PMATH 367 Notes

Fall 2024

Based on Professor Blake Madill's Lectures

Contents

Lecture 1, 2024/09/04

1 Topological Spaces

1.1 Basic Notations

Motivation. Recall from analysis that:

1. $A \subseteq \mathbb{R}^n$ is closed $\iff$ $\mathbb{R}^n \setminus A$ is open.
2. $x_n \to x$ in $\mathbb{R}^n$ $\iff$ for all open set $U \subseteq \mathbb{R}^n$ with $x \in U$, $\exists N \in \mathbb{N}$ such that $n \geq N \implies x_n \in U$.
3. $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous $\iff$ $f^{-1}(U)$ is open in $\mathbb{R}^n$ for all open $U \subseteq \mathbb{R}^m$.
4. $A \subseteq \mathbb{R}^n$ is compact $\iff$ every open cover of A has a finite subcover.

Big Idea: All these concepts from analysis can be stated using open sets!

Recall. If X is a set, we define:

$$\mathcal{P}(X) = \{A : A \subseteq X\}$$

to be the power set of X.

Definition. Let X be a set. We say $\mathcal{T} \subseteq \mathcal{P}(X)$ is a **topology** on X if:

1. $\emptyset, X \in \mathcal{T}$.
2. If I is an index set and $A_\alpha \in \mathcal{T}$ for all $\alpha \in I$, then $\bigcup_{\alpha \in I} A_\alpha \in \mathcal{T}$. (Arbitrary Union)
3. If $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$. (Finite Intersection)

We call $(X, \mathcal{T})$ a **topological space**. Moreover, we call the elements of $\mathcal{T}$ the **open sets** of X. And the **closed sets** of X are $X \setminus A$ for $A \in \mathcal{T}$.

Big Idea: Topology is the study of topological spaces. It is the area of math which studies concepts like open and closed sets, continuity, compactness and connectedness.

Example 1.1. Let $X = \{a, b, c\}$. Define:

$$\mathcal{T}_1 = \{\emptyset, X, \{a, b\}, \{c\}\}$$

$$\mathcal{T}_2 = \{\emptyset, X, \{a, b\}, \{b, c\}, \{b\}\}$$

Then both $\mathcal{T}_1$ and $\mathcal{T}_2$ are topology on X.

Example 1.2. Let (X, d) be a metric space, then:

$$\mathcal{T} = \{U \subseteq X : \forall x \in U, \exists r > 0, B_r(x) \subseteq U\}$$

is the metric topology on X.

Example 1.3. In the Example 1.1, it can be shown that $\mathcal{T}_1$ is not a metric topology. That is, there is no metric d on X such that the open sets in (X, d) is $\mathcal{T}_1$. Suppose there is a metric d on X, then there is $r_1, r_2, r_3 > 0$ such that:

$$B_{r_1}(a) = \{a\},\ B_{r_2}(b) = \{b\},\ B_{r_3}(c) = \{c\}$$

Thus the metric topology would be $\mathcal{P}(X)$. But $\mathcal{T}_1$ is not $\mathcal{P}(X)$, so contradiction.

Definition. Let X be any set. $\mathcal{P}(X)$ is called the **discrete topology** and $\{\emptyset, X\}$ is called the **indiscrete topology**.

Example 1.4. Let X be a set and let:

$$\mathcal{T}_f = \{A \subseteq X : X \setminus A \text{ is finite}\} \cup \{\emptyset\}$$

is called the **finite complement topology**. Why?

1. $X \setminus X = \emptyset$, so $X \in \mathcal{T}_f$.
2. $\emptyset \in \mathcal{T}_f$ by definition.
3. $A_\alpha \in \mathcal{T}_f$ means $X \setminus A_\alpha$ is finite. Then:

$$X \setminus \bigcup_\alpha A_\alpha = \bigcap_\alpha (X \setminus A_\alpha)$$

   is also finite. Hence $\bigcup_\alpha A_\alpha \in \mathcal{T}_f$.
4. If $A, B \in \mathcal{T}_f$, then $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$. Each set is finite, so this is finite. Therefore we have $A \cap B \in \mathcal{T}_f$.

Example 1.5. Let X be any set, then:

$$\mathcal{T}_c = \{A \subseteq X : X \setminus A \text{ is at most countable}\} \cup \{\emptyset\}$$

is the **countable complement topology**.

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1.2 Bases

Definition. Let X be a set. We say $\mathcal{B} \subseteq \mathcal{P}(X)$ is a **basis for a topology on** X if:

1. For all $x \in X$ there is $B \in \mathcal{B}$ such that $x \in B$.
2. For all $x \in X$ such that $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Example 1.6. Let $X = \mathbb{R}$ and $\mathcal{B} = \{(a, b) : a < b\}$ is a basis for a topology on $\mathbb{R}$. (Open intervals).

Example 1.7. Let (X, d) be a metric space and $\mathcal{B} = \{B_r(x) : x \in X,\ r > 0\}$ is a basis for a topology on X. (All open balls).

Example 1.8. Let X be a set and $\mathcal{B} = \{\{x\} : x \in X\}$ is a basis for a topology on X.

Definition. Let $\mathcal{B}$ be a basis for a topology on X. We define the **topology generated by** $\mathcal{B}$ to be:

$$\mathcal{T}_{\mathcal{B}} = \{U \subseteq X : \forall x \in U,\ \exists B \in \mathcal{B},\ x \in B \subseteq U\}$$

Proposition 1.9. This definition is well-defined, that is, $\mathcal{T}_{\mathcal{B}}$ is a topology on X.

Proof: It suffices to check the definition.

1. $\emptyset \in \mathcal{T}_{\mathcal{B}}$ is vacuously true.
2. For all $x \in X$, we can pick any $B \in \mathcal{B}$ such that $x \in B \subseteq X$. Hence $X \in \mathcal{T}_{\mathcal{B}}$.
3. If $U_\alpha \in \mathcal{T}_B$ for $\alpha \in I$ and let $x \in \bigcup_\alpha U_\alpha$. Then $x \in U_\beta$ for some $\beta \in I$. Then there is $B \in \mathcal{B}$ such that $x \in B \subseteq U_\beta \subseteq \bigcup_\alpha U_\alpha$. Hence $\bigcup_\alpha U_\alpha \in \mathcal{T}_{\mathcal{B}}$.
4. For $U, V \in \mathcal{T}_{\mathcal{B}}$ and $x \in U \cap V$. There are $B_1, B_2 \in \mathcal{B}$ such that:

   $$x \in B_1 \subseteq U \text{ and } x \in B_2 \subseteq V$$

   So $x \in B_1 \cap B_2$. By the second condition on basis, there is $B_3 \in \mathcal{B}$ such that:

   $$x \in B_3 \subseteq B_1 \cap B_2 \subseteq U \cap V$$

   Hence $U \cap V \in \mathcal{T}_{\mathcal{B}}$.

As desired. □

Remark. For all $B \in \mathcal{B}$, we have $B \in \mathcal{T}_{\mathcal{B}}$. Since for all $x \in B$, we have $x \in B \subseteq B$.

Proposition 1.10. Let $\mathcal{B}$ be a basis for a topology on X. Then:

$$\mathcal{T}_{\mathcal{B}} = \left\{ \bigcup_{\alpha \in I} B_\alpha : B_\alpha \in \mathcal{B} \text{ for all } \alpha \in I,\ I \text{ an index set} \right\}$$

Proof: Let $\mathcal{U}$ denote the RHS. To show $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{U}$, we let $V \in \mathcal{T}_{\mathcal{B}}$. For each $x \in V$, there is $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq V$. Thus:

$$V = \bigcup_{x \in V} B_x \in \mathcal{U}$$

Therefore $V \subseteq \mathcal{U}$. Conversely, since each $B_\alpha \in \mathcal{T}_{\mathcal{B}}$ and $\mathcal{T}_{\mathcal{B}}$ is a topology, we have $\mathcal{U} \subseteq \mathcal{T}_{\mathcal{B}}$. □

Example 1.11. Let $X = \mathbb{R}$ and $\mathcal{B} = \{(a, b) : a < b\}$. Then $\mathcal{T}_{\mathcal{B}}$ is the metric/standard topology.

Example 1.12. If (X, d) is a metric space and $\mathcal{B} = \{B_r(x) : x \in X,\ r > 0\}$ = all open balls. Then $\mathcal{T}_{\mathcal{B}}$ is the metric topology.

