PMATH 440 Notes

Analytic Number Theory

Fall 2025

Based on Professor Michael Rubinstein's Lectures

Contents

Lecture 1, 2025/09/04

1 Introduction

Topics covered in this course

(1). Summation methods (summation by parts, Euler-Maclaurin Summation, Poisson Summation, Dirichlet Hyperbola).

(2). Dirichlet series and Dirichlet divisor problem.

(3). Riemann zeta function ζ. Meromorphic continuation (ζ has a pole at $s = 1$) and functional equation.

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} \quad \text{for} \quad \mathrm{Re}(s) > 1$$

(4). Prime Number Theorem. If $\pi(x)$ = number of prime numbers $\leq x$, then

$$\pi(x) \sim \int_2^x \frac{1}{\log t}\,\mathrm{d}t \sim \frac{x}{\log x}$$

(5). Dirichlet's Theorem. If $0 \neq a, b \in \mathbb{Z}$ and $\gcd(a, b) = 1$, there are infinitely many prime numbers of the form $ak + b$ for $k \in \mathbb{Z}$. For example, there are infinitely many primes of the form $4k + 1$.

(6). More Complex analysis. Gamma function, Weierstrass products and possibly linear fractional transformations and modular forms.

We first introduce some asymptotic notations.

Definition. We say that $f(x) \sim g(x)$ as $x \to \infty$ if

$$\lim_{x\to a} \frac{f(x)}{g(x)} = 1$$

The Prime Number Theorem says $\pi(x) \sim \dfrac{x}{\log x}$ as $x \to \infty$, which is equivalent to

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1$$

Example. By the Stirling's approximation, we know

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \text{ as } n \to \infty$$

Definition. Let f, g be defined on (a subset of) $\mathbb{R}$ and g be a real-valued. We write $f(x) = \mathcal{O}(g(x))$ as $x \to \infty$, where g is real-valued, if there exists $c > 0$ such that $|f(x)| \leq cg(x)$ for all $x > x_0$.

Example. $\sin(x) = \mathcal{O}(1)$ as $x \to \infty$ since sin is bounded.

Example. By the Stirling's formula we have

$$n! = \mathcal{O}\left(\sqrt{n}\left(\frac{n}{e}\right)^n\right) \quad \text{and} \quad n! = \mathcal{O}\left(\frac{n^{n+1}}{e^n}\right)$$

The first one implies the second one because $\sqrt{n} = \mathcal{O}(n)$.

Definition. We write $f(x) = o(g(x))$ as $x \to a$ if

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0$$

In most cases we will take $a = \infty$ or $a = -\infty$. This means "$f(x)$ is much smaller than $g(x)$ near a".

Example. By the Stirling's formula we have

$$\lim_{n\to\infty} \frac{n!}{\frac{n^{n+1}}{e^n}} = \lim_{n\to\infty} \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\frac{n^{n+1}}{e^n}} = \lim_{n\to\infty} \frac{\sqrt{2\pi}}{\sqrt{n}} = 0$$

It follows that $n! = o(n^{n+1}/e^n)$ as $n \to \infty$.

Remark (Vinogradov's notation). We can also write $f(x) = \mathcal{O}(g(x))$ as $f(x) \ll g(x)$.

Remark. When we write $f(x) = g(x) + \mathcal{O}(h(x))$ to mean $f(x) - g(x) = \mathcal{O}(h(x))$.

2 Summation Methods

2.1 Partial Summation

This method is the discrete version of integration by parts.

Theorem 2.1 (Partial Summation). Let $f : \mathbb{N} \to \mathbb{C}$ and $g : \mathbb{R} \to \mathbb{C}$ be continuously differentiable on $[1, x]$. Then, for all $x \geq 1$ we have

$$\sum_{1 \leq n \leq x} f(n)g(n) = \left(\sum_{1 \leq n \leq x} f(n)\right) g(x) - \int_1^x \sum_{1 \leq n \leq t} f(n)g'(t) \, \mathrm{d}t \tag{1}$$

Proof. Consider the term $f(n)g(n)$, we note

$$f(n)g(x) - f(n)\int_n^x g'(t) \, \mathrm{d}t = f(n)g(x) - f(n)(g(x) - g(n)) = f(n)g(n) \tag{2}$$

This equality is obtained by looking at the terms that have to do with $f(n)$ in (1). Then summing the equation (2) over $1 \leq n \leq x$ gives us (1). $\square$

Example (Harmonic Series). Consider $\sum_{1\le n\le x}\frac{1}{n}$. Take $f(n)=1$ and $g(x)=\frac{1}{x}$. Then by the Partial summation formula we have

$$\sum_{1\le n\le x}\frac{1}{n}=\left(\sum_{1\le n\le x}f(n)\right)g(x)-\int_1^x\sum_{1\le n\le t}f(n)g'(t)\ \mathrm{d}t=\frac{\lfloor x\rfloor}{x}+\int_1^x\frac{\lfloor t\rfloor}{t^2}\mathrm{d}t$$

Here note that

$$\lfloor x\rfloor:=\sum_{1\le n\le x}1=\text{the largest integer}\le x$$

and using this we define

$$\{x\}:=x-\lfloor x\rfloor=\text{the fractional part of } x$$

For example $\lfloor\pi\rfloor=3$ and $\{\pi\}=0.1415926535897\cdots$. Therefore

$$\begin{aligned}\sum_{1\le n\le x}\frac{1}{n}&=\frac{x-\{x\}}{x}+\int_1^x\frac{t-\{t\}}{t^2}\mathrm{d}t\\&=1-\frac{\{x\}}{x}+\int_1^x\frac{1}{t}-\frac{\{t\}}{t^2}\ \mathrm{d}t\\&=1+\log x-\int_1^x\frac{\{t\}}{t^2}\ \mathrm{d}t+\mathcal{O}\left(\frac{1}{x}\right)\end{aligned}$$

Now we analyze this integral

$$\int_1^x\frac{\{t\}}{t^2}\mathrm{d}t=\underbrace{\int_1^\infty\frac{\{t\}}{t^2}\mathrm{d}t}_{<\infty}-\underbrace{\int_x^\infty\frac{\{t\}}{t^2}\mathrm{d}t}_{\mathcal{O}(\frac{1}{x})}$$

The estimation of second integral is by bounding $\{t\}/t^2$ by $1/t^2$. Therefore

$$\sum_{1\le n\le x}\frac{1}{n}=\log x+\underbrace{1-\int_1^\infty\frac{\{t\}}{t^2}\mathrm{d}t}_{:=\gamma}+\mathcal{O}\left(\frac{1}{x}\right)$$

This constant γ is called the Euler's constant, that is,

$$\lim_{x\to\infty}\left(\sum_{1\le n\le x}\frac{1}{n}-\log x\right)=\gamma$$

Conjecture 2.2. The Euler's constant γ is irrational.

Lecture 2, 2025/09/09

Remark. Let $\lambda_1 < \lambda_2 < \cdots$ be a sequence of natural numbers, then

$$\sum_{\lambda_n \le x} f(\lambda_n)g(\lambda_n) = \left(\sum_{\lambda_n \le x} f(\lambda_n)\right) g(x) - \int_1^x \left(\sum_{\lambda_n \le x} f(\lambda_n)\right) g'(t)\ \mathrm{d}t$$

This is a generalization of the usual Partial summation formula. The proof is similar. Note that for all $n \ge 1$ we have

$$f(\lambda_n)g(\lambda_n) = f(\lambda_n)g(x) - \int_{\lambda_n}^x f(\lambda_n)g'(t)\ \mathrm{d}t$$

Then summing over all n with $\lambda_n \le x$ we obtain the formula.

Example (Factorial). Now let's study the asymptotic of the factorial $m!$ as $m \to \infty$. Since the partial summation only works for sum and $m!$ is a product, we can take the log and consider $\log(m!)$. Let $f(m) = 1$ and let $g(x) = \log(x)$. By the partial summation formula we have

$$\begin{aligned} \log(m!) = \sum_{1 \le n \le m} \log(m) &= m \log m - \int_1^m \frac{\lfloor t \rfloor}{t}\ \mathrm{d}t \\ &= m \log m - \int_1^m \frac{t - \{t\}}{t}\ \mathrm{d}t \\ &= m \log m - (m-1) + \int_1^m \frac{\{t\}}{t}\ \mathrm{d}t \end{aligned}$$

Now we need to estimate the integral and get an (rough) upper and lower bound for it.

$$0 < \int_1^m \frac{\{t\}}{t} \mathrm{d}t < \int_1^m \frac{\mathrm{d}t}{t} = \log m$$

Therefore

$$m \log m - (m-1) < \log(m!) < (m+1)\log m - (m-1)$$

Exponentiating this inequality gives

$$\frac{m^m}{e^{m-1}} < m! < \frac{m^{m+1}}{e^{m-1}}$$

This is a weaker result than the Stirling's formula.

Remark. The prime counting function is

$$\pi(x) = \sum_{p \le x} 1 = \text{number of primes} \le x$$

For the Riemann zeta function on $\mathrm{Re}(s) > 1$ we have the following identity

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}$$

This is called the Euler's product. Expand the right hand side and by the unique factorization of integers we have the equality. It is sometimes more natural to study the sum of log of primes. We define the function

$$\theta(x) := \sum_{p \le x} \log p$$

The Prime Number Theorem states that $\pi(x) \sim x/\log x$. In fact we have the following proposition.

Proposition 2.3. We have

$$\theta(x) \sim x \iff \pi(x) \sim \frac{x}{\log x}$$

Proof. $(\Rightarrow)$. Assume $\theta(x) \sim x$. Note that

$$\pi(x) = \sum_{p \le x} 1 = \sum_{p \le x} \log p \cdot \frac{1}{\log p}$$

Let $f(x) = \log x$ and $g(x) = 1/\log x$. By Partial summation we have

$$\pi(x) = \underbrace{\frac{\theta(x)}{\log x}}_{\sim \frac{x}{\log x}} + \int_2^x \theta(t) \cdot \frac{\mathrm{d}t}{(\log t)^2 t}$$

Now we note that since $\theta(x) \sim x$, we know $\theta(x) = \mathcal{O}(x)$ so that

$$\int_2^x \theta(t) \cdot \frac{\mathrm{d}t}{(\log t)^2 t} = \mathcal{O}\left(\int_2^x \frac{\mathrm{d}t}{(\log t)^2} \right)$$

But then we have

$$\int_2^x \frac{\mathrm{d}t}{(\log t)^2} = \int_2^{x^{1/2}} \frac{\mathrm{d}t}{(\log t)^2} + \int_{x^{1/2}}^x \frac{\mathrm{d}t}{(\log t)^2}$$

The first integrand is $\mathcal{O}(1)$ so the integral is $\mathcal{O}(x^{1/2})$, for the second integral we use the bound

$$\int_{x^{1/2}}^x \frac{\mathrm{d}t}{(\log t)^2} = \mathcal{O}\left(\frac{x}{(\log x)^2} \right)$$

Combine all of these, we have

$$\pi(x) = \frac{\theta(x)}{\log x} + \mathcal{O}\left(\frac{x}{(\log x)^2} \right)$$

and therefore

$$\frac{\pi(x)}{x/\log x} = \frac{\theta(x)}{\log x} \cdot \frac{\log x}{x} + \mathcal{O}\left(\frac{1}{\log x}\right) = \frac{\theta(x)}{x} + \mathcal{O}\left(\frac{1}{\log x}\right) = 1 + o(1)$$

($\Leftarrow$). Assume the PNT. Let $f(x) = 1$ and $g(x) = \log x$ when x is prime and 0 otherwise.

$$\theta(x) = \sum_{p \le x} \log x = \pi(x) \log x - \int_2^x \frac{\pi(t)}{t} \mathrm{d}t$$

Thus $\theta(x) \sim x$ after some work. □

Lecture 3, 2025/09/11

Example (Meromorphic Continuation of ζ). Recall the zeta function $\zeta(s)$ is only defined for $\mathrm{Re}(s) > 1$ and is equal to

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

This series converges absolutely if $\mathrm{Re}(s) > 1$ and uniformly in any half plane $\mathrm{Re}(s) \ge X_0 > 1$. We want to extend this function to the half plane $\mathrm{Re}(s) > 0$ using partial summation. We let $f(n) = 1$ and let $g(t) = t^{-s}$, then by the Partial summation formula

$$\sum_{1 \le n \le x} \frac{1}{n^s} = \frac{\lfloor x \rfloor}{x^s} + s \int_1^x \frac{\lfloor t \rfloor}{t^{s+1}} \, \mathrm{d}t$$

Here t^s is defined using the principal branch of logartihm, which is defined on $\mathbb{C} \setminus (\infty, 0]$.

$$\begin{aligned} \sum_{1 \le n \le x} \frac{1}{n^s} &= \frac{x - \{x\}}{x^s} + s \int_1^x \frac{(t - \{t\})}{t^s} \, \mathrm{d}t \\ &= \frac{x - \{x\}}{x^s} + \int_1^x \frac{1}{t^s} \, \mathrm{d}t - s \int_1^x \frac{\{t\}}{t^{s+1}} \, \mathrm{d}t \end{aligned}$$

By taking $x \to \infty$ we have

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{\{t\}}{t^{s+1}} \, \mathrm{d}t = \frac{1}{s-1} + 1 - s \int_1^{\infty} \frac{\{t\}}{t^{s+1}} \, \mathrm{d}t \tag{$*$}$$

The RHS is analytic on $\mathrm{Re}\, s > 0$ except for a simple pole at $s = 1$ with residue 1 because the improper integral $\int_1^\infty t^{-r} \, \mathrm{d}t$ converges when $r > 1$. Note that the function

$$\int_1^{\infty} \frac{\{t\}}{t^{s+1}} \, \mathrm{d}t$$

on the RHS is analytic by Leibniz's rule (differentiation under the integral sign). Equation ($*$) allows us to extend the domain of ζ to $\mathrm{Re}\, s > 0$, with a pole at 1.

2.2 Euler-Maclaurin Summation and Bernoulli Polynomials

This summation method looks at sums of the form

$$\sum_{a<n\leq b} g(n)$$

where $a, b \in \mathbb{Z}$ are integers and $a < b$. By the Partial summation formula

$$\begin{aligned}
\sum_{a<n\leq b} g(n) &= (b-a)g(b) - \int_a^b ([t]-a)g'(t)\ \mathrm{d}t \\
&= bg(b) - ag(b) - \int_a^b tg'(t)\ \mathrm{d}t + a\int_a^b g'(t)\ \mathrm{d}t + \int_a^b \{t\}g'(t)\ \mathrm{d}t \\
&= bg(b) - ag(b) - \underbrace{\int_a^b tg'(t)\ \mathrm{d}t}_{\star} + ag(b) - ag(a) + \int_a^b \{t\}g'(t)\ \mathrm{d}t
\end{aligned}$$

By integration by parts we have

$$\star = \int_a^b tg'(t) = tg(t)|_a^b - \int_a^b g(t)\ \mathrm{d}t = bg(b) - ag(a) - \int_a^b g(t)\ \mathrm{d}t$$

Therefore

$$\begin{aligned}
\sum_{a<n\leq b} g(n) &= bg(b) - ag(b) - \left(bg(b) - ag(a) - \int_a^b g(t)\ \mathrm{d}t\right) + ag(b) - ag(a) + \int_a^b \{t\}g'(t)\ \mathrm{d}t \\
&= \int_a^b g(t)\ \mathrm{d}t + \int_a^b \{t\}g'(t)\ \mathrm{d}t
\end{aligned}$$

Now let us analyze the integral of $\{t\}g'(t)$. Note that on average $\{t\} = 1/2$, we can write

$$\{t\} = \frac{1}{2} + \left(\{t\} - \frac{1}{2}\right)$$

We have

$$\int_a^b \{t\}g'(t)\ \mathrm{d}t = \frac{1}{2}(g(b)-g(a)) + \int_a^b \left(\{t\} - \frac{1}{2}\right) g'(t)\ \mathrm{d}t$$

Before we continue, we need Bernoulli polynomials.

Definition. We define $B_0(x) = 1$. For $k \geq 1$ we recursively define $B_k(x)$ so that

$$B_k'(x) = kB_{k-1}(x) \quad \text{and} \quad \int_0^1 B_k(x)\ \mathrm{d}x = 0$$

The polynomial $B_k(x)$ is called the k**-th Bernoulli polynomials**.

Example ($B_1(x)$). Let $k = 1$. Then

$$B_1(x) = \int B_0(x)\ \mathrm{d}x = x + B_1$$

Then because

$$\int_0^1 B_1(x)\ \mathrm{d}x = \left[\frac{x^2}{2} + B_1 x\right]_0^1 = \frac{1}{2} + B_1 = 0$$

we know that $B_1 = -\frac{1}{2}$ and $B_1(x) = x - \frac{1}{2}$.

Example ($B_2(x)$). Let $k = 2$. Then

$$B_2(x) = \int 2B_1(x)\ \mathrm{d}x = x^2 - x + B_2$$

Then because

$$\int_0^1 B_2(x)\ \mathrm{d}x = \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + B_2 x\right]_0^1 = \frac{1}{3} - \frac{1}{2} + B_2 = 0$$

we know that $B_2 = \frac{1}{6}$ and $B_2(x) = x^2 - x + \frac{1}{6}$.

Definition. For $k \geq 0$ we define $B_k = B_k(0)$ to be the **k-th Bernoulli number**.

Lecture 4, 2025/09/16

Proposition 2.4. For $k \geq 0$ we have

(a). The difference equation: $\dfrac{B_{k+1}(x+1) - B_{k+1}(x)}{k+1} = x^k$

(b). Expansion in terms of Bernoulli numbers: $B_k(x) = \displaystyle\sum_{m=0}^{k} \binom{k}{m} B_{k-m} x^m = \sum_{m=0}^{k} \binom{k}{m} B_m x^{k-m}$

(c). The functional equation: $B_k(x) = (-1)^k B_k(1-x)$

(d). Specials values: $B_k(1) = \begin{cases} (-1)^k B_k(0) & \text{if } k \geq 0 \\ 0 & \text{if } k \text{ is odd and } k \geq 3 \\ 1/2 & \text{if } k = 1 \end{cases}$

(e). Recursion of Bernoulli numbers: $\displaystyle\sum_{m=0}^{k-1} \binom{k}{m} B_m = 0$

(f). Generating Function: $F(x,t) := \displaystyle\sum_{k=0}^{\infty} B_k(x) \frac{z^k}{k!} = \frac{ze^{zx}}{e^z - 1}$

Remark. By (a) we note that

$$B_k(0) = B_k(1) \text{ for all } k \geq 0$$

This will be useful later.

Proposition 2.5. We have the following Fourier series expansions

$$B_1(\{x\}) = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n} \quad \text{for } x \notin \mathbb{Z}$$
$$B_k(\{x\}) = -k!\sum_{n\neq 0}\frac{e^{2\pi inx}}{(2\pi in)^k} \quad \text{for } k \geq 2$$

Now we return to our discussion on Euler-Maclaurin summation. We need to study the integral

$$\int_a^b \left(\{t\} - \frac{1}{2}\right) g'(t)\ \mathrm{d}t = \int_a^b B_1(\{t\})g'(t)\ \mathrm{d}t$$

Note that $B_1(\{t\}) = t - \frac{1}{2} - n$ if $n \leq t < n+1$. Hence

$$\int_a^b B_1(\{t\})g'(t)\ \mathrm{d}t = \left(\int_a^{a+1} + \cdots + \int_{b-1}^b\right) B_1(\{t\})g'(t)\ \mathrm{d}t$$

Now let us look at the integral on each interval $[n, n+1]$.

$$\int_n^{n+1} B_1(\{t\})g'(t)\ \mathrm{d}t = \int_n^{n+1}\left(t - \frac{1}{2} - n\right) g'(t)\ \mathrm{d}t$$

Let $u = g'(t)$ and $dv = B_1(\{t\})\ \mathrm{d}t$. Apply integration by parts we have

$$\begin{aligned}\int_n^{n+1} B_1(\{t\})g'(t)\ \mathrm{d}t &= \left[\frac{1}{2}B_2(\{t\})g'(t)\right]_n^{n+1} - \int_n^{n+1}\frac{1}{2}B_2(\{t\})g''(t)\ \mathrm{d}t \\ &= \frac{1}{2}\left(B_2(1) - B_2(0)\right)\left(g'(n+1) - g'(n)\right) - \frac{1}{2}\int_n^{n+1} B_2(\{t\})g''(t)\ \mathrm{d}t\end{aligned}$$

We can now apply the same method to the integral $\int_n^{n+1} B_2(\{t\})g''(t)\ \mathrm{d}t$. Keep doing it, say K times, then we ge the Euler-Maclaurin summation formula.

