Selberg's Sieve

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Contents

1 The Sieve of Eratosthenes

Sieves are used to bound the size of a set after elements with certain "undesirable" properties have been removed. A basic example of a sieve is the method of inclusion-exclusion which gives an exact count for the number of elements in a set.

Suppose we are given $A = [1, x] \cap \mathbb{Z}$, the set of integers $\leq x$. We want to find all prime numbers in A. The following lemma gives us a neat way to do it.

Lemma 1.1. Let $N \in \mathbb{N}$ be a positive integer and $n \in \mathbb{Z}$ with $2 \leq n \leq N$. If n is composite, then there is a prime divisor $p \mid n$ such that $p \leq \sqrt{N}$.

Proof: Suppose all prime divisors are $> \sqrt{N}$. Since n is composite, it means n has at least two prime factors (counting multiplicities), say p and q. Then $pq \mid n$ so $pq \leq n$, but

$$pq > \sqrt{N}\sqrt{N} = N \geq n$$

contradiction. □

Let $z = \sqrt{N}$. This lemma tells us, if we can remove all the multiples of the primes in $[1, z]$ in A, then the elements that remain are prime numbers between $[z] + 1$ and N (We do not get all primes $\leq N$ because the primes $\leq z$ are also removed). Also, note that 1 is not removed because it is not divisible by any primes.

Let $\pi(N, z)$ denote the number of integer $\leq N$ that is not divisible by any primes $\leq z$, then:

$$\pi(N) = \pi(z) + \pi(N, z) - 1 \tag{1.1}$$

Let us see an example.

Example. Find the number of primes in $S = [1, 40]$.

Note that $z = [\sqrt{40}] = 6$ and the primes $\leq z$ are $2, 3, 5$. Now let us remove the multiples of 2,3,5.

Let $A = 2\mathbb{Z} \cap S$ and $B = 3\mathbb{Z} \cap S$ and $C = 5\mathbb{Z} \cap S$ be integers that ARE divisible by $2, 3, 5$ in S. We wish to determine the size of the set

$$P = S \setminus (A \cup B \cup C)$$

It suffices to determine the size of $A \cup B \cup C$. We can do this by the inclusion-exclusion

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap B| - |B \cap C| + |A \cap B \cap C|$$

The size of each individual set is easy to determine

$$\begin{aligned} |A| &= [40/2] = 20 \\ |B| &= [40/3] = 13 \\ |C| &= [40/5] = 8 \\ |A \cap B| &= [40/6] = 6 \\ |A \cap C| &= [40/10] = 4 \\ |B \cap C| &= [40/15] = 2 \\ |A \cap B \cap C| &= [40/30] = 1 \end{aligned}$$

Then, the number of integers ≤ 40 that are not divisible by $2, 3$ or 5 is

$$40 - (20 + 13 + 8 - 6 - 4 - 2 + 1) = 10$$

Hence $\pi(40, z) = 10$, by (1.1) we have:

$$\pi(40) = \pi(z) + \pi(40, z) - 1 = 3 + 10 - 1 = 12$$

As desired!

This method can be generalized in following ways:

1. Instead of doing sieve on the set $[1, N] \cap \mathbb{Z}$, we can do it on an arbitrary set.
2. Instead of choosing $z = \sqrt{N}$, we can choose z to be any suitable positive real number, then instead of equality in (1.1), we would be an inequality.

We make same definitions first.

Definition. Let A be a finite subset of $\mathbb{N}$, P a set of primes and $z > 0$ some real number. Define

$$P_z = \prod_{\substack{p \in P \\ p < z}} p$$

For each $d \mid P_z$, let $A_d = \{a \in A : d \mid a\}$. We define

$$S(A, P, z) = \left| \left(A \setminus \bigcup_{p \mid P_z} A_p \right) \right| \tag{1.2}$$

to be the size of the set of all $a \in A$ that are not divisible by p for all $p < z$. Another way to write (1.2) is this: Note that $a \in A$ is not divisible by any $p < z$ if and only if $(a, P_z) = 1$. Hence:

$$S(A, P, z) = \sum_{\substack{a \in A \\ (n, P_z) = 1}} 1 \tag{1.3}$$

Let us see how to generalize (1.1) using this:

