Algebraic Diagonals and Asymptotics of Bivariate Generating Functions

Peiran Tao

Department of Combinatorics and Optimization
University of Waterloo

July 29th, 2025

Overview

1. Notation
2. Algebraic Generating Functions and Diagonals
3. Asymptotics of Bivariate Generating Functions

Notation

1. $\mathbb{K}$ = a field of characteristic zero (usually $\mathbb{R}$ or $\mathbb{C}$).
2. $\mathbb{K}[[z]]$ = the ring of formal power series over $\mathbb{K}$ in z.

$$\mathbb{K}[[z]] = \left\{ \sum_{n \geq 0} a_n z^n : a_n \in \mathbb{K} \right\}$$

3. $\mathbb{K}[[x, y]]$ = the ring of formal power series over $\mathbb{K}$ in x, y.

$$\mathbb{K}[[x, y]] = \left\{ \sum_{i,j \geq 0} a_{i,j} x^i y^j : a_{i,j} \in \mathbb{K} \right\}$$

I. Algebraic Generating Functions and Diagonals

Generating Functions

Given a combinatorial class $(\mathcal{A}, \omega)$, we can define its generating function

$$A(z) := \sum_{n \geq 0} a_n z^n$$

where $a_n :=$ the number of elements in $\mathcal{A}$ that have weight n.

Generating Functions

Given a combinatorial class $(\mathcal{A}, \omega)$, we can define its generating function

$$A(z) := \sum_{n \geq 0} a_n z^n$$

where $a_n :=$ the number of elements in $\mathcal{A}$ that have weight n.

Example

Let $\mathcal{A}$ be the strings in $\{1, 2, 3\}$ that avoid 11 and 23. For example

$$1222132, \; 12, \; 132$$

The weight on $\mathcal{A}$ counts the number of 1. By the *transfer matrix method* we can show that

$$A(z) = \frac{1+z}{1-2z-z^2+z^3}$$

Algebraic Power Series

A formal power series $A(z) \in \mathbb{K}[[z]]$ is called **algebraic** if

$$P(z, A(z)) = 0$$

for some polynomial $P(z, y) \in \mathbb{K}[z, y]$.

Algebraic Power Series

A formal power series $A(z) \in \mathbb{K}[[z]]$ is called **algebraic** if

$$P(z, A(z)) = 0$$

for some polynomial $P(z, y) \in \mathbb{K}[z, y]$.

Example

Let $T(z)$ be the Catalan generating function, then

$$zT(z)^2 - T(z) + 1 = 0$$

So $P(z, T(z)) = 0$ for $P(z, y) = yz^2 - y + 1$.

Diagonals

Let $F(x, y) \in \mathbb{K}[[x, y]]$ be a bivariate formal power series, write

$$F(x, y) = \sum_{i,j \geq 0} f_{i,j} x^i y^j$$

The **diagonal** of F is the univariate formal power series in $\mathbb{K}[[t]]$

$$(\Delta F)(t) := \sum_{n \geq 0} f_{n,n} t^n$$

Diagonals

Theorem

If $F(x, y) \in \mathbb{K}[[x, y]]$ *is a rational function then* $(\Delta F)(t)$ *is algebraic.*

In other word, there exists $P(t, y) \in \mathbb{K}[t, y]$ *such that* $P(t, \Delta F(t)) = 0$.

Diagonals

Theorem

If $F(x,y) \in \mathbb{K}[[x,y]]$ *is a rational function then* $(\Delta F)(t)$ *is algebraic.*

In other word, there exists $P(t,y) \in \mathbb{K}[t,y]$ *such that* $P(t, \Delta F(t)) = 0$.

Bostan et al. developed an algorithm to efficiently compute $P(t,y)$. We implemented this algorithm in SageMath.

Input: A rational function $F(x,y) \in \mathbb{K}[[x,y]]$.

Output: A polynomial $P(t,y) \in \mathbb{K}[t,y]$ such that $P(t, \Delta F(t)) = 0$.

Idea of the Algorithm

Fact 1. There is a set $\{\alpha_1(t), \ldots, \alpha_n(t)\}$ such that $\Delta F(t)$ is a sum of c elements from this set.

Each $\alpha_i(t)$ is an algebraic formal series in t determined by the "residues" of a certain function.

Idea of the Algorithm

Fact 1. There is a set $\{\alpha_1(t), \ldots, \alpha_n(t)\}$ such that $\Delta F(t)$ is a sum of c elements from this set.

Each $\alpha_i(t)$ is an algebraic formal series in t determined by the "residues" of a certain function.

Construct the polynomial

$$\Sigma(y,t) = \prod_{i_1 < \cdots < i_c} (y - (\alpha_{i_1}(t) + \cdots + \alpha_{i_c}(t)))$$

Fact 2. $\Sigma(y,t) \in \mathbb{K}[y,t]$. (Galois Theory)

Algorithm

The algorithm consists of two steps.

1. Compute the residues $\{\alpha_1(t), \dots, \alpha_n(t)\}$ using resultants.
2. Compute the polynomial $\Sigma(y, t)$.