Theorem 2.6 (Euler-Maclaurin Summation). Let $K \in \mathbb{N}$ and $g : \mathbb{R} \to \mathbb{C}$ such that $g^{(K)}$ exists, then for $a < b$ in $\mathbb{N}$ we have

$$\sum_{a<n\leq b} g(n) = \int_a^b g(t)\ \mathrm{d}t + \sum_{k=1}^{K}\frac{(-1)^k B_k}{k!}(g^{(k-1)}(b) - g^{(k-1)}(a)) + \frac{(-1)^{K+1}}{K!}\int_a^b B_K(\{t\})g^{(K)}(t)\ \mathrm{d}t$$

Example (Sum of powers). We claim that for $r \geq 1$ and $N \geq 1$ we have

$$\sum_{n=1}^{N} n^r = \frac{B_{r+1}(N+1) - B_{r+1}(1)}{r+1}$$

We will apply Euler-Maclaurin summation formula and properties of Bernoulli polynomials to prove it. Let $g(t) = t^r$ then $g^{(r)}(t) = 0$ and

$$g^{(m)}(N) - g^{(m)}(0) = \begin{cases} r(r-1)\cdots(r-m+1)N^{r-m} & \text{if } m \leq r-1 \\ 0 & \text{if } m \geq r \end{cases}$$

Let $a = 0$ and $b = N$ and $K = r$, then by Euler-Maclaurin summation we have

$$\begin{aligned}
\sum_{n=1}^{N} n^r &= \int_0^N t^r \, \mathrm{d}t + \sum_{k=1}^{r} \frac{(-1)^k B_k}{k!} r(r-1)\cdots(r-k+2)N^{r-k+1} \\
&= \int_0^N t^r \, \mathrm{d}t + \sum_{k=1}^{r} \frac{(-1)^k B_k}{r-k+1} \binom{r}{k} N^{r-k+1} \\
&= \int_0^N \sum_{k=0}^{r} (-1)^k B_k \binom{r}{k} t^{r-k} \, \mathrm{d}t && (t^r \text{ is the } k=0 \text{ term}) \\
&= \int_0^N (-1)^r \sum_{k=0}^{r} B_k \binom{r}{k} (-t)^{r-k} \, \mathrm{d}t && ((-1)^r(-1)^{r-k} = (-1)^k) \\
&= \int_0^N (-1)^r B_r(-t) \, \mathrm{d}t = \int_0^N B_r(t+1) \, \mathrm{d}t && (\text{property (b) and (c)}) \\
&= \frac{B_{r+1}(N+1) - B_{r+1}(1)}{r+1}
\end{aligned}$$

Example. The Euler-Maclaurin summation formula can be used to obtain an analytic continuation of $\zeta(s)$ as far to the left as we want, and also provides a useful expansion. Consider

$$\sum_{n=1}^{N} n^{-s} = 1 + \sum_{n=2}^{N} n^{-s}$$

Let $K \in \mathbb{N}$, we want to extend $\zeta(s)$ to the region $\operatorname{Re} s > -K+1$. Let $a = 1$ and $b = N$ and $g(x) = x^{-s}$. Note that

$$g^{(m)}(x) = (-1)^m s(s+1)\cdots(s+m-1)x^{-s-m}$$

for $m \geq 1$. By Euler-Maclaurin summation formula we have

$$\sum_{2\leq n\leq N} n^{-s} = \int_1^N t^{-s}\,\mathrm{d}t - \sum_{k=1}^{K} \frac{(-1)^k B_k}{k!}(-1)^k (s+k-2)\cdots(s+1)s(N^{-s-k+1}-1)$$
$$+ \int_1^N B_K(\{t\})\frac{(-1)^{K+1}}{K!}(-1)^K (s+K-1)\cdots(s+1)st^{-s-K}\,\mathrm{d}t \qquad (*)$$

Note that we have

$$\frac{(s+k-2)\cdots(s+1)s}{k!} = \frac{1}{k}\frac{(s+k-2)\cdots(s+1)s}{(k-1)!} = \frac{1}{k}\binom{s+k-2}{k-1}$$

It follows that

$$\sum_{2\leq n\leq N} n^{-s} = \int_1^N t^{-s}\,\mathrm{d}t - \sum_{k=1}^{K}\frac{B_k}{k}\binom{s+k-2}{k-1}(N^{-s-k+1}-1) - \binom{s+K-1}{K}\int_1^N B_K(\{t\})t^{-s-K}\,\mathrm{d}t$$

Taking $N \to \infty$ we have that

$$\zeta(s) = 1 + \sum_{n\geq 2} n^{-s} = 1 + \frac{s}{1-s} + B_1 + \sum_{k=2}^{\infty}\frac{B_k}{k}\binom{s+k-2}{k-1} - \binom{s+K-1}{K}\int_1^\infty B_K(\{t\})t^{-s-K}\,\mathrm{d}t$$

for $\operatorname{Re} s > 1$. The RHS is meromorphic on the region $\operatorname{Re} s > -K+1$, with a pole at $s = 1$. This gives us a meromorphic continuation to $\operatorname{Re} s > -K+1$ using the quantity on the RHS.

Remark. Since we can choose K arbitrarily large, we have a meromorphic continuation of $\zeta(s)$ to the entire $\mathbb{C}$ except for a pole at $s = 1$. Note that we can do this because all the meromorphic continuation to $\operatorname{Re} s > -K+1$ agree on the open set $\operatorname{Re} s > 1$, so they are all equal by the identity theorem for analytic functions.

Lecture 5, 2025/09/18

Example (Harmonic Series). We can now apply the Euler-Maclaurin summation to give a better estimate of the asymptotic of the harmonic series. Let $s = 1$ and $K = 3$, then apply $(*)$ above

$$\begin{aligned}\sum_{1\leq n\leq N}\frac{1}{n} &= 1 + \int_1^N t^{-1}\,\mathrm{d}t + \frac{1}{2}\left(\frac{1}{N}-1\right) + \frac{1}{12}\left(-\frac{1}{N^2}+1\right) - \int_1^N \frac{B_3(\{t\})}{t^4}\,\mathrm{d}t \\ &= \log N + 1 + \frac{1}{2N} - \frac{1}{2} - \frac{1}{12N^2} + \frac{1}{12} - \int_1^\infty \frac{B_3(\{t\})}{t^4}\,\mathrm{d}t + \int_N^\infty \frac{B_3(\{t\})}{t^4}\,\mathrm{d}t \\ &= \log N + \underbrace{\frac{7}{12} - \int_1^\infty \frac{B_3(\{t\})}{t^4}\,\mathrm{d}t}_{\gamma_K} + \frac{1}{2N} - \frac{1}{12N^2} + \mathcal{O}\left(\frac{1}{N^3}\right)\end{aligned}$$

Here we note that for an arbitrary K we would have

$$\sum_{1\leq n\leq N}\frac{1}{n}=\log N+\gamma_K+\left(\text{some terms of }N^{-1},\cdots,N^{-k}\right)=\log N+\gamma_K+\mathcal{O}\left(\frac{1}{N}\right)$$

If follows that

$$\gamma_K=\lim_{N\to\infty}\left(\sum_{1\leq n\leq N}\frac{1}{n}-\log N\right)$$

Hence γ_K is independent of the choice of K. Therefore we can write $\gamma_K=\gamma$ and this is the Euler's constant we saw in Lecture 1.

Example (Stirling's formula). We now apply Euler-Maclaurin summation to derive the Stirling's formula. Let $g(t)=\log t$ and $g^{(k)}(t)=(-1)^{k+1}(k-1)!t^{-k}$. Hence

$$\log(n!)=\sum_{1<m\leq n}\log(m)=\int_1^n\log(t)\ \mathrm{d}t+\frac{\log n}{2}+\sum_{k=2}^{K+1}\frac{B_k}{k(k-1)}\left(\frac{1}{n^{k-1}}-1\right)+\frac{1}{K+1}\int_1^n\frac{B_{K+1}(\{t\})}{t^{K+1}}\ \mathrm{d}t$$

The integral of $\log t$ is $t\log t-t$, so the first integral evaluates to $n\log n-n+1$. For the other integral, we have

$$\int_1^n\frac{B_{K+1}(\{t\})}{t^{K+1}}\ \mathrm{d}t=\int_1^\infty\frac{B_{K+1}(\{t\})}{t^{K+1}}\ \mathrm{d}t-\int_n^\infty\frac{B_{K+1}(\{t\})}{t^{K+1}}\ \mathrm{d}t$$

Note that $|B_{K+1}(\{t\})|$ is bounded because $\{t\}\in[0,1]$ for all $t\in\mathbb{R}$. Hence the first integral converges to a constant and the second integral is $\mathcal{O}(n^{-K})$. Collecting all the constant together as c, then

$$\log(n!)=n\log n-n+\frac{\log n}{2}+c+\sum_{k=2}^{K+1}\frac{B_k}{k(k-1)n^{k-1}}+\mathcal{O}(n^{-K})$$

As in the previous example, the constant c also does not depend on K. In fact, $c=\log(2\pi)/2$. Now, we can take $K=1$ and get

$$\begin{aligned}\log(n!)&=\left(n+\frac{1}{2}\right)\log n-n+\frac{\log 2\pi}{2}+\frac{B_2}{2n}+\mathcal{O}\left(\frac{1}{n}\right)\\&=\left(n+\frac{1}{2}\right)\log n-n+\log\sqrt{2\pi}+\mathcal{O}\left(\frac{1}{n}\right)\end{aligned}$$

Let $r(n)$ represents this error term so that $r(n)=\mathcal{O}(1/n)$. Exponentiating both sides give

$$n!=\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\cdot e^{r(n)}$$

There is $N>0$ and $M>0$ such that $r(n)\leq\frac{M}{n}$ for $n\geq N$. Hence

$$e^{r(n)}\leq e^{M/n}\quad\text{for }n\geq N$$

Now we have $e^{r(n)} = 1 + (e^{r(n)} - 1)$ and $e^{r(n)} - 1 = o(1)$. Therefore

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}(1 + o(1)) = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} + \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \cdot o(1)$$

It follows that

$$\lim_{n\to\infty} \frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} = \lim_{n\to\infty} 1 + o(1) = 1$$

Hence we obtain the Stirling's formula

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \text{ as } n \to \infty$$

3 Arithmetic Functions and Dirichlet Series

3.1 Multiplicative Functions

Definition. A function $f : \mathbb{N} \to \mathbb{C}$ is called an **arithmetic function**.

Definition. We say $f : \mathbb{N} \to \mathbb{C}$ is **multiplicative** if $f(mn) = f(m)f(n)$ when $\gcd(m, n) = 1$. We say f is **completely multiplicative** if $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$.

Example. The function $f(n) = 1$ for all $n \in \mathbb{N}$ is completely multiplicative.

Example. Let $z \in \mathbb{C}$. The function $f(n) = n^z$ is completely multiplicative.

Definition. Let $z \in \mathbb{C}$. We define the **divisor function** to be

$$\sigma_z : \mathbb{N} \to \mathbb{C} \text{ by } \sigma_z(n) := \sum_{d|n} d^z$$

is a multiplicative function. As an example, note that $d \mid 12 \iff d \in \{1, 2, 3, 4, 6, 12\}$ and

$$\sigma_z(12) = 1 + 2^z + 3^z + 4^z + 6^z + 12^z = (1 + 2^z + 4^z)(1 + 3^z) = \sigma_z(4)\sigma_z(3)$$

More generally, let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ written in its prime factorization. Then we have $d \mid n$ if and only if $d = p_1^{\beta_1} \cdots p_k^{\beta_k}$ where each $\beta_i \in \{0, \cdots, \alpha_i\}$. Then

$$\sigma_z(n) = \sum_{d|p_1^{\alpha_1}\cdots p_k^{\alpha_k}} d^z = \left(1 + p_1^z + \cdots + p_1^{\alpha_1 z}\right) \cdots \left(1 + p_k^z + \cdots + p_k^{\alpha_k z}\right) = \sigma_z(p_1^{\alpha_1}) \cdots \sigma_z(p_k^{\alpha_k})$$

This implies $\sigma_z(mn) = \sigma_z(m)\sigma_z(n)$ for $\gcd(m, n) = 1$. For $z = 0$ we denote $\sigma_0(n) = d(n)$, which counts the number of divisors of n.

Remark. If $f : \mathbb{N} \to \mathbb{C}$ is multiplicative. Define

$$F : \mathbb{N} \to \mathbb{C} \ \text{ by } \ F(n) := \sum_{d|n} f(d)$$

Similar to the methods above, for $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ we have

$$F(n) = \big(1 + f(p_1) + \cdots + f(p_1^{\alpha_1})\big) \cdots \big(1 + f(p_k) + \cdots + f(p_k^{\alpha_k})\big) = F(p_1^{\alpha_1}) \cdots F(p_k^{\alpha_k})$$

Therefore F is also multiplicative. Hence the sum (over divisors) of a multiplicative function is also a multiplicative function.

Lecture 6, 2025/09/23

Definition. Define **Euler's totient function** φ by

$$\varphi(n) = \text{number of } 1 \leq r \leq n \text{ with } (r, n) = 1$$

For example $\varphi(12) = 4$ because $1, 5, 7, 11$ are coprime to it. We claim that φ is multiplicative. Let $m, n \in \mathbb{N}$ with $(m, n) = 1$, then note that

$$(r, mn) = 1 \iff (r, m) = 1 \ \text{ and } \ (r, n) = 1$$

The condition $(r, m) = 1$ gives $\varphi(m)$ possible residue classes mod m and $(r, n) = 1$ gives $\varphi(n)$ possible residue classes mod n. By Chinese Remainder theorem, $(r, mn) = 1$ gives $\varphi(m)\varphi(n)$ possible residue classes mod mn. It follows that $\varphi(mn) = \varphi(m)\varphi(n)$.

Another way to see it is that for rings R, T we have $(R \oplus T)^\times \cong R^\times \oplus T^\times$. By Chinese Remainder theorem we have the isomorphism $\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$. Hence

$$(\mathbb{Z}/mn\mathbb{Z})^\times \cong (\mathbb{Z}/m\mathbb{Z})^\times \oplus (\mathbb{Z}/n\mathbb{Z})^\times$$

A residue class $a \pmod k$ is a unit if and only if $(a, k) = 1$. Hence $|(\mathbb{Z}/n\mathbb{Z})^\times| = \varphi(n)$. Taking the cardinality gives

$$\varphi(mn) = |(\mathbb{Z}/mn\mathbb{Z})^\times| = |(\mathbb{Z}/m\mathbb{Z})^\times| \cdot |(\mathbb{Z}/n\mathbb{Z})^\times| = \varphi(m)\varphi(n)$$

Hence, if $p_1, \cdots, p_k$ are distinct primes and $\alpha_i \geq 0$ we have

$$\varphi(p_1^{\alpha_1} \cdots p_k^{\alpha_k}) = \prod_{i=1}^{k} \varphi(p_i^{\alpha_i})$$

Now let us compute $\varphi(p^a)$ for a prime p and $a \geq 1$. Note that $\varphi(p) = p - 1$ and if $a \geq 1$, then

$$\{1 \leq n \leq p^a : (n, p^a) \neq 1\} = \{p, 2p, \cdots, (p^{a-1} - 1)p, p^a\} \ \text{ has } p^{a-1} \text{ elements}$$

It follows that the number of $n \leq p^a$ with $(n, p^a) = 1$ is

$$\varphi(p^a) = p^a - p^{a-1} = p^{a-1}(p-1)$$

In general, if $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ and each $\alpha_k \geq 1$, then

$$\varphi(n) = \prod_{i=1}^{k} \varphi(p_i^{\alpha_i}) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1) = \prod_{i=1}^{k} p_i^{\alpha_i} \cdot \frac{p_i - 1}{p_i} = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

This proved the following theorem

Theorem 3.1. If $n = 1$ then $\varphi(1) = 1$. If $n \geq 2$ then $\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$.

Remark. This can be proved using a probabilistic argument. An integer $a \leq n$ satisfies $(a, n) = 1$ if and only if $p \nmid a$ for all $p \mid n$. The probability that $p \nmid n$ is $1 - p^{-1}$. Multiplying all of them gives the probability that $(a, n) = 1$. Then multiplying by n gives the number of such a.

Example. If $n = 15$ then its prime divisors are 3 and 5, so

$$\varphi(15) = 15 \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{5}\right) = 15 \cdot \frac{2}{3} \cdot \frac{4}{5} = 8$$

If $n = 12$ then its prime divisors are 2 and 3, so

$$\varphi(12) = 12 \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) = 12 \cdot \frac{1}{2} \cdot \frac{2}{3} = 4$$

Now let us consider the sum of totient functions $\sum_{d|n} \varphi(d)$.

Example. For example, if $n = 12$ then

$$\sum_{d|12} \varphi(d) = \varphi(1) + \varphi(2) + \varphi(3) + \varphi(4) + \varphi(6) + \varphi(12) = 1 + 1 + 2 + 2 + 2 + 4 = 12$$

In general, if $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ then by the multiplicativity we have

$$\begin{aligned}
\sum_{d|n} \varphi(d) &= \sum_{i_1=1}^{\alpha_1} \cdots \sum_{i_k=1}^{\alpha_k} \varphi(p_1^{i_1} \cdots p_k^{i_k}) = \prod_{i=1}^{k} \sum_{d|p_i^{\alpha_i}} \varphi(d) = \prod_{i=1}^{k} (1 + \varphi(p_i) + \cdots + \varphi(p_i^{\alpha_i})) \\
&= \prod_{i=1}^{k} (1 + (p_i - 1) + (p_i^2 - p_i) + \cdots + (p_i^{\alpha_i} - p_i^{\alpha_i - 1})) = \prod_{i=1}^{k} p_i^{\alpha_i} = n
\end{aligned}$$

This proved that

Theorem 3.2. If $n \geq 1$ then $\displaystyle\sum_{d|n} \varphi(d) = n$.

Remark. Here is another proof of Theorem 3.2 that uses group theory. Consider the additive group $G = \mathbb{Z}/n\mathbb{Z}$. By the fundamental theorem of cyclic group, we know G has a unique subgroup C_d of order d for all $d \mid n$, generated by n/d. Also we know $x \in C_d$ is a generator of C_d if and only if $\gcd(x, n/d) = 1$. There are $\varphi(n/d)$ such x. Moreover, since every $x \in G$ is a generator for C_d for a unique d, we have the following

$$n = |G| = \sum_{d|n} |\{\text{generators of } C_d\}| = \sum_{d|n} \varphi\left(\frac{n}{d}\right) = \sum_{d|n} \varphi(d)$$

Definition. For $n \in \mathbb{N}$ we let $\nu(n) :=$ number of distinct prime factors of n.

Example. $\nu(10) = \nu(20) = 2$ because $10 = 2 \times 5$ and $20 = 2^2 \times 5$.

Definition. For $n \in \mathbb{N}$ we define the **Möbius function** μ to be

$$\mu(n) = \begin{cases} (-1)^{\nu(n)} & \text{if } n \text{ is squarefree} \\ 0 & \text{otherwise} \end{cases}$$

Note that μ is multiplicative. Let $p_1, \cdots, p_k$ be distinct primes with $\alpha_i \geq 1$. Then

$$\mu(p_1^{\alpha_1} \cdots p_k^{\alpha_k}) = \begin{cases} (-1)^k & \text{if all } \alpha_i = 1 \\ 0 & \text{otherwise} \end{cases}$$

and one the other hands

$$\mu(p_1^{\alpha_1}) \cdots \mu(p_k^{\alpha_k}) = \begin{cases} (-1)^k & \text{if all } \mu(p_i^{\alpha_i}) = -1 \iff \alpha_i = 1 \\ 0 & \text{otherwise} \iff \text{one of } \alpha_i \geq 2 \end{cases}$$

It follows that μ is a multiplicative function.

Now let us look at the sum $\sum_{d|n} \mu(d)$. If $n = 12$ then

$$\sum_{d|12} \mu(d) = \mu(1) + \mu(2) + \mu(3) + \mu(4) + \mu(6) + \mu(12) = 1 - 1 - 1 + 0 + 1 = 0$$

In general we have the following theorem

Theorem 3.3. Let $n \in \mathbb{N}$, then $\displaystyle\sum_{d|n} \mu(d) = \begin{cases} 0 & \text{if } n > 1 \\ 1 & \text{if } n = 1 \end{cases}$

Proof. If $n = 1$ then it is trivial. Assume $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ with $\alpha_i \geq 1$. Then

$$\begin{aligned} \sum_{d|n} \mu(d) &= \sum_{i_1=1}^{\alpha_1} \cdots \sum_{i_k=1}^{\alpha_k} \mu(p_1^{i_1} \cdots p_k^{i_k}) = \prod_{i=1}^{k} \sum_{d|p_i^{\alpha_i}} \mu(d) \\ &= \prod_{i=1}^{k} (1 + \mu(p_i) + \cdots + \mu(p_i^{\alpha_i})) = \prod_{i=1}^{k} (1 - 1 + 0 + \cdots + 0) = 0 \end{aligned}$$

This completes the proof. □

3.2 Dirichlet Series and Abscissa of Convergence

Definition. Let $f : \mathbb{N} \to \mathbb{C}$ be an arithmetic function. For $s = \sigma + it \in \mathbb{C}$ we let

$$D(f; s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

be the **Dirichlet series** associated to the function f. When does this series converge absolutely?

$$\sum_{n=1}^{\infty} \left| \frac{f(n)}{n^s} \right| = \sum_{n=1}^{\infty} \frac{|f(n)|}{|n^{\sigma+it}|} = \sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\sigma}}$$

If RHS converges for given σ_1, then it also converges for any $\sigma > \sigma_1$. The **Abscissa of absolute convergence** of $D(f; s)$ is defined as

$$\sigma_0 := \inf \left\{ \sigma \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\sigma}} \text{ convegers} \right\}$$

If $\sigma > \sigma_0$, then the series $\sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\sigma}}$ converges. If $\sigma < \sigma_0$, then the series $\sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\sigma}}$ diverges. If $\sigma = \sigma_0$, then the series $\sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\sigma}}$ may or may not converge.

Example. If $1(n) = 1$ is constant, then $D(1, s) = \zeta(s)$ is the zeta function. The Abscissa of convergence is $\sigma_0 = 1$. At $\sigma = 1$, this does not converge.

Example. Let $f(n) = (\log n)^{-2}$. Then

$$D(f, s) = \sum_{n=2}^{\infty} \frac{1}{n^s (\log n)^2}$$

This has Abscissa of convergence $\sigma_0 = 1$ and it converges at $\sigma = 1$. At $\sigma = 1$ we can compare it with the convergent integral

$$\int_2^{\infty} \frac{1}{t(\log t)^2} \, \mathrm{d}t = \int_{\log 2}^{\infty} \frac{1}{u^2} \, \mathrm{d}u < \infty$$

Example. Let $f(n) = (-1)^{n-1}$ for $n \in \mathbb{N}$. The η function is defined by

$$\eta(s) := D(f, s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots$$

This has Abscissa of convergence $\sigma_0 = 1$ but it converges conditionally if $\sigma > 0$ (this can be proved using partial summation).

