PMATH 367 Notes

Introduction to Topology

Fall 2024

Based on Professor Blake Madill's Lectures

Contents

Lecture 1, 2024/09/04

1 Topological Spaces

1.1 Basic Notations

Motivation. Recall from analysis that:

1. $A \subseteq \mathbb{R}^n$ is closed $\iff$ $\mathbb{R}^n \setminus A$ is open.
2. $x_n \to x$ in $\mathbb{R}^n$ $\iff$ for all open set $U \subseteq \mathbb{R}^n$ with $x \in U$, $\exists N \in \mathbb{N}$ such that $n \geq N \implies x_n \in U$.
3. $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous $\iff$ $f^{-1}(U)$ is open in $\mathbb{R}^n$ for all open $U \subseteq \mathbb{R}^m$.
4. $A \subseteq \mathbb{R}^n$ is compact $\iff$ every open cover of A has a finite subcover.

Big Idea: All these concepts from analysis can be stated using open sets!

Recall. If X is a set, we define:

$$\mathcal{P}(X) = \{A : A \subseteq X\}$$

to be the power set of X.

Definition. Let X be a set. We say $\mathcal{T} \subseteq \mathcal{P}(X)$ is a **topology** on X if:

1. $\emptyset, X \in \mathcal{T}$.
2. If I is an index set and $A_\alpha \in \mathcal{T}$ for all $\alpha \in I$, then $\bigcup_{\alpha \in I} A_\alpha \in \mathcal{T}$. (Arbitrary Union)
3. If $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$. (Finite Intersection)

We call $(X, \mathcal{T})$ a **topological space**. Moreover, we call the elements of $\mathcal{T}$ the **open sets** of X. And the **closed sets** of X are $X \setminus A$ for $A \in \mathcal{T}$.

Big Idea: Topology is the study of topological spaces. It is the area of math which studies concepts like open and closed sets, continuity, compactness and connectedness.

Example 1.1. Let $X = \{a, b, c\}$. Define:

$$\mathcal{T}_1 = \{\emptyset, X, \{a, b\}, \{c\}\}$$

$$\mathcal{T}_2 = \{\emptyset, X, \{a, b\}, \{b, c\}, \{b\}\}$$

Then both $\mathcal{T}_1$ and $\mathcal{T}_2$ are topology on X.

Example 1.2. Let (X, d) be a metric space, then:

$$\mathcal{T} = \{U \subseteq X : \forall x \in U, \exists r > 0, B_r(x) \subseteq U\}$$

is the metric topology on X.

Example 1.3. In the Example 1.1, it can be shown that $\mathcal{T}_1$ is not a metric topology. That is, there is no metric d on X such that the open sets in (X, d) is $\mathcal{T}_1$. Suppose there is a metric d on X, then there is $r_1, r_2, r_3 > 0$ such that:

$$B_{r_1}(a) = \{a\},\ B_{r_2}(b) = \{b\},\ B_{r_3}(c) = \{c\}$$

Thus the metric topology would be $\mathcal{P}(X)$. But $\mathcal{T}_1$ is not $\mathcal{P}(X)$, so contradiction.

Definition. Let X be any set. $\mathcal{P}(X)$ is called the **discrete topology** and $\{\emptyset, X\}$ is called the **indiscrete topology**.

Example 1.4. Let X be a set and let:

$$\mathcal{T}_f = \{A \subseteq X : X \setminus A \text{ is finite}\} \cup \{\emptyset\}$$

is called the **finite complement topology**. Why?

1. $X \setminus X = \emptyset$, so $X \in \mathcal{T}_f$.
2. $\emptyset \in \mathcal{T}_f$ by definition.
3. $A_\alpha \in \mathcal{T}_f$ means $X \setminus A_\alpha$ is finite. Then:

$$X \setminus \bigcup_\alpha A_\alpha = \bigcap_\alpha (X \setminus A_\alpha)$$

   is also finite. Hence $\bigcup_\alpha A_\alpha \in \mathcal{T}_f$.
4. If $A, B \in \mathcal{T}_f$, then $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$. Each set is finite, so this is finite. Therefore we have $A \cap B \in \mathcal{T}_f$.

Example 1.5. Let X be any set, then:

$$\mathcal{T}_c = \{A \subseteq X : X \setminus A \text{ is at most countable}\} \cup \{\emptyset\}$$

is the **countable complement topology**.

Lecture 2, 2024/09/06

1.2 Bases and Subbases

Definition. Let X be a set. We say $\mathcal{B} \subseteq \mathcal{P}(X)$ is a **basis for a topology on** X if:

1. For all $x \in X$ there is $B \in \mathcal{B}$ such that $x \in B$.
2. For all $x \in X$ such that $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Example 1.6. Let $X = \mathbb{R}$ and $\mathcal{B} = \{(a, b) : a < b\}$ is a basis for a topology on $\mathbb{R}$. (Open intervals).

Example 1.7. Let (X, d) be a metric space and $\mathcal{B} = \{B_r(x) : x \in X,\ r > 0\}$ is a basis for a topology on X. (All open balls).

Example 1.8. Let X be a set and $\mathcal{B} = \{\{x\} : x \in X\}$ is a basis for a topology on X.

Definition. Let $\mathcal{B}$ be a basis for a topology on X. We define the **topology generated by** $\mathcal{B}$ to be:

$$\mathcal{T}_{\mathcal{B}} = \{U \subseteq X : \forall x \in U,\ \exists B \in \mathcal{B},\ x \in B \subseteq U\}$$

Proposition 1.9. This definition is well-defined, that is, $\mathcal{T}_{\mathcal{B}}$ is a topology on X.

Proof: It suffices to check the definition.

1. $\emptyset \in \mathcal{T}_{\mathcal{B}}$ is vacuously true.
2. For all $x \in X$, we can pick any $B \in \mathcal{B}$ such that $x \in B \subseteq X$. Hence $X \in \mathcal{T}_{\mathcal{B}}$.
3. If $U_\alpha \in \mathcal{T}_B$ for $\alpha \in I$ and let $x \in \bigcup_\alpha U_\alpha$. Then $x \in U_\beta$ for some $\beta \in I$. Then there is $B \in \mathcal{B}$ such that $x \in B \subseteq U_\beta \subseteq \bigcup_\alpha U_\alpha$. Hence $\bigcup_\alpha U_\alpha \in \mathcal{T}_{\mathcal{B}}$.
4. For $U, V \in \mathcal{T}_{\mathcal{B}}$ and $x \in U \cap V$. There are $B_1, B_2 \in \mathcal{B}$ such that:

   $$x \in B_1 \subseteq U \text{ and } x \in B_2 \subseteq V$$

   So $x \in B_1 \cap B_2$. By the second condition on basis, there is $B_3 \in \mathcal{B}$ such that:

   $$x \in B_3 \subseteq B_1 \cap B_2 \subseteq U \cap V$$

   Hence $U \cap V \in \mathcal{T}_{\mathcal{B}}$.

As desired. $\square$

Remark. For all $B \in \mathcal{B}$, we have $B \in \mathcal{T}_{\mathcal{B}}$. Since for all $x \in B$, we have $x \in B \subseteq B$.

Proposition 1.10. Let $\mathcal{B}$ be a basis for a topology on X. Then:

$$\mathcal{T}_{\mathcal{B}} = \left\{ \bigcup_{\alpha \in I} B_\alpha : B_\alpha \in \mathcal{B} \text{ for all } \alpha \in I,\ I \text{ an index set} \right\}$$

Proof: Let $\mathcal{U}$ denote the RHS. To show $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{U}$, we let $V \in \mathcal{T}_{\mathcal{B}}$. For each $x \in V$, there is $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq V$. Thus:

$$V = \bigcup_{x \in V} B_x \in \mathcal{U}$$

Therefore $V \in \mathcal{U}$. Conversely, since each $B_\alpha \in \mathcal{T}_{\mathcal{B}}$ and $\mathcal{T}_{\mathcal{B}}$ is a topology, we have $\mathcal{U} \subseteq \mathcal{T}_{\mathcal{B}}$. $\square$

Example 1.11. Let $X = \mathbb{R}$ and $\mathcal{B} = \{(a, b) : a < b\}$. Then $\mathcal{T}_{\mathcal{B}}$ is the metric/standard topology.

Example 1.12. If (X, d) is a metric space and $\mathcal{B} = \{B_r(x) : x \in X,\ r > 0\}$ = all open balls. Then $\mathcal{T}_{\mathcal{B}}$ is the metric topology.

Example 1.13. Let X be a set and $\mathcal{B} = \{\{x\} : x \in X\}$. Then $\mathcal{T}_{\mathcal{B}} = \mathcal{P}(X)$ is the discrete topology.

Example 1.14. Let $X = \mathbb{R}$ and $\mathcal{B}' = \{[a, b) : a < b\}$ is a basis for a topology on $\mathbb{R}$. Let $\mathcal{T}' = \mathcal{T}_{\mathcal{B}'}$ and let $\mathcal{T}$ be the metric topology on $\mathbb{R}$. Note that

$$\mathcal{T} \subsetneq \mathcal{T}'$$

since $[0, 1) \in \mathcal{T}' \setminus \mathcal{T}$, so $\mathcal{T}' \neq \mathcal{T}$. Also $(a, b) \in \mathcal{T}'$ since $(a, b) = \bigcup_{n \in \mathbb{N}} \left[a + \frac{1}{n}, b\right) \in \mathcal{T}'$. We call $\mathcal{T}'$ the **lower limit topology** on $\mathbb{R}$.

Question: How do we build a basis for a topology?

Definition. Let X be a set. We say $S \subseteq \mathcal{P}(X)$ is a **subbasis** for a topology on X if $X = \bigcup_{A \in S} A$.

Definition. The topology generated by S is:

$$\mathcal{T}_S = \left\{ \bigcup_{\alpha} (A_{\alpha_1} \cap \cdots \cap A_{\alpha_n}) : n \in \mathbb{N}, A_{\alpha_i} \in S \right\}$$

Proposition 1.15. Let S be a subbasis for a topology on X. Then:

$$\mathcal{B} = \{A_1 \cap \cdots \cap A_n : n \in \mathbb{N}, A_i \in S\}$$

is a basis for a topology on X. In particular, $\mathcal{T}_S = \mathcal{T}_{\mathcal{B}}$ is a topology on X.

Lecture 3, 2024/09/09

Proof: Since S is a subbasis we have:

$$X = \bigcup_{A \in S} A$$

$x \in X$ implies $x \in A$ for some $A \in S$. Since $A \in \mathcal{B}$, this proves the first axiom of a basis. Now, say:

$$x \in (A_1 \cap \cdots \cap A_n) \cap (B_1 \cap \cdots \cap B_m) \in \mathcal{B}$$

So the second axiom of a basis holds trivially. Therefore $\mathcal{B}$ is a basis. $\square$

1.3 Subspaces

Definition. Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Define:

$$\mathcal{T}_A = \{A \cap U : U \in \mathcal{T}\}$$

Then $\mathcal{T}_A$ is a topology on A, called **subspace topology** on A. We say A is a subspace of X.

Proposition 1.16. Let $(X, \mathcal{T})$ and $A \subseteq X$. If $\mathcal{B}$ is a basis for $\mathcal{T}$, then:

$$\mathcal{B}' = \{A \cap B : B \in \mathcal{B}\}$$

is a basis for $\mathcal{T}_A$.

Example 1.17. Let $A = [0, 2] \subseteq \mathbb{R}$. Then $(1, 3) \cap A = (1, 2]$ is open in A but not in $\mathbb{R}$.

Proposition 1.18. Let $(X, \mathcal{T})$ be a topological space and $U \in \mathcal{T}$. If $A \subseteq U$ is open in U, then A is open in X.

Why? Well, A is open in U means $A = U \cap V$ for some $V \in \mathcal{T}$. Hence $A \in \mathcal{T}$. $\square$

Proposition 1.19. Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. The closed subsets of A are exactly the sets of the form $A \cap C$ where C is closed in X.

Proof: Suppose $C \subseteq A$ is closed, so $A \setminus C$ is open in A, which means $A \setminus C = A \cap U$ for some $U \in \mathcal{T}$. Therefore we have:

$$C = A \setminus (A \setminus C) = A \setminus (A \cap U) = A \cap (X \setminus U)$$

Here $X \setminus U$ is closed in X. Conversely, if C is closed in X, then $X \setminus C \in \mathcal{T}$. We want to prove $A \cap C$ is closed in A, indeed:

$$A \setminus (A \cap C) = A \cap (X \setminus C) \in \mathcal{T}_A$$

As desired. $\square$

Example 1.20. Let $A = [0, 1] \cup (2, 3) \subseteq \mathbb{R}$. Then:

$$[0, 1] = A \cap (-1, 3/2) = A \cap [0, 1]$$

This means $[0, 1]$ is both open and closed in A. We say it is **clopen** in A.

Remark. Let $(X, \mathcal{T})$ be a topological space.

1. $\emptyset, X$ are closed.
2. Closed sets are "closed" under arbitrary intersections.
3. Closed sets are "closed" under finite unions.

Definition. Let $(X, \mathcal{T})$ and $A \subseteq X$. The **closure** of A is:

$$\overline{A} = \bigcap \{C \subseteq X : A \subseteq C, C \text{ closed in } X\}$$

which is the intersection of all closed sets containing A. It is the smallest closed set containing A. The **interior** of A is:

$$\text{int}(A) = \bigcup \{U \in \mathcal{T} : U \subseteq A\}$$

It is largest open set contained in A. Note that:

$$\text{int}A \subseteq A \subseteq \overline{A}$$

Definition. For $(X, \mathcal{T})$. If $x \in X$ and $U \in \mathcal{T}$ with $x \in U$, we say U is a **neighborhood** of x.

Proposition 1.21. Let $(X, \mathcal{T})$ and $A \subseteq X$. Then:

$$x \in \overline{A} \iff U \cap A \neq \emptyset$$

for any neighborhood U of x.

Proof: $(\Rightarrow)$. Let $x \in \overline{A}$, suppose for a contradiction that there is $U \in \mathcal{T}$ with $x \in U$ but $U \cap A = \emptyset$. Then we have $A \subseteq X \setminus U$, which is closed. Hence by the minimality of $\overline{A}$ we have $\overline{A} \subseteq X \setminus U$. Then:

$$x \in \overline{A} \subseteq X \setminus U \quad \text{and} \quad x \in U$$

Which is a contradiction.

$(\Leftarrow)$. Suppose $x \in X$ such that for all $x \in U \in \mathcal{T}$ we have $U \cap A \neq \emptyset$. Let $C \subseteq X$ be closed such that $A \subseteq C$. Then $X \setminus C$ is open. Suppose for a contradiction that $x \notin C$, then $x \in X \setminus C$. Hence $A \cap X \setminus C \neq \emptyset$. But $A \subseteq C$! This is a contradiction. $\square$

Lecture 4, 2024/09/11

Definition. Let $(X, \mathcal{T})$ and $A \subseteq X$. We say $x \in X$ is a **limit point** of A if every neighborhood of x intersects A at a point different from x.

Example 1.22. Let $X = \mathbb{R}$ and $A = (0, 1) \cup \{2\}$. Then $0, \frac{1}{2}, 1$ are limit points of A. And 2 is not a limit point of A, because $(1.5, 2.5)$ does not contain anything in A except for 2.

Notation. Let $(X, \mathcal{T})$ and $A \subseteq X$. We denote:

$$A' = \{x \in X : x \text{ is a limit point of } A\}$$

to be the set of all limit points of A.

Proposition 1.23. Let $(X, \mathcal{T})$ and $A \subseteq X$. Then $\overline{A} = A \cup A'$.

Proof: $(\supseteq)$. This is trivial.

$(\subseteq)$. Let $x \in \overline{A}$ and suppose $x \in U \in \mathcal{T}$. Thus $U \cap A \neq \emptyset$.

1. If $x \in U \cap A$, then $x \in A$.
2. If $x \notin U \cap A$, then $x \in A'$.

As desired. □

Corollary 1.24. Let $(X, \mathcal{T})$ and $A \subseteq X$. Then A is closed if and only if $A' \subseteq A$.

Proof: $A' \subseteq A \iff A = \overline{A} \iff A$ is closed. □

1.4 Hausdorff Spaces

Definition. Let $(X, \mathcal{T})$ be a topological space. We say X is **Hausdorff** if for all $x \neq y \in X$, we can find $U, V \in \mathcal{T}$ such hat $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.

That is, given two distinct points, we can find two open sets that separate them.

Remark. All metric topologies are Hausdorff. For $x \neq y$, we can set $\epsilon = d(x, y)$. Then:

$$x \in B_{\epsilon/2}(x) \text{ and } y \in B_{\epsilon/2}(y)$$

and these two balls are disjoint.

Example 1.25. Let $X = \{a, b, c\}$ and $\mathcal{T} = \{\emptyset, X, \{a, b\}, \{c\}\}$. This is NOT Hausdorff because $a \neq b$ but there is no open sets that separate them.

Example 1.26. Consider $(\mathbb{R}, \mathcal{T}_f)$, the finite complement topology. This is NOT Hausdorff. Let $x \neq y$ with $x \in U$ and $y \in V$. Then:

$$\begin{aligned} x \in U &= \mathbb{R} \setminus \{x_1, \cdots, x_n\} \\ y \in V &= \mathbb{R} \setminus \{y_1, \cdots, y_m\} \end{aligned}$$

This means $U \cap V \neq \emptyset$, because $U \cap V = \mathbb{R} \setminus \{x_1, \cdots, x_n, y_1, \cdots, y_m\}$ which is infinite.

Proposition 1.27. Let $(X, \mathcal{T})$ be Hausdorff, then $\{x\}$ is closed for all $x \in X$.

Proof: Fix $x \in X$. Since X is Hausdorff, there is $x \in U_y \in \mathcal{T}$ and $y \in V_y \in \mathcal{T}$ with $U_y \cap V_y = \emptyset$. Then we have:

$$X \setminus \{x\} = \bigcup_{y \neq x} V_y$$

Hence $X \setminus \{x\}$ is open. $\square$

Proposition 1.28. Let $(X, \mathcal{T})$ be Hausdorff and $A \subseteq X$. Then $x \in X$ is a limit point of A if and only if every neighborhood of x intersects A at infinitely many points.

Proof: $(\Leftarrow)$. This is trivial.

$(\Rightarrow)$. Assume x is a limit point of A. For contradiction, assume there exists $x \in U \in \mathcal{T}$ such that $U \cap A$ is finite. Since x is a limit point, we have:

$$U \cap (A \setminus \{x\}) = \{x_1, \cdots, x_n\} \neq \emptyset$$

Then, since $\{x_1, \cdots, x_n\}$ is closed, so $V = X \setminus \{x_1, \cdots, x_n\}$ is open. And $x \in U \cap V$ (open). However:

$$A \cap (U \cap V) = \{x\} \text{ or } \emptyset$$

Either way this is a contradiction: Since x is a limit point and $U \cap V$ is a neighborhood of x, so $U \cap V$ must intersect A at a point that is different from x. $\square$

Definition. A **directed set** is a partially ordered set $(I, \leq)$ such that whenever $a, b \in I$, there exists $c \in I$ such that $a \leq c$ and $b \leq c$.

Definition. Let $(X, \mathcal{T})$ be a topological space. A **net** in X is a pair (I, f), where I is a directed set and $f : I \to X$ is a function. We denote $f(i)$ by x_i and say $(x_i) = (x_i)_{i \in I}$ is a net in X.