Example 1.13. Let X be a set and $\mathcal{B} = \{\{x\} : x \in X\}$. Then $\mathcal{T}_{\mathcal{B}} = \mathcal{P}(X)$ is the discrete topology.

Example 1.14. Let $X = \mathbb{R}$ and $\mathcal{B}' = \{[a, b) : a < b\}$ is a basis for a topology on $\mathbb{R}$. Let $\mathcal{T}' = \mathcal{T}_{\mathcal{B}'}$ and let $\mathcal{T}$ be the metric topology on $\mathbb{R}$. Note that

$$\mathcal{T} \subsetneq \mathcal{T}'$$

since $[0, 1) \in \mathcal{T}' \setminus \mathcal{T}$, so $\mathcal{T}' \neq \mathcal{T}$. Also $(a, b) \in \mathcal{T}'$ since $(a, b) = \bigcup_{n \in \mathbb{N}} \left[a + \frac{1}{n}, b\right) \in \mathcal{T}'$. We call $\mathcal{T}'$ the **lower limit topology** on $\mathbb{R}$.

Question: How do we build a basis for a topology?

Definition. Let X be a set. We say $S \subseteq \mathcal{P}(X)$ is a **subbasis** for a topology on X if $X = \bigcup_{A \in S} A$.

Definition. The topology generated by S is:

$$\mathcal{T}_S = \left\{ \bigcup_{\alpha} (A_{\alpha_1} \cap \cdots \cap A_{\alpha_n}) : n \in \mathbb{N}, A_{\alpha_i} \in S \right\}$$

Proposition 1.15. Let S be a subbasis for a topology on X. Then:

$$\mathcal{B} = \{A_1 \cap \cdots \cap A_n : n \in \mathbb{N}, A_i \in S\}$$

is a basis for a topology on X. In particular, $\mathcal{T}_S = \mathcal{T}_{\mathcal{B}}$ is a topology on X.

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Proof: Since S is a subbasis we have:

$$X = \bigcup_{A \in S} A$$

$x \in X$ implies $x \in A$ for some $A \in S$. Since $A \in \mathcal{B}$, this proves the first axiom of a basis. Now, say:

$$x \in (A_1 \cap \cdots \cap A_n) \cap (B_1 \cap \cdots \cap B_m) \in \mathcal{B}$$

So the second axiom of a basis holds trivially. Therefore $\mathcal{B}$ is a basis. $\square$

1.3 Subspaces

Definition. Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Define:

$$\mathcal{T}_A = \{A \cap U : U \in \mathcal{T}\}$$

Then $\mathcal{T}_A$ is a topology on A, called **subspace topology** on A. We say A is a subspace of X.

Proposition 1.16. Let $(X, \mathcal{T})$ and $A \subseteq X$. If $\mathcal{B}$ is a basis for $\mathcal{T}$, then:

$$\mathcal{B}' = \{A \cap B : B \in \mathcal{B}\}$$

is a basis for $\mathcal{T}_A$.

Example 1.17. Let $A = [0, 2] \subseteq \mathbb{R}$. Then $(1, 3) \cap A = (1, 2]$ is open in A but not in $\mathbb{R}$.

Proposition 1.18. Let $(X, \mathcal{T})$ be a topological space and $U \in \mathcal{T}$. If $A \subseteq U$ is open in U, then A is open in X.

Why? Well, A is open in U means $A = U \cap V$ for some $V \in \mathcal{T}$. Hence $A \in \mathcal{T}$. $\square$

Proposition 1.19. Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. The closed subsets of A are exactly the sets of the form $A \cap C$ where C is closed in X.

Proof: Suppose $C \subseteq A$ is closed, so $A \setminus C$ is open in A, which means $A \setminus C = A \cap U$ for some $U \in \mathcal{T}$. Therefore we have:

$$C = A \setminus (A \setminus C) = A \setminus (A \cap U) = A \cap (X \setminus U)$$

Here $X \setminus U$ is closed in X. Conversely, if C is closed in X, then $X \setminus C \in \mathcal{T}$. We want to prove $A \cap C$ is closed in A, indeed:

$$A \setminus (A \cap C) = A \cap (X \setminus C) \in \mathcal{T}_A$$

As desired. $\square$

Example 1.20. Let $A = [0, 1] \cup (2, 3) \subseteq \mathbb{R}$. Then:

$$[0, 1] = A \cap (-1, 3/2) = A \cap [0, 1]$$

This means $[0, 1]$ is both open and closed in A. We say it is **clopen** in A.

1.4 Closed Sets

Remark. Let $(X, \mathcal{T})$ be a topological space.

1. $\emptyset, X$ are closed.
2. Closed sets are "closed" under arbitrary intersections.
3. Closed sets are "closed" under finite unions.

Definition. Let $(X, \mathcal{T})$ and $A \subseteq X$. The **closure** of A is:

$$\overline{A} = \bigcap \{C \subseteq X : A \subseteq C, C \text{ closed in } X\}$$

which is the intersection of all closed sets containing A. It is the smallest closed set containing A. The **interior** of A is:

$$\text{int}(A) = \bigcup \{U \in \mathcal{T} : U \subseteq A\}$$

It is largest open set contained in A. Note that:

$$\text{int}A \subseteq A \subseteq \overline{A}$$

Definition. For $(X, \mathcal{T})$. If $x \in X$ and $U \in \mathcal{T}$ with $x \in U$, we say U is a **neighborhood** of x.

Proposition 1.21. Let $(X, \mathcal{T})$ and $A \subseteq X$. Then:

$$x \in \overline{A} \iff U \cap A \neq \emptyset$$

for any neighborhood U of x.

Proof: $(\Rightarrow)$. Let $x \in \overline{A}$, suppose for a contradiction that there is $U \in \mathcal{T}$ with $x \in U$ but $U \cap A = \emptyset$. Then we have $A \subseteq X \setminus U$, which is closed. Hence by the minimality of $\overline{A}$ we have $\overline{A} \subseteq X \setminus U$. Then:

$$x \in \overline{A} \subseteq X \setminus U \quad \text{and} \quad x \in U$$

Which is a contradiction.

$(\Leftarrow)$. Suppose $x \in X$ such that for all $x \in U \in \mathcal{T}$ we have $U \cap A \neq \emptyset$. Let $C \subseteq X$ be closed such that $A \subseteq C$. Then $X \setminus C$ is open. Suppose for a contradiction that $x \notin C$, then $x \in X \setminus C$. Hence $A \cap X \setminus C \neq \emptyset$. But $A \subseteq C$! This is a contradiction. $\square$

Lecture 4, 2024/09/11

Definition. Let $(X, \mathcal{T})$ and $A \subseteq X$. We say $x \in X$ is a **limit point** of A if every neighborhood of x intersects A at a point different from x.

Example 1.22. Let $X = \mathbb{R}$ and $A = (0,1) \cup \{2\}$. Then $0, \frac{1}{2}, 1$ are limit points of A. And 2 is not a limit point of A, because $(1.5, 2.5)$ does not contain anything in A except for 2.

Notation. Let $(X, \mathcal{T})$ and $A \subseteq X$. We denote:

$$A' = \{x \in X : x \text{ is a limit point of } A\}$$

to be the set of all limit points of A.

Proposition 1.23. Let $(X, \mathcal{T})$ and $A \subseteq X$. Then $\overline{A} = A \cup A'$.

Proof: $(\supseteq)$. This is trivial.

$(\subseteq)$. Let $x \in \overline{A}$ and suppose $x \in U \in \mathcal{T}$. Thus $U \cap A \neq \emptyset$.

1. If $x \in U \cap A$, then $x \in A$.
2. If $x \notin U \cap A$, then $x \in A'$.

As desired. $\square$

Corollary 1.24. Let $(X, \mathcal{T})$ and $A \subseteq X$. Then A is closed if and only if $A' \subseteq A$.

Proof: $A' \subseteq A \iff A = \overline{A} \iff A$ is closed. $\square$

1.5 Hausdorff Spaces

Definition. Let $(X, \mathcal{T})$ be a topological space. We say X is **Hausdorff** if for all $x \neq y \in X$, we can find $U, V \in \mathcal{T}$ such hat $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.

That is, given two distinct points, we can find two open sets that separate them.

Remark. All metric topologies are Hausdorff. For $x \neq y$, we can set $\epsilon = d(x, y)$. Then:

$$x \in B_{\epsilon/2}(x) \text{ and } y \in B_{\epsilon/2}(y)$$

and these two balls are disjoint.

