PMATH 352 Notes

Winter 2024

Contents

Lecture 1, 2024/01/08

1 Complex Numbers

Definition Informally, we define the **complex number** $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$. Note that spatially we have $\mathbb{C} \cong \mathbb{R}^2$.

We define the addition and $\mathbb{R}$-scalar multiplication like vectors in $\mathbb{R}^2$.

The multiplication is devised from 3 rules:

1. $\mathbb{R}$-multiplication.
2. $i^2 = -1$ in $\mathbb{R}$.
3. Distributive Law.

Let us see how to do this explicitly. For $x + iy, u + iv \in \mathbb{C}$ we have:

$$\begin{aligned}
(x+iy)(u+iv) &= x(u+iv) + iy(u+iv) && \text{(By 3)} \\
&= xu + ixv + iyu + i^2yv && \text{(By 3)} \\
&= (xu - yv) + i(xv + yu) && \text{(By 2)}
\end{aligned}$$

Note that $\mathbb{R}$ is commutative so if we define $ix = xi$, this makes $\mathbb{C}$ commutative.

We can define $\mathbb{C}$ in two alternative models:

Model (1). Let $\mathbb{R}[x]$ be the ring of real polynomials in x and $(x^2 + 1)$ is the ideal generated by $x^2 + 1$. Then:

$$\mathbb{C} = \mathbb{R}[x]/(x^2 + 1)$$

This is achieved by the homomorphism $\phi : \mathbb{R}[x] \to \mathbb{C}$ by $\phi(f(x)) = f(i)$, and we can show that the kernel is $(x^2 + 1)$, so the result follows from the universal property of quotients. Here we have:

$$i \mapsto \underbrace{x + (x^2 + 1)}_{\text{coset}}$$

and for all $a \in \mathbb{R}$ we have:

$$a \mapsto a + (x^2 + 1)$$

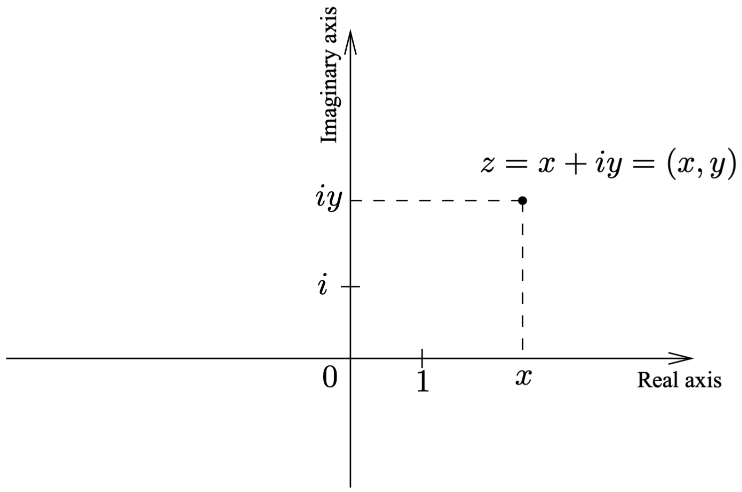
Model (2). We all like linear algebra: In $M_2(\mathbb{R})$, we can define:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

We will have an isomorphism from $\mathbb{C}$ to $\text{Span}_{\mathbb{R}}\{I, J\}$ by:

$$x + iy \mapsto xI + yJ = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

Geometrically we all know what complex plane looks like:



Definition Let $z = x+iy$ and $w = u+iv \in \mathbb{C}$. Define the **real part** and **imaginary part** of z to be $\text{Re}z = x$ and $\text{Im}z = y$. The **conjugate** of z is $\overline{z} = x - iy$.

There are some basic properties:

1. $z + \overline{z} = 2\text{Re}z$ and $z - \overline{z} = 2i\text{Im}z$.
2. $z\overline{z} = x^2 + y^2$ and $\overline{zw} = \overline{z}\overline{w}$.
3. $\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{\overline{z}} = z$.

Definition The **modulus** of $z = x + iy$ is $|z| = \sqrt{x^2 + y^2}$ and equivalently $|z| = \sqrt{z\overline{z}}$. Note that $\sqrt{a}$ is only defined if $a \in \mathbb{R}$ and it denotes the unique non-negative $b \in \mathbb{R}$ such that $b^2 = a$.

Note that $|z| = 0$ if and only if $z = 0$.

Some properties of modulus:

1. Multiplicative: $|zw| = |z||w|$.
2. Triangle Inequality: $|z + w| \leq |z| + |w|$.

Proof: (1) We have:

$$|zw|^2 = zw\overline{zw} = z\overline{z}w\overline{w} = |z|^2|w|^2$$

Taking the square root on both side, we are done.

(2) Note that $|z+w|^2 = (z+w)(\overline{z}+\overline{w})$, so:

$$\begin{aligned} |z+w|^2 &= z\overline{z} + w\overline{z} + z\overline{w} + w\overline{w} \\ &= |z|^2 + 2\text{Re}(w\overline{z}) + |w|^2 \\ &\leq |z|^2 + 2|w\overline{z}| + |w|^2 \\ &= |z|^2 + 2|wz| + |w|^2 \\ &= (|z| + |w|)^2 \end{aligned}$$

Taking square roots, we get $|z+w| \leq |z| + |w|$, as desired. $\square$

We remark that in the triangle inequality, we get an equality $\iff$ $\text{Re}(w\overline{z}) = |wz|$ $\iff$ $w\overline{z} \geq 0$ in $\mathbb{R}$ $\iff$ there exists $s, t > 0$ such that $sw = tz$. Geometrically, it means the inequality is an equality $\iff$ z and w "points in the same direction".

Remark **Distance (Metric)** in $\mathbb{C}$ by $d(z,w) = |z-w|$. Since we have triangle inequality and $|z-w| = |w-z|$, this is indeed a distance function on $\mathbb{C}$.

Multiplicative Inverse: For $z \neq 0$, we have:

$$z \cdot \frac{\overline{z}}{|z|^2} = 1 \implies z^{-1} = \frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}$$

Therefore, $\mathbb{C}$ is a field!! Yeah!

2 Topology in $\mathbb{C}$

Note that $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$ geometrically, so all the topology stuff are basically the same thing.

Definition Let $z_0 \in \mathbb{C}$ and $r \geq 0$ in $\mathbb{R}$. The **open disc** of radius r centred at z_0 is:

$$D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

The **punctured disc** is $D_0(z_0, r) = D(z_0, r) \setminus \{z_0\}$.

A set $U \subseteq \mathbb{C}$ is **open** if for all $z_0 \in U$, there exists $r > 0$ such that $D(z_0, r) \subseteq U$.

A set $F \subseteq \mathbb{C}$ is **closed** if its complement $\mathbb{C} \setminus F$ is open in $\mathbb{C}$.

The **closure** of a set $G \subseteq \mathbb{C}$ to be:

$$\begin{aligned}\overline{G} &= \bigcap \{F \subseteq \mathbb{C} : G \subseteq F, F \text{ closed}\} \\ &= \{z \in \mathbb{C} : \forall r > 0, D(z, r) \cap G \neq \emptyset\}\end{aligned}$$

The closure is the smallest closed subset of $\mathbb{C}$ that contains F.

Lecture 2, 2024/01/10

3 Differentiability in $\mathbb{C}$

Definition Let $G \subseteq \mathbb{C}$ and $f : G \to \mathbb{C}$ be a function. Let $z_0 \in \overline{G}$ and $L \in \mathbb{C}$. We say the **limit** of f at z_0 is L and we write:

$$\lim_{z \to z_0} f(z) = L$$

if for all $\epsilon > 0$, there exists $\delta = \delta_\epsilon > 0$ such that:

$$f(D_0(z_0, \delta)) \subseteq D(L, \epsilon)$$

OR equivalently: for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $z \in G$:

$$0 < |z - z_0| < \delta \implies |f(z) - L| < \epsilon$$

Note that if the limit exists, it is unique.

Definition For $f : G \subseteq \mathbb{C} \to \mathbb{C}$ and $z_0 \in G$, we say f is **continuous** at z_0 if $\lim\limits_{z \to z_0} f(z) = f(z_0)$. We say f is continuous on G if f is continuous at every $z_0 \in G$.

Definition Let $U \subseteq \mathbb{C}$ be open, let $f : U \to \mathbb{C}$, we say f is **complex differentiable** (or $\mathbb{C}$-differentiable) at $z_0 \in U$ if the limit:

$$\lim_{h \to 0} Q(h) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \quad \text{exists}$$

The domain of $Q(h)$ is:

$$(U - z_0) \setminus 0 = \{z - z_0 : z \in U\} \setminus \{0\}$$

Here $U - z_0$ is the **translation of set**. This domain is open and contains the punctured disc $D_0(0, \delta)$ for $\delta > 0$ small enough.

We say f is **holomorphic** on U if f is $\mathbb{C}$-differentiable at each $z_0 \in U$.

If f is $\mathbb{C}$-differentiable at z_0, so $\lim_{h \to 0} Q(h)$ exists. We define the unique limit to be the **derivative** of f at z_0, denoted by $f'(z_0)$.

Lemma 3.1 Let $U \subseteq \mathbb{C}$ be open and $f : U \to \mathbb{C}$ and $z_0 \in U$. Then f is differentiable at z_0 if and only if there is $a \in \mathbb{C}$ and $r > 0$ and $E : D(0, r) \to \mathbb{C}$ such that $\lim_{h \to 0} E(h) = 0$ and:

$$f(z_0 + h) - f(z_0) - ah = hE(h) \tag{1}$$

(1) is called the **linear approximation** of f at z_0.

Proof: $(\Rightarrow)$. Let $a = f'(z_0)$ and define:

$$E(h) = \frac{f(z_0 + h) - f(z_0)}{h} - f'(z_0)$$

by definition of differentiability, we are done.

$(\Leftarrow)$. Divide equation (1) by h and take modulus we have:

$$\left| \frac{f(z_0 + h) - f(z_0)}{h} - a \right| = |E(h)|$$

Since $\lim_{h \to 0} E(h) = 0$, we have:

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} - a = 0$$

It follows that $f'(z_0) = a$, as desired. $\square$

Proposition 3.2 (Chain Rule) Let $U, V \subseteq \mathbb{C}$ be open. Let $g : U \to \mathbb{C}$ with $g(U) \subseteq V$ and $f : V \to \mathbb{C}$. Let $z_0 \in U$ be so that g is differentiable at z_0 and f is differentiable at $g(z_0)$. Then $f \circ g$ is differentiable at z_0 with:

$$(f \circ g)'(z_0) = f'(g(z_0))g'(z_0)$$

Proof: By the linear approximation in the previous lemma, we have:

$$g(z_0 + h) - g(z_0) = (g'(z_0) + E_g(h))h \tag{1}$$
$$f(g(z_0) + H) - f(g(z_0)) = (f'(g(z_0)) - E_f(H))H \tag{2}$$

where $\lim_{h\to 0} E_g(h) = 0$ and $\lim_{H\to 0} E_f(H) = 0$. Define $F = f \circ g$. Then for $h \in D_0(0, r) \subseteq V - z_0$ and $r > 0$, we have:

$$\begin{aligned}
\frac{F(z_0+h)-F(z_0)}{h} &= \frac{f(g(z_0+h))-f(g(z_0))}{h} \\
&= \frac{f(g(z_0)+g'(z_0)+E_g(h)h)-f(g(z_0))}{h} && \text{(by (1))} \\
&= \frac{f(g(z_0)+H)-f(g(z_0))}{h} && (*) \\
&= \frac{(f'(g(z_0)+E_f(H)))H}{h} && \text{(by (2))} \\
&= \frac{(f'(g(z_0))+E_f(H))(g'(z_0)+E_g(h))h}{h}
\end{aligned}$$

In $(*)$ we simply let $H = g'(z_0) + E_g(h)h$. Now we get:

$$\frac{F(z_0+h)-F(z_0)}{h} = f'(g(z_0))g'(z_0) + f'(g(z_0))\underbrace{E_g(h)}_{\to 0} + \underbrace{E_f(H)}_{\to 0}(g'(z_0) + \underbrace{E_g(h)}_{\to 0})$$

Taking $h \to 0$, we also get $H \to 0$, so a lot of terms tend to 0 and we are left with:

$$\frac{F(z_0+h)-F(z_0)}{h} \to f'(g(z_0))g'(z_0)$$

as $h \to 0$, as desired. $\square$

Remark f is differentiable at $z_0 \implies f$ is continuous at z_0.

Proposition 3.3 Let $f, g : U \to \mathbb{C}$ and U is open. Let $z_0 \in U$ with f, g differentiable at z_0, then we have:

1. $f + g$ is differentiable at z_0 with:

$$(f+g)'(z_0) = f'(z_0) + g'(z_0)$$

2. (Product Rule). fg is differentiable at z_0 with:

$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$$

Proof: Exercise! $\square$

Lecture 3, 2024/01/12

Proposition 3.4 (Quotient Rule) Let $f, g : U \to \mathbb{C}$ and U is open. Suppose $0 \notin g(U)$ and $z_0 \in U$ so that f, g are differentiable at z_0. Then f/g is differentiable at z_0 with:

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}$$

Proof: Let $s : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by $s(z) = \frac{1}{z}$. If $w \in \mathbb{C} \setminus \{0\}$, then:

$$\frac{s(w+h) - s(w)}{h} = \frac{\frac{1}{w+h} - \frac{1}{w}}{h} = \frac{1}{h} \cdot \frac{w - (w+h)}{(w+h)w} = \frac{-1}{(w+h)w}$$

This tends to $-1/w^2$ as $h \to 0$, therefore we obtained:

$$s'(z) = \frac{1}{z^2}$$

Now by Chain Rule and Product rule we get:

$$\begin{aligned}\left(\frac{f}{g}\right)'(z_0) = \left(f \cdot \frac{1}{g}\right)'(z_0) &= f'(z_0)\frac{1}{g}(z_0) + f(z_0)\left(\frac{1}{g}\right)'(z_0) \\ &= \frac{f'(z_0)}{g(z_0)} + \frac{f(z_0)g'(z_0)}{g(z_0)^2} = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}\end{aligned}$$

As desired. $\square$

Remark Let $n \in \mathbb{Z}$. If $n \geq 0$, define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^n$. If $n < 0$, define $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by $f(z) = z^n$. Then we have:

$$f'(z) = nz^{n-1}$$

for all z in the domain. As a corollary, the **rational functions**:

$$f(z) = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$$

where $a_i, b_j \in \mathbb{C}$, are holomorphic on their domains.

Theorem 3.5 Let $U \subseteq \mathbb{C}$ be open and $f : U \to \mathbb{C}$ with $u, v : U \to \mathbb{R}$ by:

$$u(x, y) = \operatorname{Re}(f(x + iy)) \text{ and } v(x, y) = \operatorname{Im}(f(x + iy))$$

for all $x + iy \in U$. Now, let $z_0 = x_0 + iy_0 \in U$. Then f is $\mathbb{C}$-differentiable at $z_0 \in U$. if and only if each of u, v are $\mathbb{R}$-differentiable at (x_0, y_0) and satisfies the following **Cauchy-Riemann Equations**:

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \text{ and } \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$$

Proof: $(\Rightarrow)$. Consider the $\mathbb{R}$-partial derivatives:

$$\begin{aligned}\frac{\partial f}{\partial x}(z_0) &= \lim_{\substack{t \to 0 \\ t \in \mathbb{R}}} \frac{f(x_0 + t + iy_0) - f(x_0 + iy_0)}{t} \\ &= \lim_{\substack{t \to 0 \\ t \in \mathbb{R}}} \frac{f(z_0 + t) - f(z_0)}{t} = f'(z_0) \qquad (1)\end{aligned}$$

Since f is $\mathbb{C}$-differentiable so the limit exists. This means no matter which path we approach 0, we would get the same limit. Now we compute $\frac{\partial f}{\partial x}$ in a different way:

$$\begin{aligned}
\frac{\partial f}{\partial x}(z_0) &= \lim_{\substack{t\to 0\\ t\in\mathbb{R}}} \frac{f(x_0+t+iy_0)-f(x_0+iy_0)}{t} \\
&= \lim_{\substack{t\to 0\\ t\in\mathbb{R}}} \frac{u(x_0+t,y_0)-u(x_0,y_0)}{t} + i\frac{v(x_0+t,y_0)-v(x_0,y_0)}{t} \\
&= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0) \qquad (2)
\end{aligned}$$

We can compute $\frac{\partial f}{\partial y}$ in two different ways:

$$\begin{aligned}
\frac{\partial f}{\partial y}(z_0) &= \lim_{\substack{s\to 0\\ s\in\mathbb{R}}} \frac{f(x_0+i(y_0+s))-f(x_0+iy_0)}{s} \\
&= \lim_{\substack{t\to 0\\ t\in i\mathbb{R}}} \frac{f(x_0+iy_0+t)-f(x_0+iy_0)}{-it} \qquad (\text{set } t=is) \\
&= i\lim_{\substack{t\to 0\\ t\in i\mathbb{R}}} \frac{f(x_0+t+iy_0)-f(x_0+iy_0)}{t} = if'(z_0) \qquad (3)
\end{aligned}$$

Similar to (2) we can get:

$$\frac{\partial f}{\partial y}(z_0) = \frac{\partial u}{\partial y}(x_0,y_0) + i\frac{\partial v}{\partial y}(x_0,y_0) \qquad (4)$$

Combine (1) + (2) and (3) + (4) gives:

$$\begin{aligned}
f'(z_0) &= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0) \\
if'(z_0) &= \frac{\partial i}{\partial y}(x_0,y_0) + i\frac{\partial v}{\partial y}(x_0,y_0)
\end{aligned}$$

Mutliply by $-i$ to the second equation so that two equations are equal, then we get:

$$\frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) - i\frac{\partial u}{\partial y}(x_0,y_0)$$

Equating the real and imaginary parts, we get the Cauchy-Riemann Equation as desired. To show u and v are $\mathbb{R}$-differentiable, first note that:

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0)$$

Now for $h = h_1 + ih_2 \neq 0$, we have:

$$\begin{aligned}
&\operatorname{Re}(f(z_0+h) - f(z_0) - f'(z_0)h) \\
&= u(x_0+h_1, y_0+h_2) - u(x_0,y_0) - \operatorname{Re}(f'(z_0)h) \\
&= u(x_0+h_1, y_0+h_2) - u(x_0,y_0) - \left[\frac{\partial u}{\partial x}(x_0,y_0)h_1 - \frac{\partial v}{\partial x}(x_0,y_0)h_2\right] \\
&= u(x_0+h_1, y_0+h_2) - u(x_0,y_0) - \left[\frac{\partial u}{\partial x}(x_0,y_0)h_1 + \frac{\partial u}{\partial y}(x_0,y_0)h_2\right] \qquad (*)
\end{aligned}$$

The last equality uses CR Equation. Since f is $\mathbb{C}$-differentiable, so $(*) \to 0$ as $h \to 0$. Thus u is $\mathbb{R}$-differentiable by definition. Similarly:

$$\begin{aligned}
&\operatorname{Im}(f(z_0+h) - f(z_0) - f'(z_0)h) \\
&= v(x_0+h_1, y_0+h_2) - v(x_0,y_0) - \left[\frac{\partial v}{\partial y}(x_0,y_0)h_2 + \frac{\partial v}{\partial x}(x_0,y_0)h_1\right] \qquad (**)
\end{aligned}$$

Here $(**) \to 0$ as $h \to 0$. Thus v is $\mathbb{R}$-differentiable, as desired.

($\Leftarrow$). By the RHS of $(*)$ and $(**)$, we see that:

$$\operatorname{Re}(f(z_0+h) - f(z_0) - f'(z_0)h) \quad \text{and} \quad \operatorname{Im}(f(z_0+h) - f(z_0) - f'(z_0)h)$$

tend to 0 when dividing by $|h|$ as $h \to 0$. It follows that f is $\mathbb{C}$-differentiable. $\square$

Lecture 4, 2024/01/15

4 Power Series

Lemma 4.1 Let $(c_k)_{k=0}^{\infty} \subseteq \mathbb{C}$ be a complex sequence and $L = \limsup_{k\to\infty} |c_k|^{1/k}$. Define:

$$R = \begin{cases} 1/L & \text{if } 0 < L < \infty \\ 0 & \text{if } L = \infty \\ \infty & \text{if } L = 0 \end{cases}$$

Then we have:

1. If $z \in \mathbb{C}$ with $|z| < R$, then:
$$\lim_{n\to\infty} \sum_{k=0}^{n} c_k z^k = \sum_{k=0}^{\infty} c_k z^k \text{ converges in } \mathbb{C}$$
2. If $z \in \mathbb{C}$ with $|z| > R$, then:
$$\sum_{k=0}^{\infty} c_k z^k \text{ diverges in } \mathbb{C}$$

Proof: (1). Let $z \in \mathbb{C}$ such that $|z| < R$ ($L < \infty$). And $r = |z| < R$ and we choose $\epsilon > 0$ small enough so that:

$$\rho = (L+\epsilon)r = \left(\frac{1}{R} + \epsilon\right) r < 1$$

Let $n_\epsilon \in \mathbb{N}$ be so that for all $k \geq n_\epsilon$:

$$|c_k|^{1/k} < L + \epsilon$$

Now, if $n > m \geq n_\epsilon$ we have:

$$\begin{aligned}\left|\sum_{k=m}^{n} c_k z^k\right| &\leq \sum_{k=m}^{n} |c_k||z|^k = \sum_{k=m}^{n} |c_k| r^k < \sum_{k=m}^{n} (L+\epsilon)^k r^k \\ &= \sum_{k=m}^{n} \rho^k = \rho^m \cdot \frac{1-\rho^{n-m}}{1-\rho} < \frac{\rho^m}{1-\rho} \to 0\end{aligned}$$

as $m \to \infty$. Hence the partial sum $(\sum_{k=0}^{n} c_k z^k)_{n=0}^{\infty}$ is a Cauchy sequence, so the series $\sum_{k=0}^{\infty} c_k z^k$ converges in $\mathbb{C}$.

