# PMATH 440 Notes

Analytic Number Theory

Fall 2025

Based on Professor Michael Rubinstein's Lectures

# Contents

Lecture 1, 2025/09/04

# 1 Introduction

Topics covered in this course

(1). Summation methods (summation by parts, Euler-Maclaurin Summation, Poisson Summation, Dirichlet Hyperbola).

(2). Dirichlet series and Dirichlet divisor problem.

(3). Riemann zeta function $\zeta$. Meromorphic continuation ($\zeta$ has a pole at $s = 1$) and functional equation.

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} \quad \text{for} \quad \mathrm{Re}(s) > 1$$

(4). Prime Number Theorem. If $\pi(x) =$ number of prime numbers $\leq x$, then

$$\pi(x) \sim \int_2^x \frac{1}{\log t}\,\mathrm{d}t \sim \frac{x}{\log x}$$

(5). Dirichlet's Theorem. If $0 \neq a, b \in \mathbb{Z}$ and $\gcd(a, b) = 1$, there are infinitely many prime numbers of the form $ak + b$ for $k \in \mathbb{Z}$. For example, there are infinitely many primes of the form $4k + 1$.

(6). More Complex analysis. Gamma function, Weierstrass products and possibly linear fractional transformations and modular forms.

We first introduce some asymptotic notations.

**Definition.** We say that $f(x) \sim g(x)$ as $x \to \infty$ if

$$\lim_{x\to a} \frac{f(x)}{g(x)} = 1$$

The Prime Number Theorem says $\pi(x) \sim \dfrac{x}{\log x}$ as $x \to \infty$, which is equivalent to

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1$$

**Example.** By the Stirling's approximation, we know

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad \text{as } n \to \infty$$

**Definition.** Let $f, g$ be defined on (a subset of) $\mathbb{R}$ and $g$ be a real-valued. We write $f(x) = \mathcal{O}(g(x))$ as $x \to \infty$, where $g$ is real-valued, if there exists $c > 0$ such that $|f(x)| \leq cg(x)$ for all $x > x_0$.

**Example.** $\sin(x) = \mathcal{O}(1)$ as $x \to \infty$ since sin is bounded.

**Example.** By the Stirling's formula we have

$$n! = \mathcal{O}\left(\sqrt{n}\left(\frac{n}{e}\right)^n\right) \quad \text{and} \quad n! = \mathcal{O}\left(\frac{n^{n+1}}{e^n}\right)$$

The first one implies the second one because $\sqrt{n} = \mathcal{O}(n)$.

**Definition.** We write $f(x) = o(g(x))$ as $x \to a$ if

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0$$

In most cases we will take $a = \infty$ or $a = -\infty$. This means "$f(x)$ is much smaller than $g(x)$ near $a$".

**Example.** By the Stirling's formula we have

$$\lim_{n \to \infty} \frac{n!}{\frac{n^{n+1}}{e^n}} = \lim_{n \to \infty} \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\frac{n^{n+1}}{e^n}} = \lim_{n \to \infty} \frac{\sqrt{2\pi}}{\sqrt{n}} = 0$$

It follows that $n! = o(n^{n+1}/e^n)$ as $n \to \infty$.

**Remark (Vinogradov's notation).** We can also write $f(x) = \mathcal{O}(g(x))$ as $f(x) \ll g(x)$.

**Remark.** When we write $f(x) = g(x) + \mathcal{O}(h(x))$ to mean $f(x) - g(x) = \mathcal{O}(h(x))$.

# 2 Summation Methods

## 2.1 Partial Summation

This method is the discrete version of integration by parts.

