PMATH 440 Notes

Analytic Number Theory

Fall 2025

Based on Professor Michael Rubinstein's Lectures

Contents

Lecture 1, 2025/09/04

1 Introduction

Topics covered in this course

(1). Summation methods (summation by parts, Euler-Maclaurin Summation, Poisson Summation, Dirichlet Hyperbola).

(2). Dirichlet series and Dirichlet divisor problem.

(3). Riemann zeta function ζ. Meromorphic continuation (ζ has a pole at $s = 1$) and functional equation.

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} \quad \text{for} \quad \mathrm{Re}(s) > 1$$

(4). Prime Number Theorem. If $\pi(x) =$ number of prime numbers $\leq x$, then

$$\pi(x) \sim \int_2^x \frac{1}{\log t}\,\mathrm{d}t \sim \frac{x}{\log x}$$

(5). Dirichlet's Theorem. If $0 \neq a, b \in \mathbb{Z}$ and $\gcd(a, b) = 1$, there are infinitely many prime numbers of the form $ak + b$ for $k \in \mathbb{Z}$. For example, there are infinitely many primes of the form $4k + 1$.

(6). More Complex analysis. Gamma function, Weierstrass products and possibly linear fractional transformations and modular forms.

We first introduce some asymptotic notations.

Definition. We say that $f(x) \sim g(x)$ as $x \to \infty$ if

$$\lim_{x\to a} \frac{f(x)}{g(x)} = 1$$

The Prime Number Theorem says $\pi(x) \sim \dfrac{x}{\log x}$ as $x \to \infty$, which is equivalent to

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1$$

Example. By the Stirling's approximation, we know

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad \text{as } n \to \infty$$

Definition. Let f, g be defined on (a subset of) $\mathbb{R}$ and g be a real-valued. We write $f(x) = \mathcal{O}(g(x))$ as $x \to \infty$, where g is real-valued, if there exists $c > 0$ such that $|f(x)| \leq cg(x)$ for all $x > x_0$.

Example. $\sin(x) = \mathcal{O}(1)$ as $x \to \infty$ since sin is bounded.

Example. By the Stirling's formula we have

$$n! = \mathcal{O}\left(\sqrt{n}\left(\frac{n}{e}\right)^n\right) \quad \text{and} \quad n! = \mathcal{O}\left(\frac{n^{n+1}}{e^n}\right)$$

The first one implies the second one because $\sqrt{n} = \mathcal{O}(n)$.

Definition. We write $f(x) = o(g(x))$ as $x \to a$ if

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0$$

In most cases we will take $a = \infty$ or $a = -\infty$. This means "$f(x)$ is much smaller than $g(x)$ near a".

Example. By the Stirling's formula we have

$$\lim_{n \to \infty} \frac{n!}{\frac{n^{n+1}}{e^n}} = \lim_{n \to \infty} \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\frac{n^{n+1}}{e^n}} = \lim_{n \to \infty} \frac{\sqrt{2\pi}}{\sqrt{n}} = 0$$

It follows that $n! = o(n^{n+1}/e^n)$ as $n \to \infty$.

Remark (Vinogradov's notation). We can also write $f(x) = \mathcal{O}(g(x))$ as $f(x) \ll g(x)$.

Remark. When we write $f(x) = g(x) + \mathcal{O}(h(x))$ to mean $f(x) - g(x) = \mathcal{O}(h(x))$.

2 Summation Methods

2.1 Partial Summation

This method is the discrete version of integration by parts.

Theorem 2.1 (Partial Summation). Let $f : \mathbb{N} \to \mathbb{C}$ and $g : \mathbb{R} \to \mathbb{C}$ be continuously differentiable on $[1, x]$. Then, for all $x \geq 1$ we have

$$\sum_{1 \leq n \leq x} f(n)g(n) = \left(\sum_{1 \leq n \leq x} f(n)\right) g(x) - \int_1^x \sum_{1 \leq n \leq t} f(n)g'(t) \text{ dt} \tag{1}$$

Proof. Consider the term $f(n)g(n)$, we note

$$f(n)g(x) - f(n)\int_n^x g'(t) \text{ dt} = f(n)g(x) - f(n)(g(x) - g(n)) = f(n)g(n) \tag{2}$$

This equality is obtained by looking at the terms that have to do with $f(n)$ in (1). Then summing the equation (2) over $1 \leq n \leq x$ gives us (1). $\square$

Example (Harmonic Series). Consider $\sum_{1\le n\le x}\frac{1}{n}$. Take $f(n)=1$ and $g(x)=\frac{1}{x}$. Then by the Partial summation formula we have

$$\sum_{1\le n\le x}\frac{1}{n}=\left(\sum_{1\le n\le x}f(n)\right)g(x)-\int_1^x\sum_{1\le n\le t}f(n)g'(t)\ \mathrm{d}t=\frac{\lfloor x\rfloor}{x}+\int_1^x\frac{\lfloor t\rfloor}{t^2}\mathrm{d}t$$

Here note that

$$\lfloor x\rfloor:=\sum_{1\le n\le x}1=\text{the largest integer}\le x$$

and using this we define

$$\{x\}:=x-\lfloor x\rfloor=\text{the fractional part of } x$$

For example $\lfloor\pi\rfloor=3$ and $\{\pi\}=0.1415926535897\cdots$. Therefore

$$\begin{aligned}\sum_{1\le n\le x}\frac{1}{n}&=\frac{x-\{x\}}{x}+\int_1^x\frac{t-\{t\}}{t^2}\mathrm{d}t\\&=1-\frac{\{x\}}{x}+\int_1^x\frac{1}{t}-\frac{\{t\}}{t^2}\ \mathrm{d}t\\&=1+\log x-\int_1^x\frac{\{t\}}{t^2}\ \mathrm{d}t+\mathcal{O}\left(\frac{1}{x}\right)\end{aligned}$$

Now we analyze this integral

$$\int_1^x\frac{\{t\}}{t^2}\mathrm{d}t=\underbrace{\int_1^\infty\frac{\{t\}}{t^2}\mathrm{d}t}_{<\infty}-\underbrace{\int_x^\infty\frac{\{t\}}{t^2}\mathrm{d}t}_{\mathcal{O}(\frac{1}{x})}$$

The estimation of second integral is by bounding $\{t\}/t^2$ by $1/t^2$. Therefore

$$\sum_{1\le n\le x}\frac{1}{n}=\log x+\underbrace{1-\int_1^\infty\frac{\{t\}}{t^2}\mathrm{d}t}_{:=\gamma}+\mathcal{O}\left(\frac{1}{x}\right)$$

This constant γ is called the Euler's constant, that is,

$$\lim_{x\to\infty}\left(\sum_{1\le n\le x}\frac{1}{n}-\log x\right)=\gamma$$

Conjecture 2.2. The Euler's constant γ is irrational.

Lecture 2, 2025/09/09

Remark. Let $\lambda_1 < \lambda_2 < \cdots$ be a sequence of natural numbers, then

$$\sum_{\lambda_n \le x} f(\lambda_n)g(\lambda_n) = \left(\sum_{\lambda_n \le x} f(\lambda_n)\right) g(x) - \int_1^x \left(\sum_{\lambda_n \le x} f(\lambda_n)\right) g'(t) \ \mathrm{d}t$$

This is a generalization of the usual Partial summation formula. The proof is similar. Note that for all $n \ge 1$ we have

$$f(\lambda_n)g(\lambda_n) = f(\lambda_n)g(x) - \int_{\lambda_n}^x f(\lambda_n)g'(t) \ \mathrm{d}t$$

Then summing over all n with $\lambda_n \le x$ we obtain the formula.

Example (Factorial). Now let's study the asymptotic of the factorial $m!$ as $m \to \infty$. Since the partial summation only works for sum and $m!$ is a product, we can take the log and consider $\log(m!)$. Let $f(m) = 1$ and let $g(x) = \log(x)$. By the partial summation formula we have

$$\begin{aligned}
\log(m!) = \sum_{1 \le n \le m} \log(m) &= m \log m - \int_1^m \frac{\lfloor t \rfloor}{t} \ \mathrm{d}t \\
&= m \log m - \int_1^m \frac{t - \{t\}}{t} \ \mathrm{d}t \\
&= m \log m - (m-1) + \int_1^m \frac{\{t\}}{t} \ \mathrm{d}t
\end{aligned}$$

Now we need to estimate the integral and get an (rough) upper and lower bound for it.

$$0 < \int_1^m \frac{\{t\}}{t} \mathrm{d}t < \int_1^m \frac{\mathrm{d}t}{t} = \log m$$

Therefore

$$m \log m - (m-1) < \log(m!) < (m+1) \log m - (m-1)$$

Exponentiating this inequality gives

$$\frac{m^m}{e^{m-1}} < m! < \frac{m^{m+1}}{e^{m-1}}$$

This is a weaker result than the Striling's formula.

Remark. The prime counting function is

$$\pi(x) = \sum_{p \le x} 1 = \text{number of primes} \le x$$

For the Riemann zeta function on $\mathrm{Re}(s) > 1$ we have the following identity

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

This is called the Euler's product. Expand the right hand side and by the unique factorization of integers we have the equality. It is sometimes more natural to study the sum of log of primes. We define the function

$$\theta(x) := \sum_{p \le x} \log p$$

The Prime Number Theorem states that $\pi(x) \sim x/\log x$. In fact we have the following proposition.

