotics of Bivariate Generating Functions

Peiran Tao

Department of Combinatorics and Optimization
University of Waterloo

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Overview

1. Notation

2. Algebraic Generating Functions and Diagonals

3. Asymptotics of Bivariate Generating Functions

Notation

1. $\mathbb{K} =$ a field of characteristic zero (usually $\mathbb{R}$ or $\mathbb{C}$).
2. $\mathbb{K}[[z]] =$ the ring of formal power series over $\mathbb{K}$ in z.

$$\mathbb{K}[[z]] = \left\{ \sum_{n \geq 0} a_n z^n : a_n \in \mathbb{K} \right\}$$

3. $\mathbb{K}[[x, y]] =$ the ring of formal power series over $\mathbb{K}$ in x, y.

$$\mathbb{K}[[x, y]] = \left\{ \sum_{i,j \geq 0} a_{i,j} x^i y^j : a_{i,j} \in \mathbb{K} \right\}$$

I. Algebraic Generating Functions and Diagonals

Generating Functions

Given a combinatorial class $(\mathcal{A}, \omega)$, we can define its generating function

$$A(z) := \sum_{n \geq 0} a_n z^n$$

where $a_n :=$ the number of elements in $\mathcal{A}$ that have weight n.

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Example

Let $\mathcal{A}$ be the strings in $\{1, 2, 3\}$ that avoid 11 and 23. For example

$$1222132, \; 12, \; 132$$

The weight on $\mathcal{A}$ counts the number of 1. By the *transfer matrix method* we can show that

$$A(z) = \frac{1 + z}{1 - 2z - z^2 + z^3}$$

Algebraic Power Series

A formal power series $A(z) \in \mathbb{K}[[z]]$ is called **algebraic** if

$$P(z, A(z)) = 0$$

for some polynomial $P(z, y) \in \mathbb{K}[z, y]$.

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Example

Let $T(z)$ be the Catalan generating function, then

$$zT(z)^2 - T(z) + 1 = 0$$

So $P(z, T(z)) = 0$ for $P(z, y) = yz^2 - y + 1$.

Diagonals

Let $F(x,y) \in \mathbb{K}[[x,y]]$ be a bivariate formal power series, write

$$F(x,y) = \sum_{i,j \geq 0} f_{i,j} x^i y^j$$

For $d = (r,s) \in \mathbb{N}^2$ the **d-diagonal** of F is the univariate formal power series in $\mathbb{K}[[t]]$

$$(\Delta_d F)(t) := \sum_{n \geq 0} f_{nr,ns} t^n$$

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If $d = (1, 1)$ we say

$$(\Delta F)(t) := (\Delta_d F)(t) = \sum_{n \geq 0} f_{n,n} t^n$$

is the **main diagonal** of F.

Diagonals

Theorem

If $F(x,y) \in \mathbb{K}[[x,y]]$ *is a rational function then* $(\Delta F)(t)$ *is algebraic.*

In other word, there exists $P(t,y) \in \mathbb{K}[t,y]$ *such that* $P(t,(\Delta F)(t)) = 0$*.*

Diagonals

Theorem

If $F(x,y) \in \mathbb{K}[[x,y]]$ is a rational function then $(\Delta F)(t)$ is algebraic.
In other word, there exists $P(t,y) \in \mathbb{K}[t,y]$ such that $P(t,(\Delta F)(t)) = 0$.

Bostan et al. (2015) [2] developed an algorithm to efficiently compute this polynomial $P(t,y)$. We implemented this algorithm in SageMath.

Input: A rational function $F(x,y) \in \mathbb{K}[[x,y]]$.

Output: A polynomial $P(t,y) \in \mathbb{K}[t,y]$ such that $P(t,(\Delta F)(t)) = 0$.

Idea of the Algorithm

Fact 1. There is a set $\{\alpha_1(t), \ldots, \alpha_n(t)\}$ such that $(\Delta F)(t)$ is a sum of c elements from this set.

Each $\alpha_i(t)$ is an algebraic formal series in t determined by the "residues" of a certain function.

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Fact 1. There is a set $\{\alpha_1(t), \ldots, \alpha_n(t)\}$ such that $(\Delta F)(t)$ is a sum of c elements from this set.

Each $\alpha_i(t)$ is an algebraic formal series in t determined by the "residues" of a certain function.

