Chapter 1: Manifolds.

Lec 1

§ 1.1 Coordinate Charts

The concept of manifold starts with the notion of coordinate charts.

Def: A coordinate chart on a set X is a subset $U \subseteq X$ together with a bijection $\varphi: U \to \varphi(U) \underset{\text{open}}{\subseteq} \mathbb{R}^n$ onto an open subset of $\mathbb{R}^n$

This allows us to parametrize points $x \in U$ by n coordinates.

$$\varphi(x) = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

EX: (1). $X = \mathbb{R}^n$, $U = X = \mathbb{R}^n$. $\varphi = id$

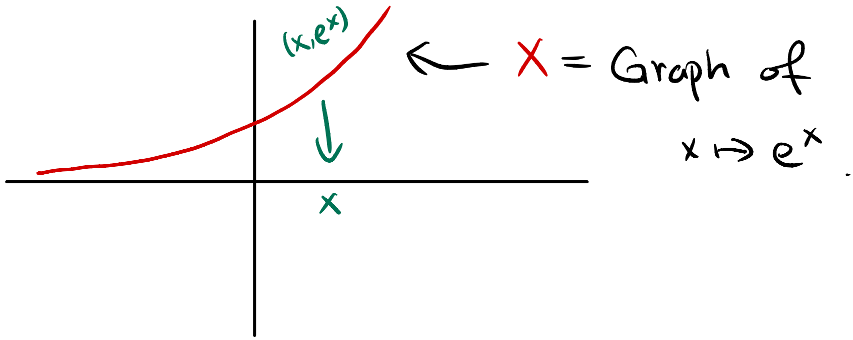
Also, for any $U \subseteq \mathbb{R}^n$ open, can take $\varphi: U \to U$ by $x \mapsto x$.

(2). For $D \underset{\text{open}}{\subseteq} \mathbb{R}^n$ and $f: D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ a map (not necc cts)

Set $X = \{(x, f(x)) : x \in U\} \subseteq \mathbb{R}^n \times \mathbb{R}^m$

the graph of f.

E.g: $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = e^x$

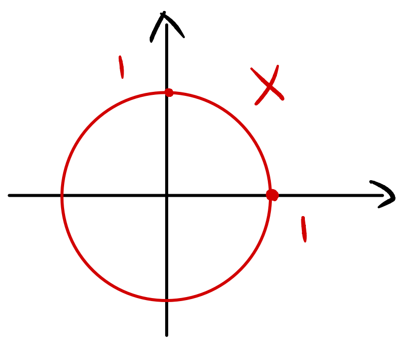


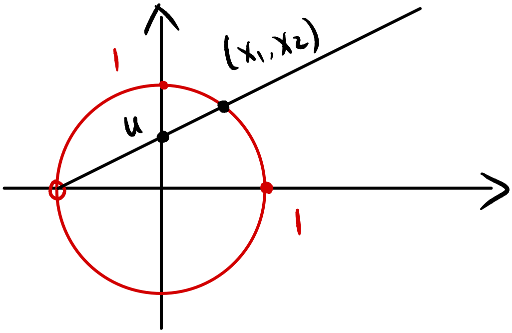
Set $\varphi : X \to D \subseteq \mathbb{R}^n$ by $(x, y) \mapsto x$

This is a coordinate chart.

