

# Algebraic Diagonals and Asymptotics of Bivariate Generating Functions

Peiran Tao

Department of Combinatorics and Optimization  
University of Waterloo

July 31st, 2025



# Overview

1. Notation
2. Algebraic Generating Functions and Diagonals
3. Asymptotics of Bivariate Generating Functions

# Notation

1.  $\mathbb{K}$  = a field of characteristic zero (usually  $\mathbb{R}$  or  $\mathbb{C}$ ).
2.  $\mathbb{K}[[z]]$  = the ring of formal power series over  $\mathbb{K}$  in  $z$ .

$$\mathbb{K}[[z]] = \left\{ \sum_{n \geq 0} a_n z^n : a_n \in \mathbb{K} \right\}$$

3.  $\mathbb{K}[[x, y]]$  = the ring of formal power series over  $\mathbb{K}$  in  $x, y$ .

$$\mathbb{K}[[x, y]] = \left\{ \sum_{i, j \geq 0} a_{i, j} x^i y^j : a_{i, j} \in \mathbb{K} \right\}$$

# I. Algebraic Generating Functions and Diagonals

# Generating Functions

Given a combinatorial class  $(\mathcal{A}, \omega)$ , we can define its generating function

$$A(z) := \sum_{n \geq 0} a_n z^n$$

where  $a_n :=$  the number of elements in  $\mathcal{A}$  that have weight  $n$ .

## Example

Let  $\mathcal{A}$  be the strings in  $\{1, 2, 3\}$  that avoid 11 and 23. For example

1222132, 12, 132

The weight on  $\mathcal{A}$  counts the number of 1. By the *transfer matrix method* we can show that

$$A(z) = \frac{1 + z}{1 - 2z - z^2 + z^3}$$

# Algebraic Power Series

A formal power series  $A(z) \in \mathbb{K}[[z]]$  is called **algebraic** if

$$P(z, A(z)) = 0$$

for some polynomial  $P(z, y) \in \mathbb{K}[z, y]$ .

## Example

Let  $T(z)$  be the Catalan generating function, then

$$zT(z)^2 - T(z) + 1 = 0$$

So  $P(z, T(z)) = 0$  for  $P(z, y) = yz^2 - y + 1$ .

# Diagonals

Let  $F(x, y) \in \mathbb{K}[[x, y]]$  be a bivariate formal power series, write

$$F(x, y) = \sum_{i, j \geq 0} f_{i, j} x^i y^j$$

For  $d = (r, s) \in \mathbb{N}^2$  the  **$d$ -diagonal** of  $F$  is the univariate formal power series in  $\mathbb{K}[[t]]$

$$(\Delta_d F)(t) := \sum_{n \geq 0} f_{nr, ns} t^n$$

If  $d = (1, 1)$  we say

$$(\Delta F)(t) := (\Delta_d F)(t) = \sum_{n \geq 0} f_{n, n} t^n$$

is the **main diagonal** of  $F$ .

# Diagonals

## Theorem

*If  $F(x, y) \in \mathbb{K}[[x, y]]$  is a rational function then  $(\Delta F)(t)$  is algebraic.  
In other word, there exists  $P(t, y) \in \mathbb{K}[t, y]$  such that  $P(t, (\Delta F)(t)) = 0$ .*

Bostan et al. (2015) [2] developed an algorithm to efficiently compute this polynomial  $P(t, y)$ . We implemented this algorithm in SageMath.

**Input:** A rational function  $F(x, y) \in \mathbb{K}[[x, y]]$ .

**Output:** A polynomial  $P(t, y) \in \mathbb{K}[t, y]$  such that  $P(t, (\Delta F)(t)) = 0$ .



# Idea of the Algorithm

**Fact 1.** There is a set  $\{\alpha_1(t), \dots, \alpha_n(t)\}$  such that  $(\Delta F)(t)$  is a sum of  $c$  elements from this set.