**Theorem 2.1 (Partial Summation).** Let $f : \mathbb{N} \to \mathbb{C}$ and $g : \mathbb{R} \to \mathbb{C}$ be continuously differentiable on $[1, x]$. Then, for all $x \geq 1$ we have

$$\sum_{1 \leq n \leq x} f(n)g(n) = \left(\sum_{1 \leq n \leq x} f(n)\right) g(x) - \int_1^x \sum_{1 \leq n \leq t} f(n)g'(t) \text{ dt} \tag{1}$$

**Proof.** Consider the term $f(n)g(n)$, we note

$$f(n)g(x) - f(n)\int_n^x g'(t) \text{ dt} = f(n)g(x) - f(n)(g(x) - g(n)) = f(n)g(n) \tag{2}$$

This equality is obtained by looking at the terms that have to do with $f(n)$ in (1). Then summing the equation (2) over $1 \leq n \leq x$ gives us (1). $\square$

**Example (Harmonic Series).** Consider $\sum_{1\le n\le x}\frac{1}{n}$. Take $f(n)=1$ and $g(x)=\frac{1}{x}$. Then by the Partial summation formula we have

$$\sum_{1\le n\le x}\frac{1}{n}=\left(\sum_{1\le n\le x}f(n)\right)g(x)-\int_1^x\sum_{1\le n\le t}f(n)g'(t)\ \mathrm{d}t=\frac{\lfloor x\rfloor}{x}+\int_1^x\frac{\lfloor t\rfloor}{t^2}\mathrm{d}t$$

Here note that

$$\lfloor x\rfloor:=\sum_{1\le n\le x}1=\text{the largest integer}\le x$$

and using this we define

$$\{x\}:=x-\lfloor x\rfloor=\text{the fractional part of } x$$

For example $\lfloor\pi\rfloor=3$ and $\{\pi\}=0.1415926535897\cdots$. Therefore

$$\begin{aligned}\sum_{1\le n\le x}\frac{1}{n}&=\frac{x-\{x\}}{x}+\int_1^x\frac{t-\{t\}}{t^2}\mathrm{d}t\\&=1-\frac{\{x\}}{x}+\int_1^x\frac{1}{t}-\frac{\{t\}}{t^2}\ \mathrm{d}t\\&=1+\log x-\int_1^x\frac{\{t\}}{t^2}\ \mathrm{d}t+\mathcal{O}\left(\frac{1}{x}\right)\end{aligned}$$

Now we analyze this integral

$$\int_1^x\frac{\{t\}}{t^2}\mathrm{d}t=\underbrace{\int_1^\infty\frac{\{t\}}{t^2}\mathrm{d}t}_{<\infty}-\underbrace{\int_x^\infty\frac{\{t\}}{t^2}\mathrm{d}t}_{\mathcal{O}(\frac{1}{x})}$$

The estimation of second integral is by bounding $\{t\}/t^2$ by $1/t^2$. Therefore

$$\sum_{1\le n\le x}\frac{1}{n}=\log x+\underbrace{1-\int_1^\infty\frac{\{t\}}{t^2}\mathrm{d}t}_{:=\gamma}+\mathcal{O}\left(\frac{1}{x}\right)$$

This constant $\gamma$ is called the Euler's constant, that is,

$$\lim_{x\to\infty}\left(\sum_{1\le n\le x}\frac{1}{n}-\log x\right)=\gamma$$

**Conjecture 2.2.** The Euler's constant $\gamma$ is irrational.

Lecture 2, 2025/09/09

**Remark.** Let $\lambda_1 < \lambda_2 < \cdots$ be a sequence of natural numbers, then

$$\sum_{\lambda_n \le x} f(\lambda_n) g(\lambda_n) = \left( \sum_{\lambda_n \le x} f(\lambda_n) \right) g(x) - \int_1^x \left( \sum_{\lambda_n \le x} f(\lambda_n) \right) g'(t) \, \mathrm{d}t$$

This is a generalization of the usual Partial summation formula. The proof is similar. Note that for all $n \ge 1$ we have

$$f(\lambda_n) g(\lambda_n) = f(\lambda_n) g(x) - \int_{\lambda_n}^x f(\lambda_n) g'(t) \, \mathrm{d}t$$

Then summing over all $n$ with $\lambda_n \le x$ we obtain the formula.