Remark. The Riemann Hypothesis is equivalent to

$$\sum_{n \le N} \mu(n) = \mathcal{O}(N^{1/2+\epsilon})$$

for $\epsilon > 0$. The Dirichlet series $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$ has abscissa of convergence equal to 1. Assuming the RH, the series converges conditionally when $\sigma > 1/2$.

Lecture 7, 2025/09/25

Theorem 3.4 (Uniqueness of Dirichlet Coefficients). Let F, G be Dirichlet series defined by

$$F(s) = \sum_{n=1}^{\infty} f(n) n^{-s} \quad \text{and} \quad G(s) = \sum_{n=1}^{\infty} g(n) n^{-s}$$

Assume that F and G converge absolutely on $\operatorname{Re} s > \sigma_0$. If $F(s) = G(s)$ for all $\operatorname{Re} s > \sigma_0$, then we have $f(n) = g(n)$ for all $n \in \mathbb{N}$.

Proof. Argue by contradiction. Let $h(n) = f(n) - g(n)$. Say $h(n) \neq 0$ for at least one $n \in \mathbb{N}$. Let N be the smallest such n. Hence $h(N) \neq 0$ and $h(n) = 0$ for $1 \le n < N$. Thus

$$0 = F(s) - G(s) = \sum_{n=1}^{\infty} \frac{f(n) - g(n)}{n^s} = \frac{h(N)}{N^s} + \sum_{n=N+1}^{\infty} \frac{h(n)}{n^s} \quad \text{for } \operatorname{Re} s > \sigma$$

Our goal is to show $h(N) = 0$ so we get a contradiction. Rearranging the equation gives

$$h(N) = -N^s \sum_{n=N+1}^{\infty} \frac{h(n)}{n^s}$$

Wrtie $s = \sigma + it$, hence we have

$$|h(N)| \le N^{\sigma} \sum_{n=N+1}^{\infty} \frac{|h(n)|}{n^{\sigma}}$$

For $n \ge N+1$ and let $c \in \mathbb{R}$ with $\sigma > c > \sigma_0$. Then $n^{\sigma} = n^{\sigma - c} n^c \ge (N+1)^{\sigma - c} n^c$. Hence

$$|h(N)| \le N^{\sigma} \sum_{n=N+1}^{\infty} \frac{|h(n)|}{n^{\sigma}} \le N^{\sigma} \sum_{n=N+1}^{\infty} \frac{|h(n)|}{(N+1)^{\sigma - c} n^c} = \left(\frac{N}{N+1} \right)^{\sigma} (N+1)^c \sum_{n=N+1} \frac{|h(n)|}{n^c}$$

Here $(N+1)^c$ is bounded and the sum $\sum_{n \ge N+1} \frac{|h(n)|}{n^c}$ is bounded, as it is the tail of a convergent series. By taking $\sigma \to \infty$, the RHS tends to 0. Therefore $h(N) = 0$. $\square$

3.3 Product of Dirichlet Series

Consider the Dirichlet series $F(s) = D(f; s)$ and $G(s) = D(g; s)$ that converge absolutely on $\sigma > \sigma_0$. For $\sigma > \sigma_0$ consider their product

$$F(s)G(s) = \left(\sum_{d=1}^{\infty} \frac{f(d)}{d^s}\right)\left(\sum_{m=1}^{\infty} \frac{g(m)}{m^s}\right)$$

Expaning this product gives us

$$f(1)g(1) + \frac{f(1)g(2) + f(2)g(1)}{2^s} + \frac{f(1)g(3) + f(3)g(1)}{3^s} + \frac{f(1)g(4) + f(2)g(2) + f(4)g(1)}{4^s} + \cdots$$

Note that the coefficient of n^{-s} term in $F(s)G(s)$ are sum of $f(d)g(m)$ for $dm = n$. Therefore

$$F(s)G(s) = \sum_{n=1}^{\infty}\left(\sum_{dm=n} f(d)g(m)\right)\frac{1}{n^s} = \sum_{n=1}^{\infty}\left(\sum_{d|n} f(d)g\left(\frac{n}{d}\right)\right)\frac{1}{n^s}$$

Example. Consider the zeta function $\zeta(s) = D(1; s)$. Then

$$\zeta(s)^2 = \sum_{n=1}^{\infty}\left(\sum_{d|n} 1 \cdot 1\right)\frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}$$

Let $k \in \mathbb{N}$, then we have

$$\zeta(s)^k = \sum_{n=1}^{\infty}\left(\sum_{a_1 \cdots a_k = n} 1\right)\frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}$$

where

$$d_k(n) := \sum_{a_1 \cdots a_k = n} 1 = \begin{array}{c}\text{number of ways to write } n \text{ as a} \\ \text{product of } k \text{ natural numbers}\end{array}$$

Example. Consider the product $\zeta(s)\zeta(s-z)$ for some $z \in \mathbb{C}$. Then

$$\zeta(s-z) = \sum_{n=1}^{\infty} \frac{n^z}{n^s} \quad \text{for } \operatorname{Re} s > \operatorname{Re} z + 1$$

Here $f(n) = 1$ and $g(n) = n^z$, so we have

$$\zeta(s)\zeta(s-z) = \sum_{n=1}^{\infty}\left(\sum_{d|n} d^z\right)\frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{\sigma_z(n)}{n^s} = D(\sigma_z; s)$$

Example (Euler Products). Euler proved that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \quad \text{for } \operatorname{Re} s > 1$$

The reason this is true is that, for every n^{-s} on the LHS by unique factorization this term appears exactly once on the RHS.

Now consider the reciprocal of this infinite product

$$\prod_p \left(1 - \frac{1}{p^s}\right) = \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right) \cdots = 1 - \frac{1}{2^s} - \frac{1}{3^s} - \frac{1}{5^s} + \frac{(-1)^2}{6^s} + \cdots$$

On the RHS we are only getting n^{-s} for squarefree n's and the coefficients is $(-1)^{\nu(n)}$ where $\nu(n)$ is the number of prime factors of n. By the definition of Möbius function we have

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

and this series converges absolutely for $\operatorname{Re} s > 1$. Thus

$$1 = \frac{1}{\zeta(s)} \cdot \zeta(s) = \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right) = \sum_{n=1}^{\infty} \left(\sum_{d|n} \mu(d)\right) \frac{1}{n^s}$$

By the Uniqueness of Dirichlet series we have

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

which is exactly Theorem 3.3!!

Example (Dirichlet series of $\varphi(n)$). Note that if $\sigma > 2$ then we have

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^\sigma} \leq \sum_{n=1}^{\infty} \frac{n}{n^\sigma} = \sum_{n=1}^{\infty} \frac{1}{n^{\sigma-1}} \quad \text{converges}$$

On the other hand, if $\sigma \leq 2$ then

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^\sigma} \geq \sum_p \frac{p-1}{p^\sigma} \quad \text{diverges because } \sum_p \frac{1}{p} \text{ diverges}$$

Hence the Abscissa of convergence is 2 and the series diverges at 2. But

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \prod_p \left(1 + \frac{\varphi(p)}{p^s} + \frac{\varphi(p^2)}{p^{2s}} + \cdots\right) \quad \text{for } \operatorname{Re} s > 2$$

by the unique factorization and multiplicativity of $\varphi(n)$. Using the formula $\varphi(p^a) = p^{a-1}(p-1)$

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \prod_p \left(1 + \frac{p-1}{p^s} + \frac{p(p-1)}{p^{2s}} + \cdots\right) = \prod_p \left(1 + \frac{p-1}{p^s}\left(1 + \frac{p}{p^s} + \frac{p^2}{p^{2s}} + \cdots\right)\right)$$

$$= \prod_p \left(1 + \frac{p-1}{p^s} \cdot \left(1 - \frac{1}{p^{s-1}}\right)^{-1}\right) = \prod_p \frac{1-p^{-s}}{1-p^{-(s-1)}} = \frac{\zeta(s-1)}{\zeta(s)} \quad \text{for } \operatorname{Re} s > 2$$

Therefore $D(\varphi; s) = \zeta(s-1)\zeta(s)^{-1}$ for $\operatorname{Re} s > 2$.

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By moving $\zeta(s)$ to the other side, for $\operatorname{Re} s > 2$ we have

$$\left(\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right) = \left(\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s}\right)\zeta(s) = \zeta(s-1) = \sum_{n=1}^{\infty} \frac{n}{n^s}$$

Expaning the LHS gives us

$$\sum_{n=1}^{\infty}\left(\sum_{d|n}\varphi(d)\right)\frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{n}{n^s}$$

By the Uniqueness of Dirichlet series we have

$$\sum_{d|n}\varphi(d) = n$$

which is exactly Theorem 3.2.

Example. Consider the Dirichlet series for $|\mu(n)|$, which is an indicator function for squarefree numbers. Hence for $\operatorname{Re} s > 1$ we have

$$\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} = \prod_p \left(1 + \frac{|\mu(p)|}{p^s}\right) = \prod_p \left(1 + \frac{1}{p^s}\right)$$

Now using the famous equality that $1 + x = (1-x^2)/(1-x)$ we have

$$\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} = \prod_p \frac{1-p^{-2s}}{1-p^{-s}} = \frac{\zeta(s)}{\zeta(2s)}$$

Let $f, g : \mathbb{N} \to \mathbb{C}$ be arithmetic functions. Assume that

$$g(n) = \sum_{d|n} f(n)$$

is a divisor sum of f. We claim that f can be written as a divisor sum in terms of g! Observe that

$$\left(\sum_{n=1}^{\infty} \frac{f(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right) = \sum_{n=1}^{\infty}\left(\sum_{d|n} f(d)\right)\frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

Recall that $\zeta(s)^{-1}$ is the Dirichlet series of $\mu(n)$. Hence

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \frac{1}{\zeta(s)}\sum_{n=1}^{\infty} \frac{g(n)}{n^s} = \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{g(n)}{n^s}\right) = \sum_{n=1}^{\infty}\left(\sum_{d|n} g(d)\mu\left(\frac{n}{d}\right)\right)\frac{1}{n^s}$$

Now by the Uniqueness of Dirichlet series we have

$$f(n) = \sum_{d|n} g(d)\mu\left(\frac{n}{d}\right)$$

[Here we are assuming that the series above converge absolutely on some half plane $\operatorname{Re} s > \sigma_0$]. The above argument works back work. Hence we have the following result.

Theorem 3.5 (Möbius Inversion). Let $f, g : \mathbb{N} \to \mathbb{C}$ be arithmetic functions, then

$$g(n) = \sum_{d|n} f(d) \iff f(n) = \sum_{d|n} g(d)\mu\left(\frac{n}{d}\right)$$

for all $n \in \mathbb{N}$.

Proof. The "proof" we had above is not too rigorous. It just gives the big picture why Möbius Inversion is true. For example, what if the Dirichlet series $D(f; s)$ has no abscissa of convergence? For example if $f(n) = e^n$, then

$$D(f; s) = \sum_{n=1}^{\infty} \frac{e^n}{n^s}$$

does not converge absolutely for any value of $s \in \mathbb{C}$! There are two approaches.

4 Order of arithmetic functions

4.1 Order of the Divisor Function

Theorem 4.1 (Order of $d(n)$). Let $\epsilon > 0$, then $d(n) = \mathcal{O}_\epsilon(n^\epsilon)$.

Remark. Here the notation $\mathcal{O}_\epsilon(\cdot)$ means the constant is dependent on ϵ. In other words, for all $\epsilon > 0$ there exist $N_\epsilon \in \mathbb{N}$ and $c_\epsilon > 0$ such that $d(n) \leq c_\epsilon n^\epsilon$ for $n > N_\epsilon$.

Proof. Let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, where p_i are distinct primes and $\alpha_i \geq 1$. Hence

$$d(n) = (1+\alpha_1)\cdots(1+\alpha_k)$$

A funny way to see this equality is the following. Let $f(n) = 1$ for all n, then

$$d(n) = \sum_{d|n} 1 = \sum_{d|n} f(d) = \prod_{i=1}^{k}(1 + f(p_i) + \cdots + f(p_i^{\alpha_i})) = \prod_{i=1}^{k}(1 + \underbrace{1 + \cdots + 1}_{\alpha_i \text{ times}}) = \prod_{i=1}^{k}(1+\alpha_i)$$

Now let $\epsilon > 0$ be arbitray, then

$$\frac{d(n)}{n^\epsilon} = \left(\prod_{i=1}^{k} \frac{1}{p_i^{\alpha_i \epsilon}}\right)\left(\prod_{i=1}^{k}(\alpha_i + 1)\right) = \prod_{i=1}^{k} \frac{\alpha_i + 1}{p_i^{\alpha_i \epsilon}}$$

Our goal is to show the RHS is bounded as $n \to \infty$. Note that if $p > 2^{1/\epsilon}$ then we have

$$p^{\alpha\epsilon} > 2^\alpha = (1+1)^\alpha \geq \alpha + 1$$

for all $\alpha \in \mathbb{N}$. If $p < 2^{1/\epsilon}$ (there are only finitely many such p for given ϵ) we have

$$\frac{\alpha+1}{p^{\alpha\epsilon}} \leq \frac{\alpha+1}{2^{\alpha\epsilon}}$$

is bounded (as a function of α, for given ϵ). Hence

$$\frac{d_n}{n^\epsilon} = \left(\prod_{p_i > 2^{1/\epsilon}} \frac{\alpha_i+1}{p_i^{\alpha_i\epsilon}}\right)\left(\prod_{p_i < 2^{1/\epsilon}} \frac{\alpha_i+1}{p_i^{\alpha_i\epsilon}}\right) \leq \underbrace{\left(\prod_{p_i > 2^{1/\epsilon}} \frac{\alpha_i+1}{\alpha_i+1}\right)}_{\leq 1}\underbrace{\left(\prod_{p_i < 2^{1/\epsilon}} \frac{\alpha_i+1}{2^{\alpha_i\epsilon}}\right)}_{\text{bounded by } C_\epsilon}$$

The bounded constant depends on ϵ, so we proved that $d_n \leq C_\epsilon n^\epsilon$ for $n \to \infty$, as desired. □

Remark. This theorem tells us $d(n)$ grows slower than any power of n. We have the following results regarding the average order of the divisor function $d(n)$.

Theorem 4.2 (Average Order of $d(n)$). For $x \geq 1$ we have

$$\sum_{n \leq x} d(n) = x \log x + \mathcal{O}(x)$$

Theorem 4.3 (Dirichlet). For $x \geq 1$ we have

$$\sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + \mathcal{O}(x^{1/2})$$

Conjecture 4.4. Instead of $\mathcal{O}(x^{1/2})$, it is conjectured that for all $\epsilon > 0$ we have

$$\sum_{n\le x} d(n) = x\log x + (2\gamma - 1)x + \mathcal{O}_\epsilon(x^{1/4+\epsilon})$$

Proof of Theorem 4.2. Note that

$$\begin{aligned}\sum_{n\le x} d(n) &= \sum_{n\le x}\sum_{d\mid n} 1 = \sum_{n\le x}\sum_{dm=n} 1 = \sum_{dm\le x} 1 \\ &= \sum_{d\le x}\sum_{m\le x/d} 1 = \sum_{d\le x}\left\lfloor \frac{x}{d}\right\rfloor = \sum_{d\le x}\frac{x}{d} - \left\{\frac{x}{d}\right\} \\ &= x\sum_{d\le x}\frac{1}{d} - \sum_{d\le x}\left\{\frac{x}{d}\right\} = x\sum_{d\le x}\frac{1}{d} + \mathcal{O}(x) \\ &= x(\log x + \gamma + \mathcal{O}(1/x)) + \mathcal{O}(x) \\ &= x\log x + \mathcal{O}(x)\end{aligned}$$

As desired. □

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Proof of Dirichlet's Theorem. Using the Dirichlet Hyperbola method we have

$$\begin{aligned}d(n) &= \sum_{d\le x}\sum_{m\le x/d} 1 = 2\sum_{d\le\sqrt{x}}\sum_{m\le x/d} 1 - \sum_{d\le\sqrt{x}}\sum_{m\le\sqrt{x}} 1 \\ &= 2\sum_{d\le\sqrt{x}}\left\lfloor\frac{x}{d}\right\rfloor - \lfloor x\rfloor^2 = 2\sum_{d\le\sqrt{x}}\left\lfloor\frac{x}{d}\right\rfloor - (\sqrt{x} - \{\sqrt{x}\})^2 \\ &= 2x\sum_{d\le\sqrt{x}}\frac{1}{d} - 2\sum_{d\le\sqrt{x}}\left\{\frac{x}{d}\right\} - x + \mathcal{O}(\sqrt{x}) \\ &= 2x\sum_{d\le\sqrt{x}}\frac{1}{d} - x + \mathcal{O}(\sqrt{x})\end{aligned}$$

Now apply the theorem $\sum_{d\le y} = y\log y + \gamma + \mathcal{O}(1/y)$ to $y = \sqrt{x}$, we have

$$\begin{aligned}d(n) &= 2x\left(\log\sqrt{x} + \gamma + \mathcal{O}(1/\sqrt{x})\right) - x + \mathcal{O}(\sqrt{x}) \\ &= x\log x + (2\gamma - 1)x + \mathcal{O}(\sqrt{x})\end{aligned}$$

This completes the proof. □

Now let's look at the average order of the Euler totient function $\varphi(n)$. Recall that by Möbius Inversion we have

$$\varphi(n) = \sum_{d\mid n}\mu(d)\cdot\frac{n}{d}$$

Therefore we have

$$\begin{aligned}
\sum_{n\le x}\varphi(n) &= \sum_{n\le x}\sum_{d|n}\mu(d)\cdot\frac{n}{d} = \sum_{d\le x}\sum_{m\le x/d}\mu(d)m = \sum_{d\le x}\mu(d)\sum_{m\le x/d}m \\
&= \sum_{d\le x}\frac{\mu(d)}{2}\cdot\left(\left\lfloor\frac{x}{d}\right\rfloor+1\right)\left\lfloor\frac{x}{d}\right\rfloor \\
&= \sum_{d\le x}\frac{\mu(d)}{2}\left[\frac{1}{2}\left(\frac{x}{d}\right)^2+\mathcal{O}\left(\frac{x}{d}\right)\right] \\
&= \frac{x^2}{2}\sum_{d\le x}\frac{\mu(d)}{d^2}+\mathcal{O}\left(x\sum_{d\le x}\frac{1}{d}\right) \\
&= \frac{x^2}{2}\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}-\frac{x^2}{2}\sum_{d>x}\frac{\mu(d)}{d^2}+\mathcal{O}(x\log x)
\end{aligned}$$

Now let's look at these two infinite series. Recall that

$$\frac{1}{\zeta(s)}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}\quad\text{for }\operatorname{Re}s>1$$

Let $s=2$, we know that $\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}=\zeta(2)^{-1}=6/\pi^2$. On the other hand

$$\left|\sum_{d>x}\frac{\mu(d)}{d^2}\right|\le\sum_{d>x}\frac{1}{d^2}=\mathcal{O}\left(\frac{1}{x}\right)$$

The bound $\mathcal{O}(1/x)$ is obtained by comparing this tail with the integral $\int_1^{\infty}t^{-2}\,\mathrm{d}t$. Collecting all these information together gives us

Theorem 4.5 (Average Order of $\varphi(n)$). For $x\ge 1$ we have

$$\sum_{n\le x}\varphi(n)=\frac{3x^2}{\pi^2}+\mathcal{O}(x\log x)$$

Definition. We define the **von Mangöldt Function** by

$$\Lambda(n)=\begin{cases}\log p & \text{if } n=p^a \text{ for some } p\\ 0 & \text{otherwise}\end{cases}$$

Where does this function come from? Let's look at the derivative of $\zeta(s)$. We know

$$\zeta(s)=\prod_p\left(1-\frac{1}{p^s}\right)^{-1}\quad\text{for }\operatorname{Re}s>1$$

Taking the logarithm gives us

$$\log\zeta(s) = -\sum_p \log\left(1-\frac{1}{p^s}\right) = \sum_p\sum_{k=1}^\infty \frac{1}{kp^{ks}}$$

Defferentiating both sides gives

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_p\sum_{k=1}^\infty \frac{-k\log p}{kp^{ks}} = -\sum_p\sum_{k=1}^\infty \frac{\log p}{p^{ks}} = \sum_{n=1}^\infty \frac{a_n}{n^s}$$

In this sum, note that $a_n \neq 0$ if and only if $n = p^a$ is a prime power. In that case $a_n = -\log p$. Therefore we conclude that

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^\infty \frac{\Lambda(n)}{n^s}$$

Proposition 4.6. For $\operatorname{Re} s > 1$ we have

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^\infty \frac{\Lambda(n)}{n^s}$$

Theorem 4.7. For $x \geq 1$ we have

$$\sum_{n\leq x}\frac{\Lambda(n)}{n} = \log x + \mathcal{O}(1)$$

Proof. We will look at $\log(m!)$. For example if $m = 10$ then $10! = 2^{3+2+1}3^{1+3}5^{1+1}7$. Then

$$\log(10!) = (3+2+1)\log 2 + (3+1)\log 3 + 2\log 5 + \log 7$$

Reall the Legendre's formula for $\nu_p(n!)$ says that

$$\nu_p(n!) = \sum_{r=1}^\infty \left\lfloor\frac{n}{p^r}\right\rfloor = \left\lfloor\frac{n}{p}\right\rfloor + \left\lfloor\frac{n}{p^2}\right\rfloor + \left\lfloor\frac{n}{p^3}\right\rfloor + \cdots$$

Therefore we have

$$\begin{aligned}\log(m!) &= \log\left(\prod_{p\leq m} p^{\nu_p(m!)}\right) = \sum_{p\leq m}\nu_p(m!)\log p = \sum_{p\leq m}\sum_{r=1}^\infty\left\lfloor\frac{m}{p^r}\right\rfloor\log p \\ &= \sum_{p^r\leq m}\left\lfloor\frac{m}{p^r}\right\rfloor\log p = \sum_{n\leq m}\left\lfloor\frac{m}{n}\right\rfloor\Lambda(n) = m\sum_{n\leq m}\frac{\Lambda(n)}{n} - \underbrace{\sum_{n\leq m}\left\{\frac{m}{n}\right\}\Lambda(n)}_{R(m)}\end{aligned}$$

It follows that

$$\sum_{n\leq m}\frac{\Lambda(n)}{n} = \frac{\log(m!)}{m} + \frac{R(m)}{m}$$

By Stirling's formula we know $\log(m!) = m\log m + \mathcal{O}(m)$. For the error term we have

$$\begin{aligned} R(m) &< \sum_{p^r \le m} \log p = \sum_{p \le m} \log p + \sum_{p^2 \le m} \log p + \sum_{p^3 \le m} \log p + \cdots \\ &\le \sum_{p \le m} \log p + (\log m)m^{1/2} + (\log m)m^{1/3} + \cdots + (\log m)m^{1/k} \\ &\le \sum_{p \le m} \log p + (\log m)m^{1/2} \cdot \frac{\log m}{\log 2} \end{aligned}$$

where k is minimal such that $p^k < m$ is impossible, which means $2^k \le (\log m)/(\log 2)$. By the Theorem 4.8 below we can show that $\sum_{p \le m} \log p = \mathcal{O}(m)$, hence

$$R(m) = \mathcal{O}(m) + \mathcal{O}\left((\log m)^2 m^{1/2}\right) = \mathcal{O}(m)$$

Therefore it follows that

$$\sum_{n \le m} \frac{\Lambda(n)}{n} = \frac{m\log m + \mathcal{O}(m)}{m} + \frac{\mathcal{O}(m)}{m} = \log m + \mathcal{O}(1)$$

This completes the proof. By Adapting this proof, we can replace $m \in \mathbb{N}$ with $x \in \mathbb{R}$. $\square$

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4.2 An Elementary Bound on $\theta(x)$

We now look at some elementary bounds on $\pi(x)$ and some arithmetic functions. First we will obtain an elementary upper bound for $\theta(x) = \mathcal{O}(x)$. The PNT is equivalent to $\theta(x) \sim x$.

