Selberg's Sieve

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1 Introduction

Recall in the Sieve of Eratosthenes, we have the setup:

Definition. Let A be a finite subset of $\mathbb{N}$. Let P be a set of primes and let $z > 0$ be a real number. Define:

$$S(A, P, z) = \sum_{\substack{a \in A \\ (a, P(z)) = 1}} 1$$

where:

$$P(z) = \prod_{\substack{p \in P \\ p < z}} p$$

With these setup, we can deduce that:

$$S(A, P, z) = \sum_{a \in A} \sum_{d \mid (a, P(z))} \mu(d) \tag{1.1}$$

using the property of the Möbius function that:

$$\sum_{d \mid n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Selberg came up with this brilliant ideal to replace $\sum \mu(d)$ in (1.1) with a quadratic form, chosen optimally to make the result minimal. That is, let $(\lambda_d) \subseteq \mathbb{R}$ be a sequence such that $\lambda_1 = 1$, then:

$$\sum_{d \mid n} \mu(d) \leq \left(\sum_{d \mid n} \lambda_d \right)^2 \tag{1.2}$$

because the LHS is at most 1.

Recall the following setup we used to estimate $\pi(x)$. Let:

$$\pi(x, z) = \{n \leq x : p \mid n \Rightarrow p \geq z\}$$

be the number of $1 \leq n \leq x$ that are not divisible by any prime $p < z$. If we let $A = [1, x] \cap \mathbb{Z}$ and $P =$ all primes, then:

$$\pi(x, z) = S(A, P, z)$$

Then we have:

$$\pi(x, z) = \sum_{\substack{n \leq x \\ p \mid n \Rightarrow p \geq z}} 1 = 1 + \sum_{\substack{1 < n \leq z \\ p \mid n \Rightarrow p \geq z}} 1 + \sum_{\substack{z < n \leq x \\ p \mid n \Rightarrow p \geq z}} 1$$

The first sum is clearly 0. The second sum certainly counts all prime numbers p with $z < p \leq x$ and the number of such primes is $\pi(x) - \pi(z)$, hence:

$$\pi(x, z) \geq 1 + \pi(x) - \pi(z)$$

Rearrange them and use the fact that $\pi(z) \leq z$, we have:

$$\pi(x) \leq 1 + z + \pi(x, z) \tag{1.3}$$

Now it suffices to bound $\pi(x, z) = S(A, P, z)$. Let us see how to do this in full generality, then we come back to this problem.

2 Main Theorem

As always, let A, P, z be given as usual. For each $p \in P$, define:

$$A_p = \{a \in A : p \mid a\}$$

Moreover, for all squarefree integer d composed of primes in P, define $A_d = \bigcap_{p|d} A_p$. Suppose there is a multiplicative function f with $f(p) > 1$ for all $p \in P$, and for all d we have:

$$|A_d| = \frac{X}{f(d)} + R_d$$

to be the estimation of $|A_d|$, where X is an estimation of A and R_d is the error term.

Theorem 2.1 (**Selberg's Sieve).** With the setting above. Let f_1 be the unique function such that:

$$f(n) = \sum_{d|n} f_1(d)$$

Also, we define:

$$V(z) = \sum_{\substack{d \leq z \\ d|P(z)}} \frac{\mu^2(d)}{f_1(d)} \tag{2.1}$$

Then we have:

$$S(A, P, z) \leq \frac{X}{V(z)} + \left(\sum_{\substack{d_1, d_2 \leq z \\ d_1, d_2 | P(z)}} |R_{[d_1, d_2]}| \right) \tag{2.2}$$

Lemma 2.2. Let f_1, f_2 be a multiplicative function and d_1, d_2 be positive squarefree integers, then:

$$f([d_1, d_2]) f((d_1, d_2)) = f(d_1) f(d_2) \tag{2.3}$$

Proof of Selberg's Sieve: Let (λ_d) be a sequence of real numbers with $\lambda_1 = 1$ and $\lambda_d = 0$ for all $d > z$. Then by (1.2) we have:

$$S(A, P, z) = \sum_{\substack{a \in A \\ (a, P(z)) = 1}} 1 = \sum_{a \in A} \sum_{d | (a, P(z))} \mu(d) \leq \sum_{a \in A} \left(\sum_{d | (a, P(z))} \lambda_d \right)^2$$

References

[1] Cojocaru, A.C. and Murty, M.R., An Introduction to Sieve Methods and their Applications. London Mathematical Society 66. Cambridge University Press, 2006.