**Example (Factorial).** Now let's study the asymptotic of the factorial $m!$ as $m \to \infty$. Since the partial summation only works for sum and $m!$ is a product, we can take the log and consider $\log(m!)$. Let $f(m) = 1$ and let $g(x) = \log(x)$. By the partial summation formula we have

$$\begin{aligned} \log(m!) = \sum_{1 \le n \le m} \log(m) &= m \log m - \int_1^m \frac{\lfloor t \rfloor}{t} \, \mathrm{d}t \\ &= m \log m - \int_1^m \frac{t - \{t\}}{t} \, \mathrm{d}t \\ &= m \log m - (m-1) + \int_1^m \frac{\{t\}}{t} \, \mathrm{d}t \end{aligned}$$

Now we need to estimate the integral and get an (rough) upper and lower bound for it.

$$0 < \int_1^m \frac{\{t\}}{t} \mathrm{d}t < \int_1^m \frac{\mathrm{d}t}{t} = \log m$$

Therefore

$$m \log m - (m-1) < \log(m!) < (m+1) \log m - (m-1)$$

Exponentiating this inequality gives

$$\frac{m^m}{e^{m-1}} < m! < \frac{m^{m+1}}{e^{m-1}}$$

This is a weaker result than the Striling's formula.

**Remark.** The prime counting function is

$$\pi(x) = \sum_{p \le x} 1 = \text{number of primes} \le x$$

For the Riemann zeta function on $\mathrm{Re}(s) > 1$ we have the following identity

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

This is called the Euler's product. Expand the right hand side and by the unique factorization of integers we have the equality. It is sometimes more natural to study the sum of log of primes. We define the function

$$\theta(x) := \sum_{p \le x} \log p$$

The Prime Number Theorem states that $\pi(x) \sim x/\log x$. In fact we have the following proposition.

**Proposition 2.3.** We have

$$\theta(x) \sim x \iff \pi(x) \sim \frac{x}{\log x}$$

**Proof.** $(\Rightarrow)$. Assume $\theta(x) \sim x$. Note that

$$\pi(x) = \sum_{p \le x} 1 = \sum_{p \le x} \log p \cdot \frac{1}{\log p}$$

Let $f(x) = \log x$ and $g(x) = 1/\log x$. By Partial summation we have

$$\pi(x) = \underbrace{\frac{\theta(x)}{\log x}}_{\sim \frac{x}{\log x}} + \int_2^x \theta(t) \cdot \frac{\mathrm{d}t}{(\log t)^2 t}$$

Now we note that since $\theta(x) \sim x$, we know $\theta(x) = \mathcal{O}(x)$ so that

$$\int_2^x \theta(t) \cdot \frac{\mathrm{d}t}{(\log t)^2 t} = \mathcal{O}\left(\int_2^x \frac{\mathrm{d}t}{(\log t)^2}\right)$$

But then we have

$$\int_2^x \frac{\mathrm{d}t}{(\log t)^2} = \int_2^{x^{1/2}} \frac{\mathrm{d}t}{(\log t)^2} + \int_{x^{1/2}}^x \frac{\mathrm{d}t}{(\log t)^2}$$

The first integrand is $\mathcal{O}(1)$ so the integral is $\mathcal{O}(x^{1/2})$, for the second integral we use the bound

$$\int_{x^{1/2}}^x \frac{\mathrm{d}t}{(\log t)^2} = \mathcal{O}\left(\frac{x}{(\log x)^2}\right)$$

Combine all of these, we have

$$\pi(x) = \frac{\theta(x)}{\log x} + \mathcal{O}\left(\frac{x}{(\log x)^2}\right)$$

and therefore

$$\frac{\pi(x)}{x/\log x} = \frac{\theta(x)}{\log x} \cdot \frac{\log x}{x} + \mathcal{O}\left(\frac{1}{\log x}\right) = \frac{\theta(x)}{x} + \mathcal{O}\left(\frac{1}{\log x}\right) = 1 + o(1)$$