Proposition 2.3. We have

$$\theta(x) \sim x \iff \pi(x) \sim \frac{x}{\log x}$$

Proof. $(\Rightarrow)$. Assume $\theta(x) \sim x$. Note that

$$\pi(x) = \sum_{p \le x} 1 = \sum_{p \le x} \log p \cdot \frac{1}{\log p}$$

Let $f(x) = \log x$ and $g(x) = 1/\log x$. By Partial summation we have

$$\pi(x) = \underbrace{\frac{\theta(x)}{\log x}}_{\sim \frac{x}{\log x}} + \int_2^x \theta(t) \cdot \frac{\mathrm{d}t}{(\log t)^2 t}$$

Now we note that since $\theta(x) \sim x$, we know $\theta(x) = \mathcal{O}(x)$ so that

$$\int_2^x \theta(t) \cdot \frac{\mathrm{d}t}{(\log t)^2 t} = \mathcal{O}\left(\int_2^x \frac{\mathrm{d}t}{(\log t)^2}\right)$$

But then we have

$$\int_2^x \frac{\mathrm{d}t}{(\log t)^2} = \int_2^{x^{1/2}} \frac{\mathrm{d}t}{(\log t)^2} + \int_{x^{1/2}}^x \frac{\mathrm{d}t}{(\log t)^2}$$

The first integrand is $\mathcal{O}(1)$ so the integral is $\mathcal{O}(x^{1/2})$, for the second integral we use the bound

$$\int_{x^{1/2}}^x \frac{\mathrm{d}t}{(\log t)^2} = \mathcal{O}\left(\frac{x}{(\log x)^2}\right)$$

Combine all of these, we have

$$\pi(x) = \frac{\theta(x)}{\log x} + \mathcal{O}\left(\frac{x}{(\log x)^2}\right)$$

and therefore

$$\frac{\pi(x)}{x/\log x} = \frac{\theta(x)}{\log x} \cdot \frac{\log x}{x} + \mathcal{O}\left(\frac{1}{\log x}\right) = \frac{\theta(x)}{x} + \mathcal{O}\left(\frac{1}{\log x}\right) = 1 + o(1)$$

($\Leftarrow$). Assume the PNT. Let $f(x) = 1$ and $g(x) = \log x$ when x is prime and 0 otherwise.

$$\theta(x) = \sum_{p \le x} \log x = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t} \mathrm{d}t$$

Thus $\theta(x) \sim x$ after some work. $\square$

Lecture 3, 2025/09/11

Example (Meromorphic Continuation of ζ). Recall the zeta function $\zeta(s)$ is only defined for $\mathrm{Re}(s) > 1$ and is equal to

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

This series converges absolutely if $\mathrm{Re}(s) > 1$ and uniformly in any half plane $\mathrm{Re}(s) \ge X_0 > 1$. We want to extend this function to the half plane $\mathrm{Re}(s) > 0$ using partial summation. We let $f(n) = 1$ and let $g(t) = t^{-s}$, then by the Partial summation formula

$$\sum_{1 \le n \le x} \frac{1}{n^s} = \frac{\lfloor x \rfloor}{x^s} + s\int_1^x \frac{\lfloor t \rfloor}{t^{s+1}} \,\mathrm{d}t$$

Here t^s is defined using the principal branch of logartihm, which is defined on $\mathbb{C} \setminus (\infty, 0]$.

$$\begin{aligned}\sum_{1 \le n \le x} \frac{1}{n^s} &= \frac{x - \{x\}}{x^s} + s\int_1^x \frac{(t - \{t\})}{t^s} \,\mathrm{d}t \\ &= \frac{x - \{x\}}{x^s} + \int_1^x \frac{1}{t^s} \,\mathrm{d}t - s\int_1^x \frac{\{t\}}{t^{s+1}} \,\mathrm{d}t\end{aligned}$$

By taking $x \to \infty$ we have

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{\infty} \frac{\{t\}}{t^{s+1}} \,\mathrm{d}t = \frac{1}{s-1} + 1 - s\int_1^{\infty} \frac{\{t\}}{t^{s+1}} \,\mathrm{d}t \tag{$*$}$$

The RHS is analytic on $\mathrm{Re}\, s > 0$ except for a simple pole at $s = 1$ with residue 1 because the improper integral $\int_1^\infty t^{-r} \,\mathrm{d}t$ converges when $r > 1$. Note that the function

$$\int_1^{\infty} \frac{\{t\}}{t^{s+1}} \,\mathrm{d}t$$

on the RHS is analytic by Leibniz's rule (differentiation under the integral sign). Equation ($*$) allows us to extend the domain of ζ to $\mathrm{Re}\, s > 0$, with a pole at 1.

2.2 Euler-Maclaurin Summation and Bernoulli Polynomials

This summation method looks at sums of the form

$$\sum_{a<n\leq b} g(n)$$

where $a, b \in \mathbb{Z}$ are integers and $a < b$. By the Partial summation formula

$$\begin{aligned}\sum_{a<n\leq b} g(n) &= (b-a)g(b) - \int_a^b ([t]-a)g'(t)\ \mathrm{d}t \\ &= bg(b) - ag(b) - \int_a^b tg'(t)\ \mathrm{d}t + a\int_a^b g'(t)\ \mathrm{d}t + \int_a^b \{t\}g'(t)\ \mathrm{d}t \\ &= bg(b) - ag(b) - \underbrace{\int_a^b tg'(t)\ \mathrm{d}t}_{\star} + ag(b) - ag(a) + \int_a^b \{t\}g'(t)\ \mathrm{d}t\end{aligned}$$

By integration by parts we have

$$\star = \int_a^b tg'(t) = tg(t)|_a^b - \int_a^b g(t)\ \mathrm{d}t = bg(b) - ag(a) - \int_a^b g(t)\ \mathrm{d}t$$

Therefore

$$\begin{aligned}\sum_{a<n\leq b} g(n) &= bg(b) - ag(b) - \left(bg(b) - ag(a) - \int_a^b g(t)\ \mathrm{d}t\right) + ag(b) - ag(a) + \int_a^b \{t\}g'(t)\ \mathrm{d}t \\ &= \int_a^b g(t)\ \mathrm{d}t + \int_a^b \{t\}g'(t)\ \mathrm{d}t\end{aligned}$$

Now let us analyze the integral of $\{t\}g'(t)$. Note that on average $\{t\} = 1/2$, we can write

$$\{t\} = \frac{1}{2} + \left(\{t\} - \frac{1}{2}\right)$$

We have

$$\int_a^b \{t\}g'(t)\ \mathrm{d}t = \frac{1}{2}(g(b)-g(a)) + \int_a^b \left(\{t\} - \frac{1}{2}\right) g'(t)\ \mathrm{d}t$$

Before we continue, we need Bernoulli polynomials.

Definition. We define $B_0(x) = 1$. For $k \geq 1$ we recursively define $B_k(x)$ so that

$$B_k'(x) = kB_{k-1}(x) \quad \text{and} \quad \int_0^1 B_k(x)\ \mathrm{d}x = 0$$

The polynomial $B_k(x)$ is called the k**-th Bernoulli polynomials**.

Example ($B_1(x)$)**.** Let $k = 1$. Then

$$B_1(x) = \int B_0(x)\ \mathrm{d}x = x + B_1$$

Then because

$$\int_0^1 B_1(x)\ \mathrm{d}x = \left[\frac{x^2}{2} + B_1 x\right]_0^1 = \frac{1}{2} + B_1 = 0$$

we know that $B_1 = -\frac{1}{2}$ and $B_1(x) = x - \frac{1}{2}$.

Example ($B_2(x)$)**.** Let $k = 2$. Then

$$B_2(x) = \int 2B_1(x)\ \mathrm{d}x = x^2 - x + B_2$$

Then because

$$\int_0^1 B_2(x)\ \mathrm{d}x = \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + B_2 x\right]_0^1 = \frac{1}{3} - \frac{1}{2} + B_2 = 0$$

we know that $B_2 = \frac{1}{6}$ and $B_2(x) = x^2 - x + \frac{1}{6}$.

Definition. For $k \geq 0$ we define $B_k = B_k(0)$ to be the **k-th Bernoulli number**.

Lecture 4, 2025/09/16

Proposition 2.4. For $k \geq 0$ we have

(a). The difference equation: $\dfrac{B_{k+1}(x+1) - B_{k+1}(x)}{k+1} = x^k$

(b). Expansion in terms of Bernoulli numbers: $B_k(x) = \displaystyle\sum_{m=0}^{k} \binom{k}{m} B_{k-m} x^m = \sum_{m=0}^{k} \binom{k}{m} B_m x^{k-m}$

(c). The functional equation: $B_k(x) = (-1)^k B_k(1-x)$

(d). Specials values: $B_k(1) = \begin{cases} (-1)^k B_k(0) & \text{if } k \geq 0 \\ 0 & \text{if } k \text{ is odd and } k \geq 3 \\ 1/2 & \text{if } k = 1 \end{cases}$

(e). Recursion of Bernoulli numbers: $\displaystyle\sum_{m=0}^{k-1} \binom{k}{m} B_m = 0$

(f). Generating Function: $F(x,t) := \displaystyle\sum_{k=0}^{\infty} B_k(x) \frac{z^k}{k!} = \frac{z e^{zx}}{e^z - 1}$

Remark. By (a) we note that

$$B_k(0) = B_k(1) \quad \text{for all } k \geq 0$$

This will be useful later.

Proposition 2.5. We have the following Fourier series expansions

$$B_1(\{x\}) = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n} \quad \text{for } x \notin \mathbb{Z}$$
$$B_k(\{x\}) = -k!\sum_{n\neq 0}\frac{e^{2\pi inx}}{(2\pi in)^k} \quad \text{for } k \geq 2$$

Now we return to our discussion on Euler-Maclaurin summation. We need to study the integral

$$\int_a^b \left(\{t\} - \frac{1}{2}\right) g'(t)\ \mathrm{d}t = \int_a^b B_1(\{t\})g'(t)\ \mathrm{d}t$$

Note that $B_1(\{t\}) = t - \frac{1}{2} - n$ if $n \leq t < n+1$. Hence

$$\int_a^b B_1(\{t\})g'(t)\ \mathrm{d}t = \left(\int_a^{a+1} + \cdots + \int_{b-1}^b\right) B_1(\{t\})g'(t)\ \mathrm{d}t$$

Now let us look at the integral on each interval $[n, n+1]$.

$$\int_n^{n+1} B_1(\{t\})g'(t)\ \mathrm{d}t = \int_n^{n+1}\left(t - \frac{1}{2} - n\right) g'(t)\ \mathrm{d}t$$

Let $u = g'(t)$ and $dv = B_1(\{t\})\ \mathrm{d}t$. Apply integration by parts we have

$$\begin{aligned}\int_n^{n+1} B_1(\{t\})g'(t)\ \mathrm{d}t &= \left[\frac{1}{2}B_2(\{t\})g'(t)\right]_n^{n+1} - \int_n^{n+1}\frac{1}{2}B_2(\{t\})g''(t)\ \mathrm{d}t \\ &= \frac{1}{2}\left(B_2(1) - B_2(0)\right)\left(g'(n+1) - g'(n)\right) - \frac{1}{2}\int_n^{n+1} B_2(\{t\})g''(t)\ \mathrm{d}t\end{aligned}$$

We can now apply the same method to the integral $\int_n^{n+1} B_2(\{t\})g''(t)\ \mathrm{d}t$. Keep doing it, say K times, then we ge the Euler-Maclaurin summation formula.

