PMATH 351 Notes

Real Analysis

Winter 2025

Based on Professor Kevin Hare's Lectures

Contents

Lecture 1, 2025/01/06

1 Metric Spaces

1.1 Normed Vector Spaces

Definition. Let V be a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We say $\|\cdot\| : V \to \mathbb{R}$ is a **norm** if:

(i). For all $v \in V$ we have $\|v\| = 0 \iff v = 0$.

(ii). For all $v \in V$ and $\lambda \in \mathbb{K}$ we have $\|\lambda v\| \leq |\lambda| \|v\|$.

(iii). For all $v, w \in V$ we have $\|v + w\| \leq \|v\| + \|w\|$.

A vector space, combined with a norm, is called a **normed vector space**.

Example. Let $V = \mathbb{R}^n$. Define a map $\|\cdot\|_1 : \mathbb{R}^n \to \mathbb{R}$ by:

$$\|v\|_1 = \|(x_1, \cdots, x_n)\|_1 = |x_1| + \cdots + |x_n|$$

Clearly property 1 and 2 holds. To see property 3 we have:

$$\begin{aligned}
\|(x_1, \cdots, x_n) + (y_1, \cdots, y_n)\|_1 &= |x_1 + y_1| + \cdots + |x_n + y_n| \\
&\leq |x_1| + |y_1| + \cdots + |x_n| + |y_n| && (\triangle \text{ inequality in } \mathbb{R}) \\
&= \|(x_1, \cdots, x_n)\|_1 + \|(y_1, \cdots, y_n)\|_1
\end{aligned}$$

Hence $\|\cdot\|_1$ defines a norm on $V = \mathbb{R}^n$.

Example. Let $V = \mathbb{R}^n$ again. Define $\|\cdot\|_\infty : \mathbb{R}^n \to \mathbb{R}$ by:

$$\|v\|_\infty = \|(x_1, \cdots, x_n)\|_\infty = \max(|x_1|, \cdots, |x_n|)$$

This also defines a norm on $\mathbb{R}^n$.

Example. What does the unit ball $B = \{v \in V : \|v\| \leq 1\}$ look like? Take $V = \mathbb{R}^2$.

Note. It is possible to extend these two norms to infinite dimensional vector spaces if we are being careful. Both of the norms above are examples of p-norms, for $1 \leq p \leq \infty$.

Example. Let $V = \mathbb{R}[x]$ be a vector over $\mathbb{R}$. Define $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on V by:

$$\|f|_1 = \int_0^1 |f(x)| \, \mathrm{d}x \quad \text{and} \quad \|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$$

The three properties are satisfied by these two norms. Note these norms can be defined beyond polynomials if we are careful.

Theorem 1.1 (Minkowski). Let $1 \leq p < \infty$ be a real number.

(i). We define:

$$\ell_p = \left\{ (x_n)_{n=1}^{\infty} \subseteq \mathbb{C} : \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} < \infty \right\}$$

Then the map $\|\cdot\|_p : \ell_p \to \mathbb{R}$ defined by:

$$\|(x_n)\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}$$

defines a norm on ℓ_p. This is called the ℓ_p**-space**.

(ii). Let $\mathcal{C}[a, b]$ be the set of continuous functions on $[a, b]$. Then:

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, \mathrm{d}x \right)^{1/p}$$

defines a norm. Define $L^p[a, b] = \{f \in \mathcal{C}[a, b] : \|f\|_p < \infty\}$, called the L^p**-space**.

Proof. Note for $p \geq 1$, define a map $\varphi(x) = |x|^p$ and φ is convex on $\mathbb{R}$. We will prove part 2 first. Assume $f, g \in \mathcal{C}[a, b]$ and $f, g \neq 0$. If $f = 0$ or $g = 0$ the triangle inequality is easy to prove.

$$\begin{aligned}
\|f + g\|_p^p = \int_a^b |f(x) + g(x)|^p \, \mathrm{d}x &= \int_a^b \left| \|f\|_p \cdot \frac{f}{\|f\|_p} + \|g\|_p \cdot \frac{g}{\|g\|_p} \right|^p \mathrm{d}x \\
&= (\|f\|_p + \|g\|_p)^p \int_a^b \left| \underbrace{\frac{\|f\|_p}{\|f\|_p + \|g\|_p}}_{\alpha} \cdot \frac{f}{\|f\|_p} + \underbrace{\frac{\|g\|_p}{\|f\|_p + \|g\|_p}}_{1-\alpha} \cdot \frac{g}{\|g\|_p} \right|^p \mathrm{d}x
\end{aligned}$$

Note that $\alpha \in [0, 1]$, we can rewrite the above quantity as:

$$\begin{aligned}
I &:= (\|f\|_p + \|g\|_p)^p \int_a^b \left| \alpha \cdot \frac{f}{\|f\|_p} + (1 - \alpha) \cdot \frac{g}{\|g\|_p} \right|^p \mathrm{d}x \\
&= (\|f\|_p + \|g\|_p)^p \int_a^b \varphi \left(\alpha \cdot \frac{f}{\|f\|_p} + (1 - \alpha) \cdot \frac{g}{\|g\|_p} \right)^p \mathrm{d}x
\end{aligned}$$

Recall $\varphi(x) = |x|^p$ is convex, we have:

$$\begin{aligned}
I &\leq (\|f\|_p + \|g\|_p)^p \left(\alpha \int_a^b \left| \frac{f}{\|f\|_p} \right|^p \mathrm{d}x + (1 - \alpha) \int_a^b \left| \frac{g}{\|g\|_p} \right|^p \mathrm{d}x \right) \\
&= (\|f\|_p + \|g\|_p)^p (\alpha + 1 - \alpha) = (\|f\|_p + \|g\|_p)^p
\end{aligned}$$

This proved that:

$$\|f+g\|_p^p \leq (\|f\|_p + \|g\|_p)^p \implies \|f+g\|_p \leq \|f\|_p + \|g\|_p$$

Part 1 (ℓ_p-space) are proved in the similar way by replacing integral with sum. □

Lecture 2, 2025/01/08

1.2 Metric Spaces

Definition. Let X be a non-empty set. A **distance (metric)** on X is a function $d : X \times X \to [0, \infty)$ such that:

(i). For all $x, y \in X$ we have $d(x, y) = 0 \iff x = y$.

(ii). For all $x, y \in X$ we have $d(x, y) = d(y, x)$.

(iii). For all $x, y, z \in X$ we have $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (X, d) is called a **metric space**. We just say X is a metric space if d is understood.

Example. Let $(X, \|\cdot\|)$ be a normed vector space, then $d(x, y) = \|x - y\|$ is a metric on X. Clearly $d(x, y) = 0 \iff \|x - y\| = 0 \iff x - y = 0 \iff x = y$. Property (ii) is also trivial. For property (iii) we have:

$$d(x, z) = \|x - z\| = \|x - y + y - z\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z)$$

Example (Graph metric). Let (X, E) be a graph where X is the vertex set. The set of **paths** from x to y is:

$$P_{xy} = \{(x = x_1, x_2, \cdots, x_n = y) : (x_i, x_{i+1}) \in E\}$$

Define a **weight** function $\omega : E \to (0, \infty)$. Then:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ \min\{\omega(x_1, x_2) + \cdots + \omega(x_{n-1}, x_n) \text{ for } (x_1, \cdots, x_n) \in P_{xy}\} & \text{otherwise} \end{cases}$$

This distance basically measures the shortest path from x to y, with weight on the edge.

Example (Trivial metric). Let X be a non-empty set, define:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Exercise: It is easy to verify that this is a distance function on X.

Example (p-adic metric on $\mathbb{Q}$). Let p be a fixed prime in $\mathbb{N}$. By unique factorization, every $q \in \mathbb{Q}$ can be uniquely written as:

$$q = p^n \frac{a}{b}$$

where $n \in \mathbb{Z}$ and $a \in \mathbb{Z}$ and $0 \neq b \in \mathbb{N}$ with $\gcd(a, b) = 1$. Define the **p-adic norm** by:

$$|q|_p = \begin{cases} p^{-n} & \text{if } q \neq 0 \text{ and } n \text{ is from above} \\ 0 & \text{if } q = 0 \end{cases}$$

Exercise: For $q, r \in \mathbb{Q}$ we have:

$$|q + r|_p \leq \max\{|q|_p,\ |r|_p\} \leq |q|_p + |r|_p$$

Take $p = 3$ and $q = 1/6$ and $r = 2/9$, then:

$$\begin{aligned} |q|_3 &= \left|3^{-1} \cdot \frac{1}{2}\right|_3 = 3^{-(-1)} = 3 \\ |r|_3 &= \left|3^{-2} \cdot \frac{2}{1}\right|_3 = 3^{-(-2)} = 9 \\ |q + r|_3 &= \left|\frac{3+4}{18}\right|_3 = \left|3^{-2} \cdot \frac{7}{2}\right|_3 = 9 = \max\{3, 9\} \end{aligned}$$

Define the **p-adic metric** on $\mathbb{Q}$ by:

$$d_p(q, r) = |q - r|_p$$

Exercise: This clearly defined a metric on $\mathbb{Q}$.

Example. Consider $\{0, 1\}^{\mathbb{N}} = \{(b_n)_{n=1}^{\infty} : b_n \in \{0, 1\}\}$. Take $b, c \in \mathbb{N}$ then define:

$$d(b, c) := \begin{cases} 0 & \text{if } b = c \\ \frac{1}{2^n} \text{ for } b = \min\{i \in \mathbb{N} : b_i \neq c_i\} & \text{otherwise} \end{cases}$$

Exericse: d is a metric on $\{0, 1\}^{\mathbb{N}}$, we may call this product metric. Now we define:

$$\rho(b, c) = \sum_{n=1}^{\infty} \frac{d(b_n, c_n)}{2^n} \qquad \text{(always converges)}$$

Fact (Exercise): $d(b, c) \leq \rho(b, c) \leq 2d(b, c)$.

Definition. Let (X, d) be a metric space. If $\emptyset \neq Y \subseteq X$, we make Y a metric space by defining $d_Y : Y \times Y \to \mathbb{R}$ by $d_Y(x, y) = d(x, y)$ for $x, y \in Y$. [This is just the restriction $d|_{Y \times Y}$] This is called the **relativized metric** on Y.

Definition. Let X be a non-empty set and d_1, d_2 be metrics on X. We say d_1 is **equivalent** to d_2 if there exist $c, C > 0$ such that:

$$cd_1(x,y) \leq d_2(x,y) \leq Cd_1(x,y)$$

for all $x, y \in X$. Exercise: This is an equivalence relation on the set of metrics on X.

Example. Let $X = \mathbb{R}^n$ and $1 \leq p < \infty$. Define a metric:

$$d_p(x,y) = \|x-y\|_p = \left(\sum_{k=1}^{n} |x_i - y_i|^p\right)^{1/p}$$

and define $d_\infty(x,y) = \max\{|x_k - y_k| : k \in \{1, \cdots, n\}\}$. Let $x \in \mathbb{R}^n$, say $\|x\|_\infty = x_j$ for some j. Then we note that:

$$\|x\|_\infty = |x_j| = (|x_j|^p)^{1/p} \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} = \|x\|_p \leq \left(\sum_{k=1}^{n} |x_j|^p\right)^{1/p} = n^{1/p}\|x\|_\infty$$

To summarize we have:

$$\|x\|_\infty \leq \|x\|_p \leq n^{1/p}\|x\|_\infty$$

Hence $\|\cdot\|_\infty$ and $\|\cdot\|_p$ are equivalent norms for all $1 \leq p < \infty$. By equivalence, $\|\cdot\|_p$ are all equivalent norms on $\mathbb{R}^n$ for $1 \leq p \leq \infty$.

Lecture 3, 2025/01/10

1.3 Topology of Metric Spaces

Definition. Let (X,d) be a metric space. Take $x \in X$ and $r > 0$. Define an **open ball** centered at x with radius r to be:

$$B_r(x) := b_r(x) := B(x,r) := \{y \in X : d(x,y) < r\}$$

Similarly we define a **closed ball** as:

$$\overline{B}_r(x) := \overline{b}_r(x) := \overline{B}(x,r) := \{y \in X : d(x,y) \leq r\}$$

Definition. Let (X,d) be a metric space. Let $N \subseteq X$ with some $x \in X$. We say N is a **neighborhood** of x if there exists $r > 0$ such that $B_r(x) \subseteq N$.

Definition. Let (X,d) be a metric space. We say $N \subseteq X$ is **open** if N is a neighborhood of x for all $x \in N$. We say N is **closed** if $X \setminus N$ is open.

Example. Let $X = \mathbb{R}$ with usual Euclidean metric. Then (a, b) is open for all $a < b$ in $\mathbb{R}$. The empty set $\emptyset$ and $\mathbb{R}$ are open.

Remark. In general, in a metric space (X, d), the set X is trivially open and the empty set $\emptyset$ is vacuously open. Note that $X \setminus X = \emptyset$ and $X \setminus \emptyset = X$. Hence $X, \emptyset$ are both open and closed.

Example. Let (X, d) be a metric space where d is the discrete metric. Every subset $N \subseteq X$ is open! Why? Take $r = 1/2$ and $x \in N$, then $B_{1/2}(x) = \{x\} \subseteq N$. Similarly every subset is closed.

Question: Consider the metric space $(\mathbb{Q}, d_3)$, where d_3 is the 3-adic metric. What do the open sets look like?

Theorem 1.2 (Union of Open sets). Let (X, d) be a metric space. Let $\{X_i\}_{i \in I}$ be a collection of open sets, then $\bigcup_{i \in I} X_i$ is an open set.

Proof. Let $x \in \bigcup_{i \in I} X_i$, then $x \in X_{i_0}$ for some $i_0 \in I$. Since X_{i_0} is open, there is $r > 0$ such that $B_r(x) \subseteq X_{i_0}$. Hence:

$$B_r(x) \subseteq X_{i_0} \subseteq \bigcup_{i \in I} X_i$$

It follows that $\bigcup_{i \in I} X_i$ is open, as desired. $\square$

Corollary 1.3 (Intersection of Closed sets). Let (X, d) be a metric space and $\{X_i\}_{i \in I}$ a collection of closed sets. Then $\bigcap_{i \in I} X_i$ is closed.

Proof. Take complement using De Morgan's Law and apply the above theorem. $\square$

Question: If $\{X_i\}_{i \in I}$ is a collection of open sets, waht can we say about $\bigcap_{i \in I} X_i$?

(i). If $|I| = n < \infty$, then this intersection is open. Consider $\{X_1, \cdots, X_n\}$. Take $x \in \bigcap_{i=1}^n X_i$, then $x \in X_i$ for all i, so there is $r_i > 0$ such that $B(x, r_i) \subseteq X_i$ for all i. Take $r = \min\{r_1, \cdots, r_n\}$, then $B(x, r) \subseteq \bigcap_{i=1}^n X_i$, hence open.

(ii). If $|I| > |\mathbb{N}|$, this may fail. For example, take $X_n = (\frac{-1}{n}, \frac{1}{n})$. Then $\bigcap_{n=1}^\infty X_n = \{0\}$, not open.

Proposition 1.4. Finite intersection of open sets is open and finite union of closed sets is closed.

Definition. Let (X, d) be a metric sapce and $(x_n)_{n=1}^\infty$ be a sequence in X. Let $x \in X$. We say the sequence $(x_n)_{n=1}^\infty$ **converges** to x if $\lim_{n \to \infty} d(x, x_n) = 0$. Equivalently, for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$:

$$n \geq N \implies d(x, x_n) < \epsilon$$

In this case we can write $\lim_{n \to \infty} x_n = x$.

Example. Let $X = \mathbb{Q}$ with Euclidean metric. Let $(x_n)_{n=1}^{\infty}$ be $x_n = 1/n$. This converges to 0.

Example. Let $X = \mathbb{Q}$, consider the sequence defined by:

$$x_n = \text{truncation of } \pi \text{ to the } n\text{-th decimal place}$$

For example $x_1 = 3,\ x_2 = 3.1,\ x_3 = 3.14$ and so on. This sequence "converges" to π, but $\pi \notin \mathbb{Q}$ so this sequence does not converge in $\mathbb{Q}$! It converges in $\mathbb{R}$.

Example. Let (X, d) with the discrete metric. A sequence (a_n) is convergent if and only if it is eventually constant. That is, there is $N \in \mathbb{N}$ such that $x_n = X_N$ for all $n \geq N$. In this case the limit is just $\lim\limits_{n\to\infty} x_n = X_N$.

Example. Consider $(\mathbb{Q}, d_3)$, the 3-adic metric. Consider the two sequences:

$$(x_n)_{n=1}^{\infty} = \left(\frac{1}{n}\right)_{n=1}^{\infty} \quad \text{and} \quad (y_n)_{n=1}^{\infty} \quad \text{by} \quad y_n = \begin{cases} 2 & \text{if } n = 1 \\ 2 + 3y_{n-1} & \text{if } n \geq 2 \end{cases}$$

For the first sequence (x_n), it has a subsequence $(3^{-k})_{k=1}^{\infty}$ and $d(0, 3^{-k}) = 3^k \to \infty$. Hence (x_n) does not converge (We defer the actual proof of this when we see Cauchy sequence). For the second sequence, we see that:

$$y_n = 2 + 2 \cdot 3 + 2 \cdot 3^2 + \cdots + 2 \cdot 3^{n-1} = 3^n - 1$$

Hence $d(-1, y_n) = \| - 3^n\|_3 = \frac{1}{3^n} \to 0$ so that $\lim\limits_{n\to\infty} y_n = -1$.

Lecture 4, 2025/01/13

Definition. Let (X, d) be a metric space and $A \subseteq X$. The **closure** of A in X is the smallest closed set in X that contains A. We denote the closure of A by $\text{cl}(A)$ or $\overline{A}$. In other word, $\overline{A}$ is the intersection of all closed sets that contain A.

Example. The closure of a closed set is itself.

Example. Consider the metric space $(\mathbb{R}, d)$ with the usual Euclidean metric. Then $\overline{\mathbb{Q}} = \mathbb{R}$.

Example. Consider $\mathbb{R}$ again with discrete metric, then $\overline{\mathbb{Q}} = \mathbb{Q}$. (because every set is closed in this topology).

Example. Consider $(\mathbb{Q}, d)$ with the 3-adic metric. We can show that $\mathbb{Z}$ is not closed. Define a sequence $(x_n)_{n=1}^{\infty}$ by $x_n = \sum_{k=0}^{n} 9^k$. This sequence has a limit in $\mathbb{Q}$ but not in $\mathbb{Z}$. How do we "guess" the limit of this sequence? Notice that:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{for all} \quad \|x\| < 1$$

In this case $\|9\|_3 = 1/9 < 1$, hence plugging in 9 shows the limit of (x_n) is $-1/8 \notin \mathbb{Z}$. [This is NOT a rigorous proof for now! This just allows us to guess the limit and we can then use the ϵ thing to prove the limit]. Hence $\mathbb{Z}$ does not contain a limit point, which means it is not closed (by the theorem below).

Theorem 1.5. A closed set contains all of its limit points. That is, if $A \subseteq X$ is closed and $(x_n)_{n=1}^{\infty}$ is a sequence in A, then whenever $\lim_{n\to\infty} x_n = x \in X$ exists, we must have $x \in A$.

1.4 Continuous Functions

Definition. Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is **continuous at** $x_0 \in X$ if for all $\epsilon > 0$ there is $\delta > 0$ such that for all $x \in X$ with $d(x, x_0) < \delta$, then $\rho(f(x), f(x_0)) < \epsilon$. [Equivalently we have $f(B_\delta(x_0)) \subseteq B_\epsilon(\epsilon)$.]

Example. Let (X, d) be a metric space with discrete metric. Let $f : X \to Y$ with (Y, ρ) a metric space. Then f is continuous at every $x_0 \in X$. Why? For any $\epsilon > 0$, pick $\delta = 1/2$. Then $d(x, x_0) < 1/2$ implies $d(x, x_0) = 0$ and $x = x_0$. Hence $\rho(f(x), f(x_0)) = 0 < \epsilon$.

Definition. Let (X, d) and (Y, ρ) be metric spaces.

(i). We say $f : X \to Y$ is **continuous on** X if it is continuous at all $x_0 \in X$.

(ii). We say $f : X \to Y$ is **uniformly continuous on** X if for all $\epsilon > 0$, there is $\delta > 0$ such that for all $x, y \in X$:

$$d(x, y) < \delta \implies \rho(f(x), f(y)) < \epsilon$$

That is, the choice of $\delta > 0$ is independent of $x, y \in X$. The usual continuity means for any $x, y \in X$ we can choose a $\delta > 0$ for them, but in this case there is one choice of $\delta > 0$ that works for all $x, y \in X$.

Note. In the Example above, we see that f is in fact uniformly continuous.

Example. Let $f : (0, 1) \to (0, \infty)$ with Euclidean metric given by $f(x) = 1/x$. This function is continuous but NOT uniformly continuous. To see it is continuous, fix $x_0 \in (0, 1)$ and let $\epsilon > 0$. We then pick $\delta > 0$ to be:

$$\delta = \min\left\{\frac{x_0}{2}, \frac{\epsilon x_0^2}{2}\right\}$$

Then, if $|x - x_0| < \delta$, we have:

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \left|\frac{x_0 - x}{x x_0}\right| < \frac{\epsilon \cdot x_0^2/2}{(x_0/2)x_0} = \epsilon$$

It follows that f is continuous on $(0,1)$. To see it is NOT uniformly continuous, assume it is. Take $\epsilon = 1$, then there is $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x) - f(y)| < 1$. However, pick $N \in \mathbb{N}$ large enough so that $1/N - 1/(N+1) < \delta$, then:

$$1 > \left| f\left(\frac{1}{N}\right) - f\left(\frac{1}{N+1}\right) \right| = \left| \frac{1}{1/N} - \frac{1}{1/(N+1)} \right| = 1$$

This is a contradiction, hence f is NOT uniformly continuous.

Definition. Let X, Y be metric spaces. We say $f : X \to Y$ is **sequentially continuous at** $x_0 \in X$ if for all sequence $(x_n)_{n=1}^{\infty}$ in X we have:

$$\lim_{n\to\infty} x_n = x_0 \implies \lim_{n\to\infty} f(x_n) = f(x_0)$$

We say f is **sequentially continuous** if it is sequentially continuous at every $x_0 \in X$.

Theorem 1.6. Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$. The followings are equivalent:

(i). f is continuous.

(ii). For all open sets $V \subseteq Y$ we have $f^{-1}(V)$ is open in X.

(iii). f is sequentially continuous.

Proof. We will prove (i) $\implies$ (ii) $\implies$ (i) and (i) $\implies$ (iii) $\implies$ (i).

(i) $\implies$ (ii). Assume f is continuous. Let $V \subseteq Y$ be open. We want to show $f^{-1}(V)$ is open in X. If $f^{-1}(V) = \emptyset$, done. Otherwise pick $x_0 \in f^{-1}(V)$, then $f(x_0) \in V$. Then is $\epsilon > 0$ such that $B_\epsilon(f(x_0)) \subseteq V$. Since f is continuous at x_0, there is $\delta > 0$ such that $f(B_\delta(x_0)) \subseteq B_\epsilon(f(x_0)) \subseteq V$. Hence $B_\delta(x_0) \subseteq f^{-1}(V)$ and therefore $f^{-1}(V)$ is open in X.

(i) $\implies$ (iii). Assume f is continuous at x_0 and $(x_n)_{n=1}^{\infty}$ is a sequence with $x_n \to x_0$. Pick $\epsilon > 0$, since f is continuous there is $\delta > 0$ such that $d(x, x_0) < \delta$ implies $\rho(f(x), f(x_0)) < \epsilon$. Now pick $N \in \mathbb{N}$ so that $d(x_n, x_0) < \delta$ for $n \geq N$. Hence if $n \geq N$ we have $\rho(f(x_n), f(x_0)) < \epsilon$.

Lecture 5, 2025/01/15

(ii) $\implies$ (i). Fix $x_0 \in X$ and let $\epsilon > 0$. Consider the open set $V = B_\epsilon(f(x_0))$. Since we are assuming (ii), we know $f^{-1}(V)$ is open and $x_0 \in f^{-1}(V)$. Therefore there is $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(V)$. Therefore we have $f(B_\delta(x_0)) \subseteq B_\epsilon(f(x_0))$, which proved f is continuous at x_0.

