PMATH 450 Notes

Fall 2024

Based on Professor Laurent Marcoux's Lectures

Contents

Lecture 1, 2024/09/04

1 Riemann Integration in Banach Spaces

Notation. We use $\mathbb{K}$ to denote $\mathbb{R}$ or $\mathbb{C}$.

Definition. Let X be a vector space over $\mathbb{K}$. A **semi-norm** on X is a function $\nu : X \to \mathbb{R}$ satisfying:

(1) $\nu(x) \geq 0$ for all $x \in X$.

(2) $\nu(x+y) \leq \nu(x) + \nu(y)$ for all $x, y \in X$.

(3) $\nu(kx) = |k|\nu(x)$ for $k \in \mathbb{K}$ and $x \in X$.

Note that if $z = 0 \in X$, then $z = 0 \cdot z$, so $\nu(z) = 0\nu(z) = 0$. In addition, if the semi-norm ν also satisfies:

(4) $\nu(x) = 0$ implies $x = 0$.

Then we we say ν is a **norm** on X. We say (X, ν) is a **normed linear space (NLS)**. We tend to write $\|x\|$ instead of $\nu(x)$ if ν is a norm.

Example 1.1. Let X be a compact Hausdorff space. Consider:

$$V = C(X, \mathbb{K}) = \{f : X \to \mathbb{K} : f \text{ is continuous}\}$$

For each $y \in X$, the map:

$$\nu_y : C(X, \mathbb{K}) \to \mathbb{R} \text{ by } f \mapsto |f(y)|$$

defineds a semi-norm on $C(X, \mathbb{K})$. But unless X is a singleton set, ν_y is NOT a norm.

For $\Omega \subseteq X$, we may consider the map:

$$\nu_\Omega : C(X, \mathbb{K}) \to \mathbb{R} \text{ by } f \mapsto \sup_{y \in \Omega} |f(y)|$$

If Ω is dense in X, then $\nu_\Omega = \nu_X$ is a norm on $C(X, \mathbb{K})$ denoted by $\|\cdot\|_{\sup}$.

Remark. Every NLS is a metric space using the metric induced by the norm. If $(X, \|\cdot\|)$ is a NLS, define $d : X^2 \to \mathbb{R}$ by $(x, y) \mapsto \|x - y\|$.

Definition. A **Banach Space** is a NLS which is complete with respect to the metric induced by the norm.

Example 1.2 (Banach Spaces).

(1) Consider $X = \mathbb{K}$ with norm $|\cdot|$ given by the absolute value.

(2) Let $N \in \mathbb{N}$ and let $X = \mathbb{K}^N$. We can define a variety of nroms on X making X a Banach Space.

(a) $\|(x_1, \cdots, x_N)\|_1 = \sum_{k=1}^{N} |x_k|$.

(b) $\|(x_1, \cdots, x_N)\|_\infty = \max_{1 \le k \le N} |x_k|$.

(c) $\|(x_1, \cdots, x_N)\|_p = \left(\sum_{k=1}^{N} |x_k|^p \right)^{1/p}$ for $1 \le p < \infty$.

Exercise: $\lim_{p\to\infty} \|(x_1, \cdots, x_N)\|_p = \|(x_1, \cdots, x_N)\|_\infty$.

(3) If X is a compact Hausdorff space, then $(C(X, \mathbb{K}), \|\cdot\|_{\sup})$ is a Banach Space. We can also define, for $f \in C([0,1], \mathbb{K})$ that:

$$\|f\|_1 = \int_0^1 |f|$$

This defines a norm on $C([0,1], \mathbb{K})$, but $C([0,1], \|\cdot\|_1)$ is NOT complete.

(4) Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be NLS over $\mathbb{K}$. We may define for a linear map $T : X \to Y$ that:

$$\|T\| = \sup_{\|x\|_X = 1} \|Tx\|_Y$$

We say T is bounded if $\|T\| < \infty$. We set:

$$\begin{aligned} B(X, Y) &= \{T : X \to Y \text{ linear} : \|T\| < \infty\} \\ &= \{T : X \to Y \text{ linear} : T \text{ is continuous}\} \end{aligned}$$

$B(X, Y)$ is complete if and only if Y is complete. And $\|\cdot\| : B(X, Y) \to \mathbb{R}$ by $T \mapsto \|T\|$ is a norm on $B(X, Y)$.