Example 1.29. A sequence is just a net (I, f) where $I = \mathbb{N}$.

Definition. Let $(X, \mathcal{T})$ be a topological space and let $(x_i)_{i \in I}$ be a net in X. We say (x_i) **converges to** $x \in X$, written as $x_i \to x$, if for every open $U \in \mathcal{T}$ with $x \in U$, there exists $j \in I$ such that $x_i \in U$ for all $i \geq j$. We call x a **limit** of (x_i).

Proposition 1.30. Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. For $x \in X$:

$$x \in A \iff \text{there exists a net } (x_i)_{i \in I} \text{ in } A \text{ with } x_i \to x$$

Proof: Assignment 1. $\square$

2 Continuity

2.1 Basic Properties

Definition. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces. We say $f : X \to Y$ is **continuous** if:

$$f^{-1}(U) = \{x \in X : f(x) \in U\} \in \mathcal{T}$$

for all $U \in \mathcal{U}$.

Proposition 2.1. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ and $f : X \to Y$. If $\mathcal{B}$ is a basis for $\mathcal{U}$, then f is continuous if and only if $f^{-1}(B) \in \mathcal{T}$ for all $B \in \mathcal{B}$.

Proof: $(\Rightarrow)$. This is trivial.

$(\Leftarrow)$. Suppose $f^{-1}(B)$ for all $B \in \mathcal{B}$. Now, to show f is continuous, let $U \in \mathcal{U}$. Write $U = \bigcup_{i \in I} B_i$ for $B_i \in \mathcal{B}$, as $\mathcal{B}$ is a basis. Then:

$$f^{-1}(U) = f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i) \in \mathcal{T}$$

As desired. $\square$

Remark. The same result is true for a subbasis. (Exercise).

Proposition 2.2. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces. A function $f : X \to Y$ is continuous if and only if $f(x_i) \to f(x)$ whenever (x_i) is net in X such that $x_i \to x \in X$.

Proof: Assignment 1. $\square$

Proposition 2.3. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ and $f : X \to Y$, TFAE:

1. f is continuous.
2. For all $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$.
3. For all closed $C \subseteq Y$ we have $f^{-1}(C)$ is closed in X.

Example 2.4. Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \arctan(x)$. This is super continuous. Let $A = \mathbb{R}$, then:

$$f(\overline{A}) = f(\mathbb{R}) = \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

and so that:

$$\overline{f(A)} = \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

So the inclusion in 2 above does not have to be an equality.

Lecture 5, 2024/09/13

Proof: (1) $\implies$ (2). Suppose f is continuous. Let $y = f(x) \in f(\overline{A})$ where $x \in \overline{A}$. Let $U \in \mathcal{U}$ with $y \in U$. Then $x \in f^{-1}(U) \in \mathcal{T}$. Since $x \in \overline{A}$, there is $a \in A$ such that $a \in f^{-1}(U)$. Hence $f(a) \in U$ and $f(a) \in f(A)$. Hence $y \in \overline{f(A)}$.

(2) $\implies$ (3). Assume $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$. Let $C \subseteq Y$ be closed and let $D = f^{-1}(C)$. Let $x \in \overline{D}$, so we have:

$$f(x) \in f(\overline{D}) \subseteq \overline{f(D)} \subseteq \overline{C} = C$$

Therefor $x \in f^{-1}(C) = D$. Hence $\overline{D} \subseteq D$, so D is closed.

(3) $\implies$ (1). Let $U \in \mathcal{U}$, so $Y \setminus U$ is closed. So:

$$f^{-1}(Y \setminus U) = f^{-1}(Y) \setminus f^{-1}(U) = X \setminus f^{-1}(U)$$

This is closed by assumption of (3), hence $f^{-1}(U)$ is open in X. □

2.2 Homeomorphisms

Definition. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces and $f : X \to Y$. We say f is a **homeomorphism** if f is continuous and f^{-1} is also continuous.

Example 2.5. Let $X = \mathbb{Z}$ and $\mathcal{T} = \mathcal{P}(\mathbb{N}) \cup \{\mathbb{Z}\}$. Let $f : X \to X$ by $f(x) = x - 1$. This is clearly bijective. If $A \subseteq \mathbb{N}$, then $f^{-1}(A) \subseteq \mathbb{N}$ and $f^{-1}(\mathbb{Z}) = \mathbb{Z}$. Hence f is continuous. Let $g = f^{-1}$ and $g(x) = x + 1$. Note that $\{1\} \in \mathcal{T}$, BUT $g^{-1}(\{1\}) = \{0\} \notin \mathcal{T}$. Therefore $g = f^{-1}$ is not continuous.

Example 2.6. Let $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Let $f : [0, 1) \to S^1$ by:

$$f(x) = (\cos(2\pi x), \sin(2\pi x))$$

Here $[0, 1)$ has the subspace topology from the standard topology of $\mathbb{R}$ and S^1 has the subspace topology from $\mathbb{R}^2$. So f is continuous and bijective. Note that $[0, 1/4)$ is open in $[0, 1)$, then:

$$(f^{-1})^{-1}([0, 1/4)) = f([0, 1/4))$$

which is not open. Hence f^{-1} is not continuous.

Big Ideas: $[0, 1)$ and S^1 have topological/geometrical differences and so there cannot exist a homeomorphism between them. For instance:

1. S^1 is compact but $[0, 1)$ is not compact.
2. Imagine removing a point from $[0, 1)$ and “disconnecting the interval”. But removing only one point on S^1 cannot disconnect S^1.

Definition. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ and $f : X \to Y$. We say f is an **open map** if $f(U) \in \mathcal{U}$ for all $U \in \mathcal{T}$. That is, the image of every open set in X is an open set in Y.

Remark. $f : X \to Y$ is a homeomorphism if and only if:

1. f is bijective.
2. f is continuous.
3. f is an open map.

Why? This is just because $(f^{-1})^{-1}(U) = f(U)$.

Big Idea: Suppose $f : X \to Y$ is a homeomorphism.

1. Points: Every $y \in Y$ is of the form $y = f(x)$ for a unique $x \in X$. So Y is a relabelling of X.
2. Open sets: The elements V of $\mathcal{U}$ are exactly $V = f(U)$ for a unique $U \in \mathcal{T}$. Why: If $U \in \mathcal{T}$, then $f(U) \in \mathcal{U}$. If $V \in \mathcal{U}$, then $f^{-1}(V) \in \mathcal{T}$ and $V = f(f^{-1}(V))$. So $\mathcal{U}$ is a relabelling of $\mathcal{T}$.

This suggests that $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ are the same topological spaces up to the relabelling f.

Remark. Let $f : \mathbb{R} \to (\frac{-\pi}{2}, \frac{\pi}{2})$ by $f(x) = \arctan(x)$. This is a homeomorphism.

Lecture 6, 2024/09/16

Notation. In what follows X, Y, Z are topological spaces.

Proposition 2.7. If $f : X \to Y$ is constant, then f is continuous.

Proof: Say $f(x) = y_0$ for all $x \in X$. If $U \subseteq Y$ is open, then:

$$f^{-1}(U) = \begin{cases} X & \text{if } y_0 \in U \\ \emptyset & \text{if } y_0 \notin U \end{cases}$$

As desired. $\square$

Proposition 2.8. For $A \subseteq X$, the map $i : A \to X$ by $i(x) = x$ is continuous.

Proof: If $U \subseteq X$ is open, then:

$$i^{-1}(U) = \{x \in A : i(x) = x \in U\} = A \cap U$$

and this is open by the definition of subspace topology. $\square$

Proposition 2.9. Say $f : X \to Y$ and $g : Y \to Z$ are continuous. Then $g \circ f : X \to Z$ is continuous.

Proof: If $U \subseteq Z$ is open, then:

$$V := (g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$$

Here $g^{-1}(U)$ is open since g is continuous, thus V is open since f is continuous. □

Proposition 2.10. If $f : X \to Y$ is continuous and $A \subseteq X$, then $f|_A : A \to Y$ is continuous.

Proof: Notice that:

$$(f|_A)^{-1}(U) = A \cap f^{-1}(U)$$

which is open in A. □

Proposition 2.11. Let $f : X \to Y$ be continuous.

1. If $f(X) \subseteq Z \subseteq Y$, then $f : X \to Z$ is also continuous.
2. If $Y \subseteq Z$, then $f : X \to Z$ is also continuous.

Proof: Homework. □

Proposition 2.12. Suppose $X = \bigcup_\alpha U_\alpha$ is a union of open sets. Let $f : X \to Y$. If $f|_{U_\alpha}$ is continuous for all α, then f is continuous.

Proof: If $V \subseteq Y$ is open, then:

$$f^{-1}(V) = f^{-1}(V) \cap X = \bigcup_\alpha (f^{-1}(V) \cap U_\alpha) = \bigcup_\alpha (f|_{U_\alpha})^{-1}(V)$$

which is open. □

Definition. We say $f : X \to Y$ is continuous at $x \in X$ if for all open set $V \subseteq Y$ with $f(x) \in V$, there exists an open set $U \subseteq X$ with $x \in U$ such that $f(U) \subseteq V$.

Proposition 2.13. $f : X \to Y$ is continuous if and only if f is continuous at all $x \in X$.

Proof: ($\Rightarrow$). Suppose f is continuous and fix $x \in X$. If V is a neighborhood of $f(x)$, then $f^{-1}(V)$ is a neighborhood of x. Moreover, $f(f^{-1}(V)) \subseteq V$.

($\Leftarrow$). Suppose f is continuous at every $x \in X$. Let $V \subseteq Y$ be open. Let $x \in f^{-1}(V)$. By assumption, there is U_x open such that $x \in U_x$ and $f(U_x) \subseteq V$, so $U_x \subseteq f^{-1}(V)$. Thus:

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} \{x\} \subseteq \bigcup_{x \in f^{-1}(V)} U_x \subseteq f^{-1}(V)$$

Hence $f^{-1}(V)$ is open. □

Proposition 2.14 (Pasting Lemma). Let $X = A \cup B$ where A, B are closed. If $f : A \to Y$ and $g : B \to Y$ are continuous and $f(x) = g(x)$ for all $x \in A \cap B$, then the natural $h = f \divideontimes g : X \to Y$ is continuous.

Proof: Let $C \subseteq Y$ be closed. Then:

$$h^{-1}(C) = \underbrace{f^{-1}(C)}_{\text{closed in } A} \cup \underbrace{g^{-1}(C)}_{\text{closed in } B} \qquad \text{(Homework)}$$

Since A, B are closed in X, so both $f^{-1}(C)$ and $g^{-1}(C)$ are closed in X. Hence $h^{-1}(C)$ is closed. $\square$

Goal: Make new topologies from old topologies.

See Assignment 2 that:

1. Let X be a set and $\mathcal{F} = \{f_\alpha : X \to Y_\alpha : \alpha \in A\}$. Say $(Y_\alpha, \mathcal{T}_\alpha)$ are topological spaces. Then:

   $$\mathcal{B} = \{f_{\alpha_1}^{-1}(U_1) \cap \cdots \cap f_{\alpha_n}^{-1}(U_n) : U_i \in \mathcal{T}_{\alpha_i}\}$$

   is a basis for a topology on X.
2. Then $\mathcal{T} = \mathcal{T}_\mathcal{B}$ is called the **initial topology on X induced by $\mathcal{F}$**. It is the smallest topology on X that makes every f_α continuous.
3. In X, a net $x_i \to x$ if and only if $f_\alpha(x_i) \to f_\alpha(x)$ for all $\alpha \in A$.
4. $g : Z \to X$ is continuous if and only if $f_\alpha \circ g : Z \to Y_\alpha$ is continuous for all $\alpha \in A$.

Lecture 7, 2024/09/18

2.3 Product Topology

Definition. Let $(X_\alpha, \mathcal{T}_\alpha)$ for $\alpha \in A$ be a collection of topological spaces, consider:

$$X = \prod_{\alpha \in A} X_\alpha = \left\{ f : A \to \bigcup_{\alpha \in A} X_\alpha : f(\alpha) \in X_\alpha \right\}$$

The **product topology** on X is the initial topology generated by:

$$\mathcal{F} = \{\pi_\alpha : X \to X_\alpha : \alpha \in A\} \quad \text{where} \quad \pi_\alpha(f) = f(\alpha)$$

We call π_α the α-th projection. The product topology is the smallest topology on X which makes each projection π_α continuous.

Example 2.15. Consider the simple case $(X, \mathcal{T})$ and $(Y, \mathcal{U})$, then:

$$\begin{aligned} X \times Y &= \{f : \{1,2\} \to X \cup Y : f(1) \in X,\ f(2) \in Y\} \\ &= \{(x,y) : x \in X,\ y \in Y\} \end{aligned}$$

where $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$.

Example 2.16 (Box Topology). Let $(X_\alpha, \mathcal{T}_\alpha)$ with $\alpha \in A$. Then:

$$\mathcal{B}_b = \left\{ \prod_{\alpha \in A} U_\alpha : U_\alpha \in \mathcal{T}_\alpha \right\}$$

is a basis for a topology on $X = \prod_{\alpha \in A} X_\alpha$.

Investigation: How do these two topologies differ? By A2:

$$\mathcal{B}_p = \{\pi_{\alpha_1}^{-1}(U_1) \cap \cdots \cap \pi_{\alpha_n}^{-1}(U_n)\} \quad \text{and} \quad \mathcal{B}_b = \left\{ \prod_{\alpha} U_\alpha : U_\alpha \in \mathcal{T}_\alpha \right\}$$

Note that:

$$\pi_{\alpha_1}^{-1}(U_1) \cap \cdots \cap \pi_{\alpha_n}^{-1}(U_n) = \left\{ x \in \prod_\alpha X_\alpha : \pi_{\alpha_i}(x) \in U_{\alpha_i},\ 1 \le i \le n \right\} = \prod_\alpha V_\alpha$$

where:

$$V_\alpha = \begin{cases} U_{\alpha_i} & \text{if } \alpha = \alpha_i \\ X_\alpha & \text{if } \alpha \neq \alpha_i \end{cases}$$

Conclusions: We can conclude that:

1. $\mathcal{B}_p = \left\{ \prod_\alpha U_\alpha : \text{ all but finitely many } U_\alpha = X_\alpha \right\}$.
2. $\mathcal{B}_p \subseteq \mathcal{B}_b$ implies product topology $\subseteq$ box topology.
3. If A (the index set) is finite, then product $=$ box topology.

Example 2.17 (Warning). Let $X = \prod_{n \in \mathbb{N}} \mathbb{R}$ and $f : \mathbb{R} \to X$ by $f(t) = (t, t, \cdots)$. Then for all $n \in \mathbb{N}$:

$$\pi_n \circ f : \mathbb{R} \to \mathbb{R} \ \text{ is continuous and satisfies } \ \pi_n(f(t)) = t$$

By A2, f is continuous with respect to the product topology. If:

$$B = (-1, 1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{3}, \frac{1}{3}\right) \times \cdots$$

is in the box topology. But $f^{-1}(B) = \{0\}$ is not open in $\mathbb{R}$. Therefore f is not continuous with respect to the box topology (Box topology is too big!)

2.4 Quotient Topology

Notation. Let X be a set and let $\sim$ be an equivalence relation on X, that is:

(1) For all $x \in X$, $x \sim x$.

(2) For all $x, y \in X$, $x \sim y \implies y \sim x$.

(3) For all $x, y, z \in X$, $x \sim y, y \sim z \implies x \sim z$.

Then for $x \in X$, we let $[x] = \{y \in X : y \sim x\}$ be the equivalence class containing x. And let:

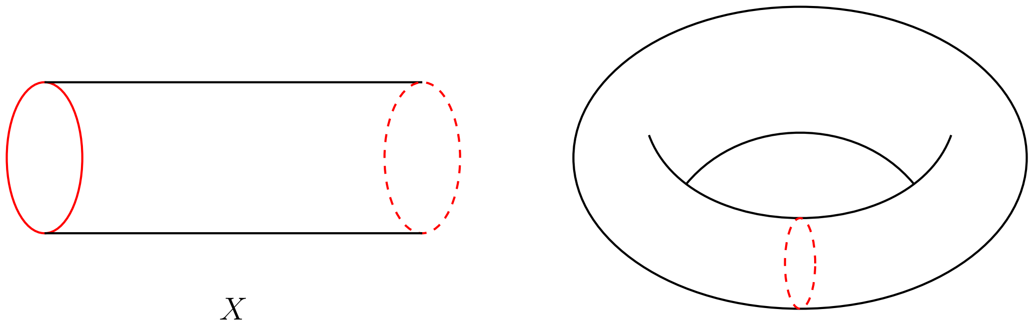
$$X/\sim = \{[x] : x \in X\}$$

Example 2.18. Let $X = [0, 1]$ and define $x \sim y \iff x = y$ or $x, y \in \{0, 1\}$. That is, we define 0,1 to be equivalent and all the other points are only equivalent to itself.

Example 2.19. If $X = [0, 1]$ and $\sim$ as above, then $X/\sim$ will be a circle, as we identify the endpoints of the line to one point, so it is like we glue them together.

$X = [0, 1]$ $X/\sim$

Example 2.20. Let $X = [0, 1] \times S^1$, where $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Then X is a cylinder. If we identify the two end circles (glue them together), we get a torus (donut).



X $X/\sim$

Example 2.21. Let $X = S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, a sphere in $\mathbb{R}^3$. If we identify the two poles, then the sphere collapses.

Lecture 8, 2024/09/20

Proposition 2.22. Let $(X, \mathcal{T})$ and $X/\sim$ be the quotient. Consider the quotient map:

$$q : X \to X/\sim \quad \text{by} \quad x \mapsto [x]$$

Then the collection of set:

$$Q = \{U \subseteq X/\sim : q^{-1}(U) \in \mathcal{T}\}$$

is a topology on $X/\sim$ called the **quotient topology on** $X/\sim$. And it is the largest topology on $X/\sim$ such that q is continuous.

Proof: We have $q^{-1}(\emptyset) = \emptyset$ and $q^{-1}(X/\sim) = X$. If $U_\alpha \in Q$, then:

$$q^{-1}\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha q^{-1}(U_\alpha) \in \mathcal{T}$$

Similarly if $U, V \in Q$ then:

$$q^{-1}(U \cap V) = q^{-1}(U) \cap q^{-1}(V) \in \mathcal{T}$$

Therefore Q is a topology on $X/\sim$. $\square$

Proposition 2.23. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$. A function $f : X/\sim \to Y$ is continuous if and only if the map $f \circ q : X \to Y$ is continuous.

Proof: $(\Rightarrow)$. Since both f and q are continuous, $f \circ q$ is continuous.

$(\Leftarrow)$. Suppose $f \circ q$ is continuous, for $U \in \mathcal{U}$:

$$(f \circ q)^{-1}(U) = q^{-1}(f^{-1}(U)) \in \mathcal{T}$$

By definition of the quotient topology, we must have $f^{-1}(U) \in Q$. $\square$

Theorem 2.24 (Universal Property of Quotients). Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces and let $\sim$ be an equivalence relation on X. For every continuous $f : X \to Y$, which is constant on equivalence classes, there exists a unique function $\overline{f} : X/\sim \to Y$ such that $f = \overline{f} \circ q$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/\sim \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

It turns out that this unique function $\overline{f}$ must be continuous!

