PMATH 464 Notes

Intro to Algebraic Geometry

Winter 2024

Based on Professor Changho Han's Lectures

Contents

1 Affine Algebraic Sets

1.1 Zero-Sets and Ideals

Note. In this course, k is an algebrically closed base field with characteristic 0. Let $\mathbb{N}$ = set of non-negative integers. We also assume all rings are commutative with 1.

Definition. For $n \in \mathbb{N}$, the **affine n-space** over k, denoted $\mathbb{A}_k^n$ (or just $\mathbb{A}^n$), is the set k^n.

We will look at polynomials and their zero sets on $\mathbb{A}^n$. We will use the notation:

$$k[\mathbb{A}^n] := k[x_1, \cdots, x_n]$$

to denote the polynomial ring on $\mathbb{A}^n$. We will see later where this notation comes from.

Definition. The **affine algebraic set** corresponding to the set $S \subseteq k[x_1, \cdots, x_n]$ is:

$$V(S) = \{x \in \mathbb{A}^n : f(x) = 0 \text{ for all } f \in S\}$$

This is also called the **zero set** of S (a set of polynomials) in $\mathbb{A}^n$. Moreover, we say $X \subseteq \mathbb{A}^n$ is an affine algebraic set if $X = V(S)$ for some $S \subseteq k[\mathbb{A}^n]$.

Remark. If $S = \{f_1, \cdots, f_m\}$ is finite, we just write $V(S) = V(f_1, \cdots, f_m)$.

Example. Since k is algebraically closed, we have:

$$V(\{0\}) = V(\emptyset) = \mathbb{A}^n \quad \text{and} \quad V(k[x_1, \cdots, x_n]) = \emptyset$$

Example. In $\mathbb{A}^n$ we have $V(x_1 - a_1, \cdots, x_n - a_n) = \{(a_1, \cdots, a_n)\} \subseteq \mathbb{A}^n$. This means singleton sets are algebraic sets.

Example. If $n = 2$, then $V(x^2 + y^2 - 1)$ is a circle in $\mathbb{A}^2 = k^2$. If $k = \mathbb{C}$, then

$$(\sqrt{2}, i) \in V(x^2 + y^2 - 1) \in \mathbb{C}^2$$

However $|\sqrt{2}|^2 + |i|^2 = 3 \neq 1$, which means:

$$V(x^2 + y^2 - 1) \neq S^3 = \{(x, y) \in \mathbb{C}^2 : |x|^2 + |y|^2 = 1\}$$

Example. If $n = 3$, the set $V(y - x^2, z - x^3)$ is called the **affine twisted cubic** (defined over k).

Lemma 1.1. If $S_1 \subseteq S_2 \subseteq k[\mathbb{A}^n]$, then $V(S_1) \supseteq V(S_2)$.

Proof. Let $x \in V(S_2)$, then $f(x) = 0$ for all $f \in S_2$. Since $S_1 \subseteq S_2$, we know $f(x) = 0$ for all $f \in S_1$ as well. Therefore $V(S_2) \subseteq V(S_1)$. □

Example. $\mathbb{Z} \subseteq \mathbb{A}^1$ is NOT an affine algebraic set! Suppose $\mathbb{Z} = V(S)$ for some $S \subseteq k[x]$, then $S \neq \emptyset$. Take $p(x) \in S$, then by the lemma we have $V(S) \subseteq V(p)$. However, $p(x) \in k[x]$ has only finitely many roots. It follows that $V(p)$ is finite, so $\mathbb{Z}$ is finite as well, contradiction.

Definition. Let $X \subseteq \mathbb{A}^n$, the **ideal of** X is:

$$I(X) = \{f \in k[x_1, \cdots, x_n] : f(x) = 0 \text{ for all } x \in X\}$$

which is indeed an ideal of $k[x_1, \cdots, x_n]$.