Theorem 2.6 (Euler-Maclaurin Summation). Let $K \in \mathbb{N}$ and $g : \mathbb{R} \to \mathbb{C}$ such that $g^{(K)}$ exists, then for $a < b$ in $\mathbb{N}$ we have

$$\sum_{a<n\leq b} g(n) = \int_a^b g(t)\ \mathrm{d}t + \sum_{k=1}^{K}\frac{(-1)^k B_k}{k!}(g^{(k-1)}(b) - g^{(k-1)}(a)) + \frac{(-1)^{K+1}}{K!}\int_a^b B_K(\{t\})g^{(K)}(t)\ \mathrm{d}t$$

Example (Sum of powers). We claim that for $r \geq 1$ and $N \geq 1$ we have

$$\sum_{n=1}^{N} n^r = \frac{B_{r+1}(N+1) - B_{r+1}(1)}{r+1}$$

We will apply Euler-Maclaurin summation formula and properties of Bernoulli polynomials to prove it. Let $g(t) = t^r$ then $g^{(r)}(t) = 0$ and

$$g^{(m)}(N) - g^{(m)}(0) = \begin{cases} r(r-1)\cdots(r-m+1)N^{r-m} & \text{if } m \leq r-1 \\ 0 & \text{if } m \geq r \end{cases}$$

Let $a = 0$ and $b = N$ and $K = r$, then by Euler-Maclaurin summation we have

$$\begin{aligned}
\sum_{n=1}^{N} n^r &= \int_0^N t^r \, \mathrm{d}t + \sum_{k=1}^{r} \frac{(-1)^k B_k}{k!} r(r-1)\cdots(r-k+2)N^{r-k+1} \\
&= \int_0^N t^r \, \mathrm{d}t + \sum_{k=1}^{r} \frac{(-1)^k B_k}{r-k+1} \binom{r}{k} N^{r-k+1} \\
&= \int_0^N \sum_{k=0}^{r} (-1)^k B_k \binom{r}{k} t^{r-k} \, \mathrm{d}t && (t^r \text{ is the } k=0 \text{ term}) \\
&= \int_0^N (-1)^r \sum_{k=0}^{r} B_k \binom{r}{k} (-t)^{r-k} \, \mathrm{d}t && ((-1)^r(-1)^{r-k} = (-1)^k) \\
&= \int_0^N (-1)^r B_r(-t) \, \mathrm{d}t = \int_0^N B_r(t+1) \, \mathrm{d}t && (\text{property (b) and (c)}) \\
&= \frac{B_{r+1}(N+1) - B_{r+1}(1)}{r+1}
\end{aligned}$$

Example. The Euler-Maclaurin summation formula can be used to obtain an analytic continuation of $\zeta(s)$ as far to the left as we want, and also provides a useful expansion. Consider

$$\sum_{n=1}^{N} n^{-s} = 1 + \sum_{n=2}^{N} n^{-s}$$

Let $K \in \mathbb{N}$, we want to extend $\zeta(s)$ to the region $\operatorname{Re} s > -K+1$. Let $a = 1$ and $b = N$ and $g(x) = x^{-s}$. Note that

$$g^{(m)}(x) = (-1)^m s(s+1)\cdots(s+m-1)x^{-s-m}$$

for $m \geq 1$. By Euler-Maclaurin summation formula we have

$$\sum_{2\leq n\leq N} n^{-s} = \int_1^N t^{-s}\,\mathrm{d}t - \sum_{k=1}^{K} \frac{(-1)^k B_k}{k!}(-1)^k(s+k-2)\cdots(s+1)s(N^{-s-k+1}-1) + \int_1^N B_K(\{t\})\frac{(-1)^{K+1}}{K!}(-1)^K(s+K-1)\cdots(s+1)st^{-s-K}\,\mathrm{d}t \tag{$*$}$$

Note that we have

$$\frac{(s+k-2)\cdots(s+1)s}{k!} = \frac{1}{k}\frac{(s+k-2)\cdots(s+1)s}{(k-1)!} = \frac{1}{k}\binom{s+k-2}{k-1}$$

It follows that

$$\sum_{2\leq n\leq N} n^{-s} = \int_1^N t^{-s}\,\mathrm{d}t - \sum_{k=1}^{K}\frac{B_k}{k}\binom{s+k-2}{k-1}(N^{-s-k+1}-1) - \binom{s+K-1}{K}\int_1^N B_K(\{t\})t^{-s-K}\,\mathrm{d}t$$

Taking $N \to \infty$ we have that

$$\zeta(s) = 1 + \sum_{n\geq 2} n^{-s} = 1 + \frac{s}{1-s} + B_1 + \sum_{k=2}^{\infty}\frac{B_k}{k}\binom{s+k-2}{k-1} - \binom{s+K-1}{K}\int_1^{\infty} B_K(\{t\})t^{-s-K}\,\mathrm{d}t$$

for $\operatorname{Re} s > 1$. The RHS is meromorphic on the region $\operatorname{Re} s > -K+1$, with a pole at $s = 1$. This gives us a meromorphic continuation to $\operatorname{Re} s > -K+1$ using the quantity on the RHS.

Remark. Since we can choose K arbitrarily large, we have a meromorphic continuation of $\zeta(s)$ to the entire $\mathbb{C}$ except for a pole at $s = 1$. Note that we can do this because all the meromorphic continuation to $\operatorname{Re} s > -K+1$ agree on the open set $\operatorname{Re} s > 1$, so they are all equal by the identity theorem for analytic functions.

Lecture 5, 2025/09/18

Example (Harmonic Series). We can now apply the Euler-Maclaurin summation to give a better estimate of the asymptotic of the harmonic series. Let $s = 1$ and $K = 3$, then apply $(*)$ above

$$\begin{aligned}\sum_{1\leq n\leq N}\frac{1}{n} &= 1 + \int_1^N t^{-1}\,\mathrm{d}t + \frac{1}{2}\left(\frac{1}{N}-1\right) + \frac{1}{12}\left(-\frac{1}{N^2}+1\right) - \int_1^N \frac{B_3(\{t\})}{t^4}\,\mathrm{d}t \\ &= \log N + 1 + \frac{1}{2N} - \frac{1}{2} - \frac{1}{12N^2} + \frac{1}{12} - \int_1^{\infty}\frac{B_3(\{t\})}{t^4}\,\mathrm{d}t + \int_N^{\infty}\frac{B_3(\{t\})}{t^4}\,\mathrm{d}t \\ &= \log N + \underbrace{\frac{7}{12} - \int_1^{\infty}\frac{B_3(\{t\})}{t^4}\,\mathrm{d}t}_{\gamma_K} + \frac{1}{2N} - \frac{1}{12N^2} + \mathcal{O}\left(\frac{1}{N^3}\right)\end{aligned}$$

Here we note that for an arbitrary K we would have

$$\sum_{1\le n\le N}\frac{1}{n}=\log N+\gamma_K+\left(\text{some terms of } N^{-1},\cdots,N^{-k}\right)=\log N+\gamma_K+\mathcal{O}\left(\frac{1}{N}\right)$$

If follows that

$$\gamma_K=\lim_{N\to\infty}\left(\sum_{1\le n\le N}\frac{1}{n}-\log N\right)$$

Hence γ_K is independent of the choice of K. Therefore we can write $\gamma_K=\gamma$ and this is the Euler's constant we saw in Lecture 1.

Example (Stirling's formula). We now apply Euler-Maclaurin summation to derive the Stirling's formula. Let $g(t)=\log t$ and $g^{(k)}(t)=(-1)^{k+1}(k-1)!t^{-k}$. Hence

$$\log(n!)=\sum_{1<m\le n}\log(m)=\int_1^n\log(t)\ \mathrm{d}t+\frac{\log n}{2}+\sum_{k=2}^{K+1}\frac{B_k}{k(k-1)}\left(\frac{1}{n^{k-1}}-1\right)+\frac{1}{K+1}\int_1^n\frac{B_{K+1}(\{t\})}{t^{K+1}}\ \mathrm{d}t$$

The integral of $\log t$ is $t\log t-t$, so the first integral evaluates to $n\log n-n+1$. For the other integral, we have

$$\int_1^n\frac{B_{K+1}(\{t\})}{t^{K+1}}\ \mathrm{d}t=\int_1^\infty\frac{B_{K+1}(\{t\})}{t^{K+1}}\ \mathrm{d}t-\int_n^\infty\frac{B_{K+1}(\{t\})}{t^{K+1}}\ \mathrm{d}t$$

Note that $|B_{K+1}(\{t\})|$ is bounded because $\{t\}\in[0,1]$ for all $t\in\mathbb{R}$. Hence the first integral converges to a constant and the second integral is $\mathcal{O}(n^{-K})$. Collecting all the constant together as c, then

$$\log(n!)=n\log n-n+\frac{\log n}{2}+c+\sum_{k=2}^{K+1}\frac{B_k}{k(k-1)n^{k-1}}+\mathcal{O}(n^{-K})$$

As in the previous example, the constant c also does not depend on K. In fact, $c=\log(2\pi)/2$. Now, we can take $K=1$ and get

$$\begin{aligned}\log(n!)&=\left(n+\frac{1}{2}\right)\log n-n+\frac{\log 2\pi}{2}+\frac{B_2}{2n}+\mathcal{O}\left(\frac{1}{n}\right)\\&=\left(n+\frac{1}{2}\right)\log n-n+\log\sqrt{2\pi}+\mathcal{O}\left(\frac{1}{n}\right)\end{aligned}$$

Let $r(n)$ represents this error term so that $r(n)=\mathcal{O}(1/n)$. Exponentiating both sides give

$$n!=\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\cdot e^{r(n)}$$

There is $N>0$ and $M>0$ such that $r(n)\le\frac{M}{n}$ for $n\ge N$. Hence

$$e^{r(n)}\le e^{M/n}\quad\text{for } n\ge N$$

Now we have $e^{r(n)} = 1 + (e^{r(n)} - 1)$ and $e^{r(n)} - 1 = o(1)$. Therefore

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}(1 + o(1)) = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} + \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \cdot o(1)$$

It follows that

$$\lim_{n\to\infty} \frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} = \lim_{n\to\infty} 1 + o(1) = 1$$

Hence we obtain the Stirling's formula

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \ \text{ as } n \to \infty$$

3 Arithmetic Functions and Dirichlet Series

3.1 Multiplicative Functions

Definition. A function $f : \mathbb{N} \to \mathbb{C}$ is called an **arithmetic function**.