($\Leftarrow$). Assume the PNT. Let $f(x) = 1$ and $g(x) = \log x$ when $x$ is prime and 0 otherwise.

$$\theta(x) = \sum_{p \leq x} \log x = \pi(x) \log x - \int_2^x \frac{\pi(t)}{t} \mathrm{d}t$$

Thus $\theta(x) \sim x$ after some work. $\square$

Lecture 3, 2025/09/11

**Example (Meromorphic Continuation of $\zeta$).** Recall the zeta function $\zeta(s)$ is only defined for $\mathrm{Re}(s) > 1$ and is equal to

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

This series converges absolutely if $\mathrm{Re}(s) > 1$ and uniformly in any half plane $\mathrm{Re}(s) \geq X_0 > 1$. We want to extend this function to the half plane $\mathrm{Re}(s) > 0$ using partial summation. We let $f(n) = 1$ and let $g(t) = t^{-s}$, then by the Partial summation formula

$$\sum_{1 \leq n \leq x} \frac{1}{n^s} = \frac{\lfloor x \rfloor}{x^s} + s \int_1^x \frac{\lfloor t \rfloor}{t^{s+1}} \, \mathrm{d}t$$

Here $t^s$ is defined using the principal branch of logartihm, which is defined on $\mathbb{C} \setminus (\infty, 0]$.

$$\begin{aligned}
\sum_{1 \leq n \leq x} \frac{1}{n^s} &= \frac{x - \{x\}}{x^s} + s \int_1^x \frac{(t - \{t\})}{t^s} \, \mathrm{d}t \\
&= \frac{x - \{x\}}{x^s} + \int_1^x \frac{1}{t^s} \, \mathrm{d}t - s \int_1^x \frac{\{t\}}{t^{s+1}} \, \mathrm{d}t
\end{aligned}$$

By taking $x \to \infty$ we have

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{\{t\}}{t^{s+1}} \, \mathrm{d}t = \frac{1}{s-1} + 1 - s \int_1^\infty \frac{\{t\}}{t^{s+1}} \, \mathrm{d}t \qquad (*)$$

The RHS is analytic on $\mathrm{Re}\, s > 0$ except for a simple pole at $s = 1$ with residue 1 because the improper integral $\int_1^\infty t^{-r} \, \mathrm{d}t$ converges when $r > 1$. Note that the function

$$\int_1^\infty \frac{\{t\}}{t^{s+1}} \, \mathrm{d}t$$

on the RHS is analytic by Leibniz's rule (differentiation under the integral sign). Equation $(*)$ allows us to extend the domain of $\zeta$ to $\mathrm{Re}\, s > 0$, with a pole at 1.

## 2.2 Euler-Maclaurin Summation and Bernoulli Polynomials

This summation method looks at sums of the form

$$\sum_{a<n\leq b} g(n)$$

where $a, b \in \mathbb{Z}$ are integers and $a < b$. By the Partial summation formula

$$\begin{aligned}\sum_{a<n\leq b} g(n) &= (b-a)g(b) - \int_a^b ([t]-a)g'(t)\ \mathrm{d}t \\ &= bg(b) - ag(b) - \int_a^b tg'(t)\ \mathrm{d}t + a\int_a^b g'(t)\ \mathrm{d}t + \int_a^b \{t\}g'(t)\ \mathrm{d}t \\ &= bg(b) - ag(b) - \underbrace{\int_a^b tg'(t)\ \mathrm{d}t}_{\star} + ag(b) - ag(a) + \int_a^b \{t\}g'(t)\ \mathrm{d}t\end{aligned}$$

