

PMATH 351 Notes

Real Analysis

Winter 2025

Based on Professor Kevin Hare's Lectures

Contents

1	Metric Spaces	4
1.1	Normed Vector Spaces	4
1.2	Metric Spaces	6
1.3	Topology of Metric Spaces	8
1.4	Continuous Functions	11
1.5	Finite dimensional normed vector spaces	13
1.6	Completeness	14
1.7	Completeness of \mathbb{R}	17
1.8	Limits of continuous functions	19
2	More Metric Topology	22
2.1	Compactness	22
2.2	Countable and Uncountable Sets	29
2.3	Compactness and Continuity	30
2.4	Cantor Set	31
2.5	Compact sets in $\mathcal{C}(X)$	32
2.6	Connectedness	35
2.7	Bonus Cantor Set Stuff	37
3	Completeness	42
3.1	Baire Category Theorem	42
3.2	Nowhere Differentiable Functions	44
3.3	Contraction Mapping Principle	47
3.4	Newton's Method	50
3.5	Metric Completion	50
3.6	The Real Numbers	54
3.7	The p -adic Numbers	58
4	Approximation Theory	61
4.1	Polynomial Approximation	61
4.2	Stone-Weierstrass Theorem	64
4.3	Best Approximation	67

5	Differential Equations	68
5.1	Global Solutions of ODEs	70
5.2	Local Solutions	70

1 Metric Spaces

1.1 Normed Vector Spaces

Definition. Let V be a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We say $\|\cdot\| : V \rightarrow \mathbb{R}$ is a **norm** if:

- (i). For all $v \in V$ we have $\|v\| = 0 \iff v = 0$.
- (ii). For all $v \in V$ and $\lambda \in \mathbb{K}$ we have $\|\lambda v\| \leq |\lambda| \|v\|$.
- (iii). For all $v, w \in V$ we have $\|v + w\| \leq \|v\| + \|w\|$.

A vector space, combined with a norm, is called a **normed vector space**.

Example. Let $V = \mathbb{R}^n$. Define a map $\|\cdot\|_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ by:

$$\|v\|_1 = \|(x_1, \dots, x_n)\|_1 = |x_1| + \dots + |x_n|$$

Clearly property 1 and 2 holds. To see property 3 we have:

$$\begin{aligned} \|(x_1, \dots, x_n) + (y_1, \dots, y_n)\|_1 &= |x_1 + y_1| + \dots + |x_n + y_n| \\ &\leq |x_1| + |y_1| + \dots + |x_n| + |y_n| \quad (\triangle \text{ inequality in } \mathbb{R}) \\ &= \|(x_1, \dots, x_n)\|_1 + \|(y_1, \dots, y_n)\|_1 \end{aligned}$$

Hence $\|\cdot\|_1$ defines a norm on $V = \mathbb{R}^n$.

Example. Let $V = \mathbb{R}^n$ again. Define $\|\cdot\|_\infty : \mathbb{R}^n \rightarrow \mathbb{R}$ by:

$$\|v\|_\infty = \|(x_1, \dots, x_n)\|_\infty = \max(|x_1|, \dots, |x_n|)$$

This also defines a norm on \mathbb{R}^n .

Example. What does the unit ball $B = \{v \in V : \|v\| \leq 1\}$ look like? Take $V = \mathbb{R}^2$.

Note. It is possible to extend these two norms to infinite dimensional vector spaces if we are being careful. Both of the norms above are examples of p -norms, for $1 \leq p \leq \infty$.

Example. Let $V = \mathbb{R}[x]$ be a vector over \mathbb{R} . Define $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on V by:

$$\|f\|_1 = \int_0^1 |f(x)| \, dx \quad \text{and} \quad \|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$$

The three properties are satisfied by these two norms. Note these norms can be defined beyond polynomials if we are careful.

Theorem 1.1 (Minkowski). Let $1 \leq p < \infty$ be a real number.

(i). We define:

$$\ell_p = \left\{ (x_n)_{n=1}^\infty \subseteq \mathbb{C} : \left(\sum_{n=1}^\infty |x_n|^p \right)^{1/p} < \infty \right\}$$

Then the map $\|\cdot\|_p : \ell_p \rightarrow \mathbb{R}$ defined by:

$$\|(x_n)\|_p := \left(\sum_{n=1}^\infty |x_n|^p \right)^{1/p}$$

defines a norm on ℓ_p . This is called the ℓ_p -space.

(ii). Let $\mathcal{C}[a, b]$ be the set of continuous functions on $[a, b]$. Then:

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{1/p}$$

defines a norm. Define $L^p[a, b] = \{f \in \mathcal{C}[a, b] : \|f\|_p < \infty\}$, called the L^p -space.

Proof. Note for $p \geq 1$, define a map $\varphi(x) = |x|^p$ and φ is convex on \mathbb{R} . We will prove part 2 first. Assume $f, g \in \mathcal{C}[a, b]$ and $f, g \neq 0$. If $f = 0$ or $g = 0$ the triangle inequality is easy to prove.

$$\begin{aligned} \|f + g\|_p^p &= \int_a^b |f(x) + g(x)|^p dx = \int_a^b \left| \|f\|_p \cdot \frac{f}{\|f\|_p} + \|g\|_p \cdot \frac{g}{\|g\|_p} \right|^p dx \\ &= (\|f\|_p + \|g\|_p)^p \int_a^b \left| \underbrace{\frac{\|f\|_p}{\|f\|_p + \|g\|_p}}_{\alpha} \cdot \frac{f}{\|f\|_p} + \underbrace{\frac{\|g\|_p}{\|f\|_p + \|g\|_p}}_{1-\alpha} \cdot \frac{g}{\|g\|_p} \right|^p dx \end{aligned}$$

Note that $\alpha \in [0, 1]$, we can rewrite the above quantity as:

$$\begin{aligned} I &:= (\|f\|_p + \|g\|_p)^p \int_a^b \left| \alpha \cdot \frac{f}{\|f\|_p} + (1 - \alpha) \cdot \frac{g}{\|g\|_p} \right|^p dx \\ &= (\|f\|_p + \|g\|_p)^p \int_a^b \varphi \left(\alpha \cdot \frac{f}{\|f\|_p} + (1 - \alpha) \cdot \frac{g}{\|g\|_p} \right)^p dx \end{aligned}$$

Recall $\varphi(x) = |x|^p$ is convex, we have:

$$\begin{aligned} I &\leq (\|f\|_p + \|g\|_p)^p \left(\alpha \int_a^b \left| \frac{f}{\|f\|_p} \right|^p dx + (1 - \alpha) \int_a^b \left| \frac{g}{\|g\|_p} \right|^p dx \right) \\ &= (\|f\|_p + \|g\|_p)^p (\alpha + 1 - \alpha) = (\|f\|_p + \|g\|_p)^p \end{aligned}$$

This proved that:

$$\|f + g\|_p^p \leq (\|f\|_p + \|g\|_p)^p \implies \|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Part 1 (ℓ_p -space) are proved in the similar way by replacing integral with sum. \square

Lecture 2, 2025/01/08

1.2 Metric Spaces

Definition. Let X be a non-empty set. A **distance (metric)** on X is a function $d : X \times X \rightarrow [0, \infty)$ such that:

- (i). For all $x, y \in X$ we have $d(x, y) = 0 \iff x = y$.
- (ii). For all $x, y \in X$ we have $d(x, y) = d(y, x)$.
- (iii). For all $x, y, z \in X$ we have $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (X, d) is called a **metric space**. We just say X is a metric space if d is understood.

Example. Let $(X, \|\cdot\|)$ be a normed vector space, then $d(x, y) = \|x - y\|$ is a metric on X . Clearly $d(x, y) = 0 \iff \|x - y\| = 0 \iff x - y = 0 \iff x = y$. Property (ii) is also trivial. For property (iii) we have:

$$d(x, z) = \|x - z\| = \|x - y + y - z\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z)$$

Example (Graph metric). Let (X, E) be a graph where X is the vertex set. The set of **paths** from x to y is:

$$P_{xy} = \{(x = x_1, x_2, \dots, x_n = y) : (x_i, x_{i+1}) \in E\}$$

Define a **weight** function $\omega : E \rightarrow (0, \infty)$. Then:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ \min\{\omega(x_1, x_2) + \dots + \omega(x_{n-1}, x_n) \text{ for } (x_1, \dots, x_n) \in P_{xy}\} & \text{otherwise} \end{cases}$$

This distance basically measures the shortest path from x to y , with weight on the edge.

Example (Trivial metric). Let X be a non-empty set, define:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Exercise: It is easy to verify that this is a distance function on X .

Example (p -adic metric on \mathbb{Q}). Let p be a fixed prime in \mathbb{N} . By unique factorization, every $q \in \mathbb{Q}$ can be uniquely written as:

$$q = p^n \frac{a}{b}$$

where $n \in \mathbb{Z}$ and $a \in \mathbb{Z}$ and $0 \neq b \in \mathbb{N}$ with $\gcd(a, b) = 1$. Define the **p -adic norm** by:

$$|q|_p = \begin{cases} p^{-n} & \text{if } q \neq 0 \text{ and } n \text{ is from above} \\ 0 & \text{if } q = 0 \end{cases}$$

Exercise: For $q, r \in \mathbb{Q}$ we have:

$$|q + r|_p \leq \max\{|q|_p, |r|_p\} \leq |q|_p + |r|_p$$

Take $p = 3$ and $q = 1/6$ and $r = 2/9$, then:

$$\begin{aligned} |q|_3 &= \left| 3^{-1} \cdot \frac{1}{2} \right|_3 = 3^{-(-1)} = 3 \\ |r|_3 &= \left| 3^{-2} \cdot \frac{2}{1} \right|_3 = 3^{-(-2)} = 9 \\ |q + r|_3 &= \left| \frac{3+4}{18} \right|_3 = \left| 3^{-2} \cdot \frac{7}{2} \right|_3 = 9 = \max\{3, 9\} \end{aligned}$$

Define the **p -adic metric** on \mathbb{Q} by:

$$d_p(q, r) = |q - r|_p$$

Exercise: This clearly defined a metric on \mathbb{Q} .

Example. Consider $\{0, 1\}^{\mathbb{N}} = \{(b_n)_{n=1}^{\infty} : b_n \in \{0, 1\}\}$. Take $b, c \in \mathbb{N}$ then define:

$$d(b, c) := \begin{cases} 0 & \text{if } b = c \\ \frac{1}{2^n} & \text{for } b = \min\{i \in \mathbb{N} : b_i \neq c_i\} \quad \text{otherwise} \end{cases}$$

Exercise: d is a metric on $\{0, 1\}^{\mathbb{N}}$, we may call this product metric. Now we define:

$$\rho(b, c) = \sum_{n=1}^{\infty} \frac{d(b_n, c_n)}{2^n} \quad (\text{always converges})$$

Fact (Exercise): $d(b, c) \leq \rho(b, c) \leq 2d(b, c)$.

Definition. Let (X, d) be a metric space. If $\emptyset \neq Y \subseteq X$, we make Y a metric space by defining $d_Y : Y \times Y \rightarrow \mathbb{R}$ by $d_Y(x, y) = d(x, y)$ for $x, y \in Y$. [This is just the restriction $d|_{Y \times Y}$] This is called the **relativized metric** on Y .

Definition. Let X be a non-empty set and d_1, d_2 be metrics on X . We say d_1 is **equivalent** to d_2 if there exist $c, C > 0$ such that:

$$cd_1(x, y) \leq d_2(x, y) \leq Cd_1(x, y)$$

for all $x, y \in X$. Exercise: This is an equivalence relation on the set of metrics on X .

Example. Let $X = \mathbb{R}^n$ and $1 \leq p < \infty$. Define a metric:

$$d_p(x, y) = \|x - y\|_p = \left(\sum_{k=1}^n |x_k - y_k|^p \right)^{1/p}$$

and define $d_\infty(x, y) = \max\{|x_k - y_k| : k \in \{1, \dots, n\}\}$. Let $x \in \mathbb{R}^n$, say $\|x\|_\infty = x_j$ for some j . Then we note that:

$$\|x\|_\infty = |x_j| = (|x_j|^p)^{1/p} \leq \left(\sum_{k=1}^n |x_k|^p \right)^{1/p} = \|x\|_p \leq \left(\sum_{k=1}^n |x_j|^p \right)^{1/p} = n^{1/p} \|x\|_\infty$$

To summarize we have:

$$\|x\|_\infty \leq \|x\|_p \leq n^{1/p} \|x\|_\infty$$

Hence $\|\cdot\|_\infty$ and $\|\cdot\|_p$ are equivalent norms for all $1 \leq p < \infty$. By equivalence, $\|\cdot\|_p$ are all equivalent norms on \mathbb{R}^n for $1 \leq p \leq \infty$.

Lecture 3, 2025/01/10

1.3 Topology of Metric Spaces

Definition. Let (X, d) be a metric space. Take $x \in X$ and $r > 0$. Define an **open ball** centered at x with radius r to be:

$$B_r(x) := b_r(x) := B(x, r) := \{y \in X : d(x, y) < r\}$$

Similarly we define a **closed ball** as:

$$\overline{B}_r(x) := \overline{b}_r(x) := \overline{B}(x, r) := \{y \in X : d(x, y) \leq r\}$$

Definition. Let (X, d) be a metric space. Let $N \subseteq X$ with some $x \in X$. We say N is a **neighborhood** of x if there exists $r > 0$ such that $B_r(x) \subseteq N$.

Definition. Let (X, d) be a metric space. We say $N \subseteq X$ is **open** if N is a neighborhood of x for all $x \in N$. We say N is **closed** if $X \setminus N$ is open.

Example. Let $X = \mathbb{R}$ with usual Euclidean metric. Then (a, b) is open for all $a < b$ in \mathbb{R} . The empty set \emptyset and \mathbb{R} are open.

Remark. In general, in a metric space (X, d) , the set X is trivially open and the empty set \emptyset is vacuously open. Note that $X \setminus X = \emptyset$ and $X \setminus \emptyset = X$. Hence X, \emptyset are both open and closed.

Example. Let (X, d) be a metric space where d is the discrete metric. Every subset $N \subseteq X$ is open! Why? Take $r = 1/2$ and $x \in N$, then $B_{1/2}(x) = \{x\} \subseteq N$. Similarly every subset is closed.

Question: Consider the metric space (\mathbb{Q}, d_3) , where d_3 is the 3-adic metric. What do the open sets look like?

Theorem 1.2 (Union of Open sets). Let (X, d) be a metric space. Let $\{X_i\}_{i \in I}$ be a collection of open sets, then $\bigcup_{i \in I} X_i$ is an open set.

Proof. Let $x \in \bigcup_{i \in I} X_i$, then $x \in X_{i_0}$ for some $i_0 \in I$. Since X_{i_0} is open, there is $r > 0$ such that $B_r(x) \subseteq X_{i_0}$. Hence:

$$B_r(x) \subseteq X_{i_0} \subseteq \bigcup_{i \in I} X_i$$

It follows that $\bigcup_{i \in I} X_i$ is open, as desired. \square

Corollary 1.3 (Intersection of Closed sets). Let (X, d) be a metric space and $\{X_i\}_{i \in I}$ a collection of closed sets. Then $\bigcap_{i \in I} X_i$ is closed.

Proof. Take complement using De Morgan's Law and apply the above theorem. \square

Question: If $\{X_i\}_{i \in I}$ is a collection of open sets, what can we say about $\bigcap_{i \in I} X_i$?

(i). If $|I| = n < \infty$, then this intersection is open. Consider $\{X_1, \dots, X_n\}$. Take $x \in \bigcap_{i=1}^n X_i$, then $x \in X_i$ for all i , so there is $r_i > 0$ such that $B(x, r_i) \subseteq X_i$ for all i . Take $r = \min\{r_1, \dots, r_n\}$, then $B(x, r) \subseteq \bigcap_{i=1}^n X_i$, hence open.

(ii). If $|I| > |\mathbb{N}|$, this may fail. For example, take $X_n = (-\frac{1}{n}, \frac{1}{n})$. Then $\bigcap_{n=1}^{\infty} X_n = \{0\}$, not open.

Proposition 1.4. Finite intersection of open sets is open and finite union of closed sets is closed.

Definition. Let (X, d) be a metric sapce and $(x_n)_{n=1}^{\infty}$ be a sequence in X . Let $x \in X$. We say the sequence $(x_n)_{n=1}^{\infty}$ **converges** to x if $\lim_{n \rightarrow \infty} d(x, x_n) = 0$. Equivalently, for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$:

$$n \geq N \implies d(x, x_n) < \epsilon$$

In this case we can write $\lim_{n \rightarrow \infty} x_n = x$.

Example. Let $X = \mathbb{Q}$ with Euclidean metric. Let $(x_n)_{n=1}^\infty$ be $x_n = 1/n$. This converges to 0.

Example. Let $X = \mathbb{Q}$, consider the sequence defined by:

$$x_n = \text{truncation of } \pi \text{ to the } n\text{-th decimal place}$$

For example $x_1 = 3$, $x_2 = 3.1$, $x_3 = 3.14$ and so on. This sequence “converges” to π , but $\pi \notin \mathbb{Q}$ so this sequence does not converge in \mathbb{Q} ! It converges in \mathbb{R} .

Example. Let (X, d) with the discrete metric. A sequence (a_n) is convergent if and only if it is eventually constant. That is, there is $N \in \mathbb{N}$ such that $x_n = X_N$ for all $n \geq N$. In this case the limit is just $\lim_{n \rightarrow \infty} x_n = X_N$.

Example. Consider (\mathbb{Q}, d_3) , the 3-adic metric. Consider the two sequences:

$$(x_n)_{n=1}^\infty = \left(\frac{1}{n}\right)_{n=1}^\infty \quad \text{and} \quad (y_n)_{n=1}^\infty \quad \text{by} \quad y_n = \begin{cases} 2 & \text{if } n = 1 \\ 2 + 3y_{n-1} & \text{if } n \geq 2 \end{cases}$$

For the first sequence (x_n) , it has a subsequence $(3^{-k})_{k=1}^\infty$ and $d(0, 3^{-k}) = 3^k \rightarrow \infty$. Hence (x_n) does not converge (We defer the actual proof of this when we see Cauchy sequence). For the second sequence, we see that:

$$y_n = 2 + 2 \cdot 3 + 2 \cdot 3^2 + \cdots + 2 \cdot 3^{n-1} = 3^n - 1$$

Hence $d(-1, y_n) = \| -3^n \|_3 = \frac{1}{3^n} \rightarrow 0$ so that $\lim_{n \rightarrow \infty} y_n = -1$.

Lecture 4, 2025/01/13

Definition. Let (X, d) be a metric space and $A \subseteq X$. The **closure** of A in X is the smallest closed set in X that contains A . We denote the closure of A by $\text{cl}(A)$ or \bar{A} . In other word, \bar{A} is the intersection of all closed sets that contain A .

Example. The closure of a closed set is itself.

Example. Consider the metric space (\mathbb{R}, d) with the usual Euclidean metric. Then $\bar{\mathbb{Q}} = \mathbb{R}$.

Example. Consider \mathbb{R} again with discrete metric, then $\bar{\mathbb{Q}} = \mathbb{Q}$. (because every set is closed in this topology).

Example. Consider (\mathbb{Q}, d) with the 3-adic metric. We can show that \mathbb{Z} is not closed. Define a sequence $(x_n)_{n=1}^\infty$ by $x_n = \sum_{k=0}^n 9^k$. This sequence has a limit in \mathbb{Q} but not in \mathbb{Z} . How do we “guess” the limit of this sequence? Notice that:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{for all } \|x\| < 1$$

In this case $\|9\|_3 = 1/9 < 1$, hence plugging in 9 shows the limit of (x_n) is $-1/8 \notin \mathbb{Z}$. [This is NOT a rigorous proof for now! This just allows us to guess the limit and we can then use the ϵ thing to prove the limit]. Hence \mathbb{Z} does not contain a limit point, which means it is not closed (by the theorem below).

Theorem 1.5. A closed set contains all of its limit points. That is, if $A \subseteq X$ is closed and $(x_n)_{n=1}^\infty$ is a sequence in A , then whenever $\lim_{n \rightarrow \infty} x_n = x \in X$ exists, we must have $x \in A$.

1.4 Continuous Functions

Definition. Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \rightarrow Y$ is **continuous at** $x_0 \in X$ if for all $\epsilon > 0$ there is $\delta > 0$ such that for all $x \in X$ with $d(x, x_0) < \delta$, then $\rho(f(x), f(x_0)) < \epsilon$. [Equivalently we have $f(B_\delta(x_0)) \subseteq B_\epsilon(f(x_0))$.]

Example. Let (X, d) be a metric space with discrete metric. Let $f : X \rightarrow Y$ with (Y, ρ) a metric space. Then f is continuous at every $x_0 \in X$. Why? For any $\epsilon > 0$, pick $\delta = 1/2$. Then $d(x, x_0) < 1/2$ implies $d(x, x_0) = 0$ and $x = x_0$. Hence $\rho(f(x), f(x_0)) = 0 < \epsilon$.

Definition. Let (X, d) and (Y, ρ) be metric spaces.

- (i). We say $f : X \rightarrow Y$ is **continuous on** X if it is continuous at all $x_0 \in X$.
- (ii). We say $f : X \rightarrow Y$ is **uniformly continuous on** X if for all $\epsilon > 0$, there is $\delta > 0$ such that for all $x, y \in X$:

$$d(x, y) < \delta \implies \rho(f(x), f(y)) < \epsilon$$

That is, the choice of $\delta > 0$ is independent of $x, y \in X$. The usual continuity means for any $x, y \in X$ we can choose a $\delta > 0$ for them, but in this case there is one choice of $\delta > 0$ that works for all $x, y \in X$.

Note. In the Example above, we see that f is in fact uniformly continuous.

Example. Let $f : (0, 1) \rightarrow (0, \infty)$ with Euclidean metric given by $f(x) = 1/x$. This function is continuous but NOT uniformly continuous. To see it is continuous, fix $x_0 \in (0, 1)$ and let $\epsilon > 0$. We then pick $\delta > 0$ to be:

$$\delta = \min \left\{ \frac{x_0}{2}, \frac{\epsilon x_0^2}{2} \right\}$$

Then, if $|x - x_0| < \delta$, we have:

$$\left| \frac{1}{x} - \frac{1}{x_0} \right| = \left| \frac{x_0 - x}{xx_0} \right| < \frac{\epsilon \cdot x_0^2/2}{(x_0/2)x_0} = \epsilon$$

It follows that f is continuous on $(0, 1)$. To see it is NOT uniformly continuous, assume it is. Take $\epsilon = 1$, then there is $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < 1$. However, pick $N \in \mathbb{N}$ large enough so that $1/N - 1/(N+1) < \delta$, then:

$$1 > \left| f\left(\frac{1}{N}\right) - f\left(\frac{1}{N+1}\right) \right| = \left| \frac{1}{1/N} - \frac{1}{1/(N+1)} \right| = 1$$

This is a contradiction, hence f is NOT uniformly continuous.

Definition. Let X, Y be metric spaces. We say $f : X \rightarrow Y$ is **sequentially continuous at** $x_0 \in X$ if for all sequence $(x_n)_{n=1}^\infty$ in X we have:

$$\lim_{n \rightarrow \infty} x_n = x_0 \implies \lim_{n \rightarrow \infty} f(x_n) = f(x_0)$$

We say f is **sequentially continuous** if it is sequentially continuous at every $x_0 \in X$.

Theorem 1.6. Let (X, d) and (Y, ρ) be metric spaces and $f : X \rightarrow Y$. The followings are equivalent:

- (i). f is continuous.
- (ii). For all open sets $V \subseteq Y$ we have $f^{-1}(V)$ is open in X .
- (iii). f is sequentially continuous.

Proof. We will prove (i) \implies (ii) \implies (i) and (i) \implies (iii) \implies (i).

(i) \implies (ii). Assume f is continuous. Let $V \subseteq Y$ be open. We want to show $f^{-1}(V)$ is open in X . If $f^{-1}(V) = \emptyset$, done. Otherwise pick $x_0 \in f^{-1}(V)$, then $f(x_0) \in V$. Then is $\epsilon > 0$ such that $B_\epsilon(f(x_0)) \subseteq V$. Since f is continuous at x_0 , there is $\delta > 0$ such that $f(B_\delta(x_0)) \subseteq B_\epsilon(f(x_0)) \subseteq V$. Hence $B_\delta(x_0) \subseteq f^{-1}(V)$ and therefore $f^{-1}(V)$ is open in X .