(3). Let $X = S^1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$

$= \text{unit circle in } \mathbb{R}^2$

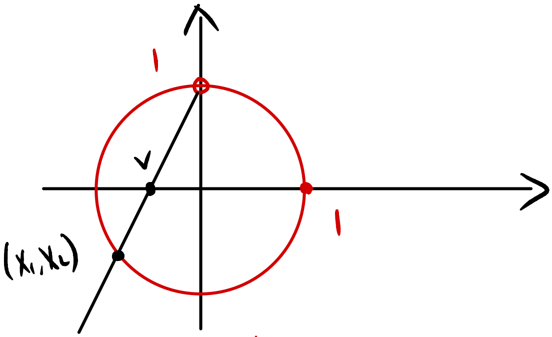




$\varphi_1 : U_1 \to \mathbb{R}$

$$(x_1, x_2) \longmapsto \frac{x_2}{x_1 + x_2} = u$$

$U_1 = S^1 \setminus \{(-1,0)\}$

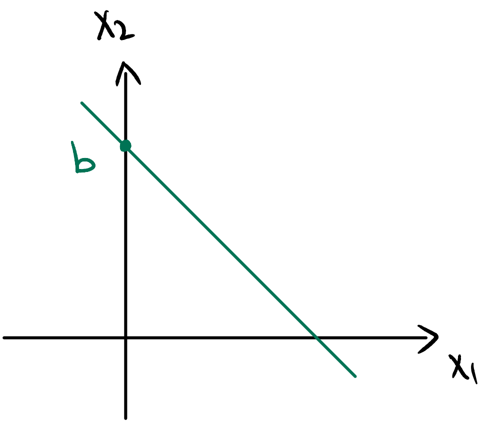


$\varphi_2 : U_2 \to \mathbb{R}$

$$(x_1, x_2) \longmapsto \frac{x_1}{1 - x_2} = v$$

$U_2 = S^1 \setminus \{(0,1)\}$

(4). Let $X = \{ \text{straight lines in } \mathbb{R}^2 \}$



Non-vertical line are of the form :

$$x_2 = \underbrace{m}_{\text{slope}} x_1 + \underbrace{b}_{\text{y-intercept}}$$

Set $U_1 = \{\text{non-vertical lines}\}$

$$= \{x_2 = m x_1 + b : m, b \in \mathbb{R}\}$$

$\varphi_1 : U_1 \to \mathbb{R}^2$ by $x_2 = m x_1 + b \mapsto (m, b)$

Set $U_2 = \{\text{non-horizontal lines}\}$

$$= \{x_1 = \underset{\substack{\uparrow \\ \text{slope}}}{\tilde{m}} x_2 + \underset{\substack{\uparrow \\ x\text{-intercept}}}{\tilde{b}} : \tilde{m}, \tilde{b} \in \mathbb{R}\}$$

$\varphi_2 : U_2 \to \mathbb{R}^2$ by $x_1 = \tilde{m} x_2 + \tilde{b} \mapsto (\tilde{m}, \tilde{b})$

As we have seen, a set X can be covered by more than one chart. We want these charts to satisfy some consistency conditions.

<u>Def</u>: An n-dimensional (smooth) atlas on X is a collection of coordinate charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ such that

(i). X is covered by U_α's : $X = \bigcup_{\alpha \in A} U_\alpha$

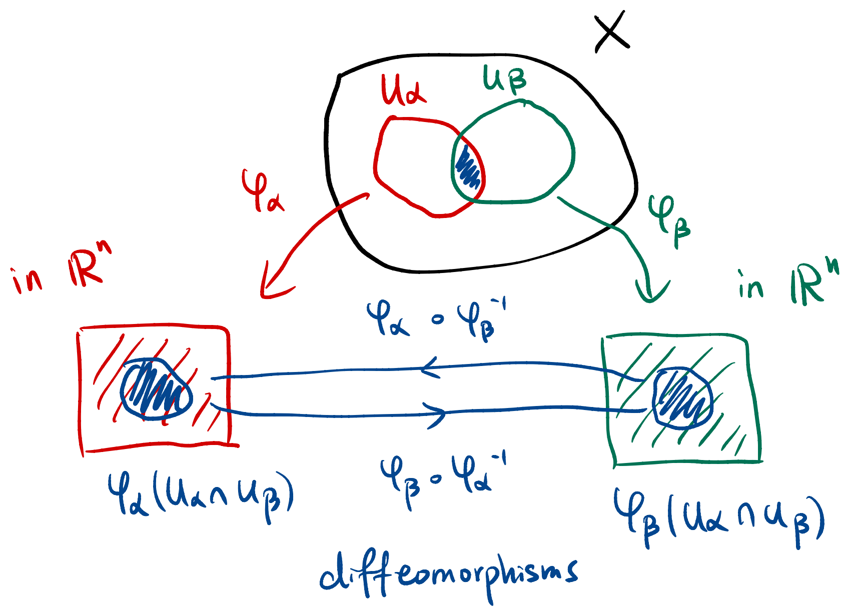
$$\varphi_\alpha : U_\alpha \longrightarrow \varphi_\alpha(U_\alpha) \underset{\text{open}}{\subseteq} X$$

(ii). $\forall \alpha, \beta \in A$, $\varphi_\alpha(U_\alpha \cap U_\beta) \underset{\text{open}}{\subseteq} \mathbb{R}^n$

(iii). $\forall \alpha, \beta \in A$, the map

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \underset{\substack{\cap \\ \mathbb{R}^n}}{\varphi_\beta(U_\alpha \cap U_\beta)} \longrightarrow \underset{\substack{\cap \\ \mathbb{R}^n}}{\varphi_\alpha(U_\alpha \cap U_\beta)}$$

is smooth ($\mathcal{C}^\infty$)



Remark: (1). Since the indices are running through all $\alpha, \beta \in A$, we in fact have

$$\varphi_\alpha \circ \varphi_\beta^{-1} \text{ is a diffeomorphism.}$$

(2). The maps $\varphi_\alpha \circ \varphi_\beta^{-1}$ are change of coordinate functions.