Each  $\alpha_i(t)$  is an algebraic formal series in  $t$  determined by the “residues” of a certain function.

Construct the polynomial

$$\Sigma(y, t) = \prod_{i_1 < \dots < i_c} (y - (\alpha_{i_1}(t) + \dots + \alpha_{i_c}(t)))$$

**Fact 2.**  $\Sigma(y, t) \in \mathbb{K}[y, t]$ . (Galois Theory)

# Fact 1

Note that

$$\begin{aligned}(\Delta F)(t) &= \sum_{n \geq 0} f_{n,n} t^n \\&= [y^{-1}] \sum_{n,m \geq 0} f_{n,m} t^n y^{m-n-1} \\&= [y^{-1}] \frac{1}{y} F\left(\frac{t}{y}, y\right) \\&= \sum_{\substack{y_i(t) \in \mathcal{P} \\ \text{val}(y_i(t)) > 0}} \underbrace{\text{Residue} \left( \frac{1}{y} F\left(\frac{t}{y}, y\right), y = y_i(t) \right)}_{\alpha_i}\end{aligned}$$

where  $\mathcal{P} = \{y_1(t), \dots, y_n(t)\}$  is the “pole set” of  $\frac{1}{y} F(\frac{t}{y}, y)$ .

$$c = \#\{y(t) \in \mathcal{P} \mid \text{val}(y(t)) > 0\}$$

# Algorithm

The algorithm consists of two steps.

1. Compute the residues  $\{\alpha_1(t), \dots, \alpha_n(t)\}$  using resultants.
2. Compute the polynomial  $\Sigma(y, t)$ .

# An Example

Let  $\mathcal{A}$  be the combinatorial class of bicolored supertrees, then  $A(t)$  is the main diagonal of the rational function

$$\frac{P(x, y)}{Q(x, y)} = \frac{2x^2y(2x^5y^2 - 3x^3y + x + 2x^2y - 1)}{x^5y^2 + 2x^2y - 2x^3y + 4y + x - 2}$$

```
[22]: P = 2*x^2*y*(2*x^5*y^2 - 3*x^3*y + x + 2*x^2*y - 1)
      Q = x^5*y^2 + 2*x^2*y - 2*x^3*y + 4*y + x - 2
      AlgebraicDiagonal(P,Q)
```

```
[22]: y^4 - 2*y^3 + (2*t + 1)*y^2 - 2*t*y + 4*t^3
```

In fact, we have  $A(t) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4t + 4t\sqrt{1 - 4t}}$ .

## II. Asymptotics of Bivariate Generating Functions

# Bivariate Generating Functions

Consider a rational bivariate generating function

$$F(x, y) = \frac{P(x, y)}{Q(x, y)} = \sum_{n, m \geq 0} f_{n, m} x^n y^m \in \mathbb{C}[[x, y]]$$

# An example

## Example

Let  $b_{n,k}$  be the number of binary strings of length  $n$  and has  $k$  zeros

$$\begin{aligned} B(x, y) &= \sum_{n,k \geq 0} b_{n,k} x^n y^k \\ &= \sum_{n \geq 0} \left( \sum_{k=0}^n \binom{n}{k} y^k \right) x^n \\ &= \sum_{n \geq 0} (1+y)^n x^n \\ &= \frac{1}{1 - x(1+y)} \end{aligned}$$

# Asymptotics

In general it is hard to find a closed form formula for  $f_{n,m}$  for  $n, m \geq 0$ .

Instead, we try to **find the asymptotics** of the coefficient sequence of a diagonal  $(\Delta_d F)(t)$  for some  $d = (r, s) \in \mathbb{N}^2$ .

That is, we want to find the asymptotics of the sequence

$$(f_{nr,ns})_{n \geq 0} = \{f_{0,0}, f_{r,s}, f_{2r,2s}, \dots\}$$

as  $n \rightarrow \infty$ .