Definition. We say $f : \mathbb{N} \to \mathbb{C}$ is **multiplicative** if $f(mn) = f(m)f(n)$ when $\gcd(m, n) = 1$. We say f is **completely multiplicative** if $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$.

Example. The function $f(n) = 1$ for all $n \in \mathbb{N}$ is completely multiplicative.

Example. Let $z \in \mathbb{C}$. The function $f(n) = n^z$ is completely multiplicative.

Definition. Let $z \in \mathbb{C}$. We define the **divisor function** to be

$$\sigma_z : \mathbb{N} \to \mathbb{C} \ \text{ by } \ \sigma_z(n) := \sum_{d|n} d^z$$

is a multiplicative function. As an example, note that $d \mid 12 \iff d \in \{1, 2, 3, 4, 6, 12\}$ and

$$\sigma_z(12) = 1 + 2^z + 3^z + 4^z + 6^z + 12^z = (1 + 2^z + 4^z)(1 + 3^z) = \sigma_z(4)\sigma_z(3)$$

More generally, let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ written in its prime factorization. Then we have $d \mid n$ if and only if $d = p_1^{\beta_1} \cdots p_k^{\beta_k}$ where each $\beta_i \in \{0, \cdots, \alpha_i\}$. Then

$$\sigma_z(n) = \sum_{d|p_1^{\alpha_1}\cdots p_k^{\alpha_k}} d^z = \left(1 + p_1^z + \cdots + p_1^{\alpha_1 z}\right) \cdots \left(1 + p_k^z + \cdots + p_k^{\alpha_k z}\right) = \sigma_z(p_1^{\alpha_1}) \cdots \sigma_z(p_k^{\alpha_k})$$

This implies $\sigma_z(mn) = \sigma_z(m)\sigma_z(n)$ for $\gcd(m, n) = 1$. For $z = 0$ we denote $\sigma_0(n) = d(n)$, which counts the number of divisors of n.

Remark. If $f : \mathbb{N} \to \mathbb{C}$ is multiplicative. Define

$$F : \mathbb{N} \to \mathbb{C} \text{ by } F(n) := \sum_{d|n} f(d)$$

Similar to the methods above, for $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ we have

$$F(n) = \big(1 + f(p_1) + \cdots + f(p_1^{\alpha_1})\big) \cdots \big(1 + f(p_k) + \cdots + f(p_k^{\alpha_k})\big) = F(p_1^{\alpha_1}) \cdots F(p_k^{\alpha_k})$$

Therefore F is also multiplicative. Hence the sum (over divisors) of a multiplicative function is also a multiplicative function.

Lecture 6, 2025/09/23

Definition. Define **Euler's totient function** φ by

$$\varphi(n) = \text{number of } 1 \le r \le n \text{ with } (r, n) = 1$$

For example $\varphi(12) = 4$ because $1, 5, 7, 11$ are coprime to it. We claim that φ is multiplicative. Let $m, n \in \mathbb{N}$ with $(m, n) = 1$, then note that

$$(r, mn) = 1 \iff (r, m) = 1 \text{ and } (r, n) = 1$$

The condition $(r, m) = 1$ gives $\varphi(m)$ possible residue classes mod m and $(r, n) = 1$ gives $\varphi(n)$ possible residue classes mod n. By Chinese Remainder theorem, $(r, mn) = 1$ gives $\varphi(m)\varphi(n)$ possible residue classes mod mn. It follows that $\varphi(mn) = \varphi(m)\varphi(n)$.

Another way to see it is that for rings R, T we have $(R \oplus T)^\times \cong R^\times \oplus T^\times$. By Chinese Remainder theorem we have the isomorphism $\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$. Hence

$$(\mathbb{Z}/mn\mathbb{Z})^\times \cong (\mathbb{Z}/m\mathbb{Z})^\times \oplus (\mathbb{Z}/n\mathbb{Z})^\times$$

A residue class $a \pmod k$ is a unit if and only if $(a, k) = 1$. Hence $|(\mathbb{Z}/n\mathbb{Z})^\times| = \varphi(n)$. Taking the cardinality gives

$$\varphi(mn) = |(\mathbb{Z}/mn\mathbb{Z})^\times| = |(\mathbb{Z}/m\mathbb{Z})^\times| \cdot |(\mathbb{Z}/n\mathbb{Z})^\times| = \varphi(m)\varphi(n)$$

Hence, if $p_1, \cdots, p_k$ are distinct primes and $\alpha_i \ge 0$ we have

$$\varphi(p_1^{\alpha_1} \cdots p_k^{\alpha_k}) = \prod_{i=1}^{k} \varphi(p_i^{\alpha_i})$$

Now let us compute $\varphi(p^a)$ for a prime p and $a \ge 1$. Note that $\varphi(p) = p - 1$ and if $a \ge 1$, then

$$\{1 \le n \le p^a : (n, p^a) \ne 1\} = \{p, 2p, \cdots, (p^{a-1} - 1)p, p^a\} \text{ has } p^{a-1} \text{ elements}$$

It follows that the number of $n \le p^a$ with $(n, p^a) = 1$ is

$$\varphi(p^a) = p^a - p^{a-1} = p^{a-1}(p-1)$$

In general, if $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ and each $\alpha_k \ge 1$, then

$$\varphi(n) = \prod_{i=1}^{k} \varphi(p_i^{\alpha_i}) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1) = \prod_{i=1}^{k} p_i^{\alpha_i} \cdot \frac{p_i - 1}{p_i} = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) = n \prod_{p \mid n} \left(1 - \frac{1}{p}\right)$$

This proved the following theorem

Theorem 3.1. If $n = 1$ then $\varphi(1) = 1$. If $n \ge 2$ then $\varphi(n) = n \prod_{p \mid n} \left(1 - \frac{1}{p}\right)$.

Remark. This can be proved using a probabilistic argument. An integer $a \le n$ satisfies $(a, n) = 1$ if and only if $p \nmid a$ for all $p \mid n$. The probability that $p \nmid n$ is $1 - p^{-1}$. Multiplying all of them gives the probability that $(a, n) = 1$. Then multiplying by n gives the number of such a.

Example. If $n = 15$ then its prime divisors are 3 and 5, so

$$\varphi(15) = 15 \left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) = 15 \cdot \frac{2}{3} \cdot \frac{4}{5} = 8$$

If $n = 12$ then its prime divisors are 2 and 3, so

$$\varphi(12) = 12 \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = 12 \cdot \frac{1}{2} \cdot \frac{2}{3} = 4$$

Now let us consider the sum of totient functions $\sum_{d \mid n} \varphi(d)$.

Example. For example, if $n = 12$ then

$$\sum_{d \mid 12} \varphi(d) = \varphi(1) + \varphi(2) + \varphi(3) + \varphi(4) + \varphi(6) + \varphi(12) = 1 + 1 + 2 + 2 + 2 + 4 = 12$$

In general, if $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ then by the multiplicativity we have

$$\begin{aligned}
\sum_{d \mid n} \varphi(d) &= \sum_{i_1 = 1}^{\alpha_1} \cdots \sum_{i_k = 1}^{\alpha_k} \varphi(p_1^{i_1} \cdots p_k^{i_k}) = \prod_{i=1}^{k} \sum_{d \mid p_i^{\alpha_i}} \varphi(d) = \prod_{i=1}^{k} (1 + \varphi(p_i) + \cdots + \varphi(p_i^{\alpha_i})) \\
&= \prod_{i=1}^{k} (1 + (p_i - 1) + (p_i^2 - p_i) + \cdots + (p_i^{\alpha_i} - p_i^{\alpha_i - 1})) = \prod_{i=1}^{k} p_i^{\alpha_i} = n
\end{aligned}$$

This proved that

Theorem 3.2. If $n \geq 1$ then $\displaystyle\sum_{d|n} \varphi(d) = n$.

Remark. Here is another proof of Theorem 3.2 that uses group theory. Consider the additive group $G = \mathbb{Z}/n\mathbb{Z}$. By the fundamental theorem of cyclic group, we know G has a unique subgroup C_d of order d for all $d \mid n$, generated by n/d. Also we know $x \in C_d$ is a generator of C_d if and only if $\gcd(x, n/d) = 1$. There are $\varphi(n/d)$ such x. Moreover, since every $x \in G$ is a generator for C_d for a unique d, we have the following

$$n = |G| = \sum_{d|n} |\{\text{generators of } C_d\}| = \sum_{d|n} \varphi\left(\frac{n}{d}\right) = \sum_{d|n} \varphi(d)$$

Definition. For $n \in \mathbb{N}$ we let $\nu(n) :=$ number of distinct prime factors of n.

Example. $\nu(10) = \nu(20) = 2$ because $10 = 2 \times 5$ and $20 = 2^2 \times 5$.

Definition. For $n \in \mathbb{N}$ we define the **Möbius function** μ to be

$$\mu(n) = \begin{cases} (-1)^{\nu(n)} & \text{if } n \text{ is squarefree} \\ 0 & \text{otherwise} \end{cases}$$

Note that μ is multiplicative. Let $p_1, \cdots, p_k$ be distinct primes with $\alpha_i \geq 1$. Then

$$\mu(p_1^{\alpha_1} \cdots p_k^{\alpha_k}) = \begin{cases} (-1)^k & \text{if all } \alpha_i = 1 \\ 0 & \text{otherwise} \end{cases}$$

and one the other hands

$$\mu(p_1^{\alpha_1}) \cdots \mu(p_k^{\alpha_k}) = \begin{cases} (-1)^k & \text{if all } \mu(p_i^{\alpha_i}) = -1 \iff \alpha_i = 1 \\ 0 & \text{otherwise } \iff \text{ one of } \alpha_i \geq 2 \end{cases}$$

It follows that μ is a multiplicative function.