Example 1.25. Let $X = \{a, b, c\}$ and $\mathcal{T} = \{\emptyset, X, \{a, b\}, \{c\}\}$. This is NOT Hausdorff because $a \neq b$ but there is no open sets that separate them.

Example 1.26. Consider $(\mathbb{R}, \mathcal{T}_f)$, the finite complement topology. This is NOT Hausdorff. Let $x \neq y$ with $x \in U$ and $y \in V$. Then:

$$x \in U = \mathbb{R} \setminus \{x_1, \cdots, x_n\}$$
$$y \in V = \mathbb{R} \setminus \{y_1, \cdots, y_m\}$$

This means $U \cap V \neq \emptyset$, because $U \cap V = \mathbb{R} \setminus \{x_1, \cdots, x_n, y_1, \cdots, y_m\}$ which is infinite.

Proposition 1.27. Let $(X, \mathcal{T})$ be Hausdorff, then $\{x\}$ is closed for all $x \in X$.

Proof: Fix $x \in X$. Since X is Hausdorff, there is $x \in U_y \in \mathcal{T}$ and $y \in V_y \in \mathcal{T}$ with $U_y \cap V_y = \emptyset$. Then we have:

$$X \setminus \{x\} = \bigcup_{y \neq x} V_y$$

Hence $X \setminus \{x\}$ is open. $\square$

Proposition 1.28. Let $(X, \mathcal{T})$ be Hausdorff and $A \subseteq X$. Then $x \in X$ is a limit point of A if and only if every neighborhood of x intersects A at infinitely many points.

Proof: $(\Leftarrow)$. This is trivial.

$(\Rightarrow)$. Assume x is a limit point of A. For contradiction, assume there exists $x \in U \in \mathcal{T}$ such that $U \cap A$ is finite. Since x is a limit point, we have:

$$U \cap (A \setminus \{x\}) = \{x_1, \cdots, x_n\} \neq \emptyset$$

Then, since $\{x_1, \cdots, x_n\}$ is closed, so $V = X \setminus \{x_1, \cdots, x_n\}$ is open. And $x \in U \cap V$ (open). However:

$$A \cap (U \cap V) = \{x\} \text{ or } \emptyset$$

Either way this is a contradiction: Since x is a limit point and $U \cap V$ is a neighborhood of x, so $U \cap V$ must intersect A at a point that is different from x. $\square$

2 Continuity

2.1 Basic Properties

Definition. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces. We say $f : X \to Y$ is **continuous** if:

$$f^{-1}(U) = \{x \in X : f(x) \in U\} \in \mathcal{T}$$

for all $U \in \mathcal{U}$.

Proposition 2.1. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ and $f : X \to Y$. If $\mathcal{B}$ is a basis for $\mathcal{U}$, then f is continuous if and only if $f^{-1}(B) \in \mathcal{T}$ for all $B \in \mathcal{B}$.

Proof: $(\Rightarrow)$. This is trivial.

$(\Leftarrow)$. Suppose $f^{-1}(B)$ for all $B \in \mathcal{B}$. Now, to show f is continuous, let $U \in \mathcal{U}$. Write $U = \bigcup_{i \in I} B_i$ for $B_i \in \mathcal{B}$, as $\mathcal{B}$ is a basis. Then:

$$f^{-1}(U) = f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i) \in \mathcal{T}$$

As desired. $\square$

Remark. The same result is true for a subbasis. (Exercise).

Proposition 2.2. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ and $f : X \to Y$, TFAE:

1. f is continuous.
2. For all $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$.
3. For all closed $C \subseteq Y$ we have $f^{-1}(C)$ is closed in X.

Example 2.3. Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \arctan(x)$. This is super continuous. Let $A = \mathbb{R}$, then:

$$f(\overline{A}) = f(\mathbb{R}) = \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

and so that:

$$\overline{f(A)} = \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

So the inclusion in 2 above does not have to be an equality.

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Proof: (1) $\implies$ (2). Suppose f is continuous. Let $y = f(x) \in f(\overline{A})$ where $x \in \overline{A}$. Let $U \in \mathcal{U}$ with $y \in U$. Then $x \in f^{-1}(U) \in \mathcal{T}$. Since $x \in \overline{A}$, there is $a \in A$ such that $a \in f^{-1}(U)$. Hence $f(a) \in U$ and $f(a) \in f(A)$. Hence $y \in \overline{f(A)}$.

(2) $\implies$ (3). Assume $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$. Let $C \subseteq Y$ be closed and let $D = f^{-1}(C)$. Let $x \in \overline{D}$, so we have:

$$f(x) \in f(\overline{D}) \subseteq \overline{f(D)} \subseteq \overline{C} = C$$

Therefor $x \in f^{-1}(C) = D$. Hence $\overline{D} \subseteq D$, so D is closed.

(3) $\implies$ (1). Let $U \in \mathcal{U}$, so $Y \setminus U$ is closed. So:

$$f^{-1}(Y \setminus U) = f^{-1}(Y) \setminus f^{-1}(U) = X \setminus f^{-1}(U)$$

This is closed by assumption of (3), hence $f^{-1}(U)$ is open in X. $\square$

2.2 Homeomorphisms

Definition. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces and $f : X \to Y$. We say f is a **homeomorphism** if f is continuous and f^{-1} is also continuous.

Example 2.4. Let $X = \mathbb{Z}$ and $\mathcal{T} = \mathcal{P}(\mathbb{N}) \cup \{\mathbb{Z}\}$. Let $f : X \to X$ by $f(x) = x - 1$. This is clearly bijective. If $A \subseteq \mathbb{N}$, then $f^{-1}(A) \subseteq \mathbb{N}$ and $f^{-1}(\mathbb{Z}) = \mathbb{Z}$. Hence f is continuous. Let $g = f^{-1}$ and $g(x) = x + 1$. Note that $\{1\} \in \mathcal{T}$, BUT $g^{-1}(\{1\}) = \{0\} \notin \mathcal{T}$. Therefore $g = f^{-1}$ is not continuous.

Example 2.5. Let $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Let $f : [0, 1) \to S^1$ by:

$$f(x) = (\cos(2\pi x), \sin(2\pi x))$$

Here $[0, 1)$ has the subspace topology from the standard topology of $\mathbb{R}$ and S^1 has the subspace topology from $\mathbb{R}^2$. So f is continuous and bijective. Note that $[0, 1/4)$ is open in $[0, 1)$, then:

$$(f^{-1})^{-1}([0, 1/4)) = f([0, 1/4))$$

which is not open. Hence f^{-1} is not continuous.

Big Ideas: $[0, 1)$ and S^1 have topological/geometrical differences and so there cannot exist a homeomorphism between them. For instance:

1. S^1 is compact but $[0, 1)$ is not compact.
2. Imagine removing a point from $[0, 1)$ and "disconnecting the interval". But removing only one point on S^1 cannot disconnect S^1.

Definition. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ and $f : X \to Y$. We say f is an **open map** if $f(U) \in \mathcal{U}$ for all $U \in \mathcal{T}$. That is, the image of every open set in X is an open set in Y.

Remark. $f : X \to Y$ is a homeomorphism if and only if:

1. f is bijective.
2. f is continuous.
3. f is an open map.

Why? This is just because $(f^{-1})^{-1}(U) = f(U)$.

Big Idea: Suppose $f : X \to Y$ is a homeomorphism.

1. Points: Every $y \in Y$ is of the form $y = f(x)$ for a unique $x \in X$. So Y is a relabelling of X.
2. Open sets: The elements V of $\mathcal{U}$ are exactly $V = f(U)$ for a unique $U \in \mathcal{T}$. Why: If $U \in \mathcal{T}$, then $f(U) \in \mathcal{U}$. If $V \in \mathcal{U}$, then $f^{-1}(V) \in \mathcal{T}$ and $V = f(f^{-1}(V))$. So $\mathcal{U}$ is a relabelling of $\mathcal{T}$.

This suggests that $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ are the same topological spaces up to the relabelling f.

Remark. Let $f : \mathbb{R} \to (\frac{-\pi}{2}, \frac{\pi}{2})$ by $f(x) = \arctan(x)$. This is a homeomorphism.

Lecture 6, 2024/09/16

Notation. In what follows X, Y, Z are topological spaces.

Proposition 2.6. If $f : X \to Y$ is constant, then f is continuous.

Proof: Say $f(x) = y_0$ for all $x \in X$. If $U \subseteq Y$ is open, then:

$$f^{-1}(U) = \begin{cases} X & \text{if } y_0 \in U \\ \emptyset & \text{if } y_0 \notin U \end{cases}$$

As desired. $\square$

Proposition 2.7. For $A \subseteq X$, the map $i : A \to X$ by $i(x) = x$ is continuous.