# Asymptotics

Assume  $F = P/Q \in \mathbb{K}[[x, y]]$  is a rational function (hence  $Q(0, 0) \neq 0$ )

By the **Cauchy's Integral Formula**, for  $\epsilon > 0$  small enough we have

$$\begin{aligned} f_{nr, ns} &= \frac{1}{(2\pi i)^2} \int_{T(\epsilon, \epsilon)} \frac{F(x, y)}{x^{rn+1} y^{sn+1}} \, dx \, dy \\ &= \frac{1}{(2\pi i)^2} \int_{T(\epsilon, \epsilon)} \underbrace{\frac{P(x, y)}{xyQ(x, y)} \cdot x^{-nr} y^{-ns}}_{\omega_F} \, dx \, dy \end{aligned} \quad (1)$$

where  $T(\epsilon, \epsilon) = \{(x, y) \in \mathbb{C}^2 : |x| = |y| = \epsilon\}$ .

Our goal is to estimate this integral (1).

# Singular Variety

When we compute integrals, we are interested in the singularities.

The function  $F = P/Q$  has singularities (poles) at the zeros of  $Q$ .

$$\mathcal{V} := \mathcal{V}(Q) := \{(x, y) \in \mathbb{C}^2 : Q(x, y) = 0\}$$

is called the **singular variety** of  $F$ .

# Estimate the integral

1. Deform the torus  $T(\epsilon, \epsilon)$  to another torus to lower the modulus of the integrand  $\omega_F$  as much as possible.
2. Reduce the integral to a *residue integral* on some cycle  $\mathcal{C}$ .
3. Understand the *homology class* of  $\mathcal{C}$ . We want to find a representative

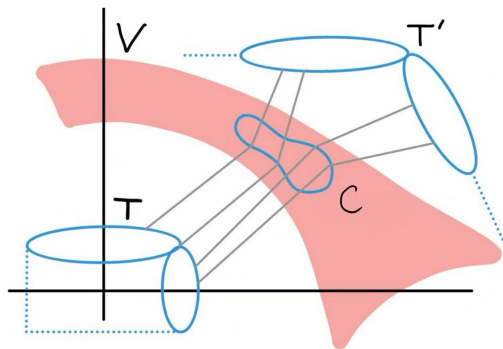
$$\kappa \in [\mathcal{C}] \in H_1(\mathcal{V})$$

that is “good” (will explain this later).

# 1. Deformation of the Contour

Let  $M > 0$  be large and let  $K$  be a homotopy from  $T(\epsilon, \epsilon)$  to  $T(\epsilon, M)$ .

In other words, we fix  $x$  and enlarge  $y$ .



Source: Page 202 of [\[1\]](#)

## 2. Reduce to residue integral

The homotopy intersects the singular variety  $\mathcal{V}(Q)$  at a cycle  $\mathcal{C}$ .

Let  $\nu$  be a “tube” around  $\mathcal{C}$ , then

$$\begin{aligned} f_{n,n} &= \frac{1}{(2\pi i)^2} \int_{\nu} \omega_F + \frac{1}{(2\pi i)^2} \int_{T(\epsilon, M)} \omega_F \\ &= \frac{1}{(2\pi i)^2} \int_{\nu} \omega_F + O(M^{1-n}) \\ &= \frac{1}{2\pi i} \int_{\mathcal{C}} \text{Res}(\omega_F) + O(M^{1-n}) \end{aligned}$$

Here  $\text{Res}(\omega_F)$  is a 1-form and  $\mathcal{C}$  is a 1-cycle in  $H_1(\mathcal{V})$ .

### 3. The homology class of $\mathcal{C}$

We want to find a good cycle  $\kappa \in [\mathcal{C}]$  that makes the calculation easy.