Example. Fix $N \in \mathbb{N}$. If $A = [1, N] \cap \mathbb{Z}$ and P be the set of all prime numbers and let $z > 0$ be arbitrary. Hence by (1.3) we obtain

$$\pi(N, z) = S(A, P, z) = \sum_{\substack{n \leq N \\ (n, P_z) = 1}} 1 = 1 + \sum_{\substack{1 < n \leq z \\ (n, P_z) = 1}} 1 + \sum_{\substack{z < n \leq N \\ (n, P_z) = 1}} 1$$

Note that the second sum is 0 as there is no $1 < n \leq z$ that is coprime with P_z. Now let us analyze the last summation, we have

$$\sum_{\substack{z < n \leq N \\ (n, P_z) = 1}} 1 \geq \sum_{z < p \leq N} 1 = \pi(N) - \pi(z)$$

because any prime $z < p \leq N$ is not divisble by any $p < z$. Therefore we have

$$\pi(N, z) \geq 1 + \pi(N) - \pi(z)$$

And hence

$$\pi(N) \leq \pi(N, z) + z - 1 \tag{1.4}$$

Therefore, if we can estimate $\pi(N, z)$, we can get an upper bound for $\pi(N)$.

We can write $S(A, P, z)$ in another form, which is some times easy to munipulate. Recall that

$$\sum_{d \mid n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$S(A, P, z) = \sum_{\substack{a \in A \\ (a, P_z) = 1}} 1 = \sum_{a \in A} \sum_{d \mid (a, P_z)} \mu(d) \tag{1.5}$$

Now we will look at (1.5) and see how we can find a way to estimate it.

Theorem 1.2.

$$\pi(x) \ll \frac{x}{\log \log x}$$

Proof: Let $A = [1, x] \cap \mathbb{Z}$, then $\pi(x, z) = S(A, P, z)$. By (1.5) we have:

$$\pi(x,z) = \sum_{a \in A} \sum_{d|(a,P_z)} \mu(d) \leq \sum_{n \leq x} \sum_{\substack{d|n \\ d|P_z}} \mu(d) = \sum_{d|P_z} \mu(d) \sum_{\substack{n \leq x \\ d|n}} 1$$

$$= \sum_{d|P_z} \mu(d) \left[\frac{x}{d}\right] = x \sum_{d|P_z} \frac{\mu(d)}{d} + O\left(\sum_{d|P_z} 1\right)$$

Note that any $d \mid P_z$ is of the form $d = p_1^{e_1} \cdots p_r^{e_r}$ with $\{p_1, \cdots, p_r\}$ the set of all primes $< z$, so $r \leq \pi(z)$ and $e_i \in \{0, 1\}$. Therefore there are at most $2^{\pi(z)}$ choices for d. Hence:

$$\pi(x,z) = x \sum_{d|P_z} \frac{\mu(d)}{d} + O(2^z) \tag{1.6}$$

Now, note that:

$$\sum_{d|P_z} \frac{\mu(d)}{d} = \prod_{p|P_z} \left(1 - \frac{1}{p}\right) = \prod_{p<z} \left(1 - \frac{1}{p}\right) \tag{1.7}$$

Using the inequality that $1 - x \leq e^{-x}$ for $x > 0$, we have:

$$\prod_{p<z} \left(1 - \frac{1}{p}\right) \leq \prod_{p<z} e^{-1/p} = \exp\left(-\sum_{p<z} \frac{1}{p}\right)$$

Recall that:

$$\sum_{p<z} \frac{1}{p} \geq \log\log z + O(1)$$

Therefore we have:

$$\prod_{p<z} \left(1 - \frac{1}{p}\right) \ll e^{-\log\log z} = \frac{1}{\log z}$$