Now let us look at the sum $\sum_{d|n} \mu(d)$. If $n = 12$ then

$$\sum_{d|12} \mu(d) = \mu(1) + \mu(2) + \mu(3) + \mu(4) + \mu(6) + \mu(12) = 1 - 1 - 1 + 0 + 1 = 0$$

In general we have the following theorem

Theorem 3.3. Let $n \in \mathbb{N}$, then $\displaystyle\sum_{d|n} \mu(d) = \begin{cases} 0 & \text{if } n > 1 \\ 1 & \text{if } n = 1 \end{cases}$

Proof. If $n = 1$ then it is trivial. Assume $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ with $\alpha_i \geq 1$. Then

$$\begin{aligned}\sum_{d|n} \mu(d) &= \sum_{i_1=1}^{\alpha_1} \cdots \sum_{i_k=1}^{\alpha_k} \mu(p_1^{i_1} \cdots p_k^{i_k}) = \prod_{i=1}^{k} \sum_{d|p_i^{\alpha_i}} \mu(d) \\ &= \prod_{i=1}^{k} (1 + \mu(p_i) + \cdots + \mu(p_i^{\alpha_i})) = \prod_{i=1}^{k} (1 - 1 + 0 + \cdots + 0) = 0\end{aligned}$$

This completes the proof. □

3.2 Dirichlet Series and Abscissa of Convergence

Definition. Let $f : \mathbb{N} \to \mathbb{C}$ be an arithmetic function. For $s = \sigma + it \in \mathbb{C}$ we let

$$D(f; s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

be the **Dirichlet series** associated to the function f. When does this series converge absolutely?

$$\sum_{n=1}^{\infty} \left| \frac{f(n)}{n^s} \right| = \sum_{n=1}^{\infty} \frac{|f(n)|}{|n^{\sigma+it}|} = \sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\sigma}}$$

If RHS converges for given σ_1, then it also converges for any $\sigma > \sigma_1$. The **Abscissa of absolute convergence** of $D(f; s)$ is defined as

$$\sigma_0 := \sup \left\{ \sigma \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\sigma}} \text{ convegers} \right\}$$

If $\sigma > \sigma_0$, then the series $\sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\sigma}}$ converges. If $\sigma < \sigma_0$, then the series $\sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\sigma}}$ diverges. If $\sigma = \sigma_0$, then the series $\sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\sigma}}$ may or may not converge.

Example. If $1(n) = 1$ is constant, then $D(1, s) = \zeta(s)$ is the zeta function. The Abscissa of convergence is $\sigma_0 = 1$. At $\sigma = 1$, this does not converge.

Example. Let $f(n) = (\log n)^{-2}$. Then

$$D(f, s) = \sum_{n=2}^{\infty} \frac{1}{n^s (\log n)^2}$$

This has Abscissa of convergence $\sigma_0 = 1$ and it converges at $\sigma = 1$. At $\sigma = 1$ we can compare it with the convergent integral

$$\int_2^{\infty} \frac{1}{t(\log t)^2} \,\mathrm{dt} = \int_{\log 2}^{\infty} \frac{1}{u^2} \,\mathrm{du} < \infty$$

Example. Let $f(n) = (-1)^{n-1}$ for $n \in \mathbb{N}$. The η function is defined by

$$\eta(s) := D(f, s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots$$

This has Abscissa of convergence $\sigma_0 = 1$ but it converges condiitonally if $\sigma > 0$ (this can be proved using partial summation).

Remark. The Riemann Hypothesis is equivalent to

$$\sum_{n \le N} \mu(n) = \mathcal{O}(N^{1/2+\epsilon})$$

for $\epsilon > 0$. The Dirichlet series $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$ has abscissa of convergence equal to 1. Assuming the RH, the series converges conditionally when $\sigma > 1/2$.

Lecture 7, 2025/09/25

Theorem 3.4 (Uniqueness of Dirichlet Coefficients). Let F, G be Dirichlet series defined by

$$F(s) = \sum_{n=1}^{\infty} f(n) n^{-s} \quad \text{and} \quad G(s) = G(s) = \sum_{n=1}^{\infty} g(n) n^{-s}$$

Assume that F and G converge absolutely on $\operatorname{Re} s > \sigma_0$. If $F(s) = G(s)$ for all $\operatorname{Re} s > \sigma_0$, then we have $f(n) = g(n)$ for all $n \in \mathbb{N}$.

Proof. Argue by contradiction. Let $h(n) = f(n) - g(n)$. Say $h(n) \neq 0$ for at least one $n \in \mathbb{N}$. Let N be the smallest such n. Hence $h(N) \neq 0$ and $h(n) = 0$ for $1 \le n < N$. Thus

$$0 = F(s) - G(s) = \sum_{n=1}^{\infty} \frac{f(n) - g(n)}{n^s} = \frac{h(N)}{N^s} + \sum_{n=N+1}^{\infty} \frac{h(n)}{n^s} \quad \text{for } \operatorname{Re} s > \sigma$$

Our goal is to show $h(N) = 0$ so we get a contradiction. Rearranging the equation gives

$$h(N) = -N^s \sum_{n=N+1}^{\infty} \frac{h(n)}{n^s}$$

Wrtie $s = \sigma + it$, hence we have

$$|h(N)| \le N^{\sigma} \sum_{n=N+1}^{\infty} \frac{|h(n)|}{n^{\sigma}}$$

For $n \ge N + 1$ and let $c \in \mathbb{R}$ with $\sigma > c > \sigma_0$. Then $n^{\sigma} = n^{\sigma - c} n^c \ge (N+1)^{\sigma - c} n^c$. Hence

$$|h(N)| \le N^{\sigma} \sum_{n=N+1}^{\infty} \frac{|h(n)|}{n^{\sigma}} \le N^{\sigma} \sum_{n=N+1}^{\infty} \frac{|h(n)|}{(N+1)^{\sigma - c} n^c} = \left(\frac{N}{N+1} \right)^{\sigma} (N+1)^c \sum_{n=N+1} \frac{|h(n)|}{n^c}$$

Here $(N+1)^c$ is bounded and the sum $\sum_{n \ge N+1} \frac{|h(n)|}{n^c}$ is bounded, as it is the tail of a convergent series. By taking $\sigma \to \infty$, the RHS tends to 0. Therefore $h(N) = 0$. $\square$

3.3 Product of Dirichlet Series

Consider the Dirichlet series $F(s) = D(f;s)$ and $G(s) = D(g;s)$ that converge absolutely on $\sigma > \sigma_0$. For $\sigma > \sigma_0$ consider their product

$$F(s)G(s) = \left(\sum_{d=1}^{\infty} \frac{f(d)}{d^s}\right)\left(\sum_{m=1}^{\infty} \frac{g(m)}{m^s}\right)$$

Expaning this product gives us

$$f(1)g(1) + \frac{f(1)g(2)+f(2)g(1)}{2^s} + \frac{f(1)g(3)+f(3)g(1)}{3^s} + \frac{f(1)g(4)+f(2)g(2)+f(4)g(1)}{4^s} + \cdots$$

Note that the coefficient of n^{-s} term in $F(s)G(s)$ are sum of $f(d)g(m)$ for $dm = n$. Therefore

$$F(s)G(s) = \sum_{n=1}^{\infty}\left(\sum_{dm=n} f(d)g(m)\right)\frac{1}{n^s} = \sum_{n=1}^{\infty}\left(\sum_{d\mid n} f(d)g\left(\frac{n}{d}\right)\right)\frac{1}{n^s}$$

Example. Consider the zeta function $\zeta(s) = D(1;s)$. Then

$$\zeta(s)^2 = \sum_{n=1}^{\infty}\left(\sum_{d\mid n} 1\cdot 1\right)\frac{1}{n^s} = \sum_{n=1}^{\infty}\frac{d(n)}{n^s}$$

Let $k \in \mathbb{N}$, then we have

$$\zeta(s)^k = \sum_{n=1}^{\infty}\left(\sum_{a_1\cdots a_k=n} 1\right)\frac{1}{n^s} = \sum_{n=1}^{\infty}\frac{d_k(n)}{n^s}$$

where

$$d_k(n) := \sum_{a_1\cdots a_k=n} 1 = \begin{array}{c}\text{number of ways to write } n \text{ as a}\\ \text{product of } k \text{ natural numbers}\end{array}$$

Example. Consider the product $\zeta(s)\zeta(s-z)$ for some $z \in \mathbb{C}$. Then

$$\zeta(s-z) = \sum_{n=1}^{\infty}\frac{n^z}{n^s} \quad \text{for } \operatorname{Re} s > \operatorname{Re} z + 1$$

Here $f(n) = 1$ and $g(n) = n^z$, so we have

$$\zeta(s)\zeta(s-z) = \sum_{n=1}^{\infty}\left(\sum_{d\mid n} d^z\right)\frac{1}{n^s} = \sum_{n=1}^{\infty}\frac{\sigma_z(n)}{n^s} = D(\sigma_z;s)$$

Example (Euler Products). Euler proved that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \quad \text{for } \operatorname{Re} s > 1$$

The reason this is true is that, for every n^{-s} on the LHS by unique factorization this term appears exactly once on the RHS.

Now consider the reciprocal of this infinite product

$$\prod_p \left(1 - \frac{1}{p^s}\right) = \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right) \cdots = 1 - \frac{1}{2^s} - \frac{1}{3^s} - \frac{1}{5^s} + \frac{(-1)^2}{6^s} + \cdots$$

On the RHS we are only getting n^{-s} for squarefree n's and the coefficients is $(-1)^{\nu(n)}$ where $\nu(n)$ is the number of prime factors of n. By the definition of Möbius function we have

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

and this series converges absolutely for $\operatorname{Re} s > 1$. Thus

$$1 = \frac{1}{\zeta(s)} \cdot \zeta(s) = \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right) = \sum_{n=1}^{\infty} \left(\sum_{d|n} \mu(d)\right) \frac{1}{n^s}$$

By the Uniqueness of Dirichlet series we have

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

which is exactly Theorem 3.3!!