Construct the polynomial

$$\Sigma(y, t) = \prod_{i_1 < \cdots < i_c} (y - (\alpha_{i_1}(t) + \cdots + \alpha_{i_c}(t)))$$

Fact 2. $\Sigma(y, t) \in \mathbb{K}[y, t]$. (Galois Theory)

Fact 1

Note that

$$
\begin{aligned}
(\Delta F)(t) &= \sum_{n\geq 0} f_{n,n} t^n \\
&= [y^{-1}] \sum_{n,m\geq 0} f_{n,m} t^n y^{m-n-1} \\
&= [y^{-1}] \frac{1}{y} F\left(\frac{t}{y}, y\right) \\
&= \sum_{\substack{y_i(t)\in\mathcal{P} \\ \mathrm{val}(y_i(t))>0}} \underbrace{\mathrm{Residue}\left(\frac{1}{y} F\left(\frac{t}{y}, y\right), y = y_i(t)\right)}_{\alpha_i}
\end{aligned}
$$

where $\mathcal{P} = \{y_1(t), \ldots, y_n(t)\}$ is the “pole set” of $\frac{1}{y}F(\frac{t}{y}, y)$.

$$c = \#\{y(t) \in \mathcal{P} \mid \mathrm{val}(y(t)) > 0\}$$

Algorithm

The algorithm consists of two steps.

1. Compute the residues $\{\alpha_1(t), \ldots, \alpha_n(t)\}$ using resultants.
2. Compute the polynomial $\Sigma(y, t)$.

An Example

Let $\mathcal{A}$ be the combinatorial class of bicolored supertrees, then $A(t)$ is the main diagonal of the rational function

$$\frac{P(x,y)}{Q(x,y)} = \frac{2x^2y(2x^5y^2 - 3x^3y + x + 2x^2y - 1)}{x^5y^2 + 2x^2y - 2x^3y + 4y + x - 2}$$

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P = 2*x^2*y*(2*x^5*y^2 - 3*x^3*y + x + 2*x^2*y - 1)
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In fact, we have $A(t) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4t + 4t\sqrt{1-4t}}.$

II. Asymptotics of Bivariate Generating Functions

Bivariate Generating Functions

Consider a rational bivariate generating function

$$F(x,y) = \frac{P(x,y)}{Q(x,y)} = \sum_{n,m \geq 0} f_{n,m} x^n y^m \in \mathbb{C}[[x,y]]$$

An example

Example

Let $b_{n,k}$ be the number of binary strings of length n and has k zeros

$$\begin{aligned} B(x,y) &= \sum_{n,k\geq 0} b_{n,k}x^n y^k \\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}y^k\right)x^n \\ &= \sum_{n\geq 0}(1+y)^n x^n \\ &= \frac{1}{1-x(1+y)} \end{aligned}$$

Asymptotics

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In general it is hard to find a closed form formula for $f_{n,m}$ for $n, m \geq 0$.

Instead, we try to **find the asymptotics** of the coefficient sequence of a diagonal $(\Delta_d F)(t)$ for some $d = (r, s) \in \mathbb{N}^2$.

That is, we want to find the asymptotics of the sequence

$$(f_{nr,ns})_{n\geq 0} = \{f_{0,0},\ f_{r,s},\ f_{2r,2s},\ \ldots\}$$

as $n \to \infty$.

Asymptotics

Assume $F = P/Q \in \mathbb{K}[[x, y]]$ is a rational function (hence $Q(0,0) \neq 0$)

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By the **Cauchy's Integral Formula**, for $\epsilon > 0$ small enough we have

$$\begin{aligned} f_{nr,ns} &= \frac{1}{(2\pi i)^2} \int_{T(\epsilon,\epsilon)} \frac{F(x,y)}{x^{rn+1} y^{sn+1}} \; \mathrm{d}x \; \mathrm{d}y \\ &= \frac{1}{(2\pi i)^2} \int_{T(\epsilon,\epsilon)} \underbrace{\frac{P(x,y)}{xyQ(x,y)} \cdot x^{-nr} y^{-ns} \mathrm{d}x \; \mathrm{d}y}_{\omega_F} \end{aligned} \tag{1}$$

where $T(\epsilon,\epsilon) = \{(x,y) \in \mathbb{C}^2 : |x| = |y| = \epsilon\}$.

Our goal is to estimate this integral (1).

Singular Variety

When we compute integrals, we are interested in the singularities.

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The function $F = P/Q$ has singularities (poles) at the zeros of Q.

$$\mathcal{V} := \mathcal{V}(Q) := \left\{ (x, y) \in \mathbb{C}^2 : Q(x, y) = 0 \right\}$$

is called the **singular variety** of F.

Estimate the integral

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1. Deform the torus $T(\epsilon, \epsilon)$ to another torus to lower the modulus of the integrand ω_F as much as possible.