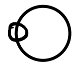
EX: (1). $X = \mathbb{R}^n$, $\varphi = id : X \to X$

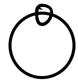
Then $\{(X, \varphi)\}$ is a n-dim smooth atlas

OR take $U \subseteq \mathbb{R}^n$ open, then $\varphi : U \xrightarrow{id} U$ gives a n-dim smooth atlas on U.

(2). In general, if X can be covered by one chart $\varphi : X \to \varphi(U) \subseteq \mathbb{R}^n$, then $\{(X, \varphi)\}$ is an n-dim smooth atlas.

Eg: Graph of fcns admit smooth atlases.

(3). $X = S^1$, $\varphi_1 : U_1 =$  $\to \mathbb{R}$

$\varphi_2 : U_2 =$  $\to \mathbb{R}$

is a 1-dim smooth atlas. (Exercise)

Def: A n-dim (smooth) manifold is a set that admits an n-dim (smooth) atlas.

Lec 2

Recall: A chart on X is a subset $U \subseteq X$ and a bijection $\varphi : U \to \varphi(U) \underset{\text{open}}{\subseteq} \mathbb{R}^n$

A smooth atlas is a collection $\{(U_\alpha, \varphi_\alpha)\}$ of charts of X such that

(i). $X = \bigcup_\alpha U_\alpha$

(ii). $\forall \alpha, \beta \in A$, $\varphi_\alpha(U_\alpha \cap U_\beta)$ open in $\mathbb{R}^n$

(iii). $\varphi_\alpha(U_\alpha \cap U_\beta) \underset{\text{smooth}}{\rightleftarrows} \varphi_\beta(U_\beta \cap U_\alpha)$

Ex: (3) from last lecture

$$\varphi_1(U_1 \cap U_2) = \mathbb{R} \setminus \{x_2 = 1\} \quad \text{open}$$

$$\varphi_2(U_1 \cap U_2) = \mathbb{R} \setminus \{x_1 = -1\} \quad \text{open}$$

Finally, $\varphi_1 \circ \varphi_2^{-1} : \varphi_2(U_1 \cap U_2) \to \varphi_1(U_1 \cap U_2)$

$$v \longmapsto \frac{v-1}{(v+1)^2} = u$$

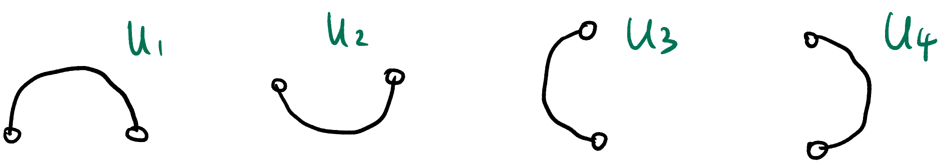
(diffeom)

is a smooth map and $\varphi_2 \circ \varphi_1^{-1}$ as well.

This checked $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ smooth atlas.

<u>other atlas</u>: $X = S^1 = \{x^2 + y^2 = 1\}$

four portions of X are graphs of some functions



$y = \sqrt{1-x^2}$ $\quad$ $y = -\sqrt{1-x^2}$ $\quad$ $x = -\sqrt{1-y^2}$ $\quad$ $x = \sqrt{1-y^2}$

then use four charts

(4). $X = \{ \text{lines in } \mathbb{R}^2 \}$, two charts

$U_1 = \{ \text{non-vertical lines} \}$

$\varphi_1 : U_1 \to \mathbb{R}^2, \quad y = mx + b \mapsto (m, b)$

$U_2 = \{ \text{non-horizontal lines} \}$

$\varphi_2 : U_2 \to \mathbb{R}^2, \quad x = m'y + b' \mapsto (m', b')$

This is a smooth atlas (exercise).

(5). Real Projective Space : $\mathbb{RP}^n$

$\mathbb{RP}^n = \{ \text{set of 1-dimensional subspaces of } \mathbb{R}^{n+1} \}$

as a vsp over $\mathbb{R}$

$= \{ \text{lines through the origin in } \mathbb{R}^{n+1} \}$

Let $L \in \mathbb{RP}^n$ be a line in $\mathbb{R}^{n+1}$. Since $0 \in L$, all we need to completely describe

L is a direction vector $0 \neq v \in \mathbb{R}^{n+1}$.