Definition. Let $(X, \|\cdot\|)$ be a Banach Space. Let $a < b$ be real numbers and $f : [a, b] \to X$ be a function. A **partition** of $[a, b]$ is a finite set:

$$P = \{a = p_0 < p_1 < \cdots < p_N = b\}$$

for some $N \ge 1$. The set of partitions of $[a, b]$ is denoted by $\mathcal{P}[a, b]$.

Definition. A **refinement** Q of a partition P of $[a, b]$ is a partition $Q \in \mathcal{P}[a, b]$ such that $P \subseteq Q$.

Definition. Given P as above, a set $P^* = \{p_1^*, \cdots, p_N^*\}$ satisfying $p_k^* \in [p_{k-1}, p_k]$ for $1 \le k \le N$ is the set of **test values** for P.

Definition. Given P and P^* above, we may define the corresponding **Riemann Sum** by:

$$S(f, P, P^*) = \sum_{k=1}^{N} f(p_k^*)(p_k - p_{k-1}) \in X$$

Remark. If $X = (\mathbb{K}, \|\cdot\|)$, these are the usual notions of Riemann sums from calculus.

Remark. In general, we have:

$$\frac{1}{b-a} S(f, P, P^*) = \sum_{k=1}^{N} \frac{p_k - p_{k-1}}{b-a} f(p_k^*) = \sum_{k=1}^{N} \alpha_k f(p_k^*)$$

where $\alpha_k = \frac{p_k - p_{k-1}}{b-a} > 0$ for $1 \leq k \leq N$ and $\sum_{k=1}^{N} \alpha_k = 1$. So $\frac{1}{b-a} S(f, P, P^*)$ is a convex combination of $f(p_k^*)$, that is, the "averaging" of the function.

Definition. Let $a < b$ be real numbers and $(X, \|\cdot\|)$ a Banach Space. Let $f : [a, b] \to X$. We say f is **Riemann integrable over** $[a, b]$ if there is $x_0 \in X$, such that for all $\epsilon > 0$, there is a partition $P \in \mathcal{P}[a, b]$ such that for all refinement Q of P and for all test values Q^* of Q, we have:

$$\|S(f, Q, Q^*) - x_0\| < \epsilon$$

Remark. If $f : [a, b] \to X$ is Riemann integrable, then $x_0 \in X$ is unique.

Proof: Suppose not, so $x_0 \neq y_0$ both satisfy the above definition. Let $\epsilon = \frac{1}{3}\|x_0 - y_0\| > 0$. Choose partition P corresponding to x_0 and partition Q corresponding to y_0 as in the definition. Let $R = P \cup Q$, so R is a refinement of both P and Q. Let R^* be any set of test values for R, then:

$$\|S(f, R, R^*) - x_0\| < \epsilon$$
$$\|S(f, R, R^*) - y_0\| < \epsilon$$

Therefore:

$$\|x_0 - y_0\| \leq \|S(f, R, R^*) - x_0\| + \|S(f, R, R^*) - y_0\| < 2\epsilon < \|x_0 - y_0\|$$

This is a contradiction. Thus x_0 is unqiue and we denote:

$$x_0 = \int_a^b f = \int_a^b f(s)\, ds$$

Theorem 1.3 (Cauchy's Criterion). Let X be a Banach space and $a < b$ be real numbers. Let $f : [a, b] \to X$ be a function. The followings are equivalent:

(a) f is Riemann integrable.

(b) For all $\epsilon > 0$, there is $R \in \mathcal{P}[a,b]$ such that if $P, Q \in \mathcal{P}[a,b]$ are refinements of R and P^*, Q^* are test values for P and Q. Then:

$$\|S(f,P,P^*) - S(f,Q,Q^*)\| < \epsilon$$

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Proof: ($\Rightarrow$). Exercise.