By integration by parts we have

$$\star = \int_a^b tg'(t) = tg(t)|_a^b - \int_a^b g(t)\ \mathrm{d}t = bg(b) - ag(a) - \int_a^b g(t)\ \mathrm{d}t$$

Therefore

$$\begin{aligned}\sum_{a<n\leq b} g(n) &= bg(b) - ag(b) - \left(bg(b) - ag(a) - \int_a^b g(t)\ \mathrm{d}t\right) + ag(b) - ag(a) + \int_a^b \{t\}g'(t)\ \mathrm{d}t \\ &= \int_a^b g(t)\ \mathrm{d}t + \int_a^b \{t\}g'(t)\ \mathrm{d}t\end{aligned}$$

Now let us analyze the integral of $\{t\}g'(t)$. Note that on average $\{t\} = 1/2$, we can write

$$\{t\} = \frac{1}{2} + \left(\{t\} - \frac{1}{2}\right)$$

We have

$$\int_a^b \{t\}g'(t)\ \mathrm{d}t = \frac{1}{2}(g(b)-g(a)) + \int_a^b \left(\{t\} - \frac{1}{2}\right) g'(t)\ \mathrm{d}t$$

Before we continue, we need Bernoulli polynomials.

**Definition.** We define $B_0(x) = 1$. For $k \geq 1$ we recursively define $B_k(x)$ so that

$$B_k'(x) = kB_{k-1}(x) \quad \text{and} \quad \int_0^1 B_k(x)\ \mathrm{d}x = 0$$

The polynomial $B_k(x)$ is called the **$k$-th Bernoulli polynomials**.

**Example ($B_1(x)$).** Let $k = 1$. Then

$$B_1(x) = \int B_0(x)\ \mathrm{d}x = x + B_1$$

Then because

$$\int_0^1 B_1(x)\ \mathrm{d}x = \left[\frac{x^2}{2} + B_1 x\right]_0^1 = \frac{1}{2} + B_1 = 0$$

we know that $B_1 = -\frac{1}{2}$ and $B_1(x) = x - \frac{1}{2}$.

**Example ($B_2(x)$).** Let $k = 2$. Then

$$B_2(x) = \int 2B_1(x)\ \mathrm{d}x = x^2 - x + B_2$$

Then because

$$\int_0^1 B_2(x)\ \mathrm{d}x = \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + B_2 x\right]_0^1 = \frac{1}{3} - \frac{1}{2} + B_2 = 0$$

we know that $B_2 = \frac{1}{6}$ and $B_2(x) = x^2 - x + \frac{1}{6}$.

**Definition.** For $k \geq 0$ we define $B_k = B_k(0)$ to be the **$k$-th Bernoulli number**.

Lecture 4, 2025/09/16

**Proposition 2.4.** For $k \geq 0$ we have

(a). The difference equation: $\dfrac{B_{k+1}(x+1) - B_{k+1}(x)}{k+1} = x^k$

(b). Expansion in terms of Bernoulli numbers: $B_k(x) = \displaystyle\sum_{m=0}^{k} \binom{k}{m} B_{k-m} x^m = \sum_{m=0}^{k} \binom{k}{m} B_m x^{k-m}$

(c). The functional equation: $B_k(x) = (-1)^k B_k(1-x)$

(d). Specials values: $B_k(1) = \begin{cases} (-1)^k B_k(0) & \text{if } k \geq 0 \\ 0 & \text{if } k \text{ is odd and } k \geq 3 \\ 1/2 & \text{if } k = 1 \end{cases}$

(e). Recursion of Bernoulli numbers: $\displaystyle\sum_{m=0}^{k-1} \binom{k}{m} B_m = 0$

(f). Generating Function: $F(x,t) := \displaystyle\sum_{k=0}^{\infty} B_k(x) \frac{z^k}{k!} = \frac{ze^{zx}}{e^z - 1}$

**Remark.** By (a) we note that

$$B_k(0) = B_k(1) \text{ for all } k \geq 0$$

This will be useful later.

