Algebraic Diagonals and Asymptotics of Bivariate Generating Functions

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Overview

1. Notations
2. Algebraic Generating Functions and Diagonals
3. Asymptotics of Bivariate Generating Functions

Notations

1. $\mathbb{K}$ = a field of characteristic zero (usually $\mathbb{R}$ or $\mathbb{C}$).
2. $\mathbb{K}[[z]]$ = the ring of formal power series over $\mathbb{K}$ in z.

$$\mathbb{K}[[z]] = \left\{ \sum_{n \geq 0} a_n z^n \mid a_n \in \mathbb{K} \right\}$$

3. $\mathbb{K}[[x, y]]$ = the ring of formal power series over $\mathbb{K}$ in x, y.

$$\mathbb{K}[[x, y]] = \left\{ \sum_{i,j \geq 0} a_{i,j} x^i y^j \mid a_{i,j} \in \mathbb{K} \right\}$$

I. Algebraic Generating Functions and Diagonals

Generating Functions

Given a combinatorial class $(\mathcal{A}, \omega)$, we can define its generating function

$$A(z) := \sum_{n \geq 0} a_n z^n$$

where $a_n := \omega^{-1}(\{n\})$, the number of elements in $\mathcal{A}$ that has weight n.

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Example

Let $\mathcal{A}$ be the strings in $\{1, 2, 3\}$ that avoid 11 and 23. For example

$$1222132, \; 12, \; 132$$

The weight on $\mathcal{A}$ counts the number of 1. By the transfer matrix method we can show that

$$A(z) = \frac{1+z}{1 - 2z - z^2 + z^3}$$

Algebraic Power Series

A formal power series $A(z) \in \mathbb{K}[[z]]$ is called **algebraic** if

$$P(z, A(z)) = 0$$

for some polynomial $P(z, y) \in \mathbb{K}[z, y]$.

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Example

Let $\mathcal{T}$ be the set of all full binary trees, then $T(z)$ satisfies

$$zT(z)^2 - T(z) + 1 = 0$$

In other word, $P(z, T(z)) = 0$ for $P(z, y) = yz^2 - y + 1$.

Diagonals

Let $F(x, y) \in \mathbb{K}[[x, y]]$ be a bivariate formal power series, write

$$F(x, y) = \sum_{i,j \geq 0} f_{i,j} x^i y^j$$

The **diagonal** of F is the univariate formal power series in $\mathbb{K}[[t]]$

$$\Delta F(t) := \sum_{n \geq 0} f_{n,n} t^n$$

Diagonals

Theorem

If $F(x,y) \in \mathbb{K}[[x,y]]$ is a rational function, then $(\Delta F)(t)$ is algebraic.

In other word, there exists $P(t,y) \in \mathbb{K}[t,y]$ such that $P(t, \Delta F(t)) = 0$.

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Theorem

If $F(x,y) \in \mathbb{K}[[x,y]]$ *is a rational function, then* $(\Delta F)(t)$ *is algebraic.*

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Bostan et al. developed an algorithm to efficiently compute $P(t,y)$. We implemented this algorithm in SageMath.

Input: A rational function $F(x,y) \in \mathbb{K}[[x,y]]$.

Output: A polynomial $P(t,y) \in \mathbb{K}[t,y]$ such that $P(t, \Delta F(t)) = 0$.

Idea of the Algorithm

Fact 1. There is a set $\{\alpha_1(t), \cdots, \alpha_n(t)\}$ such that $\Delta F(t)$ is a sum of c elements from this set.

For example, if $n = 5$ and $c = 2$. The set is $\{\alpha_1(t), \cdots, \alpha_5(t)\}$ and

$$\Delta F(t) = \alpha_1(t) + \alpha_3(t)$$

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Construct the polynomial

$$\Sigma(y, t) = \prod_{i_1 < \cdots < i_c} (y - (\alpha_{i_1}(t) + \cdots + \alpha_{i_c}(t)))$$

In the case of the example, the polynomial is

$$(y - (\alpha_1 + \alpha_2))(y - (\alpha_1 + \alpha_3)) \cdots (y - (\alpha_4 + \alpha_5))$$

Fact 2. $\Sigma(y, t) \in \mathbb{K}[y, t]$. (Galois Theory)

Fact 1

Note that

$$\begin{aligned}\Delta F(t) &= \sum_{n\geq 0} f_{n,n} t^n = [y^{-1}] \sum_{n,m\geq 0} f_{n,m} t^n y^{m-n-1} \\ &= [y^{-1}] \frac{1}{y} F\left(\frac{t}{y}, y\right) \\ &= \sum_{\substack{y_i(t)\in\mathcal{P} \\ \mathrm{val}(y_i(t))>0}} \underbrace{\mathrm{Residue}\left(\frac{1}{y} F\left(\frac{t}{y}, y\right), y = y_i(t)\right)}_{\alpha_i}\end{aligned}$$

where $\mathcal{P} = \{y_1(t), \cdots, y_n(t)\}$ is the "pole set" of $\frac{1}{y}F(\frac{t}{y}, y)$.

$$c = \#\{y(t) \in \mathcal{P} \mid \mathrm{val}(y(t)) > 0\}$$

Algorithm

The algorithm consists of two steps.