(2). Let $\epsilon > 0$ and $\rho > 0$ be so that $\rho(L-\epsilon)|z| > 1$ (if $|z| > R$). Then there are infinitely many k so that $|c_k|^{1/k} > L - \epsilon$, then:

$$|c_k z^k| = |c_k||z|^k > (L-\epsilon)^k |z|^k = \rho^k$$

and $\rho > 1$. So the series fails to converge as k-th terms fail to tend to 0 (By the Divergence test). $\square$

Definition Given $(c_k)_{k=0}^{\infty} \subseteq \mathbb{C}$ and $a \in \mathbb{C}$. The function:

$$f(z) = \sum_{k=0}^{\infty} c_k (z-a)^k$$

is defined on $D(a, R)$ where $R = 1/L$ from the last question. This is called the **power series** function. The R is called the **radius of convergence**.

Remark It can be shown that $f(z) = \sum_{k=0}^{\infty} c_k(z-a)^k$ converges uniformly on each disc $\overline{D}(a, r)$ with $r < R$. Hence f is continuous on $D(a, R) = \bigcup_{0 \leq r < R} \overline{D}(a, r)$.

Theorem 4.2 A power series function $f(z) = \sum_{k=0}^{\infty} c_k (z-a)^k$ on $D(a, R)$ is holomorphic on $D(a, R)$ with:

$$f'(z) = \sum_{k=1}^{\infty} k c_k (z-a)^{k-1} = \sum_{k=0}^{\infty} (k+1)(z-a)^k$$

for all $z \in D(a, R)$.

Proof: Notice that:

$$\limsup_{k\to\infty} |kc_k|^{1/k} = \limsup_{k\to\infty} |k|^{1/k}|c_k|^{1/k}$$

Relcall that $\lim_{k\to\infty} |k|^{1/k} = 1$. Therefore:

$$\limsup_{k\to\infty} |kc_k|^{1/k} = \limsup_{k\to\infty} |c_k|^{1/k} = L$$

Hence:

$$g(z) = \sum_{k=1}^{\infty} kc_k(z-a)^{k-1} = \begin{cases} \frac{1}{z-a}\sum_{k=1}^{\infty} kc_k(z-a)^k & \text{if } z \neq a \\ c_1 & \text{if } z = a \end{cases}$$

defines a function on $D(a,R)$. We wish to show that $f' = g$. WLOG suppose $a = 0$. If $z \in D(0,R)$, for each $n \in \mathbb{N}$, write:

$$f(z) = S_n(z) + E_n(z) = \sum_{k=0}^{n} c_k z^k + \sum_{k=n+1}^{\infty} c_k z^k$$

So $S_n'(z) = \sum_{k=1}^{n} kc_k z^{k-1}$ and $\lim\limits_{n\to\infty} S_n'(z) = g(z)$. Fix $z_0 \in D(0,R)$, let $r > 0$ so that $|z_0| < r < R$. If $h \in \mathbb{C}$ is chosen so $|z_0 + h| < r$, then first recall that:

$$a^k - b^k = (a-b)\sum_{j=0}^{k-1} a^{k-j-1}b^j$$

Thus we have:

$$|(z_0+h)^k - z_0^k| \leq |(z_0+h) - z_0| \sum_{j=0}^{k-1} |z_0+h|^{k-j-1}|z_0|^j < |h|kr^{k-1}$$

Then if $h \neq 0$ we have:

$$\begin{aligned} \left|\frac{f(z_0+h)-f(z_0)}{h} - g(z_0)\right| &\leq \left|\frac{S_n(z_0+h)-S_n(z_0)}{h} - S_n'(z_0)\right| + |S_n'(z_0) - g(z_0)| \\ &\quad + \left|\frac{E_n(z_0+h)-E_n(z_0)}{h}\right| \end{aligned} \tag{1}$$

Where:

$$\left|\frac{E_n(z_0+h)-E_n(z_0)}{h}\right| \leq \sum_{k=n+1}^{\infty} |c_k|\frac{|(z_0+h)^k - z_0^k|}{|h|} \leq \sum_{k=n+1}^{\infty} |c_k|kr^{k-1} \to 0$$

as $n \to \infty$, because this is the tail of a convergent series. Given $\epsilon > 0$, choose $n \in \mathbb{N}$ so that:

$$|S_n'(z_0) - g(z_0)| < \frac{\epsilon}{3} \quad \text{and} \quad \sum_{k=n+1}^{\infty} k|c_k|r^{k-1} < \frac{\epsilon}{3}$$

Choose $\delta > 0$ so that $h \in D_0(0, \delta)$ and $\delta \leq r$ gives:

$$\left| \frac{S_n(z_0 + h) - S_n(z_0)}{h} - S_n'(z_0) \right| < \frac{\epsilon}{3}$$

Then for such h, we see that, by (1):

$$\left| \frac{f(z_0 + h) - f(z_0)}{h} - g(z_0) \right| < \epsilon$$

As desired. $\square$

Lecture 5, 2024/01/17

Example **Exponential Function.** Define $\exp : \mathbb{C} \to \mathbb{C}$ by:

$$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

The radius of convergence is $R = \infty$, so it convergents on all of $\mathbb{C}$. This is because:

$$(k!)^{1/k} = \left(1 \cdots \left[\frac{k}{2}\right] \cdots k\right)^{1/k} \geq \left(\left[\frac{k}{2}\right]^{k/2}\right)^{1/k} = \left[\frac{k}{2}\right]^{1/2} \geq \left(\frac{k}{2} - 1\right)^{1/2} \to \infty$$

Here $[\cdot]$ is the Gaussian floor function. Two properties of exp:

1. $\exp'(z) = \exp(z)$.
2. (A1) $\exp(w + z) = \exp(w)\exp(z)$, so the notation $\exp(z) = e^z$ makes sense.

Example **Trignometric Functions.** Define $\sin, \cos : \mathbb{C} \to \mathbb{C}$ by:

$$\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} \quad \text{and} \quad \cos(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}$$

The radius of convergence for both of them is $R = \infty$. Notice that:

$$e^{iz} = \sum_{k=0}^{\infty} \frac{i^k}{k!} z^k = \sum_{\ell=0}^{\infty} \frac{i^{2\ell}}{(2\ell)!} z^{2\ell} + \sum_{\ell=0}^{\infty} \frac{i^{2\ell+1}}{(2\ell+1)!} z^{2\ell+1}$$

Warning: Here we split the infinite summation into two parts, we can do this here because we know the series converges absolutely! Otherwise this is complete crap. Note that $i^{2\ell} = (-1)^\ell$ and $i^{2l+1} = i(-1)^\ell$ Thus we have:

$$e^{iz} = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell)!} z^{2\ell} + i \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell+1)!} z^{2\ell+1} = \cos(z) + i\sin(z)$$

This is called the **Euler's Formula**. Note that this actually gives us an alternative way to define sin and cos in $\mathbb{C}$. By Euler's formula:

$$e^{iz} = \cos(z) + i\sin(z) \quad \text{and} \quad e^{-iz} = \cos(z) - i\sin(z)$$

Note that sin is odd and cos is even by the power series definition above. Thus by some rearrangements:

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

And we can use these two as the definitions if we want.

Let's look at the picture of the exponential function. If $z = x + iy$, then:

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$$

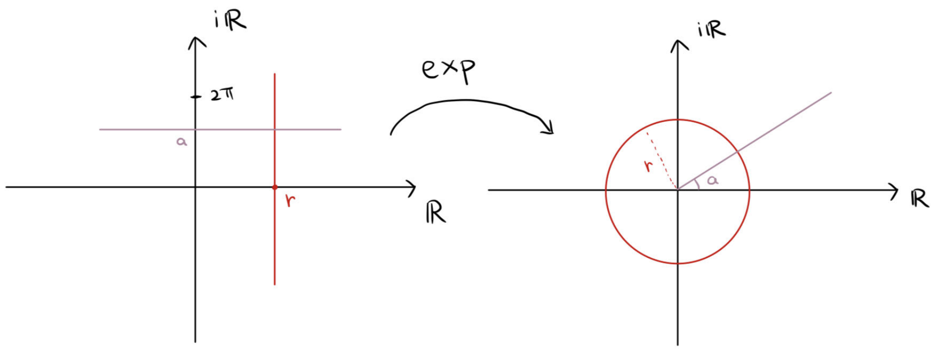
Here $x, y \in \mathbb{R}$, thus $|e^{iy}| = 1$ and $|e^z| = |e^x| = e^x > 0$. Also, exp is $2\pi i$-periodic:

$$\exp(z + 2\pi ik) = e^{z+2\pi ik} = e^z(\underbrace{\cos(2\pi k)}_{=1} + i\underbrace{\sin(2\pi k)}_{=0}) = e^z = \exp(z)$$

for all $k \in \mathbb{Z}$. By this $2\pi i$-periodicity, it is enough to consider $\exp(z)$ on the strip with $0 \le \text{Im}(z) \le 2\pi$. Let's consider some cases:

1. If we have a vertical line $z = b + ia$ where $b \in \mathbb{R}$ is fixed and $a \in \mathbb{R}$, then $\exp(z) = e^b e^{ia}$. Here the modulus is always e^b and since a is changing, so it forms a circle in $\mathbb{C}$. **(Vertical Lines map to circles)**.
2. If we have a horizontal line $z = b + ia$ where $a \in \mathbb{R}$ is fixed and $b \in \mathbb{R}$. Then $\exp(z) = e^b e^{ia}$ and the fixed $a \in \mathbb{R}$ represents the angle (argument) in the complex plane, since b is changing, this gives us a ray from 0. **(Horizontal Lines map to rays)**.

The pictures are below:



If $w \in \mathbb{C} \setminus \{0\}$, then we can write:

$$w = u + iv = |w| \left(\frac{u}{|w|} + i \frac{v}{|w|} \right)$$

Here note that:

$$\left(\frac{u}{|w|} \right)^2 + \left(\frac{v}{|w|} \right)^2 = \frac{u^2 + v^2}{|w|^2} = 1$$

There is $y \in \mathbb{R}$ so that $w = |w|(\cos y + i \sin y)$. Then let $x = \log |w|$. So:

$$w = e^x(\cos y + i \sin y) = e^{x+iy}$$

Note that y is not unique, but it is unique if we insist $y \in [\alpha, \alpha + 2\pi)$ for some $\alpha \in \mathbb{R}$, this $w = e^{x+iy}$ is called the **polar form**.

Proposition If $n \in \mathbb{N} = \{1, 2, \cdots\}$, then the equation $w^n = 1$ has exactly n solutions in $\mathbb{C}$, they are:

$$w_k = \cos\left(\frac{2k\pi}{n}\right) + i \sin\left(\frac{2k\pi}{n}\right)$$

for $k = 0, 1, \cdots, n-1$.

Proof: Let $w = e^{x+iy}$ for $y \in [0, 2\pi)$ so $w = e^x e^{iy}$. we get:

$$1 = w^n = e^{nx} e^{iny} = e^n(\cos(ny) + i \sin(ny))$$

This implies that:

$$1 = |w^n| = e^{nx} \implies x = 0$$

Therefore:

$$1 = \cos(ny) + i \sin(ny) \implies ny = 0 + 2\pi\ell$$

where $\ell \in \mathbb{Z}$. If $0 \leq y < 2\pi$, we get $y = \frac{2k\pi}{n}$ and $k = 0, 1, \cdots, n-1$. $\square$

Example Say $\mathbb{D} = D(0, 1)$, the **unit disc**. If $z \in \mathbb{D}$, then:

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$$

Note that we have:

$$(1-z) \sum_{k=0}^{n} z^k = (1 + z + \cdots + z^n) - (z + z^2 + \cdots + z^{n+1}) = 1 - z^{n+1} \to 1$$

as $n \to \infty$. Thus $\frac{1}{1-z}$ has radius of convergence $R = 1$.

If $z = -w^2$ and $w \in \mathbb{D}$, we get:

$$\frac{1}{1+w^2} = \sum_{k=0}^{\infty} (-1)^k w^{2k}$$

also with a radius of convergence $R = 1$.

Lecture 6, 2024/01/19

5 Path and Line Integrals

Definition A **p.s.c (piecewise smooth continuous) path** is a function $\gamma : [a, b] \to \mathbb{C}$ such that:

1. γ is **piecewise smooth**, that is, there are $a_0, \cdots, a_n$ with:

$$a = a_0 < a_1 < \cdots < a_n = b$$

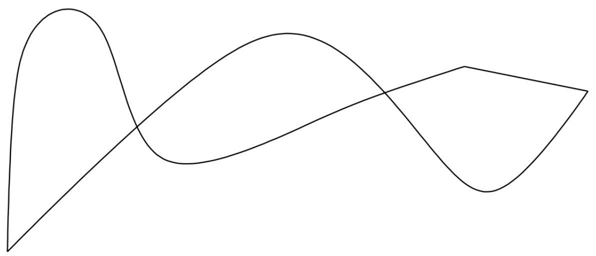
   such that $\gamma|_{(a_{i-1}, a_i)}$ is differentiable and bounded.

2. γ is continuous. Let $\gamma^* = \gamma([a, b])$ be the **image (trace)** of γ.

We say γ is **closed** if $\gamma(a) = \gamma(b)$ and γ is **simple** if

$$\gamma(s) = \gamma(t) \implies s = t \quad \text{OR} \quad s \neq t \quad \text{and} \quad s, t \in \{a, b\}$$

This curve below is not simple, but closed.



Given $\gamma : [a, b] \to \mathbb{C}$, the **backward path** is $\gamma^- : [a, b] \to \mathbb{C}$ by:

$$\gamma^-(t) = \gamma(a + b - t)$$

Clearly γ and γ^- has the same trace, but just moving in different directions.

Definition We say two paths $\gamma : [a, b] \to \mathbb{C}$ and $\lambda : [c, d] \to \mathbb{C}$ are **equivalent** if there is $\varphi : [c, d] \to [a, b]$ such that:

1. φ is continuous.
2. φ is piecewise smooth with bounded derivatives.
3. $\varphi'(t) > 0$ where differentiable (thus increasing by MVT).

so that $\lambda = \gamma \circ \varphi$.

Remark By one variable inverse function theorem, $\varphi^{-1} : [a, b] \to [c, d]$ is a piecewise smooth curve at well. Thus define $\gamma \sim \lambda$ when γ and λ are equivalent. This $\sim$ is an equivalence relation.

Definition Let $\gamma : [a, b] \to \mathbb{C}$ be a p.s.c path and $f : \gamma^* \to \mathbb{C}$ be continuous. We define the **line integral** of f over γ to be:

$$\int_\gamma f = \int_\gamma f(z)\ dz = \int_a^b f(\gamma(t))\gamma'(t)\ dt \tag{1}$$

Recall that here:

$$\gamma'(t) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{\gamma(t+h) - \gamma(t)}{h}$$

Note that in (1), the integrand in the last integral may not be definite at finitely many points, but this is fine because integrals do not care about finitely many points.

Also, we can note that:

$$\int_a^b \mathrm{Re}(f(\gamma(t))\gamma'(t))\ dt \quad \text{and} \quad \int_a^b \mathrm{Im}(f(\gamma(t))\gamma'(t))\ dt$$

are just ordinary Riemann integrals in $\mathbb{R}$ of bounded piecewise continuous functions. So we can just define:

$$\int_\gamma f = \int_a^b \mathrm{Re}(f(\gamma(t))\gamma'(t))\ dt + i\int_a^b \mathrm{Im}(f(\gamma(t))\gamma'(t))\ dt$$

Example Fix $a \in \mathbb{C}$ and $r > 0$. Define $\gamma(t) = a + re^{it}$ where $t \in [0, 2\pi]$. If $f : \gamma^* \to \mathbb{C}$ is continuous, then:

$$\int_\gamma f = \int_0^{2\pi} f(a + re^{it})ire^{it}\ dt = ir\int_0^{2\pi} f(a + re^{it})e^{it}\ dt$$

Example **(Line Segments).** Let $a, b \in \mathbb{C}$ and define:

$$[a, b] = \{(1-t)a + tb : t \in [0, 1]\}$$

If $f : [a, b] \to \mathbb{C}$ be continuous:

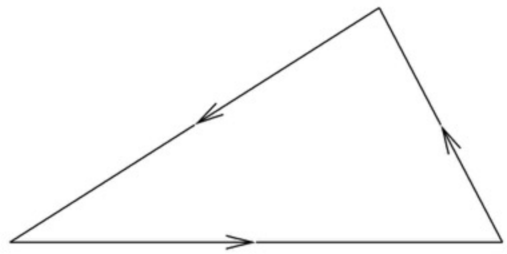
$$\int_{[a,b]} f = \int_\gamma f = \int_0^1 f((1-t)a + tb)(b-a)\ dt = (b-a) \int_0^1 f((1-t)a + tb)\ dt$$

Example (Triangles). Let $a, b, c \in \mathbb{C}$ and $T = \operatorname{conv}\{a, b, c\}$, the **convex hull** of $\{a, b, c\}$. That is, the smallest convex set that contains $\{a, b, c\}$. More explicitly:

$$T = \{sa + tb + uc : s, t, u \in [0, 1],\ \ s + t + u = 1\}$$

Boundary $\partial T = [a, b] \cup [b, c] \cup [c, a]$. Define $f : \partial T \to \mathbb{C}$ be continuous, then:

$$\int_{[a,b,c]} f = \int_{[a,b]} f + \int_{[a,c]} f + \int_{[b,c]} f$$



Proposition Let $\gamma : [a, b] \to \mathbb{C}$ be a p.s.c path and $f, g : \gamma^* \to \mathbb{C}$ be continuous, then we have the following basic properties of line integrals:

1. (Linearity). For $\alpha, \beta \in \mathbb{C}$, then:

$$\int_\gamma (\alpha f + \beta g) = \alpha \int_\gamma f + \beta \int_\gamma g$$

2. (Equivalent Paths). If $\lambda : [c, d] \to \mathbb{C}$ is equivalent to γ, then $\displaystyle\int_\gamma f = \int_\lambda f$

3. (Backward Path). $\displaystyle\int_{\gamma^-} f = -\int_\gamma f$

4. (Estimation). $\displaystyle\left|\int_\gamma f\right| \leq \max_{z \in \gamma^*} |f(z)| \int_a^b |\gamma'(t)|\ dt$. Here:

$$\ell(\gamma) = \text{length}(\gamma) = \int_a^b |\gamma'(t)|\ dt$$

is defined to be the **length** of γ.

Proof: (1). Easy exercises.

(2). Let $\varphi : [c,d] \to [a,b]$ such that $\lambda = \gamma \circ \varphi$. Let $a = a_1 < \cdots < a_n = b$ be points where γ is not differentiable. Then:

$$\begin{aligned}
\int_\lambda f = \int_{\gamma\circ\varphi} f &= \int_c^d f(\gamma\circ\varphi(s))(\gamma\circ\varphi)'(s)\ ds \\
&= \sum_{j=1}^n \int_{\varphi^{-1}(a_{j-1})}^{\varphi^{-1}(a_j)} f(\gamma\circ\varphi(s))\gamma'(\varphi(s))\varphi'(s)\ ds && \text{(Chain Rule)} \\
&= \sum_{j=1}^n \int_{a_{j-1}}^{a_j} f(\varphi(t))\gamma'(t)\ dt = \int_\gamma f && \text{(Change of variable)}
\end{aligned}$$

Good :)

(3). By chain rule we have:

$$\int_{\gamma^-} f = \int_a^b f(\gamma(a+b-t))\gamma'(a+b-t)(-1)\ dt$$

Let $s = a+b-t$ so $ds = -dt$, then:

$$\int_{\gamma^-} f = \int_a^b f(\gamma(s))\gamma'(s)\ ds = -\int_a^b f(\gamma(s))\gamma'(s)\ ds$$

Lecture 7, 2024/01/22

Example A non-example. Define $\gamma : [0,1] \to \mathbb{C}$ by:

$$\gamma(t) = \begin{cases} t + it\sin\left(\frac{1}{t}\right) & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$

This γ does not have bounded derivative, so it is NOT a p.s.c path.

Proof continued. (4). We have:

$$\begin{aligned}
\left|\int_\gamma f\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)\ dt\right| &= \left|\lim_{n\to\infty}\sum_{j=1}^n f\left(\gamma\left(a+\frac{b-a}{n}j\right)\right)\gamma'\left(a+\frac{b-a}{n}(j-1)\right)\frac{b-a}{n}\right| \\
&\leq \lim_{n\to\infty}\sum_{j=1}^n \underbrace{\left|f\left(\gamma\left(a+\frac{b-a}{n}j\right)\right)\right|}_{\leq \max_{z\in\gamma^*}|f(z)|}\cdot\left|\gamma'\left(a+\frac{b-a}{n}(j-1)\right)\right|\cdot\frac{b-a}{n} \\
&\leq \max_{z\in\gamma^*}|f(z)|\int_a^b |\gamma'(t)|\ dt && (*)
\end{aligned}$$

In $(*)$, we use max instead of sup because the maximum exists. Why? Because $[a, b]$ is compact and γ is continuous, so γ^* is compact. $\square$

Theorem (FTC for Line Integrals) Suppose $U \subseteq \mathbb{C}$ is open and $f : U \to \mathbb{C}$ is continuous and has a **primitive** $F : U \to \mathbb{C}$ such that $F' = f$ on U. Then for any p.s.c path $\gamma : [a, b] \to \mathbb{C}$ we have:

$$\int_\gamma f = F(\gamma(b)) - F(\gamma(a))$$

As a simple corollary, if γ is closed, then $\displaystyle\int_\gamma f = 0$.