(i) \implies (iii). Assume f is continuous at x_0 and $(x_n)_{n=1}^\infty$ is a sequence with $x_n \rightarrow x_0$. Pick $\epsilon > 0$, since f is continuous there is $\delta > 0$ such that $d(x, x_0) < \delta$ implies $\rho(f(x), f(x_0)) < \epsilon$. Now pick $N \in \mathbb{N}$ so that $d(x_n, x_0) < \delta$ for $n \geq N$. Hence if $n \geq N$ we have $\rho(f(x_n), f(x_0)) < \epsilon$.

Lecture 5, 2025/01/15

(ii) \implies (i). Fix $x_0 \in X$ and let $\epsilon > 0$. Consider the open set $V = B_\epsilon(f(x_0))$. Since we are assuming (ii), we know $f^{-1}(V)$ is open and $x_0 \in f^{-1}(V)$. Therefore there is $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(V)$. Therefore we have $f(B_\delta(x_0)) \subseteq B_\epsilon(f(x_0))$, which proved f is continuous at x_0 .

(iii) \implies (i). We will prove this by contrapositive. Assume f is not continuous. This means there is $x_0 \in X$ and $\epsilon > 0$ such that for all $\delta > 0$, there are $x \in X$ with $d(x, x_0) < \delta$ but $\rho(f(x), f(x_0)) \geq \epsilon$. We are going to construct a sequence $(x_n)_{n=1}^\infty \subseteq X$ using this information that breaks the sequential

continuity. For $n \in \mathbb{N}$, we choose $x_n \in X$ such that $d(x_0, x_n) < 1/n$ but $\rho(f(x_0), f(x_n)) \geq \epsilon$. Then we clearly have $x_n \rightarrow x_0$ but $f(x_n)$ does NOT converge to $f(x_0)$ as they are always at least ϵ -away from each other. \square

Theorem 1.7. Let X, Y, Z be metric spaces. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be continuous. Then $g \circ f : X \rightarrow Z$ is continuous.

Definition. Let (X, d) and (Y, ρ) be metric spaces. We can define a metric $d \times \rho$ on $X \times Y$ by:

$$(d \times \rho)((x_1, y_1), (x_2, y_2)) := d(x_1, x_2) + \rho(y_1, y_2)$$

It is easy to check that this defines a metric.

Theorem 1.8. Let X, Y, Z, W be metric spaces and $f : X \rightarrow Z$ and $g : Y \rightarrow W$ be continuous. Then $f \times g : X \times Y \rightarrow Z \times W$ by $(f \times g)(x, y) = (f(x), g(y))$ is continuous where $X \times Y$ and $Z \times W$ are equipped with the metric defined above.

Definition. Let $f : X \rightarrow Y$ where (X, d) and (Y, ρ) are metric spaces. We say f is an **isometry** if for all $x_1, x_2 \in X$ we have $d(x_1, x_2) = \rho(f(x_1), f(x_2))$.

Example. In \mathbb{R}^2 , any rotation, reflection, translation and combination of them are isometries.

Definition. Let (X, d) and (Y, ρ) be metric spaces. A map $f : X \rightarrow Y$ is called **Lipschitz** if there exists a constant $C > 0$ such that:

$$\rho(f(x_1), f(x_2)) \leq Cd(x_1, x_2)$$

for all $x_1, x_2 \in X$.

Definition. Let (X, d) and (Y, ρ) be metric spaces. A map $f : X \rightarrow Y$ is called **bi-Lipschitz** if there exist constants $C, c > 0$ such that:

$$cd(x_1, x_2) \leq \rho(f(x_1), f(x_2)) \leq Cd(x_1, x_2)$$

for all $x_1, x_2 \in X$.

Definition. Let (X, d) and (Y, ρ) be metric spaces. We say $f : X \rightarrow Y$ is a **homeomorphism** if f is a continuous bijection such that $f^{-1} : Y \rightarrow X$ is also continuous.

1.5 Finite dimensional normed vector spaces

Definition. Let V be a vector space. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are said to be **equivalent** if there are constants $c, C > 0$ such that:

$$c\|v\|_2 \leq \|v\|_1 \leq C\|v\|_2$$

for all $v \in V$. It is clear that this is an equivalence relation.

Theorem 1.9. For $n \in \mathbb{N}$, all norms in \mathbb{R}^n are equivalent. The similar result holds for \mathbb{C}^n .

Proof. It suffices to show all norms $\|\cdot\|$ are equivalent to the 1-norm $\|\cdot\|_1$. Then since equivalence norm is an equivalence relation, all norms are equivalent. A basis for \mathbb{R}^n is $\{e_1, \dots, e_n\}$, the standard basis. Let $C = \max\{\|e_1\|, \dots, \|e_n\|\}$. Let $v = (v_1, \dots, v_n) \in \mathbb{R}^n$, then:

$$\begin{aligned} \|v\| &= \|v_1 e_1 + \dots + v_n e_n\| \\ &\leq |v_1| \|e_1\| + \dots + |v_n| \|e_n\| && (\triangle\text{-inequality}) \\ &\leq C(|v_1| + \dots + |v_n|) \\ &= C\|v\|_1 \end{aligned}$$

This gives us one inequality. This also shows that $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}$ is Lipschitz, hence continuous (where \mathbb{R}^n is equipped with $\|\cdot\|_1$ norm). Define:

$$S = \{v \in \mathbb{R}^n : \|v\|_1 = 1\}$$

Since $\|\cdot\|$ is continuous on $(\mathbb{R}^n, \|\cdot\|_1)$ we have that $\|\cdot\| : S \rightarrow \mathbb{R}$ obtains its maximum and minimum. Further, the minimum is nonzero. Define $c = \min_{v \in S} \|v\| > 0$. Note for all $0 \neq v \in \mathbb{R}^n$ we have that $v/\|v\|_1 \in S$. Hence:

$$\left\| \frac{v}{\|v\|_1} \right\| \geq c \implies \|v\| \geq c\|v\|_1$$

Hence $c\|v\|_1 \leq \|v\| \leq C\|v\|_1$, as desired. \square

Lecture 6, 2025/01/17

1.6 Completeness

Definition. Let (X, d) be a metric space and $(x_n)_{n=1}^\infty$ be a sequence in X . We say $(x_n)_{n=1}^\infty$ is a **Cauchy sequence** if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$:

$$n, m \geq N \implies d(x_n, x_m) < \epsilon$$

Example. Let $(\frac{1}{n})_{n=1}^\infty$ be a sequence in \mathbb{Q} but with different metrics.

- (i). If \mathbb{Q} is equipped with the Euclidean metric. This is clearly Cauchy. To see that, let $\epsilon > 0$ pick $N > 2/\epsilon$, then for $n, m \geq N$ we have:

$$d\left(\frac{1}{n}, \frac{1}{m}\right) = \left| \frac{1}{n} - \frac{1}{m} \right| \leq \left| \frac{1}{n} \right| + \left| \frac{1}{m} \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

- (ii). If \mathbb{Q} is equipped with the discrete metric, then $d(1/n, 1/m) = 1$ for all $n, m \in \mathbb{N}$. This means this sequence is not Cauchy. (If it is Cauchy, take $\epsilon = 1/2$ then contradiction)

- (iii). If \mathbb{Q} is equipped with the 3-adic metric, then this is not a Cauchy sequence. Let $n = 3^k$ and $m = 3^\ell$ where $k \neq \ell$. Then we have:

$$d\left(\frac{1}{n}, \frac{1}{m}\right) = d\left(\frac{1}{3^k}, \frac{1}{3^\ell}\right) = 3^{\min\{k, \ell\}}$$

If we pick k, ℓ large enough, then the distance between them can be arbitrarily large. Hence this is not a Cauchy sequence.

Theorem 1.10. Let (X, d) be a metric space and let $(x_n)_{n=1}^\infty$ be a convergent sequence, then $(x_n)_{n=1}^\infty$ is a Cauchy sequence.

Proof. Say $\lim_{n \rightarrow \infty} x_n = x^* \in X$. Let $\epsilon > 0$, pick $N \in \mathbb{N}$ such that for $n \in \mathbb{N}$:

$$n \geq N \implies d(x^*, x_n) < \frac{\epsilon}{2}$$

Now pick $n, m \in \mathbb{N}$ such that $n, m \geq N$, we have:

$$d(x_n, x_m) \leq d(x_n, x^*) + d(x_m, x^*) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore $(x_n)_{n=1}^\infty$ is a Cauchy sequence. □

Example (The Converse is False). Every convergent sequence is Cauchy but not the other way around. There are Cauchy sequences that do not converge.

- (i). Let $X = \mathbb{Q}$ with the Euclidean metric. Let x_n = the truncation of π to the n -th decimal place. For example: $x_1 = 3$, $x_2 = 3.1$, $x_3 = 3.14$ and so on. This is a Cauchy sequence, but the limit does not exist (because its “limit” is π , which is not in \mathbb{Q}).
- (ii). Consider the sequence $(\frac{1}{n})_{n=2}^\infty$ with $X = (0, 1)$ with Euclidean metric. Then this is Cauchy but not convergent because $0 \notin X$.

Definition. We say a metric space (X, d) is **complete** if every Cauchy sequence in X converges.

Example (Complete Spaces).

- (i). The metric space (\mathbb{R}, d) is complete with Euclidean metric.
- (ii). Any X with the discrete metric space.

Definition. A complete normed vector space is called a **Banach Space**.

Theorem 1.11. Let (X, d) be a complete metric space. Let $Y \subseteq X$ be a subset. Then (Y, d) is a complete metric space if and only if Y is closed in X .

Proof. (\Leftarrow). Assume Y is closed in X . Let $(x_n)_{n=1}^\infty \subseteq Y$ be a cauchy sequence in Y . Hence it is also a cauchy sequence in X . Therefore $(x_n)_{n=1}^\infty$ converges to x^* in X since X is complete. However, Y is closed so it contains its limit point, which means $x^* \in Y$ and thus $(x_n)_{n=1}^\infty$ converges in Y . This proved that (Y, d) is a complete metric space.

(\Rightarrow). Assume (Y, d) is complete. To show Y is closed in X it suffices to show it contains all of its limit points. Let $(x_n)_{n=1}^\infty$ be a convergent sequence with limit $x^* \in X$. Since convergent sequences are cauchy, we know $(x_n)_{n=1}^\infty$ is cauchy. Since Y is complete, this cauchy sequence converges in Y ! This means $x^* \in Y$ and hence Y is closed. \square

Theorem 1.12. Let $1 \leq p < \infty$. Then the space $(\ell_p, \|\cdot\|_p)$ is complete.

Proof. An element in ℓ_p is already a sequence, so a sequence of elements in ℓ_p is annoying. We use the following notation.

$$x^{(n)} = \{x_1^{(n)}, x_2^{(n)}, \dots\} = \left(x_k^{(n)}\right)_{k=1}^\infty$$

where $x^{(n)} \in \ell_p$ is the n -th term in the sequence $(x^{(n)})_{n=1}^\infty$. Let $(x^{(n)})_{n=1}^\infty$ be a cauchy sequence in ℓ_p . Pick $\epsilon > 0$, hence there exists an $N \in \mathbb{N}$ such that for $n, m \in \mathbb{N}$:

$$n, m \geq N \implies d(x^{(n)}, x^{(m)}) = \left(\sum_{k=1}^\infty |x_k^{(n)} - x_k^{(m)}|^p\right)^{1/p} < \epsilon \quad (1)$$

Our goal is to find a limit point $x = (x_k)_{k=1}^\infty \in \ell_p$ of the sequence $(x^{(n)})_{n=1}^\infty$ and prove it. Fix $k \in \mathbb{N}$, we claim that $(x_k^{(n)})_{n=1}^\infty$ is a cauchy sequence in $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Indeed:

$$|x_k^{(n)} - x_k^{(m)}| = \left(|x_k^{(n)} - x_k^{(m)}|^p\right)^{1/p} \leq \left(\sum_{j=1}^\infty |x_j^{(n)} - x_j^{(m)}|^p\right)^{1/p}$$

We have seen that the RHS can be arbitrarily small by (1), hence $(x_k^{(n)})_{n=1}^\infty$ is cauchy in \mathbb{K} . Since \mathbb{K} is complete, this limit exists, we define $x_k = \lim_{n \rightarrow \infty} x_k^{(n)} \in \mathbb{K}$. We claim that:

$$\lim_{n \rightarrow \infty} x^{(n)} = x \in \ell_p$$

There are two things to prove: the limit is x and x lies in ℓ_p .

Lecture 7, 2025/01/20

(i). Pick $\epsilon > 0$, there exists an N such that for all $n, m \geq N$ we have:

$$d(x^{(n)}, x^{(m)}) = \left(\sum_{k=1}^\infty |x_k^{(n)} - x_k^{(m)}|^p\right)^{1/p} < \epsilon$$

For any $J \in \mathbb{N}$ and for all $n, m \geq N$ we have:

$$\sum_{k=1}^J |x_k^{(n)} - x_k^{(m)}| \leq \sum_{k=1}^{\infty} |x_k^{(n)} - x_k^{(m)}| < \epsilon^p$$

As this is true for all $n \geq M$, it is true as $m \rightarrow \infty$. This gives:

$$\lim_{m \rightarrow \infty} \sum_{k=1}^J |x_k^{(n)} - x_k^{(m)}|^p \leq \epsilon^p \implies \sum_{k=1}^J |x_k^{(n)} - x_k|^p < \epsilon^p$$

because $x_k^{(m)} \rightarrow x_k$ as $m \rightarrow \infty$. This result is true for all $n \geq N$, independent of the choice of J . As this is true for all J , we can take the limit as $J \rightarrow \infty$. Hence:

$$\sum_{k=1}^{\infty} |x_k^{(n)} - x_k|^p \leq \epsilon^p \tag{1}$$

It follows that $(x^{(n)} - x) \in \ell_p$ for all $n \in \mathbb{N}$, by definition. We also know $x^{(n)} \in \ell_p$, hence:

$$x = (x^{(n)} - x) + x^{(n)} \in \ell_p$$

as ℓ_p is a vector space. Inequality (1) says that for any $\epsilon > 0$ we can find $N \in \mathbb{N}$ such that $n \geq N$ implies:

$$\|x^{(n)} - x\|_p = \left(\sum_{k=1}^{\infty} |x_k^{(n)} - x_k|^p \right)^{1/p} \leq \epsilon$$

This is exactly the definition of $x^{(n)} \rightarrow x$ in ℓ_p , as desired. \square

1.7 Completeness of \mathbb{R}

Definition. Let $S \subseteq \mathbb{R}$. We say S is **bounded above** if there exists an $M \in \mathbb{R}$ such that $s \leq M$ for all $s \in S$. Similarly S is **bounded below** if there is $N \in \mathbb{R}$ such that $s \geq N$ for all $s \in S$. A set is **bounded** if it is both bounded above and below.

Example. $\mathbb{Z} \subseteq \mathbb{R}$ is not bounded above or below. $(0, 1) \subseteq \mathbb{R}$ is bounded.

Definition. Let $S \subseteq \mathbb{R}$ be bounded above. Then we say $M \in \mathbb{R}$ is the **least upper bound** if M is an upper bound for S and if $N \in \mathbb{R}$ is another upper bound for S we have $M \leq N$. We define the **greatest lower bound** similarly. We denote them by $\sup S$ and $\inf S$.

Theorem 1.13 (Least Upper Bound Property). Let $\emptyset \neq S \subseteq \mathbb{R}$ be bounded above, then S has a least upper bound.

Proof. Let $M \in \mathbb{Z}$ be an upper bound of S . Consider $M - 1$. One of two things is true. Either $M - 1$ is an upper bound or it is not. If $M - 1$ is an upper bound, replace M by $M - 1$ and repeat this argument. Eventually we will get $M \in \mathbb{Z}$ such that M is an upper bound but $M - 1$ is NOT an upper bound (This process terminates because $S \neq \emptyset$). Divide $[M - 1, M]$ into 10 subintervals.

$$\left[M - 1, M - 1 + \frac{1}{10}\right], \dots, \left[M - 1 + \frac{9}{10}, M\right]$$

We can find some $k \in \{0, \dots, 9\}$ such that $M - 1 + \frac{k}{10}$ is not an upper bound and $M - 1 + \frac{k+1}{10}$ is an upper bound. We construct u^* as the decimal sequence which is an upper bound (We have to be careful if a ring end point is a least upper bound, as we get a decimal expansion of trailing 9's but this is fine). As desired. \square

Theorem 1.14 (MCT). Let $(x_n)_{n=1}^\infty$ be a bounded, non-decreasing sequence in \mathbb{R} . Then $(x_n)_{n=1}^\infty$ converges in \mathbb{R} .

Proof. Let $x^* = \sup\{x_n : n \in \mathbb{N}\}$, this exists because $(x_n)_{n \geq 1}$ is bounded. Let $\epsilon > 0$, as x^* is the least upper bound, there exists $N \in \mathbb{N}$ such that:

$$x^* - \epsilon < x_N \leq x^*$$

Hence, for $n \geq N$ we have that $x_n \geq x_N$, which means:

$$x^* - \epsilon < x_N \leq x_n \leq x^* < x^* + \epsilon \implies |x^* - x_n| < \epsilon$$

which prvoed that $\lim_{n \rightarrow \infty} x_n = x^*$, as desired. \square

Theorem 1.15 (Bolzano-Weierstrass). Every bounded sequence $(x_n)_{n=1}^\infty$ in \mathbb{R} has a convergent subsequence (that converges in \mathbb{R}).

Proof. Just see MATH 147/247 notes, the proof idea is just bisection. \square

Lecture 8 - 17, 2025/01/22 - 2025/02/12

Lemma 1.16. Let $(x_n)_{n=1}^\infty$ be a cauchy sequence in \mathbb{R} . Then $(x_n)_{n=1}^\infty$ is bounded.

Proof. Pick $\epsilon = 1$, there is $N \in \mathbb{N}$ such that for $n, m \geq N$ we have $|x_n - x_m| < 1$. In particular $|x_n - x_N| < 1$ for all $n \geq N$. Let $M = \max\{|x_1|, \dots, |x_N| + 1\}$. Then $|x_n| \leq M$ for all $n \in \mathbb{N}$. \square

Theorem 1.17. $(\mathbb{R}, |\cdot|)$ is a complete normed space.

Proof. Let $(x_n)_{n=1}^\infty$ be a cauchy sequence in \mathbb{R} . By the above lemma, $(x_n)_{n=1}^\infty$ is bounded. By Bolzano-Weierstrass, $(x_n)_{n=1}^\infty$ has a convergent subsequence $(x_{n_k})_{k=1}^\infty$. Say $\lim_{k \rightarrow \infty} x_{n_k} = x^* \in \mathbb{R}$. We claim that $\lim_{n \rightarrow \infty} x_n = x^*$ as well. Indeed, let $\epsilon > 0$. There is $N_1 \in \mathbb{N}$ such that:

$$n, m \geq N \implies |x_n - x_m| < \frac{\epsilon}{2}$$

Find $k \in \mathbb{N}$ such that $n_k \geq N$ and $|x_{n_k} - x^*| < \epsilon/2$. Hence for $n \geq N$ we have:

$$|x_n - x^*| \leq |x_n - x_{n_k}| + |x_{n_k} - x^*| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore \mathbb{R} is complete. \square

1.8 Limits of continuous functions

Definition. Let (X, d) and (Y, ρ) be metric spaces. Let $(f_n)_{n=1}^\infty$ be a sequence of function $X \rightarrow Y$. We say $(f_n)_{n=1}^\infty$ **converges uniformly** to $f^* : X \rightarrow Y$ if for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for $n \in \mathbb{N}$ we have:

$$n \geq N \implies d^*(f_n, f^*) := \sup_{x \in X} \rho(f_n(x), f^*(x)) < \epsilon$$

Example. Let $X = [0, \frac{1}{2}]$ and $Y = \mathbb{R}$ with Euclidean metrics. Define:

$$f_n(x) = 1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

This $(f_n)_{n=1}^\infty$ is a sequence of bounded continuous functions from $X \rightarrow Y$. We claim that it converges to $f^*(x) = \frac{1}{1-x}$. Indeed, for any $n \in \mathbb{N}$ we have:

$$d^*(f_n, f^*) = \sup_{x \in [0, \frac{1}{2}]} \left| \frac{1}{1-x} - \frac{1 - x^{n+1}}{1-x} \right| = \sup_{x \in [0, \frac{1}{2}]} \frac{x^{n+1}}{|1-x|} \leq \frac{(1/2)^{n+1}}{1/2} = \left(\frac{1}{2}\right)^n$$

where the denominator is at least $1/2$ and the numerator is at most $(1/2)^{n+1}$. As $n \rightarrow \infty$ this tends to 0, which means $f_n \rightarrow f^*$ uniformly.

Theorem 1.18. Let $(f_n)_{n=1}^\infty$ be a sequence of continuous functions that converges uniformly to f^* . Then f^* is continuous.

Proof. Let $x \in X$ and $\epsilon > 0$. Pick $N \in \mathbb{N}$ such that $d^*(f^*, f_N) < \epsilon/3$. Since f_N is continuous at x , we can pick $\delta > 0$ such that:

$$d(x, y) < \delta \implies \rho(f_N(x), f_N(y)) < \frac{\epsilon}{3}$$

Therefore if $y \in X$ and $d(x, y) < \delta$, we have:

$$\begin{aligned} \rho(f^*(x), f^*(y)) &\leq \rho(f^*(x), f_N(x)) + \rho(f_N(x), f_N(y)) + \rho(f_N(y), f^*(y)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

Therefore f^* is continuous at $x \in X$, as desired. \square

Definition. Let (X, d) be a metric space. A subset $A \subseteq X$ is **bounded** if:

$$\text{diam}(A) := \sup_{x, y \in A} d(x, y) < \infty$$

We say a function $f : X \rightarrow Y$ is bounded if $f(X) \subseteq Y$ is bounded.

Definition. Let (X, d) and (Y, ρ) be metric spaces. Define:

$$\mathcal{C}^b(X, Y) = \{f : X \rightarrow Y \mid f \text{ is continuous and bounded}\}$$

The metric on $\mathcal{C}^b(X, Y)$ is the metric d^* defined by:

$$\rho^*(f, g) := \sup_{x \in X} \rho(f(x), g(x))$$

Then $(\mathcal{C}^b(X, Y), \rho^*)$ is a metric space.

Theorem 1.19. Let (f_n) be a sequence of bounded functions $f_n \in \mathcal{C}^b(X, \mathbb{K})$ that converges uniformly to f^* , then f^* is also bounded.

Theorem 1.20. Let (X, d) and (Y, ρ) be metric spaces. The metric space $(\mathcal{C}^b(X, Y), \rho^*)$ is complete if and only if (Y, ρ) is complete!

Proof. See Assignment 2. □

Theorem 1.21. Let (X, d) be a metric space. Then $\mathcal{C}^b(X, \mathbb{K})$ is complete.

Proof. Let $(f_n)_{n=1}^\infty$ be a cauchy sequence. Construct $f^* : X \rightarrow \mathbb{K}$ by:

$$f^*(x) := \lim_{n \rightarrow \infty} f_n(x)$$

Why is this well-defined? Note that for all fixed $x \in \mathbb{K}$, the sequence $(f_n(x))_{n=1}^\infty$ is a cauchy sequence in \mathbb{K} ! Since \mathbb{K} is complete, this sequence converges. We claim that $(f_n)_{n=1}^\infty$ converges uniformly to f^* . Let $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that:

$$n, m \geq N \implies d^*(f_n, f_m) = \sup_{x \in X} |f_n(x) - f_m(x)| < \frac{\epsilon}{2}$$

Let $n \geq N$ be arbitrary. Let $x \in X$ be arbitrary as well. Since $f_n(x) \rightarrow f^*(x)$, we can find $M \in \mathbb{N}$ with $M \geq N$ such that $|f_n(x) - f^*(x)| < \epsilon/2$. Then, for $n \geq N$:

$$\begin{aligned} |f_n(x) - f^*(x)| &\leq |f_n(x) - f_M(x)| + |f_M(x) - f^*(x)| \\ &\leq d^*(f_n, f_M) + |f_M(x) - f^*(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Since $x \in X$ is chosen arbitrarily, we have:

$$d^*(f_n, f^*) = \sup_{x \in X} |f_n(x) - f^*(x)| \leq \epsilon$$

Therefore $f_n \rightarrow f^*$ in the d^* metric (that is $f_n \rightarrow f^*$ uniformly in the usual sense). Hence f^* is continuous and bounded, so $f^* \in \mathcal{C}^b(X, \mathbb{R})$. \square

Theorem 1.22 (Weierstrass M-Test). Let $\zeta : X \rightarrow \mathbb{R}$ by $\zeta(a) = 0$ denote the zero function. Then we let $(f_n)_{n=1}^\infty$ be a sequence in $\mathcal{C}^b(X, \mathbb{R})$ such that there exists $M \in \mathbb{R}$ with:

$$\sum_{n=1}^{\infty} d^*(f_n, \zeta) \leq M < \infty$$

Define $g_N(x) = \sum_{n=1}^N f_n(x)$. Then $(g_N)_{N=1}^\infty$ converges to $g^* \in \mathcal{C}^b(X)$ in the d^* metric.