Proof: If $U \subseteq X$ is open, then:

$$i^{-1}(U) = \{x \in A : i(x) = x \in U\} = A \cap U$$

and this is open by the definition of subspace topology. $\square$

Proposition 2.8. Say $f : X \to Y$ and $g : Y \to Z$ are continuous. Then $g \circ f : X \to Z$ is continuous.

Proof: If $U \subseteq Z$ is open, then:

$$V := (g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$$

Here $g^{-1}(U)$ is open since g is continuous, thus V is open since f is continuous. $\square$

Proposition 2.9. If $f : X \to Y$ is continuous and $A \subseteq X$, then $f|_A : A \to Y$ is continuous.

Proof: Notice that:

$$(f|_A)^{-1}(U) = A \cap f^{-1}(U)$$

which is open in A. $\square$

Proposition 2.10. Let $f : X \to Y$ be continuous.

1. If $f(X) \subseteq Z \subseteq Y$, then $f : X \to Z$ is also continuous.
2. If $Y \subseteq Z$, then $f : X \to Z$ is also continuous.

Proof: Homework. $\square$

Proposition 2.11. Suppose $X = \bigcup_\alpha U_\alpha$ is a union of open sets. Let $f : X \to Y$. If $f|_{U_\alpha}$ is continuous for all α, then f is continuous.

Proof: If $V \subseteq Y$ is open, then:

$$f^{-1}(V) = f^{-1}(V) \cap X = \bigcup_\alpha (f^{-1}(V) \cap U_\alpha) = \bigcup_\alpha (f|_{U_\alpha})^{-1}(V)$$

which is open. $\square$

Definition. We say $f : X \to Y$ is continuous at $x \in X$ if for all open set $V \subseteq Y$ with $f(x) \in V$, there exists an open set $U \subseteq X$ with $x \in U$ such that $f(U) \subseteq V$.

Proposition 2.12. $f : X \to Y$ is continuous if and only if f is continuous at all $x \in X$.

Proof: ($\Rightarrow$). Suppose f is continuous and fix $x \in X$. If V is a neighborhood of $f(x)$, then $f^{-1}(V)$ is a neighborhood of x. Moreover, $f(f^{-1}(V)) \subseteq V$.

($\Leftarrow$). Suppose f is continuous at every $x \in X$. Let $V \subseteq Y$ be open. Let $x \in f^{-1}(V)$. By assumption, there is U_x open such that $x \in U_x$ and $f(U_x) \subseteq V$, so $U_x \subseteq f^{-1}(V)$. Thus:

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} \{x\} \subseteq \bigcup_{x \in f^{-1}(V)} U_x \subseteq f^{-1}(V)$$

Hence $f^{-1}(V)$ is open. $\square$

Proposition 2.13 (Pasting Lemma). Let $X = A \cup B$ where A, B are closed. If $f : A \to Y$ and $g : B \to Y$ are continuous and $f(x) = g(x)$ for all $x \in A \cap B$, then the natural $h = f \divideontimes g : X \to Y$ is continuous.

Proof: Let $C \subseteq Y$ be closed. Then:

$$h^{-1}(C) = \underbrace{f^{-1}(C)}_{\text{closed in } A} \cup \underbrace{g^{-1}(C)}_{\text{closed in } B} \qquad \text{(Homework)}$$

Since A, B are closed in X, so both $f^{-1}(C)$ and $g^{-1}(C)$ are closed in X. Hence $h^{-1}(C)$ is closed. $\square$

Goal: Make new topologies from old topologies.

See Assignment 2 that:

1. Let X be a set and $\mathcal{F} = \{f_\alpha : X \to Y_\alpha : \alpha \in A\}$. Say $(Y_\alpha, \mathcal{T}_\alpha)$ are topological spaces. Then:

$$\mathcal{B} = \{f_{\alpha_1}^{-1}(U_1) \cap \cdots \cap f_{\alpha_n}^{-1}(U_n) : U_i \in \mathcal{T}_{\alpha_i}\}$$

   is a basis for a topology on X.

2. Then $\mathcal{T} = \mathcal{T}_\mathcal{B}$ is called the **initial topology on X induced by $\mathcal{F}$**. It is the smallest topology on X that makes every f_α continuous.

3. In X, a net $x_i \to x$ if and only if $f_\alpha(x_i) \to f_\alpha(x)$ for all $\alpha \in A$.

4. $g : Z \to X$ is continuous if and only if $f_\alpha \circ g : Z \to Y_\alpha$ is continuous for all $\alpha \in A$.

Lecture 7, 2024/09/18

2.3 Product Topology

Definition. Let $(X_\alpha, \mathcal{T}_\alpha)$ for $\alpha \in A$ be a collection of topological spaces, consider:

$$X = \prod_{\alpha \in A} X_\alpha = \left\{ f : A \to \bigcup_{a \in A} X_\alpha : f(\alpha) \in X_\alpha \right\}$$

The **product topology** on X is the initial topology generated by:

$$\mathcal{F} = \{\pi_\alpha : X \to X_\alpha : \alpha \in A\} \text{ where } \pi_\alpha(f) = f(\alpha)$$

We call π_α the α-th projection. The product topology is the smallest topology on X which makes each projection π_α continuous.

Example 2.14. Consider the simple case $(X, \mathcal{T})$ and $(Y, \mathcal{U})$, then:

$$\begin{aligned} X \times Y &= \{f : \{1, 2\} \to X \cup Y : f(1) \in X,\ f(2) \in Y\} \\ &= \{(x, y) : x \in X,\ y \in Y\} \end{aligned}$$

where $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$.

Example 2.15 (Box Topology). Let $(X_\alpha, \mathcal{T}_\alpha)$ with $\alpha \in A$. Then:

$$\mathcal{B}_b = \left\{ \prod_{\alpha \in A} U_\alpha : U_\alpha \in \mathcal{T}_\alpha \right\}$$

is a basis for a topology on $X = \prod_{\alpha \in A} X_\alpha$.

Investigation: How do these two topologies differ? By A2:

$$\mathcal{B}_p = \{\pi_{\alpha_1}^{-1}(U_1) \cap \cdots \cap \pi_{\alpha_n}^{-1}(U_n)\} \text{ and } \mathcal{B}_b = \left\{ \prod_{\alpha} U_\alpha : U_\alpha \in \mathcal{T}_\alpha \right\}$$

Note that:

$$\pi_{\alpha_1}^{-1}(U_1) \cap \cdots \cap \pi_{\alpha_n}^{-1}(U_n) = \left\{ x \in \prod_{\alpha} X_\alpha : \pi_{\alpha_i}(x) \in U_{\alpha_i},\ 1 \le i \le n \right\} = \prod_{\alpha} V_\alpha$$

where:

$$V_\alpha = \begin{cases} U_{\alpha_i} & \text{if } \alpha = \alpha_i \\ X_\alpha & \text{if } \alpha \ne \alpha_i \end{cases}$$

Conclusions: We can conclude that:

1. $\mathcal{B}_p = \left\{ \prod_\alpha U_\alpha : \text{ all but finitely many } U_\alpha = X_\alpha \right\}$.
2. $\mathcal{B}_p \subseteq \mathcal{B}_b$ implies product topology $\subseteq$ box topology.
3. If A (the index set) is finite, then product = box topology.

Example 2.16 (Warning). Let $X = \prod_{n\in\mathbb{N}} \mathbb{R}$ and $f : \mathbb{R} \to X$ by $f(t) = (t, t, \cdots)$. Then for all $n \in \mathbb{N}$:

$$\pi_n \circ f : \mathbb{R} \to \mathbb{R} \text{ is continuous and satisfies } \pi_n(f(t)) = t$$

By A2, f is continuous with respect to the product topology. If:

$$B = (-1, 1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{3}, \frac{1}{3}\right) \times \cdots$$

is in the box topology. But $f^{-1}(B) = \{0\}$ is not open in $\mathbb{R}$. Therefore f is not continuous with respect to the box topology (Box topology is too big!)

2.4 Quotient Topology

Notation. Let X be a set and let $\sim$ be an equivalence relation on X, that is:

(1) For all $x \in X$, $x \sim x$.

(2) For all $x, y \in X$, $x \sim y \implies y \sim x$.