(iii) $\implies$ (i). We will prove this by contrapositive. Assume f is not continuous. This means there is $x_0 \in X$ and $\epsilon > 0$ such that for all $\delta > 0$, there are $x \in X$ with $d(x, x_0) < \delta$ but $\rho(f(x), f(x_0)) \geq \epsilon$. We are going to construct a sequence $(x_n)_{n=1}^{\infty} \subseteq X$ using this information that breaks the sequential continuity. For $n \in \mathbb{N}$, we choose $x_n \in X$ such that $d(x_0, x_n) < 1/n$ but $\rho(f(x_0), f(x_n)) \geq \epsilon$. Then we clearly have $x_n \to x_0$ but $f(x_n)$ does NOT converge to $f(x_0)$ as they are always at least ϵ-away from each other. $\square$

Theorem 1.7. Let X, Y, Z be metric spaces. Let $f : X \to Y$ and $g : Y \to Z$ be continuous. Then $g \circ f : X \to Z$ is continuous.

Definition. Let (X, d) and (Y, ρ) be metric spaces. We can define a metric $d \times \rho$ on $X \times Y$ by:

$$(d \times \rho)((x_1, y_1), (x_2, y_2)) := d(x_1, x_2) + \rho(y_1, y_2)$$

It is easy to check that this defines a metric.

Theorem 1.8. Let X, Y, Z, W be metric spaces and $f : X \to Z$ and $g : Y \to W$ be continuous. Then $f \times g : X \times Y \to Z \times W$ by $(f \times g)(x, y) = (f(x), g(y))$ is continuous where $X \times Y$ and $Z \times W$ are equipped with the metric defined above.

Definition. Let $f : X \to Y$ where (X, d) and (Y, ρ) are metric spaces. We say f is an **isometry** if for all $x_1, x_2 \in X$ we have $d(x_1, x_2) = \rho(f(x_1), f(x_2))$.

Example. In $\mathbb{R}^2$, any rotation, reflection, translation and combination of them are isometries.

Definition. Let (X, d) and (Y, ρ) be metric spaces. A map $f : X \to Y$ is called **Lipschitz** if there exists a constant $C > 0$ such that:

$$\rho(f(x_1), f(x_2)) \leq C d(x_1, x_2)$$

for all $x_1, x_2 \in X$.

Definition. Let (X, d) and (Y, ρ) be metric spaces. A map $f : X \to Y$ is called **bi-Lipschitz** if there exist constants $C, c > 0$ such that:

$$c d(x_1, x_2) \leq \rho(f(x_1), f(x_2)) \leq C d(x_1, x_2)$$

for all $x_1, x_2 \in X$.

Definition. Let (X, d) and (Y, ρ) be metric spaces. We say $f : X \to Y$ is a **homeomorphism** if f is a continuous bijection such that $f^{-1} : Y \to X$ is also continuous.

1.5 Finite dimensional normed vector spaces

Definition. Let V be a vector space. Two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ on V are said be to **equivalent** if there are constants $c, C > 0$ such that:

$$c\|v\|_2 \leq \|v\|_1 \leq C\|v\|_2$$

for all $v \in V$. It is clear that this is an equivalence relation.

Theorem 1.9. For $n \in \mathbb{N}$, all norms in $\mathbb{R}^n$ are equivalent. The similar result holds for $\mathbb{C}^n$.

Proof. It suffices to show all norms $\|\cdot\|$ are equivalent to the 1-norm $\|\cdot\|_1$. Then since equivalence norm is an equivalence relation, all norms are equivalent. A basis for $\mathbb{R}^n$ is $\{e_1, \cdots, e_n\}$, the standard basis. Let $C = \max\{\|e_1\|, \cdots, \|e_n\|\}$. Let $v = (v_1, \cdots, v_n) \in \mathbb{R}^n$, then:

$$\begin{aligned} \|v\| &= \|v_1 e_1 + \cdots + v_n e_n\| \\ &\leq |v_1|\|e_1\| + \cdots + |v_n|\|e_n\| && (\triangle\text{-inequality}) \\ &\leq C(|v_1| + \cdots + |v_n|) \\ &= C\|v\|_1 \end{aligned}$$

This gives us one inequality. This also shows that $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz, hence continuous (where $\mathbb{R}^n$ is equipped with $\|\cdot\|_1$ norm). Define:

$$S = \{v \in \mathbb{R}^n : \|v\|_1 = 1\}$$

Since $\|\cdot\|$ is continuous on $(\mathbb{R}^n, \|\cdot\|_1)$ we have that $\|\cdot\| : S \to \mathbb{R}$ obtains its maximum and minimum. Further, the minimum is nonzero. Define $c = \min_{v \in S} \|v\| > 0$. Note for all $0 \neq v \in \mathbb{R}^n$ we have that $v/\|v\|_1 \in S$. Hence:

$$\left\| \frac{v}{\|v\|_1} \right\| \geq c \implies \|v\| \geq c\|v\|_1$$

Hence $c\|v\|_1 \leq \|v\| \leq C\|v\|_1$, as desired. □

Lecture 6, 2025/01/17

1.6 Completeness

Definition. Let (X, d) be a metric space and $(x_n)_{n=1}^{\infty}$ be a sequence in X. We say $(x_n)_{n=1}^{\infty}$ is a **Cauchy sequence** if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$:

$$n, m \geq N \implies d(x_n, x_m) < \epsilon$$

Example. Let $(\frac{1}{n})_{n=1}^{\infty}$ be a sequence in $\mathbb{Q}$ but with different metrics.

(i). If $\mathbb{Q}$ is equipped with the Euclidean metric. This is clearly Cauchy. To see that, let $\epsilon > 0$ pick $N > 2/\epsilon$, then for $n, m \geq \mathbb{N}$ we have:

$$d\left(\frac{1}{n}, \frac{1}{m}\right) = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

(ii). If $\mathbb{Q}$ is equipped with the discrete metric, then $d(1/n, 1/m) = 1$ for all $n, m \in \mathbb{N}$. This means this sequence is not Cauchy. (If it is Cauchy, take $\epsilon = 1/2$ then contradiction) (iii). If $\mathbb{Q}$ is equipped with the 3-adic metric, then this is not a Cauchy sequence. Let $n = 3^k$ and $m = 3^\ell$ where $k \neq \ell$. Then we have:

$$d\left(\frac{1}{n}, \frac{1}{m}\right) = d\left(\frac{1}{3^k}, \frac{1}{3^\ell}\right) = 3^{\min\{k,\ell\}}$$

If we pick k, ℓ large enoguh, then the distance between them can be arbitrarily large. Hence this is not a Cauchy sequence.

Theorem 1.10. Let (X, d) be a metric space and let $(x_n)_{n=1}^\infty$ be a convergent sequence, then $(x_n)_{n=1}^\infty$ is a Cauchy sequence.

Proof. Say $\lim_{n\to\infty} x_n = x^* \in X$. Let $\epsilon > 0$, pick $N \in \mathbb{N}$ such that for $n \in \mathbb{N}$:

$$n \geq N \implies d(x^*, x_n) < \frac{\epsilon}{2}$$

Now pick $n, m \in \mathbb{N}$ such that $n, m \geq \mathbb{N}$, we have:

$$d(x_n, x_m) \leq d(x_n, x^*) + d(x_m, x^*) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore $(x_n)_{n=1}^\infty$ is a Cauchy sequence. □

Example (The Converse is False). Every convergent sequence is Cauchy but not the other way around. There are cauchy sequences that do not converge.

(i). Let $X = \mathbb{Q}$ with the Euclidean metric. Let $x_n =$ the truncation of π to the n-th decimal place. For example: $x_1 = 3$, $x_2 = 3.1$, $x_3 = 3.14$ and so on. This is a Cauchy sequence, but the limit does not exist (because its "limit" is π, which is not in $\mathbb{Q}$).

(ii). Consider the sequence $(\frac{1}{n})_{n=2}^\infty$ with $X = (0, 1)$ with Euclidean metric. Then this is Cauchy but not convergent because $0 \notin X$.

Definition. We say a metric space (X, d) is **complete** if every Cauchy sequence in X converges.

Example (Complete Spaces).

(i). The metric space $(\mathbb{R}, d)$ is complete with Euclidean metric.

(ii). Any X with the discrete metric space.

Definition. A complete normed vector space is called a **Banach Space**.

Theorem 1.11. Let (X, d) be a complete metric space. Let $Y \subseteq X$ be a subset. Then (Y, d) is a complete metric space if and only if Y is closed in X.

Proof. ($\Leftarrow$). Assume Y is closed in X. Let $(x_n)_{n=1}^{\infty} \subseteq Y$ be a cauchy sequence in Y. Hence it is also a cauchy sequence in X. Therefore $(x_n)_{n=1}^{\infty}$ converges to x^* in X since X is complete. However, Y is closed so it contains its limit point, which means $x^* \in Y$ and thus $(x_n)_{n=1}^{\infty}$ converges in Y. This proved that (Y, d) is a complete metric space.

($\Rightarrow$). Assume (Y, d) is complete. To show Y is closed in X it suffices to show it contains all of its limit points. Let $(x_n)_{n=1}^{\infty}$ be a convergent sequence with limit $x^* \in X$. Since convergent sequences are cauchy, we know $(x_n)_{n=1}^{\infty}$ is cauchy. Since Y is complete, this cauchy sequence converges in Y! This means $x^* \in Y$ and hence Y is closed. $\square$

Theorem 1.12. Let $1 \leq p < \infty$. Then the space $(\ell_p, \|\cdot\|_p)$ is complete.

Proof. An element in ℓ_p is already a sequence, so a sequence of elements in ℓ_p is annoying. We use the following notation.

$$x^{(n)} = \{x_1^{(n)},\ x_2^{(n)},\ \cdots\} = \left(x_k^{(n)}\right)_{k=1}^{\infty}$$

where $x^{(n)} \in \ell_p$ is the n-th term in the sequence $(x^{(n)})_{n=1}^{\infty}$. Let $(x^{(n)})_{n=1}^{\infty}$ be a cauchy sequence in ℓ_p. Pick $\epsilon > 0$, hence there exists an $N \in \mathbb{N}$ such that for $n, m \in \mathbb{N}$:

$$n, m \geq N \implies d(x^{(n)}, x^{(m)}) = \left(\sum_{k=1}^{\infty} |x_k^{(n)} - x_k^{(m)}|^p\right)^{1/p} < \epsilon \tag{1}$$

Our goal is to find a limit point $x = (x_k)_{k=1}^{\infty} \in \ell_p$ of the sequence $(x^{(n)})_{n=1}^{\infty}$ and prove it. Fix $k \in \mathbb{N}$, we claim that $(x_k^{(n)})_{n=1}^{\infty}$ is a cauchy sequence in $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Indeed:

$$|x_k^{(n)} - x_k^{(m)}| = \left(|x_k^{(n)} - x_k^{(m)}|^p\right)^{1/p} \leq \left(\sum_{j=1}^{\infty} |x_j^{(n)} - x_j^{(m)}|^p\right)^{1/p}$$

We have seen that the RHS can be arbitrarily small by (1), hence $(x_k^{(n)})_{n=1}^{\infty}$ is cauchy in $\mathbb{K}$. Since $\mathbb{K}$ is complete, this limit exists, we define $x_k = \lim_{n\to\infty} x_k^{(n)} \in \mathbb{K}$. We claim that:

$$\lim_{n\to\infty} x^{(n)} = x \in \ell_p$$

There are two things to prove: the limit is x and x lies in ℓ_p.

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(i). Pick $\epsilon > 0$, there exists an N such that for all $n, m \geq N$ we have:

$$d(x^{(n)}, x^{(m)}) = \left(\sum_{k=1}^{\infty} |x_k^{(n)} - x_k^{(m)}|^p\right)^{1/p} < \epsilon$$

For any $J \in \mathbb{N}$ and for all $n, m \geq N$ we have:

$$\sum_{k=1}^{J} |x_k^{(n)} - x_k^{(m)}| \leq \sum_{k=1}^{\infty} |x_k^{(n)} - x_k^{(m)}| < \epsilon^p$$

As this is true for all $n \geq M$, it is true as $m \to \infty$. This gives:

$$\lim_{m\to\infty} \sum_{k=1}^{J} |x_k^{(n)} - x_k^{(m)}|^p \leq \epsilon^p \implies \sum_{k=1}^{J} |x_k^{(n)} - x_k|^p < \epsilon^p$$

because $x_k^{(m)} \to x_k$ as $m \to \infty$. This result is true for all $n \geq N$, independent of the choice of J. As this is true for all J, we can take the limit as $J \to \infty$. Hence:

$$\sum_{k=1}^{\infty} |x_k^{(n)} - x_k|^p \leq \epsilon^p \tag{1}$$

It follows that $(x^{(n)} - x) \in \ell_p$ for all $n \in \mathbb{N}$, by definition. We also know $x^{(n)} \in \ell_p$, hence:

$$x = (x^{(n)} - x) + x^{(n)} \in \ell_p$$

as ℓ_p is a vector space. Inequality (1) says that for any $\epsilon > 0$ we can find $N \in \mathbb{N}$ such that $n \geq N$ implies:

$$\|x^{(n)} - x\|_p = \left(\sum_{k=1}^{\infty} |x_k^{(n)} - x_k|^p \right)^{1/p} \leq \epsilon$$

This is exactly the definition of $x^{(n)} \to x$ in ℓ_p, as desired. □

1.7 Completeness of $\mathbb{R}$

Definition. Let $S \subseteq \mathbb{R}$. We say S is **bounded above** if there exists an $M \in \mathbb{R}$ such that $s \leq M$ for all $s \in S$. Similarly S is **bounded below** if there is $N \in \mathbb{R}$ such that $s \geq N$ for all $s \in S$. A set is **bounded** if it is both bounded above and below.

Example. $\mathbb{Z} \subseteq \mathbb{R}$ is not bounded above or below. $(0, 1) \subseteq \mathbb{R}$ is bounded.

Definition. Let $S \subseteq \mathbb{R}$ be bounded above. Then we say $M \in \mathbb{R}$ is the **least upper bound** if M is an upper bound for S and if $N \in \mathbb{R}$ is another upper bound for S we have $M \leq N$. We define the **greatest lower bound** similarly. We denote them by $\sup S$ and $\inf S$.

Theorem 1.13 (Least Upper Bound Property). Let $\emptyset \neq S \subseteq \mathbb{R}$ be bounded above, then S has a least upper bound.

Proof. Let $M \in \mathbb{Z}$ be an upper bound of S. Consider $M-1$. One of two things is true. Either $M-1$ is an upper bound or it is not. If $M-1$ is an upper bound, replace M by $M-1$ and repeat this argument. Eventually we will get $M \in \mathbb{Z}$ such that M is an upper bound but $M-1$ is NOT an upper bound (This process terminates because $S \neq \emptyset$). Divide $[M-1, M]$ into 10 subintervals.

$$\left[M-1, M-1+\frac{1}{10}\right], \cdots, \left[M-1+\frac{9}{10}, M\right]$$

We can find some $k \in \{0, \cdots, 9\}$ such that $M-1+\frac{k}{10}$ is not an upper bound and $M-1+\frac{k+1}{10}$ is an upper bound. We construct u^* as the decimal sequence which is an upper bound (We have to be careful if a ring end point is a least upper bound, as we get a decimal expansion of trailing 9's but this is fine). As desired. $\square$

Theorem 1.14 (MCT). Let $(x_n)_{n=1}^{\infty}$ be a bounded, non-decreasing sequence in $\mathbb{R}$. Then $(x_n)_{n=1}^{\infty}$ converges in $\mathbb{R}$.

Proof. Let $x^* = \sup\{x_n : n \in \mathbb{N}\}$, this exists because $(x_n)_{n\geq 1}$ is bounded. Let $\epsilon > 0$, as x^* is the least upper bound, there exists $N \in \mathbb{N}$ such that:

$$x^* - \epsilon < x_N \leq x^*$$

Hence, for $n \geq N$ we have that $x_n \geq x_N$, which means:

$$x^* - \epsilon < x_N \leq x_n \leq x^* < x^* + \epsilon \implies |x^* - x_n| < \epsilon$$

which prvoed that $\lim\limits_{n\to\infty} x_n = x^*$, as desired. $\square$

Theorem 1.15 (Bolzano-Weierstrass). Every bounded sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}$ has a convergent subsequence (that converges in $\mathbb{R}$).

Proof. Just see MATH 147/247 notes, the proof idea is just bisection. $\square$

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Lemma 1.16. Let $(x_n)_{n=1}^{\infty}$ be a cauchy sequence in $\mathbb{R}$. Then $(x_n)_{n=1}^{\infty}$ is bounded.

Proof. Pick $\epsilon = 1$, there is $N \in \mathbb{N}$ such that for $n, m \geq N$ we have $|x_n - x_m| < 1$. In particular $|x_n - x_N| < 1$ for all $n \geq N$. Let $M = \max\{|x_1|, \cdots, |x_N| + 1\}$. Then $|x_n| \leq M$ for all $n \in \mathbb{N}$. $\square$

Theorem 1.17. $(\mathbb{R}, |\cdot|)$ is a complete normed space.

Proof. Let $(x_n)_{n=1}^{\infty}$ be a cauchy sequence in $\mathbb{R}$. By the above lemma, $(x_n)_{n=1}^{\infty}$ is bounded. By Bolzano-Weierstrass, $(x_n)_{n=1}^{\infty}$ has a convergent subsequence $(x_{n_k})_{k=1}^{\infty}$. Say $\lim\limits_{k\to\infty} x_{n_k} = x^* \in \mathbb{R}$. We claim that $\lim_{n\to\infty} x_n = x^*$ as well. Indeed, let $\epsilon > 0$. There is $N_1 \in \mathbb{N}$ such that:

$$n, m \geq N \implies |x_n - x_m| < \frac{\epsilon}{2}$$

Find $k \in \mathbb{N}$ such that $n_k \geq N$ and $|x_{n_k} - x^*| < \epsilon/2$. Hence for $n \geq N$ we have:

$$|x_n - x^*| \leq |x_n - x_{n_k}| + |x_{n_k} - x^*| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore $\mathbb{R}$ is complete. $\square$

1.8 Limits of continuous functions

Definition. Let (X, d) and (Y, ρ) be metric spaces. Let $(f_n)_{n=1}^{\infty}$ be a sequence of function $X \to Y$. We say $(f_n)_{n=1}^{\infty}$ **converges uniformly** to $f^* : X \to Y$ if for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for $n \in \mathbb{N}$ we have:

$$n \geq N \implies d^*(f_n, f^*) := \sup_{x \in X} \rho(f_n(x), f^*(x)) < \epsilon$$

Example. Let $X = [0, \frac{1}{2}]$ and $Y = \mathbb{R}$ with Euclidean metrics. Define:

$$f_n(x) = 1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

This $(f_n)_{n=1}^{\infty}$ is a sequence of bounded continuous functions from $X \to Y$. We claim that it converges to $f^*(x) = \frac{1}{1-x}$. Indeed, for any $n \in \mathbb{N}$ we have:

$$d^*(f_n, f^*) = \sup_{x \in [0, \frac{1}{2}]} \left| \frac{1}{1-x} - \frac{1 - x^{n+1}}{1-x} \right| = \sup_{x \in [0, \frac{1}{2}]} \frac{x^{n+1}}{|1-x|} \leq \frac{(1/2)^{n+1}}{1/2} = \left(\frac{1}{2}\right)^n$$

where the denominator is at least $1/2$ and the numerator is at most $(1/2)^{n+1}$. As $n \to \infty$ this tends to 0, which means $f_n \to f^*$ uniformly.

Theorem 1.18. Let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous functions that converges uniformly to f^*. Then f^* is continuous.

Proof. Let $x \in X$ and $\epsilon > 0$. Pick $N \in \mathbb{N}$ such that $d^*(f^*, f_N) < \epsilon/3$. Since f_N is continuous at x, we can pick $\delta > 0$ such that:

$$d(x, y) < \delta \implies \rho(f_N(x), f_N(y)) < \frac{\epsilon}{3}$$

Therefore if $y \in X$ and $d(x, y) < \delta$, we have:

$$\begin{aligned}\rho(f^*(x), f^*(y)) &\leq \rho(f^*(x), f_N(x)) + \rho(f_N(x), f_N(y)) + \rho(f_N(y), f^*(y)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon\end{aligned}$$

Therefore f^* is continuous at $x \in X$, as desired. $\square$

Definition. Let (X, d) be a metric space. A subset $A \subseteq X$ is **bounded** if:

$$\operatorname{diam}(A) := \sup_{x,y \in A} d(x, y) < \infty$$

We say a function $f : X \to Y$ is bounded if $f(X) \subseteq Y$ is bounded.

Definition. Let (X, d) and (Y, ρ) be metric spaces. Define:

$$\mathcal{C}^b(X, Y) = \{f : X \to Y \mid f \text{ is continuous and bounded}\}$$

The metric on $\mathcal{C}^b(X, Y)$ is the metric d^* defined by:

$$\rho^*(f, g) := \sup_{x \in X} \rho(f(x), g(x))$$

Then $(\mathcal{C}^b(X, Y), \rho^*)$ is a metric space.

Theorem 1.19. Let (f_n) be a sequence of bounded functions $f_n \in \mathcal{C}^b(X, \mathbb{K})$ that converges uniformly to f^*, then f^* is also bounded.

Theorem 1.20. Let (X, d) and (Y, ρ) be metric spaces. The metric space $(\mathcal{C}^b(X, Y), \rho^*)$ is complete if and only if (Y, ρ) is complete!

Proof. See Assignment 2. □

Theorem 1.21. Let (X, d) be a metric space. Then $\mathcal{C}^b(X, \mathbb{K})$ is complete.

Proof. Let $(f_n)_{n=1}^{\infty}$ be a cauchy sequence. Construct $f^* : X \to \mathbb{K}$ by:

$$f^*(x) := \lim_{n \to \infty} f_n(x)$$

Why is this well-defined? Note that for all fixed $x \in \mathbb{K}$, the sequence $(f_n(x))_{n=1}^{\infty}$ is a cauchy sequence in $\mathbb{K}$! Since $\mathbb{K}$ is complete, this sequence converges. We claim that $(f_n)_{n=1}^{\infty}$ converges uniformly to f^*. Let $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that:

$$n, m \geq N \implies d^*(f_n, f_m) = \sup_{x \in X} |f_n(x) - f_m(x)| < \frac{\epsilon}{2}$$

Let $n \geq N$ be arbitrary. Let $x \in X$ be arbitrary as well. Since $f_n(x) \to f^*(x)$, we can find $M \in \mathbb{N}$ with $M \geq N$ such that $|f_n(x) - f^*(x)| < \epsilon/2$. Then, for $n \geq N$:

$$\begin{aligned} |f_n(x) - f^*(x)| &\leq |f_n(x) - f_M(x)| + |f_M(x) - f^*(x)| \\ &\leq d^*(f_n, f_M) + |f_M(x) - f^*(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Since $x \in X$ is chosen arbitrarily, we have:

$$d^*(f_n, f^*) = \sup_{x \in X} |f_n(x) - f^*(x)| \leq \epsilon$$

Therefore $f_n \to f^*$ in the d^* metric (that is $f_n \to f^*$ uniformly in the usual sense). Hence f^* is continuous and bounded, so $f^* \in \mathcal{C}^b(X, \mathbb{K})$. $\square$

Theorem 1.22 (Weierstrass M-Test). Let $\zeta : X \to \mathbb{R}$ by $\zeta(a) = 0$ denote the zero function. Then we let $(f_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{C}^b(X, \mathbb{R})$ such that there exists $M \in \mathbb{R}$ with:

$$\sum_{n=1}^{\infty} d^*(f_n, \zeta) \leq M < \infty$$

Define $g_N(x) = \sum_{n=1}^{N} f_n(x)$. Then $(g_N)_{N=1}^{\infty}$ converges to $g^* \in \mathcal{C}^b(X)$ in the d^* metric.