Now, we choose $z = \log x$, then using (1.6) and (1.7) we get:

$$\pi(x,z) \ll \frac{x}{\log z} + O(2^{\log x}) \ll \frac{x}{\log\log x}$$

Lastly, using (1.4) we have:

$$\pi(x) \leq \pi(x,z) + z - 1 \ll \frac{x}{\log\log x} + \log x - 1 \ll \frac{x}{\log\log x}$$

As desired! $\square$

2 Selberg's Sieve

Selberg came up with this brilliant ideal to replace $\sum \mu(d)$ in (1.5) with a quadratic form, chosen optimally to make the result minimal. That is, let $(\lambda_d) \subseteq \mathbb{R}$ be a sequence such that $\lambda_1 = 1$, then:

$$\sum_{d|n} \mu(d) \leq \left(\sum_{d|n} \lambda_d\right)^2$$

because the LHS is at most 1.

Suppose there is a multiplicative function f with $f(p) > 1$ for all $p \in P$, and for all d we have:

$$|A_d| = \frac{X}{f(d)} + R_d \tag{2.1}$$

to be the estimation of $|A_d|$, where X is an estimation of A and R_d is the error term.

Theorem 2.1 (Selberg's Sieve). With the setting above. Let f_1 be the unique function such that:

$$f(n) = \sum_{d|n} f_1(d) \tag{2.2}$$

Also, we define:

$$V(z) = \sum_{\substack{d<z \\ d|P_z}} \frac{\mu^2(d)}{f_1(d)} \tag{2.3}$$

Then we have:

$$S(A,P,z) \leq \frac{X}{V(z)} + \left(\sum_{\substack{d_1,d_2 \leq z \\ d_1,d_2|P_z}} |R_{[d_1,d_2]}| \right) \tag{2.4}$$

Lemma 2.2. Let f_1, f_2 be a multiplicative function and d_1, d_2 be positive squarefree integers, then:

$$f([d_1,d_2])f((d_1,d_2)) = f(d_1)f(d_2) \tag{2.5}$$

Proof of Selberg's Sieve: Let (λ_d) be a sequence of real numbers with $\lambda_1 = 1$ and $\lambda_d = 0$ for all $d > z$. Then by (1.2) we have:

$$\begin{aligned} S(A,P,z) &= \sum_{\substack{a\in A \\ (a,P_z)=1}} 1 = \sum_{a\in A} \sum_{d|(a,P_z)} \mu(d) \leq \sum_{a\in A} \left(\sum_{d|(a,P_z)} \lambda_d \right)^2 = \sum_{a\in A} \left(\sum_{d_1,d_2|(a,P_z)} \lambda_{d_1}\lambda_{d_2} \right) \\ &= \sum_{d_1,d_2|P_z} \lambda_{d_1}\lambda_{d_2} \sum_{\substack{a\in A \\ d_1,d_2|a}} 1 = \sum_{d_1,d_2|P_z} \lambda_{d_1}\lambda_{d_2} \sum_{\substack{a\in A \\ [d_1,d_2]|a}} 1 = \sum_{d_1,d_2|P_z} \lambda_{d_1}\lambda_{d_2} |A_{[d_1,d_2]}| \end{aligned}$$

Now using (2.1) and (2.5) we have:

$$\begin{aligned} S(A,P,z) &= X \sum_{d_1,d_2|P_z} \frac{\lambda_{d_1}\lambda_{d_2}}{f([d_1,d_2])} + \sum_{d_1,d_2|P_z} \lambda_{d_1}\lambda_{d_2} R_{[d_1,d_2]} \\ &= X \sum_{d_1,d_2|P_z} \lambda_{d_1}\lambda_{d_2} \frac{f(d_1)f(d_2)}{f((d_1,d_2))} + \sum_{d_1,d_2|P_z} \lambda_{d_1}\lambda_{d_2} R_{[d_1,d_2]} \\ &= XT + R \end{aligned}$$

where we defined:

$$T = \sum_{d_1,d_2|P_z} \lambda_{d_1}\lambda_{d_2} \frac{f(d_1)f(d_2)}{f((d_1,d_2))} = \sum_{\substack{d_1,d_2 \leq z \\ d_1,d_2|P_z}} \lambda_{d_1}\lambda_{d_2} \frac{f(d_1)f(d_2)}{f((d_1,d_2))} \tag{2.6}$$

so that XT is our main term, and:

$$R = \sum_{d_1,d_2|P_z} \lambda_{d_1}\lambda_{d_2} R_{[d_1,d_2]} = \sum_{\substack{d_1,d_2 \le z \\ d_1,d_2|P_z}} \lambda_{d_1}\lambda_{d_2} R_{[d_1,d_2]} \tag{2.7}$$

to be our error term. Let us analyze T first. Our main term is a quadratic form in (λ_d), and remember, we want to minimize it to get a good upper bound. To do this, we will first transform it into a diagonal form.

$$\begin{aligned}
T &= \sum_{\substack{d_1,d_2 \le z \\ d_1,d_2|P_z}} \frac{\lambda_{d_1}\lambda_{d_2}}{f(d_1)f(d_2)} f((d_1,d_2)) \\
&= \sum_{\substack{d_1,d_2 \le z \\ d_1,d_2|P_z}} \frac{\lambda_{d_1}\lambda_{d_2}}{f(d_1)f(d_2)} \sum_{\delta|(d_1,d_2)} f_1(\delta) && \text{(by (2.2))} \\
&= \sum_{\substack{\delta \le z \\ \delta|P_z}} f_1(\delta) \sum_{\substack{d_1,d_2 \le z \\ d_1,d_2|P_z \\ \delta|(d_1,d_2)}} \frac{\lambda_{d_1}\lambda_{d_2}}{f(d_1)f(d_2)} \\
&= \sum_{\substack{\delta \le z \\ \delta|P_z}} f_1(\delta) u_\delta^2
\end{aligned}$$

where u_δ is defined by:

$$u_\delta = \sum_{\substack{d \le z \\ d|P_z \\ \delta|d}} \frac{\lambda_d}{f(d)} \tag{2.8}$$

Hence we have transformed our quadratic form to a diagonal form:

$$T = \sum_{\substack{\delta \le z \\ \delta|P_z}} f_1(\delta) u_\delta^2$$

By dual Möbius Inversion Formula on (2.8) we have:

$$\frac{\lambda(\delta)}{f(\delta)} = \sum_{\substack{d|P_z \\ \delta|d}} \mu\left(\frac{d}{\delta}\right) u_d \tag{2.9}$$

since $\lambda_d/f(d)$ and u_δ are well-defined on the divisor-closed set $\{\delta < z : \delta \mid P_z\}$. Let $\delta = 1$, we have:

$$1 = \frac{1}{f(1)} = \sum_{\substack{d|P_z \\ \delta|d}} \mu(d) u_d = \sum_{d|P_z} \mu(d) u_d$$

Also, by (2.8), if $\delta \ge z$, then the sum is empty since $z \le \delta < d < z$. Therefore $u_\delta = 0$ for $\delta \ge z$. Using this, we can write the above equality as:

$$\sum_{\substack{\delta \le z \\ \delta|P_z}} \mu(\delta) u_\delta = 1 \tag{2.10}$$

Therefore, we have:

$$\sum_{\substack{\delta\le z\\ \delta|P_z}} f_1(\delta)\left(u_\delta - \frac{\mu(\delta)}{f_1(\delta)V(z)}\right)^2 = \sum_{\substack{\delta\le z\\ \delta|P_z}} f_1(\delta)u_\delta^2 - 2\sum_{\substack{\delta\le z\\ \delta|P_z}} \frac{f_1(\delta)\mu(d)}{f_1(\delta)V(z)}u_\delta + \sum_{\substack{\delta\le z\\ \delta|P_z}} f_1(\delta)\frac{\mu(\delta)^2}{f_1(\delta)^2V(z)^2}$$

$$= T - \frac{2}{V(z)}\sum_{\substack{\delta\le z\\ \delta|P_z}} \mu(d)u_\delta + \frac{1}{V(z)^2}\sum_{\substack{\delta\le z\\ \delta|P_z}} \frac{\mu(\delta)^2}{f_1(\delta)}$$