Proof: Consider the map $\overline{f} : X/\sim \to Y$ by $\overline{f}([x]) = f(x)$. This function is well-defined because f is constant on equivalence classes. We have:

$$f(x) = \overline{f}([x]) = \overline{f}(q(x))$$

Therefore $f = \overline{f} \circ q$. By the previous proposition, $\overline{f}$ is continuous. If g is another such function, then for all $x \in X$:

$$g([x]) = g(q(x)) = f(x) = \overline{f}([x])$$

Hence the map $\overline{f}$ is unique. □

Example 2.25. Let $X = [0, 1]$. Define $x \sim y$ if $x = y$ or $x, y \in \{0, 1\}$.

Goal: We want to show $X/\sim$ is homeomorphic to S^1 (circle). Consider:

$$f : [0, 1] \to S^1 \text{ by } f(x) = (\cos(2\pi x), \sin(2\pi x))$$

This is continuous and surjective. Note that $f(0) = f(1)$, so it is constant on equivalence classes. By UPQ, there exists continuous $\overline{f} : X/\sim \to S^1$ where $f = \overline{f} \circ q$. We want to check $\overline{f}$ is a homeomorphism.

Surjectivity: For any $y \in S^1$, there is $x \in X$ with $f(x) = y$. Hence $\overline{f}([x]) = y$.

Injectivity: Suppose $\overline{f}([x]) = \overline{f}([y])$ then $f(x) = f(y)$, so $x = y$ or $x, y \in \{0, 1\}$. Hence:

$$x \sim y \implies [x] = [y]$$

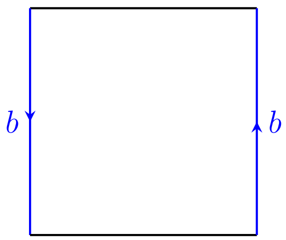
Therefore $\overline{f}$ is injective.

Lastly we want to show $g =$ inverse of $\overline{f}$ is continuous.

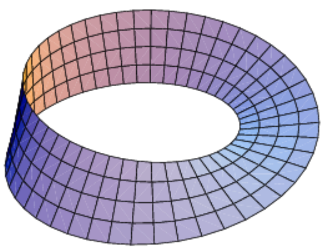
Gap: Since $[0, 1]$ is compact and q is continuous, $q(X) = X/\sim$ is compact. Since $\overline{f} : X/\sim \to S^1$ is invertible, continuous, so $X/\sim$ is compact and S^1 is Hausdorff, so $\overline{f}$ is homeomorphism. □

Remark (Culture). In topology, we rarely give such proofs. We accept proofs by picture/gluing.

Example 2.26. Let $X = [0, 1] \times [0, 1]$. If we identify the two sides in the opposite orientation, we get the **Möbius Strip**.



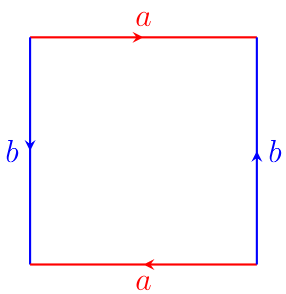
X



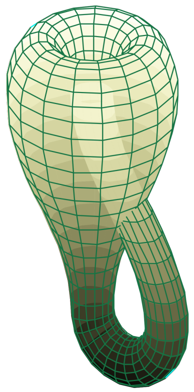
$X/\sim$

In this case, we are "gluing" two blue sides of a paper in the opposite orientation. That is, we glue the diagonal vertices together and get this Möbius strip. Think of this as twisting the paper when we are gluing them together.

Example 2.27. Let $X = [0, 1] \times [0, 1]$. If we identify one pair of sides in opposite orientation and the other one in the same orientation, we get the **Klein Bottle**.

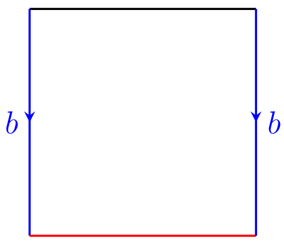


X

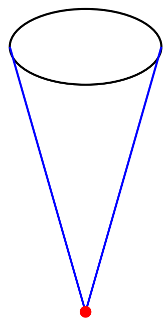


$X/\sim$

Example 2.28. Let $X = [0, 1] \times [0, 1]$. If we glue the opposite sides to get a cylinder, then identify one of the base with a point, we get a cone!



X



$X/\sim$

where the red dot in $X/\sim$ represents the bottom (red) side in X is being identified to a single point.

Lecture 9, 2024/09/23

3 Connectedness

3.1 Connected Spaces

Definition. Let $(X, \mathcal{T})$ be a topological space.

1. We say $X = U \cup V$ is a separation of X if $U, V \in \mathcal{T}$ and $U, V \neq \emptyset$ and $U \cap V = \emptyset$.
2. If a separation exists, we say X is **separated**.
3. Otherwise we say X is **connected**.

Example 3.1. Let $\mathbb{Q}$ be a topological subspace of $\mathbb{R}$, then $\mathbb{Q} = (-\infty, \pi) \cup (\pi, \infty)$, so $\mathbb{Q}$ is separated.

Example 3.2. Let $(\mathbb{R}, \mathcal{T}_f)$ with the finite complement topology. Then every two non-empty open sets intersect. So this space is connected.

Proposition 3.3. Let $(X, \mathcal{T})$, then X is connected $\iff$ clopen subsets of X are $\emptyset$ and X.

Proof: $(\Rightarrow)$. Suppose $A \subseteq X$ is clopen and $A \notin \{\emptyset, X\}$. Then $X = A \cup (X \setminus A)$, but both of these sets are open. So A is separated, contradiction.

$(\Leftarrow)$. Suppose $X = U \cup V$ is a separation. Then $U \in \mathcal{T}$ and $X \setminus U = V$ is open. Then U is clopen. Hence $U = X$ or $U = \emptyset$, which means $U \cup V$ is not a separation. $\square$

Lemma 3.4. Let $X = U \cup V$ be a separation. If $Y \subseteq X$ (subspace topology) is connected, then $Y \subseteq U$ or $Y \subseteq V$.

Proof: First, $Y = (Y \cap U) \cup (Y \cap V)$ and $Y \cap U, Y \cap V$ are open in Y and are disjoint. Since Y is connected, so $Y \cap U = \emptyset$ or $Y \cap V = \emptyset$, and thus $Y \subseteq V$ or $Y \subseteq U$. $\square$

Proposition 3.5. Let $(X, \mathcal{T})$ and $A_\alpha \subseteq X$ be connected for $\alpha \in A$. Then:

$$\bigcap_{\alpha \in A} A_\alpha \neq \emptyset \implies \bigcup_{\alpha \in A} A_\alpha \text{ is connected}$$

Example 3.6. Each $A_n = (n, n + 0.5)$ is connected, but $\bigcup_{n \in \mathbb{N}} A_n$ is separated.

Proof: Let $Y = \bigcup_{\alpha \in A} A_\alpha$ and suppose $Y = U \cup V$ is a separation. WLOG say $p \in U$, where $p \in \bigcap_{\alpha \in A} A_\alpha$. By the lemma, $A_\alpha \subseteq U$ for all $\alpha \in A$. Hence $Y \subseteq U$ and $V = \emptyset$, contradiction. $\square$

Proposition 3.7. Let $(X, \mathcal{T})$ and $A \subseteq X$ is connected. If $A \subseteq B \subseteq \overline{A}$, then B is connected. In particular, $\overline{A}$ is connected.

Proof: Suppose $B = U \cup V$ is a separation of B. Then $A \subseteq B$ and A is connected, so WLOG assume $A \subseteq U$. Thus $\overline{A} \subseteq \overline{U}$. Note that $\overline{U} \cap V = \emptyset$. Indeed, if $x \in \overline{U} \cap V$, then $U \cap V \neq \emptyset$, contradiction. So $B \cap V = \emptyset$, so $V = \emptyset$. □

Proposition 3.8. X connected and $f : X \to Y$ is continuous. Then $f(X)$ is connected.

Proof: Suppose $f(X) = U \cup V$ is a separation. Then:

$$X = \underbrace{f^{-1}(U) \cup f^{-1}(V)}_{\text{open, disjoint, non-empty}}$$

But this is a contradiction. □

Remark (Optional Reading). Let $(X_\alpha, \mathcal{T}_\alpha)$ be connected, then $\prod_{\alpha \in A} X_\alpha$ is connected with respect to the product topology.

Definition. Let $(X, \mathcal{T})$ be a topological space. Define $x \sim y$ if and only if there exists $C \subseteq X$ connected such that $x, y \in C$. Then $\sim$ is an equivalence relation.

<u>Transitivity:</u> If $x, y \in C_1$ and $y, z \in C_2$, then $x, z \in C_1 \cup C_2$. Then $C_1 \cup C_2$ is connected since $y \in C_1 \cap C_2 \neq \emptyset$ by Proposition 3.5.

The equivalence classes are called the **connected components of** X.

Remark. The components of X are pair-wise disjoint and partition X (by this equivalence relation).

Remark. If $A \subseteq X$ is connected, then $A \subseteq C$ for a unique component C.

Lecture 10, 2024/09/25

Proposition 3.9. The connected components of X are connected.

Proof: Let C be a connected component of X. Fix $x_0 \in C$. Then, for $x \in C$, we know $x \sim x_0$. There exists connected set $A_x \subseteq X$ such that $x, x_0 \in A_x$. By the remark, $A_x \subseteq C$. Therefore:

$$C = \bigcup_{x \in C} A_x \quad \text{and} \quad x_0 \in \bigcap_{x \in C} A_x \neq \emptyset$$

Whence C is connected. □

3.2 Path Connectedness

Definition. Let $(X, \mathcal{T})$ be a topological space.

1. A **path** from $a \in X$ to $b \in X$ is a continuous function:

$$f : [0,1] \to X$$

   such that $f(0) = a$ and $f(1) = b$.

2. We say X is **path connected** if for all $a, b \in X$ there exists a path from a to b in X.

Proposition 3.10. Path Connected $\implies$ Conncted.

Proof: Suppose X is path connected but $X = U \cup V$ is a separation. Take $a \in U$ and $b \in V$ and a path $f : [0,1] \to X$ from a to b. Then:

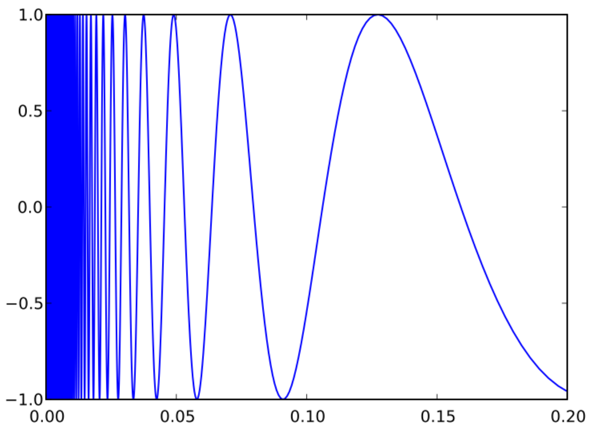
$$[0,1] = f^{-1}(X) = \underbrace{f^{-1}(U)}_{0\in} \cup \underbrace{f^{-1}(V)}_{1\in}$$

This means $[0,1]$ is separated, contradiction. $\square$

Example 3.11 (Topologist's Sine Curve). Let:

$$X = \left\{ \left(x, \sin\left(\frac{1}{x}\right)\right) : 0 \le x \le 1 \right\} \cup \{(0,0)\}$$

Let the $A = X \setminus \{(0,0)\}$. Note that A is path connected, so A is connected. Hence $\overline{A} = X$ is connected.



Now we will show X is not path connected. Suppose for a contradiction that X is path connected and let f be a path with $f(0) = (0,0)$ and $f(1) = (1/\pi, 0)$. Write:

$$f(t) = (a(t), b(t))$$

The Interdemiate Value Theorem says that there exists $0 < t < 1$ such that $a(t_1) = 2/3\pi$. Again there exists $0 < t_2 < t_1$ such that $a(t_2) = 2/5\pi$. Continue this way, there exists a decreasing sequence $(t_n) \subseteq [0, 1]$ such that:

$$a(t_n) = \frac{2}{(2n+1)\pi}$$

By MCT we have $t_n \to t \in [0, 1]$. However $b(t_n) \to b(t)$ and:

$$b(t_n) = \sin\left(\frac{(2n+1)\pi}{2}\right) = (-1)^n$$

This is a contradiction since $b(t_n)$ diverges.

4 Compactness

4.1 Compact Spaces

Definition. Let $(X, \mathcal{T})$ be a topological space.

1. An **open cover** of X is a collection $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$, for $\alpha \in A$ such that $X = \bigcup_{\alpha \in A} U_\alpha$.
2. If $B \subseteq A$ and $X = \bigcup_{\alpha \in B} U_\alpha$, we call $\{U_\alpha\}_{\alpha \in B}$ a **subcover**. If $|B| < \infty$, we call it a **finite subcover**.

Definition. Let $(X, \mathcal{T})$ be a topological space. We say X is **compact** if every open cover $\{U_\alpha\}_{a \in A}$ has a finite subcover.

Big Idea: Compactness is a bridge to finiteness ("smallness").

Example 4.1. Let $(X, \mathcal{T}_f)$ with finite complement topology. Suppose $X = \bigcup_\alpha U_\alpha$ is an open cover and each U_α is non-empty. Fix U_0, then:

$$U_0 = X \setminus \{x_1, \cdots, x_n\}$$

Say $x_i \in U_i$, then $X = U_0 \cup U_1 \cup \cdots \cup U_n$, so X is compact.

Example 4.2. Let $(\mathbb{R}, \mathcal{T}_c)$ with countable complement topology:

$$U_n = \mathbb{R} \setminus \{n, n+1, n+2, \cdots\} \text{ and } \mathbb{R} = \bigcup_{n \in \mathbb{N}} U_n$$

But this admits no finite subcover. Suppose $\mathbb{R} = U_{n_1} \cup \cdots \cup U_{n_k}$ and $n_1 < \cdots < n_k$. Then $n_k \notin \mathbb{R}$, which is a contradiction.

Lemma 4.3 (Peter's Confusion). Let $(X, \mathcal{T})$ and $A \subseteq X$. Then A is compact (under the subspace topology) if and only if for all open cover $U_\alpha \in \mathcal{T}$ of X:

$$A \subseteq \bigcup_\alpha U_\alpha \implies A \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$$

for some $\alpha_1, \cdots, \alpha_n$.

Proof: $(\Rightarrow)$. Suppose $A \subseteq \bigcup_\alpha U_\alpha$, so $A = \bigcup_\alpha (A \cap U_\alpha)$. Hence:

$$A = (A \cap U_{\alpha_1}) \cup \cdots \cup (A \cap U_{\alpha_n}) = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$$

$(\Leftarrow)$. Suppose $A = \bigcup_\alpha (A \cap U_\alpha)$, so $A \subseteq \bigcup_\alpha U_\alpha$ and hence:

$$A \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$$

Hence $A = (A \cap U_{\alpha_1}) \cup \cdots \cup (A \cap U_{\alpha_n})$. □

Lecture 11, 2024/09/27

Proposition 4.4. Let $(X, \mathcal{T})$ be compact. If $C \subseteq X$ is closed, then C is compact.

Proof: Suppose $C \subseteq \bigcup_\alpha U_\alpha$ with $U_\alpha \in \mathcal{T}$. Thus $X = (X \setminus C) \cup \bigcup_\alpha U_\alpha$. Since X is compact we have that:

$$X = (X \setminus C) \cup U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$$

Hence we have $C \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$ as desired. $\square$

Example 4.5. The converse is false! $(\mathbb{R}, \mathcal{T}_f)$ with finite-complement topology is compact. Exercise: All subsets of $\mathbb{R}$ are compact, so $\mathbb{N}$ is compact but NOT closed.

Proposition 4.6. Let $(X, \mathcal{T})$ be Hausdorff. If $K \subseteq X$ is compact, then K is closed.

Proof: Let $K \subseteq X$ be compact. We want to show that $X \setminus K$ is open. Fix $x_0 \in X \setminus K$. For all $x \in K$, there exists U_x and V_x in $\mathcal{T}$ such that $U_x \cap V_x = \emptyset$ and $x_0 \in U_x$ and $x \in V_k$. Then $K \subseteq \bigcup_{x \in K} V_x$, and since K is compact:

$$K \subseteq V_{x_1} \cup \cdots \cup V_{x_n}$$

Now consider $x_0 \in U := U_{x_1} \cap \cdots \cap U_{x_n} \in \mathcal{T}$. Notice that $x_0 \in U \subseteq X \setminus K$, hence $\operatorname{int}(X \setminus K) = X \setminus K$, so K is closed as desired. $\square$

Proposition 4.7. Let X be Hausdorff and $K \subseteq X$ be compact. For all $x \in X \setminus K$, there exists $U, V \in \mathcal{T}$ such that $x \in U$ and $K \subseteq V$ and $U \cap V = \emptyset$.

Proof: Let $x \in X \setminus K$. For each $y \in K$, since X is Hausdorff, there exist disjoint $U_y, V_y \in \mathcal{T}$ such that $x \in U_y$ and $y \in V_y$. Since K is compact, we have:

$$K \subseteq \bigcup_{y \in K} V_y \implies K \subseteq V_{y_1} \cup \cdots \cup V_{y_n} =: V$$

Define $U = U_{y_1} \cap \cdots \cap U_{y_n}$. Then U, V are both open and $x \in U$ and $K \subseteq V$. Note that if $z \in U \cap V$, then $z \in V$ means $v \in V_{y_i}$ for some i. However, we also have $z \in U_{y_i}$, which contradicts that U_{y_i} and V_{y_i} are disjoint. $\square$

Proposition 4.8. Let $(X, \mathcal{T})$ be compact and $f : X \to Y$ be continuous, then $f(X)$ is compact.

Proof: Suppose $f(X) \subseteq \bigcup_\alpha U_\alpha$ and $U_\alpha \subseteq Y$ is open. Then $X = \bigcup_\alpha f^{-1}(U_\alpha)$, hence:

$$X = f^{-1}(U_{\alpha_1}) \cup \cdots \cup f^{-1}(U_{\alpha_n}) \implies f(X) \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$$

As desired. $\square$

Proposition 4.9. Let $(X, \mathcal{T})$ be compact and $(Y, \mathcal{U})$ be Hausdorff. If $f : X \to Y$ is continuous and bijective, then f is a homeomorphism.

Proof: We want to show that if $C \subseteq X$ is closed, then $(f^{-1})^{-1}(C) = f(C)$ is closed. Since X is compact, and $C \subseteq X$ is closed, C is compact. Since f is continuous so $f(C)$ is compact. However, Y is Hausdorff and so C is also closed. $\square$

4.2 Tychonoff's Theorem

Theorem 4.10 (Tychonoff's Theorem). If $(X_\alpha, \mathcal{T}_\alpha)$ is compact for each $\alpha \in A$, then $\prod_{\alpha \in A} X_\alpha$ is compact (with resepct to the product topology).

Fact. Tychonoff's Theorem is equivalent to the axiom of choice.

Definition. Let X be a set. We say $\leq$ is a **partial order** on X if:

(a) For all $x \in X$ we have $x \leq x$.

(b) For all $x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$.

(c) For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

We call $(X, \leq)$ a **partially ordered set (poset)**. Let X be a poset.

(a) We say $A \subseteq X$ is a **chain** if for all $a, b \in A$, we have $a \leq b$ or $b \leq a$.