Example. The series of function $f(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{2^n}$ is well-defined and is continuous on $\mathbb{R}$.

2 More Metric Topology

2.1 Compactness

Definition. Let (X, d) be a metric space and $A \subseteq X$. We say $\{U_i\}_{i \in I}$ is an **open cover** of A if each U_i is open and $A \subseteq \bigcup_{i \in I} U_i$.

Definition. Let (X, d) be a metric space and $A \subseteq X$. We say A is **compact** if for every open cover $\{U_i\}_{i \in I}$ there is a finite subset $I_0 \subseteq I$ with $A \subseteq \bigcup_{i \in I_0} U_i$. This $\{U_i\}_{i \in I_0}$ is called a **finite subcover**.

Example. Let $A = \{x_1, \cdots, x_n\}$ be a finite set, then A is compact. Why? Let $\{U_i\}$ be an open cover of A. For each $j \in \mathbb{N}$ there is $i_j \in I$ such that $a \in U_{i_j}$. Hence we have:

$$A \subseteq U_{i_1} \cup \cdots \cup U_{i_n}$$

This is a finite subcover! Hence A is compact.

Example. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\} \subseteq \mathbb{R}$. We claim that A is compact. Let $\{U_i\}_{i \in I}$ be an open cover. There exists an open set U_0 such that $0 \in U_0$. Hence there is $N \in \mathbb{N}$ large enough such that $0 \in B_\epsilon(0) \subseteq U_0$, where $\epsilon = 1/N$. This means:

$$\left\{\frac{1}{n} : n \geq N+1\right\} \cup \{0\} \subseteq U_0$$

Then there are only finitely many points left, so we can use finitely many U_i to cover $\{\frac{1}{n} : n \geq N+1\}$. This gives an finite subcover of A.

Example. Let A, B be compact sets. Then $A \cup B$ is compact. Indeed, any open cover of $A \cup B$ gives an open cover for A, B. This gives a finite subcover for A, B, respectively. The union of these two finite subcovers giives a finite subcover of $A \cup B$.

Example. The set $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ is not compact! Consider the open cover:

$$\left\{\left(\frac{1}{n}, 1 + \frac{1}{n}\right) : n \in \mathbb{N}\right\}$$

This has no finite subcover. Indeed, suppose we have a finite subcollection of open sets indexed by $n_1, \cdots, n_r$. WLOG we may assume $n_1 < \cdots < n_r$. Then the union of these U_i is:

$$\left(\frac{1}{n_r}, 1 + \frac{1}{n_1}\right)$$

This clearly does not cover A.

Example. Let $A = \mathbb{R}$. Then A is not compact. The open cover $\{(-n, n) : n \in \mathbb{N}\}$ has no finite subcover. Similarly $A = \mathbb{Z}$ is not compact as well.

Proposition 2.1. Let (X, d) be a metric space. If $A \subseteq X$ is compact, then A is closed and bounded.

Proof. Assume A is not closed. There exists a subsequence $(a_n)_{n=1}^{\infty}$ in A with $a_n \to a^*$ and $a^* \notin A$. Consider the following open cover:

$$U_n = X \setminus \overline{B_{d(a_n,a^*)}(a^*)}$$

This cannot have a finite subcover, since a^* is a limit point of (a_n). Therefore A is closed. Similarly suppose A is not bounded. Fix $a \in A$. For all $N \in \mathbb{N}$ such that there exists an $a_N \in A$ such that:

$$d(a_N, a) > N$$

Consider the open cover $\{B_N(a) : N \in \mathbb{N}\}$ of A. Given a finite subset $\{N_1 < \cdots < N_r\}$, the union of these is $B_{N_r}(a)$. However, for $N = N_r + 1$ there is $a_N \in A$ such that $d(a_N, a) > N$ so $a_N \notin B_{N_r}(a)$, but $a_N \in A$. Hence this open cover has no finite subcover! Hence A is bounded. □

Definition. Let (X, d) be a metric space and $A \subseteq X$. We say A is **sequentially compact** if for every sequence $(a_n)_{n=1}^{\infty}$ of A, there is a convergent subsequence $(a_{n_k})_{k=1}^{\infty}$ with $a_{n_k} \to a^* \in A$.

Example. Let A be a finite set. This is sequentially compact. Why? For any infinite sequence of A, there exists $a \in A$ that appears infinitely many times in this sequence. Take this subsequence that only consists of a. This is a convergent subsequence.

Example. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. This is sequentially compact. Any sequence in A either has a convergent subsequence that goes to 0 or the sequence only takes on finitely many values.

Definition. Let (X, d) be a metric space. Let $A \subseteq X$ be a subset. Then (A, d_A) is a metric space, where $d_A : A \times A \to \mathbb{R}$ is the restriction of d on A. This is called the **induced metric space**. A subset $U \subseteq A$ is called **relatively open** if there exists an open set $U' \subseteq X$ such that $U = U' \cap A$.

Remark. As a metric space, the open balls of (A, d_A) are of the form:

$$B_A(a, r) = \{x \in A : d_A(x, a) < r\} = \{x \in X \cap A : d(x, a) < r\} = B_X(a, r) \cap A$$

Therefore, an open set in (A, d_A) is of the form $U' \cap A$ for open sets U' in X.

Definition. A metric space (X, d) is **compact** if every open cover of X has a finite subcover. That is, for every open cover $\{U_i : i \in I\}$, there is a finite subset $I_0 \subseteq I$ such that:

$$X = \bigcup_{i \in I_0} U_i$$

Note that this is an equality, not a subset. This is because X is our whole space, it does not sit in any bigger space.

Remark. Note that there are two notions of compactness for a subset $A \subseteq X$.

(i). A is compact as a subset of X. [This is the definition we saw above.]

(ii). A is compact as a metric space. [Note that for an open cover of A, the open sets are open sets in A! These open sets are different from the open sets in X.]

In fact, these two notions concide. Suppose (ii) is true, we want to show (i) is true. Let $\{U_i : i \in I\}$ be an open cover of A, where U_i is an open set of X for all i. Then:

$$\{U_i \cap A : i \in I\}$$

is an open cover of the metric space (A, d_A), where each $U_i \cap A$ is an open set in A. Since (A, d_A) is compact, there is a finite set $I_0 \subseteq I$ such that:

$$A = \bigcup_{i \in I_0} (U_i \cap A)$$

Then clearly we have $A \subseteq \bigcup_{i \in I_0} U_i$, an finite subcover of A (as a subset of X.)

Conversely suppose (i) is true. Let $\{U_i : i \in I\}$ be an open cover of (A, d_A), then for each $i \in I$ there is an open set $U_i' \subseteq X$ of X such that $U_i = U_i' \cap A$. Hence $\{U_i' : i \in I\}$ is an open cover of $A \subseteq X$. Since A is a compact subset of X, there is a finite $I_0 \subseteq I$ with $A \subseteq \bigcup_{i \in I_0} U_i'$. By taking the intersection with A, we have:

$$A = \bigcup_{i \in I_0} (U_i' \cap A) = \bigcup_{i \in I_0} U_i$$

Therefore (A, d_A) is compact and (ii) is true.

Definition. Let (X, d) be a metric space. A collection $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\} \subseteq X$ is said to have the **finite intersection property (FIP)** if for every finite subset $\Lambda_0 \subseteq \Lambda$ we have $\bigcap_{\lambda \in \Lambda_0} F_\lambda \neq \emptyset$.

Example. Let $X = \mathbb{R}$. Consider the collection $\{\mathbb{R} \setminus \{a\} : a \in \mathbb{R}\}$. This clearly satisfies the FIP. However, the infinite intersection:

$$\bigcap_{a \in \mathbb{R}} (\mathbb{R} \setminus \{a\}) = \emptyset$$

is empty! As we will see, this actually tells us $\mathbb{R}$ is not compact!

Definition. Let (X, d) be a metric space. A subset $A \subseteq X$ is called **cauchy** if every cauchy sequence in A converges to a point in A.

Definition. Let (X, d) be a metric space. We say $A \subseteq X$ is **totally bounded** if for all $\epsilon > 0$ there exists a finite set $F_\epsilon \subseteq X$ (called an **ϵ-net**) such that:

$$A \subseteq \bigcup_{f \in F_\epsilon} B_\epsilon(f)$$

Note that totally boundedness implies boundedness.

Remark. Note that if A is totally bounded, we may assume $F_\epsilon \subseteq A$ for all $\epsilon > 0$. Suppose for $\epsilon > 0$ we have an ϵ-net $F = \{x_1, \cdots, x_n\} \subseteq X$ of A, so:

$$A \subseteq \bigcup_{i=1}^{n} B_{\epsilon/2}(x_i)$$

We may assume $B_\epsilon(x_i) \cap A \neq \emptyset$ for all i. (If the intersection is empty we can just remove it from the ϵ-net.) Hence we may chooise $y_i \in A \cap B_{\epsilon/2}(x_i)$ for all i. Note that:

$$A \subseteq \bigcup_{i=1}^{n} B_{\epsilon}(y_i)$$

by the triangle inequality. Indeed, for any $x \in A$ we can choose $i \in \{1, \cdots, n\}$ such that $x \in B_{\epsilon/2}(x_i)$. Then we have that:

$$d(x, y_i) \leq d(x, x_i) + d(x_i, y_i) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This proved that $x \in B_\epsilon(y_i)$. Hence $\{y_1, \cdots, y_n\} \subseteq A$ is an ϵ-net for A.

Recall in $\mathbb{R}^n$, a subset is compact if and only if it is closed and bounded (Heine-Borel). We will now see that for metric spaces, there are also some easier ways to characterize compactness, and the Heine-Borel theorem for $\mathbb{R}^n$ is a special case of it.

Theorem 2.2 (Borel-Lebesgue). Let (X, d) be a metric space and $A \subseteq X$. Then the followings are equivalent:

(i). A is compact (either as a subset or a metric space, these two notions are equivalent.)

(ii). If $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\}$ is an collection of closed sets in (A, d_A) with FIP, then $\bigcap_{\lambda \in \Lambda} F_\lambda \neq \emptyset$.

(iii). A is sequentially compact.

(iv). A is complete and totally bounded.

Example. Consider $A = \mathbb{Q} \cap [0, 1]$ and $B = \mathbb{Z}$ as induced metric spaces from $(\mathbb{R}, d)$. By the Borel-Lebesgue theorem, we can show that A, B are not compact in four different ways.

(i). For A, we define the open cover:

$$\left\{\mathbb{R} \setminus \overline{B_{1/n}(1/\pi)} : n \in \mathbb{N}\right\}$$

This does not have a finite subcover. For B, the open cover $\{B_{1/2}(n) : n \in \mathbb{Z}\}$ does not have a fintie subcover as well. Hence A, B are not compact by definition.

(ii). We need to find a collection of closed sets that FIP but the intersection is empty. Let:

$$\left\{A \cap \overline{B_{1/n}(1/\pi)} : n \in \mathbb{N}\right\} \subseteq A$$
$$\{[n, \infty) \cap B : n \in \mathbb{N}\} \subseteq B$$

These two have FIP but the intersection over all $n \in \mathbb{N}$ is empty.

(iii). Let $(a_n)_{n=1}^{\infty}$ be the sequence in A such that a_n is the truncation of the decimal expansion of $1/\pi$ at the n-th place. Then $a_n \to 1/\pi$ in $\mathbb{R}$, which means any convergent subsequence of (a_n) converges to $1/\pi \notin A$. For B, the sequence $(b_n)_{n=1}^{\infty}$ by $b_n = n$ is a sequence in B that does not have a convergent subsequence.

(iv). Let $(a_n)_{n=1}^{\infty}$ be the same sequence in (iii), this is cauchy but does not converge in A. For B, consider $\epsilon = 1/2$. Then $B = \mathbb{Z}$ does not have a ϵ-net. Therefore A is not complete and B is not totally bounded.

Proof of Theorem 2.2. (i) $\Longrightarrow$ (ii). Assume (A, d_A) is a compact metric space. Let $\{F_\lambda : \lambda \in \Lambda\}$ be a collection of closed sets in A satisfying FIP. Assume for a contradiction that $\bigcap_{\lambda\in\Lambda} F_\lambda = \emptyset$. Consider the following collection of open sets in A:

$$\{U_\lambda := A \setminus F_\lambda : \lambda \in \Lambda\}$$

Note that this is an open cover for A. Since A is compact, there exists a finite subset $\Lambda_0 \subseteq \Lambda$ such that $\bigcup_{\lambda\in\Lambda_0} U_\lambda = A$. However, this implies that:

$$\bigcap_{\lambda\in\Lambda_0} F_\lambda = A \setminus \bigcup_{\lambda\in\Lambda_0} U_\lambda = A \setminus A = \emptyset$$

Since Λ_0 is finite, this contradicts to our assumption that $\{F_\lambda : \lambda \in \Lambda\}$ has FIP!

(ii) $\Longrightarrow$ (iii). Assume (ii) is true. We want to show A is sequentially compact. Let $(a_n)_{n=1}^{\infty}$ be a sequence in A. For each $k \geq 1$ we define $S_k = \{a_n : n \geq k\}$ and define the closed set:

$$F_k = \overline{S_k} = \overline{\{a_n : n \geq k\}} \subseteq A$$

to be the closure of a tail of $(a_n)_{n=1}^{\infty}$. Note that $F_{k+1} \subseteq F_k$ for all $k \geq 1$. Define $\mathcal{F} = \{F_k : k \geq 1\}$. Then $\mathcal{F}$ is a collection of closed sets in A that has FIP. It satisfies FIP because for a finite set $\{k_1 < \cdots < k_r\}$ we have:

$$F_{k_1} \cap \cdots \cap F_{k_r} = F_{k_1} \neq \emptyset$$

By our assumption we have $\bigcap_{k=1}^{\infty} F_k \neq \emptyset$. Let's pick $a^* \in \bigcap_{k=1}^{\infty} F_k$. We claim that we can find a subsequence of (a_n) that converges to a^*. First we note that:

$$B_r(a^*) \cap S_k \neq \emptyset$$

for all $r > 0$ and $k \geq 1$. This is because each $a^* \in F_k$ is closed so a^* is a limit point for every S_k. In other word, for any $r > 0$ and $k \geq 1$ we can find some a_i such that $d(a_i, a^*) < r$ and $i \geq k$. For $r = 1$ we can find $n_1 \geq 1$ with $d(a_{n_1}, a^*) < 1$. Inductively suppose we have defined $n_1, \cdots, n_r$, we can find $n_{r+1} > n_r$ such that $d(a_{n_r}, a^*) < 1/(r+1)$. Hence $(a_{n_r})_{r=1}^{\infty}$ is a subsequence that converges to $a^* \in A$. Therefore (A, d_A) is sequentially compact.

(iii) $\Longrightarrow$ (iv). Assume (A, d_A) is sequentially compact. We first show that A is complete (as a subset of X.) Let $(a_n)_{n=1}^{\infty}$ be a cauchy seqeunce in A. There exist a convergent subsequence $(a_{n_k})_{k=1}^{\infty}$ that converges to $a^* \in A$. Since $(a_n)_{n=1}^{\infty}$ is cauchy, we must have $a_n \to a^*$ as well. Hence A is complete. Now let us show that A is totally bounded. Let $\epsilon > 0$ be arbitrary. Suppose it is not, then there is $\epsilon > 0$ such that there does not exist a ϵ-net for A. First note that in the case, A must be infinite. (Any finite set is clearly totally bounded.) Let $a_1 \in A$ be arbitrary. Hence $\{a_1\}$ is not an ϵ-net. This means there exists $a_2 \in A$ such that $d(a_1, a_2) \geq \epsilon$. Now, inductively suppose we have found $a_1, \cdots, a_r$ for $r \geq 1$. Then $\{a_1, \cdots, a_r\}$ is not an ϵ-net. We can then find $a_{r+1} \in A$ such that:

$$d(a_{r+1}, a_i) \geq \epsilon \text{ for all } i \in \{1, \cdots, r\}$$

This gives us a sequence $(a_n)_{n=1}^{\infty}$ in A that has no convergent subsequence! (since for all n, m we have $d(a_n, a_m) \geq \epsilon$.) This is a contradiction, so A is totally bounded.

(iv) $\Longrightarrow$ (i). Assume (iv) is true, we want to show A is compact. Suppose for a contradiction that A is not compact as a metric space. Let $\mathcal{U} = \{U_i : i \in I\}$ be an open cover of A that does not have a finite subcover (in this case $U_i \subseteq X$ is open for all i). Since A is totally bounded, for all $n \geq 1$ there exists a $\frac{1}{n}$-net in A:

$$F_n = \{x_{n,1}, \cdots, x_{n,m_n}\}$$

such that:

$$A = \bigcup_{f \in F_n} B_{1/n}(f) = \bigcup_{f \in F_n} \overline{B_{1/n}(f)}$$

Let $n = 1$. Note that if all $\overline{B_1(f)}$ can be covered by finitely many U_i's, then A can be covered by finitely many U_i's, which is impossible. Hence there is $i_1 \in \{1, \cdots, m_n\}$ such that $\overline{B_1(x_{i_1})}$ does not have a finite subcover of $\mathcal{U}$. Let $y_1 = x_{i_1}$. Inductively suppose we have chosen $y_1, \cdots, y_k$ so that:

$$X_k := \bigcap_{i=1}^{k} \overline{B_{1/i}(y_i)}$$

has no finite subcover. Consider the sets:

$$X_{k,i} = X_k \cap \overline{B_{1/(k+1)}(x_{k+1,i})} \quad \text{for } 1 \leq i \leq m_{n+1}$$

Suppose for a contradiction that each of them has a finite subcover. However:

$$\bigcup_{i=1}^{m_{n+1}} X_{n,i} = \bigcup_{i=1}^{m_{n+1}} X_k \cap \overline{B_{1/(k+1)}(x_{k+1,i})} = X_k \cap \bigcup_{i=1}^{m_{n+1}} \overline{B_{1/(k+1)}(x_{k+1,i})} = X_k \cap A = X_k$$

This means X_k has a finite subcover, which is impossible! Hence there is i_{k+1} such that $\overline{B_{1/(k+1)}(x_{k+1,i_{k+1}})}$ does not have a finite subcover. Let $y_{k+1} = x_{k+1,i_{k+1}}$.

Note that $(y_n)_{n=1}^{\infty}$ is cauchy in A. Indeed, let $\epsilon > 0$ we choose $N > 2/\epsilon$. For all $n \geq m \geq N$ we have:

$$X_n \subseteq \overline{B_{1/m}(y_m)} \cap \overline{B_{1/n}(y_n)} \neq \emptyset$$

and X_n is non-empty set. We can pick $x \in X_n$. Then:

$$d(y_n, y_m) \leq d(y_n, x) + d(y_m, x) \leq \frac{1}{n} + \frac{1}{m} \leq \frac{2}{N} < \epsilon$$

Since A is complete, $y_n \to y^* \in A$ for some $y^* \in A$. For any $m \in \mathbb{N}$ we have:

$$d(y_m, y^*) = \lim_{n\to\infty} d(y_m, y_n) \leq \lim_{n\to\infty} \left(\frac{1}{m} + \frac{1}{n}\right) = \frac{1}{m}$$

Since $\mathcal{U}$ is a cover of A, there is $i_0 \in I$ such that $y^* \in U_{i_0}$. Since U_{i_0} is open, then is $r > 0$ such that $B_r(y^*) \subseteq U_{i_0}$. Choose $m > 2/r$, then for any $x \in X_m \subseteq \overline{B_{1/m}(y_m)}$ we have:

$$d(x, y^*) \leq d(x, y_m) + d(y_m, y^*) \leq \frac{2}{m} < r$$

Hence $X_m \subseteq U_{i_0}$. This means X_m does have a finite subcover, contradicting our construction! Therefore A is compact. □

Remark. Totally bounded is not same as bounded. There exist sets that are closed, bounded but not compact. Consider $X = \{0,1\}^{\mathbb{N}}$ with ℓ^∞ norm. It is clearly bounded since $\|x\|_\infty \leq 1$ for all $x \in X$. However, we claim that it is not totally bounded. Suppose it has an $\frac{1}{2}$-net:

$$F = \{x_1, \cdots, x_n\}$$

Then $B_{1/2}(x_i) = \{x_i\}$ because $\|x\|_\infty \in \{0,1\}$ for any $x \in X$. Hence:

$$\bigcup_{i=1}^{n} B_{1/2}(x_i) = \{x_1, \cdots, x_n\} \neq \{0,1\}^{\mathbb{N}}$$

Therefore this is not a $\frac{1}{2}$-net, contradiction. Hence $(X, \|\cdot\|_\infty)$ is bounded but NOT totally bounded!

Corollary 2.3 (Heine-Borel). A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof. Since $\mathbb{R}^n$ is compact, A is closed $\iff$ it is complete. Moreover, we claim that in $\mathbb{R}^n$, bounded implies totally bounded. Let $\epsilon \in \mathbb{N}$, we claim that there is also an ϵ-net of a bounded set A. Since A is bounded, we know $A \subseteq [-r, r]^n$ for some $r > 0$. We can cover $[-r, r]^n$ with finitely many boxes of side length $\frac{\epsilon}{2}$. Any such box can be covered by an ϵ-ball. Hence we can use finitely many ϵ-balls to cover A. Therefore A is totally bounded. Hence A is bounded $\iff$ it is totally bounded. The result follows from (iv) of Borel-Lebesgue. □

2.2 Countable and Uncountable Sets

Definition. A set X is **countable** if there is a injection $f : X \to \mathbb{N}$. A set is **denumerable** if there is a bijection $f : X \to \mathbb{N}$. We say a set is **uncountable** if it is not countable.

Example. The integers $\mathbb{Z}$ is countable because $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \cdots\}$.

Example. The rationals $\mathbb{Q} \cap [0, 1]$ is also countable because:

$$\mathbb{Q} = \left\{\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \cdots\right\}$$

Informally: Write $\frac{p}{q} \in \mathbb{Q} \cap [0, 1]$ with q in increasing order and $p \in \{1, \cdots, q\}$ such that $\gcd(p, q) = 1$. We require coprimeness so that there is no element appearing twice in the list.

Example. The set $\{0, 1\}^{\mathbb{N}}$ is uncountable. Suppose for a contradiction that it is countable. Then:

$$\{0, 1\}^{\mathbb{N}} = \{(x_{1,k})_{k=1}^{\infty}, (x_{2,k})_{k=1}^{\infty}, \cdots\}$$

Define a sequence $(x_n)_{n=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ by:

$$x_n = \begin{cases} 0 & \text{if } x_{n,n} = 1 \\ 1 & \text{if } x_{n,n} = 0 \end{cases}$$

Then $(x_n)_{n=1}^{\infty}$ is different from $(x_{n,k})_{k=1}^{\infty}$ at the n-th place for all $n \geq 1$. This is a new element in $\{0, 1\}^{\mathbb{N}}$, contradiction! Hence $\{0, 1\}^{\mathbb{N}}$ is uncountable. This method is called the **diagonal argument**: If we list out all the given $(x_{n,k})_{k=1}^{\infty}$ row by row, then our new element $(x_n)_{n=1}^{\infty}$ is constructed by changing the diagonal entries.

Example. Let A, B be sets and $f : A \to B$ be a bijection, then A is countable if and only if B is countable.

Example. Let $A \subseteq B$. If A is uncountable then so is B. If B is countable then so is A.

Example. We claim $\mathbb{R}$ is uncountable. Let $X = \{0, 1, \cdots, 9\}^{\mathbb{N}}$. By the same argument we can show that X is uncountable. Define $f : X \to \mathbb{R}$ by:

$$f((x_n)_{n=1}^{\infty}) = \sum_{k=1}^{\infty} \frac{x_k}{10^k}$$

Then $f : X \to f(X)$ is a bijection. Since $f(X) \subseteq \mathbb{R}$, we know $\mathbb{R}$ is uncountable.

Definition. Let (X, d) be a metric space. We say $A \subseteq X$ is **dense** in X if $\overline{A} = X$.

Definition. We say a metric space (X, d) is **separable** if there is a countable subset $A \subseteq X$ such that A is dense in X.