$\Rightarrow L = L_v =$ line through origin in direction $0 \neq v \in \mathbb{R}^{n+1}$

Note that $L_v = L_{\lambda v}$ for any $\lambda \in \mathbb{R}^*$

Altas: Set $U_i = \{ L_v \in \mathbb{RP}^n : v_i \neq 0 \}$

independent of scaling by $\lambda \neq 0$

For any $L \in U_i$,

$$L = L_v = L_{\frac{1}{v_i} v} \qquad \frac{1}{v_i} v \longrightarrow \left(\frac{v_0}{v_i}, \dots, 1, \dots, \frac{v_n}{v_i} \right) \in \mathbb{R}^n$$

So we can choose the direction vector $\frac{1}{v_i} v$ s.t the i-th coord $= 1$ and this is unique.

$$\varphi_i : U_i \longrightarrow \mathbb{R}^n \cong \{ x \in \mathbb{R}^{n+1} : x_i = 1 \}$$

$$L_v \longmapsto \frac{1}{v_i} v$$

This is a bijection.

Note that $\mathbb{RP}^n = U_1 \cup \cdots \cup U_n$

because directional vector $\neq 0 \Rightarrow$ at least

one of coordinate $\neq 0$.

$$\varphi_i(U_i \cap U_j) \subseteq \mathbb{R}^n \cong \{x \in \mathbb{R}^{n+1} : x_i = 1\}$$

$$L_v \longmapsto \frac{1}{v_i} v = x, \quad \begin{array}{l} x_i = 1 \\ \underline{x_j \neq 0} \end{array}$$

$\Rightarrow$ open

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

$$x \longmapsto \frac{1}{x_i} x = y$$

$$(x_j = 1, x_i \neq 0) \qquad (y_i = 1, y_j \neq 0)$$

This is smooth ✓

Hence $\{(U_i, \varphi_i) : 1 \leq i \leq n\}$ is

a smooth atlas for $\mathbb{RP}^n$.

§ 1.2 Manifolds

A manifold is a set that admits a smooth atlas. But, to have a good notion of manifold, it cannot depend on the choice of atlas.

Def: Two atlases $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ and $\{(V_i, \psi_i)\}_{i \in I}$ are compatible if their union is still an atlas.

Rmk: In particular,

$$\varphi_\alpha \circ \psi_i^{-1} : \psi_i(V_i \cap U_\alpha) \to \varphi_\alpha(V_i \cap U_\alpha)$$

is a diffeomorphism for all α and i

Also, compatibility is an equivalence relation (exercise)

Def: A smooth or differentiable structure on X is an equivalence class of atlases.

Def: A smooth manifold is a set together with a smooth structure.

Rmk: (1). To show a set X is a manifold, we just need to construct one atlas. If this atlas is n-dimensional, then X is an n-dimensional manifold.

(2). Given a smooth structure on a set X, the maximal atlas consists of the union of all the charts in the equiv. class.

Lec 3

Many books define a manifold as a top space that can be covered by charts that are "smoothy compatible" ((iii) of the def of manifold)

In fact, the atlas $\{(U_\alpha, \varphi_\alpha)\}$ induces a natural topology on X.

Ex: Let τ be a topology on X and $A \subseteq X$. Set $\tau_A = \{A \cap U : U \in \tau\}$ is a topology on A, called the relative topology on A. The closed sets in A are of the form $A \cap F$ with $F \subseteq X$ closed in X.

manifold

Suppose M is a smooth n-dim mfld. Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ be a smooth atlas in the smooth structure of M.

Def: A subset $V \subseteq M$ is said to be open if $\varphi_\alpha(V \cap U_\alpha) \underset{\text{open}}{\subseteq} \mathbb{R}^n$ for all $\alpha \in A$

Claim: (1). U_α is open in M $\forall \alpha \in A$

(2). these "open sets" define a topology on M.

<u>Proof</u> : (1). Let $\alpha \in A$, then

$$\varphi_\beta(U_\beta \cap U_\alpha) \subseteq \mathbb{R}^n \text{ is open}$$

by the def of smooth atlas.

(2). Check this is a topology.

ϕ is open b/c $\varphi_\alpha(\phi) = \phi$ is open.