Proof. Take $f, g \in I(X)$, then $(f + g)(x) = f(x) + g(x) = 0$ for all $x \in X$. Hence we have $f + g \in I(X)$. Take $f \in I(X)$ and take $h \in k[x_1, \cdots, x_n]$, then:

$$(hf)(x) = h(x)f(x) = h(x) \cdot 0 = 0$$

for all $x \in X$. Hence $hf \in I(X)$, so $I(X)$ is an ideal. □

Lemma 1.2. If $X \subseteq Y \subseteq \mathbb{A}^n$, then $I(X) \supseteq I(Y)$.

Proof. Let $f \in I(Y)$, then $f(x) = 0$ for all $x \in Y \supseteq X$, hence $f(x) = 0$ for all $x \in X$. It means $f \in I(X)$, as desired. □

Definition. Let S be a subset of a ring R, the **ideal generated by** S is:

$$RS = (S) = \left\{ \sum_{i=1}^{m} g_i f_i \in R : g_i \in R,\ f_i \in S,\ n \in \mathbb{N} \right\}$$

Proposition 1.3. Let $S \subseteq k[\mathbb{A}^n]$ and let $I = (S)$ be the ideal generated by S. Then $V(S) = V(I)$.

Proof. Note that $S \subseteq I$, so $V(I) \subseteq V(S)$. Conversely, we need to show $V(S) \subseteq V(I)$. Take $x \in V(S)$, we want to show $f(x) = 0$ for all $f \in I$. We let:

$$f = \sum_{i=1}^{m} g_i f_i$$

be an element in $I = (S)$, where $f_i \in S$ and $g_i \in k[\mathbb{A}^n]$. Hence $f_i(x) = 0$ for all i so that:

$$f(x) = \sum_{i=1}^{m} g_i(x) f_i(x) = \sum_{i=1}^{m} 0 = 0$$

It follows that $V(S) \subseteq V(I)$ as well. □

1.2 Finite Presentation

Question: Let $X = V(I)$ with $I \subseteq k[x_,1 \cdots, x_n]$ ideal, but I has too many elements! Is I finitely generated? That is, can we write $I = (f_1, \cdots, f_m)$ for some $f_i \in I$? Yes we can!

Definition. A ring R is **Noetherian** if every ideal of R is finitely generated.

Theorem 1.4 (Hilbert Basis Theorem). Let R be a ring. If R is Noetherian, then $R[x]$ is also Noetherian.

Proof. See PMATH 446. $\square$

Corollary 1.5. The ring $k[x_1, \cdots, x_n]$ is Noetherian.

Proof. The ring $k[x_1]$ is Noetherian by Hilbert Basis Theorem. Using the fact that:

$$k[x_1, \cdots, x_n] = k[x_1, \cdots, x_{n-1}][x_n]$$

and induction, we can prove the result. $\square$

Therefore, given an affine algebraic set $X \subseteq \mathbb{A}^n$, we can write $X = V(I)$ for some ideal I. Since $I \subseteq k[x_1, \cdots, x_n]$ and the ring $k[x_1, \cdots, x_n]$ is finitely Noetherian, we can write $I = (f_1, \cdots, f_m)$ for some $f_i \in I$. Hence:

$$X = V(f_1, \cdots, f_m) = V(f_1) \cap \cdots \cap V(f_m)$$

That is, every affine algebraic set X is a finite intersection of $V(f)$ for some polynomials f that vanishes on X.

1.3 Hilbert's Nullstellensatz

We saw that there is a correspondence between:

$$\{\text{subsets of } \mathbb{A}^n\} \longleftrightarrow \{\text{ideals of } k[x_1, \cdots, x_n]\}$$

by taking the operations $V(\cdot)$ and $I(\cdot)$. However, we can note that:

$$V(I(\mathbb{Z})) = V(\emptyset) = \mathbb{A}^1 \quad \text{and} \quad I(V(x^2)) = I(\{0\}) = (x)$$

This means the two operations are NOT inverses of each other, so this correspondence is NOT one-to-one. Our strategy is to restrict to some subsets of $\mathbb{A}^n$ and $k[\mathbb{A}^n]$ so that the operations are inverses of each other.