The key is to consider the middle binomial coefficient $\binom{2m}{m}$. For $m \in \mathbb{N}$ we have

$$2^{2m} = (1+1)^{2m} = \sum_{k=0}^{2m} \binom{2m}{k} \ge \binom{2m}{m} = \frac{(2m)!}{m!m!}$$

For $m = 7$ we have

$$\binom{14}{7} = \frac{14!}{7!7!} = \frac{8 \cdot 9 \cdot 10 \cdot 11 \cdot 11 \cdot 12 \cdot 13 \cdot 14}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} = 4 \cdot 3 \cdot 11 \cdot 13$$

We can see that $\binom{14}{7}$ must be divisible by primes between 8 and 14. This is because when we divide 14! by 7! and 7!, the denominator never cancels the prime between 8 and 14 on the numerator. This idea generalizes to all $m \in \mathbb{N}$. Hence we have

$$2^{2m} \ge \binom{2m}{m} = \frac{(2m)!}{m!m!} \ge \prod_{m < p \le 2m} p$$

Taking the logarithm gives

$$\sum_{m<p\leq 2m} \log p \leq 2m \log 2 \tag{$*$}$$

Note that if $2^{r-1} < x \leq 2^r$ for $r \in \mathbb{N}$, then

$$\theta(x) \leq \theta(2^r) = \sum_{2^0<p\leq 2^1} \log p + \sum_{2^1<p\leq 2^2} \log p + \sum_{2^2<p\leq 2^3} \log p + \cdots + \sum_{2^{r-1}<p\leq 2^r} \log p$$

Using the bound $(*)$ we obtained above to each sum, we get

$$\theta(x) \leq (\log 2)(2 + 4 + \cdots + 2^r) \leq 2^{r+1} \log 2 \leq 4 \log 2 \cdot 2^{r-1} \leq (4 \log 2)x$$

This gives the theorem.

Theorem 4.8. For $x \geq 1$ we have

$$\sum_{p\leq x} \log p = \mathcal{O}(x)$$

5 Prime Number Theorem

5.1 Mittag-Leffler Expansions

Example. Consider the function

$$f(z) = \frac{\pi \cos \pi z}{\sin \pi z} - \frac{1}{z}$$

It has simple poles at $0 \neq n \in \mathbb{Z}$. To compute the residues at these points we can compute the Laurent series expansion of $f(z)$ around $z = n$.

$$\cos(\pi z) = (-1)^n \left(1 - \frac{\pi^2}{2!}(z-n)^2 + \frac{\pi^4}{4!}(z-n)^4 - \cdots\right)$$

$$\sin(\pi z) = (-1)^n \left(\pi(z-n) - \frac{\pi^3}{3!}(z-n)^3 + \cdots\right)$$

Hence the residue $r_n = \operatorname{Res}(f, n)$ is

$$r_n = \lim_{z\to n} (z-n) \left(\frac{\pi \cos \pi z}{\sin \pi z} - \frac{1}{z}\right) = 1$$

Our goal is to show that

$$\frac{\pi \cos \pi z}{\sin \pi z} = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n}\right) \quad \text{for } z \in \mathbb{C} \text{ not a pole of } f$$

Proof. Let $z \in \mathbb{C}$ and z is not a pole (so $z \notin \mathbb{Z} \setminus \{0\}$). Let $R = n + 1/2$ for some $n \in \mathbb{N}$. Let γ_R be the contour whose trace is the square with four vertices $(\pm R, \pm R)$ oriented counter-clockwise. Then

$f(z)$ is bounded on γ_R because it has no poles on γ_R. Moreover, this bound is independent of R! This is basically becasue $\frac{\pi\cos\pi z}{\sin\pi z}$ is periodic with period 2 and $1/z$ is bounded by 1 away from 0 (this is not a proof, just an intuition). Anyway, say $|f(w)| < c$ for all $w \in \gamma_R$. Therefore we have

$$\frac{1}{2\pi i}\int_{\gamma_R} f(w)\left(\frac{1}{w-z}-\frac{1}{w}\right)\,\mathrm{d}w = \frac{1}{2\pi i}\int_{\gamma_R} f(w)\cdot\frac{z}{(w-z)w}\,\mathrm{d}w$$

Note that for given z we have $\frac{z}{(w-z)w} = \mathcal{O}(R^{-2})$. Hence the integral above goes to 0 as $R \to \infty$. On the other hand, we have

$$\frac{1}{2\pi i}\int_{\gamma_R} \underbrace{f(w)\left(\frac{1}{w-z}-\frac{1}{w}\right)}_{:=g(w)}\,\mathrm{d}w = \sum_{\substack{0\neq n\\ \text{inside } \gamma_R}} \operatorname{Res}\left(g(w),n\right)+\operatorname{Res}\left(g(w),0\right)+\operatorname{Res}\left(g(w),z\right)$$

The residues are equal to

$$\operatorname{Res}(g(w),n) = 1\cdot\left(\frac{1}{n-z}-\frac{1}{n}\right) \quad\text{and}\quad \operatorname{Res}(g(w),0) = -f(0) \quad\text{and}\quad \operatorname{Res}(g(w),z) = f(z)$$

Using a linear approximation of cos and sin we have $f(0) = 0$. Let $R \to \infty$ then we have

$$0 = f(z) - f(0) + \lim_{R\to\infty}\sum_{1\le n<R}\left(\frac{1}{n-z}-\frac{1}{n}+\frac{1}{-n-z}+\frac{1}{n}\right)$$

Rearrange this equality gives

$$\frac{\pi\cos\pi z}{\sin\pi z}-\frac{1}{z} = f(z) = -\sum_{n=1}^{\infty}\left(\frac{1}{n-z}+\frac{1}{-n-z}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{z-n}\right)$$

Moving $1/z$ to the RHS gives us the desired equality. $\square$

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We can keep expanding this summation

$$\begin{aligned}\frac{\pi z\cos\pi z}{\sin\pi z} &= 1+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{z-n}\right) = 1+2z\sum_{n=1}^{\infty}\frac{1}{z^2-n^2}\\ &= 1-2z^2\sum_{n=1}^{\infty}\frac{1}{n^2(1-z^2/n^2)} = 1-2\sum_{n=1}^{\infty}\frac{z^2}{n^2}\sum_{k=0}^{\infty}\left(\frac{z^2}{n^2}\right)^k\\ &= 1-2\sum_{k=1}^{\infty}z^{2k}\sum_{n=1}^{\infty}\frac{1}{n^{2k}} = 1-2\sum_{k=1}^{\infty}\zeta(2k)z^{2k}\end{aligned}$$

On the RHS we have a power series whose $[z^{2k}]$ terms are $-2\zeta(2k)$. Hence this allows us to compute $\zeta(2k)$ by expaning the LHS using Taylor series. We have

$$\begin{aligned}
\frac{\pi z\cos\pi z}{\sin\pi z} &= \pi z\left(1-\frac{(\pi z)^2}{2!}+\frac{(\pi z)^4}{4!}-\cdots\right)\left(\pi z-\frac{(\pi z)^3}{3!}+\frac{(\pi z)^5}{5!}-\cdots\right)^{-1} \\
&= \left(1-\frac{(\pi z)^2}{2!}+\frac{(\pi z)^4}{4!}-\cdots\right)\left(1-\frac{(\pi z)^2}{3!}+\frac{(\pi z)^4}{5!}-\cdots\right)^{-1} \\
&= 1+c_1z+c_2z^2+c_3z^3+\cdots
\end{aligned}$$

Hence we have

$$(1+c_1z+c_2z^2+c_3z^3+\cdots)\left(1-\frac{(\pi z)^2}{3!}+\frac{(\pi z)^4}{5!}-\cdots\right) = 1-\frac{(\pi z)^2}{2!}+\frac{(\pi z)^4}{4!}-\cdots$$

On the LHS the z^2 term is equal to

$$-\frac{(\pi z)^2}{3!}+c_2z^2 = \left(c_2-\frac{\pi^2}{6}\right)z^2$$

It follows that $c_2 = \pi^2/6-\pi^2/2 = -\pi^2/3$. It follows that

$$-2\zeta(2) = c_2 = -\frac{\pi^2}{3} \implies \zeta(2) = \frac{\pi^2}{6}$$

5.2 Eisenstein Series

We can expand this sucker in another way.

$$\frac{\pi\cos\pi z}{\sin\pi z} = \pi i\frac{e^{i\pi z}+e^{-i\pi z}}{e^{i\pi z}-e^{-i\pi z}}\cdot\frac{e^{i\pi z}}{e^{i\pi z}} = \pi i\cdot\frac{e^{2\pi iz}+1}{e^{2\pi iz}-1} = \pi i\cdot\frac{u+1}{u-1} = -\pi i(u+1)\sum_{r=0}^{\infty}u^r$$

where we set $u = e^{2\pi iz} = e^{2\pi ix-2\pi y}$ for $z = x+iy$. The expansion is valid for $|u|<1$, which is when $\operatorname{Im} z>0$. Hence, by the result we have seen last time

$$\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{z-n}\right) = -\pi i\left(1+2\sum_{r=1}^{\infty}u^r\right) \quad \text{for } |u|<1$$

Differentiating both sides term by term with respect to z (note $du/dz = 2\pi iu$) gives us

$$-\sum_{n\in\mathbb{Z}}\frac{1}{(z+n)^2} = -(2\pi i)^2\sum_{r=1}^{\infty}re^{2\pi irz}$$

Repeatedly differentiating with respect to z, for $k\in\mathbb{Z}$ and $k\geq 2$

$$\sum_{n\in\mathbb{Z}}\frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{r=1}^{\infty}r^{k-1}e^{2\pi irz} \qquad (\heartsuit)$$

This is the **Lipschitz formula**. The LHS is periodic with period 1 and the RHS is its Fourier series.

Definition. Let $\mathbb{H} = \{\operatorname{Im} z > 0\}$ be the upper half plane. Let $\tau \in \mathbb{H}$ and $k \in \mathbb{N}$. The **Eisenstein series of weight** k is

$$G_k(\tau) := \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k}$$

If k is odd then $G_k(\tau) = 0$ for all $\tau \in \mathbb{H}$. Hence we assume k is even. If $k = 2$ then it turns out that G_2 does not converge and for $k \geq 3$ it always converges. Hence we assume that k is even and $k \geq 3$.

Theorem 5.1. For $k \geq 3$, the Eisenstein series $G_k(\tau)$ converges absolutely.

Proof. Exercise. $\square$

Theorem 5.2. For $k \geq 3$, the Eisenstein series $G_k(\tau)$ is a **modular form of weight** k, meaning

$$G_k\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k G_k(\tau)$$

for all $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$ [This means the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in $\mathrm{SL}_2(\mathbb{Z})$].

Proof. Exercise. $\square$

If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ then for all $\tau \in \mathbb{H}$ we have

$$G_k(\tau+1) = G_k(\tau) \tag{1}$$

If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ then for all $\tau \in \mathbb{H}$ we have

$$G_k\left(\frac{-1}{\tau}\right) = \tau^k G_k(\tau) \tag{2}$$

Equation (1) tells us that G_k is periodic with period 1, so if we know the value of $G_k(\tau)$ on the strip $\{-1/2 \leq \operatorname{Re}\tau < 1/2\}$ then we know all $G_k(\tau)$ for all $\tau \in \mathbb{H}$. Equation (2) tells us that if we know $G_k(\tau)$ for $|\tau| > 1$ we know the value of $G_k(\tau)$ for $|\tau| < 1$. This means it suffices to study G_k on

$$\left\{\frac{-1}{2} \leq \operatorname{Re}\tau \leq \frac{1}{2} \quad \text{and} \quad |\tau| \geq 1\right\}$$

Consider for k even and $k \geq 3$, we have

$$G_k(\tau) = \underbrace{2\sum_{n=1}^{\infty}\frac{1}{n^k}}_{m=0 \text{ term}} + 2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\frac{1}{(m\tau+n)^k}$$

Apply the Lipschitz formula to $z = m\tau$, which satisfies $\operatorname{Im} z > 0$, we have

$$G_k(\tau) = 2\zeta(k) + 2\sum_{m=1}^{\infty}\frac{(-2\pi i)^k}{(k-1)!}\sum_{r=1}^{\infty} r^{k-1}e^{2\pi i r m\tau} = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e^{2\pi i n\tau}$$

where we re-indexed $n = rm$ and recall that $\sigma_{k-1}(n) = \sum_{r|n} r^{k-1}$.

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5.3 Poisson Summation Formula

Theorem 5.3 (Poisson Summation). Let $f : \mathbb{R} \to \mathbb{C}$ such that $\int_{-\infty}^{\infty} |f(t)| \, \mathrm{d}t < \infty$, where $\hat{f}$ is the **Fourier transform** of f defined by

$$\hat{f}(n) := \int_{-\infty}^{\infty} f(t) e^{-2\pi i n t} \, \mathrm{d}t$$

Suppose $F(v) := \sum_{n\in\mathbb{Z}} f(n+v)$ is uniformly convergent for $v \in [0,1]$. Then F is periodic with period 1 and F is continuous by uniform convergence. Assume that $\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$, then

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$$

To prove this theorem we need the following result of Fejér from Fourier analysis.

Theorem 5.4 (Fejér). Let $F : \mathbb{R} \to \mathbb{C}$ be a continuous function with period 1. Let

$$c_n = \int_0^1 F(x) e^{-2\pi i n x} \, \mathrm{d}x$$

be the n-th fourier coefficient of F. Assume $\sum_{n\in\mathbb{Z}} |c_n| < \infty$, then

$$F(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x} \quad \text{for all } x \in \mathbb{R}$$

Proof of Poisson summation. The m-th fourier coefficient of F is

$$\begin{aligned}
\hat{F}(m) &= \int_0^1 \sum_{n\in\mathbb{Z}} f(n+v) e^{-2\pi i n v} \, \mathrm{d}v \\
&= \sum_{n\in\mathbb{Z}} \int_0^1 f(n+v) e^{-2\pi i n v} \, \mathrm{d}v && \text{(by uniform convergence)} \\
&= \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(t) e^{-2\pi i m(t-n)} \, \mathrm{d}t && \text{(change of variable } t = n+v\text{)} \\
&= \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(t) e^{-2\pi i m t} \, \mathrm{d}t \\
&= \int_{-\infty}^{\infty} f(t) e^{-2\pi i m t} \, \mathrm{d}t = \hat{f}(m) && \text{(by convergence of } \textstyle\int_{-\infty}^{\infty} |f(t)| \, \mathrm{d}t\text{)}
\end{aligned}$$

Finally by Fejér's Theorem we have

$$\sum_{n\in\mathbb{Z}} f(n+v) = F(v) = \sum_{n\in\mathbb{Z}} \hat{F}(n)e^{2\pi inv} = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inv}$$

Setting $v = 0$ completes the proof. $\square$

Example. Fix $x > 0$. Apply Poisson summation to $f(t) = e^{-\pi t^2 x}$. Then

$$\hat{f}(n) = \int_{-\infty}^{\infty} e^{-\pi t^2 x - 2\pi int}\ \mathrm{d}t = \int_{-\infty}^{\infty} e^{-\pi(tx^{1/2}+inx^{-1/2})^2} e^{-\pi n^2 x^{-1}}\ \mathrm{d}t$$

In the second equality we completed the square for $-\pi t^2 x - 2\pi int$. Let $z = tx^{1/2} + inx^{-1/2}$ and therefore $\mathrm{d}t = x^{-1/2}\mathrm{d}z$. Then

$$\hat{f}(n) = x^{-1/2}e^{-\pi n^2 x^{-1}} \int_\lambda e^{\pi z^2}\ \mathrm{d}z$$

After the change of variable z, the integral from $-\infty$ to ∞ becomes the integral from $-\infty + inx^{-1/2}$ to $\infty + inx^{1/2}$. We call this line λ. By pushing the line λ down to the x-axis (this is valid because $e^{-\pi z^2}$ is entire), we deform the integral and

$$\hat{f}(n) = x^{-1/2}e^{-\pi n^2 x^{-1}} \int_\lambda e^{\pi z^2}\ \mathrm{d}z = x^{-1/2}e^{-\pi n^2 x^{-1}} \underbrace{\int_{-\infty}^{\infty} e^{-\pi t^2}\ \mathrm{d}t}_{=1}$$

The last integral is 1 becasue $\int_{-\infty}^{\infty} e^{-t^2}\ \mathrm{d}t = \sqrt{\pi}$. It follows that

$$\hat{f}(n) = x^{-1/2}e^{-\pi n^2 x^{-1}}$$

By Poisson summation, for $x > 0$ we have

$$\theta(x) := \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x} = x^{-1/2}\sum_{n\in\mathbb{Z}} e^{-\pi n^2 x^{-1}} = x^{-1/2}\theta\left(\frac{1}{x}\right)$$

We call $\theta(x)$ the **Jacobi theta function**.

Theorem 5.5 (Functional Equation of theta function). For $x > 0$ we have

$$\theta(x) = x^{-\frac{1}{2}}\theta\left(\frac{1}{x}\right)$$

Moreover note that $\theta(x) \sim x^{-1/2}$ as $x \to 0^+$.

5.4 Functional Equation of the Riemann Zeta Function

Definition. The **Euler Gamma function** on $\operatorname{Re} s > 0$ is defined by

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} \, \mathrm{d}t$$

Integrating by parts shows that $\Gamma(s+1) = s\Gamma(s)$ for $\operatorname{Re} s > 0$. This allows us to give a meromorphic continuation of Γ to $\operatorname{Re} s > -1$ with a pole at 0 by

$$\Gamma(s) := \frac{\Gamma(s+1)}{s} \quad \text{for } \operatorname{Re} s > -1$$

We can keep doing this because

$$\Gamma(s) = \frac{\Gamma(s+1)}{s} = \frac{\Gamma(s+2)}{s(s+1)} = \frac{\Gamma(s+3)}{s(s+1)(s+2)} = \cdots$$

This gives a meromorphic continuation of Γ to the entire $\mathbb{C}$ with poles at $-n$ for all $n \in \mathbb{Z}_{\geq 0}$ with residue at $-n$ is equal to $\frac{(-1)^n}{n!}$.

Theorem 5.6 (Functional Equation of ζ).

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{s-1}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

Proof. For $\operatorname{Re} s > 0$ we have

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) n^{-s} = \int_0^\infty x^{\frac{s}{2}-1} e^{-\pi n^2 x} \, \mathrm{d}x$$

Thus for $\operatorname{Re} s > 1$, summing over $n \in \mathbb{N}$ gives

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^\infty x^{\frac{s}{2}-1} \underbrace{\sum_{n=1}^\infty e^{-\pi n^2 x}}_{\omega(x)} \, \mathrm{d}x$$

Note that $\omega(x)$ is quite similar to $\theta(x)$. In fact $\omega(x) = \frac{1}{2}(\theta(x) - 1)$. Now

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \left(\int_0^1 + \int_1^\infty\right) x^{\frac{s}{2}-1} \omega(x) \, \mathrm{d}x$$

On $[0, 1]$ we change of variable by $y = 1/x$, so $\mathrm{d}y = -x^{-2} \, \mathrm{d}x = -y^2 \, \mathrm{d}x$ and $\mathrm{d}x = -y^{-2} \, \mathrm{d}y$.

$$\int_0^1 x^{\frac{s}{2}-1} \omega(x) \, \mathrm{d}x = \int_1^\infty y^{-\frac{s}{2}-1} \omega\left(\frac{1}{y}\right) \, \mathrm{d}y \qquad (*)$$

Now by the functional equation of $\theta(x)$ we have

$$\begin{aligned}\omega\left(\frac{1}{x}\right) &= \frac{\theta(\frac{1}{x})-1}{2} = \frac{x^{1/2}\theta(x)-1}{2} \\ &= x^{1/2}\left(\frac{\theta(x)-1}{2}\right) + \frac{x^{1/2}}{2} - \frac{1}{2} \\ &= x^{1/2}\omega(x) + \frac{x^{1/2}}{2} - \frac{1}{2} \qquad (\heartsuit)\end{aligned}$$

This gave us a functional equation for $\omega(x)$.