Proof: The proof of chain rule shows that:

$$(F \circ \gamma)'(t) = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t)$$

Here if $a = a_0 < \cdots < a_n = b$ are points where f is not differentiable at, then:

$$\begin{aligned}\int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)\ dt &= \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} f(\gamma(t))\gamma'(t)\ dt \\ &= \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} F'(\gamma(t))\ dt = \sum_{j=1}^{n} F(\gamma(a_j)) - F(\gamma(a_{j-1})) \qquad (*)\end{aligned}$$

In $(*)$ we used the usual FTC for $\mathbb{C}$ in this way:

$$(F \circ \gamma)(t) = u(t) + iv(t)$$

where $u = \mathrm{Re}(F \circ \gamma)$ and $v = \mathrm{Im}(F \circ \gamma)$ and apply FTC to them separately. Then by telescoping to $(*)$, we got the desired result. $\square$

Example Let $f(z) = \frac{1}{z}$ on $U = \mathbb{C} \setminus \{0\}$. Let $\gamma(t) = re^{it}$ for $t \in [0, 2\pi]$ and $r > 0$ fixed. Then we have:

$$\int_\gamma f = \int_0^{2\pi} \frac{ire^{it}}{re^{it}}\ dt = i\int_0^{2\pi} 1\ dt = 2\pi i \neq 0$$

What does this imply? Since γ is a closed path, by FTC, if f has primitive, then the integral should be zero. BUT $2\pi i \neq 0$, thus f has no primitive on U.

Remark If $n \in \mathbb{Z} \setminus \{-1\}$ and $g(z) = z^n$ on:

$$U = \begin{cases} \mathbb{C} & \text{if } n \geq 0 \\ \mathbb{C} \setminus \{0\} & \text{if } n < 0 \end{cases}$$

Then g has primitive $G(z) = \frac{1}{n+1} z^{n+1}$.

Hence power series $g(z) = \sum_{k=0}^{\infty} c_k(z-a)^k$ on $D(a, R)$ has a primitive, where R is its radius of convergence. The primitive is:

$$G(z) = \sum_{k=0}^{\infty} \frac{1}{k+1} c_k (z-a)^{k+1}$$

We are now ready for the awesome theorem:

Theorem (Cauchy-Gorusat) Let $U \subseteq \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be continuous on U and holomorphic on $U \setminus \{w_0\}$ for some $w_0 \in U$. IF $a, b, c \in U$ such that $T = \text{conv}\{a, b, c\} \subseteq U$, we have:

$$\int_{[a,b,c]} f = 0$$

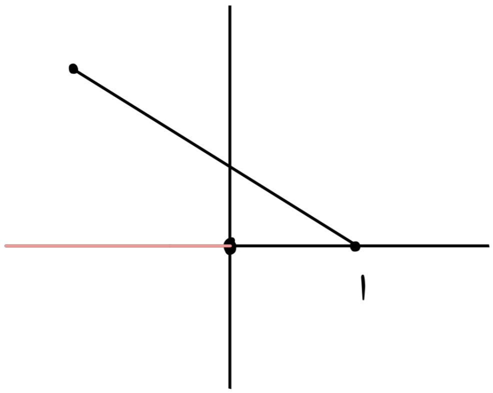
Proof: We leave the proof for next time. Let us see the power of this theorem first.

Definition A set $S \subseteq \mathbb{C}$ is **star-like** if there exists some $z_0 \in S$ such that for any $z \in S$:

$$[z_0, z] = \{(1-t)z_0 + tz : t \in [0, 1]\}$$

is contained in S.

Example Any convex set is star-like. The set $\mathbb{C} \setminus (-\infty, 0]$ is also star-like:



It is star-like around 1 (or any point on the postivie real axis).

Theorem Let $U \subseteq \mathbb{C}$ be open and star-like, then if $f : U \to \mathbb{C}$ is continuous on U and holomorphic on $U \setminus \{w_0\}$ for some $w_0 \in U$. Then f admits a primitive F on U.

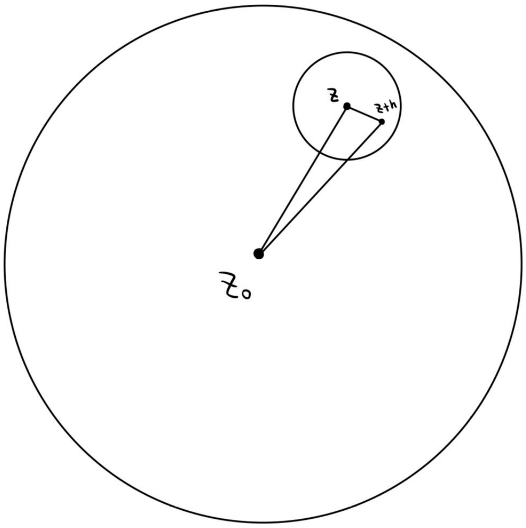
Proof: Let $z_0 \in U$ be the point in the definition of star-like. Define $F : U \to \mathbb{C}$ by:

$$F(z) = \int_{[z_0,z]} f$$

Given $z \in U$, let $r > 0$ such that $D(z,r) \subseteq U$. If $h \in \mathbb{C}$ with $0 < |h| < r$, then:

$$F(z+h) - F(z) = \int_{[z_0,z+h]} f - \int_{[z_0,z]} f \tag{1}$$

Geometrically we have:



In the diagram we saw a triangle, by Cauchy-Gorusat Theorem:

$$\int_{[z_0,z+h]} f + \int_{[z+h,z]} f + \int_{[z,z_0]} f = 0$$

By the property about backward path we have:

$$\int_{[z_0,z+h]} f - \int_{[z_0,z]} f = \int_{[z,z+h]} f \tag{2}$$

Since z is already fixed, we have:

$$\int_{[z,z+h]} f(z)\ d\zeta = f(z) \int_{[z,z+h]} 1\ d\zeta = f(z)h \tag{3}$$

By (2) and (3) we have, (1) becomes:

$$\begin{aligned} F(z+h) - F(z) - f(z)h &= \int_{[z,z+h]} f(\zeta)\ d\zeta - \int_{[z,z+h]} f(z)\ d\zeta \\ &= \int_{[z,z+h]} [f(\zeta) - f(z)]\ dz \end{aligned}$$

Dividing by h and taking the modulus we have:

$$\begin{aligned}\left|\frac{F(z+h)-F(z)}{h}-f(z)\right| &= \frac{1}{|h|}\left|\int_{[z,z+h]}[f(\zeta)-f(z)]\,dz\right| \\ &\leq \frac{1}{|h|}\max_{\zeta\in[z,z+h]}|f(\zeta)-f(z)||h| \\ &= \max_{\zeta\in[z,z+h]}|f(\zeta)-f(z)|\end{aligned}$$

Note that as $h\to 0$ we have $z+h\to z$ and since f is continuous, $|f(\zeta)-f(z)|\to 0$. Thus $F'=f$, as desired. $\square$

Lecture 8, 2024/01/24

For Cauchy-Goursat Theorem, why do we care about triangles? Because it is easy! To prove this theorem we need the following notations and facts:

1. $T=\operatorname{conv}\{a,b,c\}$ is compact.
2. $\operatorname{diam}(T)=\max\{|a-b|,|a-c|,|b-c|\}$.
3. $\ell([a,b,c])=|a-b|+|a-c|+|b-c|$.

Proof of Cauchy-Goursat: Let a',b',c' be midpoints of $[a,b]$, $[b,c]$ and $[c,a]$.

$$\int_{\partial T} f=\int_{[a,b,c]} f=\int_{[a,b]} f+\int_{[b,c]} f+\int_{[c,a]} f \tag{1}$$

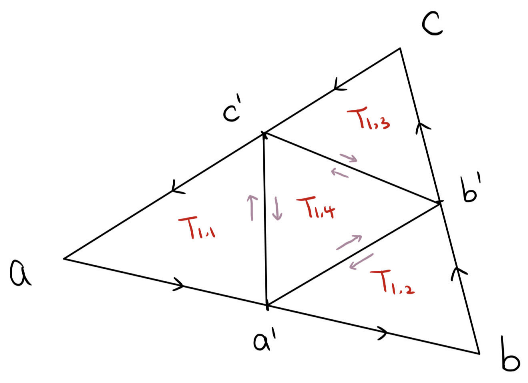
Note that, for example:

$$\int_{[a',c']} f=-\int_{[c',a']} f$$

Therefore after some cancellation we have:

$$(1)=\int_{[a,a',c']} f+\int_{[a',b,b']} f+\int_{[b',c,c']} f+\int_{[c',a',b']} f$$

We call these four triangles $T_{1,1},T_{1,2},T_{1,3},T_{1,4}$, respectively.



Then we have:

$$\int_{\partial T} f = \sum_{j=1}^{4} \int_{\partial T_{1,j}} f$$

Pick $j \in \{1,2,3,4\}$ to be the max of the four integrals:

$$\left| \int_{\partial T_{1,j}} \right| = \max_{i=1,2,3,4} \left| \int_{\partial T_{1,i}} \right|$$

Then we have:

$$\left| \int_{\partial T} f \right| \leq \sum_{i=1}^{4} \left| \int_{\partial T_{1,i} f} \right| \leq 4 \left| \int_{\partial T_{1,j}} f \right|$$

Label this particular T_{1j} by T_1. Notice that:

$$\operatorname{diam}(T_1) = \frac{1}{2}\operatorname{diam}(T) \;\text{ and }\; \ell(\partial T_1) = \frac{1}{2}\ell(\partial T)$$

Continue this way, we get a recursive sequence of triangles:

$$T \supseteq T_1 \supseteq T_2 \supseteq \cdots$$

with:

$$\left| \int_{\partial T_n} f \right| \leq 4 \left| \int_{\partial T_{n+1}} f \right|$$

and:

$$\operatorname{diam}(T_{n+1}) = \frac{1}{2}\operatorname{diam}(T_n) \;\text{ and }\; \ell(\partial T_{n+1}) = \frac{1}{2}\ell(\partial T_n)$$

By simple induction we have:

$$\left| \int_{\partial T} f \right| \leq 4^n \left| \int_{\partial T_n} f \right|$$

and:

$$\operatorname{diam}(T_n) = \frac{1}{2^n}\operatorname{diam}(T) \;\text{ and }\; \ell(\partial T_n) = \frac{1}{2^n}\ell(\partial T)$$

By nested compact set theorem, $\bigcap_{n=1}^{\infty} T_n \neq \emptyset$. In fact there is only one point z_0 in the intersection, since $\operatorname{diam}(T_n) \to 0$ as $n \to \infty$. Two cases:

1. If $z_0 \neq w_0$, so f is differentiable at z_0. By linear approximation, for $z \in U$:

   $$|f(z) - f(z_0) - (z - z_0)f'(z_0)| \leq |z - z_0||E(z - z_0)|$$

   here $E(z - z_0) \to 0$ as $z \to z_0$. Let $g(z) = f(z_0) + (z - z_0)f'(z_0)$. Thus a primitive of g is:

   $$G(z) = f(z_0)z + \frac{1}{2}(z - z_0)^2 f'(z_0)$$

so for any $n \geq 1$ we have $\displaystyle\int_{\partial T_n} g = 0$ since ∂T_n is a closed path. Thus:

$$\begin{aligned}
\left|\int_{\partial T_n} f\right| = \left|\int_{\partial T_n} (f-g)\right| &= \left|\int_{\partial T_n} [f(z) - f(z_0) - (z-z_0)f'(z_0)]\, dz\right| \\
&\leq \sup_{z\in\partial T_n} |f(z) - f(z_0) - (z-z_0)f'(z_0)|\ell(T_n) \\
&\leq \sup_{z\in\partial T_n} |z-z_0||E(z-z_0)|\ell(T_n) \\
&\leq \sup_{z\in\partial T_n} \operatorname{diam}(T_n)|E(z-z_0)| \cdot \frac{1}{2^n}\ell(T) \\
&\leq \sup_{z\in\partial T_n} \frac{1}{4^n}|E(z-z_0)|\operatorname{diam}(T)\ell(T)
\end{aligned}$$

Therefore:

$$\left|\int_{\partial T} f\right| \leq 4^n \left|\int_{\partial T_n} f\right| \leq \sup_{z\in\partial T_n} |E(z-z_0)|\operatorname{diam}(T)\ell(T)$$

Nice, the 4^n cancelled out! As $n \to \infty$ we have $E(z-z_0) \to 0$, thus $\displaystyle\int_{\partial T} f = 0$.

2. If $z_0 = w_0$, recall that $w_0 \in T_n$ for all n since $\{w_0\} = \bigcap_{n=1}^{\infty} T_n$. From above, we have that:

$$\int_{\partial T} f = \int_{\partial T_n} f \quad \text{for all} \quad n \geq 1$$

So we have:

$$\begin{aligned}
\left|\int_{\partial T} f\right| = \left|\int_{\partial T_n} f\right| &\leq \max_{z\in\partial T_n} |f(z)|\ell(T_n) \\
&\leq \max_{z\in T} |f(z)| \cdot \frac{1}{2^n}\ell(T)
\end{aligned}$$

This approaches to 0 as $n \to \infty$. It follows that:

$$\int_{\partial T} f = 0$$

As desired! □

6 Cauchy's Theorem and Applications

7 Zeros, Poles and Residues

Lecture 16, 2024/02/12

8 Winding Numbers

Note For $U \subseteq \mathbb{C}$, and:

$$U = \bigcup_{n \in \mathbb{Z}_{\geq 1}} U_n$$

where each U_n is connected and open, and $U_i \cap U_j \neq \emptyset$ for all $i \neq j$. We call U_n the **connected component** of U.

The idea is: if $z \in U$, its path component is:

$$\begin{aligned} U_z &= \{w \in U : \exists \text{ path } \gamma : [0,1] \to U \text{ with } \gamma(0) = z,\ \gamma(1) = w\} \\ &= \{w \in U : \exists \text{ p.s.c (polygonal) path } \gamma : [0,1] \to U \text{ with } \gamma(0) = z,\ \gamma(1) = w\} \end{aligned}$$

Note that $z, w \in U$, then either $U_z \cap U_w = \emptyset$ or $U_z = U_w$ (reverse path).

The **Gaussian Rationals** $\mathbb{Q}(i) = \mathbb{Q} + i\mathbb{Q}$ is dense in $\mathbb{C}$ so:

$$(\mathbb{Q} + i\mathbb{Q}) \ \cap \bigcup_{\text{dense}} \{q_k : k \in \mathbb{N}\}$$

Thus we have:

$$U = \bigcup_{z \in U} U_z = \bigcup_{k=1}^{\infty} \{q_k : k \in \mathbb{N}\}$$

We can select finitely or countable infinitely many mutually disjoint components.

If $K \subseteq \mathbb{C}$ is compact, then $\mathbb{C} \setminus K$ admits a unique component which is unique and unbounded. Note $K \subseteq D(0,R) \subseteq \overline{D}(0,R)$, let V be the component of U containing $C(0,R) = \partial \overline{D}(0,R)$.

Theorem 8.1 (Winding Numbers) Let γ be a p.s.c closed curve in $\mathbb{C}$, then $z \in \mathbb{C} \setminus \gamma^*$, the quantity:

$$W_\gamma(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0}$$

is an integer. Furthermore, $W_\gamma(z) = W_\gamma(w)$ if z, w belong to the same component of $\mathbb{C} \setminus \gamma^*$ and $W_\gamma(z) = 0$ if z is in the unbounded component.

Proof: Step (1). Parametrize $\gamma : [0,1] \to \mathbb{C}$ and we let:

$$0 = a_0 < a_1 < \cdots < a_n = 1$$

be points where $\gamma'(a_i)$ does not exist. For $z_0 \in \mathbb{C} \setminus \gamma^*$ we have:

$$\int_\gamma \frac{dz}{z - z_0} = \int_0^1 \frac{\gamma'(t)}{\gamma(t) - z_0}\, dt$$

For $t \in [0,1)$ we define:

$$\psi(t) = \exp\left(\int_0^t \frac{\gamma'(s)}{\gamma(s) - z_0}\, ds\right)$$

So we have $\psi(0) = \exp(0) = 1$ and by Chain Rule and FTC:

$$\psi'(t) = \psi(t) \cdot \frac{\gamma'(t)}{\gamma(t) - z_0} \tag{1}$$

except at $a_0 < a_1 < \cdots < a_n$. For $t \in [0,1)$, let:

$$\varphi(t) = \frac{\psi(t)}{\gamma(t) - z_0}$$

By differentiation rules and (1)

$$\begin{aligned}\varphi'(t) &= \frac{\psi'(t)}{\gamma(t) - z_0} - \psi(t) \cdot \frac{\gamma'(t)}{(\gamma(t) - z_0)^2} \\ &= \psi(t) \cdot \frac{\gamma'(t)}{(\gamma(t) - z_0)^2} - \psi(t) \cdot \frac{\gamma'(t)}{(\gamma(t) - z_0)^2} = 0\end{aligned}$$

Note that $\mathrm{Re}(\varphi)$ and $\mathrm{Im}(\varphi)$ are each constant on $[a_{j-1}, a_j]$, thus φ is constant on $[0,1]$. Therefore we get:

$$\begin{aligned}\frac{1}{\gamma(0) - z_0} &= \frac{\psi(0)}{\gamma(0) - z_0} = \varphi(0) = \varphi(1) \\ &= \frac{\exp\left(\int_0^1 \frac{\gamma'(s)}{\gamma(s) - z_0}\, ds\right)}{\gamma(1) - z_0}\end{aligned}$$

Note that $\gamma(1) = \gamma(0)$ because γ is closed, we have:

$$\exp\left(\int_0^1 \frac{\gamma'(s)}{\gamma(s) - z_0}\, ds\right) = 1$$

It follows that:

$$\int_0^1 \frac{\gamma'(s)}{\gamma(s) - z_0}\, ds = 2\pi i k \quad \text{where} \quad k \in \mathbb{Z}$$

From this we have:

$$W_\gamma(z_0) = \frac{1}{2\pi i} \int_0^1 \frac{\gamma'(s)}{\gamma(s) - z_0}\, ds = k \in \mathbb{Z}$$

Nice, this completes one part of the proof.

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Step (2). We show that $W_\gamma(z)$ is constant on components. We first show that $W_\gamma : \mathbb{C} \setminus \gamma^* \to \mathbb{Z}$ is continuous. Fix $z_0 \in \mathbb{C} \setminus \gamma^*$. Recall that γ^* is compact, so there exists $r > 0$ so that $D(z_0, r) \subseteq \mathbb{C} \setminus \gamma^*$. Now suppose $w \in D(z_0, \frac{r}{2})$, we have:

$$\begin{aligned}
|W_\gamma(z_0) - W_\gamma(w)| &= \left| \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0} - \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - w} \right| \\
&= \frac{1}{2\pi} \left| \int_\gamma \left[\frac{1}{z - z_0} - \frac{1}{z - w} \right] dz \right| \\
&= \frac{1}{2\pi} \left| \int_\gamma \frac{(z - w) - (z - z_0)}{(z - z_0)(z - w)} \, dz \right| \\
&= \frac{1}{2\pi} \left| \int_\gamma \frac{z_0 - w}{(z - z_0)(z - w)} \, dz \right|
\end{aligned}$$

For $z \in \gamma^*$ we have $|z - z_0| > r$ and $|z - w| > \frac{r}{2}$. Then:

$$\begin{aligned}
|W_\gamma(z_0) - W_\gamma(w)| &\leq \frac{1}{2\pi} \max_{z \in \gamma^*} \frac{|z_0 - w|}{|z - z_0||z - w|} \ell(\gamma) \\
&= \frac{1}{2\pi} \cdot \frac{|z_0 - w|}{r \cdot r/2} \ell(\gamma) \\
&= \frac{1}{\pi r^2} \ell(\gamma) |z_0 - w|
\end{aligned}$$

This means W_γ is a Lipschitz function, thus continuous. Let $m \in \mathbb{Z}$ and:

$$U_m = W_\gamma^{-1} \left(D \left(m, \frac{1}{2} \right) \right) = W_\gamma^{-1}(\{m\})$$

This is because W_γ is $\mathbb{Z}$-valued, so the only integer in the interval $D(m, 1/2)$ is m. Thus:

$$\mathbb{C} \setminus \gamma^* = \bigcup_{m \in \mathbb{Z}} U_m$$

where each U_m is open and $U_m \cap U_n = \emptyset$ for $m \neq n$. Thus if V is any component of $\mathbb{C} \setminus \gamma^*$, we see that $V \subseteq U_m$ for some $n \in \mathbb{Z}$, otherwise we violate the connectedness of V. Finally, let W denote the unique unbounded component of $\mathbb{C} \setminus \gamma^*$. If $|z| > \max_{\zeta \in \gamma^*} |\zeta|$, then we have:

$$\begin{aligned}
|W_\gamma(z)| &\leq \left| \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z} \right| \\
&\leq \frac{1}{2\pi} \max_{\zeta \in \gamma^*} \frac{1}{|\zeta - z|} \ell(\gamma) \\
&\leq \frac{1}{2\pi} \max_{\zeta \in \gamma^*} \frac{1}{|z| - |\zeta|} \ell(\gamma)
\end{aligned}$$

And this approaches to 0 as $|z| \to \infty$. But W_γ is constant on W, this means $W_\gamma(z) = 0$ for all $z \in W$. $\square$

Example Let $m \in \mathbb{Z} \setminus \{0\}$ and $a \in \mathbb{C}$ and $R > 0$. Let:

$$\gamma_{m,R}(t) = a + Re^{imt}$$

for $t \in [0, 2\pi]$. Exercise: If $z \in D(a, R)$ and $W_\gamma(z) = m$. Now, by Cauchy's Theorem, if $z \in \mathbb{C} \setminus \overline{D}(a, R)$, we have $W_\gamma(z) = 0$.