Example. The reals $\mathbb{R}$ with the usual metric is separable because $\overline{\mathbb{Q}} = \mathbb{R}$ and $\mathbb{Q}$ is countable.

Example. Let (X, d) with the discrete metric. Then X is separable if and only if X is countable. This is because every subset is closed (equal to their own closure), so the only dense subset is X itself. Hence X is countable if and only if X is separable.

Proposition 2.4. Let (X, d) be a metric space. If (X, d) is totally bounded, then X is separable.

Proof. For each $n \in \mathbb{N}$ there is an $\frac{1}{n}$-net of X, call it F_n. Define $F = \bigcup_{n=1}^{\infty} F_n$. Note that F is countable, being a countable union of finite sets. We claim that F is dense. Let $x \in X$ be and $\epsilon > 0$ be arbitrary. There is $N \geq 1$ such that $1/N < \epsilon$. Since F_N is an $\frac{1}{N}$-net, there is $f \in F_N$ such that $d(f, x) < 1/N < \epsilon$. Since $f \in F$, we proved that F is dense in X. $\square$

2.3 Compactness and Continuity

Proposition 2.5. Let (X, d) and (Y, ρ) be metric spaces. If $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact.

Proof. Let $\{U_i : i \in I\}$ be an open cover of $f(X)$ in Y. Since f is continuous, each $f^{-1}(U_i)$ is open. Since $f^{-1}(Y) = X$, we know $\{f^{-1}(U_i) : i \in I\}$ is an open cover of X. Since X is compact, there is a finite subcover $\{i_1, \cdots, i_n\}$. Hence:

$$f(X) \subseteq \bigcup_{k=1}^{n} U_{i_k}$$

Therefore $f(X)$ is compact. $\square$

Proposition 2.6. Let (X, d) and (Y, ρ) be metric spaces. If $f : X \to Y$ is continuous and X is compact, then f is uniformly continuous.

Proof. Let $\epsilon > 0$. For each $x \in X$ we can find $\delta_x > 0$ such that for all $y \in X$:

$$d(y, x) < \delta_x \implies \rho(f(y), f(x)) < \frac{\epsilon}{2} \tag{1}$$

Now note that $\{B_{\delta_x/2}(x) : x \in X\}$ is an open cover of X. Since X is compact, we know:

$$X = B_{\delta_{x_1}/2}(x_1) \cup \cdots \cup B_{\delta_{x_n}/2}(x_n)$$

for some $x_1, \cdots, x_n \in X$. Now define $\delta = \frac{1}{2}\min\{\delta_{x_1}, \cdots, \delta_{x_n}\} > 0$. Let $x, y \in X$ be arbitray with $d(x, y) < \delta$. Say $y \in B_{\delta_{x_b}}(x_b)$ for some $x_b \in \{x_1, \cdots, x_n\}$. However:

$$d(x, x_b) \leq d(x, y) + d(y, x_b) < \delta + \frac{1}{2}\delta_{x_b} < \frac{1}{2}\delta_{x_b} + \frac{1}{2}\delta_{x_b} < \delta_{x_b}$$

Since $d(y, x_b) < \delta_{x_b}$ as well, by (1) we have:

$$\rho(f(x), f(y)) \leq \rho(f(x), f(x_b)) + \rho(f(x_b), f(y)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence f is uniformly continuous. $\square$

2.4 Cantor Set

Construction 2.7 (Ver 1). Let $C_0 = [0, 1]$. Recursively, C_{i+1} is constructed by removing the middle third from each intervals in C_i. First we see that:

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$
$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$

We see $\{C_n\}_{n=0}^{\infty}$ has the finite intersection property and they are all compact sets. Define:

$$C^* = \bigcap_{n=0}^{\infty} C_n$$

We call C^* the **(middle-third) cantor set**. Clearly $0, 1 \in C^*$. In fact any endpoint of any C_n is in C^*. For example $1/3, 1/9, 2/27 \in C^*$. We have C^* is compact (as it is closed and bounded in $\mathbb{R}$).

Construction 2.8 (Ver 2). Equivalently we can define:

$$C^* = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{3^n} : a_n \in \{0, 2\} \right\}$$

It is the set of all real numbers that CAN be written in ternary expansion wiwthout using 1. [For example $0.1 \in C^*$ because it CAN be written as $0.222\cdots$] This shows that C^* has an uncountable number of points.

Construction 2.9 (Ver 3). The cantor set C^* is the unique non-empty compact set satisfying:

$$C^* = f_1(C^*) \cup f_2(C^*)$$

where $f_1(x) = x/3$ and $f_2(x) = x/3 + 2/3$.

Theorem 2.10. Let (X, d) be a compact metric space. There is an continuous map $f : C^* \to X$ that is surjective.

Proof. The idea is to construct $s_n : C^* \to X$ such that (s_n) is cauchy and each s_n is continuous. As $n \to \infty$ we have $s_n(C^*)$ better approximate X [produce an ϵ-net for smaller ϵ.]

For $n = 1$, construct a 1-net for X. That is, a finite set F_1 such that $X = \bigcup_{f \in F_1} B_1(f)$ [This exists since X is totally bounded.] We can assume wlog that $|F_1| = 2^{k_1}$ for some k_1. [If not power of 2, adding more points if necessary.] Now consider C_{k_1}, a union of 2^{k_1} intervals containing C^* For each $c \in C^*$, we know c is in some subinterval of C_{k_1}. We map each subinterval in C_{k_1} to a different $f \in F_1$. Let s_1 be this map. Then s_1 is continuous as it is locally constant.

For $n = 2$, construct a 1/2-net for each of each $\overline{B}_1(f_i)$, where $\{f_i\} = F_1$ from the construction of s_1. As before, we can assume that this set is a power of 2, and the same powers of 2. Say 2^{k_2} in size. For each subinterval I_i used to construct s_1, subdivide it into 2^{k_2} subintervals. As before, s_2 is continuous. We further notice $d(s_1(c), s_2(c))$ is not huge. In fact $d(s_1(c), s_2(c)) \leq 1 + \frac{1}{2}$.

We continue in this fashion, we get that:

$$d(s_n(c), s_{n+1}(c)) \leq \frac{1}{2^n}$$

We can make this arbitrarily small. Hence for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n, m \geq N$:

$$d^*(s_n, s_m) = \sup_{c \in C^*} d(s_n(c), s_m(c)) < \epsilon$$

Therefore (s_n) is a cauchy sequence. As C^* is compact and X is complete so $\mathcal{C}^b(C^*, X) = \mathcal{C}(C^*, X)$ is complete. Hence $s_n \to s^* \in \mathcal{C}(C^*, X)$. We need to show $s^*(C^*) = X$, that is, s^* is onto. Take a point x in X. This point will be distance 1 from some point in F_1. This gives us a subinterval in C^*. There exists a point in F_2 whose distance is 1/2 from x and $1 + 1/2$ from f_1. This gives a smaller subinterval. Repeating this process we get nested subintervals with non-trivial intersection with C^*. The infinite intersection is in C^*, and this intersection has $s^*(c^*) = x$, as required. $\square$

2.5 Compact sets in $\mathcal{C}(X)$

Definition. Let (X, d) be a compact metric space, we denote:

$$\mathcal{C}(X) := \mathcal{C}(X, \mathbb{R}) = \{f : X \to \mathbb{R} \mid f \text{ is continuous}\}$$

Here $\mathbb{R}$ is a metric space with the usual metric. For $f \in \mathcal{C}(X)$ we define the **uniform norm** by:

$$\|f\|_\infty := \sup\{|f(x)| : x \in X\}$$

Since X is compact, by the extreme value theorem this supremum can be achieved. So we can equivalently define it as:

$$\|f\|_\infty = \max\{|f(x)| : x \in X\}$$

Note that $(\mathcal{C}(X), \|\cdot\|_\infty)$ is a normed vector space. In fact, since $\mathbb{R}$ is complete we knew that $\mathcal{C}(X)$ is also complete. Therefore $(\mathcal{C}(X), \|\cdot\|_\infty)$ is a Banach space. Also note that $f_n \to f$ uniformly (as functions) is the same as $f_n \to f$ as sequences in the normed space $(\mathcal{C}(X), \|\cdot\|_\infty)$.

Remark. By Borel-Lebesgue we know that:

$$\begin{aligned} K \subseteq \mathcal{C}(X) \text{ is compact } &\iff K \text{ is complete and totally bounded} \\ &\iff K \text{ is closed and totally bounded} \end{aligned}$$

since closed subsets of a complete space are complete.

Example. Let $K = \{f_n(x) = x^n : n \in \mathbb{N}\} \subseteq \mathcal{C}([0,1])$. Note that every subsequence of (f_n) converges pointwise to the function $f : [0,1] \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Since f is not continuous, the sequence (f_n) does not converge in $\mathcal{C}([0,1])$. Therefore K is not sequentially compact despite being closed and bounded.

Definition. Let (X, d) be complete. A subset $F \subseteq \mathcal{C}(X)$ is called **equicontinuous at** $x \in X$ if for all $\epsilon > 0$ there is $\delta > 0$ so that for all $y \in X$:

$$d(x,y) < \delta \implies |f(x) - f(y)| < \epsilon \text{ for all } f \in F$$

We know $F \subseteq \mathcal{C}(X)$ is **equicontinuous** if it is equicontinuous at every $x \in X$. We say a subset $F \subseteq \mathcal{C}(X)$ is **uniformly equicontinuous** if for all $\epsilon > 0$ there is $\delta > 0$ such that for all $x, y \in X$:

$$d(x,y) < \delta \implies |f(x) - f(y)| < \epsilon \text{ for all } f \in F$$

That is, the choice of $\delta > 0$ does not depend on $x \in X$.

Remark. Clearly uniformly equicontinuous $\implies$ equicontinuous.

Lemma 2.11. Let (X, d) be compact. If $K \subseteq \mathcal{C}(X)$ is compact, then K is uniformly equicontinuous.

Proof. Let $\epsilon > 0$. Since K is compact, it is totally bounded and thus has a $\frac{\epsilon}{3}$-net. Say it is $F = \{f_1, \cdots, f_n\} \subseteq K$. Each f_i is continuous, thus uniformly continuous (since X is compact). For each i there is $\delta_i > 0$ such that for all $x, y \in X$:

$$d(x,y) < \delta_i \implies |f_i(x) - f_i(y)| < \frac{\epsilon}{3}$$

Let $\delta = \min\{b_1, \cdots, b_n\}$. Now let $x, y \in X$ with $d(x, y) < \delta$ and let $f \in K$ be arbitrary. We can find i such that $\|f - f_i\| < \epsilon/3$ (because F is an $\epsilon/3$-net!) Therefore we have:

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| \\ &\leq \|f - f_i\|_\infty + \frac{\epsilon}{3} + \|f - f_i\|_\infty \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

Therefore K is uniformly equicontinuous. □

Lemma 2.12. Let (X, d) be compact. Suppose $F \subseteq \mathcal{C}(X)$ is equicontinuous. Then F is uniformly equicontinuous.

Proof. Let $\epsilon > 0$. For each $x \in X$ there is $\delta_x > 0$ so that for all $y \in X$:

$$d(x, y) < \delta_x \implies |f(x) - f(y)| < \frac{\epsilon}{2} \quad \text{for all } f \in F$$

Then the collection $\{B_{\delta_x/2}(x) : x \in X\}$ is an open cover of X. Since X is compact, it has a finite subcover, indexed by $\{x_1, \cdots, x_n\}$. Let $\delta = \min\{\delta_{x_1}/2, \cdots, \delta_{x_n}/2\}$. Suppose $y_1, y_2 \in X$ and $d(y_1, y_2) < \delta$. Pick i so that $d(y_1, x_i) < \delta_{x_i}/2$. Then:

$$d(y_2, x_i) \leq d(y_2, y_1) + d(y_1, x_i) < \delta + \frac{\delta_{x_i}}{2} \leq \delta_{x_i}$$

Now we know $d(y_1, x_i) < \delta_{x_i}$ and $d(y_2, x_i) < \delta_{x_i}$. By the choice of δ_{x_i}, for all $f \in F$ we have:

$$|f(y_1) - f(y_2)| \leq |f(y_1) - f(x_i)| + |f(y_2) - f(x_i)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore F is uniformly equicontinuous. □

Theorem 2.13 (Arzela-Ascoli). Let (X, d) be a compact metric space. A subset $K \subseteq \mathcal{C}(X)$ is compact if and only if K is closed, bounded and equicontinuous.

Proof. ($\Rightarrow$). If K is compact then it is closed and bounded by Proposition 2.1. Also we know that K is equicontinuous by the lemma above.

($\Leftarrow$). Suppose K is closed, bounded and equicontinuous. Note that $\mathcal{C}(X)$ is complete and K is closed, so K is complete. It remains to show K is totally bounded. Let $\epsilon > 0$. Since K is equicontinuous, it is uniformly equicontinuous by the lemma above. There is $\delta > 0$ such that for all $f \in K$ and $x, y \in X$ we have:

$$d(x, y) < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{4} \tag{$*$}$$

Since X is compact, there is a δ-net:

$$F_X = \{x_1, \cdots, x_n\} \subseteq X \quad \text{and} \quad X \subseteq \bigcup_{i=1}^{n} B_\delta(x_i) \tag{$\dagger$}$$

Define $T : K \to (\mathbb{R}^n, \|\cdot\|_\infty)$ by:

$$T(f) = (f(x_1), \cdots, f(x_n))$$

Note that $\|T(f)\|_\infty = \max\{|f(x_i)| : 1 \leq i \leq n\} \leq \|f\|_\infty$. [Here is a bit of abusing of notation. The two $\|\cdot\|$-norm are on two different spaces.] This implies that $T(K)$ is bounded in $\mathbb{R}^n$ since K is bounded in $\mathcal{C}(X)$. This means $T(K)$ is totally bounded, thus $\overline{T(K)}$ is compact in $\mathbb{R}^n$. This means that there exists a $\epsilon/4$-net of $T(K)$:

$$F_T = \{T(f_1), \cdots, T(f_m)\} \subseteq T(K) \text{ and } T(K) \subseteq \bigcup_{i=1}^{m} B_{\epsilon/4}(f_i) \tag{††}$$

Here each $f_i \in K$. We claim that $F_K = \{f_1, \cdots, f_m\}$ is a ϵ-net for K. Indeed, let $f \in K$ be arbitrary. We can find some $j \in \{1, \cdots, m\}$ such that $\|T(f) - T(f_j)\|_\infty < \epsilon/4$ by (††). Now we let $y \in X$, we can find $i \in \{1, \cdots, n\}$ such that $d(x_i, y) < \delta$ by (†). Then:

$$|f(y) - f_j(y)| \leq \underbrace{|f(y) - f(x_i)|}_{< \frac{\epsilon}{4} \text{ by } (*)} + \underbrace{|f(x_i) - f_j(x_i)|}_{\leq \|T(f) - T(f_j)\|_\infty < \frac{\epsilon}{4}} + \underbrace{|f_j(x_i) - f_j(y)|}_{< \frac{\epsilon}{4} \text{ by } (*)} < \frac{3\epsilon}{4}$$

Since $y \in X$ is arbitrary, we have $\|f - f_j\|_\infty \leq \frac{3\epsilon}{4} < \epsilon$. This proved that $K \subseteq \bigcup_{i=1}^{m} B_\epsilon(f_j)$. Hence K is totally bounded. □

2.6 Connectedness

Definition. Let (X, d) be a metric space. We say $A \subseteq X$ is **disconnected** if there exist two open sets U, V of X such that $A \subseteq U \cup V$ and $U \cap V = \emptyset$ and $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$. We say A is **connected** if it is not disconnected.

Example. Let (X, d) be a metric space. Any finite subset $A = \{x_1, \cdots, x_n\}$ with at least two elements is disconnected. Let $r = \frac{1}{2}\min\{d(x_i, x_j) : i \neq j\} > 0$. We define open sets:

$$U = B_r(x_1) \text{ and } V = B_r(x_2) \cup \cdots \cup B_r(x_n)$$

Then $A \subseteq U \cup V$ and $U \cap V = \emptyset$ by our choice of r. Moreover $U \cap A = \{x_1\}$ and $V \cap A = \{x_2, \cdots, x_n\}$ are not empty. Therefore A is disconnected.

Example. Let X be a set with $|X| \geq 2$. Let d be the discrete metric on X. Then (X, d) is disconnected. Indeed, let $x_0 \in X$. Then $U = \{x_0\}$ is open and $V = X \setminus \{x_0\}$ is also open.

Example. The middle third cantor set is disconnected.

Example. The interval $[0, 1] \subseteq \mathbb{R}$ is connected. Assume it is disconnected by open sets U, V of $\mathbb{R}$. WLOG we may assume $0 \in U$. Let $C = \{c \in \mathbb{R} : [0, c) \subseteq U\}$. Since U is open, there is $\epsilon > 0$ so that $B_\epsilon(0) \subseteq U$. Since C is nonempty, we let $c^* = \sup C$. There are two cases.

(i). If $c^* \in U$. Then as U is open, there is $\epsilon > 0$ such that $B_\epsilon(c^*) \subseteq U$. This means $c^* + \epsilon \in U$, so we have $c^* + \epsilon \in C$. Contradiction.

(ii). If $c^* \in V$. There is $\epsilon > 0$ with $B_\epsilon(c^*) \subseteq V$. This means $B_\epsilon(c^*) \cap U = \emptyset$. However, by the definition of supremum we know $c^* - \epsilon \in C$, so $[0, c^* - \epsilon) \subseteq U$. This means $c^* - \frac{\epsilon}{2} \in U$, but we know $c^* - \frac{\epsilon}{2} \in V$ as well. Contradiction.

Theorem 2.14. Let (X, d) and (Y, ρ) be metric spaces. Suppose (X, d) is connected. If $f : X \to Y$ is continuous, then $f(X)$ is connected.

Proof. Assume $f(X)$ is disconnected, say by open sets U, V of (Y, ρ). It is easy to see that $f^{-1}(U)$ and $f^{-1}(V)$ are open sets that separate X. Contradiction. □

Theorem 2.15. Any connected subsets of $\mathbb{R}$ are intervals.

Proof. Let C be a connected set. We define:

$$a = \inf C \in \mathbb{R} \cup \{-\infty\} \quad \text{and} \quad b = \sup C \in \mathbb{R} \cup \{\infty\}$$

If $c \in \mathbb{R}$ and $a < c < b$ we must have $c \in C$. Otherwise:

$$C \subseteq \underbrace{(-\infty, c)}_{U} \cup \underbrace{(c, \infty)}_{V}$$

This gives a separation of C, contradiction. Hence we have $(a, b) \subseteq C \subseteq [a, b]$. This means C is an interval in $\mathbb{R}$. □

Definition. Let (X, d) be a metric space. We can define an equivalence relation on X by $x \sim y$ if and only if there is a connected set C containing both x, y. The equivalence classes of this relation are called **connected components**. Let $x_0 \in X$. the equivalence class that x_0 lies in is called the connected component of x_0 and it is equal to the union of all connected sets containing x_0.

Example. Let $X = [0, 1] \cup [2, 3]$ be the metric space with induced Euclidean metric. Then $[0, 1]$ and $[2, 3]$ are the connected components of X.

Definition. Let (X, d) be a metric space. We say X is **totally disconnected** if every connected component is a singleton set.

Example. Finite sets are totally disconnected.

Definition. Let (X, d) be a metric space. We say (X, d) is **path-connected** if for all $x, y \in X$ there exists a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.

Example. Let $(V, \|\cdot\|)$ be a normed space. Any convex set $C \subseteq V$ is path connected. For $x, y \in C$ we can define $f(t) = (1-t)x + ty \in C$.

Proposition 2.16. Let (X, d) be a metric space. If X is path-connected then X is connected.

Proof. Suppose $X = U \cup V$ is disconnected. Pick $x \in U$ and $y \in V$. There is a path $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$. Now:

$$[0,1] = f^{-1}(X) = f^{-1}(U) \cup f^{-1}(V)$$

Note that $0 \in f^{-1}(U)$ and $1 \in f^{-1}(V)$. It is easy to check $f^{-1}(U)$ and $f^{-1}(V)$ give a separation of $[0,1]$. This is a contradiction! $\square$

Example. The converse of this is not true. There exists connected spaces that is not path-connected. We define the following set:

$$X = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) : x > 0 \right\} \cup \{(0,0)\}$$

Then $X \subseteq \mathbb{R}^2$ is connected but not path connected.

2.7 Bonus Cantor Set Stuff

Definition. Let $n \geq 2$ and $A \subseteq \{0, 1, \cdots, n-1\}$ be a finite set. We define the **linear Cantor set**:

$$C_{A,n} = \left\{ \sum_{k=1}^{\infty} \frac{a_k}{n^k} : a_k \in A \right\}$$

Definition. Let $A \subseteq \mathbb{R}$. We define $N_\epsilon(A)$ to be the minimal number of ϵ-balls needed to cover A. The **box-counting dimension** of A is defined as:

$$\dim_B(A) = \lim_{\epsilon \to 0} \frac{-\log N_A(\epsilon)}{\log \epsilon}$$

if the limit exists. If the limit does not exist, we can take the limsup or liminf to define the **upper box dimension** and **lower box dimension**.

Example. Consider the middle third Cantor set. For $3^{-n} \leq \epsilon < 3^{-(n-1)}$, we need 2^n intervals of length $1/3^n$ to cover C. Hence:

$$\dim_B(C) = \lim_{n \to \infty} \frac{-\log 2^n}{\log 3^{-n}} = \frac{\log 2}{\log 3}$$

The box-counting dimension of C is $\log_3(2)$.

Definition. Let (X, ρ) be a metric space. For any $U \subseteq X$ we let $\operatorname{diam}(U)$ or $|U|$ denote its diameter. Let $S \subseteq X$ and let $\delta > 0$ and $d \in [0, \infty)$. We define:

$$H_\delta^d(S) = \inf\left\{\sum_{i\in I} |U_i|^d : S \subseteq \bigcup_{i\in I},\ |U_i| < \delta,\ |I| \le |\mathbb{N}|\right\}$$

Then we define:

$$H^d(S) = \lim_{\delta\to 0} H_\delta^d(S)$$

to be the **d-dimensional Hausdorff measure of S.**

Theorem 2.17. Let (X, ρ) be a metric space and $0 \le s < t < \infty$. For $A \subseteq X$ we have:

(i). If $H^s(A) < \infty$ then $H^t(A) = 0$.

(ii). If $H^t(A) > 0$ then $H^s(A) = \infty$.

Proof. It suffices to prove (i) since (ii) is just the contrapositive of (i). We have:

$$\begin{aligned}
H_\delta^t(A) &= \inf\left\{\sum_{i\in I} |U_i|^t : A \subseteq \bigcup_{i\in I} U_i,\ |U_i| < \delta,\ |I| \le |\mathbb{N}|\right\} \\
&= \inf\left\{\sum_{i\in I} |U_i|^{t-s}|U_i|^s : A \subseteq \bigcup_{i\in I} U_i,\ |U_i| < \delta,\ |I| \le |\mathbb{N}|\right\} \\
&\le \inf\left\{\sum_{i\in I} \delta^{t-s}|U_i|^s : A \subseteq \bigcup_{i\in I} U_i,\ |U_i| < \delta,\ |I| \le |\mathbb{N}|\right\} \\
&= \delta^{t-s} \inf\left\{\sum_{i\in I} |U_i|^s : A \subseteq \bigcup_{i\in I} U_i,\ |U_i| < \delta,\ |I| \le |\mathbb{N}|\right\} \\
&= \delta^{t-s} H_\delta^s
\end{aligned}$$

Suppose $H^s(A) < \infty$, we then have:

$$H^t(A) = \lim_{\delta\to 0} \delta^{t-s} H_\delta^s = H_\delta \lim_{\delta\to 0} \delta^{t-s} = 0$$

As desired. $\square$

Corollary 2.18. There is at most one $d \in [0, \infty)$ with $0 < H^d(A) < \infty$.