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Hence we have

$$\begin{aligned}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \int_0^1 x^{\frac{s}{2}-1}\omega(x)\ \mathrm{d}x + \int_1^\infty x^{\frac{s}{2}-1}\omega(x)\ \mathrm{d}x \\ &= \int_1^\infty y^{-\frac{s}{2}-1}\omega\left(\tfrac{1}{y}\right)\ \mathrm{d}y + \int_1^\infty x^{\frac{s}{2}-1}\omega(x)\ \mathrm{d}x \qquad \text{(by } (*)) \\ &= \int_1^\infty y^{-\frac{s}{2}-1}\left(y^{1/2}\omega(y) + y^{1/2} - \frac{1}{2}\right) + \int_1^\infty x^{\frac{s}{2}-1}\omega(x)\ \mathrm{d}x \qquad \text{(by } (\heartsuit)) \\ &= \int_1^\infty x^{-\frac{s}{2}-1}\left(x^{1/2}\omega(x) + x^{1/2} - \frac{1}{2}\right) + \int_1^\infty x^{\frac{s}{2}-1}\omega(x)\ \mathrm{d}x \\ &= \int_1^\infty \left(x^{\frac{s}{2}-1} + x^{\frac{1-s}{2}-1}\right)\omega(x)\ \mathrm{d}x - \frac{1}{2}\int_1^\infty x^{-\frac{s}{2}-1}\ \mathrm{d}x + \frac{1}{2}\int_1^\infty x^{\frac{1-s}{2}-1}\ \mathrm{d}x\end{aligned}$$

The first integral is invariant under $s \mapsto 1-s$. The second and third integrals are just integral of power functions, which are easy to compute:

$$-\frac{1}{2}\int_1^\infty x^{-\frac{s}{2}-1}\ \mathrm{d}x = -\frac{1}{s} \quad \text{and} \quad \frac{1}{2}\int_1^\infty x^{\frac{1-s}{2}-1}\ \mathrm{d}x = -\frac{1}{1-s}$$

Thus, we have

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_1^\infty \left(x^{\frac{s}{2}-1} + x^{\frac{1-s}{2}-1}\right)\omega(x)\ \mathrm{d}x - \frac{1}{s} - \frac{1}{1-s} \qquad (**)$$

Note that

$$\omega(x) = \sum_{n=1}^\infty e^{-\pi n^2 x} < \sum_{n=1}^\infty e^{-\pi n x} = \frac{e^{-\pi x}}{1-e^{-\pi x}}$$

Note the numerator is decaying expoentially and the numerator is tending to 1, so $\omega(x)$ is decaying to 0 expoentially. This means the integrand $(x^{\frac{s}{2}-1} + x^{\frac{1-s}{2}-1})\omega(x)$ is pretty small, allowing us to differentiate $\int_1^\infty (x^{\frac{s}{2}-1} + x^{\frac{1-s}{2}-1})\omega(x)\ \mathrm{d}x$ under the integral sign. This means $\int_1^\infty (x^{\frac{s}{2}-1} + x^{\frac{1-s}{2}-1})\omega(x)\ \mathrm{d}x$ extends to an entire function. Therefore $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ extends to a meromorphic function on $\mathbb{C}$ with poles at $s = 0, 1$. Here $\pi^{-s/2}$ is entire function and $\Gamma(z)$ is meromorphic on $\mathbb{C}$ with poles at $-\mathbb{Z}$. This measn $\Gamma(\frac{s}{2})$ is meromorphic on $\mathbb{C}$ with poles at $-2\mathbb{Z}$.

However, recall equation $(**)$ says that

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_1^\infty \left(x^{\frac{s}{2}-1} + x^{\frac{1-s}{2}-1}\right)\omega(x)\ \mathrm{d}x - \frac{1}{s} - \frac{1}{1-s}$$

This tells us $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ only have poles at $0, 1$. Hence $\zeta(s)$ have simple zeros at $s = -2, -4, -6, \cdots$ in order to cancel the extra poles of $\Gamma(s/2)$ at these points.

We know the RHS has a simple pole at $s = 1$ with residue 1, because the integral and $-1/s$ are analytic at 1, which means the only contribution to its Laurent series at 1 is $\frac{-1}{1-s} = \frac{1}{s-1}$, which has residue 1 (its coefficient). But

$$\pi^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}\right) = \pi^{-1/2} \cdot \int_0^\infty e^{-x^2}\ \mathrm{d}x = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$

Thus $\zeta(s)$ has a simple pole at $s = 1$ with residue 1. In equation $(**)$ by replacing s with $1 - s$ then we have that

$$\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) = \int_1^\infty \left(x^{\frac{1-s}{2}-1} + x^{\frac{1}{2}-1}\right)\omega(x)\ \mathrm{d}x - \frac{1}{1-s} - \frac{1}{s} = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

This completes the proof. □

Remark. To summarize this theorem.

1. $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ extends to a meromorphic function with its only poles being simple at $s = 0, 1$.

2. $\zeta(s)$ has simple zeros at $s = -2, -4, -6, \cdots$.

3. $\zeta(s)$ has a simple pole at $s = 1$ with residue 1. Why? This is because $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ has simple poles at $0, 1$. Since $\Gamma(s/2)$ already has a pole at $s = 0$, so $\zeta(s)$ has no pole at $s = 0$. But $\Gamma(s/2)$ has no pole at $s = 1$, so this pole must come from $\zeta(s)$. Also, there are no other poles of $\zeta(s)$ that are cancelled by zeros of $\pi^{-s/2}\Gamma(s/2)$ because $\pi^{-s/2}\Gamma(s/2)$ has no zeros!

4. $\zeta(s) \neq 0$ if $\operatorname{Re} s > 1$. This comes from the Euler product

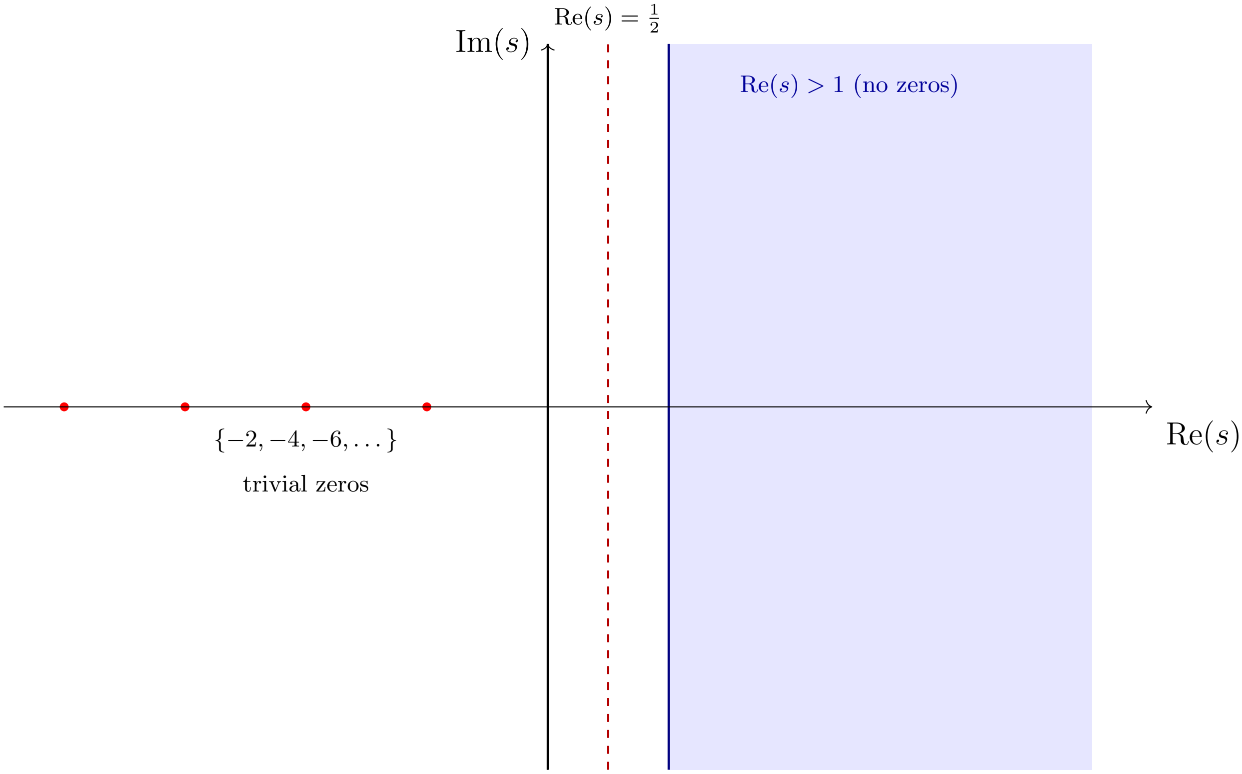
$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

None of $1 - p^{-s}$ is nonzero, so we have $\zeta(s) \neq 0$ on $\operatorname{Re} s > 1$. Then, by the functional equation we can consider the zeros of $\zeta(1-s)$, which correspond to zeros of $\zeta(s)$ on $\operatorname{Re} s < 0$. However, as long as s is not a odd natural number (so that $1-s \neq -2,-4,\cdots$) we can see that $\zeta(1-s) \neq 0$ by the functional equation. The only zeros of $\zeta(s)$ on $\operatorname{Re} s < 0$ are $-2,-4,-6,\cdots$. These are called the **trivial zeros** of $\zeta(s)$.

5. We will prove that $\zeta(s) \neq 0$ if $\operatorname{Re} = 0, 1$.

We analyzed the zeros of $\zeta(s)$ on $\operatorname{Re} s > 1$ and $\operatorname{Re} s < 0$. The trivials zeros on $\operatorname{Re} s < 0$ are at $-2,-4,\cdots$ and there are no zeros on $\operatorname{Re} s > 1$. What about the strip $0 \leq \operatorname{Re} s \leq 1$. By (v) we know there are no zeros on $\operatorname{Re} s = 0, 1$, so it remains to study the zeros on $0 < \operatorname{Re} s < 1$. Riemann conjectured that all zeros on this strip lie on the line $\operatorname{Re} s = \frac{1}{2}$.

Conjecture 5.7 (Riemann Hypothesis). If $\zeta(s) = 0$ and $0 < \operatorname{Re} s < 1$, then $\operatorname{Re} s = \frac{1}{2}$.



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Remark. Let $N(T)$ be the number of zeros of $\zeta(s)$ with $0 < \operatorname{Im} s < T$, then

$$N(T) \sim \frac{T}{2\pi} \log\left(\frac{T}{2\pi e}\right) + \frac{7}{8} + S(T)$$

where $S(T) = \mathcal{O}(\log T)$. See *Multiplicative Number Theory, Davenport.*

5.5 Proof of Prime Number Theorem

Gauss conjectued that

$$\pi(x) = \sum_{p \le x} 1 \sim \int_2^x \frac{\mathrm{d}t}{\log t} \sim \frac{x}{\log x}$$

We define the function $\theta(x) = \sum_{p\le x} \log p$ and $\psi(x) = \sum_{p^k \le x} \log p$. We showed

$$\pi(x) \sim \frac{x}{\log x} \iff \theta(x) \sim x \iff \psi(x) \sim x$$

We know that

$$\psi(x) = \theta(x) + \mathcal{O}\left(x^{1/2} + x^{1/3} + \cdots\right) = \theta(x) + \mathcal{O}\left(x^{1/2}\log x\right)$$

We also showed that $\psi(x) = \sum_{n \le x} \Lambda(n)$ and

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = \sum_{p^k} \frac{\log p}{p^{ks}} \quad \text{for } \operatorname{Re} s > 1$$

We will prove $\psi(x) \sim x$ and deduce Prime Number Theorem from it. Actually, we will prove

$$\psi_1(x) := \sum_{n \le x} \Lambda(n)(x-n) = \int_0^x \psi(t)\ \mathrm{d}t \sim \frac{x^2}{2}$$

We will first prove the equivalence of these.

Theorem 5.8. As $x \to \infty$ we have

$$\psi(x) \sim x \iff \psi_1(x) \sim \frac{x^2}{2}$$

Proof. ($\Rightarrow$). Assume $\psi(x) \sim x$, that is, $\psi(x) = x + r(x)$ where $r(x) = o(x)$. Hence

$$\psi_1(x) = \int_0^x \psi(t)\ \mathrm{d}t = \int_0^x t\ \mathrm{d}t + \int_0^x r(t)\ \mathrm{d}t = \frac{x^2}{2} + \int_0^x r(t)\ \mathrm{d}t$$

Since $r(x) = o(x)$ for any $\epsilon > 0$ there is x_0 such that $|r(x)| < \epsilon|x|$ for all $x > x_0$. Thus

$$\begin{aligned} r_1(x) := \int_0^x r(t)\ \mathrm{d}t &= \int_0^{x_0} r(t)\ \mathrm{d}t + \int_{x_0}^x r(t)\ \mathrm{d}t \\ &< \mathcal{O}_\epsilon(1) + \epsilon \int_{x_0}^x t\ \mathrm{d}t \\ &= \mathcal{O}_\epsilon(1) + \epsilon\left(\frac{x^2 - x_0^2}{2}\right) \end{aligned}$$

It is easy to show that $\frac{r_1(x)}{x^2} \to 0$ as $x \to \infty$. Hence $r_1(x) = o\left(\frac{1}{2}x^2\right)$, so $\psi_1(x) \sim \frac{1}{2}x^2$.

($\Leftarrow$). Assume $\psi_1(x) \sim \frac{1}{2}x^2$. Let $\beta > 1$, then

$$\psi(x) \leq \frac{1}{\beta x - x} \int_x^{\beta x} \psi(t)\ \mathrm{d}t$$

The RHS is the average of $\psi(t)$ on the inteval $[x, \beta x]$ and since ψ is increasing, this average is bounded below by the value at x. By the definition of $\psi_1(x)$ we have

$$\psi(x) \leq \frac{\psi_1(\beta x) - \psi_1(x)}{x(\beta - 1)}$$

Therefore

$$\frac{\psi(x)}{x} \leq \frac{1}{\beta - 1}\left(\frac{\psi_1(\beta x)}{x^2} - \frac{\psi_1(x)}{x^2}\right)$$

Since $\psi_1(x) \sim \frac{1}{2}x^2$, we know $\frac{\psi_1(x)}{x^2} \sim \frac{1}{2}$. Note

$$\frac{\psi_1(\beta x)}{x^2} = \frac{\psi_1(\beta x)\beta^2}{(\beta x)^2} \sim \frac{\beta^2}{2} \quad \text{and} \quad \frac{\psi_1(x)}{x^2} \sim \frac{1}{2}$$

Hence we have

$$\frac{\psi(x)}{x} \leq \frac{1}{\beta - 1}\left(\frac{\psi_1(\beta x)}{x^2} - \frac{\psi_1(x)}{x^2}\right) \sim \frac{1}{\beta - 1} \cdot \frac{\beta^2 - 1}{2} = \frac{\beta + 1}{2}$$

Similarly let $\alpha < 1$, consider

$$\psi(x) \geq \frac{1}{x - \alpha x} \int_{\alpha x}^{x} \psi(t)\ \mathrm{d}t = \frac{\psi_1(x) - \psi_1(\alpha x)}{x(1 - \alpha)}$$

Dividing by x again and use the same method as above

$$\frac{\psi(x)}{x} \geq \frac{1}{1 - \alpha}\left(\frac{\psi_1(x)}{x^2} - \frac{\psi_1(\alpha x)}{x^2}\right) \sim \frac{1}{1 - \alpha} \cdot \frac{1 - \alpha^2}{2} = \frac{1 + \alpha}{2}$$

Let $\alpha \to 1^-$ and $\beta \to 1^+$ to get

$$\frac{\psi(x)}{x} \sim 1 \quad \text{as } x \to \infty$$

This completes the proof. $\square$

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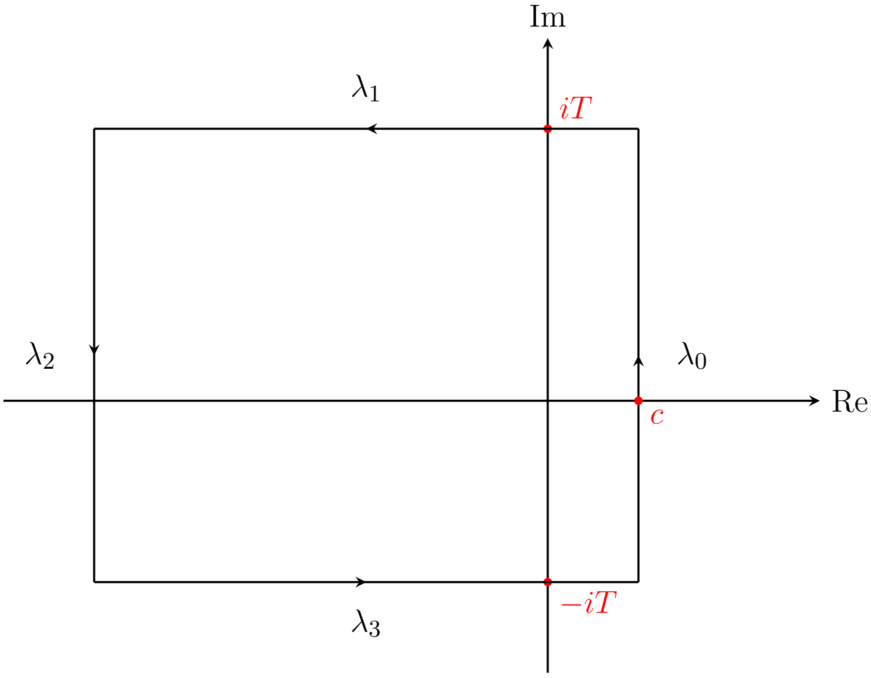
Lemma 5.9. Let $y > 0$ and $c > 0$ be real numbers. Then

$$\frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \frac{y^s}{s}\ \mathrm{d}s = \begin{cases} 1 & \text{if } y > 1 \\ 0 & \text{if } y < 1 \\ \frac{1}{2} & \text{if } y = 1 \end{cases}$$

Proof. The idea is the deform the contour to the left or right, depending on the value of y. The following diagram describes the contour in case 1.



Case 1. If $y > 1$ then y^s is huge when $\operatorname{Re} s$ is big. Hence we want to push the contour to the left. Fix $T > 0$ and let λ_0 be the path $c - iT \to c + iT$. Define $\lambda_1, \lambda_2, \lambda_3$ as well and let $\lambda = \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3$. The only pole of y^s/s is 0 and is enclosed by λ, hence

$$\frac{1}{2\pi i} \int_{\lambda} \frac{y^s}{s} \, \mathrm{d}s = \text{Residue at } 0 = 1$$

We know that

$$1 = \frac{1}{2\pi i} \int_{\lambda} \frac{y^s}{s} \, \mathrm{d}s = \frac{1}{2\pi i} \int_{\lambda_0} \frac{y^s}{s} \, \mathrm{d}s + \frac{1}{2\pi i} \int_{\lambda_1+\lambda_2+\lambda_3} \frac{y^s}{s} \, \mathrm{d}s$$

Since we are pushing the integral to the left, as T gets large we shall see that the integral over $\lambda_1, \lambda_2, \lambda_3$ are tending to 0, so the integral evalutes to 1 as desired. For λ_2 (the vertical) we have

$$\left| \frac{1}{2\pi i} \int_{\lambda_2} \frac{y^s}{s} \, \mathrm{d}s \right| = \frac{1}{2\pi} \left| \int_{-T}^{T} \frac{y^{-T+it}}{-T+it} \, \mathrm{d}t \right| \leq \frac{1}{2\pi} \frac{y^{-T}}{T} \cdot 2T \to 0 \text{ as } T \to \infty$$

The inequality is by the estimate $|\int_\gamma f| \leq \text{length}(\gamma) \cdot \sup |f|$. For λ_1

$$\left| \frac{1}{2\pi i} \int_{\lambda_1} \frac{y^s}{s} \, \mathrm{d}s \right| \leq \frac{1}{2\pi T} \int_{-T}^{c} y^\sigma \, \mathrm{d}\sigma = \frac{1}{2\pi T} \frac{1}{\log y} \left(y^c - y^{-T} \right) = \mathcal{O}_{c,y}\left(\frac{1}{T} \right) \to 0$$

This is exactly the same for λ_3. Hence

$$1 = \lim_{T\to\infty} \frac{1}{2\pi i}\int_\lambda \frac{y^s}{s}\,\mathrm{d}s = 0+0+0+\lim_{T\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{y^s}{s}\,\mathrm{d}s = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{y^s}{s}\,\mathrm{d}s$$

Case 2. If $y < 1$ then we do the same thing by pushing the integral to the right side. Then the contour does not enclose the pole 0, so the total integral is zero by Cauchy's theorem. The same estimate for $\lambda_1, \lambda_2, \lambda_3$ are true, so the integral on $\mathrm{Re} = c$ is 0.