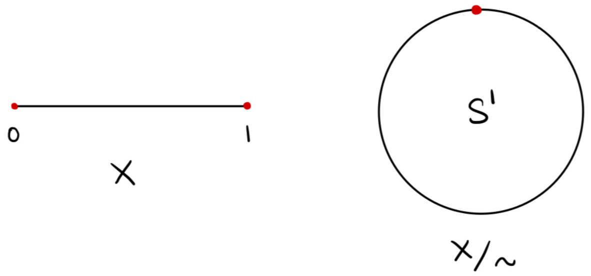
(3) For all $x, y, z \in X$, $x \sim y, y \sim z \implies x \sim z$.

Then for $x \in X$, we let $[x] = \{y \in X : y \sim x\}$ be the equivalence class containing x. And let:

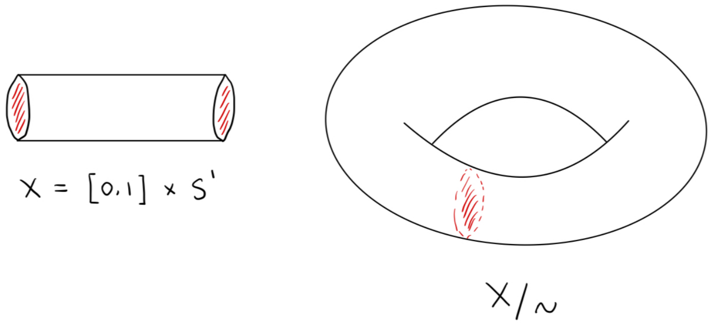
$$X/\sim = \{[x] : x \in X\}$$

Example 2.17. Let $X = [0, 1]$ and define $x \sim y \iff x = y$ or $x, y \in \{0, 1\}$. That is, we define 0,1 to be equivalent and all the other points are only equivalent to itself.

Example 2.18. If $X = [0, 1]$ and $\sim$ as above, then $X/\sim$ will be a circle, as we identify the endpoints of the line to one point, so it is like we glue them together.



Example 2.19. Let $X = [0, 1] \times S^1$, where $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Then X is a cylinder. If we identify the two end circles (glue them together), we get a torus (donut).



Example 2.20. Let $X = S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, a sphere in $\mathbb{R}^3$. If we identify the two poles, then the sphere collapses.

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Proposition 2.21. Let $(X, \mathcal{T})$ and $X/\sim$ be the quotient. Consider the quotient map:

$$q : X \to X/\sim \quad \text{by} \quad x \mapsto [x]$$

Then the collection of set:

$$Q = \{U \subseteq X/\sim : q^{-1}(U) \in \mathcal{T}\}$$

is a topology on $X/\sim$ called the **quotient topology on** $X/\sim$. And it is the largest topology on $X/\sim$ such that q is continuous.

Proof: We have $q^{-1}(\emptyset) = \emptyset$ and $q^{-1}(X/\sim) = X$. If $U_\alpha \in Q$, then:

$$q^{-1}\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha q^{-1}(U_\alpha) \in \mathcal{T}$$

Similarly if $U, V \in Q$ then:

$$q^{-1}(U \cap V) = q^{-1}(U) \cap q^{-1}(V) \in \mathcal{T}$$

Therefore Q is a topology on $X/\sim$. $\square$

Proposition 2.22. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$. A function $f : X/\sim \to Y$ is continuous if and only if the map $f \circ q : X \to Y$ is continuous.

Proof: $(\Rightarrow)$. Since both f and q are continuous, $f \circ q$ is continuous.

$(\Leftarrow)$. Suppose $f \circ q$ is continuous, for $U \in \mathcal{U}$:

$$(f \circ q)^{-1}(U) = q^{-1}(f^{-1}(U)) \in \mathcal{T}$$

By definition of the quotient topology, we must have $f^{-1}(U) \in Q$. $\square$

Theorem 2.23 (Universal Property of Quotients). Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces and let $\sim$ be an equivalence relation on X. For every continuous $f : X \to Y$, which is constant on equivalence classes, there exists a unique function $\overline{f} : X/\!\sim\, \to Y$ such that $f = \overline{f} \circ q$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/\sim \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

It turns out that this unique function $\overline{f}$ must be continuous!

Proof: Consider the map $\overline{f} : X/\!\sim\, \to Y$ by $\overline{f}([x]) = f(x)$. This function is well-defined because f is constant on equivalence classes. We have:

$$f(x) = \overline{f}([x]) = \overline{f}(q(x))$$

Therefore $f = \overline{f} \circ q$. By the previous proposition, $\overline{f}$ is continuous. If g is another such function, then for all $x \in X$:

$$g([x]) = g(q(x)) = f(x) = \overline{f}([x])$$

Hence the map $\overline{f}$ is unique. $\square$

Example 2.24. Let $X = [0, 1]$. Define $x \sim y$ if $x = y$ or $x, y \in \{0, 1\}$.

Goal: We want to show $X/\!\sim$ is homeomorphic to S^1 (circle). Consider:

$$f : [0, 1] \to S^1 \quad \text{by} \quad f(x) = (\cos(2\pi x), \sin(2\pi x))$$

This is continuous and surjective. Note that $f(0) = f(1)$, so it is constant on equivalence classes. By UPQ, there exists continuous $\overline{f} : X/\sim\, \to S^1$ where $f = \overline{f} \circ q$. We want to check $\overline{f}$ is a homeomorphism.

<u>Surjectivity:</u> For any $y \in S^1$, there is $x \in X$ with $f(x) = y$. Hence $\overline{f}([x]) = y$.

<u>Injectivity:</u> Suppose $\overline{f}([x]) = \overline{f}([y])$ then $f(x) = f(y)$, so $x = y$ or $x, y \in \{0, 1\}$. Hence:

$$x \sim y \implies [x] = [y]$$

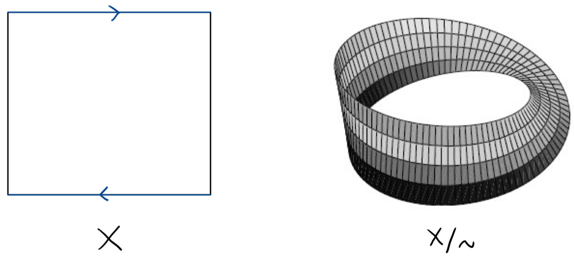
Therefore $\overline{f}$ is injective.

Lastly we want to show $g =$ inverse of $\overline{f}$ is continuous.

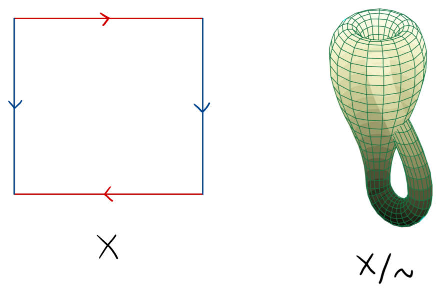
<u>Gap:</u> Since $[0, 1]$ is compact and q is continuous, $q(X) = X/\sim$ is compact. Since $\overline{f} : X/\!\sim\, \to S^1$ is invertible, continuous, so $X/\!\sim$ is compact and S^1 is Hausdorff, so $\overline{f}$ is homeomorphism. $\square$

Remark (Culture). In topology, we rarely give such proofs. We accept proofs by picture/gluing.

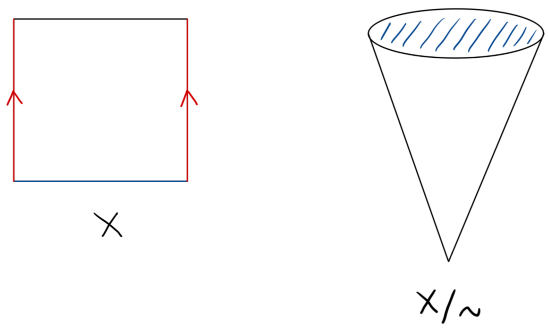
Example 2.25. Let $X = [0,1] \times [0,1]$. If we identify the two sides in the opposite orientation, we get the **Möbius Strip**.



Example 2.26. Let $X = [0,1] \times [0,1]$. If we identify one pair of sides in opposite orientation and the other one in the same orientation, we get the **Klein Bottle**.



Example 2.27. Let $X = [0,1] \times [0,1]$. If we do this we get a cone.



Lecture 9, 2024/09/23

3 Connectedness

3.1 Connected Spaces

Definition. Let $(X, \mathcal{T})$ be a topological space.

1. We say $X = U \cup V$ is a separation of X if $U, V \in \mathcal{T}$ and $U, V \neq \emptyset$ and $U \cap V = \emptyset$.