Definition A **toy path** is a closed, simple path $\gamma : [0, 1] \to \mathbb{C}$ so that $\mathbb{C} \setminus \gamma^* = U_1 \cup U_2$ where for $k = 0, 1$:

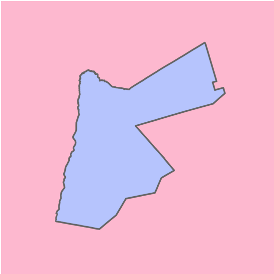
$$U_k = \{z \in \mathbb{C} \setminus \gamma^* : W_\gamma(z) = k\}$$

and U_1 and U_2 are connected.

Theorem 8.2 (Jordan Curve Theorem) If γ is a closed simple (p.s.c) path, then:

$$\mathbb{C} \setminus \gamma^* = I \cup \mathcal{O}$$

where I and $\mathcal{O}$ are connected. $I \cap \mathcal{O} = \emptyset$ and $\overline{I}$ is compact.



This theorem basically says a closed simple path can separate $\mathbb{C}$ into two disjoint parts (one interior and one exterior). This is VERY HARD to prove, so we skip it.

Example Some examples of toy curves are:

1. Circles
2. Triangles
3. Quadrilaterals
4. Semicircles
5. Dented Circles
6. Wedges
7. Keyholes

Look up their graph online if forgot what they are.

9 Homotopy

Let $\gamma_0, \gamma_1 : [0,1] \to U$ be p.s.c paths with either:

1. **Condition 1**. $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$. (Fixed endpoints)
2. **Condition 2**. Each γ_0, γ_1 is closed. (Closed path)

Definition Let $U \subseteq \mathbb{C}$ be open and $\gamma_0, \gamma_1 : [0,1] \to U$ be p.s.c paths. A p.s.c **homotopy** in U is a function:

$$H : [0,1] \times [0,1] \to U$$

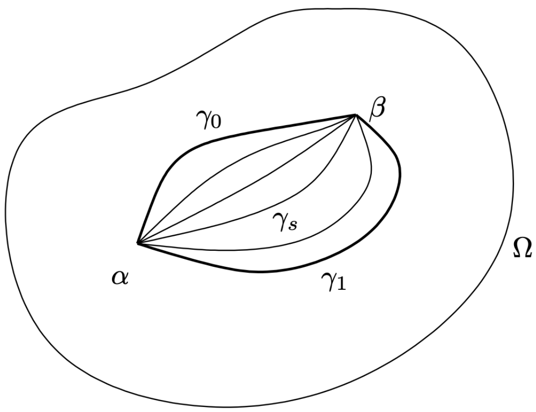
such that the followings are true:

1. H is continuous.
2. $H(0,t) = \gamma_0(t)$ and $H(1,t) = \gamma_1(t)$ for all $t \in [0,1]$.
3. If $s \in [0,1]$ then $H(s,t) = \gamma_s(t)$ defines a p.s.c path.
4. If condition 1 above is true, then:

$$\gamma_s(0) = \gamma_0(0) = \gamma_1(0) \quad \text{and} \quad \gamma_s(1) = \gamma_0(1) = \gamma_1(1)$$

   OR if condition 2 is true, then each γ_s is closed.

If a homotopy exists in U, we say γ_0 is **homotopic** to γ_1 in U.



Geometrically, γ_0 and γ_1 are homotopic if we can continuously deformed γ_0 to γ_1 and vice versa.

Example Let $U = \mathbb{C} \setminus \{0\}$ and $0 < r < R$. Let:

$$\gamma_0(t) = re^{i2\pi t} \quad \text{and} \quad \gamma_1(t) = Re^{i2\pi t}$$

where $t \in [0, 1]$. These are two circles. Define:

$$H : [0, 1] \times [0, 1] \to \mathbb{C} \quad \text{by} \quad H(s, t) = (1 - s)\gamma_0(t) + s\gamma_1(t)$$

for $(s, t) \in [0, 1]^2$. Each $\gamma_s(t)$ is a circle in between the circles γ_0 and γ_1. This is indeed a homotopy.

Example Let $U = \mathbb{C} \setminus \{a + ib, -a + ib\}$ for some $a, b > 0$ in $\mathbb{R}$. Define:

$$\gamma_0(t) = ib + Re^{i2\pi t}$$

where $R > \sqrt{a^2 + b^2} > a$. And define γ_1 to be a semicircle, which is the concatenation of:

$$[-R, R] \quad \text{and} \quad t \mapsto Re^{it}$$

for $t \in [0, \pi]$. We can reparametrize γ_1 so that:

$$\gamma_1(t) = \begin{cases} (1 - 2t)(-R) + 2tR & \text{if } t \in [0, 1/2] \\ Re^{i2\pi(2t-1)} & \text{if } t \in [1/2, 1] \end{cases}$$

Let $H(s, t) = (1 - s)\gamma_0(t) + s\gamma_1(t)$. Can check:

$$H([0, 1]^2) \subseteq \mathbb{C} \setminus \{a + ib, -a + ib\}$$

And in fact H is a homotopy.

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Theorem 9.1 If $U \subseteq \mathbb{C}$ is open and $\gamma_0, \gamma_1 : [0, 1] \to U$ are closed p.s.c path and p.s.c homotopic in U, then for any $f \in \mathcal{H}(U)$

$$\int_{\gamma_0} f = \int_{\gamma_1} f$$

Proof: Step (0). Let us do some setup first. Define $H : [0, 1] \times [0, 1] \to U$ be a homotopy linking γ_0 to γ_1:

1. H is continuous.
2. $H(0, t) = \gamma_0(t)$ and $H(1, t) = \gamma_1(t)$.
3. Each $\gamma_s(t) = H(s, t)$ is a p.s.c path.

We collect 2 consequences of the continuity of H:

(i) $K = H([0,1]^2)$ is compact. And we know:

$$\text{dist}(K, \mathbb{C} \setminus U) = \inf\{|z - w| : z \in K, w \in \mathbb{C} \setminus U\} > 0$$

since K is compact and $\mathbb{C} \setminus U$ is closed. Hence we can find $r > 0$ such that $r < \text{dist}(K, \mathbb{C} \setminus U)$.

(ii) A continuous function on a compact set is uniformly continuous, so there is a $\delta > 0$ such that:

$$\|(s,t) - (s',t')\| = \sqrt{(s-s')^2 + (t-t')^2} < \delta$$

in $[0,1] \times [0,1]$ implies that:

$$|H(s,t) - H(s',t')| < \frac{r}{2} \text{ in } U$$

Note that by (ii), if $|s - s'| < \delta$ in $[0,1]$, then for every $t \in [0,1]$ we have:

$$|\gamma_s(t) - \gamma_{s'}(t)| = |H(s,t) - H(s',t')| < \frac{r}{2}$$

That is, γ_s and γ_s' are uniformly $< r/2$ apart.

Step (1). If $|s - s'| < \delta$ in $[0,1]$, then:

$$\int_{\gamma_s} f = \int_{\gamma_s'} f$$

(A). We define the set C to be the set of all $t \in [0,1]$ such that there are:

$$0 = t_0 < \cdots < t_n = t$$

adn open discs $D_1, \cdots, D_n \subseteq U$ such that $\gamma_\sigma(t_{j-1}), \gamma_\sigma(t_j) \in D_j$ and $\sigma = s = s'$. Set $t' = \sup(C)$ and we wish to show $t' = 1$.

Corollary 9.2 If $U \subseteq \mathbb{C}$ is open and $\gamma_0, \gamma_1 : [0,1] \to U$ are p.s.c path with $\gamma_0(0) = \gamma_1(0)$ adn $\gamma_0(1) = \gamma_1(1)$ and are p.s.c homotopic in U, then for any $f \in \mathcal{H}(U)$:

$$\int_{\gamma_0} f = \int_{\gamma_1} f$$

Lecture 19, 2024/02/26

Proof of Corollary: Let $H : [0,1] \times [0,1] \to U$ be an endpoint preserving homotopy, $H(0,t) = \gamma_0(t)$ and $H(1,t) = \gamma_1(t)$. We let:

$$\tilde{H} : [0,1] \times [0,1] \text{ by } \tilde{H}(s, \cdot) = \gamma_s \cdot \gamma_1^-$$

the concatenation, and $\gamma_s(t) = H(s,t)$. Then $\tilde{H}$ is a closed p.s.c path homotopy, linking $\gamma_0 \cdot \gamma_1^-$. Then:

$$\begin{aligned} 0 &= \int_{\gamma_1} f - \int_{\gamma_1} f = \int_{\gamma_1} f + \int_{\gamma_1^-} f \\ &= \int_{\gamma_1 \cdot \gamma_1^-} f = \int_{\gamma_0 \cdot \gamma_1^-} f = \int_{\gamma_0} f - \int_{\gamma_1} f \qquad \text{(by Theorem)} \end{aligned}$$

As desired. □

Proposition 9.3 Let $U \subseteq \mathbb{C}$ be an open and star-like region. Then any p.s.c closed curve $\gamma : [0,1] \to U$ is **null-homotopic** in U, that is, there is a constant closed curve $\gamma_1(t) = z_0$, for all $t \in [0,1]$ (**null curve**) which is p.s.c homotopic to γ.

Proof: Let z_0 be a star-like base point. Let:

$$H(s,t) = (1-s)z_0 + s\gamma(t) \in [z_0, \gamma(t)] \subseteq U$$

by definition of star-like. Then H is a p.s.c homotopy with :

$$H(0,t) = \gamma(t) \quad \text{and} \quad H(1,t) = z_0$$

As desired. □

Remark Topologists call a region in which every closed path is null-homotopic a **simply connected region**.

Say $f \in \mathcal{H}(U)$, then $f \in \mathcal{H}(D(z_0, r))$ for some disc around z_0 in U, so f admits a primitive $F \in \mathcal{H}(D(z_0, r))$ and:

$$\int_{\gamma_1} f = F(z_0) - F(z_0) = 0$$

Example Consider $\gamma_r(t) = re^{it}$. We claim this is not null-homotopic. We know:

$$W_{\gamma_r}(0) = \frac{1}{2\pi i}\int_{\gamma_r} \frac{d\zeta}{\zeta} = 1$$

Since $s \mapsto \frac{1}{s} \in \mathcal{H}(\mathbb{C} \setminus 0)$, if γ_r were null-homotopic, then we would have:

$$W_{\gamma_r}(0) = \frac{1}{2\pi i}\int_{\gamma_r} \frac{d\zeta}{\zeta} = \frac{1}{2\pi i}\int_{\{z_0\}} f = 0$$

contradiction! Thus not null-homotopic.

Definition Let $U \subseteq \mathbb{C}$ be open. A **p.s.c cycle** Γ in U is a finite collection of p.s.c closed paths in U. Write:

$$\Gamma = \gamma_1 \dot{+} \cdots \dot{+} \gamma_n$$

where each γ_k is a p.s.c closed path. The trace is:

$$\Gamma^* = \gamma_1^* \cup \cdots \cup \gamma_n^*$$

there each γ_k^* is compact, so Γ^* is compact. Finally, if f is continuous on Γ^*, write:

$$\int_\Gamma f = \sum_{k=1}^n \int_{\gamma_k} f = \sum_{k=1}^n \int_0^1 f(\gamma_k(t))\gamma_k'(t)\ dt$$

here we parametrize each γ_k over $[0,1]$. Also, if $z \in \mathbb{C} \setminus \Gamma^*$, then:

$$W_\Gamma(z) = \sum_{k=1}^n W_{\gamma_k}(z)$$

Theorem 9.4 (Cauchy Integral Formula over a compact set) Let $U \subseteq \mathbb{C}$ be open and $K \subseteq U$ be compact. Then there is a p.s.c cycle Γ in $U \setminus K$ for which for any $f \in \mathcal{H}(U)$ such that:

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\ d\zeta$$

for all $z \in K$. So in particular, we have $W_\Gamma(z) = 1$ for all $z \in K$.

Proof: Let $\delta > 0$ be so that $\sqrt{2}\delta < \operatorname{dist}(K, \mathbb{C} \setminus U)$. Note that since K is compact and $\mathbb{C} \setminus U$ is closed, so their distance is strictly bigger than 0. For $k, \ell \in \mathbb{Z}$, let:

$$I_{k,\ell} = \{z \in \mathbb{C} : \operatorname{Re}(z) \in [k\delta, (k+1)\delta],\ \operatorname{Im}(z) \in [\ell\delta, (\ell+1)\delta]\}$$

Thus:

$$\operatorname{diam}(I_{k,\ell}) = \sup\{|w - z| : w, z \in I_{k,\ell}\} = \sqrt{2}\delta$$

We enumerate:

$$\{I_1, \cdots, I_m\} = \{I_m : (k,\ell) \in \mathbb{Z}^2,\ K \cap I_{k,\ell} \neq \emptyset\}$$

This is finite as K is bounded. Notice that each $I_k \subseteq U$ as:

$$\operatorname{diam}(I_k) < \operatorname{dist}(K, \mathbb{C} \setminus U)$$

Parametrize each ∂I_j for $j \in \{1, \cdots, m\}$ so that we have counter-clockwise orientation. Proof continued next time.

Lecture 20, 2024/02/28

We found non-overlapping squares $\{I_1, \cdots, I_m\}$ so:

$$K \subseteq \bigcup_{j=1}^m I_j \subseteq U$$

For each j, parametrize ∂I_j with counter-clockwise orientation. If $f \in \mathcal{H}(U)$, then:

$$f(z) = \frac{1}{2\pi i} \int_{\partial I_j} \frac{f(\zeta)}{\zeta - z} \, d\zeta \tag{A}$$

for $z \in I_j^\circ$ (interior) by Cauchy's Formula for rectanges in A4. Also:

$$0 = \frac{1}{2\pi i} \int_{\partial I_j} \frac{f(\zeta)}{\zeta - z} \, d\zeta \tag{B}$$

for all $z \in U \setminus I_j$. Why: Let $\epsilon = \operatorname{dist}(I_j, (\mathbb{C} \setminus U) \cup \{z\}) > 0$, then (can check):

1. Define $C_j := I_j + D(0, \epsilon) = \{w + w' : w \in I_j, w' \in D(0, \epsilon)\}$. Then C_j is open, convex and inside $U \setminus \{z\}$.
2. We get $g(\zeta) = \frac{f(\zeta)}{\zeta - z}$ is in $\mathcal{H}(C_j)$.

So Cauchy's Theorem (in star-like region) applies. Let:

$$\Delta = \partial I_1 \dot{+} \cdots \dot{+} \partial I_m$$

so if $z \in \bigcup_{j=1}^{m} I_j^\circ$, then (A) and (B) implies that:

$$f(z) = \frac{1}{2\pi i} \int_{\Delta} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

Let us build Γ. First we define $F = \bigcup_{j=1}^{m} I_j$ (compact). Let a_{11} be the top then leftmost corner of ∂F. Let $a_{12}, \cdots, a_{1p_1}$ be the other corners of ∂F which are met, tracing ∂F, so (we are in counterclockwise orientation) on $[a_{i,k}, a_{i,k+1}] \cap \partial F \cap \partial I_j \neq \emptyset$, we follow the same direction. If we meet a "disconnecting corner", always turn right. We now define:

$$\gamma_1 = [a_{11}, a_{12}] \dot{+} [a_{12}, a_{13}] \dot{+} \cdots \dot{+} [a_{1,p_1 - 1}, a_{1,p_1}] \dot{+} [a_{1,p_1}, a_{11}]$$

Let a_{21} be the top then leftmost corner of $\partial F \setminus \gamma_1^*$. We choose $a_{22}, \cdots, a_{2p_2}$ similarly. We choose in order so that on $\partial I_j \cap [a_{1k}, a_{1,k+1}]$ when nonempty, we follow same orientation. We get γ_2. We continue this process, making closed polygonal curves $\gamma_3, \cdots, \gamma_n$ until we exhaust ∂F. Now, let:

$$\gamma = \gamma_1 \dot{+} \cdots \dot{+} \gamma_n$$

Now we have that:

$$\int_{\Gamma} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \int_{\Delta} \frac{f(\zeta)}{\zeta - z} \, d\zeta \quad \text{for} \quad z \in \bigcup_{j=1}^{m} I_j^\circ$$

because each line segment omitted from a ∂I_j in $\Delta = \partial I_1 + \cdots + \partial I_m$ by the construction of Γ has its reverse line segment also omitted. In the cycle integral over Δ, these cancelled out. Hence from the Cauchy Integral type formula for Δ:

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta \quad\text{for}\quad z \in \bigcup_{j=1}^m I_j^\circ$$

However we have:

$$\overline{\bigcup_{j=1}^m I_j^\circ} = F \cup \bigcup_{j=1}^m I_j \supseteq K$$

And the following two maps:

$$z \mapsto f(z) \quad\text{and}\quad z \mapsto \int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

are continuous on $U \setminus \Gamma^*$ (for the second one, see the proof that W_γ is constant on components of $\mathbb{C} \setminus \gamma^*$). Thus for $z \in F^\circ$ ($K \subseteq F^\circ$ by choices of $I_1, \cdots, I_m$), we have $(z_n)_{n=1}^\infty \subseteq \bigcup_{j=1}^m I_j^\circ$ so $z = \lim\limits_{n\to\infty} z_n$. Thus:

$$f(z) = \lim_{n\to\infty} f(z_n) = \lim_{n\to\infty}\frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z_n}\,d\zeta = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

Notice that for $z \in K$ (even $z \in F^\circ$), we have:

$$W_\Gamma(z) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{\zeta - z}\,d\zeta = 1$$

Since $f = 1$ always evaluates to 1. □

Corollary 9.5 If $U \subseteq \mathbb{C}$ is open and $\emptyset \neq K \subseteq U$ and K compact, then for $z \in K$ there exists a p.s.c closed path γ in U so $W_\gamma(z) \neq 0$.

Proof: If, as in the last theorem, $\Gamma = \gamma_1 \dot{+} \cdots \dot{+} \gamma_n$.

$$1 = W_\Gamma(z) = \sum_{k=1}^n W_{\gamma_k}(z)$$

Thus $W_{\gamma_k}(z) \neq 0$ for some k. □

Lecture 21, 2024/03/01

Definition Let $U \subseteq \mathbb{C}$ be open, and γ in U be a p.s.c closed curve. We call γ **special** for U if $W_\gamma(z) = 0$ for all $z \in \mathbb{C} \setminus U$.

Remark If U is star-like (convex), then any p.s.c closed curve in U null-homotopic, hence special for U.

Example Let $U = \mathbb{C} \setminus \{0\}$, then $\gamma_0(t) = 2e^{it}$ for $t \in [0, 2\pi]$ is NOT special for U.

$$0 \in \mathbb{C} \setminus (\mathbb{C} \setminus \{0\}) \text{ and } W_{\gamma_0}(0) = 1 \neq 0$$

But $\gamma_1(t) = 5 + 3e^{-it}$ for $t \in [0, 2\pi]$ has $W_{\gamma_1}(0) = 0$. Thus γ_1 is special for U.

Definition Now let Γ be a cycle in U, we call Γ **special** in U if:

$$W_\Gamma(z) = 0 \text{ for all } z \in \mathbb{C} \setminus U$$

Example Let $U = \mathbb{C} \setminus \{0\}$ and $\gamma_2(t) = e^{-it}$ for $t \in [0, 2\pi]$. Then $\Gamma = \gamma_0 \dot{+} \gamma_2$ is special for U (γ_0 from earlier).

Why special cycles (curves) ?

Notation Let Γ be a cycle, special for U, we let:

$$U_\Gamma := W_\Gamma^{-1}(\{0\}) = \{\text{all points with zero winding number}\}$$

And note that:

$$U_\Gamma = W_\Gamma^{-1}(D(0, 1/2))$$

Thus U_Γ is open, as $W_\Gamma : \mathbb{C} \setminus \Gamma^* \to \mathbb{Z}$ is continuous. Thus:

$$K_\Gamma = \mathbb{C} \setminus U_\Gamma$$

is closed and bounded (thus compact). We have that:

$$\mathbb{C} \setminus U \subseteq U_\Gamma \qquad \text{(by specialness)}$$

It follows that:

$$K_\Gamma \subseteq U$$

We call K_Γ the **support** of Γ. (Can think of it like "interior").