2. Reduce the integral to a *residue integral* on some cycle $\mathcal{C}$.

3. Understand the *homology class* of $\mathcal{C}$. We want to find a representative

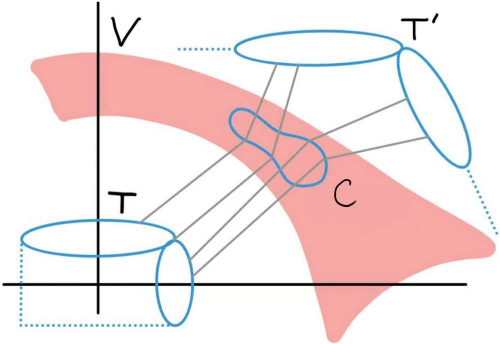
$$\kappa \in [\mathcal{C}] \in H_1(\mathcal{V})$$

   that is “good” (will explain this later).

1. Deformation of the Contour

Let $M > 0$ be large and let K be a homotopy from $T(\epsilon, \epsilon)$ to $T(\epsilon, M)$.

In other words, we fix x and enlarge y.



Source: Page 202 of [1]

2. Reduce to residue integral

The homotopy intersect the singular variety $\mathcal{V}(Q)$ at a cycle $\mathcal{C}$.

Let ν be a "tube" around $\mathcal{C}$, then

$$
\begin{aligned}
f_{n,n} &= \frac{1}{(2\pi i)^2}\int_{\nu}\omega_F + \frac{1}{(2\pi i)^2}\int_{T(\epsilon,M)}\omega_F \\
&= \frac{1}{(2\pi i)^2}\int_{\nu}\omega_F + O(M^{1-n}) \\
&= \frac{1}{2\pi i}\int_{\mathcal{C}}\mathrm{Res}(\omega_F) + O(M^{1-n})
\end{aligned}
$$

Here $\mathrm{Res}(\omega_F)$ is a 1-form and $\mathcal{C}$ is a 1-cycle in $H_1(\mathcal{V})$.

3. The homology class of $\mathcal{C}$

We want to find a good cycle $\kappa \in [\mathcal{C}]$ that makes the calculation easy.

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Note that

$$\omega_F = \frac{P(x,y)}{xyQ(x,y)} \cdot x^{-nr} y^{-ns} \, \mathrm{d}x \, \mathrm{d}y$$

$$= \frac{P(x,y)}{xyQ(x,y)} \cdot e^{nH(x,y)} \, \mathrm{d}x \, \mathrm{d}y$$

where $H : \mathbb{C}_*^2 \to \mathbb{C}$ is the multi-valued function defined by

$$H(x,y) = -r\log(x) - s\log(y)$$

and $\mathbb{C}_* := \mathbb{C} \setminus \{0\}$ is the set of nonzero complex numbers.

Height function

The real part of H is the function $h = \mathrm{Re}(H) : \mathbb{C}_*^2 \to \mathbb{R}$ by

$$h(x, y) = -r \log |x| - s \log |y|$$

The function $h|_{\mathcal{V}}$ is called a **height function** on $\mathcal{V}$. By applying ideas of *Morse theory*, we will use this function to study the variety $\mathcal{V}$.

Components

For $M > 0$ we define

$$\mathcal{V}^{>M} := \{(x,y) \in \mathcal{V} : h(x,y) > M\}$$

Let $\mathcal{V}^{>M} = R_1 \cup \cdots \cup R_n$ be the connected components. Define

$$X^{>M} := \{R_i : \forall \epsilon > 0,\ \exists (x,y) \in R_i \text{ such that } |x| < \epsilon\}$$
$$Y^{>M} := \{R_i : \forall \epsilon > 0,\ \exists (x,y) \in R_i \text{ such that } |y| < \epsilon\}$$

Each $R_i \in X^{>M}$ is called a x-**component**.

Each $R_i \in Y^{>M}$ is called a y-**component**.

Critical Points

We say $\sigma = (x_0, y_0) \in \mathcal{V}$ is a **critical point** or **saddle point** of $h|_{\mathcal{V}}$ if

$$\nabla H(\sigma) \parallel \nabla Q(\sigma)$$

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$$\nabla H(\sigma) \parallel \nabla Q(\sigma)$$

This is equivalent to

$$\det \begin{pmatrix} \nabla H(\sigma) \\ \nabla Q(\sigma) \end{pmatrix} = \det \begin{pmatrix} \frac{-r}{x_0} & \frac{-s}{y_0} \\ Q_x(x_0, y_0) & Q_y(x_0, y_0) \end{pmatrix} = 0$$

Critical Points

To find the critical points we can solve the following system

$$\begin{aligned} Q(x,y) &= 0 \\ \frac{-r}{x} Q_y(x,y) + \frac{s}{y} Q_x(x,y) &= 0 \end{aligned}$$

This can be done using *Gröbner bases*.