Example. The series of function $f(x) = \sum_{n=1}^\infty \frac{\sin(nx)}{2^n}$ is well-defined and is continuous on \mathbb{R} .

2 More Metric Topology

2.1 Compactness

Definition. Let (X, d) be a metric space and $A \subseteq X$. We say $\{U_i\}_{i \in I}$ is an **open cover** of A if each U_i is open and $A \subseteq \bigcup_{i \in I} U_i$.

Definition. Let (X, d) be a metric space and $A \subseteq X$. We say A is **compact** if for every open cover $\{U_i\}_{i \in I}$ there is a finite subset $I_0 \subseteq I$ with $A \subseteq \bigcup_{i \in I_0} U_i$. This $\{U_i\}_{i \in I_0}$ is called a **finite subcover**.

Example. Let $A = \{x_1, \dots, x_n\}$ be a finite set, then A is compact. Why? Let $\{U_i\}$ be an open cover of A . For each $j \in \mathbb{N}$ there is $i_j \in I$ such that $a \in U_{i_j}$. Hence we have:

$$A \subseteq U_{i_1} \cup \dots \cup U_{i_n}$$

This is a finite subcover! Hence A is compact.

Example. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\} \subseteq \mathbb{R}$. We claim that A is compact. Let $\{U_i\}_{i \in I}$ be an open cover. There exists an open set U_0 such that $0 \in U_0$. Hence there is $N \in \mathbb{N}$ large enough such that $0 \in B_\epsilon(0) \subseteq U_0$, where $\epsilon = 1/N$. This means:

$$\left\{ \frac{1}{n} : n \geq N+1 \right\} \cup \{0\} \subseteq U_0$$

Then there are only finitely many points left, so we can use finitely many U_i to cover $\{\frac{1}{n} : n \geq N+1\}$. This gives an finite subcover of A .

Example. Let A, B be compact sets. Then $A \cup B$ is compact. Indeed, any open cover of $A \cup B$ gives an open cover for A, B . This gives a finite subcover for A, B , respectively. The union of these two finite subcovers gives a finite subcover of $A \cup B$.

Example. The set $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ is not compact! Consider the open cover:

$$\left\{ \left(\frac{1}{n}, 1 + \frac{1}{n} \right) : n \in \mathbb{N} \right\}$$

This has no finite subcover. Indeed, suppose we have a finite subcollection of open sets indexed by n_1, \dots, n_r . WLOG we may assume $n_1 < \dots < n_r$. Then the union of these U_i is:

$$\left(\frac{1}{n_r}, 1 + \frac{1}{n_1} \right)$$

This clearly does not cover A .

Example. Let $A = \mathbb{R}$. Then A is not compact. The open cover $\{(-n, n) : n \in \mathbb{N}\}$ has no finite subcover. Similarly $A = \mathbb{Z}$ is not compact as well.

Proposition 2.1. Let (X, d) be a metric space. If $A \subseteq X$ is compact, then A is closed and bounded.

Proof. Assume A is not closed. There exists a subsequence $(a_n)_{n=1}^\infty$ in A with $a_n \rightarrow a^*$ and $a^* \notin A$. Consider the following open cover:

$$U_n = X \setminus \overline{B_{d(a_n, a^*)}(a^*)}$$

This cannot have a finite subcover, since a^* is a limit point of (a_n) . Therefore A is closed. Similarly suppose A is not bounded. Fix $a \in A$. For all $N \in \mathbb{N}$ such that there exists an $a_N \in A$ such that:

$$d(a_N, a) > N$$

Consider the open cover $\{B_N(a) : N \in \mathbb{N}\}$ of A . Given a finite subset $\{N_1 < \dots < N_r\}$, the union of these is $B_{N_r}(a)$. However, for $N = N_r + 1$ there is $a_N \in A$ such that $d(a_N, a) > N$ so $a_N \notin B_{N_r}(a)$, but $a_N \in A$. Hence this open cover has no finite subcover! Hence A is bounded. \square

Definition. Let (X, d) be a metric space and $A \subseteq X$. We say A is **sequentially compact** if for every sequence $(a_n)_{n=1}^\infty$ of A , there is a convergent subsequence $(a_{n_k})_{k=1}^\infty$ with $a_{n_k} \rightarrow a^* \in A$.

Example. Let A be a finite set. This is sequentially compact. Why? For any infinite sequence of A , there exists $a \in A$ that appears infinitely many times in this sequence. Take this subsequence that only consists of a . This is a convergent subsequence.

Example. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. This is sequentially compact. Any sequence in A either has a convergent subsequence that goes to 0 or the sequence only takes on finitely many values.

Definition. Let (X, d) be a metric space. Let $A \subseteq X$ be a subset. Then (A, d_A) is a metric space, where $d_A : A \times A \rightarrow \mathbb{R}$ is the restriction of d on A . This is called the **induced metric space**. A subset $U \subseteq A$ is called **relatively open** if there exists an open set $U' \subseteq X$ such that $U = U' \cap A$.

Remark. As a metric space, the open balls of (A, d_A) are of the form:

$$B_A(a, r) = \{x \in A : d_A(x, a) < r\} = \{x \in X \cap A : d(x, a) < r\} = B_X(a, r) \cap A$$

Therefore, an open set in (A, d_A) is of the form $U' \cap A$ for open sets U' in X .

Definition. A metric space (X, d) is **compact** if every open cover of X has a finite subcover. That is, for every open cover $\{U_i : i \in I\}$, there is a finite subset $I_0 \subseteq I$ such that:

$$X = \bigcup_{i \in I_0} U_i$$

Note that this is an equality, not a subset. This is because X is our whole space, it does not sit in any bigger space.

Remark. Note that there are two notions of compactness for a subset $A \subseteq X$.

- (i). A is compact as a subset of X . [This is the definition we saw above.]
- (ii). A is compact as a metric space. [Note that for an open cover of A , the open sets are open sets in A ! These open sets are different from the open sets in X .]

In fact, these two notions coincide. Suppose (ii) is true, we want to show (i) is true. Let $\{U_i : i \in I\}$ be an open cover of A , where U_i is an open set of X for all i . Then:

$$\{U_i \cap A : i \in I\}$$

is an open cover of the metric space (A, d_A) , where each $U_i \cap A$ is an open set in A . Since (A, d_A) is compact, there is a finite set $I_0 \subseteq I$ such that:

$$A = \bigcup_{i \in I_0} (U_i \cap A)$$

Then clearly we have $A \subseteq \bigcup_{i \in I_0} U_i$, a finite subcover of A (as a subset of X .)

Conversely suppose (i) is true. Let $\{U_i : i \in I\}$ be an open cover of (A, d_A) , then for each $i \in I$ there is an open set $U'_i \subseteq X$ of X such that $U_i = U'_i \cap A$. Hence $\{U'_i : i \in I\}$ is an open cover of $A \subseteq X$. Since A is a compact subset of X , there is a finite $I_0 \subseteq I$ with $A \subseteq \bigcup_{i \in I_0} U'_i$. By taking the intersection with A , we have:

$$A = \bigcup_{i \in I_0} (U'_i \cap A) = \bigcup_{i \in I_0} U_i$$

Therefore (A, d_A) is compact and (ii) is true.

Definition. Let (X, d) be a metric space. A collection $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\} \subseteq X$ is said to have the **finite intersection property (FIP)** if for every finite subset $\Lambda_0 \subseteq \Lambda$ we have $\bigcap_{\lambda \in \Lambda_0} F_\lambda \neq \emptyset$.

Example. Let $X = \mathbb{R}$. Consider the collection $\{\mathbb{R} \setminus \{a\} : a \in \mathbb{R}\}$. This clearly satisfies the FIP. However, the infinite intersection:

$$\bigcap_{a \in \mathbb{R}} (\mathbb{R} \setminus \{a\}) = \emptyset$$

is empty! As we will see, this actually tells us \mathbb{R} is not compact!

Definition. Let (X, d) be a metric space. A subset $A \subseteq X$ is called **cauchy** if every cauchy sequence in A converges to a point in A .

Definition. Let (X, d) be a metric space. We say $A \subseteq X$ is **totally bounded** if for all $\epsilon > 0$ there exists a finite set $F_\epsilon \subseteq X$ (called an **ϵ -net**) such that:

$$A \subseteq \bigcup_{f \in F_\epsilon} B_\epsilon(f)$$

Note that totally boundedness implies boundedness.

Remark. Note that if A is totally bounded, we may assume $F_\epsilon \subseteq A$ for all $\epsilon > 0$. Suppose for $\epsilon > 0$ we have an ϵ -net $F = \{x_1, \dots, x_n\} \subseteq X$ of A , so:

$$A \subseteq \bigcup_{i=1}^n B_{\epsilon/2}(x_i)$$

We may assume $B_\epsilon(x_i) \cap A \neq \emptyset$ for all i . (If the intersection is empty we can just remove it from the ϵ -net.) Hence we may choose $y_i \in A \cap B_{\epsilon/2}(x_i)$ for all i . Note that:

$$A \subseteq \bigcup_{i=1}^n B_\epsilon(y_i)$$

by the triangle inequality. Indeed, for any $x \in A$ we can choose $i \in \{1, \dots, n\}$ such that $x \in B_{\epsilon/2}(x_i)$. Then we have that:

$$d(x, y_i) \leq d(x, x_i) + d(x_i, y_i) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This proved that $x \in B_\epsilon(y_i)$. Hence $\{y_1, \dots, y_n\} \subseteq A$ is an ϵ -net for A .

Recall in \mathbb{R}^n , a subset is compact if and only if it is closed and bounded (Heine-Borel). We will now see that for metric spaces, there are also some easier ways to characterize compactness, and the Heine-Borel theorem for \mathbb{R}^n is a special case of it.

Theorem 2.2 (Borel-Lebesgue). Let (X, d) be a metric space and $A \subseteq X$. Then the followings are equivalent:

- (i). A is compact (either as a subset or a metric space, these two notions are equivalent.)
- (ii). If $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\}$ is an collection of closed sets in (A, d_A) with FIP, then $\bigcap_{\lambda \in \Lambda} F_\lambda \neq \emptyset$.
- (iii). A is sequentially compact.
- (iv). A is complete and totally bounded.

Example. Consider $A = \mathbb{Q} \cap [0, 1]$ and $B = \mathbb{Z}$ as induced metric spaces from (\mathbb{R}, d) . By the Borel-Lebesgue theorem, we can show that A, B are not compact in four different ways.

- (i). For A , we define the open cover:

$$\{\mathbb{R} \setminus \overline{B_{1/n}(1/\pi)} : n \in \mathbb{N}\}$$

This does not have a finite subcover. For B , the open cover $\{B_{1/2}(n) : n \in \mathbb{Z}\}$ does not have a finite subcover as well. Hence A, B are not compact by definition.

(ii). We need to find a collection of closed sets that FIP but the intersection is empty. Let:

$$\begin{aligned} \{A \cap \overline{B_{1/n}(1/\pi)} : n \in \mathbb{N}\} &\subseteq A \\ \{[n, \infty) \cap B : n \in \mathbb{N}\} &\subseteq B \end{aligned}$$

These two have FIP but the intersection over all $n \in \mathbb{N}$ is empty.

(iii). Let $(a_n)_{n=1}^\infty$ be the sequence in A such that a_n is the truncation of the decimal expansion of $1/\pi$ at the n -th place. Then $a_n \rightarrow 1/\pi$ in \mathbb{R} , which means any convergent subsequence of (a_n) converges to $1/\pi \notin A$. For B , the sequence $(b_n)_{n=1}^\infty$ by $b_n = n$ is a sequence in B that does not have a convergent subsequence.

(iv). Let $(a_n)_{n=1}^\infty$ be the same sequence in (iii), this is cauchy but does not converge in A . For B , consider $\epsilon = 1/2$. Then $B = \mathbb{Z}$ does not have a ϵ -net. Therefore A is not complete and B is not totally bounded.

Proof of Theorem 2.2. (i) \implies (ii). Assume (A, d_A) is a compact metric space. Let $\{F_\lambda : \lambda \in \Lambda\}$ be a collection of closed sets in A satisfying FIP. Assume for a contradiction that $\bigcap_{\lambda \in \Lambda} F_\lambda = \emptyset$. Consider the following collection of open sets in A :

$$\{U_\lambda := A \setminus F_\lambda : \lambda \in \Lambda\}$$

Note that this is an open cover for A . Since A is compact, there exists a finite subset $\Lambda_0 \subseteq \Lambda$ such that $\bigcup_{\lambda \in \Lambda_0} U_\lambda = A$. However, this implies that:

$$\bigcap_{\lambda \in \Lambda_0} F_\lambda = A \setminus \bigcup_{\lambda \in \Lambda_0} U_\lambda = A \setminus A = \emptyset$$

Since Λ_0 is finite, this contradicts to our assumption that $\{F_\lambda : \lambda \in \Lambda\}$ has FIP!

(ii) \implies (iii). Assume (ii) is true. We want to show A is sequentially compact. Let $(a_n)_{n=1}^\infty$ be a sequence in A . For each $k \geq 1$ we define $S_k = \{a_n : n \geq k\}$ and define the closed set:

$$F_k = \overline{S_k} = \overline{\{a_n : n \geq k\}} \subseteq A$$

to be the closure of a tail of $(a_n)_{n=1}^\infty$. Note that $F_{k+1} \subseteq F_k$ for all $k \geq 1$. Define $\mathcal{F} = \{F_k : k \geq 1\}$. Then \mathcal{F} is a collection of closed sets in A that has FIP. It satisfies FIP because for a finite set $\{k_1 < \dots < k_r\}$ we have:

$$F_{k_1} \cap \dots \cap F_{k_r} = F_{k_1} \neq \emptyset$$

By our assumption we have $\bigcap_{k=1}^\infty F_k \neq \emptyset$. Let's pick $a^* \in \bigcap_{k=1}^\infty F_k$. We claim that we can find a subsequence of (a_n) that converges to a^* . First we note that:

$$B_r(a^*) \cap S_k \neq \emptyset$$

for all $r > 0$ and $k \geq 1$. This is because each $a^* \in F_k$ is closed so a^* is a limit point for every S_k . In other word, for any $r > 0$ and $k \geq 1$ we can find some a_i such that $d(a_i, a^*) < r$ and $i \geq k$. For $r = 1$ we can find $n_1 \geq 1$ with $d(a_{n_1}, a^*) < 1$. Inductively suppose we have defined n_1, \dots, n_r , we can find $n_{r+1} > n_r$ such that $d(a_{n_r}, a^*) < 1/(r+1)$. Hence $(a_{n_r})_{r=1}^\infty$ is a subsequence that converges to $a^* \in A$. Therefore (A, d_A) is sequentially compact.

(iii) \implies (iv). Assume (A, d_A) is sequentially compact. We first show that A is complete (as a subset of X .) Let $(a_n)_{n=1}^\infty$ be a cauchy sequeunce in A . There exist a convergent subsequence $(a_{n_k})_{k=1}^\infty$ that converges to $a^* \in A$. Since $(a_n)_{n=1}^\infty$ is cauchy, we must have $a_n \rightarrow a^*$ as well. Hence A is complete. Now let us show that A is totally bounded. Let $\epsilon > 0$ be arbitrary. Suppose it is not, then there is $\epsilon > 0$ such that there does not exist a ϵ -net for A . First note that in the case, A must be infinite. (Any finite set is clearly totally bounded.) Let $a_1 \in A$ be arbitrary. Hence $\{a_1\}$ is not an ϵ -net. This means there exists $a_2 \in A$ such that $d(a_1, a_2) \geq \epsilon$. Now, inductively suppose we have found a_1, \dots, a_r for $r \geq 1$. Then $\{a_1, \dots, a_r\}$ is not an ϵ -net. We can then find $a_{r+1} \in A$ such that:

$$d(a_{r+1}, a_i) \geq \epsilon \text{ for all } i \in \{1, \dots, r\}$$

This gives us a sequence $(a_n)_{n=1}^\infty$ in A that has no convergent subsequence! (since for all n, m we have $d(a_n, a_m) \geq \epsilon$.) This is a contradiction, so A is totally bounded.

(iv) \implies (i). Assume (iv) is true, we want to show A is compact. Suppose for a contradiction that A is not compact as a metric space. Let $\mathcal{U} = \{U_i : i \in I\}$ be an open cover of A that does not have a finite subcover (in this case $U_i \subseteq X$ is open for all i). Since A is totally bounded, for all $n \geq 1$ there exists a $\frac{1}{n}$ -net in A :

$$F_n = \{x_{n,1}, \dots, x_{n,m_n}\}$$

such that:

$$A = \bigcup_{f \in F_n} B_{1/n}(f) = \bigcup_{f \in F_n} \overline{B_{1/n}(f)}$$

Let $n = 1$. Note that if all $\overline{B_1(f)}$ can be covered by finitely many U_i 's, then A can be covered by finitely many U_i 's, which is impossible. Hence there is $i_1 \in \{1, \dots, m_1\}$ such that $\overline{B_1(x_{i_1})}$ does not have a finite subcover of \mathcal{U} . Let $y_1 = x_{i_1}$. Inductively suppose we have chosen y_1, \dots, y_k so that:

$$X_k := \bigcap_{i=1}^k \overline{B_{1/i}(y_i)}$$

has no finite subcover. Consider the sets:

$$X_{k,i} = X_k \cap \overline{B_{1/(k+1)}(x_{k+1,i})} \text{ for } 1 \leq i \leq m_{k+1}$$

Suppose for a contradiction that each of them has a finite subcover. However:

$$\bigcup_{i=1}^{m_{k+1}} X_{k,i} = \bigcup_{i=1}^{m_{k+1}} X_k \cap \overline{B_{1/(k+1)}(x_{k+1,i})} = X_k \cap \bigcup_{i=1}^{m_{k+1}} \overline{B_{1/(k+1)}(x_{k+1,i})} = X_k \cap A = X_k$$

This means X_k has a finite subcover, which is impossible! Hence there is i_{k+1} such that $\overline{B_{1/(k+1)}(x_{k+1,i_{k+1}})}$ does not have a finite subcover. Let $y_{k+1} = x_{k+1,i_{k+1}}$.

Note that $(y_n)_{n=1}^\infty$ is cauchy in A . Indeed, let $\epsilon > 0$ we choose $N > 2/\epsilon$. For all $n \geq m \geq N$ we have:

$$X_n \subseteq \overline{B_{1/m}(y_m)} \cap \overline{B_{1/n}(y_n)} \neq \emptyset$$

and X_n is non-empty set. We can pick $x \in X_n$. Then:

$$d(y_n, y_m) \leq d(y_n, x) + d(y_m, x) \leq \frac{1}{n} + \frac{1}{m} \leq \frac{2}{N} < \epsilon$$

Since A is complete, $y_n \rightarrow y^* \in A$ for some $y^* \in A$. For any $m \in \mathbb{N}$ we have:

$$d(y_m, y^*) = \lim_{n \rightarrow \infty} d(y_m, y_n) \leq \lim_{n \rightarrow \infty} \left(\frac{1}{m} + \frac{1}{n} \right) = \frac{1}{m}$$

Since \mathcal{U} is a cover of A , there is $i_0 \in I$ such that $y^* \in U_{i_0}$. Since U_{i_0} is open, then is $r > 0$ such that $B_r(y^*) \subseteq U_{i_0}$. Choose $m > 2/r$, then for any $x \in X_m \subseteq \overline{B_{1/m}(y_m)}$ we have:

$$d(x, y^*) \leq d(x, y_m) + d(y_m, y^*) \leq \frac{2}{m} < r$$

Hence $X_m \subseteq U_{i_0}$. This means X_m does have a finite subcover, contradicting our construction! Therefore A is compact. \square

Remark. Totally bounded is not same as bounded. There exist sets that are closed, bounded but not compact. Consider $X = \{0, 1\}^\mathbb{N}$ with ℓ^∞ norm. It is clearly bounded since $\|x\|_\infty \leq 1$ for all $x \in X$. However, we claim that it is not totally bounded. Suppose it has an $\frac{1}{2}$ -net:

$$F = \{x_1, \dots, x_n\}$$

Then $B_{1/2}(x_i) = \{x_i\}$ because $\|x\|_\infty \in \{0, 1\}$ for any $x \in X$. Hence:

$$\bigcup_{i=1}^n B_{1/2}(x_i) = \{x_1, \dots, x_n\} \neq \{0, 1\}^\mathbb{N}$$

Therefore this is not a $\frac{1}{2}$ -net, contradiction. Hence $(X, \|\cdot\|_\infty)$ is bounded but NOT totally bounded!

Corollary 2.3 (Heine-Borel). A subset of \mathbb{R}^n is compact if and only if it is closed and bounded.

Proof. Since \mathbb{R}^n is compact, A is closed \iff it is complete. Moreover, we claim that in \mathbb{R}^n , bounded implies totally bounded. Let $\epsilon \in \mathbb{N}$, we claim that there is also an ϵ -net of a bounded set A . Since A is bounded, we know $A \subseteq [-r, r]^n$ for some $r > 0$. We can cover $[-r, r]^n$ with finitely many boxes of side length $\frac{\epsilon}{2}$. Any such box can be covered by an ϵ -ball. Hence we can use finitely many ϵ -balls to cover A . Therefore A is totally bounded. Hence A is bounded \iff it is totally bounded. The result follows from (iv) of Borel-Lebesgue. \square

2.2 Countable and Uncountable Sets

Definition. A set X is **countable** if there is a injection $f : X \rightarrow \mathbb{N}$. A set is **denumerable** if there is a bijection $f : X \rightarrow \mathbb{N}$. We say a set is **uncountable** if it is not countable.

Example. The integers \mathbb{Z} is countable because $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$.

Example. The rationals $\mathbb{Q} \cap [0, 1]$ is also countable because:

$$\mathbb{Q} = \left\{ \frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \dots \right\}$$

Informally: Write $\frac{p}{q} \in \mathbb{Q} \cap [0, 1]$ with q in increasing order and $p \in \{1, \dots, q\}$ such that $\gcd(p, q) = 1$. We require coprimeness so that there is no element appearing twice in the list.

Example. The set $\{0, 1\}^{\mathbb{N}}$ is uncountable. Suppose for a contradiction that it is countable. Then:

$$\{0, 1\}^{\mathbb{N}} = \{(x_{1,k})_{k=1}^{\infty}, (x_{2,k})_{k=1}^{\infty}, \dots\}$$

Define a sequence $(x_n)_{n=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ by:

$$x_n = \begin{cases} 0 & \text{if } x_{n,n} = 1 \\ 1 & \text{if } x_{n,n} = 0 \end{cases}$$

Then $(x_n)_{n=1}^{\infty}$ is different from $(x_{n,k})_{k=1}^{\infty}$ at the n -th place for all $n \geq 1$. This is a new element in $\{0, 1\}^{\mathbb{N}}$, contradiction! Hence $\{0, 1\}^{\mathbb{N}}$ is uncountable. This method is called the **diagonal argument**: If we list out all the given $(x_{n,k})_{k=1}^{\infty}$ row by row, then our new element $(x_n)_{n=1}^{\infty}$ is constructed by changing the diagonal entries.

Example. Let A, B be sets and $f : A \rightarrow B$ be a bijection, then A is countable if and only if B is countable.

Example. Let $A \subseteq B$. If A is uncountable then so is B . If B is countable then so is A .