Example (Dirichlet series of $\varphi(n)$). Note that if $\sigma > 2$ then we have

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^\sigma} \leq \sum_{n=1}^{\infty} \frac{n}{n^\sigma} = \sum_{n=1}^{\infty} \frac{1}{n^{\sigma-1}} \quad \text{converges}$$

On the other hand, if $\sigma \leq 2$ then

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^\sigma} \geq \sum_p \frac{p-1}{p^\sigma} \quad \text{diverges because } \sum_p \frac{1}{p} \text{ diverges}$$

Hence the Abscissa of convergence is 2 and the series diverges at 2. But

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \prod_p \left(1 + \frac{\varphi(p)}{p^s} + \frac{\varphi(p^2)}{p^{2s}} + \cdots\right) \quad \text{for } \operatorname{Re} s > 2$$

by the unique factorization and multiplicativity of $\varphi(n)$. Using the formula $\varphi(p^a) = p^{a-1}(p-1)$

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \prod_p \left(1 + \frac{p-1}{p^s} + \frac{p(p-1)}{p^{2s}} + \cdots\right) = \prod_p \left(1 + \frac{p-1}{p^s}\left(1 + \frac{p}{p^s} + \frac{p^2}{p^{2s}} + \cdots\right)\right)$$
$$= \prod_p \left(1 + \frac{p-1}{p^s} \cdot \left(1 - \frac{1}{p^{s-1}}\right)^{-1}\right) = \prod_p \frac{1-p^{-s}}{1-p^{-(s-1)}} = \frac{\zeta(s-1)}{\zeta(s)} \quad \text{for } \operatorname{Re} s > 2$$

Therefore $D(\varphi; s) = \zeta(s-1)\zeta(s)^{-1}$ for $\operatorname{Re} s > 2$.

Lecture 8, 2025/09/30

By moving $\zeta(s)$ to the other side, for $\operatorname{Re} s > 2$ we have

$$\left(\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right) = \left(\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s}\right)\zeta(s) = \zeta(s-1) = \sum_{n=1}^{\infty} \frac{n}{n^s}$$

Expaning the LHS gives us

$$\sum_{n=1}^{\infty}\left(\sum_{d|n} \varphi(d)\right)\frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{n}{n^s}$$

By the Uniqueness of Dirichlet series we have

$$\sum_{d|n} \varphi(d) = n$$

which is exactly Theorem 3.2.

Example. Consider the Dirichlet series for $|\mu(n)|$, which is an indicator function for squarefree numbers. Hence for $\operatorname{Re} s > 1$ we have

$$\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} = \prod_p \left(1 + \frac{|\mu(p)|}{p^s}\right) = \prod_p \left(1 + \frac{1}{p^s}\right)$$

Now using the famous equality that $1 + x = (1-x^2)/(1-x)$ we have

$$\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} = \prod_p \frac{1-p^{-2s}}{1-p^{-s}} = \frac{\zeta(s)}{\zeta(2s)}$$

Let $f, g : \mathbb{N} \to \mathbb{C}$ be arithmetic functions. Assume that

$$g(n) = \sum_{d|n} f(n)$$

is a divisor sum of f. We claim that f can be written as a divisor sum in terms of g! Observe that

$$\left(\sum_{n=1}^{\infty} \frac{f(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right) = \sum_{n=1}^{\infty}\left(\sum_{d|n} f(d)\right)\frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

Recall that $\zeta(s)^{-1}$ is the Dirichlet series of $\mu(n)$. Hence

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \frac{1}{\zeta(s)}\sum_{n=1}^{\infty} \frac{g(n)}{n^s} = \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{g(n)}{n^s}\right) = \sum_{n=1}^{\infty}\left(\sum_{d|n} g(d)\mu\left(\frac{n}{d}\right)\right)\frac{1}{n^s}$$

Now by the Uniqueness of Dirichlet series we have

$$f(n) = \sum_{d|n} g(d)\mu\left(\frac{n}{d}\right)$$

[Here we are assuming that the series above converge absolutely on some half plane $\operatorname{Re} s > \sigma_0$]. The above argument works back work. Hence we have the following result.

Theorem 3.5 (Möbius Inversion). Let $f, g : \mathbb{N} \to \mathbb{C}$ be arithmetic functions, then

$$g(n) = \sum_{d|n} f(d) \iff f(n) = \sum_{d|n} g(d)\mu\left(\frac{n}{d}\right)$$

for all $n \in \mathbb{N}$.

Proof. The "proof" we had above is not too rigorous. It just gives the big picture why Möbius Inversion is true. For example, what if the Dirichlet series $D(f; s)$ has no abscissa of convergence? For example if $f(n) = e^n$, then

$$D(f; s) = \sum_{n=1}^{\infty} \frac{e^n}{n^s}$$

does not converge absolutely for any value of $s \in \mathbb{C}$! There are two approaches.

4 Order of arithmetic functions

4.1 Order of the Divisor Function

Theorem 4.1 (Order of $d(n)$). Let $\epsilon > 0$, then $d(n) = \mathcal{O}_\epsilon(n^\epsilon)$.

Remark. Here the notation $\mathcal{O}_\epsilon(\cdot)$ means the constant is dependent on ϵ. In other words, for all $\epsilon > 0$ there exist $N_\epsilon \in \mathbb{N}$ and $c_\epsilon > 0$ such that $d(n) \le c_\epsilon n^\epsilon$ for $n > N_\epsilon$.

Proof. Let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, where p_i are distinct primes and $\alpha_i \geq 1$. Hence

$$d(n) = (1+\alpha_1)\cdots(1+\alpha_k)$$

A funny way to see this equality is the following. Let $f(n) = 1$ for all n, then

$$d(n) = \sum_{d|n} 1 = \sum_{d|n} f(d) = \prod_{i=1}^{k}(1 + f(p_i) + \cdots + f(p_i^{\alpha_i})) = \prod_{i=1}^{k}(1 + \underbrace{1 + \cdots + 1}_{\alpha_i \text{ times}}) = \prod_{i=1}^{k}(1+\alpha_i)$$

Now let $\epsilon > 0$ be arbitray, then

$$\frac{d(n)}{n^\epsilon} = \left(\prod_{i=1}^{k} \frac{1}{p_i^{\alpha_i \epsilon}}\right)\left(\prod_{i=1}^{k}(\alpha_i + 1)\right) = \prod_{i=1}^{k} \frac{\alpha_i + 1}{p_i^{\alpha_i \epsilon}}$$

Our goal is to show the RHS is bounded as $n \to \infty$. Note that if $p > 2^{1/\epsilon}$ then we have

$$p^{\alpha\epsilon} > 2^\alpha = (1+1)^\alpha \geq \alpha + 1$$

for all $\alpha \in \mathbb{N}$. If $p < 2^{1/\epsilon}$ (there are only finitely many such p for given ϵ) we have

$$\frac{\alpha+1}{p^{\alpha\epsilon}} \leq \frac{\alpha+1}{2^{\alpha\epsilon}}$$

is bounded (as a function of α, for given ϵ). Hence

$$\frac{d_n}{n^\epsilon} = \left(\prod_{p_i > 2^{1/\epsilon}} \frac{\alpha_i+1}{p_i^{\alpha_i\epsilon}}\right)\left(\prod_{p_i < 2^{1/\epsilon}} \frac{\alpha_i+1}{p_i^{\alpha_i\epsilon}}\right) \leq \underbrace{\left(\prod_{p_i > 2^{1/\epsilon}} \frac{\alpha_i+1}{\alpha_i+1}\right)}_{\leq 1}\underbrace{\left(\prod_{p_i < 2^{1/\epsilon}} \frac{\alpha_i+1}{2^{\alpha_i\epsilon}}\right)}_{\text{bounded by } C_\epsilon}$$

The bounded constant depends on ϵ, so we proved that $d_n \leq C_\epsilon n^\epsilon$ for $n \to \infty$, as desired. □

Remark. This theorem tells us $d(n)$ grows slower than any power of n. We have the following results regarding the average order of the divisor function $d(n)$.

Theorem 4.2 (Average Order of $d(n)$). For $x \geq 1$ we have

$$\sum_{n \leq x} d(n) = x \log x + \mathcal{O}(x)$$

Theorem 4.3 (Dirichlet). For $x \geq 1$ we have

$$\sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + \mathcal{O}(x^{1/2})$$

Conjecture 4.4. Instead of $\mathcal{O}(x^{1/2})$, it is conjectured that for all $\epsilon > 0$ we have

$$\sum_{n\leq x} d(n) = x\log x + (2\gamma - 1)x + \mathcal{O}_\epsilon(x^{1/4+\epsilon})$$

Proof of Theorem 4.2. Note that

$$\begin{aligned}
\sum_{n\leq x} d(n) &= \sum_{n\leq x}\sum_{d\mid n} 1 = \sum_{n\leq x}\sum_{dm=n} 1 = \sum_{dm\leq x} 1 \\
&= \sum_{d\leq x}\sum_{m\leq x/d} 1 = \sum_{d\leq x}\left\lfloor\frac{x}{d}\right\rfloor = \sum_{d\leq x}\frac{x}{d} - \left\{\frac{x}{d}\right\} \\
&= x\sum_{d\leq x}\frac{1}{d} - \sum_{d\leq x}\left\{\frac{x}{d}\right\} = x\sum_{d\leq x}\frac{1}{d} + \mathcal{O}(x) \\
&= x(\log x + \gamma + \mathcal{O}(1/x)) + \mathcal{O}(x) \\
&= x\log x + \mathcal{O}(x)
\end{aligned}$$

As desired. ☐

Lecture 9, 2025/10/02

Proof of Dirichlet's Theorem. Using the Dirichlet Hyperbola method we have

$$\begin{aligned}
d(n) &= \sum_{d\leq x}\sum_{m\leq x/d} 1 = 2\sum_{d\leq\sqrt{x}}\sum_{m\leq x/d} 1 - \sum_{d\leq\sqrt{x}}\sum_{m\leq\sqrt{x}} 1 \\
&= 2\sum_{d\leq\sqrt{x}}\left\lfloor\frac{x}{d}\right\rfloor - \lfloor x\rfloor^2 = 2\sum_{d\leq\sqrt{x}}\left\lfloor\frac{x}{d}\right\rfloor - (\sqrt{x} - \{\sqrt{x}\})^2 \\
&= 2x\sum_{d\leq\sqrt{x}}\frac{1}{d} - 2\sum_{d\leq\sqrt{x}}\left\{\frac{x}{d}\right\} - x + \mathcal{O}(\sqrt{x}) \\
&= 2x\sum_{d\leq\sqrt{x}}\frac{1}{d} - x + \mathcal{O}(\sqrt{x})
\end{aligned}$$