Definition. Let R be a ring and $I \subseteq R$ be an ideal. The **radical** of I is:

$$\sqrt{I} = \{f \in R : f^m \in I \text{ for some } m \in \mathbb{Z}^+\}$$

We say an ideal I is a **radical ideal** if $I = \sqrt{I}$. Note that $I \subseteq \sqrt{I}$ holds for any ideal I.

Proposition 1.6. $I(X)$ is a radical ideal for every $X \subseteq \mathbb{A}^n$.

Proof. If $f \in \sqrt{I(X)}$, then $f^m \in I(X)$ for some $m > 0$. This means $f(x)^m = 0$ for all $x \in X$, which means $f(x) = 0$ for all $x \in X$. It follows that $f \in I(X)$ so $\sqrt{I(X)} \subseteq I(X)$. $\square$

Theorem 1.7 (Hilbert's Nullstellensatz).

1. If $X \subseteq \mathbb{A}^n$ is an affine algebraic set, then $V(I(X)) = X$.
2. If $J \subseteq k[x_1, \cdots, x_n]$ is an ideal, then $I(V(J)) = \sqrt{J}$.
3. There is a inclusion-reversing correspondence:

$$\{\text{affine algebraic subsets of } \mathbb{A}^n\} \longleftrightarrow \{\text{radical ideals of } k[x_1, \cdots, x_n]\}$$

   by taking the operations $X \mapsto I(X)$ and $J \mapsto V(J)$.

Recall. Every ideal $I \subseteq k[\mathbb{A}^n]$ is contained in some maximal ideal $\mathfrak{m}$ so that $V(I) \supseteq V(\mathfrak{m})$.

Theorem 1.8. For all maximal ideal $\mathfrak{m} \subseteq k[x_1, \cdots, x_n]$, there exist $a_1, \cdots, a_n \in k$ such that:

$$\mathfrak{m} = (x_1 - a_1, \cdots, x_n - a_n)$$

Proof. See Noether's Normalization Lemma in PMATH 446. $\square$

Corollary 1.9 (Weak Nullstellensatz). Let $I \subsetneq k[x_1, \cdots, x_n]$ be an proper ideal, then $V(I) \neq \emptyset$.

Proof. Since I is proper. $I \subseteq \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. Then we have $V(\mathfrak{m}) \subseteq V(I)$. By the previous theorem we have $\mathfrak{m} = (x_1 - a_1, \cdots, x_n - a_n)$ so we have $V(\mathfrak{m}) = \{(a_1, \cdots, x_n)\} \neq \emptyset$. It follows that $V(I) \neq \emptyset$. $\square$

Proof of Hilbert's Nullstellensatz. Note that (1) and (2) implies (3).

(1). There are two inclusions. We first show $X \subseteq V(I(X))$. Take $x \in X$, by definition, $f(x) = 0$ for all $f \in I(X)$. Hence $x \in V(I(X))$. Conversely, write $X = V(J)$ for some ideal J. By (2) we have:

$$I(X) = I(V(J)) \supseteq \sqrt{J} \supseteq J$$

which follows that $V(I(X)) \subseteq V(J) = X$. WARNING: We used one inclusion of (2) before we proved it, but we will prove that inclusion of (2) independent from part (1).