Example. We claim \mathbb{R} is uncountable. Let $X = \{0, 1, \dots, 9\}^{\mathbb{N}}$. By the same argument we can show that X is uncountable. Define $f : X \rightarrow \mathbb{R}$ by:

$$f((x_n)_{n=1}^{\infty}) = \sum_{k=1}^{\infty} \frac{x_k}{10^k}$$

Then $f : X \rightarrow f(X)$ is a bijection. Since $f(X) \subseteq \mathbb{R}$, we know \mathbb{R} is uncountable.

Definition. Let (X, d) be a metric space. We say $A \subseteq X$ is **dense** in X if $\overline{A} = X$.

Definition. We say a metric space (X, d) is **separable** if there is a countable subset $A \subseteq X$ such that A is dense in X .

Example. The reals \mathbb{R} with the usual metric is separable because $\overline{\mathbb{Q}} = \mathbb{R}$ and \mathbb{Q} is countable.

Example. Let (X, d) with the discrete metric. Then X is separable if and only if X is countable. This is because every subset is closed (equal to their own closure), so the only dense subset is X itself. Hence X is countable if and only if X is separable.

Proposition 2.4. Let (X, d) be a metric space. If (X, d) is totally bounded, then X is separable.

Proof. For each $n \in \mathbb{N}$ there is an $\frac{1}{n}$ -net of X , call it F_n . Define $F = \bigcup_{n=1}^{\infty} F_n$. Note that F is countable, being a countable union of finite sets. We claim that F is dense. Let $x \in X$ be and $\epsilon > 0$ be arbitrary. There is $N \geq 1$ such that $1/N < \epsilon$. Since F_N is an $\frac{1}{N}$ -net, there is $f \in F_N$ such that $d(f, x) < 1/N < \epsilon$. Since $f \in F$, we proved that F is dense in X . \square

2.3 Compactness and Continuity

Proposition 2.5. Let (X, d) and (Y, ρ) be metric spaces. If $f : X \rightarrow Y$ is continuous and X is compact, then $f(X)$ is compact.

Proof. Let $\{U_i : i \in I\}$ be an open cover of $f(X)$ in Y . Since f is continuous, each $f^{-1}(U_i)$ is open. Since $f^{-1}(Y) = X$, we know $\{f^{-1}(U_i) : i \in I\}$ is an open cover of X . Since X is compact, there is a finite subcover $\{i_1, \dots, i_n\}$. Hence:

$$f(X) \subseteq \bigcup_{k=1}^n U_{i_k}$$

Therefore $f(X)$ is compact. \square

Proposition 2.6. Let (X, d) and (Y, ρ) be metric spaces. If $f : X \rightarrow Y$ is continuous and X is compact, then f is uniformly continuous.

Proof. Let $\epsilon > 0$. For each $x \in X$ we can find $\delta_x > 0$ such that for all $y \in X$:

$$d(y, x) < \delta_x \implies \rho(f(y), f(x)) < \frac{\epsilon}{2} \quad (1)$$

Now note that $\{B_{\delta_x/2}(x) : x \in X\}$ is an open cover of X . Since X is compact, we know:

$$X = B_{\delta_{x_1}/2}(x_1) \cup \dots \cup B_{\delta_{x_n}/2}(x_n)$$

for some $x_1, \dots, x_n \in X$. Now define $\delta = \frac{1}{2} \min\{\delta_{x_1}, \dots, \delta_{x_n}\} > 0$. Let $x, y \in X$ be arbitray with $d(x, y) < \delta$. Say $y \in B_{\delta_{x_b}}(x_b)$ for some $x_b \in \{x_1, \dots, x_n\}$. However:

$$d(x, x_b) \leq d(x, y) + d(y, x_b) < \delta + \frac{1}{2}\delta_{x_b} < \frac{1}{2}\delta_{x_b} + \frac{1}{2}\delta_{x_b} < \delta_{x_b}$$

Since $d(y, x_b) < \delta_{x_b}$ as well, by (1) we have:

$$\rho(f(x), f(y)) \leq \rho(f(x), f(x_b)) + \rho(f(x_b), f(y)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence f is uniformly continuous. □

2.4 Cantor Set

Construction 2.7 (Ver 1). Let $C_0 = [0, 1]$. Recursively, C_{i+1} is constructed by removing the middle third from each intervals in C_i . First we see that:

$$\begin{aligned} C_1 &= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \\ C_2 &= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right] \end{aligned}$$

We see $\{C_n\}_{n=0}^\infty$ has the finite intersection property and they are all compact sets. Define:

$$C^* = \bigcap_{n=0}^{\infty} C_n$$

We call C^* the **(middle-third) cantor set**. Clearly $0, 1 \in C^*$. In fact any endpoint of any C_n is in C^* . For example $1/3, 1/9, 2/27 \in C^*$. We have C^* is compact (as it is closed and bounded in \mathbb{R}).

Construction 2.8 (Ver 2). Equivalently we can define:

$$C^* = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{3^n} : a_n \in \{0, 2\} \right\}$$

It is the set of all real numbers that CAN be written in ternary expansion wiwthout using 1. [For example $0.1 \in C^*$ because it CAN be written as $0.222\dots$] This shows that C^* has an uncountable number of points.

Construction 2.9 (Ver 3). The cantor set C^* is the unique non-empty compact set satisfying:

$$C^* = f_1(C^*) \cup f_2(C^*)$$

where $f_1(x) = x/3$ and $f_2(x) = x/3 + 2/3$.

Theorem 2.10. Let (X, d) be a compact metric space. There is an continuous map $f : C^* \rightarrow X$ that is surjective.

Proof. The idea is to construct $s_n : C^* \rightarrow X$ such that (s_n) is cauchy and each s_n is continuous. As $n \rightarrow \infty$ we have $s_n(C^*)$ better approximate X [produce an ϵ -net for smaller ϵ .]

For $n = 1$, construct a 1-net for X . That is, a finite set F_1 such that $X = \bigcup_{f \in F_1} B_1(f)$ [This exists since X is totally bounded.] We can assume wlog that $|F_1| = 2^{k_1}$ for some k_1 . [If not power of 2, adding more points if necessary.] Now consider C_{k_1} , a union of 2^{k_1} intervals containing C^* . For each $c \in C^*$, we know c is in some subinterval of C_{k_1} . We map each subinterval in C_{k_1} to a different $f \in F_1$. Let s_1 be this map. Then s_1 is continuous as it is locally constant.

For $n = 2$, construct a $1/2$ -net for each of each $\overline{B}_1(f_i)$, where $\{f_i\} = F_1$ from the construction of s_1 . As before, we can assume that this set is a power of 2, and the same powers of 2. Say 2^{k_2} in size. For each subinterval I_i used to construct s_1 , subdivide it into 2^{k_2} subintervals. As before, s_2 is continuous. We further notice $d(s_1(c), s_2(c))$ is not huge. In fact $d(s_1(c), s_2(c)) \leq 1 + \frac{1}{2}$.

We continue in this fashion, we get that:

$$d(s_n(c), s_{n+1}(c)) \leq \frac{1}{2^n}$$

We can make this arbitrarily small. Hence for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n, m \geq N$:

$$d^*(s_n, s_m) = \sup_{c \in C^*} d(s_n(c), s_m(c)) < \epsilon$$

Therefore (s_n) is a cauchy sequence. As C^* is compact and X is complete so $\mathcal{C}^b(C^*, X) = \mathcal{C}(C^*, X)$ is complete. Hence $s_n \rightarrow s^* \in \mathcal{C}(C^*, X)$. We need to show $s^*(C^*) = X$, that is, s^* is onto. Take a point x in X . This point will be distance 1 from some point in F_1 . This gives us a subinterval in C^* . There exists a point in F_2 whose distance is $1/2$ from x and $1 + 1/2$ from f_1 . This gives a smaller subinterval. Repeating this process we get nested subintervals with non-trivial intersection with C^* . The infinite intersection is in C^* , and this intersection has $s^*(c^*) = x$, as required. \square

2.5 Compact sets in $\mathcal{C}(X)$

Definition. Let (X, d) be a compact metric space, we denote:

$$\mathcal{C}(X) := \mathcal{C}(X, \mathbb{R}) = \{f : X \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$$

Here \mathbb{R} is a metric space with the usual metric. For $f \in \mathcal{C}(X)$ we define the **uniform norm** by:

$$\|f\|_\infty := \sup\{|f(x)| : x \in X\}$$

Since X is compact, by the extreme value theorem this supremum can be achieved. So we can equivalently define it as:

$$\|f\|_\infty = \max\{|f(x)| : x \in X\}$$

Note that $(\mathcal{C}(X), \|\cdot\|_\infty)$ is a normed vector space. In fact, since \mathbb{R} is complete we knew that $\mathcal{C}(X)$ is also complete. Therefore $(\mathcal{C}(X), \|\cdot\|_\infty)$ is a Banach space. Also note that $f_n \rightarrow f$ uniformly (as functions) is the same as $f_n \rightarrow f$ as sequences in the normed space $(\mathcal{C}(X), \|\cdot\|_\infty)$.

Remark. By Borel-Lebesgue we know that:

$$\begin{aligned} K \subseteq \mathcal{C}(X) \text{ is compact} &\iff K \text{ is complete and totally bounded} \\ &\iff K \text{ is closed and totally bounded} \end{aligned}$$

since closed subsets of a complete space are complete.

Example. Let $K = \{f_n(x) = x^n : n \in \mathbb{N}\} \subseteq \mathcal{C}([0, 1])$. Note that every subsequence of (f_n) converges pointwise to the function $f : [0, 1] \rightarrow \mathbb{R}$ defined by:

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Since f is not continuous, the sequence (f_n) does not converge in $\mathcal{C}([0, 1])$. Therefore K is not sequentially compact despite being closed and bounded.

Definition. Let (X, d) be complete. A subset $F \subseteq \mathcal{C}(X)$ is called **equicontinuous at** $x \in X$ if for all $\epsilon > 0$ there is $\delta > 0$ so that for all $y \in X$:

$$d(x, y) < \delta \implies |f(x) - f(y)| < \epsilon \text{ for all } f \in F$$

We know $F \subseteq \mathcal{C}(X)$ is **equicontinuous** if it is equicontinuous at every $x \in X$. We say a subset $F \subseteq \mathcal{C}(X)$ is **uniformly equicontinuous** if for all $\epsilon > 0$ there is $\delta > 0$ such that for all $x, y \in X$:

$$d(x, y) < \delta \implies |f(x) - f(y)| < \epsilon \text{ for all } f \in F$$

That is, the choice of $\delta > 0$ does not depend on $x \in X$.

Remark. Clearly uniformly equicontinuous \implies equicontinuous.

Lemma 2.11. Let (X, d) be compact. If $K \subseteq \mathcal{C}(X)$ is compact, then K is uniformly equicontinuous.

Proof. Let $\epsilon > 0$. Since K is compact, it is totally bounded and thus has a $\frac{\epsilon}{3}$ -net. Say it is $F = \{f_1, \dots, f_n\} \subseteq K$. Each f_i is continuous, thus uniformly continuous (since X is compact). For each i there is $\delta_i > 0$ such that for all $x, y \in X$:

$$d(x, y) < \delta_i \implies |f_i(x) - f_i(y)| < \frac{\epsilon}{3}$$

Let $\delta = \min\{b_1, \dots, b_n\}$. Now let $x, y \in X$ with $d(x, y) < \delta$ and let $f \in K$ be arbitrary. We can find i such that $\|f - f_i\| < \epsilon/3$ (because F is an $\epsilon/3$ -net!) Therefore we have:

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| \\ &\leq \|f - f_i\|_\infty + \frac{\epsilon}{3} + \|f - f_i\|_\infty \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

Therefore K is uniformly equicontinuous. \square

Lemma 2.12. Let (X, d) be compact. Suppose $F \subseteq \mathcal{C}(X)$ is equicontinuous. Then F is uniformly equicontinuous.

Proof. Let $\epsilon > 0$. For each $x \in X$ there is $\delta_x > 0$ so that for all $y \in X$:

$$d(x, y) < \delta_x \implies |f(x) - f(y)| < \frac{\epsilon}{2} \text{ for all } f \in F$$

Then the collection $\{B_{\delta_x/2}(x) : x \in X\}$ is an open cover of X . Since X is compact, it has a finite subcover, indexed by $\{x_1, \dots, x_n\}$. Let $\delta = \min\{\delta_{x_1}/2, \dots, \delta_{x_n}/2\}$. Suppose $y_1, y_2 \in X$ and $d(y_1, y_2) < \delta$. Pick i so that $d(y_1, x_i) < \delta_{x_i}/2$. Then:

$$d(y_2, x_i) \leq d(y_2, y_1) + d(y_1, x_i) < \delta + \frac{\delta_{x_i}}{2} \leq \delta_{x_i}$$

Now we know $d(y_1, x_i) < \delta_{x_i}$ and $d(y_2, x_i) < \delta_{x_i}$. By the choice of δ_{x_i} , for all $f \in F$ we have:

$$|f(y_1) - f(y_2)| \leq |f(y_1) - f(x_i)| + |f(y_2) - f(x_i)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore F is uniformly equicontinuous. \square

Theorem 2.13 (Arzela-Ascoli). Let (X, d) be a compact metric space. A subset $K \subseteq \mathcal{C}(X)$ is compact if and only if K is closed, bounded and equicontinuous.

Proof. (\implies). If K is compact then it is closed and bounded by Proposition 2.1. Also we know that K is equicontinuous by the lemma above.

(\impliedby). Suppose K is closed, bounded and equicontinuous. Note that $\mathcal{C}(X)$ is complete and K is closed, so K is complete. It remains to show K is totally bounded. Let $\epsilon > 0$. Since K is equicontinuous, it is uniformly equicontinuous by the lemma above. There is $\delta > 0$ such that for all $f \in K$ and $x, y \in X$ we have:

$$d(x, y) < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{4} \quad (*)$$

Since X is compact, there is a δ -net:

$$F_X = \{x_1, \dots, x_n\} \subseteq X \text{ and } X \subseteq \bigcup_{i=1}^n B_\delta(x_i) \quad (\dagger)$$

Define $T : K \rightarrow (\mathbb{R}^n, \|\cdot\|_\infty)$ by:

$$T(f) = (f(x_1), \dots, f(x_n))$$

Note that $\|T(f)\|_\infty = \max\{|f(x_i)| : 1 \leq i \leq n\} \leq \|f\|_\infty$. [Here is a bit of abusing of notation. The two $\|\cdot\|$ -norm are on two different spaces.] This implies that $T(K)$ is bounded in \mathbb{R}^n since K is bounded in $\mathcal{C}(X)$. This means $T(K)$ is totally bounded, thus $\overline{T(K)}$ is compact in \mathbb{R}^n . This means that there exists a $\epsilon/4$ -net of $T(K)$:

$$F_T = \{T(f_1), \dots, T(f_m)\} \subseteq T(K) \quad \text{and} \quad T(K) \subseteq \bigcup_{i=1}^m B_{\epsilon/4}(f_i) \quad (\dagger\dagger)$$

Here each $f_i \in K$. We claim that $F_K = \{f_1, \dots, f_m\}$ is a ϵ -net for K . Indeed, let $f \in K$ be arbitrary. We can find some $j \in \{1, \dots, m\}$ such that $\|T(f) - T(f_j)\|_\infty < \epsilon/4$ by $(\dagger\dagger)$. Now we let $y \in X$, we can find $i \in \{1, \dots, n\}$ such that $d(x_i, y) < \delta$ by (\dagger) . Then:

$$|f(y) - f_j(y)| \leq \underbrace{|f(y) - f(x_i)|}_{< \frac{\epsilon}{4} \text{ by } (*)} + \underbrace{|f(x_i) - f_j(x_i)|}_{\leq \|T(f) - T(f_j)\|_\infty < \frac{\epsilon}{4}} + \underbrace{|f_j(x_i) - f_j(y)|}_{< \frac{\epsilon}{4} \text{ by } (*)} < \frac{3\epsilon}{4}$$

Since $y \in X$ is arbitrary, we have $\|f - f_j\|_\infty \leq \frac{3\epsilon}{4} < \epsilon$. This proved that $K \subseteq \bigcup_{i=1}^m B_\epsilon(f_i)$. Hence K is totally bounded. \square

2.6 Connectedness

Definition. Let (X, d) be a metric space. We say $A \subseteq X$ is **disconnected** if there exist two open sets U, V of X such that $A \subseteq U \cup V$ and $U \cap V = \emptyset$ and $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$. We say A is **connected** if it is not disconnected.

Example. Let (X, d) be a metric space. Any finite subset $A = \{x_1, \dots, x_n\}$ with at least two elements is disconnected. Let $r = \frac{1}{2} \min\{d(x_i, x_j) : i \neq j\} > 0$. We define open sets:

$$U = B_r(x_1) \quad \text{and} \quad V = B_r(x_2) \cup \dots \cup B_r(x_n)$$

Then $A \subseteq U \cup V$ and $U \cap V = \emptyset$ by our choice of r . Moreover $U \cap A = \{x_1\}$ and $V \cap A = \{x_2, \dots, x_n\}$ are not empty. Therefore A is disconnected.

Example. Let X be a set with $|X| \geq 2$. Let d be the discrete metric on X . Then (X, d) is disconnected. Indeed, let $x_0 \in X$. Then $U = \{x_0\}$ is open and $V = X \setminus \{x_0\}$ is also open.

Example. The middle third cantor set is disconnected.

Example. The interval $[0, 1] \subseteq \mathbb{R}$ is connected. Assume it is disconnected by open sets U, V of \mathbb{R} . WLOG we may assume $0 \in U$. Let $C = \{c \in \mathbb{R} : [0, c] \subseteq U\}$. Since U is open, there is $\epsilon > 0$ so that $B_\epsilon(0) \subseteq U$. Since C is nonempty, we let $c^* = \sup C$. There are two cases.

- (i). If $c^* \in U$. Then as U is open, there is $\epsilon > 0$ such that $B_\epsilon(c^*) \subseteq U$. This means $c^* + \epsilon \in U$, so we have $c^* + \epsilon \in C$. Contradiction.
- (ii). If $c^* \in V$. There is $\epsilon > 0$ with $B_\epsilon(c^*) \subseteq V$. This means $B_\epsilon(c^*) \cap U = \emptyset$. However, by the definition of supremum we know $c^* - \epsilon \in C$, so $[0, c^* - \epsilon] \subseteq U$. This means $c^* - \frac{\epsilon}{2} \in U$, but we know $c^* - \frac{\epsilon}{2} \in V$ as well. Contradiction.

Theorem 2.14. Let (X, d) and (Y, ρ) be metric spaces. Suppose (X, d) is connected. If $f : X \rightarrow Y$ is continuous, then $f(X)$ is connected.

Proof. Assume $f(X)$ is disconnected, say by open sets U, V of (Y, ρ) . It is easy to see that $f^{-1}(U)$ and $f^{-1}(V)$ are open sets that separate X . Contradiction. \square

Theorem 2.15. Any connected subsets of \mathbb{R} are intervals.

Proof. Let C be a connected set. We define:

$$a = \inf C \in \mathbb{R} \cup \{-\infty\} \quad \text{and} \quad b = \sup C \in \mathbb{R} \cup \{\infty\}$$

If $c \in \mathbb{R}$ and $a < c < b$ we must have $c \in C$. Otherwise:

$$C \subseteq \underbrace{(-\infty, c)}_U \cup \underbrace{(c, \infty)}_V$$

This gives a separation of C , contradiction. Hence we have $(a, b) \subseteq C \subseteq [a, b]$. This means C is an interval in \mathbb{R} . \square

Definition. Let (X, d) be a metric space. We can define an equivalence relation on X by $x \sim y$ if and only if there is a connected set C containing both x, y . The equivalence classes of this relation are called **connected components**. Let $x_0 \in X$. the equivalence class that x_0 lies in is called the connected component of x_0 and it is equal to the union of all connected sets containing x_0 .

Example. Let $X = [0, 1] \cup [2, 3]$ be the metric space with induced Euclidean metric. Then $[0, 1]$ and $[2, 3]$ are the connected components of X .

Definition. Let (X, d) be a metric space. We say X is **totally disconnected** if every connected component is a singleton set.

Example. Finite sets are totally disconnected.

Definition. Let (X, d) be a metric space. We say (X, d) is **path-connected** if for all $x, y \in X$ there exists a continuous function $f : [0, 1] \rightarrow X$ such that $f(0) = x$ and $f(1) = y$.

Example. Let $(V, \|\cdot\|)$ be a normed space. Any convex set $C \subseteq V$ is path connected. For $x, y \in C$ we can define $f(t) = (1-t)x + ty \in C$.

Proposition 2.16. Let (X, d) be a metric space. If X is path-connected then X is connected.

Proof. Suppose $X = U \cup V$ is disconnected. Pick $x \in U$ and $y \in V$. There is a path $f : [0, 1] \rightarrow X$ such that $f(0) = x$ and $f(1) = y$. Now:

$$[0, 1] = f^{-1}(X) = f^{-1}(U) \cup f^{-1}(V)$$

Note that $0 \in f^{-1}(U)$ and $1 \in f^{-1}(V)$. It is easy to check $f^{-1}(U)$ and $f^{-1}(V)$ give a separation of $[0, 1]$. This is a contradiction! \square

Example. The converse of this is not true. There exists connected spaces that is not path-connected. We define the following set:

$$X = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) : x > 0 \right\} \cup \{(0, 0)\}$$

Then $X \subseteq \mathbb{R}^2$ is connected but not path connected.

2.7 Bonus Cantor Set Stuff

Definition. Let $n \geq 2$ and $A \subseteq \{0, 1, \dots, n-1\}$ be a finite set. We define the **linear Cantor set**:

$$C_{A,n} = \left\{ \sum_{k=1}^{\infty} \frac{a_k}{n^k} : a_k \in A \right\}$$

Definition. Let $A \subseteq \mathbb{R}$. We define $N_{\epsilon}(A)$ to be the minimal number of ϵ -balls needed to cover A . The **box-counting dimension** of A is defined as:

$$\dim_B(A) = \lim_{\epsilon \rightarrow 0} \frac{-\log N_A(\epsilon)}{\log \epsilon}$$

if the limit exists. If the limit does not exist, we can take the limsup or liminf to define the **upper box dimension** and **lower box dimension**.

Example. Consider the middle third Cantor set. For $3^{-n} \leq \epsilon < 3^{-(n-1)}$, we need 2^n intervals of length $1/3^n$ to cover C . Hence:

$$\dim_B(C) = \lim_{n \rightarrow \infty} \frac{-\log 2^n}{\log 3^{-n}} = \frac{\log 2}{\log 3}$$

The box-counting dimension of C is $\log_3(2)$.

Definition. Let (X, ρ) be a metric space. For any $U \subseteq X$ we let $\text{diam}(U)$ or $|U|$ denote its diameter. Let $S \subseteq X$ and let $\delta > 0$ and $d \in [0, \infty)$. We define:

$$H_\delta^d(S) = \inf \left\{ \sum_{i \in I} |U_i|^d : S \subseteq \bigcup_{i \in I} U_i, |U_i| < \delta, |I| \leq |\mathbb{N}| \right\}$$

Then we define:

$$H^d(S) = \lim_{\delta \rightarrow 0} H_\delta^d(S)$$

to be the d -**dimensional Hausdorff measure** of S .

Theorem 2.17. Let (X, ρ) be a metric space and $0 \leq s < t < \infty$. For $A \subseteq X$ we have:

- (i). If $H^s(A) < \infty$ then $H^t(A) = 0$.
- (ii). If $H^t(A) > 0$ then $H^s(A) = \infty$.

Proof. It suffices to prove (i) since (ii) is just the contrapositive of (i). We have:

$$\begin{aligned} H_\delta^t(A) &= \inf \left\{ \sum_{i \in I} |U_i|^t : A \subseteq \bigcup_{i \in I} U_i, |U_i| < \delta, |I| \leq |\mathbb{N}| \right\} \\ &= \inf \left\{ \sum_{i \in I} |U_i|^{t-s} |U_i|^s : A \subseteq \bigcup_{i \in I} U_i, |U_i| < \delta, |I| \leq |\mathbb{N}| \right\} \\ &\leq \inf \left\{ \sum_{i \in I} \delta^{t-s} |U_i|^s : A \subseteq \bigcup_{i \in I} U_i, |U_i| < \delta, |I| \leq |\mathbb{N}| \right\} \\ &= \delta^{t-s} \inf \left\{ \sum_{i \in I} |U_i|^s : A \subseteq \bigcup_{i \in I} U_i, |U_i| < \delta, |I| \leq |\mathbb{N}| \right\} \\ &= \delta^{t-s} H_\delta^s(A) \end{aligned}$$

Suppose $H^s(A) < \infty$, we then have:

$$H^t(A) = \lim_{\delta \rightarrow 0} \delta^{t-s} H_\delta^s(A) = H_\delta \lim_{\delta \rightarrow 0} \delta^{t-s} = 0$$

As desired. □

Corollary 2.18. There is at most one $d \in [0, \infty)$ with $0 < H^d(A) < \infty$.