M is open b/c $\varphi_\alpha(U_\alpha \cap M) = \varphi_\alpha(U_\alpha)$ open.

Now let $\{V_i\}_{i \in I}$ be a collection of open sets in M. So that

$$\varphi_\alpha(U_\alpha \cap V_i) \underset{\text{open}}{\subseteq} \mathbb{R}^n \quad \forall \alpha, i$$

Then for any $\alpha \in A$:

$$\varphi_\alpha\Big(\bigcup_{i \in I} V_i \cap U_\alpha\Big) = \underbrace{\bigcup_{i \in I} \overbrace{\varphi_\alpha(V_i \cap U_\alpha)}^{\text{open}}}_{\text{open in } \mathbb{R}^n}$$

Similarly for the finite intersections $\boxed{/\!/}$

<u>Prop</u>: The bijection $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$ are homeomorphisms with respect to this topology on M (and the standard top)

<u>Proof</u>: Need to check φ, φ^{-1} are continuous.

* <u>φ_α^{-1} is cts</u> $\varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to U_\alpha \subseteq M$

  Let $V \subseteq M$ be open, then

  $$(\varphi_\alpha^{-1})^{-1}(V \cap U_\alpha) = \varphi_\alpha(V \cap \alpha) \text{ is } \underset{\text{by def}}{\text{open}}$$

* <u>φ_α is cts</u> $\varphi_\alpha : U_\alpha \subseteq M \to \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$

  Let $W \subseteq \mathbb{R}^n$ be open, need to check $\varphi_\alpha^{-1}(W)$ is open in M. This means

  $$\varphi_\beta(\varphi_\alpha^{-1}(W) \cap U_\beta) \underset{\text{open}}{\subseteq} \mathbb{R}^n \quad \forall \beta \in A$$

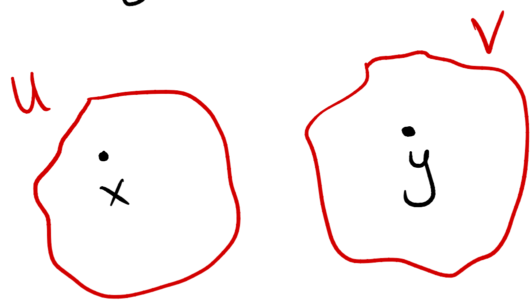
  $$\|$$

  $$\underbrace{(\varphi_\beta \circ \varphi_\alpha^{-1})}_{\text{a homeo}} \underbrace{(\underset{\text{open}}{W} \cap \underbrace{\varphi_\alpha(U_\beta \cap U_\alpha)}_{\text{open by def}})}_{\text{open in } \mathbb{R}^n} \underset{\text{open}}{\subseteq} \mathbb{R}^n$$

$\blacksquare$

In practice, we want the topology to be Hausdorff and second countable.

<u>Def</u>: A topology τ on X is Hausdorff if $\forall x \neq y \in X$, $\exists\, U, V \in \tau$ s.t $U \cap V = \phi$ and $x \in U$ and $y \in V$.



<u>Def</u>: Let τ be a topology on X. A basis $\mathcal{B}$ of τ is a collection of open sets such that

(i). $\forall x \in X$, $\exists\, V \in \mathcal{B}$ s.t $x \in V$

(ii). if $x \in U \cap V$ for $U, V \in \mathcal{B}$, there exists $W \in \mathcal{B}$ s.t $x \in W \subseteq U \cap V$

The elts of $\mathcal{B}$ are the building blocks of the topology τ.

One can show that one can recover the topology τ from $\mathcal{B}$. (exercise)

<u>Ex</u>: If $X = \mathbb{R}^n$ and τ = standard top.

$\mathcal{B} = \{$ open balls in $\mathbb{R}^n \}$ is a basis.

<u>Ex</u>: Given an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ of the mfld M, we have $\mathcal{B} = \{U_\alpha\}_{\alpha \in A}$ is a basis on the topology on M.

<u>Def</u>: A topology on X is called <u>second countable</u> if it admits a countable basis

We want the natural topology on M to also be Hausdorff and 2nd countable.

This is true if the atlas satisfies the following two more conditions

* Countably many of U_α cover M **2nd Countable**

* $\forall p, q \in M$ with $p \neq q$, then $p, q \in U_\alpha$ for some $\alpha \in A$ **OR** $\exists \alpha, \beta \in A$ with $U_\alpha \cap U_\beta = \phi$ and $p \in U_\alpha$, $q \in U_\beta$. **Haus**