Definition. Same setting as above. We define the **Hausdorff dimension** of A to be:

$$\dim_H(A) = \sup\{d \in [0, \infty) : H^d(A) = \infty\} = \inf\{d \in [0, \infty) : H^d(A) = 0\}$$

Example. Let $A = \mathbb{Q} \cap [0,1]$. We need $\lceil \epsilon^{-1} \rceil$ many ϵ-balls to cover A, as $\mathbb{Q}$ is dense in $\mathbb{R}$. Hence:

$$\dim_B(A) = \lim_{\epsilon \to 0} \frac{-\log\lceil \epsilon^{-1} \rceil}{\log \epsilon} = 1$$

However, we claim the Hausdorff dimension is 0. Consider:

$$\begin{aligned} H_\delta^0(A) &= \inf\left\{ \sum_{i \in I} |U_i|^0 : A \subseteq \bigcup_{i \in I} U_i,\ |U_i| < \delta,\ |I| \leq |\mathbb{N}| \right\} \\ &= \inf\left\{ \sum_{i \in I} |U_i|^t : A \subseteq \bigcup_{i \in I} U_i,\ |U_i| < \delta,\ I \text{ finite} \right\} \\ &= \inf\left\{ |I| : A \subseteq \bigcup_{i \in I} U_i,\ |U_i| < \delta,\ I \text{ finite} \right\} \\ &= \left\lceil \frac{1}{\delta} \right\rceil \end{aligned}$$

Then we have $H^0(A) = \lim_{\delta \to 0} H_\delta^0(A) = \infty$. Let $d > 0$, we wish to show that $H^d(A) = 0$. To do this it suffices to show for all $\epsilon > 0$ and $\delta > 0$ we have $H_\delta^d(A) \leq \epsilon$. Since A is countable, we can enumerate $A = \{r_n : n \geq 1\}$. For each $n \geq 1$ let:

$$\epsilon_n = \min\left\{ \delta,\ \frac{1}{2}\left(\frac{\epsilon}{2^n}\right)^{1/d} \right\} > 0$$

Then let $U_n = B_{\epsilon_n}(r_n)$ and $|U_n| \leq \left(\frac{\epsilon}{2^n}\right)^{1/d}$. Hence we have:

$$\sum_{n=1}^{\infty} |U_n|^d \leq \sum_{n=1}^{\infty} \left(\left(\frac{\epsilon}{2^n}\right)^{1/d} \right)^d = \epsilon$$

Hence $H^d(A) = 0$ for all $d > 0$, so $\dim_H(A) = \inf\{d \geq 0 : H^d(A) = 0\} = 0$.

Proposition 2.19. For any linear Cantor set $C_{A,n}$ we have $\dim_B(C_{A,n}) = \dim_H(C_{A,n})$.

Proposition 2.20. Let $A, B \subseteq \mathbb{R}$, then:

$$\dim_H(A \cup B) = \max\{\dim_H(A), \dim_H(B)\}$$

Proposition 2.21. Let $A, B \subseteq \mathbb{R}$, then:

$$\dim_H(A + B) \leq \dim_H(A) + \dim_H(B)$$

Proposition 2.22. Let $\emptyset \neq A \subseteq \mathbb{R}^n$, then $0 \leq \dim_H(A) \leq n$.

Example. From A4 we saw that $C + C = [0, 2]$, where C is the middle-third Cantor set. That is:

$$C_{\{0,2\},3} + C_{\{0,2\},3} = [0, 2]$$

We know the box counting dimension is $\log_3(2)$, so $\dim_H(C_{\{0,2\},3}) = \log_3(2)$ as well.

Example. What is the dimension of $C_{\{0,3\},4}$ and the dimension of $C_{\{0,3\},4} + C_{\{0,3\},4}$? In general, we need 2^n intervals of length 4^{-n} to cover $C_{\{0,3\},4}$, so:

$$\dim_B(C_{\{0,3\},4}) = \lim_{n\to\infty} \frac{\log 2^n}{\log 4^{-n}} = \frac{1}{2} = \dim_H(C_{\{0,3\},4})$$

What does $C_{\{0,3\},4} + C_{\{0,3\},4}$ looks like?

$$\begin{aligned} C_{\{0,3\},4} + C_{\{0,3\},4} &= \left\{ \sum_{k=1}^{\infty} \frac{a_k + b_k}{4^k} : a_k, b_k \in \{0, 3\} \right\} = \left\{ \sum_{k=1}^{\infty} \frac{c_k}{4^k} : c_k \in \{0, 3, 6\} \right\} \\ &= \left\{ \sum_{k=1}^{\infty} \frac{c_k}{4^k} : c_k \in \left\{0, \frac{3}{2}, 3\right\} \right\} + \left\{ \sum_{k=1}^{\infty} \frac{c_k}{4^k} : c_k \in \left\{0, \frac{3}{2}, 3\right\} \right\} \\ &= 2C_{\{0,\frac{3}{2},3\},4} \end{aligned}$$

We see that:

$$\dim_H(C_{\{0,\frac{3}{2},3\},4}) = \dim_B(C_{\{0,\frac{3}{2},3\},4}) = \frac{\log 3}{\log 4} < 1$$

Theorem 2.23. Let $C_{A,n}$ be a linear Cantor set. If $\dim_H(C_{A,n}) < \frac{1}{2}$ then $C_{A,n} + C_{A,n} \neq [0, 2]$.

Proof. By Proposition 2.21 we have:

$$\dim_H(C_{A,n} + C_{A,n}) \leq \dim_H(C_{A,n}) + \dim_H(C_{A,n}) < 1$$

However $\dim_H([0, 2]) = 1$. Hence $C_{A,n} + C_{A,n} \neq [0, 2]$. $\square$

Example. Let $C \subseteq \mathbb{R}^n$ be a perfect and totally disconnected set with $\dim_H(C) < \frac{1}{2}$. Then $C + C$ is a perfect and totally disconnected set.

Theorem 2.24. Let $C_{A,n}$ be a linear Cantor set, then:

$$C_{A,n} = \bigcup_{a\in A} S_a(C_{A,n})$$

where $S_a : \mathbb{R} \to \mathbb{R}$ is defined by $S_a(x) = \frac{x+a}{n}$.

Proof. Note that we have:

$$\begin{aligned}
C_{A,n} &= \left\{ \frac{a_1}{n} + \sum_{k=2}^{\infty} \frac{a_k}{n^k} : a_k \in A \right\} \\
&= \bigcup_{a\in A} \left\{ \frac{a}{n} + \left\{ \sum_{k=2}^{\infty} \frac{a_k}{n^k} : a_k \in A \right\} \right\} \\
&= \bigcup_{a\in A} \left\{ \frac{a}{n} + \frac{1}{n} \left\{ \sum_{k=1}^{\infty} \frac{a_k}{n^k} : a_k \in A \right\} \right\} \\
&= \bigcup_{a\in A} \frac{a}{n} + \frac{1}{n} C_{A,n} \\
&= \bigcup_{a\in A} S_a(C_{A,n})
\end{aligned}$$

As desired. □

Theorem 2.25. Let $A \subseteq \{0, \cdots, n-1\}$ and $0, n-1 \in A$. Define:

$$B := A + A = \{0 = b_0 < b_1 < \cdots < b_k = 2n-2\}$$

Then $C_{A,n} + C_{A,n} = [0,2]$ if and only if $b_i - b_{i-1} \leq 2$ for all $1 \leq i \leq k$.

Proof. Note that we have:

$$C_{A,n} + C_{A,n} = \left\{ \sum_{r=1}^{\infty} \frac{a_r + c_r}{n^r} : a_r, c_r \in A \right\} = \left\{ \sum_{r=1}^{\infty} \frac{b_r}{n^r} : b_r \in B \right\} = C_{B,n}$$

Then $b_i - b_{i-1} \leq 2$ for all i if and only if $[0,2] = \bigcup_{i=0}^{k} S_{b_i}(C_{B,n}) = C_{B,n}$. □

Definition. A **Cantorval** is a compact subset of $\mathbb{R}$ with non-empty interior such that none of its connected components are isolated.

Fact. Let $A \subseteq \{0, \cdots, n-1\}$ and $0, n-1 \in A$. Exactly one of the followings is true:

1. $C_{A,n} + C_{A,n} = [0,2]$.
2. $C_{A,n} + C_{A,n}$ is a totally disconnected and perfect set.
3. $C_{A,n} + C_{A,n}$ ia a Cantorval.

Lecture 19, 2025/02/24

3 Completeness

3.1 Baire Category Theorem

Definition. Let (X, d) be a metric space. We say $A \subseteq X$ is **nowhere dense** if $\text{int}(\overline{A}) = \emptyset$.

Example. Consider $(\mathbb{R}, d)$ with Euclidean metric. A singleton is nowhere dense. The integers $\mathbb{Z}$ is nowhere dense. Rationals $\mathbb{Q}$ is NOT nowhere dense, as $\overline{\mathbb{Q}} = \mathbb{R}$. The Cantor set is nowhere dense.

Example. Consider the metric space (X, d) where d is the discrete metric. Any non-empty set is NOT nowhere dense because every $A \subseteq X$ is both open and closed, so:

$$\text{int}(\overline{A}) = \text{int}(A) = A \neq \emptyset$$

The only nowhere dense subset of (X, d) is $\emptyset$.

Lemma 3.1. Let (X, d) be a metric space. If $A \subseteq X$ is nowhere, then $X \setminus \overline{A}$ is open and dense.

Proof. Since $\overline{A}$ is closed, clearly $X \setminus \overline{A}$ is open. Suppose $x \notin X \setminus \overline{A}$ and let $\epsilon > 0$. We want to find $y \in X \setminus \overline{A}$ such that $y \in B_\epsilon(x)$, which proves that $X \setminus \overline{A}$ is dense in X. Since $x \notin X \setminus \overline{A}$, we know $x \in \overline{A}$. Since A is nowhere dense, $\text{int}(\overline{A}) = \emptyset$. Hence we can find $y \notin \overline{A}$ such that $y \in B_\epsilon(x)$, which means $y \in X \setminus \overline{A}$, as desired. □

Definition. Let (X, d) be a metric space. We say $A \subseteq X$ is **first category (meagre)** if we can write A as a countable union of nowhere dense sets. That is:

$$A = \bigcup_{n=1}^{\infty} K_n$$

where each $K_n \subseteq X$ is nowhere dense. When X is first category as a set, then we also say (X, d) is first category. Otherwise we say A is **second category**.

Example. Consider $(\mathbb{R}, d)$ with the usual metric. Any nowhere dense set is first category. The rationals $\mathbb{Q}$ is first category because it is the countable union of $q \in \mathbb{Q}$.

Question: Is $\mathbb{R}$, with the usual metric, first category?

Answer: It is not first category (not obvious) by the Baire Category Theorem.

Theorem 3.2 (Baire Category Theorem). Any non-empty complete metric space (X, d) is second category.

Example. The reals $\mathbb{R}$ is not first category. The ℓ^p spaces for $1 \leq p < \infty$ are not first category. This does not apply to $(\mathbb{Q}, d)$ with the Euclidean metric sicne it is not complete.

Corollary 3.3. Let (X, d) be a non-empty complete metric space with $X = \bigcup_{n=1}^{\infty} K_n$, then there is $n \geq 1$ such that $\text{int}(\overline{K}_n) \neq \emptyset$.

Proof. We know X is not first category by the BCT, so one of K_n is not nowhere dense. □

Proof of BCT. Assume $(K_n)_{n=1}^{\infty}$ is a sequence of nowhere dense sets, we want to show $X \neq \bigcup_{n=1}^{\infty} K_n$ by constructing $x^* \in X$ such that $x^* \notin K_n$ for all $n \geq 1$. Pick any $x_0 \in X$ and $r_0 > 0$. Consider $\overline{B_{r_0}(x_0)}$. Since K_1 is nowhere dense, we can find $x_1 \in \overline{B_{r_0}(x_0)}$ and $r_1 < r_0/2$ such that:

$$\overline{B_{r_1}(x_1)} \cap K = \emptyset \quad \text{and} \quad \overline{B_{r_1}(x_1)} \subseteq \overline{B_{r_0}(x_0)}$$

We repeat this process. Suppose we have defined x_n and r_n, we find x_{n+1} and r_{n+1} such that $x_{n+1} \in \overline{B_{r_n}(x_n)}$ and $r_n < r_{n-1}/2$ with:

$$\overline{B_{r_{n+1}}(x_{n+1})} \cap K = \emptyset \quad \text{and} \quad \overline{B_{r_{n+1}}(x_{n+1})} \subseteq \overline{B_{r_n}(x_n)}$$

We claim that $(x_n)_{n=1}^{\infty}$ is cauchy and its limit x^* satisfies our desired property. Let $m > n$, notice:

$$d(x_n, x_m) \leq r_n < \frac{r_{n-1}}{2} < \cdots < \frac{r_0}{2^n}$$

Therefore $(x_n)_{n=1}^{\infty}$ is cauchy. Since (X, d) is complete, we let $\lim\limits_{n \to \infty} x_n = x^* \in X$. Note that $(x_n)_{n=k}^{\infty}$ is a sequence in $\overline{B_{r_k}(x_k)}$ for all $k \geq 1$ and each such closed ball is closed. Therefore:

$$x^* = \lim_{n \to \infty} x_n \in \overline{B_{r_k}(x_k)}$$

Hence $x^* \in \overline{B_{r_n}(x_n)}$ for all $n \geq 1$. Hence $x^* \notin K_n$ for all $n \geq 1$, as desired. □

Lecture 20, 2025/02/26

Definition. Let (X, d) be a metric space. We say $A \subseteq X$ is a G_δ **set** if there exist a countable sequence of open sets $U_n \subseteq X$ such that $A = \bigcap_{n=1}^{\infty} U_n$.

Example. Any open set is a G_δ set by definition.

Example. The irrational numbers are a G_δ set. Note that $\mathbb{Q}$ is countable, so:

$$\mathbb{R} \setminus \mathbb{Q} = \bigcap_{r \in \mathbb{Q}} (\mathbb{R} \setminus \{r\})$$

Definition. Let (X, d) be a metric space. We say $A \subseteq X$ is an F_σ **set** it there is a countable sequence of closed sets $C_n \subseteq X$ such that $A = \bigcup_{n=1}^{\infty} C_n$.

Remark. Note that A is G_δ if and only if A^c is F_σ.

Example. Any closed set is a F_σ set.

Example. The interval $A = (0,1)$ is an F_σ set because $(0,1) = \bigcup_{n=2}^{\infty} \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$.

Example. Note that $\mathbb{Q} = \bigcup_{r \in \mathbb{Q}} \{r\}$ is F_σ. However, we claim that $\mathbb{Q}$ is NOT a G_δ set! Assume for a contradiction that $\mathbb{Q}$ is a G_δ set, say:

$$\mathbb{Q} = \bigcap_{n=1}^{\infty} U_n$$

where each $U_n \subseteq \mathbb{R}$ is an open set. This means $\mathbb{Q} \subseteq U_n$ for all $n \geq 1$. Hence each U_n is an open denset set. Then $\mathbb{R} \setminus U_n$ is closed and nowhere dense. This means:

$$\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n=1}^{\infty} (\mathbb{R} \setminus U_n)$$

is a union of nowhere dense sets! Hence $\mathbb{R} \setminus \mathbb{Q}$ is first category. Since $\mathbb{Q}$ is first category, we have:

$$\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$$

is first category, being the union of two sets that are first category. Since $\mathbb{R}$ is complete, it is second category by BCT. Contradiction.

3.2 Nowhere Differentiable Functions

For this section we consider the space:

$$\mathcal{C}[0,1] = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$$

We will show that “most” functions $f \in \mathcal{C}[0,1]$ are nowhere differentiable!

Definition. Let $f \in \mathcal{C}[0,1]$. We say f is **Lipschitz at** $x_0 \in X$ if there is $K \in \mathbb{R}$ (dependent on x_0) such that for all $x \in [0,1]$ we have:

$$|f(x_0) - f(x)| \leq K|x_0 - x|$$

We say f is **Lipschitz** if the choice of K is independent of x_0.

Lemma 3.4. Let $f \in \mathcal{C}[0,1]$ and $x_0 \in [0,1]$. Assume $f'(x_0)$ exists, then f is Lipschitz at x_0.

Proof. Let $c_1 = |f'(x_0)| \geq 0$. This implies that:

$$c_1 = \lim_{x \to x_0} \left| \frac{f(x) - f(x_0)}{x - x_0} \right|$$

There exists $\delta > 0$ (small enough such that $(x_0-\delta, x_0+\delta) \subseteq [0,1]$) such that for all $x \in (x_0-\delta, x_0+\delta)$ we have that:

$$\left|\frac{f(x)-f(x_0)}{x-x_0}\right| \leq c_1 + 1$$

Consider the function $h(x) = \frac{f(x)-f(x_0)}{x-x_0}$ on the set $[0, x_0-\delta] \cup [x_0+\delta, 1]$. Note that h is continuous on this compact set, hence it is bounded on it. Let $c_2 \in \mathbb{R}$ such that for all $x \in [0, x_0-\delta] \cup [x_0+\delta, 1]$ we have that:

$$\left|\frac{f(x)-f(x_0)}{x-x_0}\right| \leq c_2$$

Let $K = \max\{c_1+1, c_2\} > 0$, then for all $x \in [0,1]$ we have:

$$\left|\frac{f(x)-f(x_0)}{x-x_0}\right| \leq K \implies |f(x)-f(x_0)| \leq K|x-x_0|$$

As desired. □

Lemma 3.5. Let $f \in \mathcal{C}[0,1]$ be Lipschitz at $x_0 \in [0,1]$ with constant K. Then for all $a, b \in [0,1]$ with $a \leq x_0 \leq b$ we have:

$$|f(a)-f(b)| \leq K|a-b|$$

Proof. Since $a \leq x_0 \leq b$ we have $|a-x_0| + |x_0-b| = |a-b|$. Then:

$$|f(a)-f(b)| \leq |f(a)-f(x_0)| + |f(x_0)+f(b)| \leq K(|a-x_0|+|x_0-b|) = K|a-b|$$

As desired. □

Example. Define a function $f : [0,1] \to \mathbb{R}$ by $f(x) = \sum_{n=1}^{\infty} 2^{-n}\cos(\pi 10^n x)$. We claim that $f \in \mathcal{C}[0,1]$ but nowhere differentiable! It suffices to show it is the limit of a sequence of continuous functions. For each $N \geq 1$ let:

$$f_N(x) = \sum_{n=1}^{N} 2^{-n}\cos(\pi 10^n x)$$

Then each $f_N \in \mathcal{C}[0,1]$. We claim that $(f_N)_{N=1}^{\infty}$ is cauchy. Let $N > M$, we have:

$$\|f_N - f_M\|_\infty = \sup_{x\in[0,1]}\left|\sum_{n=M+1}^{N} 2^{-n}\cos(\pi 10^n x)\right| \leq \sum_{n=M+1}^{N} 2^{-n} \to 0$$

because the series $\sum_{n=1}^{\infty} 2^{-n} = 1$ converges, its tail goes to 0. Therefore $(f_N)_{N=1}^{\infty}$ is cauchy and since $\mathcal{C}[0,1]$ is complete, it converges to $f \in \mathcal{C}[0,1]$. To show it is nowhere differentiable, it suffices to show it is not Lipschitz at any $x_0 \in [0,1]$. Write $x_0 = \sum_{k=1}^{\infty} \frac{a_k}{10^k}$ in base 10. Suppose f is Lipschitz at x_0

with constant $K \in \mathbb{R}$. Let $N \geq 1$ (to be chosen later), we define:

$$x_L = \sum_{k=1}^{N} \frac{a_k}{10^k} \quad \text{and} \quad x_R = x_L + \frac{1}{10^N}$$

We consider the difference between $f(x_R)$ and $f(x_L)$. Note that:

$$\cos(x) - \epsilon \leq \cos(x + \epsilon) \leq \cos(x) + \epsilon \tag{1}$$

for $\epsilon > 0$ small. By (1), for $1 \leq k \leq N$ we have:

$$\cos(\pi 10^k (x_L + 10^{-N})) - \cos(\pi 10^k x_L) = \pi 10^{-N+k}$$

For $k > N$, note that $10^k x_L$ and $10^k x_R$ are integers so:

$$\cos(\pi 10^k (x_L + 10^{-N})) - \cos(\pi 10^k x_L) = 0$$

With some work, this gives:

$$|f(x_L) - f(x_R)| \geq (5^N + \text{small stuff})|x_L - x_R|$$

If we pick N so that $5^N > K$ then this gives a contradiction.

Lecture 21, 2025/03/03

Theorem 3.6. Consider $\mathcal{C}[0,1]$. The set of $f \in \mathcal{C}[0,1]$ that are Lipschitz at at least one point are first category.

Proof. For each $k \geq 1$ we define:

$$A_k = \{f \in \mathcal{C}[0,1] : f \text{ is Lipschitz somewhere with constant } k\}$$

We see that $A_k \subseteq A_{k+1}$ for all $k \geq 1$. The set of functions that are Lipschitz somewhere is:

$$L = \bigcup_{k=1}^{\infty} A_k$$

We want to show L is first category. It suffices to show every A_k is nowhere denese. We first claim that A_k is closed for all $k \geq 1$. Let $(f_n)_{n=1}^{\infty}$ be a cauchy sequence in A_k. Since $C[0,1]$ is complete, we know $f_n \to f^*$ in $\mathcal{C}[0,1]$. We need to show $f^* \in A_k$. Let $(x_n)_{n=1}^{\infty}$ be a sequence such that f_n is Lipschitz with some constant k at x_n, As $[0,1]$ is compact, it will have a convergent subsequence. Say $(x_{n_i})_{i=1}^{\infty}$ converges to x^* in $[0,1]$. We claim that f^* is Lipschitz at x^* with constant k. For any $x \in [0,1]$ and $i \geq 1$ we have:

$$\begin{aligned}
|f^*(x) - f^*(x^*)| &= |f^*(x) - f_{n_i}(x) + f_{n_i}(x) - f_{n_i}(x_{n_i}) + f_{n_i}(x_{n_i}) - f_{n_i}(x^*) + f_{n_i}(x^*) - f^*(x^*)| \\
&\leq |f^*(x) - f_{n_i}(x)| + |f_{n_i}(x) - f_{n_i}(x_{n_i})| + |f_{n_i}(x_{n_i}) - f_{n_i}(x^*)| + |f_{n_i}(x^*) - f^*(x^*)| \\
&< \|f^* - f_{n_i}\|_{\infty} + k|x - x_{n_i}| + k|x_{n_i} - x^*| + \|f_{n_i} - f^*\|_{\infty}
\end{aligned}$$

By taking the limit as $i \to \infty$ we know $\|f^* - f_{n_i}\|_\infty \to 0$ and $\|x_{n_i} - x^*\| \to 0$, so we have:

$$|f^*(x) - f^*(x^*)| \leq k|x - x^*|$$

Since $x \in [0,1]$ is arbitrary, this proved that f^* is Lipschitz at x^* with constant k. Therefore $f^* \in A_k$ and A_k is closed. To show A_k is nowhere dense, it suffices to show $\text{int}(A_k) = \emptyset$ as A_k is closed. This means for all $\epsilon > 0$ and $f \in A_k$ we can find some $g \notin A_k$ with $\|f - g\|_\infty < \epsilon$. In fact, we can find g that is nowhere differentiable with $\|f - g\|_\infty < \epsilon$.

Pick $f \in A_k$ and $\epsilon > 0$. There is a polynomial function $p(x)$ such that $\|f - p\|_\infty < \frac{\epsilon}{2}$. Such a polynomial exists by the Stone-Weierstrass theorem that we will see later. By the example we did last lecture, we have a function that is differentiable nowhere, call it h. Recall that the h we constructed last time has $\|h\|_\infty \leq 1$. Now $p + \frac{\epsilon}{2}h$ is differentiable nowhere and:

$$\left\|f - \left(p + \frac{\epsilon}{2}h\right)\right\|_\infty \leq \|f - p\|_\infty + \frac{\epsilon}{2}\|h\|_\infty < \epsilon$$

This proved that $\text{int}(A_k) = \emptyset$, thus L is first category. $\square$

Corollary 3.7. The set of functions that are differentiable somewhere is first category in $\mathcal{C}[0,1]$.