2. If a separation exists, we say X is **separated**.

3. Otherwise we say X is **connected**.

Example 3.1. Let $\mathbb{Q}$ be a topological subspace of $\mathbb{R}$, then $\mathbb{Q} = (-\infty, \pi) \cup (\pi, \infty)$, so $\mathbb{Q}$ is separated.

Example 3.2. Let $(\mathbb{R}, \mathcal{T}_f)$ with the finite complement topology. Then every two non-empty open sets intersect. So this space is connected.

Proposition 3.3. Let $(X, \mathcal{T})$, then X is connected $\iff$ clopen subsets of X are $\emptyset$ and X.

Proof: $(\Rightarrow)$. Suppose $A \subseteq X$ is clopen and $A \notin \{\emptyset, X\}$. Then $X = A \cup (X \setminus A)$, but both of these sets are open. So A is separated, contradiction.

$(\Leftarrow)$. Suppose $X = U \cup V$ is a separation. Then $U \in \mathcal{T}$ and $X \setminus U = V$ is open. Then U is clopen. Hence $U = X$ or $U = \emptyset$, which means $U \cup V$ is not a separation. $\square$

Lemma 3.4. Let $X = U \cup V$ be a separation. If $V \subseteq X$ (subspace topology) is connected, then $Y \subseteq U$ or $Y \subseteq V$.

Proof: First, $Y = (Y \cap U) \cup (Y \cap V)$ and $Y \cap U, Y \cap V$ are open in Y and are disjoint. Since Y is connected, so $Y \cap U = \emptyset$ or $Y \cap V = \emptyset$, and thus $Y \subseteq V$ or $Y \subseteq U$. $\square$

Proposition 3.5. Let $(X, \mathcal{T})$ and $A_\alpha \subseteq X$ be connected for $\alpha \in A$. Then:

$$\bigcap_{\alpha \in A} A_\alpha \neq \emptyset \implies \bigcup_{\alpha \in A} A_\alpha \text{ is connected}$$

Example 3.6. Let $A_n = (n, n + 0.5)$ is connected, but $\bigcup_{n \in \mathbb{N}} A_n$ is separated.

Proof: Let $Y = \bigcup_{\alpha \in A} A_\alpha$ and suppose $Y = U \cup V$ is a separation. WLOG say $p \in U$, where $p \in \bigcap_{\alpha \in A} A_\alpha$. By the lemma, $A_\alpha \subseteq U$ for all $\alpha \in A$. Hence $Y \subseteq U$ and $V = \emptyset$, contradiction. $\square$

Proposition 3.7. Let $(X, \mathcal{T})$ and $A \subseteq X$ is connected. If $A \subseteq B \subseteq \overline{A}$, then B is connected. In particular, $\overline{A}$ is connected.

Proof: Suppose $B = U \cup V$ is a separation of B. Then $A \subseteq B$ and A is connected, so WLOS $A \subseteq U$. Thus $\overline{A} \subseteq \overline{U}$. Note that $\overline{U} \cap V = \emptyset$. Indeed, if $x \in \overline{U} \cap V$, then $U \cap V \neq \emptyset$, contradiction. So $B \cap V = \emptyset$, so $V = \emptyset$. $\square$

Proposition 3.8. X connected and $f : X \to Y$ is continuous. Then $f(X)$ is connected.

Proof: Suppose $f(X) = U \cup V$ is a separation. Then:

$$X = \underbrace{f^{-1}(U) \cup f^{-1}(V)}_{\text{open, disjoint, non-empty}}$$

But this is a contradiction. $\square$

Remark (Optional Reading). Let $(X_\alpha, \mathcal{T}_\alpha)$ be connected, then $\prod_{\alpha \in A} X_\alpha$ is connected with respect to the product topology.

Definition. Let $(X, \mathcal{T})$ be a topological space. Define $x \sim y$ if and only if $C \subseteq X$ connected such that $x, y \in C$. Then $\sim$ is an equivalence relation.

Transitivity: If $x, y \in C_1$ and $y, z \in C_2$, then $x, z \in C_1 \cup C_2$. Then $C_1 \cup C_2$ is connected since $y \in C_1 \cap C_2 \neq \emptyset$ by Proposition 3.5.

The equivalence classes are called the **connected components of** X.

Remark. The components of X are pair-wise disjoint and partition X (by this equivalence relation).

Remark. If $A \subseteq X$ is connected, then $A \subseteq C$ for a unique component C.

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Proposition 3.9. The connected components of X are connected.

Proof: Let C be a connected component of X. Fix $x_0 \in C$. Then, for $x \in C$, we know $x \sim x_0$. There exists connected set $A_x \subseteq X$ such that $x, x_0 \in A_x$. By the remark, $A_x \subseteq C$. Hence:

$$C = \bigcup_{x \in C} A_x \quad \text{and} \quad x_0 \in \bigcap_{x \in C} A_x \neq \emptyset$$

Hence C is connected. $\square$

3.2 Path Connectedness

Definition. Let $(X, \mathcal{T})$ be a topological space.

1. A **path** from $a \in X$ to $b \in X$ is a continuous function:

$$f : [0,1] \to X$$

   such that $f(0) = a$ and $f(1) = b$.

2. We say X is **path connected** if for all $a, b \in X$ there exists a path from a to b in X.

Proposition 3.10. Path Connected $\implies$ Conncted.

Proof: Suppose X is path connected but $X = U \cup V$ is a separation. Take $a \in U$ and $b \in V$ and a path $f : [0,1] \to X$ from a to b. Then:

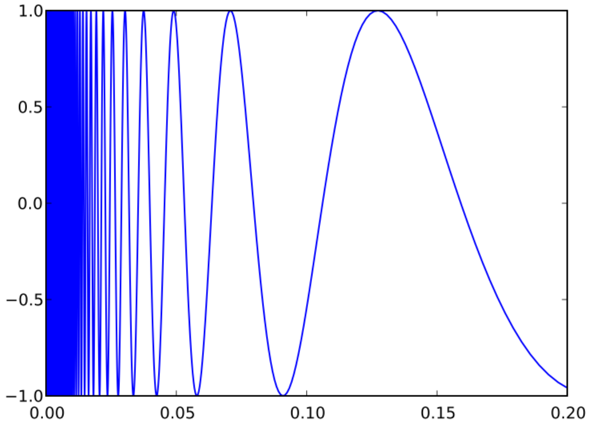
$$[0,1]^{-1} = \underbrace{f^{-1}(U)}_{0 \in} \cup \underbrace{f^{-1}(V)}_{1 \in}$$

Contradiction. $\square$

Example 3.11 (Topologist's Sine Curve). Let:

$$X = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) : 0 \leq x \leq 1 \right\} \cup \{(0,0)\}$$

Let the $A = X \setminus \{(0,0)\}$. Note that A is path connected, so A is connected. Hence $\overline{A} = X$ is connected.



Now we will show X is not path connected. Suppose for a contradiction that X is path connected and let f be a path with $f(0) = (0,0)$ and $f(1) = (1/\pi, 0)$. Write:

$$f(t) = (a(t), b(t))$$

The Interdemiate Value Theorem says that there exists $0 < t < 1$ such that $a(t_1) = 2/3\pi$. Again there exists $0 < t_2 < t_1$ such that $a(t_2) = 2/5\pi$. Continue this way, there exists a decreasing sequence $(t_n) \subseteq [0,1]$ such that:

$$a(t_n) = \frac{2}{(2n+1)\pi}$$

By MCT we have $t_n \to t \in [0,1]$. However $b(t_n) \to b(t)$ and:

$$b(t_n) = \sin\left(\frac{(2n+1)\pi}{2}\right) = (-1)^n$$

This is a contradiction since $b(t_n)$ diverges.

4 Compactness

4.1 Compact Spaces

Definition. Let $(X, \mathcal{T})$ be a topological space.

1. An **open cover** of X is a collection $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$, for $\alpha \in A$ such that $X = \bigcup_{\alpha \in A} U_\alpha$.
2. If $B \subseteq A$ and $X = \bigcup_{\alpha \in B} U_\alpha$, we call $\{U_\alpha\}_{\alpha \in B}$ a **subcover**. If $|B| < \infty$, we call it a **finite subcover**.

Definition. Let $(X, \mathcal{T})$ be a topological space. We say X is **compact** if every open cover $\{U_\alpha\}_{a \in A}$ has a finite subcover.

Big Idea: Compactness is a bridge to finiteness ("smallness").