Theorem 9.6 (Fubini's Theorem) Let $\gamma, \delta : [0, 1] \to \mathbb{C}$ be p.s.c paths and:

$$g : \gamma^* \times \delta^* \to \mathbb{C} \text{ be continuous}$$

Then we have:

$$\begin{aligned}\int_\delta \int_\gamma g(z,\zeta)\ dz\ d\zeta &= \int_0^1 \left(\int_0^1 g(\gamma(t),\delta(s))\gamma'(t)dt \right) \delta'(s)\ ds && \text{(Iterated Integrals)}\\ &= \int_0^1 \int_0^1 g(\gamma(t),\delta(s))\gamma'(t)\delta'(s)\ dt\ ds \\ &= \int_0^1 \int_0^1 g(\gamma(t),\delta(s))\delta'(t)\gamma'(t)\ ds\ dt && (*)\\ &= \int_\gamma \int_\delta g(z,\zeta)\ d\zeta\ dz\end{aligned}$$

$(*)$ is by the Real-Valued Fubini's theorem on real and imaginary part!. $\square$

Theorem 9.7 (Special Cauchy's Theorem) Let $U \subseteq \mathbb{C}$ be open and Δ be a cycle in U which is special for U. Then if $f \in \mathcal{H}(U)$:

$$\int_\Delta f = 0$$

Proof: Let K_Δ be the supporting set for Δ. That is, $K_\Delta = \mathbb{C} \setminus U_\Delta$ and $U_\Delta = W_\Delta^{-1}(\{0\})$. Since Δ is special for U, we have that $K_\Delta \subseteq U$. Now, Cauchy Integral Formula on a compact set, there is a cycle Γ in $U \setminus K_\Delta$ so that for $f \in \mathcal{H}(U)$ and $z \in K_\Delta$ and $\Delta^* \subseteq K_\Delta$, we have:

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\ d\zeta$$

Let $\Gamma = \gamma_1 + \cdots + \gamma_n$ and $\Delta = \delta_1 + \cdots + \delta_m$ where γ_k and δ_j are p.s.c closed path. Then we have:

$$\begin{aligned}\int_\Delta f = \int_\Delta f(z)\ dz = \int_\Delta \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\ d\zeta \\ &= \frac{1}{2\pi i} \sum_{j=1}^m \sum_{k=1}^n \int_{\delta_j} \int_{\gamma_k} \frac{f(\zeta)}{\zeta - z}\ d\zeta\ dz \\ &= \frac{1}{2\pi i} \sum_{k=1}^b \sum_{j=1}^m \int_{\gamma_k} \int_{\delta_j} \frac{f(\zeta)}{\zeta - z}\ dz\ d\zeta && \text{(Fubini)}\\ &= -\int_\Gamma \underbrace{\left(\frac{1}{2\pi i} \int_\Delta \frac{1}{z - \zeta}\ dz \right)}_{W_\Delta(\zeta)} f(\zeta)\ d\zeta\end{aligned}$$

and $\zeta \in \Gamma^* \subseteq U \setminus K_\Delta = U \cap U_\Delta$. Therefore:

$$\int_\Delta f = 0$$

As desired. $\square$

Remark There are few assumptions about U, so we make some assumptions on Γ.

Remark "Special" assumption is important. Consider $U = \mathbb{C} \setminus \{0\}$ and $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$. $0 \in \mathbb{C} \setminus U$ but:

$$0 \neq 1 = W_\gamma(0) = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta}$$

But if $g(\zeta) = \frac{1}{\zeta}$, we see $\displaystyle\int_\gamma g \neq 0$.

Lecture 22, 2024/03/04

Theorem 9.8 (Special Cauchy Integral Formula) Let $U \subseteq \mathbb{C}$ be open, Γ a p.s.c cycle in U which is special for U. Then for $f \in \mathcal{H}(U)$ and $z \in U \setminus \Gamma^*$, we get:

$$W_\Gamma(z) f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

Proof: Fix $z \in U \setminus \Gamma^*$, define:

$$g : U \to \mathbb{C} \ \text{ by } \ g(\zeta) = \begin{cases} \frac{f(\zeta) - f(z)}{\zeta - z} & \text{if } \zeta \neq z \\ f'(z) & \text{if } \zeta = z \end{cases}$$

Then g is holomorphic on $U \setminus \{z\}$ and continuous at z. So by CGM method, g is holomorphic on U. Then by Special Cauchy Theorem:

$$0 = \int_\Gamma g = \int_\Gamma \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta = \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta - \underbrace{\int_\Gamma \frac{1}{\zeta - z}\, d\zeta}_{2\pi i W_\Gamma(z)} \cdot f(z)$$

By rearranging this equality we get the desired result. $\square$

Theorem 9.9 (Special Residue Theorem) Let $U \subseteq \mathbb{C}$ be open and $P \subseteq U$ be a subset with no cluster points inside of U and let $f \in \mathcal{H}(U \setminus P)$ with each $z \in P$ a pole for f. Then if Γ is a cycle in $U \setminus P$ and special for U, then:

$$\int_\Gamma f = 2\pi i \sum_{z \in P} W_\Gamma(z) \text{Res}_z f$$

we will show this sum is always a finite sum.

Proof: Recall that $U_\Gamma = W_\Gamma^{-1}(\{0\})$ is open and unbounded and $K_\Gamma = \mathbb{C} \setminus U_\Gamma$ is compact and:

$$\Gamma^* \subseteq K_\Gamma \subseteq U \tag{Special}$$

Then, by assumption on P, we know $P \cap K_\Gamma$ is finite, because otherwise it would have a cluster point in $K_\Gamma \subseteq U$. Denote:

$$P \cap K_\Gamma = \{z_1, \cdots, z_n\}$$

be distinct points. For $j = 1, \cdots, n$, if $r_j > 0$ so that $D_0(z_j, r_j) \subseteq U \setminus P$ (Since $U \setminus P$ is open), by Order of Pole theorem, we have:

$$f(z) = \underbrace{\frac{a_{j,-m_j}}{(z-z_j)^{m_j}} + \cdots + \frac{a_{j,-1}}{z-z_j}}_{:=p_j(z)} + g_j(z)$$

where $z \in D_0(z_j, r_j)$ and $g_j \in \mathcal{H}(D_0(z_j, r_j))$ (power series). We call $p_j(z)$ the principal part at z_j and note that $p_j \in \mathcal{H}(U \setminus \{z_j\})$. Then define:

$$h(z) = f(z) - \sum_{j=1}^{n} p_j(z)$$

for $z \in U \setminus \{z_1, \cdots, z_n\}$, but:

$$\lim_{z \to z_j} h(z) = g_j(z_j) - \sum_{\substack{k=1 \\ k \neq j}}^{n} p_k(z_k)$$

This h extends to a holomorphic function on U by CGM method. Thus by Special Cauchy Theorem, we have:

$$0 = \int_\Gamma h = \int_\Gamma f - \sum_{j=1}^{n} \int_\Gamma p_j$$

Let $j = 1, \cdots, n$ and we consider:

$$\int_\Gamma p_j = \int_\Gamma p_j(z)\ dz = \sum_{k=1}^{m_j} \int_\Gamma \frac{a_{j,-k}}{(z-z_j)^k}\ dz \tag{1}$$

If $k \geq 2$, then $z \mapsto \frac{1}{(z-z_j)^k}$ admits a primitive on $U \setminus \{z_j\}$, thus:

$$\int_\Gamma p_j = \int_\Gamma \frac{a_{j,-1}}{z-z_j} = \underbrace{\int_\Gamma \frac{1}{z-z_j}\ dz}_{2\pi i W_\Gamma(z_j)} \cdot \underbrace{a_{j,-1}}_{\mathrm{Res}_{z_j} f}$$

Put this all together gives:

$$\int_\Gamma f = 2\pi i \sum_{j=1}^{n} W_\Gamma(z_j) \mathrm{Res}_{z_j} f = 2\pi i \sum_{z \in P} W_\Gamma(z) \mathrm{Res}_z f$$

the last equality is because $W_\Gamma(z) = 0$ if $z \in U \setminus K_\Gamma \subseteq U_\Gamma$. □

In practice we do not really use toy cycles, we use toy curves instead.

10 Laurent Series

Notation Let $0 \leq r < R \leq \infty$ and $a \in \mathbb{C}$, the **annulus** centered at a with inner radius r and outer radius R is:

$$A(a; r, R) = \{z \in \mathbb{C} : r < |z - a| < R\}$$

which is $D(a, R) \setminus \overline{D}(a, R)$. It is just a donut.

If $r = 0$ and $R < \infty$, then:

$$A(a; 0, R) = D_0(a, R)$$

is just a punctured disc.

Definition A **Laurent Series** is a function $f : A(a; r, R) \to \mathbb{C}$ by:

$$f(z) = \sum_{k=-\infty}^{\infty} c_k(z - a)^k$$

which converges on $A(a; r, R)$. We can also write:

$$f(z) = \underbrace{\sum_{k=1}^{\infty} \frac{c_{-k}}{(z - a)^k}}_{\text{principal part}} + \underbrace{\sum_{k=0}^{\infty} c_k(z - a)^k}_{\text{residual part}}$$

Note We have the following:

$$R \leq \left\{\text{radius of convergence of } \sum_{k=0}^{\infty} c_k(z - a)^k\right\}$$

and that:

$$\frac{1}{r} \leq \left\{\text{radius of convergence of } \sum_{k=1}^{\infty} c_{-k}(z - a)^{-k}\right\}$$

Thus, the series converges uniformly on compact subsets of $A(a; r, R)$.

Lecture 23, 2024/03/06

Theorem 10.1 (Laurent Series) Let $a \in \mathbb{C}$ and $0 \leq r < R \leq \infty$. If $f \in \mathcal{H}(A(a; r, R))$, then for $z \in A(a; r, R)$, there exists $(c_k)_{k=-\infty}^{\infty} \subseteq \mathbb{C}$ such that:

$$f(z) = \sum_{k=-\infty}^{\infty} c_k(z - a)^k$$

where c_k is given by:

$$c_k = \begin{cases} \frac{1}{2\pi i}\displaystyle\int_{\gamma_{r'}} f(\zeta)(\zeta-a)^{|k|-1}\,d\zeta & \text{if } k<0 \\ \\ \frac{1}{2\pi i}\displaystyle\int_{\gamma_{R'}} \frac{f(\zeta)}{(\zeta-a)^{k+1}}\,d\zeta & \text{if } k\geq 0 \end{cases}$$

where $r < r' < |z-a| < R' < R$ and $\gamma_{r'}$ and $\gamma_{R'}$ are parametrized on $[0,2\pi]$ as circles in the usual way.

Proof: Notice for r' and R' above that $\Gamma = \gamma_{R'} + \gamma_{r'}^-$, then:

$$A(a;r',R') = W_\Gamma^{-1}(\{1\})$$

and that:

$$\mathbb{C}\setminus A(a;r,R) = \begin{cases} D(a,r)\cup(\mathbb{C}\setminus D(a,R)) & \text{if } 0<r<R<\infty \\ \{0\}\cup(\mathbb{C}\setminus D(a,R)) & \text{if } 0=r<R<\infty \\ \overline{D}(a,r) & \text{if } 0<r<R=\infty \\ \{0\} & \text{if } 0=r<R=\infty \end{cases}$$

and we have:

$$\mathbb{C}\setminus A(a;r,R) \subseteq W_\Gamma^{-1}(\{0\})$$

So that Γ is special for $A(a;r,R)$. By Cauchy's Integral Formula:

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta-z}\,d\zeta = \frac{1}{2\pi i}\left[\int_{\gamma_{R'}} \frac{f(\zeta)}{\zeta-z}\,d\zeta - \int_{\gamma_{r'}} \frac{f(\zeta)}{\zeta-z}\,d\zeta\right]$$

where:

$$\frac{1}{\zeta-z} = \frac{1}{(\zeta-a)-(z-a)} = \begin{cases} \dfrac{-1}{z-a}\cdot\dfrac{1}{1-\frac{\zeta-a}{z-a}} = -\dfrac{1}{z-a}\displaystyle\sum_{k=0}^\infty\left(\frac{\zeta-a}{z-a}\right)^k & \text{if } |\zeta-a|<|z-a| \\ \\ \dfrac{1}{\zeta-a}\cdot\dfrac{1}{1-\frac{z-a}{\zeta-a}} = \dfrac{1}{\zeta-a}\displaystyle\sum_{k=0}^\infty\left(\frac{z-a}{\zeta-a}\right)^k & \text{if } |z-a|<|\zeta-a| \end{cases}$$

Thus, for $r' < |z-a| < R'$, we havem since series converges uniformly on compact set $\gamma_{R'}^*$ and $\gamma_{r'}^*$, we can interchange order of integrations and summations:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\left[\int_{\gamma_{R'}} f(z)\sum_{k=0}^\infty \frac{(z-a)^k}{(\zeta-a)^{k+1}}\,d\zeta + \int_{\gamma_{r'}} f(\zeta)\sum_{k=0}^\infty \frac{(\zeta-a)^k}{(z-a)^{k+1}}\,d\zeta\right] \\ &= \frac{1}{2\pi i}\left[\sum_{k=0}^\infty \int_{\gamma_{R'}} \frac{f(\zeta)}{(\zeta-a)^{k+1}}\,d\zeta\cdot(z-a)^k + \sum_{k=0}^\infty \int_{\gamma_{r'}} f(\zeta)(\zeta-a)^k\,d\zeta\cdot\frac{1}{(z-a)^{k+1}}\right] \end{aligned}$$

Now just reindex the principal part. $\square$

11 Simply Connected Regions

Let's look at simple connectivity.

Definition Let $U \subseteq \mathbb{C}$ be open, a **hole** for U is a nonempty subset $K \subseteq \mathbb{C} \setminus U$ such that K is compact and $U \cup K$ is open.

Example Here are some examples of holes:

1. Let $K = \{0\}$, then K is a hole for $\mathbb{C} \setminus \{0\}$.
2. For $0 < r < R = \infty$ and $a \in \mathbb{C}$, then $\overline{D}(a, r)$ is a hole for $A(a; r, R)$.

Lemma 11.1 Let $U \subseteq \mathbb{C}$ be open. The followings are equivalent:

(i) U admits a hole K.

(ii) There are $z \in \mathbb{C} \setminus U$ and a p.s.c closed curve γ in U so $W_\gamma(z) \neq 0$. That is, there exists p.s.c closed path that are not special.

Proof: (i) $\implies$ (ii). Let $V = U \cup K$ so V is open. The corollary to Cauchy's Integral Formula over a compact set provides a p.s.c closed path γ in $V \setminus K$ and some $z \in K$, so that $W_\gamma(z) \neq 0$.

(ii) $\implies$ (i). Let $z \in \mathbb{C} \setminus U$ and γ a p.s.c closed path and $m = W_\gamma(z) \neq 0$. Let $U_m = W_\gamma^{-1}(\{m\}) = W_\gamma^{-1}(D(m, 1/2))$, so U_m is open (since W_γ cotninuous on $\mathbb{C} \setminus \gamma^*$). Also U_m is bounded (by winding number theorem). We define:

$$K = U_m \setminus U$$

and $z \in K$ so that $K \neq \emptyset$. Then $K \subseteq U_m$ and hence bounded. We wish to see that K is closed. Use the sequential characterization of closed set, suppose $(z_n) \subseteq K$ such that $z = \lim_{n \to \infty} z_n$ with $z \in \mathbb{C}$. First, $\mathbb{C} \setminus U$ is closed and $z \in \mathbb{C} \setminus U$. Second:

$$W_\gamma(z) = \lim_{n \to \infty} W_\gamma(z_n) = m$$

since W_γ is continuous on U_m. And:

$$z \in (\mathbb{C} \setminus U) \cap U_m = U_m \setminus U = K$$

so K is closed. Thus K is compact by Heine-Borel. Finally:

$$U \cup K = U \cup (U_m \setminus U) = U_m \cup U \quad \text{is open}$$

Hence K is a hole. $\square$

Lecture 24, 2024/03/08

Theorem 11.2 (Simple Connectivity) Let $U \subseteq \mathbb{C}$ be open, then consider followings:

0. Any p.s.c closed path γ in U is null-homotopic.
1. (Special). Any closed p.s.c path γ in U is special for U.
2. U admits no hole.
3. (Cauchy's Theorem). For any closed p.s.c path in U, for any $f \in \mathcal{H}(U)$:
$$\int_\gamma f = 0$$
4. (Primitive exists). Any $f \in \mathcal{H}(U)$ admits a primitive on U.
5. (Logarithm). If $f \in \mathcal{H}(U)$ and $0 \notin f(U)$, then there exists a function $g \in \mathcal{H}(U)$ such that $f = \exp(g)$.

Then conditions (1), (2), (3) and (4) are equivalent, each is implied by (0).

FACT: All $(0) - (5)$ are equivalent, the proof requires Riemann Mapping Theorem.

Proof: $(0) \implies (1)$. Notice that $g(\zeta) = \frac{1}{\zeta - z}$ satisfies $g \in \mathcal{H}(U)$ if $z \in \mathbb{C} \setminus U$. Hence for a p.s.c closed path γ in U, we have:

$$W_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int_{\{z_0\}} \frac{d\zeta}{\zeta - z} = 0$$

the last equality is by the theorem on homotopy and line integrals. Hence γ is special.

$(1) \iff (2)$. Contrapositive of a lemma from last lecture.

$(2) \implies (3)$. This is the Special Cauchy's Theorem.

$(3) \implies (4)$. For z_0, if U_{z_0} is the connected component of z_0, then for $z \in U_{z_0}$ and any p.s.c paths $\gamma_0, \gamma_1 : [0,1] \to U$ (into U_{z_0}) so:

$$\gamma_0(0) = z_0 = \gamma_1(0) \quad \text{and} \quad \gamma_0(1) = z = \gamma_1(1)$$

we have that $\gamma_0 \cdot \gamma_1^-$ is a p.s.c closed path, so for $f \in \mathcal{H}(U)$:

$$\int_{\gamma_0} f - \int_{\gamma_1} f_1 = \int_{\gamma_0} + \int_{\gamma_1^-} = \int_{\gamma_0 \cdot \gamma_1^-} f = 0$$

last equality comes from (3). Thus we define $F : U_{z_0} \to \mathbb{C}$ by:

$$F(z) = \int_\gamma f$$

where γ is any p.s.c path with $\gamma(0) = z_0$ and $\gamma(1) = z$, and see the proof of Morera's Theorem that $F'(z) = f(z)$. We similarly define F on each (path) component (that it, we choose representatives $z_1, z_2 \cdots$ in different components by axiom of choice, and build F).

(4) $\implies$ (5). Since $0 \notin f(U)$ we get $1/f \in \mathcal{H}(U)$, by chain rule. Also $f' \in \mathcal{H}(U)$, if follows that $f'/f \in \mathcal{H}(U)$. By (4), there is a primitive F of f'/f on U. Let $z_0, z_1, \cdots$ be representatives of th path connected components of U. For each z_j, let $c_j \in \mathbb{C}$ be so that:

$$\exp(c_j) = f(z_j) \exp(-F(z_j))$$

and that:

$$\exp(F(z) + c_j) = f(z) \quad \text{for} \quad z \in U_{z_j}$$

Let $g : U \to \mathbb{C}$ such that:

$$g(z) = F(z) + c_j$$

where $z \in U_{z_j}$ for $j = 0, 1, 2, \cdots$. That is, $\exp(g(z_j)) = f(z_j)$. We compute for $h = \frac{f}{\exp(g)} = f \exp(-g)$. So:

$$h' = f' \exp(-g) + f \exp(-g)(-g') = f' \exp(-g) - f \exp(-g) \cdot \frac{f'}{f} = 0$$

Thus $h' = 0$, which means h is constant on each component, with:

$$h(z_j) = f(z_j) \exp(-g(z_j)) = 1$$

by our construction of g, thus $h = 1$. It follows that $\exp(g) = f$.

(5) $\implies$ (1). Let $z_0 \in \mathbb{C} \setminus U$. Let $z_0 \in \mathbb{C} \setminus U$, so $f(z) = z - z_0$ satisfies $0 \notin f(U)$. Then, by proof of last implication:

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_0}$$

admits a primitive g on U. Now, if γ is any closed p.s.c path in U, we have:

$$W_\gamma(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_0} = 0$$

by FTC of line integral. □

Remark If we have (5), then for any $f \in \mathcal{H}(U)$ with $0 \notin f(U)$ with g as in (5), the function h defined by:

$$h = \exp\left(\frac{1}{2}g\right)$$

satisfies $h^2 = f$ (Existence of square root). Now suppose U is connected, the existence of square roots get us towards the theorem:

Theorem 11.3 (Riemann Mapping Theorem) If $U \subsetneq \mathbb{C}$ satisfies for each $f \in \mathcal{H}(U)$ with $0 \notin f(U)$, an $h \in \mathcal{H}(U)$ and $h^2 = f$, then there is $F \in \mathcal{H}(U)$ such that:

$$F(U) = \mathbb{D} = D(0, 1)$$

and F is invertible with $F^{-1} : \mathbb{D} \to U$ is holomorphic.

Note Star-like implies (0) holds, which implies "simply connected", (1) to (4).

If there is a star-like open V and $F : V \to U$ be invertible and F and F^{-1} are both holomorphic, then U is simply connected.