3.3 Contraction Mapping Principle

Theorem 3.8 (Contraction Mapping Principle). Let (X,d) be a complete metric space. Let $f : X \to X$ be Lipschitz with constant $K < 1$ (such function is called a **contraction**). Then:

(i). There is a unique $x^* \in X$ with $f(x^*) = x^*$. [Existence and Uniqueness of fixed point]

(ii). For any $x_0 \in X$, we can construct a sequence $(x_n)_{n=1}^\infty$ by $x_{n+1} = f(x_n)$. Then $(x_n)_{n=1}^\infty$ is cauchy and we have $x_n \to x^*$.

Example. Let $X = \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{1}{2}x + \frac{1}{2}$. Note that $x^* = 1$ is a fixed point. Let $x_0 = 2$ then $x_1 = 1 + 1/2 = 3/2$ and $x_2 = 1 + 1/4 = 5/4$. In general for $n \geq 0$ we have $x_n = 1 + 1/2^n$ and it is easy to see that $x_n \to 1$.

Proof. Pick $x_0 \in X$ and define $(x_n)_{n=1}^\infty$ by $x_{n+1} = f(x_n)$ for $n \geq 0$. We claim that $(x_n)_{n=1}^\infty$ is Cauchy. Let $d(x_0, x_1) = c \geq 0$. Then:

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq K \cdot d(x_n, x_{n-1}) \leq \cdots \leq K^n \cdot d(x_1, x_0) = K^n \cdot c$$

In general, if $m > n$ we have:

$$d(x_m, x_n) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq c\sum_{i=n}^{m-1} K^i \leq c\sum_{i=n}^{\infty} K^i$$

Since $K < 1$, this is a tail of a convergent series $\sum_{i=1}^{\infty} K^i$. Hence this $d(x_m, x_n) \to 0$ as $n \to \infty$. Therefore $(x_n)_{n=1}^{\infty}$ is cauchy, as desired. Since X is complete, $x_n \to x^*$ in X. Hence:

$$f(x^*) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x^*$$

Assume we have two fixed points x^* and y^* in X, then:

$$d(x^*, y^*) = d(f(x^*), f(y^*)) \leq K \cdot d(x^*, y^*)$$

Hence $d(x^*, y^*) = 0$, so the fixed point is unique. □

Lecture 22, 2025/03/05

Example (Logistic Equation). For $\lambda \in [0, 4]$ we define $f_\lambda : [0, 1] \to \mathbb{R}$ by $f_\lambda(x) = \lambda x(1 - x)$. This is used in population dynamics. Here λ represents the birth rate and x represents the current population and $1 - x$ represents the impact of limited resources.

Let $\lambda \in [0, 1)$, then we have:

$$\begin{aligned} |f_\lambda(x) - f_\lambda(y)| &= |\lambda x(1 - x) - \lambda y(1 - y)| \\ &= |\lambda x(1 - x - y) - \lambda y(1 - x - y)| \\ &= \lambda |x - y||1 - x - y| \\ &< \lambda |x - y| && (|1 - x - y| \in [0, 1]) \end{aligned}$$

This means f_λ is Lipschitz with constant $\lambda < 1$. Hene it has a unique attractive fixed point satisfies $x^* = \lambda x^*(1 - x^*)$. In fact $x^* = 0$. This species is headng for extinction.

Theorem 3.9. Let $f : \mathbb{R} \to \mathbb{R}$ with continuous derivative. Suppose $p \in \mathbb{R}$ is a fixed point of f such that $|f'(p)| < 1$. Then there exists $a, b \in \mathbb{R}$ with $a < p < b$ such that $f : [a, b] \to [a, b]$ is Lipschitz on $[a, b]$ with constant $K < 1$.

Proof. There exists an interval $[a, b]$ such that $|f'(x)| \leq K < 1$ (with $|f'(p)| < K < 1$). This is because we have a continuous derivative. We see for all $x, y \in [a, b]$ there is $c \in [x, y]$ by the mean value theorem that:

$$\left|\frac{f(x) - f(y)}{x - y}\right| = |f'(c)| \leq K$$

Hence f is Lipschitz on $[a, b]$ with constant K. □

Example. Consider the Logistic equation again. Consider $f_\lambda(x) = \lambda x(1 - x)$ with $\lambda \in (1, 3)$. This has two fixed points, $x = 0$ or $x = 1 - \lambda^{-1}$. We have $f'_\lambda(x) = \lambda - 2\lambda x$. At $x = 0$ we have $f'_\lambda(x) = \lambda > 1$, so 0 is NOT an attractive fixed point. At $x^* = 1 - 1/\lambda$ then we have:

$$f'(x^*) = \lambda - 2\lambda\left(1 - \frac{1}{\lambda}\right) = 2 - \lambda < 1$$

Hence there is $a, b \in \mathbb{R}$ with $a < 1 - \lambda^{-1} < b$ such taht for all $x_0 \in [a, b]$ we have $x_n \to 1 - \lambda^{-1}$, with $x_n = f_\lambda(x_{n-1})$. This means we have a stable population.

Definition. Let K be the set of all compacts sets in $\mathbb{R}^n$. For $A, B \in K$ we define:

$$\begin{aligned} d_H(A, B) &= \inf\{\epsilon > 0 : A \subseteq B + B_\epsilon(0), B \subseteq A + B_\epsilon(0)\} \\ &= \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\} \end{aligned}$$

This is known as the **Hausdorff metric** on K.

Example. Let $A = [0, 1]$ and $B = [\frac{1}{3}, \frac{3}{2}]$. Note that:

$$A \subseteq B + B_{1/3+\epsilon}(0) \quad \text{and} \quad B \subseteq A + B_{1/2+\epsilon}(0)$$

for all $\epsilon > 0$. Hence the $d_H(A, B) = \frac{1}{2}$.

Remark. This indeed gives us a metric. [Exercise]

Construction 3.10. Let $n = 1$ and K denote the set of compact sets in $\mathbb{R}$. Consider the following map $S : K \to K$ defined by:

$$S(A) = \frac{1}{3}A \cup \left(\frac{1}{3}A + \frac{2}{3}\right)$$

If $A = [1, 2]$ then $S(A) = [\frac{1}{3}, \frac{2}{3}] \cup [1, \frac{4}{3}]$. If we let $A_0 = A$ and define $A_n = S(A_{n-1})$, then A_n converges to the cantor set in (K, d_H). In fact, we can prove S is Lipschitz with constant $1/3$ and thus by the contraction mapping principle, it has a unqiue fixed point C, the cantor set.

Let $A, B \in K$ be arbitrary. Say $d_H(A, B) = c > 0$ (if $A = B$ then trivial). This means $A \subseteq B + B_c(0)$ and $B \subseteq A + B_c(0)$. Note that:

$$S(A) = \frac{1}{3}A \cup \left(\frac{1}{3}A + \frac{2}{3}\right) \quad \text{and} \quad S(B) = \frac{1}{3}B \cup \left(\frac{1}{3}B + \frac{2}{3}\right)$$

Since $A \subseteq B + B_c(0)$ we have:

$$\frac{1}{3}A \subseteq \frac{1}{3}(B + B_c(0)) = \frac{1}{3}B + B_{c/e}(0) \quad \text{and} \quad \frac{1}{3}B \subseteq \frac{1}{3}A + B_{c/3}(0)$$

Similarly we have:

$$\begin{aligned} \frac{1}{3}B &\subseteq \frac{1}{3}A + B_{c/3}(0) \\ \frac{1}{3}A + \frac{2}{3} &\subseteq \frac{1}{3}B + \frac{2}{3} + B_{c/3}(0) \\ \frac{1}{3}B + \frac{2}{3} &\subseteq \frac{1}{3}A + \frac{2}{3} + B_{c/3}(0) \end{aligned}$$

Hence we have $d_H(S(A), S(B)) = \frac{c}{3} = \frac{1}{3}d_H(A, B)$. With some work, we can show K is complete. Therefore the map S has a unique attractive fixed point, which is the Cantor set!

Lecture 23, 2025/03/07

3.4 Newton's Method

We see from the previous result taht if g has a fixed point $g(p) = p$ adn $|g'(p)| = \lambda < 1$, then p is an attractive fixed point, within a interval around p. Moreover:

$$|g^{(n)}(x) - p| \leq \lambda^n |x - p| \qquad \text{(approximately)}$$

We see that the smaller λ is , the faster the convergence is. This implies that $\lambda = 0$ is ideal. This is what is explained by Newton's method.

Theorem 3.11. Let f be twice continuously differentiable such that $f(p) = 0$ and $f'(p) \neq 0$ for some $p \in \mathbb{R}$. Define g by:

$$g(x) = x - \frac{f(x)}{f'(x)}$$

Then $g(p) = p$ and $g'(p) = 0$.

Proof. It is easy to see that $g(p) = p$. Moreover,

$$g'(p) = 1 - \frac{f'(p)f'(p) - f''(p)f(p)}{(f'(p))^2} = 1 - 1 = 0$$

As desired. $\square$

Corollary 3.12. If we start sufficently close to p, then we are attracted to p.

Consider the Taylor polynomial of g around $x = p$, we have:

$$\begin{aligned} g(x) &= g(p) + g'(p)(x-p) + \frac{g''(p)}{2!}(x-p)^2 + \cdots \\ &= p + 0 + C(x-p)^2 + \cdots \end{aligned}$$

That is, if $x = p + \epsilon$, then $g(x) \approx p + C\epsilon^2$. [**Quadratic convergence**]

3.5 Metric Completion

Definition. Let (X, d) be a metric space. We say (Y, ρ) is a **completion** of (X, d) if (Y, ρ) is a complete space and there exists an **isometry** $J : X \to Y$ (that is, $\rho(Jx, Jy) = d(x, y)$ for all $x, y \in X$) such that $\overline{JX} = Y$ (JX is dense in Y).

Remark. Our goal is to show that for a metric space (X, d),

1. The completion of (X, d) always exists.

2. The completion of (X, d) is unique (up to isometric isomorphism).

3. Show the completion of $\mathbb{Q}$ is $\mathbb{R}$.

4. Discuss the completion of $\mathbb{Q}$ with p-adic metric.

Theorem 3.13. Every metric space has a completion.

Proof. Let (X, d) be a metric space. Recall that:

$$\mathcal{C}^b(X) = \{f : X \to \mathbb{R} \mid f \text{ continuous and bounded}\}$$

is a complete metric space (in fact normed) with norm $\|f\|_\infty = \sup_{x\in X} |f(x)|$. We will find a closed subset $Y \subseteq \mathcal{C}^b(X)$ (then Y is complete) and an isometry $J : X \to Y$ such that JX is dense in Y.

Fix $x_0 \in X$. We define $J : X \to \mathcal{C}^b(X)$ by $J(x) = f_x$, where:

$$f_x(y) = d(x, y) - d(x_0, y)$$

We claim that these functions have the desired property. Clearly each f_x is continuous, as $d(x, \cdot)$ and $d(x_0, \cdot)$ are both continuous. To see they are bounded, we note that:

$$f_x(y) = d(x, y) - d(x_0, y) \leq d(x, x_0)$$

by the triangle inequality. For fixed $x \in X$, we know $d(x, x_0)$ is a constnat. Hence f_x is bounded above. Similarly we have:

$$f_x(y) = d(x, y) - d(x_0, y) \geq -d(x, x_0)$$

by the triangle inequality again. Hence $f_x \in \mathcal{C}^b(X)$ for all $x \in X$. Now we claim $J : x \mapsto f_x$ is an isometry. Indeed, let $x, z \in X$ we have:

$$\begin{aligned}
\|f_x - f_z\|_\infty &= \sup_{y\in X} |d(x, y) - d(x_0, y) - d(z, y) + d(x_0, y)| \\
&= \sup_{y\in X} |d(x, y) - d(z, y)| \\
&\leq \sup_{y\in X} d(x, z) = d(x, z)
\end{aligned}$$

This bound can be achieved at $y = x$, so we get $\|f_x - f_z\|_\infty = d(x, z)$. Let $Y = \overline{JX}$ in $\mathcal{C}^b(X)$, so Y is complete, being a closed subset of a complete space. By construction we have $\overline{JX} = Y$, so JX is dense in Y. Since J is an isometry, $(Y, \|\cdot\|_\infty)$ is a completion of X. $\square$

Example. Let $X = \mathbb{Q} \cap [0, 1]$ with usual metric d. Let $x_0 = \frac{1}{2}$. We can approach $\pi/10$ with a cauchy sequence $(x_n)_{n=1}^\infty$ in (X, d).

Lecture 24, 2025/03/10

Construction 3.14. Let (X, d) be a metric space. We define:

$$Z = \{\text{cauchy sequences in } X\}$$

We define a psuedo-metric $\tilde{\rho}$ on X by:

$$\tilde{\rho}((x_n), (y_n)) = \lim_{d \to \infty} d(x_n, y_n)$$

for $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in Z. This is possbily NOT a metric because (for example the distance between $(1, 0, \cdots)$ and $(0, 0, \cdots)$ is zero but they are different).

We say two Cauchy sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are **equivalent** if $\tilde{\rho}((x_n), (y_n)) = 0$. Let:

$$Y = Z/\sim = \{\text{equivalence classes of cauchy sequences in } X\}$$

Let $J : X \to Y$ by $J(x) = [(x_n)_{n=1}^{\infty}]$ where $x_n = x$ for all $n \geq 1$. Then J and Y have all the desired properties. Hence Y is a completion of X.

Theorem 3.15. Let (X, d) be a metric space with completion (Y, ρ) and $J : X \to Y$. Let (Z, σ) be a complete metric space and $f : X \to Z$ is uniformly continuous. Then there is a unique uniformly continuous map $\tilde{f} : Y \to Z$ with $\tilde{f}(J(x)) = f(x)$ for all $x \in X$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "J"'] & Z \\ Y \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

Corollary 3.16. Let (X, d) be a metric space with completion (Y, ρ) and (Z, σ) given by $J_Y : X \to Y$ and $J_Z : X \to Z$, respectively. Then J_Y and J_Z can be extended to isometries $\tilde{J}_Y : Z \to Y$ and $\tilde{J}_Z : Y \to Z$ such that $\tilde{J}_Z$ and $\tilde{J}_Y$ are inverses of each other.

Proof. By Theorem 3.15, we have these two diagrams:

$$\begin{tikzcd} X \arrow[r, "J_Z"] \arrow[d, "J_Y"'] & Z \\ Y \arrow[ur, "\tilde{J}_Z"'] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "J_Y"] \arrow[d, "J_Z"'] & Y \\ Z \arrow[ur, "\tilde{J}_Y"'] & \end{tikzcd}$$

Hence $\tilde{J}_Z$ and $\tilde{J}_Y$ exist. Since $J_Z = \tilde{J}_Z \circ J_Y$ and $J_Y = \tilde{J}_Y \circ J_Z$, it is not hard to see $\tilde{J}_Y$ and $\tilde{J}_Z$ are inverses of each other. Now we want to show $\tilde{J}_Y$ and $\tilde{J}_Z$ are isometries. We first show $\tilde{J}_Z : Y \to Z$

is an isometry. Let $a, b \in Y$ be arbitrary. We want to show that:

$$\rho(a,b) = \sigma(\tilde{J}_Z(a), \tilde{J}_Z(b))$$

Since $J_Y(X)$ is dense in Y, we can find sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ in X such that:

$$a = \lim_{n\to\infty} J_Y(a_n) \quad \text{and} \quad b = \lim_{n\to\infty} J_Y(b_n)$$

Now, by continuity we have that:

$$\begin{aligned}\rho(a,b) &= \lim_{n\to\infty} \rho(J_Y(a_n), J_Y(b_n)) = \lim_{n\to\infty} d(a_n, b_n) = \lim_{n\to\infty} \sigma(J_Z(a_n), J_Z(b_n)) \\ &= \lim_{n\to\infty} \sigma(\tilde{J}_Z(J_Y(a_n)), \tilde{J}_Z(J_Y(b_n))) = \sigma(\tilde{J}_Z(a), \tilde{J}_Z(b))\end{aligned}$$

Hence $\tilde{J}_Z$ is an isometry. By the same argument, $\tilde{J}_Y$ is an isometry. $\square$

Proof of Theorem 3.15. Step 1. Let $(x_n)_{n=1}^{\infty}$ be a cauchy sequence in X, we claim $(f(x_n))_{n=1}^{\infty}$ is a cauchy sequence in Z. Let $\epsilon > 0$. Since $f : X \to Z$ is uniformly continuous, there is $\delta > 0$ such that for $x, y \in X$:

$$d(x,y) < \delta \implies \sigma(f(x), f(y)) < \epsilon$$

Since $(x_n)_{n=1}^{\infty}$ is cauchy, we can find $N \geq 1$ such that for all $n, m \geq N$ we have $d(x_n, x_m) < \delta$. Hence for $n, m \geq N$ weh ave $\sigma(f(x_n), f(x_m)) < \epsilon$. Therefore $(f(x_n))_{n=1}^{\infty}$ is cauchy in (Z, σ).

Step 2. Let $y \in Y$. Since JX is dense in Y, we can find a cauchy sequence $(x_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} J(x_n) = y$. We define:

$$\tilde{f}(y) = \lim_{n\to\infty} f(x_n)$$

Since Z is complete, this limit exists (shown in Step 1 that $(f(x_n))_{n=1}^{\infty}$ is cauchy). Now we need to check this definition is well-defined. That is, we need to show this definition does not depend on the choice of the cauchy sequences. If we chose two different cauchy sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in X such that:

$$y = \lim_{n\to\infty} J(x_n) = \lim_{n\to\infty} J(y_n)$$

Construct a new Cauchy sequence $(z_n)_{n=1}^{\infty} = (x_1, y_1, x_2, y_2, \cdots)$. We see that:

$$\tilde{f}(y) = \lim_{n\to\infty} f((z_n)_{n=1}^{\infty}) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n)$$

Therefore $\tilde{f}$ does not depend on the choices of cauchy sequences. Hence $\tilde{f}$ is well-defined. Moreover, for $x \in X$ we can pick the constant cauchy sequence $(x)_{n=1}^{\infty}$ with $y = J(x)$. Then we have:

$$\tilde{f}(y) = \lim_{n\to\infty} f(x) = f(x)$$

Therefore we have $\tilde{f}(J(x)) = f(x)$ for all $x \in X$.

Step 3. We need to show $\tilde{f}$ is uniformly continuous. Pick $\epsilon > 0$. Pick $\delta > 0$ such that for $x, y \in X$ with $d(x, y) < \delta$ we have $\sigma(f(x), f(y)) < \epsilon$. Pick points in Y with $\rho(y_1, y_2) < \delta$. Find cauchy sequence $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ in X such taht $J(a_n) \to y_1$ nad $J(b_n) \to y_2$. We can pick $N \geq 1$ sufficently large so that $d(a_n, b_n) < \delta$ for $n > N$. Hence $\sigma(f(a_n), f(b_n)) < \epsilon$ for $n > N$. Therefore:

$$\sigma(\tilde{f}(y_1), \tilde{f}(y_2)) \leq \epsilon$$

by taking the limit. Therefore $\tilde{f}$ is uniformly continuous. □

Lecture 25, 2025/03/12

3.6 The Real Numbers

Definition. A **field** is a bunch of things you can add and mulitply and subtract and divide (by nonzero elements).

Example. Rational, real and complex numbers are fields. $\mathbb{Z}/p\mathbb{Z}$ is a field for prime p. The rational functions $\mathbb{R}(x)$ is a field.

Definition. We say a field $\mathbb{F}$ is an **ordered field** if we can write $\mathbb{F}$ with a disjoint union:

$$\mathbb{F} = \mathbb{P} \sqcup \{0\} \sqcup (-\mathbb{P})$$

such that $a, b \in \mathbb{P}$ impleis $a + b \in \mathbb{P}$ and $ab \in \mathbb{P}$. Think of $\mathbb{P}$ as the set of positive elements in $\mathbb{F}$.

Lemma 3.17. If $\mathbb{F}$ is an ordered field, then $1 \in \mathbb{P}$.

Proof. If $1 \in \mathbb{P}$ then we are done. If $1 \in -\mathbb{P}$, then $-1 \in \mathbb{P}$. Hence $1 = (-1)(-1) \in \mathbb{P}$. This is a contradiction. □

Example. The reals is an ordered field. Let $\mathbb{P} = \{x \in \mathbb{R} : x > 0\}$.

Example. By the same logic, the rationals $\mathbb{Q}$ is an ordered field with $\mathbb{P} = \{x \in \mathbb{Q} : x > 0\}$.

Example. The complex numbers $\mathbb{C}$ cannot be made into an order field! Suppose there is $\mathbb{P}$. Assume that $i \in \mathbb{P}$, then $-1 = i \cdot i \in \mathbb{P}$ and thus $1 \in -\mathbb{P}$. This contradicts the previous Lemma. Similarly if $i \in -\mathbb{P}$ we would get another contradiction.

Example. $\mathbb{Z}/p\mathbb{Z}$ is not an ordered field. Assume it is $1 \in \mathbb{P}$. Then $p \cdot 1 = 1 + \cdots + 1 = 0 \in \mathbb{P}$. This is a contradiction.

Example. $\mathbb{R}(x)$ is an ordered field! We define:

$$\mathbb{P} = \left\{ \frac{p(x)}{q(x)} \in \mathbb{R}(x) : \text{there is } T \in \mathbb{R} \text{ such that } \frac{p(t)}{q(t)} > 0 \text{ for all } t \geq T \right\}$$

Not hard to check that $\mathbb{P}$ has the desired property.

Definition. Let $\mathbb{F}$ be an ordered field with $\mathbb{F} = \mathbb{P} \sqcup \{0\} \sqcup (-\mathbb{P})$. We define $a < b$ if $b - a \in \mathbb{P}$. We can define $a \leq b$ if $a = b$ or $a < b$. It is easy to check $<$ defines a total order of $\mathbb{F}$.

Example. In $\mathbb{R}(x)$ with $\mathbb{P}$ above, we have $\frac{1}{x} < 1$ because $1 - \frac{1}{x} > 0$ for $x \geq 2$.

Definition. We say an ordered field $\mathbb{F}$ has the **least upper bound property (LUBP)** if for all $\emptyset \neq S \subseteq \mathbb{F}$ that has an upper bound (there is $M \in \mathbb{F}$ with $s \leq M$ for all $s \in S$), there exists a **least upper bound** $x \in \mathbb{F}$ in the sense that if $y < x$ then y is NOT an upper bound of S.

Example. We have seen that $\mathbb{R}$ has LUBP by Theorem 1.13

Example. $\mathbb{Q}$ does not have LUBP. Take $S = \{x \in \mathbb{Q} : x^2 < 2\}$. This does not have a supremum.

Example. The set of rational functions $\mathbb{R}(x)$ does NOT have the LUBP. Take:

$$S = \left\{ \frac{a}{x} : a \in \mathbb{R},\ x > 0 \right\}$$

This has an upper bound but does not have a least upper bound.

Notation. Let $\mathbb{F}$ be an ordered field. For $n \in \mathbb{N}$ we define:

$$n := \underbrace{1 + \cdots + 1}_{n \text{ times}} \in \mathbb{P}$$

Definition. We say an ordered field $\mathbb{F}$ is **Archimedean** if for any $x > 0$ there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < x$.

Example. $\mathbb{R}$ and $\mathbb{Q}$ are archimedean.

Example. The raional functions $\mathbb{R}(x)$ is NOT archimedean. Note that $\frac{1}{x} < \frac{1}{n}$ for all $n \in \mathbb{N}$.

Theorem 3.18. Let $\mathbb{F}$ be an ordered field. Then:

(a). There is a nonzero field homomorphism $\phi : \mathbb{Q} \to \mathbb{F}$. [This means ϕ is injecitve, so $\mathbb{F}$ contains a copy of $\mathbb{Q}$] and $\mathbb{Q} \cap \mathbb{P} = \{x \in \mathbb{Q} : x > 0\}$.

(b). If $\mathbb{F}$ has the LUBP then $\mathbb{F}$ is archimedean.

(c). If $\mathbb{F}$ is archimedean and $x < y$, then there exists $\frac{m}{n} < \mathbb{Q}$ such that $x < \frac{m}{n} < y$.