By (2.10) and (2.3), the above sum is equal to:

$$T - \frac{2}{V(z)} + \frac{1}{V(z)} = T - \frac{1}{V(z)}$$

Therefore we have:

$$T = \sum_{\substack{\delta\le z\\ \delta|P_z}} f_1(\delta)\left(u_\delta - \frac{\mu(\delta)}{f_1(\delta)V(z)}\right)^2 + \frac{1}{V(z)} \tag{2.11}$$

Note that since $\sum_{d|n} f_1(d) = f(n)$, by Möbius inverison we have:

$$f_1(n) = \sum_{d|n}\mu\left(\frac{n}{d}\right)f(d)$$

so when $n = p$ is prime:

$$f_1(p) = \mu(p)f(p) + \mu(1)f(1) = f(p) - 1 > 0$$

By multiplicativity, $f_1(d) > 0$ for all d. Therefore the first sum in (2.10) is always non-negative, so T is minimized when the sum is 0, which is when:

$$u_\delta = \frac{\mu(\delta)}{f_1(\delta)V(z)} \tag{2.12}$$

because $f_1(d)$ is always positive. The minimal value of T is $1/V(z)$.

Now let us look at the error term R. By (2.12) and (2.9) we have:

$$V(z)\lambda_\delta = f(\delta)\sum_{\substack{d\le z\\ d|P_z\\ \delta|d}} \frac{\mu(d/\delta)\mu(d)}{f_1(\delta)} = f(\delta)\sum_{\substack{t\le z/\delta\\ t|P_z\\ (t,\delta)=1}} \frac{\mu^2(t)\mu(\delta)}{f_1(t)f_1(\delta)}$$

$$= \mu(\delta)\left(\sum_{p|\delta}\frac{f(p)}{f_1(p)}\right)\sum_{\substack{t\le z/\delta\\ t|P_z\\ (t,\delta)=1}} \frac{\mu^2(t)}{f_1(t)}$$

$$= \mu(\delta)\left(\sum_{p|\delta}\left(1 + \frac{1}{f_1(p)}\right)\right)\sum_{\substack{t\le z/\delta\\ t|P_z\\ (t,\delta)=1}} \frac{\mu^2(t)}{f_1(t)}$$

Therefore we ge t $|V(z)||\lambda_\delta| \le |V(z)|$ so $|\lambda_\delta| \le 1$. Hence:

$$R = O\left(\sum_{\substack{d_1,d_2\le z\\ d_1,d_2|P_z}} |\lambda_{d_1}\lambda_{d_2}||R_{[d_1,d_2]}|\right) = \left(\sum_{\substack{d_1,d_2\le z\\ d_1,d_2|P_z}} |R_{[d_1,d_2]}|\right)$$

As desired. □

3 Applications

In order to use Selberg's Sieve, we need to find a lower bound on $V(z)$. So we have the following lemma:

Lemma 3.1. Let $\tilde{f}$ be a completely multiplicative function such that $\tilde{f}(p) = f(p)$ for all primes p. We have:

$$V(z) \ge \sum_{\substack{\delta\le z\\ p|\delta\Rightarrow p|P_z}} \frac{1}{\tilde{f}(\delta)} \tag{3.1}$$

In Theorem 1.2 we gave an upper bound for $\pi(x)$, now using Selberg's Sieve it turns out we can give a better upper bound for $\pi(x)$.