Definition. Same setting as above. We define the **Hausdorff dimension** of A to be:

$$\dim_H(A) = \sup\{d \in [0, \infty) : H^d(A) = \infty\} = \inf\{d \in [0, \infty) : H^d(A) = 0\}$$

Example. Let $A = \mathbb{Q} \cap [0, 1]$. We need $\lceil \epsilon^{-1} \rceil$ many ϵ -balls to cover A , as \mathbb{Q} is dense in \mathbb{R} . Hence:

$$\dim_B(A) = \lim_{\epsilon \rightarrow 0} \frac{-\log \lceil \epsilon^{-1} \rceil}{\log \epsilon} = 1$$

However, we claim the Hausdorff dimension is 0. Consider:

$$\begin{aligned} H_\delta^0(A) &= \inf \left\{ \sum_{i \in I} |U_i|^0 : A \subseteq \bigcup_{i \in I} U_i, |U_i| < \delta, |I| \leq |\mathbb{N}| \right\} \\ &= \inf \left\{ \sum_{i \in I} |U_i|^t : A \subseteq \bigcup_{i \in I} U_i, |U_i| < \delta, I \text{ finite} \right\} \\ &= \inf \left\{ |I| : A \subseteq \bigcup_{i \in I} U_i, |U_i| < \delta, I \text{ finite} \right\} \\ &= \left\lceil \frac{1}{\delta} \right\rceil \end{aligned}$$

Then we have $H^0(A) = \lim_{\delta \rightarrow 0} H_\delta^0(A) = \infty$. Let $d > 0$, we wish to show that $H^d(A) = 0$. To do this it suffices to show for all $\epsilon > 0$ and $\delta > 0$ we have $H_\delta^d(A) \leq \epsilon$. Since A is countable, we can enumerate $A = \{r_n : n \geq 1\}$. For each $n \geq 1$ let:

$$\epsilon_n = \min \left\{ \delta, \frac{1}{2} \left(\frac{\epsilon}{2^n} \right)^{1/d} \right\} > 0$$

Then let $U_n = B_{\epsilon_n}(r_n)$ and $|U_n| \leq \left(\frac{\epsilon}{2^n} \right)^{1/d}$. Hence we have:

$$\sum_{n=1}^{\infty} |U_n|^d \leq \sum_{n=1}^{\infty} \left(\left(\frac{\epsilon}{2^n} \right)^{1/d} \right)^d = \epsilon$$

Hence $H^d(A) = 0$ for all $d > 0$, so $\dim_H(A) = \inf\{d \geq 0 : H^d(A) = 0\} = 0$.

Proposition 2.19. For any linear Cantor set $C_{A,n}$ we have $\dim_B(C_{A,n}) = \dim_H(C_{A,n})$.

Proposition 2.20. Let $A, B \subseteq \mathbb{R}$, then:

$$\dim_H(A \cup B) = \max\{\dim_H(A), \dim_H(B)\}$$

Proposition 2.21. Let $A, B \subseteq \mathbb{R}$, then:

$$\dim_H(A + B) \leq \dim_H(A) + \dim_H(B)$$

Proposition 2.22. Let $\emptyset \neq A \subseteq \mathbb{R}^n$, then $0 \leq \dim_H(A) \leq n$.

Example. From A4 we saw that $C + C = [0, 2]$, where C is the middle-third Cantor set. That is:

$$C_{\{0,2\},3} + C_{\{0,2\},3} = [0, 2]$$

We know the box counting dimension is $\log_3(2)$, so $\dim_H(C_{\{0,2\},3}) = \log_3(2)$ as well.

Example. What is the dimension of $C_{\{0,3\},4}$ and the dimension of $C_{\{0,3\},4} + C_{\{0,3\},4}$? In general, we need 2^n intervals of length 4^{-n} to cover $C_{\{0,3\},4}$, so:

$$\dim_B(C_{\{0,3\},4}) = \lim_{n \rightarrow \infty} \frac{\log 2^n}{\log 4^{-n}} = \frac{1}{2} = \dim_H(C_{\{0,3\},4})$$

What does $C_{\{0,3\},4} + C_{\{0,3\},4}$ looks like?

$$\begin{aligned} C_{\{0,3\},4} + C_{\{0,3\},4} &= \left\{ \sum_{k=1}^{\infty} \frac{a_k + b_k}{4^k} : a_k, b_k \in \{0, 3\} \right\} = \left\{ \sum_{k=1}^{\infty} \frac{c_k}{4^k} : c_k \in \{0, 3, 6\} \right\} \\ &= \left\{ \sum_{k=1}^{\infty} \frac{c_k}{4^k} : c_k \in \left\{0, \frac{3}{2}, 3\right\} \right\} + \left\{ \sum_{k=1}^{\infty} \frac{c_k}{4^k} : c_k \in \left\{0, \frac{3}{2}, 3\right\} \right\} \\ &= 2C_{\{0, \frac{3}{2}, 3\}, 4} \end{aligned}$$

We see that:

$$\dim_H(C_{\{0, \frac{3}{2}, 3\}, 4}) = \dim_B(C_{\{0, \frac{3}{2}, 3\}, 4}) = \frac{\log 3}{\log 4} < 1$$

Theorem 2.23. Let $C_{A,n}$ be a linear Cantor set. If $\dim_H(C_{A,n}) < \frac{1}{2}$ then $C_{A,n} + C_{A,n} \neq [0, 2]$.

Proof. By Proposition 2.21 we have:

$$\dim_H(C_{A,n} + C_{A,n}) \leq \dim_H(C_{A,n}) + \dim_H(C_{A,n}) < 1$$

However $\dim_H([0, 2]) = 1$. Hence $C_{A,n} + C_{A,n} \neq [0, 2]$. □

Example. Let $C \subseteq \mathbb{R}^n$ be a perfect and totally disconnected set with $\dim_H(C) < \frac{1}{2}$. Then $C + C$ is a perfect and totally disconnected set.

Theorem 2.24. Let $C_{A,n}$ be a linear Cantor set, then:

$$C_{A,n} = \bigcup_{a \in A} S_a(C_{A,n})$$

where $S_a : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $S_a(x) = \frac{x+a}{n}$.

Proof. Note that we have:

$$\begin{aligned}
C_{A,n} &= \left\{ \frac{a_1}{n} + \sum_{k=2}^{\infty} \frac{a_k}{n^k} : a_k \in A \right\} \\
&= \bigcup_{a \in A} \left\{ \frac{a}{n} + \left\{ \sum_{k=2}^{\infty} \frac{a_k}{n^k} : a_k \in A \right\} \right\} \\
&= \bigcup_{a \in A} \left\{ \frac{a}{n} + \frac{1}{n} \left\{ \sum_{k=1}^{\infty} \frac{a_k}{n^k} : a_k \in A \right\} \right\} \\
&= \bigcup_{a \in A} \frac{a}{n} + \frac{1}{n} C_{A,n} \\
&= \bigcup_{a \in A} S_a(C_{A,n})
\end{aligned}$$

As desired. □

Theorem 2.25. Let $A \subseteq \{0, \dots, n-1\}$ and $0, n-1 \in A$. Define:

$$B := A + A = \{0 = b_0 < b_1 < \dots < b_k = 2n-2\}$$

Then $C_{A,n} + C_{A,n} = [0, 2]$ if and only if $b_i - b_{i-1} \leq 2$ for all $1 \leq i \leq k$.

Proof. Note that we have:

$$C_{A,n} + C_{A,n} = \left\{ \sum_{r=1}^{\infty} \frac{a_r + c_r}{n^r} : a_r, c_r \in A \right\} = \left\{ \sum_{r=1}^{\infty} \frac{b_r}{n^r} : b_r \in B \right\} = C_{B,n}$$

Then $b_i - b_{i-1} \leq 2$ for all i if and only if $[0, 2] = \bigcup_{i=0}^k S_{b_i}(C_{B,n}) = C_{B,n}$. □

Definition. A **Cantorval** is a compact subset of \mathbb{R} with non-empty interior such that none of its connected components are isolated.

Fact. Let $A \subseteq \{0, \dots, n-1\}$ and $0, n-1 \in A$. Exactly one of the followings is true:

1. $C_{A,n} + C_{A,n} = [0, 2]$.
2. $C_{A,n} + C_{A,n}$ is a totally disconnected and perfect set.
3. $C_{A,n} + C_{A,n}$ is a Cantorval.

3 Completeness

3.1 Baire Category Theorem

Definition. Let (X, d) be a metric space. We say $A \subseteq X$ is **nowhere dense** if $\text{int}(\overline{A}) = \emptyset$.

Example. Consider (\mathbb{R}, d) with Euclidean metric. A singleton is nowhere dense. The integers \mathbb{Z} is nowhere dense. Rationals \mathbb{Q} is NOT nowhere dense, as $\overline{\mathbb{Q}} = \mathbb{R}$. The Cantor set is nowhere dense.

Example. Consider the metric space (X, d) where d is the discrete metric. Any non-empty set is NOT nowhere dense because every $A \subseteq X$ is both open and closed, so:

$$\text{int}(\overline{A}) = \text{int}(A) = A \neq \emptyset$$

The only nowhere dense subset of (X, d) is \emptyset .

Lemma 3.1. Let (X, d) be a metric space. If $A \subseteq X$ is nowhere, then $X \setminus \overline{A}$ is open and dense.

Proof. Since \overline{A} is closed, clearly $X \setminus \overline{A}$ is open. Suppose $x \notin X \setminus \overline{A}$ and let $\epsilon > 0$. We want to find $y \in X \setminus \overline{A}$ such that $y \in B_\epsilon(x)$, which proves that $X \setminus \overline{A}$ is dense in X . Since $x \notin X \setminus \overline{A}$, we know $x \in \overline{A}$. Since A is nowhere dense, $\text{int}(\overline{A}) = \emptyset$. Hence we can find $y \notin \overline{A}$ such that $y \in B_\epsilon(x)$, which means $y \in X \setminus \overline{A}$, as desired. \square

Definition. Let (X, d) be a metric space. We say $A \subseteq X$ is **first category (meagre)** if we can write A as a countable union of nowhere dense sets. That is:

$$A = \bigcup_{n=1}^{\infty} K_n$$

where each $K_n \subseteq X$ is nowhere dense. When X is first category as a set, then we also say (X, d) is first category. Otherwise we say A is **second category**.

Example. Consider (\mathbb{R}, d) with the usual metric. Any nowhere dense set is first category. The rationals \mathbb{Q} is first category because it is the countable union of $q \in \mathbb{Q}$.

Question: Is \mathbb{R} , with the usual metric, first category?

Answer: It is not first category (not obvious) by the Baire Category Theorem.

Theorem 3.2 (Baire Category Theorem). Any non-empty complete metric space (X, d) is second category.

Example. The reals \mathbb{R} is not first category. The ℓ^p spaces for $1 \leq p < \infty$ are not first category. This does not apply to (\mathbb{Q}, d) with the Euclidean metric since it is not complete.

Corollary 3.3. Let (X, d) be a non-empty complete metric space with $X = \bigcup_{n=1}^{\infty} K_n$, then there is $n \geq 1$ such that $\text{int}(\overline{K_n}) \neq \emptyset$.

Proof. We know X is not first category by the BCT, so one of K_n is not nowhere dense. \square

Proof of BCT. Assume $(K_n)_{n=1}^{\infty}$ is a sequence of nowhere dense sets, we want to show $X \neq \bigcup_{n=1}^{\infty} K_n$ by constructing $x^* \in X$ such that $x^* \notin K_n$ for all $n \geq 1$. Pick any $x_0 \in X$ and $r_0 > 0$. Consider $\overline{B_{r_0}}(x_0)$. Since K_1 is nowhere dense, we can find $x_1 \in \overline{B_{r_0}}(x_0)$ and $r_1 < r_0/2$ such that:

$$\overline{B_{r_1}}(x_1) \cap K_1 = \emptyset \quad \text{and} \quad \overline{B_{r_1}}(x_1) \subseteq \overline{B_{r_0}}(x_0)$$

We repeat this process. Suppose we have defined x_n and r_n , we find x_{n+1} and r_{n+1} such that $x_{n+1} \in \overline{B_{r_n}}(x_n)$ and $r_n < r_{n+1}/2$ with:

$$\overline{B_{r_{n+1}}}(x_{n+1}) \cap K_{n+1} = \emptyset \quad \text{and} \quad \overline{B_{r_{n+1}}}(x_{n+1}) \subseteq \overline{B_{r_n}}(x_n)$$

We claim that $(x_n)_{n=1}^{\infty}$ is cauchy and its limit x^* satisfies our desired property. Let $m > n$, notice:

$$d(x_n, x_m) \leq r_n < \frac{r_{n-1}}{2} < \dots < \frac{r_0}{2^n}$$

Therefore $(x_n)_{n=1}^{\infty}$ is cauchy. Since (X, d) is complete, we let $\lim_{n \rightarrow \infty} x_n = x^* \in X$. Note that $(x_n)_{n=k}^{\infty}$ is a sequence in $\overline{B_{r_k}}(x_k)$ for all $k \geq 1$ and each such closed ball is closed. Therefore:

$$x^* = \lim_{n \rightarrow \infty} x_n \in \overline{B_{r_k}}(x_k)$$

Hence $x^* \in \overline{B_{r_n}}(x_n)$ for all $n \geq 1$. Hence $x^* \notin K_n$ for all $n \geq 1$, as desired. \square

Lecture 20, 2025/02/26

Definition. Let (X, d) be a metric space. We say $A \subseteq X$ is a G_{δ} **set** if there exist a countable sequence of open sets $U_n \subseteq X$ such that $A = \bigcap_{n=1}^{\infty} U_n$.

Example. Any open set is a G_{δ} set by definition.

Example. The irrational numbers are a G_{δ} set. Note that \mathbb{Q} is countable, so:

$$\mathbb{R} \setminus \mathbb{Q} = \bigcap_{r \in \mathbb{Q}} (\mathbb{R} \setminus \{r\})$$

Definition. Let (X, d) be a metric space. We say $A \subseteq X$ is an F_{σ} **set** if there is a countable sequence of closed sets $C_n \subseteq X$ such that $A = \bigcup_{n=1}^{\infty} C_n$.

Remark. Note that A is G_δ if and only if A^c is F_σ .

Example. Any closed set is a F_σ set.

Example. The interval $A = (0, 1)$ is an F_σ set because $(0, 1) = \bigcup_{n=2}^{\infty} [\frac{1}{n}, 1 - \frac{1}{n}]$.

Example. Note that $\mathbb{Q} = \bigcup_{r \in \mathbb{Q}} \{r\}$ is F_σ . However, we claim that \mathbb{Q} is NOT a G_δ set! Assume for a contradiction that \mathbb{Q} is a G_δ set, say:

$$\mathbb{Q} = \bigcap_{n=1}^{\infty} U_n$$

where each $U_n \subseteq \mathbb{R}$ is an open set. This means $\mathbb{Q} \subseteq U_n$ for all $n \geq 1$. Hence each U_n is an open denset set. Then $\mathbb{R} \setminus U_n$ is closed and nowhere dense. This means:

$$\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n=1}^{\infty} (\mathbb{R} \setminus U_n)$$

is a union of nowhere dense sets! Hence $\mathbb{R} \setminus \mathbb{Q}$ is first category. Since \mathbb{Q} is first category, we have:

$$\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$$

is first category, being the union of two sets that are first category. Since \mathbb{R} is complete, it is second category by BCT. Contradiction.

3.2 Nowhere Differentiable Functions

For this section we consider the space:

$$\mathcal{C}[0, 1] = \{f : [0, 1] \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$$

We will show that “most” functions $f \in \mathcal{C}[0, 1]$ are nowhere differentiable!

Definition. Let $f \in \mathcal{C}[0, 1]$. We say f is **Lipschitz at** $x_0 \in X$ if there is $K \in \mathbb{R}$ (dependent on x_0) such that for all $x \in [0, 1]$ we have:

$$|f(x_0) - f(x)| \leq K|x_0 - x|$$

We say f is **Lipschitz** if the choice of K is independent of x_0 .

Lemma 3.4. Let $f \in \mathcal{C}[0, 1]$ and $x_0 \in [0, 1]$. Assume $f'(x_0)$ exists, then f is Lipschitz at x_0 .

Proof. Let $c_1 = |f'(x_0)| \geq 0$. This implies that:

$$c_1 = \lim_{x \rightarrow x_0} \left| \frac{f(x) - f(x_0)}{x - x_0} \right|$$

There exists $\delta > 0$ (small enough such that $(x_0 - \delta, x_0 + \delta) \subseteq [0, 1]$) such that for all $x \in (x_0 - \delta, x_0 + \delta)$ we have that:

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| \leq c_1 + 1$$

Consider the function $h(x) = \frac{f(x) - f(x_0)}{x - x_0}$ on the set $[0, x_0 - \delta] \cup [x_0 + \delta, 1]$. Note that h is continuous on this compact set, hence it is bounded on it. Let $c_2 \in \mathbb{R}$ such that for all $x \in [0, x_0 - \delta] \cup [x_0 + \delta, 1]$ we have that:

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| \leq c_2$$

Let $K = \max\{c_1 + 1, c_2\} > 0$, then for all $x \in [0, 1]$ we have:

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| \leq K \implies |f(x) - f(x_0)| \leq K|x - x_0|$$

As desired. \square

Lemma 3.5. Let $f \in \mathcal{C}[0, 1]$ be Lipschitz at $x_0 \in [0, 1]$ with constant K . Then for all $a, b \in [0, 1]$ with $a \leq x_0 \leq b$ we have:

$$|f(a) - f(b)| \leq K|a - b|$$

Proof. Since $a \leq x_0 \leq b$ we have $|a - x_0| + |x_0 - b| = |a - b|$. Then:

$$|f(a) - f(b)| \leq |f(a) - f(x_0)| + |f(x_0) - f(b)| \leq K(|a - x_0| + |x_0 - b|) = K|a - b|$$

As desired. \square

Example. Define a function $f : [0, 1] \rightarrow \mathbb{R}$ by $f(x) = \sum_{n=1}^{\infty} 2^{-n} \cos(\pi 10^n x)$. We claim that $f \in \mathcal{C}[0, 1]$ but nowhere differentiable! It suffices to show it is the limit of a sequence of continuous functions. For each $N \geq 1$ let:

$$f_N(x) = \sum_{n=1}^N 2^{-n} \cos(\pi 10^n x)$$

Then each $f_N \in \mathcal{C}[0, 1]$. We claim that $(f_N)_{N=1}^{\infty}$ is Cauchy. Let $N > M$, we have:

$$\|f_N - f_M\|_{\infty} = \sup_{x \in [0, 1]} \left| \sum_{n=M+1}^N 2^{-n} \cos(\pi 10^n x) \right| \leq \sum_{n=M+1}^N 2^{-n} \rightarrow 0$$

because the series $\sum_{n=1}^{\infty} 2^{-n} = 1$ converges, its tail goes to 0. Therefore $(f_N)_{N=1}^{\infty}$ is Cauchy and since $\mathcal{C}[0, 1]$ is complete, it converges to $f \in \mathcal{C}[0, 1]$. To show it is nowhere differentiable, it suffices to show it is not Lipschitz at any $x_0 \in [0, 1]$. Write $x_0 = \sum_{k=1}^{\infty} \frac{a_k}{10^k}$ in base 10. Suppose f is Lipschitz at x_0

with constant $K \in \mathbb{R}$. Let $N \geq 1$ (to be chosen later), we define:

$$x_L = \sum_{k=1}^N \frac{a_k}{10^k} \quad \text{and} \quad x_R = x_L + \frac{1}{10^N}$$

We consider the difference between $f(x_R)$ and $f(x_L)$. Note that:

$$\cos(x) - \epsilon \leq \cos(x + \epsilon) \leq \cos(x) + \epsilon \quad (1)$$

for $\epsilon > 0$ small. By (1), for $1 \leq k \leq N$ we have:

$$\cos(\pi 10^k(x_L + 10^{-N})) - \cos(\pi 10^k x_L) = \pi 10^{-N+k}$$

For $k > N$, note that $10^k x_L$ and $10^k x_R$ are integers so:

$$\cos(\pi 10^k(x_L + 10^{-N})) - \cos(\pi 10^k x_L) = 0$$

With some work, this gives:

$$|f(x_L) - f(x_R)| \geq (5^N + \text{small stuff})|x_L - x_R|$$

If we pick N so that $5^N > K$ then this gives a contradiction.

Lecture 21, 2025/03/03

Theorem 3.6. Consider $\mathcal{C}[0, 1]$. The set of $f \in \mathcal{C}[0, 1]$ that are Lipschitz at at least one point are first category.

Proof. For each $k \geq 1$ we define:

$$A_k = \{f \in \mathcal{C}[0, 1] : f \text{ is Lipschitz somewhere with constant } k\}$$

We see that $A_k \subseteq A_{k+1}$ for all $k \geq 1$. The set of functions that are Lipschitz somewhere is:

$$L = \bigcup_{k=1}^{\infty} A_k$$

We want to show L is first category. It suffices to show every A_k is nowhere dense. We first claim that A_k is closed for all $k \geq 1$. Let $(f_n)_{n=1}^{\infty}$ be a cauchy sequence in A_k . Since $\mathcal{C}[0, 1]$ is complete, we know $f_n \rightarrow f^*$ in $\mathcal{C}[0, 1]$. We need to show $f^* \in A_k$. Let $(x_n)_{n=1}^{\infty}$ be a sequence such that f_n is Lipschitz with some constant k at x_n . As $[0, 1]$ is compact, it will have a convergent subsequence. Say $(x_{n_i})_{i=1}^{\infty}$ converges to x^* in $[0, 1]$. We claim that f^* is Lipschitz at x^* with constant k . For any $x \in [0, 1]$ and $i \geq 1$ we have:

$$\begin{aligned} |f^*(x) - f^*(x^*)| &= |f^*(x) - f_{n_i}(x) + f_{n_i}(x) - f_{n_i}(x_{n_i}) + f_{n_i}(x_{n_i}) - f_{n_i}(x^*) + f_{n_i}(x^*) - f^*(x^*)| \\ &\leq |f^*(x) - f_{n_i}(x)| + |f_{n_i}(x) - f_{n_i}(x_{n_i})| + |f_{n_i}(x_{n_i}) - f_{n_i}(x^*)| + |f_{n_i}(x^*) - f^*(x^*)| \\ &< \|f^* - f_{n_i}\|_{\infty} + k|x - x_{n_i}| + k|x_{n_i} - x^*| + \|f_{n_i} - f^*\|_{\infty} \end{aligned}$$

By taking the limit as $i \rightarrow \infty$ we know $\|f^* - f_{n_i}\|_\infty \rightarrow 0$ and $\|x_{n_i} - x^*\| \rightarrow 0$, so we have:

$$|f^*(x) - f^*(x^*)| \leq k|x - x^*|$$

Since $x \in [0, 1]$ is arbitrary, this proved that f^* is Lipschitz at x^* with constant k . Therefore $f^* \in A_k$ and A_k is closed. To show A_k is nowhere dense, it suffices to show $\text{int}(A_k) = \emptyset$ as A_k is closed. This means for all $\epsilon > 0$ and $f \in A_k$ we can find some $g \notin A_k$ with $\|f - g\|_\infty < \epsilon$. In fact, we can find g that is nowhere differentiable with $\|f - g\|_\infty < \epsilon$.

