PMATH 367 Notes

Fall 2024

Based on Professor Blake Madill's Lectures

Contents

Lecture 1, 2024/09/04

1 Topological Spaces

Motivation. Recall from analysis that:

(1) $A \subseteq \mathbb{R}^n$ is closed $\iff$ $\mathbb{R}^n \setminus A$ is open.

(2) $x_n \to x$ in $\mathbb{R}^n$ $\iff$ for all open set $U \subseteq \mathbb{R}^n$ with $x \in U$, $\exists N \in \mathbb{N}$ such that $n \geq N \implies x_n \in U$.

(3) $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous $\iff$ $f^{-1}(U)$ is open in $\mathbb{R}^n$ for all open $U \subseteq \mathbb{R}^m$.

(4) $A \subseteq \mathbb{R}^n$ is compact $\iff$ every open cover of A has a finite subcover.

Big Idea: All these concepts from analysis can be stated using open sets!

Recall. If X is a set, we define:

$$P(X) = \{A : A \subseteq X\}$$

to be the power set of X.

Definition. Let X be a set. We say $T \subseteq P(X)$ is a **topology** on X if:

(1) $\emptyset, X \in T$.

(2) If I is an index set and $A_\alpha \in T$ for all $\alpha \in I$, then $\bigcup_{\alpha \in I} A_\alpha \in T$. (Arbitrary Union)

(3) If $A, B \in T$, then $A \cap B \in T$. (Finite Intersection)

We call (X, T) a **topological space**. Moreover, we call the elements of T the **open sets** of X. And the **closed sets** of X are $X \setminus X$ for $A \in T$.

Big Idea: Topology is the study of topological spaces. It is the area of math which studies concepts like open and closed sets, continuity, compactness and connectedness.

Example 1.1. Let $X = \{a, b, c\}$. Define:

$$T_1 = \{\emptyset, X, \{a, b\}, \{c\}\}$$

$$T_2 = \{\emptyset, X, \{a, b\}, \{b, c\}, \{b\}\}$$

Then both T_1 and T_2 are topology on X.

Example 1.2. Let (X, d) be a metric space, then:

$$T = \{U \subseteq X : \forall\, x \in U, \exists\, r > 0, B_r(x) \subseteq U\}$$

is the metric topology on X.

Example 1.3. In the Example 1.1, it can be shown that T_1 is not a metric topology. That is, there is no metric d on X such that the open sets in (X, d) is T_1. Suppose there is a metric d on X, then there is $r_1, r_2, r_3 > 0$ such that:

$$B_{r_1}(a) = \{a\},\ B_{r_2}(b) = \{b\},\ B_{r_3}(c) = \{c\}$$

Thus the metric topology would be $P(X)$. But T_1 is not $P(X)$, so contradiction.

Definition. Let X be any set. $P(X)$ is called the **discrete topology** and $\{\emptyset, X\}$ is called the **indiscrete topology**.

Example 1.4. Let X be a set and let:

$$T_f = \{A \subseteq X : X \setminus A \text{ is finite}\} \cup \{\emptyset\}$$

is called the **finite complement topology**. Why?

(1) $X \setminus X = \emptyset$, so $X \in T_f$.

(2) $\emptyset \in T_f$ by definition.

(3) $A_\alpha \in T_f$ means $X \setminus A_\alpha$ is finite. Then:

$$X \setminus \bigcup_\alpha A_\alpha = \bigcap_\alpha (X \setminus A_\alpha)$$

is also finite. Hence $\bigcup_\alpha A_\alpha \in T_f$.

(4) If $A, B \in T_f$, then $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$. Each set is finite, so this is finite. Therefore we have $A \cap B \in T_f$.

Example 1.5. Let X be any set, then:

$$T_c = \{A \subseteq X : X \setminus A \text{ is at most countable}\} \cup \{\emptyset\}$$

is the **countable complemenet topology**.

Lecture 2, 2024/09/06y

Definition. Let X be a set. We say $\mathcal{B} \subseteq P(X)$ is a **basis** for a topology on X if:

(1) For all $x \in X$ there is $B \in \mathcal{B}$ such that $x \in B$.

(2) For all $x \in X$ such that $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Example 1.6. Let $X = \mathbb{R}$ and $\mathcal{B} = \{(a, b) : a < b\}$ is a basis for a topology on $\mathbb{R}$. (Open intervals).

Example 1.7. Let (X, d) be a metric space and $\mathcal{B} = \{B_r(x) : x \in X,\ r > 0\}$ is a basis for a topology on X. (All open balls).

Example 1.8. Let X be a set and $\mathcal{B} = \{\{x\} : x \in X\}$ is a basis for a topology on X.