Lecture 25, 2024/03/11

Theorem 11.4 (Argument Principle) Let $U \subseteq \mathbb{C}$ be open, define $P \subseteq U$ be a set with no cluster points in U. Let $f \in \mathcal{H}(U \setminus P)$, so each $z \in P$ is a pole for f. Let γ be a toy curve in $U \setminus (P \cup Z(f))$, where $Z(f)$ is the zero set of f in U, such that γ is special for U. Let $U_1 = W_\gamma^{-1}(\{1\}) \subseteq U$ by speciality. Then:

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = NZ_\gamma(f) - NP_\gamma(f)$$

where $NZ_\gamma(f)$ and $NP_\gamma(f)$ are:

$$NZ_\gamma(f) = \text{number of zeros of } f \text{ (counting multiplicities) in } U_1$$
$$NP_\gamma(f) = \text{number of poles of } f \text{ with order in } U_1$$

Proof: The Residue Theorem tells us that:

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = \sum_{j=1}^{n} \mathrm{Res}_{z_j}\left(\frac{f'}{f}\right)$$

where $\{z_1, \cdots, z_n\} = P \cap Z(f) \cap U_1$. The "logarithmic" derivative:

$$\frac{(gh)'}{gh} = \frac{gh' + g'h}{gh} = \frac{g'}{g} + \frac{h'}{h}$$

Since $f(z_j) = 0$, by Order of Zero theorem, there is $m_j \in \mathbb{N}$ such that:

$$f(z) = (z - z_j)^{m_j} g(z)$$

where g is holomorphic and $g(z_j) \neq 0$. Then:

$$\frac{f'(z)}{f(z)} = \frac{m_j(z - z_j)^{m_j - 1}}{(z - z_j)^{m_j}} + \frac{g'(z)}{g(z)} = \frac{m_j}{z - z_j} + \frac{g'(z)}{g(z)}$$

since $g(z) \neq 0$ at $z = z_j$, the residue is $\text{Res}_{z_j}(f'/f) = m_j$. By the Order of Pole theorem, there is $m_j \in \mathbb{N}$ and $g(z_j) \neq 0$ with g holomorphic:

$$f(z) = \frac{g(z)}{(z - z_j)^{m_j}} = (z - z_j)^{-m_j} g(z)$$

So that:

$$\frac{f'(z)}{f(z)} = \frac{-m_j(z - z_j)^{-m_j - 1}}{(z - z_j)^{-m_j}} + \frac{g'(z)}{g(z)} = -\frac{m_j}{z - z_j} + \frac{g'(z)}{g(z)}$$

Therefore $\text{Res}_{z_j}(f'/f) = -m_j$. □

Theorem 11.5 (Rouche Theorem) Let $U \subseteq \mathbb{C}$ be open and connected and $f \in \mathcal{H}(U)$ be non-constant. If $g \in \mathcal{H}(U)$ and γ a toy curve in U such that:

$$|f(z) - g(z)| < |f(z)| \text{ for all } z \in \gamma^*$$

Then $NZ_\gamma(f) = NZ_\gamma(g)$, that is, f and g have the same number of zeros in U.

Proof: Note that for $z \in \gamma^*$, we have $|f(z)| > 0$, so γ is in $U \setminus Z(f)$. Thus $f \circ \gamma$ is a curve in $\mathbb{C} \setminus \{0\}$ and by argument principle, we have:

$$NZ_\gamma(f) = \frac{1}{2\pi i} \int_\gamma \frac{f'}{f} = \frac{1}{2\pi i} \int_0^1 \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t))}\, dt = \frac{1}{2\pi i} \int_{f \circ \gamma} \frac{d\zeta}{\zeta} \tag{1}$$

Note that $z \in \gamma^* \cap Z(g)$, then we would have:

$$|f(z) - g(z)| = |f(z) - 0| = |f(z)|$$

violating the assumptions. So, as above, $g \circ \gamma$ is a p.s.c closed curve in $\mathbb{C} \setminus \{0\}$. Hence, as above:

$$NZ_\gamma(g) = \frac{1}{2\pi i} \int_\gamma \frac{g'}{g} = \frac{1}{2\pi i} \int_{g \circ \gamma} \frac{d\zeta}{\zeta} \tag{2}$$

We shall see that $f \circ \gamma$ and $g \circ \gamma$ are homotopic (as closed paths) in $\mathbb{C} \setminus \{0\}$. Define, for $s \in [0,1]$, the convex combination:

$$\begin{aligned}\gamma_s(t) &= (1-s)(f \circ \gamma)(t) + s(g \circ \gamma)(t) \\ &= (f \circ \gamma)(t) + s[(g \circ \gamma)(t) - (f \circ \gamma)(t)]\end{aligned}$$

Then for $t \in [0,1]$ (parameter for γ), we have:

$$\begin{aligned}|\gamma_s(t)| &= |f(\gamma(t)) + s[g(\gamma(t)) - f(\gamma(t))]| \\ &\geq |f(\gamma(t))| - s|g(\gamma(t)) - f(\gamma(t))| \\ &> |f(\gamma(t))| - s|f(\gamma(t))| && \text{(by assumption)} \\ &\geq 0\end{aligned}$$

So γ_s is a closed p.s.c path in $\mathbb{C} \setminus \{0\}$. Thus $f \circ \gamma$ is homotopic to $g \circ \gamma$ we have:

$$\int_{f \circ \gamma} \frac{d\zeta}{\zeta} = \int_{g \circ \gamma} \frac{d\zeta}{\zeta}$$

By (1) and (2), we get the desired result. □

Example Fundamental Theorem of Algebra (Another proof). Let:

$$p(z) = a_0 + \cdots + a_n z^n \in \mathbb{C}[x]$$

with $a_n \neq 0$. Then $p(z)$ has n zeros in $\mathbb{C}$, counting multiplicities. Choose a large enough $R > 0$ so that:

$$R^n > |a_0| + |a_1|R + \cdots + |a_{n-1}|R^{n-1}$$

Let $\gamma(t) = Re^{it}$ for $t \in [0, 2\pi]$. Then check that:

$$|p(z) - z^n| < |z^n|$$

on $z \in C(0,R) = \gamma^*$. Thus $NZ_\gamma(p) = NZ_\gamma(z \mapsto z^n) = n$. [And we know $p(z)$ contains no more than n zeros by basic algebra.]

Lecture 26, 2024/03/13

Theorem 11.6 (Open Mapping) Let $U \subseteq \mathbb{C}$ be connected and open and $f \in \mathcal{H}(U)$ be non-constant. Then $f(U)$ is open in $\mathbb{C}$.

Remark If $V \subseteq U$ is open, then $f|_V \in \mathcal{H}(U)$, so $f(V)$ is open as weel. That is, f is an **open map**.

Proof of Open Mapping Theorem: Let $w_0 \in f(U)$, we will find $\epsilon > 0$ such that $D(w_0, \epsilon) \subseteq f(U)$. Since $w_0 \in f(U)$, write $w_0 = f(z_0)$ for some $z_0 \in U$. Let $g(z) = f(z) - w_0$ so that $g(z_0) = 0$ and $g \in \mathcal{H}(U)$. By Zero sets theorem, there is $r > 0$ so g admits no other zeros in $D(z_0, r)$. Thus if $\delta \in (0, r)$, then:

$$m = \min_{z \in C(z_0, \delta)} |f(z) - w_0| > 0$$

Find $\epsilon \in (0, m)$, we will show that $D(w_0, \epsilon) \subseteq f(U)$. Fix $w \in D(w_0, \epsilon)$ and let $h(z) = f(z) - w$. Then for $z \in C(z_0, \delta)$, we have:

$$|h(z) - g(z)| = |w - w_0| < \epsilon < m \leq |g(z)|$$

So by Rouche's theorem, h has same number of zeros in $D(z_0, \delta)$ as g, namely at least one. Hence there is z with:

$$z \in D(z_0, \delta) \subseteq D(z_0, r) \subseteq U$$

so that $0 = h(z) = f(z) - w$, that is, $w = f(z)$. Since w is picked arbitrarily in $D(w_0, \epsilon)$, we see that:

$$D(w_0, \epsilon) \subseteq f(D(z_0, \delta)) \subseteq f(U)$$

As desired. $\square$

Theorem 11.7 (Maximum Modulus Principle) Let $U \subseteq \mathbb{C}$ be connected and open. Let $f \in \mathcal{H}(U)$ be non-constant, if $\emptyset \neq V \subseteq U$ is open, with $\overline{V}$ compact. Then for $z_0 \in V$ we have:

$$|f(z_0)| < \max_{z \in \partial V} |f(z)|$$

[Moral: Never a maximum modulus point in the interior.]

Proof #1: Let $M = \max_{z \in \overline{V}} |f(z)| > 0$ since $V \neq \emptyset$. If there were $z_0 \in V$ such that $f(z_0) = M$. Then by open mapping theorem, there is $\epsilon > 0$ such that $D(f(z_0), \epsilon) \subseteq f(V)$. However:

$$\left| f(z_0) + \frac{f(z_0)}{|f(z_0)|} \cdot \frac{\epsilon}{2} \right| = |f(z_0)| \left| 1 + \frac{\epsilon}{2|f(z_0)|} \right| > |f(z_0)| = M$$

But:

$$\left| f(z_0) + \frac{f(z_0)}{|f(z_0)|} \cdot \frac{\epsilon}{2} - f(z_0) \right| < \epsilon$$

so for some $z \in V$, we have:

$$f(z) = f(z_0) + \frac{f(z_0)}{|f(z_0)|} \cdot \frac{\epsilon}{2}$$

with $|f(z)| > M$, contradiction. $\square$

Lemma 11.8 If $h : [0, 2\pi] \to [0, \infty]$ be continuous, then:

$$\frac{1}{2\pi} \int_0^{2\pi} h(t)\ dt = \max_{t \in [0,2\pi]} h(t)$$

implies that h is constant.

Proof #2 of Maximum Modulus Theorem: Suppose $z_0 \in V$ be such that:

$$|f(z_0)| = \max_{z \in \overline{V}} |f(z)| = M$$

Let $d = \operatorname{dist}(z_0, \partial V) > 0$. For $r \in (0, d)$, we have Cauchy's Integral Formula:

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{z - z_0}\ dz$$

where $\gamma_r(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$. Then:

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{z_0 + re^{it} - z_0} \cdot ire^{it}\ dt \\ &= \frac{1}{2\pi i} \int_0^{2\pi} f(z_0 + re^{it})\ dt \end{aligned}$$

So that:

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})| dt \leq \frac{1}{2\pi} \int_0^{2\pi} M\ dt = M$$

By the lemma, $|f(z)| = M$ for all $z \in C(z_0, r) = \gamma_r^*$. This happens for every $r \in (0, d)$, so $|f|$ is constant on:

$$\{z_0\} \cup \bigcup_{r \in (0,d)} C(z_0, r) = D(z_0, d)$$

So, by A1 Q4, f is constant on $D(z_0, d)$. But by Zeros Set Theorem, f is constant in U, contradicting the assumption. $\square$

Lecture 27, 2024/03/15

Theorem 11.9 (Injectivity) Let $U \subseteq \mathbb{C}$ be open and $f \in \mathcal{H}(U)$ be injective on U. Then we have:

(i) $f'(z) \neq 0$ for all $z \in U$.

(ii) $f^{-1} \in \mathcal{H}(f(U))$. [Recall that $f(U)$ is open as f is non-constant.]

Proof: (i). For contradiction, assume there is $z_0 \in U$ with $f'(z_0) = 0$. Find $R > 0$ such that $\overline{D}(z_0, R) \subseteq U$ and $f'(z) \neq 0$ for all $D_0(z_0, R)$. [This R exists by zero sets theorem that z_0 is an isolated zero for f', as f is non-constant.] By corollary to Cauchy's Integral Formula, for $z \in D(z_0, R)$ we have:

$$\begin{aligned}
f(z) &= f(z_0) + \sum_{k=2}^{\infty} c_k (z - z_0)^k && (f'(z_0) = 0) \\
&= f(z_0) + \sum_{k=n}^{\infty} c_k (z - z_0)^k && (n \geq 2,\ c_n \neq 0) \\
&= f(z_0) + c_n (z - z_0)^n + G(z)
\end{aligned}$$

where $G(z) = (z - z_0)^{n+1} g(z)$ where $g \in \mathcal{H}(D(z_0, R))$ and $g(z_0) \neq 0$. Let us find $r \in (0, R)$ so:

$$|c_n| > rM \quad \text{where} \quad M = \max_{z \in \overline{D}(z_0, R)} |g(z)|$$

so $|c_n| r^n > r^{n+1} M$. Then let $w \in \mathbb{C} \setminus \{0\}$ be so:

$$|c_n| r^n - |w| > r^{n+1} M$$

Write $H(z) = f(z) - f(z_0) - w$, we have:

$$H(z) = \underbrace{c_n (z - z_0)^n - w}_{F(z)} + G(z)$$

Now for $z \in C(z_0, r) \subseteq \overline{D}(z_0, R)$ (where M is defined) Then:

$$\begin{aligned}
|F(z) - H(z)| &= |F(z) - [f(z) - f(z_0) - w]| = |G(z)| \\
&= |z - z_0|^{n+1} |g(z)| \leq r^{n+1} M \\
&< |c_n| r^n - |w| && \text{(by choice of } r, w) \\
&= |c_n| |z - z_0|^n - |w| \\
&\leq |c_n (z - z_0)^n - w| = |F(z)|
\end{aligned}$$

Note that F has at least $n \geq 2$ zeros (with order) in $D(z_0, r)$, since:

$$0 = c_n (z - z_0)^n - w \implies \frac{w}{c_n} = (z - z_0)^w \quad \text{where } |w/c_n| < r^2$$

Thus by Rouche's Theorem, H has at least two zeros in $D(z_0, r)$. But:

$$H(z_0) = f(z_0) - f(z_0) - w = -w \neq 0$$

and $H'(z) = f'(z) \neq 0$ on $D(z_0, r) \subseteq D(z_0, R)$. So the zeros of H at distinct, each of order 1. This implies that $f(z) = f(z_0) + w$ has at least 2 solutions, contradicting the assumption that f is injective.

(ii). Since $f(V)$ is open for each open $V \subseteq U$ (open mapping theorem), we see that for open $V \subseteq U = f^{-1}(f(U))$ and:

$$V = f^{-1}(f(V)) = \{w \in \mathbb{C} : f^{-1}(w) \in f(U)\}$$

For $W \subseteq f(U)$ open, then $V = f^{-1}(W) \subseteq U$ is open and $W = f(V)$. So we see that f^{-1} is continuous by A1 Q1. If $w, w_0 \in f(U)$, so $w = f(z)$ adn $w_0 = f(z_0)$ where $z, z_0 \in U$. And $z - z_0 = f^{-1}(w) - f^{-1}(w_0)$. Then:

$$\frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - f(z_0)} = \frac{1}{\frac{f(z)-f(z_0)}{z-z_0}}$$

Taking $w \to w_0$ we know $z \to z_0$ by continuity of f^{-1}, this tends to $\frac{1}{f'(z_0)}$. Since $f'(z_0) \neq 0$, we thus proved that f^{-1} is holomorphic on U. $\square$

If $U \subseteq \mathbb{C}$ is open and connected, a **branch of logarithm** is $L : U \to \mathbb{C}$ such that $\exp(L(z)) = z$. By the last theorem, $L \in \mathcal{H}(U)$. By Simple Connectedness Theorem, we know such L always exists if U is simply connected (for example, star-like).

BUT this is not unique. If L is a branch of logarithm, so is $L + i2\pi k$ for $k \in \mathbb{Z}$.

Lecture 28, 2024/03/18

12 Automorphisms of $\mathbb{D}$ and $\mathbb{C}$

Let $\mathbb{D} = D(0, 1) \subseteq \mathbb{C}$.

Theorem 12.1 (Maximum Modulus Principle on $\mathbb{D}$) If $f \in \mathcal{H}(\mathbb{D})$, then either:

1. $|f(z)| < \sup_{z\in\mathbb{D}} |f(z)|$ or
2. f is constant.

Proof: If f is non-constant, then $f(\mathbb{D})$ is open, by Open Mapping Theorem. Hence the supremum is never achived. $\square$

Definition A **rotation** of $\mathbb{D}$ is any map of the form $f(z) = e^{i\theta} z$ where $\theta \in \mathbb{R}$.

Lemma 12.2 (Schwarz's Lemma) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then we have:

(i) $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$.

(ii) $|f'(0)| \leq 1$.

(iii) If equality holds in (i) for some $z_0 \in \mathbb{D} \setminus \{0\}$, or equality holds in (ii), then f is a rotation.

Proof: (i). Since $f(0) = 0$, we have Maclaurin series on $\mathbb{D}$:

$$f(z) = \sum_{k=1}^{\infty} c_k z^k = zg(z)$$

where $g(z) = \sum_{k=1}^{\infty} c_k z^{k-1}$ and $g \in \mathcal{H}(\mathbb{D})$. If $0 < |z| = r < 1$, then:

$$|g(z)| = \left|\frac{f(z)}{z}\right| = \frac{|f(z)|}{r} \leq \frac{1}{r}$$

So MMP shows that $|g(z)| \leq \frac{1}{r}$ for all $z \in \overline{D}(0, r)$. This holds for all $r \in (0, 1)$, so if $z \in \mathbb{D} \setminus \{0\}$, then:

$$\left|\frac{f(z)}{z}\right| \leq |g(z)| \leq 1$$

which gives (i).

(ii). Notice that:

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} g(z) = g(0)$$

where g is from the proof above. Then $|f'(0)| = |g(0)| \leq 1$.

(iii). If $|f(z_0)| = |z_0|$ for some $z_0 \in \mathbb{D} \setminus \{0\}$, then $g(z_0) = \frac{g(z_0)}{z_0}$ so g attains its maximum/supremum inside $\mathbb{D}$. Thus by MMP, $g(z) = c$ and $|c| = |g(z_0)| = 1$. So $c = e^{i\theta}$ for some $\theta \in \mathbb{R}$. Thus if $z \in \mathbb{D} \setminus \{0\}$, then:

$$\frac{f(z)}{z} = e^{i\theta} \implies f(z) = ze^{i\theta}$$

so f is a rotation. If $|f'(0)| = 1$, then by proof of (ii), then g gains its maximum at 0, so by MMP $g(z) = c$, and apply the argument above and f is a rotation. $\square$

Theorem 12.3 (Automorphisms of $\mathbb{D}$) We have:

(i) Let $w \in \mathbb{D}$, then the map $\psi_w(z) = \frac{w-z}{1-\overline{w}z}$ for $z \in \mathbb{D}$ satisfies:

$$\psi_w(\mathbb{D}) = \mathbb{D} \quad \text{and} \quad \psi_w^{-1} = \psi_w \quad \text{and} \quad \psi_w(0) = w$$

(ii) If $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic bijection, then:

$$f(z) = e^{i\theta}\psi_w(z)$$

for some $\theta \in \mathbb{R}$ and $w \in \mathbb{D}$.

Proof: (i). Since $|w| < 1$, so $\psi_w \in \mathcal{H}(D(0, 1/|w|))$ (stare at denominator of ψ_w), and $\overline{\mathbb{D}} \subseteq D(0, 1/|w|)$. If $z \in \partial\mathbb{D} = C(0,1)$, then since $z\overline{z} = 1$:

$$\psi_w(z) = \frac{w - z}{1 - \overline{w}z} = z \cdot \frac{w\overline{z} - 1}{1 - \overline{w} - z} = z \cdot \frac{\overline{\overline{w}z - 1}}{-(\overline{w}z - 1)}$$

So $|\psi_w(z)| = 1$. Since ψ_w is non-constant, MMP tells us that $\psi_w(\mathbb{D}) \subseteq \mathbb{D}$.

Now, if $z \in \mathbb{D}$, we just compute:

$$\psi_w \circ \psi_w(z) = \frac{w - \frac{w-z}{1-\overline{w}z}}{1 - \overline{w} \cdot \frac{w-z}{1-\overline{w}z}} = \frac{(1 - \overline{w}z)w - (w - z)}{(1 - \overline{w}z) - \overline{w}(w - z)} = \frac{z - |w|^2 z}{1 - |w|^2} = z$$

Therefore $\psi_w = \psi_w^{-1}$. Since $\psi_w(\mathbb{D}) \subseteq \mathbb{D}$, we have $\psi_w(\mathbb{D}) = \mathbb{D}$ since $\psi_w = \psi_w^{-1}$. Finally $\psi_w(0) = w$ is trivial.

(ii). Since f is a bijection, so $f^{-1} : \mathbb{D} \to \mathbb{D}$ exists. Let $w = f^{-1}(0)$ so $0 = f(w)$. Let $g = f \circ \psi_w$ so that:

$$g(0) = f(\psi_w(0)) = f(w) = 0$$

Also, $g^{-1} = \psi_w^{-1} \circ f^{-1} = \psi_w \circ f^{-1}$ is the inverse of g. We have:

$$g : \mathbb{D} \to \mathbb{D},\ g(0) = 0 \ \text{ and } \ g^{-1} : \mathbb{D} \to \mathbb{D},\ g^{-1}(0) = 0$$

By Schwarz's Lemma, we see $|g(z)| \leq |z|$ and $|g^{-1}(z')| \leq |z'|$ for $z, z' \in \mathbb{D}$ and letting $z' = g(z)$, and applying g^{-1} gives $|z| \leq |g(z)|$, so $|z| = |g(z)|$. Then (iii) of Schwarz's Lemma tells us $g(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$. Thus:

$$f(z) = (g \circ \psi_w)(z) = e^{i\theta}\psi_w(z)$$

As desired. $\square$

Lecture 29, 2024/03/20

Lemma 12.4 Let $z_0 \in \mathbb{C}$ and $r > 0$. Let $f \in \mathcal{H}(D_0(z_0, r))$ be bounded on $D_0(z_0, r)$. Then z_0 is a removable singularity.