Theorem 3.2.

$$\pi(x) \ll \frac{x}{\log x}$$

Proof: Let $A = [1,x]\cap\mathbb{Z}$ and $P =$ all primes and $z > 0$. We have:

$$A_d = \{n \le x : d \mid n\} \implies |A_d| = \left[\frac{x}{d}\right] = \frac{x}{d} + \left\{\frac{x}{d}\right\}$$

Therefore let $X = x$ and $f(d) = d$ and $R_d = \left\{\frac{x}{d}\right\}$. Therefore since $\sum_{d|n} f_1(d) = n$, we have $f_1(d) = \phi(d)$.

$$V(z) = \sum_{\substack{d\le z\\ d|P_z}} \frac{\mu^2(d)}{\phi(d)} \ge \sum_{\substack{d\le z\\ d|P_z}} \frac{\mu^2(d)}{d} = \sum_{d\le z}\frac{1}{d} - \sum_{d\le z}{}' \frac{1}{d}$$

where the sum $\sum'$ is over all non-squarefree integers d. Since:

$$\sum_{d\le z}\frac{1}{d} = \log z + O(1)$$

and also notice that:

$$\sum_{d\le z}{}'\frac{1}{d} \le \frac{1}{4}\sum_{d\le z/4}\frac{1}{d}$$

It follows that:

$$V(z) = \sum_{\substack{d\le z\\ d|P_z}} \frac{\mu^2(d)}{\phi(d)} \gg \log z$$

Hence by Selberg's Sieve we have:

$$\pi(x,z) = S(A,P,z) \ll \frac{x}{\log z} + z^2$$

here the error term is $\ll z^2$ since $R_d \ll 1$. Pick:

$$z = \left(\frac{x}{\log x}\right)^{1/2}$$

Note that $\log z \gg \log x$, and $z^2 = x/\log x$, so we have:

$$\pi(x,z) \ll \frac{x}{\log x}$$

Hence, combined with (1.3) it follows that:

$$\pi(x) \ll 1 + \left(\frac{x}{\log x}\right)^{1/2} + \frac{x}{\log x} \ll \frac{x}{\log x}$$

As desired! $\square$

Definition. We say a prime p is a **twin prime** if $p+2$ is also a prime. Let $\pi_2(x)$ denote the number of twin primes $\leq x$.

Fix $x > 0$. Define $A = \{n(n+2) : n \leq x\}$ and let P be the set of all primes. Let $z > 0$. Let us look at what $S(A,P,z)$ counts. Note that $n(n+2)$ is counted in $S(A,P,z)$ if:

$$\begin{aligned}(n(n+2), P_z) = 1 &\iff p \nmid n(n+2) \text{ for all } p < z \\ &\iff p \nmid n \text{ and } p \nmid n+2 \text{ for all } p < z\end{aligned}$$

Therefore, if p is a twin prime and $p > z$, then $p(p+2)$ is counted in $S(A,P,z)$. Also, note that if an integer can be expressed as $n(n+2)$ for $n > 0$, then this expression is unique! So we can correspond $p(p+2)$ to p. This means, $S(A,P,z)$ counts all $p(p+2)$ for all twin primes $p > z$ iff $S(A,P,z)$ counts all twin prime p with $p > z$. Therefore we have:

$$\pi_2(x) - \pi_2(z) \leq S(A,P,z)$$

It follows that:

$$\pi_2(x) \leq S(A,P,z) + z \tag{3.3}$$

Now, it all boils down to estimate $S(A,P,z)$. Recall that, we need:

1. Estimation of $|A|$.
2. Estimation of $|A_d|$ for $d \mid P_z$.
3. Lower bound of $V(z)$.

The first one is easy, we have $|A| = [x]$, so let $X = x$. The second part is a little tricky, let us make a definition first:

Definition. For $n \in \mathbb{N}$, let $\Omega(n)$ denote the number of prime divisors of n, counting multiplicities. Also, let $\tau(n)$ denote the number of divisors of n and $\tau_1(n)$ denote the number of odd divisors of n:

$$\tau(n) = \sum_{d|n} 1 \quad \text{and} \quad \tau_1(n) = \sum_{\substack{d|n \\ (d,2)=1}} 1$$

For example, for $n = 2^2 \cdot 3$, we have $\Omega(n) = 3$ and $\tau(n) = 6$ and $\tau_1(n) = 1$.