(2). Two inclusions. If $f \in J$, then $f(x) = 0$ for all $x \in V(J)$, so $f \in I(V(J))$. Hence $J \subseteq I(V(J))$. Also, $I(V(J))$ is radical, so $\sqrt{J} \subseteq I(V(J))$. Here is the result we used in (1)! Now we want to show $I(V(J)) \subseteq \sqrt{J}$. By Hilbert Basis:

$$J = (f_1, \cdots, f_m)$$

for some $f_i \in k[x_1, \cdots, x_n]$. Cleary $0 \in \sqrt{J}$. Therefore let $h \in I(V(J)) \setminus 0$, by part (a) we have:

$$V(h) \supseteq V(I(V(J))) = V(J)$$

Then consider the ideal $\tilde{J} = (f_1, \cdots, f_m, x_{h+1}h - 1) \subseteq k[x_1, \cdots, x_n, x_{n+1}]$. If $y = (y_1, \cdots, y_{n+1}) \in V(\tilde{J})$, then we have $(y_1, \cdots, y_n) \in V(J)$. However $V(h) \supseteq V(J)$, so $h(y_1, \cdots, y_n) = 0$ and thus:

$$y_{n+1}h(y_1, \cdots, y_n) - 1 = 0 - 1 \neq 0$$

It follows that $y \notin V(\tilde{J})$, contradiction! Hence $V(\tilde{J}) = \emptyset$, which implies $\tilde{J} = k[x_1, \cdots, x_n, x_{n+1}]$ by Weak Nullstellensatz. Then $1 \in \tilde{J}$, so we can write:

$$1 = \sum_{i=1}^{m} \alpha_i f_i + \beta(x_{n+1}h - 1)$$

for some $\alpha_i, \beta \in k[x_1, \cdots, x_n, x_{n+1}]$. Let us work in $k(x_1, \cdots, x_n, x_{n+1})$ and set $x_{n+1} = 1/h$. Then:

$$1 = \sum_{i=1}^{m} \alpha_i \left(x_1, \cdots, x_n, \frac{1}{h}\right) f_i + \beta\left(h \cdot \frac{1}{h} - 1\right) = \sum_{i=1}^{m} \alpha_i \left(x_1, \cdots, x_n, \frac{1}{h}\right) f_i$$

Here the rational function $\alpha_i\left(x_1, \cdots, x_n, \frac{1}{h}\right)$ has numerator h^{n_i} for some $n_i \geq 0$. Hence there exists $N \geq 0$ such that multiplying by h^N clears the denominators and get:

$$h^N = \underbrace{\sum_{i=1}^{m} h^N \alpha_i \left(x_1, \cdots, x_n, \frac{1}{h}\right) f_i}_{\in k[x_1, \cdots, x_n]}$$

It follows that $h^N \in J$ and thus $h \in \sqrt{J}$. It proved $I(V(J)) \subseteq \sqrt{J}$. $\square$

Corollary 1.10. There is a one-to-one correspondence:

$$\{\text{points in } \mathbb{A}^n\} \longleftrightarrow \{\text{maximal ideals of } k[x_1, \cdots, x_n]\}$$

by Weak Nullstellensatz.

Corollary 1.11. Let $I \subseteq k[x_1, \cdots, x_n]$ be an ideal, then:

$$\sqrt{I} = \bigcap_{\substack{I \subseteq \mathfrak{m} \\ \mathfrak{m} \text{ maximal}}} \mathfrak{m}$$

In particular, any radical ideal I is equal to the intersection of all maximal ideal above it.

2 Affine Varieties

We start from some informal discussions. Recall from Calculus 3 (MATH 247) and Differential Geometry (PMATH 365) that a **space** consists of set, topolgy and functions.

Example. We know $\mathbb{R}^n$ is a space. The set is $\mathbb{R}^n$. The topology on $\mathbb{R}^n$ is the usual Euclidean topology. Functions on $\mathbb{R}^n$ are differential functions $f : \mathbb{R}^n \to \mathbb{R}$. These also induce notion of topology and functions on any subset $X \subseteq \mathbb{R}^n$.

Goal: We want to define topolgy and functions on affine algebraic sets.

2.1 Zariski Topology

Definition. A **topology** on a set X is a set $\mathcal{C}_X$ of subsets of X such that:

1. $\emptyset \in \mathcal{C}_X$ and $X \in \mathcal{C}_X$.
2. If $A, B \in \mathcal{C}_X$, then $A \cup B \in \mathcal{C}_X$.
3. If $(A_i)_{i\in I}$ is a collection of elements in $\mathcal{C}_X$, then $\bigcap_{i\in I} A_i \in \mathcal{C}_X$.