Proof: Let $M = \sup_{z \in D_0(z_0,r)} |f(z)| < \infty$. Then for $z \in D_0(z_0, r)$:

$$0 \leq |(z - z_0)f(z)| = |z - z_0||f(z)| \leq |z - z_0|M$$

Therefore:

$$\lim_{z \to z_0} (z - z_0) f(z) = 0$$

Thus either $\lim\limits_{z \to z_0} f(z)$ exists, wthen we are done. Or z_0 is a simple (order 1) pole of f, but with residue 0, which puts us in the case above (Order of Pole). $\square$

Theorem 12.5 (Casorati-Weierstrass) Let $z_0 \in \mathbb{C}$ and $R > 0$. Let $f \in \mathcal{H}(D_0(z_0, R))$. If z_0 is an essential singularity, then $f(D_0(z_0, r))$ is dense in $\mathbb{C}$ for any $r \in (0, R]$.

Picard: In fact, $f(D_0(z_0, r)) \supseteq \mathbb{C} \setminus \{w_0\}$ for some $w_0 \in \mathbb{C}$. For example, for $f(z) = \exp(1/z)$, then $f(D_0(0, r)) = \mathbb{C} \setminus \{0\}$.

Proof: Suppose for some $r \in (0, R]$, that $f(D_0(z_0, r))$ is not dense. Hence there is $w \in \mathbb{C}$ and $\delta > 0$ so $D(w, \delta) \cap f(D_0(z_0, r)) = \emptyset$ by definition. Let:

$$g(z) = \frac{1}{f(z) - w}$$

so that:

$$|g(z)| = \frac{1}{|f(z) - w|} \leq \frac{1}{\delta}$$

for $z \in D_0(z_0, r)$. Hence, by the lemma, z_0 is a removable singularity for g. Extending g to $D(z_0, r)$. If $g(z_0) = \lim_{z \to z_0} g(z) \neq 0$, then:

$$f(z) = \frac{1}{g(z)} + w$$

for $z \in D(z_0, r')$ where $0 < r' \leq r$ and has z_0 as a removable singularity, contradicting the assumptions. Thus we consider $g(z_0) = 0$. Using the order of zero theorem:

$$\frac{1}{f(z) - w} = g(z) = (z - z_0)^m h(z)$$

for some $m \in \mathbb{N}$ and $h \in \mathcal{H}(D(z_0, r))$ with $h(z_0) \neq 0$, so:

$$(z - z_0)^m f(z) = \frac{1}{h(z)} - (z - z_0)^m w$$

Taking $z \to z_0$, this tends to $1/h(z_0)$, which exists, again contradiction. $\square$

Theorem 12.6 (Automorphisms of $\mathbb{C}$) If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and injective, then there are $a, b \in \mathbb{C}$ and $a \neq 0$ such that:

$$f(z) = az + b$$

for all $z \in \mathbb{C}$.

Proof: Since f is non-constant, Open Mapping Theorem provides for $R > 0$ that $f(D(0,R))$ is open. Since f is injective:

$$f(D(0,R)) \cap f(\mathbb{C} \setminus \overline{D}(0,R)) = \emptyset$$

Define $V = f(\mathbb{C} \setminus \overline{D}(0,R))$ and V is not dense in $\mathbb{C}$ by definition. Also if V is unbounded, since if V were bounded, then f is bounded, which contradicts Liouville's Theorem, as f is non-constant. Let $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by $g(z) = f(1/z)$, so:

$$g(D_0(0,1/R)) = f(\mathbb{C} \setminus \overline{D}(0,R)) = V$$

is not dense in $\mathbb{C}$, so 0 is a pole of g by Casorati-Weierstrass. Hence for some $n \in \mathbb{N}$:

$$\lim_{z \to 0} z^n g(z) \quad \text{exists}$$

Hence:

$$\lim_{|z| \to \infty} \frac{f(z)}{z^n} = \lim_{|z| \to \infty} \frac{1}{z^n} g\left(\frac{1}{z}\right) = \lim_{w \to 0} w^n g(w) \quad \text{exists}$$

which, by a Liouville's Theorem argument, we know f is a polynomial function. By injectivity, f has only one zero $f(z_0) = 0$. Furthermore, by the Injectivity Theorem, $f'(z_0) \neq 0$. Hence this unique zero is of order 1. $\square$

Hence $\text{Aut}(\mathbb{C}) = \{z \mapsto az + b : a, b \in \mathbb{C},\ a \neq 0\}$. This is in fact solvable.

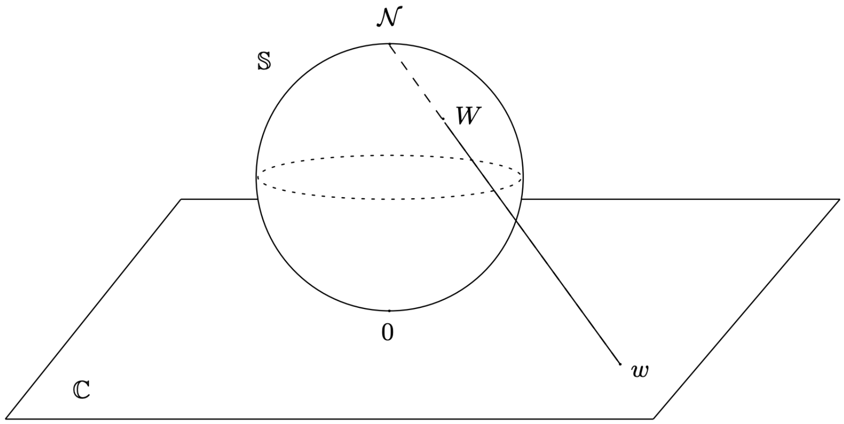
13 Riemann Sphere and Mobius Maps

We define $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. Let $\mathbb{C} \to \mathbb{R}^3$ be a inclusion by:

$$z \mapsto (\operatorname{Re} z, \operatorname{Im} z, 0)$$

So we can identity $\mathbb{C}$ as a subset of $\mathbb{R}$. Then let:

$$\begin{aligned} \mathbb{S} :&= \left\{(X,Y,Z) \in \mathbb{R}^3 : X^2 + Y^2 + \left(Z - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2\right\} \\ &= \{(X,Y,Z) \in \mathbb{R}^3 : X^2 + Y^2 = (1-Z)Z\} \end{aligned}$$



where $\mathcal{N} = (0,0,1)$ is the **north pole**. If $W = (X, Y, Z)$, then:

$$w = \left(\frac{X}{1-Z}, \frac{Y}{1-Z}, 0\right) \in \mathbb{R}^3 \quad \text{and} \quad Z \neq 1$$

Which corresponds to the complex number $\frac{X}{1-Z} + i\frac{Y}{1-Z}$. Consider the convex combination for $(X, Y, Z) \in \mathbb{S}$:

$$(1-Z)\left(\frac{X}{1-Z}, \frac{Y}{1-Z}, 0\right) + Z(0,0,1) = (X, Y, Z) \in \mathbb{S}$$

Defien $P : \mathbb{S} \to \mathbb{C}_\infty$ by:

$$P(X, Y, Z) = \begin{cases} \frac{X}{1-Z} + i\frac{Y}{1-Z} & \text{if } Z \neq 1 \\ \infty & \text{if } Z = 1 \end{cases}$$

Compute P^{-1}. Let $z \in \mathbb{C}$ with $x = \text{Re}(z)$ and $y = \text{Im}(z)$. Solve for $t \in [0,1]$ with:

$$(1-t)(x, y, 0) + t(0,0,1) \in \mathbb{S}$$

That is, we need:

$$((1-t)x)^2 + ((1-t)y)^2 = (1-t)t$$

which implies that:

$$(1-t)^2(x^2 + y^2) = (1-t)t$$

Then:

$$(1-t)(x^2+y^2) = t \iff \begin{cases} \text{if } t = 1, \text{ we are at } \mathcal{N} = (0,0,1) \\ \text{if } t < 1, \; x^2 + y^2 = (x^2+y^2+1)t \implies t = \frac{|z|^2}{1+|z|^2} \end{cases}$$

Notice that:

$$1 - \frac{|z|^2}{1+|z|^2} = \frac{1}{1+|z|^2}$$

Then we have:

$$P^{-1}(z) = \begin{cases} (0,0,1) & \text{if } z = \infty \\ \left(\frac{\mathrm{Re}(z)}{1+|z|^2}, \frac{\mathrm{Im}(z)}{1+|z|^2}, \frac{|z|^2}{1+|z|^2}\right) & \text{if } z \in \mathbb{C} \end{cases}$$

We assign topololy to $\mathbb{C}_\infty$ allowing P to be a homeomorphism.

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Define $\mathrm{Aut}(U) = \{f : U \to U : f \text{ holomorphic and bijective}\}$

Inversion Theorem $\implies$ f^{-1} holomorphic.

Example Some examples:

1. Automorphism on $\mathbb{D}$:

$$\mathrm{Aut}(\mathbb{D}) = \{f = e^{i\theta}\psi_w : \theta \in \mathbb{R},\ w \in \mathbb{D}\}$$

   where $\psi_w(z) = \frac{w-z}{1-\overline{w}z}$.

2. Automorphism on $\mathbb{C}$:

$$\mathrm{Aut}(\mathbb{C}) = \{z \mapsto az + b : a, b \in \mathbb{C}, a \neq 0\}$$

The general linear group (group of invertible matrices) is:

$$GL_2(\mathbb{C}) = \left\{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C}) : \det(A) = 0\right\}$$

Cramer's Rule for $n = 2$ tells us that:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Definition If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$, for $z \in \mathbb{C}_\infty$ we define:

$$\mu_A(z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \neq \infty,\ z \neq -d/c \\ \frac{a}{c} & \text{if } z = \infty,\ c \neq 0 \\ \infty & \text{if } z = \infty,\ c = 0 \\ \infty & \text{if } z = -d/c \end{cases}$$

Then $\mu_A : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is called the **Mobius map** from A.

Example (Some Mobius Maps)

1. If $A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, then:
$$\mu_A(z) = z + b$$
is a translation.

2. If $w \in \partial\mathbb{D} = C(0,1)$, that is, $|w| = 1$. Let $A = \begin{pmatrix} w & 0 \\ 0 & 1 \end{pmatrix}$, we have:
$$\mu_A(z) = wz$$
is a rotation.

3. For $r > 0$ in $\mathbb{R}$, define $A = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$ we have:
$$\mu_A(z) = rz$$
is an amplification (if $r > 1$) and contraction if $r < 1$. If $r = 1$, then μ_A is the identity map.

4. If $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then:
$$\mu_A(z) = \begin{cases} \frac{1}{z} & \text{if } z \neq 0 \\ \infty & \text{if } z = 0 \\ 0 & \text{if } z = \infty \end{cases}$$
is the inverse. And if $B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, then:
$$\mu_B(z) = \frac{i}{iz} = \frac{1}{z}$$
note that $\det B = 1$ so $B \in SL_2(\mathbb{C})$, the special linear group.

Remark Some facts:

1. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ both invertible (in $GL_2(\mathbb{C})$), then:
$$AB = \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix}$$

then $\det(AB) = \det(A)\det(B) \neq 0$, so $AB \in GL_2(\mathbb{C})$. Then:

$$\mu_A \circ \mu_B = \mu_{AB}$$

holds (exercise). So we know that $\mu_A^{-1} = \mu_{A^{-1}}$. So μ_A is invertible and we know its inverse.

2. For $\alpha \in \mathbb{C}$ we have $\alpha I = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$, then $\mu_{\alpha I}(z) = z$, that is, $\mu_{\alpha I}$ is the identity map. Hence if $\alpha \in \mathbb{C} \setminus \{0\}$ and $B \in GL_2(\mathbb{C})$, then:

$$\mu_{\alpha B} = \mu_{(\alpha I)B} = \mu_{\alpha I} \circ \mu_B = \mu_B$$

Often, we parametrize Mobius maps over $SL_2(\mathbb{C})$. The **projective special linear group**:

$$PSL_2(\mathbb{C}) = SL_2(\mathbb{C})/\{I, -I\}$$

the quotient group by the normal subgroup $\{I, -I\}$, parametrizes Mobius maps.

Note We have the followings:

1. $\mathrm{Aut}(\mathbb{C}) = \{\mu_A : A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{C}, a \neq 0\}$

2. $\mathrm{Aut}(\mathbb{D})$ as Mobius maps. Let:

$$G = \left\{ \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C},\ |\alpha|^2 - |\beta|^2 = 1 \right\}$$

Note that G is a subgroup of $SL_2(\mathbb{C}) \subseteq GL_2(\mathbb{C})$ (exericse). Let $z \in \mathbb{C}$ and $A = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in G$, so $|\alpha| > |\beta| \geq 0$, so that:

$$\mu_A(z) = \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}} = \frac{\overline{\alpha}}{\beta} \cdot \frac{z + \frac{\beta}{\alpha}}{\left(\overline{\frac{\beta}{\alpha}}\right) z + 1} = \frac{\overline{\alpha}}{\alpha} \cdot \frac{z - w}{-\overline{w} z + 1}$$

where we let $w = -\beta/\alpha$. Then let $\theta \in \mathbb{R}$ so $e^{i\theta} = -\frac{\overline{\alpha}}{\alpha}$ and:

$$\mu_A(z) = -\frac{\overline{\alpha}}{\alpha} \cdot \frac{w - z}{1 - \overline{w} z} = e^{i\theta} \psi_w(z)$$

as in the automorphism of $\mathbb{D}$ theorem. Thus:

$$\mathrm{Aut}(\mathbb{D}) = \{e^{i\theta} \psi_w : \theta \in \mathbb{R},\ w \in \mathbb{R}\} = \left\{ \mu_A : A = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in G \right\}$$

Theorem 13.1 (Automorphisms of Riemann Sphere) If $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is continuous on $\mathbb{C}_\infty$ and bijective, with $z_0 = f^{-1}(\infty)$ and $f|_{\mathbb{C}\setminus\{0\}} \in \mathcal{H}(\mathbb{C}\setminus\{z_0\})$. Then $f = \mu_A$ for some $A \in GL_2(\mathbb{C})$.

Proof: (I). If $f(\infty) = \infty$, then $f|_\mathbb{C}$ is entire on $\mathbb{C}$ and is injective and thus by the automorphism of $\mathbb{C}$, we know $f(z) = az + b$, which is a Mobius map.

(II). Otherwise, let $z_0 = f^{-1}(\infty)$ and:

$$B = \begin{pmatrix} z_0 & b \\ 1 & -z_0 \end{pmatrix}$$

where $b \in \mathbb{C}$ is chosen so that $-z_0^2 - b \neq 0$. Thus:

$$\mu_B(z) = \frac{z_0 z + b}{z - z_0}$$

then $\mu_B(z_0) = \infty$ and $\mu_B(\infty) = z_0$. Let $g = f \circ \mu_B$, so that g is bijective and $g(\infty) = \infty$ and g is holomorphic on $\mathbb{C} \setminus \{z_0\}$. Also g is continuous at z_0 by the continuity of f at ∞. So z_0 is a removable singularity for g, hence by CGM method we can extend g to a map on $\mathbb{C}$. Thus by (I), we know $g = \mu_A$ where $A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, therefore we have:

$$f = g \circ \mu_B^{-1} = \mu_{AB^{-1}}$$

As desired. $\square$

Lecture 31, 2024/03/25

14 Conformal Maps

Definition Let $U, V \subseteq \mathbb{C}$ be open. A **conformal map** is a function $f : U \to V$ such that:

1. f is holomorphic on U
2. f is bijective.

Inversion Theorem $\implies$ $f^{-1} : V \to U$ is also holomorphic.

Definition First, let:

$$\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\} \text{ (open upper half plane)}$$

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \text{ (open unit disc)}$$

Notice for $z \in \mathbb{C}$:

$$z \in \mathbb{H} \iff |z - i| < |z + i|$$

that is, z is closer to i than it is to $-i$. And:

$$|z - i| < |z + i| \iff \left|\frac{z-i}{z+i}\right| < 1 \iff \left|\frac{z-i}{z+i}\right| \in \mathbb{D}$$

Define the **Cayley map** as follow. Define matrix:

$$C = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

then we know:

$$C^{-1} = \frac{1}{2i}\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

Then it follows that:

$$\mu_C(z) = \frac{z-i}{z+i} \text{ and } \mu_C^{-1}(z) = \mu_{C^{-1}}(z) = \frac{z+1}{i(z-1)}$$

Note $\mu_C(i) = 0$, and the upper half plane $\mathbb{H}$ is mapped to $\mathbb{D}$ by μ_C. Where the boundary line mapped to the boundary of $\mathbb{D}$, indeed:

$$|\mu_C(x)| = \left|\frac{x-i}{x+i}\right| = 1$$

for $x \in \mathbb{R}$. Also, we have $\lim_{|x|\to\infty} \mu_C(x) = 1$.

Theorem 14.1 (Automorphism of $\mathbb{H}$) If $f : \mathbb{H} \to \mathbb{H}$ is a conformal map. Then $f = \mu_A$ for some $A \in SL_2(\mathbb{R})$ (special linear $\mathbb{R}$-matrices).

Proof: Let C be the Cayley matrix above. So:

$$\mu_C(\mathbb{H}) = \mathbb{D} \text{ and } \mu_{C^{-1}}(\mathbb{D}) = \mathbb{H}$$

Then $\mu_C \circ f \circ \mu_{C^{-1}}$ is an automorphism of $\mathbb{D}$. (Automorphism = Conformal maps of set to itself). So:

$$\mu_C \circ f \circ \mu_{C^{-1}} = \mu_A, \ A = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$$

where $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 - |\beta|^2 = 1$. Thus:

$$f = \mu_{C^{-1}} \circ \mu_A \circ \mu_C = \mu_{C^{-1}AC}$$

We recall that:

$$C^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

Thus:

$$\begin{aligned}
C^{-1}AC &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} \alpha+\overline{\beta} & \beta+\overline{\alpha} \\ i(\alpha-\overline{\beta}) & i(\beta-\overline{\alpha}) \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} \alpha+\overline{\beta}+\beta+\overline{\alpha} & i(\beta+\overline{\alpha}-(\alpha+\overline{\beta})) \\ i(\alpha-\overline{\beta}+\beta-\overline{\alpha}) & \alpha-\overline{\beta}-(\beta-\overline{\alpha}) \end{pmatrix} \\
&= \begin{pmatrix} \operatorname{Re}(\alpha)+\operatorname{Re}(\beta) & -\operatorname{Im}(\beta)+\operatorname{Im}(\alpha) \\ -\operatorname{Im}(\alpha)+\operatorname{Im}(\beta) & \operatorname{Re}(\alpha)-\operatorname{Re}(\beta) \end{pmatrix}
\end{aligned}$$

which is a real matrix in $GL_2(\mathbb{R})$! Note that:

$$\det A = \det(CAC^{-1}) = 1$$

so this matrix is in $SL_2(\mathbb{R})$. We get all of $SL_2(\mathbb{R})$:

$$\text{If } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$$

Solve:

$$\begin{aligned}
a &= \operatorname{Re}(\alpha)+\operatorname{Re}(\beta) \\
b &= \operatorname{Im}(\alpha)-\operatorname{Im}(\beta) \\
c &= -\operatorname{Im}(\alpha)+\operatorname{Im}(\beta) \\
d &= \operatorname{Re}(\alpha)-\operatorname{Re}(\beta)
\end{aligned}$$

As desired. □

Notice for $A \in SL_2(\mathbb{R})$, then $\mu_A(\mathbb{R}) = \mathbb{R} \cup \{0\}$. That is, "boundary" of $\mathbb{H}$ is preserved. Notice, since μ_A preserves $\mathbb{R} \cup \{\infty\}$ and is continuous, so should preserve each side of the sphere, being continuous.

Example (Some important conformal maps)

1. **Affine maps**, defined on $\mathbb{C}$.
   (a) **Rotations:** $R_\theta(z) = e^{i\theta}z$ for $\theta \in \mathbb{R}$.
   (b) **Amplification/Contraction:** $h_r(z) = rz$ for $r > 0$.
   (c) **Trnaslation:** $\tau_b(z) = z + b$ for $b \in \mathbb{C}$.

2. **Exponents and Logarithms:**

$$S = \{z \in \mathbb{C} : \alpha < \operatorname{Im}(z) < \beta\}$$

where $\alpha < \beta \in \mathbb{R}$, and $\beta - \alpha \leq 2\pi$.

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3. Cayley map: $\mu_C(z) = \frac{z+i}{z-i}$. And it maps $\mathbb{H}$ maps to $\mathbb{D}$, the right-upper quarter to the lower half of the circle. It maps the strip between 0 and i to the complement of the circle. See picture.

4. **Powers and Principal roots:** For $n \in \mathbb{N}$ and $\alpha \in (0, \pi/n)$. See picture.

Example Find a conformal map between:

$$S = \{z \in \mathbb{C} : 0 < \operatorname{Re}(z) < 1\} \ \text{ and } \ \mathbb{D} \cap \mathbb{H} = \{z \in \mathbb{C} : |z| < 1, \ \operatorname{Im}(z) > 0\}$$

See picture, the result is, for $z \in S$:

$$\begin{aligned}
z &\mapsto e^{i\pi/2} z = iz \\
&\mapsto e^{iz} && \text{(exp)} \\
&\mapsto e^{-1/2} e^{iz} = e^{i(z-1/2)} && (R_{\frac{1}{2}}) \\
&\mapsto e^{i\frac{\pi}{2}(z-1/2)} && \text{(principal power)} \\
&\mapsto e^{i\frac{\pi}{2}} e^{i\frac{\pi}{2}(z-1/2)} = e^{i\frac{\pi}{2}z} \\
&\mapsto \frac{e^{i\frac{\pi}{2}z} - i}{e^{i\frac{\pi}{2}z} + i} && (\mu_C) \\
&\mapsto e^{-i\pi} \cdot \frac{e^{i\frac{\pi}{2}z} - i}{e^{i\frac{\pi}{2}z} + i} = \frac{i - e^{i\frac{\pi}{2}z}}{e^{i\frac{\pi}{2}z} + i}
\end{aligned}$$

15 Harmonic Functions

Let $U \subseteq \mathbb{C} \cong \mathbb{R}^2$ be open. Let:

$$C^2(U) = \{u : U \to \mathbb{R} : \text{all second order partials are continuous}\}$$

Mixed Partial Theorem:

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

for u that is C^2.