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This can be done using *Gröbner bases*.

Let $\Sigma = \{\sigma \in \mathcal{V} : \sigma \text{ is a critical point}\}$ be the set of all critical points.

We say $c \in \mathbb{R}$ is a **critical value** if $c = h(\sigma)$ for some $\sigma \in \Sigma$.

Assumptions

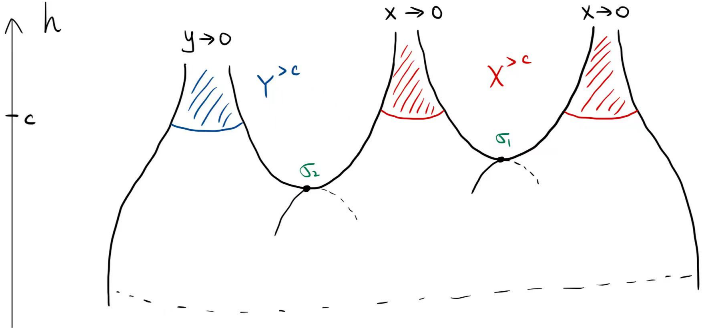
1. We assume Σ is finite.

2. We assume $\mathcal{V}$ is smooth. That is, $\nabla Q(p) \neq 0$ for all $p \in \mathcal{V}$.

Critical Points

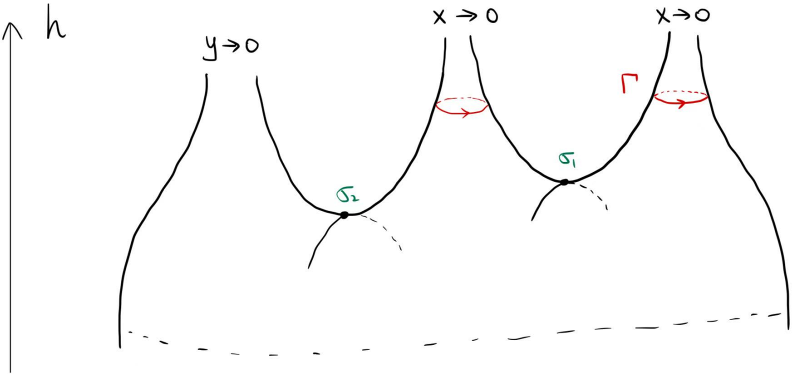
We can use the height function to visualize the variety $\mathcal{V}$.

Note that $h(x, y) \to \infty$ if $x \to 0$ or $y \to 0$.



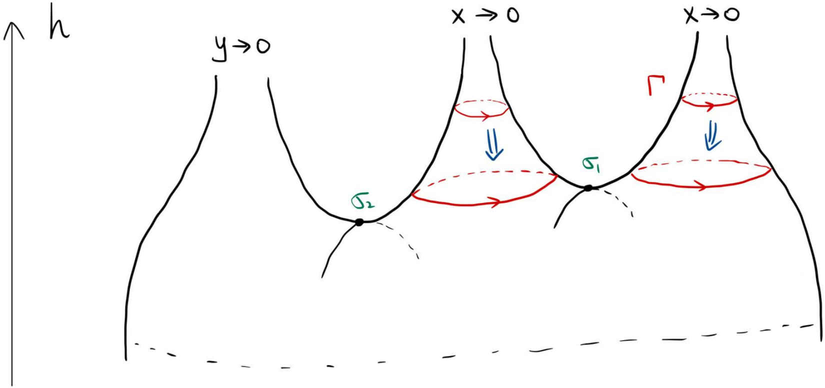
3. The homology class of $\mathcal{C}$

Fact: The cycle $\mathcal{C}$ is homogolous to a cycle Γ, which consists of disjoint cycles, one in each x-component.