We say $A \subseteq X$ is **closed** if $A \in \mathcal{C}_X$ and $U \subseteq X$ is **open** if $X \setminus U$ is closed.

Definition. A **topological space** is a set X equipped with a topology $\mathcal{C}_X$.

Definition. The collection of affine algebraic sets of $\mathbb{A}^n$ is a topology on $\mathbb{A}^n$, called the **Zariski topology**.

Proposition 2.1. The Zariski topology is indeed a topology. That is:

1. If $I, J \subseteq k[x_1, \cdots, x_n]$ are ideals, then $V(I) \cup V(J) = V(IJ) = V(I \cap J)$.
2. If $(I_j)_{j\in J}$ is a collection of ideals of $k[x_1, \cdots, x_n]$, then:

$$\bigcap_{j\in J} V(I_j) = V\left(\sum_{j\in J} I_j\right)$$

Proof. (1). Clearly $I \supseteq IJ$ by definition, then $V(I) \subseteq V(IJ)$. Therefore we get:

$$V(I) \cup V(J) \subseteq V(IJ)$$

Similarly we have $V(I)\cup V(J) \subseteq V(I\cap J)$. For the other inclusion, we can consider the contrapositive. Suppose $x \notin V(I) \cup V(J)$, then there is $f \in I$ and $g \in J$ such that $f(x) \neq 0 \neq g(x)$. Hence $f(x)g(x) \neq 0$ so $x \notin V(fg) \supseteq V(IJ)$. It follows that:

$$V(I) \cup V(J) \supseteq V(IJ)$$

Then we have $V(I) \cup V(J) \supseteq V(IJ) \supseteq V(I \cap J)$, which proved the result.

(2). Let $I = \sum_{j \in J} I_j$. For all $j \in J$, we have $I_j \subseteq I$, thus $V(I_j) \supseteq V(I)$. It follows that $V(I) \subseteq \bigcap_{j \in J} V(I_j)$. Conversely, let $x \in \bigcap_{j \in J} V(I_j)$. We claim that $x \in V(I)$. Let $f \in I$, then:

$$f = \sum_{i=1}^{n} \alpha_i f_{n_i}$$

for some $\alpha_i \in k[x_1, \cdots, x_n]$ and $f_i \in I_i$ for some $n_i \in J$. Then $f(x) = 0$ as well since each $f_{n_i}(x) = 0$, hence we get $x \in V(I)$ as desired. $\square$

Example. In $\mathbb{A}^2$ with $k[\mathbb{A}^2] = k[x, y]$. Then:

$$V(y - x^2) \cap V(y) = V(y, y - x^2) = V(x^2, y) = V(x, y) = \{(0, 0)\}$$

Geometrically this makes sense. The parabola intersects x-axis at the origin.

Example. Let us look at the Zariski topolgoy on $\mathbb{A}^1$ when $k = \mathbb{C}$. Since $\mathbb{C}[x]$ is a PID, for any ideal $I \subseteq \mathbb{C}[x]$ we have:

$$V(I) = \begin{cases} \mathbb{A}_{\mathbb{C}} = \mathbb{C} & \text{if } I = (0) \\ \text{some finite set} & \text{if } I = (f) \neq (0) \end{cases}$$

This means closed sets of $\mathbb{A}_{\mathbb{C}}$ are $\mathbb{C}$ and all finite subsets. The unit ball $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is open in Euclidean topology but NOT in Zariski topology because $\mathbb{A}_{\mathbb{C}} \setminus B$ is not finite nor $\mathbb{C}$.

Definition. Let X be a topological space with topology $\mathcal{C}_X$ and $Y \subseteq X$. Define:

$$\mathcal{C}_Y = \{Y \cap A : A \in \mathcal{C}_X\}$$

Then $\mathcal{C}_Y$ is a topology on Y and is called the **subspace topology** on Y. We say Y is a **subspace** of X.