Definition We define the **Harmonic functions** on U to be functions in the set:

$$\mathrm{Har}(U) = \left\{ u \in C^2(U) : \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \right\}$$

where $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is the **Homogeneous Laplace Equation**.

Note As a linear space, we note that $\mathrm{Har}(U)$ is a $\mathbb{R}$-linear space.

Recall Cauchy-Riemann Equations. For $f \in \mathcal{H}(U)$ with $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$ so that $f = u + iv$. And for $z \in U$ with $z = x + iy$:

$$\begin{aligned} f'(z) &= \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(z+h) - f(z)}{h} = \frac{\partial f}{\partial x}(z) = \frac{\partial u}{\partial x}(z) + i \frac{\partial v}{\partial x}(z) \\ &= \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(z+ih) - f(z)}{ih} = \frac{1}{i} \frac{\partial f}{\partial y}(z) = -i \left(\frac{\partial u}{\partial y}(z) + i \frac{\partial v}{\partial y}(z) \right) \end{aligned}$$

Combine them together gives:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial v}{\partial y}$$

on U. Exercise: Since f is infinitely differentiable, we have $u, v \in C^\infty(U)$.

Lecture 33, 2024/04/01

Proposition 15.1 Let $U \subseteq \mathbb{C} \cong \mathbb{R}^2$ be open, connected, $u \in C^2(U)$ be $\mathbb{R}$-valued.

1. If there is $f \in \mathcal{H}(U)$ so that $u = \mathrm{Re}(f)$, then $u \in \mathrm{Har}(U)$.
2. If $u \in \mathrm{Har}(U)$, then define $g : U \to U$ by:

$$g(z) = \frac{\partial u}{\partial x}(z) - i \frac{\partial u}{\partial y}(z)$$

   then g is holomorphic on U.
3. If U is simply connected and $u \in \mathrm{Har}(U)$, then there is $f \in \mathcal{H}(U)$, such that $u = \mathrm{Re}(f)$.

Proof: (1). Let $f \in \mathcal{H}(U)$ and $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$. Then $u, v \in C^\infty(U) \subseteq C^2(U)$ as f is infinitely differentiable. We have, by Cauchy-Riemann:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial}{\partial y} \frac{\partial u}{\partial y} = \frac{\partial}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial}{\partial y} \frac{\partial v}{\partial x} \\ &= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0 \qquad \text{(Mixed Partials)} \end{aligned}$$

This proved (1).

(2). Write $\tilde{u} = \frac{\partial u}{\partial x}$ and $\tilde{v} = -\frac{\partial u}{\partial y}$, so:

$$g = \tilde{u} + i\tilde{v}$$

with $\tilde{u}, \tilde{v} \in C^1(U)$. Check Cauchy-Riemann for g:

$$\frac{\partial \tilde{u}}{\partial x} = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} = \frac{\partial \tilde{v}}{\partial y}$$

the second equality is by Laplacian equation $u_x^2 + u_y^2 = 0$. Also:

$$\frac{\partial \tilde{u}}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial \tilde{v}}{\partial x}$$

Thus g satisfies Cauchy-Riemann and then $g \in \mathcal{H}(U)$.

(3). By simple connectivity, the g defined in (2) admits a primitive $\tilde{f}$. Then:

$$\frac{\partial(\mathrm{Re}(\tilde{f}))}{\partial x} = \mathrm{Re}\left(\frac{\partial \tilde{f}}{\partial x}\right) = \mathrm{Re}(g) = \frac{\partial u}{\partial x}$$

The second equality is because this is partial of a $\mathbb{R}$-function. Also:

$$\frac{\partial(\mathrm{Re}(\tilde{f}))}{\partial y} = \mathrm{Re}\left(\frac{\partial \tilde{f}}{\partial y}\right) = -\mathrm{Re}(i\tilde{f}) = \frac{\partial u}{\partial y}$$

The second equality by the proof of the Cauchy-Riemann equations. Write $w = \mathrm{Re}(\tilde{f}) - u$, so:

$$\frac{\partial w}{\partial x} = 0 \quad \text{and} \quad \frac{\partial w}{\partial y} = 0$$

By A1 Q4, we see that $w = C$ for some constant C. Let $f = \tilde{f} - C$, and we get $\mathrm{Re}(f) = u$ as desired. $\square$

Definition If $u = \mathrm{Re}(f)$ for some $f \in \mathcal{H}(U)$, where $U \subseteq \mathbb{C}$ open. Then $v = \mathrm{Im}(f)$ is called the **harmonic conjugate** of f.

Example Let $u : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ by $u(x+iy) = \log(x^2+y^2)$. Note that $\mathbb{C} \setminus \{0\}$ is not simply connected. Restrict domain to $\mathbb{C} \setminus (-\infty, 0]$. This is star-like and we have a principal branch of logarithm here. Notice that:

$$u(x+iy) = \log(x^2+y^2) = 2\log|x+iy|$$

so $u = \text{Re}(2\text{Log})$, here Log is the principal branch of logarithm. Therefore:

$$u \in \text{Har}(\mathbb{C} \setminus (-\infty, 0])$$

We can compute its harmonic conjugate. $v(x+iy) = 2\theta$ if:

$$x + iy = |x+iy|e^{i2\theta}$$

where $\theta \in (-\pi, \pi)$. Then:

$$x + iy = |x+iy|(\cos(2\theta) + i\sin(2\theta))$$

It implies that:

$$\cos(2\theta) = \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad \sin(2\theta) = \frac{y}{\sqrt{x^2+y^2}}$$

Some cases here:

$$\begin{cases} x \neq 0 & \implies \frac{y}{x} = \tan(2\theta) \\ y \neq 0 & \implies \frac{x}{y} = \cos(2\theta) \end{cases}$$

and it follows that:

$$v(x+iy) = \begin{cases} 2\,\text{arccot}(x/y) & \text{if } y > 0 \\ 2\arctan(y/x) & \text{if } x > 0 \\ 2\,\text{arccot}(x/y) - \pi & \text{if } y < 0 \end{cases}$$

Corollary 15.2 Let $U \subseteq \mathbb{C}$ be open and connected, and $h \in \mathcal{H}(U)$ be non-constant and $u \in \text{Har}(h(U))$. Recall that $h(U)$ is open by Open Mapping Theorem. Then $u \circ h$ is harmonic on U.

Proof: Let $z_0 \in U$, and $R > 0$ with $D(h(z_0), R) \subseteq h(U)$. Note that $D(h(z_0), R)$ is simply connected. Hence by previous theorem, there is $f \in \mathcal{H}(D(h(z_0), R))$, so $u = \text{Re}(f)$. Then:

$$u \circ h = \text{Re}(f \circ h) \in \text{Har}(U)$$

As desired. $\square$

Proposition 15.3 (Mean Value Property) Let $U \subseteq \mathbb{C}$ be open and connected, and $u \in \text{Har}(U)$. If $z_0 \in U$ and $R > 0$ so $\overline{D}(z_0, R) \subseteq U$. Then we have:

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + Re^{i\theta})\, d\theta \tag{1}$$

$$u(z_0) = \frac{1}{\pi R^2}\iint\limits_{\overline{D}(z_0,R)} u(x+iy)\, d(x,y) \tag{2}$$

Proof: (1). Let $0 < \delta < \mathrm{dist}(\overline{D}(z_0, R), \mathbb{C} \setminus U)$, so:

$$\overline{D}(z_0, R) \subseteq \underbrace{D(z_0, R+\delta)}_{\text{simply connected}}$$

Hence there is $f \in \mathcal{H}(D(z_0, R), R+\delta)$ so $\mathrm{Re}(f) = u$. By Cauchy's Integral Formula:

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_{C(z_0,R)} \frac{f(z)}{z - z_0}\, dz \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{i\theta}) Rie^{i\theta}}{(z_0 + Re^{i\theta} - z_0)}\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta})\, d\theta \end{aligned}$$

Take the real part of both sides, done.

(2). By change of variable (polar coordinate), we have:

$$\begin{aligned} \iint\limits_{\overline{D}(z_0,R)} u(x+iy)d(x,y) &= \int_0^{2\pi} \int_0^R u(z_0 + re^{i\theta}) r\ dr\ d\theta \\ &= \int_0^R \left[\int_0^{2\pi} u(z_0 + re^{i\theta})\ d\theta \right] r\ dr \\ &= \int_0^R 2\pi u(z_0) r\ dr \qquad \text{(by (1))} \\ &= \pi R^2 u(z_0) \end{aligned}$$

As desired. □

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Theorem 15.4 (Liouville's Theorem for Harmonic Functions) Let $u \in \mathrm{Har}(U)$ be bounded. then u is a constant function.

Proof: Fix $z_0 \in \mathbb{C}$ and a large $R > 0$. Then:

$$\begin{aligned} u(z_0) - u(0) &= \frac{1}{\pi R^2} \iint\limits_{\overline{D}(z_0,R)} u(x+iy)\ d(x,y) - \frac{1}{\pi R^2} \iint\limits_{\overline{D}(0,R)} u(x+iy)\ d(x,y) \\ &= \frac{1}{\pi R^2} \iint_S u(x+iy)\ d(x,y) \end{aligned}$$

where S is the symmetric difference of $\overline{D}(z_0, R)$ and $\overline{D}(0, R)$, more explicitly:

$$\begin{aligned} S &= (\overline{D}(z_0, R) \setminus \overline{D}(0, R)) \cup (\overline{D}(0, R) \setminus \overline{D}(z_0, R)) \\ &= \overline{D}(z_0, R)\ \Delta\ \overline{D}(0, R) \subseteq \underbrace{\overline{D}(0, R + |z_0|) \setminus \overline{D}(0, R)}_{\text{annulus}} \cup A' \end{aligned}$$

Call the annulus A, and the area is:

$$\text{Area}(A) = \pi(R + |z_0|)^2 - \pi R^2 = 2\pi|z_0|R + \pi|z_0|^2$$

And the area of A' is the same as the area of A. Thus, if $M = \sup_{z\in\mathbb{C}} |u(z)| < \infty$ (by assumption), then:

$$\begin{aligned}
|u(z_0) - u(0)| &\leq \frac{1}{\pi R^2}\iint_S |u(x+iy)|\ d(x,y) \\
&\leq \frac{1}{\pi R^2}\iint_{A\cup A'} M\ d(x,y) = \frac{2\,\text{Area}(A)}{\pi R^2}\cdot M \\
&= \frac{2\pi|z_0|R + \pi|z_0|^2}{\pi R^2}\cdot 2M \to 0
\end{aligned}$$

As $R \to \infty$, therefore $u(z_0) = u(0)$, as desired. □

Theorem 15.5 (Maximum Principle) Let $U \subseteq \mathbb{C}$ be open and connected and $u \in \text{Har}(U)$. If u is non-constant, then u does NOT achieve its maximum (minimum) value inside the interior of U.

Proof: Let $M = \sup_{z\in U} u(z)$. If $M = \infty$ we are done. Thus assume $M \in \mathbb{R}$. Let:

$$V = \{z \in U : u(z) = M\} = u^{-1}(\{M\})$$

Note that $U \setminus V = u^{-1}(\mathbb{R}\setminus\{M\})$ is open as u is continuous. If there exists $z_0 \in V$, let $R > 0$ be so that $\overline{D}(z_0, R) \subseteq U$. Then:

$$\begin{aligned}
M = u(z_0) &= \frac{1}{\pi R^2}\iint\limits_{\overline{D}(z_0,R)} u(x+iy)\ d(x,y) \\
&\leq \frac{1}{\pi R^2}\iint\limits_{\overline{D}(z_0,R)} M\ d(x,y) = M
\end{aligned}$$

By something like Lemma 11.8, we have $u(x+iy) = M$ for all $x+iy \in \overline{D}(z_0, R)$, as u is continuous. Thus

$$z_0 \in V \implies D(z_0, R) \subseteq \overline{D}(z_0, R) \subseteq V$$

so V is open. Thus V is open and $U \setminus V$ is open, with $U = V \cup (U \setminus V)$. SInce U is connected, one of V or $U \setminus V$ is empty. Since u is non-constant, it forces $V = \emptyset$. We can replace u with $-u$ to see that the minimum is not achieved as well. □

Question: Could Mean Value Property $\implies$ Harmonic? (YES!)

Theorem 15.6 (Dirichlet Problem on Disc) Let $\beta : \partial\mathbb{D} \to \mathbb{R}$ be continuous, there exists $u \in C(\mathbb{D})$ (continuous on disc) so:

$$u|_{\partial\mathbb{D}} = \beta \quad \text{and} \quad u|_{\mathbb{D}} \in \mathrm{Har}(\mathbb{D})$$

The second equality is a PDE (Parital Differential Equation) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Proof: Idea: A plausible formula. If such u exists, then for $r \in (0,1)$:

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{it})\,dt \tag{$*$}$$

by the Mean Value Property. And we would like to show:

$$u(0) \to \frac{1}{2\pi}\int_0^{2\pi} \beta(e^{it})\,dt \tag{1}$$

as $r \to 1$. Accepting this, let for $z \in \mathbb{D}$, we have $\psi_z(w) = \frac{w-z}{1-\overline{z}w}$ with $\psi_z(0) = z$ and $\psi_z(z) = 0$. And $\psi_z^{-1} = \psi_z$, and $\psi_z(\mathbb{D}) = \mathbb{D}$. Thus, we can apply (1) to $u \circ \psi_z$:

$$u(z) = u \circ \psi_z(0) = \frac{1}{2\pi}\int_0^{2\pi} \beta(\underbrace{\psi_z(e^{it})}_{(2)})\,dt$$

For (2), inspect that $\psi_z(e^{i\theta}) \in \partial\mathbb{D}$ and $\theta \in \mathbb{R}$.

As suggest we define for $z \in \mathbb{D}$, the **Poisson Transformation of** β:

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \beta(\psi_z(e^{it}))\,dt$$

(I). Let use see that $u \in \mathrm{Har}(\mathbb{D})$. Change of variables, $z \in \mathbb{D}$ is fixed:

$$e^{i\theta} = \psi_z(e^{it}) \implies \psi_z(e^{i\theta}) = e^{it}$$

Take the derivative with respect to t on both side:

$$\psi_z'(e^{i\theta})ie^{i\theta} \cdot \frac{d\theta}{dt} = ie^{it}$$

then:

$$\psi_z'(w) = \frac{(\overline{z}w - 1) - (w - z)}{(\overline{z}w-1)^2} = \frac{|z|^2 - 1}{(\overline{z}w-1)^2}$$

and note that $e^{it} = \psi_z(e^{i\theta})$. Thus:

$$\psi_z'(e^{i\theta})\overline{\psi_z(e^{it})}e^{i\theta} \cdot \frac{d\theta}{dt} = 1$$

And:

$$\mathrm{LHS} = \frac{|z^2 - 1}{(\overline{z}e^{it}-1)^2} \cdot \frac{e^{-i\theta} - \overline{z}}{ze^{-i\theta} - 1} e^{i\theta} \cdot \frac{d\theta}{dt} = \frac{|z|^2 - 1}{|z - e^{i\theta}|^2} \cdot \frac{d\theta}{dt}$$

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Now, notice that:

$$\begin{aligned}\frac{z+e^{i\theta}}{z-e^{i\theta}} &= \frac{(z+e^{i\theta})(\overline{z}+e^{-i\theta})}{|z-e^{i\theta}|^2} \\ &= \frac{z\overline{z}+ze^{-i\theta}+\overline{z}e^{i\theta}+1}{|z-e^{i\theta}|^2} \\ &= \frac{|z|^2-1+i\,\mathrm{Im}(\overline{z}e^{i\theta})}{|z-e^{i\theta}|^2}\end{aligned}$$

Then, note that:

$$\frac{|z|^2-1}{|z-e^{i\theta}|^2} = \mathrm{Re}\left(\frac{z+e^{i\theta}}{z-e^{i\theta}}\right)$$

Thus, if $z \in \mathbb{D}$, we have:

$$\begin{aligned}u(z) &= \frac{1}{2\pi}\int_0^{2\pi} \beta(\psi_z(e^{it}))\ dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \beta(e^{i\theta})\frac{|z^2|-1}{|z-e^{it}|^2}\cdot\frac{d\theta}{dt}\cdot dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \beta(e^{i\theta})\,\mathrm{Re}\left(\frac{z+e^{i\theta}}{z-e^{i\theta}}\right)\ d\theta \\ &= \mathrm{Re}\underbrace{\left(\frac{1}{2\pi}\int_0^{2\pi} \beta(e^{it})\frac{z+e^{i\theta}}{z-e^{i\theta}}\ d\theta\right)}_{f(z)}\end{aligned}$$

Let us verify that $f \in \mathcal{H}(\mathbb{D})$. Let T be a triangle in $\mathbb{D}$, we consider:

$$\begin{aligned}\int_{\partial T} f(z)\ dz &= \frac{1}{2\pi}\int_{\partial T}\int_0^{2\pi} \beta(e^{i\theta})\frac{z+e^{i\theta}}{z-e^{i\theta}}\ d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \beta(e^{i\theta})\int_{\partial T}\frac{z+e^{i\theta}}{z-e^{i\theta}}\ dz\ d\theta = 0\end{aligned}$$

by Cauchy's Theorem, since $(z+e^{i\theta})/(z-e^{i\theta})$ is holomorphic. Thus by Morera's Theorem, $f \in \mathcal{H}(\mathbb{D})$. Hence $u = \mathrm{Re}(f) \in \mathrm{Har}(\mathbb{D})$.

(II). Let use consider $z \in \partial\mathbb{D}$, that is, $|z| = 1$.

$$\psi_z(e^{it}) = \frac{e^{it}-z}{1-\overline{z}e^{it}} = z\cdot\frac{e^{it}-z}{e^{it}-z} = z$$

unless $z = e^{it}$. And:

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \beta(\psi_z(e^{it}))\ dt = \frac{1}{2\pi}\int_0^{2\pi} \beta(z)\ dt = \beta(z)$$

As desired. $\square$

Definition Let $U \subseteq \mathbb{C}$ be open, we say $u : U \to \mathbb{R}$ has **Mean Value Property** if for any $z \in U$ and $r < 0$ so that $\overline{D}(z_0, r) \subseteq U$, then:

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it})\ dt$$

Recall that Harmonic $\implies$ MVP.

Lemma 15.7 Let $v : \overline{D} \to \mathbb{R}$ be continuous with:

1. v has the mean value property.
2. $v|_{\partial \mathbb{D}} = 0$.

Proof: By (the proof of) the maximum modulus principle, we see that v is non-constant, then v admits no maximum in $D(z, r)$. This violates assumption that $v|_{\partial \mathbb{D}} = 0$. So v must be constant on $\overline{D}$, that is, 0. $\square$

Theorem 15.8 Let $U \subseteq \mathbb{C}$ be connected and openb, and $u : U \to \mathbb{R}$ be continuous with MVP. Then u is harmonic on U.

Proof: (I). Suppose $\overline{\mathbb{D}} \subseteq U$, and we wil see that $u|_{\mathbb{D}} \in \text{Har}(\mathbb{D})$. Since u is continuous on $U \supseteq \overline{D} \supseteq \partial \mathbb{D}$, we know u is continuous on $\partial \mathbb{D}$. We consider the Poisson transform:

$$\tilde{u} = \frac{1}{2\pi} \int_0^{2\pi} u(\psi_z(e^{it}))\ dt$$

Then $\tilde{u}|_{\mathbb{D}} \in \text{Har}(\mathbb{D})$ and $\tilde{u}|_{\mathbb{D}} = u$ and $\tilde{u}$ is continuous on $\mathbb{D}$. Let $v = u - \tilde{u}$ and it satisfies $v|_{\partial \mathbb{D}} = 0$ and v has the mean value property. Lemma 15.7 tells us $v = 0$. That is, $u = \tilde{u}$ is harmonic on $\mathbb{D}$.

(II). Let $z_0 \in U$ and $R > 0$ so that $\overline{D}(z_0, R) \subseteq U$, we will show taht $u|_{D(z_0,R)}$ is harmonic. Hence u is harmonic on U. Let $\mu : \overline{\mathbb{D}} \to \overline{D}(z_0, R)$ by $\mu(z) = Rz + z_0$. If $z \in \mathbb{D}$ and $r \in (0, 1)$, we have:

$$\begin{aligned}\frac{1}{2\pi} \int_0^{2\pi} (u \circ \mu)(z + re^{it})\ dt &= \frac{1}{2\pi} \int_0^{2\pi} u\left(Rz + Rre^{it} + z_0\right)\ dt \\ &= u(Rz + z_0) = (u \circ \mu)(z)\end{aligned}$$

Thus $u \circ \mu \in \text{Har}(\mathbb{D})$, by (I). Then:

$$u = u \circ \mu \circ \mu^{-1} \in \text{Har}(D(z_0, R))$$

since μ^{-1} is holomorphic. $\square$