Note that

$$\begin{aligned}\omega_F &= \frac{P(x, y)}{xyQ(x, y)} \cdot x^{-nr} y^{-ns} \, dx \, dy \\ &= \frac{P(x, y)}{xyQ(x, y)} \cdot e^{nH(x, y)} \, dx \, dy\end{aligned}$$

where  $H : \mathbb{C}_*^2 \rightarrow \mathbb{C}$  is the multi-valued function defined by

$$H(x, y) = -r \log(x) - s \log(y)$$

and  $\mathbb{C}_* := \mathbb{C} \setminus \{0\}$  is the set of nonzero complex numbers.

# Height function

The real part of  $H$  is the function  $h = \operatorname{Re}(H) : \mathbb{C}_*^2 \rightarrow \mathbb{R}$  defined by

$$h(x, y) = -r \log |x| - s \log |y|$$

The function  $h|_{\mathcal{V}}$  is called a **height function** on  $\mathcal{V}$ . By applying ideas of *Morse theory*, we will use this function to study the variety  $\mathcal{V}$ .

# Components

For  $M > 0$  we define

$$\mathcal{V}^{>M} := \{(x, y) \in \mathcal{V} : h(x, y) > M\}$$

Let  $\mathcal{V}^{>M} = R_1 \cup \dots \cup R_n$  be the connected components. Define

$$X^{>M} := \{R_i : \forall \epsilon > 0, \exists (x, y) \in R_i \text{ such that } |x| < \epsilon\}$$

$$Y^{>M} := \{R_i : \forall \epsilon > 0, \exists (x, y) \in R_i \text{ such that } |y| < \epsilon\}$$

Each  $R_i \in X^{>M}$  is called a ***x*-component**.

Each  $R_i \in Y^{>M}$  is called a ***y*-component**.



# Critical Points

We say  $\sigma = (x_0, y_0) \in \mathcal{V}$  is a **critical point** or **saddle point** of  $h|_{\mathcal{V}}$  if

$$\nabla H(\sigma) \parallel \nabla Q(\sigma)$$

This is equivalent to

$$\det \begin{pmatrix} \nabla H(\sigma) \\ \nabla Q(\sigma) \end{pmatrix} = \det \begin{pmatrix} \frac{-r}{x_0} & \frac{-s}{y_0} \\ Q_x(x_0, y_0) & Q_y(x_0, y_0) \end{pmatrix} = 0$$

# Critical Points

To find the critical points we can solve the following system

$$Q(x, y) = 0$$

$$\frac{-r}{x}Q_y(x, y) + \frac{s}{y}Q_x(x, y) = 0$$

This can be done using *Gröbner bases*.

Let  $\Sigma = \{\sigma \in \mathcal{V} : \sigma \text{ is a critical point}\}$  be the set of all critical points.

We say  $c \in \mathbb{R}$  is a **critical value** if  $c = h(\sigma)$  for some  $\sigma \in \Sigma$ .

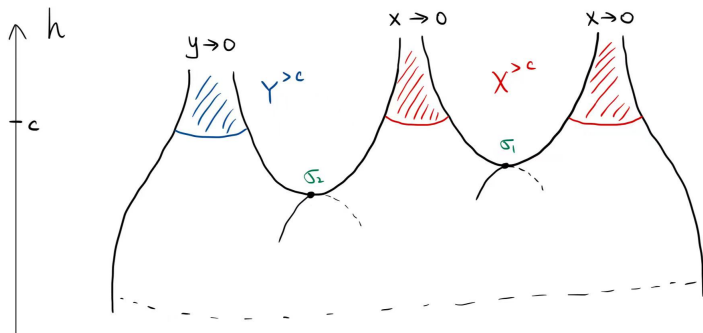
# Assumptions

1. We assume  $\Sigma$  is finite.
2. We assume  $\mathcal{V}$  is smooth. That is,  $\nabla Q(p) \neq 0$  for all  $p \in \mathcal{V}$ .