Proposition 2.2. Given $X = V(J) \subseteq \mathbb{A}^n$. Then $Y \subseteq X$ (with subspace topological of X) is closed if and only if $Y = V(J')$ for some ideal J' with $J \subseteq J'$.

Proof. ($\Rightarrow$). If Y is closed, then $Y = X \cap Y'$ for some Y' closed in $\mathbb{A}^n$. That is, $Y' = V(J_1)$ for some ideal J_1. Then:

$$Y = V(J) \cap V(J_1) = V(J + J_1) = V(J')$$

where we defined $J' = J + J_1$. Then $J \subseteq J'$, as desired.

($\Leftarrow$). We have $Y = V(J') \subseteq V(J) = X$. Since $Y = V(J')$, by definition Y is closed in $\mathbb{A}^n$. Hence $Y = Y \cap X$ is closed in X by the definition of subspace topology. $\square$

2.2 Irreducibility

Example. Note that we have:

$$V(xy) = V(x) \cup V(y)$$

However, $V(x) \neq V(I) \cup V(J)$ for any ideal I, J unless $V(x) = V(I)$ or $V(J)$. This is because $V(x)$ is homeomorphic to $\mathbb{A}^1$ as topological spaces and $\mathbb{A}^1 \neq B_1 \cup B_2$ with closed subsets $B_1, B_2 \subsetneq \mathbb{A}^1$.

Definition. A topological space X is **reducible** if there exist $Y_1, Y_2 \subsetneq X$ closed subsets such that $X = Y_1 \cup Y_2$. We say a non-empty topological space X is **irreducible** if it is not reducible.

Proposition 2.3. An affine algebraic set X is irreducible if and only if $I(X)$ is a prime ideal.

Proof. ($\Rightarrow$). If $I(X)$ is a proper ideal but not prime, then there exist $f, g \notin I(X)$ but $fg \in I(X)$. Hence $X \subseteq V(fg)$. Since $f, g \notin I(X)$ we have:

$$X \cap V(f) \subsetneq X \quad \text{and} \quad X \cap V(g) \subsetneq X$$

Now it follows that:

$$(X \cap V(f)) \cup (X \cap V(g)) = X \cap (V(f) \cap V(g)) = X \cap V(fg) = X$$

It follows from definition that X is reducible.

($\Leftarrow$). Assume $X = X_1 \cup X_2$ is reducible, where $X_1, X_2 \subsetneq X$ are proper closed subsets. By Nullstellensatz we know that $I(X_1), I(X_2) \supsetneq I(X)$. There exists $f \in I(X_1) \setminus I(X)$ and $g \in I(X_2) \setminus I(X)$. Now we have:

$$V(fg) = V(f) \cup V(g) \supseteq X_1 \cup X_2 = X$$

Hence $fg \in I(X)$ but $f, g \notin I(X)$. This proved $I(X)$ is not a prime ideal. $\square$

Example. $\mathbb{A}^n$ is irreducible because $I(\mathbb{A}^n) = 0$ is prime in $k[x_1, \cdots, x_n]$.

Example. For any $\ell \geq 0$, the space $V(x_{\ell+1}, \cdots, x_n) \subseteq \mathbb{A}^n$ is irreducible because the corresponding ideal is $(x_{\ell+1}, \cdots, x_n)$, which is prime.

Lemma 2.4. A ring R is Noetherian if and only if R satisfies the ascending chain condition [every ascending chain of idels in R stabilizes. That is, if $I_1 \subseteq I_2 \subseteq \cdots$ is an increasing sequence of ideals in R, then there is $N \geq 1$ such that $I_n = I_N$ for all $n \geq N$.]