Pick $f \in A_k$ and $\epsilon > 0$. There is a polynomial function $p(x)$ such that $\|f - p\|_\infty < \frac{\epsilon}{2}$. Such a polynomial exists by the Stone-Weierstrass theorem that we will see later. By the example we did last lecture, we have a function that is differentiable nowhere, call it h . Recall that the h we constructed last time has $\|h\|_\infty \leq 1$. Now $p + \frac{\epsilon}{2}h$ is differentiable nowhere and:

$$\left\| f - \left(p + \frac{\epsilon}{2}h \right) \right\|_\infty \leq \|f - p\|_\infty + \frac{\epsilon}{2}\|h\|_\infty < \epsilon$$

This proved that $\text{int}(A_k) = \emptyset$, thus L is first category. □

Corollary 3.7. The set of functions that are differentiable somewhere is first category in $\mathcal{C}[0, 1]$.

3.3 Contraction Mapping Principle

Theorem 3.8 (Contraction Mapping Principle). Let (X, d) be a complete metric space. Let $f : X \rightarrow X$ be Lipschitz with constant $K < 1$ (such function is called a **contraction**). Then:

- (i). There is a unique $x^* \in X$ with $f(x^*) = x^*$. [Existence and Uniqueness of fixed point]
- (ii). For any $x_0 \in X$, we can construct a sequence $(x_n)_{n=1}^\infty$ by $x_{n+1} = f(x_n)$. Then $(x_n)_{n=1}^\infty$ is Cauchy and we have $x_n \rightarrow x^*$.

Example. Let $X = \mathbb{R}$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = \frac{1}{2}x + \frac{1}{2}$. Note that $x^* = 1$ is a fixed point. Let $x_0 = 2$ then $x_1 = 1 + 1/2 = 3/2$ and $x_2 = 1 + 1/4 = 5/4$. In general for $n \geq 0$ we have $x_n = 1 + 1/2^n$ and it is easy to see that $x_n \rightarrow 1$.

Proof. Pick $x_0 \in X$ and define $(x_n)_{n=1}^\infty$ by $x_{n+1} = f(x_n)$ for $n \geq 0$. We claim that $(x_n)_{n=1}^\infty$ is Cauchy. Let $d(x_0, x_1) = c \geq 0$. Then:

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq K \cdot d(x_n, x_{n-1}) \leq \cdots \leq K^n \cdot d(x_1, x_0) = K^n \cdot c$$

In general, if $m > n$ we have:

$$d(x_m, x_n) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq c \sum_{i=n}^{m-1} K^i \leq c \sum_{i=n}^{\infty} K^i$$

Since $K < 1$, this is a tail of a convergent series $\sum_{i=1}^{\infty} K^i$. Hence this $d(x_m, x_n) \rightarrow 0$ as $n \rightarrow \infty$. Therefore $(x_n)_{n=1}^{\infty}$ is cauchy, as desired. Since X is complete, $x_n \rightarrow x^*$ in X . Hence:

$$f(x^*) = f\left(\lim_{n \rightarrow \infty} x_n\right) = \lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = x^*$$

Assume we have two fixed points x^* and y^* in X , then:

$$d(x^*, y^*) = d(f(x^*), f(y^*)) \leq K \cdot d(x^*, y^*)$$

Hence $d(x^*, y^*) = 0$, so the fixed point is unique. \square

Lecture 22, 2025/03/05

Example (Logistic Equation). For $\lambda \in [0, 4]$ we define $f_\lambda : [0, 1] \rightarrow \mathbb{R}$ by $f_\lambda(x) = \lambda x(1 - x)$. This is used in population dynamics. Here λ represents the birth rate and x represents the current population and $1 - x$ represents the impact of limited resources.

Let $\lambda \in [0, 1)$, then we have:

$$\begin{aligned} |f_\lambda(x) - f_\lambda(y)| &= |\lambda x(1 - x) - \lambda y(1 - y)| \\ &= |\lambda x(1 - x - y) - \lambda y(1 - x - y)| \\ &= \lambda |x - y| |1 - x - y| \\ &< \lambda |x - y| \quad (|1 - x - y| \in [0, 1]) \end{aligned}$$

This means f_λ is Lipschitz with constant $\lambda < 1$. Hence it has a unique attractive fixed point satisfies $x^* = \lambda x^*(1 - x^*)$. In fact $x^* = 0$. This species is heading for extinction.

Theorem 3.9. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ with continuous derivative. Suppose $p \in \mathbb{R}$ is a fixed point of f such that $|f'(p)| < 1$. Then there exists $a, b \in \mathbb{R}$ with $a < p < b$ such that $f : [a, b] \rightarrow [a, b]$ is Lipschitz on $[a, b]$ with constant $K < 1$.

Proof. There exists an interval $[a, b]$ such that $|f'(x)| \leq K < 1$ (with $|f'(p)| < K < 1$). This is because we have a continuous derivative. We see for all $x, y \in [a, b]$ there is $c \in [x, y]$ by the mean value theorem that:

$$\left| \frac{f(x) - f(y)}{x - y} \right| = |f'(c)| \leq K$$

Hence f is Lipschitz on $[a, b]$ with constant K . \square

Example. Consider the Logistic equation again. Consider $f_\lambda(x) = \lambda x(1 - x)$ with $\lambda \in (1, 3)$. This has two fixed points, $x = 0$ or $x = 1 - \lambda^{-1}$. We have $f'_\lambda(x) = \lambda - 2\lambda x$. At $x = 0$ we have $f'_\lambda(x) = \lambda > 1$, so 0 is NOT an attractive fixed point. At $x^* = 1 - 1/\lambda$ then we have:

$$f'(x^*) = \lambda - 2\lambda \left(1 - \frac{1}{\lambda}\right) = 2 - \lambda < 1$$

Hence there is $a, b \in \mathbb{R}$ with $a < 1 - \lambda^{-1} < b$ such that for all $x_0 \in [a, b]$ we have $x_n \rightarrow 1 - \lambda^{-1}$, with $x_n = f_\lambda(x_{n-1})$. This means we have a stable population.

Definition. Let K be the set of all compact sets in \mathbb{R}^n . For $A, B \in K$ we define:

$$\begin{aligned} d_H(A, B) &= \inf\{\epsilon > 0 : A \subseteq B + B_\epsilon(0), B \subseteq A + B_\epsilon(0)\} \\ &= \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\} \end{aligned}$$

This is known as the **Hausdorff metric** on K .

Example. Let $A = [0, 1]$ and $B = [\frac{1}{3}, \frac{3}{2}]$. Note that:

$$A \subseteq B + B_{1/3+\epsilon}(0) \quad \text{and} \quad B \subseteq A + B_{1/2+\epsilon}(0)$$

for all $\epsilon > 0$. Hence the $d_H(A, B) = \frac{1}{2}$.

Remark. This indeed gives us a metric. [Exercise]

Construction 3.10. Let $n = 1$ and K denote the set of compact sets in \mathbb{R} . Consider the following map $S : K \rightarrow K$ defined by:

$$S(A) = \frac{1}{3}A \cup \left(\frac{1}{3}A + \frac{2}{3}\right)$$

If $A = [1, 2]$ then $S(A) = [\frac{1}{3}, \frac{2}{3}] \cup [1, \frac{4}{3}]$. If we let $A_0 = A$ and define $A_n = S(A_{n-1})$, then A_n converges to the cantor set in (K, d_H) . In fact, we can prove S is Lipschitz with constant $1/3$ and thus by the contraction mapping principle, it has a unique fixed point C , the cantor set.

Let $A, B \in K$ be arbitrary. Say $d_H(A, B) = c > 0$ (if $A = B$ then trivial). This means $A \subseteq B + B_c(0)$ and $B \subseteq A + B_c(0)$. Note that:

$$S(A) = \frac{1}{3}A \cup \left(\frac{1}{3}A + \frac{2}{3}\right) \quad \text{and} \quad S(B) = \frac{1}{3}B \cup \left(\frac{1}{3}B + \frac{2}{3}\right)$$

Since $A \subseteq B + B_c(0)$ we have:

$$\frac{1}{3}A \subseteq \frac{1}{3}(B + B_c(0)) = \frac{1}{3}B + B_{c/3}(0) \quad \text{and} \quad \frac{1}{3}B \subseteq \frac{1}{3}A + B_{c/3}(0)$$

Similarly we have:

$$\begin{aligned} \frac{1}{3}B &\subseteq \frac{1}{3}A + B_{c/3}(0) \\ \frac{1}{3}A + \frac{2}{3} &\subseteq \frac{1}{3}B + \frac{2}{3} + B_{c/3}(0) \\ \frac{1}{3}B + \frac{2}{3} &\subseteq \frac{1}{3}A + \frac{2}{3} + B_{c/3}(0) \end{aligned}$$

Hence we have $d_H(S(A), S(B)) = \frac{c}{3} = \frac{1}{3}d_H(A, B)$. With some work, we can show K is complete. Therefore the map S has a unique attractive fixed point, which is the Cantor set!

3.4 Newton's Method

We see from the previous result that if g has a fixed point $g(p) = p$ and $|g'(p)| = \lambda < 1$, then p is an attractive fixed point, within a interval around p . Moreover:

$$|g^{(n)}(x) - p| \leq \lambda^n |x - p| \quad (\text{approximately})$$

We see that the smaller λ is, the faster the convergence is. This implies that $\lambda = 0$ is ideal. This is what is explained by Newton's method.

Theorem 3.11. Let f be twice continuously differentiable such that $f(p) = 0$ and $f'(p) \neq 0$ for some $p \in \mathbb{R}$. Define g by:

$$g(x) = x - \frac{f(x)}{f'(x)}$$

Then $g(p) = p$ and $g'(p) = 0$.

Proof. It is easy to see that $g(p) = p$. Moreover,

$$g'(p) = 1 - \frac{f'(p)f'(p) - f''(p)f(p)}{(f'(p))^2} = 1 - 1 = 0$$

As desired. □

Corollary 3.12. If we start sufficiently close to p , then we are attracted to p .

Consider the Taylor polynomial of g around $x = p$, we have:

$$\begin{aligned} g(x) &= g(p) + g'(p)(x - p) + \frac{g''(p)}{2!}(x - p)^2 + \cdots \\ &= p + 0 + C(x - p)^2 + \cdots \end{aligned}$$

That is, if $x = p + \epsilon$, then $g(x) \approx p + C\epsilon^2$. [**Quadratic convergence**]

3.5 Metric Completion

Definition. Let (X, d) be a metric space. We say (Y, ρ) is a **completion** of (X, d) if (Y, ρ) is a complete space and there exists an **isometry** $J : X \rightarrow Y$ (that is, $\rho(Jx, Jy) = d(x, y)$ for all $x, y \in X$) such that $\overline{JX} = Y$ (JX is dense in Y).

Remark. Our goal is to show that for a metric space (X, d) ,

1. The completion of (X, d) always exists.

2. The completion of (X, d) is unique (up to isometric isomorphism).
3. Show the completion of \mathbb{Q} is \mathbb{R} .
4. Discuss the completion of \mathbb{Q} with p -adic metric.

Theorem 3.13. Every metric space has a completion.

Proof. Let (X, d) be a metric space. Recall that:

$$\mathcal{C}^b(X) = \{f : X \rightarrow \mathbb{R} \mid f \text{ continuous and bounded}\}$$

is a complete metric space (in fact normed) with norm $\|f\|_\infty = \sup_{x \in X} |f(x)|$. We will find a closed subset $Y \subseteq \mathcal{C}^b(X)$ (then Y is complete) and an isometry $J : X \rightarrow Y$ such that JX is dense in Y .

Fix $x_0 \in X$. We define $J : X \rightarrow \mathcal{C}^b(X)$ by $J(x) = f_x$, where:

$$f_x(y) = d(x, y) - d(x_0, y)$$

We claim that these functions have the desired property. Clearly each f_x is continuous, as $d(x, \cdot)$ and $d(x_0, \cdot)$ are both continuous. To see they are bounded, we note that:

$$f_x(y) = d(x, y) - d(x_0, y) \leq d(x, x_0)$$

by the triangle inequality. For fixed $x \in X$, we know $d(x, x_0)$ is a constant. Hence f_x is bounded above. Similarly we have:

$$f_x(y) = d(x, y) - d(x_0, y) \geq -d(x, x_0)$$

by the triangle inequality again. Hence $f_x \in \mathcal{C}^b(X)$ for all $x \in X$. Now we claim $J : x \mapsto f_x$ is an isometry. Indeed, let $x, z \in X$ we have:

$$\begin{aligned} \|f_x - f_z\|_\infty &= \sup_{y \in X} |d(x, y) - d(x_0, y) - d(z, y) + d(x_0, y)| \\ &= \sup_{y \in X} |d(x, y) - d(z, y)| \\ &\leq \sup_{y \in X} d(x, z) = d(x, z) \end{aligned}$$

This bound can be achieved at $y = x$, so we get $\|f_x - f_z\|_\infty = d(x, z)$. Let $Y = \overline{JX}$ in $\mathcal{C}^b(X)$, so Y is complete, being a closed subset of a complete space. By construction we have $\overline{JX} = Y$, so JX is dense in Y . Since J is an isometry, $(Y, \|\cdot\|_\infty)$ is a completion of X . \square

Example. Let $X = \mathbb{Q} \cap [0, 1]$ with usual metric d . Let $x_0 = \frac{1}{2}$. We can approach $\pi/10$ with a cauchy sequence $(x_n)_{n=1}^\infty$ in (X, d) .

Construction 3.14. Let (X, d) be a metric space. We define:

$$Z = \{\text{cauchy sequences in } X\}$$

We define a psuedo-metric $\tilde{\rho}$ on Z by:

$$\tilde{\rho}((x_n), (y_n)) = \lim_{d \rightarrow \infty} d(x_n, y_n)$$

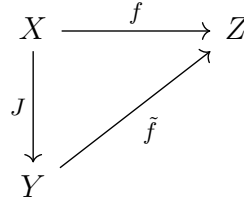
for $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in Z . This is possbily NOT a metric because (for example the distance between $(1, 0, \dots)$ and $(0, 0, \dots)$ is zero but they are different).

We say two Cauchy sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are **equivalent** if $\tilde{\rho}((x_n), (y_n)) = 0$. Let:

$$Y = Z / \sim = \{\text{equivalence classes of cauchy sequences in } X\}$$

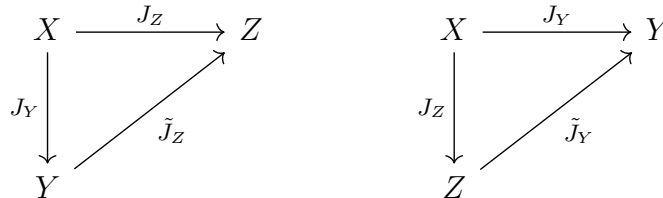
Let $J : X \rightarrow Y$ by $J(x) = [(x_n)_{n=1}^{\infty}]$ where $x_n = x$ for all $n \geq 1$. Then J and Y have all the desired properties. Hence Y is a completion of X .

Theorem 3.15. Let (X, d) be a metric space with completion (Y, ρ) and $J : X \rightarrow Y$. Let (Z, σ) be a complete metric space and $f : X \rightarrow Z$ is uniformly continuous. Then there is a unique uniformly continuous map $\tilde{f} : Y \rightarrow Z$ with $\tilde{f}(J(x)) = f(x)$ for all $x \in X$.



Corollary 3.16. Let (X, d) be a metric space with completion (Y, ρ) and (Z, σ) given by $J_Y : X \rightarrow Y$ and $J_Z : X \rightarrow Z$, respectively. Then J_Y and J_Z can be extended to isometries $\tilde{J}_Y : Z \rightarrow Y$ and $\tilde{J}_Z : Y \rightarrow Z$ such that \tilde{J}_Z and \tilde{J}_Y are inverses of each other.

Proof. By Theorem 3.15, we have these two diagrams:



Hence \tilde{J}_Z and \tilde{J}_Y exist. Since $J_Z = \tilde{J}_Z \circ J_Y$ and $J_Y = \tilde{J}_Y \circ J_Z$, it is not hard to see \tilde{J}_Y and \tilde{J}_Z are inverses of each other. Now we want to show \tilde{J}_Y and \tilde{J}_Z are isometries. We first show $\tilde{J}_Z : Y \rightarrow Z$

is an isometry. Let $a, b \in Y$ be arbitrary. We want to show that:

$$\rho(a, b) = \sigma(\tilde{J}_Z(a), \tilde{J}_Z(b))$$

Since $J_Y(X)$ is dense in Y , we can find sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ in X such that:

$$a = \lim_{n \rightarrow \infty} J_Y(a_n) \quad \text{and} \quad b = \lim_{n \rightarrow \infty} J_Y(b_n)$$

Now, by continuity we have that:

$$\begin{aligned} \rho(a, b) &= \lim_{n \rightarrow \infty} \rho(J_Y(a_n), J_Y(b_n)) = \lim_{n \rightarrow \infty} d(a_n, b_n) = \lim_{n \rightarrow \infty} \sigma(J_Z(a_n), J_Z(b_n)) \\ &= \lim_{n \rightarrow \infty} \sigma(\tilde{J}_Z(J_Y(a_n)), \tilde{J}_Z(J_Y(b_n))) = \sigma(\tilde{J}_Z(a), \tilde{J}_Z(b)) \end{aligned}$$

Hence \tilde{J}_Z is an isometry. By the same argument, \tilde{J}_Y is an isometry. \square

Proof of Theorem 3.15. Step 1. Let $(x_n)_{n=1}^\infty$ be a cauchy sequence in X , we claim $(f(x_n))_{n=1}^\infty$ is a cauchy sequence in Z . Let $\epsilon > 0$. Since $f : X \rightarrow Z$ is uniformly continuous, there is $\delta > 0$ such that for $x, y \in X$:

$$d(x, y) < \delta \implies \sigma(f(x), f(y)) < \epsilon$$

Since $(x_n)_{n=1}^\infty$ is cauchy, we can find $N \geq 1$ such that for all $n, m \geq N$ we have $d(x_n, x_m) < \delta$. Hence for $n, m \geq N$ we have $\sigma(f(x_n), f(x_m)) < \epsilon$. Therefore $(f(x_n))_{n=1}^\infty$ is cauchy in (Z, σ) .

Step 2. Let $y \in Y$. Since JX is dense in Y , we can find a cauchy sequence $(x_n)_{n=1}^\infty$ such that $\lim_{n \rightarrow \infty} J(x_n) = y$. We define:

$$\tilde{f}(y) = \lim_{n \rightarrow \infty} f(x_n)$$

Since Z is complete, this limit exists (shown in Step 1 that $(f(x_n))_{n=1}^\infty$ is cauchy). Now we need to check this definition is well-defined. That is, we need to show this definition does not depend on the choice of the cauchy sequences. If we chose two different cauchy sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ in X such that:

$$y = \lim_{n \rightarrow \infty} J(x_n) = \lim_{n \rightarrow \infty} J(y_n)$$

Construct a new Cauchy sequence $(z_n)_{n=1}^\infty = (x_1, y_1, x_2, y_2, \dots)$. We see that:

$$\tilde{f}(y) = \lim_{n \rightarrow \infty} f((z_n)_{n=1}^\infty) = \lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} f(y_n)$$

Therefore \tilde{f} does not depend on the choices of cauchy sequences. Hence \tilde{f} is well-defined. Moreover, for $x \in X$ we can pick the constant cauchy sequence $(x)_{n=1}^\infty$ with $y = J(x)$. Then we have:

$$\tilde{f}(y) = \lim_{n \rightarrow \infty} f(x) = f(x)$$

Therefore we have $\tilde{f}(J(x)) = f(x)$ for all $x \in X$.

Step 3. We need to show \tilde{f} is uniformly continuous. Pick $\epsilon > 0$. Pick $\delta > 0$ such that for $x, y \in X$ with $d(x, y) < \delta$ we have $\sigma(f(x), f(y)) < \epsilon$. Pick points in Y with $\rho(y_1, y_2) < \delta$. Find cauchy sequence $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ in X such that $J(a_n) \rightarrow y_1$ and $J(b_n) \rightarrow y_2$. We can pick $N \geq 1$ sufficiently large so that $d(a_n, b_n) < \delta$ for $n > N$. Hence $\sigma(f(a_n), f(b_n)) < \epsilon$ for $n > N$. Therefore:

$$\sigma(\tilde{f}(y_1), \tilde{f}(y_2)) \leq \epsilon$$

by taking the limit. Therefore \tilde{f} is uniformly continuous. \square

Lecture 25, 2025/03/12

3.6 The Real Numbers

Definition. A **field** is a bunch of things you can add and multiply and subtract and divide (by nonzero elements).

Example. Rational, real and complex numbers are fields. $\mathbb{Z}/p\mathbb{Z}$ is a field for prime p . The rational functions $\mathbb{R}(x)$ is a field.

Definition. We say a field \mathbb{F} is an **ordered field** if we can write \mathbb{F} as a disjoint union:

$$\mathbb{F} = \mathbb{P} \sqcup \{0\} \sqcup (-\mathbb{P})$$

such that $a, b \in \mathbb{P}$ implies $a + b \in \mathbb{P}$ and $ab \in \mathbb{P}$. Think of \mathbb{P} as the set of positive elements in \mathbb{F} .

Lemma 3.17. If \mathbb{F} is an ordered field, then $1 \in \mathbb{P}$.

Proof. If $1 \in \mathbb{P}$ then we are done. If $1 \in -\mathbb{P}$, then $-1 \in \mathbb{P}$. Hence $1 = (-1)(-1) \in \mathbb{P}$. This is a contradiction. \square

Example. The reals is an ordered field. Let $\mathbb{P} = \{x \in \mathbb{R} : x > 0\}$.

Example. By the same logic, the rationals \mathbb{Q} is an ordered field with $\mathbb{P} = \{x \in \mathbb{Q} : x > 0\}$.

Example. The complex numbers \mathbb{C} cannot be made into an order field! Suppose there is \mathbb{P} . Assume that $i \in \mathbb{P}$, then $-1 = i \cdot i \in \mathbb{P}$ and thus $1 \in -\mathbb{P}$. This contradicts the previous Lemma. Similarly if $i \in -\mathbb{P}$ we would get another contradiction.

Example. $\mathbb{Z}/p\mathbb{Z}$ is not an ordered field. Assume it is $1 \in \mathbb{P}$. Then $p \cdot 1 = 1 + \cdots + 1 = 0 \in \mathbb{P}$. This is a contradiction.

Example. $\mathbb{R}(x)$ is an ordered field! We define:

$$\mathbb{P} = \left\{ \frac{p(x)}{q(x)} \in \mathbb{R}(x) : \text{there is } T \in \mathbb{R} \text{ such that } \frac{p(t)}{q(t)} > 0 \text{ for all } t \geq T \right\}$$

Not hard to check that \mathbb{P} has the desired property.

Definition. Let \mathbb{F} be an ordered field with $\mathbb{F} = \mathbb{P} \sqcup \{0\} \sqcup (-\mathbb{P})$. We define $a < b$ if $b - a \in \mathbb{P}$. We can define $a \leq b$ if $a = b$ or $a < b$. It is easy to check $<$ defines a total order of \mathbb{F} .

Example. In $\mathbb{R}(x)$ with \mathbb{P} above, we have $\frac{1}{x} < 1$ because $1 - \frac{1}{x} > 0$ for $x \geq 2$.

Definition. We say an ordered field \mathbb{F} has the **least upper bound property (LUBP)** if for all $\emptyset \neq S \subseteq \mathbb{F}$ that has an upper bound (there is $M \in \mathbb{F}$ with $s \leq M$ for all $s \in S$), there exists a **least upper bound** $x \in \mathbb{F}$ in the sense that if $y < x$ then y is NOT an upper bound of S .

Example. We have seen that \mathbb{R} has LUBP by Theorem 1.13

Example. \mathbb{Q} does not have LUBP. Take $S = \{x \in \mathbb{Q} : x^2 < 2\}$. This does not have a supremum.

Example. The set of rational functions $\mathbb{R}(x)$ does NOT have the LUBP. Take:

$$S = \left\{ \frac{a}{x} : a \in \mathbb{R} \right\}$$

This has an upper bound but does not have a least upper bound.

Notation. Let \mathbb{F} be an ordered field. For $n \in \mathbb{N}$ we define:

$$n := \underbrace{1 + \cdots + 1}_{n \text{ times}} \in \mathbb{P}$$

Definition. We say an ordered field \mathbb{F} is **Archimedean** if for any $x > 0$ there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < x$.

Example. \mathbb{R} and \mathbb{Q} are archimedean.

Example. The rational functions $\mathbb{R}(x)$ is NOT archimedean. Note that $\frac{1}{x} < \frac{1}{n}$ for all $n \in \mathbb{N}$.