Proof. (a). For $m, n \in \mathbb{Q}$ with $m, n > 0$ we define:

$$\phi\left(\frac{m}{n}\right) = (\underbrace{1 + \cdots + 1}_{m \text{ times}})(\underbrace{1 + \cdots + 1}_{n \text{ times}})^{-1}$$

For $q \in \mathbb{Q}$ with $q < 0$ we just define $\phi(-q) = -\phi(q)$. Easy to check ϕ is nonzero (since $\phi(1) = 1 \neq 0$). Hence ϕ is injecitve and $\mathbb{F}$ contains a copy of $\mathbb{Q}$.

(b). Assume $\mathbb{F}$ has the LUBP. We define:

$$J = \{x \in \mathbb{P} : nx < 1 \text{ for all } n \in \mathbb{N}\}$$

If $J = \emptyset$ then for all $x \in \mathbb{P}$ there is $n \in \mathbb{N}$ such that $nx > 1$, so $x > 1/n$ as required. Suppose J is not empty, then J is bounded above by 1. By the LUBP, it has a least upper bound y. Pick $x_1, x_2 \in J$. Then $2nx_1, 2nx_2 < 1$ for all $n \in \mathbb{N}$. Hence:

$$n(x_1 + x_2) < 1$$

for all $n \in \mathbb{N}$. This means $x_1 + x_2 \in J$. Therefore $x_1 + x_2 \leq y$ and $x_1 \leq y - x_2$ for all $x_1 \in J$. Hence x_1 is a better upper bound for J, meaning y is not the least upper bound. This is a contradiction, so $J = \emptyset$. Therefore $\mathbb{F}$ is archimedean. □

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Definition. Let $\mathbb{F}$ and $\mathbb{K}$ be ordered fields. A map $\gamma : \mathbb{F} \to \mathbb{K}$ is an **embedding** if γ is a field homomorphism and preserves order. That is, $\gamma(a) \leq \gamma(b)$ whenever $a \leq b$.

Theorem 3.19. Let $\mathbb{F}$ be an Archimedean ordered field and $\mathbb{K}$ is a complete ordered field. Then there is an embedding from $\mathbb{F}$ to $\mathbb{K}$.

Example. Both $\mathbb{Q}$ and $\mathbb{R}$ are Archimedean and $\mathbb{R}, \mathbb{C}, \mathbb{R}(x)$ are complete.

Proof of Theorem 3.19. Both $\mathbb{F}$ and $\mathbb{K}$ are ordered fields. Hence they contain a copy of $\mathbb{Q}$. Let us call them $\mathbb{Q}_{\mathbb{F}}$ and $\mathbb{Q}_{\mathbb{K}}$, respectively. We define:

$$\gamma_0 : \mathbb{Q}_{\mathbb{F}} \to \mathbb{Q}_{\mathbb{K}} \quad \text{by} \quad \gamma_0(r_{\mathbb{F}}) = r_{\mathbb{K}}$$

For $f \in \mathbb{F}$ we define $S_f = \{r \in \mathbb{Q}_{\mathbb{F}} : r < f\}$. Note that S_f is bounded above, hence $\gamma_0(S_f)$ is bounded above. Now we define:

$$\gamma(f) := \sup\{\gamma_0(r) : \gamma_0(r) \in \gamma_0(S_f)\}$$

here the supremum is taken in $\mathbb{K}$. As $\mathbb{K}$ is complete we have $\gamma(f) \in \mathbb{K}$. This defined a map $\gamma : \mathbb{F} \to \mathbb{K}$. Let $f_1, f_2 \in \mathbb{F}$ with $f_1 < f_2$. We know there exists $r \in \mathbb{Q}_{\mathbb{F}}$ such that $f_1 < r < f_2$. This tells us that:

$$\gamma_0(s) < \gamma_0(r) \quad \text{for all} \quad s \in S_{f_1} \quad \text{and} \quad \gamma_0(r) < \gamma_0(s) \quad \text{for some} \quad s \in S_{f_2}$$

Hence $\gamma(f_1) < \gamma(f_2)$. That is, γ preserves order. With loss of generality, assume $0 < f_1, f_2$. Now:

$$\begin{aligned} S_{f_1} + S_{f_2} &= \{r_1 < f_1 : r_1 \in \mathbb{Q}_{\mathbb{F}}\} + \{r_2 < f_2 : r_2 \in \mathbb{Q}_{\mathbb{F}}\} \\ &= \{r_1 + r_2 : r_1 < f_1,\ r_2 < f_2,\ r_1, r_2 \in \mathbb{Q}_{\mathbb{F}}\} \\ &= \{r_3 : r_3 < f_1 + f_2,\ r_3 \in \mathbb{Q}_{\mathbb{F}}\} \\ &= S_{f_1 + f_2} \end{aligned}$$

This gives $\gamma(f_1) + \gamma(f_2) = \gamma(f_1 + f_2)$. Now:

$$\begin{aligned} S_{f_1} \cdot S_{f_2} &= (\{0 \le r_1 < f_1 : r_1 \in \mathbb{Q}_{\mathbb{F}} \cap \mathbb{P}\} \cup (-\mathbb{P})) \cdot (\{0 \le r_2 < f_2 : r_2 \in \mathbb{Q}_{\mathbb{F}} \cap \mathbb{P}\} \cup (-\mathbb{P})) \\ &= \{0 \le r_1 \cdot r_2 : r_1 < f_1,\ r_2 < f_2,\ r_1, r_2 \in \mathbb{Q}_{\mathbb{F}} \cap \mathbb{P}\} \cup \{0\} \cup (-\mathbb{P}) \\ &= \{0 \le r_1 \cdot r_2 < f_1 \cdot f_2 : r_1, r_2 \in \mathbb{Q}_{\mathbb{F}} \cap (\mathbb{P} \cup \{0\})\} \cup (-\mathbb{P}) \\ &= S_{f_1 f_2} \end{aligned}$$

Hence we have $\gamma(f_1 f_2) = \gamma(f_1)\gamma(f_2)$. This defined a field homomorphism $\mathbb{F} \to \mathbb{K}$. Since this is nonzero, it is an injection (an embedding). □

Corollary 3.20. There is a unique complete Archimedean ordered field, up to isomorphism.

Proof. Assume $\mathbb{K}$ and $\mathbb{F}$ are both Archimedean complete ordered field. By Theorem 3.19, there are order preserving homomorphisms:

$$\gamma_0 : \mathbb{K} \to \mathbb{F} \quad \text{and} \quad \gamma_1 : \mathbb{F} \to \mathbb{K}$$

Since γ_0 is identity on $\mathbb{Q}_{\mathbb{K}}$ and γ_1 is identity on $\mathbb{Q}_{\mathbb{F}}$, we know $\gamma_0 \circ \gamma_1$ is identity on $\mathbb{Q}_{\mathbb{F}}$. By the completeness of $\mathbb{F}$, the map $\gamma_0 \circ \gamma_1$ is identity on $\mathbb{F}$. □

Definition. We call this unique complete Archimedean ordered field $\mathbb{R}$, the **real numbers**.

Remark. This only proved the uniqueness of such complete archimedean ordered field, but we have not constructedsuch field yet. Now we are going to provide a detailed construction of real numbers.

Definition. We say $\emptyset \neq C \subseteq \mathbb{Q}$ is a **cut** if for all $x \in C$ we have $y \in C$ for all $y < x$. Further, we requires that $C \neq \mathbb{Q}$.

Example. $C = \{x \in \mathbb{Q} : x < 7\}$ is a cut. [This represents the real number 7.]

Example. $C = \{x \in \mathbb{Q} : x \le 0 \ \text{ or } \ x^2 < 2\}$ is a cut. [This represents the real number $\sqrt{2}$.]

Definition. Let C_1, C_2 be cuts. We define $C_1 < C_2$ if $C_1 \subsetneq C_2$. We define $C_1 \leq C_2$ if $C_1 \subseteq C_2$.

Theorem 3.21. Let $\mathcal{R} = \{C \subseteq \mathbb{Q} : C \text{ is a cut}\}$. Then $(\mathcal{R}, \leq)$ has the least upper bound property.

Proof. Let $\mathcal{S} \subseteq \mathcal{R}$. Suppose there is $P \in \mathcal{R}$ such that $R < P$ for all $S \in \mathcal{S}$. We claim that $\mathcal{S}$ has a least upper bound. We define:

$$E = \bigcup_{S \in \mathcal{S}} S$$

We see that if $y \in E$ then $y \in S$ for some $S \in \mathcal{S}$. If $x < y$ then $x \in S \subseteq E$. This proved that E is a cut. As $S \subseteq E$ for all $S \in \mathcal{S}$, hence E is an upper bound of $\mathcal{S}$. We claim that E is the least upper bound. Suppose for a contradiction that $F < E$ is also an upper bound. Since $F < E$ there is $r \in E \setminus F$. Then as $r \in E$, there exists $S \in \mathcal{S}$ such that $r \in S$. But $S \subsetneq F$, which means $S \nleq F$. This means F is not an upper bound, contradiction. Therefore E is the least upper bound. $\square$

Construction 3.22. We need to show how we can write $\mathcal{R}$ as a field, and show it is complete. For cuts $C_1, C_2 \in \mathcal{R}$, we define:

$$C_1 + C_2 := \{c_1 + c_2 : c_1 \in C_1,\ c_2 \in C_2\}$$

It is easy to check that $C_1 + C_2$ is a cut. If $0 \in C_1, C_2$ we define:

$$C_1 \cdot C_2 := \{c_1 c_2 : c_1, c_2 \geq 0,\ c_1 \in C_1,\ c_2 \in C_2\} \cup (-\mathbb{Q})$$

The other cases are similar to define. [The idea of dedekind is that we want to define a real number α as the cut $\{r \in \mathbb{Q} : r < \alpha\}$. The multiplication is intuitive but tricky to write down.] With some work we can show $\mathcal{R}$ is a complete Archimedean ordered field.

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We define a metric on $\mathcal{R}$ by:

$$d(C_1, C_2) := \max\left(\sup_{c_1 \in C_1} \inf_{c_2 \in C_2} |c_1 - c_2|,\ \sup_{c_2 \in C_2} \inf_{c_1 \in C_1} |c_1 - c_2|\right)$$

Consider the map $\gamma : \mathbb{Q} \to \mathcal{R}$ by $\gamma(q) = \{r \in \mathbb{Q} : r < q\}$. Then γ is an embedding of $\mathbb{Q}$ into $\mathcal{R}$ such that $\gamma(\mathbb{Q})$ is dense in $\mathcal{R}$. Hence $\mathcal{R}$ is a completion of $\mathbb{Q}$.

3.7 The p-adic Numbers

Let p be a prime number. We know $(\mathbb{Z}, |\cdot|_p)$ and $(\mathbb{Q}, |\cdot|_p)$ are not complete. We can get this by either the Baire category theorem or a counting argument.

Definition. We call the completion of $(\mathbb{Z}, |\cdot|_p)$ the **p-adic integers**.

Definition. We call the completion of $(\mathbb{Q}, |\cdot|_p)$ the p**-adic numbers**.

Theorem 3.23. Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in $(\mathbb{Z}, |\cdot|_p)$. Either $\lim x_n = 0$ in p-adic norm or for all $k \geq 1$, the sequence $(a_n \pmod{p^k})_{n=1}^{\infty}$ is an eventually constant sequence in $\mathbb{Z}/p^k\mathbb{Z}$.

Proof. If $\lim x_n = 0$ then we are done. Suppose not. Pick $k \geq 1$ arbitray. Then $|\cdot|_p : (\mathbb{Q}, |\cdot|_p) \to \mathbb{R}$ is a continuous function. Hence $(|x_n|_p)_{n=1}^{\infty}$ is a cauchy sequence in $\mathbb{R}$ that does not converge to 0. We know that $|\cdot|_p$ takes on values of the form p^r for $r \in \mathbb{Z}$. Therefoe $(|x_n|_p)_{n=1}^{\infty}$ is eventually constant in $\mathbb{R}$. Say $|x_n|_p = p^{-N}$ for n large enough and some $N \geq 0$. Let $\epsilon < p^{-k}$. There exists N_0 such that $|x_n - x_m|_p < p^{-k}$ for all $n, m \geq N_0$. This means $p^k \mid (x_n - x_m)$ so $x_n \equiv x_m \pmod{p^k}$. ☐

Theorem 3.24. For all $a \in \mathbb{Z}_p$ there exists $a_0 \in \{0, 1, \cdots, p-1\}$ such that $|a - a_0|_p \leq \frac{1}{p}$.

Proof. We know $\mathbb{Z}$ is dense in $\mathbb{Z}_p$. Pick $k \in \mathbb{Z}$ such that $|k - a| \leq \frac{1}{p}$. Pick $a_0 \in \{0, \cdots, p-1\}$ such that $k \equiv a_0 \pmod{p}$. This gives:

$$|a - a_0|_p \leq |a - a_k|_p + |k - a_0|_p \leq \frac{1}{p} + \frac{1}{p} = \frac{2}{p}$$

If $p \geq 3$, then this gives that $|a - a_0| \leq \frac{1}{p}$ (as the norm is 1 or less than $1/p$). If $p = 2$ we need to do more work, but it is still easy. ☐

Corollary 3.25. Let $a \in \mathbb{Z}_p$. There exists $a_0, a_1, \cdots, a_n \in \{0, \cdots, p-1\}$ such that:

$$|a - (a_0 + a_1 p + \cdots + a_n p^n)|_p \leq \frac{1}{p^{n+1}}$$

Remark. This justifies write $a = \sum_{n=0}^{\infty} a_n p^n$ with $a_n \in \{0, \cdots, n-1\}$ for $a \in \mathbb{Z}_p$.

Corollary 3.26. For each $a \in \mathbb{Z}_p$ there is a sequence $(a_n)_{n=0}^{\infty}$ with $a_n \in \{0, \cdots, p-1\}$ such that:

$$a = \sum_{n=0}^{\infty} a_n p^n = \lim_{N \to \infty} x_N$$

where $x_N := \sum_{n=0}^{N} a_n p^n$ and $(x_N)_{n=0}^{\infty}$ is cauchy in $|\cdot|_p$.

Example. Let $p = 3$. Note that:

$$-1 = \sum_{n=0}^{\infty} 2 \cdot 3^n = 2 + 2 \cdot 3 + 2 \cdot 3^2 + \cdots$$

What about $\alpha = \sum_{n=0}^{\infty} 2 \cdot 3^n + \sum_{n=0}^{\infty} 2 \cdot 3^n$? The N-th partial sum is

$$1 + \sum_{n=1}^{N} 2 \cdot 3^n + 3^{N+1}$$

Taking $N \to \infty$ shows that the associated sequence is $(1, 2, 2, 2, \cdots)$.

Example. Let $p = 3$ again. Consider the multiplication $\alpha = (\sum_{n=0}^{\infty} 2 \cdot 3^n)(\sum_{n=0}^{\infty} 2 \cdot 3^n)$. The product of the N-partial sums is equal to:

$$1 + (3^{N+1} - 2) \cdot 3^{N+1} \equiv 1 \pmod{3^{N+1}}$$

Hence the associated sequence is $(1, 0, 0, 0, \cdots)$.

Theorem 3.27. $\mathbb{Z}_p$ is compact for all prime p.

Proof. We know that $\mathbb{Z}_p$ is complete, as it is the completion of $(\mathbb{Z}, |\cdot|_p)$. We also know $\mathbb{Z}$ is totally bounded by an assignment. Pick $\epsilon > 0$ and $k \geq 0$ such that $p^{-k} < \epsilon$. For every $a \in \mathbb{Z}_p$ we can find $a_i \in \{0, \cdots, p-1\}$ such that:

$$|a - (a_0 + a_1 p + \cdots + a_k p^k)|_p \leq \frac{1}{p^{k+1}} < \epsilon$$

There are p^{k+1} choices for $a_0, \cdots, a_k$, hence this gives a ϵ-net. □

Remark. Addition and multiplication on $\mathbb{Q}_p$ are similar because we can always write $\alpha \in \mathbb{Q}_p$ as:

$$\alpha = \sum_{n=-N}^{\infty} a_n p^n$$

for $a_n \in \{0, \cdots, p-1\}$ and some $N \geq 0$.

Remark. By an assignment we showed how to invert an element in $\mathbb{Z}_p$ with $a_0 \neq 0$ (and hence $\mathbb{Q}_p$). Hence $\mathbb{Q}_p$ is a field. We also show that $\sqrt{-2} \in \mathbb{Q}_3$ and more generally $\sqrt{1-p} \in \mathbb{Q}_p$ for $p \geq 3$. This means that $\mathbb{Q}_p$ is NOT an ordered field. Note that $p = 2$ is a special case to consider, and is typically a special case for any p-adic problems.

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4 Approximation Theory

4.1 Polynomial Approximation

When we proved that the set of functions that are differentiable somewhere was first categroy, we first approximated a random function by a differentiable function and then added a small non-differentiable function to it. We can find such a function if we can approximate it by a polynomial. There are three methods we will discuss, but the first two do not work.

Method (Taylor Polynomials). Say f is some function, then we know that:

$$f(x) \approx \sum_{n=0}^{N} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Here $f^{(n)}(c)$ is the n-th derivative of f at c. This has the following problems:

(1). We need f to have lots of derivatives. This means it is useless for non-differentiable functions.

(2). This only really converges inside its disk of convergence. Take $f(x) = (1+x^2)^{-1}$. Its Taylor series converges for $|x| < 1$. Also, take:

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

This only converges at $x = 0$, so its Taylor series is useless.

Method (Lagrange Polynomials). Let f be some function and $x_0, x_1, \cdots, x_n$ be a collection of distinct points (in the domain of f). Define:

$$P_k(x) = \prod_{i \neq k} \left(\frac{x - x_i}{x_k - x_i} \right)$$

This is a polynomial in x. Note that $P_k(x_j) = \delta_{kj}$ for all k, j. Define:

$$P(x) = \sum_{i=0}^{n} f(x_i) P_i(x)$$

This has the property that $P(x_i) = f(x_i)$ for $i = 0, 1, \cdots, n$. One would hope that the more points one uses, the better the approximation. Consider:

$$f : [0,1] \to \mathbb{R} \text{ by } f(x) = \frac{1}{1+25x^2}$$

In this case the Lagrange polynomial does not approximate f at all.

Theorem 4.1 (Weierstrass Approximation Theorem). Let $f : [0,1] \to \mathbb{R}$ be continuous. For every $\epsilon > 0$ there exists a polynomial $p(x) \in \mathbb{R}[x]$ such that:

$$\|f - p\|_\infty = \sup_{x\in[0,1]} |f(x) - p(x)| < \epsilon$$

Proof. Assume WLOG that $f(0) = f(1) = 0$. If $f(0) = a$ and $f(1) = b$ we could consider the polynomial $g(x) = f(x) - a + (a - b)x$. Consider:

$$Q_n(x) = \begin{cases} (1-x^2)^n c_n & \text{if } x \in [-1,1] \\ 0 & \text{otherwise} \end{cases}$$

where for each $n \geq 1$ we define:

$$c_n = \left(\int_{-1}^{1} (1-x^2)^n \, dx \right)^{-1}$$

Hence we have $\int_{-1}^{1} Q_n(x) \, dx = 1$. Now we define functions q_n by:

$$q_n(x) = \int_{-1}^{1} f(x+t) Q_n(t) \, dt$$

where $f(z) = 0$ if $z \notin [0,1]$. We claim that $q_n(x)$ is a polynomial in x and $\|q_n - f\|_\infty \to 0$ as $n \to \infty$. Our plan to prove this claim is as follows:

1. Estimate c_n for each $n \geq 1$.
2. For $\delta > 0$ we have $\lim\limits_{n\to\infty} \int_{\delta}^{1} Q_n(x) \, dx = 0$. This implies $\int_{-1}^{-\delta} Q_n(x) \, dx \to 0$ since Q_n is even.
3. Show $q_n(x)$ is a polynomial.

Step 1. By trig substitution with $x = \sin(u)$ we have $dx = \cos(u) \, du$. Then:

$$\begin{aligned} \int_{-1}^{1} (1-x^2)^n \, dx &= \int_{-\pi/2}^{\pi/2} (1 - \sin^2(u))^n \cos u \, du = \int_{-\pi/2}^{\pi/2} \cos^{2n+1}(u) \, du \\ &= 2 \int_{0}^{\pi/2} \cos^{2n+1}(u) \, du = 2 \left(\frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdots (2n-1)(2n+1)} \right) \geq \frac{2}{2n+1} \end{aligned}$$

This gives the estimation that $c_n \leq \frac{2n+1}{2} \leq 2n+1$.

Step 2. Fix $\delta > 0$. Then we have:

$$I_n = \int_{\delta}^{1} (1-x^2)^n c_n \, dx \leq \int_{\delta}^{1} (1-\delta^2)^n (2n+1) \, dx \leq (1-\delta^2)^n (2n+1)$$

We see that $(1-\delta^2)^n(2n+1)$ goes to 0 as $n \to \infty$ (using ratio test). Hence $I_n \to 0$ as $n \to \infty$.

Step 3. By substitution with $u = x + t$ we have:

$$q_n(x) = \int_{-1}^{1} f(x+t)Q_n(t)\ dt = \int_{-1+x}^{1+x} f(u)Q_n(u-x)\ du$$

We are assuming that $f(u) = 0$ for all $u \notin [0,1]$, hence:

$$q_n(x) = \int_0^1 f(u)Q(u-x)\ du$$

This is a polynomial! (Consider what happens to individual x^j term in the polynomial $Q(u-x)$). For example, we have that:

$$\int_0^1 f(u) \sum_{i,j} a_{ij}x^iu^j\ du = \sum_i \left(\int_0^1 \sum_j a_{ij} f(u)u^j\ du \right) x^i$$

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Since $f \in \mathcal{C}[0,1]$ is continuous on $[0,1]$, it is uniformly continuous. Pick $\delta > 0$ such that for all $x, y \in [0,1]$ we have:

$$|x-y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{2}$$

We also know that f is bounded. There is $M > 0$ such that $|f(x)| < M$ for all $x \in [0,1]$. Now pick n sufficently large so that:

$$\int_{-1}^{-\delta} 2MQ_n(t)\ dt + \int_\delta^1 2MQ_n(t)\ dt < \frac{\epsilon}{2} \tag{$*$}$$

Pick $x \in [0,1]$. Notice that:

$$|q_n(x) - f(x)| = \left| \int_{-1}^1 f(t+x)Q_n(t)\ dt - f(x)\int_{-1}^1 Q_n(t)\ dt \right| = \left| \int_{-1}^1 (f(x+t) - f(x))Q_n(t)\ dt \right|$$

Now we estimate two integrals.

$$A = \left| \int_{-\delta}^{\delta} (f(x+t) - f(x))Q_n(t)\ dt \right| \le \int_{-\delta}^{\delta} |f(x+t) - f(x)|Q_n(t)\ dt \le \frac{\epsilon}{2} \int_{-\delta}^{\delta} Q_n(t)\ dt = \frac{\epsilon}{2}$$

By the choice of n in equation $(*)$ we have:

$$B = \left| \int_{-1}^{-\delta} \underbrace{(f(x+t) - f(x))}_{\le 2M} Q_n(t)\ dt + \int_\delta^1 \underbrace{(f(x+t) - f(x))}_{\le 2M} Q_n(t)\ dt \right| < \frac{\epsilon}{2}$$

Now, by the triangle inequality we obtain that:

$$|q_n(x) - f(x)| = \left| \int_{-1}^{1} (f(x+t) - f(x))Q_n(t)\ dt \right| \leq A + B \leq \epsilon$$

Since $x \in [0,1]$ is arbitrary, we have $\|q_n - f\| \leq \epsilon$. As desired. □

4.2 Stone-Weierstrass Theorem

Definition. Let X be a compact metric space. Then $\mathcal{C}(X) = \{f : X \to \mathbb{R} \mid f \text{ continuous}\}$ is a vector space over $\mathbb{R}$. We say $\mathcal{A} \subseteq \mathcal{C}(X)$ is an **algebra** if $\mathcal{A}$ is a vector subspace of $\mathcal{C}(X)$ and for all $f, g \in \mathcal{A}$ we have $fg \in \mathcal{A}$.