II. Asymptotics of Bivariate Generating Functions

Bivariate Generating Functions

Consider a rational bivariate generating function

$$F(x,y) = \frac{P(x,y)}{Q(x,y)} = \sum_{n,m \geq 0} f_{n,m} x^n y^m \in \mathbb{C}[[x,y]]$$

Bivariate Generating Functions

Consider a rational bivariate generating function

$$F(x,y) = \frac{P(x,y)}{Q(x,y)} = \sum_{n,m\geq 0} f_{n,m} x^n y^m \in \mathbb{C}[[x,y]]$$

Example

Let $b_{n,k}$ be the number of binary strings of length n and has k zeros

$$B(x,y) = \sum_{n,k\geq 0} b_{n,k} x^n y^k = \sum_{n\geq 0} \left(\sum_{k=0}^{n} \binom{n}{k} y^k \right) x^n$$

Asymptotics

It is hard to find a closed form formula for $f_{n,m}$ for $n, m \geq 0$, instead we try to **find the asymptotics** of the diagonal sequence $(f_{n,n})$.

Asymptotics

It is hard to find a closed form formula for $f_{n,m}$ for $n, m \geq 0$, instead we try to **find the asymptotics** of the diagonal sequence $(f_{n,n})$.

Assume $F = P/Q \in \mathbb{K}[[x, y]]$ is a rational function (hence $Q(0, 0) \neq 0$)

Asymptotics

It is hard to find a closed form formula for $f_{n,m}$ for $n, m \geq 0$, instead we try to **find the asymptotics** of the diagonal sequence $(f_{n,n})$.

Assume $F = P/Q \in \mathbb{K}[[x, y]]$ is a rational function (hence $Q(0, 0) \neq 0$)

By the **Cauchy's Integral Formula**, for $\epsilon > 0$ small enough we have

$$\begin{aligned} f_{n,n} &= \frac{1}{(2\pi i)^2} \int_{T(\epsilon,\epsilon)} \frac{F(x, y)}{x^{n+1} y^{n+1}} \, \mathrm{d}x \, \mathrm{d}y \\ &= \frac{1}{(2\pi i)^2} \int_{T(\epsilon,\epsilon)} \underbrace{\frac{P(x, y)}{Q(x, y)} \cdot \frac{\mathrm{d}x \, \mathrm{d}y}{x^{n+1} y^{n+1}}}_{\omega} \end{aligned}$$

where $T(\epsilon, \epsilon) = \{(x, y) \in \mathbb{C}^2 : |x| = |y| = \epsilon\}$.

Singularities

When we compute integrals, we are interested in the singularities.

Singularities

When we compute integrals, we are interested in the singularities.

The function $F = P/Q$ has singularities (poles) at the zeros of Q.

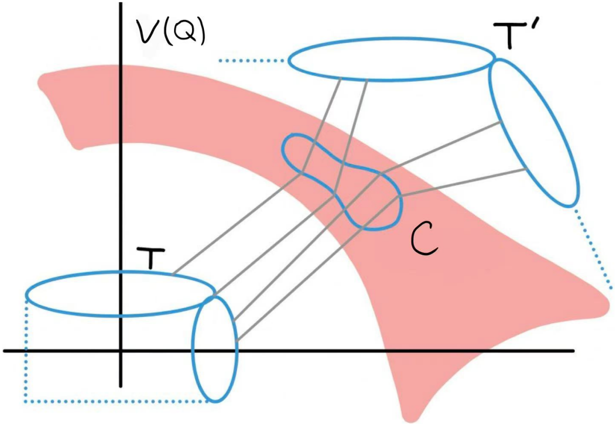
$$\mathcal{V}(Q) = \left\{(x, y) \in \mathbb{C}^2 : Q(x, y) = 0\right\}$$

is called the **singular variety** of F.

Deformation of the Contour

Let $M > 0$ be large and let K be a homotopy from $T(\epsilon, \epsilon)$ to $T(\epsilon, M)$.

In other words, we fix x and enlarge y.



Deformation of the Contour

The homotopy intersect the singular variety $\mathcal{V}(Q)$ at a cycle $\mathcal{C}$.

We then have

$$\begin{aligned} f_{n,n} &= \frac{1}{(2\pi i)^2} \int_{\mathcal{C}} \omega + \frac{1}{(2\pi i)^2} \int_{T(\epsilon,M)} \omega \\ &= \frac{1}{(2\pi i)^2} \int_{\mathcal{C}} \omega + O(M^{1-n}) \\ &= \frac{1}{2\pi i} \int_{\mathcal{N}} \mathrm{Res}(\omega) + O(M^{1-n}) \end{aligned}$$

where $\mathcal{N} = \alpha_1\gamma_1 + \cdots + \alpha_r\gamma_r$ is a sum of cycles in $\mathbb{C}$.

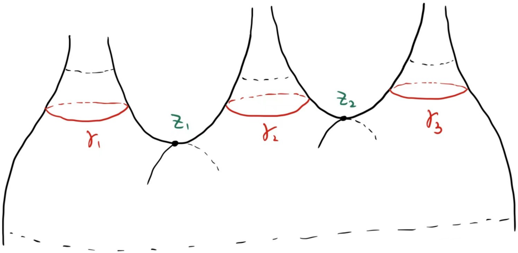
Determine the contributing points

Therefore

$$\frac{1}{2\pi i}\int_{\mathcal{N}} \mathrm{Res}(\omega) = \sum_{i=1}^{r} \alpha_i \int_{\gamma_i} \mathrm{Res}(\omega)$$

DeVries developed an algorithm to determine which cycles γ_i contribute the most to the integral, and thus determines the asymptotics of $f_{n,n}$.

We are working on to imporve the algorithm.



Thank you!