Example 4.1. Let $(X, \mathcal{T}_f)$ with finite complement topology. Suppose $X = \bigcup_\alpha U_\alpha$ is an open cover and each U_α is non-empty. Fix U_0, then:

$$U_0 = X \setminus \{x_1, \cdots, x_n\}$$

Say $x_i \in U_i$, then $X = U_0 \cup U_1 \cup \cdots \cup U_n$, so X is compact.

Example 4.2. Let $(\mathbb{R}, \mathcal{T}_c)$ with countable complement topology:

$$U_n = \mathbb{R} \setminus \{n, n+1, n+2, \cdots\} \text{ and } \mathbb{R} = \bigcup_{n \in \mathbb{N}} U_n$$

But this admits no finite subcover. Suppose $\mathbb{R} = U_{n_1} \cup \cdots \cup U_{n_k}$ and $n_1 < \cdots < n_k$. Then $n_k \notin \mathbb{R}$, which is a contradiction.

Lemma 4.3 (Peter's Confusion). Let $(X, \mathcal{T})$ and $A \subseteq X$. Then A is compact (under the subspace topology) if and only if for all open cover $U_\alpha \in \mathcal{T}$ of X:

$$A \subseteq \bigcup_\alpha U_\alpha \implies A \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$$

for some $\alpha_1, \cdots, \alpha_n$.

Proof: $(\Rightarrow)$. Suppose $A \subseteq \bigcup_\alpha U_\alpha$, so $A = \bigcup_\alpha (A \cap U_\alpha)$. Hence:

$$A = (A \cap U_{\alpha_1}) \cup \cdots \cup (A \cap U_{\alpha_n}) = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$$

$(\Leftarrow)$. Suppose $A = \bigcup_\alpha (A \cup U_\alpha)$, so $A \subseteq \bigcup_\alpha U_\alpha$ and hence:

$$A \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$$

Hence $A = (A \cap U_{\alpha_1}) \cup \cdots \cup (A \cap U_{\alpha_n})$. $\square$

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Proposition 4.4. Let $(X, \mathcal{T})$ be compact. If $C \subseteq X$ is closed, then C is compact.

Proof: Suppose $C \subseteq \bigcup_\alpha U_\alpha$ with $U_\alpha \in \mathcal{T}$. Thus $X = (X \setminus C) \cup \bigcup_\alpha U_\alpha$. Since X is compact we have that:

$$X = (X \setminus C) \cup U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$$

Hence we have $C \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$ as desired. □

Example 4.5. $(R, \mathcal{T}_f)$ with countable-finite topology is compact. Exercise: All subsets of $\mathbb{R}$ are compact, so $\mathbb{N}$ is compact but NOT closed.

Proposition 4.6. Let $(X, \mathcal{T})$ be Hausdorff. If $K \subseteq X$ is compact, then K is closed.

Proof: Let $K \subseteq X$ be compact. We want to show that $X \setminus K$ is open. Fix $x_0 \in X \setminus K$. For all $x \in K$, there exists U_x and V_x in $\mathcal{T}$ such that $U_x \cap V_x = \emptyset$ and $x_0 \in U_x$ and $x \in V_k$. Then $K \subseteq \bigcup_{x \in K} V_x$, and since K is compact:

$$K \subseteq V_{x_1} \cup \cdots \cup V_{x_n}$$

Now consider $x_0 \in U := U_{x_1} \cap \cdots \cap U_{x_n} \in \mathcal{T}$. Notice that $x_0 \in U \subseteq X \setminus K$, hence $\operatorname{int}(X \setminus K) = X \setminus K$, so K is closed as desired. □

Proposition 4.7. Let X be Hausdorff and $K \subseteq X$ be compact. For all $x \in X \setminus K$, there exists $U, V \in \mathcal{T}$ such that $x \in U$ and $K \subseteq V$ and $U \cap V = \emptyset$.

Proposition 4.8. Let $(X, \mathcal{T})$ be compact and $f : X \to Y$ be continuous, then $f(X)$ is compact.

Proof: Suppose $f(X) \subseteq \bigcup_\alpha U_\alpha$ and $U_\alpha \subseteq Y$ is open. Then $X = \bigcup_\alpha f^{-1}(U_\alpha)$, hence:

$$X = f^{-1}(U_{\alpha_1}) \cup \cdots \cup f^{-1}(U_{\alpha_n}) \implies f(X) \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$$

As desired. □

Proposition 4.9. Let $(X, \mathcal{T})$ be compact and $(Y, \mathcal{U})$ be Hausdorff. If $f : X \to Y$ is continuous and bijective, then f is a homeomorphism.

Proof: We want to show that if $C \subseteq X$ is closed, then $(f^{-1})^{-c}(C) = f(C)$ is closed. Since X is compact, and $C \subseteq X$ is closed, C is compact. Since f is continuous so $f(C)$ is compact. However, Y is Hausdorff and so C is also closed.

4.2 Tychonoff's Theorem

Theorem 4.10 (Tychonoff's Theorem). If $(X_\alpha, \mathcal{T}_\alpha)$ is compact for each $\alpha \in A$, then $\prod_{\alpha \in A} X_\alpha$ is compact (with resepct to the product topology).

Fact. Tychonoff's Theorem is equivalent to the axiom of choice.

Definition. Let X be a set. We say $\leq$ is a **partial order** on X if:

(a) For all $x \in X$ we have $x \leq x$.

(b) For all $x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$.

(c) For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

We call $(X, \leq)$ a **partially ordered set (poset)**. Let X be a poset.

(a) We say $A \subseteq X$ is a **chain** if for all $a, b \in A$, we have $a \leq b$ or $b \leq a$.

(b) We say $x \in X$ is **maximal** if and only if for all $y \in X$, $x \leq y \implies x = y$.

(c) Let $A \subseteq X$ be a chain, an **upper bound** for A is any $x \in X$ such that $a \leq x$ for all $a \in A$.

Theorem 4.11 (Zorn's Lemma). Let $(X, \leq)$ be a poset. If every chain of X has an upper bound, then X has a maximal element.

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Definition. Let $\mathcal{C} \subseteq \mathcal{P}(X)$. We have $\mathcal{C}$ has the **finite intersection property (FIP)** for all $F_1, \cdots, F_n \in \mathcal{C}$ we have $F_1 \cap \cdots \cap F_n \neq \emptyset$.

Proposition 4.12. Let $(X, \mathcal{T})$. Then X is compact if and only only whenever $\mathcal{C}$ is a family of closed sets in X having FIP, we have $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

Proof: ($\Rightarrow$). Homework.

($\Leftarrow$). Suppose X satisfies the condition on such families of closed sets. Consider:

$$X = \bigcup_{U_\alpha \in U} U_\alpha \implies \emptyset = \bigcap_{U_a \in U} (X \setminus U_\alpha)$$

Therefore $\emptyset = (X \setminus U_{\alpha_1}) \cap \cdots \cap (X \setminus U_{\alpha_n})$, hence:

$$X = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$$

As desired. $\square$

Lemma 4.13. Let $(X, \mathcal{T})$. Let $\mathcal{C} \subseteq \mathcal{P}(X)$ be a family of closed sets having the FIP. There exists $\mathcal{C} \subseteq \mathcal{F} \subseteq \mathcal{P}(X)$ which is maximal with respect having the FIP.

Proof: Let $Y = \{\mathcal{K} \subseteq \mathcal{P}(X) : \mathcal{C} \subseteq \mathcal{K},\ \mathcal{K} \text{ has the FIP}\}$ and order Y via $\subseteq$. Note that $\mathcal{C} \in Y$ so $Y \neq \emptyset$. Let $S \subseteq Y$ be a chain. Consider $Z = \bigcup_{A \in S} A$. Note that $\mathcal{C} \subseteq Z$ since $Z \subseteq A$ for all $A \in S$.

Claim: Z has the FIP.

Proof (Claim): Let $F_1, \cdots, F_n \subseteq Z$. Say each $F_i \in A_i \in S$. Since S is a chain, WLOG suppose $A_i \subseteq A_1$ for all $i \in \{1, \cdots, n\}$. Then $F_1, \cdots, F_n \in A_1$. Then:

$$F_1 \cap \cdots \cap F_n \neq \emptyset$$

since $A_1 \in Y$ has the FIP. Therefore Z has the FIP. (QED Claim)

By the claim, Z is an upper bound for the chain S. Hence by Zorn's Lemma, Y has a maximal element $\mathcal{F}$ as desired. $\square$

Lemma 4.14. Let $(X, \mathcal{T})$. Let $\mathcal{C} \subseteq \mathcal{F} \subseteq \mathcal{P}(X)$ be as before.