Now apply the theorem $\sum_{d\leq y} = y\log y + \gamma + \mathcal{O}(1/y)$ to $y = \sqrt{x}$, we have

$$\begin{aligned}
d(n) &= 2x\left(\log\sqrt{x} + \gamma + \mathcal{O}(1/\sqrt{x})\right) - x + \mathcal{O}(\sqrt{x}) \\
&= x\log x + (2\gamma - 1)x + \mathcal{O}(\sqrt{x})
\end{aligned}$$

This completes the proof. ☐

Now let's look at the average order of the Euler totient function $\varphi(n)$. Recall that by Möbius Inversion we have

$$\varphi(n) = \sum_{d\mid n}\mu(d)\cdot\frac{n}{d}$$

Therefore we have

$$\begin{aligned}
\sum_{n\le x}\varphi(n) &= \sum_{n\le x}\sum_{d\mid n}\mu(d)\cdot\frac{n}{d} = \sum_{d\le x}\sum_{m\le x/d}\mu(d)m = \sum_{d\le x}\mu(d)\sum_{m\le x/d} m \\
&= \sum_{d\le x}\frac{\mu(d)}{2}\cdot\left(\left\lfloor\frac{x}{d}\right\rfloor+1\right)\left\lfloor\frac{x}{d}\right\rfloor \\
&= \sum_{d\le x}\frac{\mu(d)}{2}\left[\frac{1}{2}\left(\frac{x}{d}\right)^2+\mathcal{O}\left(\frac{x}{d}\right)\right] \\
&= \frac{x^2}{2}\sum_{d\le x}\frac{\mu(d)}{d^2}+\mathcal{O}\left(x\sum_{d\le x}\frac{1}{d}\right) \\
&= \frac{x^2}{2}\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}-\frac{x^2}{2}\sum_{d>x}\frac{\mu(d)}{d^2}+\mathcal{O}(x\log x)
\end{aligned}$$

Now let's look at these two infinite series. Recall that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} \quad \text{for } \operatorname{Re} s>1$$

Let $s=2$, we know that $\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}=\zeta(2)^{-1}=6/\pi^2$. On the other hand

$$\left|\sum_{d>x}\frac{\mu(d)}{d^2}\right| \le \sum_{d>x}\frac{1}{d^2} = \mathcal{O}\left(\frac{1}{x}\right)$$

The bound $\mathcal{O}(1/x)$ is obtained by comparing this tail with the integral $\int_1^\infty t^{-2}\,\mathrm{d}t$. Collecting all these information together gives us

Theorem 4.5 (Average Order of $\varphi(n)$). For $x\ge 1$ we have

$$\sum_{n\le x}\varphi(n) = \frac{3x^2}{\pi^2}+\mathcal{O}(x\log x)$$

Definition. We define the **von Mangöldt Function** by

$$\Lambda(n) = \begin{cases}\log p & \text{if } n=p^a \text{ for some } p\\ 0 & \text{otherwise}\end{cases}$$

Where does this function come from? Let's look at the derivative of $\zeta(s)$. We know

$$\zeta(s) = \prod_p\left(1-\frac{1}{p^s}\right)^{-1} \quad \text{for } \operatorname{Re} s>1$$

Taking the logarithm gives us

$$\log \zeta(s) = -\sum_p \log\left(1 - \frac{1}{p^s}\right) = \sum_p \sum_{k=1}^{\infty} \frac{1}{kp^{ks}}$$

Defferentiating both sides gives

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_p \sum_{k=1}^{\infty} \frac{-k\log p}{kp^{ks}} = -\sum_p \sum_{k=1}^{\infty} \frac{\log p}{p^{ks}} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

In this sum, note that $a_n \neq 0$ if and only if $n = p^a$ is a prime power. In that case $a_n = -\log p$. Therefore we conclude that

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

Proposition 4.6. For $\operatorname{Re} s > 1$ we have

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

Theorem 4.7. For $x \geq 1$ we have

$$\sum_{n\leq x} \frac{\Lambda(n)}{n} = \log x + \mathcal{O}(1)$$

Proof. We will look at $\log(m!)$. For example if $m = 10$ then $10! = 2^{3+2+1}3^{1+3}5^{1+1}7$. Then

$$\log(10!) = (3+2+1)\log 2 + (3+1)\log 3 + 2\log 5 + \log 7$$

Reall the Legendre's formula for $\nu_p(n!)$ says that

$$\nu_p(n!) = \sum_{r=1}^{\infty} \left\lfloor \frac{n}{p^r} \right\rfloor = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots$$

Therefore we have

$$\begin{aligned}\log(m!) &= \log\left(\prod_{p\leq m} p^{\nu_p(m!)}\right) = \sum_{p\leq m} \nu_p(m!)\log p = \sum_{p\leq m}\sum_{r=1}^{\infty} \left\lfloor \frac{m}{p^r} \right\rfloor \log p \\ &= \sum_{p^r\leq m} \left\lfloor \frac{m}{p^r} \right\rfloor \log p = \sum_{n\leq m} \left\lfloor \frac{m}{n} \right\rfloor \Lambda(n) = m\sum_{n\leq m} \frac{\Lambda(n)}{n} - \underbrace{\sum_{n\leq m} \left\{ \frac{m}{n} \right\} \Lambda(n)}_{R(m)}\end{aligned}$$

It follows that

$$\sum_{n\leq m} \frac{\Lambda(n)}{n} = \frac{\log(m!)}{m} + \frac{R(m)}{m}$$

By Stirling's formula we know $\log(m!) = m\log m + \mathcal{O}(m)$. For the error term we have

$$\begin{aligned} R(m) &< \sum_{p^r \le m} \log p = \sum_{p \le m} \log p + \sum_{p^2 \le m} \log p + \sum_{p^3 \le m} \log p + \cdots \\ &\le \sum_{p \le m} \log p + (\log m) m^{1/2} + (\log m) m^{1/3} + \cdots + (\log m) m^{1/k} \\ &\le \sum_{p \le m} \log p + (\log m) m^{1/2} \cdot \frac{\log m}{\log 2} \end{aligned}$$

where k is minimal such that $p^k < m$ is impossible, which means $2^k \le (\log m)/(\log 2)$. By the Theorem 4.8 below we can show that $\sum_{p \le m} \log p = \mathcal{O}(m)$, hence

$$R(m) = \mathcal{O}(m) + \mathcal{O}\left((\log m)^2 m^{1/2}\right) = \mathcal{O}(m)$$

Therefore it follows that

$$\sum_{n \le m} \frac{\Lambda(n)}{n} = \frac{m\log m + \mathcal{O}(m)}{m} + \frac{\mathcal{O}(m)}{m} = \log m + \mathcal{O}(1)$$

This completes the proof. By Adapting this proof, we can replace $m \in \mathbb{N}$ with $x \in \mathbb{R}$. $\square$

Lecture 10, 2025/10/07

4.2 An Elementary Bound on $\theta(x)$

We now look at some elementary bounds on $\pi(x)$ and some arithmetic functions. First we will obtain an elementary upper bound for $\theta(x) = \mathcal{O}(x)$. The PNT is equivalent to $\theta(x) \sim x$.

The key is to consider the middle binomial coefficient $\binom{2m}{m}$. For $m \in \mathbb{N}$ we have

$$2^{2m} = (1+1)^{2m} = \sum_{k=0}^{2m} \binom{2m}{k} \ge \binom{2m}{m} = \frac{(2m)!}{m!m!}$$

For $m = 7$ we have

$$\binom{14}{7} = \frac{14!}{7!7!} = \frac{8 \cdot 9 \cdot 10 \cdot 11 \cdot 11 \cdot 12 \cdot 13 \cdot 14}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} = 4 \cdot 3 \cdot 11 \cdot 13$$

We can see that $\binom{14}{7}$ must be divisible by primes between 8 and 14. This is because when we divide 14! by 7! and 7!, the denominator never cancels the prime between 8 and 14 on the numerator. This idea generalizes to all $m \in \mathbb{N}$. Hence we have

$$2^{2m} \ge \binom{2m}{m} = \frac{(2m)!}{m!m!} \ge \prod_{m < p \le 2m} p$$

Taking the logarithm gives

$$\sum_{m<p\leq 2m} \log p \leq 2m \log 2 \tag{$*$}$$

Note that if $2^{r-1} < x \leq 2^r$ for $r \in \mathbb{N}$, then

$$\theta(x) \leq \theta(2^r) = \sum_{2^0<p\leq 2^1} \log p + \sum_{2^1<p\leq 2^2} \log p + \sum_{2^2<p\leq 2^3} \log p + \cdots + \sum_{2^{r-1}<p\leq 2^r} \log p$$

Using the bound $(*)$ we obtained above to each sum, we get

$$\theta(x) \leq (\log 2)(2+4+\cdots+2^r) \leq 2^{r+1}\log 2 \leq 4\log 2 \cdot 2^{r-1} \leq (4\log 2)x$$

This gives the theorem.

Theorem 4.8. For $x \geq 1$ we have

$$\sum_{p\leq x} \log p = \mathcal{O}(x)$$

4.3 Mittag-Leffler Expansions

Example. Consider the function

$$f(z) = \frac{\pi\cos\pi z}{\sin\pi z} - \frac{1}{z}$$

It has simple poles at $0 \neq n \in \mathbb{Z}$. To compute the residues at these points we can compute the Laurent series expansion of $f(z)$ around $z = n$.

$$\cos(\pi z) = (-1)^n \left(1 - \frac{\pi^2}{2!}(z-n)^2 + \frac{\pi^4}{4!}(z-n)^4 - \cdots\right)$$
$$\sin(\pi z) = (-1)^n \left(\pi(z-n) - \frac{\pi^3}{3!}(z-n)^3 + \cdots\right)$$

Hence the residue $r_n = \operatorname{Res}(f, n)$ is

$$r_n = \lim_{z\to n}(z-n)\left(\frac{\pi\cos\pi z}{\sin\pi z} - \frac{1}{z}\right) = 1$$

Our goal is to show that

$$\frac{\pi\cos\pi z}{\sin\pi z} = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z-n} + \frac{1}{z+n}\right) \quad \text{for } z \in \mathbb{C} \text{ not a pole of } f$$

Proof. Let $z \in \mathbb{C}$ and z is not a pole (so $z \notin \mathbb{Z} \setminus \{0\}$). Let $R = n + 1/2$ for some $n \in \mathbb{N}$. Let γ_R be the contour whose trace is the square with four vertices $(\pm R, \pm R)$ oriented counter-clockwise. Then $f(z)$ is bounded on γ_R because it has no poles on γ_R. Moreover, this bound is independent of R!