For $d \mid P_z$, let $N(d)$ denote the number of solutions to $n(n+2) = 0$ in $\mathbb{Z}/d\mathbb{Z}$. Then:

$$|A_d| = \frac{[x]}{d} N(d)$$

This is because, $[x]/d$ represents the number of times a complete list of representatives mod d appears in A. Eahc time, there are $N(d)$ solutions. Let:

$$R_d = |A_d| - \frac{x}{d}N(d) = \frac{[x]}{d}N(d) - \frac{x}{d}N(d) = \frac{-\{x\}}{d}N(d)$$

It follows that $|R_d| \leq 1$. Using the notations from Selberg's Sieve, we can define $X = x$ and $f(d) = d/N(d)$. Now let us analyze $N(d)$. If $d = 2$, then $N(d) = 1$. Otherwise write $d = p_1 \cdots p_r$ with $r = \omega(d)$ the number of prime divisors. Then to solve $n(n+2) \equiv 0 \pmod 2$, it is enough to solve:

$$n(n+2) \equiv 0 \pmod{p_1}$$
$$\vdots$$
$$n(n+2) \equiv 0 \pmod{p_r}$$

Each congruence has 1 or 2 solutions. If p_i is odd then there are two and if $p_i = 2$ then there is 1. Therefore, by Chines Remainder theorem, there are $2^{\omega(d)}$ or $2^{\omega(d)-1}$ solutions! Hence we have:

$$|R_d| \leq N(d) \leq 2^{\omega(d)} \tag{3.4}$$

Now we would like to use Lemma 3.1 to analyze $V(z)$. Note that:

$$f(p) = \begin{cases} p & \text{if } p = 2 \\ \frac{p}{2} & \text{if } p > 2 \end{cases}$$

Let $\tilde{f}$ be the completely multiplicative function with $\tilde{f}(p) = f(p)$ for prime p. Then, the lemma says:

$$V(z) \geq \sum_{\substack{n \leq z \\ p|\delta \Rightarrow p|P_z}} \frac{1}{\tilde{f}(n)} = \sum_{n \leq z} \frac{1}{\tilde{f}(n)}$$

Let us analyze $\tilde{f}(n)$ for $n \leq z$.

Now, back to $\tilde{f}(n)$. We write $n = 2^s p_1 \cdots p_m$ where p_i are odd primes, not necessarily distinct. Then:

$$\tilde{f}(n) = f(2)^s f(p_1) \cdots f(p_m) = 2^s \cdot \frac{p_1}{2} \cdots \frac{p_m}{2} = \frac{n}{2^m}$$

Therefore we have:

$$\frac{1}{\tilde{f}(n)} = \frac{2^m}{n} \geq \frac{\tau_1(n)}{n}$$

Now, let us analyze $\sum \tau_1(n)$. Note that we have:

$$\begin{aligned}\sum_{n \leq z} \tau_1(n) &= \sum_{n \leq z} \sum_{\substack{d|n \\ (d,2)=1}} 1 = \sum_{\substack{d \leq z \\ (d,2)=1}} \sum_{\substack{n \leq z \\ d|n}} 1 = \sum_{\substack{d \leq z \\ (d,2)=1}} \left[\frac{z}{d}\right] \\ &\geq \sum_{\substack{d \leq z \\ (d,2)=1}} \frac{z}{d} - \sum_{\substack{d \leq z \\ (d,2)=1}} 1 \\ &\geq z \sum_{\substack{d \leq z \\ (d,2)=1}} \frac{1}{d} - z\end{aligned}$$

Now let us analyze this sum. By partial summation with:

$$a_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \quad \text{and} \quad f(t) = \frac{1}{t}$$