Theorem 3.18. Let \mathbb{F} be an ordered field. Then:

- (a). There is a nonzero field homomorphism $\phi : \mathbb{Q} \rightarrow \mathbb{F}$. [This means ϕ is injective, so \mathbb{F} contains a copy of \mathbb{Q}] and $\mathbb{Q} \cap \mathbb{P} = \{x \in \mathbb{Q} : x > 0\}$.
- (b). If \mathbb{F} has the LUBP then \mathbb{F} is archimedean.
- (c). If \mathbb{F} is archimedean and $x < y$, then there exists $\frac{m}{n} \in \mathbb{Q}$ such that $x < \frac{m}{n} < y$.

Proof. (a). For $m, n \in \mathbb{Q}$ with $m, n > 0$ we define:

$$\phi\left(\frac{m}{n}\right) = \underbrace{(1 + \cdots + 1)}_{m \text{ times}} \underbrace{(1 + \cdots + 1)}_{n \text{ times}}^{-1}$$

For $q \in \mathbb{Q}$ with $q < 0$ we just define $\phi(-q) = -\phi(q)$. Easy to check ϕ is nonzero (since $\phi(1) = 1 \neq 0$). Hence ϕ is injective and \mathbb{F} contains a copy of \mathbb{Q} .

(b). Assume \mathbb{F} has the LUBP. We define:

$$J = \{x \in \mathbb{P} : nx < 1 \text{ for all } n \in \mathbb{N}\}$$

If $J = \emptyset$ then for all $x \in \mathbb{P}$ there is $n \in \mathbb{N}$ such that $nx > 1$, so $x > 1/n$ as required. Suppose J is not empty, then J is bounded above by 1. By the LUBP, it has a least upper bound y . Pick $x_1, x_2 \in J$. Then $2nx_1, 2nx_2 < 1$ for all $n \in \mathbb{N}$. Hence:

$$n(x_1 + x_2) < 1$$

for all $n \in \mathbb{N}$. This means $x_1 + x_2 \in J$. Therefore $x_1 + x_2 \leq y$ and $x_1 \leq y - x_2$ for all $x_1 \in J$. Hence x_1 is a better upper bound for J , meaning y is not the least upper bound. This is a contradiction, so $J = \emptyset$. Therefore \mathbb{F} is archimedean. \square

Lecture 26, 2025/03/14

Definition. Let \mathbb{F} and \mathbb{K} be ordered fields. A map $\gamma : \mathbb{F} \rightarrow \mathbb{K}$ is an **embedding** if γ is a field homomorphism and preserves order. That is, $\gamma(a) \leq \gamma(b)$ whenever $a \leq b$.

Theorem 3.19. Let \mathbb{F} be an Archimedean ordered field and \mathbb{K} is a complete ordered field. Then there is an embedding from \mathbb{F} to \mathbb{K} .

Example. Both \mathbb{Q} and \mathbb{R} are Archimedean and $\mathbb{R}, \mathbb{C}, \mathbb{R}(x)$ are complete.

Proof of Theorem 3.19. Both \mathbb{F} and \mathbb{K} are ordered fields. Hence they contain a copy of \mathbb{Q} . Let us call them $\mathbb{Q}_{\mathbb{F}}$ and $\mathbb{Q}_{\mathbb{K}}$, respectively. We define:

$$\gamma_0 : \mathbb{Q}_{\mathbb{F}} \rightarrow \mathbb{Q}_{\mathbb{K}} \text{ by } \gamma_0(r_{\mathbb{F}}) = r_{\mathbb{K}}$$

For $f \in \mathbb{F}$ we define $S_f = \{r \in \mathbb{Q}_{\mathbb{F}} : r < f\}$. Note that S_f is bounded above, hence $\gamma_0(S_f)$ is bounded above. Now we define:

$$\gamma(f) := \sup\{\gamma_0(r) : \gamma_0(r) \in \gamma_0(S_f)\}$$

here the supremum is taken in \mathbb{K} . As \mathbb{K} is complete we have $\gamma(f) \in \mathbb{K}$. This defined a map $\gamma : \mathbb{F} \rightarrow \mathbb{K}$. Let $f_1, f_2 \in \mathbb{F}$ with $f_1 < f_2$. We know there exists $r \in \mathbb{Q}_{\mathbb{F}}$ such that $f_1 < r < f_2$. This

tells us that:

$$\gamma_0(s) < \gamma_0(r) \text{ for all } s \in S_{f_1} \text{ and } \gamma_0(r) < \gamma_0(s) \text{ for some } s \in S_{f_2}$$

Hence $\gamma(f_1) < \gamma(f_2)$. That is, γ preserves order. With loss of generality, assume $0 < f_1, f_2$. Now:

$$\begin{aligned} S_{f_1} + S_{f_2} &= \{r_1 < f_1 : r_1 \in \mathbb{Q}_{\mathbb{F}}\} + \{r_2 < f_2 : r_2 \in \mathbb{Q}_{\mathbb{F}}\} \\ &= \{r_1 + r_2 : r_1 < f_1, r_2 < f_2, r_1, r_2 \in \mathbb{Q}_{\mathbb{F}}\} \\ &= \{r_3 : r_3 < f_1 + f_2, r_3 \in \mathbb{Q}_{\mathbb{F}}\} \\ &= S_{f_1+f_2} \end{aligned}$$

This gives $\gamma(f_1) + \gamma(f_2) = \gamma(f_1 + f_2)$. Now:

$$\begin{aligned} S_{f_1} \cdot S_{f_2} &= (\{0 \leq r_1 < f_1 : r_1 \in \mathbb{Q}_{\mathbb{F}} \cap \mathbb{P}\} \cup (-\mathbb{P})) \cdot (\{0 \leq r_2 < f_2 : r_2 \in \mathbb{Q}_{\mathbb{F}} \cap \mathbb{P}\} \cup (-\mathbb{P})) \\ &= \{0 \leq r_1 \cdot r_2 : r_1 < f_1, r_2 < f_2, r_1, r_2 \in \mathbb{Q}_{\mathbb{F}} \cap \mathbb{P}\} \cup \{0\} \cup (-\mathbb{P}) \\ &= \{0 \leq r_1 \cdot r_2 < f_1 \cdot f_2 : r_1, r_2 \in \mathbb{Q}_{\mathbb{F}} \cap (\mathbb{P} \cup \{0\})\} \cup (-\mathbb{P}) \\ &= S_{f_1 f_2} \end{aligned}$$

Hence we have $\gamma(f_1 f_2) = \gamma(f_1) \gamma(f_2)$. This defined a field homomorphism $\mathbb{F} \rightarrow \mathbb{K}$. Since this is nonzero, it is an injection (an embedding). \square

Corollary 3.20. There is a unique complete Archimedean ordered field, up to isomorphism.

Proof. Assume \mathbb{K} and \mathbb{F} are both Archimedean complete ordered field. By Theorem 3.19, there are order preserving homomorphisms:

$$\gamma_0 : \mathbb{K} \rightarrow \mathbb{F} \text{ and } \gamma_1 : \mathbb{F} \rightarrow \mathbb{K}$$

Since γ_0 is identity on $\mathbb{Q}_{\mathbb{K}}$ and γ_1 is identity on $\mathbb{Q}_{\mathbb{F}}$, we know $\gamma_0 \circ \gamma_1$ is identity on $\mathbb{Q}_{\mathbb{F}}$. By the completeness of \mathbb{F} , the map $\gamma_0 \circ \gamma_1$ is identity on \mathbb{F} . \square

Definition. We call this unique complete Archimedean ordered field \mathbb{R} , the **real numbers**.

Remark. This only proved the uniqueness of such complete archimedean ordered field, but we have not constructed such field yet. Now we are going to provide a detailed construction of real numbers.

Definition. We say $\emptyset \neq C \subseteq \mathbb{Q}$ is a **cut** if for all $x \in C$ we have $y \in C$ for all $y < x$. Further, we requires that $C \neq \mathbb{Q}$.

Example. $C = \{x \in \mathbb{Q} : x < 7\}$ is a cut. [This represents the real number 7.]

Example. $C = \{x \in \mathbb{Q} : x \leq 0 \text{ or } x^2 < 2\}$ is a cut. [This represents the real number $\sqrt{2}$.]

Definition. Let C_1, C_2 be cuts. We define $C_1 < C_2$ if $C_1 \subsetneq C_2$. We define $C_1 \leq C_2$ if $C_1 \subseteq C_2$.

Theorem 3.21. Let $\mathcal{R} = \{C \subseteq \mathbb{Q} : C \text{ is a cut}\}$. Then (\mathcal{R}, \leq) has the least upper bound property.

Proof. Let $\mathcal{S} \subseteq \mathcal{R}$. Suppose there is $P \in \mathcal{R}$ such that $R < P$ for all $S \in \mathcal{S}$. We claim that \mathcal{S} has a least upper bound. We define:

$$E = \bigcup_{S \in \mathcal{S}} S$$

We see that if $y \in E$ then $y \in S$ for some $S \in \mathcal{S}$. If $x < y$ then $x \in S \subseteq E$. This proved that E is a cut. As $S \subseteq E$ for all $S \in \mathcal{S}$, hence E is an upper bound of \mathcal{S} . We claim that E is the least upper bound. Suppose for a contradiction that $F < E$ is also an upper bound. Since $F < E$ there is $r \in E \setminus F$. Then as $r \in E$, there exists $S \in \mathcal{S}$ such that $r \in S$. But $S \subsetneq F$, which means $S \not\leq F$. This means F is not an upper bound, contradiction. Therefore E is the least upper bound. \square

Construction 3.22. We need to show how we can write \mathcal{R} as a field, and show it is complete. For cuts $C_1, C_2 \in \mathcal{R}$, we define:

$$C_1 + C_2 := \{c_1 + c_2 : c_1 \in C_1, c_2 \in C_2\}$$

It is easy to check that $C_1 + C_2$ is a cut. If $0 \in C_1, C_2$ we define:

$$C_1 \cdot C_2 := \{c_1 c_2 : c_1, c_2 \geq 0, c_1 \in C_1, c_2 \in C_2\} \cup (-\mathbb{Q})$$

The other cases are similar to define. [The idea of dedekind is that we want to define a real number α as the cut $\{r \in \mathbb{Q} : r < \alpha\}$. The multiplication is intuitive but tricky to write down.] With some work we can show \mathcal{R} is a complete Archimedean ordered field.

Lecture 27, 2025/03/17

We define a metric on \mathcal{R} by:

$$d(C_1, C_2) := \max \left(\sup_{c_1 \in C_1} \inf_{c_2 \in C_2} |c_1 - c_2|, \sup_{c_2 \in C_2} \inf_{c_1 \in C_1} |c_1 - c_2| \right)$$

Consider the map $\gamma : \mathbb{Q} \rightarrow \mathcal{R}$ by $\gamma(q) = \{r \in \mathbb{Q} : r < q\}$. Then γ is an embedding of \mathbb{Q} into \mathcal{R} such that $\gamma(\mathbb{Q})$ is dense in \mathcal{R} . Hence \mathcal{R} is a completion of \mathbb{Q} .

3.7 The p -adic Numbers

Let p be a prime number. We know $(\mathbb{Z}, |\cdot|_p)$ and $(\mathbb{Q}, |\cdot|_p)$ are not complete. We can get this by either the Baire category theorem or a counting argument.

Definition. We call the completion of $(\mathbb{Z}, |\cdot|_p)$ the p -adic integers.

Definition. We call the completion of $(\mathbb{Q}, |\cdot|_p)$ the **p -adic numbers**.

Theorem 3.23. Let $(x_n)_{n=1}^\infty$ be a Cauchy sequence in $(\mathbb{Z}, |\cdot|_p)$. Either $\lim x_n = 0$ in p -adic norm or for all $k \geq 1$, the sequence $(a_n \pmod{p^k})_{n=1}^\infty$ is an eventually constant sequence in $\mathbb{Z}/p^k\mathbb{Z}$.

Proof. If $\lim x_n = 0$ then we are done. Suppose not. Pick $k \geq 1$ arbitray. Then $|\cdot|_p : (\mathbb{Q}, |\cdot|_p) \rightarrow \mathbb{R}$ is a continuous function. Hence $(|x_n|_p)_{n=1}^\infty$ is a cauchy sequence in \mathbb{R} that does not converge to 0. We know that $|\cdot|_p$ takes on values of the form p^r for $r \in \mathbb{Z}$. Therefoe $(|x_n|_p)_{n=1}^\infty$ is eventually constant in \mathbb{R} . Say $|x_n|_p = p^{-N}$ for n large enough and some $N \geq 0$. Let $\epsilon < p^{-k}$. There exists N_0 such that $|x_n - x_m|_p < p^{-k}$ for all $n, m \geq N_0$. This means $p^k \mid (x_n - x_m)$ so $x_n \equiv x_m \pmod{p^k}$. \square

Theorem 3.24. For all $a \in \mathbb{Z}_p$ there exists $a_0 \in \{0, 1, \dots, p-1\}$ such that $|a - a_0|_p \leq \frac{1}{p}$.

Proof. We know \mathbb{Z} is dense in \mathbb{Z}_p . Pick $k \in \mathbb{Z}$ such that $|k - a| \leq \frac{1}{p}$. Pick $a_0 \in \{0, \dots, p-1\}$ such that $k \equiv a_0 \pmod{p}$. This gives:

$$|a - a_0|_p \leq |a - k|_p + |k - a_0|_p \leq \frac{1}{p} + \frac{1}{p} = \frac{2}{p}$$

If $p \geq 3$, then this gives that $|a - a_0| \leq \frac{1}{p}$ (as the norm is 1 or less than $1/p$). If $p = 2$ we need to do more work, but it is still easy. \square

Corollary 3.25. Let $a \in \mathbb{Z}_p$. There exists $a_0, a_1, \dots, a_n \in \{0, \dots, p-1\}$ such that:

$$|a - (a_0 + a_1p + \dots + a_np^n)|_p \leq \frac{1}{p^{n+1}}$$

Remark. This means we can write $a = \sum_{n=0}^\infty a_np^n$ with $a_n \in \{0, \dots, p-1\}$ for $a \in \mathbb{Z}_p$.

Corollary 3.26. For each $a \in \mathbb{Z}_p$ there is a sequence $(a_n)_{n=0}^\infty$ with $a_n \in \{0, \dots, p-1\}$ such that:

$$a = \sum_{n=0}^\infty a_np^n = \lim_{N \rightarrow \infty} x_N$$

where $x_N := \sum_{n=0}^N a_np^n$ and $(x_N)_{n=0}^\infty$ is cauchy in $|\cdot|_p$.

Example. Let $p = 3$. Note that:

$$-1 = \sum_{n=0}^\infty 2 \cdot 3^n = 2 + 2 \cdot 3 + 2 \cdot 3^2 + \dots$$

What about $\alpha = \sum_{n=0}^\infty 2 \cdot 3^n + \sum_{n=0}^\infty 2 \cdot 3^n$? The N -th partial sum is

$$1 + \sum_{n=1}^N 2 \cdot 3^n + 3^{N+1}$$

Taking $N \rightarrow \infty$ shows that the associated sequence is $(1, 2, 2, 2, \dots)$.

Example. Let $p = 3$ again. Consider the multiplication $\alpha = (\sum_{n=0}^{\infty} 2 \cdot 3^n)(\sum_{n=0}^{\infty} 2 \cdot 3^n)$. The product of the N -partial sums is equal to:

$$1 + (3^{N+1} - 2) \cdot 3^{N+1} \equiv 1 \pmod{3^{N+1}}$$

Hence the associated sequence is $(1, 0, 0, 0, \dots)$.

Theorem 3.27. \mathbb{Z}_p is compact for all prime p .

Proof. We know that \mathbb{Z}_p is complete, as it is the completion of $(\mathbb{Z}, |\cdot|_p)$. We also know \mathbb{Z} is totally bounded by an assignment. Pick $\epsilon > 0$ and $k \geq 0$ such that $p^{-k} < \epsilon$. For every $a \in \mathbb{Z}_p$ we can find $a_i \in \{0, \dots, p-1\}$ such that:

$$|a - (a_0 + a_1p + \dots + a_kp^k)|_p \leq \frac{1}{p^{k+1}} < \epsilon$$

There are p^{k+1} choices for a_0, \dots, a_k , hence this gives a ϵ -net. □

Remark. Addition and multiplication on \mathbb{Q}_p are similar because we can always write $\alpha \in \mathbb{Q}_p$ as:

$$\alpha = \sum_{n=-N}^{\infty} a_n p^n$$

for $a_n \in \{0, \dots, p-1\}$ and some $N \geq 0$.

Remark. By an assignment we showed how to invert an element in \mathbb{Z}_p with $a_0 \neq 0$ (and hence \mathbb{Q}_p). Hence \mathbb{Q}_p is a field. We also show that $\sqrt{-2} \in \mathbb{Q}_3$ and more generally $\sqrt{1-p} \in \mathbb{Q}_p$ for $p \geq 3$. This means that \mathbb{Q}_p is NOT an ordered field. Note that $p = 2$ is a special case to consider, and is typically a special case for any p -adic problems.

4 Approximation Theory

4.1 Polynomial Approximation

When we proved that the set of functions that are differentiable somewhere was first category, we first approximated a random function by a differentiable function and then added a small non-differentiable function to it. We can find such a function if we can approximate it by a polynomial. There are three methods we will discuss, but the first two do not work.

Method (Taylor Polynomials). Say f is some function, then we know that:

$$f(x) \approx \sum_{n=0}^N \frac{f^{(n)}(c)}{n!} (x - c)^n$$

Here $f^{(n)}(c)$ is the n -th derivative of f at c . This has the following problems:

- (1). We need f to have lots of derivatives. This means it is useless for non-differentiable functions.
- (2). This only really converges inside its disk of convergence. Take $f(x) = (1 + x^2)^{-1}$. Its Taylor series converges for $|x| < 1$. Also, take:

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

This only converges at $x = 0$, so its Taylor series is useless.

Method (Lagrange Polynomials). Let f be some function and x_0, x_1, \dots, x_n be a collection of distinct points (in the domain of f). Define:

$$P_k(x) = \prod_{i \neq k} \left(\frac{x - x_i}{x_k - x_i} \right)$$

This is a polynomial in x . Note that $P_k(x_j) = \delta_{kj}$ for all k, j . Define:

$$P(x) = \sum_{i=0}^n f(x_i) P_i(x)$$

This has the property that $P(x_i) = f(x_i)$ for $i = 0, 1, \dots, n$. One would hope that the more points one uses, the better the approximation. Consider:

$$f : [0, 1] \rightarrow \mathbb{R} \quad \text{by} \quad f(x) = \frac{1}{1 + 25x^2}$$

In this case the Lagrange polynomial does not approximate f at all.

Theorem 4.1 (Weierstrass Approximation Theorem). Let $f : [0, 1] \rightarrow \mathbb{R}$ be continuous. For every $\epsilon > 0$ there exists a polynomial $p(x) \in \mathbb{R}[x]$ such that:

$$\|f - p\|_\infty = \sup_{x \in [0, 1]} |f(x) - p(x)| < \epsilon$$

Proof. Assume WLOG that $f(0) = f(1) = 0$. If $f(0) = a$ and $f(1) = b$ we could consider the function $g(x) = f(x) - a + (a - b)x$. Consider:

$$Q_n(x) = \begin{cases} (1 - x^2)^n c_n & \text{if } x \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}$$

where for each $n \geq 1$ we define:

$$c_n = \left(\int_{-1}^1 (1 - x^2)^n dx \right)^{-1}$$

Hence we have $\int_{-1}^1 Q_n(x) dx = 1$. Now we define functions q_n by:

$$q_n(x) = \int_{-1}^1 f(x + t) Q_n(t) dt$$

where $f(z) = 0$ if $z \notin [0, 1]$. We claim that $q_n(x)$ is a polynomial in x and $\|q_n - f\|_\infty \rightarrow 0$ as $n \rightarrow \infty$. Our plan to prove this claim is as follows:

1. Estimate c_n for each $n \geq 1$.
2. For $\delta > 0$ we have $\lim_{n \rightarrow \infty} \int_\delta^1 Q_n(x) dx = 0$. This implies $\int_{-1}^{-\delta} Q_n(x) dx \rightarrow 0$ since Q_n is even.
3. Show $q_n(x)$ is a polynomial.

Step 1. By trig substitution with $x = \sin(u)$ we have $dx = \cos(u) du$. Then:

$$\begin{aligned} \int_{-1}^1 (1 - x^2)^n dx &= \int_{-\pi/2}^{\pi/2} (1 - \sin^2(u))^n \cos u du = \int_{-\pi/2}^{\pi/2} \cos^{2n+1}(u) du \\ &= 2 \int_0^{\pi/2} \cos^{2n+1}(u) du = 2 \left(\frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdots (2n-1)(2n+1)} \right) \geq \frac{2}{2n+1} \end{aligned}$$

This gives the estimation that $c_n \leq \frac{2n+1}{2} \leq 2n+1$.

Step 2. Fix $\delta > 0$. Then we have:

$$I_n = \int_\delta^1 (1 - x^2)^n c_n dx \leq \int_\delta^1 (1 - \delta^2)^n (2n+1) dx \leq (1 - \delta^2)^n (2n+1)$$

We see that $(1 - \delta^2)^n(2n + 1)$ goes to 0 as $n \rightarrow \infty$ (using ratio test). Hence $I_n \rightarrow 0$ as $n \rightarrow \infty$.

Step 3. By substitution with $u = x + t$ we have:

$$q_n(x) = \int_{-1}^1 f(x+t)Q_n(t) dt = \int_{-1+x}^{1+x} f(u)Q_n(u-x) du$$

We are assuming that $f(u) = 0$ for all $u \notin [0, 1]$, hence:

$$q_n(x) = \int_0^1 f(u)Q(u-x) du$$

This is a polynomial! (Consider what happens to individual x^j term in the polynomial $Q(u-x)$). For example, we have that:

$$\int_0^1 f(u) \sum_{i,j} a_{ij} x^i u^j du = \sum_i \left(\int_0^1 \sum_j a_{ij} f(u) u^j du \right) x^i$$

Lecture 29, 2025/03/21

Since $f \in \mathcal{C}[0, 1]$ is continuous on $[0, 1]$, it is uniformly continuous. Pick $\delta > 0$ such that for all $x, y \in [0, 1]$ we have:

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{2}$$

We also know that f is bounded. There is $M > 0$ such that $|f(x)| < M$ for all $x \in [0, 1]$. Now pick n sufficiently large so that:

$$\int_{-1}^{-\delta} 2MQ_n(t) dt + \int_{\delta}^1 2MQ_n(t) dt < \frac{\epsilon}{2} \quad (*)$$

Pick $x \in [0, 1]$. Notice that:

$$|q_n(x) - f(x)| = \left| \int_{-1}^1 f(t+x)Q_n(t) dt - f(x) \int_{-1}^1 Q_n(t) dt \right| = \left| \int_{-1}^1 (f(x+t) - f(x))Q_n(t) dt \right|$$

Now we estimate two integrals.

$$A = \left| \int_{-\delta}^{\delta} (f(x+t) - f(x))Q_n(t) dt \right| \leq \int_{-\delta}^{\delta} |f(x+t) - f(x)|Q_n(t) dt \leq \frac{\epsilon}{2} \int_{-\delta}^{\delta} Q_n(t) dt = \frac{\epsilon}{2}$$

By the choice of n in equation $(*)$ we have:

$$B = \left| \int_{-1}^{-\delta} \underbrace{(f(x+t) - f(x))}_{\leq 2M} Q_n(t) dt + \int_{\delta}^1 \underbrace{(f(x+t) - f(x))}_{\leq 2M} Q_n(t) dt \right| < \frac{\epsilon}{2}$$

Now, by the triangle inequality we obtain that:

$$|q_n(x) - f(x)| = \left| \int_{-1}^1 (f(x+t) - f(x))Q_n(t) dt \right| \leq A + B \leq \epsilon$$

Since $x \in [0, 1]$ is arbitrary, we have $\|q_n - f\| \leq \epsilon$. As desired. \square

4.2 Stone-Weierstrass Theorem

Definition. Let X be a compact metric space. Then $\mathcal{C}(X) = \{f : X \rightarrow \mathbb{R} \mid f \text{ continuous}\}$ is a vector space over \mathbb{R} . We say $\mathcal{A} \subseteq \mathcal{C}(X)$ is an **algebra** if \mathcal{A} is a vector subspace of $\mathcal{C}(X)$ and for all $f, g \in \mathcal{A}$ we have $fg \in \mathcal{A}$.