(b) We say $x \in X$ is **maximal** if and only if for all $y \in X$, $x \leq y \implies x = y$.

(c) Let $A \subseteq X$ be a chain, an **upper bound** for A is any $x \in X$ such that $a \leq x$ for all $a \in A$.

Theorem 4.11 (Zorn's Lemma). Let $(X, \leq)$ be a poset. If every chain of X has an upper bound, then X has a maximal element.

Lecture 12, 2024/09/30

Definition. Let $\mathcal{C} \subseteq \mathcal{P}(X)$. We have $\mathcal{C}$ has the **finite intersection property (FIP)** for all $F_1, \cdots, F_n \in \mathcal{C}$ we have $F_1 \cap \cdots \cap F_n \neq \emptyset$.

Proposition 4.12. Let $(X, \mathcal{T})$. Then X is compact if and only only whenever $\mathcal{C}$ is a family of closed sets in X having FIP, we have $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

Proof: $(\Rightarrow)$. Homework.

$(\Leftarrow)$. Suppose X satisfies the condition on such families of closed sets. Consider:

$$X = \bigcup_{U_\alpha \in U} U_\alpha \implies \emptyset = \bigcap_{U_\alpha \in U} (X \setminus U_\alpha)$$

Therefore $\emptyset = (X \setminus U_{\alpha_1}) \cap \cdots \cap (X \setminus U_{\alpha_n})$, hence:

$$X = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$$

As desired. $\square$

Lemma 4.13. Let $(X, \mathcal{T})$. Let $\mathcal{C} \subseteq \mathcal{P}(X)$ be a family of closed sets having the FIP. There exists $\mathcal{C} \subseteq \mathcal{F} \subseteq \mathcal{P}(X)$ which is maximal with respect having the FIP.

Proof: Let $Y = \{\mathcal{K} \subseteq \mathcal{P}(X) : \mathcal{C} \subseteq \mathcal{K},\ \mathcal{K} \text{ has the FIP}\}$ and order Y via $\subseteq$. Note that $\mathcal{C} \in Y$ so $Y \neq \emptyset$. Let $S \subseteq Y$ be a chain. Consider $Z = \bigcup_{A \in S} A$. Note that $\mathcal{C} \subseteq Z$ since $C \subseteq A$ for all $A \in S$.

Claim: Z has the FIP.

Proof (Claim): Let $F_1, \cdots, F_n \subseteq Z$. Say each $F_i \in A_i \in S$. Since S is a chain, WLOG suppose $A_i \subseteq A_1$ for all $i \in \{1, \cdots, n\}$. Then $F_1, \cdots, F_n \in A_1$. Then:

$$F_1 \cap \cdots \cap F_n \neq \emptyset$$

since $A_1 \in Y$ has the FIP. Therefore Z has the FIP. (QED Claim)

By the claim, Z is an upper bound for the chain S. Hence by Zorn's Lemma, Y has a maximal element $\mathcal{F}$ as desired. $\square$

Lemma 4.14. Let $(X, \mathcal{T})$. Let $\mathcal{C} \subseteq \mathcal{F} \subseteq \mathcal{P}(X)$ be as before.

(1) $\mathcal{F}$ is closed under finite intersections.

(2) If $A \subseteq X$ intersects every $F \in \mathcal{F}$, then $A \in \mathcal{F}$.

Proof: (1). Let $F_1, \cdots, F_n \in \mathcal{F}$, then $\mathcal{F} \cup \{F_1 \cap \cdots \cap F_n\}$ has the FIP. By the maximality of $\mathcal{F}$ we get $\mathcal{F} \cup \{F_1 \cap \cdots \cap F_n\} = \mathcal{F}$ and $F_1 \cap \cdots \cap F_n \in \mathcal{F}$.

(2). Note that $\mathcal{F} \cup \{A\}$ has the FIP, then $A \in \mathcal{F}$. $\square$

Proof (Tychonoff Theorem): Let $\mathcal{C}$ be a family of closed sets in $X = \prod X_\alpha$ having the FIP. Consider $\mathcal{C} \subseteq \mathcal{F}$ maximal with respect to FIP. Define:

$$\mathcal{A}_\alpha = \{\pi_\alpha(F) : F \in \mathcal{F}\}$$

Claim 1: $\mathcal{A}_a$ has the FIP.

Proof (Claim 1): Suppose $\pi_\alpha(F_1) \cap \cdots \cap \pi_\alpha F(_n) = \emptyset$, so:

$$F_1 \cap \cdots \cap F_n \subseteq \pi_\alpha^{-1}(\pi_\alpha(F_1)) \cap \cdots \cap \pi_\alpha^{-1}(\pi_\alpha(F_n)) = \pi_\alpha^{-1}(\emptyset) = \emptyset$$

This is a contradiction. (QED Claim 1)

Claim 2: The intersection $\bigcap_{A\in\mathcal{A}_\alpha}\overline{A}\neq\emptyset$.

Proof (Claim 2): Suppose for a contradiction $\bigcap_{A\in\mathcal{A}_\alpha}\overline{A}=\emptyset$, then by the compactness of X_α:

$$X_\alpha = X_\alpha\setminus\bigcap_{A\in\mathcal{A}_\alpha}\overline{A}=\bigcup_{A\in\mathcal{A}_\alpha}(X_\alpha\setminus\overline{A})=(X_\alpha\setminus\overline{A_1})\cup\cdots\cup(X_\alpha\setminus\overline{A_n})$$

Hence $\overline{A_1}\cap\cdots\cap\overline{A_n}=\emptyset$ and $A_1\cap\cdots\cap A_n=\emptyset$. Contradiction. (QED Claim 2)

By Claim 2, let $p_\alpha\in\bigcap_{A\in\mathcal{A}_\alpha}\overline{A}$. Consider $p\in X$ such that $\pi_\alpha(p)=p_\alpha$ for all α.

Claim 3: We have $p\in\bigcap_{F\in\mathcal{F}}\overline{F}$.

Lecture 13, 2024/10/02

Note that if we proved Claim 3, then we have:

$$p\in\bigcap_{F\in\mathcal{F}}\overline{F}\subseteq\bigcap_{C\in\mathcal{C}}\overline{C}=\bigcap_{C\in\mathcal{C}}C$$

and we are done.

Proof (Claim 3): Let $F\in\mathcal{F}$, we want to show $p\in\overline{F}$. We need to show every neighborhood of p intersects F. Suppose U is a neighborhood of p, then:

$$U=\pi_{\alpha_1}^{-1}(U_{\alpha_1})\cap\cdots\cap\pi_{\alpha_n}^{-1}(U_{\alpha_n})$$

where $U_{\alpha_i}\in\mathcal{T}_{\alpha_i}$. For $i=1,\cdots,n$ we have:

$$\pi_{\alpha_i}(p)=p_{\alpha_i}\in U_{\alpha_i}$$

For all $A\in\mathcal{A}_{\alpha_i}$, we have $p_{\alpha_i}\in U_{\alpha_i}\cap\overline{A}$. Thus $U_{\alpha_i}\cap A\neq\emptyset$. Then for all $F\in\mathcal{F}$, there exists $z\in U_{\alpha_i}\cap\pi_{\alpha_i}(F)$. Say $z=\pi_{\alpha_i}(f)$ for some $f\in F$. Then $f\in\pi_{\alpha_i}^{-1}(U_{\alpha_i})\cap F$. Therefore, for all $F\in\mathcal{F}$, we have $F\cap\pi_{\alpha_i}^{-1}(U_{\alpha_i})\neq\emptyset$. By (2) of Lemma 4.14, we have $\pi_{\alpha_i}^{-1}(U_{\alpha_i})\in\mathcal{F}$. By (1) of Lemma 4.14, we have $U\in\mathcal{F}$. Then for all $F\in\mathcal{F}$, $U\in\mathcal{F}$ so $U\cap F\neq\emptyset$ by FIP. Thus:

$$p\in\bigcap_{F\in\mathcal{F}}\overline{F}$$

Therefore we are done. (QED Claim 3). Hence we finished the proof. $\square$

5 Countability and Separation

5.1 Countability

Definition. Let $(X, \mathcal{T})$ be a topological space and fix $x \in X$. A **basis at** $x \in X$ is a collection $\mathcal{B}$ of neighborhoods of x such that whenever $x \in U \in \mathcal{T}$, then there exists $B \in \mathcal{B}$ such that $B \subseteq U$.

Definition. We say $(X, \mathcal{T})$ is **first countable** if for all $x \in X$, there exists a countable basis at x.

Example 5.1. Let (X, d) be a metric space. Fix $x \in X$, then:

$$\mathcal{B}_x = \{B_q(x) : q \in \mathbb{Q}^+\}$$

is a countable basis for x.

Idea: $(X, \mathcal{T})$ is first countable if and only if x has a strong relationship with countability.

Proposition 5.2. Let $(X, \mathcal{T})$ be first countable and $A \subseteq X$.

1. $x \in \overline{A} \iff$ there exists a sequence $(a_n) \subseteq A$ such that $a_n \to x$.
2. $f : X \to Y$ is continuous $\iff$ for all sequence (x_n), we have $x_n \to x$ in X implies $f(x_n) \to f(x)$.

Proof: (1). $(\Leftarrow)$. See Assignment 1 because every sequence is a net.

$(\Rightarrow)$. Suppose $x \in \overline{A}$. Let $\mathcal{B} = \{B_n\}_{n=1}^{\infty}$ be a basis at x. This countable basis exists because $(X, \mathcal{T})$ is first countable. Take $a_1 \in B_1 \cap A$ and $a_2 \in B_1 \cap B_2 \cap A$. In general, choose:

$$a_n \in B_1 \cap \cdots \cap B_n \cap A$$

We claim that $a_n \to x$. Let $U \in \mathcal{T}$ with $x \in U$, there exists $N \in \mathbb{N}$ such that $B_N \subseteq U$. For all $n \geq N$ we get $a_n \in B_n \subseteq B_N \subseteq U$. As desired.

(2). $(\Leftarrow)$. See Assignment 1 again.

$(\Rightarrow)$. Let $A \subseteq X$, we want to use Proposition 2.2 to prove f is continuous. We claim that:

$$f(\overline{A}) \subseteq \overline{f(A)}$$

Let $y \in f(\overline{A})$ so that $y = f(x)$ with $x \in \overline{A}$. By 1, there exists $(a_n) \subseteq A$ such that $a_n \to x$. Then $f(a_n) \to f(x)$ with each $f(a_n) \in f(A)$. Hence $y \in \overline{f(A)}$. $\square$

Definition. Let $(X, \mathcal{T})$ be a topological space. We say X is **second countable** if X has a countable basis. That is, there is a basis $\mathcal{B}$ with $|\mathcal{B}| \leq |\mathbb{N}|$.

Proposition 5.3. Second countable $\implies$ First countable.

Proof: Let $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ be a basis. For $x \in X$ define:

$$\mathcal{B}_x = \{B_n : x \in B_n\}$$

is a basis at x. $\square$

Example 5.4. Consider $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$, this is the metric topology by the discrete metric on $\mathbb{R}$:

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Therefore $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ is first countable. However, every basis for $\mathbb{R}$ must contain all $\{x\}$ for all $x \in \mathbb{R}$. Thus every basis for $\mathbb{R}$ is uncountable, so $\mathbb{R}$ is not second countable.

Definition. Let $(X, \mathcal{T})$ and $A \subseteq X$. We say A is **dense** in X if $\overline{A} = X$.

Definition. Let $(X, \mathcal{T})$. We say X is **separable** if X has a countable, dense subset.

Example 5.5. $\mathbb{Q} \subseteq \mathbb{R}$ and $\overline{\mathbb{Q}} = \mathbb{R}$, so $\mathbb{R}$ is separable.

Definition. We say $(X, \mathcal{T})$ is **Lindelöf** if every open cover of X has a countable subcover.

Proposition 5.6. If $(X, \mathcal{T})$ is second countable, then X is separable and Lindelöf.

Remark. The lower limit topology on $\mathbb{R}$ is separable and Lindelöf but not second countable.

Lecture 14, 2024/10/07

Proof of Proposition 5.6: Let $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ be a countable basis for X. We may assume that each $B_n \neq \emptyset$. We will prove the two conclusions separately.

Claim 1: X is separable.

Proof (Claim 1): Take $x_n \in B_n$ and consider $D = \{x_n : n \in \mathbb{N}\}$. Let $x \in X$ and let $x \in U$ for some open set U. This means $x \in B_n \subseteq U$ for some $n \in \mathbb{N}$ by the definition of basis. Hence $x_n \in U \cap D$ and $x \in \overline{D}$. Hence $\overline{D} = X$. Therefore X is separable. (QED Claim 1)

Claim 2: X is Lindelöf.

Proof (Claim 2): Suppose $X = \bigcup_\alpha U_\alpha$ is an open cover. Let:

$$I = \{n \in \mathbb{N} : \exists\, \alpha \text{ such that } B_n \subseteq U_\alpha\}$$

Say $B_n \subseteq U_{\alpha_n}$ for all $n \in I$. We claim that:

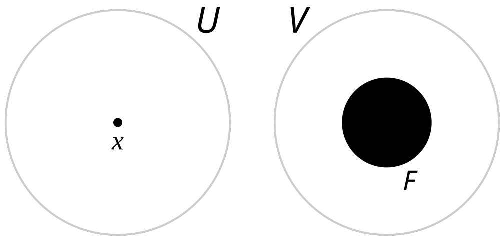
$$X = \bigcup_{n \in I} U_{\alpha_n}$$

This is a countable union since $|I| \leq |\mathbb{N}|$. Take $x \in X$, so there exists α such that $x \in U_\alpha$. So there exists $n \in \mathbb{N}$ such that $x \in B_n \subseteq U_\alpha$. Therefore $n \in I$ and thus $x \in U_{\alpha_n}$. As desired. $\square$

5.2 Separation

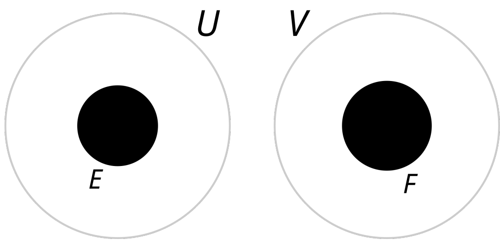
Note (Convention). For this section, we assume every topological space $(X, \mathcal{T})$ is T_1. That is, all singleton set $\{x\}$ is closed in X.

Definition. We say $(X, \mathcal{T})$ is **regular** if for all $x \in X$ and for all $F \subseteq X$ closed with $x \notin F$, then there exists disjoint open $U, V \in \mathcal{T}$ such that $x \in U$ and $F \subseteq V$.



This means we can separate a point and a closed set using disjoint open sets.

Definition. We say $(X, \mathcal{T})$ is **normal** if for all disjoint closed $E, F \subseteq X$, there exists disjoint $U, V \in \mathcal{T}$ such that $E \subseteq U$ and $F \subseteq V$.



This means we can separate two disjoint closed sets using disjoint open sets.

Remark. Using our convetion (every space is T_1), then:

$$\text{Normal} \implies \text{Regular} \implies \text{Hausdorff}$$

Recall. If $(X, \mathcal{T})$ is compact and Hausdorff, then X is regular.

Lemma 5.7. Let $(X, \mathcal{T})$ be a topological space.

(1) X is regular $\iff$ $\forall x \in X$ and all $x \in U \in \mathcal{T}$, there exists $x \in V \in \mathcal{T}$ such that $\overline{V} \subseteq U$.

(2) X is normal $\iff$ $\forall A \subseteq X$ closed and $A \subseteq U \in \mathcal{T}$, there exists $A \subseteq V \in \mathcal{T}$ such that $\overline{V} \subseteq U$.

Proof: (1). ($\Rightarrow$). Suppose X is regular. Let $x \in X$ and let U be a neighborhood of x. Consider $C = X \setminus U$. By Regularity, there exist disjoint $V, W \in \mathcal{T}$ such that $x \in V$ and $C \subseteq W$. We claim that $\overline{V} \subseteq U$. Indeed, suppose $z \in \overline{V}$ but $z \notin U$. Then $z \in C \subseteq W$ and so $V \cap W \neq \emptyset$. Contradiction.

($\Leftarrow$). Let $x \in X$ and let $C \subseteq X$ be closed with $x \notin C$. Then $U := X \setminus C$ is open with $x \in U$. By assumption, there exists $x \in V \in \mathcal{T}$ such that $\overline{V} \subseteq U$. Let $W = X \setminus \overline{V}$, which is open. Note that $W \cap V = \emptyset$. Finally we have:

$$C = X \setminus U \subseteq X \setminus \overline{V} = W$$

(2). Very similar to (1). □

Proposition 5.8. Every metric space is normal.

Proof: Homework. □

Proposition 5.9. If X is compact and Hausdorff, then X is normal.

Proof: Let A, B be disjoint closed sets. Since X is regular, for all $x \in A$, there exists $x \in U_x \in \mathcal{T}$ and $B \subseteq V_x \in \mathcal{T}$ with $U_x \cap V_x = \emptyset$. Then $A \subseteq \bigcup_{x \in X} U_x$. Since A is closed in a comapct space X, we know A is compact. Therefore:

$$A \subseteq U_{x_1} \cup \cdots \cup U_{x_n}$$

for some $x_i \in X$. Finally $B \subseteq V_{x_1} \cap \cdots \cap V_{x_n}$. □

Remark. See A4. If $(X, \mathcal{T})$ is compact and Hausdorff, then:

$$X \text{ second countable} \iff X \text{ is metrizable (“super normal”)}$$

Here, X is **metrizable** means we can define a metric d on X such that $\mathcal{T}$ is equal to the metric topology coming from d.

Lecture 15, 2024/10/09

Fact (Unusable). Regular and second countable $\implies$ metrizable.

5.3 Urysohn's Lemma

Theorem 5.10 (Urysohn's Lemma). Let $(X, \mathcal{T})$ be normal and A, B be disjoint closed sets. There exists a continuous map $f : X \to [0, 1]$ such that $f|_A = 0$ and $f|_B = 1$.

Proof: Let $P = \{p_n\}_{n=1}^{\infty}$ be an enumeration of $\mathbb{Q} \cap [0, 1]$ with $p_1 = 1$ and $p_2 = 0$.

Step 1: For $p, q \in P$, we will construct $U_p, U_q \in \mathcal{T}$ such that:

$$p < q \implies \overline{U_p} \subseteq U_q \tag{$*$}$$

We proceed by induction. Set $U_1 = X \setminus B$. By normality, there exists $U_0 \in \mathcal{T}$ such that:

$$A \subseteq U_0 \subseteq \overline{U}_0 \subseteq U_1$$

Suppose $P_n = \{p_1, \cdots, p_n\}$ and that for all $p \in P_n$, there exists $U_p \in \mathcal{T}$ such that:

$$p, q \in P_n \ \text{ and } \ p < q \implies \overline{U_p} \subseteq U_q$$

Let $r = p_{n+1}$. Say $p, q \in P_n$ are closest such that $p < r < q$. Note that $\overline{U}_p \subseteq U_q$. By normality, there exists $U_r \in \mathcal{T}$ such that $\overline{U}_p \subseteq U_r$ and $\overline{U}_r \subseteq U_q$. For $s \in P_n$,

$$\begin{aligned} s \leq p &\implies \overline{U}_s \subseteq \overline{U}_p \subseteq U_r \\ s \geq q &\implies \overline{U}_r \subseteq U_q \subseteq \overline{U}_q \subseteq U_s \end{aligned}$$

This proved $(*)$ for p_{n+1}. By induction, such $U_p \in \mathcal{T}$ exists for all $p \in P$.