($\Leftarrow$). For each $n \geq 1$, we can choose a partition R_n of $[a,b]$ as in (b) corresponding to $\epsilon_n = 1/n$. Set for $m \geq 1$ that:

$$W_m = \bigcup_{n=1}^{m} R_n$$

So that W_m is a common refinement of $R_1, \cdots, R_m$. Observe that $m \geq n \geq N \geq 1$, then W_m and W_n are both refinements of R_N. Fix a set W_k^* of test values for W_k where $k \geq 1$. By (b) we have:

$$\|S(f,W_m,W_m^*) - S(f,W_n,W_n^*)\| < \frac{1}{N}$$

Set $x_k = S(f,W_k,W_k^*)$ for $k \geq 1$. Then for $m \geq n \geq N \geq 1$ we have $\|x_m - x_n\| < 1/N$, so (x_n) is a Cauchy sequence in X. Since X is complete, $x_0 = \lim x_n$ exists. We claim that:

$$x_0 = \int_a^b f$$

and hence f is Riemann integrable. Let $\epsilon > 0$, choose $N \geq 1$ such that:

(i) $1/N < \epsilon/2$.

(ii) For $m \geq N$ we have $\|x_0 - x_m\| < \epsilon/2$.

Consider $R = W_N$. If P is any refinement of W_N and P^* is any set of test values for P, then P is a refinement of R_N and W_N is a refinement of R_N. So:

$$\begin{aligned}
\|x_0 - S(f,P,P^*)\| &\leq \|x_0 - x_N\| + \|X_N - S(f,P,P^*)\| \\
&\leq \frac{\epsilon}{2} + \|S(f,W_N,W_N^*) - S(f,P,P^*)\| \\
&\leq \frac{\epsilon}{2} + \frac{1}{N} < \epsilon
\end{aligned}$$

As desried. $\square$

Theorem 1.4. If $(X, \|\cdot\|)$ is a Banach space. Let $a < b \in \mathbb{R}$ and $f : [a,b] \to X$. If f is continuous, then f is Riemann integrable.

Proof: Since $(X, \|\cdot\|)$ be a Banach space. Since $[a,b]$ is compact, we know f is continuous implies it is uniformly continuous. Let $\epsilon > 0$, choose $\delta > 0$ such that for $r, s \in [a,b]$:

$$|r-s| < \delta \implies \|f(r) - t(s)\| < \frac{\epsilon}{2(b-a)}$$

Let $R = \{a = r_0 < r_1 < \cdots < r_N = b\} \in \mathcal{P}[a,b]$ with $\max\limits_{1 \le k \le N} |r_k - r_{k-1}| < \delta$. Let P be a refinement of R and choose:

$$0 = k_0 < k_1 < \cdots < k_N = r_n = b$$

such that $P_{k_j} = r_j$ for $0 \le j \le N$. Let $R^* = \{r_1^*, \cdots, r_N^*\}$ be a set of test values for R and $P^* = \{P_1^*, \cdots, P_{k_N}^*\}$ be a set of test values for P. Now:

$$\begin{aligned}
S(f, R, R^*) &= \sum_{j=1}^{N} f(r_j^*)(r_j - r_{j-1}) = \sum_{j=1}^{N} f(r_j^*)(P_{k_j} - P_{k_{j-1}}) \\
&= \sum_{j=1}^{N} f(r_j^*) \sum_{i=k_{j-1}+1}^{k_j} (P_i - P_{i-1}) \\
&= \sum_{j=1}^{N} \sum_{i=k_j+1}^{k_j} f(r_j^*)(P_i - P_{i-1})
\end{aligned}$$

while that:

$$S(f, R, R^*) = \sum_{j=1}^{N} \sum_{i=k_j+1}^{k_j} f(P_i^*)(P_i - P_{i-1})$$

Hence we have:

$$\|S(f, R, R^*) - S(f, P, P^*)\| \le \sum_{j=1}^{N} \sum_{i=k_j+1}^{k_j} \|f(r_j^*) - f(P_i^*)\|(P_i - P_j)$$