Example. Let X be any compact space. Then $\mathcal{A} = \{\text{constant functions}\}$ is an algebra.

Example. The set of polynomials is an algebra in $\mathcal{C}[0, 1]$. Moreover, even degree Polynomials are an algebra. Odd degree polynomials do not form an algebra because $x \cdot x$ is even.

Example. Differentiable functions form an algebra.

Example. Let $X = [0, 1]$. Then $\{f \in \mathcal{C}[0, 1] : f(0) = f(1)\}$ is an algebra.

Example. Even polynomials (polynomials that are also even functions) is an algebra.

Definition. For two functions $f, g : X \rightarrow \mathbb{R}$ we define $f \vee g : X \rightarrow \mathbb{R}$ and $f \wedge g : X \rightarrow \mathbb{R}$ by:

$$(f \vee g)(x) := \max(f(x), g(x)) \quad \text{and} \quad (f \wedge g)(x) = \min(f(x), g(x))$$

We also write $f \vee g = \max(f, g)$ and $f \wedge g = \min(f, g)$.

Definition. Let X be compact and $\mathcal{A} \subseteq \mathcal{C}(X)$ be an algebra. We say \mathcal{A} is a **vector lattice** if for all $f, g \in \mathcal{A}$ we have $f \vee g \in \mathcal{A}$ and $f \wedge g \in \mathcal{A}$.

Example. Constant functions is a vector lattice.

Example. Let $P = \text{polynomials on } [0, 1]$. This is not a vector lattice. Note $x \vee 0 = |x|$ is not a polynomial. However $\overline{P} = \mathcal{C}[0, 1]$ by the Weierstrass approximation theorem. Hence \overline{P} is a lattice.

Definition. Let X be compact and $\mathcal{A} \subseteq \mathcal{C}(X)$ is an algebra. We say \mathcal{A} **separates points** if for all $x, y \in X$ with $x \neq y$ there is $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

Example. Constant functions do not separate points.

Example. Polynomials separate points because x separates points.

Example. Even functions in $\mathcal{C}[-1, 1]$ do not separate points.

Definition. Let X be compact and $\mathcal{A} \subseteq \mathcal{C}(X)$ be an algebra. We say \mathcal{A} **vanishes at** $x \in X$ if for all $f \in \mathcal{A}$ we have $f(x) = 0$.

Example. Constant functions $X \rightarrow \mathbb{R}$ do not vanish anywhere (because 1 does not vanish anywhere).

Example. The algebra $\mathcal{A} = \text{span}_{\mathbb{R}}\{x^{2n} : n \geq 1\}$ vanishes at 0.

Example. Let $P =$ set of polynomials on $[0, 1]$. Then P is an algebra and \overline{P} is a vector lattice. Also P separates points and P does not vanish anywhere.

Theorem 4.2 (Stone-Weierstrass). Let X be a compact metric space. Let $\mathcal{A} \subseteq \mathcal{C}(X)$ be an algebra that separates points and does not vanish anywhere. Then \mathcal{A} is dense in $\mathcal{C}(X)$.

Our plan of the proof is the followings:

1. If \mathcal{A} is an algebra, then \mathcal{A} in $\mathcal{C}(X)$ is a vector lattice.
2. If \mathcal{A} separates points then for all $x, y \in X$ with $x \neq y$ and $a, b \in \mathbb{R}$, there exists $f \in \mathcal{A}$ such that $f(x) = a$ and $f(y) = b$.
3. For each $a \in X$ and $\epsilon > 0$, we can find $g_a \in \overline{\mathcal{A}}$ such that $g_a(x) > f(x) - \epsilon$ and $g_a(a) = f(a)$.
4. Using these g_a , we can find g such that $f(x) + \epsilon > g(x) > f(x) - \epsilon$ for all $x \in X$.

Lecture 30, 2025/03/24

Lemma 4.3. Let X be compact and $\mathcal{A} \subseteq \mathcal{C}(X)$ be an algebra. Then $\overline{\mathcal{A}}$ is a closed algebra and a vector lattice.

Proof. Clearly $\overline{\mathcal{A}}$ is closed and it is easy to check it is an algebra. Recall that $\overline{\mathcal{A}}$ is a vector lattice if for all $f, g \in \overline{\mathcal{A}}$ we have $f \vee g$ and $f \wedge g \in \overline{\mathcal{A}}$. We will first show that if $f \in \overline{\mathcal{A}}$ then $|f| \in \overline{\mathcal{A}}$. Now take $f \in \overline{\mathcal{A}}$, we know $L := \|f\|_{\infty} < \infty$. By the Weierstrass approximation theorem we know polynomials are dense in $\mathcal{C}[-L, L]$. Since $g(x) = |x| \in \mathcal{C}[-L, L]$, there is a sequence of polynomials $(p_n)_{n=1}^{\infty}$ such that $p_n \rightarrow g$ uniformly on $[-L, L]$. Notice $p_n(0) \rightarrow 0$ as $n \rightarrow \infty$. Let $q_n = p_n - p_n(0)$, then we have $q_n \rightarrow g$ as well. Note that $q_n(f) \in \mathcal{A}$ and $q_n(f) \rightarrow |f|$. It follows that $|f| \in \overline{\mathcal{A}}$. Now:

$$\begin{aligned} \max(f, g) &= \frac{f+g}{2} + \frac{|f-g|}{2} \in \overline{\mathcal{A}} \\ \min(f, g) &= \frac{f+g}{2} - \frac{|f-g|}{2} \in \overline{\mathcal{A}} \end{aligned}$$

This proved that $\overline{\mathcal{A}}$ is a vector lattice. □

Lemma 4.4. Let X be compact and $\mathcal{A} \subseteq \mathcal{C}(X)$ be an algebra. Further, assume \mathcal{A} separates points and vanishes nowhere. For all $x, y \in X$ with $x \neq y$ and any $c, d \in \mathbb{R}$ we can find $f \in \mathcal{A}$ such that $f(x) = c$ and $f(y) = d$.

Proof. Let $x, y \in X$ with $x \neq y$. Let $g \in \mathcal{A}$ such that $g(x) \neq g(y)$. Write $g(x) = a$ and $g(y) = b$ and $a \neq b$. Hence they are not both 0. WLOG assume $b \neq 0$.

Case 1. Assume $a = 0$. Since \mathcal{A} does not vanish at x , there exists $h \in \mathcal{A}$ such that $h(x) \neq 0$. Set:

$$f(z) = \frac{c}{h(x)}h(z) + \left(\frac{d}{g(y)} - \frac{c \cdot h(y)}{h(x)g(y)} \right) g(z)$$

Evaluate this function f at x we get $f(x) = c$ and $f(y) = d$.

Case 2. Assume $a \neq 0$. Consider the function:

$$\tilde{g}(z) = g(z) - \frac{g(z)^2}{g(x)}$$

Then $\tilde{g}(x) = 0$ and $\tilde{g}(y) \neq 0$. Now apply case 1. □

Proof of Theorem 4.2. Let $f \in \mathcal{C}(X)$ be arbitrary and $\epsilon > 0$. Fix $a \in X$. For each $a \neq x \in X$ there is a function $h_x \in \mathcal{A}$ such that:

$$h_x(a) = f(a) \quad \text{and} \quad h_x(x) = f(x)$$

by Lemma 4.4 applying to $c = f(a)$ and $d = f(x)$. Define:

$$U_x = \{z \in X : h_x(z) > f(z) - \epsilon\}$$

Notice that $x \in U_x$ and $a \in U_x$. Also note that U_x is open because:

$$U_x = (h_x - f)^{-1}((-\epsilon, \infty))$$

Note that $\{U_x\}_{x \in X}$ is an open cover of X . As X is compact, we have a finite subcover:

$$\{U_{x_1}, \dots, U_{x_n}\}$$

Take $g_a = \max(h_{x_1}, \dots, h_{x_n}) \in \overline{\mathcal{A}}$, as $\overline{\mathcal{A}}$ is a vector lattice. Notice that:

$$g_a(a) = f(a) \quad \text{and} \quad g_a(z) > f(z) - \epsilon \quad \text{for all } z \in X$$

Let $V_a = \{z \in X : g_a(z) < f(z) + \epsilon\}$. We see that $a \in V_a$ and each V_a is open. As before $\{V_a\}_{a \in X}$ is an open cover. We have a finite subcover $\{V_{a_1}, \dots, V_{a_k}\}$. Take $g = \min(g_{a_1}, \dots, g_{a_k})$. We see that $g(z) > f(z) - \epsilon$ for all $z \in X$ by the properties of g_{a_i} . Further $g(z) < f(z) + \epsilon$ by properties of U_{a_i} . Hence $\|g - f\|_\infty < \epsilon$. Since $f \in \mathcal{C}(X)$ and $\epsilon > 0$ are arbitray, we proved $\overline{\mathcal{A}} = \mathcal{C}(X)$. □

4.3 Best Approximation

Notation. For $n \geq 0$ let $\mathbb{P}_n[x]$ denote the polynomials of degree at most n .

Definition. For X compact and $f \in \mathcal{C}(X)$ and $n \geq 1$ we define:

$$E_n(f) = \inf_{p \in \mathbb{P}_n[x]} \|f - p\|_\infty$$

Note that $T : \mathbb{P}_n[x] \rightarrow \mathbb{R}$ by $p \mapsto \|f - p\|_\infty$ is a continuous function. Consider $S \subseteq \mathbb{P}_n[x]$ such that:

$$S = \{p \in \mathbb{P}_n[x] : \|p\|_\infty \leq 4\|f\|_\infty\}$$

Then $S \subseteq \mathbb{P}_n[x]$ is compact and the polynomial $0 \in S$. By restriction, $T : S \rightarrow \mathbb{R}$ is continuous on a compact set! Hence there exists $p^* \in S$ such that:

$$\|p^* - f\|_\infty = \inf_{p \in S} T(p) = \inf_{p \in S} \|p - f\|_\infty \leq \|f\|_\infty$$

If $p \in \mathbb{P}_n[x] \setminus S$ then we have $\|p - f\|_\infty \geq 2\|f\|_\infty$. Hence:

$$\|p^* - f\|_\infty = \inf_{p \in \mathbb{P}_n[x]} \|f - p\|_\infty = E_n(f) \quad (*)$$

We say p^* is a **best approximation** of f of degree n .

Definition. We say a function $g \in \mathcal{C}[a, b]$ satisfies the **equioscillation property** of degree n if there exists $(n+2)$ points $x_1 < \dots < x_{n+2}$ in $[a, b]$ with:

$$g(x_i) = (-1)^i \|g\|_\infty \quad \text{or} \quad g(x_i) = (-1)^{i+1} \|g\|_\infty$$

for all $i \in \{1, \dots, n+2\}$.

Theorem 4.5. Let $n \geq 1$ and $f \in \mathcal{C}[a, b]$. Assume $p \in \mathbb{P}_n[x]$ such that $g := f - p$ satisfies the equioscillation property of degree n . Then p is a best approximation of f , that is, $\|f - p\|_\infty = E_n(f)$.

Proof. Assume p is not a best approximation. There exists another polynomial $r(x)$ that gives a better approximation. We know $q = r - p$ is a polynomial of degree at most n . Let x_1, \dots, x_{n+2} such that $g(x_i) = (-1)^i \|g\|_\infty$. We see that $g(x_i)$ and $q(x_i)$ must have the same sign, otherwise:

$$|g(x_i) - q(x_i)| > |g(x_i)| = \|g\|_\infty$$

which is a contradiction since $|g(x_i) - q(x_i)| < \|g\|_\infty$ as it is a better approximation. That is, $q(x)$ has $(n+2)$ sign changes. As $\deg(q) \leq n$ we have $q(x) = 0$. This proves $p(x)$ is a best approximation. \square

Theorem 4.6. If $p \in \mathbb{P}_n[x]$ is a best approximation of f , then $g = f - p$ satisfies the equioscillation property of degree n .

Theorem 4.7. Best approximations are unique.

5 Differential Equations

Example. Consider the differential equation $y' = x^2 + 1$. This is a boring differential equation. We can just find an anti-derivative for $x^2 + 1$.

Example. Consider $(y')^2 + 1 = 0$, an first order DE with no real solutions. $[y' = \pm i]$

Example. Consider the second order DE $y'' = -y$. We see that $y(x) = a \sin(x)$ and $y(x) = b \cos(x)$ are both solutions. In fact $\text{span}\{\sin(x), \cos(x)\}$ is the set of all solutions to this DE.

Goal: We wish to use the contraction mapping principle to show that certain families of first order DEs have a solution, and that the solution is unique.

Example. Consider the differential equation $y' = -1 + y/2$ on $x \in [-1, 1]$ with $y(0) = 1$. How do we solve it? We have:

$$\int_0^x y'(t) dt = \int_0^x -1 + \frac{y(t)}{2} dt$$

By the FTC this gives us:

$$y(x) = y(0) - x + \int_0^x \frac{y(t)}{2} dt = 1 - x + \int_0^x \frac{y(t)}{2} dt$$

Consider $T : \mathcal{C}[-1, 1] \rightarrow \mathcal{C}[-1, 1]$ given by $Tf(x) = 1 - x + \int_0^x f(t)/2 dt$. We claim that T is Lipschitz with constant < 1 . Indeed, we have:

$$\begin{aligned} \|Tf - Tg\|_\infty &= \sup_{x \in [-1, 1]} \left| \frac{1}{2} \int_0^x (f(t) - g(t)) dt \right| \\ &\leq \sup_{x \in [-1, 1]} \left| \frac{1}{2} \int_0^x |f(t) - g(t)| dt \right| \\ &\leq \frac{1}{2} \sup_{x \in [-1, 1]} \int_0^x \|f - g\|_\infty dt \\ &= \frac{1}{2} \|f - g\|_\infty \end{aligned}$$

By the contraction mapping principle, T has a unique fixed point. Moreover, for all $f \in \mathcal{C}[-1, 1]$ the sequence $(T^n f)_{n=0}^\infty$ converges to this unique fixed point! Let $f_0 = 0$, then:

$$\begin{aligned} f_1(x) &= (Tf_0)(x) = 1 - x + \frac{1}{2} \int_0^x 0 dt = 1 - x \\ f_2(x) &= (Tf_1)(x) = 1 - x + \frac{1}{2} \int_0^x (1 - t) dt = 1 - \frac{x}{2} - \frac{x^2}{4} \end{aligned}$$

By some computation we can see that:

$$\begin{aligned} f_3(x) &= (Tf_2)(x) = 1 - \frac{x}{2} - \frac{x^2}{4} - \frac{x^3}{8} \\ f_4(x) &= (Tf_3)(x) = 2 - \left(1 + \frac{x}{2} + \frac{(x/2)^2}{2!} + \frac{(x/2)^3}{3!}\right) - \frac{x^4}{192} \end{aligned}$$

If we continue, we obtain $f^*(x) = 2 - e^{x/2}$ is the fixed point and this is a solution to our DE.

Remark. The key observation is we could construct $T : \mathcal{C}(X) \rightarrow \mathcal{C}(X)$ that was Lipschitz and contractive. We will show for a large family of first order DE, something “like this” will happen!

Definition. Let I_1 and I_2 be intervals. We say $\varphi : I_1 \times I_2 \rightarrow \mathbb{R}$ is **Lipschitz in y** if there is $L \geq 0$ such that for any fixed $x \in I_1$ and all $y_1, y_2 \in I_2$:

$$|\varphi(x, y_1) - \varphi(x, y_2)| \leq L|y_1 - y_2|$$

Example. For $y' = 1 + x^2$ we can take $\varphi(x, y) = 1 + x^2$. This is clearly Lipschitz in y with constant $L = 0$. This is because $|\varphi(x, y_1) - \varphi(x, y_2)| = 0$ for any y_1, y_2 .

Example. For $y' = -1 + y/2$ we let $\varphi(x, y) = -1 + y/2$. Then φ is Lipschitz in y with constant $1/2$.

Lemma 5.1. Let $I_1 = [a, b]$. Let $\varphi : I_1 \times \mathbb{R} \rightarrow \mathbb{R}$ be Lipschitz in y with constant L . Let $c \in [a, b]$ and define:

$$T : \mathcal{C}[a, b] \rightarrow \mathcal{C}[a, b] \quad \text{by} \quad Tf(x) = c_0 + \int_0^x \varphi(t, f(t)) dt$$

where $y(c) = c_0$ is the initial condition. Let $f, g \in \mathcal{C}[a, b]$. If there exists M and k such that:

$$|f(x) - g(x)| \leq \frac{M|x - c|^k}{k!} \quad \text{for all } x \in [a, b]$$

Then for all $x \in [a, b]$ we have:

$$|Tf(x) - Tg(x)| \leq \frac{LM|x - c|^{k+1}}{(k+1)!}$$

Note. For all $f, g \in \mathcal{C}[a, b]$ there is such constant $M = \|f - g\|_\infty$ and $k = 0$, so:

$$|f(x) - g(x)| \leq \sup_{x \in [a, b]} |f(x) - g(x)| = \|f - g\|_\infty = \frac{M|x - c|^0}{0!}$$

By iterating this we get that:

$$|T^k f(x) - T^k g(x)| \leq \frac{L^k |x - c|^k}{k!}$$

Taking $k \rightarrow \infty$ we get $\frac{L^k |x - c|^k}{k!} \rightarrow 0$. Hence there is k_0 large enough such that:

$$L_0 := \frac{L^{k_0} |x - c|^{k_0}}{k_0!} < 1$$

It follows that T^{k_0} is a contraction with constant $L_0 < 1$.

5.1 Global Solutions of ODEs

Proof of Lemma 5.1. Assume (1) holds. Then:

$$\begin{aligned}
 |Tf(x) - Tg(x)| &= \left| c_0 + \int_c^x \varphi(t, f(t)) dt - c_0 - \int_c^x \varphi(t, g(t)) dt \right| \\
 &= \left| \int_c^x \varphi(t, f(t)) - \varphi(t, g(t)) dt \right| \\
 &\leq \int_c^x L|f(t) - g(t)| dt && \text{(Lipschitz in } y) \\
 &\leq \int_c^x \frac{LM|t - c|^k}{k!} dt && \text{(by (1))} \\
 &= \frac{LM|x - c|^{k+1}}{(k+1)!}
 \end{aligned}$$

As desired. \square

Theorem 5.2 (Global Picard Theorem). If $\varphi : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz in y and $c \in [a, b]$ then there exists a unique solution to $y'(x) = \varphi(x, y(x))$ with $y(c) = c_0$ in $\mathcal{C}[a, b]$.

Proof. Use $T : \mathcal{C}[a, b] \rightarrow \mathcal{C}[a, b]$ as in the previous lemma. We know from Lemma 5.1 that:

- (1) Take $k = 0$ and $M = \|f - g\|_\infty$, it satisfies the condition of the previous lemma. Hence:

$$|Tf(x) - Tg(x)| \leq L\|f - g\|_\infty|x - c|$$

- (2) By a different corollary there exists k_0 such that $T^{(k_0)}$ is a contraction on $\mathcal{C}[a, b]$. This gives a unique fixed point to T . Hence we have a unique solution to the DE. \square

5.2 Local Solutions

There are solutions where φ is not Lipschitz in y , where φ is “nice enough” that we can still do something to find a unique solution. The problem occurs as $y \in \mathbb{R}$ and \mathbb{R} is big.

Example. Consider $y' = -2xy^2$ with $y(0) = 1$. This has solution $y(x) = \frac{1}{1+x^2}$. The associated function here is $\varphi(x, y) = -2xy^2$, which is not Lipschitz in y for $x \neq 0$. Let $[a, b] = [-1/4, 1/4]$ and:

$$\mathcal{C}'[a, b] = \left\{ f \in \mathcal{C}[a, b] : |f(x) - 1| \leq \frac{1}{2}, x \in [a, b] \right\} \subseteq \mathcal{C}[a, b]$$

Let $c = 0$ then $c_0 = y(0) = 1$. Let $T : \mathcal{C}'[a, b] \rightarrow \mathcal{C}'[a, b]$ by:

$$Tf(x) = 1 + \int_0^x \varphi(t, f(t)) dt = 1 - 2 \int_0^x t f(t)^2 dt$$

We claim this is well-defined, that is, $Tf \in \mathcal{C}'[a, b]$ for all $f \in \mathcal{C}'[a, b]$. We also need to show T is Lipschitz. Assume $|f(x) - 1| \leq 1/2$ for all $x \in [a, b]$. This implies $|f(x)| \leq 3/2$. Then:

$$|Tf(x) - 1| = \left| 1 - 2 \int_0^x tf(t)^2 dt - 1 \right| \leq \int_0^{1/4} 2t \left(\frac{3}{2}\right)^2 dt = \frac{9}{64} < \frac{1}{2}$$

Hence $Tf \in \mathcal{C}'[a, b]$. Consider the Lipschitz constant on φ . For fixed x we have:

$$\left| \frac{\partial}{\partial y} \varphi(x, y) \right| = 4|xy| \leq \frac{3}{2}$$

as $x \in [-1/4, 1/4]$ and $y \in [1/2, 3/2]$. Using the same trick before there is k_0 such that $T^{(k_0)}$ is a contraction.

Definition. We say $\varphi : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is **locally Lipschitz** in y if for all $(x_0, y_0) \in [a, b] \times \mathbb{R}$ there exists $h > 0$ such that φ is Lipschitz on $[x_0 - h, x_0 + h] \times [y_0 - h, y_0 + h]$ in y .

Lemma 5.3. Let $\varphi : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ be locally Lipschitz in y on a convex compact set K . Then φ is Lipschitz on K .

Proof. For every (x, y) we can find a neighborhood where φ is Lipschitz in y (on this neighborhood). This gives an open cover of K . We can find a finite subcover. Pick L to be the worst constant from this finite set. With some work we can finish the proof. \square

Lecture 34, 2025/04/02

Theorem 5.4 (Local Picard's Theorem). Suppose $\varphi : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and locally Lipschitz on $[a, b] \times [c_0 - R, c_0 + R]$. Then the DE $y' = \varphi(x, y)$ with $y(a) = c_0$ has a solution on $[a, a + h]$ with $h = \min(b - a, R/\|\varphi\|)$, where:

$$\|\varphi\| := \sup_{\substack{x \in [a, b] \\ y \in [c_0 - R, c_0 + R]}} |\varphi(x, y)|$$

Proof. Take $T : \mathcal{C}[a, a + h] \rightarrow \mathcal{C}[a, a + h]$ by:

$$Tf(x) = c_0 + \int_a^x \varphi(t, f(t)) dt$$

As before, take $\mathcal{C}' \subseteq \mathcal{C}[a, a + h]$ by:

$$\mathcal{C}' = \{f \in \mathcal{C}[a, a + h] : \|f - c_0\|_\infty \leq R\}$$

We need to show $T : \mathcal{C}' \rightarrow \mathcal{C}'$ and T is Lipschitz in y on \mathcal{C}' . We see \mathcal{C}' is a compact set. As φ and T are locally Lipschitz in y on a compact and convex set, φ and T' are Lipschitz on \mathcal{C}' . To see that

$T : \mathcal{C}' \rightarrow \mathcal{C}'$, note for $f \in \mathcal{C}'$ we have:

$$\begin{aligned} |Tf(x) - c_0| &= \left| c_0 + \int_a^x \varphi(t, f(t)) \, dt - c_0 \right| \\ &= \left| \int_a^x \varphi(t, f(t)) \, dt \right| \\ &\leq h \cdot \|\varphi\| \leq R \end{aligned}$$

Hence $T : \mathcal{C}' \rightarrow \mathcal{C}'$. Using the same trick as before there is k_0 such that T^{k_0} is contractive. Hence there exists a solution. \square

Remark. We can often use the solution on $[a, a + h]$, and use local Picard to extend this (using $[a + h, b]$ and $y(a + h) = c_0$ for the DE). Sometimes this blows up, but often it is for a good reason.

Remark. We did this analysis for $y : \mathbb{R} \rightarrow \mathbb{R}$. We could have done something similar for $y : \mathbb{R} \rightarrow \mathbb{R}^n$.

Remark. We can modify higher order DEs to look like first order DEs with more parts.