Step 2: For $p \in \mathbb{Q}$, we set:

$$U_p = \begin{cases} \emptyset & \text{if } p < 0 \\ X & \text{if } p > 1 \end{cases}$$

Therefore, for all $p, q \in \mathbb{Q}$ we have $p < q \implies \overline{U_p} \subseteq U_q$.

Step 3: For $x \in X$, we let:

$$Q(x) = \{p \in \mathbb{Q} : x \in U_p\}$$

Note that $Q(x) \subseteq [0, \infty)$ and $(1, \infty) \subseteq Q(x)$ by Step 2. Let $f(x) = \inf Q(x)$. This exists since $Q(x)$ is bounded below by 0.

Step 4: We will show $f : X \to \mathbb{R}$ by $f(x) = \inf Q(x)$ is continuous and $f|_A = 0$ and $f|_B = 1$.

1. If $x \in A$, then $x \in U_0$ and $0 \in Q(x)$, so $f(x) = 0$ and $f|_A = 0$.
2. If $x \in B$, then for $p < 1$ we have $\overline{U}_p \subseteq U_1 = X \setminus B$, so $x \notin U_p$. This means $Q(x)$ does not contain any $p \in \mathbb{Q}$ with $p < 1$. Since we know $(1, \infty) \subseteq Q(x)$, we must have $f(x) = \inf Q(x) = 1$ and thus $f|_B = 1$.

Now we must prove f is continuous. First we note that:

a) $x \in \overline{U}_r \implies x \in U_s$ for some $r < s \implies f(x) \leq r$.

b) $x \notin U_r \implies x \notin U_s$ for $s < r \implies f(x) \geq r$.

Fix $x_0 \in X$ with $f(x_0) \in (c, d) \subseteq \mathbb{R}$. Choose $p, q \in \mathbb{Q}$ such that:

$$c < p < f(x_0) < q < d$$

Let $U = U_q \setminus \overline{U_p}$ so that $x_0 \in U$ by (a) and (b) above. We must prove $f(U) \subseteq (c, d)$. For $x \in U$:

$$x \in U_q \subseteq \overline{U}_q \implies f(x) \leq q \qquad \text{(by (a))}$$
$$x \notin \overline{U}_p \implies x \notin U_p \implies f(x) \geq p \qquad \text{(by (b))}$$

Therefore $f(x) \in [p, q] \subseteq (c, d)$, so $f(u) \subseteq (c, d)$ and f is continuous. □

Remark. We can extend this to $[a, b]$. (Homework)

Theorem 5.11 (Tietze Extension Theorem). Let $(X, \mathcal{T})$ be normal and $A \subseteq X$ is closed. Every continuous $f : A \to \mathbb{R}$ can be extended to a continuous function $f : X \to \mathbb{R}$.

Lemma 5.12. Let $(X, \mathcal{T})$ be normal and $A \subseteq X$. If $f : A \to [-r, r]$ is continuous, then there exists continuous $g : X \to \mathbb{R}$ such that:

1. For all $x \in X$, $|g(x)| \leq \frac{r}{3}$.
2. For all $a \in A$, $|g(a) - f(a)| < \frac{2r}{3}$.

Lecture 16, 2024/10/11

Proof: Define the sets:

$$I_1 = \left[-r, \frac{-r}{3}\right] \quad \text{and} \quad I_2 = \left[\frac{-r}{3}, \frac{r}{3}\right] \quad \text{and} \quad I_3 = \left[\frac{r}{3}, r\right]$$

Consider $C = f^{-1}(I_1)$ and $D = f^{-1}(I_3)$, these are closed and disjoint. By Urysohn's Lemma, there exists a continuous map $g : X \to [-\frac{r}{3}, \frac{r}{3}]$ such that:

$$g|_C = -\frac{r}{3} \quad \text{and} \quad g|_D = \frac{r}{3}$$

Then for all x we have $|g(x)| \leq \frac{r}{3}$ and for all $a \in A$:

a) If $a \in C$, then $f(a), g(a) \in I_1$.

b) If $a \in D$, then $f(a), g(a) \in I_3$.

c) If $a \notin C \cup D$, then $f(a), g(a) \in I_2$.

Since the length of I_1, I_2, I_3 are all $2r/3$, we are done. □

Proof (Tietze Extension Theorem): Consider the following cases.

Case 1: $f(A) \subseteq [-1, 1]$. By the lemma, there exists a continuous $g_1 : X \to \mathbb{R}$ such that:

$$|g_1(x)| \leq \frac{1}{3} \quad \text{and} \quad |f(a) - g_1(a)| \leq \frac{2}{3}$$

Then $(f - g_1) : A \to [\frac{-2}{3}, \frac{2}{3}]$. The lemma implies there exists continuous $g_2 : X \to \mathbb{R}$ such that:

$$|g_2(x)| \leq \frac{1}{3} \text{ and } |f(a) - g_1(a) - g_2(a)| \leq \left(\frac{2}{3}\right)^2$$

Then $(f - g_1 - g_2) : A \to [-(\frac{2}{3})^2, (\frac{2}{3})^2]$. Keep doing this, for all $n \in \mathbb{N}$, there exists continuous function $g_n : X \to \mathbb{R}$ such that:

$$|g_n(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} \text{ and } |f(a) - g_1(a) - \cdots - g_n(a)| \leq \left(\frac{2}{3}\right)^n$$

Now consider $g : X \to \mathbb{R}$ given by:

$$g(x) = \sum_{n=1}^{\infty} g_n(x)$$

which converges by comparison with the series:

$$\sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} = 1$$

By the Weierstrass M-Test, g is continuous. Now we defined a function on X and we want to check this g extends f. Indeed, for all $a \in A$:

$$\left|f(a) - \sum_{i=1}^{n} g_i(a)\right| \leq \left(\frac{2}{3}\right)^n \to 0$$

It follows that $f(a) = g(a)$, hence $g|_A = f$. Notice that the range of g is contained in $[-1, 1]$, that is:

$$g : X \to [-1, 1] \text{ since } |g(x)| \leq \sum_{n=1}^{\infty} |g_n(x)| \leq 1$$

<u>Case 2:</u> $f(A) \subseteq (-1, 1)$. By Case 1, we can extend f to a continuous $g : X \to [-1, 1]$. Consider $D = g^{-1}(\{-1, 1\})$, which is closed. Then $g(A) = f(A) \subseteq (-1, 1)$. Hence:

$$A \cap D = A \cap g^{-1}(\{-1, 1\}) = \emptyset$$

By Urysohn's Lemma, there exists continuous $\varphi : X \to [0, 1]$ with $\varphi|_D = 0$ and $\varphi|_A = 1$. Consider $h : X \to (-1, 1)$ by $h(x) = \varphi(x)g(x)$. For $a \in A$:

$$h(a) = \varphi(a)g(a) = g(a) = f(a)$$

<u>Case 3:</u> $f : A \to \mathbb{R}$, the general case. Let $\psi : \mathbb{R} \to (-1, 1)$ be a homeomorphism. Then:

$$\psi \circ f : A \to (-1, 1) \text{ is continuous}$$

By Case 2, we may extend $\psi \circ f$ to a continuous map $g : X \to (-1, 1)$. Then $\psi^{-1} \circ g : X \to \mathbb{R}$ is continuous. For all $a \in A$ we have:

$$\psi^{-1}(g(a)) = \psi^{1}(\psi(f(a))) = f(a)$$

Hence $\psi^{-1} \circ g$ extends f, as desired. $\square$

5.4 Topological Manifolds

Goal: We want to use Urysohn's Lemma to study topological manifolds.

Definition. For $m \in \mathbb{N}$, an m-**manifold** is a Hausdorff, second-countable space $(X, \mathcal{T})$ such that for all $x \in X$, there exists $x \in U \in \mathcal{T}$ such that U is homeomorphic to a subset of $\mathbb{R}^m$.

Definition. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$. An **embedding** of X into Y is a continuous injective $f : X \to Y$ such that $f^{-1} : f(X) \to X$ is also continuous. That is, $f : X \to f(X)$ is a homeomorphism.

Theorem 5.13. If X is a compact m-manifold, then X can be embedded in $\mathbb{R}^N$ for some N.

Lecture 17, 2024/10/21

Definition. Let $(X, \mathcal{T})$ and $f : X \to \mathbb{R}$. The **support** of f is $\mathrm{Supp}(f) = \{x \in X : f(x) \neq 0\}$.

Definition. Let $(X, \mathcal{T})$. Let $U_1, \cdots, U_n$ be a cover of X. A **partition of unity** dominated by $\{U_i\}$ is a family of continuous $\varphi_i : X \to [0, 1]$ for $i \in \{1, \cdots, n\}$ such that:

1. $\mathrm{Supp}(P_i) \subseteq U_i$.
2. For all $x \in X$ we have $\sum_{i=1}^{n} \varphi_i(x) = 1$.

Proposition 5.14. Let $(X, \mathcal{T})$ be normal. If $U_1, \cdots, U_n$ is a finite open cover of X, then there is a partition of unity dominated by $\{U_i\}$.

Proof: First, we find an open cover $V_1, \cdots, V_n$ of X such that $\overline{V_i} \subseteq U_i$. To start, let:

$$A = \underbrace{X \setminus (U_2 \cup \cdots U_n)}_{\text{closed}} \subseteq U_1$$

By the normality lemma, there is $V_1 \in \mathcal{T}$ such that $A \subseteq V_1 \subseteq \overline{V_1} \subseteq U_1$. Note that $V_1, U_2, \cdots, U_n$ covers X. Inductively, assume:

$$V_1, \cdots, V_{k-1}, U_k, \cdots, U_n \text{ covers } X$$

with $\overline{V_i} \subseteq U_i$. Now we let:

$$A = X \setminus (V_1 \cup \cdots \cup V_{k-1} \cup U_{k+1} \cup \cdots \cup U_n) \subseteq U_k$$

Similarly we find $V_K \in \mathcal{T}$ such that $A \subseteq V_k \subseteq \overline{V_k} \subseteq U_k$ and $V_1, \cdots, V_k, U_{k+1}, \cdots, U_n$ covers X. By induction, such a cover $\{V_1, \cdots, V_n\}$ exists. Once again, let $W_1, \cdots, W_n$ be a cover of X such that $\overline{W_i} \subseteq V_i$. Note that $\overline{W_i}$, $X \setminus V_i$ are disjoint, closed sets. By Urysohn, there exist continuous maps $\psi_i : X \to [0,1]$ with:

$$\psi_i|_{\overline{W_i}} = 1 \quad \text{and} \quad \psi_i|_{X \setminus V_i} = 0$$

Therefore $\operatorname{Supp}(\psi_i) \subseteq \overline{V_i} \subseteq U_i$. Suppose $\{W_i\}$ covers X, then $\sum_{j=1}^n \psi_j(x) > 0$. For $i \in \{1, \cdots, n\}$, we define $\varphi_i : X \to [0,1]$ by:

$$\varphi_i(x) = \frac{\psi_i(x)}{\sum_{j=1}^n \psi_j(x)}$$

Therefore $\operatorname{Supp}(\varphi_i) = \operatorname{Supp}(\psi_i) \subseteq U_i$ and $\sum_{i=1}^n \varphi_i(x) = 1$. $\square$

Proof of Theorem 5.13: For each $x \in X$, suppose $x \in U_x \in \mathcal{T}$ such that U_x is homeomorphic to a subset of $\mathbb{R}^m$. That is, there is an embedding $g_x : U_x \to \mathbb{R}^m$ for each $x \in X$. Since X is compact:

$$X = \bigcup_{x \in X} U_x \implies X = U_1 \cup \cdots \cup U_n$$

that is, there exists an open cover $U_1, \cdots, U_n$ of X and embeddedings $g_i : U_i \to \mathbb{R}^m$. Since X is normal, let $\varphi_1, \cdots, \varphi_n$ be a parition dominated by $\{U_i\}$. Let $A_i = \operatorname{Supp}(\varphi_i) \subseteq U_i$. Consider the maps $h_i : X \to \mathbb{R}^m$ given by:

$$h_i(x) = \begin{cases} \varphi_i(x) g_i(x) & \text{if } x \in U_i \\ 0 & \text{if } x \in X \setminus A_i \end{cases}$$

Note that if $x \in U_i \cap (X \setminus A_i)$, then $\varphi_i(x) = 0$. By a lemma, $h_i|_{U_i}$ and $h_i|_{X \setminus A_i}$ are continuous, which implies h_i is continuous. Define:

$$F : X \to \mathbb{R}^n \times \mathbb{R}^{mn} = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}} \times \underbrace{\mathbb{R}^m \times \cdots \times \mathbb{R}^m}_{n \text{ times}}$$

by $F(x) = (\varphi_1(x), \cdots \varphi_n(x), h_1(x), \cdots, h_n(x))$. Then $\pi_i \circ F$ is continuous for all i, hence F is continuous. Since X is compact, we only need to prove F is injective. Suppose $x, y \in X$ such that $F(x) = F(y)$, since $\sum \varphi_i(x) = 1$, there exists i such that $\varphi_i(x) > 0$. Therefore $\varphi_i(x) = \varphi_i(y) > 0$, so $x, y \in A_i \subseteq U_i$. Thus, for that same i, we have $h_i(x) = h_i(y)$. Hence $\varphi_i(x) g_i(x) = \varphi_i(y) g_i(y)$, which implies $g_i(x) = g_i(y)$ and thus $x = y$. Since $\mathbb{R}^m$ is Hausdorff and X is compact, so X is homeomorphic to $F(X) \subseteq \mathbb{R}^m$. $\square$

Fact. In general we have:

1. If X is a compact m-manifold, then X embeds into $\mathbb{R}^{2m+1}$.
2. If X is compact and smooth m-manifold, then X embeds into $\mathbb{R}^{2m}$, and there is X such that X does not embed into $\mathbb{R}^{2m-1}$.

Lecture 18, 2024/10/23

6 The Fundamental Group

6.1 Paths and Homotopy

Goal: Classify spaces (surfaces) in terms of how paths in the spaces behave. We will introduce algebraic concepts to deal with hard geometry.

Idea: We will try to continuously deform one path into another.

Definition. Let $(X, \mathcal{T})$ be a topological space and $x_0, x_1 \in X$. A **homotopy of path** in X is a family of $f_t : [0,1] \to X$ where $t \in [0,1]$ such that:

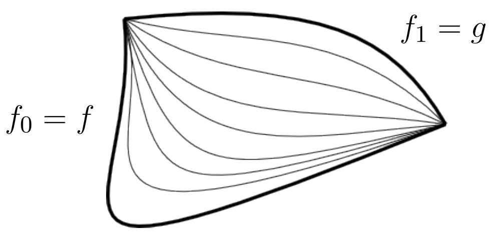
(i) For all t we have $f_t(0) = x_0$ and $f_t(1) = x_1$.

(ii) The map $F : [0,1] \times [0,1] \to X$ by $F(s,t) = f_t(s)$ is continuous.

Remark. Say $s_i \to s$ in $[0,1]$, then for all $t \in [0,1]$ we have $F(s_i, t) \to F(s,t)$.

$$f_t(s_i) \to f_t(s) \implies f_t \text{ is continuous} \implies f_t \text{ is a path in } X$$

Definition. Let $(X, \mathcal{T})$ and $f, g : [0,1] \to X$ be paths with $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$. We say f and g are **homotopic**, denoted by $f \simeq g$ if and only if there is a homtopy of paths $\{f_t\}$ such that $f_0 = f$ and $f_1 = g$.



Example 6.1. Let $X \subseteq \mathbb{R}^n$b e convex and f_0, f_1 paths in X (same endpoints). Let:

$$f_t(s) = (1-t)f_0(s) + tf_1(s)$$

so we have $f_0 \simeq f_1$.

Proposition 6.2. Let $(X, \mathcal{T})$ and $x_0, x_1 \in X$, let:

$$P = \{f : [0,1] \to X : f \text{ continuous and } f(0) = x_0,\ f(1) = x_1\}$$

Then $\simeq$ is an equivalence relation on P.

Proof: (1). Let $f \in P$, then f_t is a homotopy from f to f.

(2). Suppose $f \simeq g$ via f_t, then $g \simeq f$ via f_{1-t}.

(3). Suppose $f \simeq g$ via f_t and $g \simeq h$ via g_t. Consider:

$$h_t(s) = \begin{cases} f_{2t}(s) & \text{if } 0 \leq t \leq \frac{1}{2} \\ g_{2t-1}(s) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

Note that $h_{1/2}(s) = f_1(s) = g(s)$ and $h_{1/2}(s) = g_0(s) = g(s)$. Then h_t is a homotopy from $h_0 = f$ to $h_1 = h$. This proved $f \simeq h$. □

Notation. Let $[f]$ be the **homotopy** class of f.

Definition. Let $(X, \mathcal{T})$ and $f, g : [0,1] \to X$ be continuous with $f(1) = g(0)$, we define the **product path** $f \cdot g : [0,1] \to X$ by:

$$(f \cdot g)(s) = \begin{cases} f(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

Proposition 6.3. If $f_0 \simeq f_1$ via f_t and $g_0 \simeq g_1$ via g_t, then $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ via $f_t \cdot g_t$.

Proof: Homework. □

6.2 Fundamental Groups

Big Idea: We should be able to tell topological spaces apart using their homotopy classes. In fact, we will be able to just use loops to tell spaces apart.

Definition. Let $(X, \mathcal{T})$. A **loop** in X is a path $f : [0,1] \to X$ such that $f(0) = f(1)$.

Definition. Let $(X, \mathcal{T})$ and $x_0 \in X$. We call:

$$\pi_1(X, x_0) = \{[f] : f : [0,1] \to X \text{ loop and } f(0) = f(1) = x_0\}$$

is called the **fundamental group of** X **at the base point** x_0.

Theorem 6.4. The fundamental group $\pi_1(X, x_0)$ is indeed a group via the operation:

$$[f][g] = [f \cdot g]$$

Remark. By A5, this operation is well-defined.

Lecture 19, 2024/10/25

Main Tool: Reparametrization. Consider $f : [0,1] \to X$ a path and $\varphi : [0,1] \to [0,1]$ continuous with $\varphi(0) = 0$ and $\varphi(1) = 1$. Then $f \circ \varphi : [0,1] \to X$ is a path with endpoints $f(0)$ and $f(1)$ such that $f \circ \varphi$ is homotopic to f. Why? Define:

$$\varphi_t(s) = (1-t)\varphi(s) + ts$$

Then $\{f \circ \varphi_t\}$ is a homotopy from $f \circ \varphi_0 = f \circ \varphi$ to $f \circ \varphi_1 = f$.