Proof. ($\Rightarrow$). Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain. Define the ideal $I = \bigcup_{i=1}^{\infty} I_n$. This is an ideal of R because I_i is an ascending chain. Since R is Noetherian, $I = (f_1, \cdots, f_\ell)$ for some $f_i \in I$. There is $N \in \mathbb{N}$ such that $f_1, \cdots, f_\ell \in I_N$. Hence $I_n = I_N$ for all $n \geq N$.

($\Leftarrow$). Suppose there exists an ideal $I \subseteq R$ that is not finitely generated. Then $(f_1) \neq R$ for any $f_1 \in I$. There is $f_2 \in R \setminus (f_1)$ and $(f_1, f_2) \neq R$. Inductively we get a sequence $(f_k)_{k=1}^{\infty}$ in R such that $f_{k+1} \notin (f_1, \cdots, f_k)$ for any $k \geq 1$. Therefore:

$$(f_1) \subsetneq (f_1, f_2) \subsetneq (f_1, f_2, f_3) \subsetneq \cdots$$

is an ascending chain of ideals in R that does not stabilize. Contradiction. $\square$

Definition. A topological space X is **Noetherian** if it satisfies the descending chain condition, which means every descending chain of closed sets stabilizes.

Remark. By Lemma 2.4, we know $\mathbb{A}^n$ is Noetherian because for every descending chain of closed sets in $\mathbb{A}^n$:

$$X_1 \supseteq X_2 \supseteq \cdots$$

we can take $I(\cdot)$ and get an ascending chain of ideals:

$$I(X_1) \subseteq I(X_2) \subseteq \cdots$$

in $k[x_1, \cdots, x_n]$. Since $k[x_1, \cdots, x_n]$ is Noetherian, this chain stabilizes at $I(X_N)$, which means the original chain stabilizes at X_N.

Theorem 2.5. Let X be a Noetherian topological space. For any closed subset $Y \subseteq$ there exists a unique irreducible decomposition of $Y = Y_1 \cup \cdots \cup Y_m$, where $Y_i \subset Y$ is irreducible for all $i \in \{1, \cdots, m\}$ and $Y_i \nsubseteq Y_j$ for all $i \neq j$.

Proof. Let $\Sigma = \{$closed subsets of X that does not admit the irreducible decomposition$\}$. We want to show $\Sigma = \emptyset$. Assume $\Sigma \neq \emptyset$, then Σ must have a minimal element (with respect to inclusion). Why? If there is no minimal element, we can find a descending chain that does not stabilize. This contradicts the assumption that X is Noetherian. Let Y be this minimal element. If Y is irreducible, then it admits an irreducible decomposition $Y = Y$. If $Y = Y_1 \cup Y_2$ is reducible, then by the minimality of Y we have $Y_1, Y_2 \notin \Sigma$. Therefore $Y_1 = U_1 \cup \cdots \cup U_n$ and $Y_2 = V_1 \cup \cdots \cup V_m$ admit irreducible decompositions. Hence $Y = U_1 \cup \cdots \cup U_n \cup V_1 \cup \cdots \cup V_m$ admits an irreducible decomposition. This is a contradiction. The uniqueness is proved in A2. $\square$

Definition. Let X be a Noetherian topological space. Each X_i in the irreducible decomposition $X = X_1 \cup \cdots \cup X_m$ is called an **irreducible component** of X.

Example. In $\mathbb{A}^2$ we have $V(xy) = V(x) \cup V(y)$. Geometrically, $V(xy)$ is the xy-axis, which is the union of the x-axis and the y-axis.

Remark. If $X = X_1 \cup \cdots \cup X_m$ is the irreducible decomposition. The irreducible components X_i are the *largest* irreducible subset of X. To see this, we let Y be an irreducible subset of X, then:

$$Y = Y \cap X = (Y \cap X_1) \cup \cdots \cup (Y \cap X_m)$$

By the irreducibility of Y, there is n such that $Y \cap X_n = Y$. Therefore $Y \subseteq X_n$. In the setting of algebraic sets, an irreducible component of X corresponds to a *minimal* prime ideal containing $I(X)$.