# Critical Points

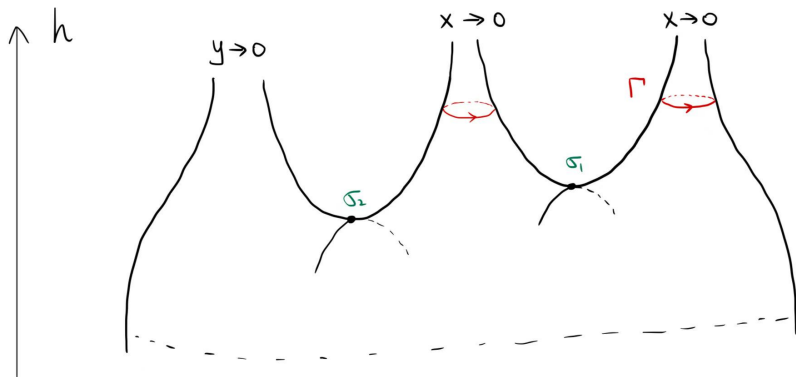
We can use the height function to visualize the variety  $\mathcal{V}$ .

Note that  $h(x, y) \rightarrow \infty$  if  $x \rightarrow 0$  or  $y \rightarrow 0$ .



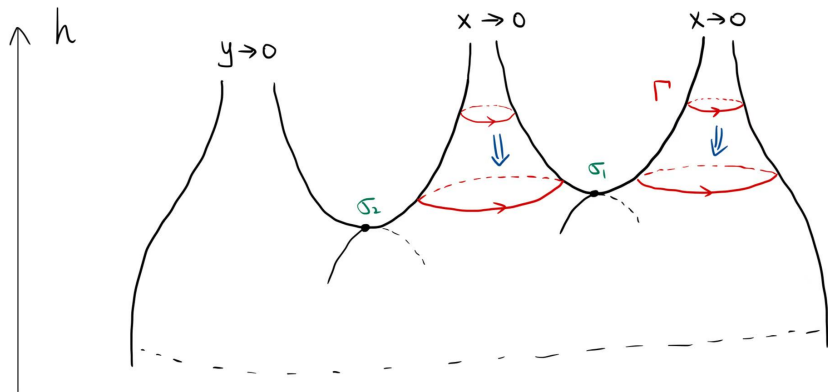
### 3. The homology class of $\mathcal{C}$

**Fact:** The cycle  $\mathcal{C}$  is homologous to a cycle  $\Gamma$ , which consists of disjoint cycles, one in each  $x$ -component.



### 3. The homology class of $\mathcal{C}$

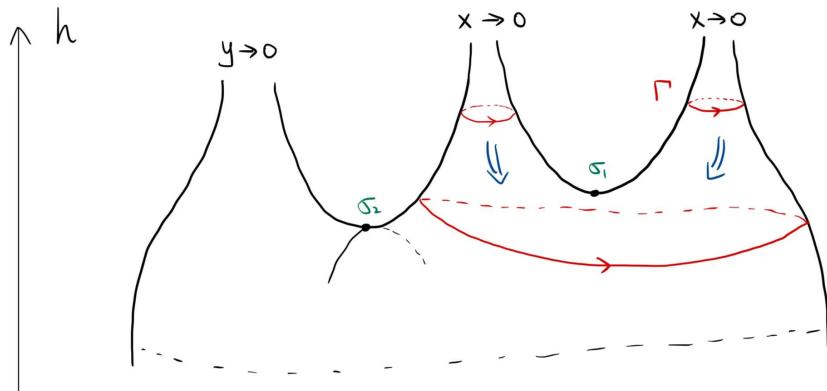
We can push the cycle  $\Gamma$  down



### 3. The homology class of $\mathcal{C}$

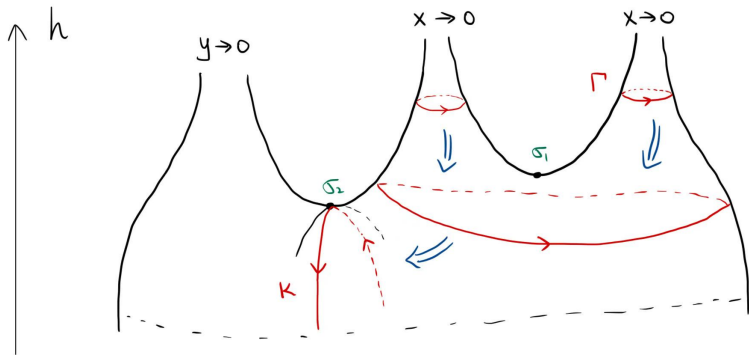
In this case, the two cycles “merge” to one bigger cycle.