Case 3. Let $y = 1$. Then we have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{s}\,\mathrm{ds}$$

Consider the line segment $c - iT \to c + iT$ and then take $T \to \infty$. Let $s = c + it$ for $t \in (-T, T)$, then we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{1}{s}\,\mathrm{ds} &= \frac{1}{2\pi}\int_{-T}^{T}\frac{\mathrm{d}t}{c+it} = \frac{1}{2\pi}\int_{-T}^{T}\frac{c-it}{c^2+t^2}\,\mathrm{d}t\\ &= \frac{1}{2\pi}\int_{-T}^{T}\frac{c}{c^2+t^2}\,\mathrm{d}t - \frac{i}{2\pi}\int_{-T}^{T}\frac{t}{c^2+t^2}\,\mathrm{d}t\end{aligned}$$

The second integral is 0 because the integrand is odd. And

$$\begin{aligned}A := \frac{1}{2\pi}\int_{-T}^{T}\frac{c}{c^2+t^2}\,\mathrm{d}t &= \frac{c}{2\pi c^2}\int_{-T}^{T}\frac{1}{1+(t/c)^2}\,\mathrm{d}t = \frac{1}{2\pi}\int_{-T/c}^{T/c}\frac{1}{1+x^2}\,\mathrm{d}x \qquad (x = t/c)\\ &= \frac{1}{\pi}\arctan\left(\frac{T}{c}\right) \to \frac{1}{\pi}\cdot\frac{\pi}{2} = \frac{1}{2} \text{ as } T\to\infty\end{aligned}$$

It follows that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{s}\,\mathrm{ds} = \frac{1}{2}$$

An alternative way to compute A is to construct a contour $\gamma(t) = Te^{it}$ for $0 \le t \le \pi$ and $[-T, T]$. Then there is a pole at ic enclosed by the contour. Then apply Residue theorem. The integral over the arc γ is tending to 0, so the residue is equal to the integral on $[-T, T]$, which is exactly the integral we want. □

Remark. We also use the following notation

$$\int_{(c)} f := \int_{c-i\infty}^{c+i\infty} f$$

for $c \in \mathbb{R}$ to denote the integral on the line $\mathrm{Re}\, z = c$.

Theorem 5.10. For $y > 0$ and $c > 0$ we have

$$\frac{1}{2\pi i}\int_{(c)} \frac{y^s}{s(s+1)}\,\mathrm{d}s = \begin{cases} 1-\frac{1}{y} & \text{if } y \geq 1 \\ 0 & \text{if } y \leq 1 \end{cases}$$

Proof. Similar to the lemma. The difference is that now we have two poles at $0, -1$. □

Now let's start the proof of the Prime Number Theorem. Recall that our goal is to show

$$\psi_1(x) = \sum_{n\leq x} \Lambda(n)(x-n) \sim \frac{x^2}{2}$$

Using Theorem 5.10 by setting $c > 1$ and $y = x/n \geq 1$ we have

$$\psi_1(x) = \sum_{n\leq x} \Lambda(n)(x-n) = x\sum_{n\leq x}\Lambda(n)\left(1-\frac{n}{x}\right) = x\sum_{n=1}^{\infty}\frac{\Lambda(n)}{2\pi i}\int_{(c)}\frac{(x/n)^s}{s(s+1)}\,\mathrm{d}s$$

We change the sum from $n \to \infty$ because

$$\frac{\Lambda(n)}{2\pi i}\int_{(c)}\frac{(x/n)^s}{s(s+1)}\,\mathrm{d}s = \begin{cases} \Lambda(n)\left(1-\frac{n}{x}\right) & \text{if } n \leq x \\ 0 & \text{if } n \geq x \end{cases}$$

By swapping the summation and integral

$$\psi_1(x) = x\sum_{n=1}^{\infty}\frac{\Lambda(n)}{2\pi i}\int_{(c)}\frac{(x/n)^s}{s(s+1)}\,\mathrm{d}s = \frac{x}{2\pi i}\int_{(c)}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\frac{x^s}{s(s+1)}\,\mathrm{d}s$$

We can do this because we know $\sum_{n\geq 1}\frac{\Lambda(n)}{n^s}$ has Abscissa of convergence for $\operatorname{Re} s > 1$ and we know $c > 1$, so this series converge absolutely on the c-line.

Lecture 16, 2025/11/04

Our goal in this lecture is to prove the Prime Number Theorem $\psi_1(x) \sim \frac{1}{2}x^2$ as $x \to \infty$, which is equivalent to prove that for all $\epsilon > 0$

$$\left|\frac{\psi(x)}{x} - \frac{1}{2}\right| < \epsilon$$

for all x sufficiently large.

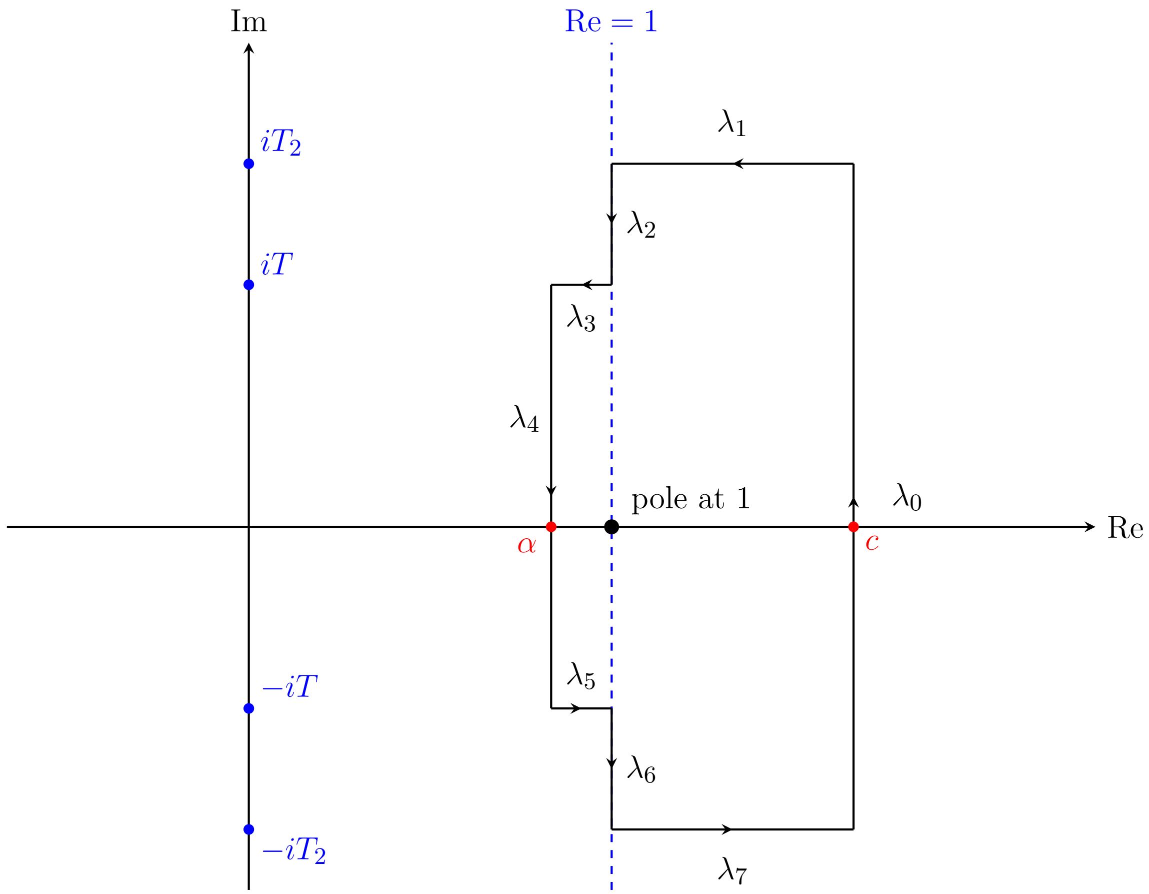
Proof of Prime Number Theorem. In the above setting, we set $c > 1$ and then

$$\begin{aligned}\frac{1}{x^2}\psi_1(x) &= \frac{1}{x^2}\frac{x}{2\pi i}\int_{(c)}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\frac{x^s}{s(s+1)}\,\mathrm{d}s \\ &= -\frac{1}{2\pi i}\int_{(c)}\frac{\zeta'(s)}{\zeta(s)}\frac{x^{s-1}}{s(s+1)}\,\mathrm{d}s\end{aligned}$$

Now we want to use the property of $\zeta(s)$ to study this integral. We want the RHS to be something about $1/2$, and where can it possibly come from? Note that the poles of ζ'/ζ come from zero and poles of ζ and ζ' has a pole of order 2 at $s=1$ because ζ has a pole of order 1 at $s=1$. Note that

$$\operatorname{Res}\left(-\frac{\zeta'(s)}{\zeta(s)}\frac{x^{s-1}}{s(s+1)}, s=1\right)=\frac{1}{2}$$

This pole comes from a pole of ζ, what about zeros of ζ? We claim that ζ has no zeros on the boundary $\operatorname{Re} s = 0, 1$! In fact there is no zero in a small strip $\{\alpha < \operatorname{Re} s \leq 1\}$, so we can consider the following contour. Here T_2 can depend on x and T is fixed.



We let $T_2 \to \infty$ and that

$$-\frac{1}{2\pi i}\int_{\lambda_0}\frac{\zeta'(s)}{\zeta(s)}\frac{x^{s-1}}{s(s+1)}\,\mathrm{ds} \to \frac{1}{x^2}\psi_1(x)$$

We will show that

$$-\frac{1}{2\pi i}\int_{\lambda_1+\cdots+\lambda_7}\frac{\zeta'(s)}{\zeta(s)}\frac{x^{s-1}}{s(s+1)}\,\mathrm{ds} \to 0 \text{ as } x \to \infty$$

The only pole λ enclose is $s = 1$ with residue $1/2$, so the residue theorem tells us

$$-\frac{1}{2\pi i}\int_{\lambda}\frac{\zeta'(s)}{\zeta(s)}\frac{x^{s-1}}{s(s+1)}\,\mathrm{ds} = \frac{1}{2}$$

Taking $T_2 \to \infty$ and $x \to \infty$ yields

$$\frac{1}{x^2}\psi_1(x) \sim \frac{1}{2}$$

This will complete the proof. It remains to prove the claims.

Theorem 5.11. Write $s = \sigma + it$. Assume $\sigma \geq 1$ and $|t| \geq 2$, then

1. We have $\zeta'(s) = \mathcal{O}((\log|t|)^2)$.
2. We have $\zeta(s) \neq 0$ and $\frac{1}{\zeta(s)} = \mathcal{O}((\log|t|)^7)$.

Part A. As $T_2 \to \infty$, using (1) and (2) of Theorem 5.11, for given $c > 1$ and $\epsilon > 0$ and x,

$$\left|-\frac{1}{2\pi i}\int_{\lambda_1}\frac{\zeta'(s)}{\zeta(s)}\frac{x^{s-1}}{s(s+1)}\,\mathrm{ds}\right| < \frac{\epsilon}{1000}$$

for all T_2 sufficiently large with respect to x and c. It is because

$$\frac{\zeta'(s)}{\zeta(s)s(s+1)} = \mathcal{O}\left(\frac{(\log T_2)^{7+2}}{T_2^2}\right)$$

Choose $c = 2$ and $T_2(x)$ so that this integrand is $< \epsilon/1000$. The same works for λ_7.

Part B. Consider λ_2 or λ_6. On these two lines $s = 1 + it$ and $|x^{s-1}| = 1$ and thus

$$\frac{\zeta'(s)}{\zeta(s)}\frac{1}{s(s+1)} = \mathcal{O}\left(\frac{(\log|t|)^{7+2}}{t^2}\right)$$

by (1) and (2) of Theorem 5.11. Then

$$-\frac{1}{2\pi i}\int_{\lambda_2}\frac{\zeta'(s)}{\zeta(s)}\frac{x^{s-1}}{s(s+1)}\,\mathrm{d}s \text{ is convergent as } T_2 \to \infty \text{ and bounded with respect to } x$$

Can make LHS as small as we wish (for all x), say $< \frac{\epsilon}{1000}$ by taking T sufficiently large and independent of x.

Part C. Consider λ_3, λ_5. Let M be the maximum of

$$\left|\frac{1}{2\pi}\frac{\zeta'(s)}{\zeta(s)}\frac{1}{s(s+1)}\right| \text{ on } \lambda_3 + \lambda_4 + \lambda_5$$

Therefore we have

$$\begin{aligned}\left|\frac{-1}{2\pi i}\int_{\lambda_3}\frac{\zeta'(s)}{\zeta(s)}\frac{x^{s-1}}{s(s+1)}\,\mathrm{d}s\right| &\le M\int_\alpha^1 x^{\sigma-1}\,\mathrm{d}\sigma = \frac{M}{x}\int_\alpha^1 x^\sigma\,\mathrm{d}\sigma\\ &= \frac{M}{x}\left(\frac{x-e^{\alpha\log x}}{\log x}\right) < \frac{M}{\log x}\end{aligned}$$

For given T and α, we know M is fixed. We can make the RHS $< \epsilon/1000$ for all x sufficiently large (for given $\epsilon > 0$). The same estimate works for λ_5 as well.

Part D. Finally consider λ_4. We have

$$\left|\frac{-1}{2\pi i}\int_{\lambda_4}\frac{\zeta'(s)}{\zeta(s)}\frac{x^{s-1}}{s(s+1)}\,\mathrm{d}s\right| \le \text{length}(\lambda_4)\cdot M = (Mx^{\alpha-1})\cdot 2T$$

Here $|x^{s-1}| = x^{\alpha-1}$ is fixed and $Mx^{\alpha-1}$ is the maximum of the interand on λ_4. Hence we can make the RHS $< \epsilon/1000$ for all x sufficiently large (for given $\epsilon > 0$).

Combining the four parts, we proved that the integral on $\lambda_1 + \cdots + \lambda_7$ can be made sufficiently small as $x, T_2(x) \to \infty$. This completes the proof of PNT (assuming Theorem 5.11). ☐

Lecture 17, 2025/11/06

We will prove part (a) of Theorem 5.11 as part of the following theorem

Theorem 5.12. Theorem 5.12 For $s = \sigma + it$ and $\sigma \ge 1$ and $t \ge 2$, then

$$\zeta(s) = \mathcal{O}(\log t) \quad\text{and}\quad \zeta'(s) = \mathcal{O}((\log t)^2)$$

Proof. Recall that for $\operatorname{Re} s > 0$ we have

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}}\,\mathrm{d}t \tag{1}$$

Furthermore, recall (from Lecture 3) by partial summation we have

$$\begin{aligned}\sum_{n\le X}\frac{1}{n^s} &= \frac{\lfloor X\rfloor}{X^s} + s\int_1^X \frac{\lfloor t\rfloor}{t^{s+1}}\,\mathrm{d}s\\ &= \frac{s}{s-1} - \frac{s}{s-1}\frac{1}{X^{s-1}} - s\int_1^X\frac{\{t\}}{t^{s+1}}\,\mathrm{d}t + \frac{1}{X^{s-1}} - \frac{\{X\}}{X^s}\\ &= \frac{s}{s-1} - \frac{1}{(s-1)X^{s-1}} - s\int_1^X\frac{\{t\}}{t^{s+1}}\,\mathrm{d}t - \frac{\{X\}}{X^s}\end{aligned} \tag{2}$$

Let's compare (1) and (2): the limit and the partial sum. Consider (1) − (2), we have

$$\zeta(s) - \sum_{n \le X} \frac{1}{n^s} = -s \int_X^\infty \frac{\{t\}}{t^{s+1}} \, \mathrm{d}t + \frac{1}{(s-1)X^{s-1}} + \frac{\{X\}}{X^s} \tag{3}$$

Remember, our goal is to estimate $\zeta(s)$ on the region $\sigma \ge 1$ and $t \ge 2$. Let's look at each term.

1. If $\sigma \ge 1$ then $\dfrac{\{X\}}{X^s} = \mathcal{O}\left(\dfrac{1}{X}\right)$ for any $t \in \mathbb{R}$.
2. If $\sigma \ge 1$ and $t \ge 0$ then $\dfrac{1}{(s-1)X^{s-1}} = \mathcal{O}\left(\dfrac{1}{t}\right)$.
3. For the first term, if $\sigma > 0$ and $t \in \mathbb{R}$ then

$$\left| -s \int_X^\infty \frac{\{y\}}{y^{s+1}} \, \mathrm{d}y \right| \le |s| \int_X^\infty \frac{1}{y^{\sigma+1}} \, \mathrm{d}y = \frac{|s|}{-\sigma} y^{-\sigma} \Big|_X^\infty = \frac{|s|}{\sigma} X^{-\sigma}$$

Thus for $\sigma \ge 1$ and $t \ge 2$ we have

$$\zeta(s) = \sum_{n \le X} \frac{1}{n^s} + \mathcal{O}\left(\frac{|s|}{\sigma X^\sigma} + \frac{1}{t} + \frac{1}{X} \right)$$

Note that $|\sum_{n \le X} \frac{1}{n^s}| \le \sum_{n \le X} \frac{1}{n^\sigma} = \mathcal{O}(\log X)$ because $\sigma \ge 1$. Note

$$\frac{|s|}{\sigma} = \frac{\sqrt{\sigma^2 + t^2}}{\sigma} = \sqrt{1 + \left(\frac{t}{\sigma}\right)^2} \le 1 + \frac{t}{\sigma} = \mathcal{O}(t)$$

Therefore we have $\frac{|s|}{\sigma X^\sigma} = \mathcal{O}\left(\frac{t}{X}\right)$ for $\sigma \ge 1$ and $t \ge 2$. Set $X = t$, then

$$\zeta(s) = \mathcal{O}(\log t) + \mathcal{O}\left(\frac{t}{t} + \frac{1}{t} + \frac{1}{t} \right) = \mathcal{O}(\log t)$$

Differentiating (3) with respect to s gives

$$\begin{aligned}
\zeta'(s) &= -\sum_{n \le X} \frac{\log n}{n^s} - \int_X^\infty \frac{\{t\}}{t^{s+1}} \, \mathrm{d}t + s \int_X^\infty \frac{\{t\} \log t}{t^{s+1}} \, \mathrm{d}t \\
&= \mathcal{O}((\log X)^2) + \mathcal{O}\left(\frac{1}{X}\right) + \mathcal{O}\left(\frac{|s|}{\sigma X^\sigma} \log X \right) \\
&= \mathcal{O}((\log t)^2)
\end{aligned}$$

where last step is by setting $X = t$. This completes the proof. □

Proof of Theorem 5.11. We already did part (a), so it suffices to show part (b), which is the difficult part. The following is Hadamard's approach. The important inequality

$$3 + 4\cos\theta + \cos 2\theta \ge 0$$

It is because $3 + 4\cos\theta + \cos 2\theta = 2 + 4\cos\theta + 2\cos^2\theta = (2 + 1 + \cos\theta)^2 \geq 0$. Recall

$$\log\zeta(\sigma + it) = \sum_p \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{p^{m\sigma + imt}} = \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{m\sigma}} \exp(-imt\log p)$$

Exponentiating both sides and rewrite $\exp(-imt\log p)$ in standard form

$$\zeta(\sigma + it) = \exp\left(\sum_p \sum_{m=1}^{\infty} \frac{\cos(mt\log p) - i\sin(mt\log p)}{mp^{m\sigma}}\right)$$

Hence

$$|\zeta(\sigma + it)| = \exp\left(\sum_p \sum_{m=1}^{\infty} \frac{\cos(mt\log p)}{mp^{m\sigma}}\right)$$

Lecture 18, 2025/11/11

We will prove $\zeta(1 + it) \neq 0$ for all $t \in \mathbb{R}$. Assume $\zeta(1 + it_0) = 0$ for some $0 \neq t_0 \in \mathbb{R}$, and we want to get some contradiction. We want to use that magic inequality. The idea is that we can multiply $|\zeta(\sigma + it)|$'s together for different t. Consider the expression

$$\zeta(\sigma)^3\zeta(\sigma + it)^4\zeta(\sigma + 2it)$$

For $\sigma > 1$ and $t \in \mathbb{R}$, in modulus we have

$$|\zeta(\sigma)^3\zeta(\sigma + it)^4\zeta(\sigma + 2it)| = \exp\left(\sum_p \sum_{m=1}^{\infty} \frac{3 + 4\cos(mt\log p) + \cos(2mt\log p)}{mp^{m\sigma}}\right)$$

By the magic inequality, we know $3 + 4\cos(mt\log p) + \cos(2mt\log p) \geq 0$, so the RHS ≥ 1.

$$\text{For } \sigma > 1 \text{ and } t \in \mathbb{R} \quad |\zeta(\sigma)^3\zeta(\sigma + it)^4\zeta(\sigma + 2it)| \geq 1 \tag{4}$$

However, since $\zeta(1 + it_0) = 0$ consider the expression

$$\zeta(\sigma)^3\zeta(\sigma + it_0)^4\zeta(\sigma + 2it_0)$$

We know $\zeta(\sigma)^3$ has a pole of order 3 at $\sigma = 1$ and $\zeta(\sigma + it_0)^4$ has a zero of order at least 4 at $\sigma = 1$. Finally $\zeta(\sigma + 2i)$ has order ≥ 0 (not pole, maybe zero, we don't care). Together it means

$$\lim_{\sigma \to 1^+} \zeta(\sigma)^3\zeta(\sigma + it_0)^4\zeta(\sigma + 2it_0) = 0$$

This contradicts (4). Hence $\zeta(1 + it) \neq 0$ for all $t \in \mathbb{R}$. Now it remains to prove

$$\frac{1}{\zeta(s)} = \mathcal{O}((\log t)^7) \quad \text{for } \sigma \geq 1 \text{ and } t \geq 2$$

If $\sigma \geq 2$ we want to show $\zeta(s)^{-1}$ is bounded.

$$\begin{aligned}\left|\frac{1}{\zeta(s)}\right| = \left|\prod_p\left(1-\frac{1}{p^s}\right)\right| &\leq \prod_p\left(1+\frac{1}{p^\sigma}\right) \leq \prod_p\left(1+\frac{1}{p^\sigma}+\frac{1}{p^{2\sigma}}+\cdots\right)\\ &= \zeta(\sigma) = 1+\frac{1}{2^\sigma}+\frac{1}{3^\sigma}+\cdots \leq 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+ = \zeta(2)\end{aligned}$$

Hence $\zeta(s)^{-1} = \mathcal{O}(1)$ when $\sigma \geq 2$ and $t \in \mathbb{R}$. Next, let $1 < \sigma \leq 2$. Then by (4)

$$\underbrace{(\sigma-1)^3\zeta(\sigma)^3}_{(*)}|\zeta(\sigma+it)|^4\underbrace{|\zeta(\sigma+2it)|}_{(**)} \geq (\sigma-1)^3$$

We know $(*)$ is bounded on $(1,2]$ because ζ has a simple pole at 1 so $(\sigma-1)\zeta(\sigma)$ cancels the pole. Hence $(\sigma-1)\zeta(\sigma)$ is continuous on $(1,2]$ and $(\sigma-1)\zeta(\sigma) \to 1$ as $\sigma \to 1^+$.