2.3 Regular Functions

So far, affine algebraic sets and Zariski topology are defined in terms of polynomials. It makes sense to define functions in terms of polynomials!

Definition. Let $X \subseteq \mathbb{A}^n$ be an affine algebraic set. A function $f : X \to k = \mathbb{A}$ is called **regular** if there exists $g \in k[x_1, \cdots, x_n]$ such that $f(x) = g(x)$ for all $x \in X$.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook, "\iota"'] & & \mathbb{A} \\ & \mathbb{A}^n \arrow[ur, "g"'] & \end{tikzcd}$$

Example. Consider $X = V(xy - 1)$ in $\mathbb{A}^2$. The map $f : X \to \mathbb{A}$ by $f(x, y) = y$ is regular. The range of this function misses the point 0.

To define a regular function $g : X \to \mathbb{A}$, we note that g and $g + h$ are the same function on X for any $h \in I(X)$. This is because $h(a) = 0$ for all $a \in X$.

Definition. The **coordinate ring** of an affine algebraic set $X \subseteq \mathbb{A}^n$ is:

$$k[X] := k[x_1, \cdots, x_n]/I(X)$$

This is the ring of regular functions on X.

Example. In $\mathbb{A}^n$, we have $I(\mathbb{A}^n) = (0)$. Therefore we have $k[\mathbb{A}^n] = k[x_1, \cdots, x_n]$. This explained our notation from the beginning.

Example. We know $\mathbb{A}^0 = \{0\}$ is a singleton point, so $k[\mathbb{A}^0] = k$.

Proposition 2.6. An affine algebraic set X is irreducible if and only if $k[X]$ is a domain.

Proof. X is irreducible $\iff$ $I(X)$ is prime $\iff$ $k[X]$ is a domain. $\square$

Remark. Using the coordinate ring $k[X]$, we can directly characterize affine algebraic sets.

Definition. For a subset $S \subseteq k[X]$, we define:

$$V_X(S) := \{x \in X : f(x) = 0 \text{ for all } f \in S\}$$

to be the **affine algebraic subset** of X defined by S.

Definition. For a subset $Y \subseteq X$, we define:

$$I_X(Y) := \{f \in k[X] : f(y) = 0 \text{ for all } y \in Y\}$$

to be the **ideal** of Y in X.

Theorem 2.7 (Relative Nullstellensatz). Let X be an affine algebraic set, there is a inclusion-reversing 1-1 correspondence:

$$\{\text{affine algebraic subsets of } X\} \longleftrightarrow \{\text{radical ideals of } k[X]\}$$

by taking $I_X(\cdot)$ and $V_X(\cdot)$.

Proof. Use the fact that there is a 1-1 correspondence between radical ideals of $k[X]$ and radical ideals of $k[\mathbb{A}^n]$ containing $I(X)$. $\square$

Definition. Let R be a ring. An R**-algebra** is a ring S with a ring homomorphism $\varphi_S : R \to S$.

Definition. Let S, T be R-algebras (with ring homomorphisms φ_S and φ_T). We say a map $\varphi : S \to T$ is an R**-algebra homomorphism** if φ is a ring homomorphism and $h \circ \varphi_S = \varphi_T$.

$$\begin{tikzcd} S \arrow[rr, "\varphi"] & & T \\ & R \arrow[ul, "\varphi_S"] \arrow[ur, "\varphi_T"'] & \end{tikzcd}$$

Definition. An R-algebra S is **finitely generated** if there is $m \in \mathbb{N}$ and a surjective R-algebra homomorphism $f : R[x_1, \cdots, x_m] \to S$. In other word:

$$S \cong R[x_1, \cdots, x_m]/\ker f$$

is a quotient of a polynomial ring over R.

3 Structure of Affine Varieties

4 Projective Varieties

5 Classical Algebraic Geometry

6 Smoothness

7 Curves