**Proposition 2.5.** We have the following Fourier series expansions

$$B_1(\{x\}) = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{n} \quad \text{for } x \notin \mathbb{Z}$$

$$B_k(\{x\}) = -k! \sum_{n \neq 0} \frac{e^{2\pi i n x}}{(2\pi i n)^k} \quad \text{for } k \geq 2$$

Now we return to our discussion on Euler-Maclaurin summation. We need to study the integral

$$\int_a^b \left(\{t\} - \frac{1}{2}\right) g'(t) \, \mathrm{d}t = \int_a^b B_1(\{t\}) g'(t) \, \mathrm{d}t$$

Note that $B_1(\{t\}) = t - \frac{1}{2} - n$ if $n \leq t < n+1$. Hence

$$\int_a^b B_1(\{t\}) g'(t) \, \mathrm{d}t = \left(\int_a^{a+1} + \cdots + \int_{b-1}^b\right) B_1(\{t\}) g'(t) \, \mathrm{d}t$$

Now let us look at the integral on each interval $[n, n+1]$.

$$\int_n^{n+1} B_1(\{t\}) g'(t) \, \mathrm{d}t = \int_n^{n+1} \left(t - \frac{1}{2} - n\right) g'(t) \, \mathrm{d}t$$

Let $u = g'(t)$ and $dv = B_1(\{t\}) \, \mathrm{d}t$. Apply integration by parts we have

$$\begin{aligned}\int_n^{n+1} B_1(\{t\}) g'(t) \, \mathrm{d}t &= \left[\frac{1}{2} B_2(\{t\}) g'(t)\right]_n^{n+1} - \int_n^{n+1} \frac{1}{2} B_2(\{t\}) g''(t) \, \mathrm{d}t \\ &= \frac{1}{2} \left(B_2(1) - B_2(0)\right) \left(g'(n+1) - g'(n)\right) - \frac{1}{2} \int_n^{n+1} B_2(\{t\}) g''(t) \, \mathrm{d}t\end{aligned}$$

We can now apply the same method to the integral $\int_n^{n+1} B_2(\{t\}) g''(t) \, \mathrm{d}t$. Keep doing it, say $K$ times, then we ge the Euler-Maclaurin summation formula.

**Theorem 2.6 (Euler-Maclaurin Summation).** Let $K \in \mathbb{N}$ and $g : \mathbb{R} \to \mathbb{C}$ such that $g^{(K)}$ exists, then for $a < b$ in $\mathbb{N}$ we have

$$\sum_{a < n \leq b} g(n) = \int_a^b g(t) \, \mathrm{d}t + \sum_{k=1}^{K} \frac{(-1)^k B_k}{k!} (g^{(k-1)}(b) - g^{(k-1)}(a)) + \frac{(-1)^{K+1}}{K!} \int_a^b B_K(\{t\}) g^{(K)}(t) \, \mathrm{d}t$$

**Example (Sum of powers).** We claim that for $r \geq 1$ and $N \geq 1$ we have

$$\sum_{n=1}^{N} n^r = \frac{B_{r+1}(N+1) - B_{r+1}(1)}{r+1}$$

We will apply Euler-Maclaurin summation formula and properties of Bernoulli polynomials to prove it. Let $g(t) = t^r$ then $g^{(r)}(t) = 0$ and

$$g^{(m)}(N) - g^{(m)}(0) = \begin{cases} r(r-1)\cdots(r-m+1)N^{r-m} & \text{if } m \leq r-1 \\ 0 & \text{if } m \geq r \end{cases}$$