Proof of Theorem 6.4: (1). Associativity. Let $[f], [g], [h] \in \pi_1(X, x_0)$, then:

$$(f \cdot g) \cdot h(s) = \begin{cases} (f \cdot g)(2s) & \text{for } 0 \le s \le \frac{1}{2} \\ h(2s-1) & \text{for } \frac{1}{2} \le s \le 1 \end{cases} = \begin{cases} f(4s) & \text{for } 0 \le s \le \frac{1}{4} \\ g(4s-1) & \text{for } \frac{1}{4} \le s \le \frac{1}{2} \\ h(2s-1) & \text{for } \frac{1}{2} \le s \le 1 \end{cases}$$

and we have:

$$f \cdot (g \cdot h)(s) = \begin{cases} f(2s) & \text{for } 0 \le s \le \frac{1}{2} \\ (g \cdot h)(2s-1) & \text{for } \frac{1}{2} \le s \le 1 \end{cases} = \begin{cases} f(2s) & \text{for } 0 \le s \le \frac{1}{2} \\ g(4s-2) & \text{for } \frac{1}{2} \le s \le \frac{3}{4} \\ h(4s-3) & \text{for } \frac{3}{4} \le s \le 1 \end{cases}$$

Consider $\varphi : [0,1] \to [0,1]$ defined by:

$$\varphi(s) = \begin{cases} \frac{1}{2}s & \text{for } 0 \le s \le \frac{1}{2} \\ s - \frac{1}{4} & \text{for } \frac{1}{2} \le s \le \frac{3}{4} \\ 2s-1 & \text{for } \frac{3}{4} \le s \le 1 \end{cases}$$

Therefore we have $((f \cdot g) \cdot h) \circ \varphi = f \cdot (g \cdot h)$. Hence $([f][g])[h] = [f]([g][h])$.

(2). Identity. Let C be the constant loop by $C(s) = x_0$ and let $[f] \in \pi_1(X, x_0)$. We want to show $[f][C] = [C][f] = [f]$. First we have:

$$(f \cdot C)(s) = \begin{cases} f(2s) & \text{for } 0 \le s \le \frac{1}{2} \\ x_0 & \text{for } \frac{1}{2} \le s \le 1 \end{cases}$$

Consider $\varphi : [0,1] \to [0,1]$ by:

$$\varphi(s) = \begin{cases} 2s & \text{for } 0 \le s \le \frac{1}{2} \\ 1 & \text{for } \frac{1}{2} \le s \le 1 \end{cases}$$

Then $f \circ \varphi = f \cdot C$ which proved that $[f][C] = [f \cdot C] = [f]$. Now we want to show $[C][f] = [f]$ as well. Indeed:

$$(C \cdot f)(s) = \begin{cases} x_0 & \text{for } \ 0 \leq s \leq \frac{1}{2} \\ f(2s-1) & \text{for } \ \frac{1}{2} \leq s \leq 1 \end{cases}$$

Consider $\varphi : [0,1] \to [0,1]$ by:

$$\varphi(s) = \begin{cases} 0 & \text{for } \ 0 \leq s \leq \frac{1}{2} \\ 2s-1 & \text{for } \ \frac{1}{2} \leq s \leq 1 \end{cases}$$

Hence $f \circ \varphi = C \cdot f$, so $[C][f] = [f]$.

(3). Inverses. Let $[f] \in \pi_1(X, x_0)$ and consider $\bar{f}(s) = f(1-s)$. Then:

$$(f \cdot \overline{f})(s) = \begin{cases} f(2s) & \text{for } \ 0 \leq s \leq \frac{1}{2} \\ \overline{f}(2s-1) & \text{for } \ \frac{1}{2} \leq s \leq 1 \end{cases} = \begin{cases} f(2s) & \text{for } \ 0 \leq s \leq \frac{1}{2} \\ f(2-2s) & \text{for } \ \frac{1}{2} \leq s \leq 1 \end{cases}$$

Now consider:

$$f_t(s) = \begin{cases} f(s) & \text{for } \ s \in [0, 1-t] \\ f(1-t) & \text{for } \ s \in [1-t, 1] \end{cases}$$

and we let $g_t(s) = f_t(1-s)$. Then $h_t = f_t \cdot g_t$ is a homotopy of path from:

$$h_0(s) = (f_0 \cdot g_0)(s) = \begin{cases} f_0(2s) & \text{for } \ 0 \leq s \leq \frac{1}{2} \\ g_0(2s-1) & \text{for } \ \frac{1}{2} \leq s \leq 1 \end{cases} = \begin{cases} f(2s) & \text{for } \ 0 \leq s \leq \frac{1}{2} \\ \overline{f}(2s-1) & \text{for } \ \frac{1}{2} \leq s \leq 1 \end{cases} = f \cdot \overline{f}$$

to the path:

$$h_1(s) = (f_1 \cdot g_1)(s) = \begin{cases} f_1(2s) & \text{for } \ 0 \leq s \leq \frac{1}{2} \\ g_1(2s-1) & \text{for } \ \frac{1}{2} \leq s \leq 1 \end{cases} = \begin{cases} f(0) & \text{for } \ 0 \leq s \leq \frac{1}{2} \\ f(0) & \text{for } \ \frac{1}{2} \leq s \leq 1 \end{cases} = C$$

It follows that $[f][\overline{f}] = [C]$. By replacing f with $\overline{f}$, we get $[\overline{f}][f] = C$. □

Lecture 20, 2024/10/28

Example 6.5. Let $X \subseteq \mathbb{R}^m$ be convex and $x_0 \in X$, then $\pi_1(X, x_0) = \{[c]\} = \{1\}$ (trivial group).

Question: How does the choice of x_0 affect $\pi_1(X, x_0)$?

Proposition 6.6. If h is a path from x_0 to x_1, then $h\overline{h} = 1 = \overline{h}h$.

Proof: Homework. □

Proposition 6.7. Let $(X, \mathcal{T})$ be a topological space. Let $h : [0,1] \to X$ be a path from x_0 to x_1, then the map:

$$\varphi_h : \pi_1(X, x_1) \to \pi_1(X, x_0) \text{ by } [f] \mapsto [h \cdot f \cdot \overline{h}]$$

is a group isomorphism.

Proof: For $[f], [g] \in \pi_1(X, x_1)$ we have:

$$\begin{aligned}\varphi_h([f \cdot g]) &= [h \cdot f \cdot g \cdot \overline{h}] \\ &= [h \cdot f \cdot \overline{h} \cdot h \cdot g \cdot \overline{h}] \\ &= [h \cdot f \cdot \overline{h}] \cdot [h \cdot g \cdot \overline{h}] \\ &= \varphi_h([f]) \cdot \varphi_h([g])\end{aligned}$$

We claim that φ_h has a inverse $\varphi_{\overline{h}}$. Indeed:

$$(\varphi_h \circ \varphi_{\overline{h}})([f]) = \varphi_h([\overline{h} \cdot f \cdot h]) = [h \cdot \overline{h} \cdot f \cdot h \cdot \overline{h}] = [f]$$

Similarly $\varphi_{\overline{h}} \circ \varphi_h = \text{id}$ as well. Hence φ_h is an isomorphism. □

Proposition 6.8. If f is a path from x_0 to x_1 and $c = x_0$ and $d = x_1$ are constant loops, then:

$$c \cdot f \simeq f \quad \text{and} \quad f \cdot d \simeq f$$

Corollary 6.9. If X is path connected, then for all $x_0, x_1 \in X$ we have $\pi_1(X, x_0) \cong \pi_1(X, x_1)$. In this case, we just write $\pi_1(X)$ to denote the fundamental group.

Definition. Let $(X, \mathcal{T})$ be a topological space. We say X is **simply connected** if X is path connected and $\pi_1(X) = \{1\}$.

Example 6.10. The circle $D^2 = \{(x, y) : x^2 + y^2 \leq 1\}$ is simply connected.

Example 6.11. Convex implies simply connected.

Proposition 6.12. Let $(X, \mathcal{T})$. Then X is simply connected if and only if there is a unique homotopy class connecting any two $x_0, x_1 \in X$.

Proof: $(\Leftarrow)$. Trivial.

$(\Rightarrow)$. Now suppose X is simply connected. Fix $x_0, x_1 \in X$ and paths f, g from x_0 to x_1. Then:

$$[f \cdot \overline{g}] = [1] \quad \text{and} \quad [\overline{g} \cdot g] = [1]$$

hence $f \simeq f \cdot (\bar{g} \cdot g) \simeq (f \cdot \bar{g}) \cdot g \simeq g$. As desired. □

Lecture 21, 2024/10/30

Sad Fact: Computing the fundamental group $\pi_1(X)$ is hard!

Theorem 6.13. The map $\varphi : \mathbb{Z} \to \pi_1(S^1)$ by $\varphi(n) = [w_n]$, where:

$$w_n(s) = (\cos(2\pi ns), \sin(2\pi ns))$$

is an isomorphism.

Let $H = \{(\cos(2\pi ns), \sin(2\pi ns), s) : s \in \mathbb{R}\}$ be the helix. Our goal is to "lift" paths in S^1 to paths in H. Let $P : \mathbb{R} \to S^1$ by $p(s) = (\cos(2\pi s), \sin(2\pi s))$. Then $w_n = p \circ \tilde{w}_n$, where $\tilde{w}_n : [0,1] \to \mathbb{R}$ by $\tilde{w}_n(s) = ns$. We say that $\tilde{w}_n$ is a lift of w_n.

Proof: We first prove that φ is a group homomorphism. If $\tilde{f} : [0,1] \to \mathbb{R}$ is a path from 0 to n in $\mathbb{R}$, then $\tilde{f} \simeq \tilde{w}_n$ (since $\mathbb{R}$ is simply connected). Therefore:

$$\varphi(n) = [w_n] = [p \circ \tilde{w}_n] = [p \circ \tilde{f}] \qquad \text{(by A5)}$$

Consider $\tau_m : \mathbb{R} \to \mathbb{R}$ by $\tau_m(x) = x + m$ for $m \in \mathbb{Z}$. Then $\tilde{w}_m \cdot (\tau_m \circ \tilde{w}_n)$ is a path in $\mathbb{R}$ from 0 to $m + n$. So:

$$\varphi(m+n) = p \circ (\tilde{w}_m \cdot (\tau_m \circ \tilde{w}_n))$$

Well, note that we have:

$$\begin{aligned} p \circ (\tilde{w}_m \cdot (\tau_m \circ \tilde{w}_n))(s) &= \begin{cases} p(\tilde{w}_m(2s)) \\ p(\tau_m(\tilde{w}_n(2s-1))) \end{cases} \\ &= \begin{cases} p(\tilde{w}_m(2s)) \\ p(\tilde{w}_n(2s-1) + m) \end{cases} \\ &= \begin{cases} p(2ms) \\ p(2ns - n + m) \end{cases} \\ &= \begin{cases} p(2ms) \\ p(2ns - n) \end{cases} \end{aligned}$$

We also have:

$$\begin{aligned} (w_m \cdot w_n)(s) &= \begin{cases} w_m(2s) \\ w_n(2s-1) \end{cases} \\ &= \begin{cases} p(\tilde{w}_m(2s)) \\ p(\tilde{w}_n(2s-1)) \end{cases} \\ &= \begin{cases} p(2ms) \\ p(2ns - n) \end{cases} \end{aligned}$$

Therefore $\varphi(m+n) = [w_m \cdot w_n] = [w_m][w_n] = \varphi(m)\varphi(n)$. Now we need to show φ is a bijection. We will do this later. $\square$

Proposition 6.14. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$. Then:

$$\pi_1(X \times Y, (x, y)) \cong \pi_1(X, x) \times \pi_1(X, y)$$

Proof: Consider $\varphi : \pi_1(X \times Y, (x, y)) \to \pi_1(X, x) \times \pi_1(Y, y)$ given by:

$$\varphi([f]) = ([g], [h])$$

where $f : [0, 1] \to X \times Y$ by $f(s) = (g(s), h(s))$. We can also write $g = \pi_1 \circ f$ and $h = \pi_2 \circ f$. First we prove φ is a homomorphism. Let $[f_1], [f_2] \in \pi_1(X \times Y, (x, y))$. Say:

$$f_1 = (g_1, h_1) \quad \text{and} \quad f_2 = (g_2, h_2)$$

Hence $f_1 \cdot f_2 = (g_1 \cdot g_2, h_1 \cdot h_2)$. Therefore:

$$\varphi([f_1][f_2]) = ([g_1 \cdot g_2], [h_1 \cdot h_2]) = ([g_1], [h_1])([g_2], [h_2]) = \varphi([f_1])\varphi([f_2])$$

Showing φ is surjective is easy. Now we look at injectivity. Suppose $\varphi([f_1]) = \varphi([f_2])$. Then:

$$([g_1], [h_1]) = ([g_2], [h_2]) \implies g_1 \simeq g_2 \ \text{ via } \ G_t \ \text{ and } \ h_1 \simeq h_2 \ \text{ via } \ H_t$$

Thus $f_1 \simeq f_2$ via $f_t(s) = (G_t(s), H_t(s))$. Hence $[f_1] = [f_2]$. $\square$

Example 6.15. Consider the torus X (donut). It has fundamental group:

$$\pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}$$

Example 6.16. Consider $[0, 1] \times S^1$ (hollow cylinder), it has fundamental group:

$$\pi_1([0, 1] \times S^1) \cong \pi_1([0, 1]) \times \pi_1(S^1) \cong \{1\} \times \mathbb{Z} \cong \mathbb{Z}$$

Example 6.17. Consider $[0, 1] \times D^2$ (cylinder), it has fundamental group:

$$\pi_1([0, 1] \times D^2) \cong \{1\} \times \{1\} = \{1\}$$

Lecture 22, 2024/11/04

Theorem 6.18 (FTA). Every non-constant $p(z) \in \mathbb{C}[z]$ has a root.

Proof: Suppose $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ By contrapositive, suppose that $p(z)$ has no root in $\mathbb{C}$, we want to show $p(z)$ is constant. For $r \geq 0$, consider:

$$f_r(s) = \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}$$

This is a family of loops at 1 in $S^1 \subseteq \mathbb{C}$. Note that $f_0(s) = 1$. Therefore $[f_r] = [1]$. Let us say:

$$\pi_1(S^1) = \langle [w] \rangle \cong \mathbb{Z}$$

Take $r > 1$ with $r > |a_{n-1}| + \cdots + |a_1| + |a_0|$. For $z \in \mathbb{C}$ with $|z| = r$, we have:

$$\begin{aligned} |z|^n = r^n = r \cdot r^{n-1} &> (|a_{n-1}| + \cdots + |a_1| + |a_0|)|z|^{n-1} \\ &\geq |a_{n-1}||z|^{n-1} + \cdots + |a_1||z|^1 + |a_0| \\ &\geq |a_{n-1}z^{n-1} + \cdots + a_1 z + a_0| \end{aligned} \tag{Δ}$$

For $0 \leq t \leq 1$, consider:

$$p_t(z) = z^n + t(a_{n-1}z^{n-1} + \cdots + a_1 z + a_0)$$

These have no roots in the circle $|z| = r$. Then:

$$\frac{p_t(re^{2\pi i s})/p_t(r)}{|p_t(re^{2\pi i s})/p_t(r)|}$$

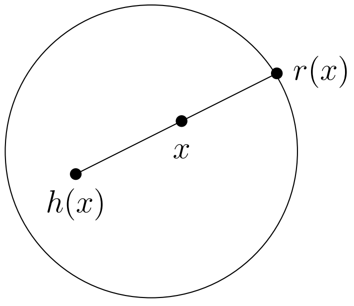
provided a homotopy between f_r and w^n. Hence we have:

$$[w]^n = [f_r] = [1]$$

which implies $n = 0$, hence $p(z)$ is a constant polynomial. $\square$

Theorem 6.19 (Brouwer Fixed Point). Every continuous $H : D^2 \to D^2$ has a fixed point.

Proof: Suppose $h : D^2 \to D^2$ is continuous but has no fixed points. Consider the continuous map $r : D^2 \to S^1$ given by:



That is, $r(x)$ is the intersection of S^1 and the line segment joining x and $h(x)$. Here we implictly used the fact that h has no fixed points, if there is a fixed point a, then the line between a and $h(a) = a$ is just a point and we cannot find its intersection with S^1. For all $x \in S^1$ we have $r(x) = x$. Let $[f_0] \in \pi_1(S^1)$. There exists a homotopy f_t in D^2 from f_0 to $f_1 = x_0$ (base point). By A5, we have that $r \circ f_t$ is a homotopy in S^1 from $r \circ f_0 = f_0$ to $r \circ f_1 = x_0$. It follows that $[f_0] = [1]$. Since $[f_0]$ is chosen arbitrary, hence $\pi_1(S^1) = \{1\}$. Contradiction. $\square$

Definition. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces. Let $f, g : X \to Y$ be continuous maps. A **homotopy** (not just of paths!) is a family of maps $f_t : X \to Y$ with $t \in [0, 1]$ such that $f_0 = f$ and $f_1 = g$ and the map $F : X \times [0, 1] \to Y$ by $F(x, t) = f_t(x)$ is continuous in the product topology. If such a homotopy exists, we say f, g are **homotopic**, and denoted by $f \simeq g$.

Remark. The big idea is that $f \simeq g$ if and only if f can be continuously deformed into g. This is also an equivalence relation.

Proposition 6.20. Let X, Y be topological spaces. Let p be a path in X from x_0 to x_1. If $f, g : X \to Y$ are homotopic, then $f \circ p \simeq g \circ p$.

Proof: Assignment 5. □

Definition. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces. We say X and Y are **homotopy equivalent** if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that:

$$f \circ g \simeq \mathrm{id}_Y \quad \text{and} \quad g \circ f \simeq \mathrm{id}_X$$

If X, Y are homtopy equivalent, we denote it by $X \simeq Y$.

Remark. The big idea is that $X \simeq Y$ if and only if X can be continuously deformed into Y and Y can be continuously deformed back into X. For instance, the donut is homotopy equivalent to the coffee cup. Notice that homeomorphic spaces are homotopy equivalent.

Proposition 6.21. Suppose $X \simeq Y$ via f, g. Let $x_0 \in X$, then:

$$\pi_1(X, x_0) \cong \pi_1(Y, f(x_0))$$

Proof: Assignment 5. □

6.3 van Kampen's Theorem

Goal: We want to compute $\pi_1(X)$ when $X = A \cup B$ and $\pi_1(A)$ and $\pi_1(B)$ are easier to compute.

Main Tool: Free products of grups.

Definition. For each $\alpha \in A$, let G_α be a group. The **free product** of the G_α's is:

$$\prod_{\alpha \in A}^{\star} G_\alpha = \left\{ \underbrace{g_1 g_2 \cdots g_m}_{\text{words}} : m \geq 0,\ g_i \in G_{\alpha_i},\ \alpha_i \neq \alpha_{i+1} \right\}$$

The group operation is by:

$$(g_1 \cdots g_m)(h_1 \cdots h_n) = g_1 g_2 \cdots g_m h_1 h_2 \cdots h_n$$

The identity of the free product G_α is the empty word 1.

Example 6.22. Say $a_i, b_i \in G_i$ for $i = 1, 2, 3$. Then:

$$(a_1a_2a_3)(a_3^{-1}b_2b_1b_3) = a_1a_2a_3a_3^{-1}b_2b_1b_3 = a_1(a_2b_2)b_1b_3$$

Example 6.23. Say $G = \langle x \rangle \cong C_4$ and $H = \langle y \rangle \cong C_3$. Then:

$$G * H = \langle x, y \mid x^4 = y^3 = 1 \rangle$$

Lecture 23, 2024/11/06

Example 6.24. Let $G = \langle x \rangle \cong C_2$ and $H = \langle y \rangle \cong C_2$. Then:

$$G * H = \langle x, y \mid x^2 = y^2 = 1 \rangle$$

Example 6.25. $\mathbb{Z} \cong \langle x \rangle = \{x^n : n \in \mathbb{Z}\}$ is the **free group** on one generator.