But for $k_{j-1} + 1 \le i \le k_j$ we have $|P_i^* - r_j^*| \le r_j - r_{j-1} < \delta$, so:

$$\|f(r_j^*) - f(P_i^*)\| < \frac{\epsilon}{2(b-a)}$$

Hence:

$$\|S(f, R, R^*) - S(f, P, P^*)\| < \sum_{j=1}^{N} \sum_{i=k_j+1}^{k_j} \frac{\epsilon}{2(b-a)}(P_i - P_j) = \frac{\epsilon}{2(b-a)} \sum_{j=1}^{N} (r_j - r_{j-1}) = \frac{\epsilon}{2}$$

Similarly, if Q is any refinement of R with test values Q^* of Q, then $\|S(f, R, R^*) - S(f, Q, Q^*)\| < \epsilon/2$.

Therefore we have:

$$\begin{aligned}\|S(f,P,P^*)-S(f,Q,Q^*)\| &\leq \|S(f,P,P^*)-S(f,R,R^*)\|+\|S(f,R,R^*)-S(f,Q,Q^*)\| \\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon\end{aligned}$$

Hence f is Riemann integrable by Cauchy's Criterion. $\square$

Definition. Let $E \subseteq \mathbb{R}$, the **characteristic/indicator function** of E is $\chi_E : \mathbb{R} \to \mathbb{R}$ by:

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Example 1.5. Let $E = \mathbb{Q} \cap [0,1]$. Let $f = \chi_E|_{[0,1]}$, that is, $f : [0,1] \to \mathbb{R}$ with:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Recall that this is called the **Dirichlet function**. We claim that f is NOT Riemann integrable.

Proof: Suppose f is integrable. Let R be an arbitrary partition of $[0,1]$. Let $P = Q = R$ so P and Q are refinements of $R = \{0 = r_0 < r_1 < \cdots < r_N = 1\}$. For $1 \leq k \leq N$, we pick sets of test values of P and Q such that:

$$p_k^* \in [r_{k-1}, r_k] \cap \mathbb{Q} \text{ and } q_k^* \in [r_{k-1}, r_k] \setminus \mathbb{Q}$$

So that $f(p_k^*) = 1$ and $f(q_k^*) = 0$ for $1 \leq k \leq N$. Then:

$$\begin{aligned} S(f,P,P^*) &= \sum_{j=1}^{N} f(p_j^*)(r_j - r_{j-1}) = \sum_{j=1}^{N}(r_j - r_{j-1}) = 1 \\ S(f,Q,Q^*) &= \sum_{j=1}^{N} f(q_j^*)(r_j - r_{j-1}) = \sum_{j=1}^{N} 0 = 0 \end{aligned}$$

Hence $S(f,P,P^*) - S(f,Q,Q^*) = 1$. By the Cauchy's Criterion, f cannot be Riemann integrable. $\square$

Note. Clearly $\mathbb{Q} \cap [0,1]$ is denumerable, write $\mathbb{Q} \cap [0,1] = \{q_1, q_2, \cdots\}$. Let $E_n = \{q_1, \cdots, q_n\}$ be the first n rational numbers in this sequence. Let $f_n = \chi_{E_n}|_{[0,1]}$, that is:

$$f_n : [0,1] \to \mathbb{R} \text{ by } f_n(x) = \begin{cases} 1 & \text{if } x \in E_n = \{q_1, \cdots, q_n\} \\ 0 & \text{otherwise} \end{cases}$$

Ech f_n is 0 for all but finitely many points, hence Riemann integrable with an $\int f_n = 0$. Moreover, we have $f_1 \leq f_2 \leq \cdots$ and for each $x \in [0,1]$:

$$\lim_{n\to\infty} f_n(x) = \chi_E|_{[0,1]}$$

That is, f_n converges pointwise to $\chi_E|_{[0,1]}$. However:

$$\underbrace{\int_0^1 \chi_E|_{[0,1]}}_{DNE} = \int_0^1 \lim_{n\to\infty} f_n \neq \lim_{n\to\infty} \int_0^1 f_n = 0$$

The first integral DNE by Example 1.5. This is a deficiency of the Riemann integral! We cannot change the order of limit and integral when we have a sequence of Riemann integrable function that converges pointwise. (If convegers uniformly, we can).

Lecture 3, 2024/09/09