Example. Let X be any compact space. Then $\mathcal{A} = \{\text{constant functions}\}$ is an algebra.

Example. The set of polynomials is an algbera in $\mathcal{C}[0,1]$. Moreover, even degree Polynomials are an algebra. Odd degree polynomials do not form an algebra because $x \cdot x$ is even.

Example. Differentiable functions form an algebra.

Example. Let $X = [0,1]$. Then $\{f \in \mathcal{C}[0,1] : f(0) = f(1)\}$ is an algebra.

Example. Even polynomials (polynomials that are also even functions) is an algebra.

Definition. For two functions $f, g : X \to \mathbb{R}$ we define $f \vee g : X \to \mathbb{R}$ and $f \wedge g : X \to \mathbb{R}$ by:

$$(f \vee g)(x) := \max(f(x), g(x)) \quad \text{and} \quad (f \wedge g)(x) = \min(f(x), g(x))$$

We also write $f \vee g = \max(f, g)$ and $f \wedge g = \min(f, g)$.

Definition. Let X be compact and $\mathcal{A} \subseteq \mathcal{C}(X)$ be an algebra. We say $\mathcal{A}$ is a **vector lattice** if for all $f, g \in \mathcal{A}$ we have $f \vee g \in \mathcal{A}$ and $f \wedge g \in \mathcal{A}$.

Example. Constant functions is a vector lattice.

Example. Let P = polynomials on $[0,1]$. This is not a vector lattice. Note $x \vee 0 = |x|$ is not a polynomial. However $\overline{P} = \mathcal{C}[0,1]$ by the Weierstrass approximation theorem. Hence $\overline{P}$ is a lattice.

Definition. Let X be compact and $\mathcal{A} \subseteq \mathcal{C}(X)$ is an algbera. We say $\mathcal{A}$ **separates points** if for all $x, y \in X$ with $x \neq y$ there is $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

Example. Constant functions do not separate points.

Example. Polynomials separate points because x separates points.

Example. Even functions in $\mathcal{C}[-1,1]$ do not separate points.

Definition. Let X be compact and $\mathcal{A} \subseteq \mathcal{C}(X)$ be an algebra. We say $\mathcal{A}$ **vanishes at** $x \in X$ if for all $f \in \mathcal{A}$ we have $f(x) = 0$.

Example. Constant functions $X \to \mathbb{R}$ do not vanish anywhere (because 1 does not vanish anywhere).

Example. The algebra $\mathcal{A} = \operatorname{span}_{\mathbb{R}}\{x^{2n} : n \geq 1\}$ vanishes at 0.

Example. Let $P =$ set of polynomials on $[0, 1]$. Then P is an algbera and $\overline{P}$ is a vector lattice. Also P separate points and P does not vanish anywhere.

Theorem 4.2 (Stone-Weierstrass). Let X be a compact metric space. Let $\mathcal{A} \subseteq \mathcal{C}(X)$ be an algebra that separates points and does not vanish anywhere. Then $\mathcal{A}$ is dense in $\mathcal{C}(X)$.

Our plan of the proof is the followings:

1. If $\mathcal{A}$ is an algebra, then $\mathcal{A}$ in $\mathcal{C}(X)$ is a vector lattice.
2. If $\mathcal{A}$ separates points then for all $x, y \in X$ with $x \neq y$ and $a, b \in \mathbb{R}$, there exists $f \in \mathcal{A}$ such that $f(x) = a$ and $f(y) = b$.
3. For each $a \in X$ and $\epsilon > 0$, we can find $g_a \in \overline{\mathcal{A}}$ such that $g_a(x) > f(x) - \epsilon$ and $g_a(a) = f(a)$.
4. Using these g_a, we can find g such that $f(x) + \epsilon > g(x) > f(x) - \epsilon$ for all $x \in X$.

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Lemma 4.3. Let X be compact and $\mathcal{A} \subseteq \mathcal{C}(X)$ be an algebra. Then $\overline{\mathcal{A}}$ is a closed algebra and a vector lattice.

Proof. Clearly $\overline{\mathcal{A}}$ is closed and it is easy to check it is an algebra. Recall that $\overline{\mathcal{A}}$ is a vector lattice if for all $f, g \in \overline{\mathcal{A}}$ we have $f \vee g$ and $f \wedge g \in \overline{\mathcal{A}}$. We will first show that if $f \in \overline{\mathcal{A}}$ then $|f| \in \overline{\mathcal{A}}$. Now take $f \in \overline{\mathcal{A}}$, we know $L := \|f\|_\infty < \infty$. By the Weierstrass approximation theorem we know polynomials are dense in $\mathcal{C}[-L, L]$. Since $g(x) = |x| \in \mathcal{C}[-L, L]$, there is a sequence of polynomials $(p_n)_{n=1}^\infty$ such that $p_n \to g$ uniformly on $[-L, L]$. Notice $p_n(0) \to 0$ as $n \to \infty$. Let $q_n = p_n - p_n(0)$, then we have $q_n \to g$ as well. Note that $q_n(f) \in \mathcal{A}$ and $q_n(f) \to |f|$. It follows that $|f| \in \overline{\mathcal{A}}$. Now:

$$\max(f, g) = \frac{f + g}{2} + \frac{|f - g|}{2} \in \overline{\mathcal{A}}$$
$$\min(f, g) = \frac{f + g}{2} - \frac{|f - g|}{2} \in \overline{\mathcal{A}}$$

This proved that $\overline{\mathcal{A}}$ is a vector lattice. □

Lemma 4.4. Let X be compact and $\mathcal{A} \subseteq \mathcal{C}(X)$ be an algebra. Further, assume $\mathcal{A}$ separates points and vanishes nowhere. For all $x, y \in X$ with $x \neq y$ and any $c, d \in \mathbb{R}$ we can find $f \in \mathcal{A}$ such that $f(x) = c$ and $f(y) = d$.

Proof. Let $x, y \in X$ with $x \neq y$. Let $g \in \mathcal{A}$ such that $g(x) \neq g(y)$. Write $g(x) = a$ and $g(y) = b$ and $a \neq b$. Hence they are not both 0. WLOG assume $b \neq 0$.

Case 1. Assume $a = 0$. Since $\mathcal{A}$ does not vanish at x, there exists $h \in \mathcal{A}$ such that $h(x) \neq 0$. Set:

$$f(z) = \frac{c}{h(x)}h(z) + \left(\frac{d}{g(y)} - \frac{c \cdot h(y)}{h(x)g(y)}\right)g(z)$$

Evaluate this function f at x we get $f(x) = c$ and $f(y) = d$.

Case 2. Assume $a \neq 0$. Consider the function:

$$\tilde{g}(z) = g(z) - \frac{g(z)^2}{g(x)}$$

Then $\tilde{g}(x) = 0$ and $\tilde{g}(y) \neq 0$. Now apply case 1. $\square$

Proof of Theorem 4.2. Let $f \in \mathcal{C}(X)$ be arbitrary and $\epsilon > 0$. Fix $a \in X$. For each $a \neq x \in X$ there is a function $h_x \in \mathcal{A}$ such that:

$$h_x(a) = f(a) \ \text{ and } \ h_x(x) = f(x)$$

by Lemma 4.4 applying to $c = f(a)$ and $d = f(x)$. Define:

$$U_x = \{z \in X : h_x(z) > f(z) - \epsilon\}$$

Notice that $x \in U_x$ and $a \in U_x$. Also note that U_x is open because:

$$U_x = (h_x - f)^{-1}((-\epsilon, \infty))$$

Note that $\{U_x\}_{x \in X}$ is an open cover of X. As X is compact, we have a finite subcover:

$$\{U_{x_1}, \cdots, U_{x_n}\}$$

Take $g_a = \max(h_{x_1}, \cdots, h_{x_n}) \in \overline{\mathcal{A}}$, as $\overline{\mathcal{A}}$ is a vector lattice. Notice that:

$$g_a(a) = f(a) \ \text{ and } \ g_a(z) > f(z) - \epsilon \text{ for all } z \in X$$

Let $V_a = \{z \in X : g_a(z) < f(z) + \epsilon\}$. We see that $a \in V_a$ and each V_a is open. As before $\{V_a\}_{\alpha \in X}$ is an open cover. We have a finite subcover $\{V_{a_1}, \cdots, V_{a_k}\}$. Take $g = \min(g_{a_1}, \cdots, g_{a_k})$. We see that $g(z) > f(z) - \epsilon$ for all $z \in X$ by the properties of g_{a_i}. Further $g(z) < f(z) + \epsilon$ by properties of U_{a_i}. Hence $\|g - f\|_\infty < \epsilon$. Since $f \in \mathcal{C}(X)$ and $\epsilon > 0$ are arbitray, we proved $\overline{\mathcal{A}} = \mathcal{C}(X)$. $\square$

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4.3 Best Approximation

Notation. For $n \geq 0$ let $\mathbb{P}_n[x]$ denote the polynomials of degree at most n.

Definition. For X compact and $f \in \mathcal{C}(X)$ and $n \geq 1$ we define:

$$E_n(f) = \inf_{p \in P_n[x]} \|f - p\|_\infty$$

Note that $T : \mathbb{P}_n[x] \to \mathbb{R}$ by $p \mapsto \|f - p\|_\infty$ is a continuous function. Consider $S \subseteq \mathbb{P}_n[x]$ such that:

$$S = \{p \in \mathbb{P}_n[x] : \|p\|_\infty \leq 4\|f\|_\infty\}$$

Then $S \subseteq \mathbb{P}_n[x]$ is compact and the polynomial $0 \in S$. By restriction, $T : S \to \mathbb{R}$ is continuous on a compact set! Hence there exists $p^* \in S$ such that:

$$\|p^* - f\|_\infty = \inf_{p \in S} T(p) = \inf_{p \in S} \|p - f\|_\infty \leq \|f\|_\infty$$

If $p \in \mathbb{P}_n[x] \setminus S$ then we have $\|p - f\|_\infty \geq 2\|f\|_\infty$. Hence:

$$\|p^* - f\|_\infty = \inf_{p \in \mathbb{P}_n[x]} \|f - p\|_\infty = E_n(f) \tag{$*$}$$

We say p^* is a **best approximation** of f of degree n.

Definition. We say a function $g \in \mathcal{C}[a, b]$ satisfies the **equioscilation property** of degree n if there exists $(n + 2)$ points $x_1 < \cdots < x_{n+2}$ in $[a, b]$ with:

$$g(x_i) = (-1)^i \|g\|_\infty \quad \text{or} \quad g(x_i) = (-1)^{i+1} \|g\|_\infty$$

for all $i \in \{1, \cdots, n + 2\}$.

Theorem 4.5. Let $n \geq 1$ and $f \in \mathcal{C}[a, b]$. Assume $p \in \mathbb{P}_n[x]$ such that $g := f - p$ satisfies the equioscilation property of degree n. Then p is a best approximation of f, that is, $\|f - p\|_\infty = E_n(f)$.

Proof. Assume p is not a best approximation. There exists another polynomial $r(x)$ that gives a better approximation. We know $q = r - p$ is a polynomial of degree at most n. Let $x_1, \cdots, x_{n+2}$ such that $g(x_i) = (-1)^i \|g\|_\infty$. We see that $g(x_i)$ and $q(x_i)$ must have the same sign, otherwise:

$$|g(x_i) - q(x_i)| > |g(x_i)| = \|g\|_\infty$$

which is a contradiction since $|g(x_i) - q(x_i)| < \|g\|_\infty$ as it is a better approximation. That is, $q(x)$ has $(n+2)$ sign changes. As $\deg(q) \leq n$ we have $q(x) = 0$. This proves $p(x)$ is a best approximation. $\square$

Theorem 4.6. If $p \in \mathbb{P}_n[x]$ is a best approximation of f, then $g = f - p$ satisfies the equioscilation property of degree n.

Theorem 4.7. Best approximations are unique.

Lecture 32, 2025/03/28

5 Differential Equations

Example. Consider the differential equation $y' = x^2 + 1$. This is a boring differential equation. We can just find an anti-derivative for $x^2 + 1$.

Example. Consider $(y')^2 + 1 = 0$, an first order DE with no real solutions. $[y' = \pm i]$

Example. Consider the second order DE $y'' = -y$. We see that $y(x) = a\sin(x)$ and $y(x) = b\cos(x)$ are both solutions. In fact $\operatorname{span}\{\sin(x), \cos(x)\}$ is the set of all solutions to this DE.

Goal: We wish to use the contraction mapping principle to show that certain families of first order DEs have a solution, and that the solution is unique.

Example. Consider the differential equation $y' = -1 + y/2$ on $x \in [-1, 1]$ with $y(0) = 1$. How do we solve it? We have:

$$\int_0^x y'(t)\ dt = \int_0^x -1 + \frac{y(t)}{2}\ dt$$

By the FTC this gives us:

$$y(x) = y(0) - x + \int_0^x \frac{y(t)}{2}\ dt = 1 - x + \int_0^x \frac{y(t)}{2}\ dt$$

Consider $T : \mathcal{C}[-1, 1] \to \mathcal{C}[-1, 1]$ given by $Tf(x) = 1 - x + \int_0^x f(t)/2\ dt$. We claim that T is Lipschitz with constant < 1. Indeed, we have:

$$\begin{aligned}
\|Tf - Tg\|_\infty &= \sup_{x\in[-1,1]} \left| \frac{1}{2} \int_0^x (f(t) - g(t))\ dt \right| \\
&\leq \sup_{x\in[-1,1]} \left| \frac{1}{2} \int_0^x |f(t) - g(t)|\ dt \right| \\
&\leq \frac{1}{2} \sup_{x\in[-1,1]} \int_0^x \|f - g\|_\infty\ dt \\
&= \frac{1}{2} \|f - g\|_\infty
\end{aligned}$$

By the contraction mapping principle, T has a unique fixed point. Moreover, for all $f \in \mathcal{C}[-1, 1]$ the sequence $(T^n f)_{n=0}^\infty$ converges to this unique fixed point! Let $f_0 = 0$, then:

$$\begin{aligned}
f_1(x) &= (Tf_0)(x) = 1 - x + \frac{1}{2} \int_0^x 0\ dt = 1 - x \\
f_2(x) &= (Tf_1)(x) = 1 - x + \frac{1}{2} \int_0^x (1 - t)\ dt = 1 - \frac{x}{2} - \frac{x^2}{4}
\end{aligned}$$

By some computation we can see that:

$$f_3(x) = (Tf_2)(x) = 1 - \frac{x}{2} - \frac{x^2}{4} - \frac{x^3}{8}$$
$$f_4(x) = (Tf_3)(x) = 2 - \left(1 + \frac{x}{2} + \frac{(x/2)^2}{2!} + \frac{(x/2)^3}{3!}\right) - \frac{x^4}{192}$$

If we continue, we obtain $f^*(x) = 2 - e^{x/2}$ is the fixed point and this is a solution to our DE.

Remark. The key observation is we could construct $T : \mathcal{C}(X) \to \mathcal{C}(X)$ that was Lipschitz and contractive. We will show for a large family of first order DE, something "like this" will happen!

Definition. Let I_1 and I_2 be intervals. We say $\varphi : I_1 \times I_2 \to \mathbb{R}$ is **Lipschitz in** y if there is $L \geq 0$ such that for any fixed $x \in I_1$ and all $y_1, y_2 \in I_2$:

$$|\varphi(x, y_1) - \varphi(x, y_2)| \leq L|y_1 - y_2|$$

Example. For $y' = 1 + x^2$ we can take $\varphi(x, y) = 1 + x^2$. This is clearly Lipschitz in y with constant $L = 0$. This is becasue $|\varphi(x, y_1) - \varphi(x, y_2)| = 0$ for any y_1, y_2.

Example. For $y' = -1 + y/2$ we let $\varphi(x, y) = -1 + y/2$. Then φ is Lipschitz in y with constant $1/2$.

Lemma 5.1. Let $I_1 = [a, b]$. Let $\varphi : I_1 \times \mathbb{R} \to \mathbb{R}$ be Lipschitz in y with constant L. Let $c \in [a, b]$ and define:

$$T : \mathcal{C}[a, b] \to \mathcal{C}[a, b] \text{ by } Tf(x) = c_0 + \int_0^x \varphi(t, f(t))\, dt$$

where $y(c) = c_0$ is the initial condition. Let $f, g \in \mathcal{C}[a, b]$. If there exists M and k such that:

$$|f(x) - g(x)| \leq \frac{M|x - c|^k}{k!} \text{ for all } x \in [a, b]$$

Then for all $x \in [a, b]$ we have:

$$|Tf(x) - Tg(x)| \leq \frac{LM|x - c|^{k+1}}{(k + 1)!}$$

Note. For all $f, g \in \mathcal{C}[a, b]$ there is such constant $M = \|f - g\|_\infty$ and $k = 0$, so:

$$|f(x) - g(x)| \leq \sup_{x \in [a,b]} |f(x) - g(x)| = \|f - g\|_\infty = \frac{M|x - c|^0}{0!}$$

By iterating this we get that:

$$|T^k f(x) - T^k g(x)| \leq \frac{L^k|x - c|^k}{k!}$$

Taking $k \to \infty$ we get $\frac{L^k|x-c|^k}{k!} \to 0$. Hence there is k_0 large enough such that:

$$L_0 := \frac{L^{k_0}|x - c|^{k_0}}{k_0!} < 1$$

It follows that T^{k_0} is a contraction with constant $L_0 < 1$.

Lecture 33, 2025/03/31

5.1 Global Solutions of ODEs

Proof of Lemma 5.1. Assume (1) holds. Then:

$$\begin{aligned}
|Tf(x) - Tg(x)| &= \left| c_0 + \int_c^x \varphi(t, f(t))\ dt - c_0 - \int_c^x \varphi(t, g(t))\ dt \right| \\
&= \left| \int_c^x \varphi(t, f(t)) - \varphi(t, g(t))\ dt \right| \\
&\leq \int_c^x L|f(t) - g(t)|\ dt && \text{(Lipschitz in } y\text{)} \\
&\leq \int_c^x \frac{LM|t-c|^k}{k!}\ dt && \text{(by (1))} \\
&= \frac{LM|x-c|^{k+1}}{(k+1)!}
\end{aligned}$$

As desired. □

Theorem 5.2 (Global Picard Theorem). If $\varphi : [a,b] \times \mathbb{R} \to \mathbb{R}$ is Lipschitz in y and $c \in [a,b]$ then there exists a unique solution to $y'(x) = \varphi(x, y(x))$ with $y(c) = c_0$ in $\mathcal{C}[a,b]$.

Proof. Use $T : \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ as in the previous lemma. We know from Lemma 5.1 that:

(1) Take $k = 0$ and $M = \|f - g\|_\infty$, it satisfies the condition of the previous lemma. Hence:

$$|Tf(x) - Tg(x)| \leq L\|f - g\|_\infty |x - c|$$

(2) By a different corollary there exists k_0 such that $T^{(k_0)}$ is a contraction on $\mathcal{C}[a,b]$. This gives a unique fixed point to T. Hence we have a unique solution to the DE. □

5.2 Local Solutions

There are solutions where φ is not Lipschitz in y, where φ is "nice enough" that we can still do something to find a unique solution. The problem occurs as $y \in \mathbb{R}$ and $\mathbb{R}$ is big.

Example. Consider $y' = -2xy^2$ with $y(0) = 1$. This has solution $y(x) = \frac{1}{1+x^2}$. The associated function here is $\varphi(x,y) = -2xy^2$, which is not Lipschitz in y for $x \neq 0$. Let $[a,b] = [-1/4, 1/4]$ and:

$$\mathcal{C}'[a,b] = \left\{ f \in \mathcal{C}[a,b] : |f(x) - 1| \leq \frac{1}{2},\ x \in [a,b] \right\} \subseteq \mathcal{C}[a,b]$$

Let $c = 0$ then $c_0 = y(0) = 1$. Let $T : \mathcal{C}'[a,b] \to \mathcal{C}'[a,b]$ by:

$$Tf(x) = 1 + \int_0^x \varphi(t, f(t))\ dt = 1 - 2\int_0^x tf(t)^2\ dt$$

We claim this is well-define, that is, $Tf \in \mathcal{C}'[a,b]$ for all $f \in \mathcal{C}'[a,b]$. We also need to show T is Lipschitz. Assume $|f(x) - 1| \leq 1/2$ for all $x \in [a,b]$. This implies $|f(x)| \leq 3/2$. Then:

$$|Tf(x) - 1| = \left|1 - 2\int_0^x tf(t)^2 \, dt - 1\right| \leq \int_0^{1/4} 2t\left(\frac{3}{2}\right)^2 \, dt = \frac{9}{64} < \frac{1}{2}$$

Hence $Tf \in \mathcal{C}'[a,b]$. Consider the Lipschitz constant on φ. For fixed x we have:

$$\left|\frac{\partial}{\partial y}\varphi(x,y)\right| = 4|xy| \leq \frac{3}{2}$$

as $x \in [-1/4, 1/4]$ and $y \in [1/2, 3/2]$. Using the same trick before there is k_0 such that $T^{(k_0)}$ is a contraction.

Definition. We say $\varphi : [a,b] \times \mathbb{R} \to \mathbb{R}$ is **locally Lipschitz** in y is for all $(x_0, y_0) \in [a,b] \times \mathbb{R}$ there exists $h > 0$ such that φ is Lipschitz on $[x_0 - h, x_0 + h] \times [y_0 - h, y_0 + h]$ in y.

Lemma 5.3. Let $\varphi : [a,b] \times \mathbb{R} \to \mathbb{R}$ be locally Lipschitz in y on a convex compact set K. Then φ is Lipschitz on K.

Proof. For every (x,y) we can find a neighborhood where φ is Lipschitz in y (on this neighborhood). This gives an open cover of K. We can find a finite subcover. Pick L to be the worst constant from this finite set. With some work we can finish the proof. □

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Theorem 5.4 (Local Picard's Theorem). Suppose $\varphi : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous and locally Lipschitz on $[a,b] \times [c_0 - R, c_0 + R]$. Then the DE $y' = \varphi(x,y)$ with $y(a) = c_0$ has a solution on $[a, a+h]$ with $h = \min(b-a, R/\|\varphi\|)$, where:

$$\|\varphi\| := \sup_{\substack{x \in [a,b] \\ y \in [c_0 - R, c_0 + R]}} |\varphi(x,y)|$$

Proof. Take $T : \mathcal{C}[a, a+h] \to \mathcal{C}[a, a+h]$ by:

$$Tf(x) = c_0 + \int_a^x \varphi(t, f(t)) \, dt$$

As before, take $\mathcal{C}' \subseteq \mathcal{C}[a, a+h]$ by:

$$\mathcal{C}' = \{f \in \mathcal{C}[a, a+h] : \|f - c_0\|_\infty \leq R\}$$

We need to show $T : \mathcal{C}' \to \mathcal{C}'$ and T is Lipschitz in y on $\mathcal{C}'$. We see $\mathcal{C}'$ is a compact set. As φ and T are locally Lipschitz in y on a compact and convex set, φ and T' are Lipschitz on C'. To see that

$T : \mathcal{C}' \to \mathcal{C}'$, note for $f \in \mathcal{C}'$ we have:

$$\begin{aligned} |Tf(x) - c_0| &= \left| c_0 + \int_a^x \varphi(t, f(t))\ dt - c_0 \right| \\ &= \left| \int_a^x \varphi(t, f(t))\ dt \right| \\ &\leq h \cdot \|\varphi\| \leq R \end{aligned}$$

Hence $T : \mathcal{C}' \to \mathcal{C}'$. Using the same trick as before there is k_0 such that T^{k_0} is contractive. Hence there exists a solution. $\square$

Remark. We can often use the solution on $[a, a+h]$, and use local Picard to extend this (using $[a+h, b]$ and $y(a+h) = c_0$ for the DE). Sometimes this blows up, but often it is for a good reason.

Remark. We did this analysis for $y : \mathbb{R} \to \mathbb{R}$. We could have done something similar for $y : \mathbb{R} \to \mathbb{R}^n$.

Remark. We can modify higher order DEs to look like first order DEs with more parts.