Example 6.26. Suppose $G = \langle x \rangle \cong \mathbb{Z}$ and $H = \langle y \rangle \cong \mathbb{Z}$. Then:

$$G * H = \langle x, y \rangle$$

the free group on 2 generators.

Example 6.27. Suppose $G = \langle x \rangle \cong \mathbb{Z}$ and $H = \langle y \rangle \cong C_4$. Then:

$$G * H = \langle x, y \mid y^4 = 1 \rangle$$

Example 6.28. Let $G = \langle x \rangle \cong C_2$ and $H = D_4 = \langle r, s \mid r^4 = s^2 = 1,\ rs = sr^{-1} \rangle$. Then:

$$G * H = \langle x, r, s \mid x^2 = r^4 = s^2 = 1,\ rs = sr^{-1} \rangle$$

Remark. Let $\varphi_\alpha : G_\alpha \to H$ be a group homomorphism for $\alpha \in A$. These homomorphisms extend uniquely to a homomorphism $\varphi : \prod_\alpha^* G_\alpha \to H$ by:

$$\varphi(g_1 \cdots g_n) = \varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n)$$

where $g_i \in G_{\alpha_i}$ for all $i \in \{1, \cdots, n\}$.

Theorem 6.29 (van Kampen). Suppose $X = A \cup B$, where $A, B \in \mathcal{T}$ are path connected and $A \cap B \neq \emptyset$ is also path connected. Take $x_0 \in A \cap B$. Then:

1. The natural homomorphisms $i_1 : \pi_1(A) \to \pi_1(X)$ and $i_2 : \pi_1(B) \to \pi_1(X)$ induces a surjective homomorphism $\Phi : \pi_1(A) * \pi_1(B) \to \pi_1(X)$.

2. If $N = \ker \Phi$, then by the first isomorphism theorem we have:

$$(\pi_1(A) * \pi_1(B))/N \cong \pi_1(X)$$

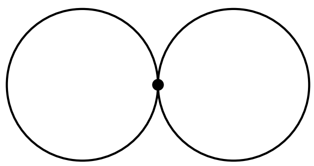
   Moreover, define $j_1 : \pi_1(A \cap B) \to \pi_1(A)$ and $j_2 : \pi_1(A \cap B) \to \pi_1(B)$ be the natural homomorphisms. Then N is the normal subgroup generated by $j_1(w)j_2(w)^{-1}$ for $w \in \pi_1(A \cap B)$.

$$
\begin{tikzcd}
 & \pi_1(A) \arrow[d, "i_1"] \arrow[dr] & \\
\pi_1(A\cap B) \arrow[ur, "j_1"] \arrow[dr, "j_2"'] & \pi_1(X) & \pi_1(A)*\pi_1(B) \arrow[l, dashed, "\Phi"'] \arrow[dl] \\
 & \pi_1(B) \arrow[u, "i_2"'] &
\end{tikzcd}
$$

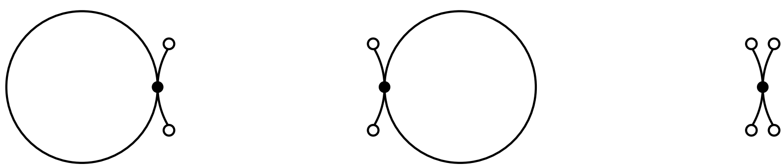
Corollary 6.30. Say $X = A \cup B$ and $A, B \in \mathcal{T}$ are path connected and $A \cap B$ is non-empty and simply connected. Then $\pi_1(X) \cong \pi_1(A) * \pi_1(B)$.

Example 6.31. Let $X = S^n$ where $n \geq 2$. Fix $x \in X$. Let $A = X \setminus \{x\}$ and $B = X \setminus \{-x\}$ be path connected. And $A \cap B \neq \emptyset$ is also path connected. By A6 we have $A, B \cong \mathbb{R}^n$. Hence $\pi_1(A), \pi_2(B)$ are trivial. Hence $\pi_1(X) = \{1\}$, so X is simply connected.

Example 6.32. Let X be the infinity symbol (called $S^1 \vee S^1$) in $\mathbb{R}^2$.



Define A, B and $A \cap B$ as followings:



A $\qquad$ B $\qquad$ $A \cap B$

Note that we do not include the endpoints in the arc because we need A, B to be open! Here, both A, B are homotopy equivalent to a circle, and $A \cap B$ is just homotopy equivalent to a point! Hence $\pi_1(A) \cong \pi_1(B) \cong \mathbb{Z}$ and $\pi_1(A \cap B) = \{1\}$. By van Kampen's theorem we have:

$$\pi_1(X) \cong \pi_1(A) * \pi_1(B) \cong \mathbb{Z} * \mathbb{Z}$$

Let a be the loop that loops 1 cycle around the left circle and let b be the loop that loops 1 cycle around the right circle. Then $\pi_1(A) = \langle [a] \rangle$ and $\pi_1(B) = \langle [b] \rangle$. This result just says that every loop in X is of the form $a^{n_1} b^{m_1} \cdots a^{n_k} b^{n_k}$ (note that $ab \neq ba$)!

Example 6.33. Let $X = S^1 \times S^1$ be the torus. We have seen that $\pi_1(X) = \mathbb{Z} \times \mathbb{Z}$. Let us try to use van Kampen's theorem to compute it to double check :)

Lecture 24, 2024/11/08

Proof of van Kampen: We start from a claim.

Claim 1: $\Phi : \pi_1(A) * \pi_1(B) \to \pi_1(X)$ is surjective.

Proof (Claim 1): Let $[f] \in \pi_1(X)$. For all $s \in [0, 1]$, we know f is continuous at s and so there exists $s \in V_s \subseteq [0, 1]$ open such that:

$$f(V_s) \subseteq A \ \text{ or } \ f(V_s) \subseteq B$$

WLOG suppose V_s is an interval with $f(\overline{V}_s) \subseteq A$ or $f(\overline{V}_s) \subseteq B$. Since $[0, 1] = \bigcup V_s$ and $[0, 1]$ is compact, $[0, 1] = V_{s_1} \cup \cdots \cup V_{s_m}$ gives a partition:

$$0 = t_0 < t_1 < \cdots < t_m = 1$$

such that $f([t_{i-1}, t_i]) \subseteq A_i$, where $A_i \in \{A, B\}$.

Lecture 25, 2024/11/11

If $f_i = f|_{[t_{i-1},t_i]}$, then $f = f_1 \cdots f_m$. Let g_i be a path in $A_i \cap A_{i+1}$ from x_0 to $f(t_i)$. (Here, $A_i \cap A_{i+1} \in \{A, B, A \cap B\}$ and all of them are path connected). Then:

$$\underbrace{(f_1 \cdot \overline{g_1})}_{\substack{f_1(0)\to\overline{g_1}(1) \\ x_0\to x_0 \\ \text{loop in } A_1}} \cdot \underbrace{(g_1 \cdot f_2 \cdot \overline{g_2})}_{\substack{g_1(0)\to f_2(1)\to\overline{g_2}(1) \\ x_0\to f(t_2)\to x_0 \\ \text{loop in } A_2}} \cdots \underbrace{(g_{m-1} \cdot f_m)}_{\text{loop in } A_m}$$

Therefore we have:

$$\Phi([f_1 \cdot \overline{g_1}][g_1 \cdot f_2 \cdot \overline{g_2}]) \cdots [g_{m-1} \cdot f_m] = [f]$$

It follows that Φ is onto. (QED Claim 1).

Let N be the normal subgroup of $\pi_1(A) * \pi_1(B)$ generated by $j_1(w)j_2(w)^{-1}$ for $w \in \pi_1(A \cap B)$. Note that $N \subseteq \ker \Phi$. If $N = \ker \Phi$, then:

$$(\pi_1(A) * \pi_1(B))/N \cong \pi_1(X)$$

by the first isomorphism theorem.

Claim 2: There is an isomorphism $(\pi_1(A) * \pi_1(B))/N \cong \pi_1(X)$.

Definition. For $[f] \in \pi_1(X)$, a **factorization** of $[f]$ is $[f_1] \cdots [f_k] \in \pi_1(A) * \pi_1(B)$ such that:

1. f_i is a loop in A or a loop in B.
2. $f \simeq f_1 \cdots f_k$ in X.

We say two factorizations of $[f]$ are equivalent if they are related to each other by a sequence of moves of the form:

(a) If possible, $[f_i][f_{i+1}] \to [f_i f_{i+1}]$ or vice versa.

(b) $[f_i]$ can be viewed as in $\pi_1(A)$ or $\pi_1(B)$ when f_i is a loop in $A \cap B$.

A move of type (a) does not change the element in $\pi_1(A) * \pi_1(B)$. A move of type (b) does not change the element in $Q = (\pi_1(A) * \pi_1(B))/N$. By the UPQ for groups,

$$\begin{tikzcd}
\pi_1(A) * \pi_1(B) \arrow[r, "\Phi"] \arrow[d, "q"'] & \pi_1(X) \\
Q \arrow[ur, "\overline{\Phi}"'] &
\end{tikzcd}$$

Since $N \subseteq \ker \Phi$, we get a surjective homomorphism $\overline{\Phi} : Q \to \pi_1(X)$.

Big Fact: Any two factorizations of $[f] \in \pi_1(X)$ are equivalent. Therefore $\overline{\Phi}$ is injective! Why?

$$\overline{\Phi}([f_1] \cdots [f_k]N) = \overline{\Phi}([g_1] \cdots [g_\ell]N) \implies \Phi([f_1] \cdots [f_k]) = \Phi([g_1] \cdots [g_\ell])$$

So they rae the same factorizations of some $[f]$, so:

$$[f_1] \cdots [f_k]N = [g_1] \cdots [g_\ell]N$$

Hence $\overline{\Phi}$ is injective and an isomorphism. $\square$

Lecture 26, 2024/11/13

7 Covering Spaces

7.1 Covering Spaces

Big Idea: Algebraic information of $\pi_1(X)$ should translate to geometric information of X. In fact, we can obtain correspondence like Galois correspondence.

Definition. Let $(X, \mathcal{T})$ be a topological space. A **covering space** of X is a space $\tilde{X}$ together with a continuous $p : \tilde{X} \to X$ such that:

1. There is an open cover $\{U_a\}_{\alpha \in A}$ of X such that for all $\alpha \in A$:

$$p^{-1}(U_\alpha) = \bigsqcup_i V_{\alpha_i}$$

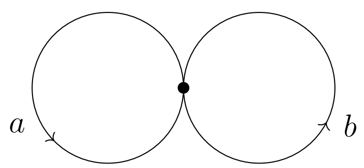
   is a disjoint union by $V_{\alpha_i} \subseteq \tilde{X}$.

2. For all $\alpha \in A$, the restriction $V_{\alpha_i} \to U_\alpha$ is a homeomorphism.

Remark. This means for all $x \in X$, there exists U_α with $x \in U_\alpha$ such that $p^{-1}(U_\alpha)$ is a disjoint union of open sets.

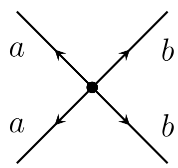
Remark. It is possible that $p^{-1}(U_\alpha) = \emptyset$, so p does not have to be surjective! If $p^{-1}(U_\alpha) \neq \emptyset$ for all $\alpha \in A$, then for all $x \in X$ there is U_α with $x \in U_\alpha$. Since $p^{-1}(U_\alpha) = \bigcup V_{\alpha_i}$ and $p : V_{\alpha_i} \to U_\alpha$ is a homeomorphism, there exists $v \in V_{\alpha_i}$ such that $p(v) = x$.

Example 7.1. Consider $X = S^1 \vee S^1$. We can view it as a digraph with one vertex and 2 edges.



Let $\tilde{X}$ be another digraph where:

1. Each vertex has 4 edges incident to it (as a graph).
2. Each edge can be labelled in the form:



with a-in, a-out, b-in and b-out (same as X). This is called a 2-oriented graph.

Then $\tilde{X}$ will be a covering space of X with $p : \tilde{X} \to X$ by gluing all vertices together and glues the a-edges together and b-edges together according to their orientation.

Definition. Let X, Y be topological spaces. Let $p : \tilde{X} \to X$ be a covering space, a **lift** of continuous function $f : Y \to X$ is a continuous function $\tilde{f} : Y \to \tilde{X}$ such that $p \circ \tilde{f} = f$.

$$\begin{tikzcd} \tilde{X} \arrow[r, "p"] & X \\ & Y \arrow[u, "f"'] \arrow[ul, dotted, "\tilde{f}"] \end{tikzcd}$$

Example 7.2. Define $p : \mathbb{R} \to S^1$ by $p(t) = (\cos(2\pi t), \sin(2\pi t))$. This is a covering space (helix). For all $n \in \mathbb{Z}$ define the map $w_n : [0,1] \to S^1$ by $w_n(t) = (\cos(2\pi n t), \sin(2\pi n t))$. Then $\tilde{w}_n : [0,1] \to \mathbb{R}$ by $\tilde{w}_n(t) = nt$ is a lift of w_n for all $n \in \mathbb{Z}$.

Lecture 27, 2024/11/15

Proposition 7.3 (Big Fact). Let X, Y be topological spaces and $p : \tilde{X} \to X$ be a covering space. Let $F : Y \times [0,1] \to X$ be continuous such that $\tilde{F} : Y \times \{0\} \to \tilde{X}$ is a lift of $F : Y \times \{0\} \to X$. Then there is a unqiue lift $\tilde{F} : Y \times [0,1] \to \tilde{X}$ of F.

Proof: Page 31 of Hatcher. $\square$

Proposition 7.4. Let $p : \tilde{X} \to X$ be a covering space. If $f_t : Y \to X$ is a homotopy and f_0 lifts to $\tilde{f}_0$, then there exists an unique homotopy $\tilde{f}_t : Y \to \tilde{X}$ such that $\tilde{f}_t$ lifts each f_t.

Proof: Let $F(s,t) = f_t(s)$. By the big fact, done. $\square$

Corollary 7.5. Let $p : \tilde{X} \to X$ and let $f : I := [0,1] \to X$ be a path starting at x_0. Suppose $p(\tilde{x}_0) = x_0$. Then there is a unique path $\tilde{f} : I \to \tilde{X}$ which lifts f and starts at $\tilde{x}_0$.

Corollary 7.6. Let $p : \tilde{X} \to X$. A lift of a constant path is constant.

Remark. Let $p : \tilde{X} \to X$. Suppose $f_t : I \to X$ is a homotopy of paths and assume $\tilde{f}_0$ lifts f_0. We know there exists a unique lift homotopy $\tilde{f}_t : I \to \tilde{X}$.

Question: But is $\tilde{f}_t$ a homotopy of paths?

Consider $g : I \to X$ by $g(t) = f_t(0)$. Let $\tilde{g}$ be a lift of g, then $\tilde{g}$ is constant. In particular, $\tilde{f}_t(0)$ is constant. Same for $\tilde{f}_t(1)$. So the answer is YES!

Proposition 7.7. Let $p : \tilde{X} \to X$ be a covering space and $p(\tilde{x}_0) = x_0$. Then the map:

$$p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0) \text{ by } p_*([g]) = [p \circ g]$$

is injective. Moreover, we have:

$$p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = \{[f] : \tilde{f} \text{ is a loop starting at } \tilde{x}_0\}$$

Proof: We will prove it with two claims.

Claim 1: p_* is injective.

Proof (Claim 1): Suppose $[\tilde{f}] \in \ker(p_*)$ so that if $f = p \circ \tilde{f}$, then $[f] = [1]$. Let f_t be a homotopy in X from $f_0 = f$ to $f_1 = x_0$. Then there exists a unique $\tilde{f}_t$ homotopy which lifts f_t. Moreover, $\tilde{f}_t$ goes from $\tilde{f}$ to $\tilde{x}_0$, which is constant. Hence $[\tilde{f}] = [1]$. (QED Claim 1)

Claim 2: $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = \{[f] : \tilde{f} \text{ is a loop starting at } \tilde{x}_0\}$.

Proof (Claim 2): Take $[f] \in \pi_1(X, x_0)$ such that f lifts to $\tilde{f}$, where $\tilde{f}$ is a loop at $\tilde{x}_0$. Hence we have:

$$[f] = [p \circ \tilde{f}] = p_*([\tilde{f}])$$

Finally, assume that $[g] = p_*([\tilde{f}])$, then $[g] = [p \circ \tilde{f}]$. Hence $[g] = [f]$. It follows that:

$$[f] \in \{[f] : \tilde{f} \text{ is a loop starting at } \tilde{x}_0\}$$

As desired. $\square$

Remark. if $p : \tilde{X} \to X$ is a covering space. Let $x \in X$ and take U open with $x \in U$ such that $p^{-1}(U) = \bigsqcup_{i \in I} V_i$. We know $p : V_i \to U$ is a homeomorphism. Then for all $y \in U$, there exists a unique $x_i \in V_i$ with $p(x_i) = y_i$. Hence $|p^{-1}(\{y\})| = |I|$. So $\varphi : X \to \mathbb{N} \cup \{0, \infty\}$ with $\varphi(x) = |p^{-1}(\{x\})|$ is locally constant.

Lecture 28, 2024/11/18

Recall. Let $p : \tilde{X} \to X$ be a covering space and $p(\tilde{x}_0) = x_0$. Define:

$$G = \pi_1(X, x_0) \text{ and } H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$$

Therefore:

$$H = \{[f] \in \pi_1(X, x_0) : \text{the unique lift } \tilde{f} \text{ starting at } \tilde{x}_0 \text{ is a loop}\}$$

Therefore $H \cong \pi_1(\tilde{X}, \tilde{x}_0)$.

Proposition 7.8. Let $p : \tilde{X} \to X$ be a covering space and assume $X, \tilde{X}$ are path-connected. Let $G = \pi_1(X, x_0)$ and let $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Then:

$$[G : H] = \text{number of sheets in the covering}$$

Proof: Let $[g] \in \pi_1(X, x_0)$ and let $\tilde{g}$ be the unique lift of g starting at x_0. For $h \in H$, its unique lift $\tilde{h}$ starting at $\tilde{x}_0$ is a loop. Hence $\tilde{h} \cdot \tilde{g}$ starts at $\tilde{x}_0$ and ends where $\tilde{g}$ ends. Consider:

$$\Phi : G/H \to p^{-1}(\{x_0\}) \text{ by } \Phi(H[g]) = \tilde{g}(1)$$

Note that $p(\tilde{g}(1)) = g(1) = x_0$. We claim that Φ is well-defined.

$$H[g_1] = H[g_2] \implies [g_1][g_2]^{-1} =: [h] \in H$$

Hence $[g_1] = [h \cdot g_2]$ and $\tilde{g}_1(1) = \tilde{g}_2(1)$.

Claim 1: Φ is surjective.

Proof (Claim 1): Let $\tilde{y} \in p^{-1}(\{x_0\})$, then $p(\tilde{y}) = x_0$. Since $\tilde{X}$ is path connected, let $\tilde{g}$ be a path in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{y}$. Then $g := p \circ \tilde{g}$ is a path in X from $p(\tilde{x}_0) = x_0$ to $p(\tilde{y}) = x_0$. Hence $[g] \in G$ and:

$$\Phi(H[g]) = \tilde{g}(1) = \tilde{y}$$

It follows that Φ is surjective. (QED Claim 1).