Topologically, this is done by “attaching” some disks, which does not change the homology class.



### 3. The homology class of $\mathcal{C}$

We keep pushing the cycle down, until it get “stuck” at a critical point.



This is the desired cycle  $\kappa$ . This is good because we can apply the *saddle point method* near this saddle point.



# The idea

Which critical points do the cycles get stuck at?

It depends on the components “near” this critical point.

If all components near it are  $x$ -components, then all the cycles merge.

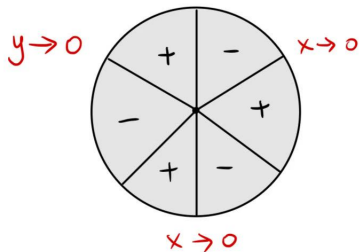
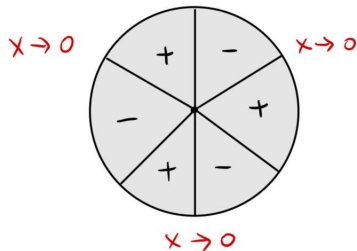
If one of the components is a  $y$ -component, then the cycle gets stuck.

The critical points with the largest height such that the cycles get stuck dominate the asymptotics.

# Some possible cases

In the above picture, there are only two components “near”  $\sigma_1$  and  $\sigma_2$ .

Sometimes there are more components.



# The algorithm

DeVries (2011) [3] developed an algorithm to find this cycle  $\kappa$ .

We are trying to improve the algorithm and make it practical.

# Step 1

Define the set  $\mathbb{W} = \emptyset$  and let  $c = -\infty$ .

List the critical value in order of decreasing height  $\sigma_1, \dots, \sigma_n$  so

$$h(\sigma_1) \geq \dots \geq h(\sigma_n)$$

## Step 2

Iterate from  $i = 1$  to  $i = n$ .

- (a). Compute the order  $k$  of  $\sigma_i$ , the number of components near it. Call these components  $C_1, \dots, C_k$ .
- (b). For each  $C_j$ , follow an ascending path from  $\sigma_i$  to  $C_j$  and check if  $C_j$  is an  $x$ -component or  $y$ -component.
- (c). If one of  $C_j$  is a  $y$ -component, add  $\sigma_i$  to the set  $\mathbb{W}$  and let  $c = h(\sigma_i)$ . Then go to Step 3.

## Step 3

Perform Step 2 to each  $\sigma_i$  such that  $h(\sigma_i) = c$ .

This is because if we already found one  $\sigma \in \mathbb{W}$ , then all the other critical points with lower height do not matter. Thus we iterate through the other critical points with this height  $c$ .

# Conclusion

The algorithm ends with a set  $\mathbb{W}$  and  $c \in \mathbb{R}$ .

If  $\mathbb{W} = \emptyset$  then  $c = -\infty$ , so the asymptotics decay super-exponentially.

Otherwise,  $\mathbb{W}$  is the set of **contributing points** that dominate the asymptotics.

# Bibliography

- [1] Robin Pemantle, Mark C. Wilson, Stephen Melczer (2024). Analytic Combinatorics in Several Variables, 2nd Edition. Cambridge University Press.
- [2] Bostan, A., Dumont, L., Salvy, B.: Algebraic diagonals and walks: algorithms, bounds, complexity. J. Symbolic Comput. 83, 68-92 (2017)
- [3] T. DeVries. "Algorithms for bivariate singularity analysis". PhD thesis. University of Pennsylvania, 2011



Thank you!