(1) $\mathcal{F}$ is clsoed under finite intersections.

(2) If $A \subseteq X$ intersects every $F \in \mathcal{F}$, then $A \in \mathcal{F}$.

Proof: (1). Let $F_1, \cdots, F_n \in \mathcal{F}$, then $\mathcal{F} \cup \{F_1 \cap \cdots F_n\}$ has the FIP. By the maximality of $\mathcal{F}$ we get $\mathcal{F} \cup \{F_1 \cap \cdots F_n\} = \mathcal{F}$ and $F_1 \cap \cdots F_n \in \mathcal{F}$.

(2). Note that $\mathcal{F} \cup \{A\}$ has the FIP, then $A \in \mathcal{F}$. $\square$

Proof (Tychonoff Theorem): Let $\mathcal{C}$ be a family of closed sets in $X = \prod X_\alpha$ having the FIP. Consider $\mathcal{C} \subseteq \mathcal{F}$ maximal with respect to FIP. Define:

$$\mathcal{A}_\alpha = \{\pi_\alpha(F) : F \in \mathcal{F}\}$$

Claim 1: $\mathcal{A}_a$ has the FIP.

Proof (Claim 1): Suppose $\pi_\alpha(F_1) \cap \cdots \cap \pi_\alpha F(_n) = \emptyset$, so:

$$F_1 \cap \cdots \cap F_n \subseteq \pi_\alpha^{-1}(\pi_\alpha(F_1)) \cap \cdot \cap \pi_\alpha^{-1}(\pi_\alpha(F_n)) = \pi_\alpha^{-1}(\emptyset) = \emptyset$$

This is a contradiction. (QED Claim 1)

Claim 2: The intersection $\bigcap_{A \in \mathcal{A}_\alpha} \overline{A} \neq \emptyset$.

Proof (Claim 2): Suppose for a contradiction $\bigcap_{A \in \mathcal{A}_\alpha} \overline{A} = \emptyset$, then:

$$X_\alpha = \bigcup_{A \in \mathcal{A}_\alpha} (X_\alpha \setminus \overline{A}) = (X_\alpha \setminus \overline{A_1}) \cup \cdots \cup (X_\alpha \setminus \overline{A_n})$$

Hence $\overline{A_1} \cap \cdots \cap \overline{A_n} = \emptyset$ and $A_1 \cap \cdots \cap A_n = \emptyset$. Contradiction. (QED Claim 2)

By Claim 2, let $p_\alpha \in \bigcap_{A \in \mathcal{A}_\alpha} \overline{A}$. Consider $p \in X$ such that $\pi_\alpha(p) = p_\alpha$ for all α.

Claim 3: We have $p \in \bigcap_{F \in \mathcal{F}} \overline{F}$.

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Note that if we proved Claim 3, then we have:

$$p \in \bigcap_{F \in \mathcal{F}} \overline{F} \subseteq \bigcap_{C \in \mathcal{C}} C$$

and we are done.

Proof (Claim 3): Suppose $p \in U$ and:

$$U = \pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n})$$

where $U_{\alpha_i} \in \mathcal{T}_{\alpha_i}$. For $i = 1, \cdots, n$ we have:

$$\pi_{\alpha_i}(P) = P_{\alpha_i} \in U_{\alpha_i}$$

For all $A \in \mathcal{A}_{\alpha_i}$, we have $P_{\alpha_i} \in U_{\alpha_i} \cap \overline{A}$. Thus $U_{\alpha_i} \cap A \neq \emptyset$. Then for all $F \in \mathcal{F}$, there exists $z \in U_{\alpha_i} \cap \pi_{\alpha_i}(F)$. Say $z = \pi_{\alpha_i}(f)$ for some $f \in F$. Then $f \in \pi_{\alpha_i}^{-1}(U_{\alpha_i}) \cap F$. Therefore, for all $F \in \mathcal{F}$, we have $F \cap \pi_{\alpha_i}^{-1}(U_{\alpha_i}) \neq \emptyset$. By (2) of Lemma 4.14, we have $\pi_{\alpha_i}^{-1}(U_{\alpha_i}) \in \mathcal{F}$. By (1) of Lemma 4.14, we have $U \in \mathcal{F}$. Then for all $F \in \mathcal{F}$, $U \in \mathcal{F}$ so $U \cap F \neq \emptyset$ by FIP. Thus:

$$p \in \bigcap_{F \in \mathcal{F}} \overline{F}$$

Therefore we are done. (QED Claim 3). Hence we finished the proof. $\square$

5 Countability and Separation

Definition. Let $(X, \mathcal{T})$ be a topological space and fix $x \in X$. A **basis at** $x \in X$ is a collection $\mathcal{B}$ of neighborhoods of x such that whenever $x \in U \in \mathcal{T}$, then there exists $B \in \mathcal{B}$ such that $B \subseteq U$.

Definition. We say $(X, \mathcal{T})$ is **first countable** if for all $x \in X$, there exists a countable basis at x.

Example 5.1. Let (X, d) be a metric space. Fix $x \in X$, then:

$$\mathcal{B}_x = \{B_q(x) : q \in \mathbb{Q}^+\}$$

is a countable basis for x.

Idea: $(X, \mathcal{T})$ is first countable if and only if x has a strong relationship with countability.

Proposition 5.2. Let $(X, \mathcal{T})$ be first countable and $A \subseteq X$.

1. $x \in \overline{A} \iff$ there exists $(a_n) \subseteq A$ such that $a_n \to x$.
2. $f : X \to Y$ is continuous $\iff$ $x_n \to x$ in X implies $f(x_n) \to f(x)$.

Proof: (1). ($\Leftarrow$). See Assignment 1 because every sequence is a net.

($\Rightarrow$). Suppose $x \in \overline{A}$. Let $\mathcal{B} = \{B_n\}_{n=1}^{\infty}$ be a basis at x. This countable basis exists because $(X, \mathcal{T})$ is first countable. Take $a_1 \in B_1 \cap A$ and $a_2 \in B_1 \cap B_2 \cap A$. In general, choose:

$$a_n \in B_1 \cap \cdots \cap B_n \cap A$$

We claim that $a_n \to x$. Let $U \in \mathcal{T}$ with $x \in U$, there exists $N \in \mathbb{N}$ such that $B_N \subseteq U$. For all $n \geq N$ we get $a_n \in B_n \subseteq B_N \subseteq U$. As desired.

(2). ($\Leftarrow$). See Assignment 1 again.

($\Rightarrow$). Let $A \subseteq X$, we want to use Proposition 2.2 to prove f is continuous. We claim that:

$$f(\overline{A}) \subseteq \overline{f(A)}$$

Let $y \in f(A)$ so that $y = f(x)$ with $x \in \overline{A}$. By 1, there exists $(a_n) \subseteq A$ such that $a_n \to x$. Then $f(a_n) \to f(x)$ with each $f(a_n) \in f(A)$. Hence $y \in \overline{f(A)}$. $\square$

Definition. Let $(X, \mathcal{T})$ be a topological space. We say X is **second countable** if X has a countable basis. That is, there is a basis $\mathcal{B}$ with $|\mathcal{B}| \leq |\mathbb{N}|$.

Proposition 5.3. Second countable $\implies$ First countable.

Proof: Let $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ be a basis. For $x \in X$ define:

$$\mathcal{B}_x = \{B_n : x \in B_n\}$$

is a basis at x. $\square$

Example 5.4. Consider $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$, this is the metric topology by the discrete metric on $\mathbb{R}$:

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Therefore $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ is first countable. However, every basis for $\mathbb{R}$ must contain all $\{x\}$ for all $x \in \mathbb{R}$. Thus every basis for $\mathbb{R}$ is uncountable, so $\mathbb{R}$ is not second countable.

Definition. Let $(X, \mathcal{T})$ and $A \subseteq X$. We say A is **dense** in X if $\overline{A} = X$.

Definition. Let $(X, \mathcal{T})$. We say X is **separable** if X has a countable, dense subset.

Example 5.5. $\mathbb{Q} \subseteq \mathbb{R}$ and $\overline{\mathbb{Q}} = \mathbb{R}$, so $\mathbb{R}$ is separable.

Definition. We say $(X, \mathcal{T})$ is **Lindelöf** if every open cover of X has a countable subcover.

Proposition 5.6. If $(X, \mathcal{T})$ is second countable, then X is separable and Lindelöf.

Remark. The lower limit topology on $\mathbb{R}$ is separable and Lindelöf but not second countable.