Claim 2: Φ is injective.

Proof (Claim 2): Suppose $\Phi(H[g_1]) = \Phi(H[g_2])$, then $\tilde{g}_1(1) = \tilde{g}_2(1)$. Hence $\tilde{g}_1 \cdot \overline{\tilde{g}_2}$ is a loop at $\tilde{x}_0$. Hence $[g_1][g_2]^{-1} \in H$ and $H[g_1] = H[g_2]$. (QED Claim 2).

Since Φ is a bijection, we are done. $\square$

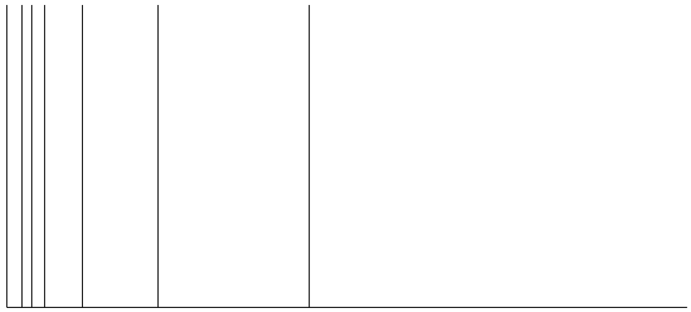
Example 7.9. Let $p : S^1 \subseteq \mathbb{C} \to S^1$ by $p(z) = z^2$. For all $x \in S^1$ we have $|p^{-1}(\{x\})| = 2$, so it is a 2-sheeted cover. Let $G = \pi_1(S^1, 1)$ and $H = p_*(\pi_1(S^1, 1))$. By the proposition we know $[G : H] = 2$. Since $G = \mathbb{Z}$, we know $H = 2\mathbb{Z}$.

Definition. Let $(X, \mathcal{T})$ be a topological space. We say X is **locally path-connected** if for all $x \in X$, for all $x \in U \in \mathcal{T}$, there exists $V \in \mathcal{T}$ such that V is path-connected and $x \in V \subsetneq U$.

Example 7.10. Let $K = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ and define:

$$C = (\{0\} \times [0,1]) \cup ([0,1] \times \{0\}) \cup (K \times [0,1])$$

This is path-connected but not locally path-connected.



Proposition 7.11 (Lifting Criterion). Let $p : \tilde{X} \to X$ and $p(\tilde{x}_0) = x_0$. Let Y be path-connected and locally path-connected. Let $f : Y \to X$ be continuous such that $f(y_0) = x_0$. Then a lift $\tilde{f} : Y \to \tilde{X}$ with $\tilde{f}(y_0) = \tilde{x}_0$ exists if and only if:

$$f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$$

In other words, the following diagram commutes.

$$\begin{tikzcd} (Y, y_0) \arrow[r, "f"] \arrow[dr, dashed, "\tilde{f}"'] & (X, x_0) \\ & (\tilde{X}, \tilde{x}_0) \arrow[u, "p"'] \end{tikzcd}$$

Why: $(\Rightarrow)$. Assume such $\tilde{f}$ exists, then:

$$f_*([g]) = [f \circ g] = [p \circ \tilde{f} \circ g] = p_*[\tilde{f} \circ g]$$

Lecture 29, 2024/11/20

Proposition 7.12 (Unique Lifting Property). Suppose $p : \tilde{X} \to X$ is a covering space and Y is connected, and $f : Y \to X$ is continuous. Suppose $\tilde{f}_1, \tilde{f}_2 : Y \to \tilde{X}$ are lifts of f. If $\tilde{f}_1$ and $\tilde{f}_2$ agree at a point, then $\tilde{f}_1 = \tilde{f}_2$.

Proof: Let $y \in Y$ and let $U \subseteq X$ be a neighborhood of $f(y)$ such that $p^{-1}(U) = \bigsqcup_i V_i$ and $p : V_i \to U$ is a homeomorphism. Let $\tilde{U}_1$ and $\tilde{U}_2$ be the V_i's containing $\tilde{f}_1(y)$ and $\tilde{f}_2(y)$, respectively. By continuity of $\tilde{f}_1$ and $\tilde{f}_2$ we may find an open neighborhood N of y such that $\tilde{f}_1(N) \subseteq \tilde{U}_1$ and $\tilde{f}_2(N) \subseteq \tilde{U}_2$. Two cases.

<u>Case 1:</u> $\tilde{f}_1(y) \neq \tilde{f}_2(y)$. Suppose there is $n \in \mathbb{N}$ such that $\tilde{f}_1(n) = \tilde{f}_2(n)$. Since $\tilde{f}_1(n) \in \tilde{U}_1$ and $\tilde{f}_2(n) \in \tilde{U}_2$, we know $\tilde{U}_1 = \tilde{U}_2$. We know $p : \tilde{U}_1 \to U$ is a bijection. Hence $p \circ \tilde{f}_1(y) \neq p \circ \tilde{f}_2(y)$,

so $f(y) \neq f(y)$, contradiction! Hence, for all $n \in \mathbb{N}$, we have $\tilde{f}_1(n) \neq \tilde{f}_2(n)$. Therefore the set $\{y \in Y : \tilde{f}_1(y) \neq \tilde{f}_2(y)\}$ is open.

Case 2: $\tilde{f}_1(y) = \tilde{f}_2(y)$. Then $\tilde{U}_1 = \tilde{U}_2$. So for all $n \in N$ we have $\tilde{f}_1(n), \tilde{f}_2(n) \in \tilde{U}_1$. Since $p \circ \tilde{f}_1(n) = p \circ \tilde{f}_2(n)$, so $\tilde{f}_1(n) = \tilde{f}_2(n)$. Therefore $\{y \in Y : f_1(y) = f_2(y)\}$ is open.

Since $\tilde{f}_1, \tilde{f}_2$ agree at a point and $\{y : \tilde{f}_1(y) = \tilde{f}_2(y)\}$ is open. If $\tilde{f}_1 \neq \tilde{f}_2$, then $\{y \in Y : \tilde{f}_1(y) = \tilde{f}_2(y)\}$ is clopen. Since Y is connected we have $\{y \in Y : \tilde{f}_1(y) = \tilde{f}_2(y)\} = Y$. Contradiction. $\square$

Definition. We say $(X, \mathcal{T})$ is **semilocally simply connected (SSC)** if for all $x \in X$, there is $x \in U \in \mathcal{T}$ such that $i_* : \pi_1(U, x) \to \pi_1(X, x)$ is trivial.

Example 7.13. If U is simply connected, then U is SSC.

Construction. Let X be path-connected and locally path-connected and SSC. Fix $x_0 \in X$. Consider the set:

$$\tilde{X} = \{[\gamma] : \gamma \text{ path in } X \text{ starting at } x_0\}$$

Now we let:

$$\mathcal{U} = \{U \subseteq X \text{ open, path-connected with } \pi_1(U) \to \pi_1(X) \text{ trivial}\}$$

For $U \in \mathcal{U}$ and $[\gamma] \in \tilde{X}$, let:

$$U_{[\gamma]} = \{[\gamma \cdot \rho] : \rho \text{ is a path in } U \text{ starting at } \gamma(1)\}$$

Then $\mathcal{B} = \{U_{[\gamma]} : U \in \mathcal{U},\ [\gamma] \in \tilde{X}\}$ is a basis for a topology on $\tilde{X}$. In Lecture 31, we will prove $\tilde{X}$ is simply connected ($\pi_1(\tilde{X}) = \{1\}$) and $p : \tilde{X} \to X$ by $p([\gamma]) = \gamma(1)$ is a covering space.

Lecture 30, 2024/11/22

7.2 Galois Correspondence of Covering Spaces

Proposition 7.14. Let X be path-connected and locally path-connected and SSC. For all subgroup $H \subseteq \pi_1(X, x_0)$, there exists a covering $p : X_H \to X$ such that $p_*\pi(\tilde{X}_H, \tilde{x}_0) = H$ for some basepoint $\tilde{x}_0$ with $p(\tilde{x}_0) = x_0$.

Proof: Let $\tilde{X} = \{[\gamma] : \gamma \text{ path starting at } x_0\}$ be the simply connected covering spaces of X we constructed last time. Consider the equivalence relation on $\tilde{X}$ given by:

$$[\gamma_1] \sim [\gamma_2] \iff \gamma_1(1) = \gamma_2(1) \text{ and } [\gamma_1 \overline{\gamma}_2] \in H$$

Let $X_H = \tilde{X}/\sim$ be the quotient space.

Observation: Suppose $[\gamma_1], [\gamma_2] \in \tilde{X}$ satisfying $\gamma_1(1) = \gamma_2(1)$. Let $\gamma_1(1) \in U$ be path-connected such that $\pi_1(U) \to \pi_1(X)$ is trivial. For a path ρ in U starting at $\gamma_1(1)$:

$$\begin{aligned} [\gamma_1 \cdot \rho] \sim [\gamma_2 \cdot \rho] &\iff [\gamma_1 \cdot \rho \cdot \overline{\gamma_2 \cdot \rho}] \in H \\ &\iff [\gamma_1 \cdot \rho \cdot \overline{\rho} \cdot \overline{\gamma}_2] \in H \\ &\iff [\gamma_1 \cdot \overline{\gamma}_2] \in H \end{aligned}$$

Therefore $[\gamma_1] \sim [\gamma_2] \iff$ every point in $U_{[\gamma_1]}$ is equivalent to the corresponding point in $U_{[\gamma_2]}$. This means that $p : X_H \to X$ by $p([\gamma]) = \gamma(1)$ is a covering space.

Let $\tilde{x}_0 = [x_0] \in X_H$ be a constant loop. Consider some $[\gamma] \in \pi_1(X, x_0)$ and let $\tilde{\gamma}$ be its unique lift to X_H starting to $\tilde{x}_0$. For $0 \leq t \leq 1$, let γ_t be the path in X from x_0 to $\gamma(t)$ which follows γ. Then $t \mapsto [\gamma_t]$ is a path in X_H from $[x_0]$ to $[\gamma]$. Also, $p([\gamma_t]) = \gamma_t(1) = \gamma(t)$. This path is $\tilde{\gamma}$! Hence:

$$\begin{aligned} [\gamma] \in p_*\pi_1(X_H, \tilde{x}_0) &\iff \tilde{\gamma} \text{ ends at } \tilde{x}_0 \\ &\iff [\gamma] = \tilde{x}_0 \text{ in } X_H \\ &\iff [\gamma] \in H \end{aligned}$$

As desired. $\square$

Remark. X_H is path-connected because $\tilde{X}$ is path-connected.

Definition. Let $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ be covering spaces of X. An isomorphism between p_1 and p_2 is a homeomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ such that $p_1 = p_2 \circ f$.

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Proposition 7.15. Let X be path-connected, locally path-connected. Then $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ be path-connected covering spaces of X. Fix $\tilde{x}_1 \in p_1^{-1}(\{x_0\})$ and $\tilde{x}_2 \in p_2^{-1}(\{x_0\})$. Then p_1, p_2 are isomorphic via an isomorphism $\tilde{x}_1 \mapsto \tilde{x}_2$ (basepoint preserving) if and only if:

$$p_{1*}\pi_1(\tilde{X}_1, \tilde{x}_1) = p_{2*}\pi_1(\tilde{X}_2, \tilde{x}_2)$$

Proof: $(\Rightarrow)$. If f is such a homeomorphism, then $p_1 = p_2 \circ f$ and $p_2 = p_1 \circ f^{-1}$. Immediately, the two subgroups are equal. Suppose the two subgroups are equal.

($\Leftarrow$). Suppose the two subgroups are equal. By the lifting criterion, we can make two lifts:

$$\underbrace{\tilde{p}_1 : \tilde{X}_1 \to \tilde{X}_2}_{\tilde{x}_1 \mapsto \tilde{x}_2} \quad \text{and} \quad \underbrace{\tilde{p}_2 : \tilde{X}_2 \to \tilde{X}_1}_{\tilde{x}_2 \mapsto \tilde{x}_1}$$

Note that $p_2 \circ \tilde{p}_1 = p_1$ and $p_1 \circ \tilde{p}_2 = p_2$. Consider $\tilde{p}_1 \circ \tilde{p}_2 : \tilde{X}_2 \to \tilde{X}_2$. Hence:

$$p_2 \circ \tilde{p}_1 \circ \tilde{p}_2 = p_1 \circ \tilde{p}_2 = p_2$$

Therefore $\tilde{p}_1 \circ \tilde{p}_2$ is a lift of p_2. Also, $\tilde{p}_2 \circ \tilde{p}_1$ is a lift of p_1. Also $1 : \tilde{X}_1 \to \tilde{X}_1$ (the identity map) is a lift of p_1 and $1 : \tilde{X}_2 \to \tilde{X}_2$ is a lift of p_2. However:

$$\begin{aligned} &\tilde{p}_1 \circ \tilde{p}_2(\tilde{x}_2) = p_1(\tilde{x}_1) = \tilde{x}_2 = 1(\tilde{x}_2) \\ &\tilde{p}_2 \circ \tilde{p}_1(\tilde{x}_1) = 1(\tilde{x}_2) \end{aligned}$$

By the uniqueness of lifts, we know $\tilde{p}_1 \circ \tilde{p}_2 = 1$ and $\tilde{p}_2 \circ \tilde{p}_1 = 1$. $\square$

The correspondence between $\mathcal{H} = \{H \subseteq \pi_1(X, x_0)\}$ and:

$$\mathcal{C} = \{\text{basepoint preserving isomorphicm classes of path-connected } p : \tilde{X} \to X\}$$

by the map $\mathcal{C} \to \mathcal{H} : p \mapsto p_* \pi_1(\tilde{X}, \tilde{x}_0)$ is called the Galois correspondence of covering spaces.

Lecture 31, 2024/11/25

Recall. Recall the construction from Lecture 29. Let X be path-connected and locally path-connected and SSC. Fix $x_0 \in X$. Consider the set:

$$\tilde{X} = \{[\gamma] : \gamma \text{ path in } X \text{ starting at } x_0\}$$

Now we let:

$$\mathcal{U} = \{U \subseteq X \text{ open, path-connected with } \pi_1(U) \to \pi_1(X) \text{ trivial}\}$$

For $U \in \mathcal{U}$ and $[\gamma] \in \tilde{X}$, let:

$$U_{[\gamma]} = \{[\gamma \cdot \rho] : \rho \text{ is a path in } U \text{ starting at } \gamma(1)\}$$

Then $\mathcal{B} = \{U_{[\gamma]} : U \in \mathcal{U},\ [\gamma] \in \tilde{X}\}$ is a basis for a topology on $\tilde{X}$.

We will now prove that $\tilde{X}$ is simply connected ($\pi_1(\tilde{X}) = \{1\}$) and $p : \tilde{X} \to X$ by $p([\gamma]) = \gamma(1)$ is a covering space.

Lemma 7.16. If $[\gamma_1] \in U_{[\gamma_2]}$, then $U_{[\gamma_1]} = U_{[\gamma_2]}$.

Proof: Suppose $[\gamma_1] = [\gamma_2 \cdot \rho]$, where ρ is a path in U. If $[\gamma_1 \cdot \rho_1] \in U_{[\gamma_1]}$, then:

$$[\gamma_1 \cdot \rho_1] = [\gamma_2 \cdot \rho \cdot \rho_1] = [\gamma_2 \cdot (\rho \cdot \rho_1)] \in U_{[\gamma_2]}$$

Conversely if $[\gamma_2 \cdot \rho_2] \in U_{[\gamma_2]}$, then:

$$[\gamma_2 \cdot \rho_2] = [\gamma_2 \cdot \rho \cdot \overline{\rho} \cdot \rho_2] = [\gamma_1 \cdot (\overline{\rho} \cdot \rho_2)] \in U_{[\gamma_1]}$$

As desired. □

Lemma 7.17. $\mathcal{B} = \{U_{[\gamma]} : U \in \mathcal{U},\ [\gamma] \in \tilde{X}\}$ forms a basis for a topology on $\tilde{X}$.

Proof: Let $[\gamma] \in \tilde{X}$. Since X is SSC, there exists $\gamma(1) \in U \in \mathcal{U}$ and so $[\gamma] \in U_{[\gamma]}$. Suppose $[\gamma] \in U_{[\gamma_1]} \cap V_{[\gamma_2]}$. By Lemma 7.16, $U_{[\gamma]} = U_{[\gamma_1]}$ and $V_{[\gamma]} = V_{[\gamma_2]}$. Let W be open and path-connected with $W \subseteq U \cap V$ and $\gamma(1) \in W$. Then:

$$[\gamma] \in W_{[\gamma]} \subseteq U_{[\gamma_1]} \cap V_{[\gamma_2]} = U_{[\gamma]} \cap V_{[\gamma]}$$

Therefore $\mathcal{B}$ is a basis. □

Lemma 7.18. $p : U_{[\gamma]} \to U$ is a homeomorphism.

Proof: We will first prove injectivity.

$$p([\gamma\rho_1]) = p([\gamma\rho_2]) \implies \gamma\rho_1(1) = \gamma\rho_2(1) \implies [\gamma\rho_1] = [\gamma\rho_2]$$

Since U is path-connected, p is surjective. Let $V \subseteq U$ be open and let $[\gamma'] \in U_{[\gamma]}$ with endpoint in V. We claim that $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma']}$. Take $[\tilde{\gamma}] \in p^{-1}(V) \cap U_{[\gamma]}$. Then:

$$\begin{aligned} [\tilde{\gamma}] &= [\gamma\rho_1] \text{ where } \rho_1 \text{ path in } U \text{ which ends in } V \\ &= [\gamma'\rho_2] \text{ where } \rho_2 \text{ path in } V && \text{(by Lemma 7.16)} \\ &\in V_{[\gamma']} \end{aligned}$$

Take $[\gamma'\rho] \in V_{[\gamma']}$, then $[\gamma'\rho] = [\gamma\rho'\rho] \in U_{[\gamma]}$, where ρ' is a path in U. Finanly we have $p([\gamma'\rho]) \in V$. Homework: Show that if $V_{[\gamma']} \subseteq U_{[\gamma]}$, then $p(V_{[\gamma']}) = V$. □

Theorem 7.19. $p : \tilde{X} \to X$ is a simply connected covering space.

Proof: By Lemma 7.18, we know p is continuous. For $U \in \mathcal{U}$:

$$p^{-1}(U) = \bigsqcup_{[\gamma] \in \tilde{X}} U_{[\gamma]}$$

and each $U_{[\gamma]} \cong U$ by Lemma 7.18. Hence p is a covering space. Now we want to show $\tilde{X}$ is simply connected. Let $[\gamma] \in \tilde{X}$ and let γ_t be the path in X from x_0 to $\gamma(t)$. Then $t \mapsto [\gamma_t]$ is a lift to γ from $[x_0]$ to $[\gamma]$. Hence $\tilde{X}$ is path-connected. Since p_* is injective, it is enough to show that:

$$p_*\pi_1(\tilde{X}, [x_0]) = \{[x_0]\}$$

Take $[\gamma] \in p_*\pi_1(\tilde{X}, [x_0])$. Its lift to $\tilde{X}$ (before) is a loop, that is, $[x_0] = [\gamma]$. As desired. $\square$

Remark. By the Galois correspondence, $\tilde{X}$ above is the unique simply connected covering space of X, up to isomorphism. We call it the **universal cover** of X.