We know $(**)$ is $\mathcal{O}(\log t)$ by Theorem 5.12 for $\sigma \geq 1$ and $t \geq 2$, so there is $a > 0$ such that

$$|\zeta(1+2it)| < a\log t \quad \text{for all } \sigma \geq 1 \text{ and } t \geq 2$$

Thus for some $\kappa > 0$ we have

$$|\zeta(\sigma+it)| \geq \frac{\kappa(\sigma-1)^{3/4}}{(\log t)^{1/4}} \quad \text{for all } 1 < \sigma \leq 2$$

We want to estimate ζ^{-1}, so we have to flip this inequality. But the thing is, as σ gets to close to 0, the $(\sigma-1)^{3/4}$ on the denominator will blow up. Hence we want to consider two cases separately. First we let $\sigma \geq 1 + B(\log t)^{-9} =: \eta$, where B will be chosen later. Then

$$|\zeta(\sigma+it)| \geq \frac{\kappa B^{3/4}}{(\log t)^7} \tag{5}$$

so that for $1 + B(\log t)^{-9} < \sigma \leq 2$ and $t \geq 2$ we have

$$\left|\frac{1}{\zeta(\sigma+it)}\right| \leq \frac{1}{\kappa B^{3/4}}(\log t)^7$$

Next let $1 \leq \sigma \leq 1 + \eta$. Consider

$$|\zeta(\eta+it)-\zeta(\sigma+it)| = \left|\int_\sigma^\eta \zeta'(u+it)\ \mathrm{d}u\right|$$

Since $\zeta'(u+it) = \mathcal{O}((\log t)^2)$ there is $\kappa_2 > 0$ such that $|\zeta'(t+it)| \leq \kappa(\log t)^2$. We have

$$|\zeta(\eta+it)-\zeta(\sigma+it)| \leq \kappa_2(\log t)^2(\eta-\sigma) \leq \kappa_2(\log t)^2(\eta-1) = \frac{\kappa_2 B}{(\log t)^7}$$

The reverse triangle inequlity says $|z_2 - z_1| \geq |z_2| - |z_1|$. Therefore

$$|\zeta(\eta + it)| - |\zeta(\sigma + it)| \leq |\zeta(\eta + it) - \zeta(\sigma + it)|$$

Hence we have

$$|\zeta(\sigma + it)| \geq |\zeta(\eta + it)| - \frac{\kappa_2 B}{(\log t)^7} \geq \frac{\kappa B^{3/4}}{(\log t)^7} - \frac{\kappa_2 B}{(\log t)^7}$$

by (5) applying to $\eta = \sigma$. Now it's the time to choose that $B > 0$. Choose it so that

$$\kappa B^{3/4} = 2\kappa_2 B$$

More explicitly we let $B = (\kappa/2\kappa_2)^4 > 0$. Therefore

$$|\zeta(\sigma + it)| \geq \frac{\kappa B^{3/4}}{(\log t)^7} - \frac{\kappa_2 B}{(\log t)^7} = \frac{\kappa_2 B}{(\log t)^7}$$

Thus for $1 \leq \sigma \leq 1 + B(\log t)^{-9} = \eta$ and $t \geq 2$ we have

$$\left|\frac{1}{\zeta(\sigma + it)}\right| = \mathcal{O}((\log t)^7)$$

Combine this at the previous result for $\sigma \geq \eta$, we conclude that

$$\frac{1}{\zeta(\sigma + it)} = \mathcal{O}((\log t)^7)$$

for $\sigma \geq 1$ and $t \geq 2$. □

Theorem 5.13 (Explicit Formula).

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \sum_{m=1}^{\infty} \frac{x^{-2m}}{2m} - \frac{\zeta'(0)}{\zeta(0)}$$

where the sum $\sum_\rho$ is over all the non-trivial zeros of ζ.

Lecture 19, 2025/11/13

6 Dirichlet's Theorem

Example. There are infinitely many primes of the from $4k + 1$.

$$\{4k + 1 : k \in \mathbb{N}\} = \{1, 5, 9, 13, 21, 25, 29, 33, 37, 41, 45, 49, 53, 57, \cdots\}$$

There are also infinitely many primes of the from $4k + 3$.

Remark. Recall how Euclid proved the infinitude of primes: suppose finitely many and take their product and plus 1. This method can be used to prove the above example, but it fails in general.

In general, Dirichlet proved that

Theorem 6.1 (Dirichlet's Theorem). If $q \geq 2$ and $a > 0$ with $\gcd(a, q) = 1$, then

$$\{qk + a : k \in \mathbb{N}\} \text{ contains infinitely many primes}$$

How Euler proved the infinitude of primes is that

$$\sum_{n=1} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

If $s \to 1^+$ then the LHS tends to ∞, so there must be infinitely many prime. Dirichlet proved that

$$\sum_{p \equiv a \bmod q} \frac{1}{p} = \infty$$

Therefore there must be infinitely many primes $p \equiv a \pmod q$. The main tool to prove this is the Dirichlet characters.

6.1 Idea of the Proof

Example. Consider the $4k+1$ case. We consider something called the Dirichlet character (we will see the formal defintion later, the idea is that it "picks out" certain residue classes) $\chi : (\mathbb{Z}/4\mathbb{Z})^\times \to \mathbb{C}$.

	0	1	2	3
χ_1	0	1	0	1
χ_2	0	1	0	-1

where $\chi_1(1) = 1$ and $\chi_1(3) = 1$ and $\chi_2(1) = 1$ and $\chi_2(3) = -1$. We extend these to $\chi_i : \mathbb{Z} \to \mathbb{C}$ by periodicity: Set $\chi_i(n) = 0$ if $4 \mid n$ and otherwise

$$\chi_i(n) := \chi_i(n \pmod 4)$$

Consider

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

In this cases for χ_1 and χ_2 we have

$$L(s, \chi_1) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \cdots$$
$$L(s, \chi_2) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \frac{1}{9^s} - \cdots$$

Note that $L(s,\chi_1)$ converges absolutely for $\operatorname{Re} s > 1$. Also $L(s,\chi_2)$ converges absolutely for $\operatorname{Re} s > 1$ and conditionally for $\operatorname{Re} s > 0$. By Partial Summation

$$\sum_{n\le N}\frac{\chi_2(n)}{n^s} = \frac{1}{N^s}\sum_{n\le N}\chi_2(n) + s\int_1^N \frac{1}{t^{s+1}}\sum_{n\le t}\chi_2(n)\ \mathrm{d}t$$

Note that $\sum_{n\le N}\chi_2(n)$ is bounded (some cancellation). Hence taking $N\to\infty$ we have

$$\text{RHS} \to s\int_1^\infty \frac{1}{t^{s+1}}\sum_{n\le t}\chi_2(n)\ \mathrm{d}t$$

This gives an analytic continuation of $L(s,\chi_2)$ to $\operatorname{Re} s > 0$. The same trick does not work for $L(s,\chi_1)$ because there are no "cancellation". However

$$L(s,\chi_1) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots - \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \cdots\right) = \zeta(s)\left(1 - \frac{1}{2^s}\right) \tag{$*$}$$

Note that χ_1 and χ_2 are completely multiplicative. This means we can do Euler product! In general any Dirichlet character χ is completely multiplicative and

$$L(s,\chi) = \sum_{n=1}^\infty \frac{\chi(n)}{n^s} = \prod_p\left(1 + \frac{\chi(p)}{p^s} + \frac{\chi(p)^2}{p^{2s}} + \cdots\right) = \prod_p\left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \tag{1}$$

In the case $\chi = \chi_1$, we know $\chi(2) = 0$ and $\chi(p) = 1$ for all p.

$$L(s,\chi_1) = \prod_{p\text{ odd}}\left(1 - \frac{1}{p^s}\right)^{-1} = \zeta(s)\left(1 - \frac{1}{2^s}\right)$$

This gives another way to get the equality $(*)$. Taking the logarithm of (1)

$$\begin{aligned}\log L(s,\chi) &= -\sum_p \log\left(1 - \frac{\chi(p)}{p^s}\right) = \sum_p\sum_{k=1}^\infty \frac{\chi(p)^k}{kp^{ks}} \\ &= \underbrace{\sum_p \frac{\chi(p)}{p^s}}_{\text{want}} + \underbrace{\sum_p\sum_{k\ge 2}\frac{\chi(p)^k}{kp^{ks}}}_{\text{We will show }\mathcal{O}(1)}\end{aligned}$$

Recall that we want to study the sum $\sum_{p\equiv 1 \pmod 4}\frac{1}{p}$ but this sum $\sum_p \frac{\chi(p)}{p^s}$ contains more than this. What we can do it to take the linear combination of different $\log L(s,\chi)$'s to just get $\sum_{p\equiv 1 \pmod 4}$. Indeed, for $\operatorname{Re} s > 1$ we have

$$\log L(s,\chi_1) + \log L(s,\chi_2) = \sum_p \frac{\chi_1(p) + \chi_2(p)}{p^s} + \mathcal{O}(1)$$

Note that we have

$$\chi_1(p) + \chi_2(p) = \begin{cases} 2 & \text{if } p \equiv 1 \pmod 4 \\ 0 & \text{if } p \equiv 3 \pmod 4 \end{cases}$$

Therefore we have

$$\frac{\log L(s,\chi_1) + \log L(s,\chi_2)}{2} = \sum_{p \equiv 1 \pmod 4} \frac{1}{p^s} + \mathcal{O}(1)$$

Note that since the harmonic series diverges, we know

$$\lim_{s\to 1^+} \log L(s,\chi_1) = \lim_{s\to 1^+} \log\left(1 + \frac{1}{3^s} + \frac{1}{5^s} + \cdots\right) = \infty$$

On the other hand $\log L(s,\chi_2)$ tends to a constant as $s \to 1^+$.

$$L(1,\chi_2) = \int_1^\infty \frac{1}{t^2} \sum_{n\le t} \chi_2(n) \quad \text{by the partial summation}$$

Or we can just compute the it using definition

$$L(1,\chi_2) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

The constant here doesn't really matter, we just need to know $\log L(s,\chi_2)$ tends to a constant. Anyway, combining all of these tells us

$$\infty = \lim_{s\to 1^+} \frac{\log L(s,\chi_1) + \log L(s,\chi_2)}{2} = \lim_{s\to 1^+} \sum_{p \equiv 1 \pmod 4} \frac{1}{p^s} + \mathcal{O}(1)$$

It follows that there are infinitely many primes $p \equiv 1 \pmod 4$.

The same idea applies to the $3 \pmod 4$ case, we want a linear combination of χ_1 and χ_2 that cancels the $1 \pmod 4$ residue classes. Take $\chi_1 - \chi_2$. Then

$$\frac{\log L(s,\chi_1) - \log L(s,\chi_2)}{2} = \sum_{p \equiv 3 \pmod 4} \frac{1}{p^s} + \mathcal{O}(1)$$

Therefore there are infinitely many priems $p \equiv 3 \pmod 4$.

Remark. What could go wrong in the general case? What do we need to modify in the general? In general mod q there are $\varphi(q)$ many Dirichlet characters, given by

$$\chi_i : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}$$

and χ_i's are group homomorphisms. In the above case it is easy to "guess" how to take linear combinations to cancel out certain residue classes. In general this can also be done and we will see it later. In the proof above we know

$$\lim_{s\to 1^+} L(s,\chi_2) = \alpha \in \mathbb{C}$$

exists. But what if $\alpha = 0$? Then we cannot take $\log L(s, \chi_2)$. It looks like $\alpha \neq 0$ because we have $1 - 1/3 + 1/5 + \cdots$ and it is an alternating series that does not go to 0. However, in general $\chi(n)$ can be some non-real complex numbers that are root of unities. So potentially $\alpha = 0$ is possible. But luckily this never happens! We will show it cannot be zero

$$L(1, \chi) \neq 0$$

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6.2 Dirichlet Characters

Definition. Let $q \geq 2$ be an integer. A homomorphism (of multiplicative groups)

$$\chi : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$$

is called a **Dirichlet character** of $(\mathbb{Z}/q\mathbb{Z})^\times$. This means $\chi(ab) = \chi(a)\chi(b)$. We call the trivial character χ_0 the **principal character**.

Example. For $q = 10$ we have $(\mathbb{Z}/10\mathbb{Z})^\times = \{1, 3, 7, 9\}$. In general $|(\mathbb{Z}/q\mathbb{Z})^\times| = \varphi(q)$.

By our discussion in the previous section, we know the Dirichlet's Theorem is implied by the following result: Let $a \in \mathbb{Z}$ and $q \geq 2$ and $\gcd(a, q) = 1$, then

$$\sum_{p = a \bmod q} \frac{1}{p} = \infty$$

We will take a general approach that can be applied to any finite abelian groups.

Example. Let $q = 8$ then $(\mathbb{Z}/8\mathbb{Z})^\times = \{1, 3, 5, 7\}$. Clearly we know $\chi(1) = 1$ for any character χ. Note $3^2 = 5^2 = 7^2 = 1 \pmod 8$, so $\chi(3)^2 = 1$ and $\chi(3) \in \{\pm 1\}$. Simialrly $\chi(5), \chi(7) \in \{\pm 1\}$. Moreover we note that $7 = 3$ in $(\mathbb{Z}/8\mathbb{Z})^\times$, so the value of $\chi(7)$ depends on $\chi(3)$ and $\chi(5)$. There are four possibilities for $(\chi(3), \chi(5)) \in \{\pm 1\}^2$.

	1	3	5	7
χ_0	1	1	1	1
χ_1	1	-1	1	-1
χ_2	1	1	-1	-1
χ_3	1	-1	-1	1

Take the entires in the character table and make them in a matrix, we see that

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

This means the rows and columns are orthogonal! (at least for distinct rows/columns) and the dot product of a row with itself is $4 = \varphi(8)$. We will prove this in general.

Note that for $q \geq 2$ integer we have
$$a^{\varphi(q)} = 1 \pmod q$$
Therefore we have $\chi(a)^{\varphi(q)} = \chi(a^{\phi(q)}) = \chi(1) = 1$, hence each $\chi(a)$ is a $\varphi(q)$-th root of unity. Fix $a \in (\mathbb{Z}/q\mathbb{Z})^\times$, then let r be the smallest postive integer such that $a^r = 1 \pmod q$. Then $r \mid \varphi(q)$ and we know $\chi(a)$ is an r-th root of unity.

Example. Let $q = 20$ then $(\mathbb{Z}/20\mathbb{Z})^\times = \{1, 3, 7, 9, 11, 13, 17, 19\}$ and $\varphi(20) = 8$. Note

$$3^4 = 1 \pmod{20} \quad \text{and} \quad 3^k \neq 1 \pmod{20} \quad \text{for } k = 1, 2$$

Hence 4 is the order of 3 mod 20. This means $\chi(3)$ is a 4-th root of unity, so $\chi(3) \in \{1, -1, i, -i\}$. Note that $3 \cdot 7 = 1 \pmod{20}$, so $\chi(7) = \chi(3)^{-1}$. Also note $3^2 = 9$, so

	1	3	7	9
χ_0	1	1	1	1
χ_1	1	i	$-i$	-1
χ_2	1	-1	-1	1
χ_3	1	$-i$	i	-1

This part of the table is given the subgroup generated by 3. Now let's look at 11. Note $11^2 = 1 \pmod{20}$, so $\chi(11) \in \{1, -1\}$. Note $13 = 3 \cdot 11$ and $17 = 7 \cdot 11$ and $19 = 9 \cdot 11$, so we have the table

	1	3	7	9	11	13	17	19
χ_0	1	1	1	1	1	1	1	1
χ_1	1	i	$-i$	-1	1	i	$-i$	-1
χ_2	1	-1	-1	1	1	-1	-1	1
χ_3	1	$-i$	i	-1	1	$-i$	i	-1
χ_4	1	1	1	1	-1	-1	-1	-1
χ_5	1	i	$-i$	-1	-1	$-i$	i	1
χ_6	1	-1	-1	1	-1	1	-1	-1
χ_7	1	$-i$	i	-1	-1	i	$-i$	1

Algorithm 6.2. Generally, here is an algorithm to find all characters mod q.

(i). We know $\chi(1) = 1$. Let $G_1 = \{1\}$.

(ii). Given G_r for $r \geq 1$, take any $a \in (\mathbb{Z}/q\mathbb{Z})^\times$ not in G_r. Consider powers of a mod q. Let h be the smallest positive integer such that $a^h = g \in G_r$. Thus $\chi(a)^h = \chi(g)$. Hence $\chi(a)$ is an h-th root of $\chi(g)$. Therefore h possible choices for $\chi(a)$.

(iii). Let $G_{r+1} = \langle G_r, a\rangle$. Then $|G_{r+1}| = h \cdot |G_r|$. Let $u \in G_r$ and $v = a^k$ for $k = 0, \cdots, h-1$. Set

$$\chi(uv) = \chi(u)\chi(a)^k$$

where $\chi(u)$ is an h-th root of unity of $\chi(g)$.

Example. Take $(\mathbb{Z}/20\mathbb{Z})^\times$ again. We have $G_1 = \{1\}$. Then $3 \notin G_1$ and $3^4 = 1 \in G_1$. Then we have $G_2 = \langle 1, 3\rangle = \{1, 3, 7, 9\}$. For G_3 we note that $13 \notin G_2$ and $13^2 = 9 \in G_2$. Then

$$G_3 = \langle G_2, 13\rangle = \{1, 3, 7, 9, 1\cdot 13, 3\cdot 13, 7\cdot 13, 9\cdot 13\} = (\mathbb{Z}/20\mathbb{Z})^\times$$

Here $|G_3| = 2 \cdot |G_2|$ where $13^2 = 9$ is the h as in the algorithm.

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Remark. We can extend $\chi : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$ to the map

$$\chi : \mathbb{Z} \to \mathbb{C}^\times$$

by setting $\chi(n) := \chi(n + q\mathbb{Z})$ if $\gcd(n, q) = 1$ and $\chi(n) := 0$ if $\gcd(n, q) \neq 1$. Then χ is completely multiplicative. The notation

$$\sum_{n \bmod q}$$

means the sum over all $n \in (\mathbb{Z}/q\mathbb{Z})^\times$.

Proposition 6.3. Let $q \geq 2$ be an integer. The number of Dirichlet characters of $(\mathbb{Z}/q\mathbb{Z})^\times$ is $\varphi(q)$. Let χ be a Dirichlet charter, then we have

$$\sum_{n \bmod q} \chi(n) = \begin{cases} \varphi(q) & \text{if } \chi = \chi_0 \\ 0 & \text{if } \chi \neq \chi_0 \end{cases}$$

Proof. The number of Dirichlet characters follows from Algorithm 6.2. Now we assume $\chi = \chi_0$. This means

$$\sum_{n \bmod q} \chi(n) = \sum_{n \bmod q} 1 = |(\mathbb{Z}/q\mathbb{Z})^\times| = \varphi(q)$$

Assume $\chi \neq \chi_0$, then there exists $m \in (\mathbb{Z}/q\mathbb{Z})^\times$ such that $\chi(m) \neq 1$. Then

$$\sum_{n \bmod q} \chi(n) = \sum_{mn \bmod q} \chi(mn) = \sum_{n \bmod q} \chi(mn) = \chi(m) \sum_{n \bmod q} \chi(n)$$

The multiplication by m on $(\mathbb{Z}/q\mathbb{Z})^\times$ is a isomorphism, so $m(\mathbb{Z}/q\mathbb{Z})^\times = (\mathbb{Z}/q\mathbb{Z})^\times$, which explains the red equality. Since $\chi(m) \neq 1$ we must have $\sum_{n \bmod q} \chi(n) = 0$. □

Theorem 6.4 (Row Orthogonality). Let $q \in \mathbb{Z}$ with $q \geq 2$ and let χ_1 and χ_2 be two Dirichlet characters mod q. Then

$$\sum_{n \bmod q} \chi_1(n)\overline{\chi_2(n)} = \begin{cases} \varphi(q) & \text{if } \chi_1 = \chi_2 \\ 0 & \text{if } \chi_1 \neq \chi_2 \end{cases}$$

Proof. Note that the product of two characters is still a character because

$$(\chi\psi)(ab) = \chi(ab)\psi(ab) = \chi(a)\psi(a)\chi(b)\psi(b) = (\chi\psi)(a)(\chi\psi)(b)$$

for characters χ and ψ. In this case, $\overline{\chi}_2$ is a character, so $\chi := \chi_1\overline{\chi}_2$ is a Dirichlet character. Then by Theorem 6.3 we have

$$\sum_{n \bmod q} \chi_1(n)\overline{\chi_2(n)} = \sum_{n \bmod q} \chi(n) = \begin{cases} \varphi(q) & \text{if } \chi = \chi_0 \\ 0 & \text{if } \chi \neq \chi_0 \end{cases}$$

Note since $\chi_2(n)^{\varphi(q)} = 1$ we know $|\chi_2(n)| = 1$ and so $\chi_2(n)^{-1} = \overline{\chi_2(n)}$. Hence

$$\begin{aligned} \chi = \chi_0 &\iff \chi_1\overline{\chi}_2 = \chi_0 \\ &\iff \chi_1(n)\overline{\chi_2(n)} = 1 \quad \text{for all } n \\ &\iff \overline{\chi_2(n)} = \chi_1(n)^{-1} \quad \text{for all } n \\ &\iff \chi_2(n)^{-1} = \chi_1(n)^{-1} \quad \text{for all } n \\ &\iff \chi_1(n) = \chi_2(n) \quad \text{for all } n \\ &\iff \chi_1 = \chi_2 \end{aligned}$$

The result follows. □

Theorem 6.5 (Column Orthogonality). Let $q \in \mathbb{Z}$ with $q \geq 2$ and let n and m be two residue classes in $(\mathbb{Z}/q\mathbb{Z})^\times$. Then

$$\sum_{\chi \bmod q} \chi(n)\overline{\chi(m)} = \begin{cases} \varphi(q) & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Proof. We can view the character table mod q as a matrix A. The previous theorem on Row Orthogonality tells us the rows of A are orthogonal with respect to the complex inner product.

$$AA^* = \varphi(q)I$$

This means A has right inverse $\frac{1}{\varphi(q)}A^*$, which means it has left inverse $\frac{1}{\varphi(q)}A^*$. Hence

$$\frac{1}{\varphi(q)}A^* \cdot A = I \implies A^*A = \varphi(q)I$$

This proved the column orthogonality. □

Corollary 6.6. As a special case of Column Orthogonality for $m = 1$, we have

$$\sum_{\chi \bmod q} \chi(n) = \begin{cases} \varphi(q) & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

This is the important property that we will use in the proof of Dirichlet's Theorem.