3. The homology class of $\mathcal{C}$

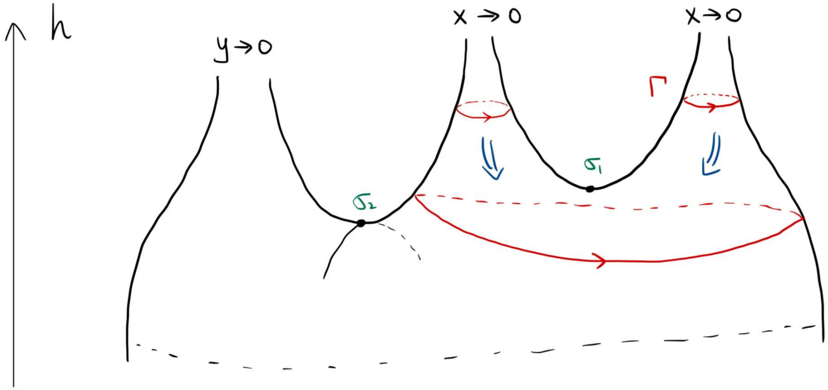
We can push the cycle Γ down



3. The homology class of $\mathcal{C}$

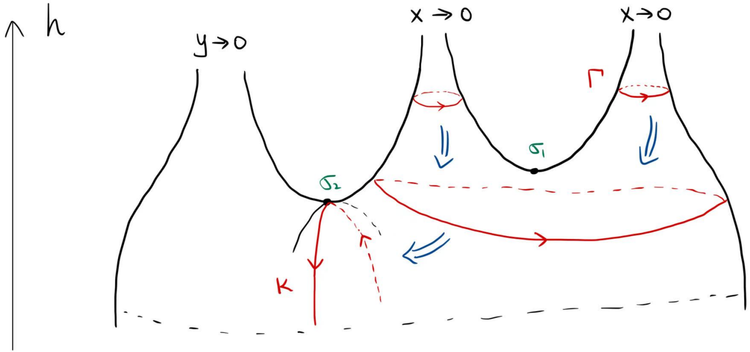
In this case, the two cycles "merge" to one bigger cycle.

Topologically, this is done by "attaching" some disks, which does not change the homology class.



3. The homology class of $\mathcal{C}$

We keep pushing the cycle down, until it get "stuck" at a critical point.



This is the desired cycle κ. This is good because we can apply the *saddle point method* near this saddle point.

The idea

Which critical points do the cycles get stucked at?

It depends on the components "near" this critical point.

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It depends on the components “near” this critical point.

If all components near it are x-components, then all the cycles merge.

If one of the components is a y-component, then the cycle gets stuck.

The idea

Which critical points do the cycles get stucked at?

It depends on the components “near” this critical point.

If all components near it are x-components, then all the cycles merge.

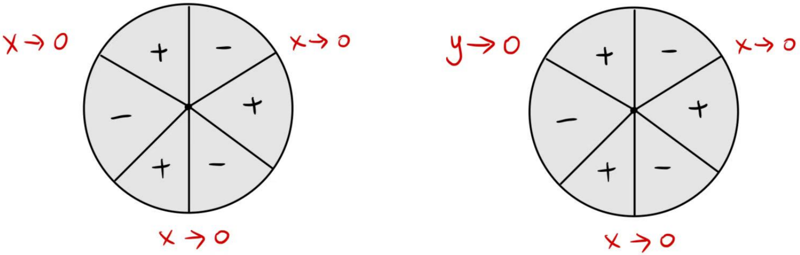
If one of the components is a y-component, then the cycle gets stuck.

The critical points with the largest height such that the cycles get stuck dominate the asymptotics.

Some possible cases

In the above picture, there are only two components "near" σ_1 and σ_2.

Sometimes there are more components.



The algorithm

DeVries (2011) [3] developed an algorithm to find this cycle κ.

We are trying to improve the algorithm and make it practical.

Step 1

Define the set $\mathbb{W} = \emptyset$ and let $c = -\infty$.

List the critical value in order of decreasing height $\sigma_1, \ldots, \sigma_n$ so

$$h(\sigma_1) \geq \cdots \geq h(\sigma_n)$$

Step 2

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(b). For each C_j, follow an ascending path from σ_i to C_j and check if C_j is an x-component or y-component.

(c). If one of C_j is a y-component, add σ_i to the set $\mathbb{W}$ and let $c = h(\sigma_i)$. Then go to Step 3.

Step 3

Perform Step 2 to each σ_i such that $h(\sigma_i) = c$.

This is because if we already found one $\sigma \in \mathbb{W}$, then all the other critical points with lower height do not matter. Thus we iterate through the other critical points with this height c.

Conclusion

The algorithm ends with a set $\mathbb{W}$ and $c \in \mathbb{R}$.

If $\mathbb{W} = \emptyset$ then $c = -\infty$, so the asymptotics decay super-exponentially.

Otherwise, $\mathbb{W}$ is the set of **contributing points** that dominate the asymptotics.

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Thank you!