This is basically becasue $\frac{\pi \cos \pi z}{\sin \pi z}$ is periodic with period 2 and $1/z$ is bounded by 1 away from 0 (this is not a proof, just an intuition). Anyway, say $|f(w)| < c$ for all $w \in \gamma_R$. Therefore we have

$$\frac{1}{2\pi i} \int_{\gamma_R} f(w) \left(\frac{1}{w-z} - \frac{1}{w} \right) \, \mathrm{d}w = \frac{1}{2\pi i} \int_{\gamma R} f(w) \cdot \frac{z}{(w-z)w} \, \mathrm{d}w$$

Note that for given z we have $\frac{z}{(w-z)w} = \mathcal{O}(R^{-2})$. Hence the integral above goes to 0 as $R \to \infty$. On the other hand, we have

$$\frac{1}{2\pi i} \int_{\gamma_R} \underbrace{f(w) \left(\frac{1}{w-z} - \frac{1}{w} \right)}_{:=g(w)} \, \mathrm{d}w = \sum_{\substack{0 \neq n \\ \text{inside } \gamma_R}} \operatorname{Res}(g(w), n) + \operatorname{Res}(g(w), 0) + \operatorname{Res}(g(w), z)$$

The residues are equal to

$$\operatorname{Res}(g(w), n) = 1 \cdot \left(\frac{1}{n-z} - \frac{1}{n} \right) \quad \text{and} \quad \operatorname{Res}(g(w), 0) = -f(0) \quad \text{and} \quad \operatorname{Res}(g(w), z) = f(z)$$

Using a linear approximation of cos and sin we have $f(0) = 0$. Let $R \to \infty$ then we have

$$0 = f(z) - f(0) + \lim_{R \to \infty} \sum_{1 \leq n < R} \left(\frac{1}{n-z} - \frac{1}{n} + \frac{1}{-n-z} + \frac{1}{n} \right)$$

Rearrange this equality gives

$$\frac{\pi \cos \pi z}{\sin \pi z} - \frac{1}{z} = f(z) = -\sum_{n=1}^{\infty} \left(\frac{1}{n-z} + \frac{1}{-n-z} \right) = \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{z-n} \right)$$

Moving $1/z$ to the RHS gives us the desired equality. $\square$

Lecture 11, 2025/10/09

We can keep expanding this summation

$$\begin{aligned}
\frac{\pi z \cos \pi z}{\sin \pi z} &= 1 + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{z-n} \right) = 1 + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2} \\
&= 1 - 2z^2 \sum_{n=1}^{\infty} \frac{1}{n^2(1 - z^2/n^2)} = 1 - 2 \sum_{n=1}^{\infty} \frac{z^2}{n^2} \sum_{k=0}^{\infty} \left(\frac{z^2}{n^2} \right)^k \\
&= 1 - 2 \sum_{k=1}^{\infty} z^{2k} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = 1 - 2 \sum_{k=1}^{\infty} \zeta(2k) z^{2k}
\end{aligned}$$

On the RHS we have a power series whose $[z^{2k}]$ terms are $-2\zeta(2k)$. Hence this allows us to compute

$\zeta(2k)$ by expaning the LHS using Taylor series. We have

$$\begin{aligned}\frac{\pi z \cos \pi z}{\sin \pi z} &= \pi z \left(1 - \frac{(\pi z)^2}{2!} + \frac{(\pi z)^4}{4!} - \cdots\right)\left(\pi z - \frac{(\pi z)^3}{3!} + \frac{(\pi z)^5}{5!} - \cdots\right)^{-1} \\ &= \left(1 - \frac{(\pi z)^2}{2!} + \frac{(\pi z)^4}{4!} - \cdots\right)\left(1 - \frac{(\pi z)^2}{3!} + \frac{(\pi z)^4}{5!} - \cdots\right)^{-1} \\ &= 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots\end{aligned}$$

Hence we have

$$(1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots)\left(1 - \frac{(\pi z)^2}{3!} + \frac{(\pi z)^4}{5!} - \cdots\right) = 1 - \frac{(\pi z)^2}{2!} + \frac{(\pi z)^4}{4!} - \cdots$$

On the LHS the z^2 term is equal to

$$-\frac{(\pi z)^2}{3!} + c_2 z^2 = \left(c_2 - \frac{\pi^2}{6}\right) z^2$$

It follows that $c_2 = \pi^2/6 - \pi^2/2 = -\pi^2/3$. It follows that

$$-2\zeta(2) = c_2 = -\frac{\pi^2}{3} \implies \zeta(2) = \frac{\pi^2}{6}$$

4.4 Eisenstein Series

We can expand this sucker in another way.

$$\frac{\pi \cos \pi z}{\sin \pi z} = \pi i \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \cdot \frac{e^{i\pi z}}{e^{i\pi z}} = \pi i \cdot \frac{e^{2\pi i z} + 1}{e^{2\pi i z} - 1} = \pi i \cdot \frac{u+1}{u-1} = -\pi i (u+1) \sum_{r=0}^{\infty} u^r$$

where we set $u = e^{2\pi i z} = e^{2\pi i x - 2\pi y}$ for $z = x + iy$. The expansion is valid for $|u| < 1$, which is when $\operatorname{Im} z > 0$. Hence, by the result we have seen last time

$$\frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{z-n}\right) = -\pi i \left(1 + 2\sum_{r=1}^{\infty} u^r\right) \quad \text{for } |u| < 1$$

Differentiating both sides term by term with respect to z (note $du/dz = 2\pi i u$) gives us

$$-\sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^2} = -(2\pi i)^2 \sum_{r=1}^{\infty} r e^{2\pi i r z}$$

Repeatedly differentiating with respect to z, for $k \in \mathbb{Z}$ and $k \geq 2$

$$\sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1} e^{2\pi i r z} \qquad (\heartsuit)$$

This is the **Lipschitz formula**. The LHS is periodic with period 1 and the RHS is its Fourier series.

Definition. Let $\mathbb{H} = \{\operatorname{Im} z > 0\}$ be the upper half plane. Let $\tau \in \mathbb{H}$ and $k \in \mathbb{N}$. The **Eisenstein series of weight** k is

$$G_k(\tau) := \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k}$$

If k is odd then $G_k(\tau) = 0$ for all $\tau \in \mathbb{H}$. Hence we assume k is even. If $k = 2$ then it turns out that G_2 does not converge and for $k \geq 3$ it always converges. Hence we assume that k is even and $k \geq 3$.

Theorem 4.9. For $k \geq 3$, the Eisenstein series $G_k(\tau)$ converges absolutely.

Theorem 4.10. For $k \geq 3$, the Eisenstein series $G_k(\tau)$ is a **modular form of weight** k, meaning

$$G_k\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k G_k(\tau)$$

for all $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$ [This means the matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is in $\mathrm{SL}_2(\mathbb{Z})$].

If $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ then for all $\tau \in \mathbb{H}$ we have

$$G_k(\tau+1) = G_k(\tau) \tag{1}$$

If $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ then for all $\tau \in \mathbb{H}$ we have

$$G_k\left(\frac{-1}{\tau}\right) = \tau^k G_k(\tau) \tag{2}$$

Equation (1) tells us that G_k is periodic with period 1, so if we know the value of $G_k(\tau)$ on the strip $\{-1/2 \leq \operatorname{Re}\tau < 1/2\}$ then we know all $G_k(\tau)$ for all $\tau \in \mathbb{H}$. Equation (2) tells us that if we know $G_k(\tau)$ for $|\tau| > 1$ we know the value of $G_k(\tau)$ for $|\tau| < 1$. This means it suffices to study G_k on

$$\left\{\frac{-1}{2} \leq \operatorname{Re}\tau \leq \frac{1}{2} \quad \text{and} \quad |\tau| \geq 1\right\}$$

Consider for k even and $k \geq 3$, we have

$$G_k(\tau) = \underbrace{2\sum_{n=1}^{\infty}\frac{1}{n^k}}_{m=0\text{ term}} + 2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\frac{1}{(m\tau+n)^k}$$

Apply the Lipschitz formula to $z = m\tau$, which satisfies $\operatorname{Im} z > 0$, we have

$$G_k(\tau) = 2\zeta(k) + 2\sum_{m=1}^{\infty}\frac{(-2\pi i)^k}{(k-1)!}\sum_{r=1}^{\infty} r^{k-1}e^{2\pi i r m\tau} = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e^{2\pi i n\tau}$$

where we re-indexed $n = rm$ and recall that $\sigma_{k-1}(n) = \sum_{r|n} r^{k-1}$.

4.5 Possion Summation Formula

Theorem 4.11 (Possion Summation). Let $f : \mathbb{R} \to \mathbb{C}$ such that $\int_{-\infty}^{\infty} |f(t)| \, \mathrm{d}t < \infty$. Then

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n)$$

where $\hat{f}$ is the **Fourier transform** of f defined by

$$\hat{f}(u) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i n t} \, \mathrm{d}t$$

Proof. We will make some assumptions that make the proof easier. Define the function

$$F(t) = \sum_{n \in \mathbb{Z}} f(n+t)$$

Then $F(t+1) = F(t)$ so F is periodic. Assume $F(t)$ converges uniformly for $t \in [0,1]$. The m-th fourier coefficient of $F(t)$ is

$$\begin{aligned} c_m &= \int_0^1 F(t) e^{-2\pi i m t} \, \mathrm{d}t = \int_0^1 \sum_{n \in \mathbb{Z}}^{\infty} f(n+t) e^{-2\pi i m t} \, \mathrm{d}t \\ &= \sum_{n \in \mathbb{Z}} \int_0^1 f(n+t) e^{-2\pi i m t} \, \mathrm{d}t = \int_{-\infty}^{\infty} f(t) e^{-2\pi i m t} \, \mathrm{d}t = \hat{f}(m) \end{aligned}$$

Set $t = 0$ to the equation $F(t) = \sum_{n \in \mathbb{Z}} c_m e^{2\pi i m t}$ we have

$$\sum_{n \in \mathbb{Z}} f(n) = F(0) = \sum_{n \in \mathbb{Z}} c_m = \sum_{n \in \mathbb{Z}} \hat{f}(n)$$

Here we assume that $\sum_{m \in \mathbb{Z}} |\hat{f}(m)| < \infty$. This completes the proof. $\square$