6.3 Proof of Dirichlet's Theorem

Recall the function $L(s, \chi)$ defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

This converges absolutely for $\operatorname{Re} s > 1$. If χ is not principal, then it converges conditionally for $\operatorname{Re} s > 0$. Let $\chi \neq \chi_0$, then by Partial Summation

$$\sum_{n \le X} \frac{\chi(n)}{n^s} = \frac{1}{X^s} \sum_{n \le X} \frac{\chi(n)}{n^s} + s \int_1^X \frac{1}{t^{s+1}} \sum_{n \le t} \chi(n) \, \mathrm{d}t$$

But we know that $|\sum_{n \le t} \chi(n)| \le \varphi(q)$ for all $t > 0$ by the equality

$$\sum_{n \bmod q} \chi(n) = 0$$

so there are some cancellation for every q steps and in each cycle the sum is at most $\varphi(q)$. If $\operatorname{Re} s > 0$ we let $X \to \infty$ and get

$$\left| \frac{1}{X^s} \sum_{n \le X} \right| \le \frac{\varphi(q)}{X^s} \to 0$$

Therefore we have

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = s \int_1^{\infty} \frac{1}{t^{s+1}} \sum_{n \le t} \chi(n) \, \mathrm{d}t$$

This is analytic in $\operatorname{Re} s > 0$ by the differentiation under the integral sign.

Theorem 6.7. For a non-principal character χ we have

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}=s\int_1^{\infty}\frac{1}{t^{s+1}}\sum_{n\le t}\chi(n)\ \mathrm{d}t$$

for $\operatorname{Re} s>0$ and it is analytic for $\operatorname{Re} s>0$.

Theorem 6.8. For any χ (principal or not), we have the Euler product

$$\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}=\prod_p\left(1+\frac{\chi(p)}{p^s}+\frac{\chi(p)^2}{p^{2s}}+\cdots\right)=\prod_p\left(1-\frac{\chi(p)}{p^s}\right)^{-1}$$

for $\operatorname{Re} s>1$.

If $\chi=\chi_0$ is princiapl, then

$$L(s,\chi)=\prod_p\left(1-\frac{\chi(p)}{p^s}\right)^{-1}=\prod_{p\nmid q}\left(1-\frac{1}{p^s}\right)^{-1}$$

because if $p\mid q$ then $\chi(p)=0$ and otherwise $\chi(p)=1$ as it is principal. This means

$$L(s,\chi_0)=\prod_{p\nmid q}\left(1-\frac{1}{p^s}\right)^{-1}=\prod_p\left(1-\frac{1}{p^s}\right)^{-1}\cdot\prod_{p\mid q}\left(1-\frac{1}{p^s}\right)=\zeta(s)\prod_{p\mid q}\left(1-\frac{1}{p^s}\right)$$

We know $\zeta(s)$ can be analytic continued to $\mathbb{C}$ with a pole at $s=1$ and $\prod_{p\mid q}\left(1-\frac{1}{p^s}\right)$ is just a constant. This gives the following theorem.

Theorem 6.9. The L-function $L(s,\chi_0)$ has a meromorphic continuation of $L(s,\chi_0)$ to $\mathbb{C}$ with a pole at $s=1$ given by

$$L(s,\chi_0)=\zeta(s)\prod_{p\mid q}\left(1-\frac{1}{p^s}\right)$$

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Proof of Dirichlet's Theorem. By Theorem 6.9 we know that

$$\lim_{s\to 1^+}L(s,\chi_0)=\infty$$

since $\zeta(s)\to\infty$ as $s\to 1^+$. Let $a\in\mathbb{Z}$ and $q\ge 2$ with $\gcd(a,q)=1$. For every character χ we have

$$\log L(s,\chi)=\sum_p\sum_{m=1}^{\infty}\frac{\chi(p)^m}{mp^{ms}}$$

for $\operatorname{Re} s > 1$ by the Euler product of $L(s,\chi)$. The sum converges absolutely for $\operatorname{Re} s > 1$.

Case 1. Assume $a = 1$. We consider

$$\sum_{\chi \bmod q} \log L(s,\chi) = \sum_{\chi \bmod q} \sum_p \sum_{m=1}^{\infty} \frac{\chi(p)^m}{mp^{ms}} = \sum_{m=1}^{\infty} \frac{1}{m} \sum_p \frac{1}{p^{ms}} \sum_{\chi \bmod q} \chi(p^m)$$

By the Column Orthogonality, we know

$$\sum_{\chi \bmod q} \chi(p^m) = \begin{cases} \varphi(q) & \text{if } p^m \equiv 1 \pmod q \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$\begin{aligned} \sum_{\chi \bmod q} \log L(s,\chi) &= \varphi(q) \sum_{m=1}^{\infty} \sum_{p^m = 1 \bmod q} \frac{1}{mp^{ms}} \\ &= \varphi(q) \sum_{p=1 \bmod q} \frac{1}{p^s} + \underbrace{\varphi(q) \sum_{m=2}^{\infty} \sum_{p^m=1 \bmod q} \frac{1}{mp^{ms}}}_{R(s)} \end{aligned}$$

Note that for $s = \sigma + it$ with $\sigma \geq 1$ we have

$$\begin{aligned} |R(s)| &\leq \sum_p \sum_{m=2}^{\infty} \frac{1}{p^{m\sigma}} = \sum_p \left(\frac{1}{p^{2\sigma}} + \frac{1}{p^{3\sigma}} + \frac{1}{p^{4\sigma}} + \cdots \right) \\ &= \sum_p \frac{1}{p^{2\sigma}} \left(1 + \frac{1}{p^{\sigma}} + \frac{1}{p^{2\sigma}} + \cdots \right) \\ &= \sum_p \frac{1}{p^{2\sigma}} \left(1 - p^{-\sigma}\right)^{-1} \\ &\leq 2 \sum_p \frac{1}{p^2} && ((1-p^{-\sigma})^{-1} \leq 2) \\ &\leq 2\zeta(2) \end{aligned}$$

Therefore $R(s) = \mathcal{O}(1)$ as $s \to 1^+$. Therefore

$$\sum_{\chi \bmod q} \log L(s,\chi) = \varphi(q) \sum_{p=1 \bmod q} \frac{1}{p^s} + R(s)$$

where $R(s) = \mathcal{O}(1)$ for $\operatorname{Re} s > 1$. We want to show that as $s \to 1^+$ the LHS tends to ∞. Let $s \to 1^+$, then $\log L(s,\chi_0) \to \infty$. However, $L(1,\chi) \neq 0$ if $\chi \neq \chi_0$. Hence $\log L(s,\chi)$ is bounded (converging to

$\log L(1,\chi))$ as $s \to 1^+$, for $\chi \neq \chi_0$. It follows that

$$\underbrace{\log(s,\chi_0)}_{\to\infty} + \underbrace{\sum_{\chi\neq\chi_0} \log L(s,\chi)}_{\mathcal{O}(1)} = \varphi(q) \sum_{p=1 \bmod q} \frac{1}{p^s} + \underbrace{R(s)}_{\mathcal{O}(1)}$$

It then follows that

$$\sum_{p=1 \bmod q} \frac{1}{p} = \infty$$

The only thing we need to prove is that "however", that is, $L(1,\chi) \neq 0$ if $\chi \neq \chi_0$. Before doing this we first look at case 2 when a is not necessarily 1.

Case 2. Assume a is not (necessarily) 1. If $p \equiv a \pmod q$ then $a^{-1}p \equiv 1 \pmod q$. The Column Orthogonality tells us

$$\sum_{\chi \bmod q} \overline{\chi(a)}\chi(n) = \begin{cases} \varphi(q) & \text{if } n \equiv a \pmod q \\ 0 & \text{otherwise} \end{cases} \tag{$*$}$$

Hence we have

$$\begin{aligned}
\sum_{\chi \bmod q} \log L(s,\chi) &= \sum_{\chi \bmod q} \sum_p \sum_{m=1}^{\infty} \frac{\chi(p)^m}{mp^{ms}} = \sum_{m=1}^{\infty} \sum_p \frac{1}{mp^{ms}} \sum_{\chi \bmod q} \chi(p^m) \\
&= \chi(a) \sum_{m=1}^{\infty} \sum_p \frac{1}{mp^{ms}} \sum_{\chi \bmod q} \overline{\chi(a)}\chi(p^m) \\
&= \chi(a)\varphi(q) \sum_{m=1}^{\infty} \sum_{p^m=a \bmod q} \frac{1}{mp^{ms}} && \text{(by $(*)$)}
\end{aligned}$$

Moving $\chi(a)$ to the LHS gives

$$\sum_{\chi \bmod q} \overline{\chi(a)} \log L(s,\chi) = \varphi(q) \sum_{p=a \bmod q} \frac{1}{p^s} + R_a(s)$$

The same analysis shows that $R_a(s) = \mathcal{O}(1)$ for $\operatorname{Re} s \geq 1$. Take $s \to 1^+$ then

$$\overline{\chi_0(a)} \log L(s,\chi_0) = \log L(s,\chi_0) \to \infty$$

For $\chi \neq \chi_0$ we have

$$\overline{\chi(a)} \log L(s,\chi) \text{ converges as } s \to 1^+$$

Hence we conclude that

$$\sum_{p=a \bmod q} \frac{1}{p} = \infty$$

This completes the proof, asumming that $L(s,\chi) \neq 0$ for $\chi \neq \chi_0$. $\square$

Now let us complete the proof of Dirichlet's theorem by proving the Non-vanishing of $L(s,\chi)$ at $s=1$, for $\chi \neq \chi_0$.

6.4 Non-vanishing of $L(1,\chi)$

Theorem 6.10. If $\chi \neq \overline{\chi}$ (in other word χ is NOT a real character) is a Dirichlet character mod q, then $L(1,\chi) \neq 0$.

Proof. Consider the sum from Case 1.

$$\sum_{\chi \bmod q} \log L(s,\chi) = \varphi(q) \sum_{m=1}^{\infty} \sum_{p^m = 1 \bmod q} \frac{1}{mp^{ms}}$$

Exponentiating both sides gives

$$I(s) := \prod_{\chi \bmod q} L(s,\chi) = \exp \underbrace{\left(\varphi(q) \sum_{m=1}^{\infty} \sum_{p^m = 1 \bmod q} \frac{1}{mp^{ms}} \right)}_{\geq 0 \text{ if } \operatorname{Re} s > 1 \text{ and } s \text{ is real}}$$

Therefore we have

$$I(s) = \prod_{\chi \bmod q} L(s,\chi) \geq 1$$

for all $s \in \mathbb{R}$ with $s > 1$. We know

$$L(s,\chi_0) = \zeta(s) \prod_{p \mid q} \left(1 - \frac{1}{p^s}\right)$$

has a simple pole at $s=1$. For $\chi \neq \chi_0$ we saw that $L(s,\chi)$ is analytic on $\operatorname{Re} s > 0$. Now, assume for a contradiction that $L(1,\chi) = 0$ for some $\chi \neq \chi_0$, then $L(1,\overline{\chi}) = 0$ as well. This means

$$I(s) = \prod_{\chi \bmod q} L(s,\chi) = L(s,\chi_0) \prod_{\chi \neq \chi_0} L(s,\chi)$$

has a order 1 pole at $s=1$ given by $L(s,\chi_0)$ and at least two zeros at $s=1$ given by $L(s,\chi)$ and $L(s,\overline{\chi})$ (which are distinct by assumption). Hence $I(s)$ must have a zero at $s=1$, which is a contradiction because $I(s) \geq 1$. $\square$

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Theorem 6.11. Let $\chi \neq \chi_0$ be a Dirichlet character mod q and $\chi = \overline{\chi}$, then $L(1,\chi) \neq 0$.

Proof. We consider

$$\zeta(s)L(s,\chi) = \left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}\right) = \sum_{n=1}^{\infty}\left(\sum_{d|n}\chi(d)\right)\frac{1}{n^s}$$

But χ is completely multiplicative, so letting $n = p_1^{\alpha_1}\cdots p_k^{\alpha_k}$, where p_j are distinct primes and $\alpha_j \geq 1$

$$c(n) := \sum_{d|n}\chi(d) = \prod_{j=1}^{k}(1+\chi(p_j)+\cdots+\chi(p_j)^{\alpha_j})$$

Note that since $\chi(p) \in \{0,1,-1\}$ (this is because χ is a real character, so $\chi(p)$ is a root of unity that is also a real number, so it must be 1 or -1. Well it can also be 0 if $p = 0 \bmod q$). we have

$$1+\chi(p)+\cdots+\chi(p)^{\alpha} = \begin{cases} 1 & \text{if } \chi(p)=0 \\ \alpha+1 & \text{if } \chi(p)=1 \\ 0 & \text{if } \chi(p)=-1 \text{ and } \alpha \text{ is odd } (*) \\ 1 & \text{if } \chi(p)=-1 \text{ and } \alpha \text{ is even} \end{cases}$$

This means $c(n) \geq 0$ for all $n \in \mathbb{N}$ because each term in the product is non-negative. Moreover, $c(n) \geq 1$ if and only if $(*)$ never happens. This never happens if and only all α_j are even. Hence

$$c(n) \geq 1 \quad \text{if } n \text{ is a perfect square} \tag{1}$$

We also see that $|c(n)| \leq \sum_{d|n} 1 = d(n) = \mathcal{O}_\epsilon(n^\epsilon)$ for $\epsilon > 0$ by Theorem 4.1 and so

$$\zeta(s)L(s,\chi) = \sum_{n=1}^{\infty}\frac{c(n)}{n^s} \quad \text{converges uniformly on } \operatorname{Re} s \geq \delta > 1$$

Assume for contradiction that $L(1,\chi) = 0$. We know $L(s,\chi)$ is analytic on $\operatorname{Re} s > 0$ (in fact it is entire, but we don't need it) and $\zeta(s)$ is analytic on $\operatorname{Re} s > 0$ with a simple pole at 1. However the simple pole of $\zeta(s)$ at 1 is cancelled by the zero of $L(s,\chi)$ at 1. Therefore we conclude that $\zeta(s)L(s,\chi)$ is analytic on $\operatorname{Re} s > 0$.

Consider the Taylor series for $\zeta(s)L(s,\chi)$ about $3/2$ (can use any $\sigma_1 > 1$).

$$\zeta(s)L(s,\chi) = \sum_{k=0}^{\infty}\left(\sum_{n=1}^{\infty}\frac{c(n)(\log n)^k}{n^{3/2}}\right)\frac{1}{k!}\left(\frac{3}{2}-s\right)^k$$

This is because

$$\frac{d^{(k)}}{ds^{(k)}}\zeta(s)L(s,\chi)\Big|_{s=3/2} = \frac{d^{(k)}}{ds^{(k)}}\sum_{n=1}^{\infty}\frac{c(n)}{n^s}\Big|_{s=3/2} = \frac{(-1)^k c(n)(\log n)^k}{n^{3/2}}$$

Thus, substituting $s = 1/2$ (or any $0 < s \leq 1/2$). This would give us a contradiction because there should be a pole at $s = 1$ so we cannot apply the Taylor series to $s = 1/2$, which is beyond $\operatorname{Re} s = 1$.

$$\zeta\left(\tfrac{1}{2}\right) L\left(\tfrac{1}{2}, \chi\right) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n=1}^{\infty} \frac{c_n (\log n)^k}{n^{3/2}}$$

By the absolute convergence, we rearrange the sum and get

$$\begin{aligned}
\zeta\left(\tfrac{1}{2}\right) L\left(\tfrac{1}{2}, \chi\right) &= \sum_{n=1}^{\infty} \frac{c(n)}{n^{3/2}} \underbrace{\sum_{k=0}^{\infty} \frac{(\log n)^k}{k!}}_{\exp(\log n) = n} = \sum_{n=1}^{\infty} \frac{c(n)}{n^{1/2}} \\
&\geq \sum_{n=1}^{\infty} \frac{c(n^2)}{n} && \text{(only consider the squares)} \\
&\geq \sum_{n=1}^{\infty} \frac{1}{n} = \infty && \text{(by (1) from above)}
\end{aligned}$$

This is a contradiction, which completes the proof. $\square$

We now present a proof of non-vanishing of $L(s, \chi)$ using an identity by Ramanujan.

Theorem 6.12. Let $\alpha, \beta \in \mathbb{C}$ and $\operatorname{Re} s > \max\{1, 1 + \alpha, 1 + \beta, 1 + \alpha + \beta\}$ then

$$\sum_{n=1}^{\infty} \frac{\sigma_\alpha(n) \sigma_\beta(n)}{n^s} = \frac{\zeta(s)\zeta(s - \alpha)\zeta(s - \beta)\zeta(s - \alpha - \beta)}{\zeta(2s - \alpha - \beta)}$$

where we recall $\sigma_\alpha(n) = \sum_{d \mid n} d^\alpha$.

Proof. This is Q2 of Assignment 2. $\square$

Take $\alpha = -\beta = ai$ for $a \in \mathbb{R}$. Then

$$\sum_{n=1}^{\infty} \frac{|\sigma_\alpha(n)|^2}{n^s} = \frac{\zeta(s)^2 \zeta(s + ai)\zeta(s - ai)}{\zeta(2s)}$$

Therefore

$$\zeta(s)^2 \zeta(s + ai)\zeta(s - ai) = \zeta(2s) \sum_{n=1}^{\infty} \frac{|\sigma_{ai}(n)|^2}{n^s}$$

Write $\zeta(2s) = \sum_{n=1}^{\infty} b(n) n^{-s}$, where $b(n) = 1$ if n is squaree and 0 otherwise. Then

$$\zeta(s)^2 \zeta(s + ai)\zeta(s - ai) = \sum_{n=1}^{\infty} \underbrace{\left(\sum_{d \mid n} b(n) |\sigma_{ai}(n/d)|^2 \right)}_{a(n)} \frac{1}{n^s}$$

Assume $0 \neq a \in \mathbb{R}$ and $\zeta(1+ai) = 0$. This would imply that $\zeta(1-ai) = 0$ as well by the formula

$$\zeta(s) = \frac{1}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}}\ \mathrm{d}t$$

Then

$$\zeta(s)^2\zeta(s+ai)\zeta(s-ai) \text{ is entire}$$

because the double zero of $\zeta(s+ai)\zeta(s-ai)$ at $s = 1$ cancels the double pole of $\zeta(s)^2$ at $s = 1$. And the pole of $\zeta(s+ai)$ at $s = 1-ai$ is cancelled by the double zero of $\zeta(s)^2$ at $s = 1-ai$ and likewise for $s = 1+ai$. Let

$$f(s) := \frac{\zeta(s)^2\zeta(s+ai)\zeta(s-ai)}{\zeta(2s)} = \sum_{n=1}^\infty \frac{|\sigma_{ai}(n)|^2}{n^s}$$

Note that $|\sigma_{ai}(n)| \leq \sum_{d|n} |d^{a_i}| \leq \sum_{d|n} 1 = d(n) = \mathcal{O}_\epsilon(n^\epsilon)$ by Theorem 4.1. Write $f(s)$ in a Taylor series about $s = 3/2$

$$f(s) = \sum_{k=0}^\infty \frac{(3/2-s)^k}{k!} \underbrace{\sum_{n=1}^\infty \frac{|\sigma_{ai}(n)|^2}{n^{3/2}}(\log n)^k}_{>0}$$

The reason this is > 0 is this: for example $|\sigma_{ai}(2)|$ and $|\sigma_{ai}(4)|$ cannot both be 0 because $1+2^{ai}$ and $1+2^{ai}+4^{ai}$ cannot both be 0. Therefore $f(1/2) > 0$. However we know $f(1/2) = 0$, so we get a contradiction.

6.5 Differentiation under the integral

Theorem 6.13. Consider the function

$$F(z) = \int_a^b f(t,z)\ \mathrm{d}t$$

where $f(t,z)$ is analytic at z (differentiable on some neighborhood of z) for all $t \in [a,b]$. Assume that $D_2 f(t,z) := \frac{\partial}{\partial z} f(t,z)$ exists and is continuous as a function of t and z. Then

$$F'(z) = \int_a^b \frac{\partial}{\partial z} f(t,z)\ \mathrm{d}t$$

Proof. (Real Variable Case). Consider

$$\frac{F(z+h)-F(z)}{h} = \int_a^b \frac{f(t,z+h)-f(t,z)}{h}\ \mathrm{d}t$$

By Mean Value Theorem the integrand is equal to $D_2(t,c)$ where c is between $z, z+h$.

$$D_2(t,c) = D_2(t,z) + \epsilon$$

Let $\epsilon \to 0$ uniformly for $t \in [a, b]$ and h sufficiently small.

(Complex Variable Case). Use the complex MVT. $\square$