1. Compute the residues $\{\alpha_1(t), \cdots, \alpha_n(t)\}$, using resultants.
2. Compute the polynomial $\Sigma(y, t)$.

II. Asymptotics of Bivariate Generating Functions

Bivariate Generating Functions

Consider a rational bivariate generating function

$$F(x,y) = \frac{P(x,y)}{Q(x,y)} = \sum_{n,m\geq 0} f_{n,m}x^n y^m \in \mathbb{C}[[x,y]]$$

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Example

Let $b_{n,k}$ be the number of binary strings of length n and has k zeros

$$B(x,y) = \sum_{n,k\geq 0} b_{n,k} x^n y^k = \sum_{n\geq 0} \left(\sum_{k=0}^{n} \binom{n}{k} y^k \right) x^n$$

Asymptotics

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Assume $F = P/Q$ is a rational function and it defines an analytic function on a neighborhood of origin in $\mathbb{C}^2$.

By the Cauchy's Integral Formula, for $\epsilon > 0$ small enough we have

$$\begin{aligned} f_{n,n} &= \frac{1}{(2\pi i)^2} \int_{T(\epsilon,\epsilon)} \frac{F(x,y)}{x^{n+1}y^{n+1}} \; \mathrm{d}x \; \mathrm{d}y \\ &= \frac{1}{(2\pi i)^2} \int_{T(\epsilon,\epsilon)} \underbrace{\frac{P(x,y)}{Q(x,y)} \cdot \frac{\mathrm{d}x \; \mathrm{d}y}{x^{n+1}y^{n+1}}}_{\omega} \end{aligned}$$

where $T(\epsilon, \epsilon) = \{(x, y) \in \mathbb{C}^2 : |x| = |y| = \epsilon\}$.

Singularities

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The function $F = P/Q$ has singularities (poles) at the zeros of Q.

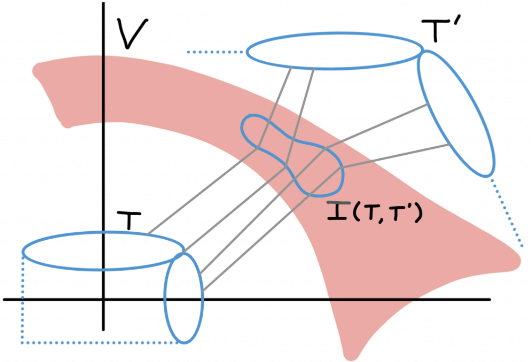
$$\mathcal{V}(Q) = \{(x, y) \in \mathbb{C}^2 : Q(x, y) = 0\}$$

is called the **singular variety** of F.

Deformation of the Contour

Let $M > 0$ be large and let K be a homotopy from $T(\epsilon, \epsilon)$ to $T(\epsilon, N)$.

In other words, we fix x and enlarge y.



Deformation of the Contour

The homotopy intersect the singular variety $\mathcal{V}(Q)$ at a cycle $\mathcal{C}$.

We then have

$$
\begin{aligned}
f_{n,n} &= \frac{1}{(2\pi i)^2} \int_{\mathcal{C}} \omega + \frac{1}{(2\pi i)^2} \int_{T(\epsilon,M)} \omega \\
&= \frac{1}{(2\pi i)^2} \int_{\mathcal{C}} \omega + O(M^{1-n}) \\
&= \frac{1}{(2\pi i)^2} \int_{\mathcal{N}} \operatorname{Res}(\omega) + O(M^{1-n})
\end{aligned}
$$

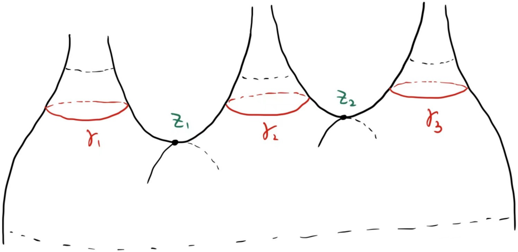
where $\mathcal{N} = \alpha_1\gamma_1 + \cdots + \alpha_r\gamma_r$ is a sum of cycles in $\mathbb{C}$.

Determine the contributing points

Therefore

$$\frac{1}{(2\pi i)^2}\int_{\mathcal{N}} \mathrm{Res}(\omega) = \sum_{i=1}^{r} \alpha_i \int_{\gamma_i} \mathrm{Res}(\omega)$$

DeVries developed an algorithm to determine which cycle γ_i contributes most to the integral, and thus determines the asymptotics of $f_{n,n}$.



Thank you!