We have that $A(z) = [z/2]$, thus:

$$\begin{aligned}\sum_{\substack{n \leq z \\ (n,2)=1}} \frac{1}{n} &= \frac{1}{z}\left[\frac{z}{2}\right] + \int_1^z \frac{\left[\frac{t}{2}\right]}{t^2}\,dt \\ &= \frac{1}{2}\int_1^z \frac{1}{t}\,dt + \frac{1}{z}\left[\frac{z}{2}\right] - \int_1^z \frac{1}{t^2}\,dt \\ &\geq \frac{1}{2}\log z - \int_1^\infty \frac{1}{t^2}\,dt \\ &= \frac{1}{2}\log z - C\end{aligned}$$

Hence, we have that:

$$\sum_{n \leq z} \tau_1(n) \geq \frac{1}{2} z \log z - (C+1)z \tag{3.5}$$

Now, we have:

$$V(z) \geq \sum_{n \leq z} \frac{1}{\tilde{f}(n)} \geq \sum_{n \leq z} \frac{\tau_1(n)}{n}$$

For the last sum, we apply partial summation and get:

$$\begin{aligned}\sum_{n \leq z} \frac{\tau_1(n)}{n} &= \frac{1}{z}\sum_{n \leq z} \tau_1(n) + \int_1^z \frac{\sum_{n \leq t} \tau_1(n)}{t^2}\,dt \\ &\geq \frac{1}{z}\left(\frac{1}{2} z \log z - (C+1)z\right) + \frac{1}{2}\int_1^z \frac{\frac{1}{2} t \log t - (C+1)t}{t^2}\,dt \\ &\geq \frac{1}{4}\log^2 z - A\log z - B\end{aligned}$$

For some real number $A, B > 0$. Thus we have:

$$V(z) \geq \frac{1}{4}\log^2 z - A\log z - B$$

which implies that:

$$V(z) \gg \frac{1}{4}\log^2 z$$

Now, we can analyze the error term:

$$\sum_{\substack{d_1,d_2 \leq z \\ d_1,d_2 | P_z}} |R_{[d_1,d_2]}| \leq \left(\sum_{\substack{d\leq z \\ d|P_z}} 2^{\omega(d)}\right)^2 \leq \left(\sum_{\substack{d\leq z \\ d \text{ squarefree}}} 2^{\omega(d)}\right)^2$$
$$\leq \left(\sum_{d\leq z} \tau(d)\right)^2 \leq (z\log z)^2$$

Hence, combine everything together we obtained:

$$\begin{aligned} \pi_2(x) &\leq S(A, P, z) + z \\ &\leq \frac{x}{V(z)} + \sum_{\substack{d_1,d_2 \leq z \\ d_1,d_2 | P_z}} |R_{[d_1,d_2]}| + z \\ &\ll \frac{4x}{\log^2 z} + (z\log z)^2 + z \end{aligned}$$

Now, we choose $z = x^{1/4}$, we obtain that:

Theorem 3.3.

$$\pi_2(x) \ll \frac{x}{\log^2 x} \tag{3.6}$$

Recall the Dirichlet Theorem says that for $(a, k) = 1$, there are infinitely many primes p such that $p \equiv a \pmod{k}$. The trick of the proof of this theorem is to prove the series

$$\sum_{p\equiv a \pmod{k}} \frac{1}{p} = \infty$$

One may wonder if this trick works for twin primes. The answer is no.

Corollary 3.4 (Brun). The sum of reciprocals of twin primes converges.

Proof: For fixed $x > 0$, consider the sum:

$$S(x) = \sum_{\substack{p\leq x \\ p+2 \text{ is prime}}} \frac{1}{p} = \sum_{n\leq x} a_n f(n)$$

where $a_n = 1$ if n is prime and $n + 2$ is prime, and 0 otherwise and $f(t) = 1/t$. Partial summation yields:

$$S(x) = \frac{A(x)}{x} + \int_2^x \frac{A(t)}{t^2}\, dt$$

where $A(x) = \sum_{n \le x} a_n = \pi_2(x)$. By Theorem 3.3 we have

$$S(x) \ll \frac{1}{\log^2 x} + \int_2^x \frac{1}{t \log^2 t} \, dt$$

The first term goes to 0, and the integral converges. Therefore $S(x)$ is bounded. $\square$