Let $a = 0$ and $b = N$ and $K = r$, then by Euler-Maclaurin summation we have

$$\begin{aligned}
\sum_{n=1}^{N} n^r &= \int_0^N t^r \, \mathrm{d}t + \sum_{k=1}^{r} \frac{(-1)^k B_k}{k!} r(r-1)\cdots(r-k+2)N^{r-k+1} \\
&= \int_0^N t^r \, \mathrm{d}t + \sum_{k=1}^{r} \frac{(-1)^k B_k}{r-k+1} \binom{r}{k} N^{r-k+1} \\
&= \int_0^N \sum_{k=0}^{r} (-1)^k B_k \binom{r}{k} t^{r-k} \, \mathrm{d}t && (t^r \text{ is the } k=0 \text{ term}) \\
&= \int_0^N (-1)^r \sum_{k=0}^{r} B_k \binom{r}{k} (-t)^{r-k} \, \mathrm{d}t && ((-1)^r(-1)^{r-k} = (-1)^k) \\
&= \int_0^N (-1)^r B_r(-t) \, \mathrm{d}t = \int_0^N B_r(t+1) \, \mathrm{d}t && (\text{property (b) and (c)}) \\
&= \frac{B_{r+1}(N+1) - B_{r+1}(1)}{r+1}
\end{aligned}$$

**Example.** The Euler-Maclaurin summation formula can be used to obtain an analytic continuation of $\zeta(s)$ as far to the left as we want, and also provides a useful expansion. Consider

$$\sum_{n=1}^{N} n^{-s} = 1 + \sum_{n=2}^{N} n^{-s}$$

Let $K \in \mathbb{N}$, we want to extend $\zeta(s)$ to the region $\operatorname{Re} s > -K+1$. Let $a = 1$ and $b = N$ and $g(x) = x^{-s}$. Note that

$$g^{(m)}(x) = (-1)^m s(s+1)\cdots(s+m-1)x^{-s-m}$$

for $m \geq 1$. By Euler-Maclaurin summation formula we have

$$\sum_{2\leq n\leq N} n^{-s} = \int_1^N t^{-s}\ \mathrm{d}t - \sum_{k=1}^{K} \frac{(-1)^k B_k}{k!}(-1)^k (s+k-2)\cdots(s+1)s(N^{-s-k+1}-1) + \int_1^N B_K(\{t\})\frac{(-1)^{K+1}}{K!}(-1)^K(s+K-1)\cdots(s+1)st^{-s-K}\ \mathrm{d}t$$

Note that we have

$$\frac{(s+k-2)\cdots(s+1)s}{k!} = \frac{1}{k}\frac{(s+k-2)\cdots(s+1)s}{(k-1)!} = \frac{1}{k}\binom{s+k-2}{k-1}$$

It follows that

$$\sum_{2\leq n\leq N} n^{-s} = \int_1^N t^{-s}\ \mathrm{d}t - \sum_{k=1}^{K}\frac{B_k}{k}\binom{s+k-2}{k-1}(N^{-s-k+1}-1) - \binom{s+K-1}{K}\int_1^N B_K(\{t\})t^{-s-K}\ \mathrm{d}t$$

Taking $N \to \infty$ we have that

$$\zeta(s) = 1 + \sum_{n\geq 2} n^{-s} = 1 + \frac{s}{1-s} + B_1 + \sum_{k=2}^{\infty}\frac{B_k}{k}\binom{s+k-2}{k-1} - \binom{s+K-1}{K}\int_1^{\infty} B_K(\{t\})t^{-s-K}\ \mathrm{d}t$$

for $\operatorname{Re} s > 1$. The RHS is meromorphic on the region $\operatorname{Re} s > -K+1$, with a pole at $s = 1$. This gives us a meromorphic continuation to $\operatorname{Re} s > -K+1$ using the quantity on the RHS.

**Remark.** Since we can choose $K$ arbitrarily large, we have a meromorphic continuation of $\zeta(s)$ to the entire $\mathbb{C}$ except for a pole at $s = 1$. Note that we can do this because all the meromorphic continuation to $\operatorname{Re} s > -K+1$ agree on the open set $\operatorname{Re} s > 1$, so they are all equal by the identity theorem for analytic functions.